# Homework 2

## Abstract Algebra I

Complete the following problems.

**Problem 1.** Determine whether each of the following binary operations is (i) associative and (ii) commutative.

(a) The operation $\star$ on $\mathbb{R}$ defined via $a \star b = a + b + ab$.

(b) The operation $\circ$ on $\mathbb{Q}$ defined via $a \circ b = \dfrac{a+b}{5}$.

(c) The operation $\circledcirc$ on $\mathbb{Z} \times \mathbb{Z}$ defined via $(a,b) \circledcirc (c,d) = (ad + bc, bd)$.

(d) The operation $\circledast$ on $\mathbb{Q} \setminus \{0\}$ defined via $a \circledast b = \dfrac{a}{b}$.

(e) The operation $\ominus$ on $\mathbb{R}/I := \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ defined via $a \ominus b = a + b - \lfloor a + b \rfloor$ (i.e., $a \ominus b$ is the fractional part of $a + b$).

**Problem 2.** Determine which of the following sets are groups under the given operation. Justify your answer.

(a) $\mathbb{Z}/n\mathbb{Z}$ under multiplication mod $n$.

(b) Set of rational numbers in lowest terms whose denominators are odd under addition.

(c) Set of rational numbers in lowest terms whose denominators are even together with 0 under addition.

(d) Set of rational numbers of absolute value less than 1 under addition.

(e) $\mathbb{R}/I$ under $\ominus$ as defined in Problem 1(e).

**Problem 3.** Let $G = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Prove one of the following.

(a) The set $G$ is a group under addition.

(b) If $H = G \setminus \{0\}$, then $H$ is a group under multiplication.

**Problem 4.** Assume $G$ is a group and let $x \in G$. Prove one of the following.

(a) If $a, b \in \mathbb{Z}$, then $x^{a+b} = x^a x^b$.

(b) If $a, b \in \mathbb{Z}$, then $(x^a)^b = x^{ab}$.

Don't forget to handle the case when either $a$ or $b$ is nonpositive.

**Problem 5.** Assume $G$ is a group and let $a, b \in G$. Is it true that $(ab)^n = a^n b^n$? If not, under what minimal conditions would it be true? Prove the statement that you think is true.

**Problem 6.** Assume $G$ is a group. Prove that if $x^2 = e$ for all $x \in G$, then $G$ is abelian.

**Problem 7.** Assume $(G,\star)$ is a group and let $H$ be a nonempty subset of $G$ that is (i) closed under $\star$ and (ii) closed under inverses (i.e., for all $h,k \in H$, (i) $hk \in H$ and (ii) $h^{-1} \in H$). Prove that $H$ is a group under $\star$ in its own right. Such a subset is called a *subgroup*.

**Problem 8.** Assume $G$ is a group. Prove that if $x \in G$ such that $x^n \neq e$ for all $n \in \mathbb{Z}^+$, then $x^i \neq x^j$ for all $i \neq j$.

**Problem 9.** Assume $G = \{e,a,b,c\}$ is a group under $\star$ with the property that $x^2 = x^4$ for all $x \in G$ (where $e$ is the identity). Complete the following *group table*, where $x \star y$ is defined to be the entry in the row labeled by $x$ and the column labeled by $y$.

| $\star$ | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | | | |
| $b$ | $b$ | | | |
| $c$ | $c$ | | | |

Is your table unique? That is, did you have to fill it out the way you did? Deduce that $G$ is abelian.

**Problem 10. (Optional)** Assume $G$ is a finite group. Prove that every element of $G$ must appear exactly once in every row and column of the group table for $G$. (Of course, we are not counting the row and column headings.)