MTH 302: Linear Algebra and Differential Equations
Section 03 Final Exam, Winter 2023

Instructions

- You are allowed to use a 3x5 inch notecard and a graphic or scientific calculator on this exam. No other technologies or implements are allowed, including the use of cell/smartphones.
- Except for multiple choice questions, please complete each item in the space below the item. Multiple choice responses will consist of circling the letter for your answer. Do not use any separate sheets of paper.
- Unless otherwise indicated, each item should consist of a correct answer that is clearly indicated (for example by putting a circle or box around it) *and* complete and correct supporting work. Answers without supporting work will not be given full credit.
- When done, return the exam to the professor or proctor.

Part 1: Concepts (16 points; 2 each)

For each item in this part, circle the letter for the one answer that seems most correct to you.

1. A system of linear equations has an augmented matrix that can be partially row-reduced to $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$. How many *equations* are in the system?

   a. 0
   b. 1
   c. 2
   d. More than 2 but finitely many
   e. Infinitely many
   f. Not possible to tell from this information

2. Using the same linear system from the previous question, how many *solutions* does the system have?
   a. 0
   b. 1
   c. 2
   d. More than 2 but finitely many
   e. Infinitely many
   f. Not possible to tell from this information

3. A system of linear equations has a free variable present if
   a. The augmented matrix for the system is upper-triangular
   b. The RREF of the augmented matrix has a row of all zeroes
   c. The RREF of the augmented matrix has a column of all zeroes
   d. The RREF of the coefficient matrix is the identity matrix
   e. None of these

4. The matrix A can be partially row-reduced to $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -5 \end{bmatrix}$. Which of the following is true about A?
   a. The columns of A are linearly independent
   b. The matrix A is invertible
   c. The columns of A span all of $\mathbb{R}^4$
   d. All of these
   e. None of these

5. If $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for A, then which of the following are also eigenvectors for A?
   a. $\mathbf{v} + \mathbf{w}$
   b. $3\mathbf{v} - \mathbf{w}$
   c. $3\mathbf{v}$
   d. $-\mathbf{w}$
   e. Both (c) and (d)
   f. All of the above
   g. Not possible to tell from this information

6. Suppose A is a matrix and $\mathbf{x}$ and $\mathbf{b}$ are vectors such that $A\mathbf{x} = \mathbf{b}$. Then
   a. $\mathbf{b}$ is a linear combination of the columns of A
   b. $\mathbf{x}$ is a linear combination of the columns of A
   c. $\mathbf{b}$ is a linear combination of the rows of A
   d. $\mathbf{x}$ is a linear combination of the rows of A
   e. None of these

7. Consider the system of differential equations given by $\mathbf{x}' = A\mathbf{x}$ where A is a 2x2 matrix whose determinant equals 1 and whose eigenvalues are 3 and 2. Which of the following is true about this system?
   a. It has no equilibrium points
   b. It has one equilibrium point, that is a sink
   c. It has one equilibrium point, that is a source
   d. It has one equilibrium point, that is a saddle
   e. It has one equilibrium point, that is a spiral
   f. It has infinitely many equilibrium points

8. Which of the following are equilibrium solutions of the differential equation $\frac{dP}{dt} = P(P - 10) + 21$?
   a. $P = 0$
   b. $P = 3$
   c. $P = 7$
   d. $P = 10$
   e. Both (a) and (d)
   f. Both (b) and (c)

Part 2: Computation (44 points)

1. (8 pts) Consider the matrices and the vector:

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \\ 5 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix} \quad v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Compute each of the following and simplify completely. If any cannot be computed, say so and then explain why.

   a. AB

   b. Av

   c. Bv

2. (8 pts) Given the initial-value problem $y' + 3t^2y = 0, y(0) = -2$, use Euler's Method to estimate the value of $y(0.75)$ using a step size of 0.25. *You may round your results to three decimal places. But no more than that!* Show all your steps.

3. (12 pts) Choose ANY TWO of the following differential equations and find their general solutions. Clearly indicate which two you are attempting; do not attempt more than two, or no credit will be given for this problem.

$$y' + y\cos(t) = 0 \qquad y' + \frac{3}{t}y = t^2 \qquad y' = 6y^2t$$

4. (8 pts) Consider the system of differential equations given by

$$\frac{dx}{dt} = -4x + \frac{1}{3}y \qquad \frac{dy}{dt} = 3x - 4y$$

a. Let $x = [x(t), y(t)]^T$. State the matrix A such that $x' = Ax$.

b. Here is some information from SymPy about the matrix A from the previous question:

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A.eigenvects()
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$$\left[\left(-5,\ 1,\ \left[\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right]\right),\ \left(-3,\ 1,\ \left[\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]\right)\right]$$

With this information, state formulas for the straight-line solutions for the system.

c. State the general solution for the system.

5. (8 pts) Find the general solution for the differential equation $y'' + 3y' = 18y$. Show all your work.

Part 3: Applications (32 points; 16 each)

1. If you are memorizing a list of words or digits over a certain span of time, let L be the fraction of the list that you have learned at time t. For example, $L = 0$ means you have not learned anything on the list, $L = 1$ means you've learned the entire list, and $L = 0.5$ means you've learned half of it. Typically, *the rate at which you learn is proportional to the fraction of the list left to learn*. (In other words, you learn faster when you are just starting out than later when there is less left to learn.) As a differential equation, this would be written as: $\frac{dL}{dt} = k(1 - L)$ where k is a proportionality constant.

   Suppose that two students are memorizing the first 500 digits of π, according to this model and it's been determined experimentally that $k = 1$.

   a. If one of the students knows one-third of the digits at time $t = 0$ and the other knows none of them at time $t = 0$, which student is learning most rapidly at this instant? Be sure to explain your reasoning.

   b. Will the student who starts out knowing none of the digits ever catch up to the student who starts out knowing one-third of the digits? Be sure to explain your reasoning.

2. A system of two tanks is connected in such a way that each of the tanks has an independent inflow that delivers salt solution to it, each has an independent outflow (drain), and each tank is connected to the other with an outflow and an inflow. The relevant information about each tank is given in the table below:

	Tank A	*Tank B*
Tank volume	100 liters	200 liters
Rate of inflow to the tank	5 liters/min	8 liters/min
Concentration of salt in inflow	7 g/liter	4 g/liter
Rate of drain outflow	4 liters/min	9 liters/min
Rates of outflows to other tank	To B: 3 liters/min	To A: 4 liters/min

Let S_A and S_B be the amount of salt in tanks A and B, respectively, at time t minutes. Set up, *but do not attempt to solve*, a system of differential equations whose solution would give S_A and S_B. (You do not need the initial quantities of salt in this problem.) Be sure to explain your work fully!

Part 4: Reflection (8 points)

For each of the three prompts below, give a brief but thoughtful response.

1. What was the most interesting thing you learned (mathematically speaking) in MTH 302 this semester, and why?
2. What was the most interesting thing you learned that *wasn't* math in MTH 302 this semester, and why?
3. If you had to do MTH 302 over again, what would you do differently the next time, and why?