Chapter 1: Logic
Sections 1.10–1.14

Dana C. Ernst

Plymouth State University
Department of Mathematics
http://oz.plymouth.edu/~dcernst

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1. Produce a candidate (either find a specific object that works or deduce that there must be such an object).
2. Assume that there are two candidates and then demonstrate that they must actually be the same.

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Notice that if we want to show that "If A, then B" is false, we must show that "There exists x such that $A(x)$ and $\sim B(x)$" is true. In other words, providing a counterexample to an implication is equivalent to proving the existence theorem "There exists x such that $A(x)$ and $\sim B(x)$".

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It is important to realize that providing a counterexample is conclusive proof that an implication is *not* true. However, checking even a large number of examples (unless you've check them ALL) *never* constitutes a proof of an implication.

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Strategy for proving implications via direct proof

1. Assume that there is an arbitrary x in the universe that satisfies the hypothesis (i.e., makes $A(x)$ true).
2. Show/deduce that x satisfies the conclusion (i.e., makes $B(x)$ true).

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Observe that $2k^2 + 3$ is an integer. Therefore, $x^2 + 7$ is odd. ∎

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Strategy for proving an implication by contrapositive

To prove $A \implies B$ by contrapositive, assume that there is an x such that $\sim B(x)$ and then show $\sim A(x)$.

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Strategy for proving an implication by contradiction

1. Assume A and $\sim B$.
2. Derive some statement P and its negation $\sim P$. (This may be harder than it sounds since it takes some skill to determine what statement P you might be able to contradict.)

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You'll explore this idea further in Exercise 1.14.2.

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Let's do an example.

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For sake of a contradiction, assume that $a > 0$ and $1/a \leq 0$. Since $1/a \leq 0$, there exists a nonnegative number b such that $1/a + b = 0$. Multiplying both sides by a yields $1 + ab = 0$,

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