Section 4.1: Functions as Relations

Quick Review

Recall 1. A *relation* R from a set A to a set B is a collection of ____________ from the set $A \times B$. That is, $R \subseteq A \times B$. We have seen many different ways of representing relations:

1. explicit list of ordered pairs;
2. a digraph;
3. a "bubble" diagram with arrows going from left to right (for an example of what I mean by this, see Figure 3.10 on page 140);
4. a graph in the xy-plane (which really is a list of ordered pairs);
5. a verbal description of what is related to what.

For this section, it will be most useful for us to deal with "bubble" diagrams and graphs in the xy-plane. Let's work out a few examples. (Note: In this section, we are *not* requiring that our relations be equivalence relations.)

Example 2. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{w, x, y, z\}$. Let S be the relation from A to B given by

$$S = \{(1, x), (2, x), (1, y), (3, w), (5, y)\}.$$

Draw the "bubble" diagram that corresponds to S.

Example 3. Let A and B be as in Example 2. Let T be the relation from A to B given by

$$T = \{(1, x), (2, x), (4, y), (3, w), (5, y)\}.$$

Draw the "bubble" diagram that corresponds to T.

Example 4. Let C be the relation from $[-1, 1]$ to $\mathbb{R}$ given by

$$xCy \text{ iff } x^2 + y^2 = 1.$$

Draw the graph in the xy-plane that corresponds to C.

In this case, the graph of C is a ____________.

Example 5. Let P be the relation from $\mathbb{R}$ to $\mathbb{R}$ given by

$$xPy \text{ iff } y = x^2.$$

Draw the graph in the xy-plane that corresponds to P.

In this case, the graph of P is a ____________.

We will refer back to each of these examples throughout the rest of this handout.

Recall 6. The *domain* of a relation R from A to B is the set $\{$____________$\}$. It is always the case that

$$Dom(R) \subseteq \underline{\hspace{3cm}},$$

and the domain of R need not be all of ____________.

Also, the *range* of a relation R from A to B is the set $\{$____________$\}$. It is always the case that

$$Rng(R) \subseteq \underline{\hspace{3cm}},$$

and the range of R need not be all of ____________.

Example 7. Determine the domain and range of each of the relations S, T, C, and P from the previous examples.

Functions

We're now ready to define what a function is.

Definition 8. A *function* (or *mapping*) *from A to B* is a relation f from A to B such that

(i) the domain of f is all of A;

(ii) if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

We write $f : A \to B$ and say "f is a function from A to B," or "f maps A to B." The set B is called the *codomain* of f. In the special case where $A = B$, we say that$f$ is a *function on A*.

Part (i) of the definition above is requiring that every element of A be related to something in B. So, in our "bubble" diagrams, every point in the left "bubble" must have an arrow leaving it. Part (ii) of the definition is requiring that every element of the domain have exactly ____________ element of the range related to it. So, in our "bubble" diagrams, at most ____________ arrow may leave a point in the domain.

Important Note 9. Some relations are functions and some are not. Since the definition of a function has two requirements, there are two ways in which a relation could fail to be a function: If R is a relation from A to B, then

1. if $Dom(R) \neq$ ____________, then R is not a function;
2. if there exists x in the domain of R such that x is related to (or mapped to) two different elements in the range, then R is not a function.

Note 10 (Missile Analogy)**.** Here is a useful analogy. Think of the domain of a function as a set of missiles and the codomain as a set of targets. A function is a description of which missiles blow up which targets. In this case, a missile can hit at most one target. However, it is possible that more than one missile may blow up the same target. Furthermore, not every target may get hit. In this case, the targets that get blown up form the range (which may be smaller than the ____________).

Example 11. There are two reasons why the relation S from Example 2 is *not* a function.

1. First, we see that ____________ $\neq A$. This violates condition (i) of the definition of function.
2. Second, we see that the point ____________ in the domain of S is related to ____________ elements of the range. This violates condition (ii) of the definition.

Example 12. Explain why the relation T from Example 3 is a function. (There are 2 things that need to be said.) Identify the domain, codomain, and range for this function.

Example 13. The relation C from Example 4 is *not* a function. Give two examples of values in the domain of C that have more than one value related to them in the range.

Note 14. Everyone should be familiar with the *vertical line test* that tests whether a graph is a function. Well, the vertical line test is applicable in this context, as well. If some vertical line hits the graph (not talking about digraph) of a relation R in more than one spot, then that means that there is a value x in the domain of R such that x is related to more than one value in the range. In this case, R is not a function. Notice that in the previous example, the graph of C fails the vertical line test.

Example 15. The relation P from Example 5 is a well-known function. In this case, the domain of P is $\mathbb{R}$. Each element $x \in Dom(P)$ has exactly one element y that it is related to. For example, the ordered pair $(2, \_\_\_)\in P$ and there is no other ordered pair in P that has first coordinate 2. The codomain of P is ____________, but the range of P is ____________. Notice that for most elements in the range, there is more than one element of the domain that is related to it. For example, $(\_\_\_, 4)$ and $(\_\_\_, 4)$ are both elements of P. Again, a missile cannot blow up more than one target, but more than one missile can blow up the same target.

Example 16. Let P be the set of all people and let N be the set of first names. Then there is a relation R_1 from P to N. An element of R_1 is an ordered pair where the first coordinate is a person and the second coordinate is their first name. Explain why R_1 is a function. Is it okay that there is more than one person named Sally?

There is also a relation R_2 from N to P. In this case, an element of R_2 is an ordered pair where the first coordinate is a name and the second coordinate is a person. Explain why R_2 is *not* a function.

Definition 17. Let $f : A \to B$ (that is, f is a function from A to B). We write $y = f(x)$ when $(x, y) \in f$. We say that y is the *image of x under f* (or *value of f at x*) and that x is a *pre-image of y under f*.

Note 18. Notice that we said *the* image and *a* pre-image. The reason for this is that there can only be a single image for a point in the domain, but there may be many pre-images for a point in the range.

Example 19. Consider the function T from Example 3. Since $(4, y) \in T$, we write $T(4) = y$. In this case, we say that y is the image of 4 under T. Also, we say that 4 is a pre-image of y under T. However, ___ is also a pre-image of y. Similarly, since $(1, x) \in T$, $T($___$) =$ ___ and we say that ___ is the image of ___ under T and ___ and ___ are the pre-images of ___ under T.

Important Note 20. If you have a "bubble" diagram for a function (like we do for the previous example), pre-images live in the __________ bubble while images live in the __________ bubble.

Example 21. Rephrase the definitions of image and pre-image in terms of the missile analogy.

Example 22. Consider the function P from Example 5. We see that $P(2) =$ ___. The image of 0 under P is ___. The image of -3 under P is ___. The pre-image of 25 is ___ and ___.

Note 23. If we have a function f that has a formula for the second coordinate in terms of the first coordinate, we often define the function by this formula.

Example 24. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f = \{(x, y) \in \mathbb{R} : y = x^3 - 2x + 1\}.$$

Recall that $(x, y) \in f$ iff $f(x) = y$. But $y = x^3 - 2x + 1$. So, we can simply define f by $f(x) =$ __________.

Note 25. When defining a function by a formula, the domain is often assumed to be the largest set that makes sense.

Example 26. Let

$$g(x) = \frac{-3}{1 - x^2}.$$

Then

$$\mathrm{Dom}(g) = \underline{\hspace{6cm}}$$

and

$$\mathrm{Rng}(g) = \underline{\hspace{6cm}}.$$