MA 2560: Calculus II (Fall 2009)
Exam 1

NAME: Solutions (2 points!)

Instructions: Answer each of the following questions completely. To receive full credit, you must *justify* each of your answers (unless stated otherwise). How you reached your answer is more important than the answer itself. If something is unclear, or if you have any questions, then please ask. Good luck!

Here are some potentially useful formulas. You should *not* expect to use all of them.

$$\frac{d}{dx}[\sinh x] = \cosh x \qquad \frac{d}{dx}[\cosh x] = \sinh x$$

$$\frac{d}{dx}[\tanh x] = \text{sech}^2 x \qquad \frac{d}{dx}[\text{sech } x] = -\text{sech } x \tanh x$$

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2} \qquad \frac{d}{dx}[\sec^{-1} x] = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\tanh^{-1} x] = \frac{1}{1-x^2} \qquad \frac{d}{dx}[\text{sech}^{-1} x] = \frac{-1}{x\sqrt{1-x^2}}$$

$$\int \sinh u \, du = \cosh u + C \qquad \int \cosh u \, du = \sinh u + C$$

$$\int \text{sech}^2 u \, du = \tanh u + C \qquad \int \text{sech } u \tanh u \, du = -\text{sech } u + C$$

$$\int \frac{1}{\sqrt{a^2-u^2}} \, du = \sin^{-1}\frac{u}{a} + C \qquad \int \frac{1}{u^2+a^2} \, du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

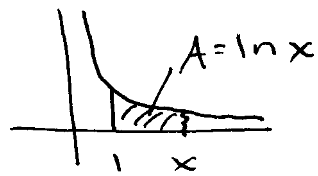
$$\int \frac{1}{u\sqrt{u^2-a^2}} \, du = \frac{1}{a}\sec^{-1}\frac{u}{a} + C \qquad \int \frac{1}{\sqrt{u^2+a^2}} \, du = \sinh^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{\sqrt{u^2-a^2}} \, du = \cosh^{-1}\frac{u}{a} + C \qquad \int \frac{1}{a^2-u^2} \, du = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C$$

1. (4 points) State the definition of the *natural log function* in terms of a definite integral and draw a picture of what this integral represents (at least for $x > 1$).

$$\ln x = \int_1^x \frac{1}{t}\,dt,$$

for $x > 0$



2. (4 points) State the definition of the *natural exponential function* in terms of the natural log function.

$y = e^x$ iff $x = \ln y$

3. (4 points) How are the natural log function and the natural exponential function related? How are their graphs related?

They are inverses of each other. Their graphs are reflections over the line $y = x$.

4. (4 points) Why do we need to restrict the domains of the 6 trigonometric functions in order to obtain (partial) inverses?

Otherwise, the graphs of the 6 trig fcns do not pass horizontal line test, and hence, do not have inverses.

5. (8 points) Let $f(x) = x + \arctan e^x$. It turns out that f is a one-to-one function, which implies that f^{-1} exists (you do *not* need to show this). Find $(f^{-1})'\left(\frac{\pi}{4}\right)$.

$$(f^{-1})'(\pi/4) = \frac{1}{f'(f^{-1}(\pi/4))}$$

$$= \frac{1}{f'(0)} \quad (\text{since } f(0) = \pi/4)$$

$$= \frac{1}{\left(1 + \frac{1}{1+1}\cdot 1\right)} = \boxed{\frac{2}{3}}$$

6. (6 points each) Differentiate each of the following functions. (You do *not* need to simply your answers.)

(a) $f(x) = \dfrac{e^{\tanh x}}{\sinh x}$ (use quotient rule)

$$f'(x) = \frac{e^{\cancel{\tanh}\,\tanh x} \cdot \operatorname{sech}^2 x \cdot \sinh x - e^{\tanh x} \cosh x}{(\sinh x)^2}$$

(b) $y = \sqrt{\arcsin e^x}$ (use chain rule)

$$\frac{dy}{dx} = \frac{1}{2}(\arcsin e^x)^{-1/2} \cdot \frac{1}{\sqrt{1 - e^{2x}}} \cdot e^x$$

7. (8 points) Use logarithmic differentiation to find the derivative of the following function. (You do *not* need to simplify your answer.)

$$y = \frac{\sqrt{1-x}}{x^{2/3}(x^2+5)^3}$$

$$\ln y = \frac{1}{2}\ln(1-x) - \frac{2}{3}\ln x - 3\ln(x^2+5)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{-1}{2(1-x)} - \frac{2}{3x} - \frac{6x}{x^2+5}$$

$$\frac{dy}{dx} = \frac{\sqrt{1-x}}{x^{2/3}(x^2+5)^3}\left[\frac{-1}{2(1-x)} - \frac{2}{3x} - \frac{6x}{x^2+5}\right]$$

8. (8 points) Find an equation of the tangent line to the graph of $f(x) = x\ln x$ at $x = e$. (Your answer does not need to be in any particular form, but you should use exact values.)

$x_0 = e$

$y_0 = f(e) = e\ln e = e$

$m = f'(e) = \left(\ln x + x\cdot\frac{1}{x}\right)\Big|_{x=e} = \ln e + 1 = 2$

Then $y - y_0 = m(x - x_0)$ becomes

$$\boxed{y - e = 2(x-e)}$$

9. (6 points) Evaluate the following limit. If the limit does not exist, specify whether it equals ∞, $-\infty$, or simply does not exist (in which case, write DNE). If the limit does exist, you should give an *exact* answer, as opposed to a decimal approximations.

$$\lim_{x \to 0^+} \arctan(\ln x)$$

First, note that $\lim_{x \to 0^+} \ln x = -\infty$. Second, note that $\lim_{u \to -\infty} \arctan(u) = -\pi/2$. Therefore, $\lim_{x \to 0^+} \arctan(\ln x) = \boxed{-\pi/2}$

10. (8 points each) Integrate each of the following indefinite or definite integrals. For the definite integrals, you should give an *exact* answer, rather than a decimal approximation.

(a) $\int_1^e \frac{\ln x}{x}\,dx$ $\qquad u = \ln x$

$$= \int_{x=1}^{x=e} \frac{u}{\cancel{x}} \cdot \cancel{x}\,du$$

$$= \frac{u^2}{2}\Big|_{x=1}^{x=e}$$

$$= \frac{(\ln x)^2}{x}\Big|_1^e = \boxed{\frac{1}{2}}$$

(b) $\int_0^2 \frac{1}{4x^2+16}$

$a = 4$

$u = 2x$

$du = 2dx$

$dx = \frac{du}{2}$

$$= \int_{x=0}^{x=2} \frac{1}{u^2+4^2}\,\frac{du}{2}$$

$$= \frac{1}{2} \cdot \frac{1}{4} \arctan\left(\frac{2x}{4}\right)\Big|_0^2$$

$$= \frac{1}{8}\left[\arctan(1) - \arctan(0)\right] = \boxed{\frac{\pi}{32}}$$

(c) $\int \frac{x}{\sqrt{16-4x^2}}\,dx$

$u = 16 - 4x^2$

$du = -8x\,dx$

$dx = \frac{du}{-8x}$

$= \int \frac{\cancel{x}}{u^{1/2}} \frac{du}{8\cancel{x}}$

$= \frac{-1}{8} \int u^{-1/2}\,du$

$= -\frac{1}{8} \cdot 2\, u^{1/2} + C$

$= \boxed{\frac{-1}{4}\sqrt{16-4x^2} + C}$

(d) $\int \frac{1}{\sqrt{16-4x^2}}\,dx$

$a = 4$

$u = 2x$

$du = 2\,dx$

$dx = \frac{du}{2}$

$= \int \frac{1}{\sqrt{4^2 - u^2}} \frac{du}{2}$

$= \frac{1}{2} \arcsin\left(\frac{2x}{4}\right) + C$

$= \boxed{\frac{1}{2}\arcsin\left(\frac{x}{2}\right) + C}$

(e) $\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$

$a = 1$

$u = e^x$

$du = e^x\,dx$

$du = \frac{dx}{e^x}$

$$= \int \frac{1}{\sqrt{u^2-1}}\,\frac{du}{e^x}$$

$$= \int \frac{1}{u\sqrt{u^2-1}}\,du$$

$$= \frac{1}{1}\operatorname{arcsec}\left(\frac{u}{1}\right) + C = \boxed{\operatorname{arcsec}(e^x) + C}$$

11. **Bonus Question:** (5 points, missing this question will not count against you) Prove that

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}.$$

(Hint: If $y = \arcsin x$, then $x = \sin y$. Now, use implicit differentitation to find dy/dx and then use your knowledge of trig.)

Since $y = \arcsin x$, $x = \sin y$. Using implicit differentiation, we get

$$1 = \cos y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$