System Dynamics and Vibrations

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Chapter 6: Two-degree-of-freedom systems
Exercises - 2

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Exercise 3

The two-degree-of-freedom system of the figure consists of two masses on a string of tension *T* vibrating in the vertical plane. Let

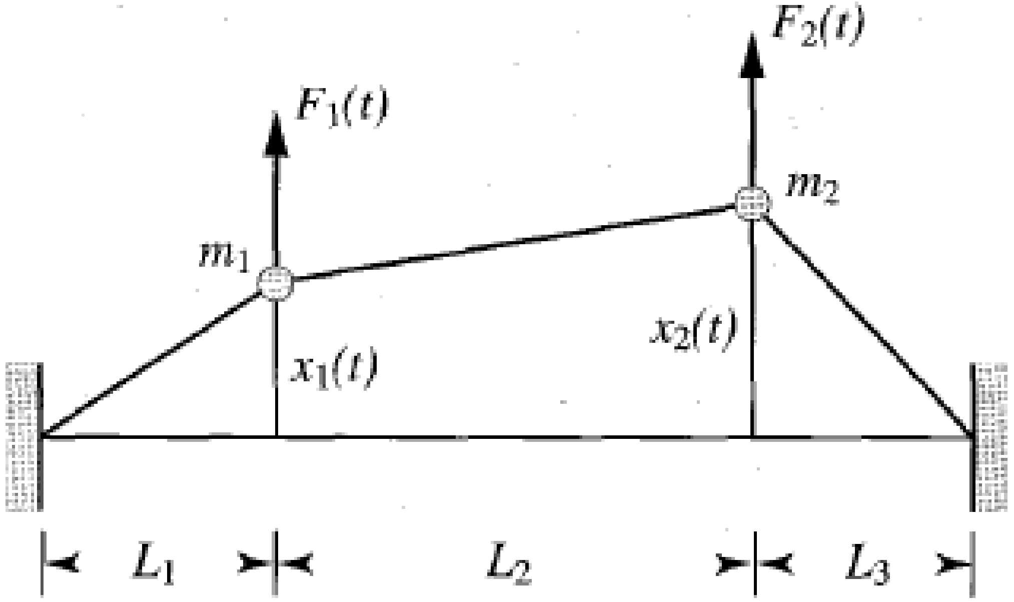
$m_1 = m$

$m_2 = 2m$

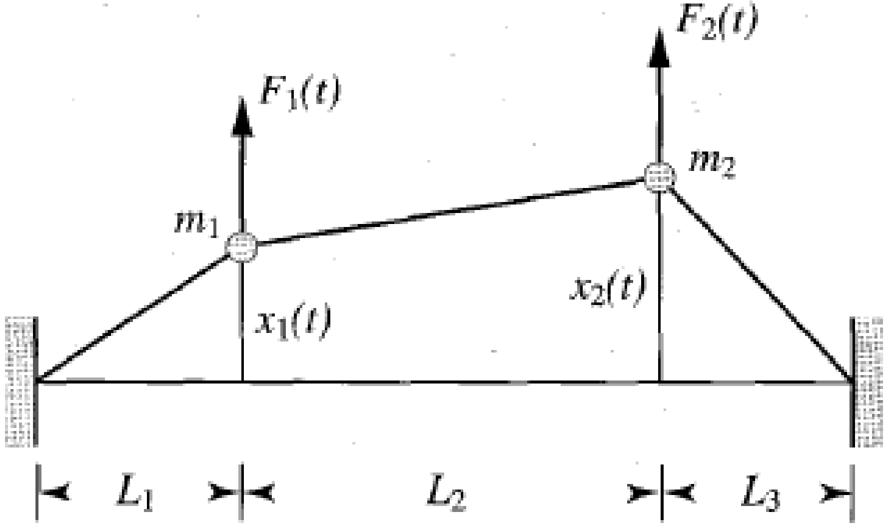
$L_1 = L_2 = L$

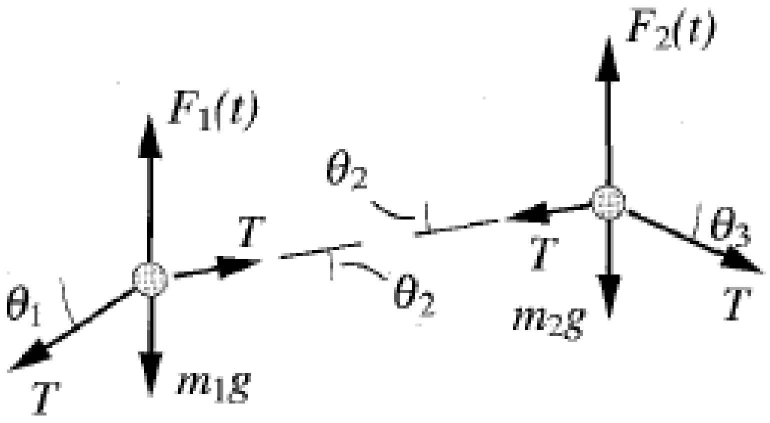
$L_3 = 0.5L$

Determine the natural modes of vibration



Exercise 3





$$m_1 \frac{d^2x_1}{dt^2} + \left(\frac{T}{L_1} + \frac{T}{L_2}\right)x_1 - \frac{T}{L_2}x_2 = F_1$$

$$m_2 \frac{d^2x_2}{dt^2} - \frac{T}{L_2}x_1 + \left(\frac{T}{L_2} + \frac{T}{L_3}\right)x_2 = F_2$$

(with the assumption that displacements are small and being x_i the vibration about the equilibrium position)

Exercise 3

$$m_1 \frac{d^2 x_1}{dt^2} + \left(\frac{T}{L_1} + \frac{T}{L_2} \right) x_1 - \frac{T}{L_2} x_2 = F_1$$

$$m_2 \frac{d^2 x_2}{dt^2} - \frac{T}{L_2} x_1 + \left(\frac{T}{L_2} + \frac{T}{L_3} \right) x_2 = F_2$$

$$\Rightarrow \begin{cases} k_{11} = \dfrac{T}{L_1} + \dfrac{T}{L_2} = \dfrac{2T}{L} \\ k_{22} = \dfrac{T}{L_2} + \dfrac{T}{L_3} = \dfrac{3T}{L} \\ k_{12} = -\dfrac{T}{L_2} = -\dfrac{T}{L} \end{cases}$$

<u>Natural frequencies:</u>

$$\begin{matrix} \omega_1^2 \\ \omega_2^2 \end{matrix} = \frac{1}{2}\left(\frac{k_{11}}{m_{11}} + \frac{k_{22}}{m_{22}} \right) \mp \frac{1}{2}\sqrt{\left(\frac{k_{11}}{m_{11}} + \frac{k_{22}}{m_{22}} \right)^2 - 4\frac{k_{11}k_{22} - k_{12}^2}{m_1 m_2}} = \begin{cases} \dfrac{T}{mL} \\ \dfrac{5T}{2mL} \end{cases}$$

Exercise 3

Mode shapes:

$$\frac{u_{21}}{u_{11}} = -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} = -\frac{\dfrac{2T}{L} - \dfrac{T}{mL} m}{-\dfrac{T}{L}} = 1$$

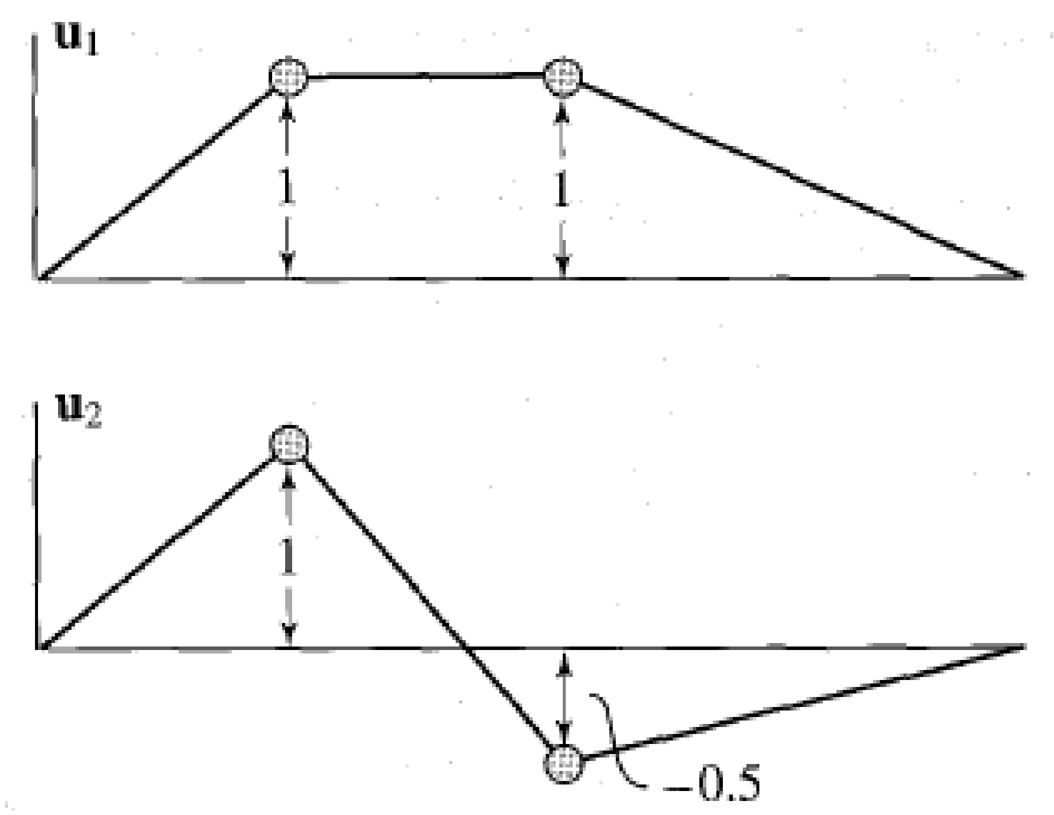
$$\frac{u_{22}}{u_{12}} = -\frac{k_{11} - \omega_2^2 m_1}{k_{12}} = -\frac{\dfrac{2T}{L} - \dfrac{5T}{2mL} m}{-\dfrac{T}{L}} = -0.5$$

Normalization of modal vectors: $u_{11} = 1,\ u_{12} = 1$

$$\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

Exercise 3

$$\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$



$$\omega_1 = \sqrt{\frac{T}{mL}}$$

$$\omega_2 = \sqrt{\frac{5T}{2mL}}$$

Exercise 4

Consider the model shown in the figure.
Let the parameters have the values:

m = 1,500 kg

I_C = 2,000 kg·m^2

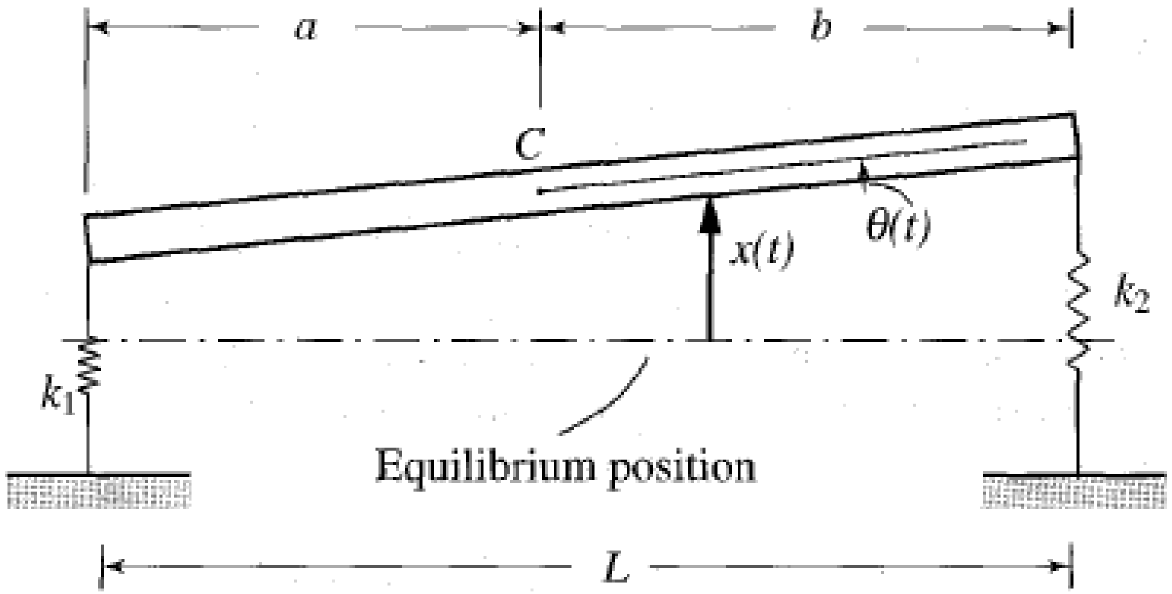
k_1 = 36,000 kg/m

k_2 = 40,000 kg/m

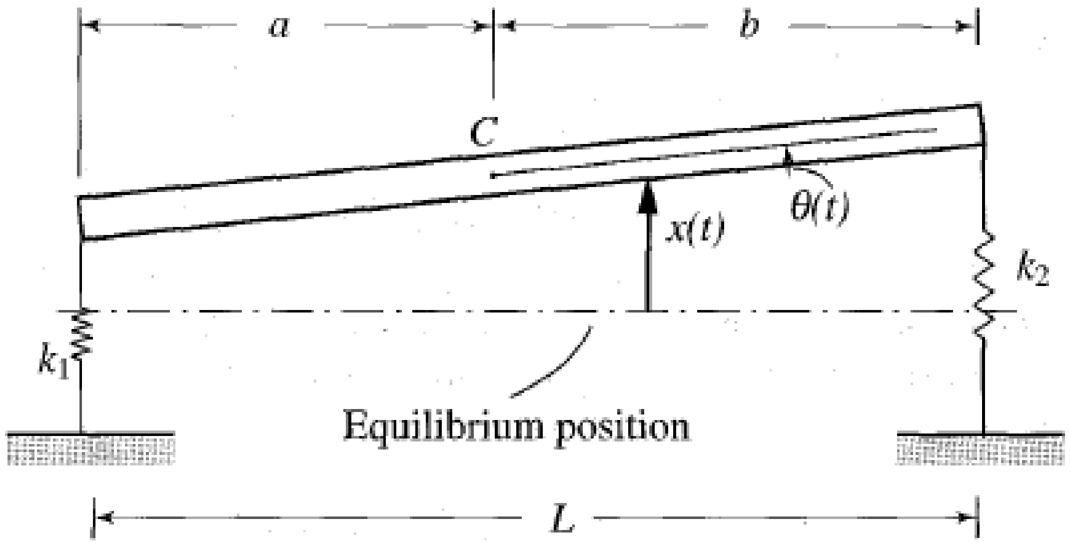
a = 1.3 m

b = 1.7 m

Calculate the natural modes of the system and write an expression for the response



Exercise 4



$$m\ddot{x}+\left(k_1+k_2\right)x-\left(k_1a-k_2b\right)\theta=0$$

$$I_C\ddot{\theta}-\left(k_1a-k_2b\right)x+\left(k_1a^2-k_2b^2\right)=0$$

Exercise 4

Equations of motion:

$$m\ddot{x}+\left(k_1+k_2\right)x-\left(k_1a-k_2b\right)\theta=0$$

$$I_C\ddot{\theta}-\left(k_1a-k_2b\right)x+\left(k_1a^2-k_2b^2\right)=0$$

$$\begin{bmatrix}1{,}500 & 0\\ 0 & 2{,}000\end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix}+\begin{bmatrix}76{,}000 & 21{,}200\\ 21{,}200 & 176{,}440\end{bmatrix}\begin{bmatrix}x\\ \theta\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Solution (harmonic):

$$x(t)=X\cos\left(\omega t-\phi\right),\quad \theta(t)=\Theta\cos\left(\omega t-\phi\right)$$

Exercise 4

Eigenvalue problem:

$$-\omega^2 \begin{bmatrix} 1,500 & 0 \\ 0 & 2,000 \end{bmatrix} \begin{bmatrix} X \\ \Theta \end{bmatrix} + \begin{bmatrix} 76,000 & 21,200 \\ 21,200 & 176,440 \end{bmatrix} \begin{bmatrix} X \\ \Theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Natural frequencies (solve the characteristic equation):

$$\det \begin{bmatrix} 76,000 - 1,500\omega^2 & 21,200 \\ 21,200 & 176,440 - 2,000\omega^2 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{matrix} \omega_1^2 \\ \omega_2^2 \end{matrix} = \begin{cases} 47\ (\text{rad/s})^2 \\ 91\ (\text{rad/s})^2 \end{cases}$$

Exercise 4

Natural modes:

$$-\omega^2\begin{bmatrix}1,500 & 0\\ 0 & 2,000\end{bmatrix}\begin{bmatrix}X\\ \Theta\end{bmatrix}+\begin{bmatrix}76,000 & 21,200\\ 21,200 & 176,440\end{bmatrix}\begin{bmatrix}X\\ \Theta\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\omega^2=\omega_1^2 \Rightarrow \frac{\Theta_1}{X_1}=-0,257\text{ rad/m}$$

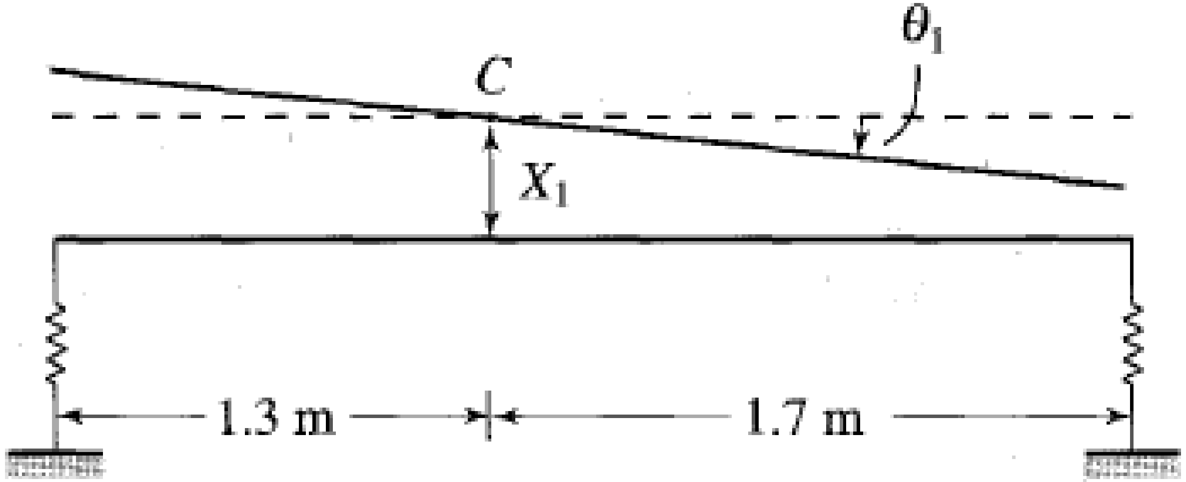
$$\omega^2=\omega_2^2 \Rightarrow \frac{\Theta_2}{X_2}=2,914\text{ rad/m}$$

$$X_1=1,\ X_2=1 \Rightarrow \mathrm{u}_1=\begin{bmatrix}X_1\\ \Theta_1\end{bmatrix}=\begin{bmatrix}1\\ -0.257\end{bmatrix},\ \mathrm{u}_2=\begin{bmatrix}X_2\\ \Theta_2\end{bmatrix}=\begin{bmatrix}1\\ 2.914\end{bmatrix}$$

Exercise 4

<u>Natural modes:</u>

$$\omega_1 = 6.83\text{rad/s}$$



$$\omega_2 = 9.18\text{rad/s}$$