Primes in other number systems

MTH 350 -- Module 4B

What's MTH 350 all about?

The algebra and arithmetic we learn in school are full of facts that students tend to accept without question. In MTH 350, we will **discover the real rules behind much of algebra**, and then try to see **how far we can bend the rules before they break**.

MTH 350 takes on two big questions in mathematics:

1. **How do arithmetic and algebra actually work?** For example, you probably know that every positive integer can be factored into a product of prime numbers, like $20 = 2 \times 2 \times 5$. But can you explain *why* this is true, for *all* positive integers?
2. **Can the rules of arithmetic and algebra be extended to mathematical objects that aren't necessarily numbers?** Take multiplication for example. We multiply numbers together, but we multiply things like matrices together too. How much of our rules for multiplication of *numbers*, work for *matrices*? Does matrix multiplication satisfy the commutative property, or the associative property? Can you "divide" matrices? Can we "factor" a matrix into a product of "prime" matrices? Is there even a sensible definition of these concepts?

Can we make a definition of “prime” that works for stuff besides integers, that becomes the normal definition of “prime” if we apply it to the integers?

Prime numbers in the even integers

ABSTRACT definition of prime

$$\mathbb{E} = \{2k \; : \; k \in \mathbb{Z}\}$$

Definition: A prime number n in **E** is a positive even integer p that cannot be written as a product of two other even integers. That is, p in **E** is prime if there do not exist even integers x,y such that p = xy.

This definition becomes the usual definition of "prime" when the set of objects is changed back to normal (with minor adjustments):

Definition: A prime number n in **Z** is a positive integer p that cannot be written as a product of two other integers. That is, p in **Z** is prime if there do not exist integers x,y such that p = xy.

What do primes in the even integers look like?

Even integer	Prime?	Even integer	Prime?
2		12	
4		30	
6		60	
8		100	
10		-24	

Extending this idea

$$n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$$

In your groups:

- Come up with a reasonable definition of a **prime number in nZ** that extends the definition of prime number in **E**. (I.e. this should become the definition for prime number in E if n = 2.)
- Choose a few specific natural numbers *n*. Find several examples of prime numbers in nZ for your choices of n and several non-examples.
- Look at the non-examples -- the numbers that are not prime. Can you factor them into a product of primes in nZ? Is the factorization unique?

Extending it some more

In your groups:

- Come up with a reasonable definition of a **prime number in R** where **R** is the set of all real numbers.
- Find, if possible, several examples of prime numbers in **R**.
- Repeat the above using **Q**, the set of all rational numbers (ie. numbers that can be written as a ratio a/b with a,b integers and b nonzero).

Extending it even more

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

These are not numbers but they do form something like a “number system”: Addition, multiplication, many of the axioms of “integer” arithmetic hold.

What would a definition of “prime” look like here?

Is the matrix on the right prime? Do prime matrices exist?

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$$

Recap

- New concept: Extending the concepts and axioms of **integer** arithmetic and algebra to sets of objects besides just the integers.
- What still works? What works if you add some restrictions? Which ones don’t even make sense or are trivial or impossible?
- The ones that do work or can work: **Abstract algebra** properties.
- **Big question moving forward: What kinds of sets of objects with operations will obey most or all of the important axioms and properties of integer arithmetic and algebra?**

Feedback:

http://gvsu.edu/s/1zN