<u>Thm 1:</u> If $x, y \in \mathbb{R}$, then $|x+y| \leq |x| + |y|$.

<u>Pf:</u> We will consider 4 cases.

Case 1: Assume $x \geq 0$ and $y \geq 0$. Then

$$|x+y| = x+y = |x| + |y|,$$

which certainly implies that

$$|x+y| \leq |x| + |y|.$$

Case 2: Assume $x < 0$ and $y < 0$. Then

$$|x+y| = -(x+y) = -x + -y = |x| + |y|,$$

which gives us

$$|x+y| \leq |x| + |y|.$$

Case 3: Assume $x \geq 0$ and $y < 0$. Here we consider two subcases.

Subcase (i): Assume $x+y \geq 0$. Then

$$|x+y| = x+y.$$

Since $y<0$, $|y| = -y$. Also, since $y<0$, we must have

$$x+y < x.$$

Furthermore, since $0 < -y$, we must have

$$x < x + -y.$$

Combining all of this together, we see that

$$\begin{aligned} |x+y| &= x+y \\ &< x \\ &< x+-y \\ &= |x|+|y|. \end{aligned}$$

So, as desired

$$|x+y| \leq |x|+|y|.$$

subcase (ii): Assume $x+y<0$. Then

$$|x+y| = -(x+y).$$

Since $y<0$, $|y|=-y$. Also, since $x \geq 0$, $-x<0$, which implies that

$$-x+-y < -y.$$

Furthurmore, since $|x| \geq 0$, we must have

$$-y < -y+|x|.$$

Combining all of this together, we see that

$$\begin{aligned} |x+y| &= -(x+y) \\ &= -x+-y \end{aligned}$$

$$< -y$$

$$< -y + |x|$$

$$= |y| + |x|.$$

So, as desired

$$|x+y| \leq |x| + |y|.$$

case 4: Lastly, assume that $x<0$ and $y \geq 0$. This case is identical to case 3, except the role's of x and y are switched. $\blacksquare$

Here's a much shorter, but probably more difficult to come up with proof.

Alternate pf: Let $x, y \in \mathbb{R}$. We see that

$$\begin{aligned} |x+y|^2 &= (x+y)^2 \\ &= x^2 + 2xy + y^2 \\ &\leq x^2 + 2|x||y| + y^2 \\ &= (|x| + |y|)^2. \end{aligned}$$

This implies that

$$|x+y| \leq |x| + |y|$$

since the square root fcn is increasing. $\blacksquare$

Thm 2: If $x, y, z \in \mathbb{Z}$ and $x+y$ and $y+z$ are both even, then $x+z$ is even.

Pf: Let $x, y, z \in \mathbb{Z}$ and assume that $x+y$ and $y+z$ are both even. Then $\exists\, k, m \in \mathbb{Z}$ s.t. $x+y = 2k$ and $y+z = 2m$. This implies that

$$x = 2k - y$$

and

$$z = 2m - y.$$

Then we see that

$$\begin{aligned} x+z &= (2k-y) + (2m-y) \\ &= 2k + 2m - 2y \\ &= 2(k+m-y). \end{aligned}$$

Note that since $k, m, y \in \mathbb{Z}$, $k+m-y \in \mathbb{Z}$, as well. therefore, $x+z$ is even. $\blacksquare$

Thm 3: If $a, b, c \in \mathbb{Z}$ and $a \mid b$ and $b \mid c$, then $a \mid c$.

Pf: Let $a, b, c \in \mathbb{Z}$ and assume that $a \mid b$ and $b \mid c$. Then $\exists\, k, m \in \mathbb{Z}$ s.t. $ak = b$ and $bm = c$. We see that

$$\begin{aligned} c &= bm \\ &= (ak)m \\ &= a(km). \end{aligned}$$

Note that since $k, m \in \mathbb{Z}$, $km \in \mathbb{Z}$, as well. Therefore, $a \mid c$. ∎

<u>Thm 4:</u> If x is an odd integer, then $8 \mid x^2 - 1$.

<u>Pf:</u> Assume that x is an odd integer. Then $\exists\ k \in \mathbb{Z}$ s.t. $x = 2k+1$. We see that

$$\begin{aligned} x^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 \\ &= 4k^2 + 4k \\ &= 4k(k+1). \end{aligned}$$

Now, observe that $k(k+1)$ must be the product of an odd integer times an even integer (not necessarily in that order). By a previous result, we then know that $k(k+1)$ is even. This implies that $\exists\ m \in \mathbb{Z}$ s.t. $k(k+1) = 2m$. Thus, we see that

$$\begin{aligned} x^2 - 1 &= 4k(k+1) \\ &= 4(2m) \\ &= 8m. \end{aligned}$$

Therefore, $8 \mid x^2 - 1$. $\blacksquare$

<u>Thm 5:</u> If x and y are positive real numbers, then

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

<u>Pf:</u> Assume x and y are positive real numbers. We see that

$$0 \leq (x-y)^2 = x^2 - 2xy + y^2.$$

But if we add $4xy$ to both sides of the inequality, we get

$$4xy \leq x^2 + 2xy + y^2,$$

which implies that

$$4xy \leq (x+y)^2$$

$$2\sqrt{xy} \leq x+y \qquad \text{(by taking square roots)}$$

$$\sqrt{xy} \leq \frac{x+y}{2},$$

as desired. $\blacksquare$