# Writeup for Class Work on §3.5 (October 2)

**Lemma 1:** For all integers $a$, if 3 divides $a^2$, then 3 divides $a$.

*Proof*: We will prove the contrapositive. So assume 3 does not divide $a$, and we will show that 3 does not divide $a^2$. Since 3 does not divide $a$, the Division Algorithm gives us two cases to consider: (1) the remainder obtained when dividing $a$ by 3 is 1, and (2) the remainder obtained when dividing $a$ by 3 is 2.

**Case 1:** Suppose the remainder obtained when dividing $a$ by 3 is 1. Then there exists an integer $q$ such that $a = 3q + 1$. We want to show that 3 does not divide $a^2$, so let us get an expression for $a^2$ first:

$$\begin{aligned} a^2 &= (3q+1)^2 \\ &= 9q^2 + 6q + 1 \\ &= 3(3q^2 + 2q) + 1 \end{aligned}$$

From the last line, closure of the set of integers under multiplication and addition tells us that $3q^2 + 2q$ is an integer, which we will call $x$. So we can write $a^2$ as $3x + 1$ with $x$ an integer. This means the remainder obtained when dividing $a^2$ by 3 is 1, and hence 3 does not divide $a^2$.

**Case 2:** Suppose the remainder obtained when dividing $a$ by 3 is 2. Then there exists an integer $q$ such that $a = 3q + 2$. We want to show that 3 does not divide $a^2$, so get an expression for $a^2$ :

$$\begin{aligned} a^2 &= (3q+2)^2 \\ &= 9q^2 + 12q + 4 \\ &= 3(3q^2 + 4q + 1) + 1 \end{aligned}$$

Closure of the set of integers under multiplication and addition tells us that $3q^2 + 4q + 1$ is an integer, which we will call $y$. So we can write $a^2$ as $3y + 1$ with $y$ an integer. This means the remainder obtained when dividing $a^2$ by 3 is 1, and hence 3 does not divide $a^2$.

Since 3 does not divide $a^2$ in either case, the lemma is proven. ■

**Proposition 1:** The number $\sqrt{3}$ is irrational.

*Proof*: For a contradiction, suppose $\sqrt{3}$ is rational. Then there exist integers $a, b$ such that

$$\sqrt{3} = \frac{a}{b}$$

and the fraction $a/b$ is in lowest form. Square both sides to get:

$$3 = \frac{a^2}{b^2}$$

Clearing fractions gives us

$$3b^2 = a^2 \tag{1}$$

From (1) we see that 3 divides $a^2$. By Lemma 1, this means 3 divides $a$, so write $a = 3k$ for some integer $k$. Substituting this expression back into (1) gives:

$$3b^2 = (3k)^2 = 9k^2 \tag{2}$$

From (2) we have $b^2 = 3k^2$. This means $3$ divides $b^2$ and so by Lemma 1, $3$ divides $b$. But this is a contradiction, because now $a$ and $b$ share a common factor of $3$, which violates our assumption that $a/b$ is in lowest form. Therefore it cannot be the case that $\sqrt{3}$ is rational; hence $\sqrt{3}$ is irrational as desired. $\blacksquare$