MTH 302: Linear algebra and differential equations

Activities for Thursday, March 16

Part 1

Look at the system of DE's

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{x}$$

The eigenvalue-eigenvector pairs of the matrix here are $\lambda_1 = -1, \mathbf{v}_1 = [-1, 1]^T$ and $\lambda_2 = 4, \mathbf{v}_2 = [2, 3]^T$. (If you are not sure how to find these using a computer, pause and try it, or ask a question.)

1. Give the two straight-line solutions for this system.
2. Write the general solution for this system.
3. Now let's think about how the solutions to this system behave over long periods of time. You can click here for the direction field or "phase plane". By plotting them, describe the long-term behavior of the particular solutions with these initial conditions: $\mathbf{x}(0) = [1, 1]^T$, $\mathbf{x}(0) = [0, 2]^T$, and $\mathbf{x}(0) = [3, 0]^T$. (Click on the "Initial points" button to enter a specific initial condition.)
4. Now plot the solution that has $\mathbf{x}(0) = [-1, 1]^T$. What's different? What do you think the long-term behavior of this solution is, based on the phase plane plot?
5. Using the general solution from question 2, solve the IVP that has $\mathbf{x}(0) = [-1, 1]^T$. What is the long-term behavior of *this* solution? You should be able to tell without graphing.
6. True or false: $(0, 0)$ is a stable equilibrium point for this system.

Part 2: Equilibrium points

We've just learned that the trajectory of a non-equilibrium solution to a linear homogeneous system of first-order DEs is dependent on the signs (positive/negative) of the eigenvalues of its matrix. Use this fact to give examples with linear, homogeneous systems of DEs that have the following characteristics:

1. The system has **two straight-line solutions**, and all non-equilibrium solutions move **away from** the origin over time
2. The system has **two straight-line solutions**, and all non-equilibrium solutions move **toward** the origin over time
3. The system has **no straight-line solutions**, and all non-equilibrium solutions move **away from** the origin over time

4. The system has **no straight-line solutions**, and all non-equilibrium solutions move **toward** the origin over time
5. (*Challenge*!) The system has **no straight line solutions**, and all non-equilibrium solutions **orbit** the origin in a circular pattern that neither approaches nor moves away from the origin.

We've already seen an example of a system that has two straight-line solutions, and its non-equilibrium solutions sometimes move toward the origin over time but sometimes move away, depending on the initial condition:

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{x}$$

One way to get these examples, if you can't find any we've already seen, is to use the `randMatrix` function in SymPy to generate some random 2×2 matrices. Every 2×2 matrix corresponds to a linear homogeneous system of DE's. Once you have your matrix, you can either plot the corresponding system in the phase plane; or you can use a computer to get information about the matrix which you can then use to tell the number of straight line solutions and classify the behaviors.

Example of use: The following generates a random 2×2 matrix with values between -10 and 10:

```
randMatrix(2,2,-10,10)
```

Part 3: Putting it all together

1. **(AA7)** Consider the system of differential equations $\mathbf{x}' = A\mathbf{x}$ given by $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$.

   (a) Determine the general solution to the system.

   (b) Find all equilibrium points to the system and classify each one (stable/sink, unstable/source, saddle point, stable spiral, unstable spiral, or center).

   (c) By hand, sketch the straight-line solutions to the system with the trajectories indicated. Then use these to plot a few nonlinear trajectories in the phase plane.

2. Consider the system of differential equations $\mathbf{x}' = A\mathbf{x}$ given by $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

   (a) Find the general solution.

   (b) Find the particular solution if $\mathbf{x}(0) = [-3, 1]^T$.

   (c) Find and classify the equilibrium points.

   (d) Sketch the straight-line solutions and a few nonlinear trajectories.