

4 Module Theory

4.1 Definitions and Examples

This section of notes roughly follows Section 10.1 in Dummit and Foote.

Let's start with the definition of a module.

Definition 4.1. Let R be a ring (not necessarily commutative nor with 1). A **left R -module** (or **left module over R**) is a set M together with

- (1) a binary operation $+$ on M under which M is an abelian group, and
- (2) an action of R on M (that is, $R \times M \rightarrow M$) denoted by rm , for all $r \in R$ and for all $m \in M$ that satisfies.
 - (a) $(r + s)m = rm + sm$ for all $r, s \in R$ and $m \in M$,
 - (b) $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$, and
 - (c) $r(m + n) = rm + rn$ all $r \in R$ and $m, n \in M$.

(d) If R has a 1 , then we also require: $1m = m$ for all $m \in M$.

We analogously define **right R -modules**. If R is commutative and M is a left R -module, then we can make it a right R -module by defining $mr = rm$ for all $r \in R$ and $m \in M$. Notice that we cannot do this in general if R is not commutative since Axiom (2b) may fail. Unless we explicitly say otherwise, all modules will be left modules. Modules satisfying Axiom (2d) are called **unital modules**. We will assume that all our modules are unital.

The axioms for a module should look familiar. If R is a field, the axioms are precisely those for a vector space over R .

We emphasize that an abelian group M may have many different R -module structures for a fixed ring R (in the same way a group G could act in many ways as a permutation group of some fixed set S).

Definition 4.2. Let R be a ring and let M be an R -module. An **R -submodule** of M is a subgroup N of M that is closed under the action of ring elements, i.e., $rn \in N$ for all $r \in R$ and $n \in N$.

As expected, submodules of M are just subsets of M that are themselves modules under the same action. In particular, if R is a field, submodules are just vector subspaces. Every R -module has at least two submodules: M and $\{0\}$. The latter is often written as just 0 and called the **trivial submodule**.

Example 4.3. Let's see some examples.

- (1) Let R be any ring. Then $M = R$ is a left R -module, where the action of a ring element on a module element is just usual ring multiplication. In this case, the submodules of $M = R$ are the left ideals of R .
- (2) A special case of the first example is what R is a field. Then R is 1-dimensional vector space over itself.
- (3) More generally, if $R = F$ is a field, every vector space over F is an F -module and vice versa. Let $n \in \mathbb{Z}^+$ and let

$$F^n = \{(a_1, \dots, a_n) \mid a_i \in F \text{ for all } i\}.$$

We can make F^n into an n -dimensional vector space by defining addition and scalar multiplication in the standard way.

- (4) Let R be a ring with 1 and let $n \in \mathbb{Z}^+$. As above, define

$$R^n = \{(a_1, \dots, a_n) \mid a_i \in R \text{ for all } i\}.$$

We can make R^n an R -module by defining addition and multiplication by elements of R in the same manner as when R was a field. The module R^n is called the **free module of rank n over R** .

- (5) The same abelian group M may have the structure of a module for several different rings R . In particular, if M is an R -module and S is a subring of R with $1_R = 1_S$, then M is

automatically an S -module. For example, the field \mathbb{R} is an \mathbb{R} -module, a \mathbb{Q} -module, and a \mathbb{Z} -module.

- (6) If M is an R -module and for some 2-sided ideal I of R , $am = 0$ for all $a \in I$ and $m \in M$, we say M is **annihilated by I** . In this case, we can make M into an (R/I) -module by defining an action of the quotient ring R/I on M . For each $m \in M$ and coset $r + I \in R/I$, define

$$(r + I)m = rm.$$

Since $am = 0$ for all $a \in I$ and $m \in M$, this is well-defined. In the special case that I is a maximal ideal in a commutative ring R and $IM = 0$, M is a vector space over the field R/I .

(7) \mathbb{Z} -modules...

(8) $F[x]$ -modules...

Theorem 4.4 (Submodule Criterion). Let R be a ring and let M be an R -module. A subset N of M is a submodule of M iff

- (1) $N \neq \emptyset$, and
- (2) $x + ry \in N$ for all $r \in R$ and $x, y \in N$.