MA4220: Number Theory (Spring 2011)
Exam 1

NAME:

Instructions

This exam is worth 15% of your overall grade and all of the problems have equal weight. For each part of the exam, read the instructions carefully.

I expect your proofs to be *well-written, neat, and organized*. You should write in *complete sentences*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version.

The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 1.41, then you should say so.
2. Unless you prove them, you cannot use any results from the textbook that we have not covered.
3. You are NOT allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are NOT allowed to copy someone else's work.
5. You are NOT allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

The exam is due to my office by 5PM on **Friday, March 11**. You should turn in this cover page and all of the work that you have decided to submit.

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

Part 1

Complete *one* of the following problems. You need to justify your answers with sufficient work and should not rely on technology.

1. Exercise 1.50 on page 21.
2. Exercise 1.54 on page 22.

Part 2

Complete any 3 of the following problems.

3. The Fibonacci sequence is an infinite sequence that begins $1, 1, 2, 3, 5, 8, 13, \ldots$. Let F_n denote the nth Fibonacci number. The sequence is defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. Prove that for any natural number n, $(F_n, F_{n+1}) = 1$.
4. Suppose p and q are prime numbers such that $q = p + 2$. (Such primes are called *twin primes.*) Also, suppose that $p > 3$. Prove that $p \equiv 2 \pmod 3$.
5. Theorem 2.28 on page 34.
6. Which of the following statements is true? Prove your assertion.
   (a) There are no integers n such that $n^2 - 1$ is prime.
   (b) There are a nonzero but finite number of integers n such that $n^2 - 1$ is prime.
   (c) There are infinitely many integers n such that $n^2 - 1$ is prime.
7. Prove that if $n + 1$ integers are chosen from the set $\{1, 2, ..., 2n\}$ then at least 2 of the chosen integers are relatively prime.*

Part 3

Here are the instructions for this portion of the exam.

- Prove both of the following theorems.
- You are required to type your proofs and submit them to me via email at `dcernst@plymouth.edu`.
- Please put each proof on its own page. If you choose to type your entire exam, I would like these problems to be in a separate file.
- Send me a PDF file and name your file according to: `Exam1Part3Last-Name.pdf`.
- Do *not* include your name anywhere on the typeset document (but you should include your last name in the filename).

These proofs will be sent to students in a number theory course at Wellesley College to be peer reviewed. You will receive a critical review of your proof from a student at Wellesley, but their critique will *not* impact your grade. Similarly, we will be reviewing proofs submitted by students from Wellesley in the near future.

Theorem 1. If $a, b \in \mathbb{Z}$, not both 0, and $k \in \mathbb{N}$, then $\gcd(ka, kb) = k \cdot \gcd(a, b)$.†

Theorem 2. Suppose that $p_1, p_2, \ldots, p_n$ are distinct primes (i.e., each p_i is prime, but $p_i \neq p_j$ for $i \neq j$). Then $\sqrt{p_1 p_2 \cdots p_n}$ is irrational.

*Make a clever use of the Fundamental Theorem of Arithmetic.

†This is Theorem 1.55 on page 22.

