Chapter 8

Cardinality

In this chapter, we will explore the notion of cardinality, which formalizes what it means for two sets to be the same "size".

8.1 Introduction to Cardinality

What does it mean for two sets to have the same "size"? If the sets are finite, this is easy: just count how many elements are in each set. Another approach would be to pair up the elements in each set and see if there are any left over. In other words, check to see if there is a one-to-one correspondence (i.e., bijection) between the two sets.

But what if the sets are infinite? For example, consider the set of natural numbers $\mathbb{N}$ and the set of even natural numbers $2\mathbb{N} := \{2n \mid n \in \mathbb{N}\}$. Clearly, $2\mathbb{N}$ is a proper subset of $\mathbb{N}$. Moreover, both sets are infinite. In this case, you might be thinking that $\mathbb{N}$ is "larger than" $2\mathbb{N}$ However, it turns out that there is a one-to-one correspondence between these two sets. In particular, consider the function $f : \mathbb{N} \to 2\mathbb{N}$ defined via $f(n) = 2n$. It is easily verified that f is both one-to-one and onto. In this case, mathematics has determined that the right viewpoint is that $\mathbb{N}$ and $2\mathbb{N}$ do have the same "size". However, it is clear that "size" is a bit too imprecise when it comes to infinite sets. We need something more rigorous.

Definition 8.1. Let A and B be sets. We say that A and B have the same **cardinality** iff there exists a one-to-one correspondence between A and B. If A and B have the same cardinality, then we write $\boxed{\operatorname{card}(A) = \operatorname{card}(B)}$.

Problem 8.2. Prove each of the following. In each case, you should create a bijection between the two sets. Briefly justify that your functions are in fact bijections.

(a) Let $A = \{a, b, c\}$ and $B = \{x, y, z\}$. Then $\operatorname{card}(A) = \operatorname{card}(B)$.

(b) Let $\mathcal{O}$ denote the set of odd natural numbers. Then $\operatorname{card}(\mathbb{N}) = \operatorname{card}(\mathcal{O})$.

(c) $\operatorname{card}(\mathbb{N}) = \operatorname{card}(\mathbb{Z})$.

(d) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Then $\text{card}((a,b)) = \text{card}((c,d))$.[1]

(e) Let $R = \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$. Then $\text{card}(\mathbb{N}) = \text{card}(R)$.

(f) Let $\mathcal{F}$ be the set of functions from $\mathbb{N}$ to $\{0,1\}$. Then $\text{card}(\mathcal{F}) = \text{card}(\mathcal{P}(\mathbb{N}))$.[2]

(g) Let A be any set. Then $\text{card}(A) = \text{card}(A \times \{x\})$.

Theorem 8.3. Let A, B, and C be sets. Then we have the following:

(a) $\text{card}(A) = \text{card}(A)$.

(b) If $\text{card}(A) = \text{card}(B)$, then $\text{card}(B) = \text{card}(A)$.

(c) If $\text{card}(A) = \text{card}(B)$ and $\text{card}(B) = \text{card}(C)$, then $\text{card}(A) = \text{card}(C)$.

In light of the previous theorem, the next result should not be surprising.

Corollary 8.4. If X is a set, then "has the same cardinality as" is an equivalence relation on $\mathcal{P}(X)$.

Theorem 8.5. Let A, B, C, and D be sets such that $\text{card}(A) = \text{card}(C)$ and $\text{card}(B) = \text{card}(D)$.

(a) If A and B are disjoint and C and D are disjoint, then $\text{card}(A \cup B) = \text{card}(C \cup D)$.

(b) $\text{card}(A \times B) = \text{card}(C \times D)$.

Given two finite sets, it makes sense to say that one set is "larger than" another provided one set contains more elements than the other. We would like to generalize this idea to handle both finite and infinite sets.

Definition 8.6. Let A and B be sets. If there is a one-to-one function (i.e., injection) from A to B, then we say that the **cardinality of** A **is less than or equal to the cardinality of** B. In this case, we write $\boxed{\text{card}(A) \leq \text{card}(B)}$.

Theorem 8.7. Let A, B, and C be sets. Then we have the following:

(a) If $A \subseteq B$, then $\text{card}(A) \leq \text{card}(B)$.

(b) If $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

(c) If $C \subseteq A$ while $\text{card}(B) = \text{card}(C)$, then $\text{card}(B) \leq \text{card}(A)$.

It might be tempting to think that the existence of a one-to-one function from a set A to a set B that is *not* onto would verify that $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(A) \neq \text{card}(B)$. While this is true for finite sets, it is not true for infinite sets as the next exercise asked you to verify.

[1] *Hint:* Try creating a linear function $f : (a,b) \to (c,d)$. Drawing a picture should help.
[2] *Hint:* Define $\phi : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ so that $\phi(f)$ outputs a subset of $\mathbb{N}$ determined by when f outputs a 1.

Exercise 8.8. Provide an example of sets A and B such that $\text{card}(A) = \text{card}(B)$ despite the fact that there exists a one-to-one function from A to B that is not onto.

Definition 8.9. Let A and B be sets. We write $\boxed{\text{card}(A) < \text{card}(B)}$ provided $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(A) \neq \text{card}(B)$.

It is important to point out that the statements $\text{card}(A) = \text{card}(B)$ and $\text{card}(A) \leq \text{card}(B)$ are symbolic ways of asserting the existence of certain types of functions from A to B. When we write $\text{card}(A) < \text{card}(B)$, we are saying something much stronger than "There exists a function $f : A \to B$ that is one-to-one but not onto." The statement $\text{card}(A) < \text{card}(B)$ is asserting that *every* one-to-one function from A to B is not onto. In general, it is difficult to prove statements like $\text{card}(A) \neq \text{card}(B)$ or $\text{card}(A) < \text{card}(B)$.

8.2 Finite Sets

In the previous section, we used the phrase "finite set" without formally defining it. Let's be a bit more precise.

Definition 8.10. For each $n \in \mathbb{N}$, define $[n] = \{1, 2, \ldots, n\}$.

For example, $[5] = \{1, 2, 3, 4, 5\}$. Notice that our notation looks just like that for the set of relatives given a relation on some set (see Definition 6.33), which is an equivalence class if the relation happens to be an equivalence relation. However, despite the similar notation, these concepts are unrelated. We will have to rely on context to keep them straight.

The next definition should coincide with your intuition about what it means for a set to be finite versus infinite.

Definition 8.11. A set A is **finite** iff $A = \emptyset$ or $\text{card}(A) = \text{card}([n])$ for some $n \in \mathbb{N}$. If $A = \emptyset$, then we say that A has **cardinality** 0 and if $\text{card}(A) = \text{card}([n])$, then we say that A has **cardinality** n. A set A is **infinite** iff A is not finite.

Let's prove a few results about finite sets.

Theorem 8.12. If A is finite and $\text{card}(A) = \text{card}(B)$, then B is finite.[3]

Theorem 8.13. For every $n \in \mathbb{N}$, every subset of $[n]$ is finite.[4]

Theorem 8.14. If A has cardinality $n \in \mathbb{N} \cup \{0\}$ and $x \notin A$, then $A \cup \{x\}$ is finite and has cardinality $n + 1$.

The previous theorem shows that adding a single element to a finite set increases the cardinality by 1. As you would expect, removing one element from a finite set decreasing the cardinality by 1.

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[3] Don't forget to consider the case when $A = \emptyset$.

[4] *Hint:* Use induction.

Theorem 8.15. If A has cardinality $n \in \mathbb{N}$, then for all $x \in A$, $A \setminus \{x\}$ is finite and has cardinality $n-1$.

The next result will not come as a surprise. The proof is not complicated, but is not immediate either. It is a consequence of Theorems 8.13 and 8.14.

Theorem 8.16. Every subset of a finite set is finite.

Theorem 8.17. If $A_1, A_2, \ldots, A_k$ is a finite collection of finite sets, then $\displaystyle\bigcup_{i=1}^{k} A_i$ is finite.[5]

The next theorem, called the Pigeonhole Principle, is surprisingly useful. It puts restrictions on when we may have a one-to-one function. The name of the theorem is inspired by the following idea: If n pigeons wish to roost in a house with k pigeonholes and $n > k$, then is must be the case that at least one hole contains more than one pigeon.

Theorem 8.18 (Pigeonhole Principle)**.** If $n, k \in \mathbb{N}$ and $f : [n] \to [k]$ with $n > k$, then f is not one-to-one.[6]

The last theorem of this section tells us that the cardinality of a proper subset of a finite set is never the same as the cardinality of the original set. It turns out that this theorem does not hold for infinite sets.

Theorem 8.19. If A is a finite set, then $\mathrm{card}(B) < \mathrm{card}(A)$ for all proper subsets B of A.

8.3 Infinite Sets

Coming soon...

8.4 Countable Sets

Coming soon...

8.5 Uncountable Sets

Coming soon...

[5] *Hint:* Use induction.
[6] *Hint:* Induct on the number of pigeons. The base case is $n = 2$.