Chapter 0: Introduction–An Essay

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Let's summarize some of the key ideas in Chapter 0 of our textbook.

- Most math courses through calculus teach students how to use established mathematical techniques to solve problem.
- A student of math must learn to discover and prove mathematical facts on his/her own.
- It takes a long time to learn how to create new mathematics (brand new or new to you).
- This book (and this course) are meant to be the beginning of this journey.

- In mathematical reasoning, logical arguments are used to deduce the consequences (call theorems) of basic assumptions (called axioms).
- Most of math is built out of sets; so, the axioms of math are the axioms of set theory (see Appendix A).
- In practice, the axioms are rarely invoked. We often work from secondary assumptions, usually definitions.
- Our assumptions (axioms or definitions) provide the starting point for careful chains of logical reasoning leading to theorems.

One of the cruxes for students just learning how to write proofs is understanding what to assume. To help us understand this issue, let's consider Euclid's five axioms of geometry (developed in 300 B.C.):

1. Two points determine a line.
2. A line segment can be extended indefinitely.
3. A point and a radius determine a circle.
4. Any two right angles are equal.
5. Given any line and any point not on that line, there is exactly one line through that point that is parallel to the first line.

From these 5 axioms, Euclid was able to deduce (rigorously!) all of the geometry in his day, and then some.

The axiomatic method became a model for mathematical thinking. Euclid's *Elements* was used as a geometry text for over 20 centuries, and today's students usually learn geometry from this approach.

But how did Euclid decide what to assume? He believed that his axioms embodied a kind of absolute truth about the plane. He had intuitive notions of planes, lines, etc., and wrote down axioms that he thought described them.

Geometers liked the first 4 axioms, but doubted whether the 5th was really necessary. Could we prove the 5th statement by assuming the first 4?

The answer is no. It turns out that axiom 5 is independent of the other 4. In the 19th century, self-consistent systems of non-Euclidean geometry were developed based on axioms 1–4 and a new axiom that contradicted Euclid's axiom 5. One possibility is:

5′. Given any line and any point not on that line, there is *more than one* line through that point that is parallel to the first line.

Non-Euclidean geometry can be interpreted as the geometry of a curved surface, with certain curves in the surface playing the role of "lines."

So, are Euclid's axioms correct/true?

It is important to realize that this is the wrong question.

In the aftermath of the discovery of non-Euclidean geometry, mathematicians came to realize that the choice of axioms is completely arbitrary as long as the collection of axioms is self-consistent (i.e., does not give rise to contradictions).

Whether a set of axioms is the "correct" one depends on what it is that you are trying to model. Euclid's axioms do a great job at modeling geometry in the plane, whereas, non-Euclidean geometry embodies the geometry of certain curved surfaces. (Note: the general theory of relativity posits that the geometry of the universe is non-Euclidean.)

So, where do we start and where do we go?

- Axioms & definitions are our starting point.
- Once we have agreed on our assumptions, we can begin to prove theorems.
- A proof is a (possibly long) chain of logical inferences by which we deduce a mathematical statement from the basic assumptions.
- We rarely exhibit the entire chain of reasoning all the way from the axioms to the statement we are trying to prove. It suffices to show that the statement follows from previously proved theorems.
- A supposed "proof" with an incorrect step in it is not a proof at all. So, it is important to avoid even small errors!

- Our defense against faulty proofs is strict adherence to the rules of deductive logic.
- At each step in an argument, these rules are used to derive a new statement based on previous ones. A step is invalid if it is not sanctioned by the rules (even if the new statement happens to be true).
- An argument consisting entirely of true statements is not valid if the connecting inferences are not justified by logic.
- Mathematical statements can be very elaborate and it often requires some skill to take them apart and understand how they fit together.

Even the apparently trivial process of negation can be tricky. Let's try to negate the following statement:

> *If people recognize you every time you go to the store, then either you live in a small town or else you are a famous celebrity.*

Here is one possible way of wording the negation:

> *People recognize you every time you go to the store and you don't live in a small town and you are not a famous celebrity.*

There is much more to mathematical language than its fancy vocabulary and notation. It is a very formal and stylized dialect.

The same connecting phrases appear over and over: "if and only if," "for every," "there exists," etc. These phrases have been carefully chosen to express logical relations with a little ambiguity as possible. Not only do the phrases matter, but the order in which they occur is crucial.

To illustrate the point, let's consider an example.

What is the difference between these two statements?

For every poison there is a chemical that is an antidote.

There is a chemical that is the antidote for every poison.

The second statement is claiming that there is a “one size fits all” chemical that is the antidote for all poisons. The first statement is not claiming something so strong.

About this book:

- This book is designed to guide students through an intro to logic, mathematical language, mathematical proof, and mathematical culture.
- BUT it is the student's journey not the instructor's or the author's.
- The only way to learn mathematics is to do mathematics.
- This book is deliberately incomplete!
- The reader must do a what a mathematician does when reading any mathematical text: take apart definitions and theorems and see "what makes them tick."
- You will have to provide proofs for most of the theorems and to construct many of the examples.

One of the many goals of this book/course is that you will be able to pick up virtually any undergraduate textbook of abstract mathematics, read it, and understand it.

This is what it means to be an independent learner!

The “exercises” in this book fall into several categories:

1. Exercises: usually straight forward. I expect you to be completing these on your own. Occasionally, I may assign and collect a particular exercise.

   If I am not collecting an exercise, you don’t necessarily need to write out a solution, but at the very least think about it. I will always answer questions about exercises.
2. Problems: a step up in difficulty from the exercises and often require you to write a proof. Scattered throughout section text and at end of chapters.

3. Theorems: These are our big results. Some theorems have complete proofs (end in $\blacksquare$). These should be thought of as examples of how to model the proofs that you write.

   Theorems with incomplete proofs or missing proofs will end with $\square$. These will usually be the proofs that you will present at the board.
4. Questions to Ponder: More difficult, sometimes open-ended and/or philosophical.