MTH 350 Problem Set 5 Due: 11:59pm Eastern, Sunday April 4

Reminders:

- Remember to review the instructions for Problem Sets before turning in work.
- **Individual** problems (second section below) that are marked with a star (★) may be included in your Proof Portfolio.

1 Team Problems

New Instructions for Team Problems: The problems given in this section may be done in teams of up to four (4) people, or done individually. You can stick with your old teams, or form new ones, or work in pairs, etc. Or, as mentioned, you can work on them individually. If you choose to work in a team, please state the person or people with whom you worked. The rules for doing problems in teams, and the procedures for grading and revising work on the problems in this section, will be the usual ones spelled out in the Instructions for MTH 350 Problem Sets document.

1. Let S denote the set of all 2×2 matrices of the form

$$\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$$

with $x, y \in \mathbb{R}$. Prove or disprove: This set under the operations of matrix addition and matrix multiplication forms a ring. (Be careful with closure — it takes some thought this time.)

2. Let R be a ring with identity, and let $x, y \in R$. Prove or disprove: If xy is a unit in R, then both x and y are units in R.

2 Individual Problems

1. Let $\mathbb{Z}_3[i]$ be defined as

$$\mathbb{Z}_3[i] = \{[a] + [b]i \, : \, [a], [b] \in \mathbb{Z}_3\} = \{[0], [1], [2], [1]i, [2]i, [1] + [1]i, [1] + [2]i, [2] + [1]i, [2] + [2]i\}$$

Using the modified complex number addition and multiplication defined in Workshop 9, it can be shown that this is a commutative ring with identity, the identity being $[1]$. Prove or disprove: $\mathbb{Z}_3[i]$ is a field. (Suggestion: Make addition and multiplication tables first to understand how the operations work.)

2. **Each** of the following are starred problems; **pick exactly one** and do it for this Problem Set. You can choose another one for your Portfolio later.

   (a) (★) Suppose that a and b belong to a commutative ring and that ab is a zero divisor. Prove that either a or b is a zero divisor.

   (b) (★) A ring element a is called *idempotent* if $a^2 = a$. (Squaring, like usual, just means multiplying by itself.) Prove that the only idempotent elements in an integral domain are the identities 0_R and 1_R. (Be careful to remember that 0_R and 1_R are **not** the integers 0 and 1.)

   (c) (★) Prove that every finite integral domain R is a field. (Hint: Let x be any nonzero element of R. Show that $x^n = 1_R$ for some natural number n. Also: The restriction that R is *finite* is necessary; are you using it in your proof?)

   (d) (★) Suppose that R is a commutative ring without zero divisors. (But not necessarily an integral domain.) Show that the characteristic of R is either zero or prime.