

4 Module Theory

4.1 Definitions and Examples

This section of notes roughly follows Section 10.1 in Dummit and Foote.

Let's start with the definition of a module.

Definition 4.1. Let R be a ring (not necessarily commutative nor with 1). A **left R -module** (or **left module over R**) is a set M together with

- (1) a binary operation $+$ on M under which M is an abelian group, and
- (2) an action of R on M (that is, $R \times M \rightarrow M$) denoted by rm , for all $r \in R$ and for all $m \in M$ that satisfies.
 - (a) $(r + s)m = rm + sm$ for all $r, s \in R$ and $m \in M$,
 - (b) $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$, and
 - (c) $r(m + n) = rm + rn$ all $r \in R$ and $m, n \in M$.

(d) If R has a 1 , then we also require: $1m = m$ for all $m \in M$.

We analogously define **right R -modules**. If R is commutative and M is a left R -module, then we can make it a right R -module by defining $mr = rm$ for all $r \in R$ and $m \in M$. Notice that we cannot do this in general if R is not commutative since Axiom (2b) may fail. Unless we explicitly say otherwise, all modules will be left modules. Modules satisfying Axiom (2d) are called **unital modules**. We will assume that all our modules are unital.

The axioms for a module should look familiar. If R is a field, the axioms are precisely those for a vector space over R .

We emphasize that an abelian group M may have many different R -module structures for a fixed ring R (in the same way a group G could act in many ways as a permutation group of some fixed set S).

Definition 4.2. Let R be a ring and let M be an R -module. An **R -submodule** of M is a subgroup N of M that is closed under the action of ring elements, i.e., $rn \in N$ for all $r \in R$ and $n \in N$.

As expected, submodules of M are just subsets of M that are themselves modules under the same action. In particular, if R is a field, submodules are just vector subspaces. Every R -module has at least two submodules: M and $\{0\}$. The latter is often written as just 0 and called the **trivial submodule**.

Example 4.3. Let's see some examples.

- (1) Let R be any ring. Then $M = R$ is a left R -module, where the action of a ring element on a module element is just usual ring multiplication. In this case, the submodules of $M = R$ are the left ideals of R .
- (2) A special case of the first example is what R is a field. Then R is 1-dimensional vector space over itself.
- (3) More generally, if $R = F$ is a field, every vector space over F is an F -module and vice versa. Let $n \in \mathbb{Z}^+$ and let

$$F^n = \{(a_1, \dots, a_n) \mid a_i \in F \text{ for all } i\}.$$

We can make F^n into an n -dimensional vector space by defining addition and scalar multiplication in the standard way.

- (4) Let R be a ring with 1 and let $n \in \mathbb{Z}^+$. As above, define

$$R^n = \{(a_1, \dots, a_n) \mid a_i \in R \text{ for all } i\}.$$

We can make R^n an R -module by defining addition and multiplication by elements of R in the same manner as when R was a field. The module R^n is called the **free module of rank n over R** .

- (5) The same abelian group M may have the structure of a module for several different rings R . In particular, if M is an R -module and S is a subring of R with $1_R = 1_S$, then M is

automatically an S -module. For example, the field \mathbb{R} is an \mathbb{R} -module, a \mathbb{Q} -module, and a \mathbb{Z} -module.

- (6) If M is an R -module and for some 2-sided ideal I of R , $am = 0$ for all $a \in I$ and $m \in M$, we say M is **annihilated by I** . In this case, we can make M into an (R/I) -module by defining an action of the quotient ring R/I on M . For each $m \in M$ and coset $r + I \in R/I$, define

$$(r + I)m = rm.$$

Since $am = 0$ for all $a \in I$ and $m \in M$, this is well-defined. In the special case that I is a maximal ideal in a commutative ring R and $IM = 0$, M is a vector space over the field R/I .

(7) \mathbb{Z} -modules...

(8) $F[x]$ -modules...

Theorem 4.4 (Submodule Criterion). Let R be a ring and let M be an R -module. A subset N of M is a submodule of M iff

- (1) $N \neq \emptyset$, and
- (2) $x + ry \in N$ for all $r \in R$ and $x, y \in N$.

Definition 4.5. Let R be a commutative ring with 1. An R -algebra is a ring A with identity together with a ring homomorphism $f : R \rightarrow A$ mapping $1_R \rightarrow 1_A$ such that the subring $f(R)$ of A is contained in the center of A (i.e., the set of all elements of A that commute with every element of A).

If A is an R -algebra, then it is easy to verify that A has a natural left and right unital R -module structure defined by $r \cdot a = a \cdot r = f(r)a$, where $f(r)a$ is just the multiplication in the ring A (which is the same as $af(r)$ since $f(r)$ lies in center). In general, it is possible for an R -algebra A to have other left (or right) R -module structures. Unless stated otherwise, we assume the natural module structure on the algebra will be assumed.

Here is an alternate definition.

Definition 4.6. Let R be a commutative ring with 1. An R -algebra is a ring A that is also an R -module such that the multiplication map $A \times A \rightarrow A$ is R -bilinear, that is,

$$r * (ab) = (r * a) \cdot b = a \cdot (rb)$$

for all $a, b \in A$ and $r \in R$, where $r \cdot$ denotes the R -action on A .

Loosely speaking, the definition above says that an R -algebra is an R -module, where we are also allowed to multiply the module elements.

Theorem 4.7. Definitions 4.5 and 4.6 are equivalent.

Example 4.8. Here are a few quick examples. Throughout assume that R is a commutative ring with 1.

- (1) Any ring with 1 is a \mathbb{Z} -algebra.
- (2) Let A be any ring with 1_A . If R is a subring of the center of A containing 1_A , then A is an R -algebra under $f(r) = r1_A$ for $r \in R$. For example, the polynomial ring $R[x_1, \dots, x_n]$ is an R -algebra.
- (3) The group ring $R[G]$ for a finite group G is an R -algebra.
- (4) If A is an R -algebra, then the R -module structure of A depends only on the subring $f(R)$ contained in center of A . If we replace R by its image $f(R)$, we see that up to ring homomorphism, every algebra A arises from a subring of the center of A that contains 1_A .
- (5) In the special case that $R = F$ is a field, F is isomorphic to its image under f , so we can identify F itself as a subring of A . So, saying that A is an algebra over a field F is the same as saying that the ring A contains the field F in its center and the identity of A and of F are the same.

Definition 4.9. If A and B are two R -algebra, an **R -algebra homomorphism** (respectively, **isomorphism**) is a ring homomorphism (respectively, isomorphism) $\phi : A \rightarrow B$ such that

- (1) $\phi(1_A) = 1_B$
- (2) $\phi(r \cdot a) = r \cdot \phi(a)$ for all $r \in R$ and $a \in A$.

4.2 Quotient Modules and Module Homomorphisms

This section of notes roughly follows Section 10.2 in Dummit and Foote.

There are no big surprises in this section. Essentially, everything works out exactly as you think it should.

Definition 4.10. Let R be a ring and let M and N be R -modules.

- (1) A map $\phi : M \rightarrow N$ is an **R -module homomorphism** if it respects the R -module structures of M and N :
 - (a) $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in M$;
 - (b) $\phi(rx) = r\phi(x)$ for all $r \in R$ and $x \in M$.
- (2) An R -module homomorphism is an **isomorphism** if it is both injective and surjective. In this case, we say that M and N are isomorphic and write $M \cong N$.
- (3) If $\phi : M \rightarrow N$ is an R -module homomorphism, define the **kernel** of ϕ via

$$\ker(\phi) := \{x \in M \mid \phi(x) = 0\}$$

and the **image** of ϕ via

$$\phi(M) := \{y \in N \mid \phi(x) = y \text{ for some } x \in M\}.$$

- (4) Let M and N be R -modules and define $\text{Hom}_R(M, N)$ to be the set of all R -module homomorphisms from M into N .

Remark 4.11. Every R -module homomorphism is always a group homomorphism of abelian groups. However, not every group homomorphism of abelian groups yields an R -module homomorphism.

Theorem 4.12. If $\phi : M \rightarrow N$ is an R -module homomorphism, then $\ker(\phi)$ is an R -submodule of M and $\phi(N)$ is an R -submodule of N .

Example 4.13. Let's see some examples.

- (1) If R is a field, then R -module homomorphisms are linear transformations.
- (2) If R is a ring and $M = R$ is a module over itself, then R -module homomorphisms (even from R to itself) need not be ring homomorphisms and vice versa. For example, when $R = \mathbb{Z}$, the \mathbb{Z} -module homomorphism $\psi : x \mapsto 2x$ is not a ring homomorphism. When $R = F[x]$ for some field F , the ring homomorphism $\phi : f(x) \mapsto f(x^2)$ is not an $F[x]$ -module homomorphism since $x^2 = \phi(x) = \phi(x \cdot 1) = x\phi(1) = x$ is a contradiction.
- (3) If R is a ring and $M = R^n$, then the **projection map** $\pi_i : R^n \rightarrow R$ given by $\phi_i(x_1, \dots, x_n) = x_i$ is a surjective R -module homomorphism with kernel equal to the submodule of n -tuples that have a zero in position i .

- (4) For \mathbb{Z} -modules, Condition (a) of being a module homomorphism forces Condition (b). This implies that \mathbb{Z} -module homomorphisms are the same as abelian group homomorphisms.
- (5) Let R be a ring, let I be a 2-sided ideal of R and suppose M and N are R -modules annihilated by I (i.e., $am = 0$ and $an = 0$ for all $a \in I$, $n \in N$, and $m \in M$). Any R -module homomorphism from N to M is then automatically a homomorphism of R/I -modules (see Example 4.3(6)). For example, if A is an additive abelian group such that for some prime p , $px = 0$ for all $x \in A$, then any group homomorphism from A to itself is a $\mathbb{Z}/p\mathbb{Z}$ -module homomorphism, i.e., a linear transformation over the field \mathbb{Z}_p . In particular, the group of all group homomorphisms of A is the group of invertible linear transformations from A to itself: $GL(A)$.

Theorem 4.14. Let M, N , and L be R -modules.

- (1) A map $\phi : M \rightarrow N$ is an R -module homomorphism iff $\phi(rx + y) = r\phi(x) + \phi(y)$ for all $x, y \in M$ and $r \in R$.
- (2) Let $\phi, \psi \in \text{Hom}_R(M, N)$. Define $\phi + \psi$ via

$$(\phi + \psi)(m) = \phi(m) + \psi(m)$$

for all $m \in M$. Then $\phi + \psi \in \text{Hom}_R(M, N)$ and with this operation $\text{Hom}_R(M, N)$ is an abelian group. If R is a commutative ring, then for $r \in R$, define $r\phi$ via

$$(r\phi)(m) = r(\phi(m))$$

for $m \in M$. Then $r\phi \in \text{Hom}_R(M, N)$ and with this action of the commutative ring R , the abelian group $\text{Hom}_R(M, N)$ is an R -module.

- (3) If $\phi \in \text{Hom}_R(L, M)$ and $\psi \in \text{Hom}_R(M, N)$, then $\psi \circ \phi \in \text{Hom}_R(L, N)$.
- (4) With addition as above and multiplication as function composition, $\text{Hom}_R(M, M)$ is a ring with 1. When R is commutative, $\text{Hom}_R(M, M)$ is an R -algebra.

Definition 4.15. The ring $\text{Hom}_R(M, M)$ is called the **endomorphism ring of M** and may be denoted by $\text{End}_R(M)$. Elements of $\text{End}_R(M)$ are called **endomorphisms**.

When R is commutative, there is a natural map from R into $\text{End}_R(M)$ given by $r \mapsto rI$, where the latter endomorphisms of M is just multiplication by r on M . The image of R is contained in the center of $\text{End}_R(M)$, so if R has an identity, $\text{End}_R(M)$ is an R -algebra. The ring homomorphism from R to $\text{End}_R(M)$ may not be injective since for some $r \in R$, we may have $rm = 0$ for all $m \in M$ (e.g., $R = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z}$, and $r = 2$).

When R is a field, this map is injective (no unit is in the kernel) and the copy of R in $\text{End}_R(M)$ is called the **subring of scalar transformations**.

Recall that if G is a group and $H \leq G$, then we can form the quotient group G/H exactly when H is a normal subgroup of G . However, if G is abelian, then every subgroup is normal. In the case of a module M , every submodule is automatically a normal subgroup of M . We wish to show that we can always form the quotient module M/N for *any* submodule N .

Theorem 4.16. Let R be a ring, let M be an R -module, and let N be a submodule of M . The (additive, abelian) quotient group M/N can be made into an R -module by defining an action of elements of R by

$$r(x + N) = (rx) + N,$$

for all $r \in R$ and $x + N \in M/N$. The natural projection $\pi : M \rightarrow M/N$ defined by $\pi(x) = x + N$ is an R -module homomorphism with kernel N .

Definition 4.17. Let A and B be submodules of the R -module M . The **sum** of A and B is the set

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Theorem 4.18. Let A and B be submodules of the R -module M . Then $A + B$ is the smallest submodule of M that contains both A and B and $A \cap B$ is the largest submodule of M that is contained in both A and B .

All the isomorphism theorems stated for groups also hold for R -modules.

Theorem 4.19 (Isomorphism Theorems for Modules). Let M and N be R -modules.

- (1) (First) Let $\phi : M \rightarrow N$ be an R -module homomorphism. Then $\ker(\phi)$ is a submodule of M and $M/\ker(\phi) \cong \phi(M)$.
- (2) (Second) If A and B are submodules of M , then $(A + B)/B \cong A/(A \cap B)$.
- (3) (Third) If A and B are submodules of M such that $A \subseteq B$, then $(M/A)/(B/A) \cong M/B$.

(4) (Fourth) Let N be a submodule of M . There is a bijection between the submodules of M that contain N and the submodules of M/N . The correspondence is given by $A \leftrightarrow A/N$, for all $A \supseteq N$. This correspondence commutes with the process of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M that contain N).