Chapter 6

Relations

6.1 Relations

Definition 6.1. An **ordered pair** is an object of the form (x, y). Two ordered pairs (x, y) and (a, b) are **equal** iff $x = a$ and $y = b$.

Definition 6.2. An n-**tuple** is an object of the form $(x_1, x_2, \ldots, x_n)$. Each x_i is referred to as the ith **component**.

Note that an ordered pair is just a 2-tuple.

Definition 6.3. If X and Y are sets, the **Cartesian product** of X and Y is defined by

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

That is, $X \times Y$ is the set of all ordered pairs where the first element is from X and the second element is from Y. The set $X \times X$ is sometimes denoted by X^2. We similarly define the Cartesian product of n sets, say $X_1, \ldots, X_n$, by

$$\prod_{i=1}^{n} X_i = X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) : \text{each } x_i \in X_i\}.$$

Example 6.4. Let $A = \{a, b, c\}$ and $B = \{☺, ☹\}$. Then

$$A \times B = \{(a, ☺), (a, ☹), (b, ☺), (b, ☹), (c, ☺), (c, ☹)\}.$$

Exercise 6.5. Using the sets A and B from the previous example, find $B \times A$.

Exercise 6.6. Using the set B from the previous examples, find $B \times B$.

Exercise 6.7. What general conclusion can you make about $X \times Y$ versus $Y \times X$? When will they be equal?

Exercise 6.8. If X and Y are both finite sets, then how many elements will $X \times Y$ have? Be as specific as possible.

Exercise 6.9. Let $A = \{1,2,3\}$, $B = \{1,2\}$, and $C = \{1,3\}$. List the elements of the set $A \times B \times C$.

Exercise 6.10. Let $A = \mathbb{N}$ and $B = \mathbb{R}$. Describe the elements of the set $A \times B$.

Exercise 6.11. Let A be the set of all differentiable functions on the open interval $(0,1)$, and let B equal the set of all derivatives of functions in A evaluated at $x = \frac{1}{2}$. Describe the elements of the set $A \times B$.

Exercise 6.12. Three space, $\mathbb{R}^3$, is a Cartesian product. Unpack the meaning of $\mathbb{R}^3$ using the Cartesian product, and write the complete set notation version.

Exercise 6.13. Let $X = [0,1]$ and let $Y = \{1\}$. Describe geometrically what $X \times Y$, $Y \times X$, $X \times X$, and $Y \times Y$ look like.

Definition 6.14. Let X and Y be sets. A **relation** from a set X to a set Y is a subset of $X \times Y$. A relation on X is a subset of $X \times X$.

Example 6.15. You may not realize it, but you are familiar with many relations. For example, on the real numbers, we have the relation $\leq$. We could say that $(3,\pi)$ is in the relation $\leq$ since $3 \leq \pi$. However, $(1,-1)$ is not in the relation since $1 \not\leq -1$. (Order matters!)

Remark 6.16. Different notations for relations are used in different contexts. When talking about relations in the abstract, we indicate that a pair (a,b) is in the relation by some notation like $a \sim b$, which is read "a is related to b."

Example 6.17. Let P_f denote the set of all people with accounts on Facebook. Define F via xFy iff x is friends with y. Then F is a relation on P_f.

We can often represent relations using graphs or digraphs. Given a finite set X and a relation $\sim$ on X, a **digraph** (short for *directed graph*) is a discrete graph having the members of X as vertices and a directed edge from x to y iff $x \sim y$.

Example 6.18. Figure 6.1 depicts a digraph that represents a relation R given by

$$R = \{(a,b),(a,c),(b,b),(b,c),(c,d),(c,e),(d,d),(d,a),(e,a)\}.$$

Exercise 6.19. Let $A = \{a,b,c\}$ and define $\sim = \{(a,a),(a,b),(b,c),(c,b),(c,a)\}$. Draw the digraph for $\sim$.

Exercise 6.20. Let $A = \{1,2,3,4,5,6\}$. Define $|$ on A via $x|y$ iff x divides y. Draw the digraph for $|$ on A.

When X or Y is infinite, it is not practical to draw a digraph. However, you are familiar with the graphs of some relations involving infinite sets.

Example 6.21. When we write $x^2 + y^2 = 1$, we are implicitly defining a relation. In particular, the relation is the set of ordered pairs (x,y) satisfying $x^2 + y^2 = 1$. In set notation:

$$\{(x,y) : x^2 + y^2 = 1\}$$

The graph of this relation in $\mathbb{R}^2$ is the standard unit circle.

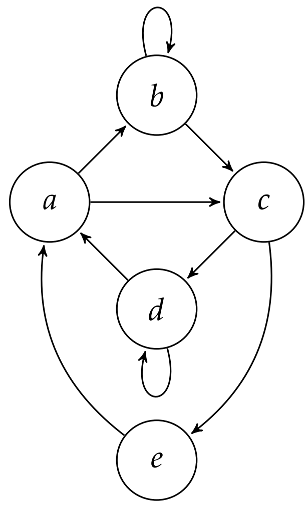

Figure 6.1: An example of a digraph for a relation.

Exercise 6.22. Define $\sim$ on $\mathbb{R}$ via $x \sim y$ iff $x \leq y$. Draw a picture of this relation in $\mathbb{R}^2$. In other words, draw all points (x, y) where $x \sim y$.

Definition 6.23. Let $\sim$ be a relation on a set A.

(a) $\sim$ is **reflexive** if for all $x \in A$, $x \sim x$ (every element is related to itself).

(b) $\sim$ is **symmetric** if for all $x, y \in A$, if $x \sim y$, then $y \sim x$.

(c) $\sim$ is **transitive** if for all $x, y, z \in A$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

Example 6.24.

(a) $\leq$ on $\mathbb{R}$ is reflexive and transitive, but not symmetric. $<$ on $\mathbb{R}$ is transitive, but not symmetric and not reflexive.

(b) If S is a set, then $\subseteq$ on $\mathcal{P}(S)$ is reflexive and transitive, but not symmetric.

(c) $=$ on $\mathbb{R}$ is reflexive, symmetric, and transitive.

Exercise 6.25. Given a finite set A and a relation $\sim$, describe what each of reflexive, symmetric, and transitive look like in terms of a digraph.

Exercise 6.26. Let P be the set of people at a party and define N via $(x, y) \in N$ iff x knows the name of y. Describe what it would mean for N to be reflexive, symmetric, and transitive.

Exercise 6.27. Determine whether each of the following relations is reflexive, symmetric, or transitive.

(a) Let P_f denote the set of all people with accounts on Facebook. Define F via xFy iff x is friends with y.

(b) Let P be the set of all people and define H via xHy iff x and y have the same height.

(c) Let P be the set of all people and define T via xTy iff x is taller than y.

(d) Consider the relation "divides" on $\mathbb{N}$.

(e) Let L be the set of lines and define $\|$ via $l_1 \| l_2$ iff l_1 is parallel to l_2.

(f) Let $C[0,1]$ be the set of continuous functions on $[0,1]$. Define $f \sim g$ iff

$$\int_0^1 |f(x)|\,dx = \int_0^1 |g(x)|\,dx.$$

(g) Define $\sim$ on $\mathbb{N}$ via $n \sim m$ iff $n+m$ is even.

(h) Define D on $\mathbb{R}$ via $(x,y) \in D$ iff $x = 2y$.

6.2 Equivalence Relations

Let $\sim$ be a relation on a set A. Recall the following definitions:

(a) $\sim$ is **reflexive** if for all $x \in A$, $x \sim x$ (every element is related to itself).

(b) $\sim$ is **symmetric** if for all $x, y \in A$, if $x \sim y$, then $y \sim x$.

(c) $\sim$ is **transitive** if for all $x, y, z \in A$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

As we've seen in the previous section of notes, these conditions are independent. That is, a relation may have some combination of these properties, but not necessarily all of them. However, we have a special name for when a relation does satisfy all three.

Definition 6.28. Let $\sim$ be a relation on a set A. Then $\sim$ is called an **equivalence relation** iff $\sim$ is reflexive, symmetric, and transitive.

Exercise 6.29. Given a finite set A and a relation $\sim$ on A, describe what the corresponding digraph would have to look like in order for $\sim$ to be an equivalence relation.

Exercise 6.30. Let $A = \{a,b,c,d,e\}$. Make up an equivalence relation on A by drawing a digraph such that a is not related to b and c is not related to b.

Exercise 6.31. Let $S = \{1,2,3,4,5,6\}$ and define

$$\sim = \{(1,1),(1,6),(2,2),(2,3),(2,4),(3,3),(3,2),(3,4),(4,4),(4,2),(4,3),(5,5),(6,6),(6,1)\}.$$

Justify that this is an equivalence relation.

Problem 6.32. Determine which of the following are equivalence relations. Some of these occurred in the last section of notes and you are welcome to use your answers from those problems.