Problem Collection for Introduction to Mathematical Reasoning

By Dana C. Ernst and Nándor Sieben
Northern Arizona University

Problem 1. Three strangers meet at a taxi stand and decide to share a cab to cut down the cost. Each has a different destination but all are heading in more-or-less the same direction. Bob is traveling 10 miles, Sally is traveling 20 miles, and Mike is traveling 30 miles. If the taxi costs \$2 per mile, how much should each contribute to the total fare? What do you think is the most common answer to this question?

Problem 2. Multiply together the numbers of fingers on each hand of all the human beings in the world—approximately 7 billion in all. What is the approximate answer?

Problem 3. Imagine a hallway with 1000 doors numbered consecutively 1 through 1000. Suppose all of the doors are closed to start with. Then some dude with nothing better to do walks down the hallway and opens all of the doors. Because the dude is still bored, he decides to close every other door starting with door number 2. Then he walks down the hall and changes (i.e., if open, he closes it; if closed, he opens it) every third door starting with door 3. Then he walks down the hall and changes every fourth door starting with door 4. He continues this way, making a total of 1000 passes down the hallway, so that on the 1000th pass, he changes door 1000. At the end of this process, which doors are open and which doors are closed?

Problem 4. Suppose you have 6 toothpicks that are exactly the same length. Can you arrange the toothpicks so that 4 identical triangles are formed? Justify your answer.

Problem 5. I have 10 sticks in my bag. The length of each stick is an integer. No matter which 3 sticks I try to use, I cannot make a triangle out of those sticks. What is the minimum length of the longest stick?

Problem 6. Imagine you have 25 pebbles, each occupying one square on a 5 by 5 chess board. Tackle each of the following variations of a puzzle.

(a) Variation 1: Suppose that each pebble must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(b) Variation 2: Suppose that all but one pebble (your choice which one) must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(c) Variation 3: Consider Variation 1 again, but this time also allow diagonal moves to adjacent squares. If this is possible, describe a solution. If this is impossible, explain why.

Problem 7. Consider an $n \times n$ chess board and variation 1 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

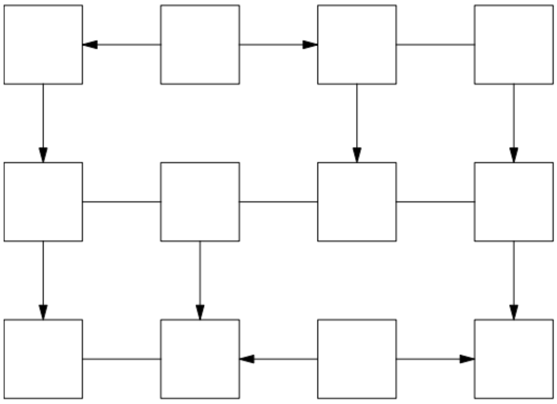
Problem 8. Consider an $n \times n$ chess board and variation 2 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 9. An ant is crawling along the edges of a unit cube. What is the maximum distance it can cover starting from a corner so that it does not cover any edge twice?

Problem 10. The grid below has 12 boxes and 15 edges connecting boxes. In each box, place one of the six integers from 1 to 6 such that the following conditions hold:

- For each possible pair of distinct numbers from 1 to 6, there is exactly one edge connecting two boxes with that pair of numbers.
- If an edge has an arrow, then it points from a box with a smaller number to a box with a larger number.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above.



Problem 11. In order to assess the reasoning skills of a newly developed android robot with artificial intelligence, the android's creator designs the following experiment. On Sunday, the creator describes the details of the experiment to the android and then turns the the android off. Once or twice, during the experiment, the android will be turned on, interviewed, and then turned back off. In addition, the creator will erase the awakening from the android's memory. On Sunday evening, a fair coin will be tossed to determine which experimental procedure to undertake:

- If the coin comes up heads, the android will be awakened and interviewed on Monday only.
- If the coin comes up tails, the android will be awakened and interviewed on both Monday and Tuesday.

In either case, the android will be awakened on Wednesday without interview and the experiment ends. Any time the android is awakened and interviewed, it will not be able to tell which day it is or whether it has been awakened before. During the interview the android is asked: "What is your credence now for the proposition that the coin landed heads?". One way to interpret "credence" in this context is the android's determination of the probability that the coin landed on heads.

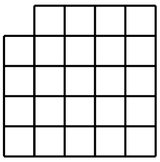
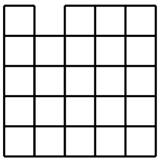
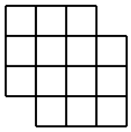
Problem 12. As a broke college student, you agree to take part in a recurring experiment. Each experiment begins on Sunday evening and ends on Wednesday morning. The experiment will be repeated 100 weeks in a row. You are told the details of the experiment in advance. Each Sunday evening, the experimenter describes the details of the experiment and then gives you a drug to put you to sleep. Once or twice, during the experiment, you will be awakened, interviewed, and then put back to sleep using a drug that includes an amnesia-inducing component that makes you forget the awakening. On Sunday evening, a fair coin will be tossed to determine which experimental procedure to undertake:

- If the coin comes up heads, you will be awakened and interviewed on Monday only.
- If the coin comes up tails, you will be awakened and interviewed on both Monday and Tuesday.

In either case, you will be awakened on Wednesday without interview and the experiment ends. Any time you are awakened and interviewed, you will not be able to tell which day it is or whether you have been awakened before. During the interview you will be asked: "Is the coin heads or tails?". You are required to respond with either "heads" or "tails". The experimenter will record whether you were correct or not, but you will not be told whether you guessed correctly. At the end of the 100th run of the experiment, you will be given $10 for each correct answer that you gave. What strategy should you employ in order to optimize your profit?

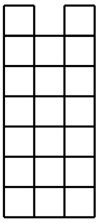
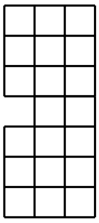
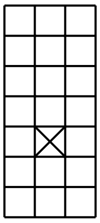
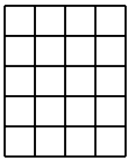
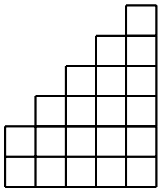
Problem 13. Four red ants and two black ants are walking along the edge of a one meter stick. The four red ants, called Albert, Bart, Debbie, and Edith, are all walking from left to right, and the two black ants, Cindy and Fred, are walking from right to left. The ants always walk at exactly one centimeter per second. Whenever they bump into another ant, they immediately turn around and walk in the other direction. And whenever they get to the end of a stick, they fall off. Albert starts at the left hand end of the stick, while Bart starts 20.2 cm from the left, Debbie is at 38.7cm, Edith is at 64.9cm and Fred is at 81.8cm. Cindy's position is not known—all we know is that he starts somewhere between Bart and Debbie. Which ant is the last to fall off the stick? And how long will it be before he or she does fall off?

Problem 14. Tile the following grids with dominoes. If a tiling is not possible, explain way.

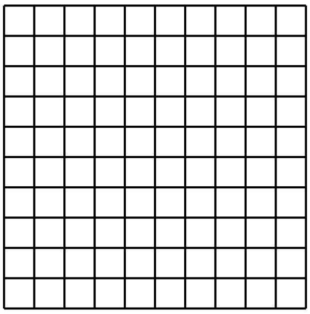
(a)  (b)  (c) 

Problem 15. Find all tetrominoes (polymonio with 4 cells).

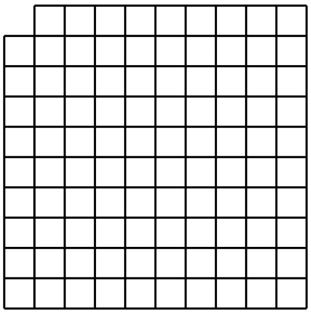
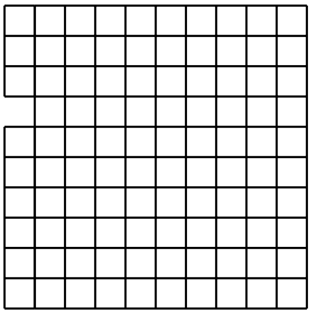
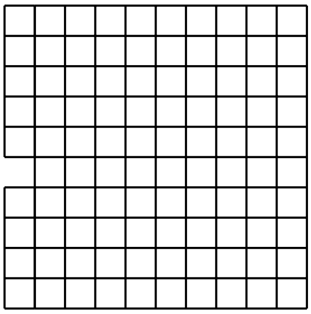
Problem 16. Tile the following grids using every tetromino exactly once. The X in (c) denotes an absence of an available square in the grid. If a tiling is not possible, explain way.

(a)  (b)  (c)  (d)  (e) 

Problem 17. Consider the 10×10 grid of squares below. Show that you can color the squares of the grid with 3 colors so that every consecutive row of 3 squares and every consecutive column of 3 squares uses all 3 colors.



Problem 18. Tile each of the grids below with trominoes that consist of 3 squares in a line. If a tiling is not possible, explain way.

(a)  (b)  (c) 

Problem 19. Take 15 poker chips or coins, divide into any number of piles with any number of chips in each pile. Arrange piles in adjacent columns. Take the top chip off every column and make a new column to the left. Repeat forever. What happens? Make conjectures about what happens when we change the number of chips.

Problem 20. The nth *triangular number* is defined via $t_n := 1+2+\cdots+n$. For example, $t_4 = 1+2+3+4 = 10$. Find a visual proof of the following fact. By "visual proof" we mean a sufficiently general picture that is convincing enough to justify the claim.

For all $n \in \mathbb{N}$, $t_n = \dfrac{n(n+1)}{2}$.

Problem 21. Let t_n denote the nth triangular number. Find both an algebraic proof and a visual proof of the following fact.

For all $n \in \mathbb{N}$, $t_n + t_{n+1} = (n+1)^2$.

Problem 22. Find a visual proof of the following fact. *Warning:* This problem is not about triangular numbers.

For $n \in \mathbb{N}$, $1+3+5+\cdots+(2n-1) = n^2$.

Problem 23. Suppose you randomly cut a stick into 3 pieces. What is the probability that you can form a triangle out of these 3 pieces?

Problem 24. Suppose you randomly pick 3 distinct points on a circle. What is the probability that the center of the circle lies in the interior of the triangle formed by these 3 points?

Problem 25. There is a plate of 40 cookies. You and your friend are going to take turns taking either 1 or 2 cookies from the plate. However, it is a faux pas to take the last cookie, so you want to make sure that you do not take the last cookie. How can you guarantee that you will never be the one taking the last cookie? What about n cookies?

Problem 26. The Sylver Coinage Game is a game in which 2 players alternately name positive integers that are not the sum of nonnegative multiples of previously named integers. The person who names 1 is the loser! Here is a sample game between A and B:

1. A opens with 5. Now neither player can name $5, 10, 15, \ldots$
2. B names 4. Now neither player can name 4, 5, 8, 9, 10, or any number greater than 11.
3. A names 11. Now the only remaining numbers are 1, 2, 3, 6, and 7.
4. B names 6. Now the only remaining numbers are 1, 2, 3, and 7.
5. A names 7. Now the only remaining numbers are 1, 2, and 3.
6. B names 2. Now the only remaining numbers are 1 and 3.
7. A names 3, leaving only 1.
8. B is forced to name 1 and loses.

If player A names 3, can you find a strategy that guarantees that the second player wins? If so, describe the strategy? If such a strategy is not possible, then explain why?

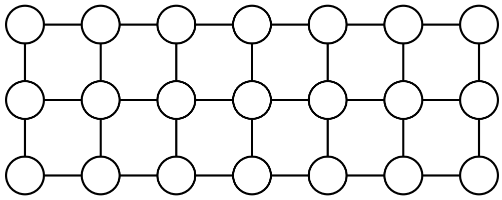
Problem 27. Suppose someone draws 20 distinct random lines in the plane. What is the maximum number of intersections of these lines?

Problem 28. A mouse eats her way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the 27 $1 \times 1 \times 1$ subcubes. If she starts at one corner and always moves to an uneaten subcube by passing through a face of a subcube, can she finish at the center of the cube?

Problem 29. An overfull prison has decided to terminate some prisoners. The jailer comes up with a game for selecting who gets terminated. Here is his scheme. 10 prisoners are to be lined up all facing the same direction. On the back of each prisoner's head, the jailer places either a black or a red dot. Each prisoner can only see the color of the dot for all of the prisoners in front of them and the prisoners do not know how many of each color there are. The jailer may use all black dots, or perhaps he uses 3 red and 7 black, but the prisoners do not know. The jailer tells the prisoners that if a prisoner can guess the color of the dot on the back of their head, they will live, but if they guess incorrectly, they will be terminated. The jailer will call on them in order starting at the back of the line. Before lining up the prisoners and placing the dots, the jailer allows the prisoners 5 minutes to come up with a plan that will maximize their survival. What plan can the prisoners devise that will maximize the number of prisoners that survive? Some more info: each prisoner can hear the answer of the prisoner behind them and they will know whether the prisoner behind them has lived or died. Also, each prisoner can only respond with the word "black" or "red." What if there are n prisoners?

Problem 30. Four prisoners are making plans to escape from jail. Their current plan requires them to cross a narrow bridge in the dark that has no handrail. In order to successfully cross the bridge, they must use a flashlight. However, they only have a single flashlight. To complicate matters, at most two people can be on the bridge at the same time. So, they will need to make multiple trips across the bridge, returning the flashlight back to the first side of the bridge by having someone walk it back. Unfortunately, they can't throw the flashlight. It takes 1, 2, 5, and 10 minutes for prisoner A, prisoner B, prisoner C, and prisoner D to cross the bridge and when two prisoners are walking together with the flashlight, it takes the time of the slower prisoner. What is the minimum total amount of time it takes all four prisoners to get across the bridge?

Problem 31. In the lattice below, we color 11 vertices points black. Prove that no matter which 11 are colored black, we always have a rectangle with black vertices (and vertical and horizontal sides).



Problem 32. Each point of the plane is colored red or blue. Show that there is a rectangle whose vertices are all the same color.

Problem 33. A certain fast-food chain sells a product called "nuggets" in boxes of $6, 9$, and 20. A number n is called *nuggetable* if one can buy exactly n nuggets by buying some number of boxes. For example, 21 is nuggetable since you can buy two boxes of six and one box of nine to get 21. Here are the first few nuggetable numbers:

$$6, 9, 12, 15, 18, 20, 21, 24, 26, 27, \ldots$$

and here are the first few non-nuggetable numbers:

$$1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots$$

What is the largest non-nuggetable number?

Problem 34. Our space ship is at a Star Base with coordinates $(1, 2)$. Our hyper drive allows us to jump from coordinates (a, b) to either coordinates $(a, a + b)$ or to coordinates $(a + b, b)$. How can we reach the impending enemy attack at coordinates $(8, 13)$?

Problem 35. Consider our Star Base from the previous problem. Recall that our hyper drive allows us to jump from coordinates (a, b) to either coordinates $(a, a + b)$ or to coordinates $(a + b, b)$. If we start at $(1, 0)$, which points in the plane can we get to by using our hyper drive? Justify your answer.

Problem 36. You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel. Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.
- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.

Your objective is to identify a single real coin that Zoltar has not taken as tribute.

Problem 37. Welcome to Circle-Dot[1]. We'll approach Circle-Dot as a game, where the object of the game is to construct a word made entirely of ∘'s and •'s. Circle-Dot begins with two words; called axioms. Using the two axioms and three rules of inference, we can create new Circle-Dot words, which are theorems in the Circle-Dot System. The process of creating Circle-Dot words using the axioms and rules of inference are proofs in the system.

On each of your "turns" in the game you can apply one of the 5 available axioms or rules to your current list of constructed Circle-Dot words to produce a new word. Also, once you have produced a new word, you can use this theorem in future "games."

Below are the axioms for Circle-Dot. Note that ∘ and • are valid symbols in the system while w and v are variables that stand for any sequence of ∘'s and •'s.

Axiom A. ∘•

Axiom B. •∘

At any time in your proof, you may quote an axiom. Below are the rules for generating new statements from known statements.

Rule 1. Given wv and vw, conclude w

Rule 2. Given w and v, conclude $w \bullet v$

Rule 3. Given $wv\bullet$, conclude $w\circ$

As an example, let's try to prove the following theorem.

Theorem C. • (just a single dot)

At the moment, the only tools we have for getting started are the axioms. As we prove theorems, we'll be able to incorporate them into our proofs, as well. To get started, let's apply Axiom A and see what that gets us. Applying Axiom A, we get ∘•. Looking at Rules 2 and 3, it should be moderately clear that they won't help us get a single dot. So, perhaps Rule 1 will be useful, but to use it, we see that we need to have wv and vw. Applying Axiom B, we get •∘. Now, if we let $w = \bullet$ and $v = \circ$, then $wv = \bullet\circ$ and $vw = \circ\bullet$. Applying Rule 1, we can conclude that • holds. Putting this altogether, we can write something like the following.

Proof of Theorem C.

1. ∘• by Axiom A
2. •∘ by Axiom B
3. • by Rule 1 (using lines 2 and 1)

Now, try proving the following theorems.

Theorem D. ∘

Theorem E. • • •

Theorem F. • • ∘

Theorem G. • ∘ ∘

Theorem H. ∘ • •∘

Theorem I. ∘ ∘ ∘∘

Theorem J. • ∘ •

Theorem K. • ∘ ∘∘

Make a conjecture about which sequences of ∘'s and •'s are theorems in the Circle-Dot system.

[1] The Circle-Dot System was developed by Ken Monks from the University of Scranton.

Problem 38. How many ways can 110 be written as the sum of 14 different positive integers? *Hint:* First, figure out what the largest possible integer could be in the sum. Note that the largest integer in the sum will be maximized when the other 13 numbers are as small as possible. Finish off the problem by doing an analysis of cases.

Problem 39. Let t_n denote the nth triangular number. Find an algebraic and a visual proof of the following fact.

For all $a, b \in \mathbb{N}$, $t_{ab} = t_a t_b + t_{a-1} t_{b-1}$.

Problem 40. We have two strings of pyrotechnic fuse. The strings do not look homogeneous in thickness but both of them have a label saying 4 minutes. So we can assume that it takes 4 minutes to burn through either of these fuses. How can we measure a one minute interval?

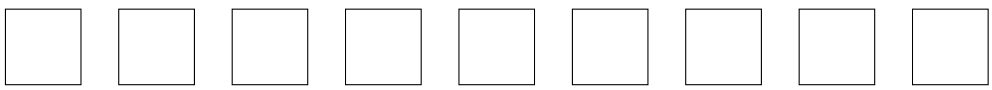
Problem 41. In the game Turnaround, you are given a permutation of the numbers from 1 to n. Your goal is to get them in the natural order $12\cdots n$. At each stage, your only option is to reverse the order of the first k places (you get to pick k at each stage). For instance, given 6375142, you could reverse the first four to get 5736142 and then reverse the first six to get 4163752. Solve the following sequence in as few moves as possible: 352614.

Problem 42. A signed permutation of the numbers 1 through n is a fixed arrangement of the numbers 1 through n, where each number can be either be positive or negative. For example, $(-2, 1, -4, 5, 3)$ is a signed permutation of the numbers 1 through 5. In this case, think of positive numbers as being right-side-up and negative numbers as being upside-down. A *reversal* of a signed permutation is the act of performing a 180-degree rotation to some consecutive subsequence of the permutation. That is, a reversal swaps the order of a subsequence of numbers while changing the sign of each number in the subsequence. Performing a reversal to a signed permutation results in a new signed permutation. For example, if we perform a reversal on the second, third, and fourth entries in $(-2, 1, -4, 5, 3)$, we obtain $(-2, -5, 4, -1, 3)$. The *reversal distance* of a signed permutation of 1 through n is the minimum number of reversals required to transform the given signed permutation into $(1, 2, \ldots, n)$. It turns out that the reversal distance of $(3, 1, 6, 5, -2, 4)$ is 5. Find a sequence of 5 reversals that transforms $(3, 1, 6, 5, -2, 4)$ into $(1, 2, 3, 4, 5, 6)$.

Problem 43. Consider a tournament with 15 teams. If every team plays every other team, how many games were played?

Problem 44. There are 8 frogs and 9 rocks on a field. The 9 rocks are laid out in a horizontal line. The 8 frogs are evenly divided into 4 green frogs and 4 brown frogs. The green frogs sit on the first 4 rocks facing right and the brown frogs sit on the last 4 rocks facing left. The fifth rock is vacant for now. Switch the places of the green and brown frogs by using the following rules:

- A frog can only jump forward
- A frog can hop to an vacant rock one place ahead
- A frog can leap over its neighbor frog to a vacant rock two places ahead



Can we generalize this problem and find how many jumps are necessary to switch n green and n brown frogs?

Problem 45. Consider a 4×4 grid with light-up squares. In the starting configuration, some subset of the squares are lit up. At each stage, a square lights up if at least two of its immediate neighbors (horizontal or vertical) were "on" during the previous stage. It's easy to see that for the starting configuration in which four squares along a diagonal of the board are lit up, the entire board is eventually covered by "on" squares. Several other simple starting configurations with four "on" squares also result in the entire board being covered. Is it possible for a starting configuration with fewer than four squares to cover the entire board? If yes, find it; if no, give a proof.

Problem 46. Consider the scenario of the previous problem, except this time suppose we have an 8×8 grid. Is it possible for a starting configuration with fewer than eight squares to cover the entire board? If yes, find it; if no, give a proof. Can you generalize to the $n \times n$ case?

Problem 47. In the game Light Up, two players alternately choose unlit squares from an $m \times n$ grid of light-up squares. The objective of the game is to be the first to light up the entire grid. At the beginning of the game, all squares are turned off. On each player's turn, the player selects any square that is currently off and then the selected square gets lit up. Moreover, additional squares get lit up if at least two of its immediate neighbors (horizontal or vertical) are lit up. This process continues until no new squares are lit up and then it is the next player's turn. The loser of the game is the player that no longer has an available square to light up. Determine which player has a winning strategy for the following grid sizes: 1×3, 1×4, 1×5, 2×2, 2×3, 3×3.