Calc Final Fall 2005

① a) $\int (x^2 + \frac{2}{x^{5/2}})dx = \int x^2 dx + 2\int x^{-5/2} dx$

$= \frac{1}{3}x^3 + 2(\frac{-2}{3}x^{-3/2}) + C$

$= \boxed{\frac{1}{3}x^3 - \frac{4}{3}x^{-3/2} + C}$

b) $\int (3e^{5x} + 7)dx = 3\int e^{5x} dx + \int 7dx$

$u = 5x$

$\frac{du}{dx} = 5$

$= 3\int e^u \frac{du}{5} + \int 7dx$

$= \boxed{\frac{3}{5}e^{5x} + 7x + C}$

c) $\int_0^{\pi/2} e^{1-\sin x} \cos x \, dx$ $\quad u = 1 - \sin x \quad \frac{du}{dx} = -\cos x$

$\int -e^u du = -e^{1-\sin x}\Big|_0^{\pi/2} = -e^{1-0} - (-e^{1-1}) = \boxed{1-e}$

d) $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$ $\quad u = \sin x \quad du/dx = \cos x$

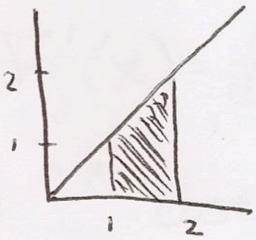
$= \int \frac{1}{u} du = \boxed{\ln|\sin x| + C}$

e) $\int_e^{e^3} \frac{1}{x(\ln x)^2}$ $\quad u = \ln x \quad \frac{du}{dx} = \frac{1}{x}$

$= \int \frac{1}{u^2} du = \frac{1}{-\ln x}\Big|_e^{e^3} = \frac{-1}{3} - (-1) = \boxed{1 - 1/3}$

② Area under $y=x$ on $[1,2]$

a)



$$\text{Area} = \left(\frac{b_1+b_2}{2}\right)h$$

$$A = \left(\frac{1+2}{2}\right)(1) = \boxed{\frac{3}{2}}$$

b) $A = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_k^*)\Delta x$

$$\Delta x = \frac{2-1}{n} = \frac{1}{n}$$

$$x_k^* = 1 + \frac{k}{n}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{k}{n}\right)\frac{1}{n} = \lim_{n\to\infty}\left(\frac{1}{n}\sum 1 + \frac{1}{n^2}\sum k\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\cdot n + \frac{1}{n^2}\,\frac{n(n+1)}{2}\right)$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{2n}\right) = 1 + 1/2 + 0 = \boxed{3/2}$$

c) $\int_1^2 x\,dx = \frac{1}{2}x^2\Big|_1^2 = \frac{1}{2}(2^2) - 1/2(1) = \boxed{3/2}$

③ $\int_2^4 (5f(x) - \frac{1}{3}g(x))dx$ given $\int_2^4 f(x)dx = \frac{1}{4}$ $\int_2^4 g(x)dx = 5$

$$\shortparallel$$

$$5\int_2^4 f(x)dx - \frac{1}{3}\int_2^4 g(x)dx = 5\left(\frac{1}{4}\right) - \frac{1}{3}(5)$$

$$= \frac{5}{4} - \frac{5}{3} = \frac{15}{12} - \frac{20}{12} = \boxed{\frac{-5}{12}}$$

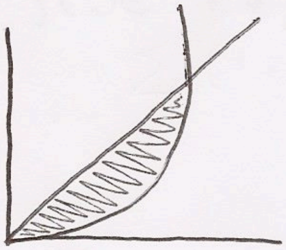
④ $F(x)=\int_{\pi/2}^{x} \cos t\,dt$

a) $F'(x)=\frac{d}{dx}\left[\int_{\pi/2}^{x} \cos t\,dt\right]=\cos x$

b) $\int_{\pi/2}^{x} \cos t\,dt = \sin t\Big|_{\pi/2}^{x} = \sin x - \sin \pi/2 = \sin x$

$\frac{d}{dx}[\sin x] = \cos x$ ✓

⑤ Area enclosed by $y=x^2$ $y=3x$



$x^2=3x$

$x^2-3x=0$

$x=0$ $x=3$

$\int_0^3 (3x-x^2)\,dx$

$=\frac{3}{2}x^2-\frac{1}{3}x^3\Big|_0^3$

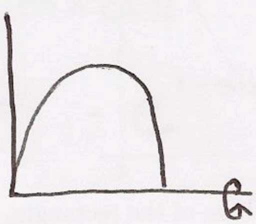
$=\frac{3}{2}(3^2)-\frac{1}{3}(3^3)-(0)$

$=\frac{27}{2}-9=\boxed{\frac{9}{2}}$

⑥ Volume of solid

region enclosed by $y=0$, $y=\sin x$, $x=0$, $x=\pi$
revolved about

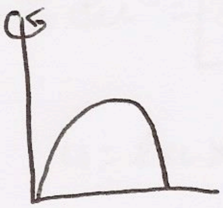
a) x-axis



washers

$\pi\int_0^{\pi} \sin^2 x\,dx$

b) y-axis



shell

$2\pi\int x\sin x\,dx$

(7) horizontal asymptote ~~when 2x-5~~

$$\lim_{x \to \pm\infty} \frac{10x}{2x} = 5 \qquad \text{(A)}$$

(8) $\lim_{x \to 6} 7 = 7$ (C)

(9) $\lim_{x \to 3^-} \frac{x}{x-3} = \frac{3}{0^-} = -\infty$ (D)

$\lim_{x \to 3^+} \frac{x}{x-3} = \frac{3}{0^+} = \infty$

(10) $f(1) = 1+1-3 = -1$ True by the Intermediate Value Thm

$f(2) = 8+2-3 = 7$

(11) $\lim_{x \to 0} \frac{1-\cos x}{\sin x} \stackrel{?}{=} 0$ True

$\frac{0}{0}$ L'Hopital $\lim_{x \to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$

(12) $y = 2x$ slope of tangent $y = 2$ $y - 6 = 2(x-3)$

point on graph $x = 3$ (A)

$y = 2(3) = 6$ $y = 2x$

(13) $\frac{dy}{dx}[e^8] = 0$ e^8 is a constant (E)

(14) $g(x) = \sqrt{x}\, f(x)$

$g'(x) = \frac{1}{2\sqrt{x}} f(x) + \sqrt{x} f'(x)$

$g'(1) = \frac{1}{2\sqrt{1}} f(1) + \sqrt{1} f'(1)$ (C) 9

$= \frac{1}{2}(8) + 5$

(15) $f(x)=x^3\cos x$ product rule (D)

$3x^2\cos x - x^3\sin x$

(16) $\dfrac{d^{71}}{dx^{71}}(\sin x) \stackrel{?}{=} \cos x$

$\boxed{\text{False}}$

$f' = \cos x$

$f'' = -\sin x$

$f''' = -\cos x$

$f'''' = \sin x$

so $\dfrac{d^n}{dx^n}(\sin x) = \cos x$ iff n can be written as $4k+1$, $k=0,1,2,\ldots$

$71 = 4(17)+3$

$f' = f^{(5)} = f^{(9)} = f^{(13)} = \ldots = \cos x$

(17) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(2^2)\left(\frac{1}{2}\right) = 8\pi$$

(C)

(18) $y = \ln(4x^2)$ chain rule

$$\frac{dy}{dx} = 8x\left(\frac{1}{4x^2}\right) = \frac{2}{x}$$

(E)

(19)

$$\frac{d}{dx}[x^3 + 3y^2] = \frac{d}{dx}[9]$$

$$3x^2 + 6y\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \frac{-3x^2}{-6y} = \frac{x^2}{2y}$$

(C)

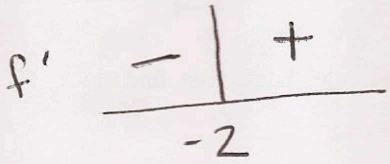
(20) $\displaystyle\lim_{x\to+\infty} \frac{\ln x}{e^x} = \frac{\infty}{\infty}$ L'Hopital $\displaystyle\lim_{x\to\infty} \frac{1/x}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = \frac{1}{\infty} = 0$

(A)

(21) $f'(x) = 2x+4$ (C)

$f'(x) = 0$ when $2x+4=0$ $[-2, \infty)$

$x = -2$



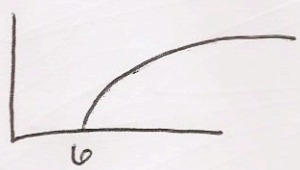
$f' > 0 \Rightarrow$

f increasing

(22) $f(x) = \sqrt{x-6}$ or look at graph (True)

$f'(x) = 1/2\sqrt{x-6}$

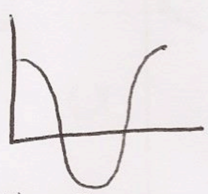
$f''(x) = \dfrac{-1}{4(x-6)^{3/2}}$



$f''(x) < 0$ $\forall$ x in domain

(23) $f(x) = \cos x$ or look at graph

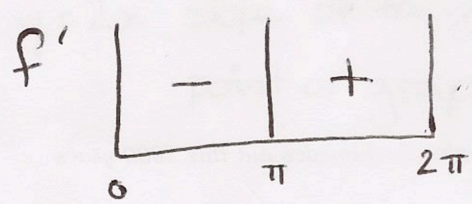
$f'(x) = -\sin x$



(B)

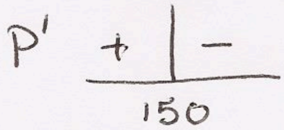
$f'(x) = 0$ $-\sin x = 0$

$x = 0,\ x = \pi,\ x = 2\pi$



(24) $P(x) = -1/10\, x^2 + 30x - 500$ $P'(x) = -1/5\, x + 30$ $P'(x) = 0$

$-1/5\, x + 30 = 0 \Rightarrow x = 150$



max!

(D)

(25) $f(x) = 1/x$ $f(1) = +1$

$f'(x) = -1/x^2$ $f(3) =$ [redacted] $\frac{1}{3}$ (A)

$f'(x) = 0$ never!

(26) False