Section 7.1: Integration by Parts

Goal

In this section, we will introduce a technique of integration called *integration by parts*.

Integration by parts

The formula for *integration by parts* is as follows.

$$\boxed{\int u\ dv = uv - \int v\ du}$$

Let's see if we can derive this formula. First, let $u(x)$ and $v(x)$ be differentiable functions of x. Recall that

$$u'(x) = \frac{du}{dx} \quad \text{and} \quad v'(x) = \frac{dv}{dx}.$$

Switching to differential form, we see that

$$du = \_\_\_\_\_ \quad \text{and} \quad dv = \_\_\_\_\_.$$

Next, by the product rule, we have

$$\frac{d}{dx}[uv] = \_\_\_\_\_.$$

If we solve for uv' in the equation above, we obtain

$$uv' = \_\_\_\_\_.$$

Now, if we integrate (with respect to x) both sides of the equation above, we see that

$$\int uv'\ dx = \_\_\_\_\_.$$

Lastly, if we replace things with the appropriate differentials and simplify the first integral on the right, we obtain the desired formula:

$$\int u\ dv = uv - \int v\ du.$$

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Important Note 1. To use integration by parts, we need to identify

(i) u;

(ii) dv (it must be something we can integrate).

Then we must find

(iii) du (by differentiation);

(iv) v (by integration).

Note that the formula for integration by parts is what one would expect if we are dealing with a definite integral:

$$\boxed{\int_a^b u\, dv = \_\_\_\_\_.}$$

Examples

Let's do some examples.

Example 2. Integrate each of the following.

(a) $\int xe^{-x}\, dx$

(b) $\int x^2 \sin x \; dx$

(c) $\int_0^1 \arctan x \; dx$

Comments

As time goes on, our proficiency at picking the correct u and dv will increase. Here is a list of "suggestions" for common integrals using integration by parts.

1. For
$$\int x^n e^{ax}\ dx, \quad \int x^n \sin ax\ dx, \quad \int x^n \cos ax\ dx$$
let $u = x^n$ and $dv = e^{ax}\ dx$, $\sin ax\ dx$, or $\cos ax\ dx$.

2. For
$$\int x^n \ln x\ dx, \quad \int x^n \arcsin ax\ dx, \quad \int x^n \arctan ax\ dx$$
let $u = \ln x$, $\arcsin ax$, or $\arctan x$ and $dv = x^n\ dx$.

3. For
$$\int e^{ax} \sin bx\ dx, \quad \int e^{ax} \cos bx\ dx$$
either choice will work and regardless of your choice, you will have to do a "feedback loop" (see next examples).

Note 3. You can use the acronym LIATE to help you choose what to let u equal when doing integration by parts.

Logarithmic functions

Inverse trigonometric functions

Algebraic functions

Trigonometric functions

Exponential functions

More Examples

Here are two more examples, both of which exhibit what I call a "feedback loop".

Example 4. Integrate each of the following.

(a) $\int e^x \cos x \; dx$

(b) $\int \sec^3 x \; dx$

