Chapter 4: Algebra at last

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Along the way, we will also introduce another powerful visualization technique, called multiplication tables.

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Let's revisit an example we have already seen to help illustrate the point.

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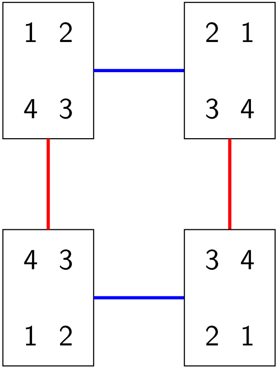

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Do you see any other paths that represent this same action?

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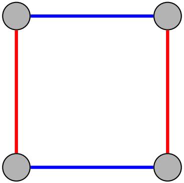

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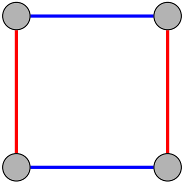

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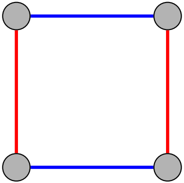

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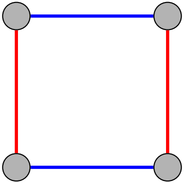

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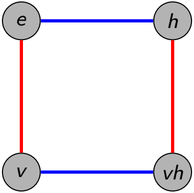

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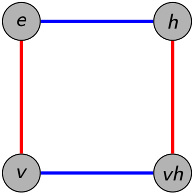

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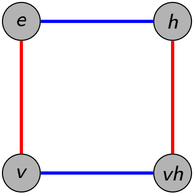

Note that we could also have labeled the node in the lower right hand corner as hv, as well. I'll emphasize this again later, but it is important to point out that this phenomenon (i.e., order of generators does not matter) does *not* always happen.

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What I mean by this is that if we want to know what a particular sequence (even really long ones!) is equal to, then we can just chase the sequence through the Cayley graph by starting at e.

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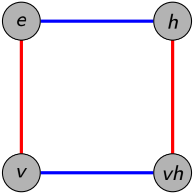

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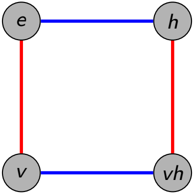

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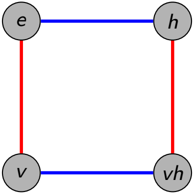

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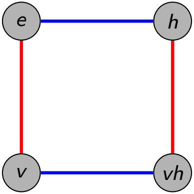

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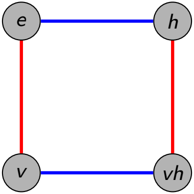

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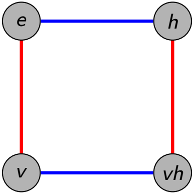

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Well, check it out! In this case, the answer is yes. Warning: not all groups have this property!!!

Group work

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1. In groups of 2–3 (try to mix the groups up again), complete the following exercises (not collected):

   - Construct the Cayley diagram with nodes labeled by actions for the group of symmetries of an equilateral triangle (assume one tip of triangle is pointing up) using:

     (i) horizontal flip (h) and 120° rotation clockwise (r) as generators.

     (ii) horizontal flip (h) and the diagonal flip that keeps the lower left corner fixed (d).

     Any observations?
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This is best illustrated by diving in and doing an example. Using our Cayley diagram from earlier, let's see if we can complete the following multiplication table for V_4 using our generators h and v.

$*$	e	v	h	vh
e				
v				
h				
vh				

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3. Now, complete Exercise 4.19(a)(b)(c). I want each group to turn in a complete solution.

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to mean "the element h is an element of the group V_4."

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The fancy way of saying this is that the set is closed under the binary operation.

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However, it is important to point out that not every table with symbols in it is going to be equal to the multiplication table for a group. Soon we will uncover a couple of features that distinguish those tables that depict groups from those that don't.

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The moral of the story is that we do not ever need to use parentheses when working with groups, but sometimes we may use them to draw our attention to a particular chunk in a sequence.

Some more group work

In groups of 2–3, complete the following exercises (not collected):

- Exercise 4.14
- Exercise 4.10(a) (see Bob)

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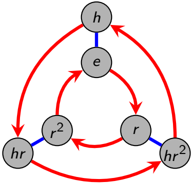

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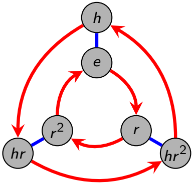

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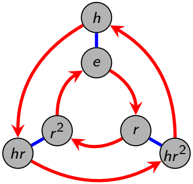

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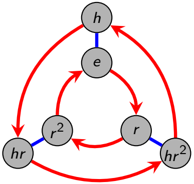

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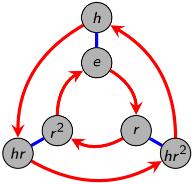

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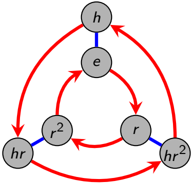

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2. Let's discuss your solutions.
3. Now, in groups of 2–3, complete Exercise 4.27(a)(b). I want each group to turn in a complete solution for both parts.

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4. Every element $g \in G$ has an inverse, g^{-1}, satisfying $g * g^{-1} = e = g^{-1} * g$.

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- group as a collection of actions
- group as a set with a binary operation

Even more group exercises

In groups of 2–3, complete Exercise 4.32. I want each group to turn in a complete solution.

Potential quiz questions

Here are some potential questions that I may ask you on tomorrow's quiz at the beginning of class:

1. What is a binary operation?
2. What is our second definition of a group?
3. Determine whether a given multiplication table represents a group.
4. State at least two properties that *all* groups share.
5. Find expression for the inverse of a group element.
6. Solve a specified group equation for a particular group.