Section 7.2: Trigonometric Integrals

Goal

As the title of the section suggests, in this section, we will be examining trigonometric integrals. More specifically, we will study techniques for evaluating integrals of the form

$$\int (\sin x)^m (\cos x)^n \, dx \quad \text{and} \quad \int (\sec x)^n (\tan x)^m \, dx.$$

Useful identities

Here are a couple of trig identities that will come in handy.

$$\boxed{\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \qquad\qquad \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)}$$

These two identities are often referred to as the *power reducing identities* for sine and cosine. Note that these identities are just the half-angle formulas in disguise.

Example 1. Integrate $\displaystyle\int \sin^2 3x \, dx$.

Guidelines for evaluating integrals involving sine and cosine

1. Check to see if ordinary u-substitution works.
2. If the power of sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then expand and integrate using u-substitution.
3. If the power of cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then expand and integrate using u-substitution.
4. If the powers of both the sine and cosine are even and nonnegative, make repeated use of the power reducing identities to convert the integrand to odd powers of cosine. Then proceed as in previous guideline.

Examples

Let's do a few examples.

Example 2. Integrate.

(a) $\displaystyle\int \sin^3 x \cos^2 x \; dx$

(b) $\displaystyle\int \cos^4 x \; dx$

Guidelines for evaluating integrals involving secant and tangent

1. Check to see if ordinary u-substitution works.
2. If the power of secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and use u-substitution to integrate.
3. If the power of tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and use u-substitution to integrate.
4. If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor. Then expand and use u-substitution to integrate.
5. If there are no tangent factors and the power of secant is odd and positive, use integration by parts (see Section 8.4).
6. If the previous guidelines do not apply, try converting to sines and cosines.

Examples

Let's do some examples.

Example 3. Integrate.

(a) $\displaystyle\int \sec^4 x \tan x \ dx$

(b) $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\ dx$

(c) $\displaystyle\int \tan^4 x\ dx$

(d) $\displaystyle\int \frac{\sec x}{\tan^2 x}\ dx$

Useful integration formulas

The following integration formulas will be useful.

$$\boxed{\int \tan x\ dx = \ln|\sec x| + C} \qquad \boxed{\int \sec x\ dx = \ln|\sec x + \tan x| + C}$$

Note that these formulas are easily verified by differentiating the right hand side.

