IMPARTIAL ACHIEVEMENT & AVOIDANCE GAMES FOR GENERATING FINITE GROUPS

ACGT Seminar at NAU

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Joint work with Bret Benesh and Nándor Sieben

Intuitive Definition

Combinatorial Game Theory (CGT) is the study of two-person games satisfying:

- Two players alternate making moves.
- No hidden information.
- No random moves.

Combinatorial games

- Chess
- Go
- Connect Four
- Nim
- Tic-Tac-Toe
- X-Only Tic-Tac-Toe

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Non-combinatorial games

- Battleship (hidden information)
- Rock-Paper-Scissors (non-alternating and random)
- Poker (hidden information and random)

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- Go
- Connect Four
- Tic-Tac-Toe

Impartial

- Nim
- X-Only Tic-Tac-Toe

Comments

- We are interested in impartial games.
- We will require that game sequence is finite and there are no ties.
- Player that moves first is called α and second player is called β.
- **Normal Play:** The last player to move wins.
- **Misère Play:** The last player to move loses.

NIM

Single-pile Nim

Start with a pile of n stones. Each player chooses at least one stone from the pile. The player that takes the last stone wins. Game is denoted $*n$ (called a **nimber**).

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Multi-pile Nim

Start with k piles consisting of $n_1, \ldots, n_k$ stones, respectively. Each player chooses at least one stone from a single pile. The player that takes the last stone wins. Denoted $*n_1 + \cdots + *n_k$.

Example

Let's play $*1 + *2 + *2$. Here's a possible sequence.

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In this case, α wins.

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Yay!

In this case, α wins.

Question

In general, is there an optimal strategy for either player?

Answer

Short answer is yes: write sizes of piles in binary, do binary addition without carry (XOR), and if possible, hand your opponent a sum of 0. If players make optimal moves, this is only possible for one of the players.

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From perspective of the player that is about to move, a P-position is a losing position while an N-position is a winning position.

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- $*1 + *1$ is a P-position
- $*1 + *2$ is an N-position
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Definition
If G and H are games, then $G+H$ is the game where each player makes a move in one of the games. Set of options:

$$\mathrm{Opt}_{G+H}(S+T) := \{Q+T \mid Q \in \mathrm{Opt}_G(S)\} \cup \{S+R \mid R \in \mathrm{Opt}_H(T)\}$$

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Theorem
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Proof

Copy cat.

Definition
$G_1 = G_2$ if and only if $G_1 + G_2$ is a P-position.

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Theorem

$G_1 = G_2$ if and only if $G_1 + H$ and $G_2 + H$ have the same outcome for all H.

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Examples

- $\text{mex}(\{0, 1, 2, 4, 5\}) = 3$
- $\text{mex}(\{1, 3\}) = 0$
- $\text{mex}(\{0, 1\}) = 2$
- $\text{mex}(\emptyset) = 0$

Definition
If G is a game, then

$$\mathrm{nim}(G) := \mathrm{mex}(\{\mathrm{nim}(Q) \mid Q \in \mathrm{Opt}(G)\}).$$

This is a recursive definition. We start computing with terminal positions (empty option set).

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Examples

- $\mathrm{nim}(*0) = \mathrm{mex}(\emptyset) = 0$
- $\mathrm{nim}(*1) = \mathrm{mex}(\{\mathrm{nim}(*0)\}) = \mathrm{mex}(\{0\}) = 1$
- $\mathrm{nim}(*2) = \mathrm{mex}(\{\mathrm{nim}(*0), \mathrm{nim}(*1)\}) = \mathrm{mex}(\{0, 1\}) = 2$
- $\mathrm{nim}(*n) = n$
- $\mathrm{nim}(*1 + *1) = \mathrm{mex}(\{\mathrm{nim}(*1)\}) = \mathrm{mex}(\{1\}) = 0$
- $$\begin{aligned}\mathrm{nim}(*1 + *2) &= \mathrm{mex}(\{\mathrm{nim}(*2), \mathrm{nim}(*1), \mathrm{nim}(*1 + *1)\}) \\ &= \mathrm{mex}(\{2, 1, 0\}) = 3\end{aligned}$$

Theorem (Sprague–Grundy)

Every game is equivalent to a single Nim pile: $G = *\,\mathrm{nim}(G)$

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We can think of nim-numbers as “isomorphism” classes of games.

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Big Picture

Fundamental problem in the theory of impartial combinatorial games is the determination of the nim-number of the game.

We can think of nim-numbers as “isomorphism” classes of games.

Theorem

2nd player β wins G if and only if $G = *0$.

Let G be a finite (possibly trivial) group.

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Generate Game

For the achievement game GEN(G):

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Generate Game

For the achievement game $\mathrm{GEN}(G)$:

- 1st player chooses any $g_1 \in G$.
- At kth turn, designated player selects $g_k \in G \setminus \{g_1, \ldots, g_{k-1}\}$ to create position $\{g_1, \ldots, g_k\}$.
- Player wins on the nth turn if $\langle g_1, \ldots, g_n \rangle = G$.

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Positions of $\mathrm{GEN}(G)$ are subsets of terminal positions, which are certain generating sets of G.

MATCH-UP

		
Name:	LeBron James	Bret Benesh
Height:	6’8”	6’5”
Weight:	260 lbs	180 lbs
Age:	32 years	>32 years
Salary:	$30.96 million/year	$0 million/year
Accolades:	3x NBA Champion 4x NBA MVP 2x Olympic gold medalist 11x NBA All-Star	Never had a cavity Sagittarius

GEN on $S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$

LeBron	P	$\langle P \rangle$	Bret

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Let G be a finite nontrivial group.

Do Not Generate Game
For the avoidance game $\mathrm{DNG}(G)$:

- 1st player chooses $g_1 \in G$ such that $\langle g_1 \rangle \neq G$.
- At the kth turn, designated player selects $g_k \in G \setminus \{g_1, \ldots, g_{k-1}\}$ such that $\langle g_1, \ldots, g_k \rangle \neq G$ to create position $\{g_1, \ldots, g_k\}$.
- Player that cannot select an element without building a generating set is loser.

Positions of $\mathrm{DNG}(G)$ are exactly the non-generating subsets of G and terminal positions are the maximal subgroups of G.

LEBRON VS BRET: GAME TWO

DNG on $S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$

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$(1,2,3)$	$\{(1,2,3)\}$	$\mathbb{Z}_3$	
	$\{(1,2,3),(1,3,2)\}$	$\mathbb{Z}_3$	$(1,3,2)$
e	$\{(1,2,3),(1,3,2),e\}$	$\mathbb{Z}_3$	

DNG on $S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$


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DNG on $D_8 = \langle r, s \rangle = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$

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r^3	$\{r^2, r^3\}$	$\mathbb{Z}_4$	

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r^3	$\{r^2, r^3\}$	$\mathbb{Z}_4$	
	$\{r^2, r^3, e\}$	$\mathbb{Z}_4$	e

LEBRON VS BRET: GAME THREE

DNG on $D_8 = \langle r, s\rangle = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$

LeBron	P	$\langle P\rangle$	Bret
	$\{r^2\}$	$\mathbb{Z}_2$	r^2
r^3	$\{r^2, r^3\}$	$\mathbb{Z}_4$	
	$\{r^2, r^3, e\}$	$\mathbb{Z}_4$	e
r	$\{r^2, r^2, e, r\}$	$\mathbb{Z}_4$	

LEBRON VS BRET: GAME THREE

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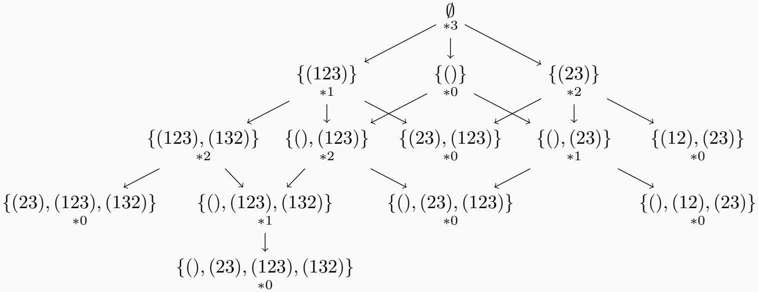
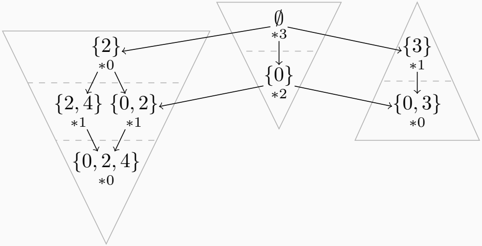
- 1988: Barnes establishes element-based criteria for who wins DNG, assorted GEN results.
- 2014: Ernst and Sieben determine nim-numbers (and hence outcomes) for cyclic, dihedral, abelian.
- 2016: Benesh, Ernst, and Sieben establish subgroup-based criteria for the determination of nim-numbers (and hence outcomes) for DNG, characterize spectrum of nim-numbers for DNG, determine nim-numbers for GEN and DNG for a variety of groups including generalized dihedral, symmetric, and alternating groups.

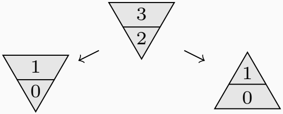
REPRESENTATIVE GAME TREES



Representative game tree for $\mathsf{GEN}(S_3) = *3$

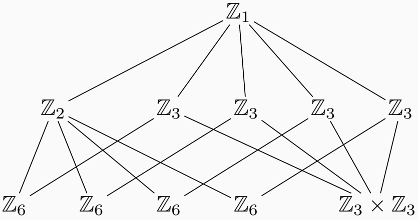


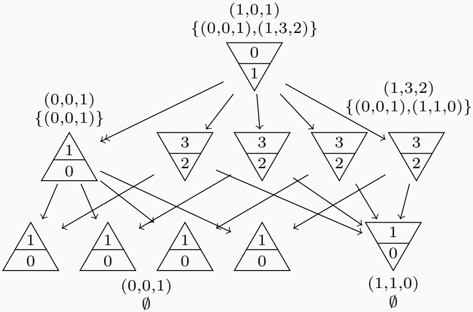
DNG($\mathbb{Z}_6$)

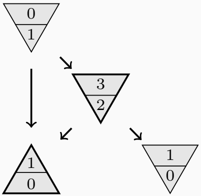


Structure diagram

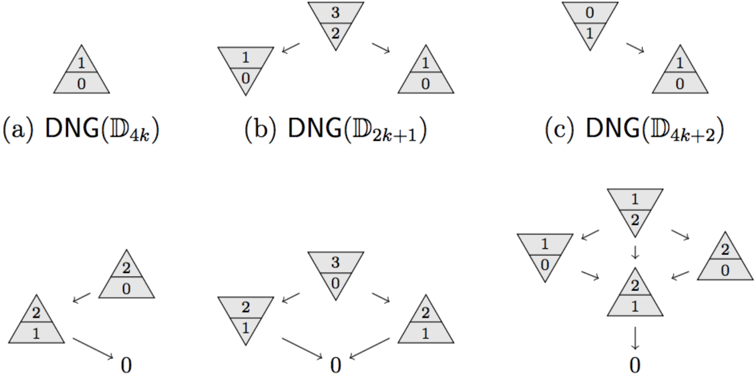
SIMPLIFIED STRUCTURE DIAGRAMS







DNG($\mathbb{Z}_6 \times \mathbb{Z}_3$)



(a) DNG($\mathbb{D}_{4k}$) (b) DNG($\mathbb{D}_{2k+1}$) (c) DNG($\mathbb{D}_{4k+2}$)

(d) GEN($\mathbb{D}_{4k}$) (e) GEN($\mathbb{D}_{2k+1}$) (f) GEN($\mathbb{D}_{4k+2}$)

Simplified structure diagrams for dihedral groups

Theorem (Ernst, Sieben)
If $n \geq 2$, then $\mathrm{nim}(\mathsf{GEN}(\mathbb{Z}_n)) = \mathrm{nim}(\mathsf{DNG}(\mathbb{Z}_n)) + 1$.

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If $n \geq 2$, then

$$\mathrm{DNG}(\mathbb{Z}_n) = \begin{cases} *1, & n = 2 \\ *1, & n \equiv_2 1 \\ *0, & n \equiv_4 0 \\ *3, & n \equiv_4 2 \end{cases}$$

and

$$\mathrm{GEN}(\mathbb{Z}_n) = \begin{cases} *2, & n = 2 \\ *2, & n \equiv_2 1 \\ *1, & n \equiv_4 0 \\ *4, & n \equiv_4 2 \end{cases}$$

Theorem (Ernst, Sieben)
For $n \geq 3$, we have

$$\mathrm{DNG}(\mathbb{D}_n) = \begin{cases} *3, & n \equiv_2 1 \\ *0, & n \equiv_2 0 \end{cases}$$

and

$$\mathrm{GEN}(\mathbb{D}_n) = \begin{cases} *3, & n \equiv_2 1 \\ *0, & n \equiv_4 0 \\ *1, & n \equiv_4 2 \end{cases}$$

Theorem (Ernst, Sieben)

If G is a finite nontrivial abelian group, then

$$\mathrm{DNG}(G) = \begin{cases} *1, & G \text{ is nontrivial of odd order} \\ *1, & G = \mathbb{Z}_2 \\ *3, & G = \mathbb{Z}_2 \times \mathbb{Z}_{2k+1} \text{ with } k \geq 1 \\ *0, & \text{else} \end{cases}$$

$$\mathrm{GEN}(G) = \begin{cases} *2, & |G| \text{ is odd and } d(G) \leq 2 \\ *1, & |G| \text{ is odd and } d(G) \geq 3 \\ *2, & G = \mathbb{Z}_2 \\ *1, & G = \mathbb{Z}_{4k} \text{ with } k \geq 1 \\ *4, & G = \mathbb{Z}_{4k+2} \text{ with } k \geq 1 \\ *1, & G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_m \times \mathbb{Z}_k \text{ for } m, k \text{ odd} \\ *0, & \text{else} \end{cases}$$

Theorem (Ernst, Sieben)

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Conjecture (In Progress)

If $|G|$ is even, then $\mathrm{GEN}(G)$ is one of $*0, *1, *2, *3, *4$.

Theorem (Benesh, Ernst, Sieben)

Let G be a finite nontrivial group.

- If all maximal subgroups are even, then $\mathrm{DNG}(G) = *0$.
- If all maximal subgroups are odd, then $\mathrm{DNG}(G) = *1$.
- If mixed maximal subgroups, then
  - If the even maximals cover G, then $\mathrm{DNG}(G) = *0$.
  - If the even maximals do not cover G, then $\mathrm{DNG}(G) = *3$.

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Using our “checklist” criteria, we have completely characterized DNG for nilpotent, generalized dihedral, generalized quaternion, symmetric, Coxeter, alternating, and some Rubik’s cube groups.

Big Picture for DNG

- The players just race to fill up one maximal subgroup M.
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Strategy

- α wants to pick an element not in any maximal subgroups of even order.
- β wants to pick an involution.

What's left to work on?

- Wrap up spectrum of GEN?
- Wrap up characterization of GEN for nilpotent groups?
- Are there nice results for products and quotients?
- Is it possible to characterize the nim-numbers of GEN in terms of covering conditions by maximal subgroups similar to what we did for DNG?
- What about other "closure systems"? We are currently tinkering with convex hulls of finitely many points in the plane.

Thanks!