Chapter 5: Five families

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Summer 2009

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Along the way, a variety of new concepts will arise, as well as some new visualization techniques.

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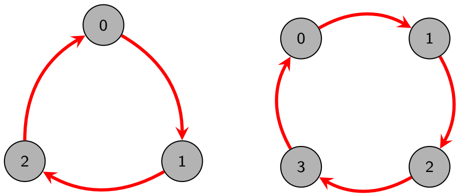

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For example, 1 and 5 generate C_6, while $1, 2, 3,$ and 4 all generate C_5.

Important: We have NOT proven this conjecture. We have only witnessed a few instances where it holds.

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If the headings on the multiplication table are arranged in the natural order, then each row is a cyclic shift to the left of the row above it.

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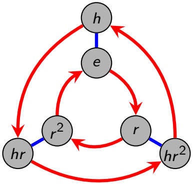

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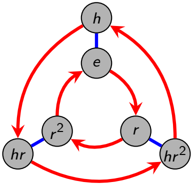

Do you see any copies of the Cayley diagram for any cyclic groups in this picture?

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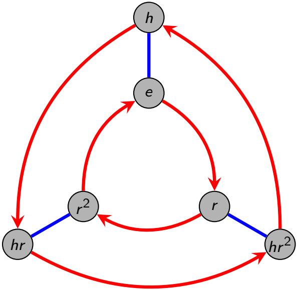
Let's work out the orbits for the remaining 5 elements of S_3.



element	orbit
e	$\{e\}$
r	$\{e, r, r^2\}$
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Another way of thinking about this is that the orbit of an element g is the collection of elements in the group that you can get to by doing g or it's inverse any number of times.

Cycle graphs

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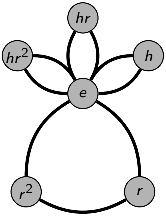

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See pages 72–73 for more examples.

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3. Now, complete Exercise 5.13(b). I want each group to turn in a complete solution.

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How can we use the Cayley diagram for a group to check to see if the corresponding group is abelian?

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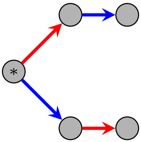
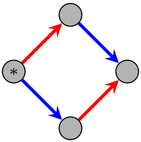
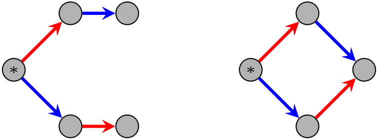




It turns out that it is enough to consider the order in which the generators are applied (Why? See Exercise 5.12). Suppose we have a group, where a and b are two of the generators and a and b are represented by red and blue arrows, respectively, in the Cayley diagram.

Commutativity requires $ab = ba$. It terms of arrows, this means that following a red arrow and then a blue arrow should put us at the same node as following a blue arrow and then a red arrow.



The pattern on the left never appears in the Cayley graph for an abelian group, whereas the pattern on the right illustrates the relation $ab = ba$.

Are cyclic groups abelian?

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How about the converse? That is, if a group is abelian, is it cyclic? The answer is no and the group V_4 provides an easy counterexample.

Let's explore a little further. The following diagrams (taken from Figure 5.9 on page 69 of *Visual Group Theory*) represent the Cayley diagrams for the groups D_4 and $C_2 \times C_4$, respectively.

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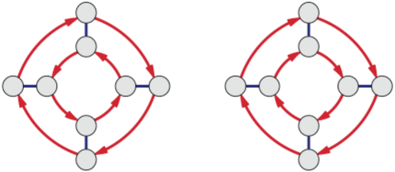

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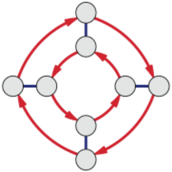

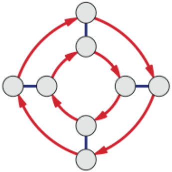

Are either one of these groups abelian?

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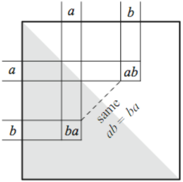

(This is Figure 5.11 on page 70 of *Visual Group Theory*.)

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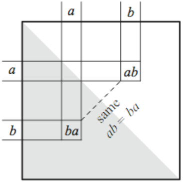

(This is Figure 5.11 on page 70 of *Visual Group Theory*.) Let's check this out in *Group Explorer*.

Dihedral groups

While cyclic groups describe objects that only have rotational symmetry, dihedral groups describe objects that have both rotational symmetry and bilateral symmetry (reflection across a midline).

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All the actions of C_n are also actions of D_n, but there are more actions than that. How many actions does D_n have?

D_n contains $2n$ actions:

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$$\underbrace{e, r, r^2, \ldots, r^{n-1}}_{\text{rotations}}, \underbrace{f, fr, fr^2, \ldots, fr^{n-1}}_{\text{reflections}}$$

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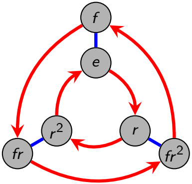

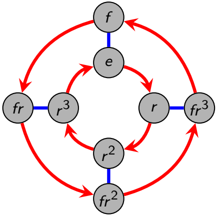

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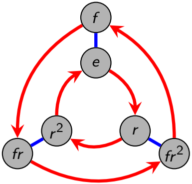

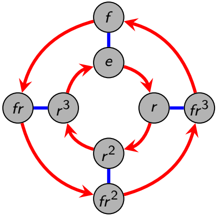

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In general, the Cayley diagram consists of an inner cycle and an outer cycle of n nodes each, where one cycle is clockwise and the other is counterclockwise.

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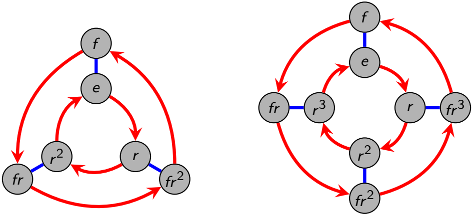

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In general, the Cayley diagram consists of an inner cycle and an outer cycle of n nodes each, where one cycle is clockwise and the other is counterclockwise. The two cycles are connected by two way arrows representing the flip.

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We can move from e to rf by walking clockwise one click and then moving to the other cycle. This is equivalent to first moving to the other cycle from e followed by $n-1$ clicks counter-clockwise, which puts us at fr^{n-1}. The relation $rf = fr^{n-1}$ will be useful to remember.

D_n consists of an r orbit (with smaller rotation orbit subsets) and n other two element flip orbits.

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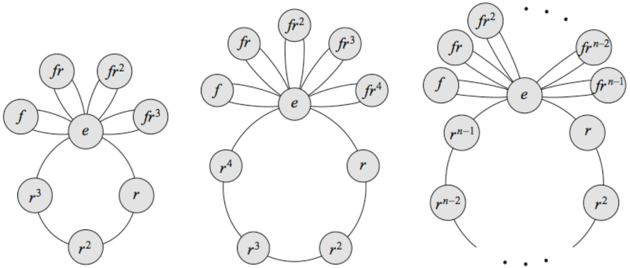

The separation of D_n into rotations and reflections is also visible in their multiplication tables.

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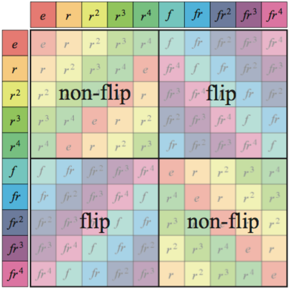
	e	r	r^2	r^3	r^4	f	fr	fr^2	fr^3	fr^4
e	e	r	r^2	r^3	r^4	f	fr	fr^2	fr^3	fr^4
r	r	r^2	r^3	r^4	e	fr^4	f	fr	fr^2	fr^3
r^2	r^2	r^3	r^4	e	r	fr^3	fr^4	f	fr	fr^2
r^3	r^3	r^4	e	r	r^2	fr^2	fr^3	fr^4	f	fr
r^4	r^4	e	r	r^2	r^3	fr	fr^2	fr^3	fr^4	f
f	f	fr	fr^2	fr^3	fr^4	e	r	r^2	r^3	r^4
fr	fr	fr^2	fr^3	fr^4	f	r^4	e	r	r^2	r^3
fr^2	fr^2	fr^3	fr^4	f	fr	r^3	r^4	e	r	r^2
fr^3	fr^3	fr^4	f	fr	fr^2	r^2	r^3	r^4	e	r
fr^4	fr^4	f	fr	fr^2	fr^3	r	r^2	r^3	r^4	e



(Figures 5.18 and 5.19 on pages 76 and 77, respectively, of *Visual Group Theory*.)

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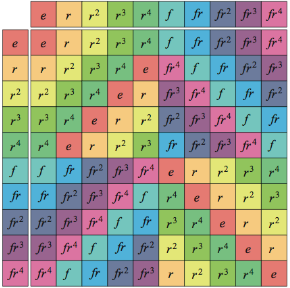

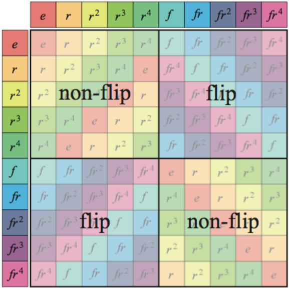

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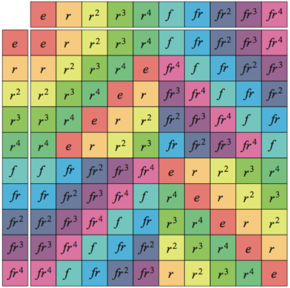
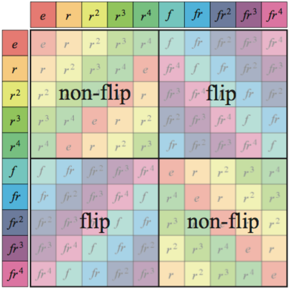


	e	r	r^2	r^3	r^4	f	fr	fr^2	fr^3	fr^4
e	e	r	r^2	r^3	r^4	f	fr	fr^2	fr^3	fr^4
r	r	r^2	r^3	r^4	e	fr^4	f	fr	fr^2	fr^3
r^2	r^2	r^3	r^4	e	r	fr^3	fr^4	f	fr	fr^2
r^3	r^3	r^4	e	r	r^2	fr^2	fr^3	fr^4	f	fr
r^4	r^4	e	r	r^2	r^3	fr	fr^2	fr^3	fr^4	f
f	f	fr	fr^2	fr^3	fr^4	e	r	r^2	r^3	r^4
fr	fr	fr^2	fr^3	fr^4	f	r^4	e	r	r^2	r^3
fr^2	fr^2	fr^3	fr^4	f	fr	r^3	r^4	e	r	r^2
fr^3	fr^3	fr^4	f	fr	fr^2	r^2	r^3	r^4	e	r
fr^4	fr^4	f	fr	fr^2	fr^3	r	r^2	r^3	r^4	e



(Figures 5.18 and 5.19 on pages 76 and 77, respectively, of *Visual Group Theory*.)

As we shall see later in the course, the partition of D_n as depicted above forms the structure of the group C_2. “Shrinking” a group in this way is called taking a quotient.

Let's explore a few more examples.

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1. In groups of 2–3, complete the following exercises (not collected):
   - Exercise 5.16(b)
   - Exercise 5.29(b)(c)

More group work

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1. In groups of 2–3, complete the following exercises (not collected):
   - Exercise 5.16(b)
   - Exercise 5.29(b)(c)
2. Let's discuss your solutions.

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A permutation is an action that rearranges a collection of things.

Because they are easy to write down and deal with, we will usually refer to permutations of positive integers (just like we did when we numbered our rectangle, etc.).

There are many ways to represent permutations, but we will use the notation illustrated by the following example.

Example

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Here are some permutations of 4 objects.

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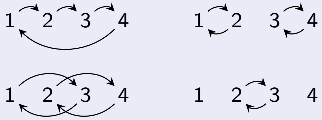

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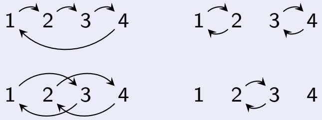

How many permutations of 4 objects are there?

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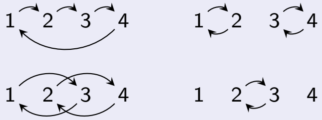

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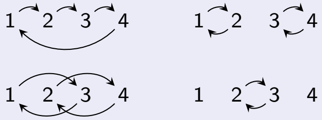

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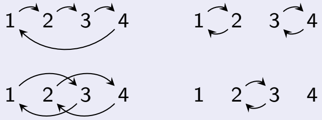

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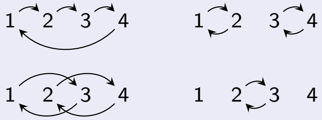

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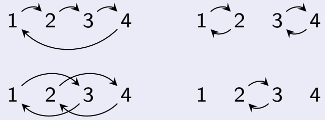

How many permutations of 4 objects are there? The answer is that there are $4! = 24$, which means that there are 24 distinct permutation pictures like above on 4 objects.

How many permutations of n objects are there? Yep, you guessed it: $n!$.

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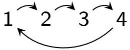

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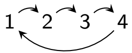

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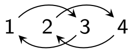

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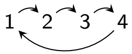
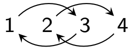


followed by

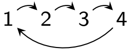


be equal to?

The first permutation rearranges the 4 objects and then we shuffle the result according to the second permutation.

In order for the collection of permutations of n objects to form a group (which is what we want!), we need to understand how to combine permutations. Let's consider an example.

What should



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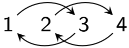

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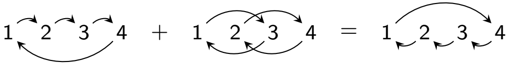

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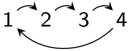

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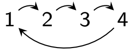
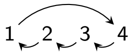


is simply



Definition

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The group of all permutations of n items is called the symmetric group (on n objects) and is denoted by S_n.

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Although the collection of *all* permutations of n items forms a group, creating a groups does not require taking all of the permutations. If we choose carefully, we can form groups by taking a subset of the permutations.

Alternating groups

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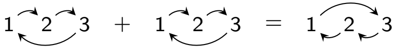
It turns out that the appropriate choice is the set of "squares" in S_n. What we mean by "square" is any element that can be written as an element of S_n times itself.

For example, since



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The group of squares from S_n is called the alternating group and is denoted A_n.

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$$\begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[r, bend left] & 3 \arrow[ll, bend left] \end{tikzcd} + \begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[r, bend left] & 3 \arrow[ll, bend left] \end{tikzcd} = \begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l, bend left] & 3 \arrow[l, bend left] \end{tikzcd}$$

The permutation

$$\begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l, bend left] & 3 \arrow[l, bend left] \end{tikzcd}$$

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Platonic solids

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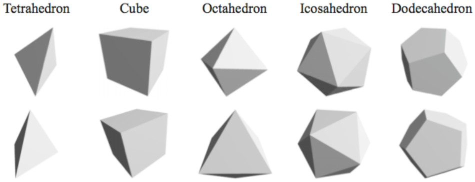
There are only 5 3-dimensional shapes all of whose faces are regular polygons that meet at equal angles. These 5 shapes are called the Platonic solids:



(Figure 5.26 on page 81 of *Visual Group Theory*.)

The groups of symmetries of the Platonic solids are as follows.

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shape	group
Tetrahedron	A_4
Cube	S_4
Octahedron	S_4
Icosahedron	A_5
Dodecahedron	A_5

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The Cayley diagrams for these 3 groups can be arranged in some very interesting configurations. In particular, the Cayley diagram for Platonic solid "blah" can be arranged on a truncated "blah", where truncated refers to cutting off some corners.

For example, here are two representations for Cayley diagrams of A_5, where the top is a truncated icosahedron and the bottom is a truncated dodecahedron.

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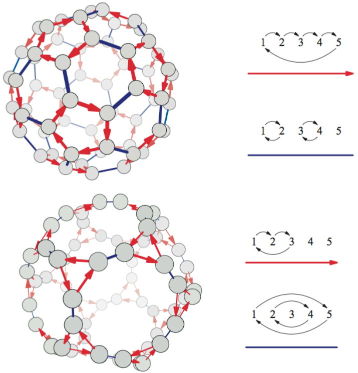

(Figure 5.29 on page 83 of *Visual Group Theory*.)

Cayley's theorem

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How do we do this?

Here is an algorithm given a Cayley diagram with n nodes:

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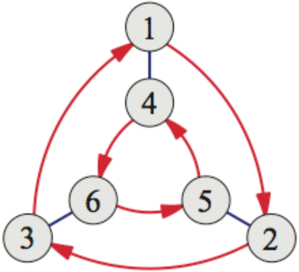

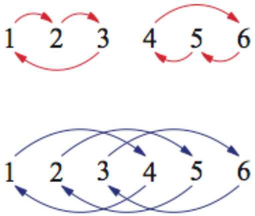

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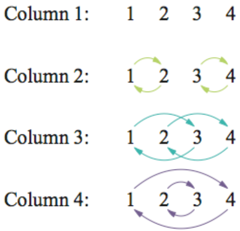
	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1



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Some more group work

Let's see Cayley's Theorem in action.

In groups of 2–3, find the permutation group for V_4 guaranteed to exist according to Cayley's theorem. Compare your answer with our original discussion of group of symmetries of the rectangle.

I want each group to turn in a complete solution.