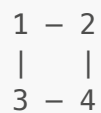


MTH 325: Fundamental properties of trees

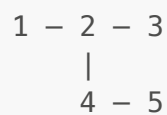
Part 1: Tree or no? (4 minutes)

For each graph below, determine if it is a tree. If not, explain why.

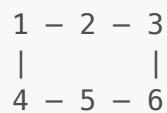
Graph A:



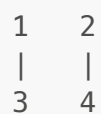
Graph B:



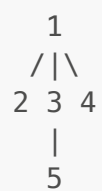
Graph C:



Graph D:



Graph E:



Question 1.1: Which of the above graphs are trees?

Question 1.2: For the non-trees, what would you need to change (add or remove edges) to make them trees?

Part 2: Counting Vertices and Edges (6-7 minutes)

Question 2.1: Based on the data collected in the spreadsheet from Class Prep, what pattern do you notice between the number of vertices and the number of edges?

Question 2.2: Make a conjecture: For a tree with n vertices, how many edges does it have?

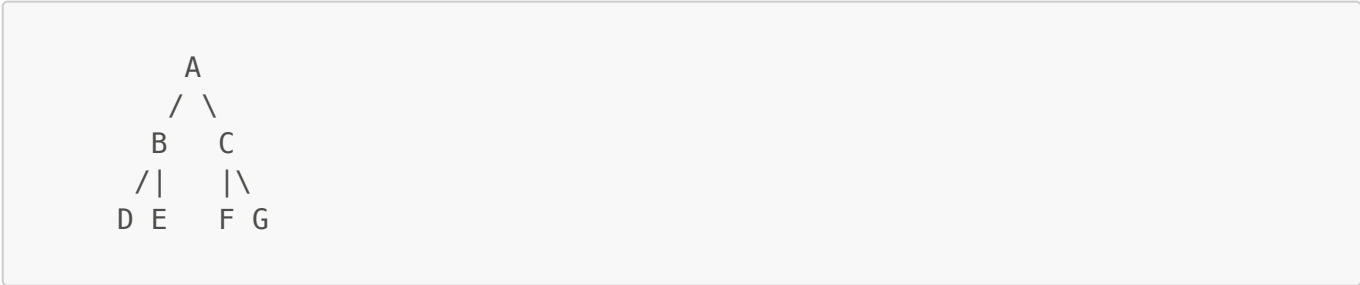
Question 2.3: Test this conjecture out by drawing some trees with specific numbers of vertices, then count the edges.

Number of Vertices (n)	Draw a Tree	Number of Edges (m)
1		
2		
3		
4		
5		
6		
10		

Question 2.4: Prove that your conjecture is true, using one of our proof techniques from class. (Which one "feels" the best?)

Part 3: Paths in Trees (5-6 minutes)

Consider this tree T:



Question 3.1: How many different paths are there from vertex A to vertex D?

Question 3.2: How many different paths are there from vertex B to vertex G?

Question 3.3: Pick any two vertices in the tree above. How many paths exist between them?

Question 3.4: Make a conjecture: In ANY tree, how many paths exist between any two vertices?

Question 3.5: Suppose a connected graph G has the property that there is exactly one path between every pair of vertices. Must G be a tree? Why or why not?

Part 4: Leaves and Internal Vertices (5-6 minutes)

Definition: A **leaf** (or pendant vertex) in a tree is a vertex of degree 1. An **internal vertex** is a vertex of degree ≥ 2 .

Question 4.1: In the tree from Part 3, identify all the leaves and all the internal vertices.

Question 4.2: Draw several different trees with exactly 8 vertices. For each tree you draw, count the number of leaves.

Your Tree	Number of Leaves
Tree 1	
Tree 2	
Tree 3	

Question 4.3: What is the minimum number of leaves a tree with 8 vertices can have?

Question 4.4: What is the maximum number of leaves a tree with 8 vertices can have?

Question 4.5: Make a conjecture: What is the minimum number of leaves that ANY tree must have? (Consider trees with different numbers of vertices)

Part 5: Alternative Characterizations (3-4 minutes)

We defined a tree as "a connected graph with no cycles." However, there are other equivalent ways to define a tree.

Question 5.1: Based on your work in Part 3, complete this statement:

"A tree is a graph in which there is exactly _____ between every pair of vertices."

Question 5.2: Based on your work in Part 2, complete this statement:

"A tree with n vertices is a connected graph with exactly _____ edges."

Question 5.3: Consider these two statements:

- Statement X: " G is a connected graph with n vertices and $n-1$ edges"
- Statement Y: " G is a tree"

Do you think X implies Y? That is, if a graph satisfies Statement X, must it be a tree?

Question 5.4: What about the reverse direction? If G is a tree (Statement Y), must it satisfy Statement X?

Part 6: Challenge Problems (If Time Permits)

Challenge 1: Prove that every tree with at least 2 vertices has at least 2 leaves.

Hint: Think about starting at any vertex and following a path as far as you can go without repeating vertices.

Challenge 2: Suppose you have a tree T with 15 vertices. You know that 3 vertices have degree 3, and 2 vertices have degree 4. How many leaves does this tree have?

Hint: Use the handshaking lemma and the relationship between vertices and edges in trees.

Challenge 3: True or False: If you remove any edge from a tree, the resulting graph is disconnected.

Challenge 4: True or False: If you add any edge to a tree (connecting two existing vertices), the resulting graph contains a cycle.