Goal: To understand rational functions where both the numerator and denominator vanish.

The key to analyzing a rational function $f(x) = \frac{P(x)}{Q(x)}$ at, or near, $x = a$ when $P(a) = 0$ and $Q(a) = 0$ is the following theorem from algebra:

The Factor Theorem. Suppose $P(x)$ is a polynomial and a is a root of $P(x)$. That is, $P(a) = 0$. Then $P(x)$ factors as $P(x) = (x - a)Q(x)$, where $Q(x)$ is another polynomial.

For example, let $P(x) = x^3 + 8$. It's easy to see that $P(-2) = 0$, since

$$P(-2) = (-2)^3 + 8 = -8 + 8 = 0,$$

so by the above theorem, $x - (-2) = x + 2$ is a factor of $P(x)$. Using long division we can EXPLICITLY factor $x + 2$ out of $P(x)$ to find that $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$.

1. Use the Factor Theorem to simplify the rational function $f(x) = \dfrac{x^2 - x - 2}{x + 1}$ and discover that its graph is a line missing a single point.

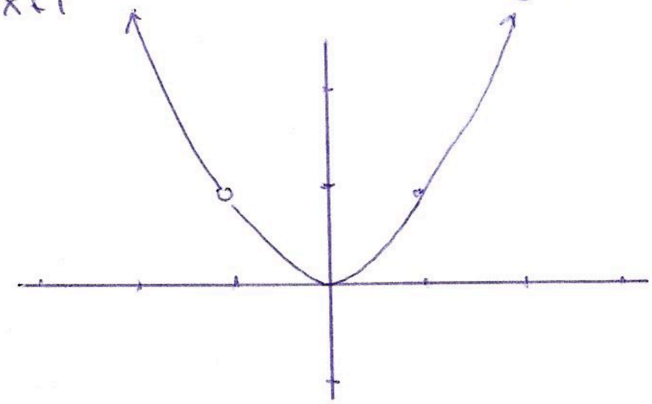
The domain of $f(x)$ is $\{x : x \neq -1\}$ so we have,

for $x \neq -1$, $f(x) = \dfrac{x^2 - x - 2}{x + 1} = \dfrac{(x+1)(x-2)}{x+1} = x - 2.$

So the graph of $f(x)$ is the line $y = x - 2$ missing the point where $x = -1$, i.e., the point $(-1, -3)$

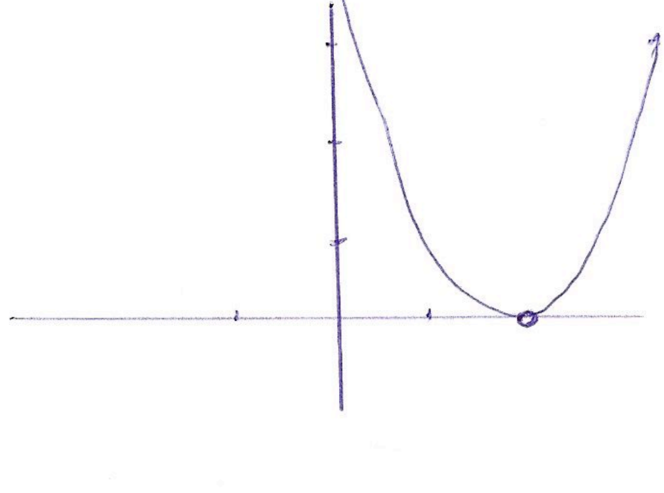
2. Use the Factor Theorem to see how to algebraically simplify each of the following rational functions. Sketch the graph for each function.

a. $f(x) = \dfrac{x^3 + x^2}{x + 1}$ The domain of $f(x)$ is $\{x : x \neq -1\}$. For $x \neq -1$, $f(x) = \dfrac{x^3 + x^2}{x+1} = \dfrac{x^2(x+1)}{x+1} = x^2$. So the graph of $f(x)$ is the parabola $y = x^2$ missing the point $(-1, 1)$



b. $g(x) = \dfrac{x^3 - 6x^2 + 12x - 8}{x - 2}$

The domain of $g(x)$ is $\{x : x \neq 2\}$. For $x \neq +2$ we have $g(x) = \dfrac{(x-2)(x^2-4x+4)}{x-2} = x^2 - 4x + 4$. So the graph of $g(x)$ is the parabola $y = x^2 - 4x + 4$ missing the point $(2, 0)$. The parabola is easy to graph if we notice that $x^2 - 4x + 4 = (x-2)^2$, and the graph of $y = (x-2)^2$ is the "shifted" parabola:



3. Use the graphs on the previous page to evaluate each of the following limits:

a. $\lim_{x\to -1} \dfrac{x^3+x^2}{x+1}$

As the x-coordinates of points on the curve (i.e. the graph of $f(x) = \dfrac{x^3+x^2}{x+1}$) become arbitrarily close to -1, their y-coordinates become arbitrarily close to 1. Therefore

$$\lim_{x\to -1} \frac{x^3+x^2}{x+1} = 1$$

b. $\lim_{x\to 2} \dfrac{x^3-6x^2+12x-8}{x-2}$

As the x-coordinates of points on the graph of $f(x) = \dfrac{x^3-6x^2+12x-8}{x-2}$ become arbitrarily close to 2, their y-coordinates become arbitrarily close to 0. Therefore

$$\lim_{x\to 2} \frac{x^3-6x^2+12x-8}{x-2} = 0$$

4. Explain how the Factor Theorem might be used to evaluate a limit $\lim_{x \to a} \frac{P(x)}{Q(x)}$, where $P(a) = 0$ and $Q(a) = 0$, without first graphing the function $f(x) = \frac{P(x)}{Q(x)}$

Since $\lim_{x \to a} \frac{P(x)}{Q(x)}$ concerns the behavior of the y-coordinates of points with x-coordinates near "a" but not equal to "a"; we can replace $\frac{P(x)}{Q(x)}$ by a simplified expression which agrees with it near $x = a$ but not at $x = a$, and compute its limit. Mathematically if $P(x) = (x-a)R(x)$ and $Q(x) = (x-a)S(x)$ then

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \lim_{x \to a} \frac{(x-a)R(x)}{(x-a)S(x)} = \lim_{x \to a} \frac{R(x)}{S(x)}$$

5. Use your observation in Problem 4 to evaluate the limit $\lim_{x \to 1} \frac{x^3 + x^2 + x - 3}{x^2 + x - 2}$.

$$x^3 + x^2 + x - 3 \Big|_{x=1} = 1 + 1 + 1 - 3 = 0$$

and

$$x^2 + x - 2 \Big|_{x=1} = 1 + 1 - 2 = 0$$

This means that $x - 1$ is a factor of both the numerator and denominator. Therefore

$$\lim_{x \to 1} \frac{x^3 + x^2 + x - 3}{x^2 + x - 2} = \lim_{x \to 1} \frac{(x-1)(x^2 + 2x + 3)}{(x-1)(x+2)}$$

$$= \lim_{x \to 1} \frac{x^2 + 2x + 3}{x + 2}$$

$$= \frac{6}{3}$$

$$= 2.$$