Chapter 2: What do groups look like?

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Let's pretend for a moment that we were interested in writing a complete solutions manual for the Rubik's Cube. Let me be more specific about what I mean.

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Let's call our solutions manual the *Big Book*. See Figure 2.1 on page 13 for a picture of what a page in the *Big Book* might look like.

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We shall not abandon the mapmaking ideas introduced by our discussion of the *Big Book* simply because the map is too large. We can use the same ideas to map out any group. In fact, we shall frequently do exactly that.

Let's try something simpler...

The Rectangle Puzzle

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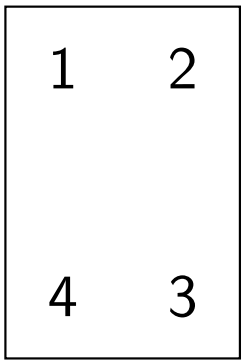
- Take a blank sheet of paper (our rectangle) and label as follows:



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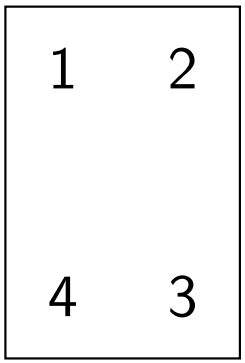

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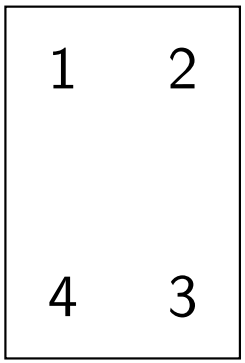

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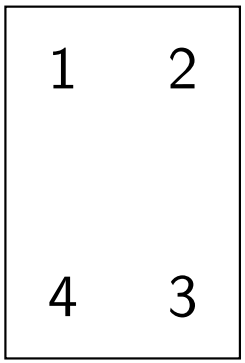

  This is the solved state of our puzzle.
- The idea of the game is to scramble the puzzle and then find a way to return the rectangle to its solved state.
- We are allowed two moves: horizontal flip and vertical flip, where "horizontal" and "vertical" refer to the motion of your hands, rather than any reference to an axis of reflection.

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Observations?

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For reference, here are the rules of a group:

Rule 1.5

There is a predefined list of actions that never changes.

Rule 1.6

Every action is reversible.

Rule 1.7

Every action is deterministic.

Rule 1.8

Any sequence of consecutive actions is also an action.

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- the group has 4 actions: the "do nothing" action, horizontal flip, vertical flip, and 180° rotation ($r = h \circ v = v \circ h$);
- the map shows us how to get from any one configuration to any other (there may be more than one way to follow the yarn).

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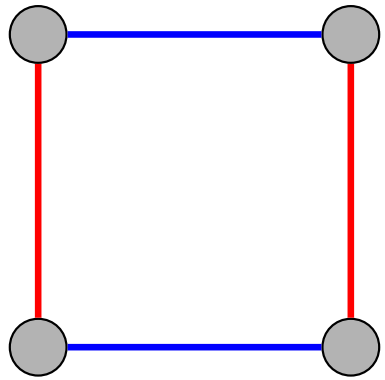
Here is one possible representation of the Cayley diagram for our Rectangle Puzzle:



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In small groups, map out the 2-Light Switch Group using paper and yarn just like we did for the Rectangle Puzzle. (I suggest using U and D to denote "light switch up" and "light switch down", respectively.)

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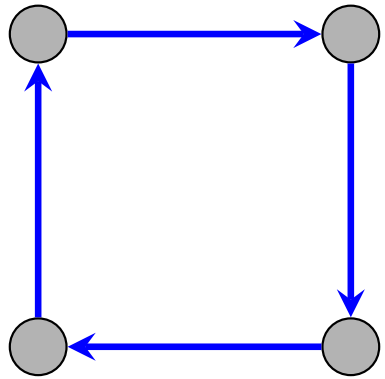

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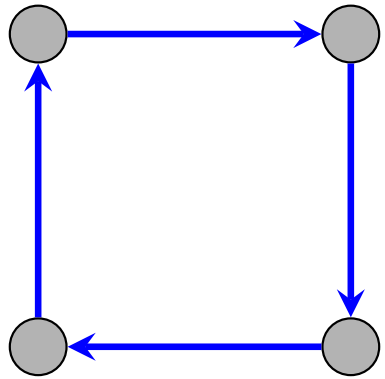

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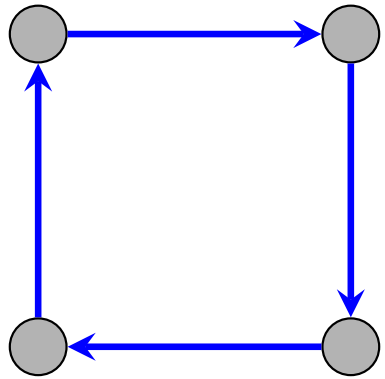

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More Group Exercises

Let's explore a few more examples.

1. In groups of 2–3 (try to mix the groups up again), complete the following exercises (not collected):
    - Exercise 2.1 (see Bob)
    - Exercise 2.3 (see Bob)
    - Exercise 2.5
    - Exercise 2.8 (see Bob)
    - Exercise 2.10
    - Exercise 2.13 (see Bob)

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   - Exercise 2.13 (see Bob)
2. I'd like each group to present their solution to one of the problems above.

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   - Exercise 2.8 (see Bob)
   - Exercise 2.10
   - Exercise 2.13 (see Bob)
2. I'd like each group to present their solution to one of the problems above.
3. Now, complete Exercise 2.18. I want each group to turn in a complete solution.

Potential quiz questions

Here are some potential questions that I may ask you on tomorrow's quiz at the beginning of class:

1. What do the arrows represent in a Cayley diagram?
2. What do the nodes represent in a Cayley diagram?
3. Draw 2 different Cayley diagrams and describe a specific set of actions (i.e., generators) that would yield the corresponding diagrams.