Section 9.9: Arc Length

Goal

In this section, we will learn how integrals can be used to find the arc length of differentiable functions.

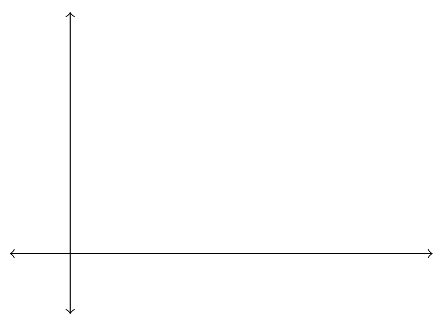
Arc length

Suppose that f is a "smooth" function (i.e., f' exists and is continuous, so that there are no sharp turns or vertical tangents on f) on a closed interval $[a, b]$.

Question 1. How could we approximate the arc length using things we know how to do?

One possible answer is to partition $[a, b]$ into equal width subintervals (as we did when we approximated area). Between each pair of adjacent points, form a line segment.

Here's the picture:



In this case,

$$\text{arc length} \approx \text{the sum of the lengths of the line segments} = \sum_{i=1}^{n} d(x_{i-1}, x_i).$$

But what is each $d(x_{i-1}, x_i)$ equal to?

$$d(x_{i-1}, x_i) =$$

We have just shown that

$$\text{arc length} \approx \sum_{i=1}^{n} \sqrt{1+[f'(c_i)]^2}\ \Delta x.$$

Well, how do you think we can get the exact value of the arc length?

$$s = \text{arc length} = \rule{6cm}{0.4pt}.$$

Therefore, the arc length of the smooth curve $y = f(x)$ over the interval $[a, b]$ is given by

$$\boxed{s = \int_a^b \sqrt{1+[f'(x)]^2}\ dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx.}$$

Sometimes $\sqrt{1+[f'(x)]^2}\ dx$ is denoted by ds, so that

$$s = \int_a^b ds.$$

Let's play with the arc length applet located at http://calculusapplets.com.

Examples

Let's do a couple of examples.

Example 2. Find the length of the curve $y = 2x^{3/2}$ over the interval $[0, 1]$.

Example 3. Prove that the circumference of the unit circle is 2π.

Example 4. Find the length of the curve $y = x^2 - \dfrac{\ln x}{8}$ over the interval $[1, e]$.

