Section 10.5: Calculus with Parametric Equations

Goal

In this section, we discuss derivatives of parametric curves and learn how to find area, arc length, and surface area in the context of parametric curves.

Derivatives of parametric curves

Suppose that

$$x = f(t)$$

$$y = g(t)$$

define a parametric curve. If we could eliminate the parameter, we would end up with something of the form

$$y = F(x)$$

(where F is *not* an antiderivative, but rather some function of x).
If we were to substitute back in, we obtain

$$\underline{\hspace{4cm}}.$$

By the chain rule (assuming F, f, and g are differentiable), we get

$$\underline{\hspace{5cm}}.$$

Now, as long as $f'(t) \neq 0$, we obtain

$$F'(x) = F'(f(t)) = \frac{\quad}{\underline{\hspace{2cm}}} = \frac{\quad}{\underline{\hspace{2cm}}}.$$

That is, we have the following theorem.

Theorem 1.

$$\boxed{\frac{dy}{dx} = \frac{\quad}{\underline{\hspace{2cm}}} \qquad \textit{(provided } dx/dt \neq 0\textit{).}}$$

Important Note 2.

1. This formula allows us to find derivative of y with respect to x without actually having to eliminate the parameter.
2. Horizontal tangents when $dy/dt = 0$ (and $dx/dt \neq 0$).
3. Vertical tangents when $dx/dt = 0$ (and $dy/dt \neq 0$).

Example 3.

(a) Define the parametric curve $C : x = t^2, y = t^3 - 3t$. Find points where C has (i) horizontal tangents, and (ii) vertical tangents.

(b) Define the parametric curve $C : x = 2\sin 2t, y = 3\sin t$. Find slope of both tangent lines at the point $(0, 0)$.

Area and arc length

Recall 4.

- Area: $A = \int_a^b y\, dx$
- Arc length: $s = \int_a^b \sqrt{1 + [dy/dx]^2}\, dx$

If a parametric curve C is given by

$$x = f(t)$$

$$y = g(t)$$

then

$$\boxed{\begin{aligned} dx &= \underline{\hspace{3cm}} \\ dy &= \underline{\hspace{3cm}} \end{aligned}}$$

and

$$\begin{aligned} \sqrt{1 + [dy/dx]^2}\, dx &= \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, f'(t)dt \\ &= \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt. \end{aligned}$$

That is,

$$\boxed{\sqrt{1 + [dy/dx]^2}\, dx = \sqrt{[dx/dt]^2 + [dy/dt]^2}\, dt}$$

If C is *smooth* (i.e., f' and g' are continuous), then by making the appropriate substitutions, we can find area and arc length in the context of parametric curves.

Important Note 5.

1. If C is on the interval $[\alpha, \beta]$, then for *area*, we integrate from either $t = \alpha \to t = \beta$ or $t = \beta \to t = \alpha$; the proper choice being the one that corresponds to traversing curve from $L \to R$.
2. For all integrals, we need to make sure we trace out curve *exactly once* from $\alpha \to \beta$ (otherwise, we may get too much or too little).

Example 6.

(a) Find area under $C : x = e^{3t}, y = e^{-t}$ on $[0, \ln 2]$.

(b) Find arc length of $C : x = \cos t, y = \sin t$ on $[0, 2\pi]$.