Name:

Names of Any Collaborators:

Instructions

This portion of Exam 3 is worth a total of 18 points and is due at the beginning of class on **Wednesday, May 3**. Your total combined score on the in-class portion and take-home portion is worth 15% of your overall grade. I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any results that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem xyz, then you should say so.
2. Unless you prove them, you cannot use any results that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

1. (4 points) Prove **one** of the following theorems.

   Theorem C.1. Define $\Psi = \{(a,b) \mid a,b \in \mathbb{Z}, b \neq 0\}$ and define $\sim$ on Ψ via $(a,b) \sim (c,d)$ iff $ad = bc$. Then $\sim$ is an equivalence relation on Ψ.

   Theorem C.2. Define the relation C on $\mathbb{R} \times \mathbb{R}$ via

   $$(x,y)C(z,w) \text{ iff } x^2 + y^2 = z^2 + w^2.$$

   Then C is an equivalence relation.

2. (2 points) For the theorem you completed in the previous problem, describe the equivalence classes in an elegant way.

3. (4 points each) For this problem, you will need the following definition.

   Definition. Suppose $f : X \to Y$ is a function. Then the **inverse image** of a set $A \subseteq \mathrm{Rng}(f)$ is defined via

   $$f^{-1}(A) = \{x \in X \mid f(x) \in A\}.$$

   In other words, the inverse image of a set A in the range is the set of all missiles that blow up targets in A. In the case that A is a single point, say $\{y\}$, we often write $f^{-1}(y)$ in place of $f^{-1}(\{y\})$. It is important to point out that f^{-1} is always a relation, but may or may not be a function. That is, $f^{-1}(y)$ may be equal to a set with more than one element in it.

   Prove **two** of the following theorems. Please put your proofs in order and make sure it is clear which theorems you are proving.

   Theorem D.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying both

   (i) $f(x+y) = f(x) + f(y)$;
   (ii) $f(-x) = -f(x)$.*

   Then f is 1-1 if and only if $f^{-1}[\{0\}] = \{0\}$.

   Theorem D.2. Let $f : X \to Y$ be a function that is onto. Then

   $$\Omega = \{f^{-1}(y) \mid y \in Y\}$$

   is a partition of X.

   Theorem D.3. Let $f : X \to Y$ be a function and let $A, B \subseteq \mathrm{Rng}(f)$. Then

   $$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

4. (4 points) Prove **one** of the following theorems.

   Theorem E.1. Suppose A and B are sets such that $\mathrm{Card}(\mathbb{N}) = \mathrm{Card}(A)$. If there exists a bijection $f : A \to B$, then $\mathrm{Card}(A) = \mathrm{Card}(B)$.

   Theorem E.2. If $f : \mathbb{N} \to A$ is an onto function, then $\mathrm{Card}(\mathbb{N}) = \mathrm{Card}(A)$.

*Actually, all we really need to assume is (i) since (ii) follows from (i), but you do not need to prove that.