Spring 2007
Calc Final

① a) $\int x^2(3-x)\,dx = \int 3x^2 - x^3\,dx = \boxed{x^3 - \frac{1}{4}x^4 + C}$

b) $\int \frac{3}{1+x^2}\,dx = 3\int \frac{1}{1+x^2}\,dx = \boxed{3\arctan x + C}$

c) $\int_0^{\pi/12} \sin 2x\,dx \quad u = 2x \quad \frac{du}{2} = dx \quad \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos(2x)\Big|_0^{\pi/12}$

$= -\frac{1}{2}\cos \pi/6 + \frac{1}{2}\cos(0)$

$= -\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}(1) = \frac{1}{2} - \frac{\sqrt{3}}{4} = \frac{2-\sqrt{3}}{4}$

d) $\int \frac{x+3}{x}\,dx = \int \frac{x}{x} + \frac{3}{x}\,dx = \int 1\,dx + 3\int \frac{1}{x}\,dx$

$= \boxed{x + 3\ln|x| + C}$

e) $\int \frac{\sin x}{\cos^2 x}\,dx \quad u = \cos x \quad \frac{du}{dx} = -\sin x \quad \frac{du}{-\sin x} = dx \quad \int \frac{\sin x}{u^2}\,\frac{du}{-\sin x}$

$= \int -u^{-2}\,du = -(-u^{-1}) = \boxed{\frac{1}{\cos x} + C}$

f) $\int_1^4 (3\sqrt{x} - x)\,dx = 3\left(\frac{2}{3}x^{3/2}\right) - \frac{1}{2}x^2\Big|_1^4$

$= 2(4^{3/2}) - \frac{1}{2}(4^2) - \left(2(1^{3/2}) - \frac{1}{2}(1^2)\right)$

$= 2\sqrt{64} - \frac{1}{2}(16) - 2 + \frac{1}{2}$

$= 16 - 8 - 2 + \frac{1}{2} = \boxed{13/2}$

g) $\int(2\omega^{54}+13\omega^{11}-45\omega+3)\,d\omega =$

$$\boxed{\frac{2}{55}\omega^{55}+\frac{13}{12}\omega^{12}-\frac{45}{2}\omega^{2}+3\omega+C}$$

h) $\int_0^2 te^{t^2}dt$ $\quad u=t^2$ $\quad \frac{du}{dt}=2t$ $\quad \frac{du}{2t}=dt$ $\quad \int te^{u}\frac{du}{2dt}=\frac{1}{2}\int e^{u}du$

$$=\frac{1}{2}e^{t^2}\Big|_0^2=\frac{1}{2}e^{4}-\frac{1}{2}e^{0}=\boxed{\frac{1}{2}(e^{4}-1)}$$

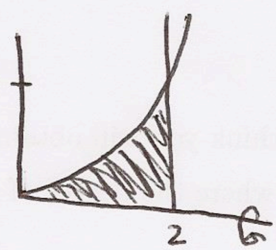
i) $\int(6\cos(u)-7\sec^2(u))\,du=\boxed{6\sin u-7\tan u+C}$

j) $\int_e^{e^4}\frac{1}{x\sqrt{\ln x}}$ $\quad u=\ln x$ $\quad \frac{du}{dx}=\frac{1}{x}$ $\quad x\,du=dx$ $\quad \int\frac{1}{x}u^{-1/2}(x\,du)=\int u^{-1/2}du$

$$=2\sqrt{\ln x}\Big|_e^{e^4}=2\sqrt{\ln e^4}-2\sqrt{\ln e}$$

$$=2\sqrt{4}-2\sqrt{1}=\boxed{1}$$

② $y=x^3$ $\quad y=0$ $\quad x=2$



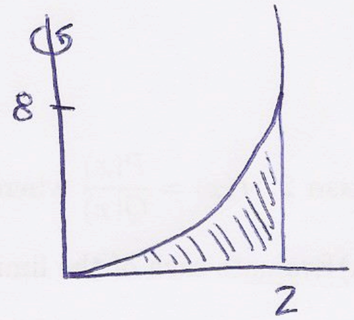
a) revolve about x-axis/washers

$$V=\pi\int_0^2(x^3)^2dx=\pi\int_0^2x^6dx$$

b) revolve about x-axis/shells

$$V=2\pi\int_0^8 y(2-\sqrt[3]{y})\,dy$$

c) revolve about y-axis / washers



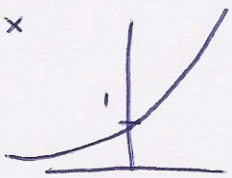
$$V = \pi \int_0^8 2^2 - (\sqrt[3]{y})^2 \, dy$$
$$= \pi \int_0^8 4 - y^{2/3} \, dy$$

d) revolve about y-axis / shell

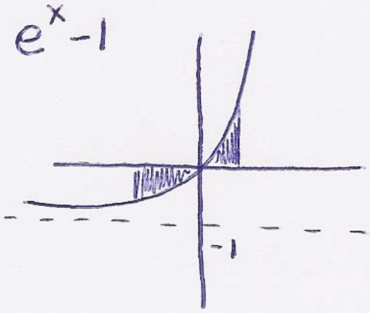
$$V = 2\pi \int_0^2 x(x^3)\,dx = 2\pi \int_0^2 x^4\,dx$$

③ Total area $y = e^x - 1$ $\quad y = 0$ on $[-1, 1]$

e^x



$e^x - 1$



$$A = -\int_{-1}^0 e^x - 1\,dx + \int_0^1 e^x - 1\,dx$$
$$= -(e^x - x\Big|_{-1}^0 + (e^x - x\Big|_0^1$$
$$= -(e^0 - 0 - (e^{-1} - (-1))$$
$$+ e^1 - 1 - (e^0 - 0)$$
$$= -(1 - \tfrac{1}{e} - 1) + e - 1 - 1 = \boxed{e + \frac{1}{e}}$$

④ $F(x) = \int_0^x \cos(t^2)\,dt$

$F'(x) = \frac{d}{dx} \int_0^x \cos(t^2)\,dt = \cos x^2$ (Fundamental Thm)

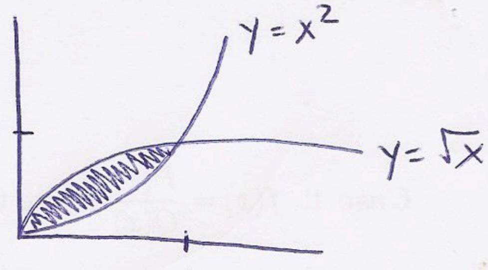
$F'(\sqrt{\pi/2}) = \cos(\sqrt{\pi/2})^2 = \cos \pi/2 = 0$

$F''(x) = \frac{d}{dx}[\cos x^2] = -2x \sin x^2$

$F''(\sqrt{\pi/2}) = -2\sqrt{\pi/2}\,\sin(\sqrt{\pi/2})^2 = -2\sqrt{\pi/2}\,\sin \pi/2$

$= -2\sqrt{\pi/2}$

(5) Area bounded by $y=\sqrt{x}$ $y=x^2$



$$A=\int_0^1 \sqrt{x}-x^2\,dx = \frac{2}{3}x^{3/2}-\frac{1}{3}x^3\Big|_0^1$$

$$=\frac{2}{3}(1^{3/2})-\frac{1}{3}(1^3)-\left(\frac{2}{3}(0^{3/2})-\tfrac{1}{3}(0^3)\right)=\frac{2}{3}-\frac{1}{3}=\boxed{\frac{1}{3}}$$

(6) tangent line has slope $4x^3 \Rightarrow f'(x)=4x^3$

curve passes through $(1,3)$ $\Rightarrow f(x)=x^4+c$

$f(1)=3$

$3=(1)^4+c \quad c=2 \qquad \boxed{f(x)=x^4+2}$

(7) a) Fundamental Thm of Calc: If f is ________ on $[a,b]$ and F is an anti-derivative on f on $[a,b]$, i.e. $F'(x)=f(x)$

then

b) Intermediate Value Thm

(8) (a) false $\int_5^{-1} f(x)dx = -\left(\int_{-1}^5 f(x)dx\right) = -\left(\int_{-1}^1 f(x)dx+\int_1^5 f(x)dx\right)$

$= -(4-1) \neq 1$

(b) false $\int_{-1}^5 f(x)-g(x)dx = \int_{-1}^5 f(x)dx - \int_{-1}^5 g(x)dx = (4-1)-(-3)$

$\neq 0$

(c) true! $\int_5^1 2f(x)dx = 2\int_5^1 f(x)dx = 2\left(-\int_1^5 f(x)dx\right)$

$=2(-(-1))=2$

$\boxed{C}$

(9) $\lim\limits_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n}(3k-1) = \lim\limits_{n\to\infty}\left(\frac{3}{n^2}\sum_{k=1}^{n} k - \frac{1}{n^2}\sum_{k=1}^{n} 1\right)$

$= \lim\limits_{n\to\infty}\left(\frac{3}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{1}{n^2}(n)\right) = \lim\limits_{n\to\infty}\left(\frac{3}{2} + \frac{3}{2n} - \frac{1}{n}\right)$

$= \boxed{\frac{3}{2}}$ A

(10) $\lim \sum \underbrace{(5x_k^{*2} - 7)}_{f(x_k^*)}\underbrace{\left(\frac{3}{n}\right)}_{\Delta x} \Rightarrow f(x) = 5x^2 - 7$

$[0,3]$

E

(11) $\lim\limits_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \frac{d}{dx}[\sin x] = \cos x$ C

(12) $f(x) = e^{-x^2}$ tangent line eqn @ $x=1$

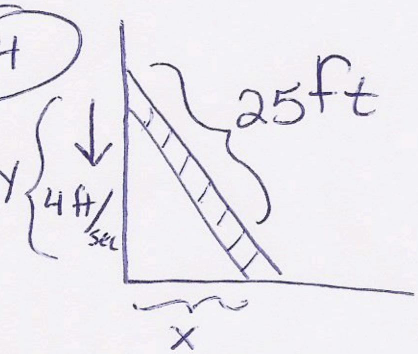
$f'(x) = -2xe^{-x^2}$ $f'(1) = -2e^{-1}$ $f(1) = e^{-1}$

$(y - e^{-1}) = -2e^{-1}(x-1)$ A

(13) $f(x) = (\ln(2x^4+3x^2+1))^{11}$ chain rule twice

$11(\ln(2x^4+3x^2+1))^{10}\left(\frac{8x^3+6x}{2x^4+3x^2+1}\right)$ D

(14)



want to find $\left.\frac{dx}{dt}\right|_{y=15}$

$x^2 + y^2 = 25$

$\frac{d}{dt}[x^2+y^2] = \frac{d}{dt}[25] \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(20)\frac{dx}{dt} + 2(15)(-4) = 0$

$40\frac{dx}{dt} = +120$

$\frac{dx}{dt} = 3$ A

(15) $f(x) = x^2 - x$ $\quad f(0) = 0 - 0 = 0$ $\quad f(1) = 1^2 - 1 = 0$

$f'(x) = 2x - 1$ $\quad f'(c) = 0$ $\quad 2c - 1 = 0 \Rightarrow c = \frac{1}{2}$

(D)

(16) $x + 2k = kx^2 + x + 1$ @ $x = 1$

$1 + 2k = k(1^2) + 1 + 1 \Rightarrow 1 + 2k = 2 + k \Rightarrow k = 1$ (A)

(17) $f(x) = \frac{1}{2}x^4 - x^2 - \frac{3}{2}$ $\quad f'(x) = 2x^3 - 2x = 0$

$0 = 2x(x^2 - 1)$ $\quad x = 0, x = \pm 1$

$f'(x)$: $-$ | $+$ | $-$ | $+$ at -1, 0, 1 $\quad$ min at $x = \pm 1$

$f(\pm 1) = \frac{1}{2} - 1 - \frac{3}{2} = -2$ (E)

(18) $\lim\limits_{x \to \infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$ L'Hopital $\lim\limits_{x \to \infty} = \frac{1/x}{1} = \lim\limits_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$

(C)

(19) $\frac{d}{dx}(e^{11}) = 0$ $\quad e^{11}$ is a constant (B)

(20) (B)

(21) $f(x) = x\tan(x)$ $\quad f'(x) = \tan x + x\sec^2 x$

$f''(x) = \sec^2 x + \sec^2 x + x(2\sec x)(\sec x \tan x)$

$= 2\sec^2 x(1 + x\tan x)$ (A)

(22) (E)

(23) $\lim\limits_{x \to 1^-} \frac{1 + x^2}{1 - x} = \frac{1+1}{0^+} = \infty$ (B)