MA 2550: Calculus I (Fall 2011)
Exam 3

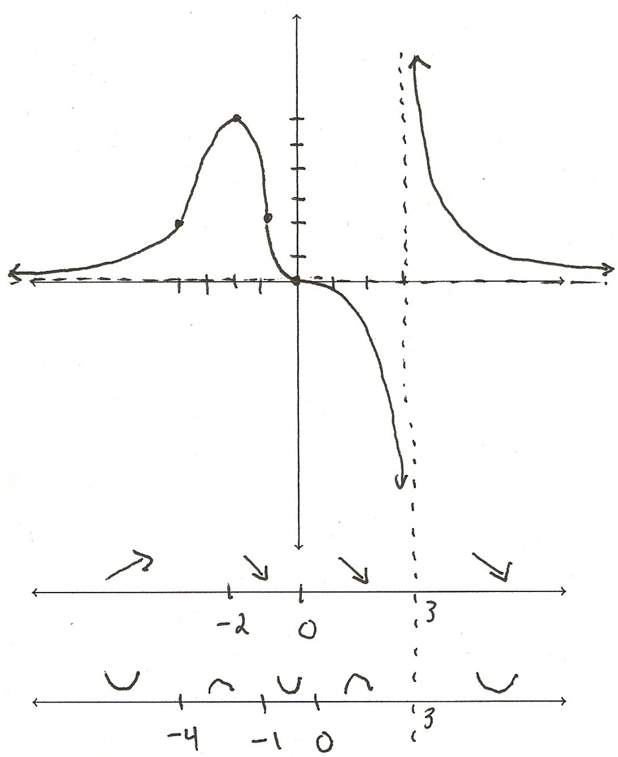
NAME: Solutions (2 points!)

Instructions: Answer each of the following questions completely. To receive full credit, you must show sufficient work for each of your answers (unless stated otherwise). How you reached your answer is more important than the answer itself. If something is unclear, or if you have any questions, then please ask. Good luck!

1. (12 points) Suppose f is a function with the following properties.

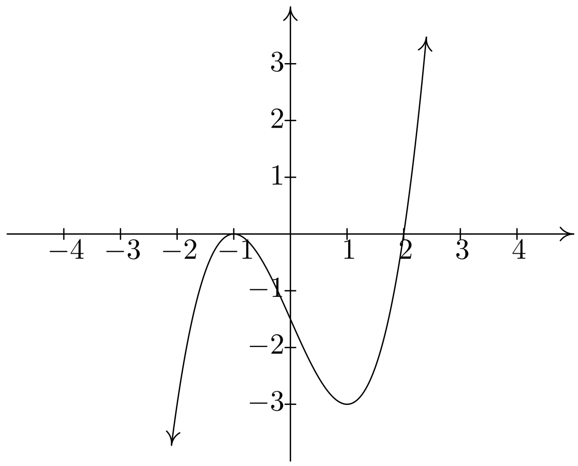
   (a) $f(-4) = 2$, $f(-2) = 5$, $f(-1) = 2$, and $f(0) = 0$
   (b) vertical asymptote at $x = 3$ such that $\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = \infty$
   (c) horizontal asymptote at $y = 0$ such that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$
   (d) $f'(-2) = 0$ and $f'(0) = 0$
   (e) $f'(x) > 0$ on $(-\infty, -2)$
   (f) $f'(x) < 0$ on $(-2, 0)$, $(0, 3)$, and $(3, \infty)$
   (g) $f''(-4) = 0$, $f''(-1) = 0$, and $f''(0) = 0$
   (h) $f''(x) > 0$ on $(-\infty, -4)$, $(-1, 0)$, and $(3, \infty)$
   (i) $f''(x) < 0$ on $(-4, -1)$ and $(0, 3)$

   Using the above information, make a sketch of the graph of f. You do *not* need to justify your answer.



Written by D.C. Ernst

2. (2 points each) Let f be a differentiable function. Suppose that the following graph is the graph of the *derivative* of f (i.e., the graph of f').



Graph of f'

(a) Find the x-coordinates of all points on the graph of f where the tangent line is horizontal.

Solution: A tangent line to f is horizontal when the derivative is 0. This occurs where the graph of f' crosses the x-axis. Therefore, the answer is $\boxed{x = -1, 2}$.

(b) Find the (open) intervals, if any, on which f is increasing.

Solution: The function f will be increasing when f' is positive. This occurs when the graph of f' is above the x-axis. Thus, the answer is $\boxed{(2, \infty)}$.

(c) Find the (open) intervals, if any, on which f is decreasing.

Solution: Similar to the previous part, f is decreasing when f' is below the x-axis. Therefore, the answer is $\boxed{(-\infty, -1) \text{ and } (-1, 2)}$ (or I would accept $(-\infty, 2)$).

(d) Find the x-coordinates, if any, where f attains a local max.

Solution: We're looking for an x value where f changes from increasing to decreasing. By by parts (b) and (c), this doesn't happen. Thus, there is $\boxed{\text{no local max}}$.

(e) Find the x-coordinates, if any, where f attains a local min.

Solution: For this part, we are looking for an x-value where f changes from decreasing to increasing. According to parts (b) and (c), this happens at $\boxed{x = 2}$.

3. (12 points) A 15 foot ladder is leaning against a wall and begins sliding down the wall. If the ladder is sliding down the wall at a rate of 1/4 foot per second when the top of ladder is 10 feet off the ground, at what rate is the bottom of the ladder sliding away from the wall? You should give an *exact* answer and be sure to label your answer with appropriate units.

Solution: Draw a right triangle and label the hypotenuse with length 15 (this is constant over time). Also, let's label the base with length x and the altitude with length y. According the Pythagorean Theorem, $x^2 + y^2 = 15^2$. At some moment in time, $y = 10$, $dy/dt = -1/4$ ft/sec, and we are looking for dx/dt. Differentiating both sides of the equation above, we see that

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0,$$

which simplifies to

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0.$$

In order to find dx/dt, we need to find x. Using the facts that $x^2 + y^2 = 15^2$ and $y = 10$, we see that $x = \sqrt{125}$. Plugging in all the known information, we get

$$\sqrt{125}\frac{dx}{dt} + 10\left(-\frac{1}{4}\right) = 0.$$

Lastly, solving for dx/dt, we obtain

$$\boxed{\frac{dx}{dt} = \frac{5}{2\sqrt{125}} \text{ ft/sec.}}$$

4. (10 points) Consider the following function.

$$f(x) = \frac{6}{5}x^{5/3} - \frac{9}{2}x^{2/3}$$

Using appropriate calculus techniques, find all critical numbers of f.

Solution: To find the critical numbers, we need to determine where the derivative of f is undefined or equal to 0. We see that

$$\begin{aligned} f'(x) &= 2x^{2/3} - 3x^{-1/3} \\ &= 2x^{2/3} - \frac{3}{x^{1/3}} \\ &= \frac{2x-3}{x^{1/3}}. \end{aligned}$$

By looking at the numerator and denominator separately, the critical numbers are $\boxed{x = 3/2, 0}$.

5. (10 points) Find *all* asymptotes (i.e., vertical and horizontal/slant) of the following function.

$$f(x) = \frac{5x^3 - 2x^2 - 1}{x^2 - 4}$$

Solution: To find the vertical asymptotes, we check to see where the denominator is 0. Solving $x^2 - 4 = 0$ gets us $\boxed{x = \pm 2}$. These are the vertical asymptotes. Since the degree of the numerator is greater than the degree of the denominator, there are no horizontal asymptotes. However, there is a slant asymptote. To find it, you need to do polynomial long division:

$$\begin{array}{r|rrr}
 & & & 5x - 2 \\ \hline
x^2 - 4 & 5x^3 - 2x^2 & & -1 \\
 & -5x^3 & +20x & \\ \hline
 & -2x^2 & +20x & -1 \\
 & 2x^2 & & -8 \\ \hline
 & & 20x & -9
\end{array}$$

This implies that the slant asymptote is $\boxed{y = 5x - 2}$.

6. (10 points) Find the derivative of the following function. You do *not* need to simplify your answer.

$$f(x) = \ln\left(\frac{\sqrt{3-x}}{x^{3/5}(x^2-4)^4}\right)$$

Solution: The easiest way (in my opinion) to solve this problem is to first use the properties of logs to expand the given expression. We see that

$$f(x) = \ln\left(\frac{\sqrt{3-x}}{x^{3/5}(x^2-4)^4}\right) = \frac{1}{2}\ln(3-x) - \frac{3}{5}\ln x - 4\ln(x^2-4).$$

Using the formula for the derivative of the natural log and the chain rule, we get

$$\boxed{f'(x) = \frac{1}{2} \cdot \frac{1}{3-x} \cdot (-1) - \frac{3}{5} \cdot \frac{1}{x} - 4 \cdot \frac{1}{x^2-4} \cdot 2x.}$$

7. (10 points) Evaluate the following limit. If the limit does not exist, specify whether it equals ∞, $-\infty$, or simply does not exist (in which case, write DNE). If the limit does exist, you should give an *exact* answer, as opposed to a decimal approximation.

$$\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right)$$

Solution: Initially, we see that this limit has an indeterminate form of $\infty \cdot 0$. Rewriting, we see that

$$\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}},$$

which has indeterminate form $\frac{0}{0}$. Applying L'Hospital's Rule, we obtain

$$\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\frac{-\pi}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \pi\cos\left(\frac{\pi}{x}\right) = \pi\cos 0 = \boxed{\pi}.$$

8. (12 points) Use appropriate calculus techniques to find the absolute maximum and absolute minimum values of the function $f(x) = 2x^3 - 3x^2 - 12x + 1$ on the interval $[1, 4]$. Sufficient work must be shown.

Solution: First we find the critical numbers of f. To this, we determine where the derivative is equal to 0 or is undefined. We see that

$$\begin{aligned} f'(x) =& 6x^2 - 6x - 12 \\ 0 =& 6(x^2 - x - 2) \\ =& 6(x-2)(x+1). \end{aligned}$$

Therefore, the critical numbers are $x = 2, -1$, but -1 is not in the given interval $[1, 4]$. Now, we need to plug the endpoints and the critical number 2 into the original function:

x	$f(x)$
1	-12
2	-19
4	33

Thus, the minimum value of the function on the given interval is $\boxed{\text{-19}}$ and the maximum value is $\boxed{33}$.

9. (12 points) If 1200 cm^2 of material is available to make a box (for holding nuggets) with a square base and an open top, find the largest possible volume of the box. Be sure to label your answer with appropriate units.

Solution: Draw a box with a square base, label the lengths of each side of the base with x, and label the height with y. We need to maximize the volume, and so our primary equation is

$$V = x^2 y.$$

We are also given the fact that there is 1200 cm^2 of material to use. Since there are 4 sides, one base, and no top, the total surface area is

$$1200 = x^2 + 4xy.$$

This is our secondary equation and can be rewritten as

$$y = \frac{1200 - x^2}{4x}.$$

Substituting this into our primary equation, we obtain

$$V = x^2 \left(\frac{1200 - x^2}{4x} \right) = \frac{1200x - x^3}{4}.$$

Given the constraints of this problem, the feasible domain is $(0, \sqrt{1200})$. Taking the derivative of our primary equation, we get

$$V' = \frac{1200 - 3x^2}{4},$$

which has critical numbers $x = \pm 20$, but only 20 is in our feasible domain. The maximum volume must occur at this value. Plugging $x = 20$ into our volume formula, we see that the maximum volume is $\boxed{4000 \text{ cm}^3}$.

