

## 4 Module Theory

### 4.1 Definitions and Examples

This section of notes roughly follows Section 10.1 in Dummit and Foote.

Let's start with the definition of a module.

**Definition 4.1.** Let  $R$  be a ring (not necessarily commutative nor with 1). A **left  $R$ -module** (or **left module over  $R$** ) is a set  $M$  together with

- (1) a binary operation  $+$  on  $M$  under which  $M$  is an abelian group, and
- (2) an action of  $R$  on  $M$  (that is,  $R \times M \rightarrow M$ ) denoted by  $rm$ , for all  $r \in R$  and for all  $m \in M$  that satisfies.
  - (a)  $(r + s)m = rm + sm$  for all  $r, s \in R$  and  $m \in M$ ,
  - (b)  $(rs)m = r(sm)$  for all  $r, s \in R$  and  $m \in M$ , and
  - (c)  $r(m + n) = rm + rn$  all  $r \in R$  and  $m, n \in M$ .
  - (d) If  $R$  has a 1, then we also require:  $1m = m$  for all  $m \in M$ .

We analogously define **right  $R$ -modules**. If  $R$  is commutative and  $M$  is a left  $R$ -module, then we can make it a right  $R$ -module by defining  $mr = rm$  for all  $r \in R$  and  $m \in M$ . Notice that we cannot do this in general if  $R$  is not commutative since Axiom (2b) may fail. Unless we explicitly say otherwise, all modules will be left modules. Modules satisfying Axiom (2d) are called **unital modules**. We will assume that all our modules are unital.

The axioms for a module should look familiar. If  $R$  is a field, the axioms are precisely those for a vector space over  $R$ .

We emphasize that an abelian group  $M$  may have many different  $R$ -module structures for a fixed ring  $R$  (in the same way a group  $G$  could act in many ways as a permutation group of some fixed set  $S$ ).

**Definition 4.2.** Let  $R$  be a ring and let  $M$  be an  $R$ -module. An  **$R$ -submodule** of  $M$  is a subgroup  $N$  of  $M$  that is closed under the action of ring elements, i.e.,  $rn \in N$  for all  $r \in R$  and  $n \in N$ .

As expected, submodules of  $M$  are just subsets of  $M$  that are themselves modules under the same action. In particular, if  $R$  is a field, submodules are just vector subspaces. Every  $R$ -module has at least two submodules:  $M$  and  $\{0\}$ . The latter is often written as just 0 and called the **trivial submodule**.

**Example 4.3.** Let's see some examples.

- (1) Let  $R$  be any ring. Then  $M = R$  is a left  $R$ -module, where the action of a ring element on a module element is just usual ring multiplication. In this case, the submodules of  $M = R$  are the left ideals of  $R$ .
- (2) A special case of the first example is what  $R$  is a field. Then  $R$  is 1-dimensional vector space over itself.

- (3) More generally, if  $R = F$  is a field, every vector space over  $F$  is an  $F$ -module and vice versa. Let  $n \in \mathbb{Z}^+$  and let

$$F^n = \{(a_1, \dots, a_n) \mid a_i \in F \text{ for all } i\}.$$

We can make  $F^n$  into an  $n$ -dimensional vector space by defining addition and scalar multiplication in the standard way.

- (4) Let  $R$  be a ring with 1 and let  $n \in \mathbb{Z}^+$ . As above, define

$$R^n = \{(a_1, \dots, a_n) \mid a_i \in R \text{ for all } i\}.$$

We can make  $R^n$  an  $R$ -module by defining addition and multiplication by elements of  $R$  in the same manner as when  $R$  was a field. The module  $R^n$  is called the **free module of rank  $n$  over  $R$** .

- (5) The same abelian group  $M$  may have the structure of a module for several different rings  $R$ . In particular, if  $M$  is an  $R$ -module and  $S$  is a subring of  $R$  with  $1_R = 1_S$ , then  $M$  is automatically an  $S$ -module. For example, the field  $\mathbb{R}$  is an  $\mathbb{R}$ -module, a  $\mathbb{Q}$ -module, and a  $\mathbb{Z}$ -module.
- (6) If  $M$  is an  $R$ -module and for some 2-sided ideal  $I$  of  $R$ ,  $am = 0$  for all  $a \in I$  and  $m \in M$ , we say  $M$  is **annihilated by  $I$** . In this case, we can make  $M$  into an  $(R/I)$ -module by defining an action of the quotient ring  $R/I$  on  $M$ . For each  $m \in M$  and coset  $r + I \in R/I$ , define

$$(r + I)m = rm.$$

Since  $am = 0$  for all  $a \in I$  and  $m \in M$ , this is well-defined. In the special case that  $I$  is a maximal ideal in a commutative ring  $R$  and  $IM = 0$ ,  $M$  is a vector space over the field  $R/I$ .

- (7)  $\mathbb{Z}$ -modules...

- (8)  $F[x]$ -modules...

**Theorem 4.4** (Submodule Criterion). Let  $R$  be a ring and let  $M$  be an  $R$ -module. A subset  $N$  of  $M$  is a submodule of  $M$  iff

- (1)  $N \neq \emptyset$ , and
- (2)  $x + ry \in N$  for all  $r \in R$  and  $x, y \in N$ .