Section 4.11: Hyperbolic Functions

Goal

In this section, we will introduce the *hyperbolic (trig) functions*, study their various properties, and most importantly, see how we can use the inverse hyperbolic functions to integrate a few more functions.

The hyperbolic functions

Definition 1. We define the *hyperbolic functions* as follows.

$$\boxed{\begin{array}{ll} \sinh x = \dfrac{e^x - e^{-x}}{2} & \operatorname{csch} x = \dfrac{1}{\sinh x} \\[2ex] \cosh x = \dfrac{e^x + e^{-x}}{2} & \operatorname{sech} x = \dfrac{1}{\cosh x} \\[2ex] \tanh x = \dfrac{\sinh x}{\cosh x} & \coth x = \dfrac{\cosh x}{\sinh x} \end{array}}$$

Note 2.

(1) We pronounce sinh, cosh, and tanh as ____________, ____________, and ____________, respectively.

(2) The trig terminology and notation stem from the fact that these functions have very similar properties to the ordinary trig functions.

Here are some identities involving the hyperbolic functions.

Theorem 3.

$$\boxed{\begin{array}{ll} \sinh(-x) = -\sinh x & \cosh(-x) = \cosh x \\[2ex] \cosh^2 x - \sinh^2 x = 1 & 1 - \tanh^2 x = \operatorname{sech}^2 x \end{array}}$$

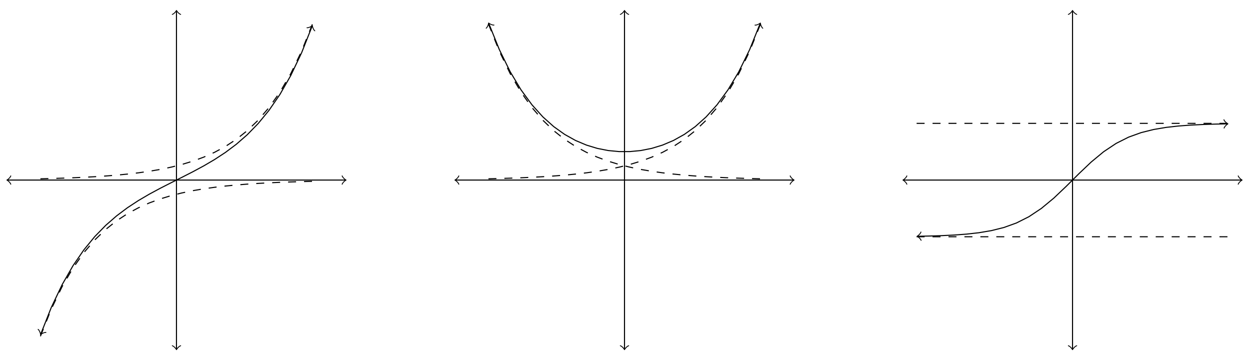
Proof. Let's prove the third identity. The proofs of the remaining ones are similar.

$$\cosh^2 x - \sinh^2 x =$$

□

Note 4. Notice that in the hyperbolic trig identities there is an occasional absence or addition of a ____________ when compared to the ordinary trig functions.

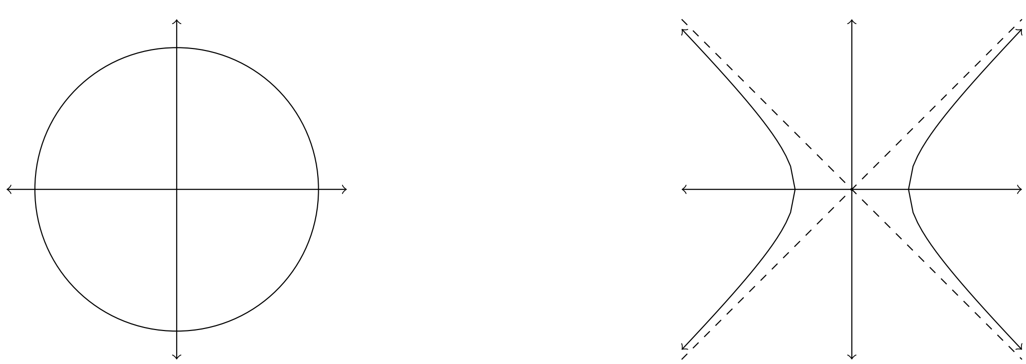
Here are the graphs of $y = \sinh x$, $y = \cosh x$, and $y = \tanh x$.



(a) $y = \sinh x$ (b) $y = \cosh x$ (c) $y = \tanh x$

Where are these asymptotes coming from?

Important Note 5. The ordinary trig functions parametrize the unit circle $x^2 + y^2 = 1$. The hyperbolic (trig) functions parametrize the unit hyperbola $x^2 - y^2 = 1$. (Now, you see the reason for the differences in minus signs.)



(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$

Since the hyperbolic functions are defined in terms of e^x and e^{-x} and these functions are differentiable, the 6 hyperbolic functions are also differentiable.

Theorem 6.

$$\frac{d}{dx}[\sinh x] = \cosh x \qquad \frac{d}{dx}[\text{csch } x] = -\text{csch } x \coth x$$

$$\frac{d}{dx}[\cosh x] = \sinh x \qquad \frac{d}{dx}[\text{sech } x] = -\text{sech } x \tanh x$$

$$\frac{d}{dx}[\tanh x] = \text{sech}^2 x \qquad \frac{d}{dx}[\coth x] = -\text{csch}^2 x$$

Example 7. Differentiate.

(a) $f(x) = \cosh(3x - 2)$

(b) $y = (\tanh x^2)^3$

Since we know the formulas for the derivatives of the hyperbolic functions, we also get the following integration formulas.

Theorem 8.

$$\int \sinh x \; dx = \qquad \int \cosh x \; dx =$$

$$\int \text{sech}^2 x \; dx = \qquad \int \text{sech } x \tanh x \; dx =$$

 Written by D.C. Ernst

Example 9. Integrate.

(a) $\int \sinh x \cosh x \; dx$

(b) $\displaystyle\int \frac{\cosh \sqrt{x}}{\sqrt{x}} \; dx$

Inverse hyperbolic functions

As with the ordinary trig functions, we can restrict the domain (if necessary) to define the inverse hyperbolic functions.

Definition 10.

$$\boxed{\begin{aligned}
&\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \text{ for } x \in \mathbb{R}\\
&\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \text{ for } x \geq 1\\
&\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \text{ for } -1 < x < 1
\end{aligned}}$$

Since each of the hyperbolic functions are differentiable, so are their (partial) inverses.

Theorem 11.

$$\frac{d}{dx}\left[\sinh^{-1}x\right] = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left[\cosh^{-1}x\right] = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\tanh^{-1}x\right] = \frac{1}{1-x^2}$$

How would we go about proving each of these formulas?

Example 12. Differentiate $f(x) = \ln\left(\tanh^{-1}x\right)$.

One of our main motivations for introducing hyperbolic functions was so that we could add a few more tools to our integration tool box.

Theorem 13.

$$\int \frac{1}{\sqrt{u^2+a^2}}\,du = \sinh^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{\sqrt{u^2-a^2}}\,du = \cosh^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{a^2-u^2}\,du = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C$$

Example 14. Integrate each of the following.

(a) $\displaystyle\int_0^1 \frac{x}{\sqrt{1+x^4}}\,dx$

(b) $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}-1}}\,dx$

(c) $\displaystyle\int \frac{1}{x(1-(\ln x)^2)}\,dx.$