MTH 302: Linear Algebra and Differential Equations

Eigenvalues and eigenvectors

2023 February 9

Housekeeping

- Miniproject 2 now available
- New tutorials to go with Miniproject 2
- We're done using Perusall for quizzing
- Class Prep for February 14 now available

Today's Goals

- Unpack activity from Class Prep
- Finding the fixed lines and scaling factors for a matrix
- What are eigenvalues and eigenvectors
- Finding eigenvalues and eigenvectors of 2×2 and 3×3 matrices by hand and on SymPy
- Quiz: LA.3, LA.4, LA.5

Review of Class Prep activity

https://www.geogebra.org/m/JP2XZpzV

Goals:

Given an $n \times n$ matrix A, think of it as an **action that is performed on $\mathbb{R}^n$. Let's find:**

- Which lines or other spaces in $\mathbb{R}^n$ are fixed in place by this action
- The scaling factor being applied to vectors on the lines that are fixed in place
- A representative vector for each line that is fixed

Example at the board

$$A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$

Overall flow:

1. Find the scaling factors first
2. Then use each scaling factor to find a vector that is scaled by that factor
3. That vector will determine a line (or other space) that is fixed with that scaling factor applied

Debrief

For $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$:

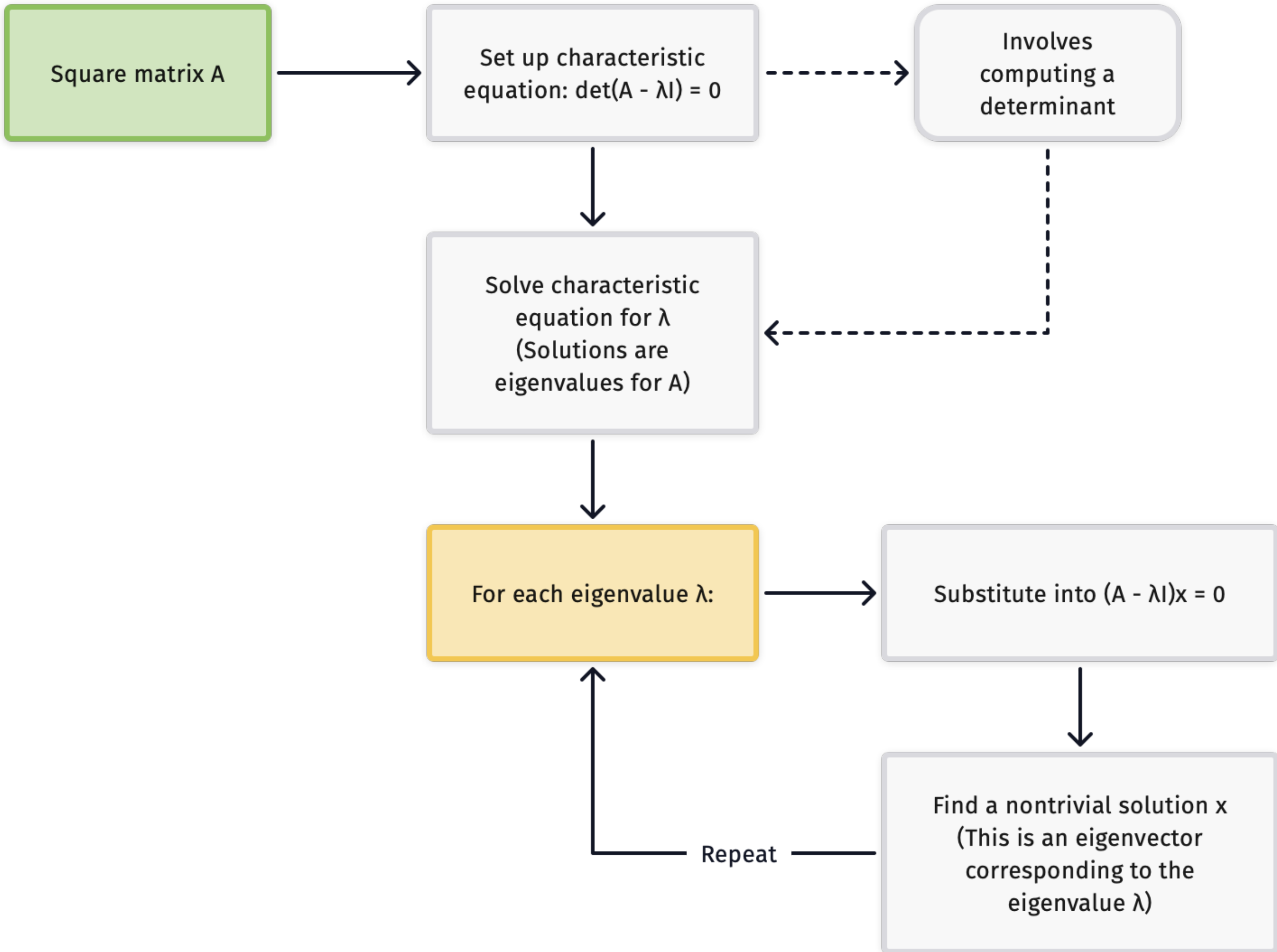
- The line in $\mathbb{R}^2$ containing $[1,1]^T$ (that is, $y = x$) is fixed in place and every vector on this line is rescaled by a factor of -2 when multiplied by A.
- The line in $\mathbb{R}^2$ containing $[-4,1]^T$ (that is, $y = -1/4x$) is fixed in place and every vector on this line is rescaled by a factor of 3 when multiplied by A.
- Note, we found these scaling factors and vectors by starting with the scaling factor, then finding the vector.

Definitions

For a given $n \times n$ matrix A, a nonzero vector $\mathbf{v}$ is said to be an **eigenvector** of A, if there exists a scalar λ such that

$$A\mathbf{v} = \lambda \mathbf{v}$$

(*English*: Multiplying $\mathbf{v}$ by A just rescales $\mathbf{v}$, but otherwise doesn't move it.) The scaling factor λ is called the **eigenvalue corresponding to the eigenvector $\mathbf{v}$**.


Square matrix A
Set up characteristic equation: det(A - λI) = 0
Involves computing a determinant
Solve characteristic equation for λ (Solutions are eigenvalues for A)
For each eigenvalue λ:
Substitute into (A - λI)x = 0
Find a nontrivial solution x (This is an eigenvector corresponding to the eigenvalue λ)
Repeat

Activity

On Part 1:

- First exercise has you go through the entire process of finding the eigenvalues and eigenvectors of a 2×2 matrix.
- Second and third exercises has you do the same with the two "weird" matrices from class prep.

Finding eigen"stuff" on SymPy

Tutorial posted to *Blackboard* > *Tutorials* > *SymPy Tutorials*

In [ ]:
```
from sympy import *
init_printing()
```

In [ ]:
```
A = Matrix([[-4,4], [-12,10]])

# The .eigenvals() method gives you just the eigenvalues along with their "multiplicity"
A.eigenvals()
```

In [ ]:
```
# The .eigenvects() method gives you the eigenvectors plus eigenvalue info
A.eigenvects()
```

In [ ]:
```
# Edge case 1
E = Matrix(2,2,[0,1,-1,0])
E.eigenvects()
```

In [ ]:
```
# Edge case 2
E = Matrix(2,2,[2,0,0,2])
E.eigenvects()
```

What happens when a matrix has an eigenvalue of 0?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

In [ ]:
```
A = Matrix([[1,0], [0,0]])
```

```
A.eigenvects()
```

What does this mean in terms of visual effects? Back to https://www.geogebra.org/m/JP2XZpzV

What about 3×3 matrices?

$$\mathbf{A} = \begin{bmatrix} -5 & -2 & 2 \\ 24 & 14 & -10 \\ 21 & 14 & -10 \end{bmatrix}$$

1. Find $\det(A - \lambda I_3)$:

In [ ]:
```
A = Matrix(3,3,[-5,-2,2,24,14,-10,21,14,-10])

# Use "s" instead of "lambda"; define as symbolic variable in SymPy
s = var("s")

# Set up the matrix whose determinant we want:
M = A - s*eye(3)
M
```

In [ ]:
```
M.det()
```

In [ ]:
```
# Using SymPy to solve; if right side is 0, only enter the left side
solve(M.det(), s)
```

In [ ]:
```
# Sub each eigenvalue back in; RREF to get nontrivial solution
AlmostThere = A + 3*eye(3)
AlmostThere
```

In [ ]:
```
AlmostThere.rref(pivots=False)
```

For class assessments:

- 2×2 matrices: Go through the whole process, find eigenvalues and corresponding eigenvectors.

- 3×3 matrices: You'll be given the eigenvalues, then go find corresponding eigenvectors; or given an upper-triangular matrix. $\rightarrow$ Why?

Application: Markov Chains

Example 1.3.1 -- What happens to the voter distribution over the long term?

In [ ]:
```
M = Matrix(3,3,[0.95, 0.03, 0.07, 0.02, 0.90, 0.13, 0.03, 0.07, 0.80])
x = Matrix([120, 110, 20])

years = 100
for i in range(years):
    x = M*x
x
```

In [ ]:
```
M.eigenvects()
```

In [ ]:
```
x
```

In [ ]:
```
L = 0.769631234107679/117.456660176547
L * x
```

Skill Quiz

- Second attempt LA.3
- First attempt LA.4
- First attempt LA.5

In [ ]:
```

```