Chapter 8: The power of homomorphisms

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The corresponding homomorphisms are called embeddings and quotient maps.

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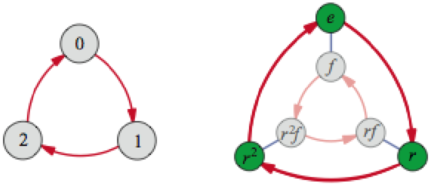

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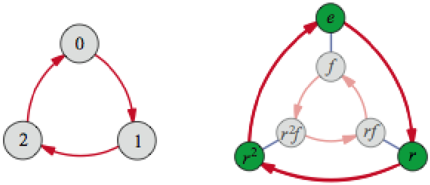

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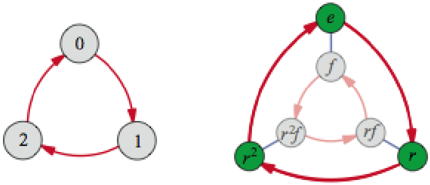

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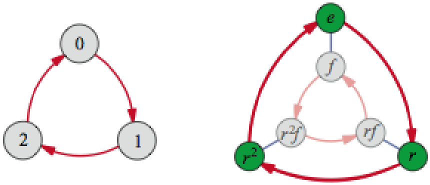

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When we say $C_3 < S_3$, what we really mean is that the structure of C_3 shows up in S_3.

In particular, there is a 1-1 correspondence between the elements in C_3 and the elements of the 3-step cyclic subgroup in S_3, and furthermore, the relationship between the corresponding nodes is the same.

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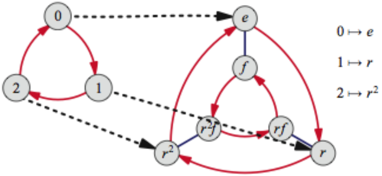

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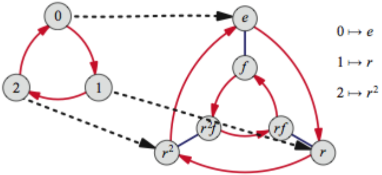

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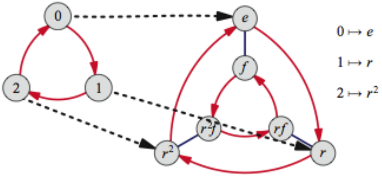

Homomorphisms are the mathematical tool for succinctly expressing precise structural correspondences. Because homomorphisms describe how elements of one group correspond to elements of another, they are a kind of function.

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Note that the operation $a \cdot b$ is occurring in the domain while $\phi(a) \cdot \phi(b)$ occurs in the codomain.

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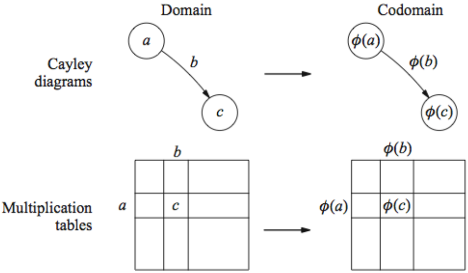

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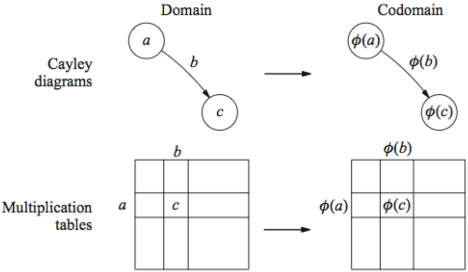

We say that the homomorphism is structure preserving.

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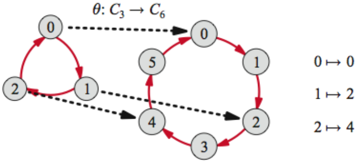

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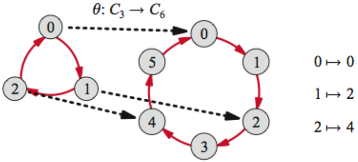

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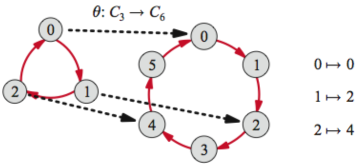
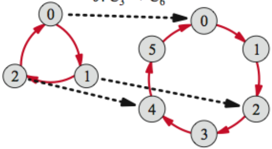


But is this really a homomorphism? We need to verify that any path from a to b in C_3 corresponds to the path from $\theta(a)$ to $\theta(b)$ in C_6.

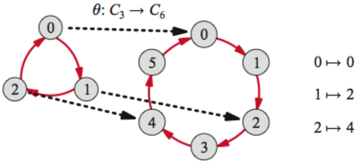
$\theta\colon C_3 \to C_6$



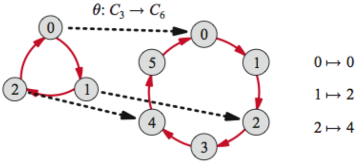
$0 \mapsto 0$

$1 \mapsto 2$

$2 \mapsto 4$



θ maps 1 to 2, but it also maps the 1-arrow in C_3 to the 2-step path representing 2 in C_6.



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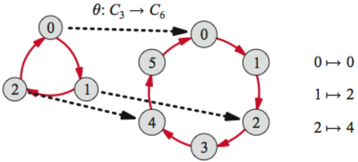

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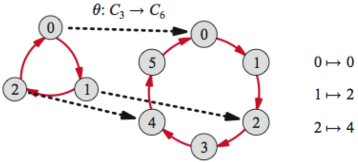

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We can think of the $\theta(1)$ arrow as the path consisting of two red arrows in succession. So, θ doubles both numbers and arrows.

What this last example illustrates to us is that if we know where a homomorphism maps *all* of the domains arrows (i.e., generators), then we know where it maps the rest of the nodes.

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For example, suppose there was another homomorphism $\theta' : C_3 \to C_6$ different from θ such that $\theta'(1) = 4.$ Using this information, we can construct the rest of the homomorphism using the fact that θ' must obey $\theta'(a + b) = \theta'(a) + \theta'(b)$.

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One consequence of this is that the identity of the domain must *always* map to the identity in the codomain (see Exercise 8.7).

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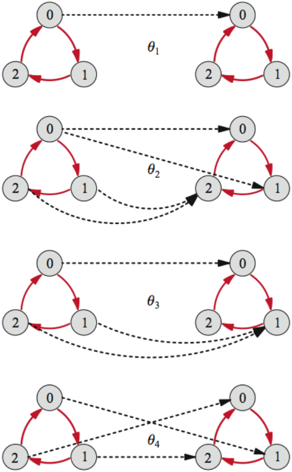

A function maps *each* element of the domain to some element of the codomain. But θ_1 ignores two elements of the domain, and thus is not a function.

A function maps each element of the domain to *one* element of the codomain. But θ_2 maps 0 to two different elements of the domain, and thus is not a function.

Although θ_3 is a function, it is not a homomorphism. In the domain, $1+1=2$, but in the codomain, $\theta_3(1)+\theta_3(1) \neq \theta_3(2)$.

Although θ_4 is a function, it is not a homomorphism. Although the image is a 3-cycle like the domain, θ_4 does not map the domain identity element to the codomain identity element. For example, $\theta_4(0)+\theta_4(1) \neq \theta_4(1)$.

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Because any embedding finds a copy of the domain in the codomain, its image is therefore the same size as its domain. So, embeddings never map 2 different elements of the domain to the same element in codomain (i.e., they are 1-1).

An embedding whose image fills the whole codomain shows us that the domain and codomain are actually the same size and have all the same structure. In this case, we say that the function is not just a homomorphism, but an isomorphism.

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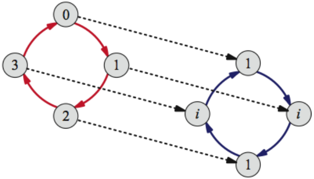

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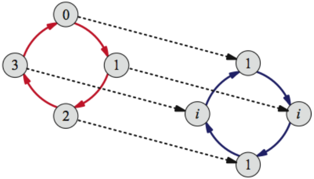

In this case, $C_4 \cong \{i, -1, i, -i\}$.

Quotient maps

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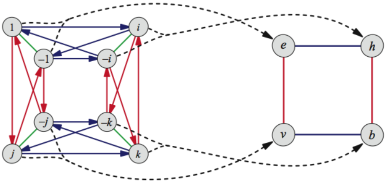
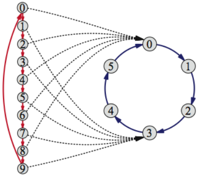
$\tau_1 : Q_4 \to V_4$



$$\tau_2 : C_{10} \to C_6 \text{ by } \tau_2(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$



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General facts about quotient maps (cont)

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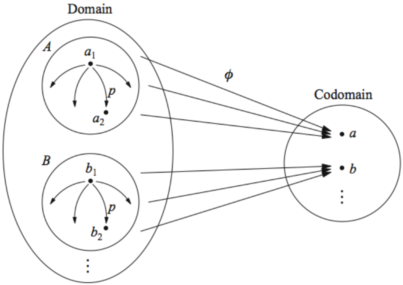

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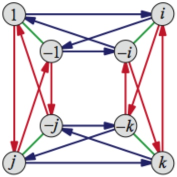

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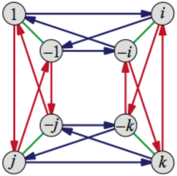

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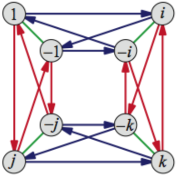

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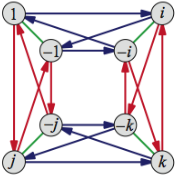

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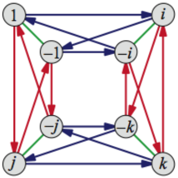

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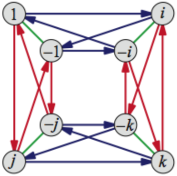

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What is $Q_4/Ker(\phi)$? Do you notice any relationship between $Q_4/Ker(\phi)$ and $Im(\phi)$?

The Fundamental Homomorphism Theorem

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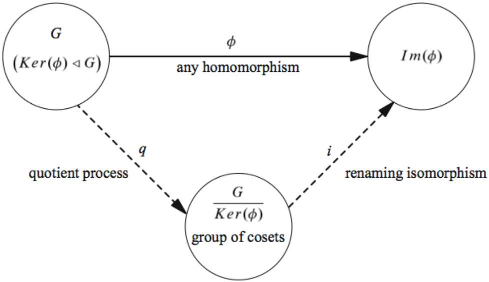

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Let's take a look at one last example.

The following figure (taken from Figure 8.18 on page 172 of *VGT*) illustrates an isomorphism between C_{12} and $\mathbb{Z}/\langle 12 \rangle$.

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