Chapter 3

Set Theory and Topology

At its essence, all of mathematics is built on set theory. In this chapter, we will introduce some of the basics of sets and their properties.

3.1 Sets

Definition 3.1. A **set** is a collection of objects called **elements**. If A is a set and x is an element of A, we write $x \in A$. Otherwise, we write $x \notin A$.

Definition 3.2. The set containing no elements is called the **empty set**, and is denoted by the symbol $\emptyset$.

If we think of a set as a box potentially containing some stuff, then the empty set is a box with nothing in it.

Definition 3.3. The language associated to sets is specific. We will often define sets using the following notation, called **set builder notation**:

$$\boxed{S = \{x \in A \mid x \text{ satisfies some condition}\}}$$

The first part "$x \in A$" denotes what type of x is being considered. The statements to the right of the colon are the conditions that x must satisfy in order to be members of the set. This notation is read as "The set of all x in A such that x satisfies some condition," where "some condition" is something specific about the restrictions on x relative to A.

Exercise 3.4. Unpack each of the following sets and see if you can find a simple description of the elements that each set contains.

(a) $A = \{x \in \mathbb{N} \mid x = 3k \text{ for some } k \in \mathbb{N}\}$

(b) $B = \{t \in \mathbb{R} \mid t^2 \leq 2\}$

(c) $C = \{t \in \mathbb{Z} \mid t^2 \leq 2\}$

(d) $D = \{m \in \mathbb{R} \mid m = 1 - \frac{1}{n}, \text{ where } n \in \mathbb{N}\}$

Exercise 3.5. Write each of the following sentences using set builder notation.

(a) The set of all real numbers less than $-\sqrt{2}$.

(b) The set of all real numbers greater than -12 and less than or equal to 42.

(c) The set of all even natural numbers.

Definition 3.6. If A and B are sets, then we say that A is a **subset** of B, written $\boxed{A \subseteq B}$, provided that every element of A is also an element of B.

Observe that $A \subseteq B$ is equivalent to "For all x (in the universe of discourse), if $x \in A$, then $x \in B$." Since we know how to deal with "for all" statements and conditional propositions, we know how to go about proving $A \subseteq B$.

Problem 3.7. Suppose that A and B are sets. Describe a skeleton proof for proving that $A \subseteq B$.

Every set always has two rather boring subsets.

Theorem 3.8. Let S be a set. Then

(a) $S \subseteq S$,

(b) $\emptyset \subseteq S$.

Exercise 3.9. List all of the subsets of $A = \{1, 2, 3\}$. Any conjectures about how many there might be for a set with n elements?

Theorem 3.10 (Transitivity of subsets)**.** Suppose that A, B, and C are sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Definition 3.11. If $A \subseteq B$, then A is called a **proper subset** provided that $A \neq B$. In this case, we may write $\boxed{A \subset B}$ or $\boxed{A \subsetneq B}$.[1]

The following definitions should look familiar from precalculus.

Definition 3.12 (Interval Notation)**.** For $a, b \in \mathbb{R}$ with $a < b$, we define the following.

(a) $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

(b) $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$

(c) $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$

(d) $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

We analogously define $[a, b)$, $(a, b]$, $[a, \infty)$, and $(-\infty, b]$.

Exercise 3.13. Provide two examples of proper subsets of the interval $[0, 1]$.

[1] *Warning:* Some books use $\subset$ to mean $\subseteq$.

Definition 3.14. Let A and B be sets in some universe of discourse U.

(a) The **union** of the sets A and B is $\boxed{A \cup B} = \{x \in U \mid x \in A \text{ or } x \in B\}$.

(b) The **intersection** of the sets A and B is $\boxed{A \cap B} = \{x \in U \mid x \in A \text{ and } x \in B\}$.

(c) The **set difference** of the sets A and B is $\boxed{A \setminus B} = \{x \in U \mid x \in A \text{ and } x \notin B\}$.

(d) The **complement of** A (relative to U) is the set $\boxed{A^c} = U \setminus A = \{x \in U \mid x \notin A\}$.

Definition 3.15. If two sets A and B have the property that $A \cap B = \emptyset$, then we say that A and B are **disjoint** sets.

Exercise 3.16. Suppose that the universe of discourse is $U = \{1,2,3,4,5,6,7,8,9,10\}$. Let $A = \{1,2,3,4,5\}$, $B = \{1,3,5\}$, and $C = \{2,4,6,8\}$. Find each of the following.

(a) $A \cap C$

(b) $B \cap C$

(c) $A \cup B$

(d) $A \setminus B$

(e) $B \setminus A$

(f) $C \setminus B$

(g) B^c

(h) A^c

(i) $(A \cup B)^c$

(j) $A^c \cap B^c$

Exercise 3.17. Suppose that the universe of discourse is $U = \mathbb{R}$. Let $A = [-3,-1)$, $B = (-2.5,2)$, and $C = (-2,0]$. Find each of the following.

(a) A^c

(b) $A \cap C$

(c) $A \cap B$

(d) $A \cup B$

(e) $(A \cap B)^c$

(f) $(A \cup B)^c$

(g) $A \setminus B$

(h) $A \setminus (B \cup C)$

(i) $B \setminus A$

Theorem 3.18. Let A and B be sets. If $A \subseteq B$, then $B^c \subseteq A^c$.

Definition 3.19. Two sets A and B are **equal**, denoted $\boxed{A = B}$, iff $A \subseteq B$ and $B \subseteq A$.

Given two sets A and B, if we want to prove $A = B$, then we have to do two separate "mini" proofs: one for $A \subseteq B$ and one for $B \subseteq A$.

Theorem 3.20. Let A and B be sets. Then $A \setminus B = A \cap B^c$.

For each of the next two theorems, you can choose to prove either part (a) or part (b).

Theorem 3.21 (DeMorgan's Law)**.** Let A and B be sets. Then

(a) $(A \cup B)^c = A^c \cap B^c$,

(b) $(A \cap B)^c = A^c \cup B^c$.

Theorem 3.22 (Distribution of Union and Intersection)**.** Let A, B, and C be sets. Then

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.