MA 3110: Logic and Proof (Spring 2009)
Exam 2 (in-class portion)

NAME: Solutions

Instructions: Answer each of the following questions completely. Your answers should be *neat* and *organized.* If something is unclear, or if you have any questions, then please ask. Good luck!

This part of Exam 2 is worth 70 points. The take-home portion of Exam 2 is worth an additional 30 points and is available on the course web page. The take-home portion of the exam is due at the beginning of class on **Monday, April 6**.

1. (4 points each) Let

   $M =$ the set of mathematicians

   $G(x)$ be the open sentence "x is a geek."

   $S(x)$ be the open sentence "x is smart."

   P be the proposition "π is my favorite number."

   (a) Provide a symbolic translation of the following English sentence using *only* (but not necessarily all) the symbols $M, G(x), S(x), P, x, \sim, \wedge, \vee, \implies, \iff, \exists, \forall, \in, (,)$.

   "All mathematicians are geeks and not smart, or π is my favorite number."

   $(\forall x \in M)(G(x) \wedge \sim S(x)) \vee P$

   (b) It turns out that this sentence is false. Negate the symbolic proposition that you found in part (a). (Just sticking a $\sim$ in front is *not* sufficient.)

   $\sim [(\forall x \in M)(G(x) \wedge \sim S(x)) \vee P]$

   $\equiv (\exists x \in M)(\sim G(x) \vee S(x)) \wedge \sim P$

2. (4 points) Provide an example of 3 sets A, B, and C such $A \cap B \cap C = \emptyset$ yet each pair is *not* disjoint. (You do *not* need to justify your answer.)

   Many possible answers. Here's one:

   $A = \{1, 2\}$

   $B = \{2, 3\}$

   $C = \{1, 3\}$

3. (4 points) Provide a counterexample to show that the following proposition is false. (You should show sufficient work to justify that your example does in fact show that the proposition is false.)

For $a, b, c \in \mathbb{Z}$, if a divides bc, then either a divides b or a divides c.

Many possible ex's. Here's one:
Let $a = 4$, $b = 2$, and $c = 6$. Then $4 \mid 2 \cdot 6 = 12$,
but $4 \nmid 2$ and $4 \nmid 6$.

4. (4 points) Provide an example of a universe U and an open sentence $P(x, y)$ such $(\forall x)(\exists y)P(x, y)$ is true, but $(\exists y)(\forall x)P(x, y)$ is false. (You do *not* need to justify your answer.)

U = people

$P(x,y)$: x is married to y

5. (3 points each) For each of the statements (a)–(d) on the left, find an equivalent symbolic proposition chosen from the list (i)–(v) on the right. Note that not every statement on the right will get used. (You do *not* need to justify your answer.)

(a) $A \not\subseteq B$ (v)

(b) $A \cap B = \emptyset$ (ii)

(c) $(A \cup B)^c \neq \emptyset$ (iii)

(d) $(A \cap B)^c = \emptyset$ (i)

(i) $(\forall x)(x \in A \wedge x \in B)$

(ii) $(\forall x)(x \in A \implies x \notin B)$

(iii) $(\exists x)(x \notin A \wedge x \notin B)$

(iv) $(\exists x)(x \in A \vee x \in B)$

(v) $(\exists x)(x \in A \wedge x \notin B)$

6. (3 points each) Let $U = \{a, b, \{b\}, c, \{a, c\}\}$ be the universe for the sets $A = \{a, b, c\}$ and $B = \{a, \{b\}\}$. Find each of the following. (Be careful: make sure you have the right number of $\{\}$ around things.)

(a) $A \cup B$

$= \boxed{\{a, b, c, \{b\}\}}$

(b) $A^c \cap B^c$

$= (A \cup B)^c = \boxed{\{\{a, c\}\}}$

(c) $A - B$

$= \boxed{\{b, c\}}$

(d) $\mathcal{P}(B)$

$= \boxed{\{\phi, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}}$

(e) $B \cap \mathcal{P}(A)$

$= \boxed{\{\{b\}\}}$

7. (3 points each) Let U be the universe consisting of all polygons with exactly 4 sides. Consider the following sets in the universe U.

$$R = \{p \in U : p \text{ is a rectangle}\}$$
$$S = \{p \in U : p \text{ is a square}\}$$
$$T = \{p \in U : p \text{ is a trapezoid}\}$$

Determine whether each of the following statements are TRUE or FALSE. Circle the correct answer. You do *not* need to justify your answer.

(a) $R \cap S^c = \emptyset$

TRUE $\boxed{\text{FALSE}}$

* (b) $R - T \neq \emptyset$

TRUE FALSE

* Didn't grade these b/c of confusion about def of trapezoid.

* (c) $T^c \subseteq R^c$

TRUE FALSE

(d) $\mathcal{P}(S) \subseteq \mathcal{P}(R)$

$\boxed{\text{TRUE}}$ FALSE

* (e) $R \cap T^c \subseteq S$

TRUE FALSE

8. (4 points each) Let f be a function and let a and L be real numbers. Consider the following definition.

   We say that the limit of $f(x)$ as x approaches a is L iff for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. *

   (a) Suppose that for some function f, the limit of $f(x)$ as x approaches a is L. Describe a general strategy of how you would go about proving this. (You do not need to actually prove anything.)

Pf: Let $\varepsilon > 0$. Choose $\delta =$ ___. Assume $0 < |x - a| < \delta$.

$\vdots$ Do stuff

Thus, $|f(x) - L| < \varepsilon$. ☒

   (b) Suppose that for some function f, the limit of $f(x)$ as x approaches a is *not* L. Describe how you would go about proving this. (Hint: think about negating the definition above and again provide a general strategy.)

Pf: Choose $\varepsilon =$ ___. Let $\delta > 0$. Assume $0 < |x - a| < \delta$.

$\vdots$ Do stuff

Thus, $|f(x) - L| \geq \varepsilon$. ☒

*You don't actually need to understand all the words in this definition to answer the questions that follow.