Visual Group Theory

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Summer 2009

Introduction

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8. OK, let's get started!

Chapter 1: What is a group?

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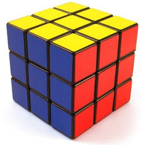

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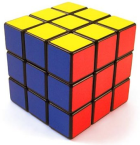

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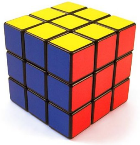

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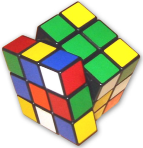

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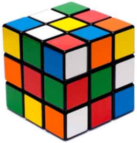

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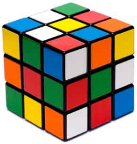

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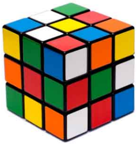

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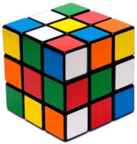

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Group theory is not primarily about numbers, but rather about patterns and symmetry; something the Rubik's Cube possesses in abundance.

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Observation 1.4

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Group theory studies the mathematical consequences of these 4 observations, which in turn will help us answer interesting questions about symmetrical objects.

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2. Helps us speak the same language, so that we know we are discussing the same objects (trapezoids. . .).
3. The rules provide the groundwork for making logical deductions, so that we can discover new facts.

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Rule 1.8

Any sequence of consecutive actions is also an action.

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Definition 1.9

A group is a system or collection of actions satisfying Rules 1.5–1.8.

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3. I'd like two groups to volunteer to discuss their answers to the two previous exercises.
4. Now, mix the groups up, so that no group stays the same. In your new groups, complete Exercise 1.8. I want each group to turn in a complete solution.

Potential quiz questions

Here are some potential questions that I may ask you on tomorrow's quiz at the beginning of class:

1. State our unofficial definition of a group by listing the 4 rules.
2. Define generators.
3. Provide 2 examples of a group. In each case, describe a set of generators.

References

I borrowed images from the following web pages:

- http://www.cunymath.cuny.edu/?page=mm
- http://www.math.cornell.edu/~mec/Winter2009/Lipa/Puzzles/lesson2.html