Chapter 8: The power of homomorphisms

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Summer 2009

Notes

Throughout the course, we've said things like:

- "This group has the same structure as that group."
- "This group is isomorphic to that group."

However, we've never really spelled out the details about what this means. In this chapter, we'll finally nail down what an isomorphism really is.

In general, we will study special types of functions between groups called homomorphisms, where isomorphisms are a specific type of homomorphism. The Greek roots "homo" and "morph" together mean "same shape."

Notes

There are two situations (and it turns out that there are only two) where homomorphisms arise:

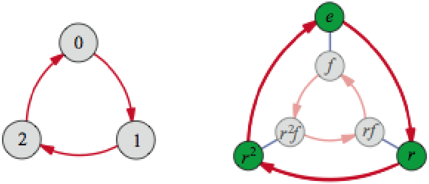
- when one group is a subgroup of another;
- when one group is a quotient group.

The corresponding homomorphisms are called embeddings and quotient maps.

Notes

Embeddings

Let's start off with an example. Consider the statement: $C_3 < S_3$. Here is a visual (taken from Figure 8.1 on page 158 of *VGT*).



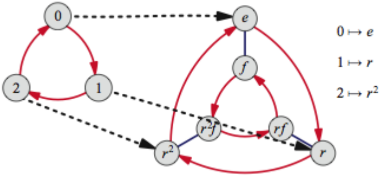
The highlighting in this figure shows that S_3 contains a 3-step cyclic subgroup, which is identical to C_3 in structure only. None of the elements of C_3 (namely 0, 1, 2) are actually in S_3.

When we say $C_3 < S_3$, what we really mean is that the structure of C_3 shows up in S_3.

Notes

In particular, there is a 1-1 correspondence between the elements in C_3 and the elements of the 3-step cyclic subgroup in S_3, and furthermore, the relationship between the corresponding nodes is the same.

The following figure (taken from Figure 8.1 on page 158 of *VGT*) illustrates this correspondence.



Homomorphisms are the mathematical tool for succinctly expressing precise structural correspondences. Because homomorphisms describe how elements of one group correspond to elements of another, they are a kind of function.

Notes

In the case of our previous example, we say that this function maps elements of C_3 to elements of S_3.

Often Greek letters are used to name maps between groups. For our example, let's use ϕ. We write $\phi : C_3 \to S_3$ to say that ϕ maps C_3 to S_3.

We use standard function notation and terminology. For example, we can write $\phi(1) = r$. In fact, there is a formula for expressing the function in our example: $\phi(n) = r^n$.

Notes

The group from which a function originates is called its domain (C_3 in our example) and the group into which the function maps is called the codomain (S_3 in our example).

The particular elements of the codomain that the function maps to are called the image of the function ($\{e, r, r^2\}$ in our example), denoted $Im(\phi)$.

It is important to note that not every function from one group to another is going to be a homomorphism. In our example, the domain and codomain respected each other's structure. We need all homomorphisms to have this same property.

Notes

Definition 8.1

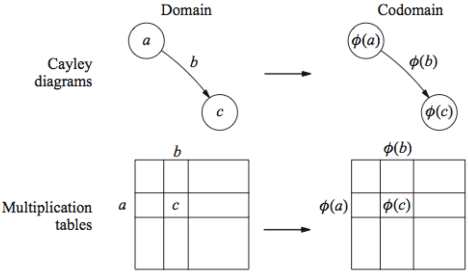
A homomorphism is a function between 2 groups that mimics the structure of it domain and codomain. The following condition expressions this requirement (stated in two different ways).

1. Cayley diagrams: If an arrow b in the domain leads from a to c, then the $\phi(b)$ arrow in the codomain must lead from the element $\phi(a)$ to $\phi(c)$.
2. Multiplication tables: If the domain multiplication table says $a \cdot b = c$, then the codomain multiplication table must say that $\phi(a) \cdot \phi(b) = \phi(c)$.

Note that the operation $a \cdot b$ is occurring in the domain while $\phi(a) \cdot \phi(b)$ occurs in the codomain.

Notes

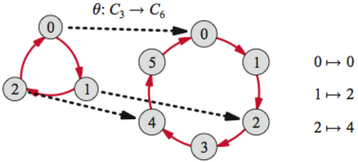
The following figure (taken from Figure 8.3 on page 159 of *VGT*) is an illustration of Definition 8.1.



We say that the homomorphism is structure preserving.

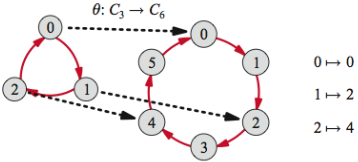
Notes

Let's walk through another example. We can express $C_3 < C_6$ using the homomorphism $\theta : C_3 \to C_6$ defined by $\theta(n) = 2n$. Here is a visual representation of θ (taken from Figure 8.4 on page 160 of *VGT*).



But is this really a homomorphism? We need to verify that any path from a to b in C_3 corresponds to the path from $\theta(a)$ to $\theta(b)$ in C_6.

Notes



θ maps 1 to 2, but it also maps the 1-arrow in C_3 to the 2-step path representing 2 in C_6. The 1 arrow traces the orbit $\{0, 1, 2\}$ in C_3 while the $\theta(1)$ path traces the corresponding orbit $\{\theta(0), \theta(1), \theta(2)\}$ in C_6, which is $\{0, 2, 4\}$.

We can think of the $\theta(1)$ arrow as the path consisting of two red arrows in succession. So, θ doubles both numbers and arrows.

Notes

What this last example illustrates to us is that if we know where a homomorphism maps *all* of the domains arrows (i.e., generators), then we know where it maps the rest of the nodes.

For example, suppose there was another homomorphism $\theta' : C_3 \to C_6$ different from θ such that $\theta'(1) = 4$. Using this information, we can construct the rest of the homomorphism using the fact that θ' must obey $\theta'(a + b) = \theta'(a) + \theta'(b)$.

We see that

$$\theta'(2) = \theta'(1 + 1) = \theta'(1) + \theta'(1) = 4 + 4 = 2$$

and

$$\theta'(0) = \theta'(1 + 2) = \theta'(1) + \theta'(2) = 4 + 2 = 0.$$

Notes

We can use the same general process to determine where a homomorphism maps all the elements of the domain by just knowing where the generators are mapped.

Say $\phi : G \to H$ and assume that $G = \langle a, b \rangle$ and we are give the value of $\phi(a)$ and $\phi(b)$. Using this information we can determine the image of any element in G. For example, for $g = aaabbbab$, we have

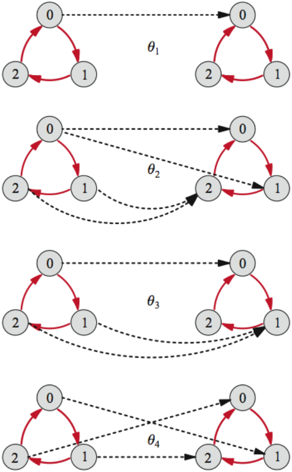
$$\phi(aaabbbab) = \phi(a)\phi(a)\phi(a)\phi(b)\phi(b)\phi(a)\phi(b).$$

Similar reasoning works for any number of generators.

One consequence of this is that the identity of the domain must *always* map to the identity in the codomain (see Exercise 8.7).

Notes

The following figure (taken from Figure 8.5 on page 161 of *VGT*) illustrates some non-homomorphisms.



A function maps *each* element of the domain to some element of the codomain. But θ_1 ignores two elements of the domain, and thus is not a function.

A function maps each element of the domain to *one* element of the codomain. But θ_2 maps 0 to two different elements of the domain, and thus is not a function.

Although θ_3 is a function, it is not a homomorphism. In the domain, $1+1=2$, but in the codomain, $\theta_3(1)+\theta_3(1)\neq\theta_3(2)$.

Although θ_4 is a function, it is not a homomorphism. Although the image is a 3-cycle like the domain, θ_4 does not map the domain identity element to the codomain identity element. For example, $\theta_4(0)+\theta_4(1)\neq\theta_4(1)$.

Notes

Any homomorphism that helps us get information about how one group is a subgroup of another is called an embedding.

Because any embedding finds a copy of the domain in the codomain, its image is therefore the same size as its domain. So, embeddings never map 2 different elements of the domain to the same element in codomain (i.e., they are 1-1).

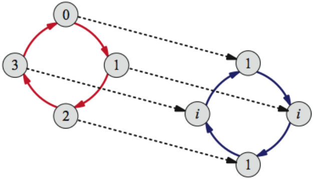
An embedding whose image fills the whole codomain shows us that the domain and codomain are actually the same size and have all the same structure. In this case, we say that the function is not just a homomorphism, but an isomorphism.

Notes

Two isomorphic groups may name their elements differently and may look different based on the layouts or choice of generators for their Cayley diagrams, but the isomorphism between them guarantees that they have the same structure.

When two groups G and H have an isomorphism between them, we say that "G and H are isomorphic." In this case, we write $G \cong H$.

The following figure equipped with some mislabeling (taken from Figure 8.8 on page 163 of *VGT*) depicts an isomorphism between C_4 and the group of complex numbers $\{1, -1, i, -i\}$.



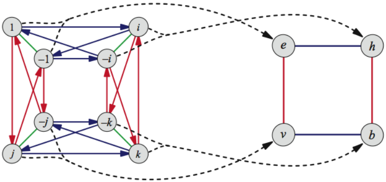
In this case, $C_4 \cong \{i, -1, i, -i\}$.

Notes

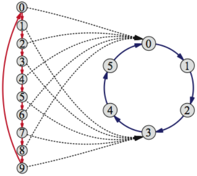
Quotient maps

Well, what happens if more than one element of domain maps to the same element of codomain (i.e., non-embeddings)? Here are some examples (taken from Figure 8.9 on page 164 of *VGT*).

$\tau_1 : Q_4 \to V_4$



$$\tau_2 : C_{10} \to C_6 \text{ by } \tau_2(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$



Notes

In the interest of time, we'll skip many of the details of this type of situation. All non-embedding homomorphisms are called quotient maps (because they correspond to our quotient process).

General facts about quotient maps

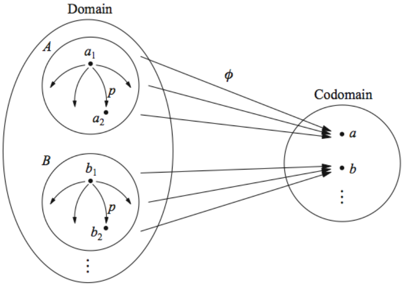
- Every cluster of domain elements that maps to the same codomain element has the same structure. That is, every non-embedding homomorphism follows a repeating pattern.
- This creates a partition of the domain into identical copies of a structure. (Sound familiar?)
- The clusters of domain elements that map to the same codomain element are actually a subgroup and its cosets.
- The cluster of elements from domain that map to the identity in codomain is the subgroup and the other clusters are the cosets. We call this subgroup the kernel of the homomorphism, denoted $Ker(\phi)$.

Notes

General facts about quotient maps (cont)

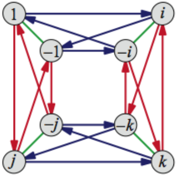
- The left cosets of $Ker(\phi)$ are also right cosets. So, $Ker(\phi)$ is a normal subgroup of the domain.
- This means that we can always form the quotient group $G/Ker(\phi)$, where G is the domain of the homomorphism ϕ.

Here is an abstract picture of what quotient maps look like in general (taken from Figure 8.10 on page 165 of *VGT*).



Notes

Let's work through an example. Define the homomorphism $\phi : Q_4 \to V_4$ via $\phi(i) = v$ and $\phi(j) = h$. ϕ's "action" on the generators i and j is enough to determine everything else we need to know. Here is the Cayley diagram for Q_4 (taken from Figure 8.12 on page 167 of *VGT*).



Let's determine:

1. the image of the rest of the elements
2. $Ker(\phi)$

What is $Q_4/Ker(\phi)$? Do you notice any relationship between $Q_4/Ker(\phi)$ and $Im(\phi)$?

Notes

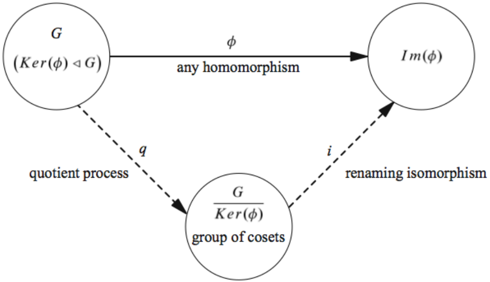
The Fundamental Homomorphism Theorem

Here is one of the crowning achievements of group theory.

Fundamental Homomorphism Theorem

If $\phi : G \to H$ is a homomorphism, then $Im(\phi) \cong G/Ker(\phi)$.

Here is an abstract illustration of the Fundamental Homomorphism Theorem (taken from Figure 8.13 on page 168 of *VGT*).



Notes

Unfortunately, we did not have time to prove all of the details leading up to this and we also don't have time to prove this theorem.

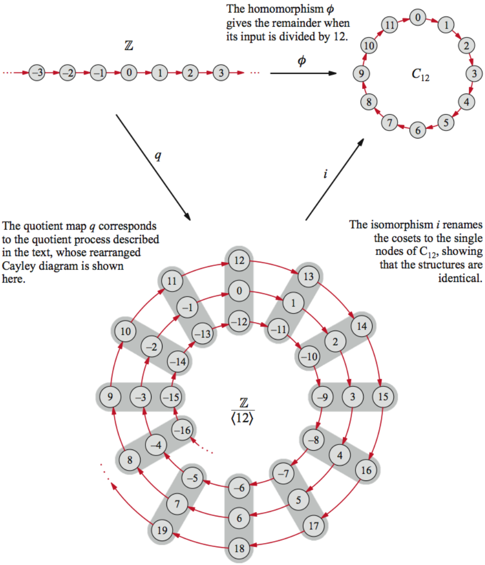
Notice that in the special case that ϕ is an embedding, $Ker(\phi) = \{e\}$, in which case the FHT says $Im(\phi) \cong G/\{e\}$. But $G/\{e\}$ is certainly isomorphic to G. So, in the case of an embedding, the FHT simply says that $Im(\phi) \cong G$.

Also, one consequence of the Fundamental Homomorphism Theorem is that $Im(\phi)$ must be a subgroup of the codomain.

Let's take a look at one last example.

Notes

The following figure (taken from Figure 8.18 on page 172 of *VGT*) illustrates an isomorphism between C_{12} and $\mathbb{Z}/\langle 12 \rangle$.



Notes