Section 1.3: Quantifiers (part 2)

Goal

We will continue our discussion of quantifiers.

More examples

Example 1. Translate each informal sentence into a symbolic sentence for the given universe.

(a) "For every odd prime x less than 10, $x^2 + 4$ is prime." ($U = \mathbb{N}$)

(b) "Some functions defined at 0 are not continuous at 0." ($U =$ all functions)

(c) "Some real numbers have a multiplicative inverse." ($U = \mathbb{R}$)

(d) "Some integers are even and some are odd." ($U = \mathbb{Z}$)

(e) "For every natural number, there is a real number greater than that natural number." ($U = \mathbb{R}$)

More on quantifiers

Definition 2. Two quantified sentences are *equivalent in a given universe* iff they have the same ____________ in that universe. Two quantified sentences are *equivalent* iff they are equivalent in _______ universe.

Example 3. Let $P(x) : x > 0$ and $Q(x) : |x| > 0$. Determine whether $(\forall x)P(x)$ and $(\forall x)Q(x)$ are equivalent in each of the following universes.

(a) $U = \mathbb{N}$

(b) $U = \{-3, -2, 1, 2, 7\}$.

We can conclude $(\forall x)P(x)$ and $(\forall x)Q(x)$ are ______________.

Theorem 4. *If $A(x)$ is an open sentence (with variable x), then*

(a) $\sim (\forall x)A(x) \equiv (\exists x)[\sim A(x)]$

(b) $\sim (\exists x)A(x) \equiv (\forall x)[\sim A(x)]$

Proof.

(a)

(b) See Exercise 4 for homework.

□

Example 5. Consider: "All primes are odd." Translate to a symbolic sentence, negate, and then translate back into an English sentence.

The unique existential quantifier

Definition 6. For an open sentence $P(x)$, the proposition $(\exists! x)P(x)$ is read "There exists a unique x such that $P(x)$" and is true iff the truth set of $P(x)$ has ____________________. $\exists!$ is called the *unique existence quantifier.*

Example 7. Consider: "There is a unique even prime." Translate into a symbolic sentence.

Theorem 8. *If $A(x)$ is an open sentence, then*

(a) $(\exists! x)A(x)$ *implies* $(\exists x)A(x)$*;*

(b) $(\exists! x)A(x)$ *is equivalent to* $(\exists x)A(x) \wedge (\forall y)(\forall z)[(A(y) \wedge A(z)) \implies y = z]$.

Proof. See Exercise 8(a) and 8(c). $\square$

Introduction to stacked quantifiers

We will spend more time discussing stacked quantifiers later, but it is worth thinking about now. What do I mean by stacked quantifiers? Here is an example.

Example 9. Let $P(x, y)$ be an open sentence in the variables x and y. How do the following propositions differ?

(a) $(\exists x)(\forall y)P(x, y)$

(b) $(\forall x)(\exists y)P(x, y)$