Chapter 3: Why study groups?

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Summer 2009

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Our choice of examples is influenced by how well they illustrate the material rather than how useful they are.

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How does symmetry relate to group? The examples of groups that we've seen so far deal with arrangements of similar things. In chapter 5, we shall uncover the following fact (we'll be more precise later):

Every group can be viewed as a collection of ways to rearrange some set of things.

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2. Consider the actions that you could perform with your hands that may rearrange the numbered parts, yet leave the object taking up the same physical space it did originally. (This collection of actions forms a group.)
3. (Optional) If you want to visualize the group, explore and map it as we did in Chapter 2 with the rectangle, etc.

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- In this context, not *every* rearrangement of the similar parts is necessarily valid. We are only allowed actions that maintain the physical integrity of the object *and* preserve its footprint. For example, we can't rip two arms off a starfish and then glue them back on in different places.

Comments (continued)

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- When selecting a set of generators, we would ideally like to select as small a set as possible. We can never choose too many generators, but we can choose too few. But having "extra" generators does nothing but clutter our Cayley diagram.

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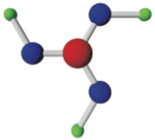
The following figure (taken from page 28 of *Visual Group Theory*) depicts a molecule of Boric acid, $B(OH)_3$.



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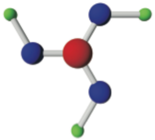

Follow the steps of Definition 3.1 to find the group that describes the symmetry of the molecule and draw a possible Cayley diagram.

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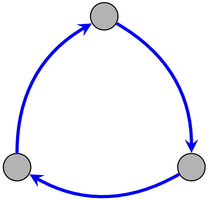

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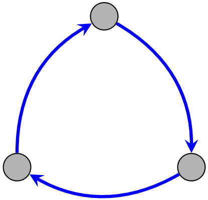

This is the cyclic group, C_3. (We'll discuss cyclic groups in Chapter 5.)

Group exercises

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3. Now, complete Exercise 3.7. I want each group to turn in a complete solution.

Crystallography

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In this case, the groups describing the symmetry of crystals are infinite. Why?

Frieze patterns

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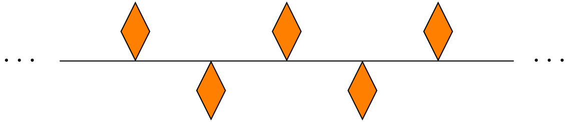

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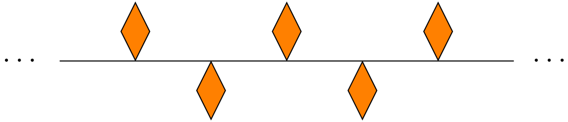

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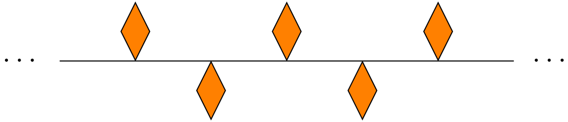

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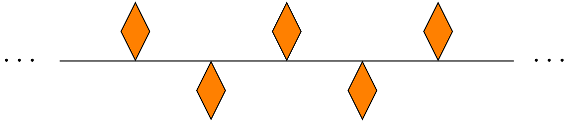

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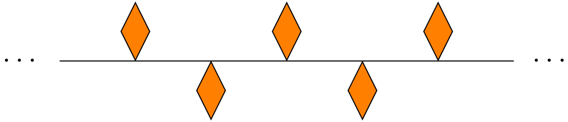

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Let's determine the group of symmetries of the frieze pattern on the previous slide and draw a possible Cayley diagram.

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- The symmetry of 2-dimensional repeating patterns, called wallpaper patterns, has also been classified. There are 17 different wallpaper groups.

More group exercises

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    - Exercise 3.11(a)
    - Exercise 3.11(b)
    - Exercise 3.11(d) (Bob may have something to say)

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2. Let's discuss your solutions.

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Dancing a figure rearranges the dancers. If they correctly obey the caller, every dance ends with the dancers back in their original positions in the square.

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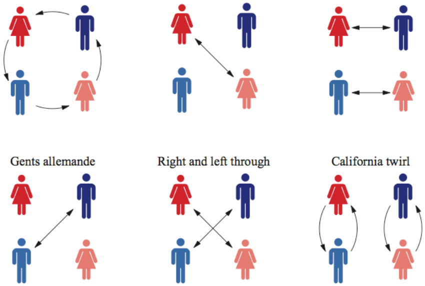

Do these 6 actions generate a group?

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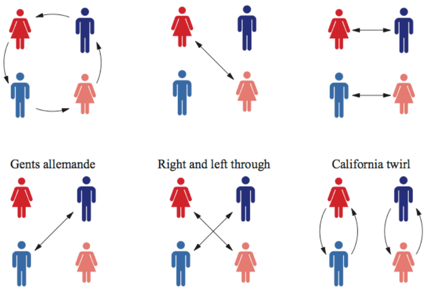

Do these 6 actions generate a group? The answer is yes (check the rules).

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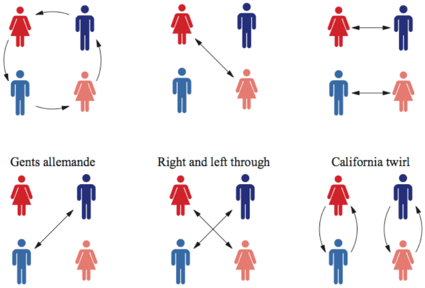

Do these 6 actions generate a group? The answer is yes (check the rules). It turns out, perhaps not surprisingly, that the group is isomorphic (i.e., same structure) as the group of symmetries of a square.

Even more group exercises

It's dance time!

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1. In a large group, complete the following exercises (not collected):
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    - Exercise 3.13 (see Bob)
    - Exercise 3.14(a)

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    - Exercise 3.14(a)
2. Let's discuss your solutions.
3. Now, in groups of 2–3, complete Exercise 3.15(a). I want each group to turn in a complete solution.

Potential quiz questions

Here are some potential questions that I may ask you on tomorrow's quiz at the beginning of class:

1. In order for an action to be a member of a group of symmetries for an object in 3-dimensions, what 2 important properties must this action have?
2. What is a glide reflection and to what kinds of objects can we apply them to?
3. Draw a Cayley diagram for a given molecule or frieze pattern.