Linear equations and systems

MTH 302 January 12

Example

Tickets to a basketball game are 25 for kids and 50 for adults. At one of the games, 2000 people attend and the total gate revenue is $70,000.

How many kids attended, and how many adults?

Let x be the number of children attending and y the number of adults. Write **two equations** that represent the two pieces of info in the second sentence.

$$\begin{aligned} x + y &= 2000 \\ 25x + 50y &= 70000 \end{aligned}$$

Linear equation

A *linear equation in* n *variables* is an equation that looks like this:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

Left side is nothing but variables multiplied by numbers and then added together. Nothing else is done to the variables.

$$\begin{aligned} x + y &= 2000 \\ 25x + 50y &= 70000 \end{aligned}$$

System of equations

A **system of** m **linear equations in** n **unknowns** (or an "$m \times n$ system") is a collection of m linear equations with n variables. A **solution** to a system is a list of specific values for the variables that makes all the equations in the system true at the same time.

A 3×5 linear system:

$$\begin{aligned} 2x_1 + 3x_2 - x_4 + 3x_5 &= 10 \\ x_2 + x_4 &= 1 \\ -x_1 + 3x_2 + 5x_3 + 2x_4 - 100x_5 &= 0 \end{aligned}$$

a. Which of the following equations are linear? Please provide a justification for your response.

1. $$2x + xy - 3y^2 = 2.$$
2. $$-2x_1 + 3x_2 + 4x_3 - x_5 = 0.$$
3. $$x = 3z - 4y.$$

b. Consider the system of linear equations:

$$\begin{aligned} x + y \quad\; &= 3 \\ y - z &= 2 \\ 2x + y + z &= 4. \end{aligned}$$

1. Is $(x, y, z) = (1, 2, 0)$ a solution?
2. Is $(x, y, z) = (-2, 1, 0)$ a solution?
3. Is $(x, y, z) = (0, -3, 1)$ a solution?
4. Can you find a solution in which $y = 0$?
5. Do you think there are other solutions? Please explain your response.

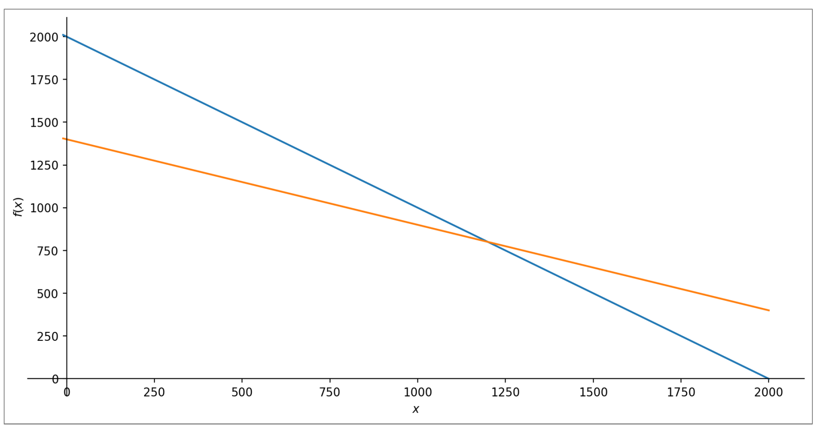
$$\begin{aligned} x + y &= 2000 \\ 25x + 50y &= 70000 \end{aligned}$$

Activity

Using whatever means you can think of, determine if this 2×2 system has a solution. If it doesn't have a solution, be ready to explain why. If it does have a solution, figure out *how many* it has, and what it is/they are.

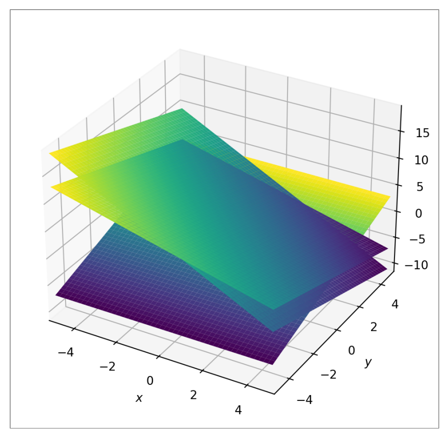
There are basically three ways to find solutions to a system

You can **graph** the equations and see if their graphs intersect. This works OK for systems with two variables:



…but for three variables, it gets weird:

$$\begin{aligned} x + y + z &= 3 \\ y - z &= 2 \\ 2x + y + z &= 4 \end{aligned}$$



But graphical intuition is very important for us

Since solutions to systems are intersections of lines or planes (ir higher-dimensional versions of these):

- A solution might have solutions (the system is **consistent**), or it might have none (it’s **inconsistent**)
- A consistent system can have either **exactly one** solution, or **infinitely many solutions**, but nothing in between.

Check this out

Example of a *consistent* system with *exactly one* solution:

$$\begin{aligned} x + y &= 2 \\ x - y &= 0 \end{aligned}$$

Example of a *consistent* system with *infinitely many* solutions:

$$\begin{aligned} x + y &= 2 \\ 3x + 3y &= 6 \end{aligned}$$

Example of an *inconsistent* system:

$$\begin{aligned} x + y &= 2 \\ x + y &= 0 \end{aligned}$$

Back to solution methods: You could also **substitute** - Pick an equation, solve for a variable, plug in to the other equations, and repeat until you have values. Tedious but doable for 2 or 3 variables.
But this? No thanks:

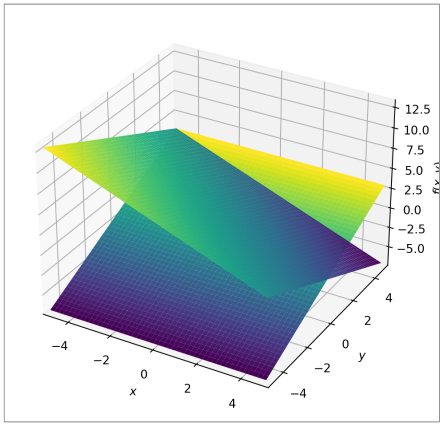
$$\begin{aligned} 2x_1 + 3x_2 - x_4 + 3x_5 &= 10 \\ x_2 + x_4 &= 1 \\ -x_1 + 3x_2 + 5x_3 + 2x_4 - 100x_5 &= 0 \end{aligned}$$

The best option is **elimination**. Works by performing a combination of three *elementary operations*:

1. *Replace* a row, with the sum of itself and a multiple of another row.
2. *Swap* any two rows.
3. *Scale* a row by multiplying both sides by a nonzero constant.

Each operation produces a system that is **equivalent** to the original one (it has the same solutions).
At the board: how this works.

$$\begin{aligned} x + y &= 3 \\ y - z &= 2 \\ 2x + y + z &= 4 \end{aligned} \implies \begin{aligned} x + z &= 1 \\ y - z &= 2 \\ 0 &= 0 \end{aligned}$$



KEY INSIGHT

The variables in the elimination process don’t matter that much. They are just there as placeholders for the coefficients. So forget them and put the coefficients and right-hand sides into an array called the **augmented matrix** for the system.

$$\begin{aligned} x + y + z &= 3 \\ y - z &= 2 \\ 2x + y + z &= 4 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 1 & 4 \end{bmatrix}$$

Now you can do elimination just with the matrix. (Board)

$$\begin{aligned} x + 2y \phantom{{}- 3z} &= 4 \\ 2x + y - 3z &= 11 \\ -3x - 2y + z &= -10 \end{aligned}$$

Activity

Convert this system into an augmented matrix. Then use a sequence of the three elementary operations to try to find a solution (there may not be one).

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

This matrix is in **reduced row echelon form (RREF)**:

- If there are any rows that are all zero, they are at the bottom.
- The first nonzero entry in a given row is 1, and it’s in a column that’s to the right of the first nonzero entry in any row above it.
- Every other entry in a column with a leading 1 is 0.

Consistent with one solution:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \end{bmatrix}$$

Consistent with infinitely many solutions:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -5 \end{bmatrix}$$

Inconsistent:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -5 \end{bmatrix}$$

Is shorthand for the system

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ x_2 - x_3 &= -5 \end{aligned}$$

To find a solution: Pick anything for x_3. Then $x_1 = 1 - 2x_3$ and $x_2 = x_3 - 5$.

- x_3 is a **free variable**
- x_1 and x_2 are **pivots** or sometimes “determined” or “dependent” variables

Activity 1.2.4. Identifying reduced row echelon matrices. Consider each of the following augmented matrices. Determine if the matrix is in reduced row echelon form. If it is not, perform a sequence of scaling, interchange, and replacement operations to obtain a row equivalent matrix that is in reduced row echelon form. Then use the reduced row echelon matrix to describe the solution space.

a. $\left[\begin{array}{ccc|c} 2 & 0 & 4 & -8 \\ 0 & 1 & 3 & 2 \end{array}\right]$.

b. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

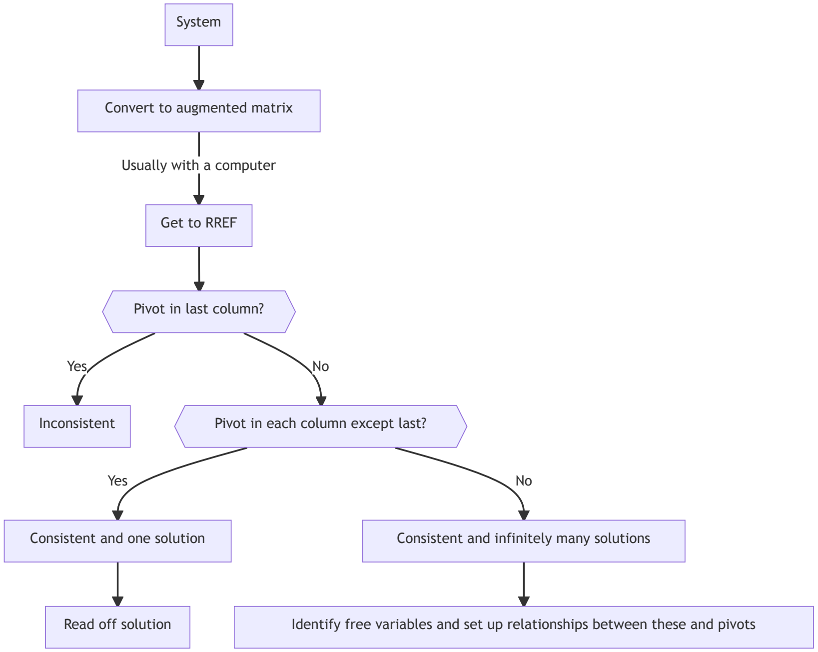
c. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$.

d. $\left[\begin{array}{ccc|c} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 2 \end{array}\right]$.

e. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

More facts about RREF

- The process of putting a matrix into RREF is called **Gauss-Jordan elimination** and it’s kind of a big deal
- *Skill LA.1: I can solve a system of linear equations by converting it into an augmented matrix and putting into reduced row echelon form*. Coming next Thursday on Skill Quiz 1 and in Practice Set 1.
- **But this is the only place you will be asked to do elimination by hand!** In all other situations you will use a computer.
- So don’t stress over doing RREF by hand, just practice until you can do it on simple systems without a bunch of mistakes. ⟶ **Optional RREF Practice set on WeBWorK**



$$\begin{cases} 2x - 3y + z = -1 \\ x - y + 2z = -3 \\ 3x + y - z = 9 \end{cases}$$

Solve this system using matrices. If there are no solutions, say so and explain how you know (using a matrix).

Next

- Remainder of today: Choose your adventure. Get 1-1 help; work on Practice Set 1; work on Startup tasks; work on RREF practice
- (Section 04 only) Please put tables back in rows
- Sunday: Complete Practice Set 1 by 11:59pm ET
- Tuesday:
  - Complete Class Prep for Jan 17 (Blackboard > Class Prep) by 11:59pm ET **Monday**
  - Focus for Tuesday: Linear combinations of vectors (and what this has to do with solutions to systems)