NAME: Solutions

HOMEWORK FOR WORKSHEET 2 MATH 1300 DUE JANUARY 25, 2008

1. Apply the approach developed in Worksheet 2 to evaluate each of the following limits:

a. $\lim_{x\to -1} \frac{x^2+4x+3}{x^2-2x-3}$. Since both the numerator and denominator vanish at $x=-1$, the both have a factor of $x-(-1) = x+1$:

$$\lim_{x\to -1} \frac{x^2+4x+3}{x^2-2x-3} = \lim_{x\to -1} \frac{(x+1)(x+3)}{(x+1)(x-3)} = \lim_{x\to -1} \frac{x+3}{x-3} = \frac{2}{-4} = -\frac{1}{2}$$

b. $\lim_{x\to 2} \frac{x^3-2x^2-4x+8}{x^2-4x+4}$. Both the numerator and denominator vanish at $x=2$, so have $(x-2)$ as a factor:

$$\lim_{x\to 2} \frac{x^3-2x^2-4x+8}{x^2-4x+4} = \lim_{x\to 2} \frac{(x-2)(x^2-4)}{(x-2)(x-2)} = \lim_{x\to 2} \frac{x^2-4}{x-2}.$$

The numerator and denominator of $\frac{x^2-4}{x-2}$ both vanish at $x=2$ as well so they can be further factored:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x+2 = 4$$

c. $\lim_{x\to -2} \frac{x^3+x^2+x+1}{x^3+3x^2+x}$. Since $x=-2$ is in the domain of this function, and it is continuous for all x in its domain,

$$\lim_{x\to -2} \frac{x^3+x^2+x+1}{x^3+3x^2+x} = \frac{(-2)^3+(-2)^2+(-2)+1}{(-2)^3+3(-2)^2+-2} = \frac{-5}{2}$$

2. Sometimes it is also possible to simplify a function of the form $f(x) = \dfrac{g(x)}{h(x)}$ in order to evaluate a limit $\lim_{x \to a} \dfrac{g(x)}{h(x)}$, when $g(a) = 0$ and $h(a) = 0$. Try to evaluate each of the following limits by first simplifying the function:

a. $\lim_{x \to 0} \dfrac{\sin^2(x)}{1 - \cos(x)}$ (Hint: use the well-known trig identity $\sin^2(x) + \cos^2(x) = 1$)

$$\frac{\sin^2(x)}{1-\cos(x)} = \frac{1-\cos^2(x)}{1+\cos(x)} = \frac{(1-\cos(x))(1+\cos(x))}{(1+\cos(x))} = 1+\cos(x)$$

for $x \neq 0$

Therefore

$$\lim_{x \to 0} \frac{\sin^2(x)}{1-\cos(x)} = \lim_{x \to 0} (1+\cos(x)) = 1+\cos(0) = 1+1 = 2$$

b. $\lim_{x \to 4} \dfrac{x-4}{\sqrt{x}-2}$ (Hint: try to factor the numerator; alternatively, multiply the numerator and denominator by the conjugate of the bottom)

Since $x - 4 = (\sqrt{x} - 2)(\sqrt{x} + 2)$ we have $\lim_{x \to 4} \dfrac{x-4}{\sqrt{x}-2} = \lim_{x \to 4} \sqrt{x} + 2 = 4$

OR

$$\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x \to 4} \frac{(x-4)}{\sqrt{x}-2}\left(\frac{\sqrt{x}-2}{\sqrt{x}-2}\right) = \lim_{x \to 4} \frac{(x-4)(\sqrt{x}-2)}{x-4} = \lim_{x \to 4} \sqrt{x}-2 = 4$$

c. $\lim_{x \to 1} \dfrac{x-1}{\sqrt[3]{x}-1}$ (Hint: use ideas similar to part b.)

We know that $t^3 - 1 = (t-1)(t^2+t+1)$ so when $t = \sqrt[3]{x}$ we have $x - 1 = (\sqrt[3]{x} - 1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)$ Therefore

$$\lim_{x \to 1} \frac{x-1}{\sqrt[3]{x}-1} = \lim_{x \to 1} \frac{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{\sqrt[3]{x}-1}$$

$$= \lim_{x \to 1} \sqrt[3]{x^2} + \sqrt[3]{x} + 1 = 1+1+1 = 3$$