Items 1—10 are multiple choice questions that address a variety of learning objectives. Please circle the ONE response you believe is most correct. You do not need to justify your answer.

1. (2 points) If an integer k is congruent to $5 \pmod 7$, it means that
   (a) The quotient obtained when dividing k by 7 is 5
   (b) The quotient obtained when dividing k by 5 is 7
   (c) The remainder obtained when dividing k by 7 is 5
   (d) The remainder obtained when dividing k by 5 is 7
   (e) The remainder obtained when dividing 7 by 5 is k

   Solution: A.

2. (2 points) The main difference between the Principle of Mathematical Induction (which we can abbreviate "PMI") and the Extended Principle of Mathematical Induction ("EPMI") is
   (a) The EPMI uses a different base case than the PMI
   (b) The EPMI allows induction over the set of real numbers
   (c) The EPMI assumes a larger induction hypothesis than the PMI
   (d) The EPMI does not use a base case
   (e) The EPMI does not use an induction hypothesis at all

   Solution: A. The EMPI works for propositions where the base case may not be 1.

3. (2 points) In the proof that 4 divides $5^n - 1$, what would be the inductive hypothesis?
   (a) Assume that 4 divides $5^1 - 1$
   (b) Assume that 4 divides $5^k - 1$ for *all* $k \in \mathbb{N}$.
   (c) Assume that 4 divides $5^k - 1$ for *some* $k \in \mathbb{N}$.
   (d) Assume that if 4 divides $5^k - 1$ for some $k \in \mathbb{N}$, then 4 divides $5^{k+1} - 1$ for all $k \in \mathbb{N}$.
   (e) Assume that if 4 divides $5^1 - 1$, then 4 divides $5^k - 1$ for all $k \in \mathbb{N}$.

   Solution: C.

4. (2 points) Suppose A and B are sets such that card$(A) = 5$ and card$(B) = 3$. Then card$(A \times B)$
   (a) Equals 8
   (b) Equals 15
   (c) Equals 125
   (d) Equals 243
   (e) Is infinite

   Solution: B. Given an element $(x, y) \in A \times B$, there are 5 choices for the first coordinate and 3 choices for the second, giving 15 possibilities in all.

5. (2 points) Let A and B be sets. If $(a, b) \notin A \times B$, it means that
   (a) $a \notin A$ and $b \notin B$

(b) $a \notin A$ or $b \notin B$
(c) $a \in A$ and $b \notin B$
(d) $a \notin A$ and $b \in B$
(e) None of the above

> **Solution:** B.

6. (2 points) Suppose T_n is the sequence defined recursively by $T_n = \frac{1}{2}T_{n-1}$ for all $n \geq 2$. Then T_{10}
   (a) Equals 0
   (b) Equals $1/2048$
   (c) Equals $1/1024$
   (d) Equals $1/512$
   (e) Is impossible to determine based on this information alone

> **Solution:** E. To determine T_{10} we would need a base case for T (that is, we'd need the value of T_2). Without knowing the base value, there's no way to tell what T_{10} is.

7. (2 points) Let A and B be sets. If $x \in (A \cap B)^c$, then
   (a) $x \notin A$ and $x \notin B$
   (b) $x \notin A$ or $x \notin B$
   (c) $x \in A - B$
   (d) $x \in B - A$
   (e) None of the above

> **Solution:** B.

8. (2 points) Which of the following must be explicitly specified when defining a function f?
   (a) The domain of f
   (b) The range of f
   (c) The codomain of f
   (d) All of the above
   (e) Just (a) and (c)

> **Solution:** E. The range of f does not need to be specified – we can tell what the range is using the definition of f.

9. (2 points) Let $f : \mathbb{N} \to \mathbb{R}$ be the function defined by $f(x) = \sqrt{x}$. Then the set of preimages of the number 4 is
   (a) $\emptyset$
   (b) $\{-2\}$
   (c) $\{2\}$
   (d) $\{16\}$
   (e) $\{-2, 2\}$

Solution: D. This is because $f(16) = 4$.

10. (2 points) Suppose f and g are functions whose domain is the set $\mathbb{R}$ of real numbers and that $f(x) = g(x)$ for all x. Then
    (a) The range of f equals the range of g
    (b) The codomain of f equals the codomain of g
    (c) $f = g$ as functions
    (d) All of the above
    (e) None of the above

Solution: D.

11. Suppose that $U = \mathbb{N}$ and define the following sets:
    - $A = \{x \in \mathbb{N} \mid x \geq 7\}$
    - $B = \{x \in \mathbb{N} \mid x \text{ is odd}\}$
    - $C = \{x \in \mathbb{N} \mid x \text{ is a multiple of } 3\}$
    - $D = \{x \in \mathbb{N} \mid x \text{ is even}\}$

    Use the roster method to list the elements of the following sets. You do not need to explain your reasoning.

    (a) (4 points) $A^c \cap B^c$

    Solution: Note that $A^c = \{x \in \mathbb{N} \mid x < 7\} = \{1, 2, 3, 4, 5, 6\}$ and B^c is just the set of all even natural numbers. Therefore
    $$A^c \cap B^c = \{2, 4, 6\}$$

    (b) (4 points) $(A \cup B) \cap C$

    Solution: The set $A \cup B$ is the set of natural numbers that are either odd or greater than or equal to 7:
    $$A \cup B = \{1, 3, 5, 7, 8, 9, 10, 11, 12, \dots\}$$
    Intersecting this set with C yields the elements of this set that are multiples of 3:
    $$(A \cup B) \cap C = \{3, 9, 12, 15, \dots\}$$

    (c) (6 points) $(A - D) \cup (B - D)$

    Solution: The set $A - D$ is the set of points in A that are not in D, that is, the points in A that are not even:
    $$A - D = \{7, 9, 11, 13, \dots\}$$
    Likewise, $B - D = B$ since B and D are disjoint. So $(A - D) \cup (B - D) = (A - D) \cup B$, the set of all points either in $A - D$ or in B:
    $$(A - D) \cup (B - D) = \{1, 3, 5, 7, 9, \dots\}$$
    This is just equal to B itself.

(d) (6 points) $B \times \{x, y\}$

Solution: This set consists of all ordered pairs (s, t) where $s \in B$ and $t \in \{x, y\}$:

$$B \times \{x, y\} = \{(1, x), (1, y), (3, x), (3, y), (5, x), (5, y), \dots\}$$

12. Define the function $F : \mathbb{N} \to \mathbb{Z}$ by defining $F(n)$ to be the n^{th} Fibonacci number.

(a) (4 points) State the domain and codomain of this function. (Be sure to label which is which.)

Solution: The domain is $\mathbb{N}$ and the codomain is $\mathbb{Z}$.

(b) (6 points) State the images of the numbers $1, 2, 3, \dots, 10$. (This is ten items to compute and state.) You do not need to show your work.

Solution: We'll summarize these in a table:

n	1	2	3	4	5	6	7	8	9	10
$F(n)$	1	1	2	3	5	8	13	21	34	55

(c) (8 points) Is the range of this function equal to its codomain? Explain.

Solution: No. For example, the number 4 will never be the output of this function because $F(4) = 3$ and $F(5) = 5$, and the function f is clearly increasing, which means it will never go back to 4 in the outputs.

(d) (8 points) Consider the function $G : \mathbb{Z} \to \mathbb{R}$ given by

$$G(n) = \frac{\left(\frac{1}{2}(1 + \sqrt{5})\right)^n - \left(\frac{1}{2}(1 - \sqrt{5})\right)^n}{\sqrt{5}}$$

It can be verified that for all natural numbers n, $G(n)$ is the n^{th} Fibonacci number (see Portfolio Problem 7a). Does this mean $F = G$ as functions? Explain.

Solution: The functions F and G are not equal because their codomains are not equal. The codomain of F is $\mathbb{Z}$ but the codomain of G is $\mathbb{R}$. You could also note that their domains are not equal, either.

13. (12 points) Choose EXACTLY ONE of the following true statements and give a formal proof.

(a) For each natural number n, 3 divides $n^3 + 23n$.

(b) For each natural number n, $4^n \equiv 1 \pmod 3$.

(c) For all integers $n > 4$, $n^2 < 2^n$.

Solution: All three of these are best proven by mathematical induction.

1. For the base case, observe that when $n = 1$, $n^3 + 23n = 1 + 23 = 24$ and this is clearly divisible by 3. So for the induction step, assume that for some natural number k, we have 3 divides $k^3 + 23k$. We want to show that 3 divides $(k + 1)^3 + 23(k + 1)$. In other words, we want to show that there is an integer q such that

$$(k + 1)^3 + 23(k + 1) = 3q \tag{1}$$

To prove this, take the left side of (1) and expand it:

$$k^3 + 3k^2 + 3k + 1 + 23k + 23$$

Rearrange and group the terms as follows:

$$(k^3 + 23k) + (3k^2 + 3k + 24) \tag{2}$$

By assumption, we know that there exists an integer a such that $k^3 + 23k = 3a$. Substituting this into (2) and factoring out 3 when possible, we obtain:

$$(k^3 + 23k) + (3k^2 + 3k + 24) = 3a + 3(k^2 + k + 8) = 3(a + k^2 + k + 8)$$

Since a and k are integers and the set of integers is closed under addition and multiplication, we have that $a+k^2+k+8$ is an integer. Therefore we have written $(k^3+23k)+(3k^2+3k+24)$ as $3q$ for an integer q (namely $q = a + k^2 + k + 8$). Therefore 3 divides $(k + 1)^3 + 23(k + 1)$ as desired.

2. For the base case, observe that when $n = 1$, we have $4^n = 4$ and this is clearly congruent to $1 \pmod 3$. So for the induction step, assume that for some natural number k, we have $4^k \equiv 1 \pmod 3$. That is, 3 divides $4^k - 1$. We want to prove that $4^{k+1} \equiv 1 \pmod 3$, that is, 3 divides $4^{k+1} - 1$. To this end, note that:

$$4^{k+1} - 1 = 4^{k+1} - 4 + 3 = 4(4^k - 1) + 3 \tag{3}$$

Since 3 divides $4^k - 1$, we may write $4^k - 1 = 3a$ for some integer a. Substituting this into (3) gives:

$$4^{k+1} - 1 = 4(3a) + 3 = 3(4a + 1)$$

Since a is an integer, so is $4a + 1$ due to the closure of the set of integers under addition and multiplication. Hence 3 divides $4^{k+1} - 1$ and therefore $4^{k+1} \equiv 1 \pmod 3$ as desired.

3. The base case here is $n = 5$. In that case, note that $n^2 = 25$ and $2^n = 32$ so obviously the proposition holds. For the inductive step, suppose $k^2 < 2^k$ for some natural number k. We want to prove that $(k + 1)^2 < 2^{k+1}$. Looking at the right side, we expand and then use the inductive hypothesis to say:

$$(k + 1)^2 = k^2 + 2k + 1 < 2^k + 2k + 1 \tag{4}$$

We will now show that $2k + 1 < k^2$. To see this, note that since $k > 4$, we have $k - 1 > 3$ and so

$$(k - 1)^2 - 2 > 0 \tag{5}$$

Expanding the left side of (5) gives:

$$0 < (k - 1)^2 - 2 = k^2 - 2k + 1 - 2 = k^2 - 2k - 1 \tag{6}$$

Adding $2k + 1$ to the far left and far right sides of this inequality gives

$$2k + 1 < k^2 \tag{7}$$

Substituting this into (4) gives:

$$\begin{aligned}
(k + 1)^2 &< 2^k + 2k + 1 \\
&< 2^k + k^2 && \text{(From (7))} \\
&< 2^k + 2^k && \text{(Induction hypothesis)} \\
&= 2 \cdot 2^k \\
&= 2^{k+1}.
\end{aligned}$$

This is what we wanted to prove.

14. (12 points) Choose EXACTLY ONE of the following true statements and give a formal proof. In all of

these, assume the sets A, B, and C are subsets of some universal set U.

(a) Prove that the sets $A \cap B$ and $A - B$ are disjoint.

Solution: We'll use the algebra of sets approach here to prove that $(A \cap B) \cap (A - B) = \emptyset$ (which is what being disjoint means):

$$\begin{aligned} (A \cap B) \cap (A - B) &= (A \cap B) \cap (A \cap B^c) && \text{Basic Property 2} \\ &= (A \cap A) \cap (B \cap B^c) && \text{Commutative and Associative Laws} \\ &= (A \cap A) \cap \emptyset && \text{Every set is disjoint with its complement} \\ &= \emptyset && \text{Property of Empty Set 1} \end{aligned}$$

(b) Prove that $A = (A - B) \cup (A \cap B)$.

Solution: Let's use algebra of sets again, starting with the right side:

$$\begin{aligned} (A - B) \cup (A \cap B) &= (A \cap B^c) \cup (A \cap B) && \text{Basic Property 2} \\ &= A \cap (B \cup B^c) && \text{Distributive Law 1} \\ &= A \cap U && \text{Property from Screencast} \\ &= A && \text{Property of the Universal Set 2} \end{aligned}$$

(c) Prove that $A \times (B - C) = (A \times B) - (A \times C)$.

Solution: We will prove this by proving $A \times (B - C) \subseteq (A \times B) - (A \times C)$ and then proving $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

($\subseteq$) Choose $(x, y) \in A \times (B - C)$. We want to show that $(x, y) \in (A \times B) - (A \times C)$. Since $(x, y) \in A \times (B - C)$, we have that $x \in A$ and $y \in B - C$. That is, $y \in B$ and $y \notin C$. Since $x \in A$ and $y \in B$, we have that $(x, y) \in A \times B$. And since $x \in A$ and $x \notin C$, we have $(x, y) \notin A \times C$. Hence $(x, y) \in (A \times B) - (A \times C)$ as desired.

($\supseteq$) Choose $(u, v) \in (A \times B) - (A \times C)$. We want to show that $(u, v) \in A \times (B - C)$. Since $(u, v) \in (A \times B) - (A \times C)$, we have that $(u, v) \in A \times B$ and $(u, v) \notin A \times C$. So $u \in A$ and $v \in B$. If $(u, v) \notin A \times C$, it means that either $u \notin A$ or $v \notin C$. Since we already know $u \in A$, we must conclude that $v \notin C$. Therefore $u \in A$, $v \in B$, and $v \notin C$. Hence $v \in B - C$, which makes $(u, v) \in A \times (B - C)$ as desired.