

Problem Collection for Introduction to Mathematical Reasoning

By Dana C. Ernst and Nándor Sieben
Northern Arizona University

Problem 1. Three strangers meet at a taxi stand and decide to share a cab to cut down the cost. Each has a different destination but all are heading in more-or-less the same direction. Bob is traveling 10 miles, Sally is traveling 20 miles, and Mike is traveling 30 miles. If the taxi costs \$2 per mile, how much should each contribute to the total fare? What do you think is the most common answer to this question?

Problem 2. Multiply together the numbers of fingers on each hand of all the human beings in the world—approximately 7 billion in all. What is the approximate answer?

Problem 3. Imagine a hallway with 1000 doors numbered consecutively 1 through 1000. Suppose all of the doors are closed to start with. Then some dude with nothing better to do walks down the hallway and opens all of the doors. Because the dude is still bored, he decides to close every other door starting with door number 2. Then he walks down the hall and changes (i.e., if open, he closes it; if closed, he opens it) every third door starting with door 3. Then he walks down the hall and changes every fourth door starting with door 4. He continues this way, making a total of 1000 passes down the hallway, so that on the 1000th pass, he changes door 1000. At the end of this process, which doors are open and which doors are closed?

Problem 4. Suppose you have 6 toothpicks that are exactly the same length. Can you arrange the toothpicks so that 4 identical triangles are formed? Justify your answer.

Problem 5. I have 10 sticks in my bag. The length of each stick is an integer. No matter which 3 sticks I try to use, I cannot make a triangle out of those sticks. What is the minimum length of the longest stick?

Problem 6. Imagine you have 25 pebbles, each occupying one square on a 5 by 5 chess board. Tackle each of the following variations of a puzzle.

- (a) Variation 1: Suppose that each pebble must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.
- (b) Variation 2: Suppose that all but one pebble (your choice which one) must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.
- (c) Variation 3: Consider Variation 1 again, but this time also allow diagonal moves to adjacent squares. If this is possible, describe a solution. If this is impossible, explain why.

Problem 7. Consider an $n \times n$ chess board and variation 1 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

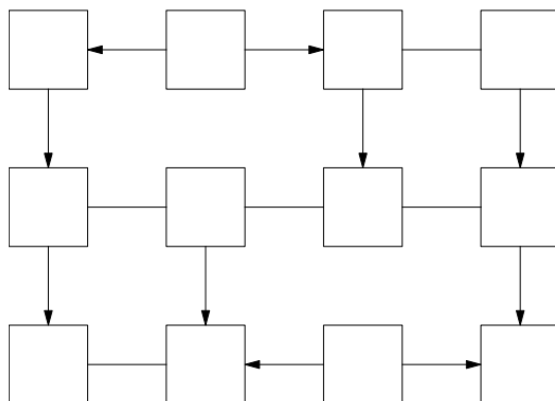
Problem 8. Consider an $n \times n$ chess board and variation 2 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 9. An ant is crawling along the edges of a unit cube. What is the maximum distance it can cover starting from a corner so that it does not cover any edge twice?

Problem 10. The grid below has 12 boxes and 15 edges connecting boxes. In each box, place one of the six integers from 1 to 6 such that the following conditions hold:

- For each possible pair of distinct numbers from 1 to 6, there is exactly one edge connecting two boxes with that pair of numbers.
- If an edge has an arrow, then it points from a box with a smaller number to a box with a larger number.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above.



Problem 11. In order to assess the reasoning skills of a newly developed android robot with artificial intelligence, the android's creator designs the following experiment. On Sunday, the creator describes the details of the experiment to the android and then turns the the android off. Once or twice, during the experiment, the android will be turned on, interviewed, and then turned back off. In addition, the creator will erase the awakening from the android's memory. On Sunday evening, a fair coin will be tossed to determine which experimental procedure to undertake:

- If the coin comes up heads, the android will be awakened and interviewed on Monday only.
- If the coin comes up tails, the android will be awakened and interviewed on both Monday and Tuesday.

In either case, the android will be awakened on Wednesday without interview and the experiment ends. Any time the android is awakened and interviewed, it will not be able to tell which day it is or whether it has been awakened before. During the interview the android is asked: "What is your credence now for the proposition that the coin landed heads?". One way to interpret "credence" in this context is the android's determination of the probability that the coin landed on heads.

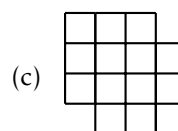
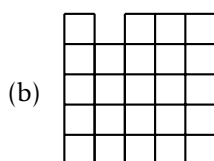
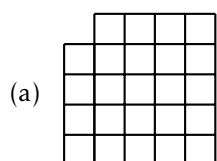
Problem 12. As a broke college student, you agree to take part in a recurring experiment. Each experiment begins on Sunday evening and ends on Wednesday morning. The experiment will be repeated 100 weeks in a row. You are told the details of the experiment in advance. Each Sunday evening, the experimenter describes the details of the experiment and then gives you a drug to put you to sleep. Once or twice, during the experiment, you will be awakened, interviewed, and then put back to sleep using a drug that includes an amnesia-inducing component that makes you forget the awakening. On Sunday evening, a fair coin will be tossed to determine which experimental procedure to undertake:

- If the coin comes up heads, you will be awakened and interviewed on Monday only.
- If the coin comes up tails, you will be awakened and interviewed on both Monday and Tuesday.

In either case, you will be awakened on Wednesday without interview and the experiment ends. Any time you are awakened and interviewed, you will not be able to tell which day it is or whether you have been awakened before. During the interview you will be asked: "Is the coin heads or tails?". You are required to respond with either "heads" or "tails". The experimenter will record whether you were correct or not, but you will not be told whether you guessed correctly. At the end of the 100th run of the experiment, you will be given \$10 for each correct answer that you gave. What strategy should you employ in order to optimize your profit?

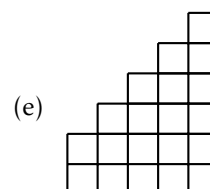
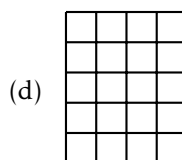
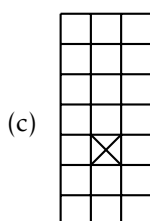
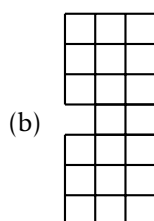
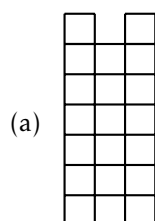
Problem 13. Four red ants and two black ants are walking along the edge of a one meter stick. The four red ants, called Albert, Bart, Debbie, and Edith, are all walking from left to right, and the two black ants, Cindy and Fred, are walking from right to left. The ants always walk at exactly one centimeter per second. Whenever they bump into another ant, they immediately turn around and walk in the other direction. And whenever they get to the end of a stick, they fall off. Albert starts at the left hand end of the stick, while Bart starts 20.2 cm from the left, Debbie is at 38.7cm, Edith is at 64.9cm and Fred is at 81.8cm. Cindy's position is not known—all we know is that he starts somewhere between Bart and Debbie. Which ant is the last to fall off the stick? And how long will it be before he or she does fall off?

Problem 14. Tile the following grids with dominoes. If a tiling is not possible, explain way.

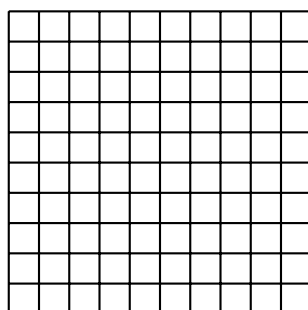


Problem 15. Find all tetrominoes (polyomino with 4 cells).

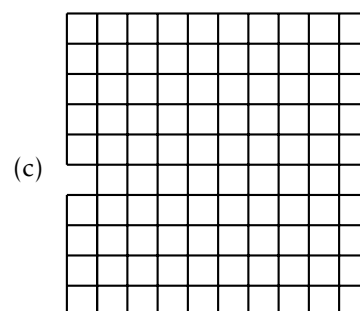
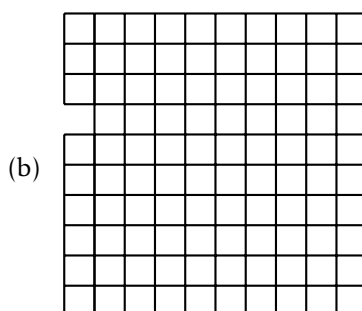
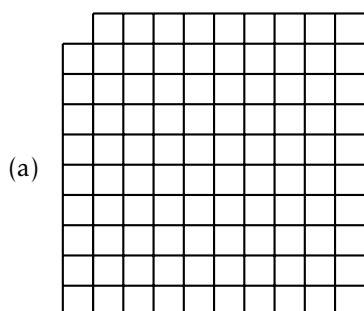
Problem 16. Tile the following grids using every tetromino exactly once. The X in (c) denotes an absence of an available square in the grid. If a tiling is not possible, explain way.



Problem 17. Consider the 10×10 grid of squares below. Show that you can color the squares of the grid with 3 colors so that every consecutive row of 3 squares and every consecutive column of 3 squares uses all 3 colors.



Problem 18. Tile each of the grids below with trominoes that consist of 3 squares in a line. If a tiling is not possible, explain way.



Problem 19. Take 15 poker chips or coins, divide into any number of piles with any number of chips in each pile. Arrange piles in adjacent columns. Take the top chip off every column and make a new column to the left. Repeat forever. What happens? Make conjectures about what happens when we change the number of chips.

Problem 20. The n th *triangular number* is defined via $t_n := 1+2+\cdots+n$. For example, $t_4 = 1+2+3+4 = 10$. Find a visual proof of the following fact. By “visual proof” we mean a sufficiently general picture that is convincing enough to justify the claim.

$$\text{For all } n \in \mathbb{N}, t_n = \frac{n(n+1)}{2}.$$

Problem 21. Let t_n denote the n th triangular number. Find both an algebraic proof and a visual proof of the following fact.

$$\text{For all } n \in \mathbb{N}, t_n + t_{n+1} = (n+1)^2.$$

Problem 22. Find a visual proof of the following fact. *Warning:* This problem is not about triangular numbers.

$$\text{For } n \in \mathbb{N}, 1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

Problem 23. Suppose you randomly cut a stick into 3 pieces. What is the probability that you can form a triangle out of these 3 pieces?