Name: ____________________________

Instructions: Welcome to your Final Exam. You may use three 3×5 notecards with notes and a calculator. You may NOT use any device that can communicate with another device. The backs of each page are blank; use them if needed. On all questions other than multiple choice, give complete and correct solutions; answers without accompanying work will be given no credit.

The test will end promptly at 2:00pm. No extensions or extra time will be given unless you have received prior permission from the instructor.

Items 1—15 are multiple choice questions that address a variety of learning objectives. Please circle the ONE response you believe is most correct. You do not need to justify your answer.

1. (2 points) Which of the following are statements?
   (a) $3^2 + 4^2 = 5^2$.
   (b) Prove or disprove that $3^2 + 4^2 = 5^2$.
   (c) $3^2 + 4^2 \neq 5^2$.
   (d) All of the above
   (e) Just (a) and (c)

2. (2 points) Suppose you wanted to prove that the following statement is FALSE: "There exists a bijection $f : \mathbb{N} \to \mathbb{Z}$". An appropriate technique for doing so would be
   (a) To give an example of a function $f : \mathbb{N} \to \mathbb{Z}$ that is a bijection
   (b) To give an example of a function $f : \mathbb{N} \to \mathbb{Z}$ that is not a bijection
   (c) To give a formal proof that all functions $f : \mathbb{N} \to \mathbb{Z}$ are bijections
   (d) To give a formal proof that all functions $f : \mathbb{N} \to \mathbb{Z}$ fail to be bijections
   (e) None of the above

3. (2 points) The negation of the statement "If A, then B" is
   (a) If A, then not B.
   (b) If not A, then B.
   (c) Not A and not B.
   (d) Not A or not B.
   (e) None of the above

4. (2 points) Under what conditions will the conditional statement $(A \wedge B) \to C$ be true?
   (a) When A is true and C is true
   (b) When B is false
   (c) When C is true
   (d) All of the above
   (e) Just (b) and (c)

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5. (2 points) Consider the statement: "For all integers k, $k^2 + 2k - 1$ is not a multiple of 4.". A mathematically correct strategy for proving this statement would be
   (a) Give an example of an integer k such that $k^2 + 2k - 1$ is not a multiple of 4.
   (b) Give an example of an integer k such that $k^2 + 2k - 1$ *is* a multiple of 4.
   (c) Assume 4 divides $k^2 + 2k - 1$ and then provide an example of where this fails, thereby arriving at a contradiction.
   (d) Assume 4 divides $k^2 + 2k - 1$ for some integer k and then arrive at a contradiction.
   (e) Both (c) and (d)

6. (2 points) Suppose that f and g are two equal functions, the domain of g is $\mathbb{Z}$, and that $g(-2) = 3$. Then based on this information alone, we can conclude that
   (a) The domain of f is $\mathbb{Z}$
   (b) The codomain of f is $\mathbb{Z}$
   (c) $f(-2) = 3$
   (d) All of the above
   (e) Just (a) and (c)

7. (2 points) Which of the following functions $f : \mathbb{Z} \to \mathbb{Z}$ is a surjection?
   (a) $f(a) = a^2$
   (b) $f(a) = a^3$
   (c) $f(a) = a \pmod{10}$
   (d) All of the above
   (e) None of the above

8. (2 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 1 + e^x + x$. This function is a bijection. The value of $f^{-1}(2)$ is
   (a) 0
   (b) $1/2$
   (c) $\dfrac{1}{3 + e^2}$
   (d) $3 + e^2$
   (e) Undefined

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9. (2 points) Let $\sim$ be the relation on $\mathbb{R}$ given by $x \sim y$ if and only if $xy \geq 0$. Then $\sim$ is
   (a) Reflexive
   (b) Symmetric
   (c) Transitive
   (d) All of the above
   (e) Just (a) and (b)

10. (2 points) Let $\sim$ be an equivalence relation on a nonempty set A and suppose $a, b \in A$. Then $[a] = [b]$
    (a) Always
    (b) Only if $a = b$
    (c) Only if $a \sim b$
    (d) Only if $[a]$ and $[b]$ are disjoint
    (e) Never

11. (2 points) To prove that a function $f : A \to B$ is an injection,
    (a) Let $a \in A$ and prove that $f(a)$ is a unique point in B.
    (b) Let $a, a' \in A$ with $a \neq a'$, and prove that $f(a) \neq f(a')$.
    (c) Let $a, a' \in A$ with $f(a) \neq f(a')$, and prove $a \neq a'$.
    (d) Let $b \in B$, and prove there exists $a \in A$ such that $f(a) = b$.
    (e) Both (b) and (c)

12. (2 points) To show that a function $f : A \to B$ is *not* a surjection,
    (a) Show that there exists a point $a \in A$ that does not map to anything in B.
    (b) Show that there exists a point $b \in B$ such that $f(a) \neq b$ for all $a \in A$.
    (c) Show that for every point $b \in B$, there is a point $a \in A$ such that $f(a) \neq b$.
    (d) Show that for every point $a \in A$, there exists a point $b \in B$ such that $f(a) = b$.
    (e) Both (b) and (c)

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13. Below are a variety of computations to carry out. Do each one. You do not need to show work, but if you make a mistake, partial credit will be awarded only if there is supporting work.

(a) (6 points) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(a) = a^3 \pmod{7}$. Fill in the following table:

a	-4	-2	0	2	4
$f(a)$					

(b) (6 points) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{6, 7\}$ Calculate the set $(A \cap B) \times C$.

(c) (6 points) Find the integers q and r guaranteed by the Division Algorithm such that $7101970 = 8q + r$.

(d) (6 points) Let $\sim$ be the relation on $\mathbb{Z}$ defined as follows: For all $a, b \in \mathbb{Z}$, $a \sim b$ if and only if $a - b$ is a multiple of 3. It can be proven that $\sim$ is an equivalence relation. Write the elements of $[1492]$ as a set in roster form.

(e) (6 points) Let $\sim$ be the relation on the set $A = \{1, 2, 3, \ldots, 10\}$ defined by $a \sim b$ if and only if a divides b. Draw the directed graph for this relation.

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14. For each of the following statements, write the negation of the statement in symbolic form in which the negation symbol ($\neg$) is not used, and then write the negation as an English sentence.

    (a) (6 points) $(\exists x \in \mathbb{R})(\cos(2x) = 2\cos x)$

    (b) (6 points) $(\forall x \in \mathbb{Z})((2|x) \Rightarrow (4|x))$ (The arrow ($\Rightarrow$) means "implies".)

15. Consider the statement: *If it is snowing and after 6:00am, then I will shovel my driveway.*

    (a) (6 points) Write a clear statement of the contrapositive of this sentence.

    (b) (6 points) Write a clear statement of the converse of this sentence.

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16. Below are several proof-related tasks that we have undertaken repeatedly during the course. For each task, give a brief but complete outline of how a correct proof would be constructed in that situation. Be sure to state explicitly **what you would assume, what you would try to prove, and a specific strategy for how you would proceed from the beginning**.

    (a) (8 points) Proving that two sets, A and B, are equal via the "choose an element" approach

    (b) (8 points) Proving the conditional statement "If P, then Q" by contradiction

    (c) (8 points) Proving that the predicate $P(n)$ is true for all natural numbers n using mathematical induction

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17. (16 points) Choose EXACTLY ONE of the following and either prove the statement or disprove it. Circle the letter of the problem you are doing.

    (a) For all integers a, b and d with $d \neq 0$, if d divides a or d divides b, then d divides ab.

    (b) For all functions $f : A \to B$ and $g : B \to C$, if $g \circ f : A \to C$ is an injection then f is an injection.

    (c) For each integer n, if n is odd, then 8 divides $n^2 - 1$.

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18. (16 points) Choose EXACTLY ONE of the following to prove. Circle the letter of the problem you are doing.

    (a) Let $\sim$ be the relation on $\mathbb{Z}$ given by $a \sim b$ if and only if $a \equiv b \pmod 3$. For all natural numbers n, prove that $[10^n] = [1]$ under this relation.

    (b) Recall that the *Fibonacci numbers* are the numbers f_n defined by $f_1 = 1$, f_2, and then $f_k = f_{k-1}+f_{k-2}$ for all $k > 1$. Prove that for each natural number n,

    $$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$$

    (c) Prove that for every natural number n,

    $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

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19. Consider the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $g(a, b) = \gcd(a, b)$, the greatest common divisor of a and b (also known as greatest common factor). For example, $g(10, 8) = 2$ and $g(5, 6) = 1$.

    (a) (10 points) Prove or disprove: The function g is injective.

    (b) (10 points) Prove or disprove: The function g is surjective.

20. (10 points) The last problem on the exam consists of three short essay questions, given on a web form. The link for this is posted to Piazza already (thread 177). Please submit that web form **by 8:00am on Thursday, December 13**.