MA3120: Linear Algebra - Spring 2012
Exam 1

Your Name:

Names of any collaborators:

Instructions

This exam is worth a total of 68 points and 20% of your overall grade. Please read the instructions for each question carefully.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Show *all* of your work and *justify* your solutions fully. If you use a calculator or computer software (e.g., Sage), be sure to write down both the input and output.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. Unless you prove them, you cannot use any results from the course notes or book that we have not yet covered.
2. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
3. You are **NOT** allowed to copy someone else's work.
4. You are **NOT** allowed to let someone else copy your work.
5. You are allowed to discuss the problems with each other and critique each other's work.

The exam is due to my office by 5PM on **Wednesday, March 7**. You should turn in this cover page and all of the work that you have decided to submit.

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

1. (2 points each) Suppose A is a coefficient matrix corresponding to a system of linear equations that is row equivalent to one of the following matrices.

$$M_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

For each of the following statements, state whether A is row equivalent to M_1, M_2, or M_3. If there is more than one correct answer, then list them all. If none of M_1, M_2, or M_3 satisfy the given conditions, then state this. You do *not* need to justify your answers.

(a) The linear system $\mathcal{LS}(A, \vec{0})$ has only the trivial solution.
(b) There exists at least one vector $\vec{b} \in \mathbb{R}^3$ such that $\mathcal{LS}(A, \vec{b})$ does *not* have a solution.
(c) The linear system $\mathcal{LS}(A, \vec{b})$ has a unique solution for all $\vec{b} \in \mathbb{R}^3$.
(d) The matrix A is singular.
(e) The matrix A is nonsingular.
(f) The span of the columns of A equals $\mathbb{R}^4$.
(g) The span of the columns of A equals $\mathbb{R}^3$.
(h) The planes corresponding to the 3 linear equations all intersect at a unique point.
(i) The planes corresponding to the 3 linear equations all intersect in a line.
(j) The planes corresponding to the 3 linear equations do not have a common intersection.
(k) The null space of A contains infinitely many vectors.
(l) There exists $\vec{b} \in \mathbb{R}^3$ such that $\vec{b} \notin \mathcal{N}(A)$.

2. (4 points) Consider the following matrix A. Doing all the computations by hand, convert A to a matrix in reduced row-echelon form. Clearly indicate which row operations you are performing.

$$A = \begin{bmatrix} 1 & 2 & -4 & -4 \\ 1 & 1 & -3 & -3 \\ -2 & -1 & 6 & 8 \end{bmatrix}$$

3. (4 points) Find all solutions to the system of equations below. Express your solution as a **set of column vectors**.

$$\begin{aligned} 2x_1 + 4x_2 + x_3 + 13x_4 &= -1 \\ 2x_1 + 4x_2 + 5x_3 + 25x_4 &= 0 \\ -2x_1 - 4x_2 \qquad - 10x_4 &= 1 \end{aligned}$$

4. (4 points) Find all solutions to the system of equations below. Express your solution as a **set of column vectors**.

$$\begin{aligned} -x_1 + 5x_2 &= 8 \\ -x_1 + 4x_2 &= 7 \\ 2x_1 - 3x_2 &= -9 \\ -1x_1 + 7x_2 &= 10 \end{aligned}$$

5. (4 points) Find all solutions to the system of equations below. Express your solution in **vector form** (i.e., as a linear combination of vectors).

$$\begin{aligned} -2x_1 - 6x_2 + x_3 + 4x_4 + 9x_5 &= 7 \\ 3x_1 + 9x_2 + 5x_3 + 7x_4 - 7x_5 &= 9 \end{aligned}$$

6. (4 points) Determine whether the following matrix B is singular or nonsingular. Justify your answer.

$$B = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 3 & 3 & -1 & 0 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 4 & 5 \end{bmatrix}$$

7. (4 points) Let $\vec{x} = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$ and let $U = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \right\}$. Is $\vec{x} \in \langle U \rangle$?

8. (4 points each) Consider the following matrix C.

$$C = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 2 & 1 & 7 & -1 \\ -3 & 4 & 6 & 7 \end{bmatrix}$$

   (a) Find a set of vectors whose span is equal to $\mathcal{N}(C)$.

   (b) Determine whether $\vec{z} = \begin{bmatrix} -7 \\ -3 \\ 2 \\ -3 \end{bmatrix}$ is an element of $\mathcal{N}(C)$.

   (c) If possible, write $\vec{z}$ as a linear combination of the vectors you found in part (a). If this is not possible, explain why.

9. (4 points) Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^m$. Prove that $\langle \{\vec{v}_1, \vec{v}_2\} \rangle = \langle \{\vec{v}_1, \vec{v}_2, 2\vec{v}_1 - 3\vec{v}_2\} \rangle$. (*Hint:* Argue that each set contains the other.)

10. (4 points) Complete either of T21 or T22 from SS.EXC. Be sure to state which one you are doing.