Chapter 7: Products and quotients

Dana C. Ernst

Plymouth State University
Department of Mathematics
http://oz.plymouth.edu/~dcernst

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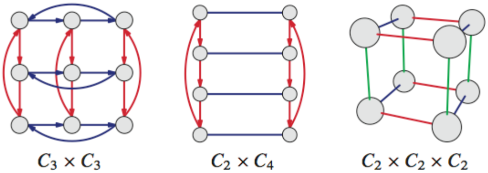

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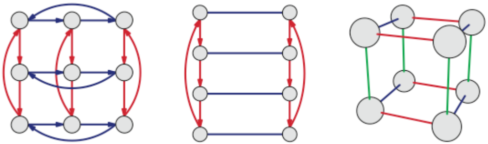

$C_3 \times C_3$ $\quad$ $C_2 \times C_4$ $\quad$ $C_2 \times C_2 \times C_2$

Do you notice anything about the orders of the product groups above?

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It'll certainly be in our best interest to work through a couple of examples.

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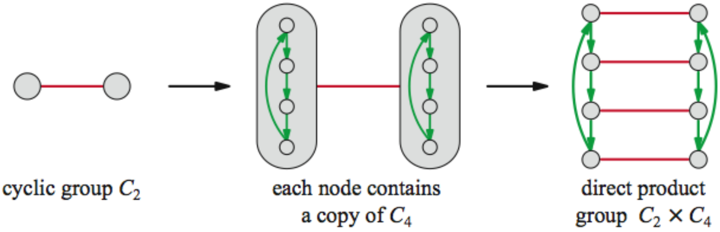

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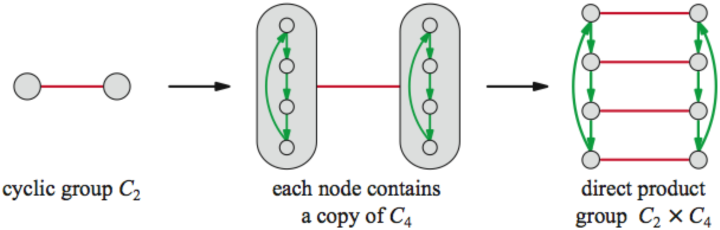

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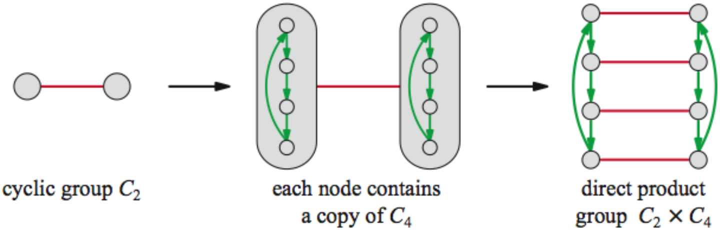

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Let's see if we can label the nodes of our Cayley diagram for $C_4 \times C_3$.

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2. Let's discuss your solutions.

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As we can see, the left and right cosets agree. Therefore, the group in $C_4 \times C_3$ that "is" C_3 is normal.

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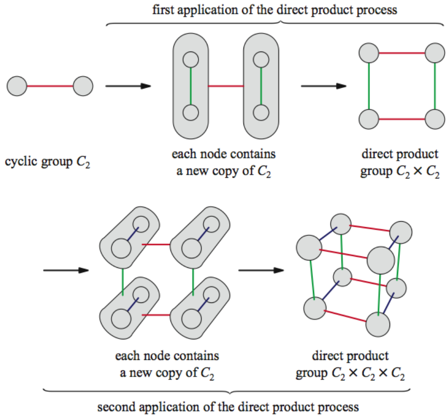
We can form direct products with more than 2 groups.

If we wanted to form the Cayley diagram for $A \times B \times C$, we could first construct the diagram for $A \times B$ and then construct the diagram for $(A \times B) \times C$. Here is the construction of $C_2 \times C_2 \times C_2$ (taken from Figure 7.6 on page 122 of *VGT*).



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One interesting observation is that for the 2 light switches, the action performed on one light switch has no impact on the other and vice versa. This phenomenon occurs in all direct products.

In a Cayley diagram for $A \times B$, following A arrows neither impacts or is impacted by the location in group B.

Imagine you are at some node (a, b) in the Cayley diagram for $A \times B$.

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The actions are: spin one or both of the wheels. Each action can be labeled by where we end up on the first wheel and where we end up on the second wheel: say (i, j).

Here is an example of a visual for more general direct products (taken from Figure 7.11 on page 125 of *VGT*) showing the element $(r^2, 5)$ in $D_4 \times C_7$.

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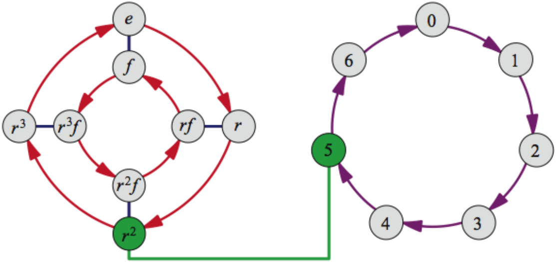

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As an example, in $D_3 \times C_4$, $(r^2, 1) * (fr, 3) = (fr^2, 0)$.

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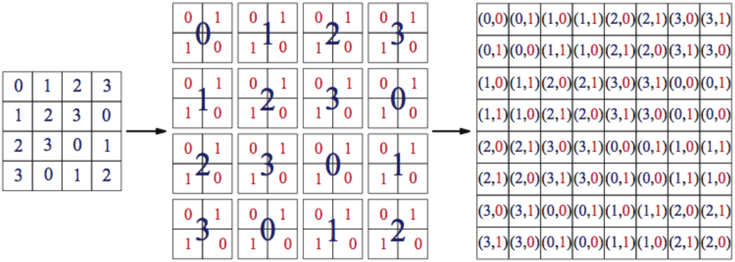

multiplication table for the group C_4 — inflate each cell to also contain a copy of the multiplication table for C_2 — join the little tables and element names to form the final direct product table

More group work

In groups of 2–3, complete all parts of Exercise 7.3. I want each group to turn in a complete solutions.

Let's discuss your solutions.

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As we did with direct products, we will first describe the quotient operation using Cayley diagrams and then we will explore some properties of the resulting group.

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3. IF the resulting diagram is a Cayley diagram of a group, then you have obtained the quotient group of G by H, denoted G/H and often read "G mod H." If not, then G cannot be divided by H.

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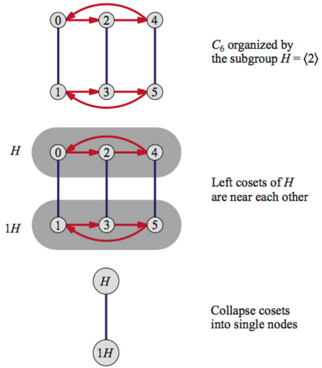

(Labeling of nodes 1, 3, 5 is wrong.)

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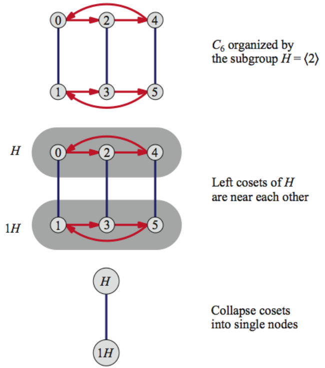

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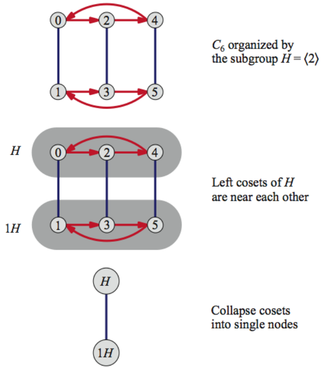

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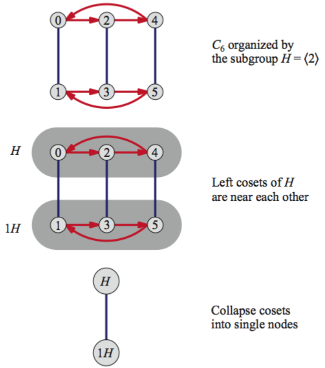

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In this example, the resulting diagram *is* a Cayley diagram. So, we can divide C_6 by $\langle 2 \rangle$. In fact, we see that $C_6/\langle 2 \rangle$ is isomorphic to C_2.

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- As one would expect, if $G = A \times B$ and we divide G by A, then the quotient group is B (it turns out that this always works; we'll see why shortly). However, the converse is not generally true. That is, if we can divide G by H, then that does not necessarily mean that G is equal to a direct product of H and the result of dividing G by H.

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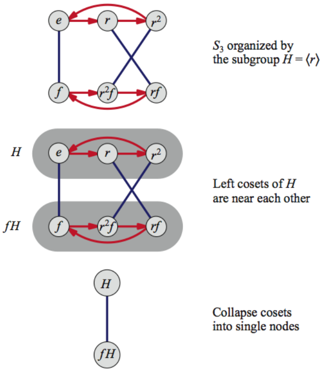

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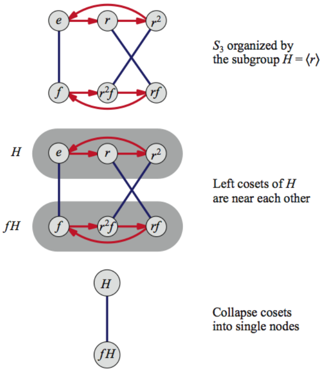

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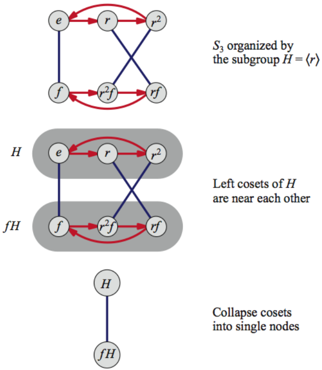

The resulting diagram is a Cayley diagram. So, S_3/C_3 makes sense and is isomorphic to C_2. However, you can tell by the inconsistent wiring of nodes in the middle step that S_3 is not a direct product of C_3 and C_2.

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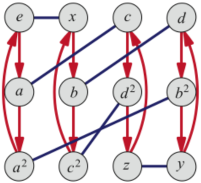

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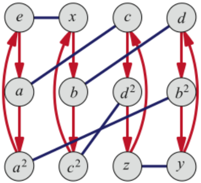

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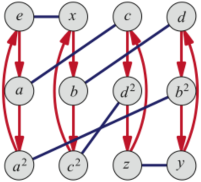

However, the subgroup $H = \langle x, z \rangle$ is not obvious from this diagram. It turns out that $H = \langle x, z \rangle$ is isomorphic to V_4.

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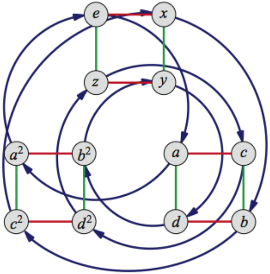

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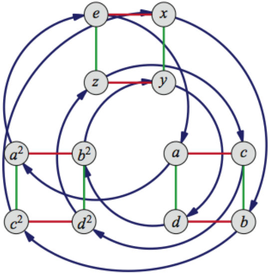

We can now see the left cosets of H clearly.

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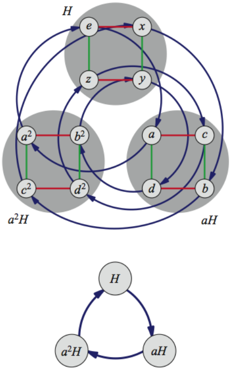

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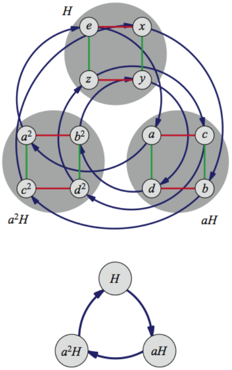

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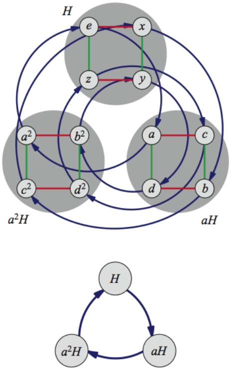

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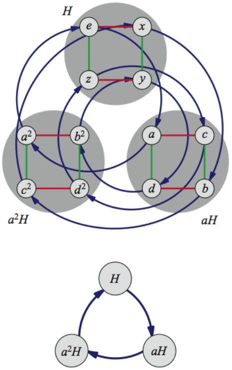

As we can see, the resulting diagram is a Cayley diagram. So, A_4/V_4 is isomorphic to C_3. However, A_4 is not isomorphic to $V_4 \times C_3$.

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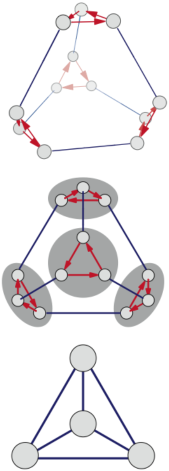
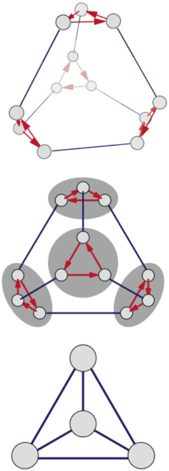
The figure on the next slide (taken from Figure 7.26 on page 138 of *VGT*) shows the result of trying to divide A_4 by $H = \langle a \rangle$.





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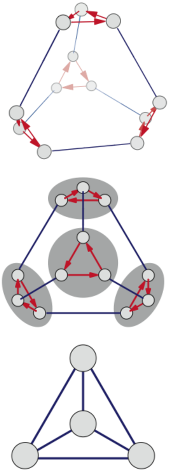
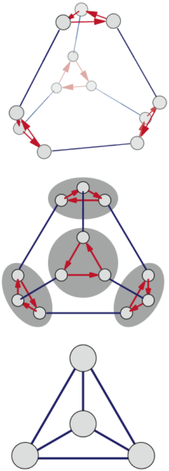


OK, so what's wrong? This diagram is *not* a Cayley diagram.



OK, so what's wrong? This diagram is *not* a Cayley diagram. It violates Rule 1.7; there is ambiguity about which blue arrow to travel anytime we leave a node.

The big question is:

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First, let's determine whether the subgroup in A_4 isomorphic to C_3 is normal or not.

Using the following Cayley diagram for A_4, the left cosets of $H = \langle a \rangle$ are easy to pick out.

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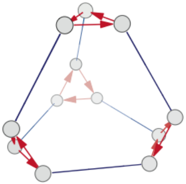

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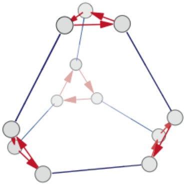

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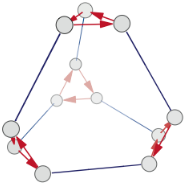

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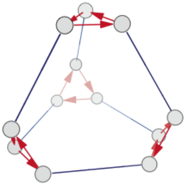

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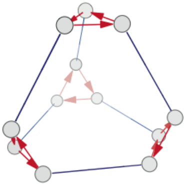

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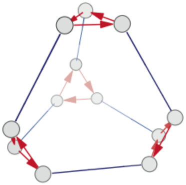

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So, $H = \langle a \rangle$ is *not* normal in A_4.

If we took the effort to check our first 3 examples, we would find out that in each case, the left cosets and right cosets coincide. So, in those examples, where G/H exists, H was normal.

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Recall that:

- each left coset gH is the set of nodes that H arrows can reach from g (which looks like a copy of H at g);
- each right coset Hg is the set of nodes to which the g arrows take the elements of H.

The following figure (taken from Figure 7.27 on page 139 of *VGT*) depicts the potential ambiguity that may arise when cosets are collapsed in the sense of Definition 7.5.

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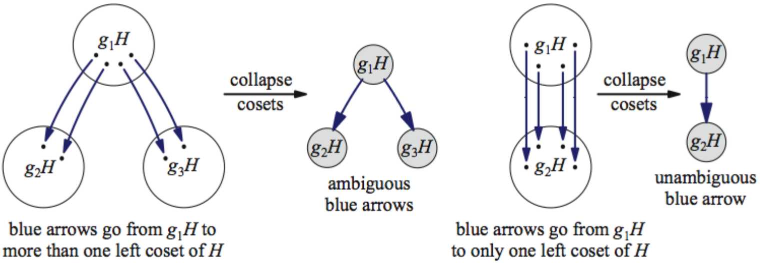

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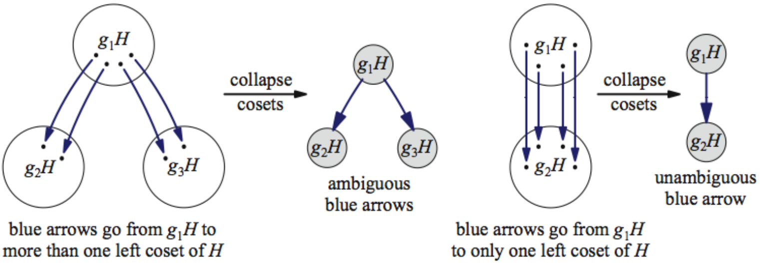

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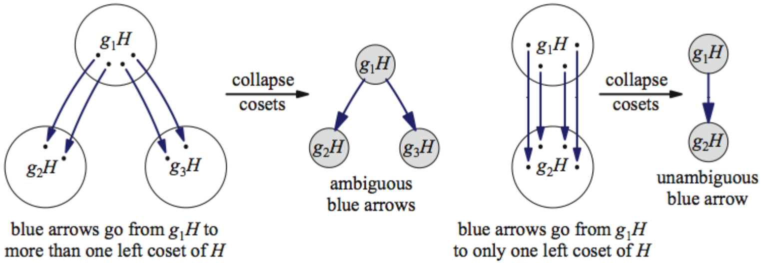

Note that the action of the blue arrows above is illustrating multiplication of a left coset on the *right* by some element. That is, the picture is showing us how left and right cosets interact.

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Theorem 7.6

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Finally, let's state the answer to our original question to when we can take a quotient.

Theorem 7.6

If $H < G$, then a quotient group G/H can be constructed only when $H \lhd G$.

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Since we begin with a regular diagram and we collapse identically structured sections distributed uniformly throughout the diagram, we end up with a regular diagram.

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3. Now, complete Exercise 7.18(f). I want each group to turn in a complete solution.

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At a minimum, we know that every $g \in H$ vote in favor of H being normal. Why? Well, since H is closed, if $g \in H$, we must have $gH = H = Hg$.

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Let's explore some possibilities for what the normalizer of a subgroup can be.

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Furthermore, the deciding factor in how a left coset will vote is simply whether it is also a right coset (because gH votes as a block exactly when $gH = Hg$).

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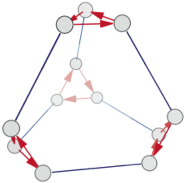

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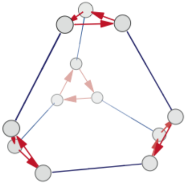

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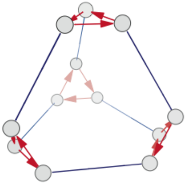

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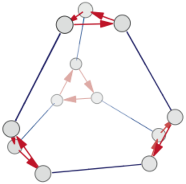

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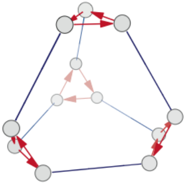

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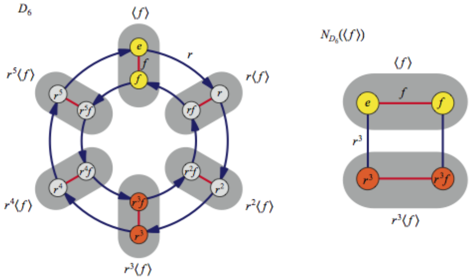

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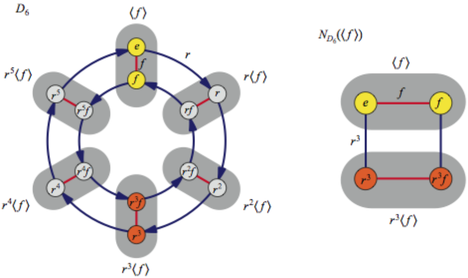

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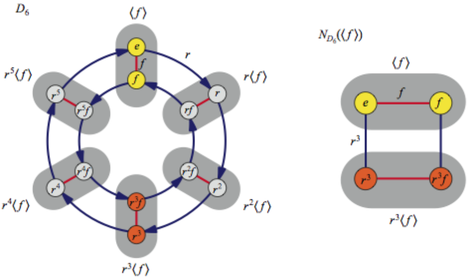

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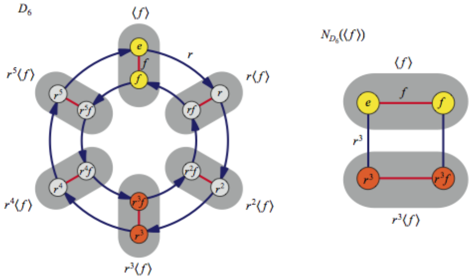

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For a proof, see pages 141–142 of *VGT*.

Comments

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$$H \lhd N_G(H) < G.$$

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Comments

- We have
$$H \lhd N_G(H) < G.$$
- The closer $N_G(H)$ is to being all of G, the closer H is to being normal.

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1. In groups of 2–3, complete the following exercises (not collected):
   - Exercise 7.25(a)
   - Exercise 7.25(b)
2. Let's discuss your solutions.
3. Now, complete Exercises 7.26(a) and 7.26(b). I want each group to turn in a complete solution for both exercises.