Chapter 3

Subgroups and Isomorphisms

For the next two sections, it would be useful to have all of the Cayley diagrams we've encountered in one place for reference. So, before continuing, gather up the following Cayley diagrams:

- $\text{Spin}_{1\times 2}$. There are 3 of these. I drew one for you in Section 2.6 and you discovered two more in Problem 2.67.
- S_2. See Problem 2.70(a).
- R_4. See Problem 2.70(b).
- V_4. See Problem 2.70(c).
- D_3. There are two of these. See Problems 2.70(d) and 2.70(e).
- S_3. See Problem 2.70(f).
- D_4. See Problem 2.70(g).

3.1 Subgroups

Problem 3.1. Recall the definition of "subset." What do you think "subgroup" means? Try to come up with a potential definition. Try not to read any further before doing this.

Problem 3.2. Examine your Cayley diagrams for D_4 (with generating set $\{r,s\}$) and R_4 (with generating set $\{r\}$) and make some observations. How are they similar and how are they different? Can you reconcile the similarities and differences by thinking about the actions of each group?

Hopefully, one of the things you noticed in the previous problem is that we can "see" R_4 inside of D_4. You may have used different colors in each case and maybe even labeled the vertices with different words, but the overall structure of R_4 is there nonetheless.

Problem 3.3. If you ignore the labels on the vertices and just pay attention to the configuration of arrows, it appears that there are two copies of the Cayley diagram for R_4 in the Cayley diagram for D_4. Isolate these two copies by ignoring the edges that correspond to the generator s. Now, paying close attention to the words that label the vertices from the original Cayley diagram for D_4, are either of these groups in their own right?

Recall that the identity must be one of the elements included in a group. If this didn't occur to you when doing the previous problem, you might want to go back and rethink your answer. Just like in the previous problem, we can often "see" smaller groups living inside larger groups. These smaller groups are called **subgroups**.

Definition 3.4. Let G be a group and let H be a subset of G. Then H is a **subgroup** of G, written $H \leq G$, provided that H is a group in its own right under the binary operation inherited from G.

The phrase "under the binary operation inherited from G" means that to combine two elements in H, we should treat the elements as if they were in G and perform the binary operation of G.

In light of Problem 3.3, we would write $R_4 \leq D_4$. The second sub-diagram of the Cayley diagram for D_4 (using $\{r, s\}$ as the generating set) that resembles R_4 cannot be a subgroup because it does not contain the identity. However, since it looks a lot like R_4, we call it a **clone** of R_4. For convenience, we also say that a subgroup is a clone of itself.

Problem 3.5. Let G be a group and let $H \subseteq G$. If we wanted to determine whether H is a subgroup of G, can we skip checking any of the axioms? Which axioms must we verify?

Let's make the observations of the previous problem a bit more formal.

Theorem 3.6 (Two Step Subgroup Test)**.** Suppose G is a group and H is a nonempty subset of G. Then $H \leq G$ if and only if (i) for all $h \in H$, $h^{-1} \in H$, as well, and (ii) H is closed under the binary operation of G.

Notice that one of the hypotheses of Theorem 3.6 is that H be nonempty. This means that if we want to prove that a certain subset H is a subgroup of a group G, then one of the things we must do is verify that H is in fact nonempty. In light of this, the "Two Step Subgroup Test" should probably be called the "Three Step Subgroup Test".

As Theorems 3.7 and 3.9 will illustrate, there are a couple of subgroups that every group contains.

Theorem 3.7. If G is a group, then $\{e\} \leq G$.

The subgroup $\{e\}$ is referred to as the **trivial subgroup**. All other subgroups are called **nontrivial**.

Problem 3.8. Let G be a group. What does the Cayley diagram for the subgroup $\{e\}$ look like? What are you using as your generating set?

Earlier, we referred to subgroups as being "smaller." However, our definition does not imply that this has to be the case.

Theorem 3.9. If G is a group, then $G \leq G$.

We refer to subgroups that are not equal to the whole group as **proper subgroups**. If H is a proper subgroup, then we may write $H < G$.

Recall Theorem 2.50 that states that if G is a group under $*$ and S is a subset of G, then $\langle S \rangle$ is also a group under $*$. Let's take this a step further.

Theorem 3.10. If G is a group and $S \subseteq G$, then $\langle S \rangle \leq G$. In particular, $\langle S \rangle$ is the smallest subgroup of G containing S.

The subgroup $\langle S \rangle$ is called the **subgroup generated by** S. In the special case when S equals a single element, say $S = \{g\}$, then

$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\},$$

which is called the (**cyclic**) **subgroup generated by** g. Every subgroup can be written in the "generated by" form. That is, if H is a subgroup of a group G, then there always exists a subset S of G such that $\langle S \rangle = H$. In particular, $\langle H \rangle = H$ for $H \leq G$, and as a special case, we have $\langle G \rangle = G$.

Problem 3.11. Consider $\text{Spin}_{1\times 2}$ with generating set $\{s_{11}, s_{22}, s_{12}\}$.

(a) Find the Cayley diagram for the subgroup $\langle s_{11} \rangle$ inside the Cayley diagram for $\text{Spin}_{1\times 2}$. Identify all of the clones of $\langle s_{11} \rangle$ inside $\text{Spin}_{1\times 2}$.

(b) Find the Cayley diagram for the subgroup $\langle s_{11}, s_{22} \rangle$ inside the Cayley diagram of $\text{Spin}_{1\times 2}$. Identify the clones of $\langle s_{11}, s_{22} \rangle$ inside $\text{Spin}_{1\times 2}$.

One of the benefits of Cayley diagrams is that they are useful for visualizing subgroups. However, recall that if we change our set of generators, we might get a very different looking Cayley diagram. The upshot of this is that we may be able to see a subgroup in one Cayley diagram for a given group, but not be able to see it in the Cayley diagram arising from a different generating set.

Problem 3.12. We currently have two different Cayley diagrams for D_3 (see Problems 2.21 and 2.55).

(a) Can you find the Cayley diagram for the trivial subgroup $\langle e \rangle$ in either Cayley diagram for D_3? Identify all of the clones of $\langle e \rangle$ in both Cayley diagrams for D_3.

(b) Can you find the Cayley diagram for the subgroup $\langle r \rangle = R_3$ in either Cayley diagram for D_3? If possible, identify all of the clones of R_3 in the Cayley diagrams for D_3.

(c) Can you find the Cayley diagrams for $\langle s \rangle$ and $\langle s' \rangle$ in either Cayley diagram for D_3? If possible, identify all of the clones of $\langle s \rangle$ and $\langle s' \rangle$ in the Cayley diagrams for D_3.

Problem 3.13. Consider D_4. Let h be the reflection of the square over the horizontal midline and let v be the reflection over the vertical midline. Which of the following are subgroups of D_4? In each case, justify your answer. If a subset is a subgroup, try to find a minimal generating set. Also, determine whether you can see the subgroups in our Cayley diagram for D_4 with generating set $\{r, s\}$.

(a) $\{e, r^2\}$

(b) $\{e, h\}$

(c) $\{e, h, v\}$

(d) $\{e, h, v, r^2\}$

Perhaps you recognized the set in part (d) of the previous problem as being the Klein four-group V_4. It follows that $V_4 \leq D_4$.

Let's introduce a group we haven't seen yet. Define the **quaternion group** to be the group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ having the Cayley diagram with generating set $\{i, j, -1\}$ given in Figure 3.1. In this case, 1 is the identity of the group.

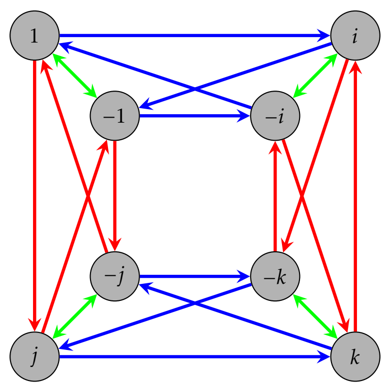

Figure 3.1. Cayley diagram for Q_8 with generating set $\{-1, i, j\}$.

Notice that I didn't mention what the actions actually do. For now, let's not worry about that. The relationship between the arrows and vertices tells us everything we need to know. Also, let's take it for granted that Q_8 actually is a group.

Problem 3.14. Consider the Cayley diagram for Q_8 given in Figure 3.1.

(a) Which arrows correspond to which generators in our Cayley diagram for Q_8?

(b) What is i^2 equal to? That is, what element of $\{1, -1, i, -i, j, -j, k, -k\}$ is i^2 equal to? How about i^3, i^4, and i^5?

(c) What are j^2, j^3, j^4, and j^5 equal to?

(d) What is $(-1)^2$ equal to?

(e) What is ij equal to? How about ji?

(f) Can you determine what k^2 and ik are equal to?

(g) Can you identify a generating set consisting of only two elements? Can you find more than one?

(h) What subgroups of Q_8 can you see in the Cayley diagram in Figure 3.1?

(i) Find a subgroup of Q_8 that you cannot see in the Cayley diagram.

Problem 3.15. Consider $(\mathbb{R}^3, +)$, where $\mathbb{R}^3$ is the set of all 3-entry row vectors with real number entries (e.g., (a, b, c) where $a, b, c \in \mathbb{R}$) and $+$ is ordinary vector addition. It turns out that $(\mathbb{R}^3, +)$ is an abelian group with identity $(0, 0, 0)$.

(a) Let H be the subset of $\mathbb{R}^3$ consisting of vectors with first coordinate 0. Is H a subgroup of $\mathbb{R}^3$? Prove your answer.

(b) Let K be the subset of $\mathbb{R}^3$ consisting of vectors whose entries sum to 0. Is K a subgroup of $\mathbb{R}^3$? Prove your answer.

(c) Construct a subset of $\mathbb{R}^3$ (different from H and K) that is *not* a subgroup of $\mathbb{R}^3$.

Problem 3.16. Consider the group $(\mathbb{Z}, +)$ (under ordinary addition).

(a) Show that the even integers, written $2\mathbb{Z} := \{2k \mid k \in \mathbb{Z}\}$, form a subgroup of $\mathbb{Z}$.

(b) Show that the odd integers are not a subgroup of $\mathbb{Z}$.

(c) Show that all subsets of the form $n\mathbb{Z} := \{nk \mid k \in \mathbb{Z}\}$ for $n \in \mathbb{Z}$ are subgroups of $\mathbb{Z}$.

(d) Are there any other subgroups besides the ones listed in part (c)? Explain your answer.

(e) For $n \in \mathbb{Z}$, write the subgroup $n\mathbb{Z}$ in the "generated by" notation. That is, find a set S such that $\langle S \rangle = n\mathbb{Z}$. Can you find more than one way to do it?

Problem 3.17. Consider the group of symmetries of a regular octagon. This group is denoted by D_8, where the operation is composition of actions. The group D_8 consists of 16 elements (8 rotations and 8 reflections). Let H be the subset consisting of the following clockwise rotations: $0°$, $90°$, $180°$, and $270°$. Determine whether H is a subgroup of D_8 and justify your answer.

Problem 3.18. Consider the groups $(\mathbb{R}, +)$ and $(\mathbb{R} \setminus \{0\}, \cdot)$. Explain why $\mathbb{R} \setminus \{0\}$ is not a subgroup of $\mathbb{R}$ despite the fact that $\mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$ and both are groups (under the respective binary operations).

Theorem 3.19. If G is an abelian group such that $H \leq G$, then H is an abelian subgroup.

Problem 3.20. Is the converse of the previous theorem true? If so, prove it. Otherwise, provide a counterexample.

As we've seen, some groups are abelian and some are not. If G is a group, then we define the **center** of G to be

$$Z(G) := \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

Notice that if G is abelian, then $Z(G) = G$. However, if G is not abelian, then $Z(G)$ will be a proper subset of G. In some sense, the center of a group is a measure of how close G is to being abelian.

Theorem 3.21. If G is a group, then $Z(G)$ is an abelian subgroup of G.

Problem 3.22. Find the center of each of the following groups.

(a) S_2

(b) V_4

(c) S_3

(d) D_3

(e) D_4

(f) R_4

(g) R_6

(h) $\mathrm{Spin}_{1\times 2}$

(i) Q_8

(j) $(\mathbb{Z},+)$

(k) $(\mathbb{R}\setminus\{0\},\cdot)$

3.2 Subgroup Lattices

One of the goals of this section is to gain better understanding of the structure of groups by studying their subgroups.

Suppose we wanted to find all of the subgroups of a finite group G. Theorems 3.7 and 3.9 tell us that $\{e\}$ and G itself are subgroups of G, but there may be others. Theorem 3.6 tells us that if we want to find other subgroups of G, we need to find nonempty subsets of G that are closed and contain all the necessary inverses. So, one method for finding subgroups would be to find all possible nonempty subsets of G and then go about determining which subsets are subgroups by verifying whether a given subset is closed under inverses and closed under the operation of G. This is likely to be fairly time consuming.

Another approach would be to utilize the fact that every subgroup H of G has a generating set. That is, if H is a subgroup of a group G, then there always exists a subset S of G such that $\langle S\rangle = H$. Given a subset S of G, $\langle S\rangle$ is guaranteed to be closed under inverses and the operation of the group G. So, we could determine all of the subgroups of G by generating groups with various subsets S of G. Of course, one drawback is that it might take a bit of effort to determine what $\langle S\rangle$ actually is. Another drawback is that two different subsets S and T may generate the same subgroup.

Let's make this a bit more concrete by exploring an example. Consider the group R_4. What are the subgroups of R_4? Since the order of R_4 is 4, we know that there are $2^4-1=15$ nonempty subsets of R_4. Some of these are subgroups, but most of them are not. Theorems 3.7 and 3.9 guarantee that $\{e\}$ and R_4 itself are subgroups of R_4. That's

2 out of 15 so far. Are there any others? Let's do an exhaustive search by playing with generating sets. We can certainly be more efficient, but below we list all of the possible subgroups we can generate using subsets of R_4. As you scan the list, you should take a moment to convince yourself that the list is accurate.

$\langle e\rangle = \{e\}$

$\langle r\rangle = \{e, r, r^2, r^3\}$

$\langle r^2\rangle = \{e, r^2\}$

$\langle r^3\rangle = \{e, r^3, r^2, r\}$

$\langle e, r\rangle = \{e, r, r^2, r^3\}$

$\langle e, r^2\rangle = \{e, r^2\}$

$\langle e, r^3\rangle = \{e, r^3, r^2, r\}$

$\langle r, r^2\rangle = \{e, r, r^2, r^3\}$

$\langle r, r^3\rangle = \{e, r, r^2, r^3\}$

$\langle r^2, r^3\rangle = \{e, r, r^2, r^3\}$

$\langle e, r, r^2\rangle = \{e, r, r^2, r^3\}$

$\langle e, r, r^3\rangle = \{e, r, r^2, r^3\}$

$\langle e, r^2, r^3\rangle = \{e, r, r^2, r^3\}$

$\langle r, r^2, r^3\rangle = \{e, r, r^2, r^3\}$

$\langle e, r, r^2, r^3\rangle = \{e, r, r^2, r^3\}$

Let's make a few observations. Scanning the list, we see only three distinct subgroups:

$$\{e\}, \{e, r^2\}, \{e, r, r^2, r^3\}.$$

Out of 15 nonempty subsets of R_4, only 3 subsets are subgroups. Our exhaustive search guarantees that these are the only subgroups of R_4. It is also worth pointing out that if a subset contains either r or r^3, then that subset generates all of R_4. The reason for this is that $\{r\}$ and $\{r^3\}$ are each minimal generating sets for R_4. More generally, observe that if we increase the size of the generating subset using an element that was already contained in the subgroup generated by the set, then we don't get anything new. For example, consider $\langle r^2\rangle = \{e, r^2\}$. Since $e \in \langle r^2\rangle$, we don't get anything new by including e in our generating set. We can state this as a general fact.

Theorem 3.23. Let G be a group and let $g_1, g_2, \ldots, g_n \in G$. If $x \in \langle g_1, g_2, \ldots, g_n\rangle$, then $\langle g_1, g_2, \ldots, g_n\rangle = \langle g_1, g_2, \ldots, g_n, x\rangle$.

In the previous theorem, we are not claiming that $\{g_1, g_2, \ldots, g_n\}$ is a generating set for G—although this may be the case. Instead, are simply making a statement about the subgroup $\langle g_1, g_2, \ldots, g_n\rangle$, whatever it may be.

We can capture the overall relationship between the subgroups of a group G using a **subgroup lattice**. Given a group G, the **lattice of subgroups** of G is the partially ordered set whose elements are the subgroups of G with the partial order relation being set inclusion. It is common to depict the subgroup lattice for a group using a **Hasse diagram**. The Hasse diagram of subgroup lattice is drawn as follows:

(1) Each subgroup H of G is a vertex.

(2) Vertices corresponding to subgroups with smaller order are placed lower in the diagram than vertices corresponding to subgroups with larger order. In particular,

the vertex for $\{e\}$ is placed at the bottom of the diagram and the vertex for G is placed at the top.

(3) There is an edge going up from H to K if $H \leq K$ and there is no subgroup L such that $H \leq L \leq K$ with $L \neq H, K$.

Notice that there is an upward path of edges in the Hasse diagram from H to K iff $H \leq K$. For convenience we will not make a distinction between the subgroup lattice for a group G and the corresponding Hasse diagram.

The Hasse diagram for the subgroup lattice for R_4 is given in Figure 3.2.

$$\begin{tikzcd}
\langle r \rangle = R_4 \arrow[d, no head] \\
\langle r^2 \rangle = \{e, r^2\} \arrow[d, no head] \\
\langle e \rangle = \{e\}
\end{tikzcd}$$

Figure 3.2. Subgroup lattice for R_4.

Let's see what we can do with $V_4 = \{e, v, h, vh\}$. Using an exhaustive search, we find that there are five subgroups:

$$\langle e \rangle = \{e\}$$

$$\langle h \rangle = \{e, h\}$$

$$\langle v \rangle = \{e, v\}$$

$$\langle vh \rangle = \{e, vh\}$$

$$\langle v, h \rangle = \langle v, vh \rangle = \langle h, vh \rangle = \{e, v, h, vh\} = V_4$$

For each subgroup above, we've used minimal generating sets to determine the subgroup. The subgroup lattice for V_4 is given in Figure 3.3. Notice that there are no edges among $\langle v \rangle, \langle h \rangle$, and $\langle vh \rangle$. The reason for this is that none of these groups are subgroups of each other.

The next two theorems provide some further insight into the overall structure of subgroups of a group.

Theorem 3.24. If G is a group such that $H, K \leq G$, then $H \cap K \leq G$. Moreover, $H \cap K$ is the largest subgroup contained in both H and K.

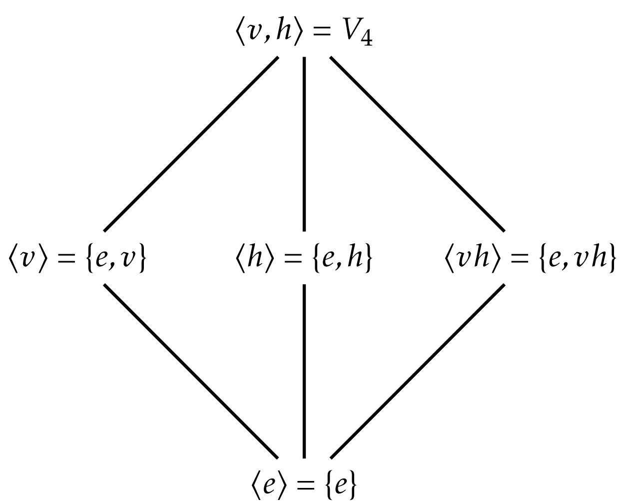

Figure 3.3. Subgroup lattice for V_4.

It turns out that we cannot simply replace “intersection” with “union” in the previous theorem

Problem 3.25. Provide an example of a group G and subgroups H and K such that $H \cup K$ is not a subgroup of G.

Theorem 3.26. If G is a group such that $H, K \leq G$, then $\langle H \cup K\rangle \leq G$. Moreover, $\langle H \cup K\rangle \leq G$ is the smallest subgroup containing both H and K.

Theorems 3.24 and 3.26 justify the use of the word “lattice” in “subgroup lattice”. In general, a partially ordered set in which every two elements have a unique **meet** (also called a **greatest lower bound** or **infimum**) and a unique **join** (also called a **least upper bound** or **supremum**). In the case of a subgroup lattice for a group G, the meet of subgroups H and K is $H \cap K$ and the join is $\langle H \cup K\rangle$. Figure 3.4 illustrates the meet (Theorem 3.24) and join (Theorem 3.26) in the case when H and K are not comparable.

In the next few problems, you are asked to create subgroup lattices. As you do this, try to minimize the amount of work it takes to come up with all the subgroups.

Problem 3.27. Find all the subgroups of $R_5 = \{e, r, r^2, r^3, r^4\}$ (where r is clockwise rotation of a regular pentagon by 72°) and then draw the subgroup lattice for R_5.

Problem 3.28. Find all the subgroups of $R_6 = \{e, r, r^2, r^3, r^4, r^5\}$ (where r is clockwise rotation of a regular hexagon by 60°) and then draw the subgroup lattice for R_6.

Problem 3.29. Find all the subgroups of $D_3 = \{e, r, r^2, s, sr, sr^2\}$ (where r and s are the usual symmetries of an equilateral triangle) and then draw the subgroup lattice for D_3.

Problem 3.30. Find all the subgroups of $S_3 = \langle s_1, s_2\rangle$ (where s_1 is the action is that swaps the positions of the first and second coins and s_2 is the action that swaps the second and third coins; see Problem 2.57) and then draw the subgroup lattice for S_3. How does your lattice compare to the one in Problem 3.29? You should look back at parts (d) and (f) of Problems 2.70 and ponder what just happened.

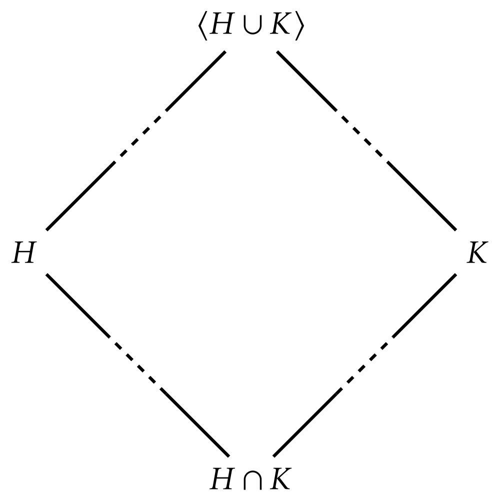

Figure 3.4. Meet and join for subgroups H and K.

Problem 3.31. Find all the subgroups of $D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$ (where r and s are the usual symmetries of a square) and then draw the subgroup lattice for D_4.

The last problem in this section is a step up in difficulty.

Problem 3.32. Find all the subgroups of $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ and then draw the subgroup lattice for Q_8.

3.3 Isomorphisms

Coming soon.