MA 2560: Calculus II (Fall 2009)
Final Exam

NAME:

Instructions: Answer each of the following questions completely. To receive full credit, you must *justify* each of your answers (unless stated otherwise). How you reached your answer is more important than the answer itself. If something is unclear, or if you have any questions, then please ask. Good luck!

Here are some potentially useful formulas. You should *not* expect to use all of them.

$$\frac{d}{dx}[\sinh x] = \cosh x \qquad \frac{d}{dx}[\cosh x] = \sinh x$$

$$\frac{d}{dx}[\tanh x] = \text{sech}\,^2 x \qquad \frac{d}{dx}[\text{sech}\ x] = -\text{sech}\ x \tanh x$$

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2} \qquad \frac{d}{dx}[\sec^{-1} x] = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\tanh^{-1} x] = \frac{1}{1-x^2} \qquad \frac{d}{dx}[\text{sech}\,^{-1} x] = \frac{-1}{x\sqrt{1-x^2}}$$

$$\int \sinh u\ du = \cosh u + C \qquad \int \cosh u\ du = \sinh u + C$$

$$\int \text{sech}\,^2 u\ du = \tanh u + C \qquad \int \text{sech}\ u \tanh u\ du = -\text{sech}\ u + C$$

$$\int \frac{1}{\sqrt{a^2-u^2}}\ du = \sin^{-1}\frac{u}{a} + C \qquad \int \frac{1}{u^2+a^2}\ du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\ du = \frac{1}{a}\sec^{-1}\frac{u}{a} + C \qquad \int \frac{1}{\sqrt{u^2+a^2}}\ du = \sinh^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{\sqrt{u^2-a^2}}\ du = \cosh^{-1}\frac{u}{a} + C \qquad \int \frac{1}{a^2-u^2}\ du = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C$$

1. (6 points) Evaluate the following limit.

$$\lim_{x \to 0^+} (\cos x)^{1/x}$$

2. (6 points each) Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{e^x}{\sqrt{1 - e^{2x}}}\, dx$

(b) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\ dx$

(c) $\displaystyle\int \sin^3 x \cos^2 x\ dx$

(d) $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx$

(e) $\displaystyle\int_1^\infty \frac{1}{4+x^2}\,dx$ (If the integral converges, give an *exact* answer.)

3. (6 points) Find the surface area of the 'surface obtained by' revolving the graph of $y = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis. You should give an *exact* answer (i.e., not a decimal approximation using your calculator).

4. (6 points) Find the area of one loop of the graph of $r = 3\sin(5\theta)$.

5. (6 points) Consider the parametric curve given by

$$x = \frac{8}{3}t^{3/2}, \qquad y = 2t - t^2$$

Find the arc length for $1 \leq t \leq 2$.

6. (4 points each) Consider the sequence determined by

$$a_n = \frac{n^2 + 17}{2n(n+1)}.$$

(a) Does the sequence converge or diverge? If the sequence converges, find its limit.

(b) Does the corresponding series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Explain your answer.

7. (6 points each) Determine whether each of the following series *converge absolutely*, *converge conditionally*, or *diverge*. In order to receive full credit, you should clearly state what test(s) you are using and verify the appropriate hypotheses. The justification of your conclusion is vastly more important than the actual conclusion.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+17}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2}$

8. (6 points) Determine the *interval* of convergence of the following power series.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^n}{n 5^n}$$

9. (4 points each) Consider the function $f(x) = \dfrac{1}{x}$.

   (a) Find the Taylor series for f centered at $x = 1$. (There are at least 2 ways to do this problem.)

   (b) Find the *radius of convergence* for the Taylor series that you found in part (a)

10. (4 points each) We will approximate $\displaystyle\int_1^2 \frac{e^x - 1}{x}\, dx$.

(a) Using the known Maclaurin series for $y = e^x$, find the 5th degree Taylor polynomial for $y = e^x$.

(b) Using your answer from part (a), find a 4th degree polynomial that approximates $f(x) = \dfrac{e^x - 1}{x}$.

(c) Using your answer from part (b), approximate $\displaystyle\int_1^2 \frac{e^x - 1}{x}\, dx$. (Round your answer to 3 decimal places.)

11. **Bonus Question:** Imagine that you are stacking an infinite number of spherical nuggets of decreasing radii on top of each other. The radii of the spheres are 1 cm, $\frac{1}{\sqrt{2}}$ cm, $\frac{1}{\sqrt{3}}$ cm, $\frac{1}{\sqrt{4}}$ cm, etc.

    (a) (2 points) Does the infinite stack of spherical nuggets have finite or infinite height? (You must justify your answer.)

    (b) (2 points) Does the infinite stack of spherical nuggets have finite or infinite total volume? Recall that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. (You must justify your answer.)

12. **Bonus Question 2:** We've repeatedly said that the alternating harmonic series converges. But what does it converge to?

    (a) (2 points) Write $y = \dfrac{1}{1+x}$ as a geometric series starting at $n = 1$. (Hint: $x = -(-x)$.)

    (b) (2 points) Integrate your answer from part (a) over the interval $[0, 1]$. (Your answer should be a familiar looking series.)

    (c) (2 points) Now, integrate the function $y = \dfrac{1}{1+x}$ on the interval $[0, 1]$. (Your answer should be a number.)

    (d) (2 points) What conclusion can you make?