System Dynamics and Vibrations

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Chapter 1: Elements of analytical dynamics
Exercises

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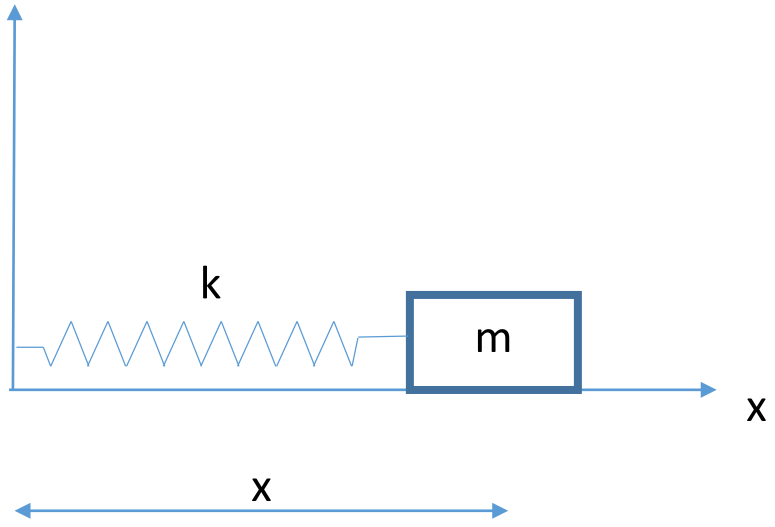
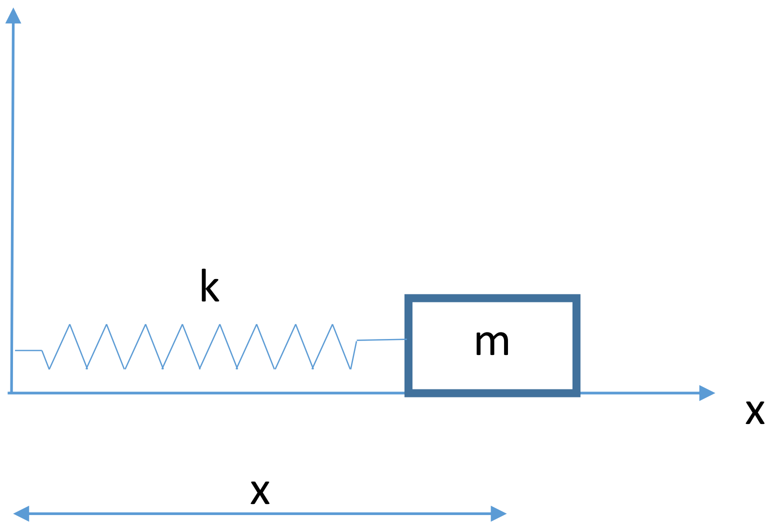
Exercise 1

- Derive the equation of motion for a simple harmonic oscillator using analytical dynamics



Exercise 1



Generalized coordinate: *x*

Kinetic energy: $$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$$

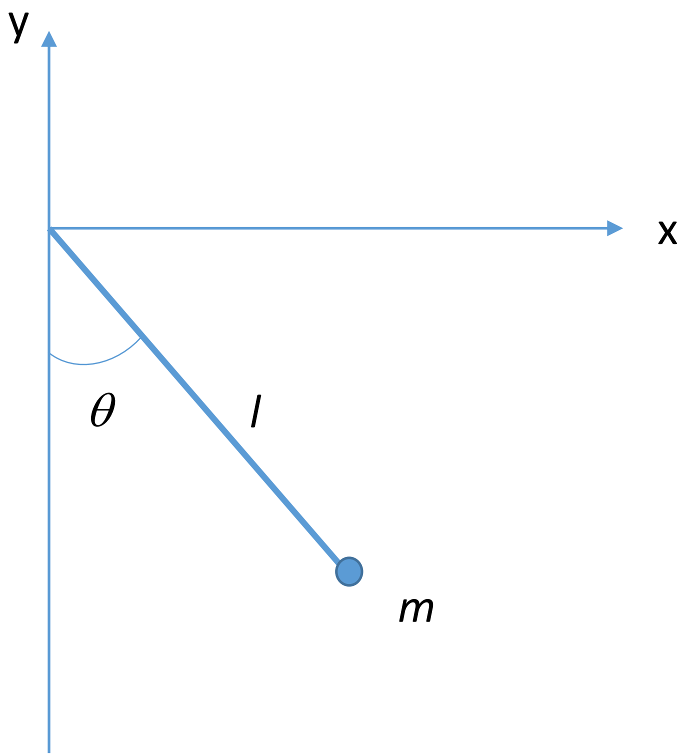
Potential energy: $$V = \frac{1}{2} k x^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_k,$$

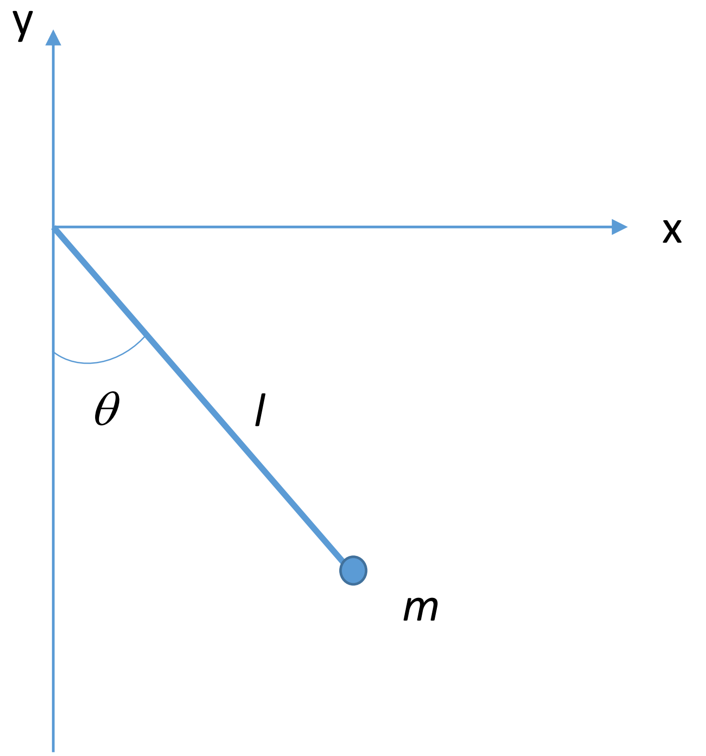
$$\left.\begin{aligned}
\frac{\partial T}{\partial \dot{x}} &= m\dot{x} \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= m\ddot{x} \\
\frac{\partial T}{\partial x} &= 0 \\
\frac{\partial V}{\partial x} &= kx
\end{aligned}\right\} \quad m\ddot{x} + kx = 0$$

Exercise 2

➢ Derive the equation of motion for a simple pendulum using analytical dynamics



Exercise 2



Generalized coordinate: θ

Coordinate transformation:

$$x = l\sin\theta \qquad \dot{x} = l\dot{\theta}\cos\theta$$

$$y = -l\cos\theta \qquad \dot{y} = l\dot{\theta}\sin\theta$$

Kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\theta}^2$$

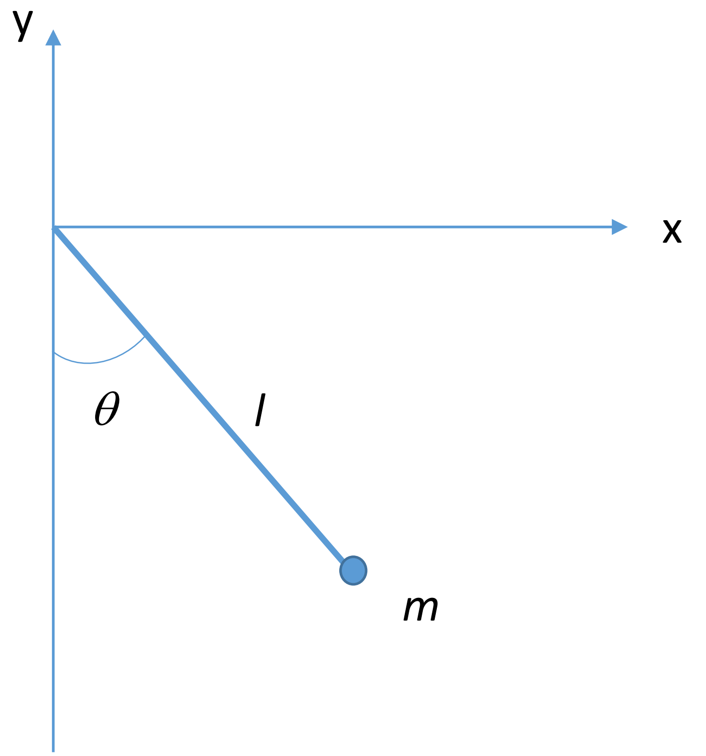
Potential energy:

$$V = mgy = -mgl\cos\theta$$

Exercise 2

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$V = -mgl\cos\theta$$



Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_k, \qquad (k = 1, 2, \ldots n)$$

$$\frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

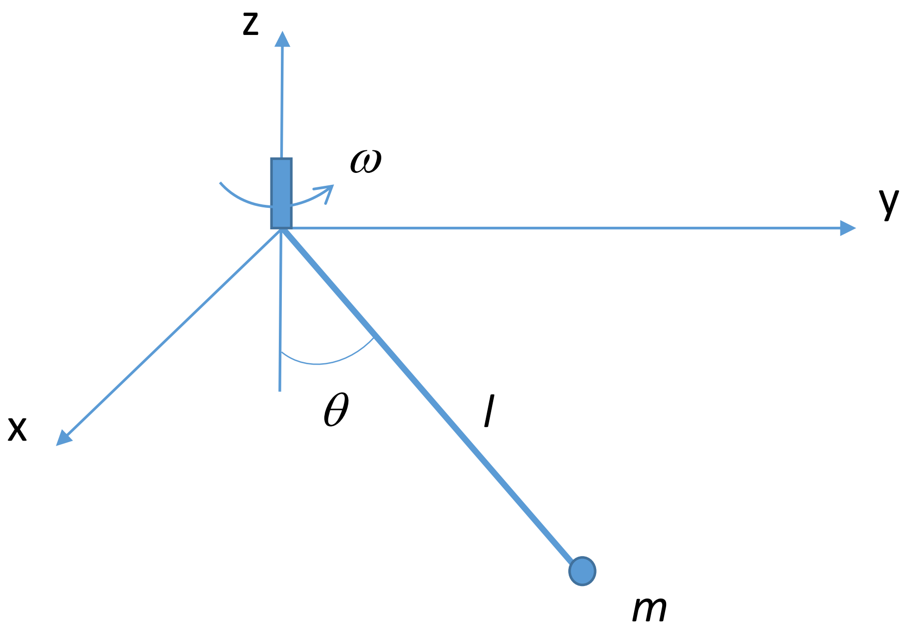
$$\frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial V}{\partial \theta} = mgl\sin\theta$$

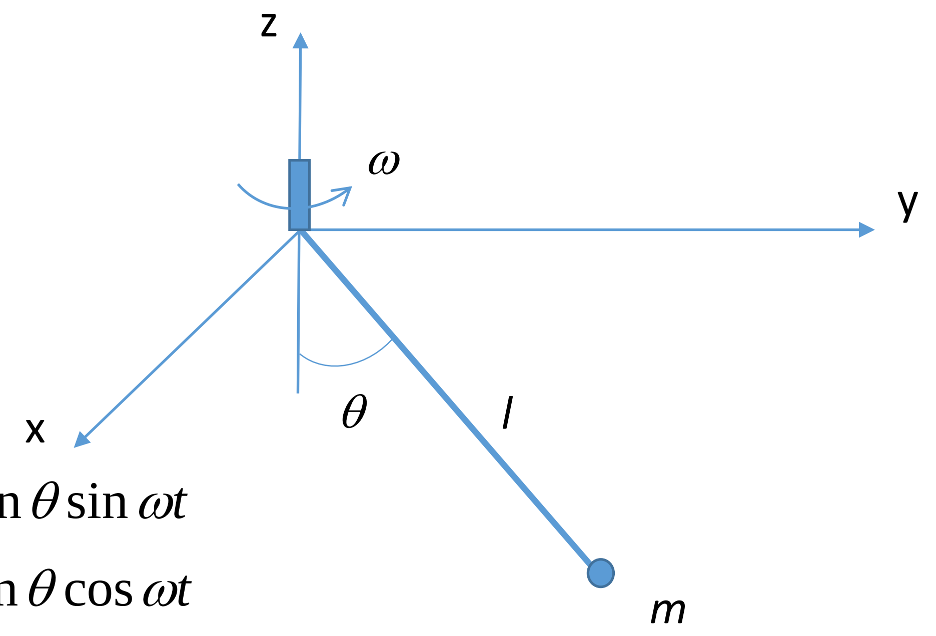
$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Exercise 3

- Derive the equation of motion for a pendulum on a rotating support



Exercise 3



Generalized coordinate: θ

Coordinate transformation:

$$x = l\sin\theta\cos\omega t \qquad \dot{x} = l\dot{\theta}\cos\theta\cos\omega t - l\omega\sin\theta\sin\omega t$$

$$y = l\sin\theta\sin\omega t \qquad \dot{y} = l\dot{\theta}\cos\theta\sin\omega t + l\omega\sin\theta\cos\omega t$$

$$z = -l\cos\theta \qquad \dot{z} = l\dot{\theta}\sin\theta$$

Kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}ml^2(\dot{\theta}^2 + \omega^2\sin^2\theta)$$

Potential energy:

$$V = mgy = -mgl\cos\theta$$

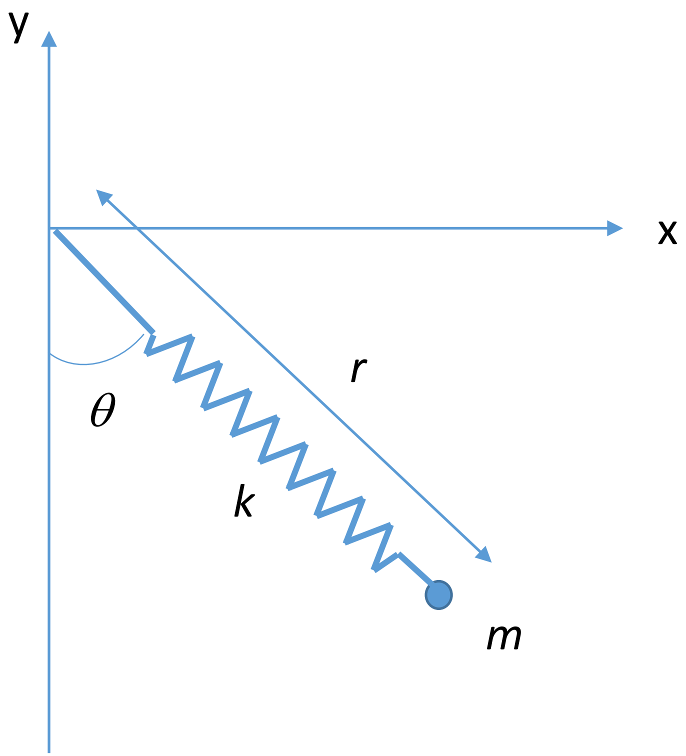
Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = \frac{d}{dt}(ml^2\dot{\theta}) - ml^2\omega^2\sin\theta\cos\theta + mgl\sin\theta =$$

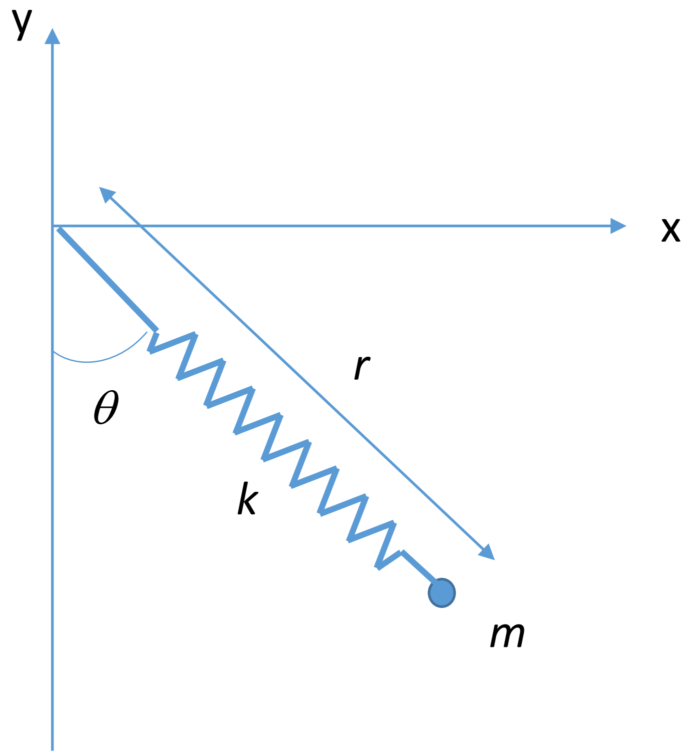
$$ml^2\ddot{\theta} - ml^2\omega^2\sin\theta\cos\theta + mgl\sin\theta = 0$$

Exercise 4

- Derive the equation of motion for a simple pendulum with spring



Exercise 4



Generalized coordinates: $r\,,\,\theta$

Coordinate transformation:

$$x = r\sin\theta \qquad \dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$y = -r\cos\theta \qquad \dot{y} = -\dot{r}\cos\theta + r\dot{\theta}\sin\theta$$

Kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Potential energy:

$$V = \frac{1}{2}k\left(r - l\right)^2 + mgy = \frac{1}{2}k\left(r - l\right)^2 - mgr\cos\theta$$

Lagrange equations:

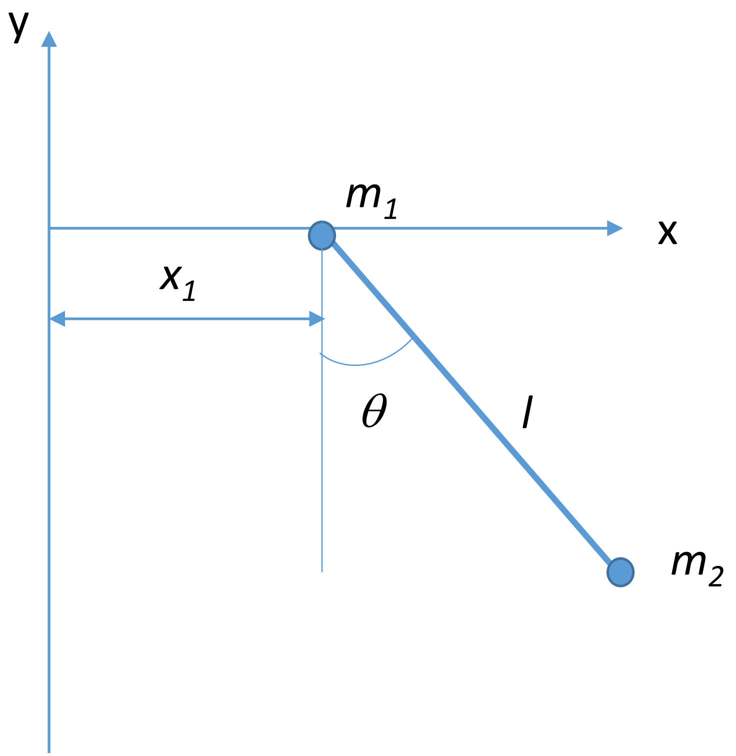
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = \frac{d}{dt}\left(mr^2\dot{\theta}\right) + mgr\sin\theta = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\sin\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} = \frac{d}{dt}\left(m\dot{r}\right) - mr\dot{\theta}^2 + k(r - l) - mg\cos\theta =$$

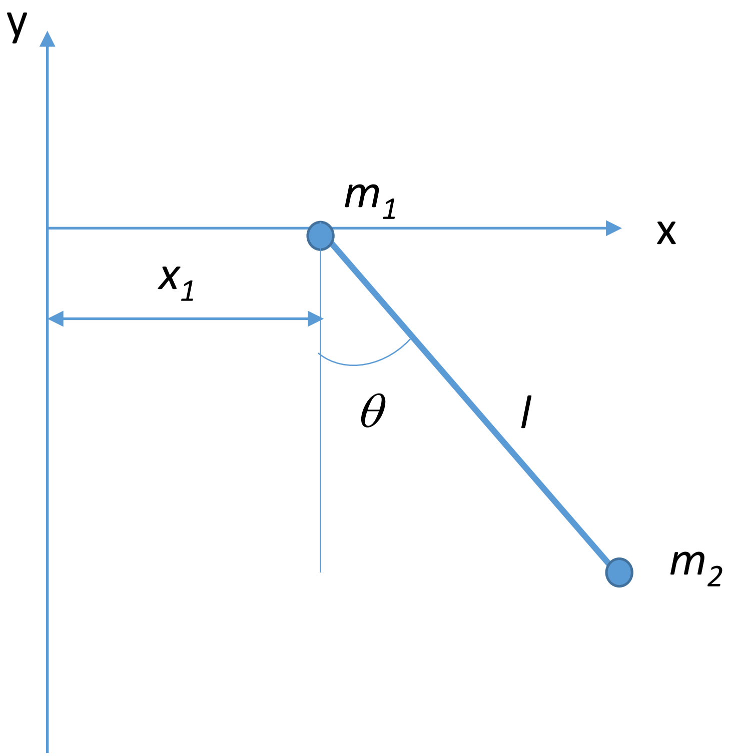
$$m\ddot{r} - mr\dot{\theta}^2 + k(r - l) - mg\cos\theta = 0$$

Exercise 5

- Derive the equation of motion for a simple pendulum with a sliding support



Exercise 5



Generalized coordinates: x_1 , θ

Coordinate transformation:

$$x_2 = x_1 + l\sin\theta \qquad \dot{x}_2 = \dot{x}_1 + l\dot{\theta}\cos\theta$$

$$y_2 = -l\cos\theta \qquad \dot{y}_2 = l\dot{\theta}\sin\theta$$

Kinetic energy:

$$T_1 = \frac{1}{2}m_1 v_1^{\,2} = \frac{1}{2}m_1(\dot{x}_1^{\,2} + \dot{y}_1^{\,2}) = \frac{1}{2}m_1\dot{x}_1^2$$

$$T_2 = \frac{1}{2}m_2\left(\dot{x}_1^2 + \dot{x}_1 l\dot{\theta}\cos\theta + 2l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right)$$

$$T = T_1 + T_2 = \frac{1}{2}\left[\left(m_1 + m_2\right)\dot{x}_1^2 + 2\dot{x}_1 m_2 l\dot{\theta}\cos\theta + m_2 l^2\dot{\theta}^2\right]$$

Potential energy:

$$V = V_1 + V_2 = 0 + m_2 g y_2 = -m_2 g l\cos\theta$$

Exercise 5

$$T=\frac{1}{2}\left[\left(m_1+m_2\right)\dot{x}_1^2+2\dot{x}_1m_2l\dot{\theta}\cos\theta+m_2l^2\dot{\theta}^2\right]$$

$$V=-m_2gl\cos\theta$$

Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right)-\frac{\partial T}{\partial x_1}+\frac{\partial V}{\partial x_1}=0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}+\frac{\partial V}{\partial \theta}=0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right)-\frac{\partial T}{\partial x_1}+\frac{\partial V}{\partial x_1}=\frac{d}{dt}\left[\left(m_1+m_2\right)\dot{x}_1+m_2l\dot{\theta}\cos\theta\right]=\left(m_1+m_2\right)\ddot{x}_1+m_2l\ddot{\theta}\cos\theta-m_2l\dot{\theta}\sin\theta=0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}+\frac{\partial V}{\partial \theta}=\frac{d}{dt}\left(\dot{x}_1m_2l\cos\theta+m_2l^2\dot{\theta}\right)+\dot{x}_1m_2l\dot{\theta}\sin\theta+m_2gl\sin\theta=$$

$$\ddot{x}_1m_2l\cos\theta-\dot{x}_1m_2l\dot{\theta}\sin\theta+m_2l^2\ddot{\theta}+\dot{x}_1m_2l\dot{\theta}\sin\theta+m_2gl\sin\theta=\ddot{x}_1m_2l\cos\theta+m_2l^2\ddot{\theta}+m_2gl\sin\theta=0$$

Exercise 7

- Derive the equation of motion for a disk rolling without sliding over an inclined plane, using analytical dynamics