Reminders:

- Remember to review the instructions for Problem Sets before turning in work.
- **Individual** problems (second section below) that are marked with a star (★) may be included in your Proof Portfolio.

1 Team Problems

The team problems this time all focus on the set $M_{2\times 2}(\mathbb{R})$, which is the set of all 2×2 matrices with real number entries. This set, together with ordinary matrix addition and multiplication, forms a number system that we can explore. This system is described in the text and you worked with it a little on Weekly Practice 7. There are more of these than usual but each one is simpler than usual in the solution.

1. Prove that mutliplication in $M_{2\times 2}(\mathbb{R})$ is associative, that is, given matrices $A, B, C \in M_{2\times 2}(\mathbb{R})$, we have $A(BC) = (AB)C$.
2. Prove or disprove: Multiplication in $M_{2\times 2}(\mathbb{R})$ is commutative.
3. Prove or disprove: $M_{2\times 2}(\mathbb{R})$ has both an additive identity and a multiplicative identity.
4. Prove or disprove: Every nonzero element of $M_{2\times 2}(\mathbb{R})$ has a multiplicative inverse.
5. Prove or disprove: Multiplicative cancellation of nonzero elements holds in $M_{2\times 2}(\mathbb{R})$. That is, for all nonzero matrices A, B, C, if $AB = AC$, then $B = C$.

2 Individual Problems

1. Redefine "multiplication" on the integers by replacing ordinary multiplication by the operation $\otimes$, which is defined by: $a \otimes b = a^b$. For example $2 \otimes 10 = 1024$.
   (a) Does the set $\mathbb{Z}$ of integers with ordinary addition and $\otimes$ as the multiplication operation form a ring? If so, then prove it; otherwise give specific reasoning as to why not.
   (b) Does the set $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ with ordinary addition and $\otimes$ as the multiplication operation form a ring? If so, then prove it; otherwise give specific reasoning as to why not.
2. **Each** of the following are starred problems; **pick exactly one** and do it for this Problem Set. You can choose another one for your Portfolio later.
   (a) (★) A **Boolean ring** R is ring in which $x^2 = x$ for all $x \in R$. Two familiar examples of Boolean rings are $\mathbb{Z}_2$ and $\mathcal{P}_n$. Prove that every element of a Boolean ring is its own additive inverse.
   (b) (★) See above for the definition of a **Boolean ring**. Prove that every Boolean ring is commutative.
   (c) (★) Let $\mathbb{R}^+$ denote the set of positive real numbers. For all $x, y \in \mathbb{R}^+$, define the operations of "addition" (denoted $\oplus$) and "multiplication" ($\otimes$) as follows:
   $$x \oplus y = xy \qquad \text{and } x \otimes y = x^{\log(y)}$$
   Reminder: $\log(y)$ means the natural logarithm of y.
      i. With these operations, does $\mathbb{R}^+$ have an additive identity? If so, what is it?
      ii. With these operations, does $\mathbb{R}^+$ have a multiplicative identity? If so, what is it?
      iii. Is $\otimes$ associative? If so, prove it; otherwise provide a specific counterexample.
      iv. Does $\otimes$ distribute over $\oplus$? If so, prove it; otherwise provide a specific counterexample.