Section 1.3: Quantifiers (part 1)

Goal

In this section, we will introduce the *existential quantifier* $\exists$ and the *universal quantifier* $\forall$.

Notation and Terminology

Important Note 1. We will be mostly dealing with propositions in a formal manner (i.e., symbolic) and we will transition to a more informal (i.e., complete sentences) without really pointing out the transition. The idea is that we want to be able to seamlessly go back and forth.

Recall that sentences like

$$x^2 - 9 = 0$$

are not ____________. This is an example of an *open sentence* (or *predicate*).

Definition 2. An *open sentence* is a sentence with one or more variables such that when the variables are replaced with particular objects, it becomes a ____________.

If P is an open sentence with variables $x_1, \ldots, x_n$, we write

$$P(x_1, \ldots, x_n).$$

Example 3.

(a) Let $P(x)$ be the open sentence $x^2 - 9 = 0$. Explain what $P(2)$ and $P(-3)$ are.

(b) Let $Q(x, y)$ denote the open sentence "x is prime and y is a multiple of 2." Find two different ordered pairs that make $Q(x, y)$ into a true statement. How about a false statement?

Definition 4. Given an open sentence, the *universe of discourse* is the set of objects available to substitute for variables. The set of objects from the universe that turn the open sentence into a true proposition is called the *truth set*.

Example 5. Again, let $P(x) : x^2 - 9 = 0$. The truth set depends on the universe. Determine the truth set for each of the following universes.

(a) $U = \mathbb{N} = \{1, 2, 3, 4, \ldots\}$

(b) $U = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Definition 6. Given a fixed universe, two open sentence $P(x)$ and $Q(x)$ are *equivalent* iff they have the same ____________ ____________.

Example 7. Let $P(x) : x^2 - 9 = 0$ and $R(x) : x = 3$. For each of the following universes, determine whether $P(x)$ and $R(x)$ are equivalent.

(a) $U = \mathbb{N} = \{1, 2, 3, 4, \ldots\}$

(b) $U = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The existential quantifier

Definition 8. For an open sentence $P(x)$, the sentence $(\exists x)P(x)$ is read "there exists x such that $P(x)$" (or, "for some x, $P(x)$") and is true iff the truth set of $P(x)$ is ____________. The symbol $\exists$ is called the *existential quantifier*.

It should be pointed out that while $P(x)$ is an open sentence, $(\exists x)P(x)$ is actually a ____________.

Example 9.

(a) Take $U = \{\text{cars}\}$. Translate $(\exists x)(x \text{ is red})$ and determine its truth value.

(b) Take $U = \mathbb{R} =$ set of real numbers. Translate $(\exists x)(x^2 - 9 = 0)$ and $(\exists x)(x^2 + 1 = 0)$ and determine their truth values.

The universal quantifier

Definition 10. For an open sentence $P(x)$, the sentence $(\forall x)P(x)$ is read "For all x, $P(x)$" (or, "For each/every x, $P(x)$") and is true iff the truth set of $P(x)$ is ____________. The symbol $\forall$ is called the *universal quantifier*.

Example 11.

(a) Take $U = \{\text{cars}\}$. Translate $(\forall x)(x \text{ is red})$ and determine its truth value.

(b) Take $U = \mathbb{R} =$ set of real numbers. Translate $(\forall x)(x^2 - 9 = 0)$ and $(\forall x)(x^2 + 1 = 0)$ and determine their truth values.

(c) Take $U = \mathbb{R} =$ set of real numbers. Translate $(\forall x)(x > x - 1)$ and determine its truth value.

More examples

Example 12. Let $U_1 = \{\text{mini cars}\}$ and $U_2 = \{\text{cars}\}$. For each universe, write a symbolic translations for the given sentences.

(a) "All minis have stripes."

(b) "Some minis have stripes."

Note 13. In general, we have:

"All $P(x)$ are $Q(x)$":	$(\forall x)(P(x) \implies Q(x))$
"Some $P(x)$ are $Q(x)$":	$(\exists x)(P(x) \wedge Q(x))$

Hidden quantifiers and abbreviations

Example 14.

(a) "Dogs have 4 legs" really means "____________dogs have 4 legs."

(b) "Professors wear jackets or sweaters with leather elbow pads" probably means "____________ professors ...."

Note 15. We use the following abbreviations:

(a) "If $x \in A$, then x has property P" can be written symbolically as $(\forall x)[(x \in A) \implies P(x)]$, but we can just abbreviate this as ________________________.

(b) Similarly, "Some $x \in A$ has property P" can be written symbolically as $(\exists x)[(x \in A) \wedge P(x)]$, but we can abbreviate this as ________________________.