The Integers: An introduction

MTH 350 -- Module 1

Today

- Review of Daily Prep 1 (polling on basic concepts)
- The Axioms of Arithmetic and Ordering -- also what even is an Axiom
- The definition of additive inverse -- also what even is a definition
- Proving the obvious, part 1
- Proving the obvious, part 2
- Recap and feedback

Which of the following is the set $\mathbb{N}$?

{1,2,3,4,...}

{0,1,2,3,4,...}

{..., -3, -2 -1, 0, 1, 2, 3, ... }

{ p/q : p, q are integers and q is not 0 }

The statement that for all integers a, b, we have a + b = b + a and ab = ba is the axiom that

- The integers are closed under addition and multiplication
- Addition and multiplication are commutative
- Addition and multiplication are associative
- Multiplication distributes over addition
- Every integer has an additive inverse

Tc 0

Let x be an integer. Then according to the definition, an additive inverse of x is

- -x
- 1/x
- Another integer y such that xy = 1
- Another integer y such that x + y = 0
- Another integer y such that x + y = y + x
- None of these

Tc 0

Let a and b be integers and suppose that a is not less than b, and b is not less than a. What can we conclude here, and why?

- a = b by the Transitivity Axiom
- a = b by the Trichotomy Axiom
- a = b by the Translation Invariance Axiom
- One of a or b (possibly both) are undefined
- a = b by the Axiom that addition is commutative

Axioms of Arithmetic and Ordering

Axioms of Integer Arithmetic

- **The integers are closed under addition and multiplication**, meaning that for all integers a and b, both $a + b$ and ab are also integers.
- **Addition and multiplication are commutative**, meaning that for all integers a and b, $a + b = b + a$ and $ab = ba$.
- **Addition and multiplication are associative**, meaning that for all integers a, b, and c, $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- **Multiplication distributes over addition**, meaning that $a(b + c) = ab + ac$ for all integers a, b, and c.
- **The integer 0 is an additive identity**, meaning that $a + 0 = a$ for every integer a.
- **The integer 1 is a multiplicative identity**, meaning that $1a = a$ for every integer a.
- **Every integer a has an additive inverse**, typically denoted $-a$; in particular, $a + (-a) = 0$ for every integer a.

Axioms:

- Are building blocks for mathematical systems; taken on faith with no proof
- Cannot be deducible from other axioms
- EVERYTHING MATTERS -- quantifiers, punctuation, etc.

Theorems/Propositions/Lemmas:

- Are facts deduced from axioms and from previously-proven results
- **Anything that is not an axiom must be proven, using axioms** (or other theorems proven from axioms)

Ordering Axioms of the Integers

The "less than" relation on the integers, denoted $<$, satisfies all of the following properties:

- **Trichotomy**: For all integers a and b, exactly one of the following is true: $a < b$, $b < a$, or $a = b$.
- **Transitivity:** For all integers a, b, and c, if $a < b$ and $b < c$, then $a < c$.
- **Translation Invariance:** For all integers a, b, and c, if $a < b$, then $a + c < b + c$.
- **Scaling:** For all integers a, b, and c, if $a < b$ and $c > 0$, then $ac < bc$.

Definitions

Definition 1.5. Let x be an integer. Then an **additive inverse** of x is an integer y such that $x + y = 0$.

Four things to do with definitions:

1. State verbatim. EVERYTHING MATTERS
2. Construct examples
3. Construct non-examples
4. Use the definition to draw conclusions or rephrase information

<u>Example</u>: To prove (whatever) is an additive inverse of (something), show that (something) + (whatever) = 0

We lean on definitions, NOT on our experiences or on what we think we learned in school. <u>Example</u>: An additive inverse for x is *denoted* -x but right now this has nothing to do with being positive or negative.

In breakout groups:

Discuss the pros and cons of each of the following potential definitions of the additive inverse of an integer x:

(a) The additive inverse of x is $-x$.
(b) The additive inverse of x is $0 - x$.
(c) The additive inverse of x is an integer y such that $x+y = 0$. (a.k.a. The official definition)
(d) The additive inverse of x is $(-1)x$.

Proving the obvious part 1

Theorem: If a is an integer, then 0a = 0.

Let a be an integer.

1. We know 0 + 0 = 0 because…
2. Multiply both sides by a to get (0+0)a = 0a.
3. On the left, we can rewrite to get 0a + 0a because…
4. Therefore 0a + 0a = 0a.
5. Since a is an integer and 0 is an integer, we know 0a is an integer because…
6. Therefore 0a has an additive inverse because…
7. Add -(0a) to both sides of 0a + 0a = 0a to get 0a + 0a + (-0a) = 0a + (-0a).
8. On the left we can group to get 0a + (0a + (-0a)) because… and this equals 0a + 0 which equals 0a because...
9. On the right we just get 0 because…
10. Therefore we have 0a = 0 which is what we wanted to show.

Proving the obvious, part 2

Activity 1.6(b): For all integers a and c, $-(ac) = a(-c)$.

Let a and c be integers. For reference "$-c$" means the additive inverse of c, and "$-(ac)$" means the additive inverse of ac.

To show that $-(ac) = a(-c)$, we need to show that $ac + a(-c) =$ ___. So, start with $ac + a(-c)$.

1. $ac + a(-c) = a(c + (-c))$ because....
2. And $a(c + (-c)) = a($___$)$
3. And this equals ___ because of....

Activity 1.6(c): For all integers a,b, and c, a(b-c) = ab - ac.

Start with the right hand side, ab - ac. We want to show this equals a(b-c).

1. ab-ac = ab + a(-c) because of ....
2. And ab + a(-c) = a(b + (-c)) because of....
3. And a(b + (-c)) = a(b-c) because of....

That’s what we wanted to show.

Activity 1.6(e): For all integers a and b, -(a+b) = -a - b.

The expression "-(a+b)" is the additive inverse of _______. To show this equals -a-b, we need to show that a + b + (-a - b) = ___. Start with the left side.

1. a + b + (-a - b) = a + b + (-a + (-b)) because....
2. a + b + (-a + (-b)) = a + ___ + b + _____ because...
3. This equals (a + ___) + (b + ____) because...
4. And this equals ___ + ____ because...
5. Finally this equals ____ because....

Therefore -(a+b) = -a - b.

Activity 1.6(f): For all integers a, -(-a) = ___.

Let a be an integer. For reference, “-(-a)” means the additive inverse of -a.

To show that -(-a) = a, we need to show that -a + ___ = ____.

So start with the left side, -a + ___.

This is equal to ___ by definition, so we’re done.

Recap

- Axioms of Arithmetic and Ordering -- basic building blocks of everything in arithmetic and algebra. Everything else is built from these.
- If it’s not an axiom and we think it’s true, we have to prove it.
- Back up every claim with an axiom, theorem, or deduction from these

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