Lemma 1. *For every natural number n,*

$$[10^n] = \begin{cases} [1] & \textit{if } n \textit{ is even} \\ [-1] & \textit{if } n \textit{ is odd} \end{cases}$$

where the equivalence classes come from the equivalence relation of integer congruence mod 11.

Proof. This is left as an exercise. Hint: Use induction. □

Theorem 2. *If n is a 5-digit integer such that the difference between its odd- and even-numbered digits is divisible by 11, then n is divisible by 11.*

Proof. Suppose n is is a 5-digit integer such that the difference between its odd- and even-numbered digits is divisible by 11. Write n in its base-10 form:

$$n = d_4 \times 10^4 + d_3 \times 10^3 + d_2 \times 10^2 + d_1 \times 10^1 + d_0 \tag{1}$$

We are assuming that 11 divides $(d_4 + d_2 + d_0) - (d_3 + d_1)$, so in terms of equivalence classes under the relation of integer congruence modulo 11, we have:

$$[(d_4 + d_2 + d_0) - (d_3 + d_1)] = [0] \tag{2}$$

We want to show 11 divides n, which we can do by proving that $[n] = [0]$ under the relation of congruence modulo 11. To this end, take equivalence classes of both sides of (1) mod 11:

$$[n] = [d_4 \times 10^4 + d_3 \times 10^3 + d_2 \times 10^2 + d_1 \times 10^1 + d_0]$$

Now using the properties of the modular arithmetic operations $\oplus$ and $\odot$ mod 11, we have:

$$\begin{aligned} [n] &= [d_4 \times 10^4 + d_3 \times 10^3 + d_2 \times 10^2 + d_1 \times 10^1 + d_0] \\ &= ([d_4] \odot [10^4]) \oplus ([d_3] \odot [10^3]) \oplus ([d_2] \odot [10^2]) \oplus ([d_1] \odot [10^1]) \oplus [d_0] \end{aligned}$$

By the Lemma, we can substitute for the classes of powers of 10:

$$\begin{aligned} [n] &= ([d_4] \odot [10^4]) \oplus ([d_3] \odot [10^3]) \oplus ([d_2] \odot [10^2]) \oplus ([d_1] \odot [10^1]) \oplus [d_0] \\ &= ([d_4] \odot [1]) \oplus ([d_3] \odot [-1]) \oplus ([d_2] \odot [1]) \oplus ([d_1] \odot [-1]) \oplus [d_0] \\ &= [d_4] \oplus [-d_3] \oplus [d_2] \oplus [-d_1] \oplus [d_0] \\ &= [d_4 + d_2 + d_0 - (d_3 + d_1)] \\ &= [0] \end{aligned}$$

The last line is true because of (2) above. Therefore $[n] = [0]$, so 11 divides n as desired. □