Chapter 6: Subgroups

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Summer 2009

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Let's begin with an example.

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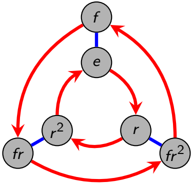

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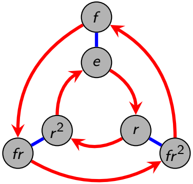

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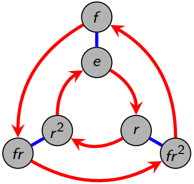

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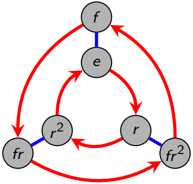

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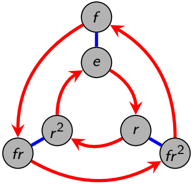

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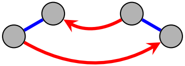

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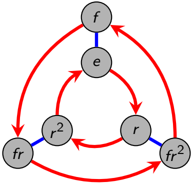

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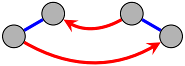

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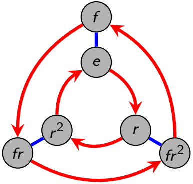

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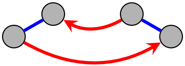

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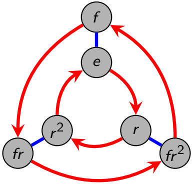

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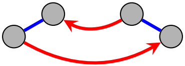

There are other patterns that permeate this diagram, as well. Do you see any? Here are a couple: $f^2 = e$, $r^3 = e$.

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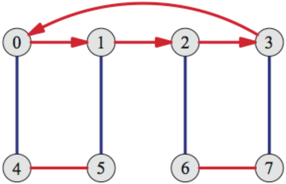

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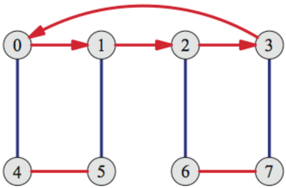

Nope. The diagram is not regular.

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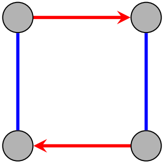

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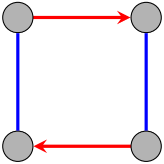

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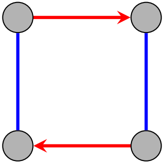

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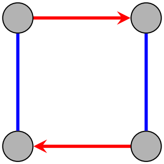

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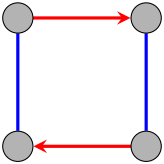

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What would go “wrong” if we tried to form a group from this diagram? If the red arrow represents action a, then a^2 is not represented in the diagram, which violates Rule 1.8.

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In fact, since $\langle r \rangle$ is really just a copy of C_3, we may be less formal and write

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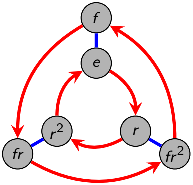

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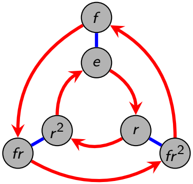

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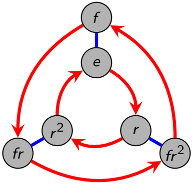

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It turns out that all of the subgroups of S_3 are just cyclic orbits, but there are many groups that have subgroups that are not cyclic orbits.

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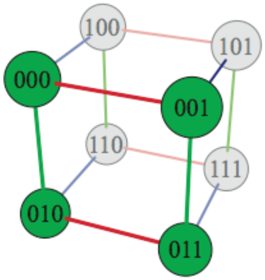

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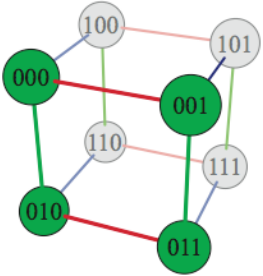

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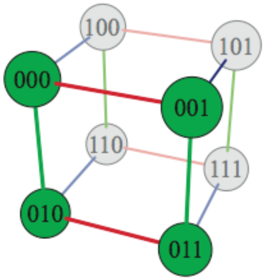

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$$\langle 001, 010 \rangle = \{000, 001, 010, 011\} < C_2 \times C_2 \times C_2.$$

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Let's take a look at $C_6 = \{0, 1, 2, 3, 4, 5\}$ in *Group Explorer* and see if we can discover all of the subgroups by experimenting with different generators for Cayley diagrams and possibly different layouts.

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$$\{e\}, \underbrace{\langle r^2\rangle, \langle f\rangle, \langle fr\rangle, \langle fr^2\rangle, \langle fr^3\rangle}_{\text{order 2}}, \underbrace{\langle r\rangle, \langle r^2, f\rangle, \langle r^2, fr\rangle}_{\text{order 4}}, D_4.$$

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Note that this algorithm works because every group (and subgroup) has a set of generators.

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3. Now, complete Exercise 6.5(a) (ignore the part about index). I want each group to turn in a complete solution.

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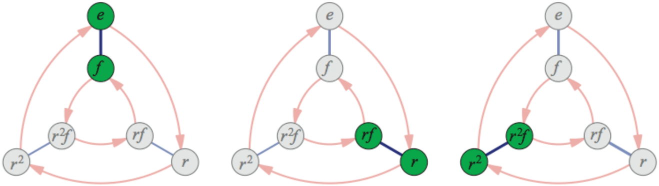

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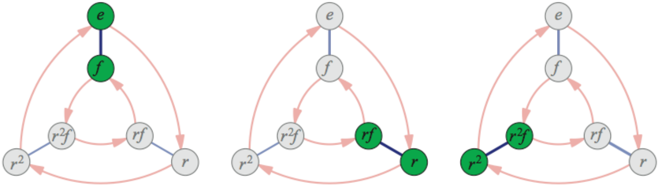

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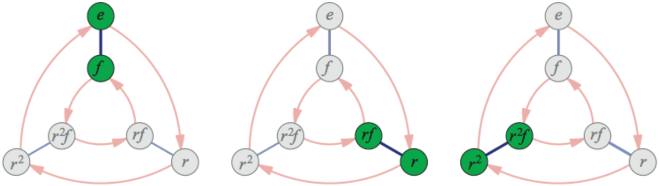
Cosets

The regularity property of Cayley diagrams implies that identical copies of the fragment of the diagram that corresponds to a subgroup appear throughout the rest of the diagram.

For example, the following figure (taken from Figure 6.6 on page 102 of *Visual Group Theory*) highlights the repeated copies of $\langle f \rangle = \{e, f\}$ in S_3.



However, only one of these copies is actually a group! Since the other two copies do *not* contain the identity, they cannot be groups.

The elements that form these repeated copies of the subgroup fragment in the Cayley diagram are called cosets.

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To be sure that we understand this concept, let's find all of the cosets of the subgroup $\langle f, r^2 \rangle = \{e, f, r^2, fr^2\}$ of D_4. Using *Group Explorer* will help us pick the right Cayley diagram and layout, so that we can "see" the cosets.

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We see that the cosets of $\langle f, r^2 \rangle$ are

$$\underbrace{\{e, f, r^2, fr^2\}}_{\text{original}}, \underbrace{\{r, r^3, fr, fr^3\}}_{\text{copy}}.$$

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The meaning of aH: start from the node a and follow *all* paths in H.

For example, for the coset $\{r, fr^2\}$ of $\langle f \rangle$ in D_3 we can write

$$r\langle f \rangle = r\{e, f\} = \{r \cdot e, r \cdot f\} = \{r, fr^2\}.$$

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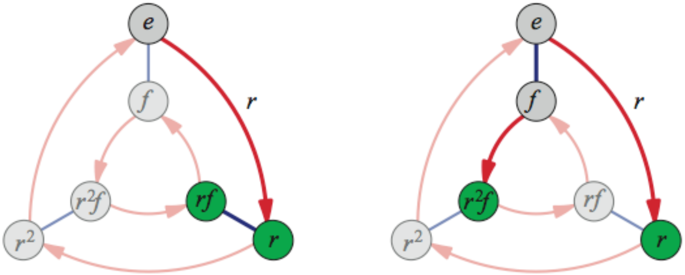
It turns out that in this example, the left cosets for $\langle f \rangle$ were different than the right cosets. Thus, they must look different in the Cayley diagram.

The left diagram below shows the left coset $r\langle f\rangle$ in S_3, the nodes that f arrows can reach after the path to r has been followed. The right diagram shows the right coset $\langle f\rangle r$ in S_3, the nodes that r arrows can reach from teh elements in $\langle f\rangle$.



(Taken from Figure 6.7 on page 104 of *Visual Group Theory*.)

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One of the most important things that we should take away from the last example is that left and right cosets are generally different.

But because they are not always different, it is worth seeing an example where they turn out to be the same.

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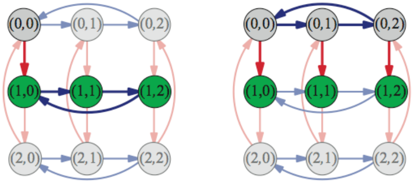

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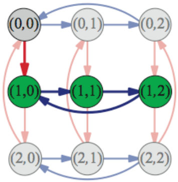

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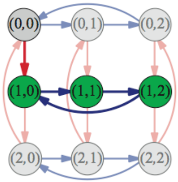

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Subgroups that satisfy $gH = Hg$ for *all* elements g in the parent group are called normal.

More group work

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1. In groups of 2–3 (try to mix the groups up again), complete the following exercises (not collected):
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2. Let's discuss your solutions.

Lagrange's Theorem

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Proof. Suppose that there exist $g \in G$ such that $g \in aH$ and $g \in bH$. By Observation 6.5, $gH = aH$ and $gH = bH$. But then we must have $aH = bH$, which shows that our arbitrary g lies in a unique coset (with possibly many different names). $\square$

The upshot of Theorem 6.7 is that we can think of a group as being composed exclusively of non-overlapping and equal size copies of any subgroup, namely that subgroup's left cosets.

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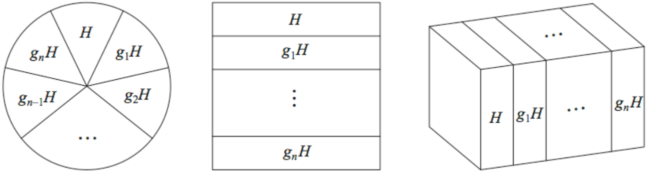

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This shows that $|H|$ divides $|G|$. $\square$

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Even more group work

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In groups of 2–3 (try to mix the groups up again), complete Exercise 6.4. I want each group to turn in a complete solution.