MTH 302: Linear Algebra and Differential Equations

Review of Week 3 + Linear independence

31 January 2023

Housekeeping

- Miniproject 1 initial deadline is this Sunday.
  - To use $k+1$ as a subscript: Put it in curly braces. So, `U_{k+1}` produces U_{k+1}
  - Remember to hit "Submit" once the link is put in
  - Submit a complete, good-faith effort by Sunday; then no deadlines on revisions if needed.

Goals

- Use Theorems 1.5.1 and 1.5.2 to draw conclusions about systems using partial information about their matrices.
- Determine whether a set of vectors is linearly independent or linearly dependent.
- Add an item to Theorems 1.5.1 and 1.5.2 to link linear independence to earlier concepts.

These notes are posted as a Jupyter notebook on the Resource Page. Also a PDF of the slides.

Q&A from Class Prep

- If a set of vectors is linearly independent then does that automatically mean that those vectors call (span?) all of $\mathbb{R}^2$?
- How does linear dependence connect to span? Like if a (set of vectors) is linearly independent does that mean it always spans all of $\mathbb{R}^2$ (or $\mathbb{R}^3$)? What is the span of a linearly dependent set? -is it just those vectors?
- Is it possible to just set every set as a homogeneous equation to find out if the set is dependent or independent?

Theorem 1.5.1

Let $\mathbf{A}$ be an $m \times n$ matrix. Let $\mathbf{b}$ be a vector in $\mathbb{R}^m$. So, $\mathbf{Ax} = \mathbf{b}$ represents a system of m linear equations in n variables. Then *the following are equivalent*:

1. The system $\mathbf{Ax} = \mathbf{b}$ is consistent.
2. The vector $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$.
3. The vector $\mathbf{b}$ is in the span of the columns of $\mathbf{A}$.
4. When the augmented matrix $[\mathbf{A}\ \mathbf{b}]$ is row-reduced, there are no rows where the first n entries are zero but the last entry is nonzero.

Theorem 1.5.2

Let $\mathbf{A}$ be an $m \times n$ matrix. Then *the following are equivalent*:

1. The system $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$.
2. Every vector $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$.
3. The span of the columns of $\mathbf{A}$ is all of $\mathbb{R}^m$.
4. $\mathbf{A}$ has a pivot position in every row. That is, when $\mathbf{A}$ is row-reduced, there are no rows of all zeros.

Polling: Using these two theorems

Go to http://pollev.com/talbert

After polling: Go to Activity for January 31, Part 1.

Break time!

Linear independence

Intuitive idea: A set of vectors is *linearly dependent*, if one of the vectors is a linear combination of the others. The dependent vector is "redundant". If none of the vectors is a linear combination of the others then the set is *linearly independent*.

Formal definition: Given a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^m$, the set is *linearly dependent* if there is a nontrivial (nonzero) solution to the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

If the only solution to this equation is the trivial (all-zeroes) solution then S is *linearly independent*.

Why the intuitive idea = formal definition

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. Let's suppose that $\mathbf{v}_k$ is a linear combination of all the others. This means there are weights $x_1, \ldots, x_{k-1}$ so that:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_{k-1}\mathbf{v}_{k-1} = \mathbf{v}_k$$

Subtract the $\mathbf{v}_k$ over to get:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_{k-1}\mathbf{v}_{k-1} - \mathbf{v}_k = \mathbf{0}$$

Now $x_1 = 0, x_2 = 0, \ldots, x_{k-1} = 0, x_k = -1$ is a nozero solution.

To determine if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent or independent:

1. Solve $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$.
2. This is the same thing as the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$ where the columns of $\mathbf{A}$ are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and the vector $\mathbf{x}$ holds the weights.
3. Is there a *nonzero* solution?
4. Yes $\rightarrow$ The set of vectors is linearly dependent

5. No (only the trivial solution) $\rightarrow$ The set of vectors is linearly independent

Example

Randomly generate 5 vectors in $\mathbb{R}^5$. Is the set linearly dependent, or linearly independent?

In [ ]:
```
from sympy import *
init_printing()
S = randMatrix(5,5,-20,20)
S
```

In [ ]:
```
# The zero vector in R5
z = Matrix([0,0,0,0,0])

# Augment S with this vector to set up the system Sx = 0
system = S.col_insert(5,z)
system
```

In [ ]:
```
system.rref(pivots=False)
```

- Exactly one solution and it's the zero vector $\rightarrow$ Linearly independent
- Otherwise linearly dependent

Example

What if it were *four* vectors in $\mathbb{R}^5$?

In [ ]:
```
S = randMatrix(5,4,-20,20)
S
```

In [ ]:
```
z = Matrix([0,0,0,0,0])
system = S.col_insert(4,z)
system.rref(pivots=False)
```

Your turn

Go to Activity for January 31, Part 2.

Work in groups and use technology to do all the computations. Debrief later.

A bigger and better theorem

(Theorem 1.6.2) Let $\mathbf{A}$ be an $m \times n$ matrix. The following are equivalent:

1. The columns of $\mathbf{A}$ are linearly independent.
2. $\mathbf{A}$ has a pivot position in every column.
3. The equation $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

Theorem 1.6.2

It gets better if A is square

(Theorem 1.6.3) Let $\mathbf{A}$ be an $n \times n$ matrix. The following are equivalent:

1. The columns of $\mathbf{A}$ are linearly independent.
2. The columns of $\mathbf{A}$ span all of $\mathbb{R}^n$.
3. $\mathbf{A}$ has a pivot position in every column.
4. $\mathbf{A}$ has a pivot position in every row.
5. The equation $\mathbf{Ax} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$.

Theorem 1.6.2

True or false? (from Class Prep quiz)

Any set of vectors that include the zero vector, is automatically linearly dependent.

If a set of vectors is linearly dependent, then one of the vectors is a scalar multiple of one of the others.

Next Time

- Matrix arithmetic and intro to matrix inverses
- Class Prep due Wednesday 11:59pm
- Skill Quiz 3: LA.1 final appearance; LA.2 second attempt; new LA.3 ("I can determine if a collection of vectors is linearly independent.")