MTH 210: Communicating in Mathematics
Proof Portfolio Problems 1—3

Problem 1

Choose either Problem 1A or Problem 1B to do. Note that Problem 1B has two parts, and you must do both parts if you choose that one.

Problem 1A: The notions of **type 0**, **type 1**, and **type 2** integers are defined in Exercise 9 of Section 1.2. Suppose that a and b are type 2 integers and look at the integer a^2+b^2. What can you conclude about the type of this integer? That is, is a^2+b^2 always type 0? Or is it always type 1? Or is it always type 2? Or can it sometimes be different types, depending on the specific values of a and b? By experimenting with different specific type 2 integers a and b, decide which of the four possibilities above is correct. If you believe one of the first three possibilities is correct, write this in the form of a conjecture that looks like:

If a and b are type 2 integers, then a^2+b^2 is a ____________ integer.

where the blank is filled in with "type 0", "type 1", or "type 2", and then give a formal mathematical proof of this conjecture. On the other hand, if you believe that a^2+b^2 can be different types depending on a and b, then say so clearly and then give specific counterexamples that show that a^2+b^2 can be different types. (Remember: a and b must be type 2 integers.)

Problem 1B: This option has two parts. You must do BOTH parts.

- Suppose a, b, and c are integers. What can we say about whether $ab+ac$ is always even or always odd? If you believe that $ab+ac$ must always be even, then write a conjecture to this effect and prove it. Similarly, if you believe $ab+ac$ is odd, write a conjecture to this effect and prove it. If you believe $ab+ac$ can be either even or odd depending on the values of a, b and c, then say so and then give specific examples that shows this.
- Suppose a is an integer and b and c are both *odd* integers. What can we say about whether $ab+ac$ is always even or always odd? If you believe that $ab+ac$ must always be even, then write a conjecture to this effect and prove it. Similarly, if you believe $ab+ac$ is odd, write a conjecture to this effect and prove it. If you believe $ab+ac$ can be either even or odd depending on the values of a, b and c, then say so and then give specific examples that shows this.

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Problem 2

Choose either Problem 2A or Problem 2B to do.

Problem 2A: In this problem, you will in some way use the Pythagorean Theorem, which we will accept to be true without proof: "For any right triangle, if its legs have length a and b and its hypotenuse length c, then $a^2 + b^2 = c^2$." Here's another important term for this problem: a *Pythagorean triple* (p, q, r) is a triple of natural numbers $p < q < r$ such that $p^2 + q^2 = r^2$. For instance, $(3, 4, 5)$ is a Pythagorean triple. Prove or disprove the following conjecture: If m is a natural number and $m \geq 2$, then $(2m, m^2 - 1, m^2 + 1)$ is a Pythagorean triple.

Problem 2B: Suppose that $f(x) = x^3 + ax^2 + bx + c$ is a cubic polynomial function with a, b, c real numbers and $a^2 > 3b$. Prove or disprove the following conjecture: The x-coordinate of the inflection point of f lies halfway between the x-coordinates of the two critical points of f.

Problem 3

Choose either Conjecture 3A or Conjecture 3B to do. If you believe the conjecture is true, give a formal mathematical proof. If you believe the conjecture is false, give a specific counterexample and prove your counterexample works. Note that both of these statements are "if and only if". If one direction of the biconditional statement is false, give a counterexample for it, but also check to see if the opposite direction is true. If so, give a proof.

Conjecture 3A: For each integer a, $a \equiv 3 \pmod 7$ if and only if $(a^2 + 5a) \equiv 3 \pmod 7$.

Conjecture 3B: For any integer k, $k^2 + 4k + 5$ is even if and only if $4|(k^2 + 2k - 1)$.