INSTRUCTIONS: Answer each of the following questions. In order to receive full credit, your answers must be **complete, legible,** and **correct.** You must also **show all of your work** and give adequate explanations where necessary. No calculators, no books, no notes are allowed on this exam.

1. (10 pts) Find the absolute extrema of $f(x) = x^3 - x^2 - x + 2$ on the interval $[-1, 2]$.

$$f'(x) = 3x^2 - 2x - 1 = 0$$

TO FIND CRITICAL POINTS

$$(3x+1)(x-1) = 0$$

$$3x+1=0 \qquad x-1=0$$

$$x = -\tfrac{1}{3} \qquad x = 1$$

CRIT PTS

$$f\left(-\tfrac{1}{3}\right) = \left(-\tfrac{1}{3}\right)^3 - \left(-\tfrac{1}{3}\right)^2 - \left(-\tfrac{1}{3}\right) + 2$$

$$= -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + 2$$

$$= -\frac{1}{27} - \frac{3}{27} + \frac{9}{27} + \frac{54}{27} = \frac{59}{27} = 2\tfrac{5}{27}$$

$$f(1) = 1^3 - 1^2 - 1 + 2 = 1$$

MIN

$$f(-1) = (-1)^3 - (-1)^2 - (-1) + 2$$

$$= -1 - 1 + 1 + 2 = 1$$

END PTS

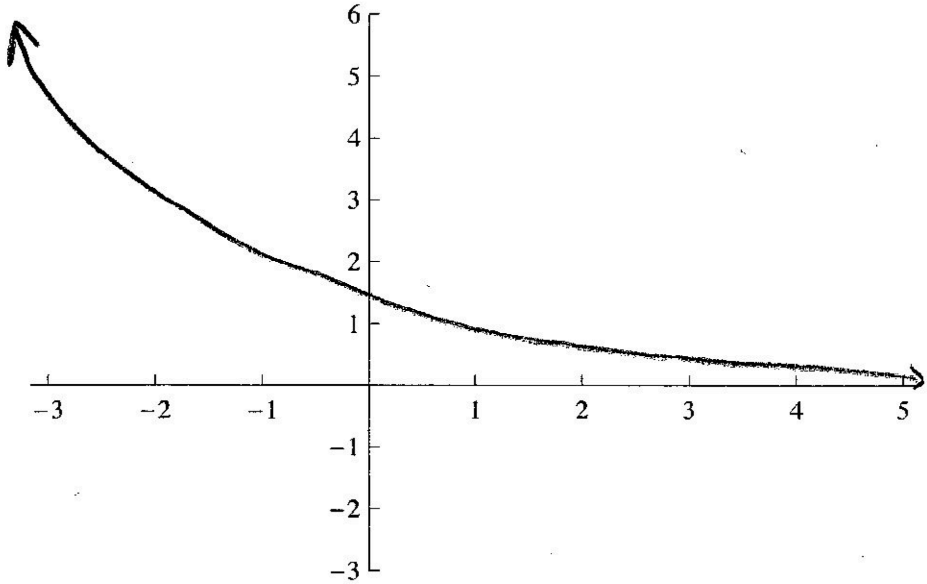
$$f(2) = 2^3 - 2^2 - 2 + 2 = 8 - 4 = 4$$

MAX

$\boxed{\text{ABSOLUTE MAX IS } 4}$, OCCURS AT $X=2$

$\boxed{\text{ABSOLUTE MIN IS } 1}$, OCCURS AT $X=1$ AND AT $X=-1$

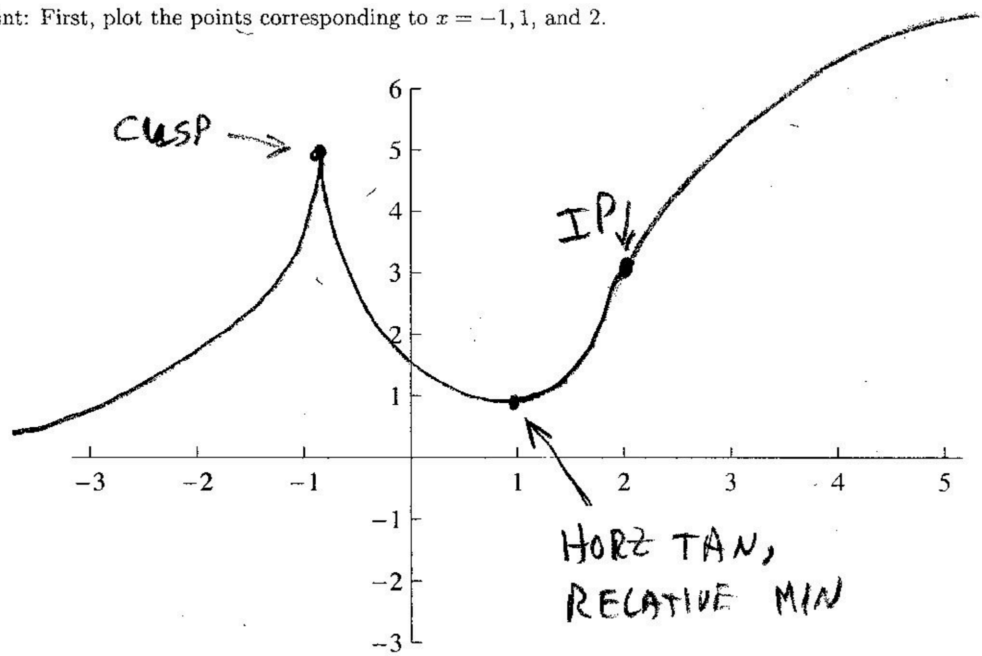
2. (a) (5 pts) Sketch the graph of a function $g(x)$ on the axes below, where $g(x)$ is continuous on $(-\infty, \infty)$, $g(x)$ is decreasing everywhere, and $g(x)$ is concave up everywhere.



(b) (5 pts) Sketch the graph of $f(x)$ on the axes below, where $f(x)$ is a continuous function on $(-\infty, \infty)$ that satisfies the following conditions:

x	$-\infty < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 2$	2	$2 < x < \infty$
$f(x)$	positive	5	positive	1	positive	3	positive
$f'(x)$	positive	DNE	negative	0	positive	2	positive
$f''(x)$	positive	DNE	positive	2	positive	0	negative

Hint: First, plot the points corresponding to $x = -1, 1$, and 2.



3. (6 pts) The function

$$f(x) = 10 - \frac{16}{x}$$

satisfies the hypotheses of the Mean Value Theorem over the interval $[2, 8]$. Find the value(s) of $x = c$ whose existence is guaranteed by the Mean Value Theorem.

$$f'(x) = \frac{d}{dx}\left[10 - \frac{16}{x}\right] = \frac{d}{dx}\left[10 - 16x^{-1}\right]$$

$$= 0 - 16(-1x^{-2})$$

$$= \frac{16}{x^2}$$

SO $f'(c) = \frac{16}{c^2}$ [CHANGE DUMMY VARIABLES FROM X TO C]

$$\frac{f(b) - f(a)}{b - a} = \frac{f(8) - f(2)}{8-2} = \frac{\left(10 - \frac{16}{8}\right) - \left(10 - \frac{16}{2}\right)}{8-2}$$

$$= \frac{(10-2) - (10-8)}{8-2} = \frac{8-2}{8-2}$$

$= 1 =$ SLOPE OF MEAN LINE

MVT ASSERTS THAT $f'(c) =$ SLOPE OF MEAN LINE FOR SOME C IN $[2, 8]$

$$\frac{16}{c^2} = 1$$

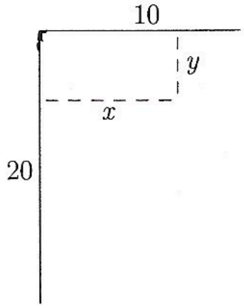
$$c^2 = 16$$

$$c = \pm 4$$

THROW AWAY $c = -4$ SINCE -4 IS NOT IN $[2, 8]$

$$\boxed{c = 4}$$

4. Suppose you want to build a enclosed, rectangular pen for your cute German Shepherd puppy. For two sides of the pen you are going to use two perpendicular stone walls in your backyard, whose total lengths are 10 ft and 20 ft, respectively, and for the other two sides you are going to use 24 ft of fencing (See drawing).



(a) (4 pts) Write an equation for the enclosed area of the pen as a function of x.

A = AREA OF PEN $= xy$ (AREA OF RECTANGLE)

$x + y = 24 \Rightarrow y = 24 - x$

SO $A = A(x) = x(24-x) = 24x - x^2$

(b) (2 pts) Determine the domain of the above area function. MUST HAVE $x \leq 10$.

BUT $x = 1$ (FOR EXAMPLE) IS NOT POSSIBLE, BECAUSE THEN $y = 24 - x = 24 - 1 = 23$, BUT y CAN'T BE THIS BIG, SINCE MUST HAVE $y \leq 20$.

$y \leq 20$ AND $y = 24 - x$ IMPLIES

$24 - x \leq 20$

$24 - 20 \leq x$

$4 \leq x$

$x \geq 4$

SO

$\boxed{\text{DOM } A = [4, 10]}$

(c) (6 pts) Find the dimensions of the pen that has maximum area.

$A'(x) = \frac{d}{dt}\left[24x - x^2\right]$

$= 24 - 2x = 0$ WHEN $x = 12$,

BUT $x = 12$ IS NOT IN DOM A

SO NO CRITICAL PTS.

TEST ENDPTS:

$A(4) = 4(24-4) = 80$

$A(10) = 10(24-10) = 140$

140 ft^2 IS MAX AREA,

OCCURS WHEN $\boxed{x = 10 \text{ ft}}$ THEN $y = 24 - x = 24 - 10 = 14$

$\boxed{y = 14 \text{ ft}}$

5. (5 pts each) Evaluate each of the following indefinite integrals.

(a) $\int x(4+x^3)\,dx$ $= \int (4x + x^4)\,dx = \int 4x\,dx + \int x^4\,dx = 4\int x\,dx + \int x^4\,dx$

$= 4 \cdot \frac{x^2}{2} + \frac{x^5}{5} + C = \boxed{2x^2 + \frac{1}{5}x^5 + C}$

(b) $\int \frac{4+2x^{3/2}}{\sqrt{x}}\,dx$ (Hint: simplify first)

$\int \frac{4+2x^{3/2}}{\sqrt{x}}\,dx = \int \left(\frac{4}{\sqrt{x}} + \frac{2x^{3/2}}{\sqrt{x}}\right)dx = \int \left(\frac{4}{x^{1/2}} + \frac{2x^{3/2}}{x^{1/2}}\right)dx$

$= \int (4x^{-1/2} + 2x)\,dx = 4 \cdot \frac{x^{1/2}}{1/2} + 2\frac{x^2}{2} + C = \boxed{8x^{1/2} + x^2 + C}$

(c) $\int \frac{\cos^3 x}{1-\sin^2 x}\,dx$ $= \int \frac{\cos^3 x}{\cos^2 x}\,dx$ SINCE $\sin^2 x + \cos^2 x = 1$
SO $\cos^2 x = 1 - \sin^2 x$

$= \int \cos x\,dx = \boxed{\sin x + C}$

(d) $\int \frac{\ln(x)}{x}\,dx$ $= \int (\ln x) \cdot \frac{1}{x}\,dx = \int u\,du = u^2/2 + C$

Let $u = \ln(x)$
THEN $du/dx = \frac{1}{x}$
$du = \frac{1}{x}\,dx$

$= \boxed{\frac{(\ln x)^2}{2} + C}$

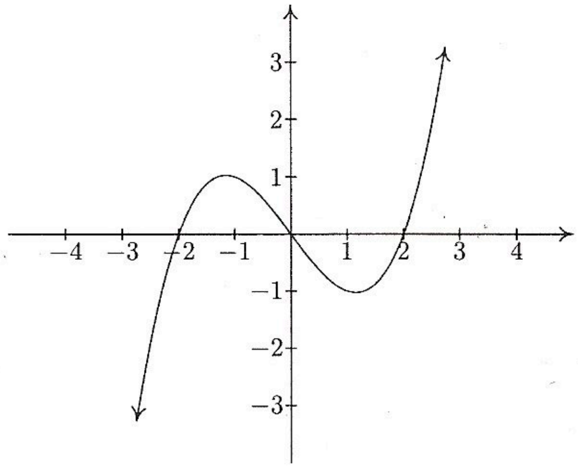
(e) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ $= \int \frac{1}{e^x + e^{-x}} \cdot (e^x - e^{-x})\,dx = \int \frac{1}{u}\,du$

Let $u = e^x + e^{-x}$
THEN $du/dx = e^x - e^{-x}$
$du = (e^x - e^{-x})\,dx$

$= \ln|u| + C$
$= \ln|e^x + e^{-x}| + C$
$= \boxed{\ln(e^x + e^{-x}) + C}$

ABSOLUTE VALUE NOT NEEDED SINCE $e^x + e^{-x} > 0$ FOR ALL REAL x

6. (3 pts each) Let $f(x)$ be continuous over $(-\infty, \infty)$. The following is the graph of the *derivative* of $f(x)$:



Graph of f'

(a) Find the x-coordinates of all points on the graph of $f(x)$ where the tangent line is horizontal.

TANGENT LINES TO THE GRAPH OF $f(x)$ OCCUR FOR THOSE X'S FOR WHICH $f'(x)=0$, FROM ABOVE GRAPH, $f'(x)=0$ FOR $\boxed{X=-2}$ AND $\boxed{X=0}$ AND $\boxed{X=2}$

(b) Find the intervals, if any, on which $f(x)$ is increasing and decreasing.

$f(x)$ IS INCR WHEN $f'(x)>0$: X IN $\boxed{(-2,0)}$ AND IN $\boxed{(2,\infty)}$

$f(x)$ IS DECR WHEN $f'(x)<0$: X IN $\boxed{(-\infty,-2)}$ AND IN $\boxed{(0,2)}$

(c) Find the x-coordinates of all the relative maxima of $f(x)$.

RELATIVE MAXIMA OF $f(x)$ OCCUR WHEN $f(x)$ CHANGES FROM INCR TO DECR

i.e. FROM $f'(x)>0$ TO $f'(x)<0$: AT $\boxed{X=0}$

(d) Find the x-coordinates of all the relative minima of $f(x)$.

RELATIVE MINIMA OF $f(x)$ OCCUR WHEN $f(x)$ CHANGES FROM DECR TO INCR

i.e. FROM $f'(x)<0$ TO $f'(x)>0$: AT $\boxed{X=-2}$ AND $\boxed{X=2}$

(e) Find the approximate x-coordinates of all of the inflection points of $f(x)$.

INFLECTION POINTS FOR $f(x)$ OCCUR WHEN THE CONCAVITY OF $f(x)$ CHANGES FROM CU TO CD OR FROM CD TO CU

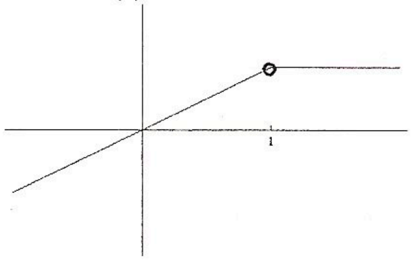
i.e. FROM $f''(x)>0$ TO $f''(x)<0$ OR FROM $f''(x)<0$ TO $f''(x)>0$

i.e. FROM $f'(x)$ INCR TO $f'(x)$ DECR OR FROM $f'(x)$ DECR TO $f'(x)$ INCR

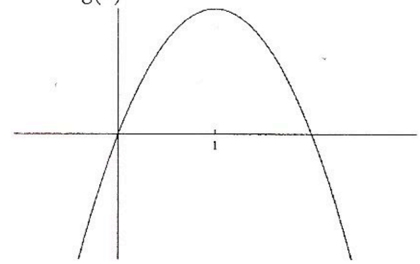
i.e. AT $\boxed{X\approx -1}$ OR AT $\boxed{X\approx 1}$

7. (4 pts each) Fill in the blanks corresponding to the graph of each function f and g with the appropriate letter below.

f(x)



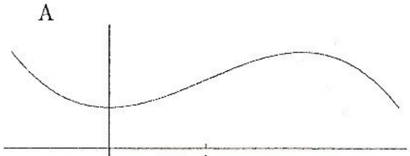
g(x)



(a) A possible antiderivative of $f(x)$ is C.

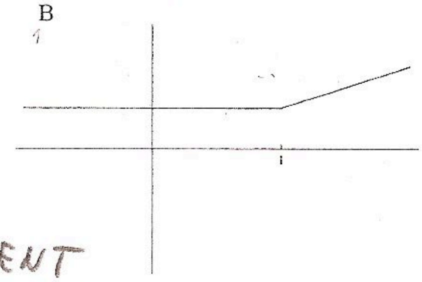
(b) A possible antiderivative of $g(x)$ is A.

A

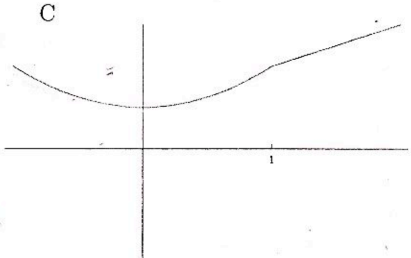


THE DERIVATIVE OF THIS FUNCTION (i.e. THE SLOPES OF THE TANGENT LINES TO THE GRAPH OF THIS FUNCTION) IS THE GRAPH OF g(x)

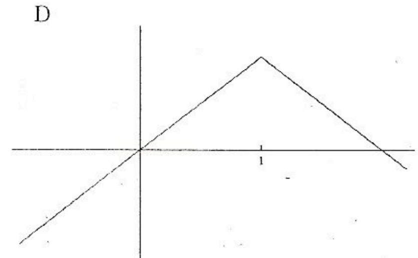
B



C



D



THE DERIVATIVE OF THIS FUNCTION (i.e. THE SLOPES OF THE TANGENT LINES TO THE GRAPH OF THIS FUNCTION) IS THE GRAPH OF f(x)

8. (8 pts) Find *all* aymptotes of the function

$$f(x) = \frac{2x^3 - 2x^2 - 1}{x - 1}.$$

$$\begin{array}{r} 2x^2 \\ x-1 \,\big)\, \overline{2x^3 - 2x^2 + 0x - 1} \\ \underline{2x^3 - 2x^2} \\ -1 \end{array}$$

SO $\dfrac{2x^3-2x^2-1}{x-1} = 2x^2 - \dfrac{1}{x-1}$

NOTE $\lim\limits_{x\to\pm\infty} \dfrac{1}{x-1} = 0$

SO $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \dfrac{2x^3-2x^2-1}{x-1} = \lim\limits_{x\to\pm\infty} 2x^2$ WHICH MEANS THAT

$f(x) \Rightarrow 2x^2$ AS $x \to \pm\infty$ SO

$\boxed{y = 2x^2 \text{ IS A CURVILINEAR ASYMPTOTE}}$

$\boxed{x = 1 \text{ IS A VERTICAL ASYMPTOTE}}$

SINCE $\lim\limits_{x\to 1} f(x) = \infty$

9. Let f be a function that has an inverse, denoted by f^{-1}. Use facts about inverse functions to answer the following questions.

THIS IS THE PROMISED PROBLEM SIMILAR TO A PROBLEM ON A PREVIOUS EXAM.

IN FACT, THIS PROBLEM IS IDENTICAL TO PROBLEM #9 ON MIDTERM EXAM #1, WITH ONLY ONE NUMBER CHANGED IN THE PROBLEM STATEMENT.

(a) (3 pts) Suppose that $f(2) = 2$ and $f(4) = 6$. Find the equation of the secant line (also called chord) to the graph of f^{-1} through the pair of points whose x-coordinates are $x = 2$ and $x = 6$.

$f(2) = 2$, SO $f^{-1}(2) = 2$

$f(4) = 6$, SO $f^{-1}(6) = 4$

SLOPE OF LINE THRU $(2,2)$ AND $(6,4)$:

$m = \dfrac{\Delta y}{\Delta x} = \dfrac{4-2}{6-2} = \dfrac{2}{4} = \dfrac{1}{2}$

EQUATION OF LINE WITH $m = \frac{1}{2}$ THRU $(2,2)$:

$\boxed{y - 2 = \tfrac{1}{2}(x-2)}$ OR $\boxed{y = \tfrac{1}{2}x + 1}$

(b) (3 pts) Explain why the graph of f and the graph of f^{-1} are symmetric about the line $y = x$.

IF (a,b) IS ON THE GRAPH OF f, THEN

$b = f(a) \Rightarrow a = f^{-1}(b) \Rightarrow (b,a)$ IS ON THE GRAPH OF f^{-1}

THE POINTS (a,b) AND (b,a) ARE SYMMETRIC IN THE LINE $y = x$, SINCE THE x & y COORDINATES OF THE TWO POINTS ARE INTERCHANGED, SO EVERY POINT (a,b) ON THE GRAPH OF f CORRESPONDS TO A SYMMETRICALLY PLACED POINT (b,a) ON THE GRAPH OF f^{-1}.