MTH 302: Linear Algebra and Differential Equations

Introduction to differential equations

2023 February 14

Housekeeping

- Miniproject 2 now available
- Application/Analysis 3 ready, due date moved to Friday
- New tutorial on Blackboard (also on Resource Page)

Today's Goals

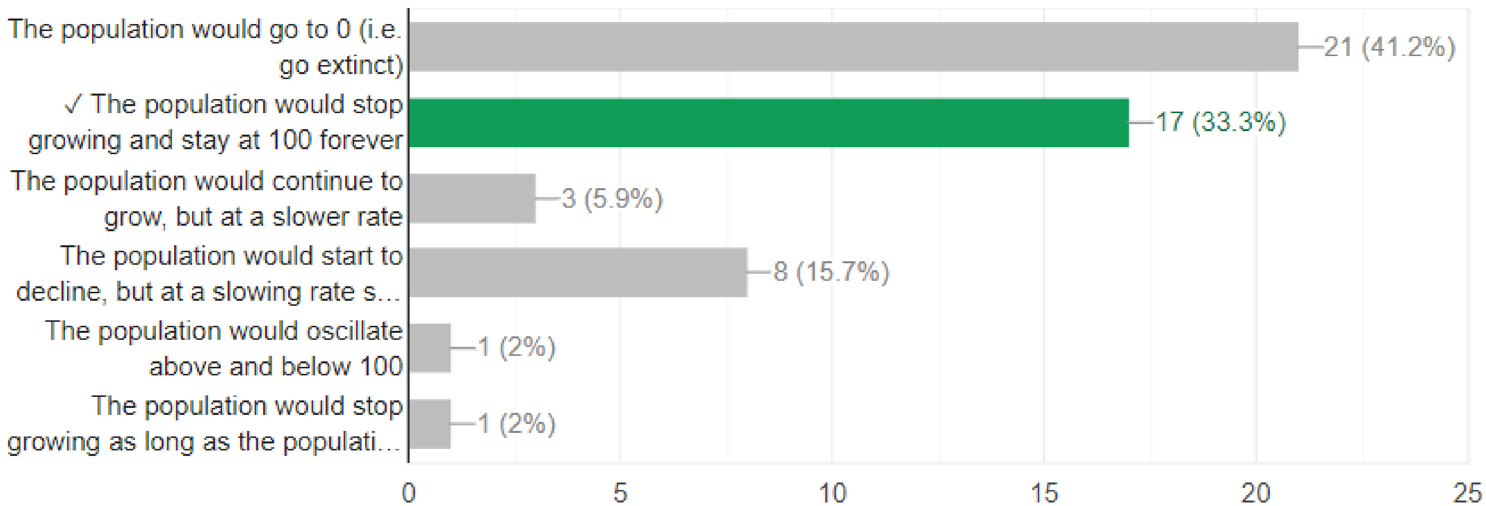
- Finish up loose ends from eigenvalue/eigenvector lesson
- Basic concepts about differential equations
- Checking solutions to DEs
- General vs. particular solutions, and initial value problems
- Slope fields and equilibrium solutions

Review of Class Prep activities

Look at the "logistic" differential equation in Section 2.1 on page 129, with the numbers filled in. (There's a "100" on the bottom of a fraction.) If P(t) ever equalled 100 -- that is, the population being modeled reached a level of 100 units -- what would happen?

Copy

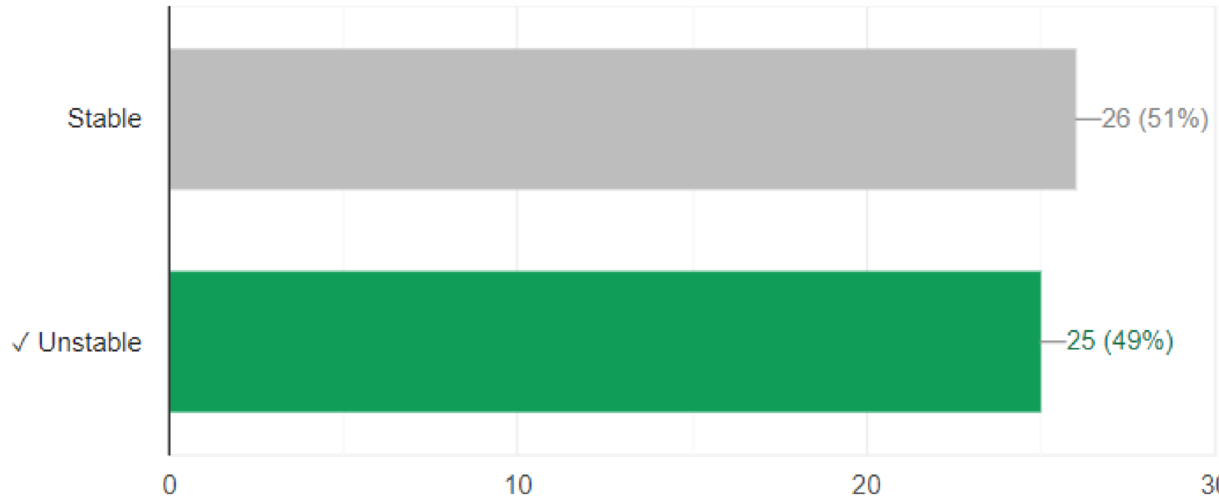
17 / 51 correct responses



Use the slope field tool to plot the slope field for y' = (y-1)(y-3)^2. The constant function y = 1 is an equilibrium solution. Is it stable, or unstable?

Copy

25 / 51 correct responses



Definitions

For a given $n \times n$ matrix A, a nonzero vector $\mathbf{v}$ is said to be an **eigenvector** of A, if there exists a scalar λ such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

(*English*: Multiplying $\mathbf{v}$ by A just rescales $\mathbf{v}$, but otherwise doesn't move it.) The scaling factor λ is called the **eigenvalue corresponding to the eigenvector** $\mathbf{v}$.

What about 3×3 matrices?

$$\mathbf{A} = \begin{bmatrix} -5 & -2 & 2 \\ 24 & 14 & -10 \\ 21 & 14 & -10 \end{bmatrix}$$

1. Find $\det(A - \lambda I_3)$:

In [ ]:
```
A = Matrix(3,3,[-5,-2,2,24,14,-10,21,14,-10])

# Use "s" instead of "lambda"; define as symbolic variable in SymPy
s = var("s")

# Set up the matrix whose determinant we want:
M = A - s*eye(3)
M
```

In [ ]:
```
M.det()
```

In [ ]:
```
# Using SymPy to solve; if right side is 0, only enter the left side
solve(M.det(), s)
```

In [ ]:
```
# Sub each eigenvalue back in; RREF to get nontrivial solution
AlmostThere = A + 3*eye(3)
AlmostThere
```

In [ ]:
```
AlmostThere.rref(pivots=False)
```

For class assessments:

- 2×2 matrices: Go through the whole process, find eigenvalues and corresponding eigenvectors.
- 3×3 matrices:
  - Given matrix and an eigenvalue, find one eigenvector that corresponds to the eigenvalue. **Or,**
  - Given matrix and a vector in $\mathbb{R}^3$, determine whether or not it's an eigenvector and if it is, state the eigenvalue.

Application: Markov Chains

Example 1.3.1 -- What happens to the voter distribution over the long term?

In [ ]:
```
M = Matrix(3,3,[0.95, 0.03, 0.07, 0.02, 0.90, 0.13, 0.03, 0.07, 0.80])
x = Matrix([120, 110, 20])

years = 100
for i in range(years):
    x = M*x
x
```

In [ ]:
```
M.eigenvects()
```

In [ ]:
```
x
```

In [ ]:
```
L = 0.769631234107679/117.456660176547
L * x
```

Differential equations

- Equations connecting a function to one or more of its derivatives.
- A system, of sorts -- where the amount of something and the rate at which the amount changes are mutually dependent.
- They arise *everywhere* in science and engineering
- A **solution** to a DE is a function

Some DE's are just integration problems in disguise:

$$\frac{dy}{dx} = x^4 + e^x$$

Solve by integrating both sides with respect to x.

Others are a little more complicated:

$$\frac{dy}{dx} = y^4 + e^x$$

What makes this different than the first one?

Activity

1. Show by checking that $y = e^{2t}$ is a solution to the second-order DE $y'' = 4y$.
2. Are there any other values of r besides $r = 2$ that make $y = e^{rt}$ a solution to this DE?
3. Use integration to solve $y' = t + \sin(t)$.
4. Use integration to solve $y'' = t^2 + 2$.

General vs. Particular solutions

Any function of the form $y(t) = \frac{t^2}{2} - \cos(t) + C$ solves $y' = t + \sin(t)$. This is the **general solution** to the DE.

But **only one** function solves the DE and passes through the point $(0, 2)$.

This function solves the **initial value problem** $y' = t + \sin(t)$, $y(0) = 2$. It is the **particular solution** to the IVP.

Activity

Find solutions to each of the IVPs:

- $y' = \dfrac{t}{t^2 + 1}, y(0) = 3$ (What integration technique is best?)
- $y'' = t^2 + 2$, $y(1) = 4$ and $y'(1) = -2$

Why did you need two initial conditions for the second one? In general how does the number of initial conditions needed, relate to the order of the DE?

Equilibrium solutions

Any **constant** function $y(t)$ that solves a DE is an **equilibrium solution** (because constant functions don't change -- they are "in equilibrium").

- Equilibrium solution(s) for $\frac{dy}{dt} = 100y$?
- For $\frac{dy}{dt} = \frac{10-y}{y^2}$?
- For $\frac{dy}{dt} = t^3 + \cos(t)$?

There are three ways to solve a DE:

- Graphical, using **slope fields**
- Symbolic, using calculus and algebra
- Numerical, using approximation techniques such as **Euler's method**

The symbolic approach is **by far** the **least** commonly used.

Slope fields

Given a DE $\frac{dy}{dt} = f(t, y)$:

- Make a rectangular grid in the ty-plane
- At each point, find the value of dy/dt at that point using the right side of the DE
- Stick a tiny tangent line/vector at that point, with that slope.

The resulting "quiver" of arrows indicates the overall flow of the particular solutions to the DE.

Use the slope field tool to explore:

- $\frac{dy}{dt} = 100y$

- $\frac{dy}{dt} = \frac{10-y}{y^2}$
- $y' = (y-1)(y-3)^2$ (← from Class Prep re: equilibrium solutions)

Activity: Exercise 7 on pp. 137-138

Next time

- Solving **linear** differential equations using calculus (and lots of it)
- Start early on the Class Prep, allow time for Calc 2 review
- Skill Quiz: LA.3 (third/final attempt), LA.4 and LA.5 (second attempt), LA.6 (eigenvalues/eigenvectors first attempt)