Chapter 2: Sets
Sections 2.1–2.3

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Notes

2.1 Sets and Set Notation

In this chapter, we will develop several set-theoretic ideas and theorems without a direct appeal to the axioms of set theory. Instead we take for granted some intermediate yet basic facts. These will act as our underlying assumptions.

We will assume:

- elementary properties of the natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

- an informal understanding of the integers

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\},$$

  the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ (rational and irrational), and the Cartesian plane $\mathbb{R}^2$.

Notes

We have a few undefined terms, such as set, element, and $\in$. We assume an intuitive understanding of these terms.

Intuitively:

- A set is a collection of things.
- An element is one of the things that lies in the set.

We write $x \in A$ if x is an element of the set A. On the other hand, if x is not an element of the set A, then we write $x \notin A$.

We can think of a set as a box containing some stuff. If we rearrange the items in the box, the contents do not change. The order of the elements in a set is immaterial.

Notes

Here are some examples that illustrate the notation we use to denote sets.

Example

- $A = \{\square, \blacksquare, \triangle\}$ (listing all the elements of a finite set)
- $E = \{2, 4, 6, 8, \ldots\}$ (listing the elements of an infinite set by establishing an easy to recognize pattern)
- $E = \{n \in \mathbb{N} : n = 2k \text{ for some } k\}$ (providing a description of the elements of a set; this one is describing an infinite set)
- $T = \{w : w \text{ is an English word that begins with } t\}$ (providing a description of the elements of a set; this one is large, but finite)

Notes

Notice that in the last two examples above, both sets were in the form $S = \{x \in X : P(x)\}$, where x is a variable, X is the universe of values that x can take on, and $P(x)$ is a predicate that describes x. The set S is collection of values from X that make $P(x)$ true.

Let's take a look at a different type of example.

Example

Consider the set $S = \{\{1, 2\}, \{2, 3, 4\}, \{5\}, 5\}$. Describe the elements of S. Is 2 an element of S? How about 5?

We can also use interval notation to denote sets (of real numbers).

Notes

Example

Interval notation:

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed interval)
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ (half-closed or half-open interval)
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ (half-closed or half-open interval)
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ (open interval)

It is convenient to be able to refer to a set with no elements (i.e., an empty box). We refer to any set with no elements as the empty set and denote it by $\emptyset$, or possibly $\{\}$.

Notes

Let's consider a couple of examples.

Example

- Consider the set

$$A = \{x \in \mathbb{R} : x^2 + 1 = 0\}.$$

  Does this set contain any elements? No. So, we write $A = \emptyset$. However, what about the set $B = \{x \in \mathbb{C} : x^2 + 1 = 0\}$ (where $\mathbb{C}$ is the set of complex numbers)? This is set is nonempty since it contains the imaginary number i.

- Consider the set

$$X = \{p : p \text{ is a person in this room whose name starts w/ x}\}.$$

  This set is equal to $\emptyset$.

Notes

2.2 Subsets

Definition 2.2.1

If A and S are sets, we say that S is a subset of A if every element of S is in A (i.e., if $x \in S$, then $x \in A$). In this case, we write $S \subseteq A$.

Example

1. Consider the set $A = \{\square, \blacksquare, \triangle\}$. Let's find all subsets of A.
2. Consider the set $B = \{\emptyset, \{\emptyset\}\}$. List all of the subsets of B.

Notice that the statement "If $x \in S$, then $x \in A$" is an implication. So, if we want to prove that $S \subseteq A$, what do we need to do?

Notes

To show $S \subseteq A$ for anything with more than a small number of elements, we should assume that an arbitrary member of the universe x is an element of S and then show that x is also a member of A.

This type of argument is called an element argument. All element arguments should start with the phrase "Let $x \in S$" (or equivalent), where S is the smaller set in question.

Theorem 2.2.2

For all sets X, $\emptyset \subseteq X$ and $X \subseteq X$.

We will prove this theorem for homework and it will definitely be one of the ones that I ask you to present.

Notes

Note that two sets are equal if they contain exactly the same elements. In other words, $A = B$ iff $A \subseteq B$ and $B \subseteq A$. (This is Definition 2.2.7 in the book and should come before the next definition.)

Definition 2.2.5

If B is a subset of X and $B \neq X$, then we say that B is a proper subset of X.

Notes

2.3 Set Operations

Just like with numbers, there are ways to combine sets together in various ways.

Definition 2.3.1

Let U be a set and let $S \subseteq U$. Define

$$S_U^C = \{x \in U : x \notin S\}.$$

The set S_U^C is called the complement of S in U.

If the set U is understood, we may just write S^C and call it the complement of S. To avoid paradoxes, we must always take complements relative to some larger (or possibly equal) set.

Notes

Example

1. Consider the sets $S = \{1, 3, 5\}$ and $U = \{1, 2, 3, 4, 5, 6\}$. What is S_U^C?
2. What is $\mathbb{N}_{\mathbb{R}}^C$?

Definition 2.3.4

Let A and B be sets.

1. $A \cup B = \{x : x \in A \text{ or } x \in B\}$ (the union of A and B)
2. $A \cap B = \{x : x \in A \text{ and } x \in B\}$ (the intersection of A and B)

Let's draw the corresponding Venn diagrams for union and intersection. See pages 44–45 for a discussion of Venn diagrams.

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Definition 2.3.6

If A and B are sets that do not intersect (i.e., $A \cap B = \emptyset$), then we say that A and B are disjoint.

Of course we can take unions and intersections of more than two sets, even an infinite collection of sets. Often it will be useful for us to index the sets when doing this. We will illustrate indexing through an example.

(For now, an intuitive understanding of indexing will suffice.)

Notes

Example 2.3.11

Consider the set of intervals

$$A = \{[0, 1/n] : n \in \mathbb{N}\}.$$

This determines the intervals

$$[0, 1],\ [0, 1/2],\ [0, 1/3], \ldots$$

There is a natural way to “index” these sets for easy reference:

$$I_1 = [0, 1], I_2 = [0, 1/2], I_3 = [0, 1/3], \ldots$$

so that $I_n = [0, 1/n]$. In this case, we say that the sets we are working with are indexed by $\mathbb{N} = \{1, 2, 3, \ldots\}$. We can write

$$A = \{I_1, I_2, I_3, \ldots\} = \{I_n : n \in \mathbb{N}\} = \{I_n\}_{n\in\mathbb{N}} = \{I_n\}_{n=1}^{\infty}.$$

Notes

Any set, finite or infinite, can be used as an indexing set. We will denote arbitrary index sets by Λ and an arbitrary index by α. See Example 2.3.12 for more discussion of the notation.

Definition 2.3.13

Suppose we have a collection of indexed sets $\{B_\alpha\}_{\alpha\in\Lambda}$.

1. The union of all the sets is denoted $\bigcup\limits_{\alpha\in\Lambda} B_\alpha$, which is read "the union over alpha in Lambda of the B-alphas."

$$\bigcup_{\alpha\in\Lambda} B_\alpha = \{x : x \in B_\alpha \text{ for some } \alpha \in \Lambda\}$$

Notes

Definition 2.3.13 (continued)

2. The intersection of all the sets is denoted $\bigcap_{\alpha \in \Lambda} B_\alpha$, which is read "the intersection over alpha in Lambda of the B-alphas."

$$\bigcap_{\alpha \in \Lambda} B_\alpha = \{x : x \in B_\alpha \text{ for all } \alpha \in \Lambda\}$$

Let's return to our previous example.

Example

We see that

$$\bigcup_{n \in \mathbb{N}} I_n = [0, \infty) = \{x \in \mathbb{R} : x \geq 0\}.$$

Notes

Example (continued)

Also, we see that

$$\bigcap_{n\in\mathbb{N}} I_n = \{0\}.$$

Notes