Chapter 6

Relations

6.1 Relations

Definition 6.1. An **ordered pair** is an object of the form (x,y). Two ordered pairs (x,y) and (a,b) are **equal** iff $x = a$ and $y = b$.

Definition 6.2. An n**-tuple** is an object of the form $(x_1, x_2, \ldots, x_n)$. Each x_i is referred to as the ith **component**.

Note that an ordered pair is just a 2-tuple.

Definition 6.3. If X and Y are sets, the **Cartesian product** of X and Y is defined by

$$X \times Y = \{(x,y) : x \in X, y \in Y\}.$$

That is, $X \times Y$ is the set of all ordered pairs where the first element is from X and the second element is from Y. The set $X \times X$ is sometimes denoted by X^2. We similarly define the Cartesian product of n sets, say $X_1, \ldots, X_n$, by

$$\prod_{i=1}^{n} X_i = X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) : \text{each } x_i \in X_i\}.$$

Example 6.4. Let $A = \{a,b,c\}$ and $B = \{☺, ☹\}$. Then

$$A \times B = \{(a,☺),(a,☹),(b,☺),(b,☹),(c,☺),(c,☹)\}.$$

Exercise 6.5. Using the sets A and B from the previous example, find $B \times A$.

Exercise 6.6. Using the set B from the previous examples, find $B \times B$.

Exercise 6.7. What general conclusion can you make about $X \times Y$ versus $Y \times X$? When will they be equal?

Exercise 6.8. If X and Y are both finite sets, then how many elements will $X \times Y$ have? Be as specific as possible.

Exercise 6.9. Let $A = \{1,2,3\}$, $B = \{1,2\}$, and $C = \{1,3\}$. List the elements of the set $A \times B \times C$.

Exercise 6.10. Let $A = \mathbb{N}$ and $B = \mathbb{R}$. Describe the elements of the set $A \times B$.

Exercise 6.11. Let A be the set of all differentiable functions on the open interval $(0,1)$, and let B equal the set of all derivatives of functions in A evaluated at $x = \frac{1}{2}$. Describe the elements of the set $A \times B$.

Exercise 6.12. Three space, $\mathbb{R}^3$, is a Cartesian product. Unpack the meaning of $\mathbb{R}^3$ using the Cartesian product, and write the complete set notation version.

Exercise 6.13. Let $X = [0,1]$ and let $Y = \{1\}$. Describe geometrically what $X \times Y$, $Y \times X$, $X \times X$, and $Y \times Y$ look like.

Definition 6.14. Let X and Y be sets. A **relation** from a set X to a set Y is a subset of $X \times Y$. A relation on X is a subset of $X \times X$.

Example 6.15. You may not realize it, but you are familiar with many relations. For example, on the real numbers, we have the relation $\leq$. We could say that $(3,\pi)$ is in the relation $\leq$ since $3 \leq \pi$. However, $(1,-1)$ is not in the relation since $1 \nleq -1$. (Order matters!)

Remark 6.16. Different notations for relations are used in different contexts. When talking about relations in the abstract, we indicate that a pair (a,b) is in the relation by some notation like $a \sim b$, which is read "a is related to b."

Example 6.17. Let P_f denote the set of all people with accounts on Facebook. Define F via xFy iff x is friends with y. Then F is a relation on P_f.

We can often represent relations using graphs or digraphs. Given a finite set X and a relation $\sim$ on X, a **digraph** (short for *directed graph*) is a discrete graph having the members of X as vertices and a directed edge from x to y iff $x \sim y$.

Example 6.18. Figure 6.1 depicts a digraph that represents a relation R given by

$$R = \{(a,b),(a,c),(b,b),(b,c),(c,d),(c,e),(d,d),(d,a),(e,a)\}.$$

Exercise 6.19. Let $A = \{a,b,c\}$ and define $\sim = \{(a,a),(a,b),(b,c),(c,b),(c,a)\}$. Draw the digraph for $\sim$.

Exercise 6.20. Let $A = \{1,2,3,4,5,6\}$. Define $|$ on A via $x|y$ iff x divides y. Draw the digraph for $|$ on A.

When X or Y is infinite, it is not practical to draw a digraph. However, you are familiar with the graphs of some relations involving infinite sets.

Example 6.21. When we write $x^2 + y^2 = 1$, we are implicitly defining a relation. In particular, the relation is the set of ordered pairs (x,y) satisfying $x^2 + y^2 = 1$. In set notation:

$$\{(x,y) : x^2 + y^2 = 1\}$$

The graph of this relation in $\mathbb{R}^2$ is the standard unit circle.

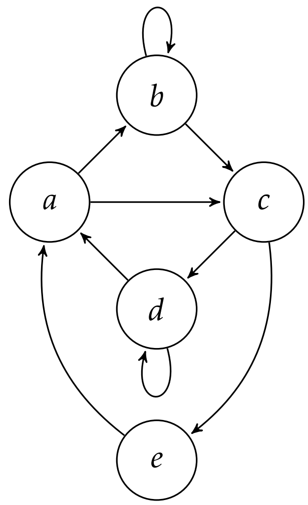

Figure 6.1: An example of a digraph for a relation.

Exercise 6.22. Define $\sim$ on $\mathbb{R}$ via $x \sim y$ iff $x \leq y$. Draw a picture of this relation in $\mathbb{R}^2$. In other words, draw all points (x,y) where $x \sim y$.

Definition 6.23. Let $\sim$ be a relation on a set A.

(a) $\sim$ is **reflexive** if for all $x \in A$, $x \sim x$ (every element is related to itself).

(b) $\sim$ is **symmetric** if for all $x, y \in A$, if $x \sim y$, then $y \sim x$.

(c) $\sim$ is **transitive** if for all $x, y, z \in A$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

Example 6.24.

(a) $\leq$ on $\mathbb{R}$ is reflexive and transitive, but not symmetric. $<$ on $\mathbb{R}$ is transitive, but not symmetric and not reflexive.

(b) If S is a set, then $\subseteq$ on $\mathcal{P}(S)$ is reflexive and transitive, but not symmetric.

(c) $=$ on $\mathbb{R}$ is reflexive, symmetric, and transitive.

Exercise 6.25. Given a finite set A and a relation $\sim$, describe what each of reflexive, symmetric, and transitive look like in terms of a digraph.

Exercise 6.26. Let P be the set of people at a party and define N via $(x,y) \in N$ iff x knows the name of y. Describe what it would mean for N to be reflexive, symmetric, and transitive.

Exercise 6.27. Determine whether each of the following relations is reflexive, symmetric, or transitive.

(a) Let P_f denote the set of all people with accounts on Facebook. Define F via xFy iff x is friends with y.

(b) Let P be the set of all people and define H via xHy iff x and y have the same height.

(c) Let P be the set of all people and define T via xTy iff x is taller than y.

(d) Consider the relation "divides" on $\mathbb{N}$.

(e) Let L be the set of lines and define $\|$ via $l_1 \| l_2$ iff l_1 is parallel to l_2.

(f) Let $C[0,1]$ be the set of continuous functions on $[0,1]$. Define $f \sim g$ iff

$$\int_0^1 |f(x)|\,dx = \int_0^1 |g(x)|\,dx.$$

(g) Define $\sim$ on $\mathbb{N}$ via $n \sim m$ iff $n+m$ is even.

(h) Define D on $\mathbb{R}$ via $(x,y) \in D$ iff $x = 2y$.

6.2 Equivalence Relations

Let $\sim$ be a relation on a set A. Recall the following definitions:

(a) $\sim$ is **reflexive** if for all $x \in A$, $x \sim x$ (every element is related to itself).

(b) $\sim$ is **symmetric** if for all $x, y \in A$, if $x \sim y$, then $y \sim x$.

(c) $\sim$ is **transitive** if for all $x, y, z \in A$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

As we've seen in the previous section of notes, these conditions are independent. That is, a relation may have some combination of these properties, but not necessarily all of them. However, we have a special name for when a relation does satisfy all three.

Definition 6.28. Let $\sim$ be a relation on a set A. Then $\sim$ is called an **equivalence relation** iff $\sim$ is reflexive, symmetric, and transitive.

Exercise 6.29. Given a finite set A and a relation $\sim$ on A, describe what the corresponding digraph would have to look like in order for $\sim$ to be an equivalence relation.

Exercise 6.30. Let $A = \{a, b, c, d, e\}$. Make up an equivalence relation on A by drawing a digraph such that a is not related to b and c is not related to b.

Exercise 6.31. Let $S = \{1, 2, 3, 4, 5, 6\}$ and define

$$\sim = \{(1,1),(1,6),(2,2),(2,3),(2,4),(3,3),(3,2),(3,4),(4,4),(4,2),(4,3),(5,5),(6,6),(6,1)\}.$$

Justify that this is an equivalence relation.

Exercise 6.32. Determine which of the following are equivalence relations. Some of these occurred in the last section of notes and you are welcome to use your answers from those problems.

(a) Let P_f denote the set of all people with accounts on Facebook. Define F via xFy iff x is friends with y.

(b) Let P be the set of all people and define H via xHy iff x and y have the same height.

(c) Let P be the set of all people and define T via xTy iff x is taller than y.

(d) Consider the relation "divides" on $\mathbb{N}$.

(e) Let L be the set of lines and define $\|$ via $l_1 \| l_2$ iff l_1 is parallel to l_2.

(f) Let $C[0,1]$ be the set of continuous functions on $[0,1]$. Define $f \sim g$ iff

$$\int_0^1 |f(x)|\, dx = \int_0^1 |g(x)|\, dx.$$

(g) Define $\sim$ on $\mathbb{N}$ via $n \sim m$ iff $n+m$ is even.

(h) Define D on $\mathbb{R}$ via $(x,y) \in D$ iff $x = 2y$.

(i) Define $\sim$ on $\mathbb{Z}$ via $a \sim b$ iff $a-b$ is a multiple of 5.

(j) Define $\sim$ on $\mathbb{R}^2$ via $(x_1,y_1) \sim (x_2,y_2)$ iff $x_1^2 + y_1^2 = x_2^2 + y_2^2$.

(k) Define $\sim$ on $\mathbb{R}$ via $x \sim y$ iff $\lfloor x \rfloor = \lfloor y \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x (e.g., $\lfloor \pi \rfloor = 3$, $\lfloor -1.5 \rfloor = -2$, and $\lfloor 4 \rfloor = 4$).

(l) Define $\sim$ on $\mathbb{R}$ via $x \sim y$ iff $|x-y| < 1$.

Definition 6.33. Let $\sim$ be a relation on a set A (not necessarily an equivalence relation) and let $x \in A$. Then we define the **set of relatives of** x via

$$[x] = \{y \in A : x \sim y\}.$$

Also, define

$$\Omega_\sim = \{[x] : x \in A\}.$$

Notice that $\Omega_\sim$ is a set of sets. In particular, an element in $\Omega_\sim$ is a subset of A (equivalently, an element of $\mathcal{P}(A)$). Other common notations for $[x]$ include $\overline{x}$ and R_x.

Exercise 6.34. Let P_f and F be as in part (a) of Exercise 6.32. Describe [Bob] (assume you know which Bob we're talking about). What is Ω_F?

Exercise 6.35. Using your digraph in Exercise 6.30, find $\Omega_\sim$ for all $x \in A$.

Exercise 6.36. Consider the relation $\leq$ on $\mathbb{R}$. If $x \in \mathbb{R}$, what is $[x]$?

Exercise 6.37. Find [1] and [2] for the relation given in part (i) of Exercise 6.32. How many different sets of relatives are there? What are they?

Exercise 6.38. Find $[x]$ for all $x \in S$ for S and $\sim$ from Exercise 6.31. Any observations?

Theorem 6.39. Suppose $\sim$ is an equivalence relation on a set A and let $a, b \in A$. Then $[a] = [b]$ iff $a \sim b$.

Theorem 6.40. Suppose $\sim$ is an equivalence relation on a set A. Then

(a) $\bigcup_{x\in A}[x] = A$, and

(b) for all $x, y \in A$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

Definition 6.41. In light of Theorem 6.40, if $\sim$ is an equivalence relation on a set A, then we refer to each $[x]$ as the **equivalence class** of x. In this case, $\Omega_\sim$ is the set of equivalence classes determined by $\sim$.

The upshot of Theorem 6.40 is that given an equivalence relation, every element lives in exactly one equivalence class. We'll see in the next section of notes that we can run this in reverse. That is, if we separate out the elements of a set so that every element is an element of exactly one subset (like the bins of my kid's toys), then this determines an equivalence relation. More on this later.

Example 6.42. The set of relatives that you found in part (i) of Exercise 6.32 is the set of equivalence classes modulo 5.

Exercise 6.43. If $\sim$ is an equivalence relation on a finite set A, then what is the connection between the equivalence classes and the corresponding digraph?

Exercise 6.44. For each of the equivalence relations in Exercise 6.32, describe the equivalence classes as best as you can.

6.3 Partitions

Remark 6.45. The upshot of Theorems 5.41 and 5.42 is that if $\sim$ is an equivalence relation on a set A, then $\sim$ breaks A up into pairwise disjoint chunks, where each chunk is some $[a]$ for $a \in A$. Furthermore, each pair of elements in the same set of relatives are related via $\sim$.

As you've probably already noticed, equivalence relations are intimately related to the following concept.

Definition 6.46. A collection Ω of nonempty subsets of a set A is said to be a **partition** of A if the elements of Ω satisfy:

1. Given $X, Y \in \Omega$, either $X = Y$ or $X \cap Y = \emptyset$ (We can't have both at the same time. Do you see why?), and
2. $\displaystyle\bigcup_{X\in\Omega} X = A$.

That is, the elements of Ω are pairwise disjoint and their union is all of A.