# System Dynamics and Vibrations

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## Chapter 5: Dynamic stability. Part I.

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# Contents

- The concept of stability. Introduction to Lyapunov's stability.
- Stability of single-degree-of-freedom systems from equilibrium

# The concept of stability. Introduction to Lyapunov’s stability

- Lyapunov stability theory was developed by Lyapunov, a Russian mathematician in 1892, and came from his doctoral dissertation.
- The Lyapunov stability theory is used to describe the stability of a dynamic system

# The concept of stability. Introduction to Lyapunov's stability

- Autonomous system: no input external to the system
- Described by differential equations

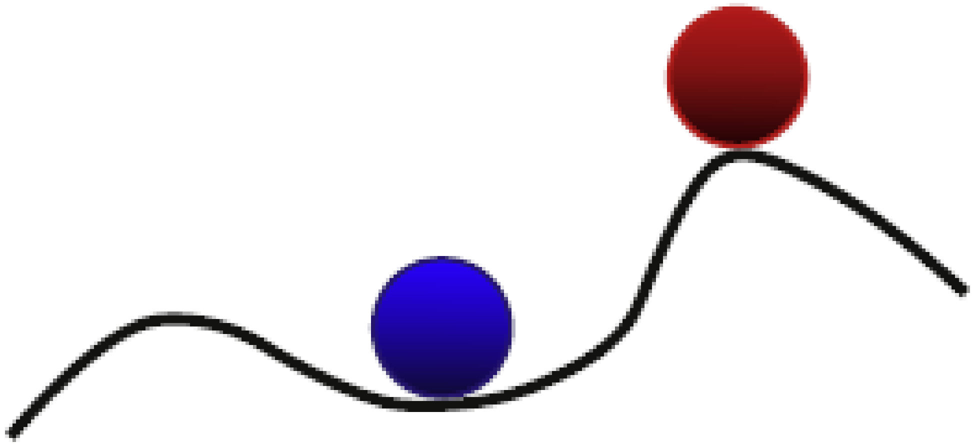
$$\dot{x} = f(x,t), \quad x(t_0) = x_0 \qquad t \in [t_0, \infty)$$

where *x* is the <u>system state</u>

# The concept of stability. Introduction to Lyapunov's stability

- Equilibrium state

$$\dot{x}_e = f(x_e, t) = 0, \quad \forall\, t \in [t_0, \infty)$$

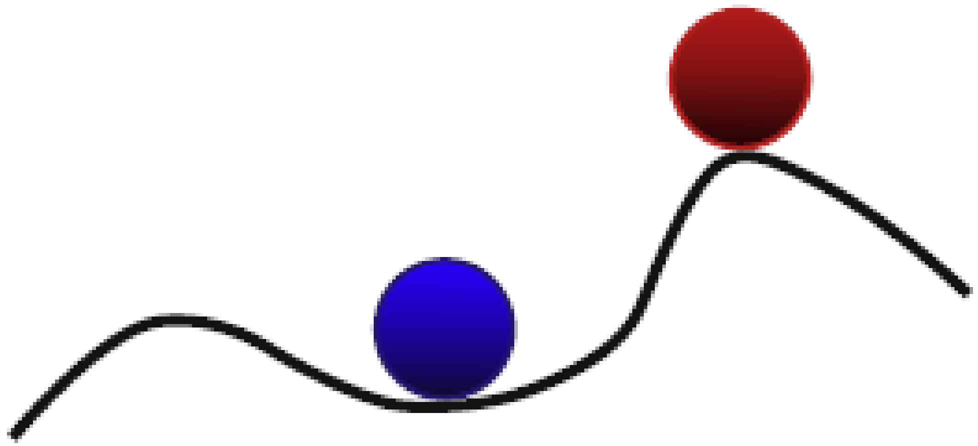


# The concept of stability. Introduction to Lyapunov's stability

- If the displacement of the ball is regarded as a state $x_e$, the derivative of the apparent state satisfies

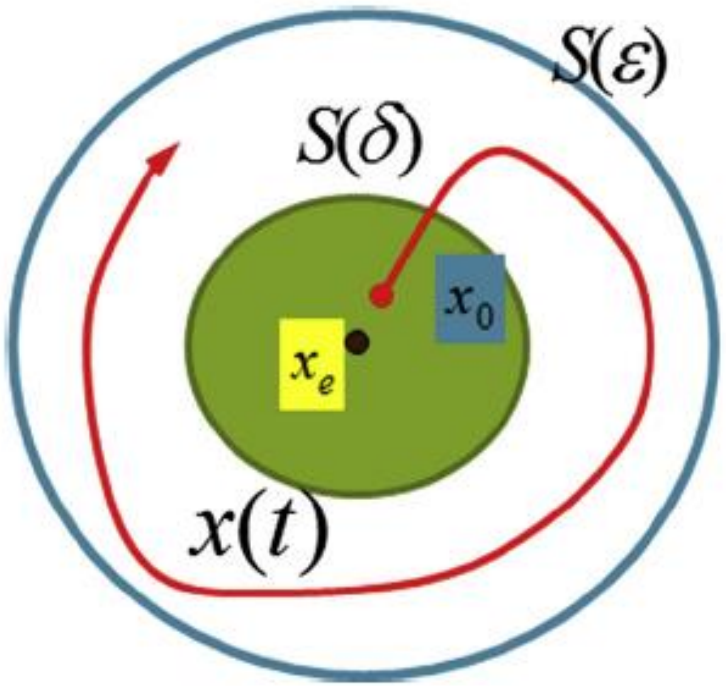
$$\dot{x}_e = f(x_e, t) = 0$$

so the two balls are all in equilibrium.



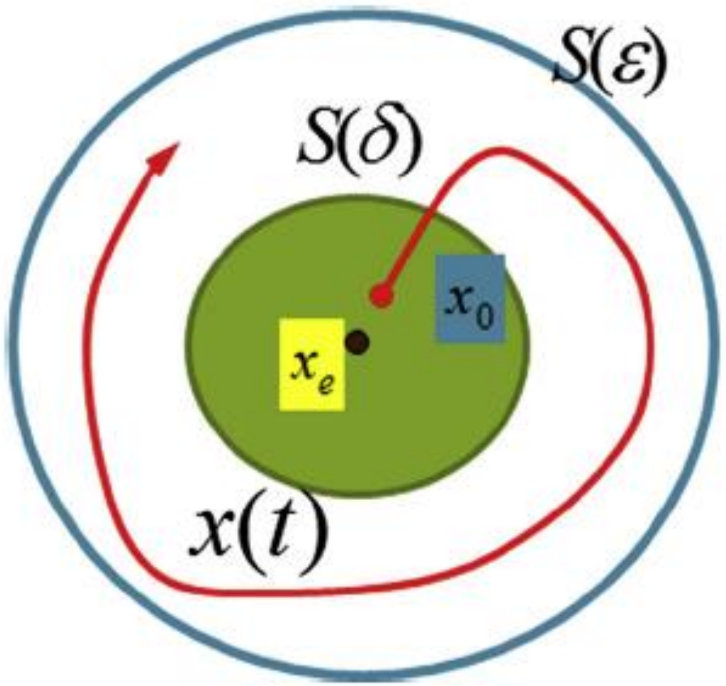
# Introduction to Lyapunov's stability

- The geometric interpretation:
  - The initial state $x_0$ is assumed to be located inside a closed region with center at the equilibrium state $x_e$ and an arbitrary radius $\delta$
  - When time $t$ tends to infinity, if the system state can be located inside a closed region with center at the equilibrium state $x_e$ and any arbitrary $\varepsilon$ as radius, then the system is stable in the Lyapunov sense.

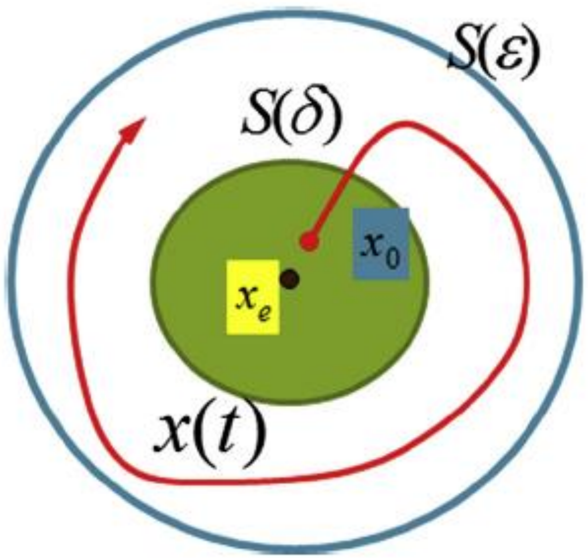


# Introduction to Lyapunov's stability

- The system trajectory:
  - At any point in $S(\delta)$, the trajectory of the system will not run out of $S(\varepsilon)$,
  - But the system trajectory does not necessarily converge to the equilibrium state $x_e$, it may not even fall into $S(\delta)$, that may stay any point outside $S(\delta)$



# Introduction to Lyapunov's stability




- Mathematical description:

The equilibrium state $x_e = 0$ of the autonomous system

$$\dot{x} = f(x,t)$$

is said Lyapunov stable at $t_0$, if for any $\varepsilon > 0$,

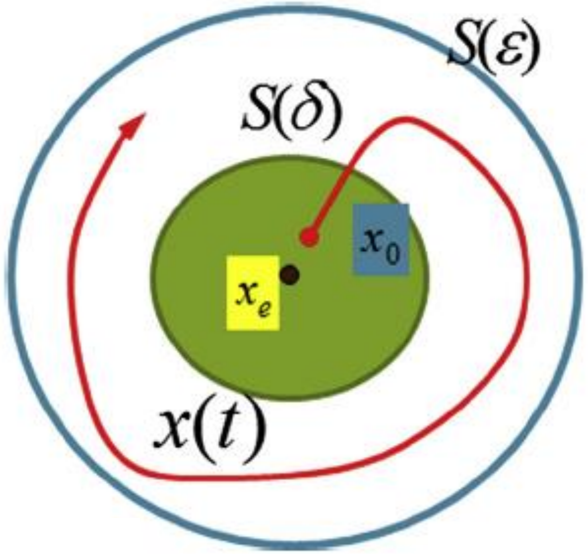
there exist $\delta(\varepsilon, t_0) > 0$ depending on $\varepsilon$ and $t_0$ that satisfies the following inequality:

$$\left\| x_0 - x_e \right\| \leq \delta\left(\varepsilon, t_0\right)$$

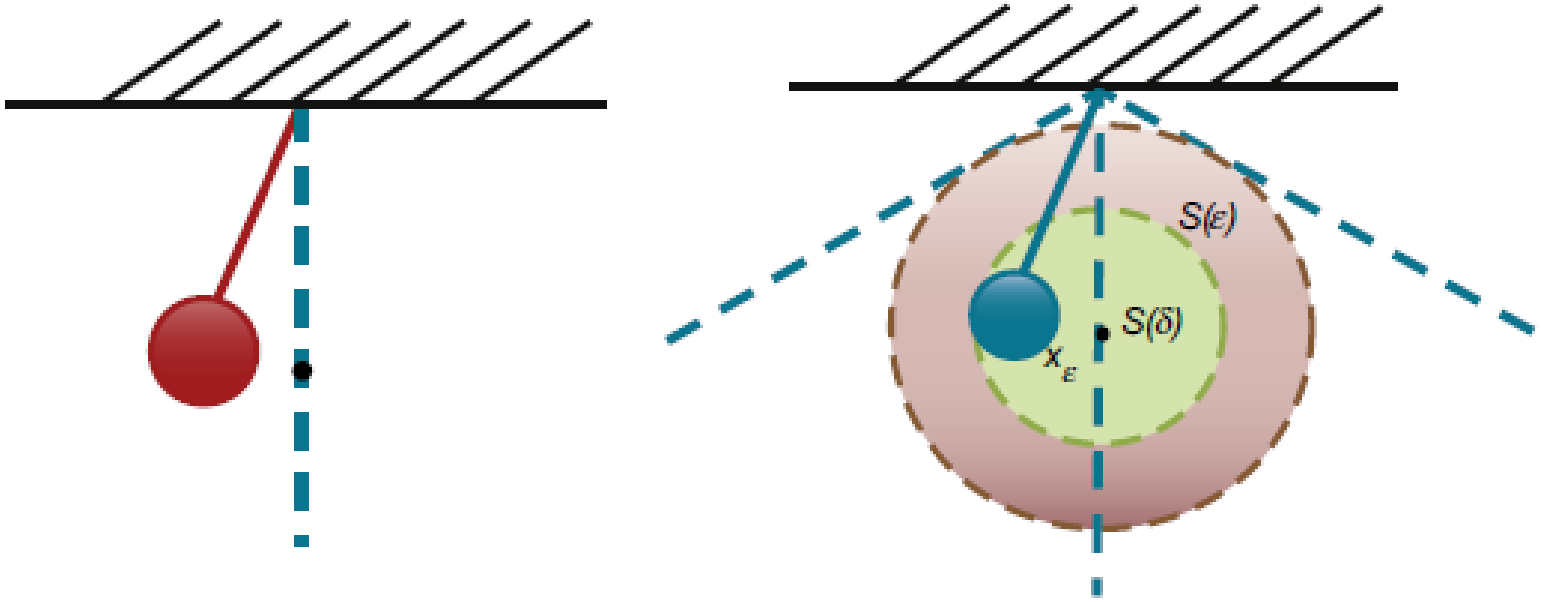
The disturbed motion of any initial state $x_0$ satisfies the inequality:

$$\left\| \phi\left(t; x_{0,t_0}\right) - x_e \right\| \leq \varepsilon, \ \ \forall t \geq t_0$$

# Introduction to Lyapunov's stability

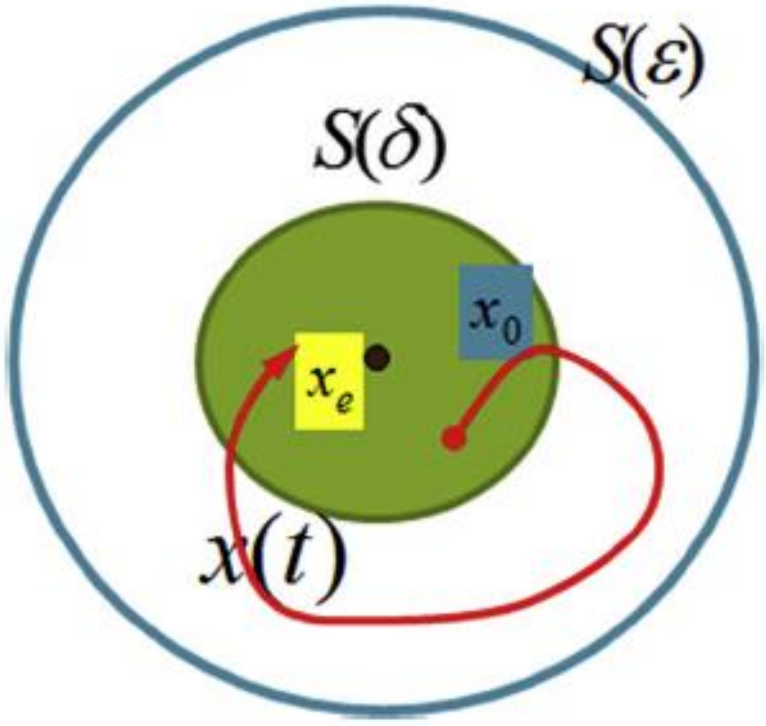



- Example

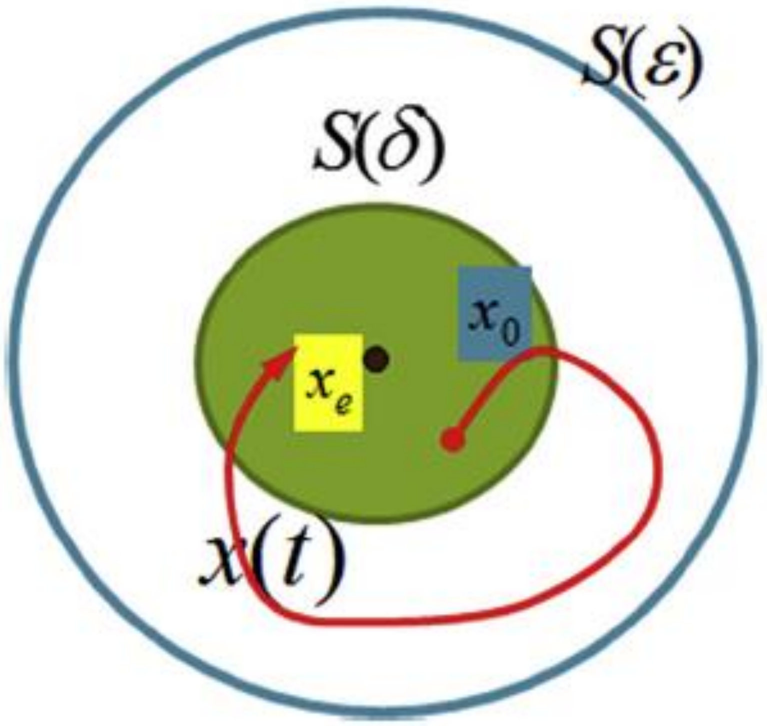


# Lyapunov asymptotic stability

- The geometric interpretation:
  - The initial state $x_0$ is assumed to be located inside a closed region with center at the equilibrium state $x_e$ and an arbitrary radius $\delta$
  - When time $t$ tends to infinity, the system state <u>converge</u> to the equilibrium state $x_e$.



# Lyapunov asymptotic stability




- The mathematical description:

The equilibrium state $x_e$ = 0 of the autonomous system

$$\dot{x} = f(x,t)$$

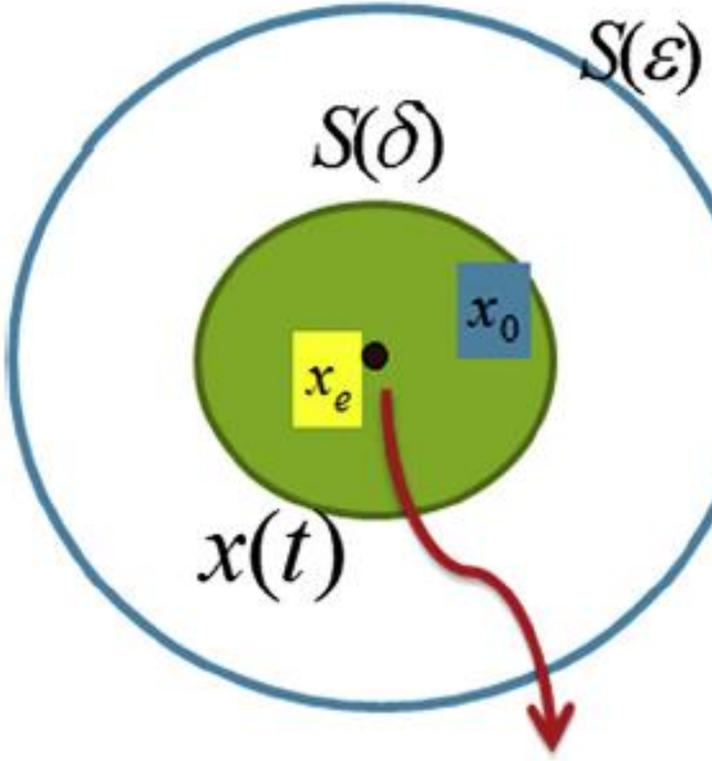
is said Lyapunov asymptotic stable at $t_0$, if:

- The disturbed motion $\phi(t; x_0, t_0)$ starting from any initial state $x_0 \in S(\delta)$ is bounded by the equilibrium state $x_e = 0$ for all $t \in [t_0, \infty)$
- The disturbed motion is asymptotically relative to the equilibrium state 1, that is,

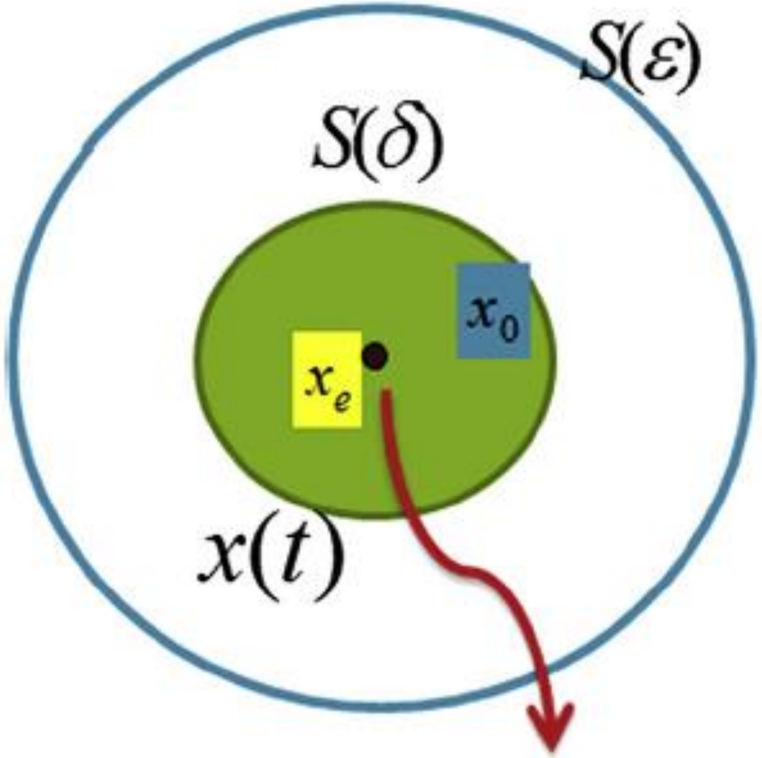
$$\lim_{t \to \infty} \phi(t; x_o, t_0) = x_e, \quad \forall x_0 \in S(\delta)$$

# Lyapunov instability

- The geometric interpretation:
  - The initial state $x_0$ is assumed to be located inside a closed region with center at the equilibrium state $x_e$ and an arbitrary radius $\delta$
  - When time *t* tends to infinity, no matter how big S(*ε)* is, or how small S($\delta$*)* is, if the system state cannot be located inside a closed region with center at the equilibrium state $x_e$ and any arbitrary *ε* as radius, then the system is instable in the Lyapunov sense.



# Lyapunov asymptotic stability




- The mathematical description:

The equilibrium state $x_e$ = 0 of the autonomous system

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is said Lyapunov instable at $t_0$, if for any $\varepsilon > 0$,

there no exist $\delta(\varepsilon, t_0) > 0$ depending on $\varepsilon$ and $t_0$ that satisfies the following

inequality: $\left\| x_0 - x_e \right\| \leq \delta\left(\varepsilon, t_0\right)$

The disturbed motion of any initial state $x_0$ satisfies the inequality:

$$\left\| \phi\left(t; x_{0,t_0}\right) - x_e \right\| \leq \varepsilon, \ \forall t \geq t_0$$

# Contents

# System differential equations of motion

$$m\ddot{y} = F(y, \dot{y})$$

- $m$ is the mass
- $F$ is in general a nonlinear function of the displacement and velocity
- General solutions to the equation are not possible
- We are interested in special solutions, to understand the system behaviour

# System differential equations of motion

$$m\ddot{y} = F(y, \dot{y})$$

- Special solution:

$$y = y_e = \text{constant}$$

$$\dot{y} = \ddot{y} = 0$$

- These constant solutions represent equilibrium points, obtained from:

$$m\ddot{y} = 0 = F(y, \dot{y}) = F(y_e, 0) \Rightarrow F(y_e, 0) = 0$$

# System differential equations of motion

- How the system behaves when disturbed from equilibrium?:
  - The system returns to the same equilibrium point ➔ <u>asymptotically stable</u>
  - The system oscillates about the same equlibrium point (without any secular trend) ➔ <u>stable</u>
  - The system moves away from the equlibrium point ➔ <u>unstable</u>

# Linearization about equilibrium points

$$m\ddot{y} = F(y, \dot{y})$$

- Let's consider a solution having the form:

$$y(t) = y_e + x(t)$$

- being *x(t)* a relatively small displacement from equilibrium
- then:

$$\dot{y}(t) = \dot{x}(t)$$

$$\ddot{y}(t) = \ddot{x}(t)$$

# Linearization about equilibrium points

- We have assumed that displacements from equilibrium are sufficiently small that the nonlinear terms can be ignored

$$m\ddot{y} = F(y, \dot{y}) \quad \Rightarrow \quad \ddot{x} + a\dot{x} + bx = 0$$

➔*linearized equation of motion about equilibrium*

*(small motions assumption)*

- The motion characteristics in the neighborhoud of equilibrium depend on parameters *a*, *b*

# Linearization about equilibrium points

$$\ddot{x} + a\dot{x} + bx = 0$$

- Linear equation with constant coefficients:

$$x(t) = Ae^{st}$$

*A:* amplitude

*s:* constant exponent

- Combining

$$\left.\begin{array}{l} m\ddot{x} + a\dot{x} + bx = 0 \\ x(t) = Ae^{st} \end{array}\right\} \quad s^2 + as + b = 0$$

# Linearization about equilibrium points

$$s^2 + as + b = 0$$

➔ Characteristic equation (algebraic equation)

- The roots are:

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

- So the solution to $m\ddot{x} + a\dot{x} + bx = 0$ is:

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

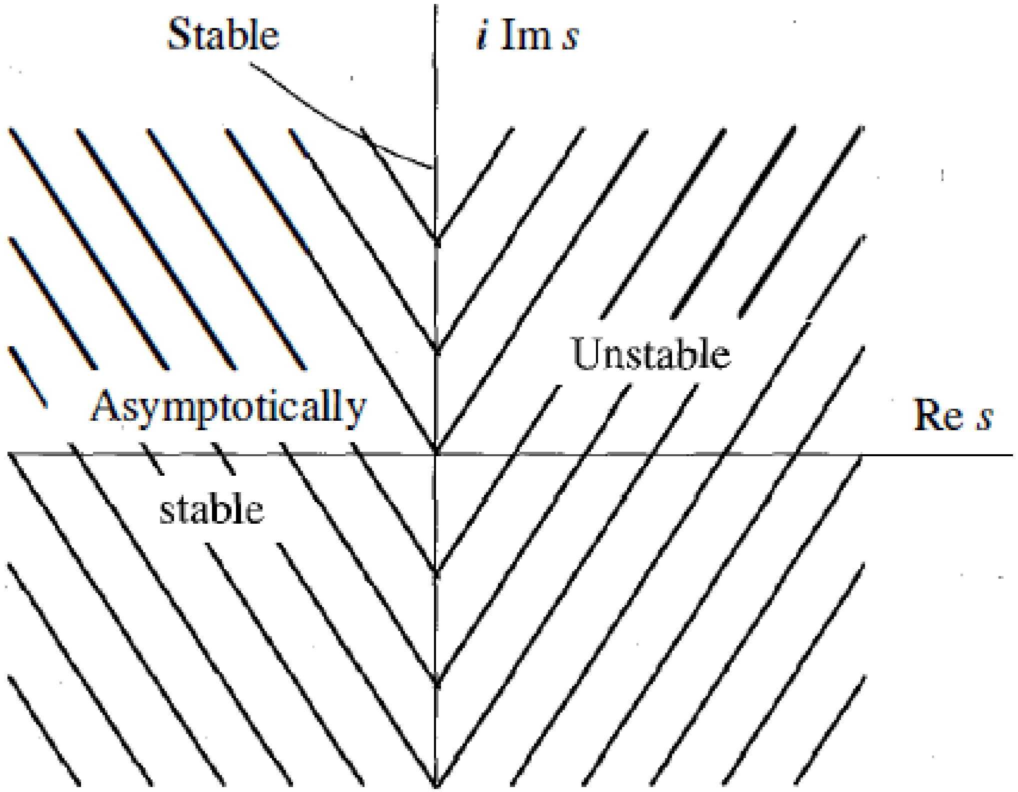
# Linearization about equilibrium points

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- The nature of the motion (around equilibrium points) depends on the values of the roots *s* (complex numbers, in general):
  - In all cases in which $s_1$ and $s_2$ are both real and negative or complex conjugates with negative real part the motion in the neighborhood of an equilibrium point is asymptotically stable
  - In all cases in which $s_1$ and $s_2$ are pure imaginary the motion is merely stable
  - If either $s_1$ or $s_2$ is real and positive, or both $s_1$ and $s_2$ are real and positive, or $s_1$ and $s_2$ are complex conjugates with positive real part, the motion is unstable

# Linearization about equilibrium points

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

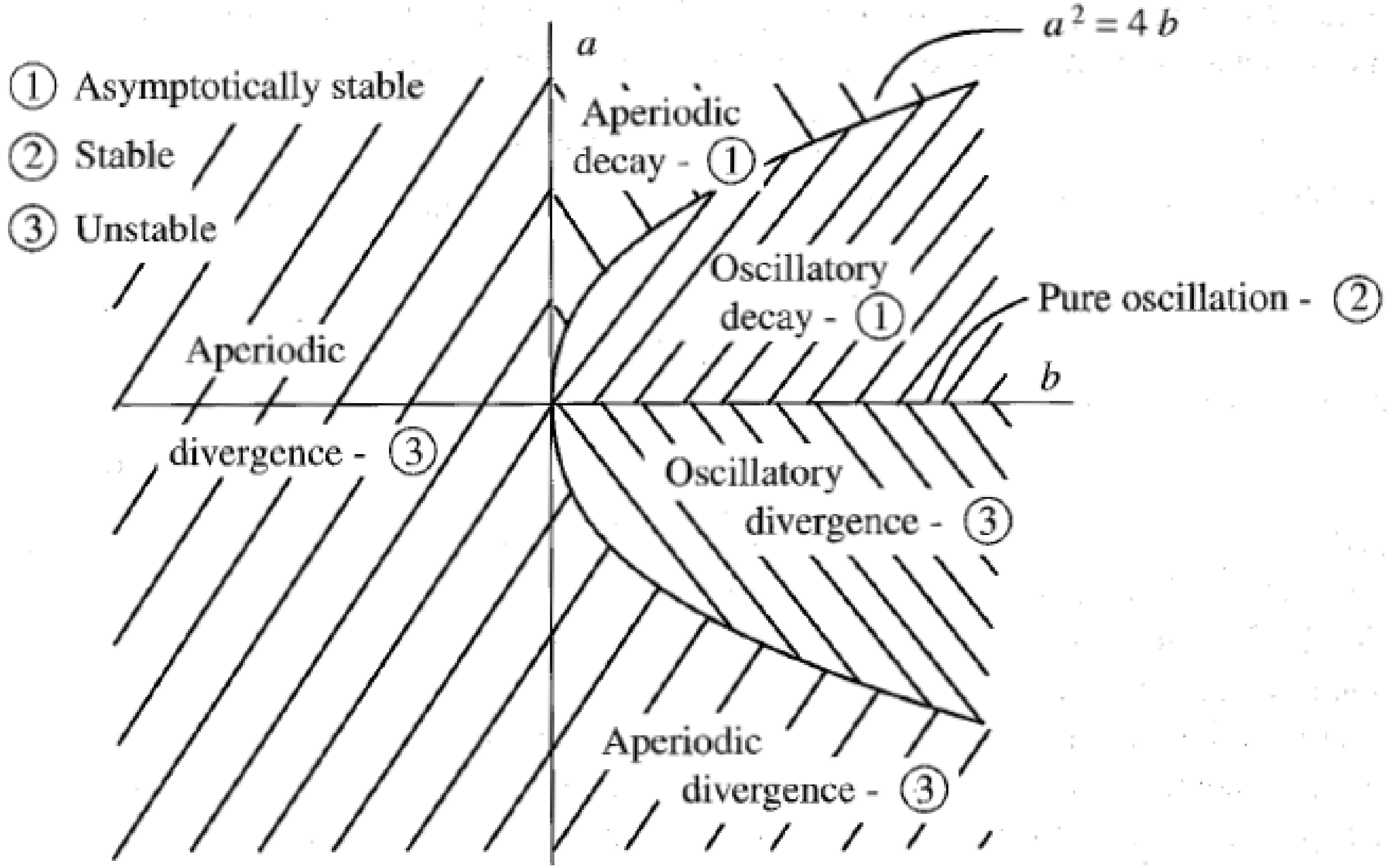


# Linearization about equilibrium points

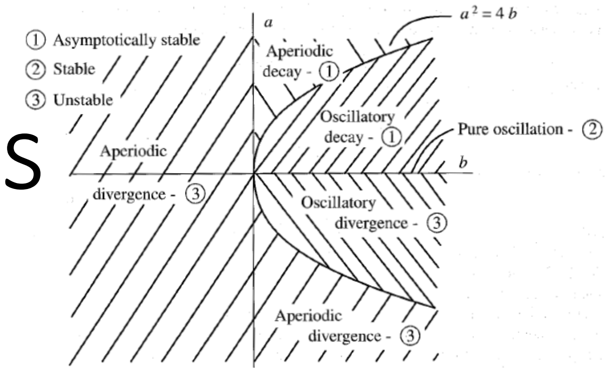
$$s^2 + as + b = 0$$

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

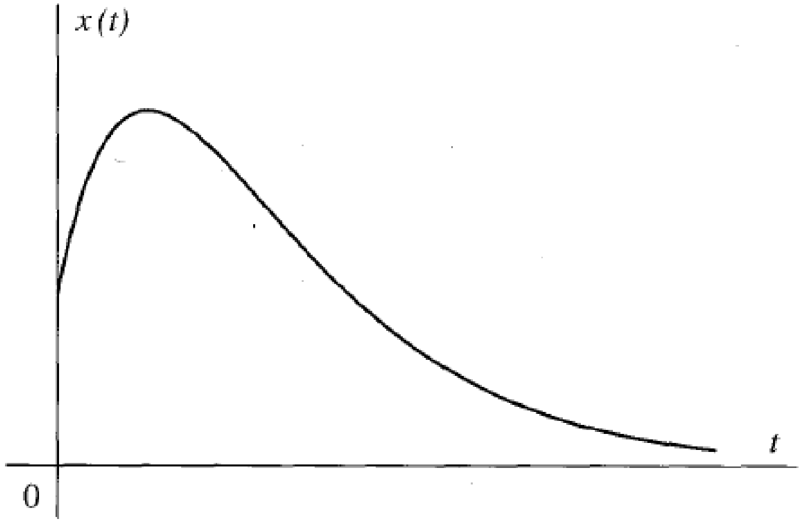


# Linearization about equilibrium points

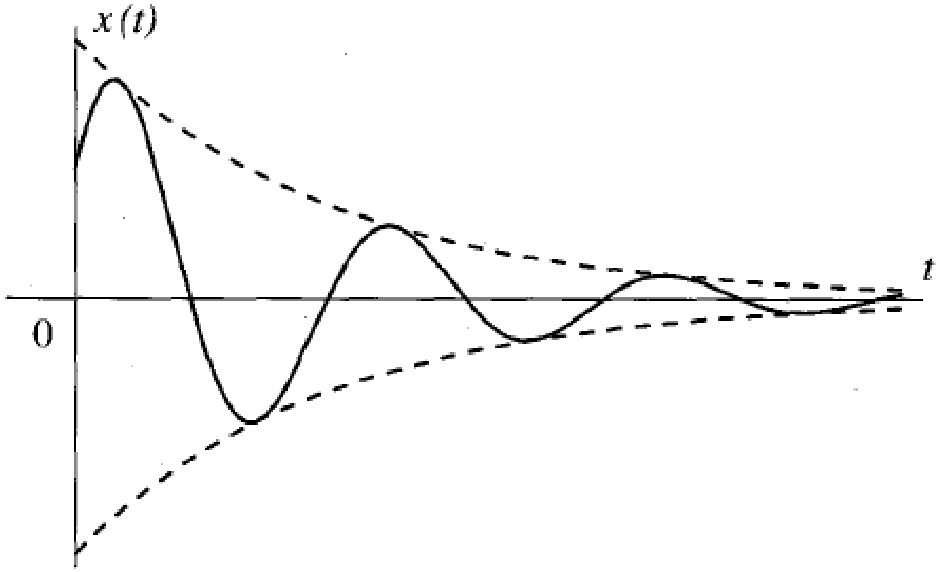



1. Asymptotically stable solution (a>0, b>0)

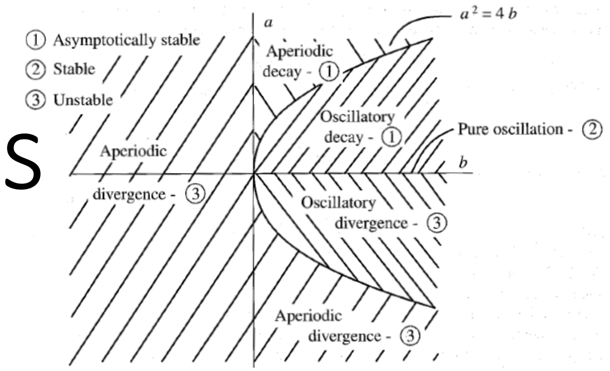
Aperiodically decay




Decaying oscillation

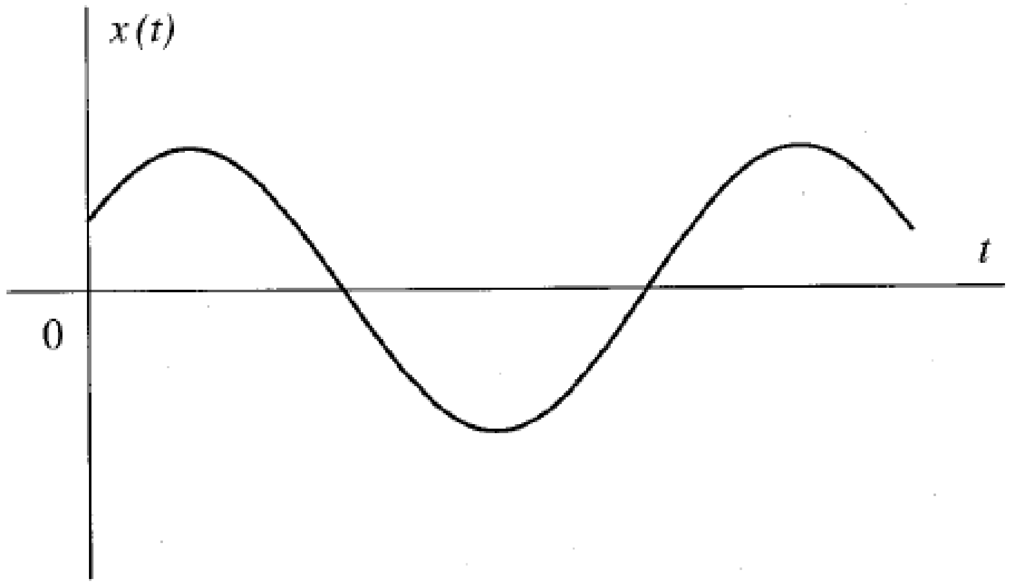


# Linearization about equilibrium points

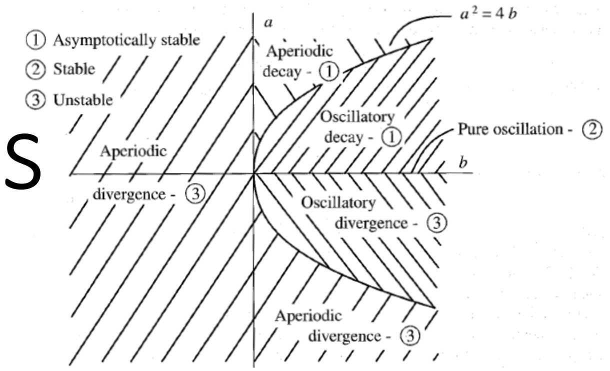



2. Stable motion (a=0, b>0)

Harmonic oscillation

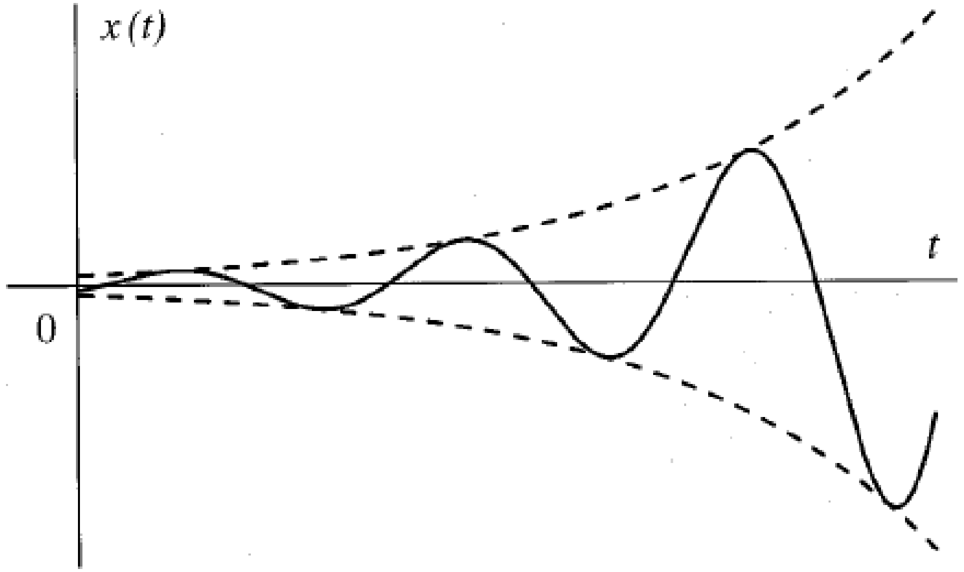


# Linearization about equilibrium points

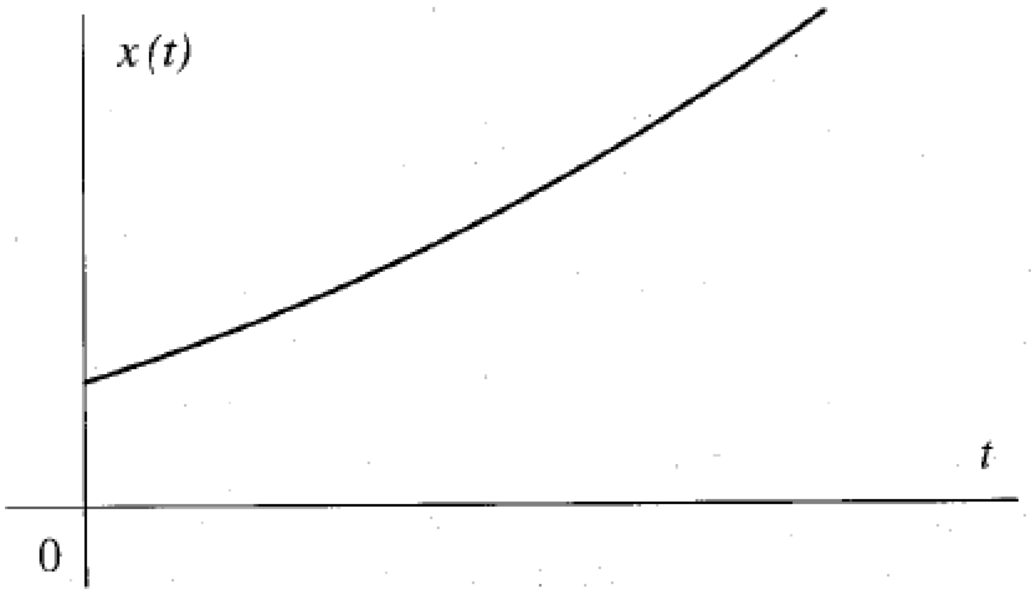



3. Unstable motion (b<0, b>0 & a<0)

Diverging oscillation




Aperiodically diverging motion



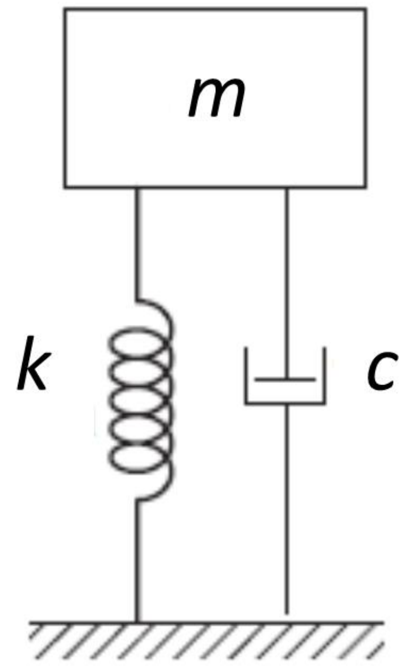
# Viscously damped SDOF

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + 2\varsigma\omega_n \dot{x} + \omega_n^2 x = 0$$

$$\omega_n = \sqrt{k/m} \qquad \varsigma = \frac{c}{2m\omega_n}$$

(viscous damping factor)



# Viscously damped SDOF

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = 0$$

$$x(0) = x_0$$

$$\dot{x}(0) = v_0$$

$$x(t) = Ae^{st}$$

$$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$$

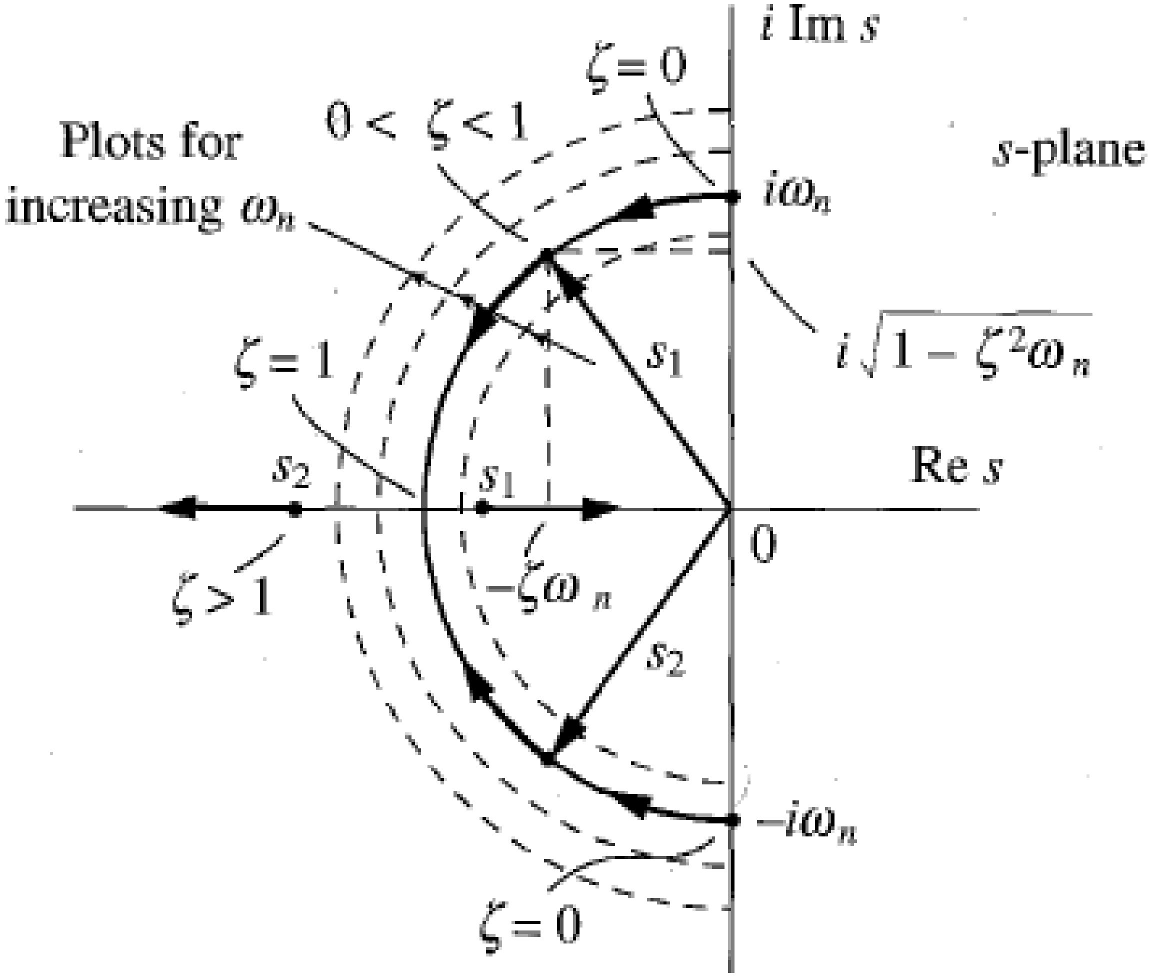
$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\varsigma\omega_n \pm \sqrt{\varsigma^2 - 1}\omega_n$$

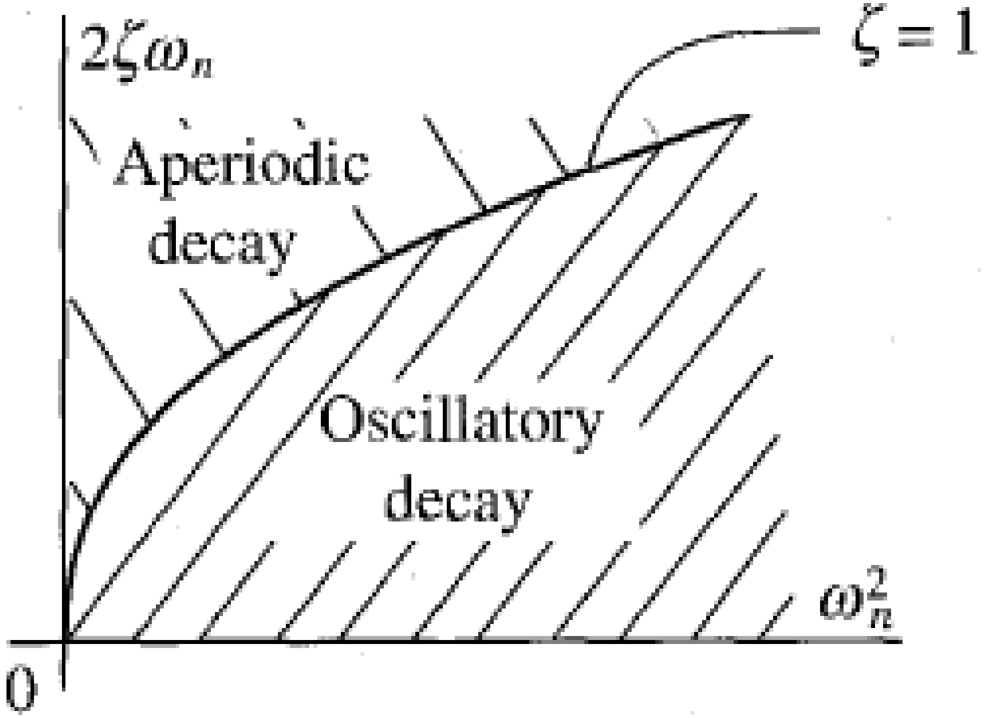
The motion depends of $s_1$, $s_2$ ➔ $\zeta$

# Viscously damped SDOF

$$x(t) = Ae^{st}$$

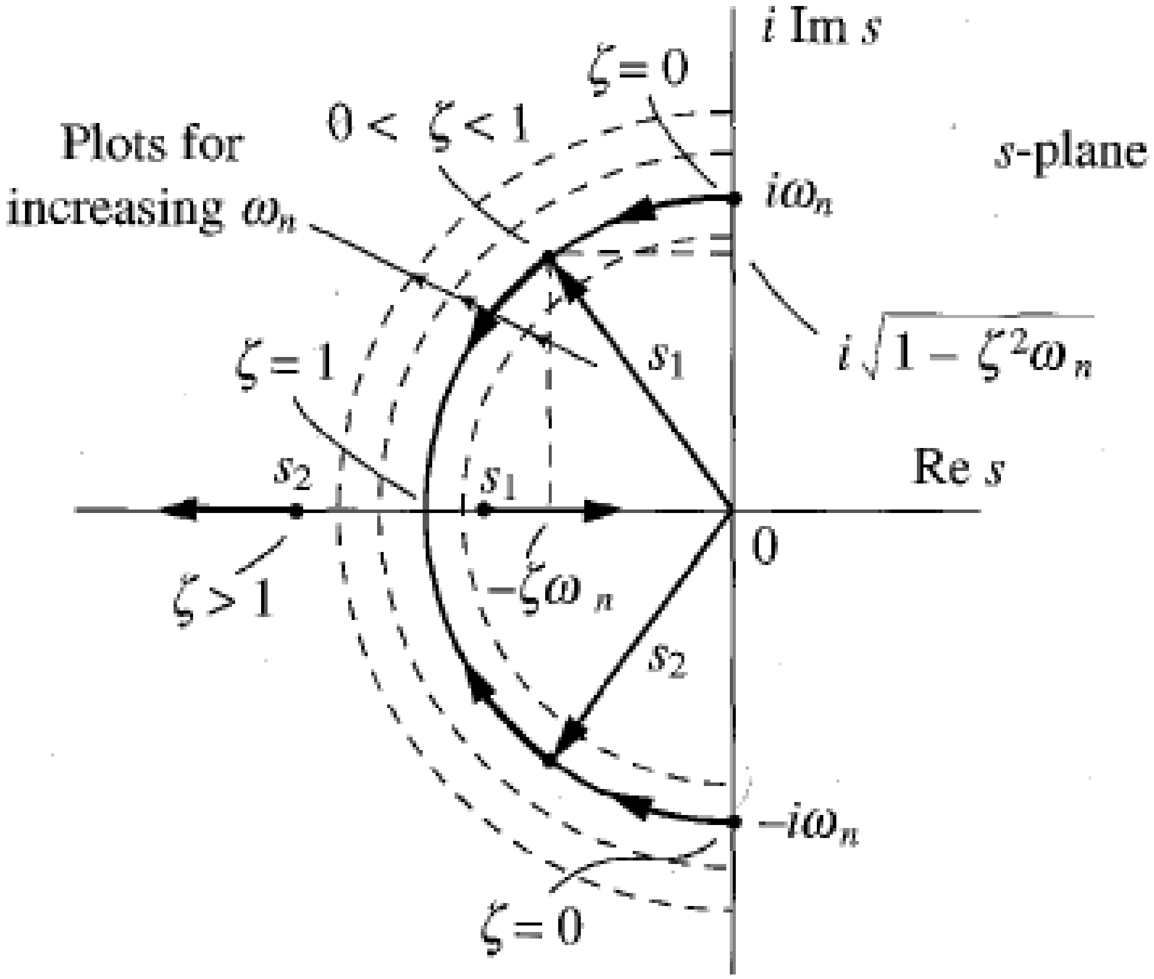
$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\varsigma\omega_n \pm \sqrt{\varsigma^2 - 1}\,\omega_n$$



# Viscously damped SDOF

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\varsigma\omega_n \pm \sqrt{\varsigma^2 - 1}\omega_n$$




$\zeta$ = 0 ➔ harmonic oscillator

$0 < \zeta < 1$ ➔ oscllatory decay (underdamping)

$\zeta$ = 1 ➔ aperiodic decay (critical damping)

$\zeta$ > 1 ➔ aperiodic decay (overdamping)

# Instability of a SDOF

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = 0$$

$$\ddot{x} + a\dot{x} + bx = 0$$

$$a = 2\varsigma\omega_n$$

$$b = \omega_n^2$$

Unstable motion (b<0, b>0 & a<0):

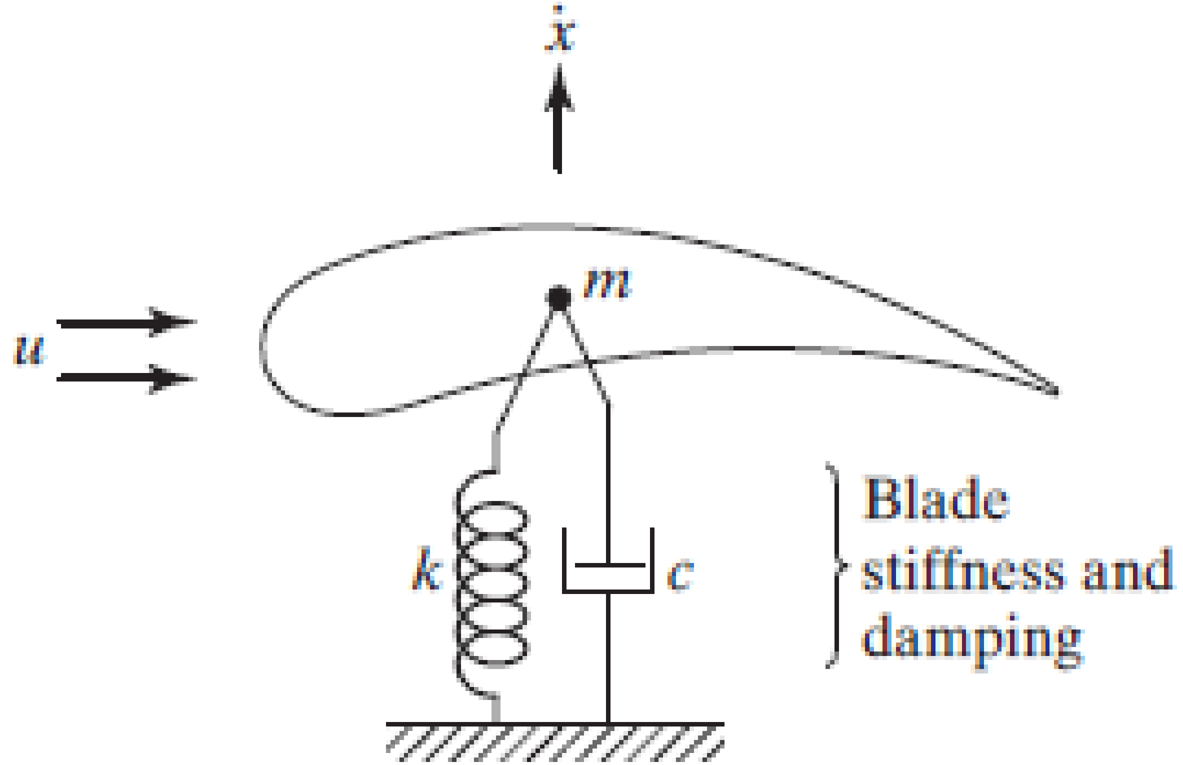
$$a = 2\varsigma\omega_n < 0 \Rightarrow \varsigma < 0$$

$$b = \omega_n^2 > 0$$

negative damping

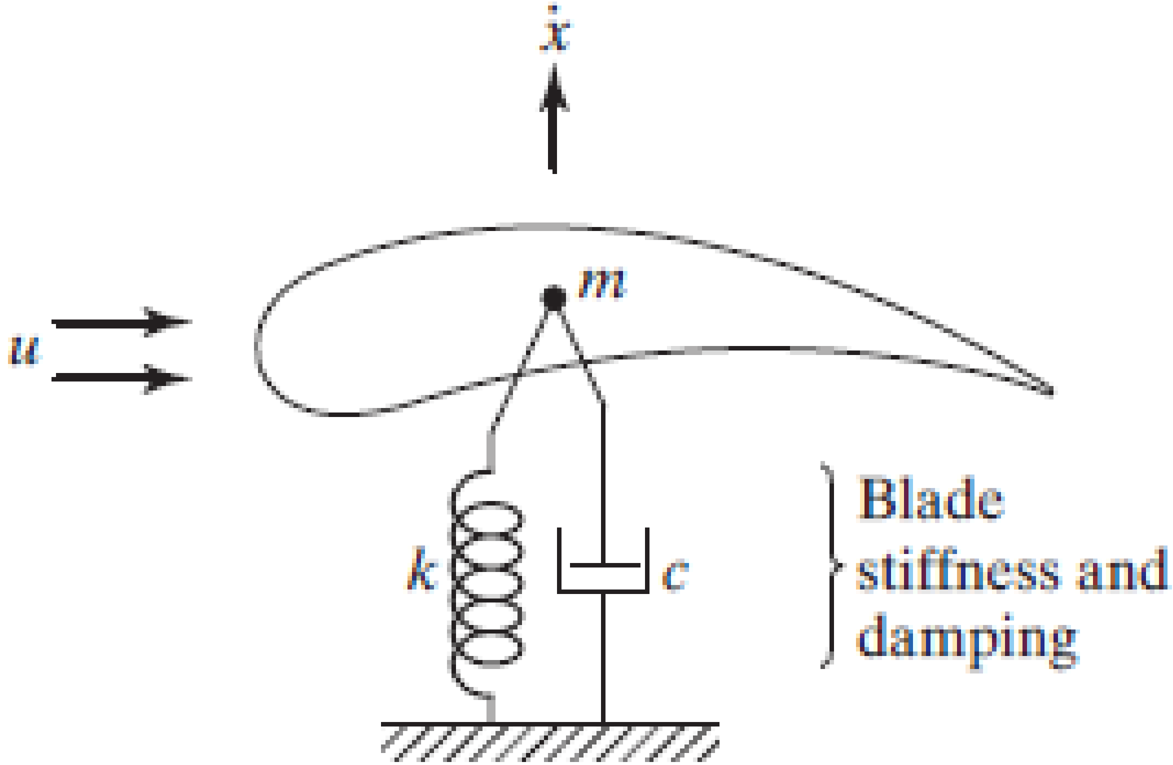
# Instability of a SDOF

- Example of negative damping: aerodynamic forces
- Exercise: find the value of the free-stream velocity *u* at which the airfoil section (SDOF) becomes instable:

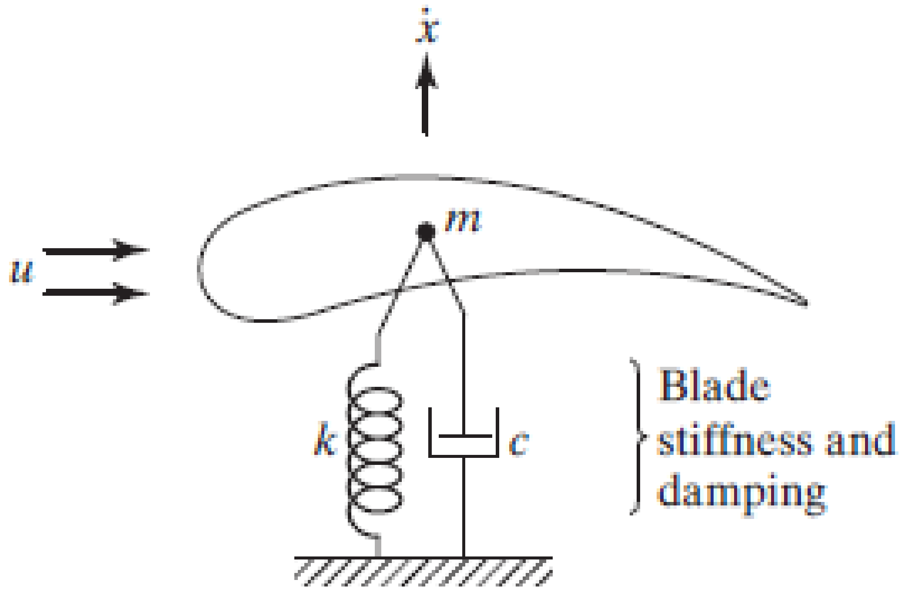


# Instability of a SDOF

- Approach: find the vertical force acting onthe airfoil (or mass *m*) and obtain the condition that leads to zero damping



# Instability of a SDOF: example




vertical force acting on the airfoil

air density

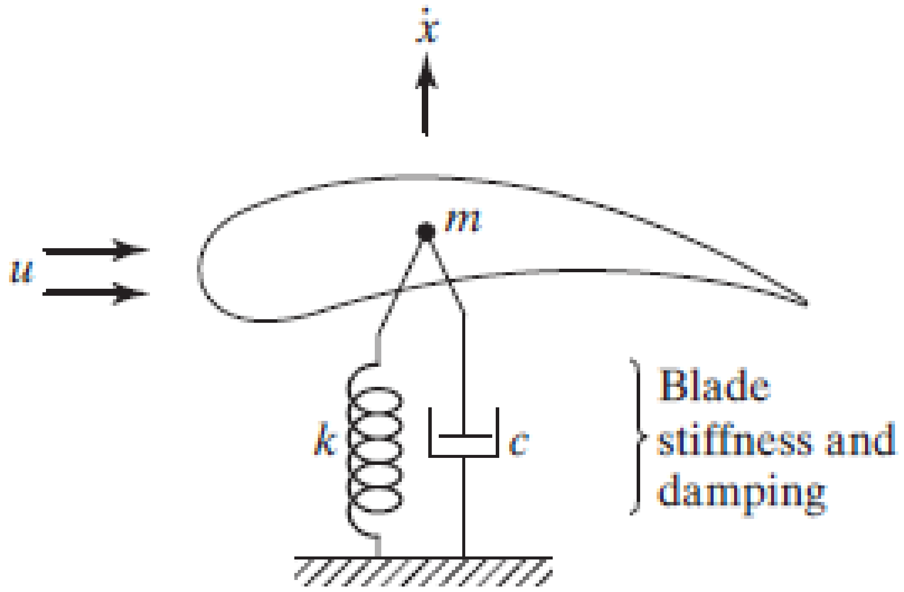
$$F = \frac{1}{2}\rho u^2 D c_x$$

vertical force coefficient

free-stream velocity

width of the cross section normal to the fluid flow direction

# Instability of a SDOF: example




relative velocity of the air

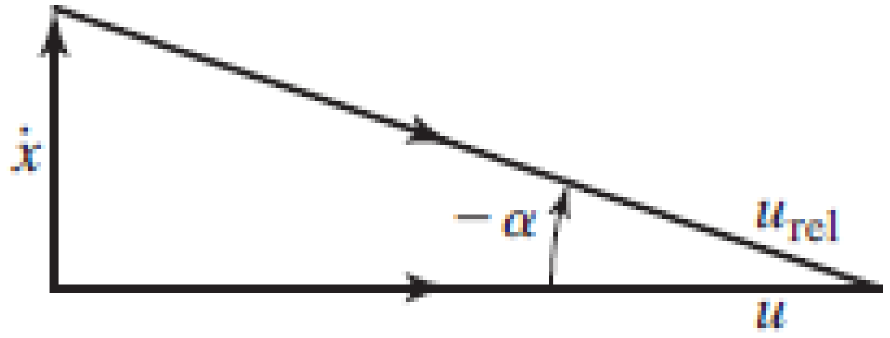
vertical force coefficient

angle of attack

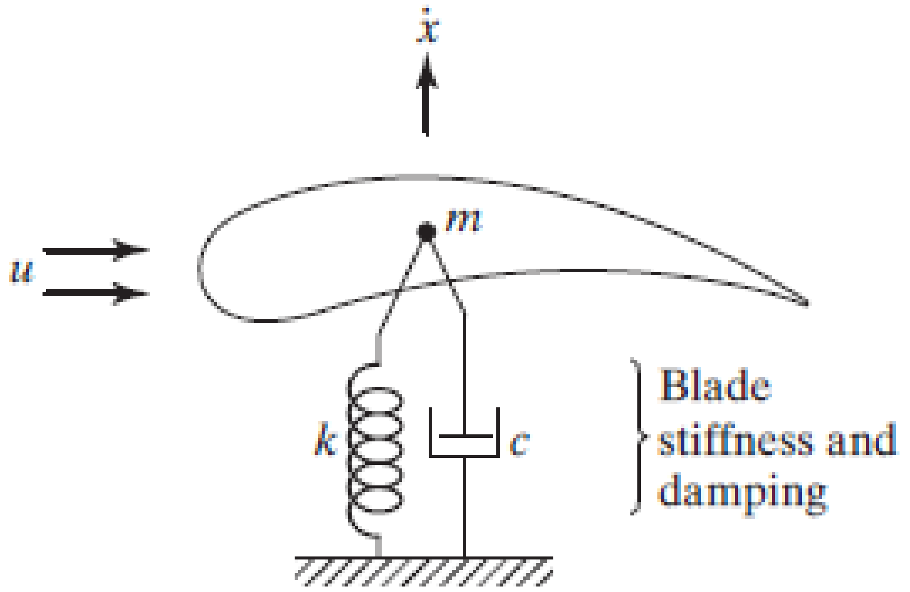
$$c_x = \frac{u_{rel}^2}{u^2}\left(c_L \cos\alpha + c_D \sin\alpha\right)$$

lift coefficient

drag coefficient



# Instability of a SDOF: example




relative velocity of the air

vertical force coefficient

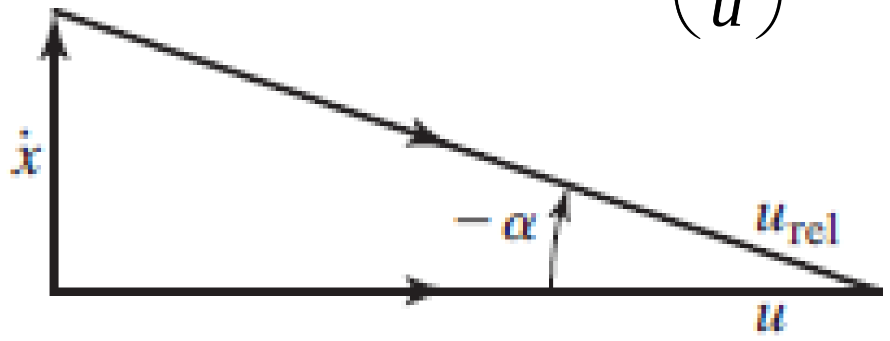
angle of attack

$$c_x = \frac{u_{rel}^2}{u^2}\left(c_L \cos\alpha + c_D \sin\alpha\right)$$

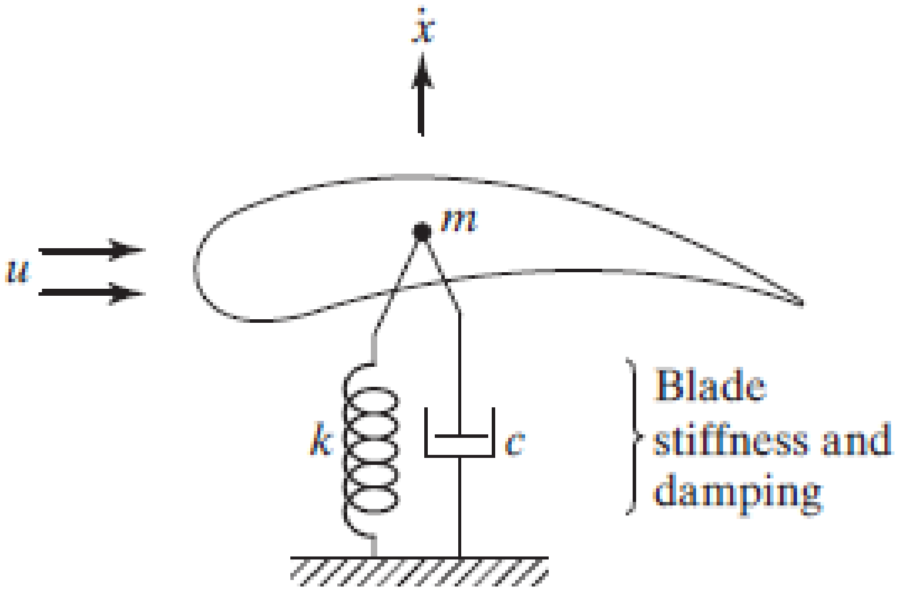
lift coefficient

drag coefficient

$$\alpha = -\tan^{-1}\left(\frac{\dot{x}}{u}\right)$$



# Instability of a SDOF: example



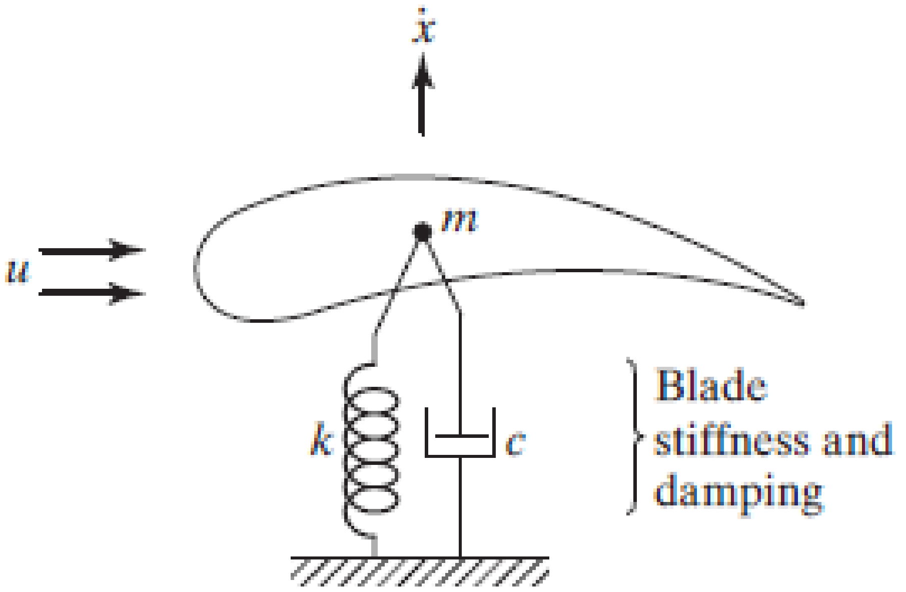

For small angles of attack:

$$
\left.
\begin{aligned}
&\alpha = -\frac{\dot{x}}{u} \\
&c_x \simeq c_x\big|_{\alpha=0} + \left.\frac{\partial c_x}{\partial \alpha}\right|_{\alpha=0} \cdot \alpha \\
&u_{rel} \simeq u
\end{aligned}
\right\}
\quad c_x = c_L \cos\alpha + c_D \sin\alpha
$$

$$
c_x = \left(c_L \cos\alpha + c_D \sin\alpha\right)\Big|_{\alpha=0} + \alpha \left.\left[\frac{\partial c_x}{\partial \alpha}\cos\alpha - c_L \sin\alpha + \frac{\partial c_D}{\partial \alpha}\sin\alpha + c_D \cos\alpha\right]\right|_{\alpha=0}
$$

$$
= c_L\big|_{\alpha=0} + \alpha \left.\frac{\partial c_x}{\partial \alpha}\right|_{\alpha=0} = c_L\big|_{\alpha=0} - \frac{\dot{x}}{u}\left\{\left.\frac{\partial c_L}{\partial \alpha}\right|_{\alpha=0} + c_D\big|_{\alpha=0}\right\}
$$

# Instability of a SDOF: example

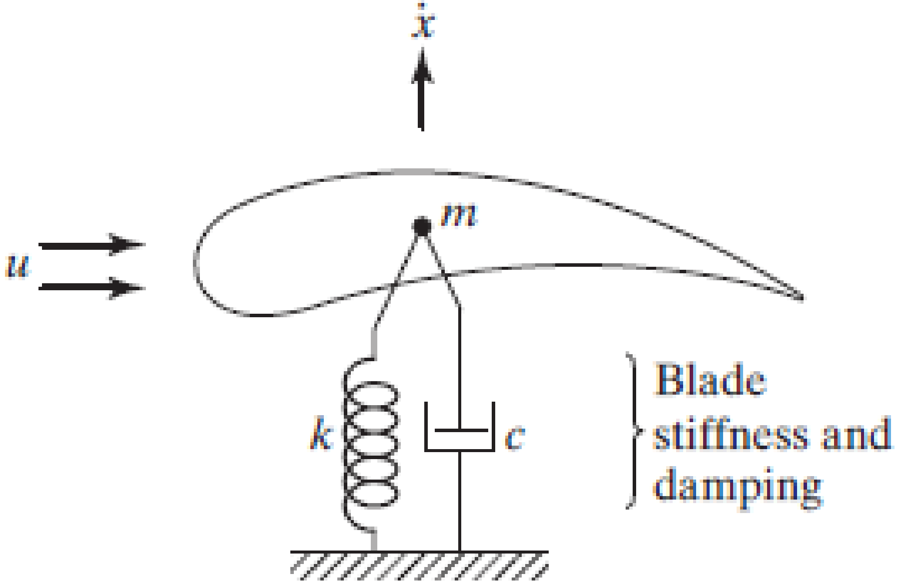



$$F = \frac{1}{2}\rho u^2 D\, c_L\Big|_{\alpha=0} - \frac{1}{2}\rho u^2 D \left.\frac{\partial c_x}{\partial \alpha}\right|_{\alpha=0} \dot{x}$$

$$m\ddot{x} + c\dot{x} + kx = F = \frac{1}{2}\rho u^2 D\, c_L\Big|_{\alpha=0} - \frac{1}{2}\rho u^2 D \left.\frac{\partial c_x}{\partial \alpha}\right|_{\alpha=0} \dot{x}$$

$$\Rightarrow \quad m\ddot{x} + \left[ c + \frac{1}{2}\rho u^2 D \left.\frac{\partial c_x}{\partial \alpha}\right|_{\alpha=0} \right]\dot{x} + kx = 0$$

# Instability of a SDOF: example




$$m\ddot{x} + \left[ c + \frac{1}{2}\rho u^2 D \left.\frac{\partial c_x}{\partial \alpha}\right|_{\alpha=0} \right]\dot{x} + kx = 0$$

$$\left[ c + \frac{1}{2}\rho u^2 D \left.\frac{\partial c_x}{\partial \alpha}\right|_{\alpha=0} \right] \leq 0 \quad \Rightarrow \quad u \geq \frac{2c}{\rho D \left.\frac{\partial c_x}{\partial \alpha}\right|_{\alpha=0}}$$