Problem Collection for Introduction to Mathematical Reasoning

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Problem 1. Three strangers meet at a taxi stand and decide to share a cab to cut down the cost. Each has a different destination but all are heading in more-or-less the same direction. Bob is traveling 10 miles, Sally is traveling 20 miles, and Mike is traveling 30 miles. If the taxi costs $2 per mile, how much should each contribute to the total fare? What do you think is the most common answer to this question?

Problem 2. Multiply together the numbers of fingers on each hand of all the human beings in the world—approximately 7 billion in all. What is the approximate answer?

Problem 3. Imagine a hallway with 1000 doors numbered consecutively 1 through 1000. Suppose all of the doors are closed to start with. Then some dude with nothing better to do walks down the hallway and opens all of the doors. Because the dude is still bored, he decides to close every other door starting with door number 2. Then he walks down the hall and changes (i.e., if open, he closes it; if closed, he opens it) every third door starting with door 3. Then he walks down the hall and changes every fourth door starting with door 4. He continues this way, making a total of 1000 passes down the hallway, so that on the 1000th pass, he changes door 1000. At the end of this process, which doors are open and which doors are closed?

Problem 4. Suppose you have 6 toothpicks that are exactly the same length. Can you arrange the toothpicks so that 4 identical triangles are formed? Justify your answer.

Problem 5. I have 10 sticks in my bag. The length of each stick is an integer. No matter which 3 sticks I try to use, I cannot make a triangle out of those sticks. What is the minimum length of the longest stick?

Problem 6. Imagine you have 25 pebbles, each occupying one square on a 5 by 5 chess board. Tackle each of the following variations of a puzzle.

(a) Variation 1: Suppose that each pebble must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(b) Variation 2: Suppose that all but one pebble (your choice which one) must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(c) Variation 3: Consider Variation 1 again, but this time also allow diagonal moves to adjacent squares. If this is possible, describe a solution. If this is impossible, explain why.

Problem 7. Consider an $n \times n$ chess board and variation 1 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 8. Consider an $n \times n$ chess board and variation 2 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 9. An ant is crawling along the edges of a unit cube. What is the maximum distance it can cover starting from a corner so that it does not cover any edge twice?

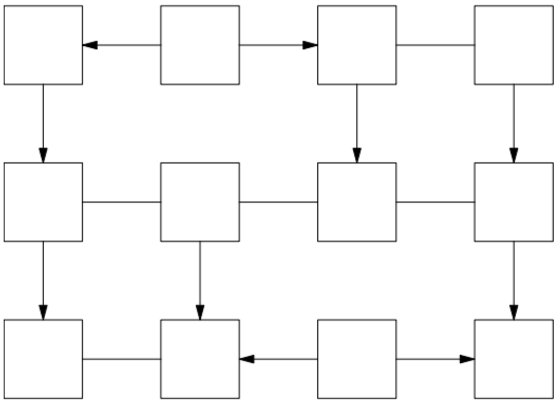
Problem 10. How many ways can 110 be written as the sum of 14 different positive integers? *Hint:* First, figure out what the largest possible integer could be in the sum. Note that the largest integer in the sum will be maximized when the other 13 numbers are as small as possible. Finish off the problem by doing an analysis of cases.

Problem 11. Four red ants and two black ants are walking along the edge of a one meter stick. The four red ants, called Albert, Bart, Debbie, and Edith, are all walking from left to right, and the two black ants, Cindy and Fred, are walking from right to left. The ants always walk at exactly one centimeter per second. Whenever they bump into another ant, they immediately turn around and walk in the other direction. And whenever they get to the end of a stick, they fall off. Albert starts at the left hand end of the stick, while Bart starts 20.2 cm from the left, Debbie is at 38.7cm, Edith is at 64.9cm and Fred is at 81.8cm. Cindy's position is not known—all we know is that he starts somewhere between Bart and Debbie. Which ant is the last to fall off the stick? And how long will it be before he or she does fall off?

Problem 12. The grid below has 12 boxes and 15 edges connecting boxes. In each box, place one of the six integers from 1 to 6 such that the following conditions hold:

- For each possible pair of distinct numbers from 1 to 6, there is exactly one edge connecting two boxes with that pair of numbers.
- If an edge has an arrow, then it points from a box with a smaller number to a box with a larger number.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above.



Problem 13. Take 15 poker chips or coins, divide into any number of piles with any number of chips in each pile. Arrange piles in adjacent columns. Take the top chip off every column and make a new column to the left. Repeat forever. What happens? Make conjectures about what happens when we change the number of chips.

Problem 14. The nth *triangular number* is defined via $t_n := 1+2+\cdots+n$. For example, $t_4 = 1+2+3+4 = 10$. Find a visual proof of the following fact. By "visual proof" we mean a sufficiently general picture that is convincing enough to justify the claim.

For all $n \in \mathbb{N}$, $t_n = \dfrac{n(n+1)}{2}$.

Problem 15. Let t_n denote the nth triangular number. Find both an algebraic proof and a visual proof of the following fact.

For all $n \in \mathbb{N}$, $t_n + t_{n+1} = (n+1)^2$.

Problem 16. Find a visual proof of the following fact. *Warning:* This problem is not about triangular numbers.

For $n \in \mathbb{N}$, $1+3+5+\cdots+(2n-1) = n^2$.

Problem 17. We have two strings of pyrotechnic fuse. The strings do not look homogeneous in thickness but both of them have a label saying 4 minutes. So we can assume that it takes 4 minutes to burn through either of these fuses. How can we measure a one minute interval?

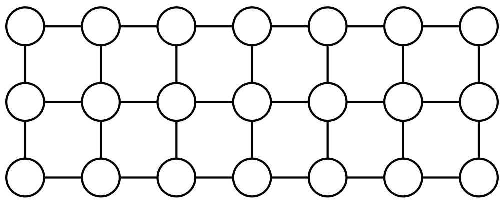
Problem 18. Suppose someone draws 20 distinct random lines in the plane. What is the maximum number of intersections of these lines?

Problem 19. A mouse eats her way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the 27 $1 \times 1 \times 1$ sub-cubes. If she starts at one corner and always moves to an uneaten sub cube, can she finish at the center of the cube?

Problem 20. An overfull prison has decided to terminate some prisoners. The jailer comes up with a game for selecting who gets terminated. Here is his scheme. 10 prisoners are to be lined up all facing the same direction. On the back of each prisoner's head, the jailer places either a black or a red dot. Each prisoner can only see the color of the dot for all of the prisoners in front of them and the prisoners do not know how many of each color there are. The jailer may use all black dots, or perhaps he uses 3 red and 7 black, but the prisoners do not know. The jailer tells the prisoners that if a prisoner can guess the color of the dot on the back of their head, they will live, but if they guess incorrectly, they will be terminated. The jailer will call on them in order starting at the back of the line. Before lining up the prisoners and placing the dots, the jailer allows the prisoners 5 minutes to come up with a plan that will maximize their survival. What plan can the prisoners devise that will maximize the number of prisoners that survive? Some more info: each prisoner can hear the answer of the prisoner behind them and they will know whether the prisoner behind them has lived or died. Also, each prisoner can only respond with the word "black" or "red." What if there are n prisoners?

Problem 21. Four prisoners are making plans to escape from jail. Their current plan requires them to cross a narrow bridge in the dark that has no handrail. In order to successfully cross the bridge, they must use a flashlight. However, they only have a single flashlight. To complicate matters, at most two people can be on the bridge at the same time. So, they will need to make multiple trips across the bridge, returning the flashlight back to the first side of the bridge by having someone walk it back. Unfortunately, they can't throw the flashlight. It takes 1, 2, 5, and 10 minutes for prisoner A, prisoner B, prisoner C, and prisoner D to cross the bridge and when two prisoners are walking together with the flashlight, it takes the time of the slower prisoner. What is the minimum total amount of time it takes all four prisoners to get across the bridge?

Problem 22. In the lattice below, we color 11 vertices points black. Prove that no matter which 11 are colored black, we always have a rectangle with black vertices (and vertical and horizontal sides).



Problem 23. Each point of the plane is colored red or blue. Show that there is a rectangle whose vertices are all the same color.

Problem 24. A certain fast-food chain sells a product called "nuggets" in boxes of 6, 9, and 20. A number n is called *nuggetable* if one can buy exactly n nuggets by buying some number of boxes. For example, 21 is nuggetable since you can buy two boxes of six and one box of nine to get 21. Here are the first few nuggetable numbers:

$$6, 9, 12, 15, 18, 20, 21, 24, 26, 27, \ldots$$

and here are the first few non-nuggetable numbers:

$$1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots$$

What is the largest non-nuggetable number?

Problem 25. Our space ship is at a Star Base with coordinates $(1, 2)$. Our hyper drive allows us to jump from coordinates (a, b) to either coordinates $(a, a + b)$ or to coordinates $(a + b, b)$. How can we reach the impending enemy attack at coordinates $(8, 13)$?

Problem 26. Consider our Star Base from the previous problem. Recall that our hyper drive allows us to jump from coordinates (a, b) to either coordinates $(a, a + b)$ or to coordinates $(a + b, b)$. If we start at $(1, 0)$, which points in the plane can we get to by using our hyper drive? Justify your answer.

Problem 27 (The Martian Artifacts)**.** Recent archaeological work on Mars discovered a site containing a pile of white spheres, each about the size of a tennis ball. A plaque near the mound states that each sphere contains a jewel that come in many different colors while strictly more than half of the spheres contain jewels of the same color. When two spheres are brought together, they both glow white if their internal jewels are the same color; otherwise, no glow. In how few tests can you find a sphere that you are certain holds a jewel of the majority color if the number of spheres in the pile is $2,3,4,5,6,7,8,9,10$, or 11? You should provide an answer with justification for each of the different values.

Problem 28. You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel. Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.
- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.

Your objective is to identify a single real coin that Zoltar has not taken as tribute.

Problem 29. The Sylver Coinage Game is a game in which 2 players alternately name positive integers that are not the sum of nonnegative multiples of previously named integers. The person who names 1 is the loser! Here is a sample game between A and B:

1. A opens with 5. Now neither player can name $5,10,15,\ldots$
2. B names 4. Now neither player can name 4, 5, 8, 9, 10, or any number greater than 11.
3. A names 11. Now the only remaining numbers are 1, 2, 3, 6, and 7.
4. B names 6. Now the only remaining numbers are 1, 2, 3, and 7.
5. A names 7. Now the only remaining numbers are 1, 2, and 3.
6. B names 2. Now the only remaining numbers are 1 and 3.
7. A names 3, leaving only 1.
8. B is forced to name 1 and loses.

If player A names 3, can you find a strategy that guarantees that the second player wins? If so, describe the strategy? If such a strategy is not possible, then explain why?

Problem 30. There is a plate of 40 cookies. You and your friend are going to take turns taking either 1 or 2 cookies from the plate. However, it is a faux pas to take the last cookie, so you want to make sure that you do not take the last cookie. How can you guarantee that you will never be the one taking the last cookie? What about n cookies?

Problem 31. Consider a 4×4 grid with light-up squares. In the starting configuration, some subset of the squares are lit up. At each stage, a square lights up if at least two of its immediate neighbors (horizontal or vertical) were "on" during the previous stage. It's easy to see that for the starting configuration in which four squares along a diagonal of the board are lit up, the entire board is eventually covered by "on" squares. Several other simple starting configurations with four "on" squares also result in the entire board being covered. Is it possible for a starting configuration with fewer than four squares to cover the entire board? If yes, find it; if no, give a proof.

Problem 32. Consider the scenario of the previous problem, except this time suppose we have an 8×8 grid. Is it possible for a starting configuration with fewer than eight squares to cover the entire board? If yes, find it; if no, give a proof. Can you generalize to the $n\times n$ case?

Problem 33. In the game Turnaround, you are given a permutation of the numbers from 1 to n. Your goal is to get them in the natural order $12\cdots n$. At each stage, your only option is to reverse the order of the first k places (you get to pick k at each stage). For instance, given 6375142, you could reverse the first four to get 5736142 and then reverse the first six to get 4163752. Solve the following sequence in as few moves as possible: 352614.

Problem 34. A signed permutation of the numbers 1 through n is a fixed arrangement of the numbers 1 through n, where each number can be either be positive or negative. For example, $(-2,1,-4,5,3)$ is a signed permutation of the numbers 1 through 5. In this case, think of positive numbers as being right-side-up and negative numbers as being upside-down. A *reversal* of a signed permutation is the act of performing a 180-degree rotation to some consecutive subsequence of the permutation. That is, a reversal swaps the order of a subsequence of numbers while changing the sign of each number in the subsequence. Performing a reversal to a signed permutation results in a new signed permutation. For example, if we perform a reversal on the second, third, and fourth entries in $(-2,1,-4,5,3)$, we obtain $(-2,-5,4,-1,3)$. The *reversal distance* of a signed permutation of 1 through n is the minimum number of reversals required to transform the given signed permutation into $(1,2,\ldots,n)$. It turns out that the reversal distance of $(3,1,6,5,-2,4)$ is 5. Find a sequence of 5 reversals that transforms $(3,1,6,5,-2,4)$ into $(1,2,3,4,5,6)$.

Problem 35. A soul swapping machine swaps the souls inside two bodies placed in the machine. Soon after the invention of the machine an unforeseen limitation is discovered: swapping only works on a pair of bodies once. Souls get more and more homesick as they spend time in another body and if a soul is not returned to its original body after a few days, it will kill its current host.

(a) Suppose Tom and Jerry swap souls and Garfield and Odie swap souls. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(b) Suppose Batman and Robin swap souls and then Robin's body and Flash utilize the machine. Argue that it is not possible to return the swapped souls to their original bodies using only Batman, Robin, and Flash.

(c) Consider the scenario of the previous problem. Suppose Wonder Woman and Superman are now available to sit in the machine after Batman, Robin, and Flash have already swapped souls. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(d) Now, suppose the soul swapping machine is used by the following pair of bodies (in the order listed): Adam and Alicia, Alicia and Gwen, Gwen and Blake. In addition, Pharrell and Miley are standing nearby. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(e) Suppose n different people have been involved in a finite sequence of soul swaps. Note that it's possible that an individual body may use the machine more than once during this soul swapping bonanza. Is it possible to return all swapped souls back to their original bodies? You may assume some innocent bystanders are nearby.

Problem 36. Two different positive numbers a and b each differ from their reciprocal by 1. What is $a+b$?

Problem 37. My Uncle Robert owns a stable with 25 race horses. He wants to know which three are the fastest. He owns a race track that can accommodate five horses at a time. What is the minimum number of races required to determine the fastest three horses?

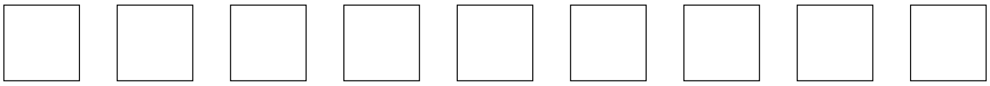
Problem 38. A father has 20 one dollar bills to distribute among his five sons. He declares that the oldest son will propose a scheme for dividing up the money and all five sons will vote on the plan. If a majority agree to the plan, then it will be implemented, otherwise dad will simply split the money evenly among his sons. Assume that all the sons act in a manner to maximize their monetary gain but will opt for evenly splitting the money, all else being equal. What proposal will the oldest son put forth, and why?

Problem 39. Imagine that in the scenario of the previous problem the father decides that after the oldest son's plan is unveiled, the second son will have the opportunity to propose a different division of funds. The sons will then vote on which plan they prefer. Assume that the sons still act to maximize their monetary gain, but will vote for the older son's plan if they stand to receive the same amount of money either way. What will transpire in this case, and why?

Problem 40. Let a,b,c,d,e,f,g,h be distinct elements in the set $\{-7,-5,-3,-2,2,4,6,13\}$. What is the minimum possible value of $(a+b+c+d)^2+(e+f+g+h)^2$?

Problem 41. There are 8 frogs and 9 rocks on a field. The 9 rocks are laid out in a horizontal line. The 8 frogs are evenly divided into 4 green frogs and 4 brown frogs. The green frogs sit on the first 4 rocks facing right and the brown frogs sit on the last 4 rocks facing left. The fifth rock is vacant for now. Switch the places of the green and brown frogs by using the following rules:

- A frog can only jump forward
- A frog can hop to an vacant rock one place ahead
- A frog can leap over its neighbor frog to a vacant rock two places ahead



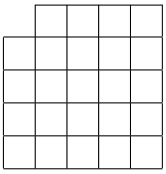
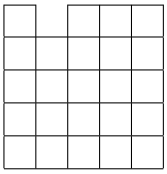
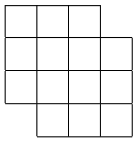
Can we generalize this problem and find how many jumps are necessary to switch n green and n brown frogs?

Problem 42. Consider a tournament with 15 teams. If every team plays every other team, how many games were played?

Problem 43 (Two Deep)**.** Consider the equation below. If a is a number, what number is it?

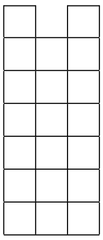
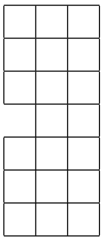
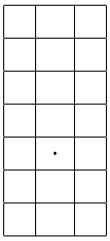
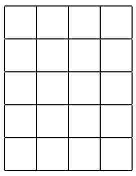
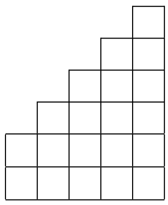
$$a = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cdots}}}}}$$

Problem 44. Tile the following grids with dominoes. If a tiling is not possible, explain way.

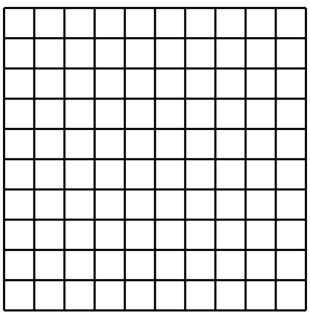
(a)  (b)  (c) 

Problem 45. Find all tetrominoes (polymonio with 4 cells).

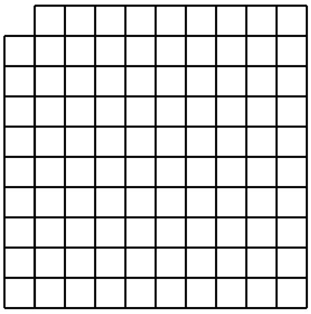
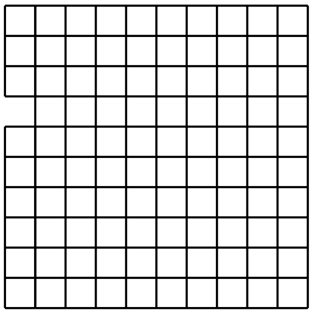
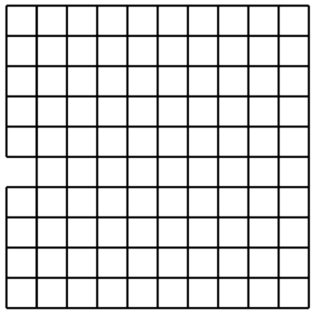
Problem 46. Tile the following grids using every tetromino exactly once. The $\cdot$ in (c) denotes an absence of an available square in the grid. If a tiling is not possible, explain way.

(a)  (b)  (c)  (d)  (e) 

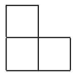
Problem 47. Consider the 10×10 grid of squares below. Show that you can color the squares of the grid with 3 colors so that every consecutive row of 3 squares and every consecutive column of 3 squares uses all 3 colors.



Problem 48. Tile each of the grids below with trominoes that consist of 3 squares in a line. If a tiling is not possible, explain way.

(a)  (b)  (c) 

Problem 49. What size rectangles can be tiled with the following tromino?



Problem 50. We have the following information about three integers:

(a) Their product is an integer;

(b) Their product is a prime;

(c) One of them is the average of the other two.

What are these numbers? *Hint*: You need to find all such triples and show that there are no others.

Problem 51. Suppose you have 12 coins, all identical in appearance and weight except for one that is either heavier or lighter than the other 11 coins. What is the minimum number of weighing one must do with a two-pan scale in order to identify the counterfeit?

Problem 52. Consider the situation in the previous problem, but suppose you have n coins. For which n is it possible to devise a procedure for identifying the counterfeit coin in only 3 weighings with a two-pan scale?

Problem 53. Let's revisit the counterfeit coin problem presented in Problems 51 and 52. In Problem 51, we discovered that we could detect the counterfeit coin in at most 3 weighings regardless of whether we knew in advance whether the counterfeit was heavier or lighter than the non-counterfeit coins. One feature of our algorithm was that after our 3 weighings, we could not only tell which coin was the counterfeit but also whether it was in fact heavier or lighter. It's certainly believable that 3 weighings is the best we can guarantee with 12 coins, but we did not prove this.

In Problem 52, we were asked to determine which number of coins we could start with and guarantee that we could identify which coin is counterfeit in at most 3 weighings. Certainly, $n = 12$ works. What about $n < 12$ and $n > 12$? It certainly seems believable that if we could handle 12 coins in 3 weighings, we could handle less. But is this true? It's not obvious at all what happens with more than 12 coins.

Let's do some exploring. Let n be the number of coins. Assume that exactly one coin is counterfeit so that the remaining $n-1$ coins are not counterfeit. Further suppose that we do not know whether the counterfeit coin is heavier or lighter than the others but we do know that the counterfeit coin is one of these. Let k denote the number of weighing used to detect the counterfeit coin. We will attempt to find a relationship between n and k. It is clear that if $n = 1$, we only have a counterfeit coin we are done without having to do any weighings. So, let's assume that $n \geq 2$. Even if you can't do one of the earlier parts, you should still try to use the results to do the later parts.

(a) Argue that n cannot be 2 so that $n \geq 3$.

(b) Suppose that on the first weighing, you take two piles of m coins where $2m < n$ and weigh them. There are two possibilities. Either the two sets of m coins balance on the scale or they don't. Let's first consider the case where the scales balance on the first weighing. In this case, the counterfeit must be one of the remaining $n-2m$ coins. We must be able to detect the counterfeit in the remaining $k-1$ weighings. Argue that

$$2(n-2m)-1 \leq 3^{k-1}. \tag{1}$$

(c) Next, let's assume that the scale was unbalanced on the first weighing when we weighed the two piles of m coins. Argue that

$$2 \cdot 2m \leq 2 \cdot 3^{k-1}. \tag{2}$$

Hint: The 2 on the righthand side comes from the fact that the heavy side of the scale may be on the left or the right.

(d) Starting with inequality (2), show that

$$2m \leq 3^{k-1} - 1. \tag{3}$$

Hint: First, simplify (2) in the obvious way and then observe that the righthand side is odd.

(e) Next, start with $2(n-2m) - 1 + 4m$ and then use (1) and (3) to show that

$$n \leq \frac{3^k - 1}{2}. \tag{4}$$

(f) Use inequality (4) to show that the number of coins must be less than or equal to 13 if we are only allowed 3 weighings?

(g) We've already seen that we could handle $n = 12$ coins in $k = 3$ weighings. However, just because we found out that the number of coins must be less than or equal to 13 if we are only allowed 3 weighings does not guarantee that we can actually pull this off. Can you adapt the strategy for 12 coins to handle 13 coins? If we can actually handle 13 coins in 3 weighings, we will show that our bound given in inequality (4) is optimal when $k = 3$. In this case, we say that the bound is "sharp."

(h) Use one of the facts above to prove that we cannot handle 12 coins in only 2 weighings.

(i) Sort out which numbers of coins we can handle when $k = 2$. Verify that your answer is correct.

Problem 54. Let P be a point inside the triangle ABC. Show that $PA + PB < CA + CB$.

Problem 55. Show that in any group of 6 students there are 3 students who know each other or 3 students who do not know each other.

Problem 56. Find all ordered pairs of real numbers (x, y) for which $\sqrt{x} + \sqrt{y} = 17$ and $x - y = 85$ without using the method of substitution.