WORKSHEET 9 **MATH 1300** **March 13, 2008**

Goal: To better understand the non-vertical asymptotes of rational function; and, in particular to explore the role of long division in understanding those asymptotes when they are not horizontal.

In this worksheet we examine rational functions, that is functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. Recall that the degree of a polynomial is the highest power of x appearing in the polynomial, e.g, the degree of the polynomial $P(x) = 2x^4 - 3x + 2$ is 4 while the degree of $Q(x) = x - 3$ is 1.

In this worksheet we will discover that the non-vertical asymptotes of a rational function will depend, in part, on the relationship between the degree of its numerator and the degree of its denominator.

IMPORTANT ASSUMPTION: Henceforth we will only consider rational functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$, where the fraction $\dfrac{P(x)}{Q(x)}$ cannot be simplified, in other words where the polynomials $P(x)$ and $Q(x)$ do not have a common factor.

Preliminary question 1. If you are presented with a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ which can be simplified, why do you think you should simplify it before you try to understand it's graph?

Preliminary question 2. What, in your own words, is an asymptote? (After you complete each of the problems below compare what you have discovered with your answer to this question.)

Case 1. $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of $Q(x)$ is greater than the degree of $P(x)$.

An example. Let $f(x) = \dfrac{-x^2 - x + 7}{2x^4 - 3x + 2}$

(a) Evaluate each of the limits: $\lim\limits_{x \to \infty} \dfrac{-x^2 - x + 7}{2x^4 - 3x + 2}$ and $\lim\limits_{x \to -\infty} \dfrac{-x^2 - x + 7}{2x^4 - 3x + 2}$.

(b) What do each of the limits in part (a) tell you about the non-vertical asymptotes of the function

$$f(x) = \frac{-x^2 - x + 7}{2x^4 - 3x + 2}?$$

Conclusion. Do you think you will obtain *exactly the same* horizontal asymptote for any rational function $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of $Q(x)$ is greater than the degree of $P(x)$. (Explain your answer.)

Case 2. $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of $Q(x)$ equals the degree of $P(x)$.

An example. Let $f(x) = \dfrac{-x^3 - x + 7}{2x^3 - 3x + 2}$

(a) Evaluate each of the limits: $\displaystyle\lim_{x\to\infty} \frac{-x^3 - x + 7}{2x^3 - 3x + 2}$ and $\displaystyle\lim_{x\to-\infty} \frac{-x^3 - x + 7}{2x^3 - 3x + 2}$.

(b) What do each of the limits in part (a) tell you about the non-vertical asymptotes of the function

$$f(x) = \frac{-x^3 - x + 7}{2x^3 - 3x + 2}?$$

Conclusion. Do you think a rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where the degree of $Q(x)$ equals the degree of $P(x)$, will always have a horizontal asymptote? (Explain your answer.)

Case 3. $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of $Q(x)$ is 1 less than the degree of $P(x)$.

An example. Let $f(x) = \dfrac{x^3 - x}{2x^2 + 2}$

(a) Evaluate each of the limits: $\displaystyle\lim_{x\to\infty} \frac{x^3 - x}{2x^2 + 2}$ and $\displaystyle\lim_{x\to-\infty} \frac{x^3 - x}{2x^2 + 2}$.

(b) What, if anything, do each of the limits in part (a) tell you about possible non-vertical asymptotes of the function

$$f(x) = \frac{x^3 - x}{2x^2 + 2}?$$

How to discover the asymptotes in the above problem. If we divide the numerator by the denominator of $f(x) = \dfrac{x^3 - x}{2x^2 + 2}$ we discover that

$$f(x) = \frac{x^3 - x}{2x^2 + 2} = \frac{1}{2}x + \frac{-2x}{2x^2 + 2}.$$

(c) Calculate each of the limits: $\displaystyle\lim_{x\to\infty} \frac{-2x}{2x^2 + 2}$ and $\displaystyle\lim_{x\to-\infty} \frac{-2x}{2x^3 + 2}$.

(d) Can you explain how your results in (c) allow you to conclude that for x with $|x|$ *very large*

$$f(x) = \frac{x^3 - x}{2x^2 + 2} \approx \frac{1}{2}x.$$

(e) Evaluate the limit

$$\lim_{x \to \infty} \frac{\frac{x^3 - x}{2x^2+2}}{\frac{1}{2}x},$$

and conclude that the approximation becomes better and better as $|x|$ becomes larger and larger. (In other words, the graph of $f(x) = \dfrac{x^3 - x}{2x^2 + 2}$ becomes closer and closer to the graph of $y = \dfrac{1}{2}x$ as $|x|$ becomes larger and larger.)

Conclusion. Do you think a rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where the degree of $Q(x)$ is 1 less than the degree of $P(x)$, will always have a line as an asymptote? (Explain your answer.)

Final question. In tonight's homework you will discover that a rational function can have a *curved* asymptote. How will this discovery force you to reconsider your original definition of an asymptote above?