System Dynamics and Vibrations

Prof. Gustavo Alonso

Chapter 1: Elements of analytical dynamics
Part 2

School of General Engineering
Beihang University (BUAA)

Introduction

- Analytical mechanics (Lagrange, variational approach)
  - Considers the **systems as a whole** ➔ reaction and constraint forces are excluded
  - Dynamics problems formulated in terms of: **kinetic energy, potential energy, virtual work of non-conservative forces**
  - Equations of motion formulated in terms of generalized coordinates and generalized forces ➔ broader and more abstract approach
  - The mathematical formulation is independent of any special system of coordinates

Contents

- Introduction
- Degrees of freedom and generalized coordinates
- The principle of virtual work
- The principle of D'Alembert
- Lagrange's equations

The principle of D’Alembert

- Principle of virtual work ➔ static equilibrium of systems
- Principle of D’Alembert ➔ extension of the principle of virtual work to dynamics (for example, vibrations)

The principle of D'Alembert

- System of N particles, of mass m_i, acted upon the applied force $\mathbf{F}_i$ and the constraint force $\mathbf{f}_i$, assuming any internal force to be negligibly small:
- Rewriting Newton's second law:

$$\mathbf{F}_i + \mathbf{f}_i - m_i \ddot{\mathbf{r}}_i = \mathbf{0}, \quad i = 1, 2, \dots N$$

➔Principle of D'Alembert

$$-m_i \ddot{\mathbf{r}}_i \quad \text{is the inertia force}$$

Dynamic problems can be solved as if they were static

The principle of D'Alembert

- The interest is to extend the principle of virtual work to the dynamical case
- The virtual work for particle m_i is:

$$(\mathbf{F}_i + \mathbf{f}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = \mathbf{0}, \quad i = 1, 2, \ldots N$$

The principle of D’Alembert

- Let’s confine ourselves to **constraint forces for which the virtual work is zero**
- Summing up over the system of particles:

$$\sum_{i=1}^{N}\left(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i\right)\cdot\delta\mathbf{r}_i = 0$$

principle of virtual work + principle of D’Alembert ➔

generalized principle of D’Alembert

(Lagrange’s version of D’Alembert’s principle)

The principle of D'Alembert

- Generalized principle of D'Alembert:

The virtual work performed by the effective forces through infinitesimal virtual displacements compatible with thesystem constraints is zero

$$\sum_{i=1}^{N}\left(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i\right)\cdot\delta\mathbf{r}_i = 0$$

$\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i$ ➔ applied force acting on particle m_i

The principle of D’Alembert

- The real interest is not just to derive the equations of motion, but
- To derive the extended Hamilton’s principle, and then:
- To derive all system equations of motion from three scalar quantities:
  - Kinetic energy
  - Potential energy
  - Virtual work of non-conservative forces
- And then to derive the Lagrange’s equations

The extended Hamilton's principle

- First case: the **<u>position vectors</u>** $\mathbf{r}_i$ (i = 1, 2, ..., N) are all **<u>independent</u>**
- Starting from $$\sum_{i=1}^{N}\left(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i\right)\cdot\delta\mathbf{r}_i = 0$$

$$\downarrow$$

$$\sum_{i=1}^{N}\mathbf{F}_i \cdot \delta\mathbf{r}_i = \overline{\delta W}$$

virtual work of all applied forces

The extended Hamilton's principle

$$\sum_{i=1}^{N}\left(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i\right)\cdot\delta\mathbf{r}_i = 0$$

$$\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i\right) = m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i + m_i\dot{\mathbf{r}}_i\cdot\delta\dot{\mathbf{r}}_i = m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i + \delta\left(\frac{1}{2}m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i\right) = m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i + \delta T_i$$

$T_i = \frac{1}{2}m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}$ is the kinetic energy of particle m_i

$$-\int_{t_1}^{t_2} m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i dt = \int_{t_1}^{t_2}\delta T_i dt - \int_{t_1}^{t_2}\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i\right)dt = \int_{t_1}^{t_2}\delta T_i dt - m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i\Big|_{t_1}^{t_2}$$

The extended Hamilton's principle

$$-\int_{t_1}^{t_2} m_i\ddot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i dt = \int_{t_1}^{t_2} \delta T_i dt - \int_{t_1}^{t_2} \frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i\right) dt = \int_{t_1}^{t_2} \delta T_i dt - m_i\dot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i \Big|_{t_1}^{t_2}$$

- The virtual displacements are arbitrary

➔ we choose them as to satisfy $\delta\mathbf{r}_i = 0$ at $t = t_1$ and $t = t_2$

$$-\int_{t_1}^{t_2} m_i\ddot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i dt = \int_{t_1}^{t_2} \delta T_i dt, \quad \delta\mathbf{r}_i = 0, \quad t = t_1, t_2; \quad i = 1, 2, \ldots N$$

- Summing up over *i*:

$$-\int_{t_1}^{t_2} \sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i dt = \int_{t_1}^{t_2} \delta T dt, \quad \delta\mathbf{r}_i = 0, \quad t = t_1, t_2; \; i = 1, 2, \ldots N$$

T is the system kinetic energy

The extended Hamilton's principle

$$\int_{t_1}^{t_2}\left(\delta T+\overline{\delta W}\right)dt=0,\qquad \delta\mathbf{r}_i=0,\quad i=1,2,\ldots N;\qquad t=t_1,t_2$$

➔ extended Hamilton's principle

- Virtual work is $\overline{\delta W}=\overline{\delta W_c}+\overline{\delta W_{nc}}=-\delta V+\overline{\delta W_{nc}}$

where *V* is the potential energy

➔ then:

$$\int_{t_1}^{t_2}\left(\delta T-\delta V+\overline{\delta W_{nc}}\right)dt=0,\qquad \delta\mathbf{r}_i=0,\quad i=1,2,\ldots N;\qquad t=t_1,t_2$$

The extended Hamilton's principle

$$\int_{t_1}^{t_2} \left(\delta T - \delta V + \overline{\delta W_{nc}} \right) dt = 0, \quad \delta \mathbf{r}_i = 0, \quad i = 1, 2, \ldots N; \quad t = t_1, t_2$$

➔ All the equations of motion can be obtained from three scalar quantities:

- Kinetic energy
- Potential energy
- Virtual work of non-conservative forces

The extended Hamilton's principle

- Second case: the **<u>position vectors</u>** $\mathbf{r}_i$ (i = 1, 2, ..., N) are **<u>not independent</u>**
- They are related by some constraint equations

$$\int_{t_1}^{t_2}\left(\delta T-\delta V+\overline{\delta W_{nc}}\right)dt=0,\quad \delta\mathbf{r}_i=0,\quad i=1,2,\ldots N;\quad t=t_1,t_2$$

The Kinetic energy, Potential energy, and Virtual work of non-conservative forces: all are <u>independent of the coordinates used</u>

➔The extended Hamilton's principle retains its form for all sets of coordinates

The extended Hamilton's principle

- Let's choose the independent generalized coordinates:

$$\int_{t_1}^{t_2}\left(\delta T-\delta V+\overline{\delta W_{nc}}\right)dt=0,\quad \delta q_k=0,\quad k=1,2,\ldots n;\quad t=t_1,t_2$$

- The extended Hamilton's principle in terms of generalized coordinates can be used to derive all system equations of motion (regardless the system is subjected to constraints or not)
- The only condition is that the **constraint forces perform no work**

The extended Hamilton's principle

- For conservative systems: $\overline{\delta W_{nc}} = 0$

➔ Hamilton's principle:

$$\int_{t_1}^{t_2} \delta L dt = 0, \quad \delta q_k = 0, \quad k = 1, 2, \ldots n; \quad t = t_1, t_2$$

$$L = T - V$$

- L is known as the Lagrangian

Contents

- Introduction
- Degrees of freedom and generalized coordinates
- The principle of virtual work
- The principle of D’Alembert
- Lagrange’s equations

Lagrange’s equations

- Lagrange’s equations can be derived directly from D’Alembert’s principle or from the Extended Hamilton’s principle (simplest)
- Kinetic energy: $T = T\left(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\right)$

$$\delta T = \sum_{k=1}^{n}\left(\frac{\delta T}{\delta q_k}\delta q_k + \frac{\delta T}{\delta \dot{q}_k}\delta \dot{q}_k\right)$$

- Potential energy: $V = V\left(q_1, q_2, \ldots, q_n\right)$

$$\delta V = \sum_{k=1}^{n}\left(\frac{\delta V}{\delta q_k}\delta q_k\right)$$

Lagrange's equations

- Virtual work of non-conservative forces: $\overline{\delta W_{nc}} = \sum_{k=1}^{n} Q_k \delta q_k$

$Q_k \quad (k = 1, 2, \ldots, n)$ ➔ generalized non-conservative forces

$$\int_{t_1}^{t_2} \left(\delta T - \delta V + \overline{\delta W_{nc}} \right) dt = \int_{t_1}^{t_2} \sum_{k=1}^{n} \left[\left(\frac{\delta T}{\delta q_k} - \frac{\delta V}{\delta q_k} + Q_k \right) \delta q_k + \frac{\delta T}{\delta \dot{q}_k} \delta \dot{q}_k \right] dt = 0$$

$$\delta q_k = 0, \quad k = 1, 2, \ldots n; \quad t = t_1, t_2$$

Lagrange's equations

- Integrating by parts:

$$\int_{t_1}^{t_2} \frac{\delta T}{\delta q_k}\delta \dot{q}_k dt = \int_{t_1}^{t_2} \frac{\delta T}{\delta \dot{q}_k}\frac{d}{dt}\delta q_k dt = \frac{\delta T}{\delta \dot{q}_k}\delta q_k \Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\delta t}{\delta \dot{q}_k}\right)\delta q_k dt$$

$$= -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_k}\right)\delta q_k dt, \qquad k = 1, 2, \ldots n;$$

It has been considered $\delta q_k = 0 \quad (k = 1, 2, \ldots n)$ at $t = t_1$ and $t = t_2$

$$\int_{t_1}^{t_2} \sum_{k=1}^{n} \left[\frac{\delta T}{\delta q_k} - \frac{\delta V}{\delta q_k} + Q_k - \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_k}\right) \right] \delta q_k dt = 0$$

Lagrange’s equations

- Since the generalized virtual displacements are arbitrary
- We assign arbitrary values to δq_1 while setting $\delta q_k = 0$

$$(k = 2, 3, \ldots n)$$

$$\int_{t_1}^{t_2} \sum_{k=1}^{n} \left[\frac{\delta T}{\delta q_k} - \frac{\delta V}{\delta q_k} + Q_k - \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_k} \right) \right] \delta q_k dt = 0$$ ➔ the coefficient of δq_1 is zero

- Repeating the argument for all $\delta q_2, \delta q_3, \ldots \delta q_n$

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_k} \right) - \frac{\delta T}{\delta q_k} + \frac{\delta V}{\delta q_k} = Q_k, \quad (k = 1, 2, \ldots n)$$

Lagrange equations