Conditional Statements and Direct Proof

Statements and Conditional Statements

What is a "statement"?

In mathematics, a **statement** (or *proposition*) is **a declarative sentence that is either true or false, but not both**. Let's unpack that:

- *Declarative sentence*: A statement makes an assertion about the truth of something. For example "All odd integers have a binary representation that ends in a 1" and "Detroit is the capital of Michigan" are both statements. But a sentence like "Detroit is a great place to visit" is not a statement (it's an opinion). Neither is a mathematical sentence like $x + 2 = 10$ because here, the truth of the sentence can't be determined because we don't know what x is. It's true for some values of x but not true for others.
- *Either true or false but not both*: There can be no ambiguity in the truth value of a statement. For example "Detroit is the capital of Michigan" is definitely false. But "Detroit is a great place to visit" isn't really either true or false; or maybe it's both.

What is a "conditional statement"?

A **conditional statement** is a statement that can be phrased in the form **"If P, then Q"** where P and Q are smaller, simpler statements. It is a statement in which there is a cause and an effect, and it claims that the effect follows from (or is "implied by") the cause. Here are some examples:

- If n is an even number, then n^2 is even.
- If if is not raining, then I am riding my bike.
- Whenever the input to $f(n)$ is even, the output is $n/2$.
- An odd input to $f(n)$ implies that the output is $3n + 1$.
- The output is $3n + 1$ when the input is odd.

Notice that the wording of a conditional statement doesn't have to be literally "If...then". There just needs to be a cause and an effect.

In mathematical notation, "If P then Q" is written $P \to Q$.

What are the parts of a conditional statement?

In the conditional statement "If P, then Q", the statement P is the **hypothesis** and Q is the **conclusion**.

Statement	Hypothesis	Conclusion
If n is an even number, then n^2 is even.	n is an even number	n^2 is even
If if is not raining, then I am riding my bike.	It is not raining	I am riding my bike
Whenever the input to $f(n)$ is even, the output is $n/2$.	The input to $f(n)$ is even	The output is $n/2$
An odd input to $f(n)$ implies that the output is $3n+1$.	The input to $f(n)$ is odd	The output is $3n+1$
The output is $3n+1$ when the input is odd.	The input is odd	The output is $3n+1$

Notice:

1. The hypothesis is not always simply the first statement you see in the sentence. (See the last example above.) The hypothesis is the "cause" and the conclusion is the "effect" that follows, regardless of where the wording places them in the sentence.
2. We don't include the word "If" as part of the hypothesis, nor do we include the word "then" as part of the conclusion.

What does it mean for a conditional statement to be true?

A conditional statement is "true" when **the conclusion follows from the hypothesis**. Whenever the hypothesis is true, the conclusion is also true. You can think of a conditional statement as being like a *promise*. A "true promise" is one where if you satsify the conditions of the promise, you will get what you were promised.

By contrast, a "false" conditional statement is like a false promise, which is one where you're promised a reward for doing something -- and then you do the thing, but don't get the reward.

Under what conditions will a conditional statement be true?

Based on what we just saw, **a conditional statement is true, if the conclusion is true whenever the hypothesis is true**. Thought of differently, **a conditional statement is false if the hypothesis is true but the conclusion is false** -- like a false promise, where you did what you were supposed to do but didn't get the reward.

We can phrase this in a truth table, with four rows, one for each possible combination of true/false for the hypothesis and conclusion:

P	Q	$P \to Q$
True	True	True
True	False	False
False	True	True
False	False	True

The result of the second row is highlighted to point out that **conditional statements are true in all cases except one: Where the hypothesis is true but the conclusion is false.**

Why is $P \to Q$ considered true, if the hypothesis P is false?

A conditional statement is true, if the conclusion follows *when the hypothesis is true*. A promise is "true" if you get what you were promised when you fulfilled the requirements of the promise.

But when the hypothesis is *false*, a conditional statement says nothing about what should happen as a "result". If you are promised a cookie if you do your homework, but then don't do your homework, then the person making the promise could still give you a cookie if they wanted -- or withhold the cookie if they wanted -- and it wouldn't make them a liar. The promise of getting a cookie if you do your homework isn't violated, because the promise only applies if you *do* your homework. If you *don't* do your homework, then anything might happen.

Direct Proof

What is a "direct proof" of a conditional statement?

Many mathematical conjectures come in the form of conditional statements, and we want to **prove** those statements. Sometimes we prove statements using mathematical induction if the statement involves a concept or process that has recursion in it. But not all statements are about recursive concepts or processes, so induction is just one of many proof techniques we might use.

When proving a *conditional* statement, it turns out there are several strategies that might work better than mathematical induction. The simplest of those strategies is called **direct proof**. Given the conditional statement $P \to Q$, a direct proof goes like this:

- First, assume that the hypothesis P is true.
- Then, use statements of definitions, computations, or previously-proven facts to explain why the conclusion Q **must also** be true.

So: Assume the hypothesis and then prove the conclusion.

Direct proof works because $P \to Q$ is automatically true if P is false; so it's safe to just assume that P is true. Making that assumption, we want to show that it is logically impossible for the conclusion *not* to follow.

How do you outline a direct proof?

Setting up the framework for a direct proof is easy:

1. First state that you are using a direct proof. "We will prove this statement directly", for example. This tells the reader what to expect.
2. State that you are assuming the hypothesis. For example, "Assume that n is an even number"; or "Suppose that the input to $f(n)$ is even."
3. The first two steps are typically the first two sentences of a direct proof. Now write the *last* sentence of the proof, which is a statement that the conclusion has been proven. For example, "Therefore n^2 is even".

The actual *proof* consists of the steps in between items 2 and 3. But the framework is always the same.

How do you construct a direct proof from the outline?

Direct proofs work by assuming the hypothesis and then building a chain of reasoning -- based on definitions, computations, and previously-known or previously-proven facts -- to arrive at the conclusion. Here is an example of how this might go.

Let's prove that **If n is odd, then n^3 is odd.

First, the outline:

- We will prove this using a direct proof.
- First assume that n is odd.
- ....
- Therefore, n^3 is odd.

The "......." above is where our chain of reasoning will go. It's sometimes helpful to view this in a table instead of a list, where we can work one line at a time, stating our steps and then being sure to provide an explanation for those steps. **It's very important to give a justification for every step in a proof** so that the reader can be fully convinced of the soundness of your argument.

Statement	Reason
Assume n is odd	Assuming the hypothesis
.... More steps will go here	
n^3 is odd	*We don't know the reason why, yet*

Let's try to add some steps to the proof. We can do this in one of two ways:

- By making a **forward step**, where we take a statement on the left that's already been established and using a definition, a computation, or a previously-proven fact to elaborate on it somehow; or
- By making a **backward step**, where we start with a statement on the left that hasn't been established yet, and rewrite or rephrase it in some way using a definition, computation, or previously-proven fact.

For example, we could make a forward step by going to line 1 and rephrasing the statement "n is odd" using a definition. That definition is:

> **Definition:** An integer n is said to be **odd** if there is some other integer, k, such that $n = 2k + 1$.

(For example, we know 21 is odd because $21 = 2(10) + 1$. Likewise 99 is odd because $99 = 2(45) + 1$.)

So let's add a forward step to the proof:

Statement	**Reason**
Assume n is odd	Assuming the hypothesis
There exists an integer k such that $n = 2k + 1$.	Definition of "odd"
.... *More steps will go here*	
n^3 is odd	*We don't know the reason why, yet*

Note that all we did was use the definition, and we stated this clearly in the right column.

What now? We could do a backwards step using the same idea:

Statement	**Reason**
Assume n is odd	Assuming the hypothesis
There exists an integer k such that $n = 2k + 1$.	Definition of "odd"
.... *More steps will go here*	
There exists an integer m such that $n^3 = 2m + 1$.	*We don't know why yet*
n^3 is odd	Definition of "odd"

Notice a few things:

- We did not use "k" in the next to last line, but "m". That's because although it says "k" in the definition statement, if we used k in both the second and next-to-last lines it would be saying that it's literally the same number k that makes n and n^3 odd. But this may not be true. The definition just uses k as a placeholder for some integer; it doesn't mean that exact letter is what you use every time.
- Once we made the backwards step, we have a justification for the last line; but we still don't have a justification for the next to last line. We have more steps to add before that happens.

What now? It may require some guesswork and dead ends. Maybe -- since we know we are trying to prove something about n^3, we could take the result of line 2 and cube it:

Statement	Reason
Assume n is odd	Assuming the hypothesis
There exists an integer k such that $n = 2k + 1$.	Definition of "odd"
Therefore $n^3 = (2k+1)^3$	Cube both sides
.... *More steps will go here*	
There exists an integer m such that $n^3 = 2m + 1$.	*We don't know why yet*
n^3 is odd	Definition of "odd"

This seems clear enough. Now maybe we could do another forward step and expand the right side of line 3:

Statement	Reason
Assume n is odd	Assuming the hypothesis
There exists an integer k such that $n = 2k + 1$.	Definition of "odd"
Therefore $n^3 = (2k+1)^3$	Cube both sides
So $n^3 = 8k^3 + 12k^2 + 6k + 1$	Algebra
.... *More steps will go here*	
There exists an integer m such that $n^3 = 2m + 1$.	*We don't know why yet*
n^3 is odd	Definition of "odd"

(The "algebra" can be done by hand or on a computer, but it's still algebra.) Now what? Perhaps we need to look ahead at the next-to-last line. We need to write n^3 as 2 times an integer, plus 1. But we can do that with the result of line 4!

Statement	Reason
Assume n is odd	Assuming the hypothesis
There exists an integer k such that $n = 2k + 1$.	Definition of "odd"
Therefore $n^3 = (2k+1)^3$	Cube both sides
So $n^3 = 8k^3 + 12k^2 + 6k + 1$	Algebra
So $n^3 = 2(4k^3 + 6k^2 + 3k) + 1$	Factoring out a 2
.... *More steps will go here*	
There exists an integer m such that $n^3 = 2m + 1$.	*We don't know why yet*
n^3 is odd	Definition of "odd"

That insight, to factor a 2 out from part of the result of line 4, only happened because we planned ahead and did a backward step earlier. Otherwise it's hard to understand how someone would think to make that step.

Lesson: Unlike doing basic math computation exercises, proof writing is not a linear process. It takes a combination of forward, backward, and sometimes lateral thinking before you arrive at a good argument. In that sense proof writing is a lot like writing nontrivial computer programs.

We have enough here to finish up the proof with one additional line:

Statement	Reason
Assume n is odd	Assuming the hypothesis
There exists an integer k such that $n = 2k + 1$.	Definition of "odd"
Therefore $n^3 = (2k+1)^3$	Cube both sides
So $n^3 = 8k^3 + 12k^2 + 6k + 1$	Algebra
So $n^3 = 2(4k^3 + 6k^2 + 3k) + 1$	Factoring out a 2
The number $4k^3 + 6k^2 + 3k$ is an integer	Adding and multiplying integers gives another integer
There exists an integer m such that $n^3 = 2m + 1$.	Set m equal to the integer from the previous line
n^3 is odd	Definition of "odd"

And now this is a very fleshed-out outline that shows why the conclusion of the conditional statement must logically follow from the hypothesis, with each step clearly articulated and justified using simple concepts and language.

A completed writeup of the proof above

To prove: If n is an odd integer, then n^3 is odd.

Proof: Let n be an odd integer. Therefore there is an integer k such that $n = 2k + 1$. Cubing both sides gives

$$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1$$

Factoring 2 from the first three terms above gives

$$n^3 = 2(4k^3 + 6k^2 + 3k) + 1$$

Now $4k^3 + 6k^2 + 3k$ is an integer because we are just adding and multiplying integers together. Call this number m. Therefore, there is an integer, m, such that $n^3 = 2m + 1$ and therefore n^3 is odd. 😎

Notes on this proof:

- Notice it's shorter than the table.
- You can vary the langauge if you want as long as it doesn't mess up the underlying logic.
- You will often see some kind of symbol at the end of a proof to tell the reader the proof is done. (I like emojis, but most mathematicians use a box: $\blacksquare$ or $\square$.)
- **Above all:** Remember that proofs are meant to be *read*, so keep your reader in mind at all times. Make things clear, simple, and engaging and don't ask the reader to do any unnecessary work.