2 Field Theory

This chapter loosely follows Chapter 13 of Dummit and Foote.

2.1 Field Extensions

We begin with a definition that you encountered on a previous homework problem.

Definition 2.1. Let R be a ring with $1 \neq 0$. We define the **characteristic** of R, denoted $\mathrm{Char}(R)$, to be the smallest positive integer n such that $n \cdot 1_R = 0$ if such an n exists and to be 0 otherwise.

Note that $n \cdot 1_R$ is an shorthand for

$$\underbrace{1_R + \cdots + 1_R}_{n \text{ terms}}.$$

The integer n may not even be in R.

Example 2.2. Here are a few quick examples.

(1) The characteristic of the ring $\mathbb{Z}/n\mathbb{Z}$ is n. In particular, if p is prime, then the field $\mathbb{Z}/p\mathbb{Z}$ has characteristic p. The polynomial ring $\mathbb{Z}/n\mathbb{Z}[x]$ also has characteristic n.

(2) The ring $\mathbb{Z}$ has characteristic 0.

(3) The fields $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ all have characteristic 0.

(4) If F is a field with characteristic 0, then $F[x]$ has characteristic 0.

The next theorem tells us what the possible characteristics are for integral domains.

Theorem 2.3. Let R be an integral domain. Then $\mathrm{Char}(R)$ is either 0 or a prime p.

Theorem 2.4. If R is an integral domain such that $\mathrm{Char}(R) = p$ (p prime), then

$$p \cdot \alpha = \underbrace{\alpha + \cdots + \alpha}_{p \text{ terms}} = 0.$$

Theorem 2.5. The characteristic of an integral domain is the same as its field of fractions.

It turns out that if F is a field, F either contains a subfield isomorphic to $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ depending on whether $\mathrm{Char}(F)$ is 0 or p (for p prime). To see why this is true, define $\phi : \mathbb{Z} \to F$ via $\phi(n) = n \cdot 1_F$, where we interpret $(-n) \cdot 1_F = -(n \cdot 1_F)$ for positive n and $0 \cdot 1_F = 0$. Then $\ker(\phi) = \mathrm{Char}(F)\mathbb{Z}$. The First Isomorphism Theorem for Rings tells us that there is an injection of either $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$ into F. This implies that F either contains a subfield isomorphic to $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$, depending on the characteristic of F. In either case, this subfield is the smallest subfield containing 1_F, which we call the **subfield generated by** 1_F.

The next definition makes sense in light of the discussion above.

Definition 2.6. The **prime subfield** of a field F is the subfield generated by 1_F (i.e., the smallest subfield of F containing 1_F).

Note that the prime subfield of F is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$.

Example 2.7. Here are a couple quick examples.

(1) The prime subfield of both $\mathbb{Q}$ and $\mathbb{R}$ is $\mathbb{Q}$.

(2) The prime subfield of the field of rational functions with coefficients from the field $\mathbb{Z}/p\mathbb{Z}$ (denoted $\mathbb{Z}/p\mathbb{Z}(x)$) is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Definition 2.8. If K is a field containing the subfield F, then K is said to be an **extension field** (or simply an **extension**) of F, denoted K/F and read "K over F" (not be be confused with quotients!). The field F is called the **base field** of the extension.

Note that every field is an extension of its prime subfield.

Note 2.9. If K/F is a field extension, then we can interpret K as a vector space over F. In this case, K is the set of vectors and the scalars are coming from F.

Definition 2.10. The **degree** (or **index**) of a field extension K/F, denoted $[K : F]$, is the dimension of K as a vector space over F (i.e., $[K : F] = \dim_F(K)$).

Example 2.11. For example, $[\mathbb{C} : \mathbb{R}] = 2$.

If we are given a polynomial $p(x)$ in $F[x]$, it is possible that $p(x)$ does not have any roots in F. It is natural to wonder if there is an extension K of F such that $p(x)$ has roots in K.

For example, consider the polynomial $x^2 + 1$ in $\mathbb{R}[x]$. We know that this polynomial does not have a root in $\mathbb{R}$. However, this polynomial has roots in $\mathbb{C}$.

Note that given any polynomial $p(x)$ in $F[x]$, any root of a factor of $p(x)$ is also a root of $p(x)$. It is enough to consider the case where $p(x)$ is irreducible.

Theorem 2.12. Let F be a field and let $p(x) \in F[x]$ be an irreducible polynomial. Then there exists a field K containing an isomorphic copy of F in which $p(x)$ has a root. Identifying F with this isomorphic copy shows that there exists an extension of F in which $p(x)$ has a root.

In the proof of the above theorem, we took $K = F[x]/(p(x))$ (where $p(x)$ is irreducible). Since F is a subfield of K, there is a basis of K as a vector space over F. The next theorem makes this explicit.

Theorem 2.13. Let F be a field and let $p(x) \in F[x]$ be an irreducible polynomial of degree n over F and let $K = F[x]/(p(x))$. Define $\theta = x \mod (p(x)) \in K$. Then the elements $1, \theta, \theta^2, \ldots, \theta^{n-1}$ are a basis for K as a vector space over F. In particular, $[K : F] = n$ and

$$K = \{a_0 + a_1\theta + \cdots + a_{n-1}\theta^{n-1} \mid a_0, a_1, \ldots, a_{n-1} \in F\},$$

which is the set of all polynomials of degree less than n in θ.

The previous theorem provides a nice description of the elements in $K = F[x]/(p(x))$ ($p(x)$ irreducible). Adding these elements is as simple as adding like terms. However, in order to be a ring, we also need to be able to multiply. The next corollary gives us some assistance in doing this.

Corollary 2.14. Let K be as in the previous theorem and let $a(\theta), b(\theta) \in K$ be two polynomials in θ of degree less than n. Then $a(\theta)b(\theta) = r(\theta)$, where $r(x)$ is the remainder of degree less than n obtained after dividing the polynomial $a(x)b(x)$ by $p(x)$ in $F[x]$.

Example 2.15. Here are a few examples.

(1) Let $p(x) = x^2 + 1$. Since $p(x)$ is irreducible over $\mathbb{R}$ and of degree 2, $\mathbb{R}[x]/(p(x))$ is a field extension of $\mathbb{R}$ of degree 2 by Theorem 2.13. In a recent homework assignment, you proved that $\mathbb{R}[x]/(p(x))$ is isomorphic to $\mathbb{C}$ (which has a basis of rank 2 over $\mathbb{R}$). As expected, $p(x)$ has a root in $\mathbb{C}$. The elements of $\mathbb{R}[x]/(p(x))$ are of the form $a + b\theta$ for $a, b \in \mathbb{R}$. Addition is defined by

$$(a + b\theta) + (c + d\theta) = (a + c) + (b + d)\theta.$$

To multiply, we use the fact that $\theta^2 + 1 = 0$, or equivalently $\theta^2 = -1$. Note that -1 is the remainder when x^2 is divided by $x^2 + 1$ in $\mathbb{R}[x]$. Then

$$\begin{aligned}(a + b\theta)(c + d\theta) &= ac + (ad + bc)\theta + bd\theta^2 \\ &= ac + (ad + bc)\theta - bd \\ &= (ac - bd) + (ad + bc)\theta\end{aligned}$$

This shouldn't come as a surprise as this is exactly how we add and multiply in $\mathbb{C}$ where we swap out θ for i. In other words, the map from $\mathbb{R}[x]/(p(x))$ to $\mathbb{C}$ defined by $a + b\theta \mapsto a + bi$ is an isomorphism. In fact, we could have defined $\mathbb{C}$ exactly as $\mathbb{R}[x]/(p(x))$ (which shows that imaginary numbers aren't so imaginary).

(2) In the example above, we could replace $\mathbb{R}$ with $\mathbb{Q}$ to obtain the field extension $\mathbb{Q}(i)$ of $\mathbb{Q}$ of degree 2 containing a root i of $x^2 + 1$.

(3) Let $p(x) = x^2 - 2$. Then $p(x)$ is irreducible over $\mathbb{Q}$ by Eisenstein's Criterion (with prime 2). We obtain a field extension of $\mathbb{Q}$ of degree 2 containing a square root θ of 2, denoted $\mathbb{Q}(\theta)$. If we denote θ by $\sqrt{2}$, the elements of this field are of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$. In this case, addition and multiplication are defined as expected.

(4) Consider $p(x) = x^3 - 2 \in \mathbb{Q}[x]$. Then $p(x)$ is irreducible over $\mathbb{Q}$ by Eisenstein's Criterion (with prime 2). Let θ be a root of $p(x)$. Then

$$\mathbb{Q}[x]/(x^3 - 2) \cong \{a + b\theta + c\theta^2 \mid a, b, c \in \mathbb{Q}\},$$

where $\theta^3 = 2$. This is an extension of degree 3. Let's find the inverse of $1 + \theta$ in this field. Since $p(x)$ is irreducible, it is relatively prime to every polynomial of smaller degree. Thus, by the Euclidean Algorithm in $\mathbb{Q}[x]$, there are polynomials $a(x)$ and $b(x)$ in $\mathbb{Q}[x]$ such that

$$a(x)(1 + x) + b(x)(x^3 - 2) = 1.$$

In the quotient field, this equation tells us that $a(\theta)$ is the inverse of $1 + \theta$ (since $b(x)(x^3 - 2) \in (p(x))$). Actually carrying out the Euclidean Algorithm yields $a(x) = \frac{1}{3}(x^2 - x + 1)$ and $b(x) = -\frac{1}{3}$. This implies that

$$(1 + \theta)^{-1} = \frac{\theta^2 - \theta + 1}{3}.$$

(5) Let $p(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0$ be an irreducible polynomial over a field F. Suppose $\theta \in K$ is a root of $p(x)$. Notice that

$$\theta(p_n \theta^{n-1} + p_{n-1}\theta^{n-2} + \cdots + p_1) = -p_0.$$

Since $p(x)$ is irreducible, $p_0 \neq 0$. This implies that

$$\theta^{-1} = -\frac{1}{p_0}(p_n\theta^{n-1} + p_{n-1}\theta^{n-2} + \cdots + p_1) \in K.$$

(6) Consider $p(x) = x^2 + x + 1 \in \mathbb{Z}/2\mathbb{Z}[x]$. In Example 1.108(4), we verified that $p(x)$ is irreducible over $\mathbb{Z}/2\mathbb{Z}$. Then

$$\mathbb{Z}/2\mathbb{Z}[x]/(p(x)) \cong \{a + b\theta \mid a, b \in \mathbb{Z}/2\mathbb{Z}\} = \mathbb{Z}/2\mathbb{Z}(x),$$

where $\theta^2 = -\theta - 1 = \theta + 1$. This is extension of $\mathbb{Z}/2\mathbb{Z}$ of degree 2. The extension field contains 4 elements. Multiplication is defined by

$$\begin{aligned}(a + b\theta)(c + d\theta) &= ac + (ad + bc)\theta + bd\theta^2 \\ &= ac + (ad + bc)\theta + bd(\theta + 1) \\ &= (ac + bd) + (ad + bc + bad)\theta.\end{aligned}$$

Definition 2.16. Let K be an extension of the field F and let $\alpha, \beta, \ldots \in K$. Then the smallest subfield of K containing both F and the elements $\alpha, \beta, \ldots$, denoted $F(\alpha, \beta, \ldots)$ is called the field **generated by** $\alpha, \beta, \ldots$ **over** F.

Definition 2.17. If the field K is the generated by a single element α over F, $K = F(\alpha)$, then K is said to be a **simple extension** of F and the element α is called a **primitive element** for the extension.

Theorem 2.18. *Let F be a field and let $p(x) \in F[x]$ be an irreducible polynomial. Suppose K is an extension field of F containing a root α of $p(x)$. Let $F(\alpha)$ denote the subfield of K generated over F by α. Then*

$$F(\alpha) = F[x]/(p(x)).$$

Note 2.19. The previous theorem tells us that any field over F in which $p(x)$ contains a root contains a subfield isomorphic to the extension of F constructed in Theorem 2.12. In addition, this field is (up to isomorphism) the smallest extension of F containing such a root.

Corollary 2.20. *Let F and $p(x)$ be as in the previous theorem and suppose $\deg(p(x)) = n$. Then*

$$F(\alpha) = \{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1} \mid a_0, a_1, \ldots, a_{n-1} \in F\} \subseteq K.$$

Example 2.21. Here are two more examples.

(1) Since $\sqrt{2}, -\sqrt{2}$ are roots of $x^2 - 2$, $\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2 - 2) \cong \mathbb{Q}(-\sqrt{2})$. Note that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} + \mid a, b \in \mathbb{Q}\}$ as we saw in an earlier example.

(2) Similarly, since $\sqrt[3]{2}$ is a root of $x^3 - 2$, $\mathbb{Q}(\sqrt[3]{2}) \cong \mathbb{Q}[x]/(x^3 - 2)$. Note that $\mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c(\sqrt[3]{2})^3 \mid a, b, c \in \mathbb{Q}\}$. The only real root of $x^3 - 2$ is $\sqrt[3]{2}$, but there are two other roots of $x^3 - 2$, namely

$$\sqrt[3]{2}\left(\frac{-1 \pm i\sqrt{3}}{2}\right).$$

The fields generated by these two roots are subfields of $\mathbb{C}$ but not $\mathbb{R}$. In both cases, the fields are isomorphic to $\mathbb{Q}[x]/(x^3 - 2)$.

Theorem 2.22. *Let $\phi : F \to F'$ be an isomorphism of fields. Then we can extend ϕ to an isomorphism from $F[x]$ to $F'[x]$. Let $p(x)$ be an irreducible polynomial in $F[x]$ and let $p'(x)$ be the corresponding irreducible polynomial in $F'[x]$. Let α be a root of $p(x)$ (in some extension of F) and let β be any root of $p'(x)$ (in some extension of F'). Then there exists an isomorphism of fields $\sigma : F(\alpha) \to F'(\beta)$ such that $\sigma(\alpha) = \beta$.*