

# Homework 11

## Abstract Algebra I

Complete the following problems. Note that you should only use results that we've discussed so far this semester.

**Problem 1.** Consider the ring  $M_2(\mathbb{R})$  (i.e., the ring of  $2 \times 2$  matrices with real number entries, where the operation is matrix multiplication). Recall that if  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , then  $\det(A) = ad - bc$ . Is  $\det$  a ring homomorphism? Justify your answer.

**Problem 2.** Define  $\phi : \mathbb{Z}_4 \rightarrow \mathbb{Z}_{12}$  via  $\phi(x) = 3x$ . Is  $\phi$  a ring homomorphism? Justify your answer.

**Problem 3.** Consider the ring  $M_2(\mathbb{Z})$ . Let  $I = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \mid a, c \in \mathbb{Z} \right\}$ . Show that  $I$  is a left ideal, but not a right ideal.

**Problem 4.** Let  $R$  be a ring. If there exists a positive integer  $n$  such that

$$\underbrace{a + a + \cdots + a}_n = 0$$

for all  $a \in R$ , then the least such positive integer is called the **characteristic** of  $R$ . If no such positive integer exists, then  $R$  is of characteristic 0. Find the characteristic of each of the following rings.

- (a)  $\mathbb{Z}_6$
- (b)  $\mathbb{Z}$
- (c)  $\mathbb{R}$

**Problem 5.** Prove **one** of the following.

- (a) Prove that the characteristic of an integral domain is either 0 or prime.
- (b) Let  $R$  be a commutative ring with prime characteristic  $p$ . Prove that if  $x, y \in R$ , then  $(x + y)^p = x^p + y^p$ .

**Problem 6.** Consider  $E = \{0, 2, 4, 6, 8\} \subseteq \mathbb{Z}_{10}$ . Find the field of fractions of  $E$  in  $\mathbb{Z}_{10}$ .

**Problem 7.** Define  $\phi : \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}$  via  $\phi(x) = 6x$ .

- (a) Prove that  $\phi$  is a ring homomorphism.
- (b) Determine whether  $\mathbb{Z}_{10}/\ker(\phi)$  is a field.
- (c) Is  $\ker(\phi)$  a maximal ideal of  $\mathbb{Z}_{10}$ ?

**Problem 8.** A **simple ring** is a ring with no nonzero proper 2-sided ideals. If  $R$  is a ring, then the **center** of  $R$  is defined to be  $Z(R) := \{x \in R \mid rx = xr \text{ for all } r \in R\}$ . Prove that the center of a simple ring with 1 is a field. *Note:* You must first show that the center is a subring.

**Problem 9.** Let  $R$  be a ring and let  $I$  be a right ideal of  $R$ . Suppose there exists an element  $a \in R$  such that  $a^2 = a$  (such an element is called **idempotent**). Let  $J = \{x \in I \mid ax = 0\}$ . Prove that  $J$  is a right ideal of  $R$ .

**Problem 10.** Let  $\phi : R \rightarrow S$  be a ring homomorphism, where  $R$  is a ring with 1, call it  $1_R$ .

- (a) Prove that  $\phi(1_R)$  is the multiplicative identity in  $\phi(R)$ .
- (b) Provide an example of a ring homomorphism where  $S$  has a multiplicative identity that is not equal to  $\phi(1_R)$  or prove that such an example does not exist.

**Problem 11.** Prove **one** of the following.

- (a) Let  $R$  be a commutative ring with 1. The **radical** of an ideal  $I$  in  $R$  is defined to be  $\sqrt{I} := \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{Z}^+\}$ . Prove that every prime ideal is radical.
- (b) Let  $R$  be a commutative ring with 1 and let  $U(R)$  be the group of units in  $R$ . Prove that  $R$  has a unique maximal ideal iff  $R \setminus U(R)$  is an ideal. *Note:* You may use Theorem 38 from our class notes.

**Problem 12.** Prove **one** of the following.

- (a) Prove that any subfield of  $\mathbb{R}$  must contain  $\mathbb{Q}$ .
- (b) Prove that a quotient of a principal ideal domain by a prime ideal is still a principal ideal domain.