Section 6.1: Areas Between Curves

Goal

We will learn how to find the area of regions bounded by curves.

Initial discussion

Recall 1. If f is a continuous function on the interval $[a, b]$, then definite integral

$$\int_a^b f \, dx$$

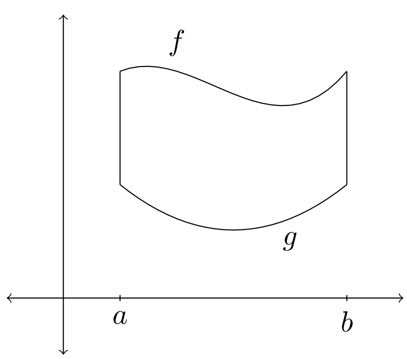
represents the *net signed area* of the region bounded by the graph of f and the x-axis. The value of the definite integral is a number, which could be positive, 0, or negative.

In this section, we want to discuss how to find *area* of regions bounded by curves. Notice that I didn't say "net signed area." In this case, our answer will always be ____________.

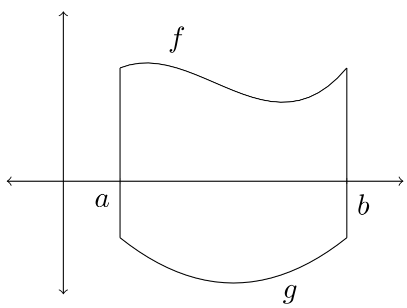
Deriving a formula

We would like to find a formula for determining the area of a region bounded by continuous functions. Let f and g be two continuous functions on the interval $[a, b]$. In each of the three following cases, let's determine an integral formula that would yield the area of the bounded region.

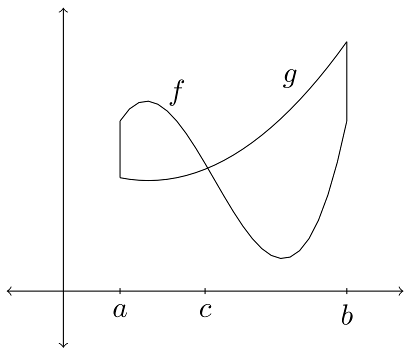
1. Case 1:



2. Case 2:



3. Case 3:



After thinking about these three cases, hopefully you can take a guess at what the formula is, in general.

Theorem 2. Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. Then the area A of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ is

$$\boxed{\int_a^b \_\_\_\_ - \_\_\_\_ \, dx}.$$

However, instead of memorizing this formula with f and g and trying to remember which is the big one and which is the small one, I would think of it this way:

$$\boxed{\int_a^b \text{top} \; - \_\_\_\_\_\_\_\_\_\_ \, dx}.$$

Here is a general strategy for attacking problems involving area between curves:

1. Draw a picture of the region you want to find the area of. If you need to, use your calculator.
2. Using your picture, determine which graph is the "top" one and which one is the "bottom" one.
3. If "top" and "bottom" switch places, as in case 3 above, you'll have to split up into multiple integrals.
4. If an interval is given, use it as your limits (unless you need to split up the integral because "top" and "bottom" switch). If an interval is not given, then you'll need to find the points of intersection of the two curves.

Question 3. How do you find the points of intersection of two curves?

OK, let's try some exercises.

Exercises

1. Find the area of the region bounded by $f(x) = 4 - x$ and $g(x) = -x^2$ on $[-1, 1]$.

2. Find the area of the region bounded by $f(x) = x - x^2$ and $g(x) = x^2$. (Hint: for this one you will need to find the points of intersection.)

3. Find the area of the region bounded by $f(x) = x^3 - x^2 - 2x$ and $g(x) = 4x$. (Hint: for this one you will need to find the points of intersection. Also, "top" and "bottom" switch places.)

Alternate formula

Sometimes it is easier to integrate with respect to y rather than x. In this case, we use

$$\boxed{\int_a^b \text{right} \; - \underline{\hspace{3cm}} \; dy},$$

where it is important to point out that a and b live on the y-axis.

One more exercise

4. Set up but do not evaluate a formula for the area of the region bounded by the curves $y = x/2$ and $y^2 = 8 - x$ given that the points of intersection are $(-8, -4)$ and $(4, 2)$.

