MTH 302: Linear Algebra and Differential Equations

Activities: 7 February 2023

Part 1: Matrix inverses

1. Use SymPy to find the inverses of each of the following matrices, or determine that they aren't invertible:

   (a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$

   (c) $\begin{bmatrix} -5 & 3 & 6 \\ 9 & 9 & -4 \\ 3 & -9 & 0 \end{bmatrix}$

2. **(AA)** Use the result of part (c) above to solve the following linear system *without doing any row reduction at all*:

$-5x_1 + 3x_2 + 6x_3 = 5$

$9x_1 + 9x_2 - 4x_3 = 7$

$3x_1 - 9x_2 = 3$

Hint: If A is the matrix from part (c) above, this system is the matrix-vector equation $A\mathbf{x} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$. Could you perform a certain operation to both sides of this equation to find $\mathbf{x}$? When you are done, **write one sentence to explain how inverses of matrices can be used to solve linear systems.**

3. Consider these three matrices:

   $$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Define these in SymPy, then create a random 3×3 matrix called A. Compute E_1A, E_2A, and E_3A. What effect does each matrix have on A when it's multiplied to A?

4. (Continuation of exercise 4) Without actually computing the inverse of each of the matrices in exercise 3, explain why you know each matrix is invertible, and make a guess as to what the inverse of each matrix is. (Then you can check your work on the computer.)

Part 2: Determinants

> Reminder: To find the determinant of `A` in SymPy, it's `A.det()`.

1. Compute the determinants of matrices (a) and (b) in Part 1 by hand. The first one is much easier than the others; why is that?
2. A square matrix is *upper-triangular* if all the entries below the "main diagonal" are zero. For example, the matrix in (a) is upper-triangular. Copy the following into a code cell (you can find it here: https://gist.github.com/RobertTalbert/0d6042b7c5d88b5a0978b29cce7a670d) and execute it:

```
def randUT(n,lower,upper):
  A = randMatrix(n,n,lower,upper)
  for j in range(n-1):
    for i in range(j+1,n):
      A[i,j] = 0
  return A
```

This is a **function** in Python, which is what we call a custom command to do a computation. This particular function generates a random $n \times n$ upper-triangular matrix with entries between `lower` and `upper`. After executing the code, try out by running this code:

```
A = randUT(5, -10, 10)
A
```

Generate several upper-triangular matrices using this code; look at each matrix after you define it; then compute its determinant. **What is a quick way to find the determinant of an upper-triangular matrix?** Study your outputs, then fill in the blank:

> If A is upper-triangular then its determinant can be found by .

3. **(AA)** What is the relationship between $\det(A)$, $\det(B)$, and $\det(AB)$? Make up some random matrices A and B, compute each of those three quantities and compare them, then make a conjecture.

4. If A is an invertible matrix, what is the relationship between $\det(A)$ and $\det(A^{-1})$? Make up some random matrices and experiment. (You are almost certain to get an invertible matrix if you randomly generate one, but you might not.)