MA 4140: Algebraic Structures (Spring 2010)
Exam 3 (take-home portion)

NAME:

Instructions

Prove any *three* of the following theorems.

This portion of Exam 3 is worth 40 points. Each of the three proofs that you complete is worth 10 points. Your written presentation of the proofs (which includes spelling, grammar, punctuation, clarity, and legibility) is worth the remaining 10 points.

I expect your proofs to be *well-written, neat, and organized*. You should write in *complete sentences*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version.

The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll be happy to help you do this.

The simple rules for this portion of the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Proposition 2.9, then you should say so.
2. You cannot use any results from the book or otherwise that we have not covered, unless you prove them.
3. You are NOT allowed to copy someone else's work.
4. You are NOT allowed to let someone else copy your work.
5. You are allowed to discuss the problems with each other and critique each other's work.

This portion of Exam 3 is due by 5PM on Friday, May 21. You should turn in this cover page and the three proofs that you have decided to submit.

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

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Theorem 1. Let H_1 and H_2 be subgroups of G_1 and G_2, respectively. Then $H_1 \times H_2 \leq G_1 \times G_2$.

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Theorem 2. Let $\phi : G \to H$ be a group homomorphism from a group G to a group H. Then for all $g \in G$, the order of $\phi(g)$ divides the order of g.

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Definition. Let G be a group. Define the *center* of G to be the set

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}.$$

In other words, the center of G is the collection of elements that commute with all the elements of G. It turns out that $Z(G) \trianglelefteq G$. (You do *not* need to prove this.)

Theorem 3. Let G be a group. If $G/Z(G)$ is cyclic, then G is abelian.

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Theorem 4. Let G be a group with order p^2, where p is prime. If H is a subgroup of G with order p, then H is normal in G.

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Theorem 5. Let $n, m \in \mathbb{Z}$. Then $(\mathbb{Z} \times \mathbb{Z})/(n\mathbb{Z} \times m\mathbb{Z}) \cong \mathbb{Z}_n \times \mathbb{Z}_m$.*

*Since $\mathbb{Z}$ is abelian, $\mathbb{Z} \times \mathbb{Z}$ is abelian. This implies that all subgroups of $\mathbb{Z} \times \mathbb{Z}$ are normal. In particular, $n\mathbb{Z} \times m\mathbb{Z} \trianglelefteq \mathbb{Z} \times \mathbb{Z}$, so that $(\mathbb{Z} \times \mathbb{Z})/(n\mathbb{Z} \times m\mathbb{Z})$ is a well-defined group.

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