MA3120: Linear Algebra - Spring 2012
Final Exam

Your Name:

Names of any collaborators:

Instructions

This exam is worth a total of 100 points and 20% of your overall grade. Please read the instructions for each question carefully.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Show *all* of your work and *justify* your solutions fully. If you use a calculator or computer software (e.g., Sage), be sure to write down both the input and output.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. Unless you prove them, you cannot use any results from the course notes or book that we have not yet covered.
2. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
3. You are **NOT** allowed to copy someone else's work.
4. You are **NOT** allowed to let someone else copy your work.
5. You are allowed to discuss the problems with each other and critique each other's work.

The exam is due to my office by 5PM on **Friday, May 18**. You should turn in this cover page and all of the work that you have decided to submit.

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

Written by D.C. Ernst

1. (2 points each) Suppose A is a matrix that is row equivalent to one of the following matrices.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad M_4 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For each of the following statements, state whether A is row equivalent to M_1, M_2, M_3, or M_4. If there is more than one correct answer, then list them all. If none of M_1, M_2, M_3, or M_4 satisfy the given conditions, then state this. If you do not have enough information to determine whether one of the given matrices has the desired property, then do not list it. You do *not* need to justify your answers.

(a) The linear system $\mathcal{LS}(A, \vec{0})$ has only the trivial solution.

(b) There exists at least one vector $\vec{b} \in \mathbb{R}^3$ such that $\mathcal{LS}(A, \vec{b})$ does *not* have a solution.

(c) The linear system $\mathcal{LS}(A, \vec{b})$ has a unique solution for all $\vec{b} \in \mathbb{R}^3$.

(d) The null space of A contains infinitely many vectors.

(e) The null space of A has a basis consisting of exactly one non-zero vector.

(f) The columns of A are linearly independent.

(g) The columns of A form a basis for $\mathbb{R}^3$.

(h) The columns of A are linearly dependent, but span all of $\mathbb{R}^3$.

(i) The column space of A has a basis consisting of exactly two non-zero vectors.

(j) The matrix-vector equation $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^3$.

(k) A is invertible.

(l) A^T has the same size at A, but is not invertible.

(m) The determinant of A exists, but $\det A \neq 0$.

(n) $\det A = 0$.

(o) If $T(\vec{x}) = A\vec{x}$, then T is neither one-to-one nor onto.

(p) If $T(\vec{x}) = A\vec{x}$, then T is onto (surjective).

(q) If $T(\vec{x}) = A\vec{x}$, then T is one-to-one (injective).

(r) If $T(\vec{x}) = A\vec{x}$, then the kernel of T contains more than the zero vector.

(s) A has 0 as an eigenvalue.

(t) A has at least one nonzero eigenvalue.

2. (4 points) The following statement is **FALSE** in general.

   If A and B are $n \times n$ matrices, then $(A + B)(A - B) = A^2 - B^2$.

   Provide a counterexample where A and B are 2×2 matrices and justify your answer by calculating both sides of the equation above.

3. (4 points) Assume that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Use this information to find $T\left(\begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}\right)$.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first reflects points over the y-axis and then rotates points $\pi/2$ radians counterclockwise.

   (a) (4 points) Find the standard matrix representation for this linear transformation.

   (b) (2 points) What single geometric transformation is T equivalent to? Briefly justify your answer.

   (c) (4 points) Using your matrix from part (a), find the eigenvalues for the matrix representation by finding the roots of the characteristic polynomial. Do your computations by hand.

   (d) (2 points) Describe geometrically why your answer in part (c) makes sense.

5. (4 points) Let

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

Show that $\det(A) = acf$.*

6. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & -2 & 2 \end{bmatrix}$$

   (a) (2 points) Use the result of Problem 5 to find the eigenvalues for A.

   (b) (4 points) For each eigenvalue of A that you found in part (a), find a basis for the corresponding eigenspace.

7. (2 points each) For each of the following functions, determine whether the function is a linear transformation. If the function is not linear, explain why. If the function is linear, you do *not* need to prove it.

   (a) Define $R : \mathbb{R} \to \mathbb{R}$ via $R(x) = \sqrt{x}$.

   (b) Define $B : \mathbb{R}^2 \to P_2$ via $B\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ax^2 + bx + 1$.

   (c) Define $L : \mathbb{R}^2 \to \mathbb{R}^2$ via $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ y - x \end{bmatrix}$.

   (d) Define $D : M_{2\times 2} \to \mathbb{R}$ via $D(A) = \det(A)$.

   (e) Define $I : P_2 \to \mathbb{R}$ via $I(f) = \int_0^1 f(x)\, dx$.

   (f) Define $S : P_2 \to P_3$ via $S(p(x)) = (x - 2)p(x)$.

*This a special case of a more general fact that says that the determinant of any lower triangular matrix has a determinant equal to the product of the entries on the diagonal.

8. Let $R : M_{2\times 2} \to P_2$ be defined via

$$R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + 2b + c - 4d) + (-a - b + 2c + d)x + (-3a - 4b + 4c + 5d)x^2.$$

It turns out that R is a linear transformation (you do not need to prove this).

(a) (4 points) Determine the kernel of R.

(b) (2 points) Is R one-to-one (injective)? Justify your answer.

9. (4 points) Let A be an $n \times n$ matrix have real eigenvalue λ. Suppose that $\vec{x}$ and $\vec{y}$ are both eigenvectors for λ. Show that $\vec{x} + \vec{y}$ is also an eigenvector for λ. Show this directly without appealing to the fact that the set of eigenvectors for λ form a subspace.

10. (4 points each) Prove any **two** of the following facts.

(a) Let $T : U \to V$ be a linear transformation, where U and V are vector spaces. Define

$$R = \{\vec{y} \in V : \text{there exists } \vec{x} \in U \text{ such that } T(\vec{x}) = \vec{y}\}.^\dagger$$

Then R is a subspace of V.

(b) Let A be an $m \times n$ matrix. If $\{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly dependent set in $\mathbb{R}^n$, then $\{A\vec{u}, A\vec{v}, A\vec{w}\}$ is linearly dependent in $\mathbb{R}^m$.

(c) Let A be an $m \times n$ matrix and define $T : \mathbb{R}^n \to \mathbb{R}^m$ via $T(\vec{x}) = A\vec{x}$. If the rank of A is n, then T is onto (surjective).‡

11. **Bonus Question:** (4 points) Imagine a baby world wide web with precisely 5 webpages, say P_1, P_2, P_3, P_4, P_5. Suppose that these pages having the following link structure:

Webpage	Links to
P_1	P_2, P_3, P_4, P_5
P_2	P_1, P_3, P_4
P_3	P_1, P_2
P_4	P_3
P_5	P_1, P_4

Using a transition matrix (not the Google matrix), compute the PageRank for this world wide web. If necessary, approximate your values to two decimal places. If you want help automating this in Sage, just ask.

† Note that this is just the range of T.

‡ Actually, the converse of this statement is also true, but I'm not asking you to prove it.