# System Dynamics and Vibrations

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## Chapter 6: Two-degree-of-freedom systems
## Part 3

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# Contents

# Response to harmonic excitations

$$M\ddot{\mathbf{x}}(t)+C\dot{\mathbf{x}}(t)+k\mathbf{x}(t)=\mathbf{F}(t)$$

$$M=\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix},\ \ C=\begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix},\ \ M=\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix},$$

$\mathbf{F}(t)=\mathbf{F}e^{i\omega t}$ → harmonic external excitation, with driving frequency $\omega$

$\mathbf{x}(t)=\mathbf{X}\left(i\omega\right)e^{i\omega t}$ → by analogy, harmonic response

# Response to harmonic excitations

The equation becomes: $$Z\left(i\omega\right)\mathbf{X}\left(i\omega\right)=\mathbf{F}$$

$$Z\left(i\omega\right)=-\omega^2 M+i\omega C+K$$

impedance matrix

$$z_{ij}\left(i\omega\right)=\omega^2 m_{ij}+i\omega c_{ij}+k_{ij},\quad i,\,j=1,2$$

And the solution: $$\mathbf{X}\left(i\omega\right)=Z^{-1}\left(i\omega\right)\mathbf{F}$$

$$Z^{-1}\left(i\omega\right)=\frac{1}{\left|Z\left(i\omega\right)\right|}\begin{bmatrix} z_{22}\left(i\omega\right) & -z_{12}\left(i\omega\right) \\ -z_{12}\left(i\omega\right) & z_{11}\left(i\omega\right) \end{bmatrix}=\frac{1}{z_{11}\left(i\omega\right)z_{22}\left(i\omega\right)-z_{12}^2\left(i\omega\right)}\begin{bmatrix} z_{22}\left(i\omega\right) & -z_{12}\left(i\omega\right) \\ -z_{12}\left(i\omega\right) & z_{11}\left(i\omega\right) \end{bmatrix}$$

# Response to harmonic excitations

The response is:

$$X_1(i\omega) = \frac{z_{22}(i\omega)F_1 - z_{12}(i\omega)F_2}{z_{11}(i\omega)z_{22}(i\omega) - z_{12}^2(i\omega)}, \quad X_2(i\omega) = \frac{-z_{12}(i\omega)F_1 + z_{11}(i\omega)F_2}{z_{11}(i\omega)z_{22}(i\omega) - z_{12}^2(i\omega)}$$

These functions are analogous to the frequency response functions we obtained for single-degree-of-freedom systems

# Response to harmonic excitations

For undamped systems:

$$z_{11}(\omega) = k_{11} - \omega^2 m_1, \quad z_{22}(\omega) = k_{22} - \omega^2 m_2, \quad z_{12}(\omega) = k_{12}$$

The impedance functions are real

➔ The response functions are also real:

$$X_1(\omega) = \frac{\left(k_{22} - \omega^2 m_2\right) F_1 - k_{12} F_2}{\left(k_{11} - \omega^2 m_1\right)\left(k_{22} - \omega^2 m_2\right) - k_{12}^2}, \quad X_2(\omega) = \frac{-k_{12} F_1 + \left(k_{11} - \omega^2 m_1\right) F_2}{\left(k_{11} - \omega^2 m_1\right)\left(k_{22} - \omega^2 m_2\right) - k_{12}^2}$$

# Response to harmonic excitations. Example

$$X_1(\omega) = \frac{\left(3k - 2m\omega^2\right)F_1}{2m^2\omega^4 - 7mk\omega^2 + 5k^2}, \quad X_2(\omega) = \frac{kF_1}{2m^2\omega^4 - 7mk\omega^2 + 5k^2}$$

$$T/L = k$$

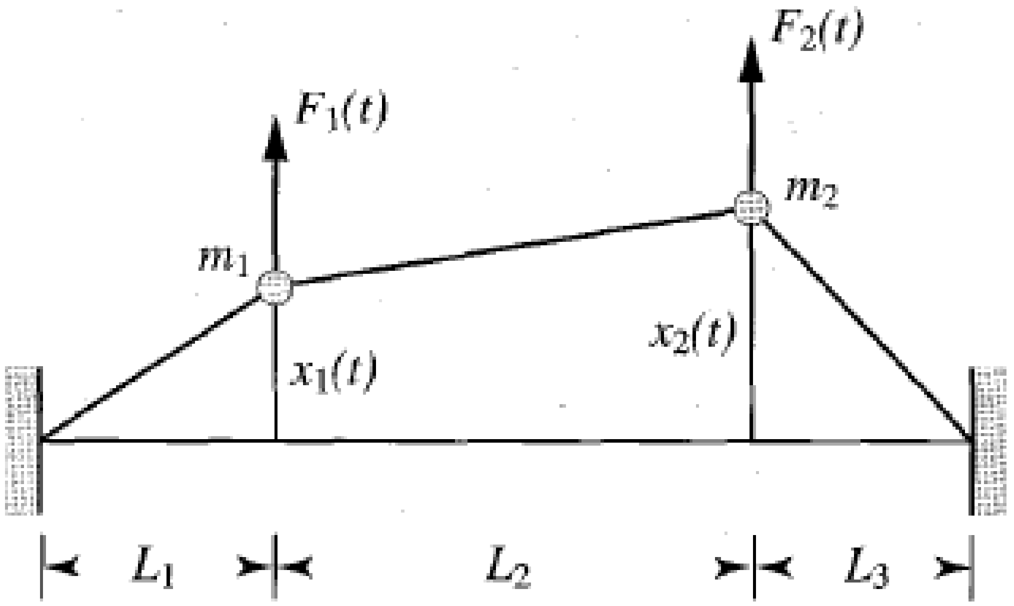
$$\Delta\left(\omega^2\right) = 2m^2\omega^4 - 7mk\omega^2 + 5k^2 = 2m^2\left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_2^2\right)$$

$$\omega_1^2 = \frac{k}{m}$$

$$\omega_2^2 = \frac{5}{2}\frac{k}{m}$$

$$X_1(\omega) = \frac{F_1}{5k}\frac{3 - 2\left(\omega/\omega_1\right)^2}{\left[1 - \left(\omega/\omega_1\right)^2\right]\left[1 - \left(\omega/\omega_2\right)^2\right]}$$

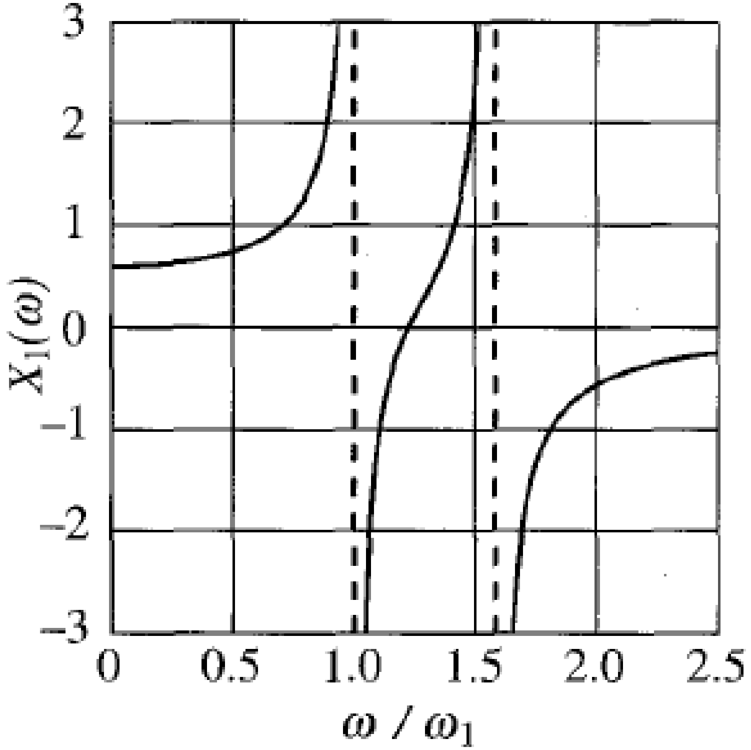
$$X_2(\omega) = \frac{F_1}{5k}\frac{1}{\left[1 - \left(\omega/\omega_1\right)^2\right]\left[1 - \left(\omega/\omega_2\right)^2\right]}$$

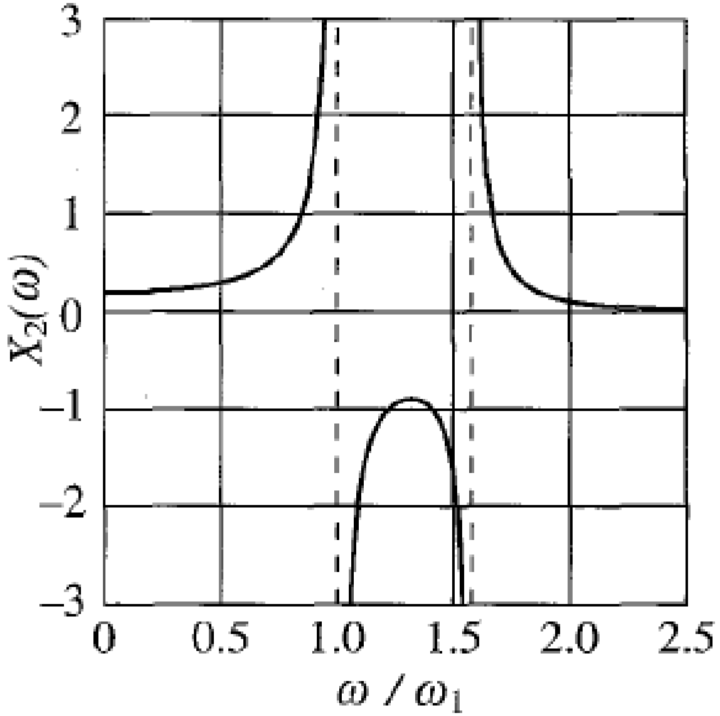


# Response to harmonic excitations. Example

$$X_1(\omega) = \frac{F_1}{5k} \frac{3 - 2(\omega/\omega_1)^2}{\left[1-(\omega/\omega_1)^2\right]\left[1-(\omega/\omega_2)^2\right]}$$

$$X_2(\omega) = \frac{F_1}{5k} \frac{1}{\left[1-(\omega/\omega_1)^2\right]\left[1-(\omega/\omega_2)^2\right]}$$



# Contents

# Multi-degree-of-freedom systems

- For two-degree-of-freedom systems, we solve the eigenvalue problem in three elementary steps:
  - Derivation of the characteristic equation
  - Solution of the characteristic equation ➔ natural frequencies
  - solution of corresponding algebraic equations ➔ natural modes
- Multi-degree-of-freedom systems ($n>2$) require more sophisticated mathematical treatment
- The fundamental concepts remain the same: coupling, orthogonality of modal vectors, modal analysis for decoupling the equations of motion

# Equations of motion for linear systems

- Let's consider a *n*-degree-of-freedom system in the neighborhood of an equilibrium position
- The motion is described by the generalized coordinates

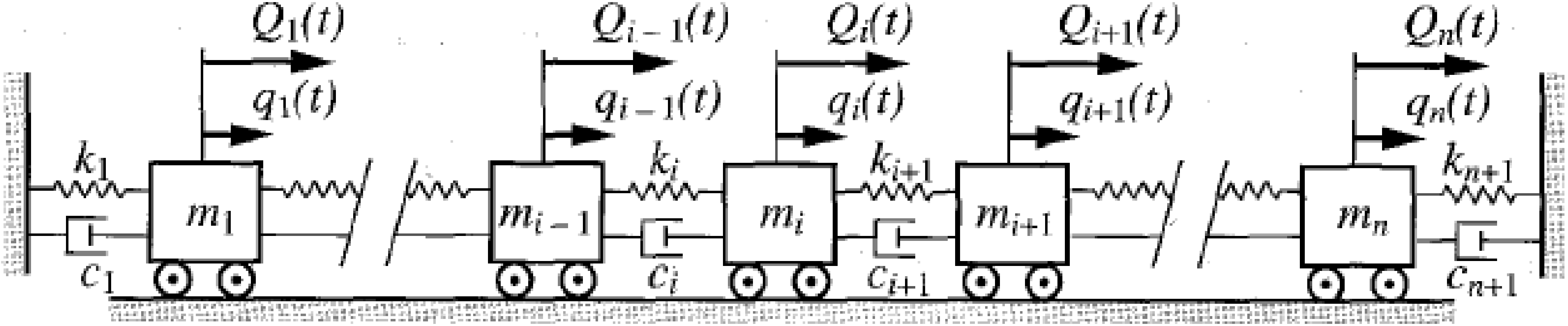
$$q_1(t),\ q_2(t),\ ...,\ q_n(t)$$

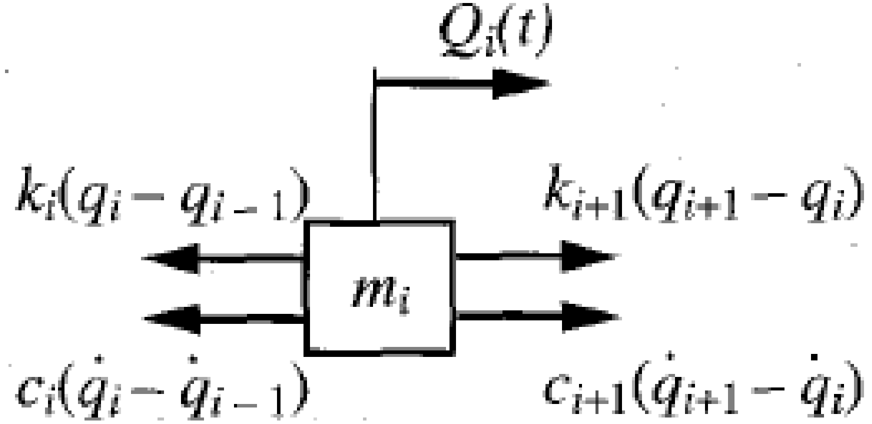
- We assume that the equilibrium position is given by the trivial solution

$$q_1 = q_2 = ... = q_n = 0$$

- Small displacements, so the relations are linear

# Equations of motion for linear systems



# Equations of motion for linear systems

- Applying Newton's second law to mass $m_i$ ($i$ = 1, 2,…,$n$)

$$Q_i(t) + c_{i+1}\left[\dot{q}_{i+1}(t) - \dot{q}_i(t)\right] + k_{i+1}\left[q_{i+1}(t) - q_i(t)\right] - c_i\left[\dot{q}_i(t) - \dot{q}_{i-1}(t)\right] - k_i\left[q_i(t) - q_{i-1}(t)\right] = m_i\ddot{q}_i(t)$$

- We can extend the equation to the full system:

$$\sum_{j=1}^{n}\left[m_{ij}\ddot{q}_j(t) + c_{ij}\dot{q}_j(t) + k_{ij}q_j(t)\right] = Q_i(t), \quad i = 1, 2, \ldots, n$$

$m_{ij}$, $c_{ij}$ and $k_{ij}$ are mass, damping and stiffness coefficients

# Equations of motion for linear systems

$$\sum_{j=1}^{n}\left[m_{ij}\ddot{q}_j(t)+c_{ij}\dot{q}_j(t)+k_{ij}q_j(t)\right]=Q_i(t),\quad i=1,2,\ldots,n$$

$$m_{ij}=\delta_{ij}m_i$$

$$c_{ij}=0,\qquad k_{ij}=0,\qquad j=1,2,\ldots,i-2,i+2,\ldots,n$$

$$c_{ij}=-c_i,\qquad k_{ij}=-k_i,\qquad j=i-1$$

$$c_{ij}=c_i+c_{i+1},\qquad k_{ij}=k_i+k_{i+1},\qquad j=i$$

$$c_{ij}=-c_{i`1},\qquad k_{ij}=-k_{i+1},\qquad j=i+1$$

Set of *n* simultaneous second-order ordinary differential equations for the displacements ➔ coupling makes the analytical solution complicated

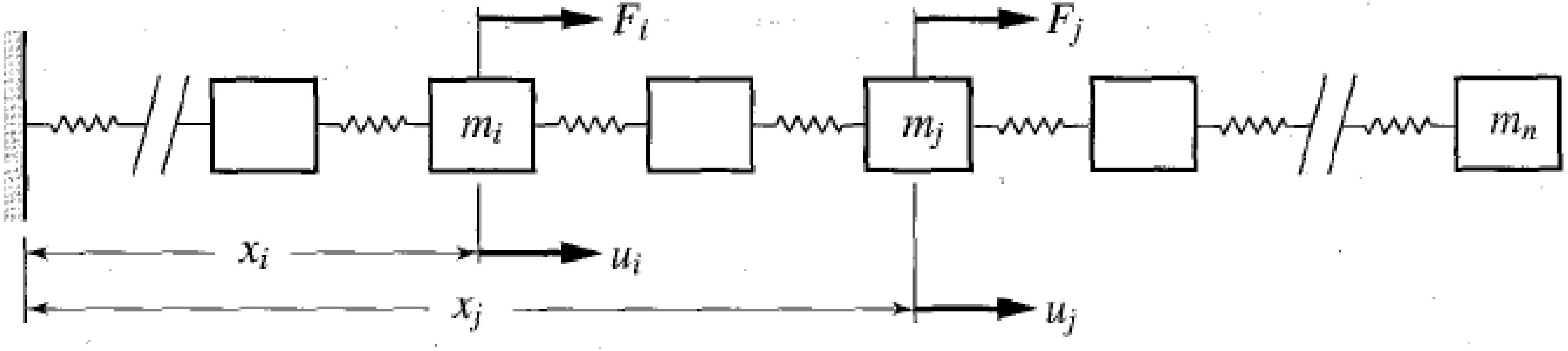
# Equations of motion for linear systems

$$M\ddot{\mathbf{q}}(t) + c\dot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{Q}(t)$$

$$\left[ m_{ij} \right] = M, \quad \left[ c_{ij} \right] = C, \quad \left[ k_{ij} \right] = K$$

$$M = M^T, \quad C = C^T, \quad K = K^T$$

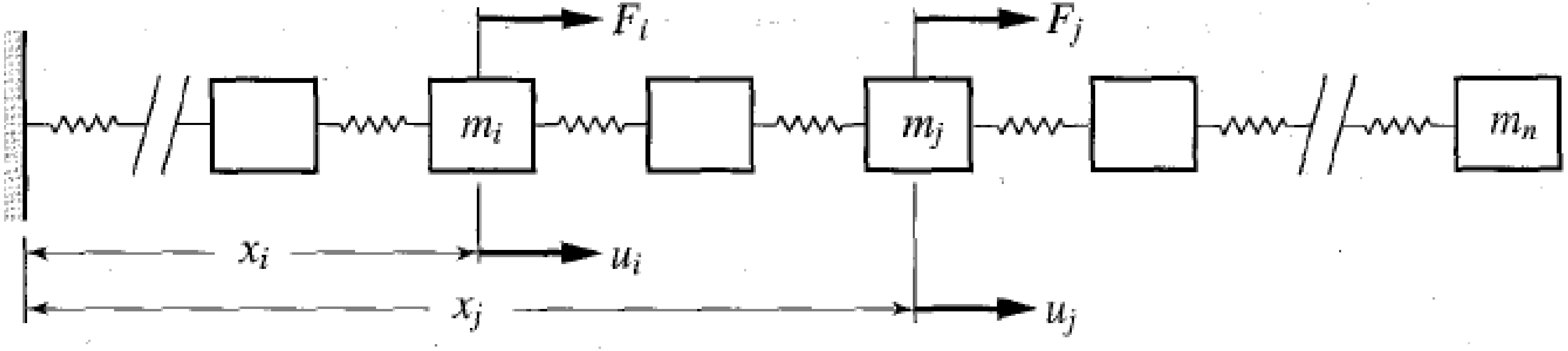
# Flexibility and stiffness influence coefficients




We define the **flexibility influence coefficient** $\boldsymbol{a_{ij}}$ as the displacement of point $x = x_i$ due to a unit force, $F_j = 1$, applied at $x = x_j$

➔ applying the principle of superposition: $$u_i = \sum_{j=1}^{n} a_{ij} F_j$$

# Flexibility and stiffness influence coefficients




By analogy, we can define the **stiffness influence coefficient $k_{ij}$** as the force required at $x = x_i$ to produce a unit displacement, $u_j = 1$, at $x = x_j$ and such that the displacements at all points for which $x \neq x_j$ are zero

$$F_i = \sum_{j=1}^{n} k_{ij} u_j$$

# Flexibility and stiffness influence coefficients

$$\left[a_{ij}\right] = A, \quad \left[k_{ij}\right] = K$$

*A* is the flexibility matrix
*K* is the stiffness matrix

$$\left.\begin{aligned} \mathbf{u} &= A\mathbf{F} \\ \mathbf{F} &= K\mathbf{u} \end{aligned}\right\} \quad A = K^{-1}$$

Influence coefficients are often determined experimentally
They can be calculated by means of the potential energy

# Potential and kinetic energy

For a single linear spring, the potential energy *V* is

$$V = \frac{1}{2}ku^2 = \frac{1}{2}Fu$$

By analogy, for the entire system the elastic potential energy (or strain energy) is

$$V = \sum_{i=1}^{n} V_i = \frac{1}{2}\sum_{i=1}^{n} F_i u_i$$

Introducing the influence coefficients it can be proved that

$$V = \frac{1}{2}\mathbf{u}^T K\mathbf{u} \qquad V = \frac{1}{2}\mathbf{F}^T A\mathbf{F}$$

# Potential and kinetic energy

The kinetic energy is simply

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{u}_i^2 \qquad\qquad T = \frac{1}{2}\dot{\mathbf{u}}^T M \dot{\mathbf{u}}$$

In this particular case, *M* (mass matrix, or inertia matrix) is diagonal, but in general *M* need not be diagonal

# Potential and kinetic energy

- The kinetic energy is always positive definite, so the mass matrix M is always positive definite.
- The stiffness matrix can be:
  - Positive definite ➔ the system is positive definite ➔ all eigenvalues are positive
  - Positive semidefinite ➔ the system is positive semidefinite ➔ all eigenvalues are nonnegative (some of then are zero) ➔ these systems are capable of moving as if they were rigid (zero frequency)

# Lagrange's equations linearized about equilibrium

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\left(\frac{\partial T}{\partial q_k}\right)+\frac{\partial V}{\partial q_k}=Q_k, \quad k=1,2,\ldots,n$$

$q_k$ generalized coordinates
$Q_k$ generalized nonconservative forces

$$T=T\left(q_1,q_2,\ldots,q_n,\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_n\right)$$

$$V=V\left(q_1,q_2,\ldots,q_n\right)$$

$$\overline{\delta W}_{nc}=\sum_{k=1}^{n} Q_k\delta q_k$$ virtual work and virtual displacements

# Lagrange's equations linearized about equilibrium

Viscous damping can be accounted for by doing:

$$Q_{kvisc} = -\frac{\partial F}{\partial q_k}, \quad k = 1, 2, \ldots, n$$

$F$ is a function of the generalized velocities known as Rayleigh's dissipation function

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \left(\frac{\partial T}{\partial q_k}\right) + \frac{\partial V}{\partial q_k} + \frac{\partial F}{\partial \dot{q}_k} = Q_k, \quad k = 1, 2, \ldots, n$$

Lagrange's equations in general are non-linear

# Lagrange’s equations linearized about equilibrium

Small motions assumption:

$$q_k(t) = q_{ek} + \tilde{q}_k(t), \quad k = 1, 2, \ldots, n$$

$$\dot{q}_k(t) = \dot{\tilde{q}}_k(t), \quad k = 1, 2, \ldots, n$$

Linearizing the kinetic energy, the potential energy and the Rayleigh’s function we can obtain the linearized equations about equilibrium:

$$\sum_{j=1}^{n} \left( m_{ij}\ddot{q}_j + c_{ij}\dot{q}_j + k_{ij}q_j \right) = Q_i, \quad i = 1, 2, \ldots, n$$

now $q_j$ represents the small perturbations from equilibrium, instead of $\tilde{q}_j$

# Lagrange's equations linearized about equilibrium

$$\sum_{j=1}^{n}\left(m_{ij}\ddot{q}_j + c_{ij}\dot{q}_j + k_{ij}q_j\right) = Q_i, \quad i = 1, 2, \ldots, n$$

$$M\ddot{\mathbf{q}}(t) + C\dot{\mathbf{q}}(t) + K\mathbf{q} = \mathbf{Q}(t)$$

generalized non-conservative forces

$$T = \frac{1}{2}\dot{\mathbf{q}}^T M\dot{\mathbf{q}} \qquad F = \frac{1}{2}\dot{\mathbf{q}}^T C\dot{\mathbf{q}}$$

$$V = \frac{1}{2}\mathbf{q}^T K\mathbf{q} \qquad \overline{\delta W}_{nc} = \mathbf{Q}^T\delta\mathbf{q}$$

# Linear transformations. Coupling

Coupling depends on the coordinates used to describe the motion
➔ it is not a basic characteristic of the system

Let's consider an **undamped** *n*-degree-of-freedom system:

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q} = \mathbf{Q}(t)$$

*M* and *K* are arbitrary, except that they are symmetric and constant
➔ if *M* or *K* are not diagonal, then the equations of motion are coupled

# Linear transformations. Coupling

We express the equations of motion in a different set of generalized coordinates $\eta_j(t) \quad (j = 1, 2, ..., n)$
such that any displacement $q_i(t) \quad (i = 1, 2, ..., n)$
is a linear combination of the coordinates $\eta_j(t)$

We consider the linear transformation: $\mathbf{q}(t) = U\boldsymbol{\eta}(t)$

*U* is the transformation matrix

# Linear transformations. Coupling

Introducing the coordinate transformation into the equations of motion we obtain:

$$M'\ddot{\eta}(t)+K'\eta(t)=N(t)$$

with $M'=U^T MU, K'=U^T KU$

$$\mathbf{N}(t)=U^T\mathbf{Q}(t)$$

Kinetic and potential energies can also be expressed in the new coordinates:

$$T=\frac{1}{2}\dot{\eta}^T(t)M'\dot{\eta}(t), V=\frac{1}{2}\eta^T(t)K'\eta(t)$$

# Linear transformations. Coupling

The object of the transformation is to produce diagonal matrices *M'* and *K'* simultaneously

➔ then the system consists of <u>independent equations of motion</u>

$$M'_{ij}\ddot{\eta}_j(t) + K'_{ij}\eta_j(t) = N_j(t), \quad j = 1, 2, ..., n$$

<u>These equations have the same structure as that of an undamped single-degree-of-freedom system and can be readily solved</u>

# Linear transformations. Coupling

- A linear transformation matrix *U* diagonalizing *M'* and *K'* simultaneously does exist ➔ the modal matrix
- The modal matrix consists of the modal vectors, or natural modes of the system
- The coordinates $\eta_j(t)$ are called natural or modal coordinates
- The procedure for solving the system of simultaneous differential equations of motion by transforming them into a set of independent equations using the modal matrix as a transformation matrix is generally referred to as modal analysis.
- How to determine the modal matrix U for a given system?

  ➔ by solving the algebraic eigenvalue problem associated with matrices *M'* and *K'*