INSTRUCTIONS: Answer each of the following questions. In order to receive full credit on the *non-multiple choice* questions your answer must be **complete**, **legible**, and **correct**. You must also **show all of your work** and give adequate explanations on the *non-multiple choice* questions. No partial credit will be given on the multiple choice questions. No calculators, no books, no notes are allowed on this exam.

1. Stanley drives from Boulder to Pancake, Colorado, which is a small town on the way to the Kansas-Colorado border. For the first hour of his trip Stanley averages 45 miles per hour; Stanley then averages 60 miles per hour for the remaining 90 miles of his trip.

   (a) (3 points) How far did Stanley drive?

DISTANCE = RATE × TIME

$d = rt$

1 HOUR @ 45 miles/hour

$d = rt = 45 \text{ miles/hour} \times 1 \text{ hour}$

$= 45 \text{ miles}$

60 miles/hour FOR 90 miles

90 miles

TOTAL: 45 miles + 90 miles = **135 miles**

   (b) (3 points) How long did Stanley drive?

1 HOUR @ 45 miles/hour

1 HOUR

60 miles/hour FOR 90 miles

$d = rt \Rightarrow t = \frac{d}{r} = \frac{90 \text{ miles}}{60 \text{ miles/hr}}$

$= 1\frac{1}{2}$ hours

TOTAL: 1 hour + $1\frac{1}{2}$ hours = **$2\frac{1}{2}$ hours**

   (c) (3 points) What was Stanley's average velocity?

$\text{AVG VEL} = \frac{\text{DIST TRAVELLED}}{\text{TIME ELAPSED}}$

$= \frac{135 \text{ miles}}{2\frac{1}{2} \text{ hours}} = \frac{135}{5/2} \text{ miles/hour}$

$= \frac{(135)(2)}{5} \text{ miles/hour} = (27)(2) \text{ miles/hour}$

$= \boxed{54 \text{ miles/hour}}$

2. (4 points each) Match each function with the correct graph. (Note that there are more graphs than functions.)

(a) $f(x) = \dfrac{x^2 - x - 6}{(x-2)(x+3)}$ Graph: B

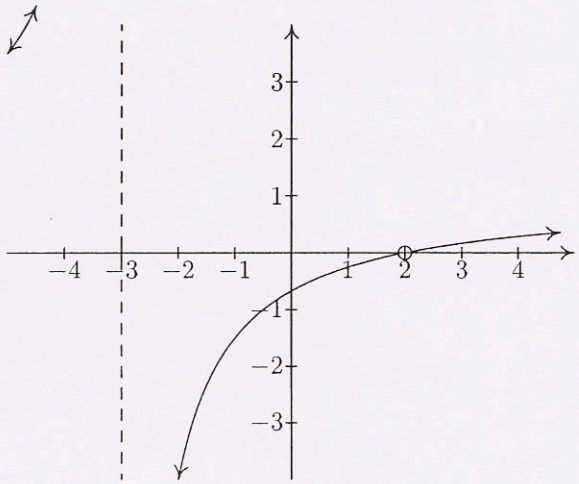
$f(x) = \dfrac{(x-3)(x+2)}{(x+3)(x-2)}$ ← VERT ASYMPTOTES AT $x=-3, 2$

(b) $g(x) = \dfrac{x^2 - 4x + 4}{(x-2)(x+3)}$ Graph: A

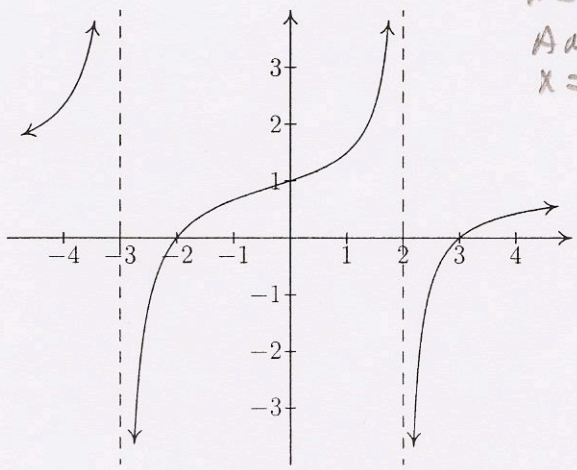
$g(x) = \dfrac{(x-2)^2}{(x-2)(x+3)} = \dfrac{x-2}{x+3}$ ← EXCEPT AT $x=2$; VERT ASYMP AT $x=-3$

(c) $h(x) = \dfrac{x^2 + x - 6}{(x-2)(x+3)}$ Graph: D

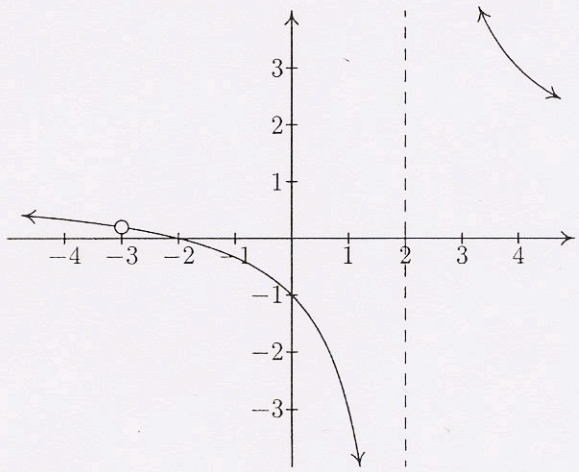
$h(x) = \dfrac{(x+3)(x-2)}{(x+3)(x-2)} = 1$ ← EXCEPT AT $x=-3$ AND $x=2$



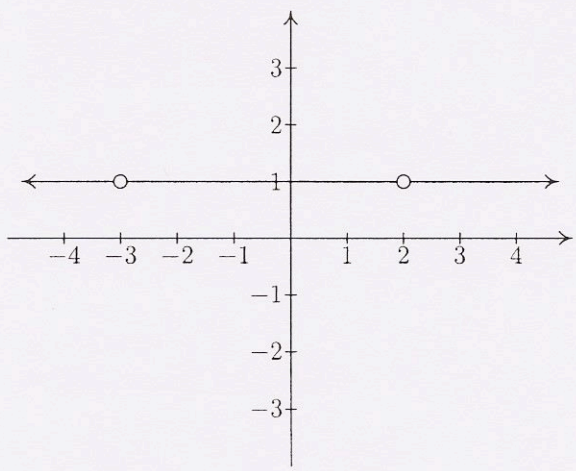
Graph A



Graph B

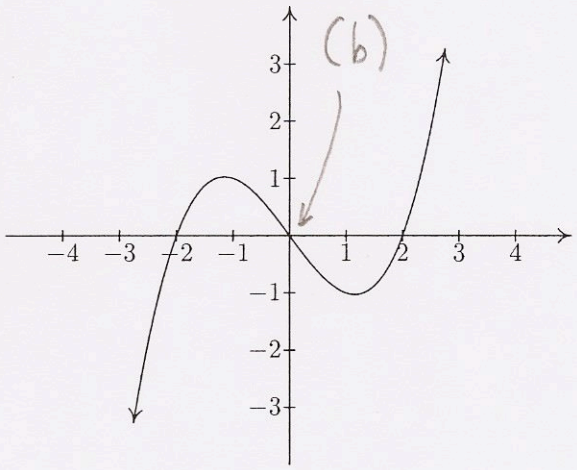


Graph C

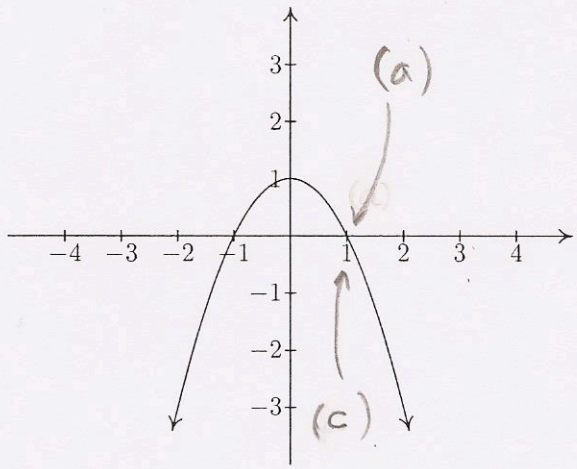


Graph D

3. (4 points each) Using the graphs below, circle the correct, or approximate, value for each of the following expressions.



Graph of f



Graph of g

(a) $g(1)$

A. 0 B. $-\frac{3}{2}$ C. -1 D. 1

(1,0) IS ON THE GRAPH OF g

(b) $f'(0)$

A. 0 B. $\frac{1}{2}$ C. -1 D. 1

THE SLOPE OF THE TAN LINE TO THE GRAPH OF f AT $x=0$ IS NEGATIVE, SINCE THE TAN LINE IS SLOPING DOWN AND TO THE RIGHT

(c) Suppose $h(x) = f(g(x))$. Then $h'(1) =$

A. 0 B. -2 C. -1 D. 2

$h'(x) = f'(g(x)) \cdot g'(x)$ [CHAIN RULE]

$h'(1) = f'(g(1)) \cdot g'(1)$

$= f'(0) \cdot (-2) = (-1)(-2) = 2$

THE ONLY POSITIVE CHOICE.

A GUESS, BUT CERTAINLY $g'(1) < 0$

(d) At how many points does $f'(x) = 0$?

A. 0 B. 1 C. 2 D. 3

$f'(x) = 0 \Rightarrow$ SLOPE OF TAN LINE IS 0

$\Rightarrow$ TAN LINE IS HORZ; THERE ARE TWO SUCH x'S ON GRAPH OF f

4. (4 points each) Circle the correct answer for each of the following problems.

(a) Let $f(x) = \sqrt{1-x^2}$. Then $f'(x) =$

A. $\dfrac{-x}{\sqrt{1-x^2}}$ B. $-2x\sqrt{1-x^2}$ C. $\dfrac{1}{\sqrt{1-x^2}}$ D. $\dfrac{x}{\sqrt{1-x^2}}$

$f(x) = (1-x^2)^{1/2}$

THEN $f'(x) = \frac{1}{2}(1-x^2)^{-1/2} \cdot \frac{d}{dx}[1-x^2] = \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} \cdot (-2x)$

$= \dfrac{-x}{\sqrt{1-x^2}}$

(b) Let $g(x) = \sin^{-1}(x)$. Then $g'(x) =$

A. $\dfrac{1}{x^2+1}$ B. $-\cos^{-1}(x)$ C. $\dfrac{1}{\sqrt{1-x^2}}$ D. $\dfrac{1}{\sin^2(x)}$

$\frac{d}{dx}[\sin^{-1}x] = \dfrac{1}{\sqrt{1-x^2}}$

(c) Let $h(x) = e^{x^2}$. Then $h'(x) =$ $e^{x^2} \cdot \frac{d}{dx}[x^2] = e^{x^2} \cdot 2x = 2x e^{x^2}$

A. e^{x^2} B. $2e^{x^2}$ C. $x^2e^{x^2}$ D. $2xe^{x^2}$

(d) Let $g(x) = \tan^2(x)$. Then $g'(x) =$

A. $\dfrac{2\sin(x)}{\cos^3(x)}$ B. $\sec^4(x)$ C. $2\sec^2(x)$ D. $2\tan(x)$

$g(x) = (\tan x)^2$

THEN $g'(x) = 2 \cdot (\tan x)^1 \cdot \frac{d}{dx}[\tan x] = 2\tan x \sec^2 x$

$= 2 \cdot \dfrac{\sin x}{\cos x} \cdot \dfrac{1}{\cos^2 x} = \dfrac{2\sin x}{\cos^3 x}$

(e) Let $h(x) = \dfrac{x-1}{x+1}$. Then $h'(x) =$

A. $\dfrac{x-1}{(x+1)^2}$ B. $\dfrac{2}{(x+1)^2}$ C. $\dfrac{x}{(x+1)^2}$ D. $\dfrac{2x}{(x+1)^2}$

$$h'(x) = \frac{d}{dx}\left[\frac{x-1}{x+1}\right] = \frac{(x+1)\cdot\frac{d}{dx}[x-1] - (x-1)\cdot\frac{d}{dx}[x+1]}{(x+1)^2}$$

$$= \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

(f) Let $f(x) = xe^x$. Then $f''(x) =$

A. xe^x B. $xe^x + e^x$ C. x^2e^x D. $xe^x + 2e^x$

$$f'(x) = \frac{d}{dx}[xe^x] = x\cdot\frac{d}{dx}[e^x] + e^x\cdot\frac{d}{dx}[x] = xe^x + e^x\cdot 1 = xe^x + e^x$$

$$f''(x) = \frac{d}{dx}[f'(x)] = \frac{d}{dx}[xe^x + e^x] = (xe^x + e^x) + e^x = xe^x + 2e^x$$

(g) Suppose $xy + y^2 = x$. Then $\dfrac{dy}{dx} =$

A. $\dfrac{1-y}{x+2y}$ B. $\dfrac{1}{\sqrt{x-xy}}$ C. $\dfrac{x-y}{x+2y}$ D. $\dfrac{1}{x+y}$

$$\frac{d}{dx}[xy + y^2] = \frac{d}{dx}[x] \Rightarrow (xy' + y\cdot 1) + 2yy' = 1$$

$$xy' + 2yy' = 1 - y$$

$$y'(x+2y) = 1-y \Rightarrow y' = \frac{1-y}{x+2y}$$

(h) $\lim_{x\to\infty} \ln(x^{1/x}) =$

A. 0 B. 1 C. e D. Does not exist

$$\lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \left(\frac{1}{x}\cdot \ln x\right) = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{L'HOP}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= 0$$

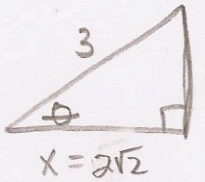
(i) $\cos\left(\sin^{-1}\left(\frac{1}{3}\right)\right) =$

A. $\cot\left(\frac{1}{3}\right)$ B. $\cos^{-1}\left(\frac{1}{3}\right)$ C. $\frac{2}{3}\sqrt{2}$ D. $\frac{2}{3}$

Let $\theta = \sin^{-1}\frac{1}{3}$

Then $\sin\theta = \sin\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

$\sin\theta = \frac{1}{3} = \frac{\text{OPP}}{\text{HYP}}$



$x^2 + 1^2 = 3^2$

$x^2 = 8$

$x = \sqrt{8} = \sqrt{4\cdot 2} = 2\sqrt{2}$

THEN $\cos\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

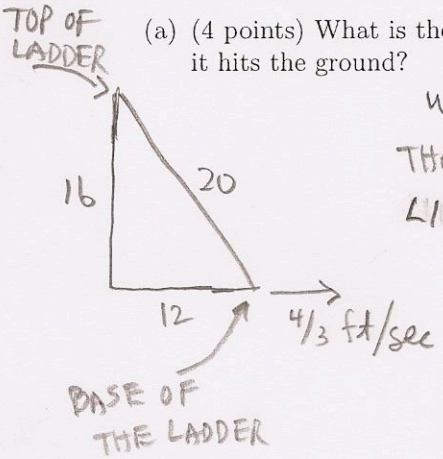
$= \cos(\theta) = \frac{\text{ADJ}}{\text{HYP}}$

$= \frac{2\sqrt{2}}{3} = \frac{2}{3}\sqrt{2}$

[NO DERIVATIVES IN THIS PROBLEM]

5. A 20 foot ladder is leaning against a wall, with its base 12 feet from the wall, so the top of the ladder is 16 feet from the ground. The base of the ladder begins to slide away from the wall at the rate of $\frac{4}{3}$ feet per second.

(a) (4 points) What is the AVERAGE rate of change of the height of the ladder from its initial position until it hits the ground?



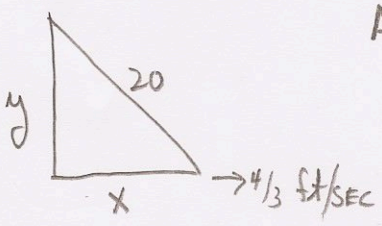
WHEN THE TOP OF THE LADDER HITS THE GROUND, THE ENTIRE 20 FT LENGTH OF THE LADDER WILL LIE HORIZONTALLY ON THE GROUND. OF THAT 20 ft, 12 ft IS ALREADY LYING ALONG THE GROUND, SO THE BASE OF THE LADDER HAS TO SLIDE $20\text{ ft} - 12\text{ ft} = 8\text{ ft}$ MORE BEFORE THE LADDER IS LYING FLAT ON THE GROUND.

SINCE DIST = RATE × TIME
$d = rt$, WE HAVE $t = \frac{d}{r} = \frac{8\text{ ft}}{4/3\text{ ft/sec}} = 6\text{ sec}$

SO IT WILL TAKE 6 SECS FOR THE TOP OF THE LADDER TO FALL THE 16 ft TO THE GROUND

SO $\underline{\text{AVG RATE OF CHANGE}} = \text{AVG VELOCITY} = \frac{\text{DIST TRAVELLED}}{\text{TIME ELAPSED}}$
$= 16\text{ ft}/6\text{ SECS} = \boxed{8/3\text{ FT/SEC}}$

(b) (6 points) When the base of the ladder is 16 feet from the wall, how fast is the top of the ladder sliding down the wall?



AT AN ARBITRARY MOMENT IN TIME, SUPPOSE THE TOP OF THE LADDER IS y FT. ABOVE THE GROUND, AND THE BASE OF THE LADDER IS x FT. FROM THE WALL.

PYTHAGOREAN THM: $x^2 + y^2 = 20^2$

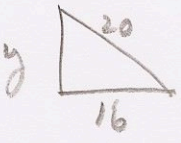
GIVEN: $\frac{dx}{dt} = \frac{4}{3}\text{ ft/sec}$

FIND: $\left.\frac{dy}{dt}\right|_{x=16\text{ ft}}$

$\frac{d}{dt}[x^2+y^2] = \frac{d}{dt}[20^2]$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{x\,dx/dt}{y}$



WHEN $x = 16\text{ ft}$

$y^2 + 16^2 = 20^2$

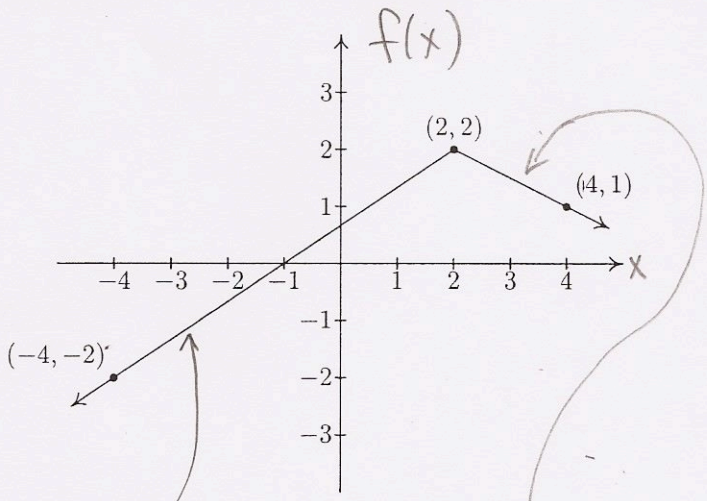
$y^2 = 20^2 - 16^2 = 400 - 256 = 144$

$y = 12\text{ ft}$

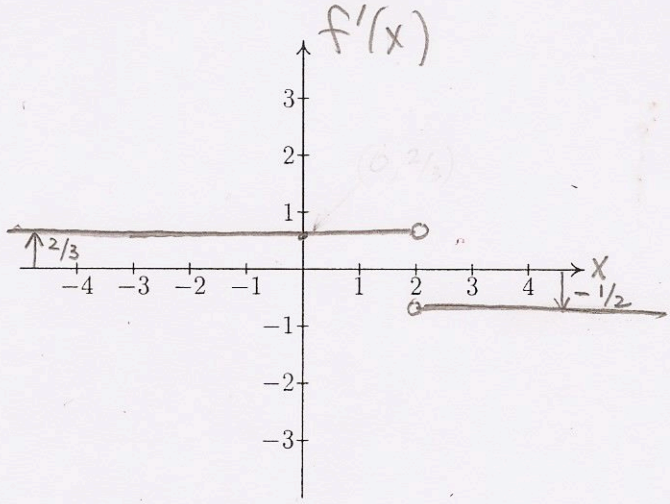
$\left.\frac{dy}{dt}\right|_{x=16\text{ ft}} = \left.-\frac{x\,dx/dt}{y}\right|_{\substack{x=16\text{ ft} \\ y=12\text{ ft} \\ dx/dt = 4/3\text{ ft/sec}}}$

$= \frac{(-16\text{ ft})(4/3\text{ ft/sec})}{12\text{ ft}} = \boxed{-16/9\text{ ft/sec}}$

6. (6 points) Below is the graph of a function. Graph its derivative on the coordinate axes provided.



Graph of f



Graph of f'

SLOPE $= \frac{\Delta y}{\Delta x} = \frac{2-(-2)}{2-(-4)}$

$= \frac{4}{6}$

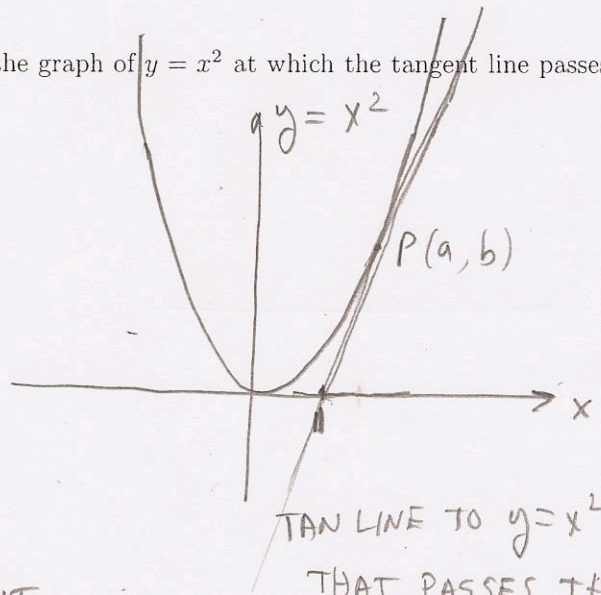
$= \boxed{\frac{2}{3} = f'(x)}$ for all x in $(-\infty, 2)$

SLOPE $= \frac{\Delta y}{\Delta x}$

$= \frac{2-1}{2-4}$

$= \frac{1}{-2} = \boxed{-\frac{1}{2} = f'(x)}$ for all x in $(2, \infty)$

7. (8 points) Find the coordinates of all points on the graph of $y = x^2$ at which the tangent line passes through the point $(1, 0)$.



LET $P(a, b)$ BE THE POINT OF TANGENCY.

SINCE P IS ON $y = x^2$, WE MUST HAVE $b = a^2$ $(y = x^2)$

SO P HAS COORDINATES $P(a, a^2)$

THE SLOPE OF THE (TANGENT) LINE THRU (a, a^2) AND $(1, 0)$ IS:

$$m = \frac{\Delta y}{\Delta x} = \frac{a^2 - 0}{a - 1} = \frac{a^2}{a-1}$$

BUT ALSO THE SLOPE OF THE TANGENT LINE IS:

$$m = \frac{d}{dx}[x^2]\Big|_{x=a} = 2x\Big|_{x=a} = 2a$$

THESE ARE EQUAL, SINCE THEY ARE THE SLOPE OF THE SAME LINE.

$$2a = \frac{a^2}{a-1}$$

$$2a(a-1) = a^2$$

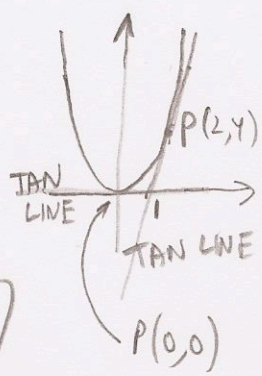
$$2a^2 - 2a = a^2$$

$$a^2 - 2a = 0$$

$$a(a-2) = 0$$

$a = 0$ THEN P HAS COORDS: $P(a, a^2) = \boxed{P(0,0)}$

$a = 2$ THEN P HAS COORDS: $P(a, a^2) = P(2, 2^2) = \boxed{P(2,4)}$



THERE ARE <u>TWO</u> POINTS OF TANGENCY!