Section 4.5: Summary of Curve Sketching

Goal

There are two goals of this section:

1. It will be important for us to have intuition about the overall shape of a graph and how that shape is related to the first and second derivatives.
2. Spending time sketching graphs is an excellent way to synthesize all of the relationships between a functions, its derivatives, infinite limits, and its overall shape.

Introduction

In general, we will be given a function and asked to sketch its graph. To do this, we will have to identify some key features of the graph (see guidelines below).

Guidelines for Sketching Graphs of Functions

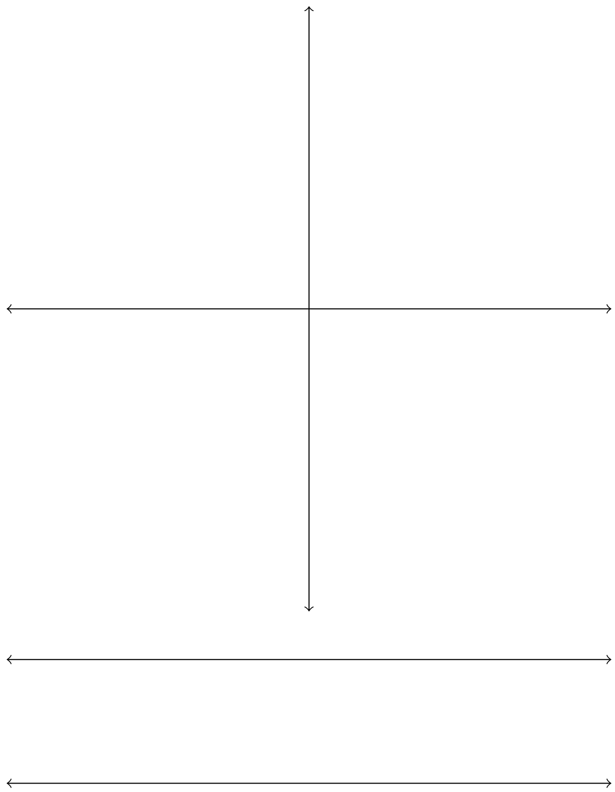
The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function.

1. Consider domain.
2. Determine whether there is symmetry about the y-axis or the origin.
3. Find x-intercepts and y-intercepts. (Finding x-intercepts is not always easy and an attempt to find them should be abandoned if too difficult.)
4. Identify vertical asymptotes.
5. Determine end behavior by computing limits of $f(x)$ as $x \to \infty$ and $x \to -\infty$ (Does graph have any horizontal or curvilinear asymptotes?).
6. Find critical numbers, determine intervals of increase and decrease, and identify any relative extrema. Plot the points corresponding to the critical numbers (to find y-values, plug the corresponding x-value into the original function).
7. Find x-values where $f''(x) = 0$ or is undefined, determine intervals of concavity, and identify any inflection points. Again, plot the corresponding points (to find y-values, plug the corresponding x-value into the original function).

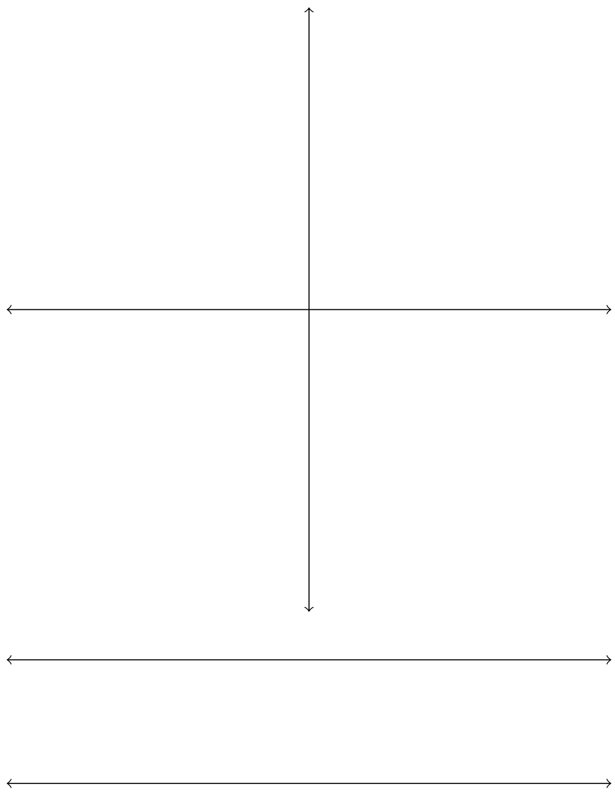
Let's start by sketching the graphs of some rational functions.

Example 1. Sketch the graph of the following functions.

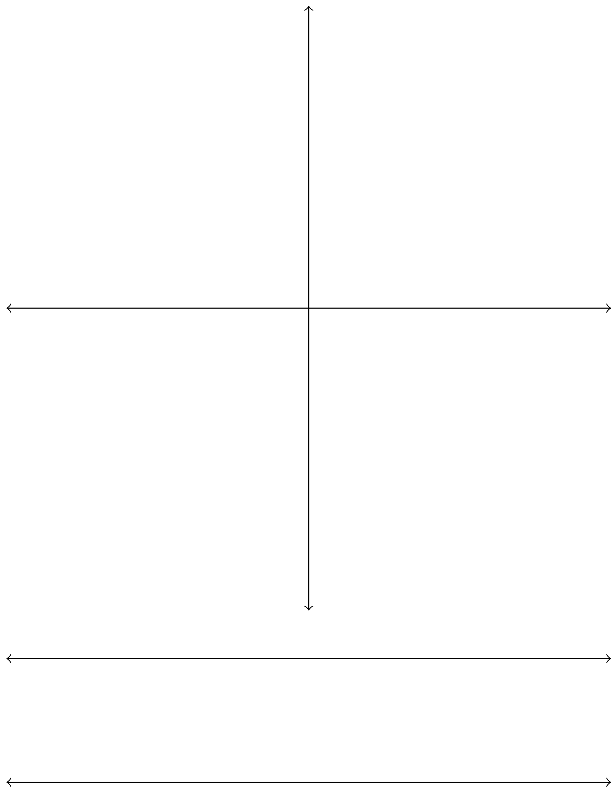
(a) $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$



(b) $g(x) = \dfrac{-x}{(x^2-1)^2}$



(c) $h(x) = x^{5/3} - 5x^{2/3}$



Curvilinear Asymptotes

Question 2. If $f(x) = p(x)/q(x)$ is a rational function such that $p(x)$ and $q(x)$ have no factors in common (i.e., the "fraction" is reduced), then when will $f(x)$ have a horizontal asymptote? When will it not?

When the degree of the numerator is ________ than the degree of the denominator, other kinds of asymptotes are possible: *curvilinear* (sometimes called *slant* or *oblique* if degree is 1). To see what these new kinds of asymptotes are, we use polynomial long division.

Theorem 3. A rational function cannot have both a horizontal asymptote and a curvilinear (including slant) asymptote. Why?

Example 4. Identify the curvilinear asymptote of $g(x) = \dfrac{x^3}{x^2+1}$ and sketch its graph.