

MA3120: Linear Algebra - Spring 2012

Exam 2

Your Name:

Names of any collaborators:

Instructions

This exam is worth a total of 78 points and 20% of your overall grade. Please read the instructions for each question carefully.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the “polished” version of potentially several drafts.

Show *all* of your work and *justify* your solutions fully. If you use a calculator or computer software (e.g., Sage), be sure to write down both the input and output.

Feel free to type up your final version. The \LaTeX source file of this exam is also available if you are interested in typing up your solutions using \LaTeX . I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. Unless you prove them, you cannot use any results from the course notes or book that we have not yet covered.
2. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
3. You are **NOT** allowed to copy someone else's work.
4. You are **NOT** allowed to let someone else copy your work.
5. You are allowed to discuss the problems with each other and critique each other's work.

The exam is due to my office by 5PM on **Wednesday, April 18**. You should turn in this cover page and all of the work that you have decided to submit.

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

1. (2 points each) Suppose A is a matrix that is row equivalent to one of the following matrices.

$$M_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

For each of the following statements, state whether A is row equivalent to M_1 , M_2 , or M_3 . If there is more than one correct answer, then list them all. If none of M_1 , M_2 , or M_3 satisfy the given conditions, then state this. You do *not* need to justify your answers.

- (a) The columns of A are linearly independent.
 - (b) The columns of A span \mathbb{R}^3 .
 - (c) The columns of A^T are linearly independent.
 - (d) The columns of A^T span \mathbb{R}^3 .
 - (e) The columns of A form a basis for \mathbb{R}^3 .
 - (f) The null space of A equals $\{\vec{0}\}$.
 - (g) The null space of A has a basis consisting of exactly one non-zero vector.
 - (h) A is invertible.
 - (i) The column space of A has a basis consisting of exactly two non-zero vectors.
 - (j) The row space of A equals \mathbb{R}^3 .
 - (k) The row space of A has a basis consisting of exactly two non-zero vectors.
 - (l) The matrix-vector equation $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^3$.
2. (4 points each) Consider the following collection of vectors.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

- (a) Find the values of h for which the vectors are linearly *dependent*. Be sure to justify your answer.
 - (b) Find the values of h for which $\langle \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \rangle = \mathbb{R}^3$. Again, be sure to justify your answer.
3. (4 points each) Consider the following matrix.

$$A = \begin{bmatrix} -1 & -2 & 2 & 1 & 5 \\ 1 & 2 & 1 & 1 & 5 \\ 3 & 6 & 1 & 2 & 7 \\ 2 & 4 & 0 & 1 & 2 \end{bmatrix}$$

- (a) Find a basis for $\mathcal{N}(A)$.
- (b) Find a basis for $\mathcal{R}(A)$.
- (c) Find a basis for $\mathcal{C}(A)$.
- (d) Using your answer from part (c), find an orthonormal basis for $\mathcal{C}(A)$.

4. The following statement is **FALSE** in general.

If A is an $m \times n$ matrix such that $A\vec{x} = \vec{0}$ has only the trivial solution, then $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^m$.

- (a) (4 points) Provide a counterexample where $m = 3$ and $n = 2$ and briefly justify your answer. Your counterexample should consist of a 3×2 matrix A and a vector \vec{b} .
- (b) (2 points) The above statement is **TRUE** if $m = n$. Briefly justify this fact.
5. (4 points) Let $p_1(x) = 1 + x^2, p_2(x) = x + x^2, p_3(x) = 1 + 2x + x^2 \in P_2$. Determine whether $\{p_1(x), p_2(x), p_3(x)\}$ is linearly independent in P_2 . You must justify your answer.
6. (4 points) Let

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 \leq x_2 \right\}.$$

Determine whether H is a subspace of \mathbb{R}^2 . Be sure to justify your answer.

7. (4 points) Let

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 2x_1 + 5x_2 - 7x_3 = 0 \right\}.$$

Determine whether W is a subspace of \mathbb{R}^3 . Be sure to justify your answer.

8. (4 points) Find a linearly independent spanning set for the following subspace of \mathbb{R}^3 .

$$X = \left\langle \left\{ \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \right\} \right\rangle$$

9. (4 points) Let

$$S = \left\{ \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 2 & 6 \end{bmatrix} \right\}$$

and let

$$M = \begin{bmatrix} 6 & 6 \\ -2 & 14 \end{bmatrix}.$$

Determine whether M is in the span of S . If so, write M as a linear combination of the matrices in S .

10. (4 points) Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for a vector space V . Prove that $\{\vec{v}_1 + \vec{v}_2 + \vec{v}_3, \vec{v}_1 + \vec{v}_2, \vec{v}_3\}$ is a spanning set for V .*
11. **Bonus Question** (4 points) Consider $\mathcal{R}(A)$ and $\mathcal{N}(A)$ from Problem 3. Prove that any vector from $\mathcal{R}(A)$ is orthogonal to any vector in $\mathcal{N}(A)$.

*It turns out that the second set is also linearly independent, but I'm not asking you to prove this fact. The upshot is that the second set is also a basis for V .