Name: ____________________________

Instructions: Welcome to your Final Exam. You may use three 3×5 notecards with notes and a calculator. You may NOT use any device that can communicate with another device. The backs of each page are blank; use them if needed. On all questions other than multiple choice, give complete and correct solutions; answers without accompanying work will be given no credit.

The test will end promptly at 2:00pm. No extensions or extra time will be given unless you have received prior permission from the instructor.

Items 1—15 are multiple choice questions that address a variety of learning objectives. Please circle the ONE response you believe is most correct. You do not need to justify your answer.

1. (2 points) Which of the following are statements?
   (a) $3^2 + 4^2 = 5^2$.
   (b) Prove or disprove that $3^2 + 4^2 = 5^2$.
   (c) $3^2 + 4^2 \neq 5^2$.
   (d) All of the above
   (e) Just (a) and (c)

   Solution: E. Both of those sentences are declarative sentences with a definite true/false value. Option B is not a declarative sentence.

2. (2 points) Suppose you wanted to prove that the following statement is FALSE: "There exists a bijection $f : \mathbb{N} \to \mathbb{Z}$". An appropriate technique for doing so would be
   (a) To give an example of a function $f : \mathbb{N} \to \mathbb{Z}$ that is a bijection
   (b) To give an example of a function $f : \mathbb{N} \to \mathbb{Z}$ that is not a bijection
   (c) To give a formal proof that all functions $f : \mathbb{N} \to \mathbb{Z}$ are bijections
   (d) To give a formal proof that all functions $f : \mathbb{N} \to \mathbb{Z}$ fail to be bijections
   (e) None of the above

   Solution: D. Proving the original statement false amounts to proving its negation is true, which is that every function $f : \mathbb{N} \to \mathbb{Z}$ is not a bijection.

3. (2 points) The negation of the statement "If A, then B" is
   (a) If A, then not B.
   (b) If not A, then B.
   (c) Not A and not B.
   (d) Not A or not B.
   (e) None of the above

   Solution: E. The correct negation is "A and not B".

4. (2 points) Under what conditions will the conditional statement $(A \wedge B) \to C$ be true?
   (a) When A is true and C is true
   (b) When B is false
   (c) When C is true

(d) All of the above

(e) Just (b) and (c)

Solution: D. Any time C is true, the entire conditional statement will be true, so options (a) and (c) are correct. When B is false, it makes the entire hypothesis false, and so the conditional statement is true.

5. (2 points) Consider the statement: “For all integers k, $k^2 + 2k - 1$ is not a multiple of 4.”. A mathematically correct strategy for proving this statement would be

(a) Give an example of an integer k such that $k^2 + 2k - 1$ is not a multiple of 4.

(b) Give an example of an integer k such that $k^2 + 2k - 1$ *is* a multiple of 4.

(c) Assume 4 divides $k^2 + 2k - 1$ and then provide an example of where this fails, thereby arriving at a contradiction.

(d) Assume 4 divides $k^2 + 2k - 1$ for some integer k and then arrive at a contradiction.

(e) Both (c) and (d)

Solution: D.

6. (2 points) Suppose that f and g are two equal functions, the domain of g is $\mathbb{Z}$, and that $g(-2) = 3$. Then based on this information alone, we can conclude that

(a) The domain of f is $\mathbb{Z}$

(b) The codomain of f is $\mathbb{Z}$

(c) $f(-2) = 3$

(d) All of the above

(e) Just (a) and (c)

Solution: E. The domains of f and g must be equal and $f(x) = g(x)$ for all x in the domain. We do not know what the codomain of g is, so we can’t say what the codomain of f is.

7. (2 points) Which of the following functions $f : \mathbb{Z} \to \mathbb{Z}$ is a surjection?

(a) $f(a) = a^2$

(b) $f(a) = a^3$

(c) $f(a) = a \pmod{10}$

(d) All of the above

(e) None of the above

Solution: E. The function in (a) is not a surjection because there is no $x \in \mathbb{Z}$ such that $f(x) = -1$. The function in (b) is not a surjection because there is no $x \in \mathbb{Z}$ such that $f(x) = 2$. The function in (c) is not a surjection because there is no $x \in \mathbb{Z}$ such that $f(x) = 11$.

8. (2 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 1 + e^x + x$. This function is a bijection. The value of $f^{-1}(2)$ is

(a) 0

(b) $1/2$

(c) $\dfrac{1}{3+e^2}$

(d) $3+e^2$

(e) Undefined

Solution: A. This is because $f(0) = 2$.

9. (2 points) Let $\sim$ be the relation on $\mathbb{R}$ given by $x \sim y$ if and only if $xy \geq 0$. Then $\sim$ is
   (a) Reflexive
   (b) Symmetric
   (c) Transitive
   (d) All of the above
   (e) Just (a) and (b)

Solution: E. The relation fails to be transitive because, for example, $-2 \sim 0$ and $0 \sim 2$, but $-2 \not\sim 2$.

10. (2 points) Let $\sim$ be an equivalence relation on a nonempty set A and suppose $a, b \in A$. Then $[a] = [b]$
    (a) Always
    (b) Only if $a = b$
    (c) Only if $a \sim b$
    (d) Only if $[a]$ and $[b]$ are disjoint
    (e) Never

Solution: C. This is Theorem 7.14(b), one of the major results of Chapter 7.

11. (2 points) To prove that a function $f : A \to B$ is an injection,
    (a) Let $a \in A$ and prove that $f(a)$ is a unique point in B.
    (b) Let $a, a' \in A$ with $a \neq a'$, and prove that $f(a) \neq f(a')$.
    (c) Let $a, a' \in A$ with $f(a) \neq f(a')$, and prove $a \neq a'$.
    (d) Let $b \in B$, and prove there exists $a \in A$ such that $f(a) = b$.
    (e) Both (b) and (c)

Solution: B. This is the definition of injection.

12. (2 points) To show that a function $f : A \to B$ is *not* a surjection,
    (a) Show that there exists a point $a \in A$ that does not map to anything in B.
    (b) Show that there exists a point $b \in B$ such that $f(a) \neq b$ for all $a \in A$.
    (c) Show that for every point $b \in B$, there is a point $a \in A$ such that $f(a) \neq b$.
    (d) Show that for every point $a \in A$, there exists a point $b \in B$ such that $f(a) = b$.
    (e) Both (b) and (c)

Solution: **B**. This is the negation of the definition of surjection.

13. Below are a variety of computations to carry out. Do each one. You do not need to show work, but if you make a mistake, partial credit will be awarded only if there is supporting work.

(a) (6 points) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(a) = a^3 \pmod 7$. Fill in the following table:

Solution:

a	-4	-2	0	2	4
$f(a)$	6	6	0	1	1

(b) (6 points) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{6, 7\}$ Calculate the set $(A \cap B) \times C$.

Solution: We have $A \cap B = \{3, 4\}$, so

$$(A \cap B) \times C = \{3, 4\} \times \{6, 7\} = \{(3, 6), (3, 7), (4, 6), (4, 7)\}$$

(c) (6 points) Find the integers q and r guaranteed by the Division Algorithm such that $7101970 = 8q + r$.

Solution: A little long division will show that 8 divides 7101970, 887746 times with a remainder of 2. So $q = 887746$ and $r = 2$.

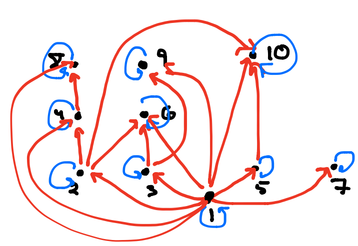
(d) (6 points) Let $\sim$ be the relation on $\mathbb{Z}$ defined as follows: For all $a, b \in \mathbb{Z}$, $a \sim b$ if and only if $a - b$ is a multiple of 3. It can be proven that $\sim$ is an equivalence relation. Write the elements of $[1492]$ as a set in roster form.

Solution: The set $[1492]$ is the set of all integers n such that $1492 - n$ is a multiple of 3 — in other words, it's the set of all integers that are congruent to 1492 mod 3. Since $1492 \pmod 3) = 1$, we have

$$[1492] = \{\dots, -8, -5, -2, 1, 4, 7, 10, 13, \dots, 1399, 1492, 1495, \dots\}$$

(e) (6 points) Let $\sim$ be the relation on the set $A = \{1, 2, 3, \dots, 10\}$ defined by $a \sim b$ if and only if a divides b. Draw the directed graph for this relation.

Solution: Here's the directed graph:



Key features: there is an arrow from 1 to everything else; every vertex has a loop; and whenever a divides b there is an arrow from a to b.

14. For each of the following statements, write the negation of the statement in symbolic form in which the negation symbol ($\neg$) is not used, and then write the negation as an English sentence.

(a) (6 points) $(\exists x \in \mathbb{R})(\cos(2x) = 2\cos x)$

> **Solution:**
> $$(\forall x \in \mathbb{R})(\cos(2x) \neq 2\cos x)$$
> For every real number x, $\cos(2x)$ does not equal $2\cos x$.

(b) (6 points) $(\forall x \in \mathbb{Z})((2|x) \Rightarrow (4|x))$ (The arrow ($\Rightarrow$) means "implies".)

> **Solution:**
> $$(\exists x \in \mathbb{Z})((2|x) \wedge (4 \not| x))$$
> There exists an integer x such that 2 divides x but 4 does not divide x.

15. Consider the statement: *If it is snowing and after 6:00am, then I will shovel my driveway.*

(a) (6 points) Write a clear statement of the contrapositive of this sentence.

> **Solution:** If I did not shovel my driveway, then either it is not snowing or it is not after 6:00am.

(b) (6 points) Write a clear statement of the converse of this sentence.

> **Solution:** If either it is not snowing or it is not after 6:00am, then I did not shovel my driveway.

16. Below are several proof-related tasks that we have undertaken repeatedly during the course. For each task, give a brief but complete outline of how a correct proof would be constructed in that situation. Be sure to state explicitly **what you would assume, what you would try to prove, and a specific strategy for how you would proceed from the beginning**.

(a) (8 points) Proving that two sets, A and B, are equal via the "choose an element" approach

> **Solution:** We would need to show that $A \subseteq B$ and $B \subseteq A$. For the first part, choose an element $x \in A$ and then prove that $x \in B$. For the second, choose an element $y \in B$ and prove that $y \in A$.

(b) (8 points) Proving the conditional statement "If P, then Q" by contradiction

> **Solution:** We assume the negation of the statement we intend to prove — that is, begin by assuming P and $\neg Q$. Then proceed through valid mathematical steps from this assumption to arrive at a contradiction, that is, two statements that have opposite truth values that are true at the same time as a result of the assumption. This forces us to reject the assumption, so the negation of the original statement is false; therefore the original statement is true.

(c) (8 points) Proving that the predicate $P(n)$ is true for all natural numbers n using mathematical induction

> **Solution:** We first prove that $P(1)$ is true, thereby establishing that there is some case in which the predicate is true. Then we assume that $P(k)$ is true for some natural number k, and then use this assumption along with valid mathematical steps to prove that $P(k+1)$ is true.

Continued →

17. (16 points) Choose EXACTLY ONE of the following and either prove the statement or disprove it. Circle the letter of the problem you are doing.

(a) For all integers a, b and d with $d \neq 0$, if d divides a or d divides b, then d divides ab.

Solution: This statement is true. Here is a proof.

Assume that either d divides a or d divides b. We will show that d divides ab. We will use two cases, depending on which integer d divides.

Case 1: $d|a$. Suppose $d|a$. This means there is an integer q such that $a = dq$. Multiplying both sides of this equation by b, we have $ab = d(bq)$. Since q and b are integers, so is bq by closure; therefore we see that d divides ab.

Case 2: $d|b$. Suppose $d|b$. This means there is an integer q such that $b = dq$. Multiplying both sides of this equation by a, we have $ab = d(aq)$. Since q and a are integers, so is aq by closure; therefore we see that d divides ab.

In either case, d divides ab as desired.

(b) For all functions $f : A \to B$ and $g : B \to C$, if $g \circ f : A \to C$ is an injection then f is an injection.

Solution: The statement is true. Here is a proof.

We will prove the contrapositive. So suppose that f is not an injection; we will show that $g \circ f$ is not an injection. Since f is not an injection, then there exist points $x_1, x_2 \in A$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. But then since $f(x_1) = f(x_2)$, we have $g(f(x_1)) = g(f(x_2))$. This means that $x_1 \neq x_2$ but $(g \circ f)(x_1) = (g \circ f)(x_2)$. Hence $g \circ f$ is not an injection, and we are done.

(c) For each integer n, if n is odd, then 8 divides $n^2 - 1$.

Solution: The statement is true. Here is a proof.

Suppose that n is odd. We will show that 8 divides $n^2 - 1$. Since n is odd, there is an integer q such that $n = 2q + 1$. Now compute $n^2 - 1$:

$$\begin{aligned} n^2 - 1 &= (2q+1)^2 - 1 \\ &= 4q^2 + 4q + 1 - 1 \\ &= 4q^2 + 4q \end{aligned}$$

Now we can proceed by two cases: One if q is even and another if q is odd.

Case 1: q is even. Suppose $q = 2k$ for some integer k. Then going back to $n^2 - 1$, we have:

$$n^2 - 1 = 4q^2 + 4q = 4(2k)^2 + 4(2k) = 16k^2 + 8k = 8(2k^2 + k)$$

Since k is an integer, so is $2k^2 + k$ by closure. Thus we see that in this case, 8 divides $n^2 - 1$.

Case 2: q is odd. Suppose $q = 2l + 1$ for some integer l. Then going back to $n^2 - 1$, we have:

$$n^2 - 1 = 4q^2 + 4q = 4(2l+1)^2 + 4(2l+1) = 16l^2 + 24l + 8 = 8(2l^2 + 3l + 1)$$

Since l is an integer, so is $2l^2 + 3l + 1$ by closure. Thus we see that in this case, 8 divides $n^2 - 1$.

In either case, we proved what we set out to prove, namely that 8 divides $n^2 - 1$.

18. (16 points) Choose EXACTLY ONE of the following to prove. Circle the letter of the problem you are doing.

(a) Let $\sim$ be the relation on $\mathbb{Z}$ given by $a \sim b$ if and only if $a \equiv b \pmod 3$. For all natural numbers n, prove that $[10^n] = [1]$ under this relation.

Solution:
Note that proving $[10^n] = [1]$ is equivalent to proving that $10^n \equiv 1 \pmod 3$ by the definition of $\sim$. We will prove this by mathematical induction. For $n = 1$, we have $10^n = 10$ which is clearly congruent to 1 mod 3. Now assume that $10^k \equiv 1 \pmod 3$ for some positive integer k. We want to show that $10^{k+1} \equiv 1 \pmod 3$. To this end we will show that 3 divides $10^{k+1} - 1$. We may rewrite $10^{k+1} - 1$ as

$$10^{k+1} - 1 = (10 \cdot 10^k) - 1 = (10 \cdot 10^k) - 10 + 9 = 10 \cdot (10^k - 1) + 9$$

By assumption, $10^k \equiv 1 \pmod 3$, and so we may write $10^k - 1 = 3q$ for some integer q. Substitution then gives

$$10^{k+1} - 1 = 10 \cdot (10^k - 1) + 9 = 10(3q) + 9 = 30q + 9 = 3(10q + 3)$$

By the closure of the set of integers under addition and multiplication, $10q + 3$ is an integer, and so we have that 3 divides $10^{k+1} - 1$, which was what we wanted.

(b) Recall that the *Fibonacci numbers* are the numbers f_n defined by $f_1 = 1$, f_2, and then $f_k = f_{k-1} + f_{k-2}$ for all $k > 1$. Prove that for each natural number n,

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$$

Solution: We proceed by mathematical induction. For $n = 1$, we have $f_1^2 = 1^2 = 1$ and $f_1 f_2 = 1 \cdot 1 = 1$. Thus the base case is established. Now assume that the equation is true for some positive integer k. We now want to prove that

$$f_1^2 + f_2^2 + \cdots + f_{k+1}^2 = f_{k+1} f_{k+2}$$

Expanding the left side of this gives:

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 &= f_k f_{k+1} + f_{k+1}^2 \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2} \end{aligned}$$

This is what we wanted to show.

(c) Prove that for every natural number n,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Solution: We proceed by mathematical induction. For $n = 1$ we have the single term $1/2$ on the left side, and

$$1 - \frac{1}{2^1} = 1 - \frac{1}{2} = \frac{1}{2}$$

on the right. These are equal, so the base case is established. Now assume the equation holds for some positive integer k. We want to show that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$$

Working with the left side, we have:

$$\begin{aligned}\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} &= \left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}\\ &= 1-\frac{2}{2^{k+1}}+\frac{1}{2^{k+1}}\\ &= 1-\frac{1}{2^{k+1}}\end{aligned}$$

This was what we wanted to show.

19. Consider the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $g(a, b) = \gcd(a, b)$, the greatest common divisor of a and b (also known as greatest common factor). For example, $g(10, 8) = 2$ and $g(5, 6) = 1$.

(a) (10 points) Prove or disprove: The function g is injective.

Solution: The function g is not injective. For example, $g(3, 4) = \gcd(3, 4) = 1$ and $g(7, 8) = \gcd(7, 8) = 1$ but of course $(3, 4) \neq (7, 8)$.

(b) (10 points) Prove or disprove: The function g is surjective.

Solution: The function g is surjective. To prove this, choose $n \in \mathbb{N}$. Then we claim that (n, n^2) maps to n. This amounts to showing that $\gcd(n, n^2) = n$. But this is true because n divides both n and n^2 and if d is any other common divisor of n and n^2, then $d|n$ so $d \leq n$. Hence n is the greatest common divisor of n and n^2, so $g(n, n^2) = n$.