

# Homework 2

## Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

**Problem 1.** Let  $R$  be a Euclidean Domain with norm  $N$  satisfying  $N(a) \leq N(ab)$  for all nonzero  $a, b \in R$ <sup>1</sup>. Prove that  $a \in R$  is a unit iff  $N(a) = N(1)$ .

**Problem 2.** Consider the Euclidean Domain  $\mathbb{Z}[i]$  with norm given by  $N(a + bi) = a^2 + b^2$ .

- (a) Find the units in  $\mathbb{Z}[i]$ .
- (b) For each of the following pairs, find  $q$  and  $r$  such that  $a = bq + r$  with  $r = 0$  or  $N(r) < N(b)$ .
  - (i)  $a = 11 + 8i, b = 1 + 2i$
  - (ii)  $a = -17 + 15i, b = 3 + i$

**Problem 3.** Consider the Euclidean Domain  $\mathbb{Q}[x]$  with norm given by  $N(p(x)) = \deg(p(x))$ .

- (a) Prove that  $(x^2 + 1, x^3 + 1) = \mathbb{Q}[x]$ .
- (b) Find polynomials  $a(x)$  and  $b(x)$  such that  $(x^2 + 1)a(x) + (x^3 + 1)b(x) = 1$ .

**Problem 4.** Let  $R$  be an integral domain and let  $u$  be a unit of  $R$ .

- (a) Prove that if  $p \in R$  is prime, then  $up$  is prime.
- (b) Prove that if  $p \in R$  is irreducible, then  $up$  is irreducible.

**Problem 5.** Consider the ring  $\mathbb{Z}[\sqrt{-5}]$ . Note that  $\mathbb{Z}[\sqrt{-5}]$  is a Euclidean Domain with norm  $N(a + b\sqrt{-5}) = a^2 + 5b^2$ .

- (a) Justify my claim in Example 1.85(3) that  $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$  are two distinct factorizations of 6 into irreducibles in  $\mathbb{Z}[\sqrt{-5}]$ .
- (b) Prove that  $1 + \sqrt{-5}$  is not prime in  $\mathbb{Z}[\sqrt{-5}]$ .

**Problem 6.** Consider the ring  $\mathbb{Z}[2\sqrt{2}]$ .

- (a) Prove that  $\mathbb{Z}[2\sqrt{2}]$  is not a UFD. *Hint:* Fiddle around with 8. You will need to justify that certain ring elements are irreducibles. One way to do this is to play with the map  $N : \mathbb{Z}[2\sqrt{2}] \rightarrow \mathbb{Z} \cup \{0\}$  given by  $N(a + 2b\sqrt{2}) = |a^2 - 8b^2|$  (the vertical bars denote absolute value). It would be useful to know  $N(rs) = N(r)N(s)$ ,  $N(r) = 1$  iff  $r$  is a unit, and  $N(r) \neq 2$  for all  $r \in R$ . If you want to use these facts, you should prove them.
- (b) If possible, give an example of an ideal of  $\mathbb{Z}[2\sqrt{2}]$  that is not principal. If not possible, briefly explain why.

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<sup>1</sup>This extra requirement on  $N$  is sometimes part of the definition of Euclidean Domain.