Theorem (Problem 3.3.2)**.** *Every natural number can be written as the sum of distinct powers of two.*

Proof. We proceed by complete induction.

Base Case: Since $1 = 2^0$, the base case holds.

Inductive Step: Let $k \in \mathbb{N}$ and assume that for all natural numbers $j \leq k$, j can be written as distinct powers of two, say $j = 2^{m_{j,1}} + 2^{m_{j,2}} + \cdots + 2^{m_{j,l_j}}$, where $m_{j,1}, m_{j,2}, \ldots, m_{j,l_j}$ are distinct nonnegative integers (that depend on j). We need to show that $k+1$ can be written as a sum of distinct powers of two. There are two possibilities: (i) k is even, or (ii) k is odd.

(i) First, assume that k is even. In this case, $m_{k,i} \neq 0$ for all $i \in \{1, \ldots, l_k\}$, otherwise k would be equal to a sum of positive powers of 2 plus 1, which would imply that k is odd. Therefore, we can write

$$k + 1 = 2^{m_{k,1}} + 2^{m_{k,2}} + \cdots + 2^{m_{k,l_k}} + 1 = 2^{m_{k,1}} + 2^{m_{k,2}} + \cdots + 2^{m_{k,l_k}} + 2^0,$$

where each power is distinct, as desired.

(ii) Now, assume that k odd. Then $k-1$ is even and by the inductive hypothesis and the argument in the first case, it must be the case that $m_{(k-1)_i} \neq 0$ for all $i \in \{(k-1)_1, \ldots, (k-1)_m\}$. Thus, we can write

$$k + 1 = (k-1) + 2 = 2^{m_{k-1,1}} + 2^{m_{k-1,2}} + \cdots + 2^{m_{k-1,l_k}} + 2^1,$$

where each power is distinct.

So, in either case, $k+1$ can be written as the sum of distinct powers of 2.

Therefore, by complete induction, every natural number can be written as a sum of distinct powers of 2. ■