Chapter 9

Homomorphisms and the Isomorphism Theorems

9.1 Homomorphisms

Let G_1 and G_2 be groups. Recall that $\phi : G_1 \to G_2$ is an isomorphism if and only if ϕ

(a) is one-to-one,

(b) is onto, and

(c) satisfies the homomorphic property.

We say that G_1 is isomorphic to G_2 and write $G_1 \cong G_2$ if such a ϕ exists. Loosely speaking, two groups are isomorphic if they have the "same structure." What if we drop the one-to-one and onto requirement?

Definition 9.1. Let $(G_1, *)$ and $(G_2, \odot)$ be groups. A function $\phi : G_1 \to G_2$ is a **homomorphism** if and only if ϕ satisfies the homomorphic property:

$$\phi(x * y) = \phi(x) \odot \phi(y)$$

for all $x, y \in G_1$. At the risk of introducing ambiguity, we will usually omit making explicit reference to the binary operations and write the homomorphic property as

$$\phi(xy) = \phi(x)\phi(y).$$

Group homomorphisms are analogous to linear transformations on vector spaces that one encounters in linear algebra.

Figure 9.1 captures a visual representation of the homomorphic property. We encountered this same representation in Figure 5.6. If $\phi(x) = x'$, $\phi(y) = y'$, and $\phi(z) = z'$ while $z' = x' \odot y'$, then the only way G_2 may respect the structure of G_1 is for

$$\phi(x * y) = \phi(z) = z' = x' \odot y' = \phi(x) \odot \phi(y).$$

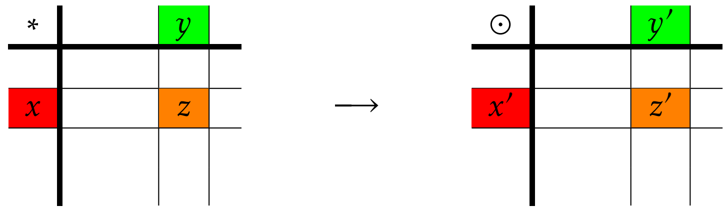

Figure 9.1

Exercise 9.2. Define $\phi : \mathbb{Z}_3 \to D_3$ via $\phi(k) = r^k$. Prove that ϕ is a homomorphism and then determine whether ϕ is one-to-one or onto. Also, try to draw a picture of the homomorphism in terms of Cayley diagrams.

Exercise 9.3. Let G and H be groups. Prove that the function $\phi : G \times H \to G$ given by $\phi(g,h) = g$ is a homomorphism. This function is an example of a **projection map**.

There is always at least one homomorphism between two groups.

Theorem 9.4. Let G_1 and G_2 be groups. Define $\phi : G_1 \to G_2$ via $\phi(g) = e_2$ (where e_2 is the identity of G_2). Then ϕ is a homomorphism. This function is often referred to as the **trivial homomorphism** or the 0-**map**.

Back in Section 5.5, we encountered several theorems about isomorphisms. However, at the end of that section we remarked that some of those theorems did not require that the function be one-to-one and onto. We collect those results here for convenience.

Theorem 9.5. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism.

(a) If e_1 and e_2 are the identity elements of G_1 and G_2, respectively, then $\phi(e_1) = e_2$.

(b) For all $g \in G_1$, we have $\phi(g^{-1}) = [\phi(g)]^{-1}$.

(c) If $H \leq G_1$, then $\phi(H) \leq G_2$, where

$$\phi(H) := \{y \in G_2 \mid \text{there exists } h \in H \text{ such that } \phi(h) = y\}.$$

Note that $\phi(H)$ is called the **image** of H. A special case is when $H = G_1$. Notice that ϕ is onto exactly when $\phi(G_1) = G_2$.

The following theorem is a consequence of Lagrange's Theorem.

Theorem 9.6. Let G_1 and G_2 be groups such that G_2 is finite and let $H \leq G_1$. If $\phi : G_1 \to G_2$ is a homomorphism, then $|\phi(G_1)|$ divides $|G_2|$.

The next theorem tells us that under a homomorphism, the order of the image of an element must divide the order of the pre-image of that element.

Theorem 9.7. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. If $g \in G_1$ such that $|g|$ is finite, then $|\phi(g)|$ divides $|g|$.

Every homomorphism has an important subset of the domain associated with it.

Definition 9.8. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. The **kernel** of ϕ is defined via

$$\ker(\phi) := \{g \in G_1 \mid \phi(g) = e_2\}.$$

The kernel of a homomorphism is analogous to the null space of a linear transformation of vector spaces.

Exercise 9.9. Identify the kernel and image for the homomorphism given in Exercise 9.2.

Exercise 9.10. What is the kernel of a trivial homomorphism (see Theorem 9.4).

Theorem 9.11. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Then $\ker(\phi) \trianglelefteq G_1$.

Theorem 9.12. Let G be a group and let $H \trianglelefteq G$. Then the map $\gamma : G \to G/H$ given by $\gamma(g) = gH$ is a homomorphism with $\ker(\gamma) = H$. This map is called the **canonical projection map**.

The upshot of Theorems 9.11 and 9.12 is that kernels of homomorphisms are always normal and every normal subgroup is the kernel of some homomorphism. It turns out that the kernel can tell us whether ϕ is one-to-one.

Theorem 9.13. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Then ϕ is one-to-one if and only if $\ker(\phi) = \{e_1\}$, where e_1 is the identity in G_1.

Remark 9.14. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Given a generating set for G_1, the homomorphism ϕ is uniquely determined by its action on the generating set for G_1. In particular, if you have a word for a group element written in terms of the generators, just apply the homomorphic property to the word to find the image of the corresponding group element.

Exercise 9.15. Suppose $\phi : Q_8 \to V_4$ is a group homomorphism satisfying $\phi(i) = h$ and $\phi(j) = v$.

(a) Find $\phi(1)$, $\phi(-1)$, $\phi(k)$, $\phi(-i)$, $\phi(-j)$, and $\phi(-k)$.

(b) Find $\ker(\phi)$.

(c) What well-known group is $Q_8/\ker(\phi)$ isomorphic to?

Exercise 9.16. Find a non-trivial homomorphism from $\mathbb{Z}_{10}$ to $\mathbb{Z}_6$.

Exercise 9.17. Find all non-trivial homomorphisms from $\mathbb{Z}_3$ to $\mathbb{Z}_6$.

Problem 9.18. Prove that the only homomorphism from D_3 to $\mathbb{Z}_3$ is the trivial homomorphism.

Exercise 9.19. Let F be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ and let D be the subset of differentiable functions on $\mathbb{R}$. It turns out that F is a group under addition of functions and D is a subgroup of F (you do not need to prove this). Define $\phi : D \to F$ via $\phi(f) = f'$ (where f' is the derivative of f). Prove that ϕ is a homomorphism. You may recall facts from calculus without proving them. Is ϕ one-to-one? Onto?

9.2 The Isomorphism Theorems

The next theorem is arguably the crowning achievement of the course.

Theorem 9.20 (The First Isomorphism Theorem)**.** Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Then

$$G_1/\ker(\phi) \cong \phi(G_1).$$

If ϕ is onto, then

$$G_1/\ker(\phi) \cong G_2.$$

Exercise 9.21. Let $\phi : Q_8 \to V_4$ be the homomorphism described in Exercise 9.15. Use the First Isomorphism Theorem to prove that $Q_8/\langle -1\rangle \cong V_4$.

Exercise 9.22. Define $\phi : S_n \to \mathbb{Z}_2$ via

$$\phi(\sigma) = \begin{cases} 0, & \sigma \text{ even} \\ 1, & \sigma \text{ odd.} \end{cases}$$

Use the First Isomorphism Theorem to prove that $S_n/A_n \cong \mathbb{Z}_2$.

Exercise 9.23. Use the First Isomorphism Theorem to prove that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6$. Attempt to draw a picture of this using Cayley diagrams.

Exercise 9.24. Use the First Isomorphism Theorem to prove that $(\mathbb{Z}_4 \times \mathbb{Z}_2)/(\{0\} \times \mathbb{Z}_2) \cong \mathbb{Z}_4$.

The next theorem is a generalization of Theorem 9.7 and follows from the First Isomorphism Theorem together with Lagrange's Theorem.

Theorem 9.25. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. If G_1 is finite, then $|\phi(G_1)|$ divides $|G_1|$.

We finish the chapter by listing a few of the remaining isomorphism theorems.

Theorem 9.26 (The Second Isomorphism Theorem)**.** Let G be a group with $H \leq G$ and $N \trianglelefteq G$. Then

(a) $HN := \{hn \mid h \in H, n \in N\} \leq G$;

(b) $H \cap N \trianglelefteq H$;

(c) $H/(H \cap N) \cong HN/N$.

Theorem 9.27 (The Third Isomorphism Theorem)**.** Let G be a group with $H, K \trianglelefteq G$ and $K \leq H$. Then $H/K \trianglelefteq G/K$ and

$$G/H \cong (G/K)/(H/K).$$

The last isomorphism theorem is sometimes called the *Lattice Isomorphism Theorem* or the *Correspondence Theorem*.

Theorem 9.28 (The Fourth Isomorphism Theorem)**.** Let G be a group with $N \trianglelefteq G$. Then there is a bijection from the set of subgroups of G that contain N onto the set of subgroups of G/N. In particular, every subgroup G is of the form H/N for some subgroup H of G containing N (namely, its pre-image in G under the canonical projection homomorphism from G to G/N.) This bijection has the following properties: for all $H, K \subseteq G$ with $N \subseteq H$ and $N \subseteq K$, we have

(a) $H \subset K$ if and only if $H/N \subset K/N$

(b) If $H \subset K$, then $|K : H| = |K/N : H/N|$

(c) $\langle H, K \rangle / N = \langle H/N, K/N \rangle$

(d) $(H \cap K)/N = H/N \cap K/N$

(e) $H \trianglelefteq G$ if and only if $H/N \trianglelefteq G/N$.