MTH 302: Linear Algebra and Differential Equations

Linear combinations and matrix-vector products

17 January 2023

Goals

- Review: What the results of RREF tell us
- Load Sympy into a Jupyter notebook; create matrices and vectors in SymPy
- Compute a linear combination of two or more vectors and state the weights
- Multiply a matrix times a vector and interpret the results as linear combinations
- Compute a matrix-vector product using a computer tool (SymPy and Jupyter notebooks)

These notes are posted as a Jupyter notebook on the Resource Page. Also a PDF of the slides.

Q&A from Class Prep

(To be filled in later)

Polling: What results of RREF tell us

Go to http://pollev.com/talbert

SymPy

Follow along by going to https://colab.research.google.com/ and open a new Jupyter notebook

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In [1]:

```
# Load the SymPy package
from sympy import *

# This makes the math output look nice
init_printing()
```

Solve this system using SymPy:

$$
\begin{aligned}
x + y + z &= 3 \\
y - z &= 2 \\
2x + y + z &= 4
\end{aligned}
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In [2]:

```
# Create the augmented matrix using the Matrix command
A = Matrix([[1,1,1,3], [0,1,-1,2], [2,1,1,4]])
A
```

Out[2]:

$$
\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 1 & 4 \end{bmatrix}
$$

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# Create the augmented matrix using the Matrix command
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A
```

Out[2]:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 1 & 4 \end{bmatrix}$$

In [3]:

```python
A.rref(pivots=False)
```

Out[3]:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

In [4]:

```python
# Vectors are just 1-column matrices entered as a row
v = Matrix([1,2,3,4])
v
```

Out[4]:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Linear combinations

A **sum** of **scalar multiples** of a collection of vectors.

Example: $\mathbf{v} = [1, 0, -1]^T$, $\mathbf{w} = [2, 5, 1]^T$

$$3\mathbf{v} - 2\mathbf{w} = 3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} -4 \\ -10 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -10 \\ -5 \end{bmatrix}$$

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In [6]:

```python
v = Matrix([1,0,-1])
w = Matrix([2,5,1])

# Don't forget the *
3*v - 2*w
```

Out[6]:

$$\begin{bmatrix} -1 \\ -10 \\ -5 \end{bmatrix}$$

Activity:

1. By hand, find the linear combination $2\mathbf{v}-\mathbf{w}$ for the vectors $\mathbf{v},\mathbf{w}$ in $\mathbb{R}^2$ randomly generated below.
2. By hand, find the linear combination $\mathbf{a}+\mathbf{b}-3\mathbf{c}$ for the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ in $\mathbb{R}^3$ randomly generated below.
3. Can you find a "linear combination" of a set of vectors with just one vector in it?
4. Can you linearly combine a vector from $\mathbb{R}^3$ with a vector from $\mathbb{R}^4$?
5. Check your work on the first two using Sympy.

In [20]:

```
v = randMatrix(2,1,-10,10)
w = randMatrix(2,1,-10,10)
a = randMatrix(3,1,-10,10)
b = randMatrix(3,1,-10,10)
c = randMatrix(3,1,-10,10)
```

In [21]:

```
# Each group will be generating vectors randomly so yours may not be these

# Find 2v - w

(v,w)
```

Out[21]:

$$\left(\begin{bmatrix}4\\-10\end{bmatrix}, \begin{bmatrix}-2\\-3\end{bmatrix}\right)$$

In [17]:

```
# Find a + b - 3c

(a,b,c)
```

Out[17]:

$$\left(\begin{bmatrix}-5\\-3\\-10\end{bmatrix}, \begin{bmatrix}-10\\-7\\-6\end{bmatrix}, \begin{bmatrix}4\\-10\\3\end{bmatrix}\right)$$

Matrix-vector products

- Given: Matrix $\mathbf{A}$, vector $\mathbf{v}$.
- The **product** $\mathbf{A}\mathbf{v}$ is a *vector* obtained through a *linear combination of the columns of* $\mathbf{A}$*, using the entries of* $\mathbf{v}$ *as the weights.*

Example:

$$\begin{bmatrix}10 & 4 & 1\\4 & 8 & 3\\5 & 8 & -9\\-4 & -3 & 1\end{bmatrix}\begin{bmatrix}-1\\-4\\3\end{bmatrix}=\begin{bmatrix}-23\\-27\\-64\\19\end{bmatrix}$$

Note: This only makes sense if *the number of columns in the matrix equals the number of entries in the vector.*

In [32]:

```
# Checking

A = Matrix(4,3,[10, 4, 1, 4, 8, 3, 5, 8, -9, -4, -3, 1])
v = Matrix(3,1,[-1,-4,3])
A*v
```

Out[32]:

$$\begin{bmatrix}-23\\-27\\-64\\19\end{bmatrix}$$

Every system of linear equations can be phrased as a *single* equation involving the product of a matrix and a vector

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$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Solving a linear system $\leftrightarrow$ solving an equation involving a matrix and a vector... a.k.a. a question about linear combinations.

Backward question: Given a randomly generated vector in $\mathbb{R}^2$, can it be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$?

In [34]:

```
v = randMatrix(2,1,-10,10)
w = randMatrix(2,1,-10,10)
x = randMatrix(2,1,-10,10)
(v,w,x)
```

Out[34]:

$$\left(\begin{bmatrix} 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -10 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -10 \end{bmatrix}\right)$$

Are there weights a, b such that $a\mathbf{v} + b\mathbf{w} = \mathbf{x}$?

Main Activity

- Posted as a Jupyter notebook on the Resource Page (look in Blackboard sidebar).
- Two versions linked: One you can download and run, one you can only view. Pick either one.
- Some portions are designated "★**AA**". These parts are to be written up individually and submitted as part of **Application/Analysis 1** due next Wednesday.

Next time

- The **span** of a set of vectors: What is it, how to determine what's in it and what it looks like; what this has to do with solving linear systems
- Complete *Class Prep for January 19* by Wednesday 11:59pm
- Also complete *Practice Set 1* on WeBWorK by Wednesday 11:59pm
- Work and ask questions on the *OPTIONAL RREF practice* set on WeBWorK as needed
- Finish the Startup Assignment if you haven't already
- **You are now able to attempt Miniproject 1** on *Markov Chains*.