Riemann Sums

1. Compute the area under $f(x) = x^2$ on the interval $[0, 1]$ using the right hand rule. Hint: first find the nth right hand estimate,

$$A_n = \sum_{k=1}^{n} f(x_k)\Delta x$$

using $x_k = a + k\Delta x$ and $\Delta x = (b - a)/n$, and then use $A = \lim A_n$ to find the total area.

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$

$x_k = 0 + k\left(\frac{1}{n}\right) = \frac{k}{n}$

$A_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n}$

$= \sum_{k=1}^{n} \frac{k^2}{n^3}$

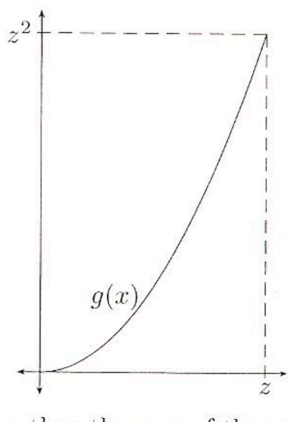
$\rightarrow A_n = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$

$= \frac{1}{n^3}\left(\frac{2n^3 + 3n^2 + n}{6}\right) = \frac{2}{6} + \frac{3}{6n} + \frac{1}{6n^2}$

$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left(\frac{2}{6} + \frac{3}{6n} + \frac{1}{6n^2}\right) = \frac{2}{6} + 0 + 0$

$\lim_{n\to\infty} A_n = A = 2/6 = \boxed{1/3}$

2. Using a Riemann sum, show that the area under the graph of $g(x) = x^2$ on $[0, z]$ is always 1/3rd the area of the rectangle with one corner at the point $(0, 0)$ and the other corner at the point (z, z^2). Hint: check out this awesome picture:



Area of rectangle

A = length × width

$= z^2 \cdot z$

$= z^3$

First calculate the area under the curve, then the area of the rectangle. *Archimedes did this 2000 years ago FTW!*

Area under curve

$\Delta x = \frac{z-0}{n} = \frac{z}{n}$

$x_k = 0 + k\left(\frac{z}{n}\right) = \frac{zk}{n}$

$A_n = \sum_{k=1}^{n} \left(\frac{zk}{n}\right)^2 \left(\frac{z}{n}\right)$

$= \sum_{k=1}^{n} \frac{z^3 k^2}{n^3}$

$\rightarrow A_n = \frac{z^3}{n^3} \sum_{k=1}^{n} k^2$

$= z^3\left(\frac{1}{n^3} \sum_{k=1}^{n} k^2\right)$

$\lim_{n\to\infty} A_n = z^3\left[\lim \frac{1}{n^3} \sum k^2\right]$

from (a) ~~we~~ we know

$\lim_{n\to\infty} \frac{1}{n^3} \sum k^2 = 1/3$

so $\lim_{n\to\infty} A_n = z^3(1/3) = \boxed{z^3/3}$

Indeed the area under the curve ($z^3/3$) is one third the area of the rectangle

$\frac{1}{3} \cdot z^3$

Integrals

3. Evaluate the following integrals directly (don't forget $+C$):

   (a) $\int x^{3/2} + x^{2/3} + 3dx$

   (b) $\int \cos(y) - \sin(y)dy$

   (c) $\int 5e^z + \ln(z)dz$

a) $\int x^{3/2} + x^{2/3} + 3\,dx = \frac{1}{5/2} x^{5/2} + \frac{1}{5/3} x^{5/3} + 3x + C = \boxed{\frac{2}{5} x^{5/2} + \frac{3}{5} x^{5/3} + 3x + C}$

b) $\int \cos y - \sin y\, dy = \sin y - (-\cos y) + C = \boxed{\sin y + \cos y + C}$

c) $\int 5e^z + \ln(z)dz = 5\int e^z dz + \int \ln(z)dz$

$= \boxed{5e^z + z\ln z - z + C}$

4. Evaluate the following integrals using substitution:

   (a) $\int \frac{\ln(1/x)}{x^2} dx$

   (b) $\int \sin(x)\sin(\cos(x)) + \sin^3(x)dx$

   (c) $\int \frac{x^2}{\sqrt{x+1}} dx$ Hint: use $u = \sqrt{x+1}$, then solve for x to find x^2 in terms of u.

a) $\int \frac{\ln(1/x)}{x^2} dx \quad u = 1/x \quad \frac{du}{dx} = -1/x^2$

$-x^2 du = dx$

$\int \frac{\ln(u)}{x^2}(-x^2 du) = \int -\ln(u) du$

$= -u\ln u + u + C$

$= \boxed{-\frac{1}{x}\ln\left(\frac{1}{x}\right) + 1/x + C}$

b) $\int \sin x \sin(\cos x)dx + \int \sin^3 x\, dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$\frac{du}{-\sin x} = dx$

$\int \sin x \sin(u) \frac{du}{-\sin x}$

$-\int \sin(u)du$

$-(-\cos u)$

$\int \sin x(1-\cos^2 x)dx$

$\int \sin x dx - \int \sin x \cos^2 x dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$\frac{du}{-\sin x} = dx$

$\int \sin x dx - \int \sin x (u^2) \frac{du}{-\sin x}$

$+ \quad \int \sin x dx + \int u^2 du$

$+ \quad -\cos x + \frac{1}{3}u^3 + C$

$\boxed{\cos(\cos x) - \cos x + \frac{1}{3}(\cos^3 x) + C}$

c) $\int \frac{x^2}{\sqrt{x+1}} \quad u = \sqrt{x+1} \quad u^2 - 1 = x$

$\frac{du}{dx} = \frac{1}{2}(x+1)^{-1/2}$

$2\sqrt{x+1}\, du = dx$

$\int \frac{(u^2-1)^2}{u}(2u\,du)$

$= \int (u^4 - 2u^2 + 1)(2\,du)$

$= 2\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right) + C$

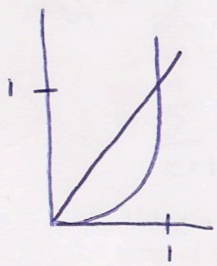
$= \boxed{\frac{2}{5}(\cancel{x}+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + 2(x+1)^{1/2} + C}$

Applications of Integrals

5. Verify problem 2 by computing $\int_0^z x^2 dx$ and comparing it to the area of the rectangle (as computed in problem 2).

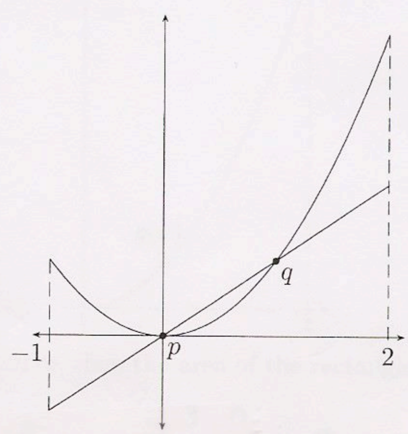
$$\int_0^z x^2 dx = \tfrac{1}{3}x^3\Big|_0^z = \tfrac{1}{3}(z)^3 - \tfrac{1}{3}(0)^3 = \boxed{\tfrac{1}{3}z^3}$$

6. Find the area between the functions $f(x) = x$ and $g(x) = x^2$ over the interval $[0, 1]$.



$$\int_0^1 (x-x^2)\,dx = \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\Big|_0^1$$
$$= \tfrac{1}{2}(1)^2 - \tfrac{1}{3}(1)^3 - \left(\tfrac{1}{2}(0)^2 - \tfrac{1}{3}(0)^3\right)$$
$$= \tfrac{1}{2} - \tfrac{1}{3} = \boxed{\tfrac{1}{6}}$$

7. Find the area between the functions $f(x) = x$ and $g(x) = x^2$ over the interval $[-1, 2]$. Hint: again, check out this excellent picture:



First find the x-coordinates of the points p and q, then write the area as the sum of multiple integrals.

$$\int_{-1}^0 (x^2-x)\,dx + \int_0^1 (x-x^2)\,dx + \int_1^2 (x^2-x)\,dx$$

$$\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\Big|_{-1}^0 + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\Big|_0^1 + \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\Big|_1^2$$

$$\left[\tfrac{1}{3}(0)^3 - \tfrac{1}{2}(0)^2 - \left(\tfrac{1}{3}(-1)^3 - \tfrac{1}{2}(-1)^2\right)\right] + \underset{\text{(from part 6)}}{\tfrac{1}{6}} + \tfrac{1}{3}(2)^3 - \tfrac{1}{2}(2)^2 - \left(\tfrac{1}{3}(1)^3 - \tfrac{1}{2}(1)\right.$$

$$0 - \left(-\tfrac{1}{3} - \tfrac{1}{2}\right) + \tfrac{1}{6} + \tfrac{8}{3} - 2 - \left(-\tfrac{1}{6}\right) = \tfrac{11}{6}$$