Chapter 1: Logic
Sections 1.6–1.9

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Notes

1.6 Compound Statements and Truth Tables

Let's continue our discussion of how to construct more complex statements (called compound statements) from "smaller" statements.

Recall that if A and B are statements, then "If A, then B" is called an implication. This is an example of a compound statement.

Symbolically, we write implications as $A \implies B$, which may also be read "A implies B." Like we did with implications, we can summarize the truth and falsehood of other compound statements in truth tables.

Notes

Recall that if A and B are predicates (i.e., contain a free variable), then $A \implies B$ is true if all possible values of the free variable(s) make the truth values for A and B not occur in the 2nd row of the truth table.

Besides implications, there are other ways that statements can be combined. Here are the important ones:

- conjunction: "A and B" denoted $A \wedge B$
- disjunction: "A or B" (inclusive or) denoted $A \vee B$
- negation: "Not A" denoted $\sim A$
- equivalence: "A if and only if B denoted $A \iff B$ (we usually abbreviate "if and only if" as "iff")

Notes

We use truth tables to define each of these logical operators.

A	B	$A \implies B$	$A \wedge B$	$A \vee B$	$\sim A$	$A \iff B$
T	T	T	T	T	F	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

By the way, why are there 4 rows in the truth table above? What if we had a more complicated compound statement involving 3 different statements?

Notes

Obviously, we can form complicated statements using various combinations of the logical operators and any number of initial statements.

1.6.3 Example

Given statements A and B, let's study the truth tables of more complicated compound statements.

1. $B \wedge \sim B$

B	$\sim B$	$B \wedge \sim B$
T		
F		

Observations? A compound statement that is always false (regardless of the truth value of the simpler statements involved) is called a contradiction.

Notes

1.6.3 Example (continued)

2. $(A \wedge \sim B) \iff \sim(A \implies B)$

A	B	$A \wedge \sim B$	$A \implies B$	$\sim(A \implies B)$	$(A \wedge \sim B) \iff \sim(A \implies B)$
T	T				
T	F				
F	T				
F	F				

Obervations? A statement that is true (regardless of the truth value of the simpler statements involved) is called a tautology.

Notes

1.7 Learning from Truth Tables

What lessons can we learn from truth tables?

Lesson 1—Tautologies

Tautologies express logical relationships that hold in any context. How would you verify that a compound statement is a tautology?

Lesson 2—What About the Converse?

The converse of an implication $A \implies B$ is the implication $B \implies A$. How are implications and their converses related?

A	B	$A \implies B$	$B \implies A$
T	T		
T	F		
F	T		
F	F		

Notes

We see that truth of the statement "If A, then B" does not imply the truth of its converse.

Consider the statement "If your name is Bob, then your name starts with B." What is the converse of this statement? We can easily see that these two statements have different truth values depending on whether you are a Bob or not.

Certainly, there are situations with specific A and B, where an implication and its converse are equivalent, but this does not happen in general. When would an implication and its converse be equivalent anyway?

Notes

Lesson 3—Equivalence and Rephrasing

Consider the truth table for $A \iff B$:

A	B	$A \iff B$
T	T	T
T	F	F
F	T	F
F	F	T

Notice that $A \iff B$ is true exactly when A and B both have the same truth value.

If A and B are predicates and $A \iff B$ is true for all possible substitutions of variables (from the appropriate universe), then we say that A and B are equivalent.

Notes

How would we show that two compound statements are equivalent? Using truth tables, there are at least two possibilities. If "blah" and "junk" are the two compound statements in question, then "blah" and "junk" are equivalent if:

1. "blah" iff "junk" is a tautology.

or

2. the last columns of the their respective truth tables are the same.

Let's take a look at an example.

Notes

Example

Show that $\sim (A \implies B)$ and $A \wedge \sim B$ are equivalent.

If two compound statements are equivalent, then essentially each statement is just a rephrasing of the other.

The phrases "A is equivalent to B", "A iff B", and "A is necessary and sufficient for B" can be used interchangeably.

Notes

1.8 Negating Statements

By looking at the truth table, we can see that the negation of a statement A has the opposite truth value as A. If A is a predicate, $\sim A$ is true exactly when A is false and false exactly when A is true.

So, if we manage to prove A, we know that $\sim A$ is false. Conversely, if we disprove A, then $\sim A$ must be true.

We can always write "It is not true that A" for $\sim A$, but often is more useful to rephrase a negation in terms of what is true rather than what is not true.

Notes

Example

Consider the predicate "x is irrational." Give two possible wordings for the negation of this predicate.

When you do the homework, you will be forced to consider the negations of several abstract statements. Here is a summary of some of the most common negations that we will encounter (some of these you are asked to prove using truth tables).

- $\sim(\sim A) \iff A$
- $\sim(A \wedge B) \iff \sim A \vee \sim B$
- $\sim(A \vee B) \iff \sim A \wedge \sim B$
- $\sim(A \implies B) \iff A \wedge \sim B$

Notes

We also need to know how to negate statements involving quantifiers (possibly many). Let's consider an example.

Example

Consider the statement "All Mini Coopers have stripes." One possible wording of the negation of this statement involves just putting a "not" in front: "Not all Mini Coopers have stripes." However, it would be useful to rephrase this statement into a more useful one. Any ideas?

Here's one: "There exists a Mini Cooper that does not have stripes."

This example illustrates what happens in general.

Notes

In general, we have:

- Negating "For all x, $P(x)$" becomes "There exists x such that $\sim P(x)$."
- Negation "There exists x such that $P(x)$" becomes "For all x, $\sim P(x)$."

We can now combine our knowledge of how to negate the logical operators and quantifiers to negate more complicated statements.

Let's try an example.

Notes

1.8.6 Example

Consider the statement "All Martians are short and bald, or my name isn't Darth Vader." Let's negate this statement. As an intermediate step, let's define some predicates and simple statements.

$$M(x) := \text{“}x \text{ is a Martian”}$$
$$S(x) := \text{“}x \text{ is short”}$$
$$B(x) := \text{“}x \text{ is bald”}$$
$$D = \text{“My name isn't Darth Vader”}$$

Using these symbols along with the appropriate quantifiers, we can rewrite the statement as

$$(\text{For all } x, M(x) \implies (S(x) \wedge B(x))) \vee D$$

Notes

1.8.6 Example (continued)

Then the negation

$$\sim [(\text{For all } x, M(x) \implies (S(x) \wedge B(x))) \vee D]$$

becomes

$$(\text{There exists } x \text{ such that } M(x) \wedge (\sim S(x) \vee \sim B(x))) \wedge \sim D$$

Now, we need to rephrase the abstract symbols back into an English sentence.

> *There exists a Martian that is either tall or has some hair, and my name is Darth Vader.*

Notes

Negating "If $A(x)$, then $B(x)$", where $A(x)$ and $B(x)$ are predicates requires a brief discussion. Recall that in this situation, it is implicitly assumed that what we really mean is "For all x, $A(x) \implies B(x)$."

So, the negation of "If $A(x)$, then $B(x)$" is "There exists x such that $A(x) \wedge \sim B(x)$."

Notes

1.9 Existence Theorems

We are finally ready to start tackling proofs. We'll first deal with existence theorems, which are theorems that assert the existence of something.

Recall the second statement from our thought experiment in Section 1.1:

There exists a real number x such that $x^3 = x$.

This is an example of an existence theorem. But how do we prove it?

Notes

All we need to do is demonstrate that there is a number that satisfies the desired property.

Proof.

$\underbrace{\text{Consider the number 1.}}_{\text{produce a candidate}}$ $\underbrace{\text{Since } 1^3 = 1\text{, the statement is true.}}_{\text{show that it does what you want}}$ □

Suppose that we wanted to prove "There exists a clacking waggler." We prove existence theorems in two steps.

1. We produce a candidate. That is, we describe an object that should be a clacking waggler. (This happens behind the scenes and is often the hard part. The proof that gets written down does not necessarily reflect all the work that went into finding the candidate.)
2. We show that our candidate actually is what we claim. In this case, we show that it is a waggler and clacks.

Notes