Module 7B: The binomial coefficient

MTH 225
19 Oct 2020

Agenda

- Review of Daily Prep activity + Q/A time
- Activities:
    - Review: The binomial coefficient and what it counts
    - The basic recurrence relation for the binomial coefficient; using it to construct **Pascal’s Triangle**
    - Mixed bag of counting problems

The binomial coefficient $\binom{n}{k}$ is

The number of bit strings of length n whose bits add up to k

The number of k-element subsets of a set with cardinality n

The coefficient on the $x^k y^{n-k}$ term when you expand $(x+y)^n$

All of the above

Tc 0

$\binom{8}{0}$ equals

- 0
- 1
- 8
- Nothing, it's undefined

To 0

$\binom{10}{4}$ equals

- $\binom{5}{2}$
- $\binom{10}{6}$
- $\binom{9}{4} + \binom{9}{3}$
- All of the above
- Both (b) and (c) but not (a)
- None of the above

To 0

$$\binom{n}{k} = \begin{cases} \text{Number of bit strings of length n, with “weight” = k} \\ \text{Number of k-element subsets of an n-element set} \\ \text{Coefficient on } x^k y^{n-k} \text{ in } (x+y)^n \\ \text{Number of ways to select k objects from a group of n objects} \end{cases}$$

binom(n,k)
if typed from the keyboard/Wolfram Alpha

```
>>> from scipy.special import binom
>>> binom(10,6)
210.0
```

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

An n-bit string with weight k either starts with a 0 or with a 1. So it's either

- A 0 followed by an (n-1)-bit string of weight k, or
- A 1 followed by an (n-1)-bit string of weight k-1.

There are binom(n-1, k) of the first kind and binom(n-1,k-1) of the second kind. There's no overlap in the kinds. So the Additive Principle says add those two numbers together.

This is a **recurrence relation.**

These all = 1 by definition

Row 0: $\binom{0}{0}$

Row 1: $\binom{1}{0} \quad \binom{1}{1}$

Row 2: $\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$

Row 3: $\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$

Row 4: $\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$

Row 5: $\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$

Row 6: $\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$

Row 7: $\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$

Row 8: $\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$

Row 9: $\binom{9}{0} \quad \binom{9}{1} \quad \binom{9}{2} \quad \binom{9}{3} \quad \binom{9}{4} \quad \binom{9}{5} \quad \binom{9}{6} \quad \binom{9}{7} \quad \binom{9}{8} \quad \binom{9}{9}$

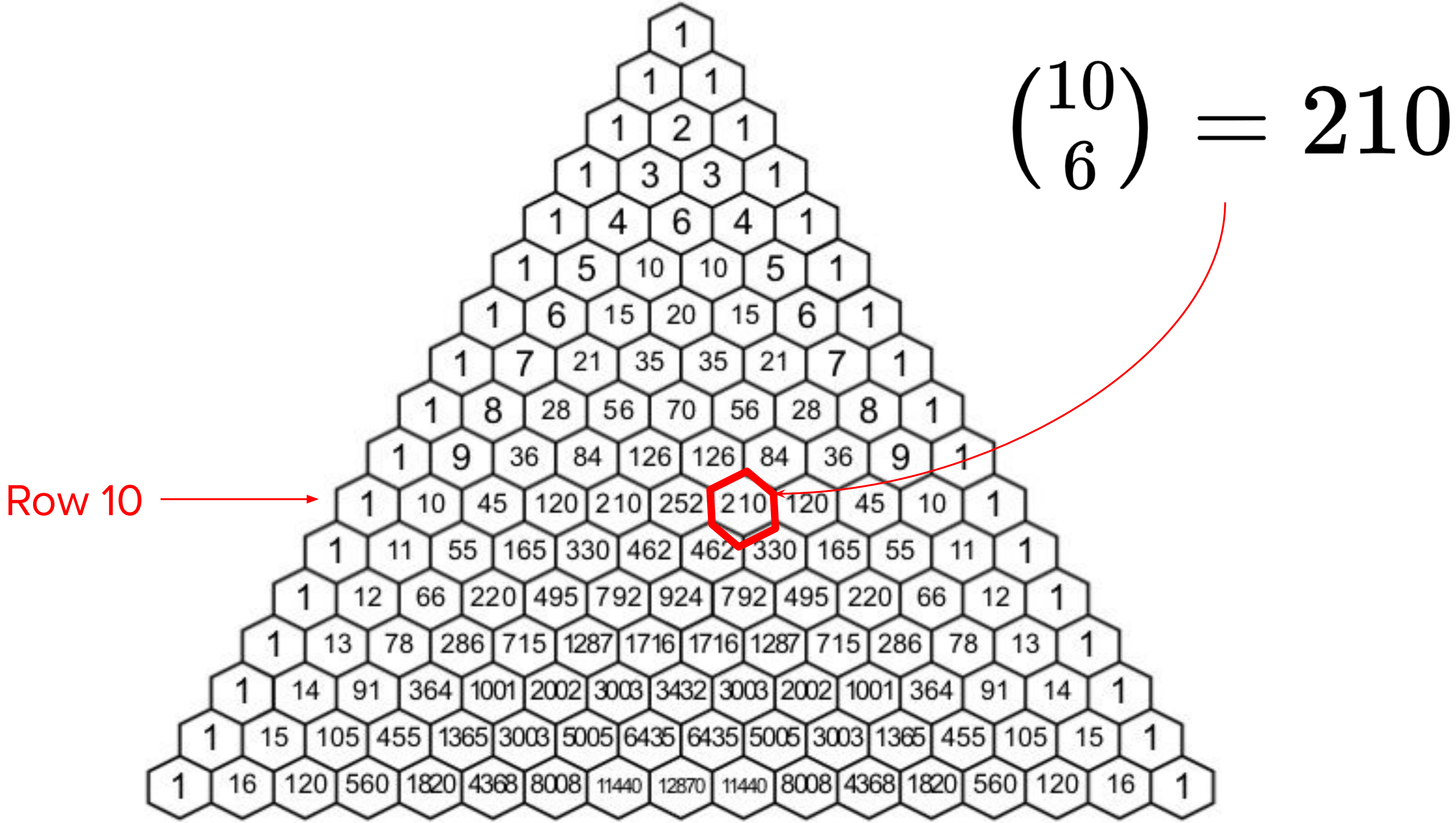
Row 10: $\binom{10}{0} \quad \binom{10}{1} \quad \binom{10}{2} \quad \binom{10}{3} \quad \binom{10}{4} \quad \binom{10}{5} \quad \binom{10}{6} \quad \binom{10}{7} \quad \binom{10}{8} \quad \binom{10}{9} \quad \binom{10}{10}$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Every other binomial coefficient is the sum of the two just above it

These also = 1 by definition

Jamboard: Filling in Pascal's Triangle with the recurrence relation


$\binom{10}{6} = 210$
Row 10
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

Mixed counting practice

Here are some simple counting problems that use tools from today and from Module 7A.

Students who sign up for a campus event are given an ID string, consisting of three letters chosen from the letters A-J. (That's ten individual letters.) How many different ID's can be created?

- 27
- 30
- 120
- 720
- 1000
- None of the above

To 0

In a group of 10 students, a group of 3 needs to be selected for Covid testing. How many different ways are there to form such a group?

- 27
- 30
- 120
- 720
- 1000
- None of the above

To 0

What makes those two counting problems different?

Why use the binomial coefficient on one, and the multiplication rule on the other?

How many bitstrings of length 8 are there, that end in a 1?

- 1
- 2
- 16
- 128
- 255
- 256
- None of the above

How many bijective functions are there from the set $\{1, 2, 3, 4, 5\}$ to itself?

- 1
- 5
- 25
- 32
- 120
- None of the above

To 0

How many *functions* are there from the set $\{1, 2, 3, 4, 5\}$ to itself, bijective or otherwise?

- 9765625
- 3125
- 32
- 120
- None of the above

How many bit strings of length 10 are there that either begin with 000 or end with 00?

- 87
- 165
- 352
- 384
- 32736
- 32768
- None of the above

Have a great day 😀

Check your info sources to stay up to speed!