北京航空航天大學
BEIHANG UNIVERSITY

HT: Convection

L9: Similarity principle & dimensional analysis

Learning Objectives:

- **Get the characteristic number by similarity principle/dimensional analysis**
- **The relationship between correlations**

§ 6-1 Similarity principle & dimensional analysis

Experiments are the main methods to investigate the heat transfer behaviors, but it comes across such difficulties when conduct experiments:

(1) Contains too much variables

$$h = f(\vec{v}, t_w, t_f, \lambda, c_p, \rho, \alpha, \eta, l)$$

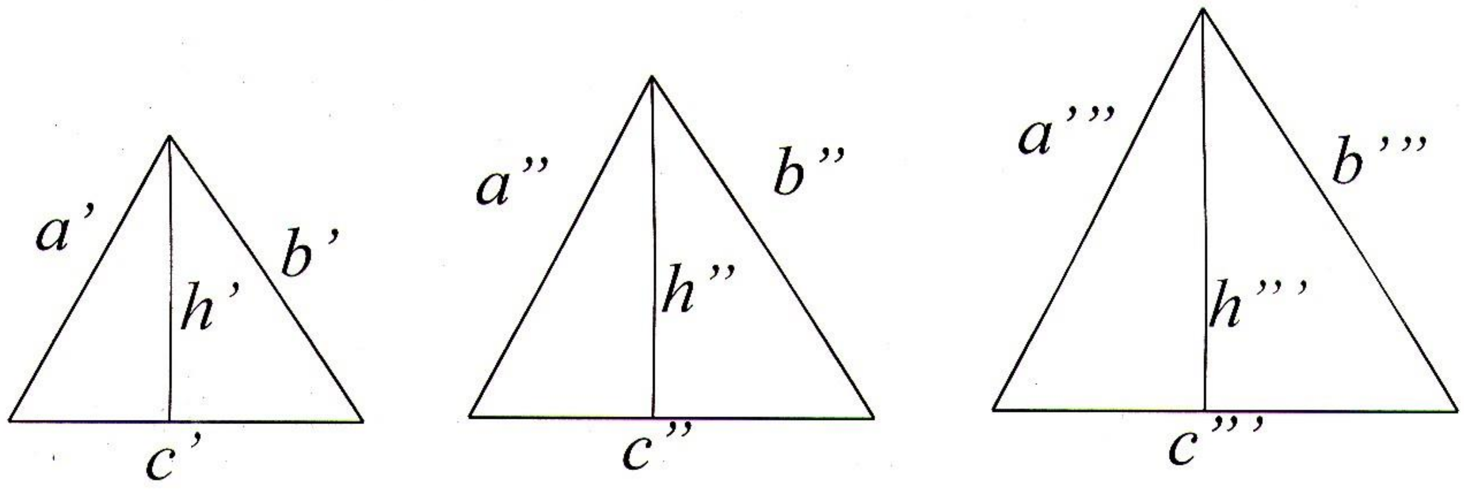
A which one is required to test?

B what kind of function/correlation of h with those parameters

(2) How to conduct EXP if the condition is too difficult to reach or the experiments is too expensive

(3) How to get a general conclusion which can be applied to others

1.Geometry similarity:



(1) (2) (3)

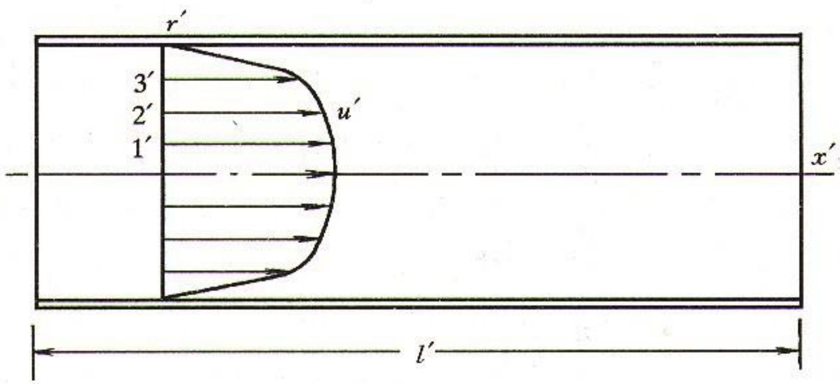
$$\frac{a'}{a''} = \frac{b'}{b''} = \frac{c'}{c''} = C_l \qquad \frac{a'}{b'''} = \frac{b'}{b'''} = \frac{c'}{c'''} = C_l''$$

$$\boxed{\frac{b'}{a'} = \frac{b''}{a''} = \frac{b'''}{a'''} = La \qquad \frac{c'}{a'} = \frac{c''}{a''} = \frac{c'''}{a'''} = Lb}$$

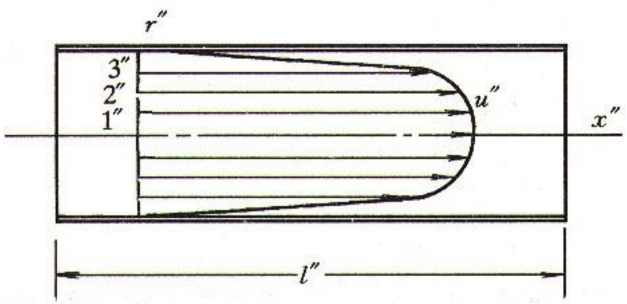
Equal

La and Lb is dimensionless and La = Lb is the necessary and sufficient condition for two triangles similar

2.Physical phenomenon similarity:



(a)



(b)

The similarity of the velocity field:

If the velocity varies along the x and r direction:

$$\frac{x1'}{x1''} = \frac{x2'}{x2''} = \frac{x3'}{x3''} = C_l$$

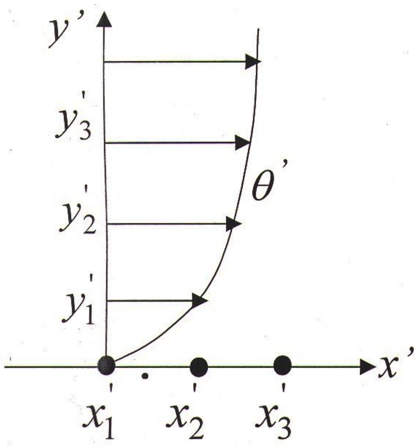
$$\frac{r1'}{r1''} = \frac{r2'}{r2''} = \frac{r3'}{r3''} = C_l$$

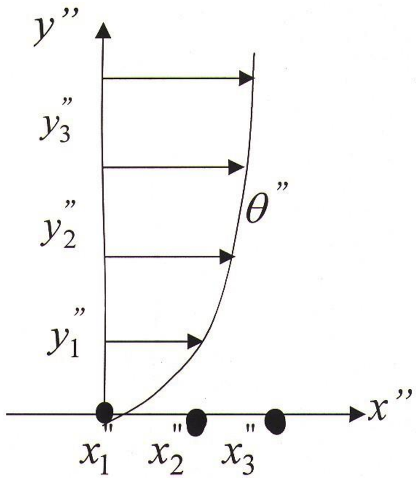
$$\frac{u1'}{u1''} = \frac{u2'}{u2''} = \frac{u3'}{u3''} = C_u$$

The velocity field is similar

Note the profile is different when the size reduces or enlarge. Why?

2.Physical phenomenon similarity:





The similarity of the temperature field:

If the temperature varies along the x and y direction:

$$\frac{x1'}{x1''} = \frac{x2'}{x2''} = \frac{x3'}{x3''} = C_l$$

$$\frac{y1'}{y1''} = \frac{y2'}{y2''} = \frac{y3'}{y3''} = C_l$$

$$\frac{\theta 1'}{\theta 1''} = \frac{\theta 2'}{\theta 2''} = \frac{\theta 3'}{\theta 3''} = C_\theta$$

The temperature field is similar

2.Physical phenomenon similarity：

Therefore, if two physical phenomenon similar, then

(1) They should follows same governing equations, including the boundary conditions。(temperature and momentum, natural and force)

(2) With same time or coordinates, the variables shows similar trend。

For convection heat transfer:

1) geometry：

$$\frac{r_{1'}}{r_1} = \frac{r_{2'}}{r_2} = C_L$$ ——几何相似倍数

2) velocity：

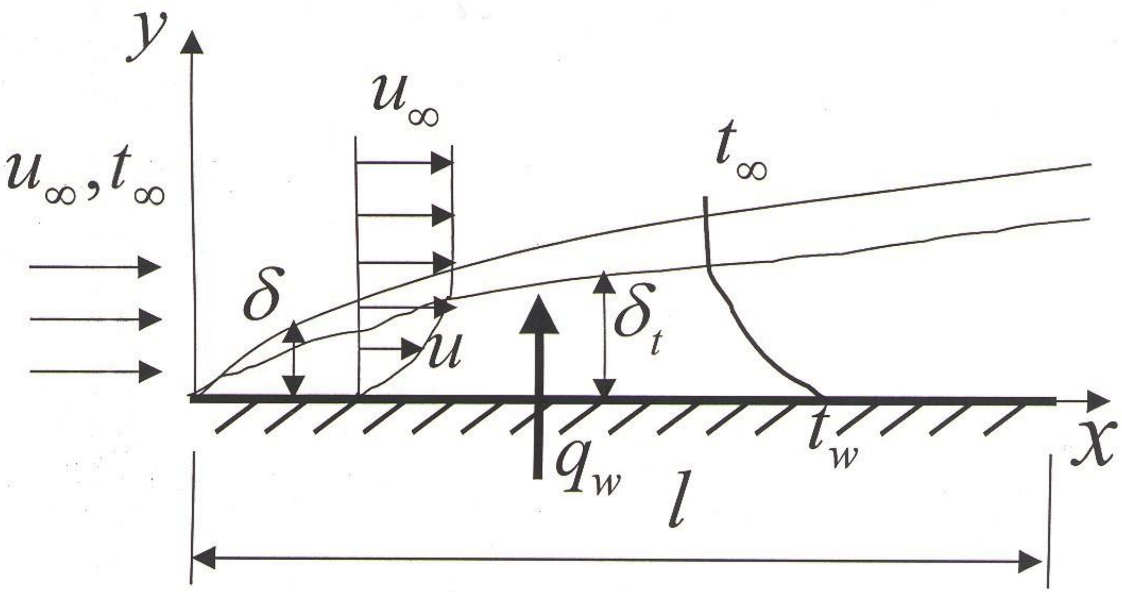
$$\frac{u_{r1'}}{u_{r1}} = \frac{u_{r2'}}{u_{r2}} = C_u$$ ——速度相似倍数

3) temperature：

$$\frac{t_{r1'}}{t_{r1}} = \frac{t_{r2'}}{t_{r2}} = C_t$$ ——温度相似倍数

3. necessary and sufficient condition of physical phenomenon

- The same characteristic number is same
- The conditions is same



Given two air flow through a plate phenomena, and they are similar. Base on the definition, they follows same governing equations.

$$h_x = -\frac{\lambda}{t_w - t_\infty}\left(\frac{\partial t}{\partial y}\right)_{w,x} \qquad \left[W/(m^2 \cdot {}^\circ C)\right]$$

4. How to get the characteristic number

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=a\frac{\partial^2 t}{\partial y^2}$$

The governing equation within the boundary layers.

$$h_x=-\frac{\lambda}{\Delta t}\left(\frac{\partial t}{\partial y}\right)_{w,x}$$

4. How to get the characteristic number

1. With two similar equations:

Case 1： $$h' = -\frac{\lambda'}{\Delta t'}\frac{\partial t'}{\partial y'}\bigg|_{y'=0} = 0$$

Case 2： $$h'' = -\frac{\lambda''}{\Delta t''}\frac{\partial t''}{\partial y''}\bigg|_{y''=0} = 0$$

and：

$$\frac{h'}{h''} = C_h' \qquad \frac{\lambda'}{\lambda''} = C_\lambda' \qquad \frac{t'}{t''} = C_t' \qquad \frac{y'}{y''} = C_y$$

therefore：

$$\frac{C_h C_y}{C_\lambda} h'' = -\frac{\lambda''}{\Delta t''}\frac{\partial t''}{\partial y''}\bigg|_{y''=0} = 0 \qquad \Rightarrow \qquad \frac{C_h C_y}{C_\lambda} = 1$$

4. How to get the characteristic number

$$\frac{C_h C_y}{C_\lambda} = 1 \quad \Rightarrow \quad \frac{h'y'}{\lambda'} = \frac{h''y''}{\lambda''} \quad \Rightarrow \quad Nu_1 = Nu_2$$

Similarly, with momentum equation： $\mathrm{Re}_1 = \mathrm{Re}_2$

with energy equation :

$$\frac{u'l'}{a'} = \frac{u''l''}{a''} \quad \Rightarrow \quad \mathrm{Pe}_1 = \mathrm{Pe}_2$$

Peclect number

$$\mathrm{Pe} = \mathrm{Pr} \cdot \mathrm{Re} \quad \Rightarrow \quad \mathrm{Pr}_1 = \mathrm{Pr}_2$$

Which indicates the same characteristic number must be same。

4. How to get the characteristic number

For natural convection heat transfer

We have a new characteristic number: grashof

$$G_r = \frac{g\alpha\Delta t l^3}{\nu^2}$$

式中：α —— dilatation coefficient K^{-1}

Gr —— Represents the ratio of buoyancy to viscous force of fluid

4. How to get the characteristic number

(2) Dimensional analysis： can applied to unknow equations with given variables

7 SI： length [m]， mass [kg]， time [s]， current[A]， Temperature[K]， mole[mol]， light intensity[cd], where we can use [L], [M], [T] and [θ] represent the length, mass time and temperature, respectively.

For a equation, the dimension for each SI is 0.

E.g.

$$0 = [LT^{-1}]^a [L]^b [ML^{-1}T^{-2}]^c [ML^{-3}]^d [ML^{-1}T^{-1}]^e$$

4. How to get the characteristic number

$$0 = [LT^{-1}]^a [L]^b [ML^{-1}T^{-2}]^c [ML^{-3}]^d [ML^{-1}T^{-1}]^e$$

For [M]: $c + d + e = 0$

For [L]: $a + b - c - 3d - e = 0$

For [T]: $-a - 2c - e = 0$

Three equations and 5 variables. If we want to solve, we need provide two already known parameters.
e.g. c=1;e=0 or c=0, e=-1.

4. How to get the characteristic number

(2) Dimensional analysis： can applied to unknow equations with given variables

a based on π theorem， if one equation contains N variables but with r dimension, it must can consist of N-r non-dimensional variables.

e.g.： a heat transfer in a cylinder tube

(1)determine the variables

$$h = f(u, d, \lambda, \eta, \rho, c_p)$$

$$\Rightarrow \qquad n = 7$$

(2)Determine r

4. How to get the characteristic number

$$h : \frac{\mathrm{kg}}{\mathrm{s}^3 \cdot K} \quad u : \frac{\mathrm{m}}{\mathrm{s}} \quad d : \mathrm{m} \quad \lambda : \frac{\mathrm{W}}{\mathrm{m} \cdot \mathrm{K}} = \frac{\mathrm{kg} \cdot \mathrm{m}}{\mathrm{s}^3 \cdot K}$$

$$\eta : Pa \cdot s = \frac{\mathrm{kg}}{\mathrm{m} \cdot \mathrm{s}} \quad \rho : \frac{\mathrm{kg}}{\mathrm{m}^3} \quad c_p : \frac{J}{kg \cdot K} = \frac{m^2}{s^2 \cdot K}$$

In our case we include: time [T], length[L], mass[M], temperature[Θ]

$\Rightarrow$ **r = 4**

4. How to get the characteristic number

$$n = 7 : h, u, d, \lambda, \eta, \rho, c_p \qquad r = 4 : [\mathrm{T}], [\mathrm{L}], [\mathrm{M}], [\Theta]$$

$\Rightarrow$ n – r = 3， therefore we must have 4 variables which includes all 4 SI and combine with other three to gain the non-dimensional number. Here we chooseu ,d ,λ ,η.

(b) Therefore we get:

$$\pi_1 = h u^{a_1} d^{b_1} \lambda^{c_1} \eta^{d_1}$$

$$\pi_2 = \rho u^{a_2} d^{b_2} \lambda^{c_2} \eta^{d_2}$$

$$\pi_3 = c_p u^{a_3} d^{b_3} \lambda^{c_3} \eta^{d_3}$$

(c)Solve the dimensional number:

$$\pi_1 = h u^{a_1} d^{b_1} \lambda^{c_1} \eta^{d_1}$$

4. How to get the characteristic number

$$\pi_1 = hu^{a_1} d^{b_1} \lambda^{c_1} \eta^{d_1}$$

$$= M^1 T^{-3} \Theta^{-1} \cdot L^{a_1} T^{-a_1} \cdot L^{b_1} \cdot M^{c_1} L^{c_1} T^{-3c_1} \Theta^{-c_1} \cdot M^{d_1} L^{-d_1} T^{-d_1}$$

$$= M^{1+c_1+d_1} T^{-3-a_1-3c_1-d_1} \Theta^{-1-c_1} \cdot L^{a_1+b_1+c_1-d_1}$$

$$\Rightarrow \begin{cases} 1 + c_1 + d_1 = 0 \\ -3 - a_1 - 3c_1 - d_1 = 0 \\ -1 - c_1 = 0 \\ a_1 + b_1 + c_1 - d_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ b_1 = 1 \\ c_1 = -1 \\ d_1 = 0 \end{cases}$$

4. How to get the characteristic number

$$\pi_1 = hu^{a_1} d^{b_1} \lambda^{c_1} \eta^{d_1} = hu^0 d^1 \lambda^{-1} \eta^0 = \frac{hd}{\lambda} = Nu$$

similarly:

$$\pi_2 = \frac{\rho u d}{\eta} = \frac{ud}{\nu} = \mathrm{Re}$$

$$\pi_3 = \frac{\eta c_p}{\lambda} = \frac{\nu}{a} = \mathrm{Pr}$$

then:

$$\boxed{Nu = f(\mathrm{Re}, \mathrm{Pr})}$$

5. The characteristic numbers

Force convection:

$$\mathrm{Nu} = f(\mathrm{Re}, \mathrm{Pr}); \qquad \mathrm{Nu}_x = f(x', \mathrm{Re}, \mathrm{Pr})$$

For other conditions：

Natural convection：

$$\mathrm{Nu} = f(\mathrm{Gr}, \mathrm{Pr})$$

Mixed convection：

$$\mathrm{Nu} = f(\mathrm{Re}, \mathrm{Gr}, \mathrm{Pr})$$

Nu — to determine（ include *h*）

Re， Pr， Gr — known

Therefore, for experiments, one just focuses on the characteristic number measurements is OK。

5. The characteristic numbers

The physical meaning of *Nu*:

$$h(t_w - t_f) = -\lambda \left.\frac{\partial t}{\partial y}\right|_{y=0}$$

$$Nu = \frac{hl}{\lambda} = -\frac{\left.\frac{\partial t}{\partial y}\right|_{y=0}}{\frac{\Delta t}{l}}$$

The temperature gradient at the wall

The temperature variation across l thickness of liquid.

Nu **reflects the real heat transferred compared with the heat by molecular diffusion. Larger** ***Nu*** **brings more intensive convection heat transfer.**

$$Nu = \frac{hl}{\lambda} = \frac{\frac{l}{\lambda}}{\frac{1}{h}} = \frac{\text{Thermal conduction resistance of liquid}}{\text{Thermal convection resistance}}$$

5. The characteristic numbers

Difference with Bi：

$$Nu = \frac{hl}{\lambda} \qquad\qquad Bi = \frac{hl}{\lambda}$$

a) The λ is solid for Bi but is liquid for Nu。

b) The *l* in *Bi* is the characteristic length of solid V/A，while for *Nu*, the *l* is the characteristic length of the contact surface.

c) The *h* in Bi is the total heat transfer coefficient，should consider radiation.

Bi: the thermal conduction resistance inside the solid / external thermal convection resistance

Nu

6.2 The application of similarity principle

Two aspects: build experimental model;
analysis experimental data.

It is difficult for some complex phenomena to be studied by experiment directly. Prototype phenomena can be studied by reduced model according to the similarity theory. This method is called model experiment.

- Therefore, the model experiment must comply with the following conditions:
- Model and prototype are the same phenomenon, the equation describing the phenomenon must be the same
- Same conditions
- Same characteristic numbers

6.2 The application of similarity principle

if " P "represents the real case， " m "represents the model case, then

$$\mathrm{Re}_P = \mathrm{Re}_m \qquad \mathrm{Pr}_P = \mathrm{Pr}_m$$

$$Nu_P = Nu_m \qquad Nu_m$$

Which can be measured in lab scale

$$\therefore \qquad h_P = \frac{\lambda_p}{L_p} Nu_m$$

To gain the real *h*

6.2 The application of similarity principle

2. Analysis of experimental data---determine the correlation of characteristic number

For each different problem, characteristic numbers can be obtained by using similarity analysis or dimensional analysis, but the relationship between these numbers and the specific function should be obtained by experimental data through experience.

In order to express the regularity of the experimental data and facilitate the application, the characteristic number correlation is usually arranged into the power function form of the established criterion.

e.g.

$$Nu = CRe^{n}Pr^{m}$$

C、n、m is constant and could be obtained by experiments

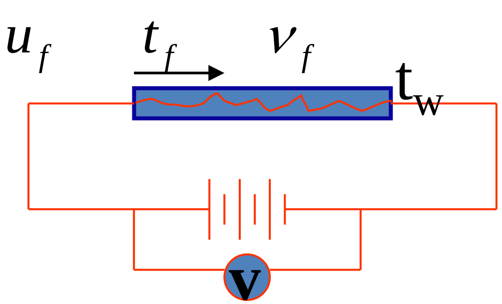
6.2 The application of similarity principle

(1) First we fix *Pr*, this can achieve by use same fluid.

Given $B = Pr^m$

then: $Nu = B Re^n$ **next we need determine B**

If： $t_w > t_f$

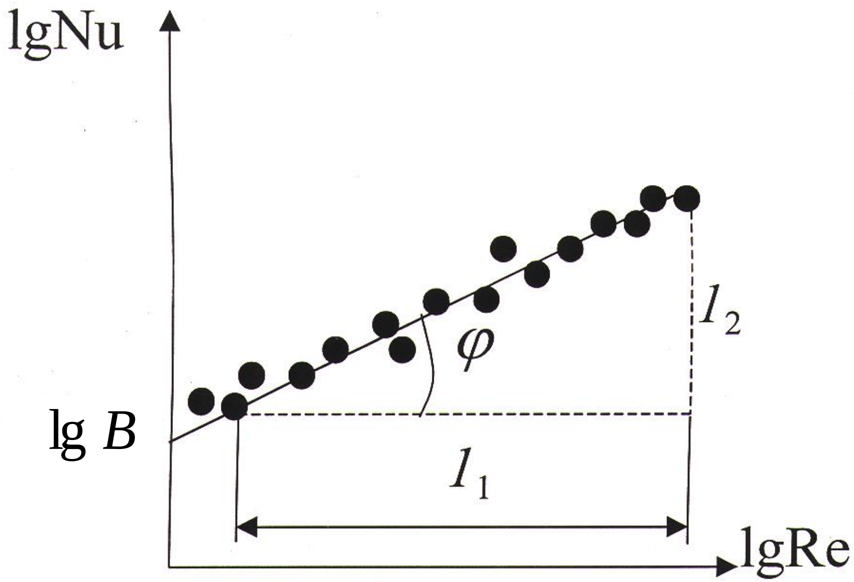


measure： u_f、t_f、t_w、**I**、**V**

With energy conservation： $\Phi = hA(t_w - t_f) = IV$

Calculate h， and $Nu = \frac{hl}{\lambda}$ $\mathrm{Re} = \frac{u_f l}{\nu_f}$ **， change I、V， obtain multiple points， and plot them in a single graph：**

6.2 The application of similarity principle



$$\mathrm{Nu} = BRe^{n}$$

$$\lg \mathrm{Nu} = \lg B + n\lg\mathrm{Re}$$

$$n = \mathrm{tg}\,\varphi = \frac{l_2}{l_1}; \qquad B = \frac{\mathrm{Nu}}{\mathrm{Re}^{n}}$$

$$n = \frac{l_2}{l_1} \approx 0.8$$

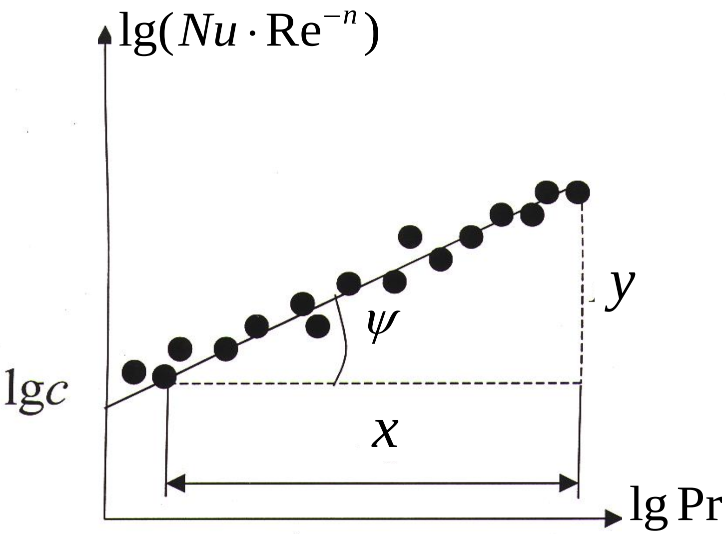
(2)Choose different Pr, determine C and m.

$$Nu = CRe^{n}Pr^{m}$$

Measure: $t_w, t_f, u_f, IV, a, \lambda, \nu$

Plot results on the graph:

6.2 The application of similarity principle



$$\frac{Nu}{\mathrm{Re}^{n}} = C\,\mathrm{Pr}^{m}$$

$$\lg(Nu\cdot \mathrm{Re}^{-n}) = \lg C + m\lg \mathrm{Pr}$$

slope $\quad m = tg\psi = \dfrac{y}{x}$

$$m = \frac{y}{x} \approx 0.4$$

C: take three points on the curve, figure out the average value。C=0.023。

$$C = \frac{Nu}{\mathrm{Re}^{0.8}\,\mathrm{Pr}^{0.4}}$$

therefore:

$$Nu = 0.023\,\mathrm{Re}^{0.8}\,\mathrm{Pr}^{0.4}$$

6.2 The application of similarity principle

(3)Reference temperature, characteristic length and characteristic velocity

a Reference temperature： **To determine the thermal properties, such as λ、ν、Pr.**

(a) Temperature of liquid： t_f

Flow through a plate： $t_f = t_\infty$

Flow in a cylinder tube： $t_f = (t'_f + t''_f)/2$

(b) Average T within TBL： $t_m = (t_w + t_f)/2$

(c) Wall temperature： t_w

We usually label the ref T with subscript， e.g.：

Nu_f、 Re_f、 Pr_f *or* Nu_m、 Re_m.

When correlate the characteristic number, the ref T mush same

6.2 The application of similarity principle

b characteristic length：

The size which has significant effect on flow and heat transfer

e.g.： flow within a tube： d

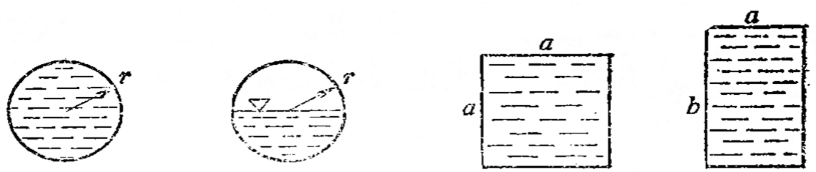
Flow through a plate: L

When flow through a passage with irregular geometries： de

$$d_e = \frac{4A_c}{P}$$

A_c —— cross-section area， m^2

P —— wetting perimeter， m



$\chi = 2\pi r$	πr	$4a$	$2(a+b)$
$R = \frac{r}{2}$	$\frac{r}{2}$	$\frac{a}{4}$	$\frac{ab}{2(a+b)}$
$d_e = 2r$	$2r$	a	$\frac{2ab}{a+b}$

6.2 The application of similarity principle

c characteristic velocity：

Re：

Through a plate or cylinder： upstream u_∞

Force within a tube u_m

Flow through tube bundle $u_{\max}$

6.2 The application of similarity principle

例 1

一换热设备的工作条件是：壁温 $t_w = 120°C$，加热$t_f = 80°C$的空气，空气流速$u = 0.5\,m/s$。采用一个全盘缩小成原设备1/5的模型来研究它的换热状况。在模型中亦对空气加热，空气温度$t_f' = 10°C$，壁面温度$t_w' = 30°C$。试问模型中流速u'应多大才能保证与原设备中的换热现象相似（模型中各量用上角码“'”标明）。

解：模型与原设备研究的是同类现象，单值性条件亦相似，所以只要已定准则Re，Pr彼此相等即可实现相似。

因空气中Pr随温度变化不大，可认为$\mathrm{Pr}' = \mathrm{Pr}$。于是需要保证的是$\mathrm{Re}' = \mathrm{Re}$。

从而

$$\frac{u'l'}{\nu'} = \frac{ul}{\nu}$$

取定性温度为流体温度与壁温的平均值，$t_m = \frac{t_w + t_f}{2}$，从附录查得$\nu = 23.13 \times 10^{-6}\,m^2/s$，$\nu' = 15.06 \times 10^{-6}\,m^2/s$

6.2 The application of similarity principle

已知$l/l' = 5$。于是模型中要求的流体流速u'为

$$u' = u\frac{\nu'}{\nu}\frac{l}{l'} = 1.63\, m/s$$

模型是实物的1/5，而模型中的流速时是实物中流速的5倍左右！