Homework 7

Abstract Algebra I

Complete the following problems. Note that you should only use results that we've discussed so far this semester.

Note: I told you that I was going to have you prove Cauchy's Theorem[1] on this homework assignment, but I decided against it. I may come back and sketch its proof later in the semester. Regardless, you can make use of this theorem as needed. (I'm not suggesting you need it on this assignment.)

Problem 1. Let G be a group and suppose H and K are both normal subgroups of G. Prove that $H \cap K$ is also a normal subgroup of G.

Problem 2. Find all normal subgroups of D_8 and then identify the isomorphism type of each of the corresponding quotient groups.

Problem 3. Let G be a group and let $N \trianglelefteq G$.

(a) Prove that if G is abelian, then G/N is abelian.

(b) Provide a counterexample to the converse of part (a).

(c) Prove that if G is cyclic, then G/N is cyclic.

(d) Provide a counterexample to the converse of part (c).

Problem 4. Let G be a group.

(a) Prove that $Z(G) \trianglelefteq G$.[2]

(b) Prove that if $G/Z(G)$ is cyclic, then G is abelian.[3]

(c) Prove that if $|G| = pq$, where p and q are primes (not necessarily distinct), then either $Z(G) = 1$ or G is abelian.

Problem 5. Let G be a group and let $N \trianglelefteq G$.

(a) Prove that if gN has finite order in G/N, then $|gN|$ is the smallest positive integer n such that $g^n \in N$.

(b) Provide an example where $|gN|$ (in G/N) is strictly smaller than $|g|$ (in G).

[1] Recall that Cauchy's Theorem tells is that if G is a finite group and p is a prime dividing $|G|$, then there exists an element in G of order p.

[2] Recall that $Z(G)$ is the center of G.

[3] This is one of my all-time favorite problems.

Problem 6. Complete **one** of the following.

(a) Define $\pi : \mathbb{R}^2 \to \mathbb{R}$ via $\pi((x,y)) = x + y$. Prove that π is a surjective homomorphism and then describe the kernel and fibers of π geometrically. What does the First Isomorphism Theorem tell us in this case?

(b) Define $\tau : \mathbb{C} \to \mathbb{C}$ via $\tau(a + bi) = a^2 + b^2$. Prove that τ is a homomorphism and find the image of τ. Describe the kernel and fibers of τ geometrically. What does the First Isomorphism Theorem tell us in this case?

Problem 7. Let $H \leq G$ and fix $g \in G$. Prove that gHg^{-1} is a subgroup of G of the same order as H. *Hint:* I suggest you utilize a homework problem or two from a previous assignment instead of starting from scratch. But starting from scratch isn't too hard either.

Problem 8. Let $H \leq G$ such that $|H| = n$.

(a) Prove that if H is the unique subgroup of order n in G, then $H \trianglelefteq G$.

(b) Provide a counterexample to the converse of part (a).

Problem 9. Prove each of the four Isomorphism Theorems as stated in class. You may consult external resources (e.g., Dummit and Foote), but you may only make use of results in your proofs that we have discussed this semester.

Problem 10. Let p be a prime and let G be a group of order $p^a m$, where p does not divide m. Assume $P \leq G$ and $N \trianglelefteq G$ such that $|P| = p^a$ and $|N| = p^a n$, where p does not divide n. Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$ for some b.