Class Work: More Methods of Proof

1. Prove that for every positive real number x, if x is irrational, then $\sqrt{x}$ is irrational. (*Hint*: What's the contrapositive of this statement? Also, make sure you know the precise definition of a rational number before you start proving anything.)

   Proof. We will prove this statement by proving the contrapositive. That is, we will show that for every positive real number x, if $\sqrt{x}$ is rational, then x is rational. To that end, suppose x is a positive real number such that $\sqrt{x}$ is rational. Then by definition there exist integers a, b with $b \neq 0$ such that
$$\sqrt{x} = \frac{a}{b} \tag{1}$$
   Squaring both sides of (1) gives
$$x = \frac{a^2}{b^2}$$
   Since $a, b \in \mathbb{Z}$ with $b \neq 0$, we know by the closure of the set of integers under multiplication that $a^2, b^2 \in \mathbb{Z}$. We also know that $b^2 \neq 0$ since $b \neq 0$. Therefore x is rational, which is what we wanted to show. □

2. Suppose we have a right triangle whose hypotenuse has length c, and the lengths of the other sides are a and b. Prove that this right triangle is isosceles if and only if its area equals $\frac{1}{4}c^2$.

   Proof. First assume that the right triangle in question is isosceles. Then by definition, $b = a$. We want to show that the area of the triangle is $\frac{1}{4}c^2$. The area of the triangle is half the base times the height. In our case, the base and height are equal, so:
$$\text{Area} = \frac{1}{2}ab = \frac{1}{2}a^2 \tag{2}$$
   Since this triangle is a right triangle, we can use the Pythagorean Theorem to relate the hypotenuse and side lengths:
$$c^2 = a^2 + b^2 = a^2 + a^2 = 2a^2 \tag{3}$$
   Solving (3) for a^2 gives
$$a^2 = \frac{1}{2}c^2$$
   Substituting this expression back into (2) gives:
$$\text{Area} = \frac{1}{2}a^2 = \frac{1}{2}\left(\frac{1}{2}c^2\right) = \frac{1}{4}c^2$$
   Therefore the area is $\frac{1}{4}c^2$, which is what we wanted to show.

   For the converse, assume that the area of the right triangle is $\frac{1}{4}c^2$. We want to show that the triangle is isosceles, that is, that $a = b$. The area of the triangle is equal to half the base times the height, that is:
$$\text{Area} = \frac{1}{2}ab \tag{4}$$

Since we are assuming that this area also equals $\frac{1}{4}c^2$, we can set these two area expressions equal to each other:

$$\frac{1}{2}ab = \frac{1}{4}c^2$$

Therefore $c^2 = 2ab$ by multiplying both sides of this by 4.

The Pythagorean Theorem says that the hypotenuse and side lengths are related by:

$$c^2 = a^2 + b^2$$

Substituting $c^2 = 2ab$ into this and using algebra, we obtain:

$$\begin{aligned} a^2 + b^2 &= 2ab \\ a^2 - 2ab + b^2 &= 0 \\ (a-b)^2 &= 0 \end{aligned}$$

Since $(a-b)^2 = 0$, it follows that $a - b = 0$, from which we obtain $a = b$ which is what we wanted. $\square$