Section 2.1: Basic concepts of set theory

Goal

As the title suggests, we will introduce some of the basic concepts of set theory.

Introduction and background

We will assume an intuitive understanding of what a *set* is:

A *set* is a specified collection of objects.

Notice that we didn't say: "A set is any collection of objects," which is a commonly used (but bad) definition.

This slightly weaker definition introduces paradoxes, the most famous of which is called *Russell's paradox* (see Exercise 2.1.18). When we say "specified", we mean that there is no ambiguity as to whether an object is in or not in a set.

It is beyond the scope of this course to "build up" set theory from the axioms. "Building" set theory from the axioms is called *axiomatic set theory* (a very important branch of mathematics). The most common axioms of set theory are the *Zermelo–Fraenkel axioms*.

Here's an example of one of the Z–F axioms:

> **Axiom of Pairing:** A collection of two objects forms a set.

Using the Z–F axioms avoids Russell's paradox.

Terminology and notation

Definition 1. The objects in a set are called *elements*.

We usually use capital letters (A, B, C, ...) for sets and lowercase letters (a, b, c, ...) for elements.

If x is an element of a set A, then we write

$$x \in A.$$

If x is not an element of A, then we write

$$x \notin A.$$

When listing the elements of a set, we surround the elements with braces. For example,

$$A = \{1, 4, 8, 9\}$$

is the set only containing the elements 1, 4, 8, and 9.

Example 2.

(a) Let D be the set of democrats.

(b) Let A be the set of prime numbers less than 17.

Note 3. Listing the elements of a set is often impractical. If there is an obvious pattern for the elements, then we can use "…". For example,

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

However, this isn't always feasible. Instead, we use *set builder notation*:

$$\{x : P(x)\},$$

where $P(x)$ is an appropriate 1-variable open sentence that describes the properties that elements of the set have and the ":" stands for "such that."

Example 4. Let $P(x)$: "x is a prime number less than 17."

Note 5.

1. The choice of variable for set builder notation is irrelevant. So,

$$\{x : P(x)\} = \{☺ : P(☺)\}.$$

2. You can't use any old $P(x)$ to define a set. Some choices produce paradoxes. So, some care must be taken in choosing defining open sentences.

Here are some common sets:

- $\mathbb{N} = \{1, 2, 3, \ldots\}$

- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Q}$ = set of rational numbers $= \{r : \exists\, p, q \in \mathbb{Z}$ such that $r = \frac{p}{q}\}$
- $\mathbb{R}$ = set of real numbers
- $(a, b) = \{x : x \in \mathbb{R}$ and $a < x < b\}$
- $(a, b] = \{x : x \in \mathbb{R}$ and $a < x \leq b\}$
- $[a, b) = \{x : x \in \mathbb{R}$ and $a \leq x < b\}$
- $[a, b] = \{x : x \in \mathbb{R}$ and $a \leq x \leq b\}$
- $(a, \infty) = \{x : x \in \mathbb{R}$ and $a < x\}$
- $(-\infty, b) = \{x : x \in \mathbb{R}$ and $x < b\}$
- $[a, \infty) = \{x : x \in \mathbb{R}$ and $a \leq x\}$
- $(-\infty, b] = \{x : x \in \mathbb{R}$ and $x \leq b\}$
- $(-\infty, \infty) = \mathbb{R}$

Note 6. When considering a set $\{x : P(x)\}$, we should always consider the universe in which the variable lives.

Example 7.

(a) $U = \mathbb{R}$ (or $\mathbb{Z}$). Then
$$\{x : x^2 - 1 = 0\} = \underline{\hspace{3cm}}.$$

(b) $U = \mathbb{N}$. Then
$$\{x : x^2 - 1 = 0\} = \underline{\hspace{3cm}}.$$

To avoid ambiguity about the universe, we can write things like
$$\{x \in \mathbb{R} : x^2 - 1 = 0\}.$$

Definition 8. The *empty set*, denoted $\emptyset$, is the set containing no elements.
$$\emptyset = \{x : x \neq x\}$$

If we think of a set as a box of stuff, the empty set is an empty box.

Example 9. Let $U = \mathbb{R}$. Then
$$\{x : x^2 + 1 = 0\} = \underline{\hspace{3cm}}.$$

Definition 10. Let A and B be sets. A is called a *subset* of B iff every element of A is also an element of B.
$$A \subseteq B \iff (\forall x)(x \in A \implies x \in B)$$

This last line tells us how to prove that $A \subseteq B$: Assume $x \in A$. Show that $x \in B$.

Example 11.

(a) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and all the intervals are subsets of $\mathbb{R}$.

(b) Let $A = \{-3, 0, 2, 4, 7\}$. Then

$$2\_\_\_\ A$$
$$\{2\}\_\_\_\ A$$
$$2, 4\_\_\_\ A$$
$$\{2, 4\}\_\_\_\ A$$
$$3\_\_\_\ A$$
$$\{-3, 0, 2, 3\}\_\_\_\ A$$

Our first set theory theorems

Theorem 12 (2.1)**.** *For any set A,*

(i) $\emptyset \subseteq A$;

(ii) $A \subseteq A$.

Proof. Let A be any set.

(i) Since $\emptyset$ contains no elements, it is vacuously true that every element of $\emptyset$ is also contained in A.

(ii) Certainly, every element of A is contained in A. So, $A \subseteq A$.

□

Theorem 13 (2.2)**.** *Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. See Exercise 2.1.12.

□

More terminology and notation

Definition 14. $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

The definition tells us how to prove that two sets are equal: show "$\subseteq$" and then show "$\supseteq$."

Example 15. Claim: If $X = \{x \in \mathbb{R} : x^2 - 7x + 12 = 0\}$ and $Y = \{3, 4\}$, then $X = Y$.

Proof.

$\square$

Definition 16. A subset A of B is called a *proper subset* iff $A \subseteq B$ but $A \neq B$. In this case, we write $A \subset B$ or $A \subsetneq B$.

Example 17. $\{1, 2, 3\}$ is a proper subset of $\{1, 2, 3, 4\}$.

Definition 18. Let A be a set. The *power set* of A is the set whose elements are the subsets of A and is denoted by $\mathcal{P}(A)$.

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

$$B \in \mathcal{P}(A) \text{ iff } B \subseteq A$$

Example 19.

(a) Let $A = \{\circ, \square, \triangle\}$. Then

$$\mathcal{P}(A) = \underline{\hspace{8cm}}.$$

(b) Let $B = \{1, \{2\}, \{1, 2\}, \{3, 4, 5\}\}$. Then

1___ B

2___ B

$\{1\}$___ B

$\{1, 2\}$___ B

$\{\{1, 2\}\}$___ B

$\{1\}$___ $\mathcal{P}(B)$

$\{\{1, 2\}\}$___ $\mathcal{P}(B)$

(c) Consider $\mathbb{N}$. Describe $\mathcal{P}(\mathbb{N})$.

More theorems

Theorem 20 (2.4)**.** *If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.*

Proof. See page 75 of book. □

Theorem 21 (2.5)**.** *Let A and B be sets. Then $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.*

Proof. □