Chapter 6: Subgroups

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Summer 2009

Notes

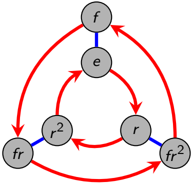
In this chapter we will introduce the concept of subgroup and begin exploring some of the rich mathematical territory that this concept opens up for us. A subgroup is some smaller group living inside a larger group.

Before we embark on this leg of our journey, we must return to a technical feature of Cayley diagrams that we temporarily ignored. This feature, called regularity, will help us visualize the new concepts that we will introduce.

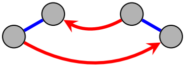
Let's begin with an example.

Notes

Consider the Cayley diagram for $S_3 = D_3$.



By following the corresponding paths, we see that $frf = r^{-1}$. Notice that this identity manifests itself throughout the diagram regardless of which node we start at. That is, the following fragment permeates throughout the diagram.



There are other patterns that permeate this diagram, as well. Do you see any? Here are a couple: $f^2 = e$, $r^3 = e$.

Notes

An algebraic equation, like $frf = r^{-1}$ in S_3, is true not just about one portion of a Cayley diagram, but it is true *across the diagram in the same way*. Cayley diagrams always have a uniform symmetry; every part of the diagram is structured like every other.

Definition 6.1

A diagram is called regular if it repeats every one of its interval patterns throughout the whole diagram, in the sense that we just discussed.

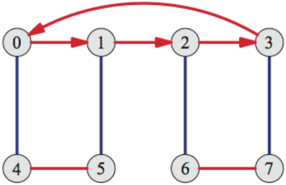
Every Cayley diagram is regular. In particular, diagrams lacking regularity do *not* represent groups (and so they are not called Cayley diagrams).

Notes

Recall that our original definition (Definition 1.9) of a group was called the "unofficial" definition of a group. One of these reasons that we called it unofficial is that technically regularity needs to be incorporated in the rules that form the definition.

We've been hinting at the regularity property of Cayley diagrams, but we haven't spelled out the details until now.

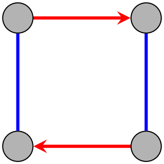
Is the following diagram a Cayley diagram for some group?



Nope. The diagram is not regular.

Notes

How about this one?



This one is tricky. The diagram looks pretty symmetrical, so you might think that it is regular, but it is not. Notice that two of the nodes have a red arrow going in and two of them have a red arrow going out.

What would go "wrong" if we tried to form a group from this diagram? If the red arrow represents action a, then a^2 is not represented in the diagram, which violates Rule 1.8.

Notes

Subgroups

Definition 6.2

When one group is completely contained in another, the smaller group is called a subgroup of the larger group. When H is a subgroup of G, we write $H < G$.

All of the orbits that we saw in Chapter 5 are subgroups.

For example, the orbit of r in S_3, $\{e, r, r^2\}$, is a cyclic subgroup of order 3 living inside S_3. We can write

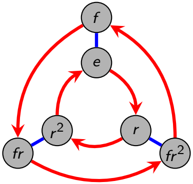
$$\langle r \rangle = \{e, r, r^2\} < S_3.$$

In fact, since $\langle r \rangle$ is really just a copy of C_3, we may be less formal and write

$$C_3 < S_3.$$

Notes

There are several other orbits in S_3 and all of them are cyclic subgroups. One of these orbits is staring at us in the Cayley diagram. Which one?



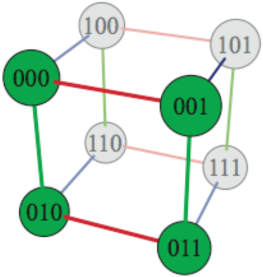
We see that

$$\langle f \rangle = \{e, f\} < S_3.$$

It turns out that all of the subgroups of S_3 are just cyclic orbits, but there are many groups that have subgroups that are not cyclic orbits.

Notes

Here is the Cayley diagram for the group $C_2 \times C_2 \times C_2$ with a copy of the subgroup V_4 highlighted (taken from Figure 6.3 on page 100 of *Visual Group Theory*.)



The group V_4 requires at least two generators and hence is not a cyclic subgroup of $C_2 \times C_2 \times C_2$. In this case, we can write

$$\langle 001, 010 \rangle = \{000, 001, 010, 011\} < C_2 \times C_2 \times C_2.$$

Notes

Every group has at least two subgroups:

1. the trivial subgroup: $\{e\}$
2. the non-proper subgroup: every group is a subgroup of itself

As we've seen, some subgroups are easy to pick out from a particular arrangement of a Cayley diagram. However, sometimes we may need to create an alternate Cayley diagram with different generators and/or different layouts for the nodes to make subgroups visually obvious.

Let's take a look at $C_6 = \{0, 1, 2, 3, 4, 5\}$ in *Group Explorer* and see if we can discover all of the subgroups by experimenting with different generators for Cayley diagrams and possibly different layouts.

Notes

What we should have discovered is that C_6 is equal to $\langle 1\rangle, \langle 5\rangle$, and $\langle 2, 3\rangle$. By looking at the corresponding Cayley diagrams, we found that the subgroups of C_6 are

$$\{e\}, \langle 2\rangle, \langle 3\rangle, C_6.$$

Now, let's use *Group Explorer* to search for the subgroups of D_4. There are 10 subgroups (some of which are isomorphic to each other):

$$\{e\}, \underbrace{\langle r^2\rangle, \langle f\rangle, \langle fr\rangle, \langle fr^2\rangle, \langle fr^3\rangle}_{\text{order 2}}, \underbrace{\langle r\rangle, \langle r^2, f\rangle, \langle r^2, fr\rangle}_{\text{order 4}}, D_4.$$

Notes

Here is a brute-force method for finding all of the subgroups of a given group G with order n:

1. we always have $\{e\}$ and G as subgroups
2. find all subgroups generated by a single element
3. find all subgroups generated by 2 elements

$$\vdots$$

n. find all subgroups generated by $n-1$ elements

Along the way, you are likely to duplicate subgroups. Also, this is horribly inefficient!

Note that this algorithm works because every group (and subgroup) has a set of generators.

Notes

Group work

Let's explore a few more examples.

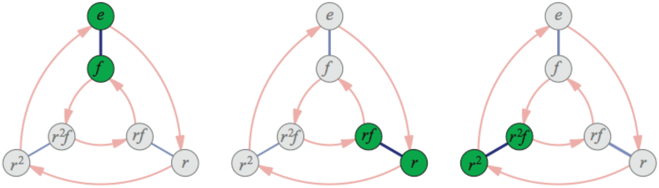
1. In groups of 2–3, complete the following exercises (not collected):
   - Exercise 6.1
   - Exercise 6.2
2. Let's discuss your solutions.
3. Now, complete Exercise 6.5(a) (ignore the part about index). I want each group to turn in a complete solution.

Notes

Cosets

The regularity property of Cayley diagrams implies that identical copies of the fragment of the diagram that corresponds to a subgroup appear throughout the rest of the diagram.

For example, the following figure (taken from Figure 6.6 on page 102 of *Visual Group Theory*) highlights the repeated copies of $\langle f \rangle = \{e, f\}$ in S_3.



However, only one of these copies is actually a group! Since the other two copies do *not* contain the identity, they cannot be groups.

Notes

The elements that form these repeated copies of the subgroup fragment in the Cayley diagram are called cosets.

To be sure that we understand this concept, let's find all of the cosets of the subgroup $\langle f, r^2 \rangle = \{e, f, r^2, fr^2\}$ of D_4. Using *Group Explorer* will help us pick the right Cayley diagram and layout, so that we can "see" the cosets.

We see that the cosets of $\langle f, r^2 \rangle$ are

$$\underbrace{\{e, f, r^2, fr^2\}}_{\text{original}}, \underbrace{\{r, r^3, fr, fr^3\}}_{\text{copy}}.$$

Notes

Now, we will list some observations concerning cosets. We will briefly justify each of these observations.

Obervation 6.3

Every subgroups has cosets, and they cover every node of the group's Cayley diagram.

This follows from the regularity of the Cayley diagram.

Obervation 6.4

Cosets can be described algebraically: we will use aH to denote the copy of H at a.

In this case, we have $aH = \{ah : a \in H\}$.

The meaning of aH: start from the node a and follow *all* paths in H.

Notes

For example, for the coset $\{r, fr^2\}$ of $\langle f \rangle$ in D_3 we can write

$$r\langle f \rangle = r\{e, f\} = \{r \cdot e, r \cdot f\} = \{r, fr^2\}.$$

Alternatively, we could have written $fr^2\langle f, r^2 \rangle$ to denote the same coset.

This leads us to the next 2 observations.

Obervation 6.5

Each coset can have more than one name.

Obervation 6.6

If $b \in aH$, then $aH = bH$.

The element that we choose to use to name the coset if called the representative. We refer to the cosets of the form aH as left cosets because we are multiplying the elements of H on the left.

Notes

There are also right cosets:

$$Ha = \{ha : h \in H\}.$$

For example, the right cosets of $\langle f \rangle$ in D_3 are

$$\langle f \rangle r = \{e, f\}r = \{e \cdot r, f \cdot r\} = \{r, fr\}$$

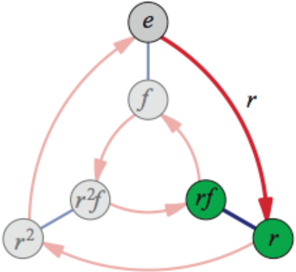
and

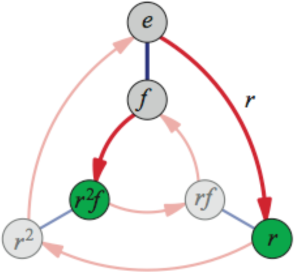
$$\langle f \rangle r^2 = \{e, f\}r^2 = \{e \cdot r^2, f \cdot r^2\} = \{r^2, fr^2\}.$$

It turns out that in this example, the left cosets for $\langle f \rangle$ were different than the right cosets. Thus, they must look different in the Cayley diagram.

Notes

The left diagram below shows the left coset $r\langle f \rangle$ in S_3, the nodes that f arrows can reach after the path to r has been followed. The right diagram shows the right coset $\langle f \rangle r$ in S_3, the nodes that r arrows can reach from teh elements in $\langle f \rangle$.





(Taken from Figure 6.7 on page 104 of *Visual Group Theory*.)

Notes

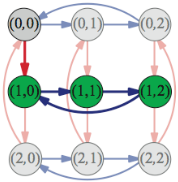
The reason that the left cosets look like copies of the subgroup while the elements of right cosets are usually scattered is that we adopted the convention that arrows represent right multiplication.

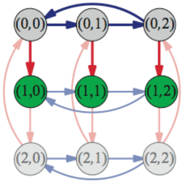
One of the most important things that we should take away from the last example is that left and right cosets are generally different.

But because they are not always different, it is worth seeing an example where they turn out to be the same.

Notes

Consider the subgroup $H = \langle (0,1) \rangle = \{(0,0),(0,1),(0,2)\}$ in the group $C_3 \times C_3$ and take $g = (1,0)$. The following figure (taken from Figure 6.9 on page 104 of *Visual Group Theory*) depicts the equality $gH = Hg$.





In this group, it turns out that $gH = Hg$ for all subgroups H and all elements g (because the group is abelian!) in $C_3 \times C_3$.

Subgroups that satisfy $gH = Hg$ for *all* elements g in the parent group are called normal.

Notes

More group work

Let's explore a few more examples.

1. In groups of 2–3 (try to mix the groups up again), complete the following exercises (not collected):
   - Exercise 6.20(a)(b) (ignore index)
   - Exercise 6.6
2. Let's discuss your solutions.

Notes

Lagrange's Theorem

In all the examples that we've seen, not only is every element in one of the cosets of a subgroup H, but each element appears in *exactly one* left or right coset. This is true in general. That is, the left (respectively, right) cosets of a subgroup H form a partition of the parent group.

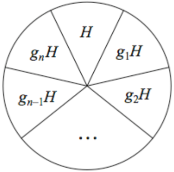
Theorem 6.7

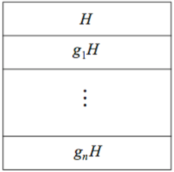
If $H < G$, then each element of G belongs to exactly one left coset of H.

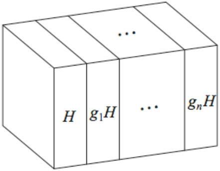
Proof. Suppose that there exist $g \in G$ such that $g \in aH$ and $g \in bH$. By Observation 6.5, $gH = aH$ and $gH = bH$. But then we must have $aH = bH$, which shows that our arbitrary g lies in a unique coset (with possibly many different names). $\square$

Notes

The upshot of Theorem 6.7 is that we can think of a group as being composed exclusively of non-overlapping and equal size copies of any subgroup, namely that subgroup's left cosets. Here are a few visualizations of this idea (taken from Figure 6.12 on page 106 of *VGT*).







Notes

We are now ready for one of our first major theorems, which is named after the Italian-born mathematician Joseph Louis Lagrange.

Lagrange's Theorem (Theorem 6.8)

Assume G is finite. If $H < G$, then the order $|H|$ of the subgroup divides the order $|G|$ of the larger group.

Proof. Suppose there are n left cosets of the subgroup H. Since all of the left cosets of H are the same size and these left cosets partition G, we must have

$$|G| = \underbrace{|H| + \cdots |H|}_{n \text{ copies}} = n|H|.$$

This shows that $|H|$ divides $|G|$. □

Definition 6.9

If $H < G$, then the index of H in G, written $[G : H]$, is how many times $|H|$ goes into $|G|$ (which is well-defined because of Lagrange's Theorem).

$$[G : H] = \frac{|G|}{|H|}$$

Note that the index of H in G is equal to the number of left (respectively, right) cosets of H.

One powerful consequence of Lagrange's Theorem is that it significantly narrows down the possibilities for subgroups. How so?

Warning: The converse of Lagrange's Theorem is not generally true. That is, just because the order of G has a divisor does not mean that there is a subgroup of that order.

Notes

Even more group work

Let's try this out.

In groups of 2–3 (try to mix the groups up again), complete Exercise 6.4. I want each group to turn in a complete solution.

Notes