MA 2560: Calculus II (Fall 2009)
Exam 2

NAME: Solutions

Instructions: Answer each of the following questions completely. To receive full credit, you must *justify* each of your answers (unless stated otherwise). How you reached your answer is more important than the answer itself. If something is unclear, or if you have any questions, then please ask. Good luck!

Here are some potentially useful formulas. You should *not* expect to use all of them.

$$\frac{d}{dx}[\sinh x] = \cosh x \qquad \frac{d}{dx}[\cosh x] = \sinh x$$

$$\frac{d}{dx}[\tanh x] = \text{sech}^2 x \qquad \frac{d}{dx}[\text{sech } x] = -\text{sech } x \tanh x$$

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2} \qquad \frac{d}{dx}[\sec^{-1} x] = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\tanh^{-1} x] = \frac{1}{1-x^2} \qquad \frac{d}{dx}[\text{sech}^{-1} x] = \frac{-1}{x\sqrt{1-x^2}}$$

$$\int \sinh u\, du = \cosh u + C \qquad \int \cosh u\, du = \sinh u + C$$

$$\int \text{sech}^2 u\, du = \tanh u + C \qquad \int \text{sech } u \tanh u\, du = -\text{sech } u + C$$

$$\int \frac{1}{\sqrt{a^2-u^2}}\, du = \sin^{-1}\frac{u}{a} + C \qquad \int \frac{1}{u^2+a^2}\, du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\, du = \frac{1}{a}\sec^{-1}\frac{u}{a} + C \qquad \int \frac{1}{\sqrt{u^2+a^2}}\, du = \sinh^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{\sqrt{u^2-a^2}}\, du = \cosh^{-1}\frac{u}{a} + C \qquad \int \frac{1}{a^2-u^2}\, du = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C$$

1. (10 points) Evaluate each of the following limits. If a limit does not exist, specify whether it equals ∞, $-\infty$, or simply does not exist (in which case, write DNE). If the limit does exist, you should give an *exact* answer, as opposed to a decimal approximations.

(a) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$ $(\infty - \infty)$

$$= \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x \to 0^+} \frac{\cos x - 1}{1 \cdot \sin x + x \cos x} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x \to 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x}$$

$$= \frac{0}{1+1-0} = \boxed{0}$$

(b) $\lim_{x \to \infty} x^{1/x}$ (∞^0)

$$\ln y = \lim_{x \to \infty} \ln x^{1/x}$$

$$= \lim_{x \to \infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1}$$

$$= 0,$$

So, $\ln y = 0$, which implies that $y = e^0 = \boxed{1}$.

2. (10 points each) Integrate each of the following indefinite or definite integrals. For the definite integrals, you should give *exact* answers (i.e., *not* decimal approximations using your calculator).

(a) $\displaystyle\int \frac{\cos^3 x}{(\sin x)^{3/2}}\,dx$

$$= \int \frac{\cos x\,(1-\sin^2 x)}{(\sin x)^{3/2}}\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$dx = \dfrac{du}{\cos x}$

$$= \int \frac{1-u^2}{u^{3/2}}\,du$$

$$= \int u^{-3/2} - u^{1/2}\,du$$

$$= -2u^{-1/2} - \frac{2u^{3/2}}{3} + C$$

$$= \boxed{\frac{-2}{\sqrt{\sin x}} - \frac{2}{3}(\sin x)^{3/2} + C}$$

(b) $\displaystyle\int_0^{\pi} x\cos x\,dx$

$u = x$ $\quad v = \sin x$

$du = dx$ $\quad dv = \cos x\,dx$

$$= uv\Big|_0^{\pi} - \int_0^{\pi} v\,du$$

$$= x\sin x\Big|_0^{\pi} - \int_0^{\pi} \sin x\,dx$$

$$= x\sin x + \cos x\Big|_0^{\pi}$$

$$= \pi\sin\pi + \cos\pi - (0\cdot\sin 0 + \cos 0)$$

$$= 0 + -1 - (0+1) = \boxed{-2}$$

(c) $\int \frac{x}{\sqrt{16-4x^2}}\,dx$

$u = 16 - 4x^2$

$du = -8x\,dx$

$dx = \frac{du}{-8x}$

$= \int u^{-1/2} \cdot \cancel{x} \frac{du}{-8\cancel{x}}$

$= \frac{-1}{8} \cdot 2u^{1/2} + C$

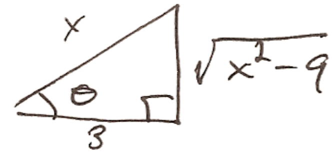
$= \boxed{\frac{-1}{4}\sqrt{16-4x^2} + C}$

(d) $\int \frac{\sqrt{x^2-9}}{x}\,dx$

$a = 3$

$u = x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$



$= \int \frac{\sqrt{9\sec^2\theta - 9}}{\cancel{3\sec\theta}} \cdot \cancel{3\sec\theta}\tan\theta\,d\theta$

$= 3\int \tan^2\theta\,d\theta$

$= 3\int \sec^2\theta - 1\,d\theta$

$= 3\left[\tan\theta - \theta\right] + C$

$= \boxed{3\left[\frac{\sqrt{x^2-9}}{3} - \operatorname{arcsec}\left(\frac{x}{3}\right)\right] + C}$

(e) $\int \frac{x^3+2x^2+5x+1}{x^2+2x+5}\,dx$

$$= \int x + \frac{1}{x^2+2x+5}\,dx$$

$$= \frac{x^2}{2} + \int \frac{1}{x^2+2x+1+5-1}\,dx$$

$$= \frac{x^2}{2} + \int \frac{1}{(x+1)^2+4}\,dx$$

$a=2$
$u=x+1$
$du=dx$

$$= \boxed{\frac{x^2}{2} + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C}$$

(f) $\int \frac{x^2-x+6}{x^3+3x}\,dx$

$$= \int \frac{x^2-x+6}{x(x^2+3)}\,dx$$

$$= \int \frac{A}{x} + \frac{Bx+C}{x^2+3}\,dx$$

$A(x^2+3)+(Bx+C)x = x^2-x+6$

$= Ax^2+3A+Bx^2+Cx$

$= (A+B)x^2+Cx+3A$

$A+B=1 \Rightarrow \boxed{B=-1}$

$\boxed{C=-1}$ $\Uparrow$

$3A=6 \Rightarrow \boxed{A=2}$

$$= \int \frac{2}{x} + \frac{-x-1}{x^2+3}\,dx$$

$$= 2\ln|x| - \int \frac{x}{x^2+3}\,dx - \int \frac{1}{x^2+3}\,dx$$

$u=x^2+3$
$du=2x\,dx$
$dx=\frac{du}{2x}$

$a=\sqrt{3}$
$u=x$
$du=dx$

$$= \boxed{2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)+C}$$

3. (10 points) Determine whether the following integral converges or diverges. If the integral converges, determine its *exact* value. If the integral diverges, explain why.

$$\int_1^\infty \frac{\ln x}{x}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx \qquad u = \ln x,\quad du = \frac{1}{x}\,dx,\quad dx = x\,du$$

$$= \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{u}{\cancel{x}}\,\cancel{x}\,du$$

$$= \lim_{t\to\infty} \left.\frac{u^2}{2}\right|_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \left.\frac{(\ln x)^2}{2}\right|_1^t = \lim_{t\to\infty}\left[\frac{(\ln t)^2}{2} - 0\right]$$

$$= \boxed{\infty} \quad \text{(diverges)}$$

4. Consider the following integral.

$$\int_0^1 \frac{\sin^2 x}{\sqrt{x}}\,dx$$

(a) (2 points) Explain why this integral is improper.

$y = \dfrac{\sin^2 x}{\sqrt{x}}$ has discont @ $x=0 \in [0,1]$.

(b) (8 points) It turns out that this integral is very difficult to compute by hand. Show that this integral converges by comparing with a larger function that converges on the same interval. (Hint: recall that $\sin^2 x \leq 1$.)

Since $\sin^2 x \leq 1$, $\dfrac{\sin^2 x}{\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$. Also,

$$\lim_{t\to 0^+} \int_t^1 x^{-1/2}\,dx = \lim_{t\to 0^+} \left. 2x^{1/2}\right|_t^1$$

$$= \lim_{t\to 0^+}\left[2 - 2\sqrt{t}\right] = 2.$$

Since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, and $\frac{\sin^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$, $\int_0^1 \frac{\sin^2 x}{\sqrt{x}}\,dx$ converges, as well.

5. **Bonus Question:** (5 points) If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, show that

$$\int_0^\infty f'(x)\,dx = -f(0).$$

$$\int_0^\infty f'(x)dx$$

$$= \lim_{t\to\infty} \int_0^t f'(x)dx$$

$$= \lim_{t\to\infty} f(x)\Big|_0^t$$

$$= \lim_{t\to\infty} \left[f(t) - f(0)\right]$$

$$= 0 - f(0) \qquad (\text{since } f(x)\to 0 \text{ as } x\to\infty)$$

$$= \boxed{-f(0)}.$$