Homework 9

Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

Problem 1. Suppose $[K : F] = 2$. Prove that K is an algebraic extension of F that is the splitting field over F for a collection of polynomials in $F[x]$ (i.e., prove that K is a normal extension).

Problem 2. Determine the Galois group of $f(x) = (x^2 - 2)(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$.

Problem 3. Determine the Galois group of the splitting field over $\mathbb{Q}$ of $g(x) = x^4 - 14x^2 + 9$.

Problem 4. Let $K = \mathbb{Q}(\sqrt[8]{2}, i)$, $F_1 = \mathbb{Q}(i)$, and $F_2 = \mathbb{Q}(\sqrt{2})$.

(a) Prove that $\mathrm{Gal}(K/F_1) \cong \mathbb{Z}_8$.

(b) Prove that $\mathrm{Gal}(K/F_2) \cong D_8$.

Problem 5. Let $f(x) \in \mathbb{Q}[x]$. Suppose that $z \in \mathbb{C}$ is a root of $f(x)$.

(a) Prove that $\overline{z}$ (complex conjugate of z) is also a root of $f(x)$.

(b) Suppose $f(x)$ has degree 3. Prove that if the Galois group of the splitting field of $f(x)$ is isomorphic to $\mathbb{Z}_3$, then $f(x)$ has only real roots.