Chapter 7: Products and quotients

Dana C. Ernst

Plymouth State University
Department of Mathematics
http://oz.plymouth.edu/~dcernst

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In the previous chapter, we looked inside groups for smaller groups lurking inside. Exploring the subgroups of a group gives us insight into the group's internal structure.

There are two main topics that we will discuss in this chapter.

1. direct products: this process will provide us with a method for making larger groups from smaller groups.
2. quotients: this process will provide us with a method for making smaller groups from larger groups.

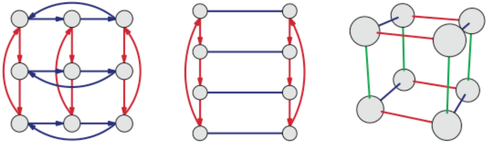
Let me mention straight away before we even describe these processes that we can *always* form a direct product of two groups. However, we cannot always take the quotient of two groups. In fact, quotients are restricted to some pretty specific circumstances as we shall see.

Notes

Every group whose name contains the symbol $\times$ can be constructed using a process called the direct product.

However, you shouldn't be fooled into thinking that the absence of this symbol means that there isn't some hidden product. As an example, it turns out that V_4 is really just $C_2 \times C_2$.

Here are some examples (take from Figure 7.1 on page 118 of *VGT*).



$C_3 \times C_3$ $\quad$ $C_2 \times C_4$ $\quad$ $C_2 \times C_2 \times C_2$

Do you notice anything about the orders of the product groups above?

Notes

Our observation on the previous slide that the order of the direct product is equal to the product of the orders of the smaller groups is true in general. That is, $|A \times B| = |A| \cdot |B|$ for (finite) groups A and B.

But what is $A \times B$?

We will first describe the direct product construction as a process for making a new Cayley diagram from two given Cayley diagrams. Then we will uncover some properties of the corresponding group.

Notes

Definition 7.1

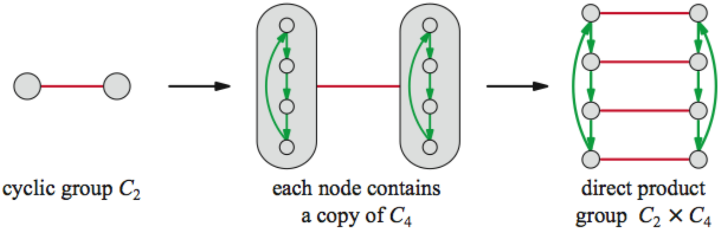
To create a Cayley diagram of $A \times B$ from Cayley diagrams of A and B, proceed as follows.

1. Begin with the Cayley diagram for A.
2. Inflate each node in the Cayley diagram of A and place in it a copy of the Cayley diagram for B. (Make sure you are using different colors for the two different Cayley diagrams.)
3. Remove the (inflated) nodes of A while using the arrows of A to connect corresponding nodes from each copy of B. That is, remove the A diagram but treat its arrows as a blueprint for how to connect corresponding nodes in the copies of B.

It'll certainly be in our best interest to work through a couple of examples.

Notes

Consider the groups C_2 and C_4. Here is an illustration (taken from Figure 7.2 on page 119 of *VGT*) that shows the construction of $C_2 \times C_4$ via Definition 7.1.



It takes quite a bit of skill to pick the right layout for the nodes to get "pretty" representations of the direct product. However, pretty or not, what really matters is the relationships among the nodes, not how they are laid out.

Notes

Let's do an example that is a bit more difficult. Create the Cayley diagram for $C_4 \times C_3$.

An important thing to consider here is whether the diagrams that we are ending up with are actually Cayley diagrams. If they aren't, then this process is stupid. Why are the resulting diagrams actually Cayley diagrams?

Definition

The group $A \times B$ whose Cayley diagram results from the procedure in Definition 7.1 is called the direct product of A and B.

We call A and B the factors of the product.

Notes

It turns out that $A \times B$ and $B \times A$ always have the same structure (Exercise 8.36 asks you to prove this). The only difference is that the Cayley diagram for one is the other "turned on its side." We say that the direct product operation is commutative.

Our construction of the Cayley diagram for $A \times B$ yielded a diagram with unlabeled nodes. How could we go about labeling the nodes?

In $A \times B$, every element is given a name of the form (a, b), where $a \in A$ and $b \in B$. In particular, a node in the Cayley diagram for $A \times B$ has 1st coordinate a if the node belonged to the inflated node a in the Cayley diagram for A. A node has 2nd coordinate b if the corresponding node inside the inflated A node was labeled b.

Let's see if we can label the nodes of our Cayley diagram for $C_4 \times C_3$.

Notes

Group work

Let's explore a few more examples.

1. In groups of 2–3, complete the following exercises (not collected):
   - Create a Cayley diagram with labeled nodes for $C_2 \times C_2$. What familiar group is this?
   - Exercise 7.4(a)
   - Exercise 7.7(a)
   - Exercise 7.2(a)
2. Let's discuss your solutions.

Notes

Recall the following definition that we mentioned at the end of the previous chapter.

Definition 7.2

A subgroup $H < G$ is called normal if each left coset of H is also a right coset of H (and vice versa). If H is normal in G, we write $H \lhd G$.

For a direct product $A \times B$ there is always at least two normal subgroups: $A \lhd A \times B$ and $B \lhd A \times B$. You will prove this in Exercise 7.12, but in the meantime, let's at least check that this is true in $C_4 \times C_3$.

Before we do this, I must point out that we are abusing notation here. Technically, A is not even a subset of $A \times B$. $A \times B$ consists of ordered pairs (a, b), whereas, A consists of singletons. We should write $A \times \{e\} \lhd A \times B$.

Notes

First, notice that the left cosets of $C_3 = \{(0,0),(0,1),(0,2)\}$ are easy to pick out. The only thing we need to check is that these coincide with the right cosets. Let's check: (I've already thought ahead of time what some good representatives might be.)

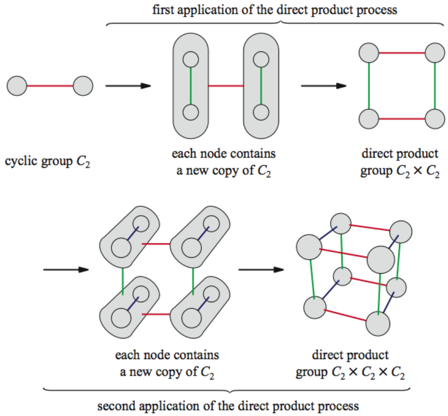
$$\{(0,0),(0,1),(0,2)\} \text{ (this is just the original)}$$
$$\{(0,0),(0,1),(0,2)\}(1,0) = \{(1,0),(1,1),(1,2)\}$$
$$\{(0,0),(0,1),(0,2)\}(2,0) = \{(2,0),(2,1),(2,2)\}$$
$$\{(0,0),(0,1),(0,2)\}(3,0) = \{(3,0),(3,1),(3,2)\}$$

As we can see, the left and right cosets agree. Therefore, the group in $C_4 \times C_3$ that "is" C_3 is normal.

Notes

We can form direct products with more than 2 groups.

If we wanted to form the Cayley diagram for $A \times B \times C$, we could first construct the diagram for $A \times B$ and then construct the diagram for $(A \times B) \times C$. Here is the construction of $C_2 \times C_2 \times C_2$ (taken from Figure 7.6 on page 122 of *VGT*).



Notes

In your group work, you learned that V_4 is isomorphic to $C_2 \times C_2$. Also, recall that V_4 is isomorphic to the 2-Light Switch Group. So, we can think of the 2-Light Switch Group as $C_2 \times C_2$ (this should be satisfying!).

One interesting observation is that for the 2 light switches, the action performed on one light switch has no impact on the other and vice versa. This phenomenon occurs in all direct products.

In a Cayley diagram for $A \times B$, following A arrows neither impacts or is impacted by the location in group B.

Notes

Imagine you are at some node (a, b) in the Cayley diagram for $A \times B$. Then we are standing at a node that was at one step in the process contained in an inflated node for A.

Following a B arrow amounts to moving to another node in $A \times B$ that was also contained in the same inflated node of A. This will only change the B coordinate of (a, b).

On the other hand, following an A arrow results in moving to another cluster of nodes that were contained in a different inflated node of A. This will only change the A coordinate of (a, b).

Notes

The moral of the story is that the direct product of two groups joins the groups, so that that act independently of each other.

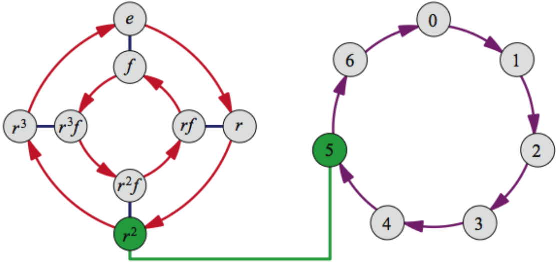
One of the benefits of this revelation is that instead of forming large and complicated Cayley diagrams for $A \times B$, we can think of an action in $A \times B$ as simply instructions for where to go in the Cayley diagram for A and where to go in the Cayley diagram for B.

Here's how I think of the direct product of two cyclic groups, say $C_n \times C_m$: Imagine a slot machine with two wheels, one with n spaces (numbered 0 through $n-1$) and the other with m spaces (numbered 0 through $m-1$).

The actions are: spin one or both of the wheels. Each action can be labeled by where we end up on the first wheel and where we end up on the second wheel: say (i, j).

Notes

Here is an example of a visual for more general direct products (taken from Figure 7.11 on page 125 of *VGT*) showing the element $(r^2, 5)$ in $D_4 \times C_7$.



Notes

One hugely important consequence of the independence of the factors in a direct product is that it tells us that the binary operation in $A \times B$ is simply done coordinate-wise.

Suppose that $(a, b), (c, d) \in A \times B$. Then

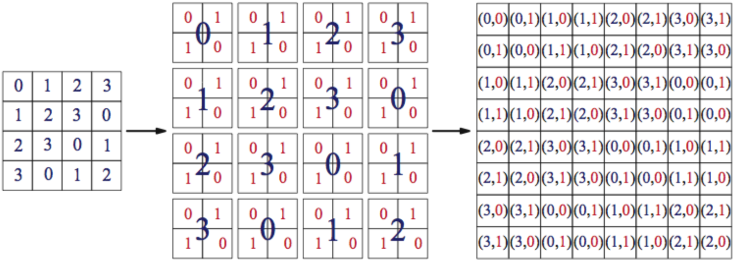
$$(a, b) * (c, d) = (ac, bd).$$

It is important to point out that our construction of $A \times B$ along with our method for labeling the nodes respects this binary operation.

As an example, in $D_3 \times C_4$, $(r^2, 1) * (fr, 3) = (fr^2, 0)$.

Notes

Direct products can also be visualized using group tables. Definition 7.3 on page 126 gives a detailed list of instructions for creating a multiplication table for $A \times B$ from the multiplication tables for A and B. However, I think you'll understand the general process after discussing this example (taken from Figure 7.12 on page 126 of *VGT*) of $C_4 \times C_2$.



0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	(3,1)
(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)	(3,1)	(3,0)
(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	(3,1)	(0,0)	(0,1)
(1,1)	(1,0)	(2,1)	(2,0)	(3,1)	(3,0)	(0,1)	(0,0)
(2,0)	(2,1)	(3,0)	(3,1)	(0,0)	(0,1)	(1,0)	(1,1)
(2,1)	(2,0)	(3,1)	(3,0)	(0,1)	(0,0)	(1,1)	(1,0)
(3,0)	(3,1)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
(3,1)	(3,0)	(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)

multiplication table for the group C_4

inflate each cell to also contain a copy of the multiplication table for C_2

join the little tables and element names to form the final direct product table

Notes

More group work

In groups of 2–3, complete all parts of Exercise 7.3. I want each group to turn in a complete solutions.

Let's discuss your solutions.

Notes

Quotients

We saw how we can use direct products to form larger groups from smaller groups. Now, we discuss the opposite procedure, which is called taking a quotient.

As we did with direct products, we will first describe the quotient operation using Cayley diagrams and then we will explore some properties of the resulting group.

Notes

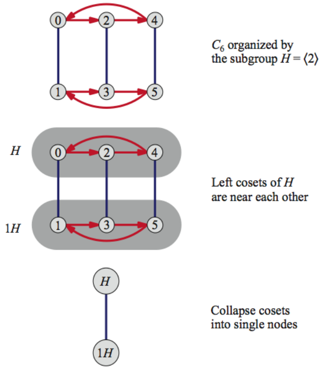
Definition 7.5

To attempt to divide a group G by one of its subgroups H, follow these steps.

1. Organize a Cayley diagram of G by H (so that we can "see" the subgroup H in the diagram for G).
2. Collapse each left coset of H into one large node. Unite those arrows that now have the same start and end nodes. This forms a new diagram with fewer nodes and arrows.
3. IF the resulting diagram is a Cayley diagram of a group, then you have obtained the quotient group of G by H, denoted G/H and often read "G mod H." If not, then G cannot be divided by H.

Notes

Here is a picture (taken from Figure 7.20 on page 133 of *VGT*) that illustrates the process of Definition 7.5 for the group $G = C_6$ and its subgroup $H = \langle 2 \rangle$.



(Labeling of nodes 1, 3, 5 is wrong.)

In this example, the resulting diagram *is* a Cayley diagram. So, we can divide C_6 by $\langle 2 \rangle$. In fact, we see that $C_6/\langle 2 \rangle$ is isomorphic to C_2.

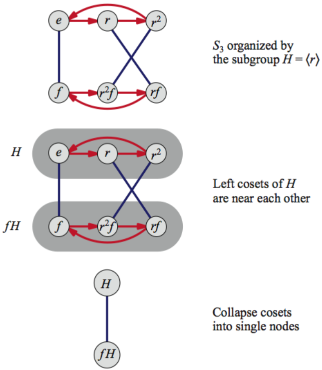
Notes

Comments

- Step 3 of Definition 7.5 says "If the new diagram is a Cayley diagram ..." It is important to point out that sometimes it won't be, in which case there is no quotient.
- *Important*: The elements of the quotient G/H (if it exists) are the cosets of H. We focus our attention on the teams rather than the individual players.
- As one would expect, if $G = A \times B$ and we divide G by A, then the quotient group is B (it turns out that this always works; we'll see why shortly). However, the converse is not generally true. That is, if we can divide G by H, then that does not necessarily mean that G is equal to a direct product of H and the result of dividing G by H.

Notes

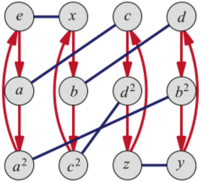
Let's take a look at another example. The following picture (taken from Figure 7.21 on page 134 of *VGT*) shows the result of dividing S_3 by $H = \langle r \rangle$.



The resulting diagram is a Cayley diagram. So, S_3/C_3 makes sense and is isomorphic to C_2. However, you can tell by the inconsistent wiring of nodes in the middle step that S_3 is not a direct product of C_3 and C_2.

Notes

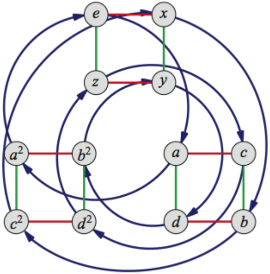
Here's another example. Consider the group A_4 and its subgroup $\langle x, z \rangle$. Recall that one possible Cayley diagram for A_4 (with generators a and x) was the following figure (taken from page 54 of *VGT*).



However, the subgroup $H = \langle x, z \rangle$ is not obvious from this diagram. It turns out that $H = \langle x, z \rangle$ is isomorphic to V_4.

Notes

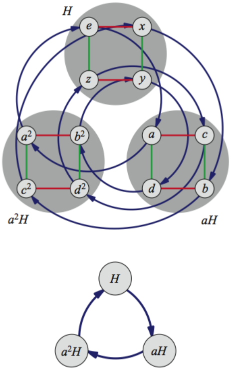
Here is a Cayley diagram for A_4 (with generators x, z, and a) organized by $H = \langle x, z \rangle$.



We can now see the left cosets of H clearly.

Notes

The following figure (taken from Figure 7.23 on page 136 of *VGT*) show the steps of Definition 7.5.



As we can see, the resulting diagram is a Cayley diagram. So, A_4/V_4 is isomorphic to C_3. However, A_4 is not isomorphic to $V_4 \times C_3$.

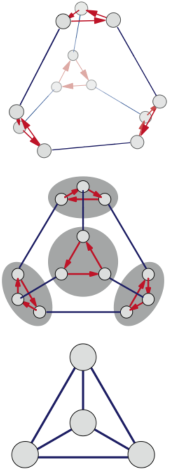
Notes

The last 3 examples may have tricked you into thinking that we can divide G by any H, but as we've already mentioned, we can't. OK, so what can go wrong?

Again, consider the group A_4. But this time, let's try to divide by its subgroup $H = \langle a \rangle$. In this case, H is a cyclic subgroup of order 3.

The figure on the next slide (taken from Figure 7.26 on page 138 of *VGT*) shows the result of trying to divide A_4 by $H = \langle a \rangle$.

Notes



OK, so what's wrong? This diagram is *not* a Cayley diagram. It violates Rule 1.7; there is ambiguity about which blue arrow to travel anytime we leave a node.

Notes

The big question is: when can we divide G by H and when can't we?

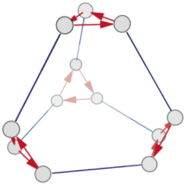
It turns out that the answer depends on whether H is normal or not.

This ought to take some convincing.

First, let's determine whether the subgroup in A_4 isomorphic to C_3 is normal or not.

Notes

Using the following Cayley diagram for A_4, the left cosets of $H = \langle a \rangle$ are easy to pick out.



Are the right cosets the same as the left cosets? The answer is no. For example, following blue arrows out of any single coset scatters the nodes.

So, $H = \langle a \rangle$ is *not* normal in A_4.

If we took the effort to check our first 3 examples, we would find out that in each case, the left cosets and right cosets coincide. So, in those examples, where G/H exists, H was normal.

Notes

However, these 4 examples do not constitute a proof; they only provide evidence that my claim is true.

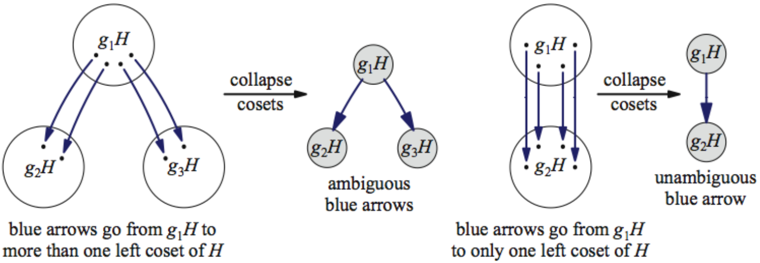
Let's see if we can gain some more insight. Consider a group G with subgroup H.

Recall that:

- each left coset gH is the set of nodes that H arrows can reach from g (which looks like a copy of H at g);
- each right coset Hg is the set of nodes to which the g arrows take the elements of H.

Notes

The following figure (taken from Figure 7.27 on page 139 of *VGT*) depicts the potential ambiguity that may arise when cosets are collapsed in the sense of Definition 7.5.



Note that the action of the blue arrows above is illustrating multiplication of a left coset on the *right* by some element. That is, the picture is showing us how left and right cosets interact.

Notes

When H is normal, $gH = Hg$ for all $g \in G$. In this case, To whichever coset one g arrow leads from H (the left coset), all g arrows lead unanimously and unambiguously (because it is also a right coset Hg).

Finally, let's state the answer to our original question to when we can take a quotient.

Theorem 7.6

If $H < G$, then a quotient group G/H can be constructed only when $H \lhd G$.

Notes

Proof. The quotient process of Definition 7.5 succeeds only when the resulting diagram is a valid Cayley diagram. Nearly all aspects of valid Cayley diagrams are guaranteed by the quotient process.

Because we begin with a diagram that has an arrow of every color exiting every node, our resulting diagram has this property, as well.

Since we begin with a regular diagram and we collapse identically structured sections distributed uniformly throughout the diagram, we end up with a regular diagram.

The only problem that can arise is ambiguity of arrow color at a given node. But we have already argued that this problem is avoided when H is normal. $\square$

Notes

Group work

Let's explore a few more examples.

1. In groups of 2–3, complete the following exercises (not collected):
    - Exercise 7.18(a)
    - Exercise 7.18(b)
2. Let's discuss your solutions.
3. Now, complete Exercise 7.18(f). I want each group to turn in a complete solution.

Notes

Some subgroups are normal and some are not. An interesting question is: if $H < G$ with H *not* normal, can we measure how far H is from being normal?

Recall that $H \lhd G$ provided that $gH = Hg$ for all $g \in G$. So, one way to answer the question above is to check how many of the $g \in G$ satisfy this requirement. Imagine that each $g \in G$ is voting as to whether H is normal.

At a minimum, we know that every $g \in H$ vote in favor of H being normal. Why? Well, since H is closed, if $g \in H$, we must have $gH = H = Hg$.

Notes

At a maximum, we would have *all* $g \in G$ voting in favor of H being normal, but this only happens when H really is normal.

There can be levels in between these 2 extremes as well.

Definition

The set of elements in G that vote in favor of H's normality is called the normalizer of H in G, denoted $N_G(H)$. That is,

$$N_G(H) = \{g \in G : gH = Hg\}.$$

Let's explore some possibilities for what the normalizer of a subgroup can be.

Notes

First, observe that if some $g \in G$ satisfies $gH = Hg$, then every element of the coset gH does, too. This follows from the fact that any member of gH (respectively, Hg) can be used as a representative of the coset.

So, $N_G(H)$ is made up of whole cosets of H. This implies that the size of $N_G(H)$ must be a multiple of $|H|$.

Furthermore, the deciding factor in how a left coset will vote is simply whether it is also a right coset (because gH votes as a block exactly when $gH = Hg$).

Notes

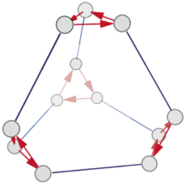
Since $N_G(H)$ is composed of left cosets of H that are also right cosets, we can describe the normalizer visually. The normalizer of H in G is made up of those copies of H that are connected to H by unanimous arrows.

We need some examples.

We saw earlier that the subgroup $H = \langle x, z \rangle$ is normal in A_4. So, $N_{A_4}(H) = A_4$.

Notes

At the other extreme, consider $\langle a \rangle$ in A_4 again.

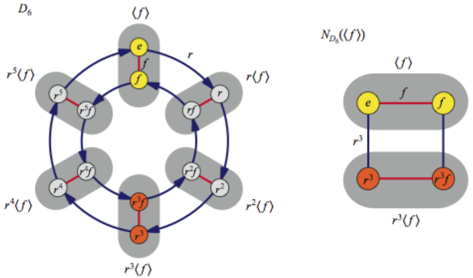


As we discussed earlier, this subgroup is *not* normal. In fact, it is as far from normal as it can possibly be.

We see that no right coset coincides with a left coset other than $\langle a \rangle$ itself. Thus, $N_{A_4}(\langle a \rangle) = \langle a \rangle$.

Notes

For our third example, consider D_6 and its subgroup $\langle f \rangle$. The following figure (taken from Figure 7.29 on page 142 of *VGT*) shows that $\langle f \rangle$ is not normal in D_6, but that its normalizer is something between $\langle f \rangle$ and D_6.



We see that $N_{D_6}(\langle f \rangle) = \{e, f, r^3, r^3 f\}$, which is isomorphic to V_4. What you should notice is that in this example, the normalizer is also a subgroup! It turns out that this is always true.

Notes

Theorem 7.7

For any $H < G$, $N_G(H) < G$, as well.

For a proof, see pages 141–142 of *VGT*.

Comments

- We have
$$H \lhd N_G(H) < G.$$
- The closer $N_G(H)$ is to being all of G, the closer H is to being normal.

Notes

More group work

Let's explore a few more examples.

1. In groups of 2–3, complete the following exercises (not collected):
    - Exercise 7.25(a)
    - Exercise 7.25(b)
2. Let's discuss your solutions.
3. Now, complete Exercises 7.26(a) and 7.26(b). I want each group to turn in a complete solution for both exercises.

Notes