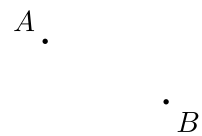
Section 10.3: Polar Coordinates

Goal

The goal of this section is to introduce the polar coordinate system and some basic polar graphs.

The polar coordinate system

Suppose you have two distinct points in the plane, labelled A and B, respectively.



Imagine a person wearing a blindfold is standing at the point A facing due East. Could you provide the blindfolded person with a list of instructions for traveling from A to B? There are many ways to do this, however, all of them require at least two pieces of information.

Here is one way: (1) walk x units East (where x is the appropriate distance), then (2) walk y units South (where y is the appropriate distance). This is exactly how rectangular coordinates work. For example, the point $(4, 3)$ is located 4 units to the right of the origin on the x-axis and 3 units up on the y-axis. This idea motivates the design of rectangular graph paper (a grid of squares).

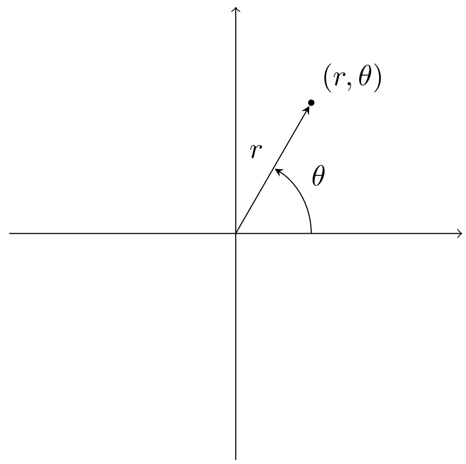
Clearly, following steps (1) and (2) does not provide us with the shortest path from A to B. In the space below, describe two steps that get the blindfolded person from A to B in the shortest distance possible (assume blindfolded person is facing due East). I'm looking for a general description; you do not need to specific exact values.

I'm hoping that what you just described above is equivalent to polar coordinates. To form the *polar coordinate system*, fix a point called the *origin* (or *pole*) and label every point in the plane with an ordered pair (r, θ), where

$r =$ directed $(+/-)$ distance from origin to point

$\theta =$ directed $(+/-)$ angle (counterclockwise if $+$, clockwise if $-$) from positive x-axis (from rectangular coordinates)

Here's the general picture, where the axes are the x and y axes of rectangular coordinates (used for reference):



In the picture above, plot the following points:

$$(-r, \theta), (-r, -\theta), (r, -\theta)$$

What would graph paper look like for the polar coordinate system? Draw a picture of it.

Important Note 1. Unlike rectangular coordinates, polar coordinates do *not* have unique representations. For example, the origin in the polar coordinate system can be labelled $(0, \theta)$ for *any* θ.

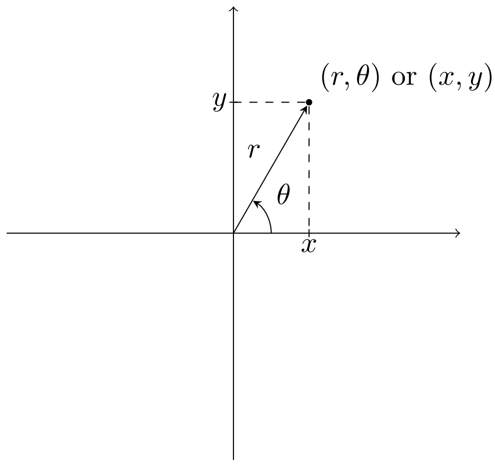
Example 2. Find two more representations for the point $(1, \pi/6)$ in polar coordinates.

In general, we have

$$\boxed{(r, \theta) = (r, \theta + 2\pi) \text{ or } (-r, \theta + \underline{\qquad})}$$

Converting between coordinate systems

It will be useful for us to be able to convert back and forth between rectangular and polar coordinates. Using the following picture, complete the following:



$x^2 + y^2 =$ __________

__________ $= r\cos\theta$

__________ $= r\sin\theta$

$\dfrac{y}{x} =$ __________

Example 3.

(a) Convert $(r, \theta) = (3, \pi/2)$ to rectangular coordinates using the formulas above.

(b) Convert $(x, y) = (-1, \sqrt{3})$ to polar coordinates using the formulas above. (Note that there are infinitely many correct answers. Try to get an answer with positive r and positive θ.)

Example 4. Convert each of the following polar equations to an equation in rectangular coordinates (i.e., an equation involving x and y) and sketch the corresponding graph.

(a) $r = 3$

(b) $r = \sec\theta$

Polar graphs

To gain more intuition, let's plot a polar graph by hand.

Example 5. Complete the following table and then use the corresponding points to sketch the graph of the polar curve $r = 1 + 2\sin\theta$.

θ	$r = 1 + 2\sin\theta$
0	
$\pi/6$	
$\pi/2$	
$5\pi/6$	
π	
$7\pi/6$	
$3\pi/2$	
$11\pi/6$	
2π	

We don't want to have to plot points all the time and often it is very difficult to convert from polar to rectangular. Thankfully, we have technology! Soon we'll take a look at some common examples of polar graphs, but first let's discuss how to sketch polar graphs using Sage, your calculator, and Wolfram|Alpha.

Sage: First, you need to define your independent variable and the function you are interested in plotting. For example, if you want to plot $r = 1 + 2\sin(t)$ for $t \in [0, 2\pi]$, you would enter the following in an empty Sage cell:

```
var('t')
r=1+2*sin(t)
```

Then to plot the function, you would enter the following in an empty Sage cell:

```
polar_plot(r,0,2*pi,aspect_ratio=1)
```

or

```
polar_plot(1+2*sin(t),0,2*pi,aspect_ratio=1)
```

where `aspect_ratio=1` is used to make sure the picture isn't "squashed."

Graphing Calculator: I'll describe the steps on a `TI-83`. First, make sure you are in `Radian` mode. Under the `MODE` menu, select `Pol` instead of `Func` (notice that `Par` is an option; this can be used for parametric curves). Now, go to the `y=` menu, where you'll be able to type in polar functions. If you hit the `GRAPH` menu, you'll be able to see your graph. If you go to the `WINDOW` menu, you can modify the range of θ values by changing θ`min=` and θ`max=`.

Wolfram|Alpha: Just type it in! For example, if you wanted to graph $r = 2\sin(3\theta)$, you would type "r=2 sin(3theta)." If you only wanted to plot this graph on the interval $[0, \pi/2]$, you would type "r=2 sin(3theta) from theta=0 to theta=pi/2."

Double check that the graph you sketched above is correct. If you made a mistake it was probably in the third and fourth quadrants.

By experimenting with specific values using Sage, your calculator, or Wolfram|Alpha, determine what each of the following polar graphs looks like in general. In each case, determine a smallest interval of θ-values necessary to sketch the entire graph (try to use positive values of θ). For example, do you only need θ to range over $[0, \pi]$ or do you need all of $[0, 2\pi]$? Assume $a > 0$.

1. Circles: In each case, determine where the center of the circle is and what the radius is.

   (a) $r = a$

   (b) $r = 2a \sin\theta$

   (c) $r = 2a \cos\theta$

2. Limacons: In each case, determine what happens when $a = b$, $a < b$, or $a > b$.

   (a) $r = a + b\sin\theta$

   (b) $r = a + b\cos\theta$

3. Rose petals: In each case, determine what happens if a is even or if a is odd (i.e., how many petals?).

   (a) $r = \sin(a\theta)$

   (b) $r = \cos(a\theta)$

