Chapter 1: Logic
Sections 1.10–1.14

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Notes

1.10 Uniqueness Theorems

Recall that an existence theorem is a theorem that assert the existence of an object. Sometimes the object that an existence theorem is claiming exists is unique (i.e., there is exactly one).

A theorem that guarantees the uniqueness of a mathematical object is called a uniqueness theorem.

Strategy for proving uniqueness theorems

1. Produce a candidate (either find a specific object that works or deduce that there must be such an object).
2. Assume that there are two candidates and then demonstrate that they must actually be the same.

Notes

Example

The equation $x^5 + 17 = 0$ has a unique real number solution.

Proof.

First, observe that $x = \sqrt[5]{-17}$ is a solution to the given equation. Now, assume that x_1 and x_2 are solutions to the given equation. Then $x_1^5 + 17 = x_2^5 + 17$, which implies that $x_1^5 = x_2^5$. Since fifth roots of real numbers are unique, it must be the case that $x_1 = x_2$. This shows that the given equation has a unique solution. ■

We actually proved something stronger than we were asked to. The claim was that there is a unique solution and we did one better by actually figuring out what that unique solution is. We won't always be able to (easily) do that.

Notes

1.11 Examples and Counterexamples

Recall that an implication "If A, then B" involving predicates A and B is true unless there is some value of the variable(s) that makes A true and B false. Providing such a value is called providing a counterexample.

Notice that if we want to show that "If A, then B" is false, we must show that "There exists x such that $A(x)$ and $\sim B(x)$" is true. In other words, providing a counterexample to an implication is equivalent to proving the existence theorem "There exists x such that $A(x)$ and $\sim B(x)$".

Notes

Example

Show that the following statement is false.

If a real number x is divisible by 3, then x is divisible by 6.

Counterexample

Consider the real number 3. We see that 3 is divisible by 3. However, 3 is not divisible by 6. ∎

It is important to realize that providing a counterexample is conclusive proof that an implication is *not* true. However, checking even a large number of examples (unless you've check them ALL) *never* constitutes a proof of an implication.

Notes

1.12 Direct Proof

Well, how do we actually go about proving an implication? We must show that all values of the variable(s) that make the hypothesis true also make the conclusion true.

We will discuss several strategies for proving implications, the first of which is called a direct proof.

Strategy for proving implications via direct proof

1. Assume that there is an arbitrary x in the universe that satisfies the hypothesis (i.e., makes $A(x)$ true).
2. Show/deduce that x satisfies the conclusion (i.e., makes $B(x)$ true).

Notes

Example

If x is an even integer, then $x^2 + 7$ is odd.

Note that by definition, x is even if there exists an integer k such that $x = 2k$ and x is odd if there exists an integer k such that $x = 2k + 1$. Any time you want to prove a statement involving even/odd, you should use this definition!

Proof.

Assume that x is an even integer. Then there exists an integer k such that $x = 2k$. We see that

$$x^2 + 7 = (2k)^2 + 7 = 4k^2 + 7 = 2(2k^2 + 3) + 1.$$

Observe that $2k^2 + 3$ is an integer. Therefore, $x^2 + 7$ is odd. ∎

Notes

1.13 Proof by Contraposition

Another common method for proving implications is called proof by contrapositive (or proof by contraposition). The statement $\sim B \implies \sim A$ is called the contrapositive of $A \implies B$.

Using truth tables, one can easily verify that an implication and its contrapositive are equivalent. So, if we want to prove an implication, we can choose to prove its contrapositive instead. This is useful if an implication has been worded in the negative.

Strategy for proving an implication by contrapositive

To prove $A \implies B$ by contrapositive, assume that there is an x such that $\sim B(x)$ and then show $\sim A(x)$.

Notes

1.14 Proof by Contradiction

Recall that if P is any statement, then $P \wedge \sim P$ is a contradiction (i.e., always false). Another technique for proving implications is called proof by contradiction.

In a proof by contradiction, we assume that the implication is false and then derive some statement P and its negation $\sim P$. Recall that $\sim (A \implies B)$ is equivalent to $A \wedge \sim B$.

Strategy for proving an implication by contradiction

1. Assume A and $\sim B$.
2. Derive some statement P and its negation $\sim P$. (This may be harder than it sounds since it takes some skill to determine what statement P you might be able to contradict.)

Notes

Why does a "proof by contradiction" actually a proof? The essense is that the statement $A \implies B$ is equivalent to

$$(A \wedge \sim B) \implies (P \wedge \sim P).$$

You'll explore this idea further in Exercise 1.14.2.

Let's do an example.

Notes

1.14.3 Example

If $a > 0$, then $1/a > 0$.

Proof.

For sake of a contradiction, assume that $a > 0$ and $1/a \leq 0$. Since $1/a \leq 0$, there exists a nonnegative number b such that $1/a + b = 0$. Multiplying both sides by a yields $1 + ab = 0$, which is equivalent to $1 = -ab$. Since $a > 0$ and $b \geq 0$, $ab \geq 0$. But then $-ab \leq 0$. This implies that $1 \leq 0$. Since we also know $1 > 0$, we have a contradiction. Therefore, we can conclude that our original assumption was false, which shows that the statement of the theorem is true. ■

Notes