Directions:

- Complete the exercises below and either write up or type up your solutions. Solutions must be submitted as PDF or Word documents, uploaded to the appropriate assignment area on Blackboard.
- If you choose to submit handwritten work, it must be neat and legible; if you do your handwritten work on paper, it must be **scanned to a PDF file** and submitted to Blackboard. Instructions and practice for scanning work to PDFs is given in the Startup Assignment. **Do not just take a picture**, **and do not submit a graphics file (JPG**, **PNG**, **etc.)** — such submissions will not be graded.
- Your work will be graded using the EMPX rubric and evaluated **not just on the basis of a right or wrong answer**, **but on the quality**, **completeness**, **and clarity of your work**. Therefore you need to show all work and explain your reasoning on each item.
- Every item must have a good-faith effort at a complete and correct response. If any item is left blank, or shows minimal effort (such as answering "I don't know"), or is significantly incomplete, the entire assignment will be graded "X" (Not Assessable) and you will have to spend a token to revise it.

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1. Recall that the ***determinant*** of a 2×2 matrix A is denoted $\det(A)$ and is defined by this formula:

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$$

   (a) Given 2×2 matrices A and B, show that $\det(AB) = \det(A) \cdot \det(B)$. (Suggestion: Write specific entries for A and B, then do the math on each side of the equals sign.)

   (b) Let S be the set of all 2×2 matrices whose determinant equals zero. Prove or give a specific counterexample: S is closed under matrix multiplication.

   (c) Prove or give a specific counterexample: S is closed under matrix addition.

2. In the ring $\mathcal{P}_n$, the multiplication is ordinary set intersection, but for the addition we use a relatively complicated operation, symmetric difference. You might be wondering why we didn't use something simpler for addition, like set union.

   (a) We showed in class and in Workshops that $\emptyset$ acts as an additive identity for symmetric difference; show that it does the same for set unions. That is, prove or explain why $A \cup \emptyset = A$ for any set A.

   (b) However, give an example that shows that if we used $\cup$ for addition in $\mathcal{P}_n$, not every element in $\mathcal{P}_n$ would have an additive inverse.

   (c) Explain why we don't use set unions for addition in $\mathcal{P}_n$, given the outcome of the previous part.

3. Let $R = \{a, b, c, d, e, f, g, h\}$, and define "addition" (+) and "multiplication" ($\cdot$) on R by the following tables:

$+$	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	a	d	c	f	e	h	g
c	c	d	a	b	g	h	e	f
d	d	c	b	a	h	g	f	e
e	e	f	g	h	a	b	c	d
f	f	e	h	g	b	a	d	c
g	g	h	e	f	c	d	a	b
h	h	g	f	e	d	c	b	a

$\cdot$	a	b	c	d	e	f	g	h
a	a	a	a	a	a	a	a	a
b	a	b	c	d	e	f	g	h
c	a	c	e	g	d	b	h	f
d	a	d	g	f	h	e	b	c
e	a	e	d	h	g	c	f	b
f	a	f	b	e	c	h	d	g
g	a	g	h	b	f	d	c	e
h	a	h	f	c	b	g	e	d

(a) Is R closed under both operations? If not, give a counterexample; if so, explain how you know. If you think the set is closed under the operations, a complete proof is unnecessary, but do more than just restate the definition of "closure" — *explain why* the definition is satisfied.

(b) Is addition associative? If you think so, give 3 examples that provide evidence. If not, give a specific counterexample.

(c) Is multiplication associative? If you think so, give 3 examples that provide evidence. If not, give a specific counterexample.

(d) Is addition commutative? If you think so, give 3 examples that provide evidence. If not, give a specific counterexample.

(e) Does multiplication distribute over addition? If you think so, give 3 examples that provide evidence. If not, give a specific counterexample.

(f) Does R have an additive identity (also known as a zero object)? If so, what is it, and how do you know this is correct? If not, explain why not.

(g) If you wrote that R does have an additive identity, then for each of the eight elements of R, list that element's additive inverse. Does every element of R have an additive inverse?

(h) Does R appear to form a ring with these two operations? If so, explain; if not, explain why not.