Chapter 1: Logic
Sections 1.1–1.5

Dana C. Ernst

Plymouth State University
Department of Mathematics
http://oz.plymouth.edu/~dcernst

Fall 2009

Notes

We defined a theorem to be a logical consequence of a collection of axioms.

In the Circle-Dot System, we had two axioms:

1. .o
2. o.

How did we go about proving our theorems in the Circle-Dot System? We had 4 rules we used to construct a logical consequence.

The goal in this chapter is to explain what the process is for constructing logical consequences in mathematics, in general.

Notes

1.1 True or False?

This section of the text lists 16 mathematical statements that we are supposed to explore as part of a "thought experiment." For each statement, we are supposed to think about whether the statement is true or fasle (one of them is unknown). In each case, we should try to come with a convincing argument.

Let's give it a (quick) try.

1. The points $(-1, 1), (2, -1)$, and $(3, 0)$ lie on a line.
2. If x is an integer, then $x^2 \geq x$.
3. If x is an integer, then $x^3 \geq x$.
4. For all real numbers x, $x^3 = x$.
5. There exists a real number x such that $x^3 = x$.
6. $\sqrt{2}$ is an irrational number.

Notes

7. If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.
8. If x is an even integer, then x^2 is an even integer.
9. Every positive integer is the sum of distinct powers of two.
10. Every positive integer is the sum of distinct powers of three.
11. If x is an integer, then x is even or x is odd.
12. If x is an integer, then x cannot be both even and odd.
13. Every integer greater than 2 can be expressed as the sum of two prime numbers. (Goldbach Conjecture)
14. There are infinitely many prime numbers.
15. For any positive real number x there exists a positive real number y such that $y^2 = x$.
16. Given three distict points in space, there is one and only one plane passing through them.

Notes

1.2 Statements & Predicates

A statement is a sentence that is either true or false, but not ambiguous.

Let's generate some examples of sentences that are statements.

How about some sentences that are not statements?

It is important to note that we do not necessarily have to have knowledge of the truth or falsehood of a statement, but only that it be unambiguous. For example, Goldbach's Conjecture (see item 13 in our thought experiment) is a statement even though no one knows whether it is true or false.

Notes

In order for a sentence to avoid ambiguity, all key phrases must be precisely and objectively defined. For example, consider "She is smart." Unless we know which "she" we are referring to and exactly what it means to be "smart," this sentence is not a statement.

The same problem can arise in mathematical sentences. Consider "$x^2 - 1 = 0$." This is a perfectly good sentence, but is it a statement? The answer is no.

However, if we replace x with a specific real number value, then this sentence is a statement. For example, "$2^2 - 1 = 0$" is a statement that happens to be false.

Notes

Sentences like "$x^2 - 1 = 0$" that contain a free variable are called predicates. Of course, we can have sentences with more than one free variable.

Variables that occur in *statements* are not free variables. For example, the statement "For every real number x, there exists a natural number y such that $y > x$" has no free variables.

Can you think of some examples of mathematical predicates?

Notes

In symbolic logic, statements and their logical relations are represented by abstract symbols.

Statements are represented by single letters. Here are some examples.

$P :=$ "Lance Armstrong is a cancer survivor."

$Q :=$ "All Mini Coopers have stripes."

(Note: we use the symbol $:=$ when we are defining something to be equal to something.)

Notes

We denote predicates with familiar function notation. Here are some examples.

$P(x) :=$ "$x^2 - 1 = 0$."

$F(x, y) :=$ "x is friends with y on Facebook."

We can always turn a predicate into a statement by assigning values to the free variables.

For example, $P(3)$ is the (false) statement "$3^2 - 1 = 0$". Also, $F(Bob, Sally)$ is now the statement "Bob is friends with Sally on Facebook." (Of course, we need to know exactly which Bob and Sally we are talking about.)

Notes

1.3 Quantification

Besides substituting specific values for free variables, there are other ways of turning predicates into statements.

One way is to make a claim about the values of the free variable that turn the predicate into a true statement. (You don't actually have to make the sentence true.)

For example, we can convert $P(x)$ from earlier into a statement in the following ways:

- "For all real numbers x, $x^2 - 1 = 0$."
- "There exists a real number x such that $x^2 - 1 = 0$."

Notes

The phrase "for all" is called the universal quantifier (often denoted $\forall$) and the phrase "there exists. . . such that" is called the existential quantifier (often denoted $\exists$), and the process of using quantifiers is called quantification.

Unless the context has been made clear, we *always* need to specify the "universe" of acceptable values for the free variables.

For example, consider the sentence "For all x, x^2 is positive" is true if our universe is the real numbers, but is not true if we consider the universe of complex numbers.

Notes

In a statement, all variables must be quantified; called bound variables. However, the order in which variables are quantified is very important.

Consider the predicate $M(x, y) :=$ "x is married to y" in the universe of all people. Explain the difference between the following statements.

- "For all x, there exists y such that x is married to y."
- "There exists x such that for all y, x is married to y."

It is worth mentioning that there are equivalent ways of wording the quantifiers. The phrases "for all," "for any," etc. are all acceptable variations for the universal quantifier. Similarly, "For some" could be used for the existential quantifier.

Notes

1.4 Mathematical Statements

Now, we begin to discuss how to build more complicated mathematical statements from simpler ones. Most statements in mathematics are of the form "If A, then B," where A and B are predicates.

Note that if A and B are predicates, then technically, "If A, then B" is also a predicate since any free variables have not been bound. However, the standard convention is to assume that the variables have been universally quantified in this context.

Definition 1.4.1

A statement of the form "If A, then B," where A and B are statements or predicates, is called an implication. A is called the hypothesis and B is called the conclusion.

Notes

Many of the statements from our thought experiment are implications. Here are a few obvious examples.

2. If x is an integer, then $x^2 \geq x$.
3. If x is an integer, then $x^3 \geq x$.
7. If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.
8. If x is an even integer, then x^2 is an even integer.
11. If x is an integer, then x is even or x is odd.
12. If x is an integer, then x cannot be both even and odd.

Notes

But there are a few not-so-obvious examples. Here is one.

4. For all real numbers x, $x^3 = x$.

This statement could be rewritten as "If x is a real number, then $x^3 = x$."

Notes

1.5 Mathematical Implication

Let's consider an example.

Example 1.5.1

If x is an integer, then $x^2 \geq x$.

Proof.

If $x = 0$, then $x^2 = x$, so $x^2 \geq x$. The same is true if $x = 1$. If $x > 1$, then $x^2 > 1 \cdot x = x$. Lastly, if $x < 0$, then $x^2 > 0 > x$. We have exhausted all possibilities for x.

$\square$

Notes

What makes this a proof? We studied all values of the variable x for which the hypothesis "x is an integer" is true and showed that for those cases the conclusion "$x^2 > x$" is also true.

Notice that we didn't consider values of x that did not meet the hypothesis even though there are such values that also satisfy the conclusion. Also, notice that we implicitly assumed universal quantification over the free variable.

Let's discuss the general situation.

Notes

Let $A(x)$ and $B(x)$ be predicates with free variable x. Now, let $P(x)$ be the predicate "If $A(x)$, then $B(x)$." We are interested in a general strategy for proving statements of the form "For all x (in some universe), $P(x)$."

The statement "For all x (in some universe), $P(x)$" is going to be true unless there is at least one value of x for which $A(x)$ is true but $B(x)$ is false.

$$P(x) \text{ is } \begin{cases} \text{true,} & \text{if A(x) and B(x) are both true.} \\ \text{false,} & \text{if A(x) is true and B(x) is false.} \\ \text{true,} & \text{if A(x) is false (regardless of the truth value of B(x)).} \end{cases}$$

Notes

We can summarize the truth and falsehood of implications in a truth table.

A	B	If A, then B
T	T	T
T	F	F
F	T	T
F	F	T

For predicates $A(x)$ and $B(x)$, "If $A(x)$, then $B(x)$" is true if for all possible values of x, the truth values of A and B fall only in the 1st, 3rd, or 4th lines of the table. It is false if even a single value of x lands us in the 2nd line.

An implication in which the hypothesis is false (lines 3 and 4) is called vacuously true.

Notes

Consider

> *"If the moon is made of green cheese, then chocolate prevents cavities."*

Is this statement true or false? Since the hypothesis is false, the statement is vacuously true.

A value of x that makes the hypothesis true and the conclusion false is called a counterexample. In order to show that an implication is false, all we need to do is provide a single counterexample.

For example, consider the false statement "If x is a real number, then $x^2 \geq x$." Provide a counterexample to this statement.

Notes