Chapter 2

Mathematics and Logic

Before you get started, make sure you've read Chapter 1, which sets the tone for the work we will begin doing here.

2.1 A Taste of Number Theory

In this section, we will work with the set of integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The purpose of this section is to get started with proving some theorems about numbers and study the properties of $\mathbb{Z}$. Because you are so familiar with properties of the integers, one of the issues that we will bump into knowing which facts about the integers we can take for granted. As a general rule of thumb, you should attempt to use the definitions provided without relying too much on your prior knowledge. We will likely need to discuss this further as issues arise.

It is important to note that we are diving in head first here. There are going to be some subtle issues that you will bump into and our goal will be to see what those issues are, and then we will take a step back and start again. See what you can do!

Recall that we use the symbol "$\in$" as an abbreviation for the phrase "is an element of" or sometimes simply "in." For example, the mathematical expression "$n \in \mathbb{Z}$" means "n is an element of the integers."

Definition 2.1. An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$.

Definition 2.2. An integer n is **odd** if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Notice that we did not define "even" as being divisible by 2. When tackling the next few theorems and problems, you should use the formal definition of even as opposed to the well-known divisibility condition. For the remainder of this section, you may assume that every integer is either even or odd but never both.

Theorem 2.3. The sum of two consecutive integers is odd.

Theorem 2.4. If n is an even integer, then n^2 is an even integer.

Problem 2.5. Prove or provide a counterexample: The sum of an even integer and an odd integer is odd.

Question 2.6. Did Theorem 2.3 need to come before Problem 2.5? Could we have used Problem 2.5 to prove Theorem 2.3? If so, outline how this alternate proof would go. Perhaps your original proof utilized the approach I'm hinting at. If this is true, can you think of a proof that does not rely directly on Problem 2.5? Is one approach better than the other?

Problem 2.7. Prove or provide a counterexample: The product of an odd integer and an even integer is odd.

Problem 2.8. Prove or provide a counterexample: The product of an odd integer and an odd integer is odd.

Problem 2.9. Prove or provide a counterexample: The product of two even integers is even.

Definition 2.10. An integer n **divides** the integer m, written $n|m$, if and only if there exists $k \in \mathbb{Z}$ such that $m = nk$. In the same context, we may also write that m **is divisible by** n.

Question 2.11. For integers n and m, how are following mathematical expressions similar and how are they different?

(a) $m|n$

(b) $\dfrac{m}{n}$

(c) m/n

In this section on number theory, we allow addition, subtraction, and multiplication of integers. In general, division is not allowed since an integer divided by an integer may result in a number that is not an integer. The upshot: don't write $\frac{m}{n}$. When you feel the urge to divide, switch to an equivalent formulation using multiplication. This will make your life much easier when proving statements involving divisibility.

Problem 2.12. Let $n \in \mathbb{Z}$. Prove or provide a counterexample: If 6 divides n, then 3 divides n.

Problem 2.13. Let $n \in \mathbb{Z}$. Prove or provide a counterexample: If 6 divides n, then 4 divides n.

Theorem 2.14. Assume $n, m, a \in \mathbb{Z}$. If $a|n$, then $a|mn$.

A theorem that follows almost immediately from another theorem is called a **corollary** (see Appendix B). See if you can prove the next result quickly using the previous theorem. Be sure to cite the theorem in your proof.

Corollary 2.15. Assume $n, a \in \mathbb{Z}$. If a divides n, then a divides n^2.

Problem 2.16. Assume $n, a \in \mathbb{Z}$. Prove or provide a counterexample: If a divides n^2, then a divides n.

Theorem 2.17. Assume $n, a \in \mathbb{Z}$. If a divides n, then a divides $-n$.

Theorem 2.18. Assume $n, m, a \in \mathbb{Z}$. If a divides m and a divides n, then a divides $m + n$.

Problem 2.19. Is the converse[1] of Theorem 2.18 true? That is, is the following statement true?

> Assume $n, m, a \in \mathbb{Z}$. If a divides $m + n$, then a divides m and a divides n.

If the statement is true, prove it. If the statement is false, provide a counterexample.

Once we've proved a few theorems, we should be on the look out to see if we can utilize any of our current results to prove new results. There's no point in reinventing the wheel if we don't have to. Try to use a couple of our previous results to prove the next theorem.

Theorem 2.20. Assume $n, m, a \in \mathbb{Z}$. If a divides m and a divides n, then a divides $m - n$.

Problem 2.21. Assume $a, b, m \in \mathbb{Z}$. Determine whether the following statement holds sometimes, always, or never. If ab divides m, then a divides m and b divides m. Justify with a proof or counterexample.

Theorem 2.22. If $a, b, c \in \mathbb{Z}$ such that a divides b and b divides c, then a divides c.

The previous theorem is referred to as **transitivity of division of integers**.

Theorem 2.23. The sum of any three consecutive integers is always divisible by three.

2.2 Introduction to Logic

After diving in head first in the last section, we'll take a step back and do a more careful examination of what it is we are actually doing.

Definition 2.24. A **proposition** (or **statement**) is a sentence that is either true or false.

For example, the sentence "All liberals are hippies" is a false proposition. However, the perfectly good sentence "$x = 1$" is *not* a proposition all by itself since we don't actually know what x is.

Exercise 2.25. Determine whether the following are propositions or not. Explain.

(a) All cars are red.

(b) Led Zeppelin is the best band of all time.

(c) If my name starts with the letter J, then my name is Joe.

(d) $x^2 = 4$.

[1] See Definition 2.37 for the formal definition of converse.

(e) There exists an x such that $x^2 = 4$.

(f) For all real numbers x, $x^2 = 4$.

(g) $\sqrt{2}$ is an irrational number.

(h) p is prime.

Given two propositions, we can form more complicated propositions using logical connectives.

Definition 2.26. Let A and B be propositions.

(a) The proposition "**not** A" is true iff[2] A is false; expressed symbolically as $\boxed{\neg A}$ and called the **negation** of A.

(b) The proposition "A **and** B" is true iff both A and B are true; expressed symbolically as $\boxed{A \wedge B}$ and called the **conjunction** of A and B.

(c) The proposition "A **or** B" is true iff at least one of A or B is true; expressed symbolically as $\boxed{A \vee B}$ and called the **disjunction** of A and B.

(d) The proposition "**If** A**, then** B" is true iff both A and B are true, or A is false; expressed symbolically as $\boxed{A \implies B}$ and called an **implication** or **conditional statement**. Note that $A \implies B$ may also be read as "A implies B" or "A only if B".

Exercise 2.27. Describe the meaning of $\neg(A \wedge B)$ and $\neg(A \vee B)$.

Exercise 2.28. Let A represent "6 is an even number" and B represent "6 is a multiple of 4." Express each of the following in ordinary English sentences and state whether the statement is true or false.

(a) $A \wedge B$

(b) $A \vee B$

(c) $\neg A$

(d) $\neg B$

(e) $\neg(A \wedge B)$

(f) $\neg(A \vee B)$

(g) $A \implies B$

Definition 2.29. A **truth table** is a table that illustrates all possible truth values for a proposition.

[2]Throughout mathematics, the phrase "if and only if" is common enough that it is often abbreviated "iff." Roughly speaking, this phrase/word means "exactly when."

Example 2.30. Let A and B be propositions. Then the truth table for the conjunction $A \wedge B$ is given by the following.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Notice that we have columns for each of A and B. The rows for these two columns correspond to all possible combinations for A and B. The third column gives us the truth value of $A \wedge B$ given the possible truth values for A and B.

Note that each proposition has two possible truth values: true or false. Thus, if a compound proposition P is built from n propositions, then the truth table for P will require 2^n rows.

Exercise 2.31. Create a truth table for each of $A \vee B$, $\neg A$, $\neg(A \wedge B)$, and $\neg A \wedge \neg B$. Feel free to add additional columns to your tables to assist you with intermediate steps.

Problem 2.32. A coach promises, "If we win tonight, then I will buy you pizza tomorrow." Determine the case(s) in which the players can rightly claim to have been lied to. Use this to help create a truth table for $A \implies B$.

Definition 2.33. Two statements P and Q are (**logically**) **equivalent**, expressed symbolically as $\boxed{P \iff Q}$ and read "P iff Q", iff they have the same truth table.

Each of the next three facts can be justified using truth tables.

Theorem 2.34. If A is a proposition, then $\neg(\neg A)$ is equivalent to A.

Theorem 2.35 (DeMorgan's Law)**.** If A and B are propositions, then $\neg(A \wedge B) \iff \neg A \vee \neg B$.

Problem 2.36. Let A and B be propositions. Conjecture a statement similar to Theorem 2.35 for the proposition $\neg(A \vee B)$ and then prove it. This is also called DeMorgan's Law.

Definition 2.37. The **converse** of $A \implies B$ is $B \implies A$.

Exercise 2.38. Provide an example of a true conditional proposition whose converse is false.

Definition 2.39. The **contrapositive** of $A \implies B$ is $\neg B \implies \neg A$.

Exercise 2.40. Let A and B represent the statements from Exercise 2.28. Express the following in ordinary English sentences.

(a) The converse of $A \implies B$.

(b) The contrapositive of $A \implies B$.

Exercise 2.41. Find the converse and the contrapositive of the following statement: "If a person lives in Flagstaff, then that person lives in Arizona."

Use truth tables to prove the following theorem.

Theorem 2.42. The implication $A \implies B$ is equivalent to its contrapositive.

The upshot of Theorem 2.42 is that if you want to prove a conditional proposition, you can prove its contrapositive instead. Try proving each of the next three theorems by proving the contrapositive of the given statement instead.

Theorem 2.43. Assume $x, y \in \mathbb{Z}$. If xy is odd, then both x and y are odd.

Theorem 2.44. Assume $x, y \in \mathbb{Z}$. If xy is even, then x or y is even.

Theorem 2.45. Assume $x \in \mathbb{Z}$. If x^2 is even, then x is even.

2.3 Negating Implications and Proof by Contradiction

So far we have discussed how to negate propositions of the form A, $A \land B$, and $A \lor B$ for propositions A and B. However, we have yet to discuss how to negate propositions of the form $A \implies B$. To begin, try proving the following result with a truth table.

Theorem 2.46. The implication $A \implies B$ is equivalent to the disjunction $\neg A \lor B$.

The next result follows quickly from Theorem 2.46 together with DeMorgan's Law.

Corollary 2.47. The proposition $\neg(A \implies B)$ is equivalent to $A \land \neg B$.

Exercise 2.48. Let A and B be the propositions "Darth Vader is a hippie" and "Sarah Palin is a liberal," respectively.

(a) Express $A \implies B$ as an English sentence involving the disjunction "or."

(b) Express $\neg(A \implies B)$ as an English sentence involving the conjunction "and."

Exercise 2.49. The proposition "If $.\overline{99} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$, then $.\overline{99} \neq 1$" is *false*. Write its (true) negation, as a conjunction.

Recall that a proposition is exclusively either true or false—it never be both.

Definition 2.50. A compound proposition that is always false is called a **contradiction**. A compound proposition that is always true is called a **tautology**.

Theorem 2.51. For any proposition A, the proposition $\neg A \land A$ is a contradiction.

Exercise 2.52. Provide an example of a tautology using arbitrary propositions and any of the logical connectives $\neg$, $\land$, and $\lor$. Prove that your example is in fact a tautology.

Suppose that we want to prove some proposition P (which might be something like $A \implies B$ or even more complicated). One approach, called **proof by contradiction**, is to assume $\neg P$ and then logically deduce a contradiction of the form $Q \wedge \neg Q$, where Q is some proposition (possibly equal to P). Since this is absurd, the assumption $\neg P$ must have been false, so P is true. The tricky part about a proof by contradiction is that it is not usually obvious what the statement Q should be.

Skeleton Proof 2.53 (Proof of P by contradiction)**.** Here is what the general structure for a proof by contradiction looks like if we are trying to prove the proposition P.

Proof. For sake of a contradiction, assume $\neg P$.

… *[Use definitions and known results to derive some Q and its negation ¬Q.]* …

This is a contradiction. Therefore, P. □

Proof by contradiction can be useful for proving statements of the form $A \implies B$, where $\neg B$ is easier to "get your hands on," because $\neg(A \implies B)$ is equivalent to $A \wedge \neg B$ (see Corollary 2.47).

Skeleton Proof 2.54 (Proof of $A \implies B$ by contradiction)**.** If you want to prove the implication $A \implies B$ via a proof by contradiction, then the structure of the proof is as follows.

Proof. For sake of a contradiction, assume A and $\neg B$.

… *[Use definitions and known results to derive some Q and its negation ¬Q.]* …

This is a contradiction. Therefore, if A, then B. □

Establish the following theorem in two ways: (i) prove the contrapositive, and (ii) prove via contradiction.

Theorem 2.55. Assume that $x \in \mathbb{Z}$. If x is odd, then 2 does not divide x. (Prove in two different ways.)

Prove the following theorem via contradiction. Afterward, consider the difficulties one might encounter when trying to prove the result more directly.

Theorem 2.56. Assume that $x, y \in \mathbb{N}$.[3] If x divides y, then $x \leq y$.

[3] $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of **natural numbers**. Some mathematicians (set theorists, in particular) include 0 in $\mathbb{N}$, but this will not be our convention. The given statement is not true if we replace $\mathbb{N}$ with $\mathbb{Z}$. Do you see why?