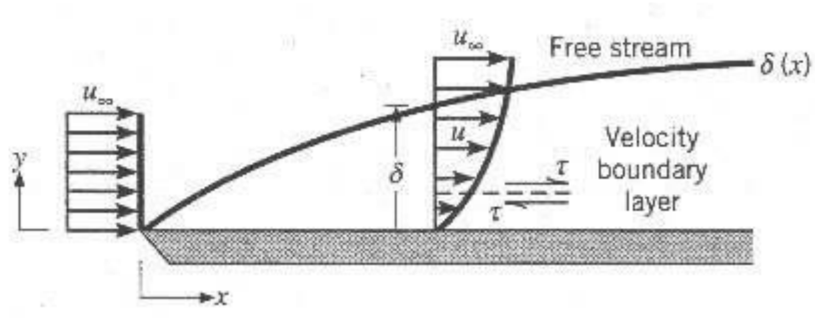
Simplication and extension of PDEs

- Boundary layer equations
- 3-D equations
- Cylindrical coordinate
- Spherical coordinate

Velocity Boundary Layers: external flow



For fluid flow over a flat plate:

- As $y \to \infty$: where u is velocity in x-direction $\boxed{u = u_\infty}$
- As $y \to 0$: (non-slip condition) $\boxed{u = 0}$
- The boundary layer thickness is defined as the value at which:
- The boundary layer thickness δ varies with x $\boxed{u(y) = 0.99u_\infty}$

- <u>Shear Stress</u>

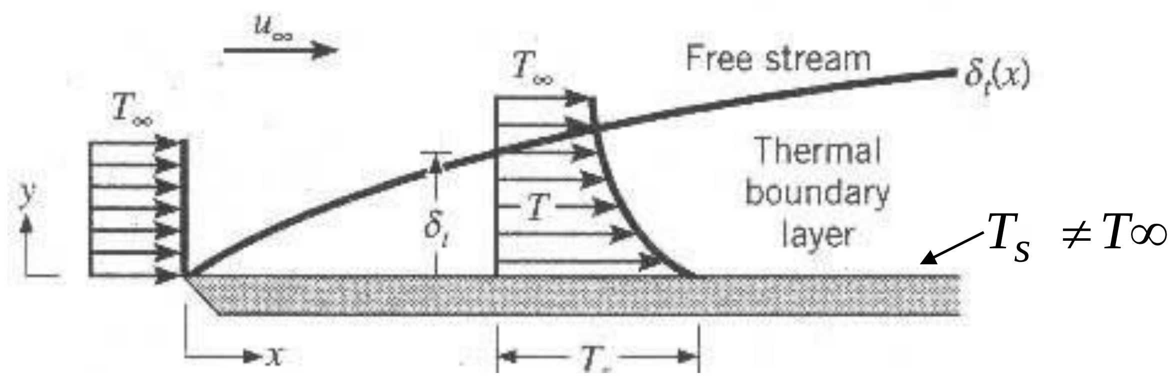
  Dynamic viscosity

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

- <u>Local friction coefficient</u>

$$C_f = \frac{\tau_s}{\rho\left(\frac{u_\infty^2}{2}\right)}$$

Thermal boundary layer: external flow



- A hot or cold plate alters the temperature distribution in the air
  - As $y \to \infty$: $T(y) = T_\infty$
  - As $y \to 0$: $T(y) = T_s$
  - The thermal boundary layer thickness is defined as the value at which:

$$\frac{T_s - T(y)}{T_s - T_\infty} = 0.99$$

  - The thermal boundary layer thickness, δ_t also varies (increases) with x

Thermal boundary layer: external flow

<u>Heat Flux</u>

- Heat flux analogous to shear stress in velocity boundary layer
- Heat flux proportional to the temperature gradient at the surface,
  AND
  since *u*(y=0) =0, energy transfer to/from fluid occurs by conduction only!

$$\boxed{q_s'' = -k_f \left.\frac{\partial T}{\partial y}\right|_{y=0}}$$

- Using Newton's law of cooling:

$$\boxed{q'' = h(T_s - T_\infty)}$$

- While δ increases with increasing x, temperature gradients in the boundary layer must decrease with increasing x.

$$\boxed{h = \frac{-k_f \left.\partial T / \partial y\right|_{y=0}}{T_s - T_\infty}}$$

- Accordingly, q_s'' and h decrease with increasing x.

Boundary Layer Approximations (1)

Velocity boundary layer

$$u >> v\ ;\ \text{and}\ \frac{\partial u}{\partial y} >> \frac{\partial u}{\partial x}\ ,\quad \frac{\partial u}{\partial y} >> \frac{\partial v}{\partial y}\ ,\quad \frac{\partial u}{\partial y} >> \frac{\partial v}{\partial y}$$

Thermal boundary layer

$$\frac{\partial T}{\partial y} >> \frac{\partial T}{\partial x}$$

The magnitude of variables in the thermal boundary layer

variables	x (main flow direction)	y	u	v	T
magnitude	1	δ	1	δ	1

Boundary Layer Approximations (2)

Conservation of mass (continuity):

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \Rightarrow \rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}=0$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0}$$

x-momentum equation:

$$0 \quad \left[\frac{\partial u}{\partial y} >> \frac{\partial u}{\partial x}\right] \qquad 0$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}\right)+\cancel{\text{"Body forces"}}$$

$$\Rightarrow \boxed{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}} \qquad where: \nu=\frac{\mu}{\rho}$$

y-momentum equation:

$$\rho\left(\cancel{u\frac{\partial v}{\partial x}}^{0}+\cancel{v\frac{\partial v}{\partial y}}^{0}\right)=-\frac{\partial P}{\partial y}+\mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}^{0}+\cancel{\frac{\partial^2 v}{\partial y^2}}^{0}\right)+\cancel{\text{"Body forces"}}^{0}$$

So: $$\boxed{\frac{\partial P}{\partial y}=0}$$

Boundary Layer Approximations (3)

Energy equation:

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial T}{\partial t}=0,\ \frac{\partial^2 T}{\partial x^2}<<\frac{\partial^2 T}{\partial y^2}$$

$$\boxed{u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}}$$

$$\alpha=\frac{k}{\rho c_p}$$

Result is 4 equations and 4 unknowns:

Unknowns are: *u*, *v*, P, and T

Since:

$$\frac{\partial \mathrm{P}}{\partial \mathrm{y}}=0 \quad \text{then} \quad \mathrm{P}=\mathrm{f}(\mathrm{x}) \ \text{ only, and } \frac{\partial \mathrm{P}}{\partial \mathrm{x}}=\frac{\mathrm{dP}}{\mathrm{dx}}$$

P(x) can be obtained from free stream flow.

Simplified boundary layer equation

$$\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + \mathrm{v}\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x} + \mathrm{v}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

Extension of PDEs in Cartesian co-ordinates

In Cartesian co-ordinates, the equations of continuity, x-wise momentum and energy for a Newtonian, Fourier fluid are:

$$\frac{\partial\rho}{\partial t}+U\frac{\partial\rho}{\partial x}+V\frac{\partial\rho}{\partial y}+W\frac{\partial\rho}{\partial z}+\rho\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)=0 \tag{1}$$

$$\begin{aligned}\rho\left[\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}\right]&=\rho X-\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left[\left(\eta-\frac{2}{3}\mu\right)\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)\right]\\&+2\frac{\partial}{\partial x}\left(\mu\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)\right]\end{aligned} \tag{2}$$

$$\begin{aligned}&\rho c_P\left[\frac{\partial T}{\partial t}+U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}+W\frac{\partial T}{\partial z}\right]\\&=\beta T\left[\frac{\partial P}{\partial t}+U\frac{\partial P}{\partial x}+V\frac{\partial P}{\partial y}+W\frac{\partial P}{\partial z}\right]+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\Phi\end{aligned} \tag{3}$$

and Φ is the rate of dissipation per volume and the other symbols have their usual meanings.

With certain additional assumptions, the equations in cylindrical polar co-ordinates are:

Extension of PDEs in cylindrical co-ordinates

With certain additional assumptions, the equations in cylindrical polar co-ordinates are:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r U_r)+\frac{1}{r}\frac{\partial}{\partial \phi}(\rho U_\phi)+\frac{\partial}{\partial z}(\rho U_z)=0 \qquad (4)$$

$$\begin{aligned}&\rho\left[\frac{\partial U_r}{\partial t}+U_r\frac{\partial U_r}{\partial r}+\frac{U_\phi}{r}\frac{\partial U_r}{\partial \phi}+U_z\frac{\partial U_r}{\partial z}-\frac{U_\phi^2}{r}\right]\\ &=\rho X_r-\frac{\partial P}{\partial r}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U_r}{\partial \phi^2}+\frac{\partial^2 U_r}{\partial z^2}-\frac{U_r}{r^2}-\frac{2}{r^2}\frac{\partial U_\phi}{\partial \phi}\right]\end{aligned} \qquad (5)$$

$$\begin{aligned}&\rho\left[\frac{\partial U_\phi}{\partial t}+U_r\frac{\partial U_\phi}{\partial r}+\frac{U_\phi}{r}\frac{\partial U_\phi}{\partial \phi}+U_z\frac{\partial U_\phi}{\partial z}+\frac{U_r U_\phi}{r}\right]=\\ &\rho X_\phi-\frac{1}{r}\frac{\partial P}{\partial \phi}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U_\phi}{\partial \phi^2}+\frac{\partial^2 U_\phi}{\partial z^2}-\frac{U_\phi}{r^2}+\frac{2}{r^2}\frac{\partial U_r}{\partial \phi}\right]\end{aligned} \qquad (6)$$

$$\rho\left[\frac{\partial U_z}{\partial t}+U_r\frac{\partial U_z}{\partial r}+\frac{U_\phi}{r}\frac{\partial U_z}{\partial \phi}+U_z\frac{\partial U_z}{\partial z}\right]=\rho X_z-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U_z}{\partial \phi^2}+\frac{\partial^2 U_z}{\partial z^2}\right] \qquad (7)$$

$$\rho c_P\left[\frac{\partial T}{\partial t}+U_r\frac{\partial T}{\partial r}+\frac{U_\phi}{r}\frac{\partial T}{\partial \phi}+U_z\frac{\partial T}{\partial z}\right]=k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}\right]+\Phi \qquad (8)$$

Solutions of simplified BL equations

- Dimensionless analysis

  π - theorem: A relationship involving *n* independent physical quantities involving *f* “fundamental” dimensions can be reduced to a relationship in *n* - *f* dimensionless groups.

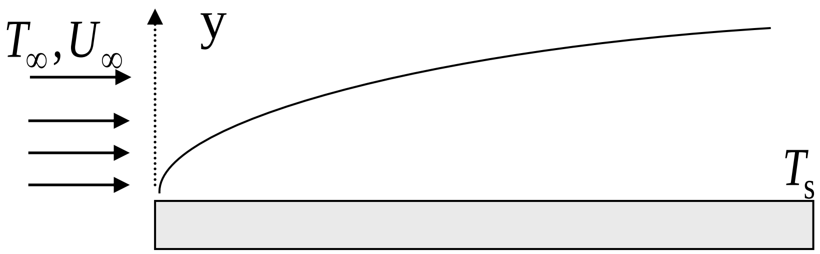
- Boundary integration method

  –Integration of momentum and energy PDE over the boundary layer, + obtain velocity and temperature profiles (assumption of polynomial form + BC met), obtaining boundary layer thickness, shear stress friction coefficient, local heat transfer coefficient and Nusselt number.

- Blasius solution
- Reynolds analogy

Blasius solution 1

- <u>Assumption:</u>
  - Steady, incompressible, laminar flow
  - Constant fluid properties
  - For flat plate,



Continuity $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

Momentum $u \dfrac{\partial u}{\partial x} + v \dfrac{\partial u}{\partial y} = \nu \dfrac{\partial^2 u}{\partial y^2}$

Energy $u \dfrac{\partial T}{\partial x} + v \dfrac{\partial T}{\partial y} = \alpha \dfrac{\partial^2 T}{\partial y^2}$

  - <u>Boundary layer equations</u>

Blasius solution 2

Define stream function, ψ(x,y))

$$u \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad v \equiv -\frac{\partial \psi}{\partial x}$$

Define new dependent and independent variables,

$$f(\eta) \equiv \frac{\psi}{u_\infty \sqrt{\nu x / u_\infty}} \qquad \eta \equiv y\sqrt{u_\infty / \nu x}$$

<u>The momentum equation</u>

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0$$

And the boundary conditions are

$$\left.\frac{df}{d\eta}\right|_{\eta=0} = f(0) = 0 \quad \text{and} \quad \left.\frac{df}{d\eta}\right|_{\eta=\infty} = 1$$

Boundary layer thickness:

$$\delta = \frac{5}{\sqrt{\frac{u_\infty}{\nu x}}} \quad \text{but, since } \mathrm{Re}_\mathrm{x} = \frac{u_\infty x}{\nu} \quad \Rightarrow \quad \delta = \frac{5\,\mathrm{x}}{\sqrt{\mathrm{Re}_x}}$$

$$\delta \propto \sqrt{\mathrm{x}} \quad \text{and} \quad \delta \propto \sqrt{\nu} \quad \text{and} \quad \delta \propto \frac{1}{\sqrt{u_\infty}}$$

Blasius solution 3

The Energy equation: $T^* \equiv [(T - T_s)/(T_\infty - T_s)]$

$$\frac{d^2T^*}{d\eta^2} + \frac{Pr}{2} f \frac{dT^*}{d\eta} = 0$$

- Boundary conditions:

$$T^*(0) = 0 \quad \text{and} \quad T^*(\infty) = 1$$

- Solution of the differential equation:

$$\left.\frac{dT^*}{d\eta}\right|_{\eta=0} = 0.332\, Pr^{1/3}$$

- Expressing the local convection coef.

$$h_x = \frac{q_s''}{T_s - T_\infty} = -\frac{T_\infty - T_s}{T_s - T_\infty} k \left.\frac{\partial T^*}{\partial y}\right|_{y=0}$$

$$h_x = k\left(\frac{u_\infty}{\nu x}\right)^{1/2} \left.\frac{dT^*}{d\eta}\right|_{\eta=0}$$

  – Then the local Nusselt number is:

$$\boxed{Nu_x \equiv \frac{h_x x}{k} = 0.332\, \mathrm{Re}_x^{1/2}\, \mathrm{Pr}^{1/3}}$$

For $0.6 \le Pr \le 50$

Blasius solution 4

- The ***Average Nusselt number*** over the whole plate can be found by integration:

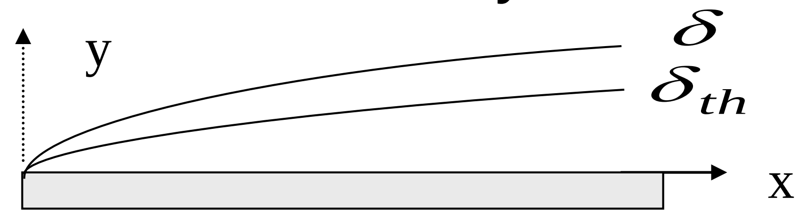
$$\overline{Nu}_x \equiv \frac{\overline{h}_x\ x}{k} = \frac{x}{k}\left[\frac{1}{x}\int_0^x h_x\ dx\right]$$

$$\boxed{\overline{Nu}_x = 0.664\,\mathrm{Re}_x^{1/2}\ \mathrm{Pr}^{1/3}}$$

For large *Pr* (oils):

For small *Pr* (liquid metals):

- Ratio of velocity to thermal boundary layer thickness:



Pr > 1000

Fluid viscosity greater than thermal diffusivity

$\therefore$

Pr < 0.1

Fluid viscosity less than thermal diffusivity

$\therefore$

Momentum and energy analogy

(Reynolds analogy)

Reynolds analogy 1

- Review of the energy and momentum equations

$$\boxed{u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\underbrace{\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2}}$$

usually small, except for high speed or highly viscous flow

$$\boxed{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}}$$

Dimensionlize two equations to see the momentum and heat transfer analogy is going …..

$$\boxed{u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\frac{1}{\mathrm{Re}_L\cdot\mathrm{Pr}}\frac{\partial^2 T^*}{\partial y^{*2}}}$$

$$\boxed{u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial P^*}{\partial x^*}+\frac{1}{\mathrm{Re}_L}\frac{\partial^2 u^*}{\partial y^{*2}}}$$

$x^* \equiv \frac{x}{L}$ and $y^* \equiv \frac{y}{L}$

$u^* \equiv \frac{u}{V}$ and $v^* \equiv \frac{v}{V}$

$T^* \equiv \frac{T-T_s}{T_\infty - T_s}$

Reynolds analogy 2

- If $dp^* / dx = 0$ and $\mathbf{Pr} = 1$, we obtain:

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L} \frac{\partial^2 T^*}{\partial y^{*2}}$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}}$$

- These two equations are of precisely the same form.

We know that if $dp^* / dx = 0$, $u_\infty = V$.

The boundary conditions for these two equations are:

$$\begin{aligned}
&\text{Wall:} && u^*(x^*, y^* = 0) = 0; \\
& && T^*(x^*, y^* = 0) = 0 \\
&\text{Freestream:} && u^*(x^*, y^* = \infty) = \frac{U_\infty(x^*)}{V} \equiv 1; \\
& && T^*(x^*, y^* = \infty) = 1
\end{aligned}$$

The boundary conditions are equivalent.

Therefore, the boundary layer velocity and temperature profiles must be of the same functional form.

Reynolds analogy 3

For the solution, the function *f* must be the same.

$$u^* = f\left(x^*, y^*, \mathrm{Re}_L\right)$$
$$\mathrm{T}^* = f\left(x^*, y^*, \mathrm{Re}_L, \mathrm{Pr}\right)$$

$$\mathrm{Nu} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = f\left(x^*, \mathrm{Re}_L, \mathrm{Pr}\right)$$

From fluid mechanics

$$C_f = \frac{\tau_s}{\rho \frac{V^2}{2}} \Rightarrow = \frac{2}{\mathrm{Re}_L} \left.\frac{\partial u^*}{\partial y^*}\right|_{y=0} = \frac{2}{\mathrm{Re}_L} f\left(x^*, \mathrm{Re}_L\right)$$

So it can be concluded that $\boxed{C_f \frac{\mathrm{Re}_L}{2} = Nu}$

Reynolds analogy 4

- Define the Stanton number St,

$$St \equiv \frac{h}{\rho V c_p} = \frac{Nu}{\mathrm{Re}\,\mathrm{Pr}}$$

- The analogy takes the form

$$\boxed{\frac{C_f}{2} = St}$$

Reynolds analogy

- The restrictions: the validity of the boundary layer approximations, $dp^*/dx = 0$ and $\mathrm{Pr} \approx 1$.

- The modified Reynolds, or Chilton-Colburn, analogy has the form

$$\frac{c_f}{2} = St\,\mathrm{Pr}^{2/3} = j \qquad (0.6 < \mathrm{Pr} < 60)$$

Colburn j factor

For laminar flow, it's only appropriate when $dp^*/dx \approx 0$

Reynolds analogy 5

- Recall the non-dimensional parameters

Reynolds # **-**
Ratio of the ***inertia*** to the ***viscous*** forces of a fluid flow

$$\frac{F_l}{F_s} \approx \frac{\rho V^2 / L}{\mu V / L^2} = \frac{\rho V L}{\mu} = \mathrm{Re}_L$$

Prandtl #
Ratio of the ***momentum*** to the ***thermal diffusivity*** in a fluid flow**.**

$$\mathrm{Pr} = \frac{c_p \mu}{k} = \frac{\nu}{\alpha}$$

For laminar boundary layers,

$$\boxed{\frac{\delta}{\delta_{th}} \approx \mathrm{Pr}^n}$$

Where n is a positive exponent.

TABLE 6.2 Selected dimensionless groups of heat transfer

Group	Definition	Interpretation
Biot number (Bi)	$\frac{hL}{k_s}$	Ratio of the internal thermal resistance of a solid to the boundary layer thermal resistance
Bond number (Bo)	$\frac{g(\rho_l - \rho_v)L^2}{\sigma}$	Ratio of gravitational and surface tension forces
Coefficient of friction (C_f)	$\frac{\tau_s}{\rho V^2/2}$	Dimensionless surface shear stress
Eckert number (Ec)	$\frac{V^2}{c_p(T_s - T_\infty)}$	Kinetic energy of the flow relative to the boundary layer enthalpy difference
Fourier number (Fo)	$\frac{\alpha t}{L^2}$	Ratio of the heat conduction rate to the rate of thermal energy storage in a solid. Dimensionless time
Friction factor (f)	$\frac{\Delta p}{(L/D)(\rho u_m^2/2)}$	Dimensionless pressure drop for internal flow
Grashof number (Gr_L)	$\frac{g\beta(T_s - T_\infty)L^3}{\nu^2}$	Measure of the ratio of buoyancy forces to viscous forces
Colburn j factor (j_H)	$St\, Pr^{2/3}$	Dimensionless heat transfer coefficient
Jakob number (Ja)	$\frac{c_p(T_s - T_{sat})}{h_{fg}}$	Ratio of sensible to latent energy absorbed during liquid–vapor phase change
Mach number (Ma)	$\frac{V}{a}$	Ratio of velocity to speed of sound
Nusselt number (Nu_L)	$\frac{hL}{k_f}$	Ratio of convection to pure conduction heat transfer
Peclet number (Pe_L)	$\frac{VL}{\alpha} = Re_L\, Pr$	Ratio of advection to conduction heat transfer rates
Prandtl number (Pr)	$\frac{c_p\mu}{k} = \frac{\nu}{\alpha}$	Ratio of the momentum and thermal diffusivities
Reynolds number (Re_L)	$\frac{VL}{\nu}$	Ratio of the inertia and viscous forces
Stanton number (St)	$\frac{h}{\rho V c_p} = \frac{Nu_L}{Re_L\, Pr}$	Modified Nusselt number
Weber number (We)	$\frac{\rho V^2 L}{\sigma}$	Ratio of inertia to surface tension forces

Recap: How to get *h*

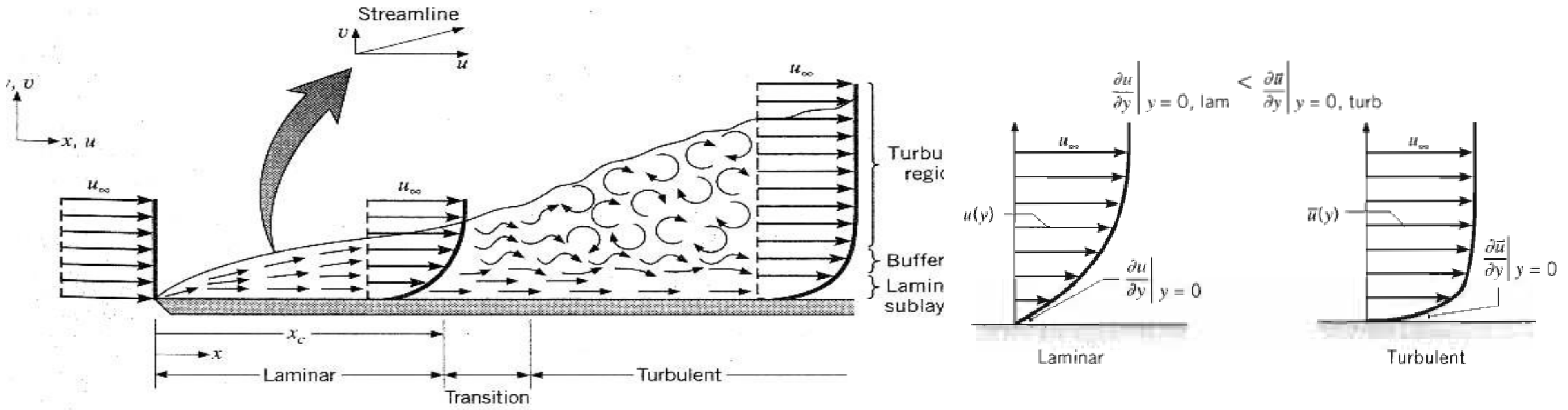
- Mathematical method
  - Derivation of partial differential equations (PDEs)
    - Solve the PDEs
      - Analytical solutions
      - Integration method
      - Numerical solutions
  - Similarity between flow and heat transfer
    - Reynolds analogy
- Experimental method
  - Dimensional analysis
  - Experiments and correlations of flow and heat transfer

Nu~f(Re Pr)

How to get h

- Understand the questions in heat transfer term
- Define flow conditions: external vs internal, laminar vs turbulent, forced vs free convection ….
- Make reasonable assumptions
- Find right correlations, usually given.
- Find properties and calculate known dimensionless groups
- Find Nu or h
- Find the solution of the quesiton

Laminar Versus Turbulent Flow



Characterization of laminar flow

- Low amount of "mixing" of fluid within the boundary layer (smooth flow)

Characterization of turbulent flow

Rec ~ 50，000

- High amount of mixing of fluid within the boundary layer (irregular flow)
- High amount of mixing means increased surface friction as well as convection transfer rates (heat and mass)

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Mach number (Ma)	$\frac{V}{a}$	Ratio of velocity to speed of sound
Nusselt number (Nu_L)	$\frac{hL}{k_f}$	Ratio of convection to pure conduction heat transfer $\longrightarrow$ k value
Peclet number (Pe_L)	$\frac{VL}{\alpha} = Re_L\, Pr$	Ratio of advection to conduction heat transfer rates
Prandtl number (Pr)	$\frac{c_p\mu}{k} = \frac{\nu}{\alpha}$	Ratio of the momentum and thermal diffusivities $\longrightarrow$ T effect
Reynolds number (Re_L)	$\frac{VL}{\nu}$	Ratio of the inertia and viscous forces $\longrightarrow$ L: Plate / pipe / non spherical
Stanton number (St)	$\frac{h}{\rho V c_p} = \frac{Nu_L}{Re_L\, Pr}$	Modified Nusselt number
Weber number (We)	$\frac{\rho V^2 L}{\sigma}$	Ratio of inertia to surface tension forces

Example question 1

EXAMPLE 6.1

Experimental results for the local heat transfer coefficient h_x for flow over a flat plate with an extremely rough surface were found to fit the relation

$$h_x(x) = ax^{-0.1}$$

where a is a coefficient $(\mathrm{W/m^{1.9} \cdot K})$ and x (m) is the distance from the leading edge of the plate.

1. Develop an expression for the ratio of the average heat transfer coefficient $\bar{h}_x$ for a plate of length x to the local heat transfer coefficient h_x at x.
2. Plot the variation of h_x and $\bar{h}_x$ as a function of x.

Example question 2

EXAMPLE 6.2

Water flows at a velocity $u_\infty = 1$ m/s over a flat plate of length $L = 0.6$ m. Consider two cases, one for which the water temperature is approximately 300 K and the other for an approximate water temperature of 350 K. In the laminar and turbulent regions, experimental measurements show that the local convection coefficients are well described by

$$h_{\text{lam}}(x) = C_{\text{lam}} x^{-0.5} \qquad h_{\text{turb}}(x) = C_{\text{turb}} x^{-0.2}$$

where x has units of m. At 300 K,

$$C_{\text{lam},300} = 395 \text{ W/m}^{1.5} \cdot \text{K} \qquad C_{\text{turb},300} = 2330 \text{ W/m}^{1.8} \cdot \text{K}$$

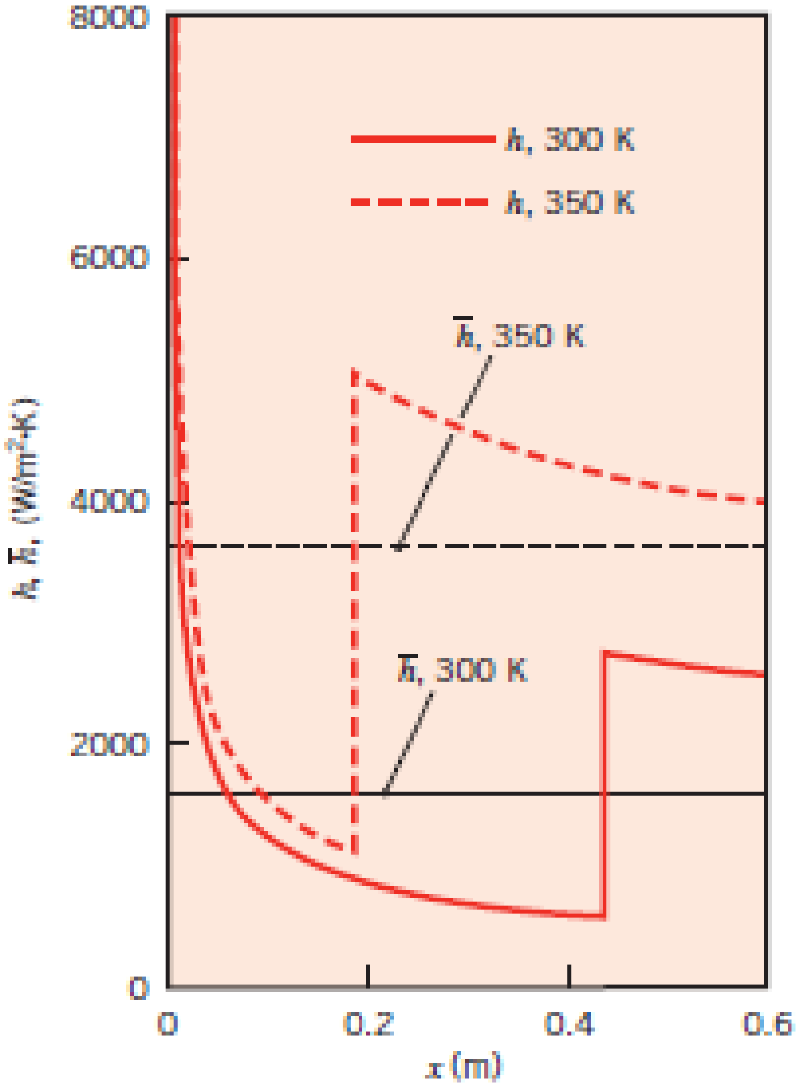
while at 350 K,

$$C_{\text{lam},350} = 477 \text{ W/m}^{1.5} \cdot \text{K} \qquad C_{\text{turb},350} = 3600 \text{ W/m}^{1.8} \cdot \text{K}$$

As is evident, the constant C depends on the nature of the flow as well as the water temperature because of the thermal dependence of various properties of the fluid.

Determine the average convection coefficient, $\overline{h}$, over the entire plate for the two water temperatures.

- Note:
  - Transition
  - T influence - property
  - h~BL thickness



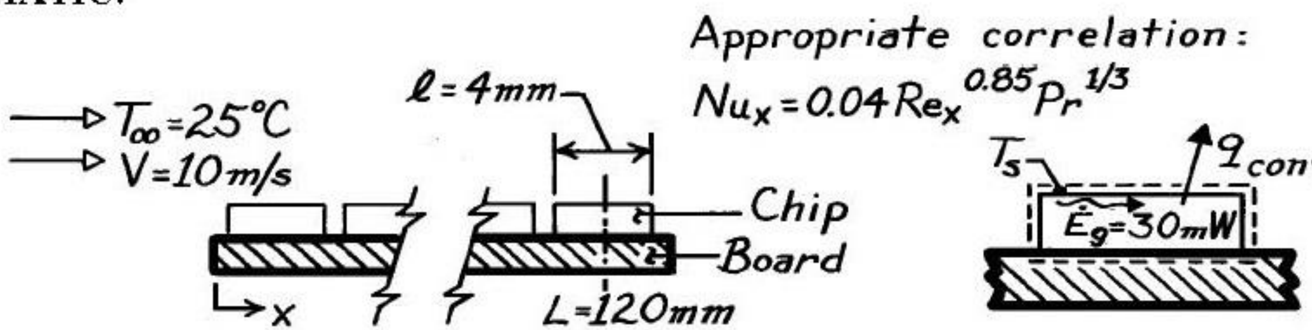
Example question 3

- Forced air at $T_\infty = 25$℃ and $V = 10 m/s$ is used to cool electronic elements on a circuit board. One such element is a chip, 4 mm by 4 mm, located 120 mm from the leading edge of the board. Experiments have revealed that flow over the board is disturbed by the elements and that convection heat transfer is correlated by an expression of the form

$$Nu_x = 0.04 \mathrm{Re}_x^{0.85} \mathrm{Pr}^{1/3}$$

Estimate the surface temperature of the chip if it is dissipating 30 mW.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Power dissipated within chip is lost by convection across the upper surface only, (3) Chip surface is isothermal, (4) The average heat transfer coefficient for the chip surface is equivalent to the local value at x = L.

PROPERTIES: *Table A-4*, Air (assume $T_s = 45°C$, $T_f = (45 + 25)/2 = 35°C = 308K$, 1atm): $\nu = 16.69 \times 10^{-6}\ m^2/s$, $k = 26.9 \times 10^{-3}\ W/m\cdot K$, $Pr = 0.703$.

ANALYSIS: From an energy balance on the chip (see above),

$$q_{conv} = \dot{E}_g = 30\,W. \tag{1}$$

Newton's law of cooling for the upper chip surface can be written as

$$T_s = T_\infty + q_{conv} / \overline{h}\, A_{chip} \tag{2}$$

where $A_{chip} = \ell^2$. Assume that the *average* heat transfer coefficient $\left(\overline{h}\right)$ over the chip surface is equivalent to the *local* coefficient evaluated at x = L. That is, $\overline{h}_{chip} \approx h_x(L)$ where the local coefficient can be evaluated from the special correlation for this situation,

$$Nu_x = \frac{h_x x}{k} = 0.04\left[\frac{Vx}{\nu}\right]^{0.85} Pr^{1/3}$$

and substituting numerical values with x = L, find

$$h_x = 0.04\frac{k}{L}\left[\frac{VL}{\nu}\right]^{0.85} Pr^{1/3}$$

$$h_x = 0.04\left[\frac{0.0269\ W/m\cdot K}{0.120\ m}\right]\left[\frac{10\ m/s \times 0.120\ m}{16.69\times10^{-6}\ m^2/s}\right]^{0.85}(0.703)^{1/3} = 107\ W/m^2\cdot K.$$

The surface temperature of the chip is from Eq. (2),

$$T_s = 25^\circ C + 30\times10^{-3}\ W/107\ W/m^2\cdot K\times(0.004m)^2 = 42.5^\circ C.$$ <

COMMENTS: (1) Note that the estimated value for T_f used to evaluate the air properties was reasonable. (2) Alternatively, we could have evaluated $\overline{h}_{chip}$ by performing the integration of the local value, h(x).

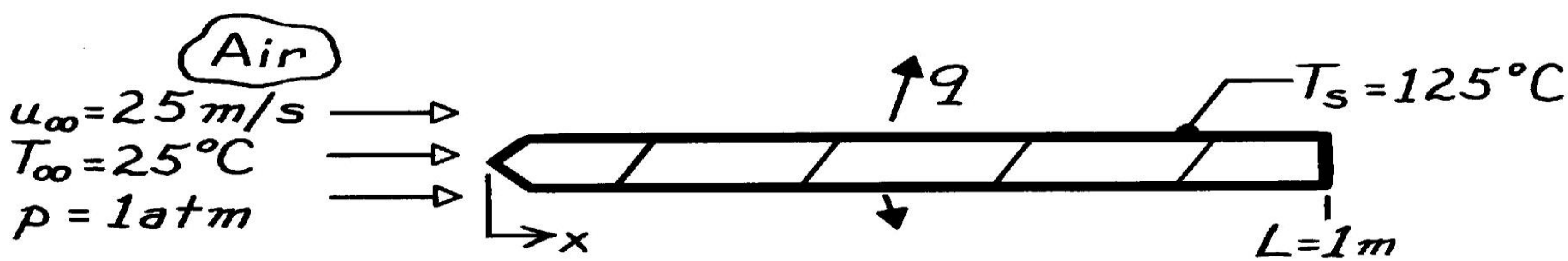
Example question 4

Consider atmospheric air at 25℃ and a velocity of 25 m/s flowing over both surfaces of a 1-m long flat plate that is maintained at 125 ℃. Determine the rate of heat transfer per unit width from the plate for values of the critical Reynolds number corresponding to 10^5, 5×10^5, and 10^6.

KNOWN: Speed and temperature of atmospheric air flowing over a flat plate of prescribed length and temperature.

FIND: Rate of heat transfer corresponding to $Re_{x,c} = 10^5$, 5×10^5 and 10^6.

SCHEMATIC:



ASSUMPTIONS: (1) Flow over top and bottom surfaces.

PROPERTIES: *Table A-4*, Air ($T_f = 348K$, 1 atm): $\rho = 1.00\ kg/m^3$, $\nu = 20.72 \times 10^{-6}\ m^2/s$, $k = 0.0299\ W/m{\cdot}K$, $Pr = 0.700$.

ANALYSIS: With

$$Re_L = \frac{u_\infty L}{\nu} = \frac{25\ m/s \times 1m}{20.72 \times 10^{-6} m^2/s} = 1.21 \times 10^6$$

the flow becomes turbulent for each of the three values of $Re_{x,c}$. Hence,

$$\overline{Nu}_L = \left(0.037\ Re_L^{4/5} - A\right) Pr^{1/3}$$
$$A = 0.037\ Re_{x,c}^{4/5} - 0.664\ Re_{x,c}^{1/2}$$

$Re_{x,c}$	10^5	5×10^5	10^6
A	160	871	1671
$\overline{Nu}_L$	2272	1641	931
$\overline{h}_L\ (W/m^2\cdot K)$	67.9	49.1	27.8
$q'\ (W/m)$	13,580	9820	5560

where $q' = 2\,\overline{h}_L L\left(T_s - T_\infty\right)$ is the total heat loss per unit width of plate.

COMMENTS: Note that $\overline{h}_L$ decreases with increasing $Re_{x,c}$, as more of the surface becomes covered with a laminar boundary layer.

Home work

- 6.1
- 6.8
- 6.10
- 7.10
- 7.35
- 7.49

- Due 18 May 2021