Section 2.3: Extended Set Operations and Families of Sets

Families of Sets

Recall that not every collection of objects forms a ____________. We did this to avoid paradoxes.

Definition 1. A *family of sets* is a collection of sets.

Note 2. Here are a few comments:

1. Most families are also sets, but not all. All of the families that we'll talk about will also be sets.
2. Any power set is an example of a family of sets.
3. We'll usually use script letters for families.

Example 3.

(a) Let $A = \{a, b\}$. Then
$$\mathcal{P}(A) = \underline{\hspace{8cm}}.$$
$\mathcal{P}(A)$ is an example of a family.

(b) Let $\mathcal{A} = \{\{1,2,3\}, \{3,4,5\}, \{3,6\}\}$. This is also a family of sets. Note that $2\_\_\{1,2,3\}$ and $\{1,2,3\}\_\_\mathcal{A}$, but $2\_\_\mathcal{A}$.

Definition 4. Let $\mathcal{A}$ be a family of sets. The *union over* $\mathcal{A}$ is
$$\bigcup_{A\in\mathcal{A}} A = \underline{\hspace{10cm}}.$$
Similarly, the *intersection over* $\mathcal{A}$ is
$$\bigcap_{A\in\mathcal{A}} A = \underline{\hspace{10cm}}.$$

Example 5.

(a) Let $\mathcal{A} = \{\{1,2,3\}, \{3,4,5\}, \{3,6\}\}$. Then
$$\bigcup_{A\in\mathcal{A}} A = \underline{\hspace{10cm}} = \underline{\hspace{3cm}}$$
and
$$\bigcap_{A\in\mathcal{A}} A = \underline{\hspace{10cm}} = \underline{\hspace{3cm}}.$$

(b) Let $\mathcal{B}$ be set of all open intervals of the form $(-b, b)$, where b is a positive real number. Then

$$\bigcup_{B\in\mathcal{B}} B = \underline{\hspace{8cm}} = \underline{\hspace{3cm}}$$

and

$$\bigcap_{B\in\mathcal{B}} B = \underline{\hspace{8cm}} = \underline{\hspace{3cm}}.$$

Note 6.

1. $x \in \bigcup_{A\in\mathcal{A}} A$ iff $(\exists\ A)(A \in \mathcal{A} \wedge x \in A)$
2. $x \in \bigcap_{A\in\mathcal{A}} A$ iff $(\forall\ A)(A \in \mathcal{A} \implies x \in A)$

Theorem 7 (2.8)**.** *For every set B in a family $\mathcal{A}$,*

(a) $\bigcap_{A\in\mathcal{A}} A \subseteq B$*;*

(b) $B \subseteq \bigcup_{A\in\mathcal{A}} A$*;*

(c) If $\mathcal{A}$ is nonempty, then

$$\bigcap_{A\in\mathcal{A}} A \subseteq \bigcup_{A\in\mathcal{A}} A.$$

Proof.

(a)

(b) See Exercise 2.3.3

(c)

$\square$

Indexing Sets

It is often useful to "tag" the sets in a family.

Definition 8. Let Δ be a nonempty set such that for each $\alpha \in \Delta$, there is a corresponding A_α. The set

$$\{A_\alpha : \alpha \in \Delta\}$$

is called an *indexed family of sets*. The set Δ is called the *index set* and each α is called an *index*.

Intuitively, you should think of each α as being a "tag" and Δ as being the collection of all "tags."

Example 9.

(a) Let $A_1 = \{1, 2\}$, $A_2 = \{2, 3, 4\}$, and $A_3 = \{3, 4, 5, 6\}$. Also, let

$$\mathcal{A} = \{A_1, A_2, A_3\}.$$

In this case, $\Delta =$ ____________.

(b) Let $\Delta = \mathbb{N}$ and for each $n \in \Delta$, define $A_n = \{n, n^2, n^3\}$. Now, let

$$\mathcal{A} = \{A_n : n \in \Delta\}.$$

Describe $\mathcal{A}$.

Here is a potentially useful analogy:

Imagine you have a filing cabinet filled with file folders. Some of the folders have a lot of stuff in them, some may be empty, and maybe some of them contain duplicate information. In this case, the cabinet is the family, each folder is a set in the family, and each item in a folder represents an element of one of the sets. The labels on the folders are the indices. If the folders are unlabeled, then the family is not indexed.

Note 10.

1. Every family of sets can be turned into an indexed family of sets if you find a large enough index set.

2. An index set may be finite or infinite.

3. Generally, we want our indexing to be unique, but it doesn't have to be. That is, a set may get "tagged" by more than one index. For example, you could have $A_3 = A_{126}$.

4. If a family is indexed by Δ, then we may write

$$\bigcup_{A\in\mathcal{A}} A = \underline{\hspace{3cm}}$$

and

$$\bigcap_{A\in\mathcal{A}} A = \underline{\hspace{3cm}}.$$

In particular, if $\Delta = \mathbb{N}$, then

$$\bigcup_{A\in\mathcal{A}} A = \underline{\hspace{3cm}}$$

and

$$\bigcap_{A\in\mathcal{A}} A = \underline{\hspace{3cm}}.$$

Example 11.

(a) Let $A_n = [0, \frac{1}{n})$. Then

$$\bigcup_{i=1}^{6} A_i = \underline{\hspace{9cm}} = \underline{\hspace{3cm}}.$$
$$\bigcup_{i=1}^{\infty} A_i = \underline{\hspace{9cm}} = \underline{\hspace{3cm}}.$$
$$\bigcap_{i=1}^{6} A_i = \underline{\hspace{9cm}} = \underline{\hspace{3cm}}.$$
$$\bigcap_{i=1}^{\infty} A_i = \underline{\hspace{9cm}} = \underline{\hspace{3cm}}.$$

(b) Let $B_x = [x^2, x^2 + 1]$ for each $x \in \mathbb{R}$. Then

$$B_{1/2} = \underline{\hspace{3cm}}$$
$$B_{-1/2} = \underline{\hspace{3cm}}$$
$$\bigcup_{x\in\mathbb{R}} B_x = \underline{\hspace{9cm}} = \underline{\hspace{3cm}}.$$
$$\bigcap_{x\in\mathbb{R}} B_x = \underline{\hspace{9cm}} = \underline{\hspace{3cm}}.$$

Theorem 12 (2.9)**.** *Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$. Then*

(a) $\displaystyle\bigcap_{\alpha\in\Delta} A_\alpha \subseteq A_\beta$ *for each* $\beta \in \Delta$*;*

(b) $\displaystyle A_\beta \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$ *for each* $\beta \in \Delta$*;*

(c) $\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^c =$ ____________________________;

(d) $\left(\bigcap_{\alpha\in\Delta} A_\alpha\right)^c =$ ____________________________;

Proof.

(a) See Exercise 2.3.5(a).

(b) See Exercise 2.3.5(a).

(c)

(d) See page 91 or Exercise 2.3.5(b).

□

Definition 13. An indexed family $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ is *pairwise disjoint* iff for all $\alpha, \beta \in \Delta$, if $A_\alpha \neq A_\beta$, then $A_\alpha \cap A_\beta = \emptyset$. (In other words, no two distinct sets intersect each other.)

Venn diagram: