Section 11.2: Series (part 2)

Goal

We will continue our introduction to series.

The harmonic series

The *harmonic series* is the series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Let's take a look at an applet at Calculus Applets (http://calculusapplets.com/) to see what this series looks like. What does it look like is happening with the series? It looks like this series is converging, but this series actually diverges.

Theorem 1. *The harmonic series diverges.*

We have the tools to prove this now, but it will be much easier to prove after we discuss the Integral Test in the next section.

A test for divergence

Important Note 2. A series may converge or diverge. However, if a series $\sum a_n$ is going to have any chance of converging, the terms of the sequence $\{a_n\}$ must get ____________ as n gets ____________.

In fact, we have the following theorem.

Theorem 3. *If the series $\sum a_n$ converges, then the sequence $\{a_n\}$ converges to ____________. That is, if $\sum a_n$ converges,*

$$\lim_{n\to\infty} a_n = \text{\_\_\_\_\_\_\_\_\_\_\_\_}.$$

It is important to note that the converse of this theorem is *not* true. That is, if a sequence $\{a_n\}$ converges to ____________, the corresponding series $\sum a_n$ does not necessarily converge.

For example, consider the harmonic series. The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to ____________, but the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ ____________.

However, the contrapositive of the previous theorem is true. (The contrapositive of a conditional statement always has the same truth value. The contrapositive is essentially the result of switching the implication and negating each half.) The contrapositive of Theorem 3 is the following statement and will be our first line of defense when determining whether a series converges or diverges.

Theorem 4 (Test for Divergence)**.** *If the sequence $\{a_n\}$ does not converge to 0, then the series $\sum a_n$ diverges.*

Example 5.

(a) Consider the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$. Since

$$\lim_{n\to\infty} \frac{n}{n+1} = \underline{\hspace{3cm}} \neq \underline{\hspace{3cm}},$$

the corresponding series must diverge (by the Divergence Test).

(b) Now, consider the series $\sum_{n=1}^{\infty} \frac{e^n}{n!}$. Since

$$\lim_{n\to\infty} \frac{e^n}{n!} = \underline{\hspace{3cm}},$$

the Divergence Test tells us nothing. It turns out that this series converges and we'll see why in a later section.

If you want to know whether a series converges or diverges, the first thing you should check is whether the corresponding sequence converges to 0. If it doesn't, then the series diverges and you are done. However, if the sequence does converge to 0, we'll need to do more work.

Some useful formulas

Occasionally, we'll need to make implicit use of the following formulas.

Theorem 6. *If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant) and $\sum(a_n \pm b_n)$, and*

(i) $\sum ca_n = c\sum a_n$

(ii) $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$