Combinatorial proof

MTH 225 -- Module 8B
30 October 2020

How mathematics actually works

Experimentation

1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32

Conjecture

The sum of numbers in row *n* is 2^n.

Proof

What is a mathematical proof? And why?

- A **logically sound** and **clearly expressed explanation** for **why** a conjecture is **always true**, under the conditions given.
- A ton of examples is not a proof (link to Colab notebook) -- why not?
- A sincerely held, strenuously asserted claim (e.g. “It’s obvious that the pattern continues”) is not a proof -- why not?
- Proofs are necessary for mathematical conjectures so that the pattern we THINK we see, can be used with confidence

Different ways to prove conjectures

- Mathematical proof is a **creative process** and hard to boil down to a few bullet points.
- There are MANY different approaches to proof (see: MTH 210).
- **Analytic** proofs: Give an explanation by using formulas and a lot of computation.
- **Combinatorial** proofs: (Used in conjectures about counting/identities)
    - Explain that **the left-hand side and right-hand side of an identity are both solutions to the same counting problem**. Or,
    - Explain that the left-hand side counts the objects in a set, the right-hand side count objects in another set, then **build a bijection** between the sets.

We’ve seen this before

Number of k-element subsets of a n-element set = Number of n-bit strings with weight k

<u>**Strategy**</u>: Build a bijection between these two (finite) sets of objects. They therefore have the same number of elements, `binom(n,k)`.

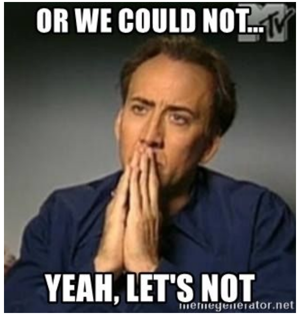
An analytical proof of our conjecture

For every natural number n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

PROOF: Let n be any natural number, and let's compute and simplify:

$$\frac{n!}{n!0!} + \frac{n!}{(n-1)!1!} + \frac{n!}{(n-2)!2!} + \cdots + \frac{n!}{(0)!n!}$$



A combinatorial proof of our conjecture

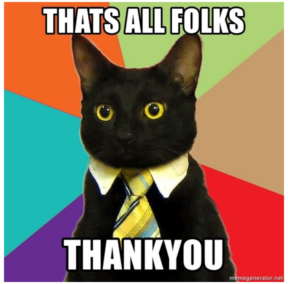
For every natural number n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

PROOF: Let n be any natural number, and let A be an n-element set. The expression on the left is:

(number of 0-element subsets of A) + (number of 1-element subsets of A) + (number of 2-element subsets of A) + … + (number of n-element subsets of A)

This is just the sum total of *all possible* subsets of A, and we already know that this number is 2^n.



ANOTHER combinatorial proof of our conjecture

For every natural number n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

PROOF: Since binom(n,k) is the coefficient on $x^k y^{n-k}$ in the expansion of $(x+y)^n$, we can write

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n}x^ny^0$$

Now just plug in x = 1 and y = 1.

Another binomial identity

Proving another identity

For any natural numbers n and k (with n ≥ k),

$$\binom{n}{k} = \binom{n}{n-k}$$

Cardinality of
The set of n-bit strings with k "1" bits

Cardinality of
The set of n-bit strings with n-k "1" bits

Come up with a bijection between these two sets.

Look at an example first: Set of 5-bit strings with weight 3 and then the set of 5-bit strings of weight 2

Formalizing the proof

- Let B(n,k) and B(n,n-k) be the sets of n-bit strings with weight k and weight n-k respectively. By definition binom(n,k) is the cardinality of the first set.
- Let f: B(n,k) → B(n, n-k) be defined as follows: the output when we plug in a bitstring returns another bit string with all the bits flipped (0 → 1, 1→ 0)
- *This is a function because…*
- *This is an injective function because…*
- *This is a surjective function because…*
- *Therefore this is a bijection, and since the sets are finite it means….*

Yet another identity

Adding squares of binomial coefficients

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

Strategy: Show that the left and the right sides count the same number of objects, and therefore they must be equal.

Number of ways to choose n objects from a set of 2n objects

- Take a set of 2n objects and divide it into two groups of n objects each, “Group A” and “Group B”. Then select n items from the whole group.
- Your selection must contain some number of items from Group A and some other number of items from Group B.
- There are ___ possible cases….

- The number of ways to encounter the first case:
- Number of ways to encounter the second case:
- …
- Number of ways to encounter the last case:
- Therefore the total number of ways to draw n items from 2n is…
- But notice….