System Dynamics and Vibrations

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Chapter 6: Two-degree-of-freedom systems
Part 1

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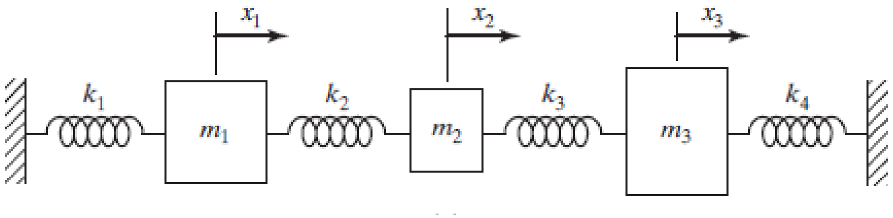
Contents

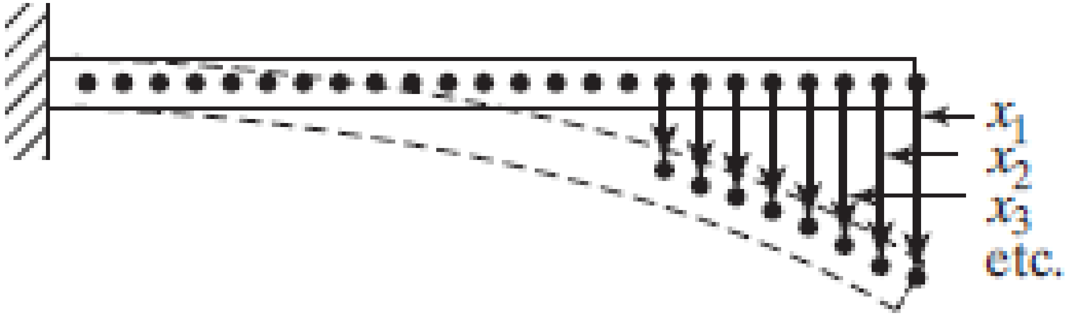
Introduction: Modeling of mechanical systems

- Physical systems are complex ➔ an exact description is not feasible
- In many cases is not even necessary
- Models represent only an approximation of actual physical systems
- Models retain all the essential dynamic characteristics of the system ➔ the behaviour predicted by the model must match the observed behaviour of the actual system

Introduction: Modeling of mechanical systems

- Models of vibrating mechanical systems
  - Lumped-parameter (discrete)
  - Distributed-parameter
  - Combination of both

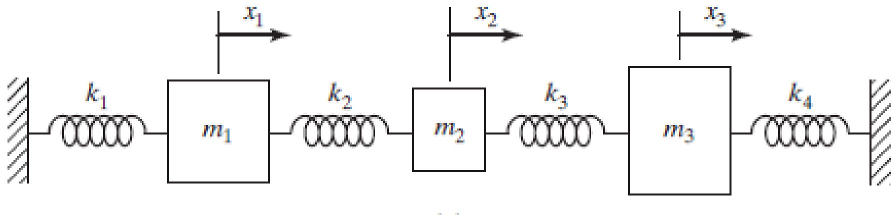




Introduction: Modeling of mechanical systems

- Models of vibrating mechanical systems
  - Lumped-parameter (discrete)

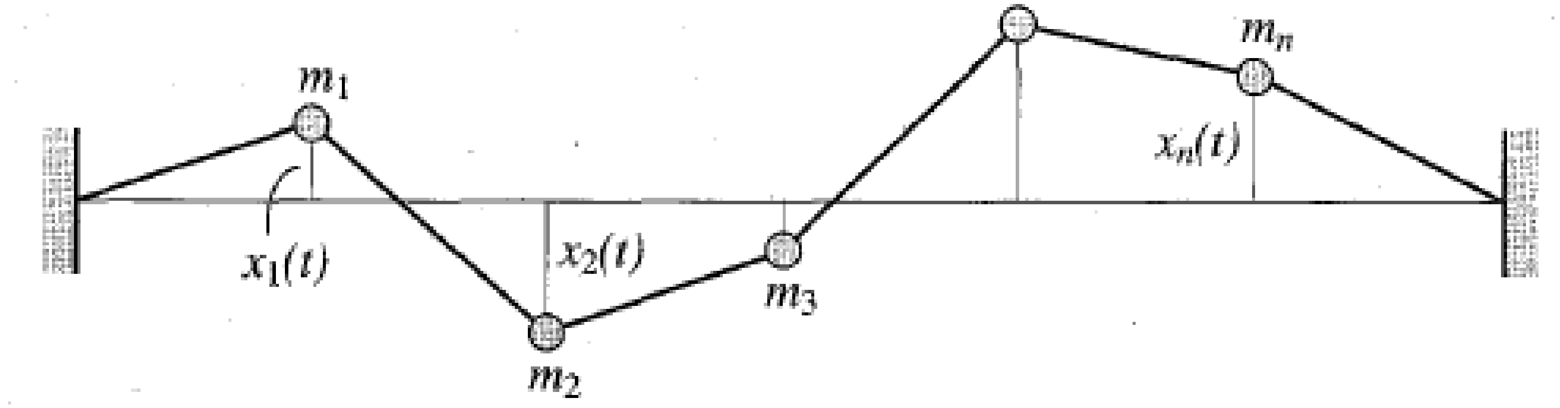
➔ <u>choice in the number of degrees of freedom</u>



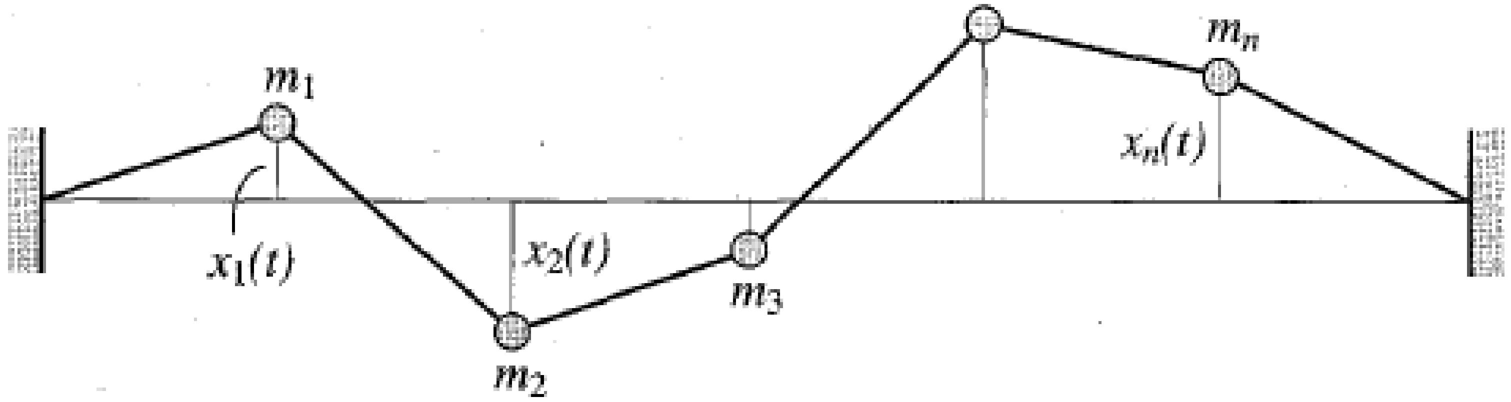
Introduction: multi-dregree-of-freedom systems

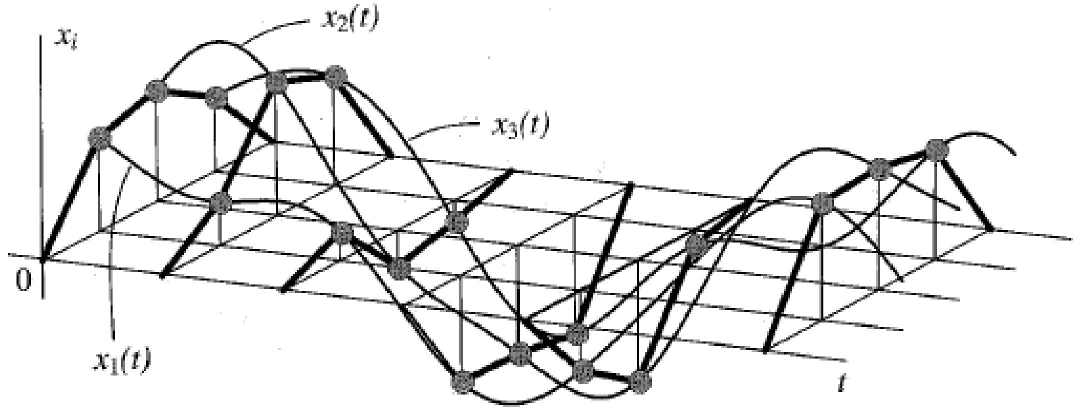
- For multi-degree-of-freedom systems natural vibration implies not only a certain natural frequency but also a certain <u>natural displacement configuration</u> assumed by the system masses during motion

Introduction: system configuration



Introduction: system configuration





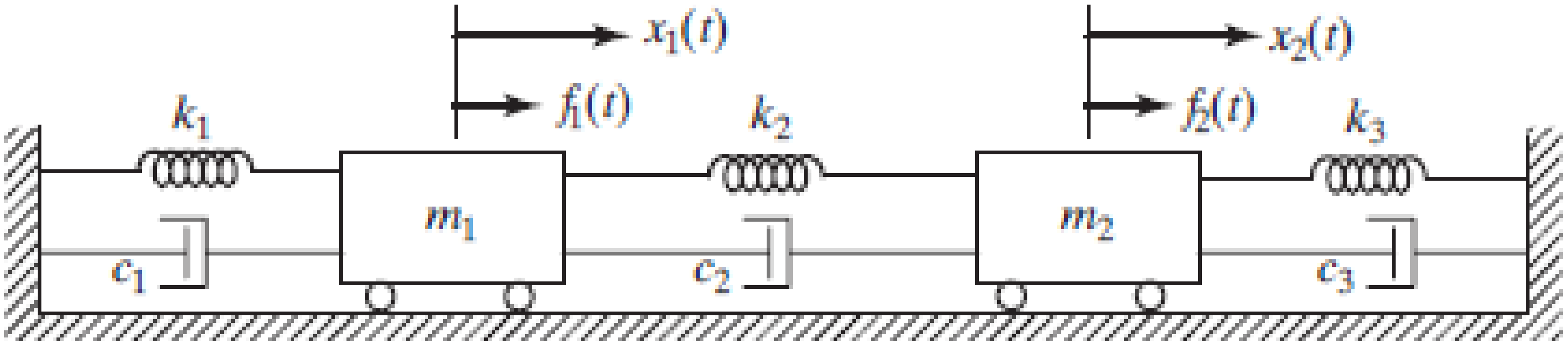
Introduction: multi-dregree-of-freedom systems

- For multi-degree-of-freedom systems natural vibration implies not only a certain natural frequency but also a certain natural displacement configuration assumed by the system masses during motion
- The system possesses <u>as many natural frequencies and natural configurations, known as natural modes of vibrations, as the number of degrees of freedom of the system</u>
- Depending on the initial excitation, the system can be made to vibrate in any of these modes independently
- The mathematical formulation for an n-degree-of-freedom system consists of n simultaneous ordinary differential equations of motion. Hence, the motion of one mass depends on the motion of the other n - 1 masses.
- For a proper choice of coordinates, known as principal coordinates, or natural coordinates, the n differential equations become independent of one another.

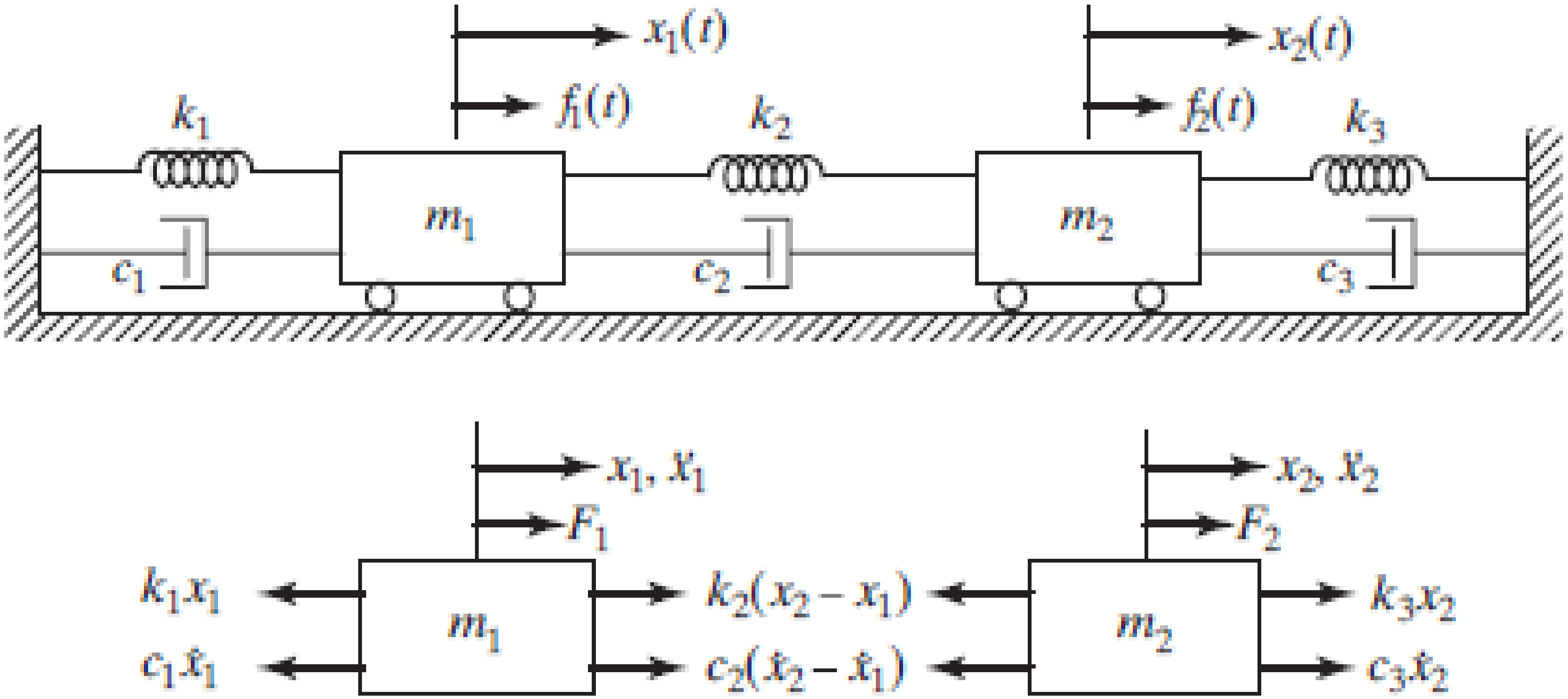
Contents

- Introduction
- <u>The equations of motion of two-degree-of-freedom systems.</u>
- Free vibration of undamped systems. Natural modes.
- Response to initial excitations.
- Orthogonality of modes. Natural coordinates.
- Systems admitting rigid-body motions.
- Systems with proportional damping.
- Response to harmonic excitations
- Introduction to multi-degree-of-freedom systems.

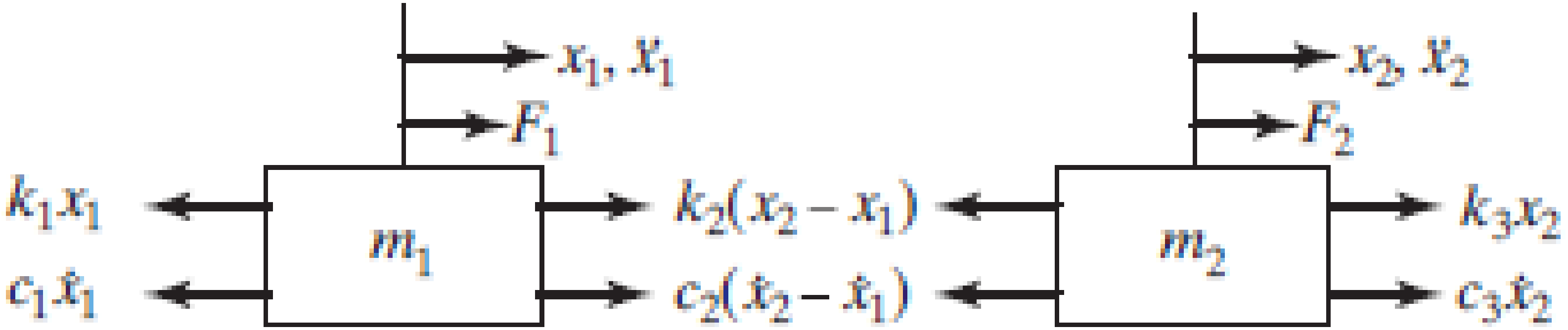
Equations of motion of 2-DOF systems



Equations of motion of 2-DOF systems



Equations of motion of 2-DOF systems



$$m_1\ddot{x}_1 + \left(c_1 + c_2\right)\dot{x}_1 - c_2\dot{x}_2 + \left(k_1 + k_2\right)x_1 - k_2 x_2 = f_1$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + \left(c_2 + c_3\right)\dot{x}_2 - k_2 x_1 + \left(k_2 + k_3\right)x_2 = f_2$$

Equations of motion of 2-DOF systems

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2 x_2 = f_1$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = f_2$$

system of two <u>coupled</u> second order differential equations

$$[m]\ddot{\vec{x}}(t) + [c]\dot{\vec{x}}(t) + [k]\vec{x}(t) = \vec{f}(t)$$

Equations of motion of 2-DOF systems

$$[m]\ddot{\vec{x}}(t)+[c]\dot{\vec{x}}(t)+[k]\vec{x}(t)=\vec{f}(t)$$

mass, damping and stiffness matrices:

$$[m]=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$[c]=\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}$$

$$[k]=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

displacementand force vectors:

$$\vec{x}(t)=\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

$$\vec{f}(t)=\begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

Equations of motion of 2-DOF systems

$$[m]\ddot{\vec{x}}(t)+[c]\dot{\vec{x}}(t)+[k]\vec{x}(t)=\vec{f}(t)$$

$$[m]=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$[c]=\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}$$

$$[k]=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

$$[m]^T=[m]$$

$$[c]^T=[c]$$

$$[k]^T=[k]$$

Equations of motion of 2-DOF systems

$$m_1\ddot{x}_1 + \left(c_1 + c_2\right)\dot{x}_1 - c_2\dot{x}_2 + \left(k_1 + k_2\right)x_1 - k_2x_2 = f_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + \left(c_2 + c_3\right)\dot{x}_2 - k_2x_1 + \left(k_2 + k_3\right)x_2 = f_2$$

initial conditions:

$$x_1(t=0) = x_1(0)$$

$$\dot{x}_1(t=0) = \dot{x}_1(0)$$

$$x_2(t=0) = x_2(0)$$

$$\dot{x}_2(t=0) = \dot{x}_2(0)$$

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- <u>Free vibration of undamped systems. Natural modes.</u>
- Response to initial excitations.
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- Systems with proportional damping.
- Response to harmonic excitations
- Introduction to multi-degree-of-freedom systems.

Free vibration of undamped systems

$$m_1\ddot{x}_1 + \cancel{\left(c_1 + c_2\right)\dot{x}_1} - \cancel{c_2\dot{x}_2} + \left(k_1 + k_2\right)x_1 - k_2 x_2 = \cancel{f_1}$$
$$m_2\ddot{x}_2 - \cancel{c_2\dot{x}_1} + \cancel{\left(c_2 + c_3\right)\dot{x}_2} - k_2 x_1 + \left(k_2 + k_3\right)x_2 = \cancel{f_2}$$

$$[m]\ddot{\vec{x}}(t) + \cancel{[c]\dot{\vec{x}}(t)} + [k]\vec{x}(t) = \cancel{\vec{f}(t)}$$

Free vibration of undamped systems

$$m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 - k_2 x_1 + \left(k_2 + k_3\right)x_2 = 0$$

$$\left[m\right]\ddot{\vec{x}}(t) + \left[k\right]\vec{x}(t) = 0$$

Free vibration of undamped systems

harmonic solutions:

$$\begin{cases} x_1(t) = X_1 \cos(\omega t + \phi) \\ x_2(t) = X_2 \cos(\omega t + \phi) \end{cases}$$

same freqeuncy and phase angle but different amplitudes

$$\left[\left\{-m_1\omega^2 + \left(k_1 + k_2\right)\right\} X_1 - k_2 X_2\right]\cos\left(\omega t + \phi\right) = 0$$

$$\left[-k_2 X_1 + \left\{-m_2\omega^2 + \left(k_2 + k_3\right)\right\} X_2\right]\cos\left(\omega t + \phi\right) = 0$$

Free vibration of undamped systems

$$x_1(t) = X_1 \cos(\omega t + \phi)$$
$$x_2(t) = X_2 \cos(\omega t + \phi)$$

$$\left[\left\{-m_1\omega^2 + \left(k_1 + k_2\right)\right\} X_1 - k_2 X_2\right] \cos\left(\omega t + \phi\right) = 0$$
$$\left[-k_2 X_1 + \left\{-m_2\omega^2 + \left(k_2 + k_3\right)\right\} X_2\right] \cos\left(\omega t + \phi\right) = 0$$

$$\Rightarrow$$

$$\left\{-m_1\omega^2 + \left(k_1 + k_2\right)\right\} X_1 - k_2 X_2 = 0$$
$$-k_2 X_1 + \left\{-m_2\omega^2 + \left(k_2 + k_3\right)\right\} X_2 = 0$$

Free vibration of undamped systems

$$x_1(t) = X_1 \cos(\omega t + \phi)$$
$$x_2(t) = X_2 \cos(\omega t + \phi)$$

$$\left[\left\{-m_1\omega^2 + \left(k_1 + k_2\right)\right\} X_1 - k_2 X_2\right] \cos\left(\omega t + \phi\right) = 0$$
$$\left[-k_2 X_1 + \left\{-m_2\omega^2 + \left(k_2 + k_3\right)\right\} X_2\right] \cos\left(\omega t + \phi\right) = 0$$

$$\left.\begin{aligned} &\left\{-m_1\omega^2 + \left(k_1 + k_2\right)\right\} X_1 - k_2 X_2 = 0 \\ &-k_2 X_1 + \left\{-m_2\omega^2 + \left(k_2 + k_3\right)\right\} X_2 = 0 \end{aligned}\right\}$$

trivial solution:

$$X_1 = X_2 = 0$$

Free vibration of undamped systems

non-trivial solution:

$$\det\begin{bmatrix} -m_1\omega^2 + \left(k_1 + k_2\right) & -k_2 \\ -k_2 & -m_2\omega^2 + \left(k_2 + k_3\right) \end{bmatrix} = 0$$

$$\Rightarrow \quad \left(m_1 m_2\right)\omega^4 - \left\{\left(k_1 + k_2\right)m_2 + \left(k_2 + k_3\right)m_1\right\}\omega^2 + \left\{\left(k_1 + k_2\right)\left(k_2 + k_3\right) - k_2^2\right\} = 0$$

characteristic equation

Free vibration of undamped systems

$$\left(m_1 m_2\right)\omega^4 - \left\{\left(k_1+k_2\right)m_2 + \left(k_2+k_3\right)m_1\right\}\omega^2 + \left\{\left(k_1+k_2\right)\left(k_2+k_3\right) - k_2^2\right\} = 0$$

solutions:

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{\left(k_1+k_2\right)m_2 + \left(k_2+k_3\right)m_1}{m_1 m_2}\right\} \mp \frac{1}{2}\left[\left\{\frac{\left(k_1+k_2\right)m_2 + \left(k_2+k_3\right)m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{\left(k_1+k_2\right)\left(k_2+k_3\right) - k_2^2}{m_1 m_2}\right\}\right]^{1/2}$$

It is possible for the system to have a non-trivial harmonic solution with <u>two natural frequencies</u>:

$$\omega_1 \text{ and } \omega_2$$

Free vibration of undamped systems. Natural modes

To determine the amplitudes X_1 and X_2 (for each of the two natural frequencies):

$$\left.\begin{aligned} r_1 &= \left\{X_2^{(1)}/X_1^{(1)}\right\} \\ r_2 &= \left\{X_2^{(2)}/X_1^{(2)}\right\} \end{aligned}\right\}$$

(because the equation is homogeneous)

$$\left\{-m_1\omega^2+\left(k_1+k_2\right)\right\}X_1-k_2X_2=0$$
$$-k_2X_1+\left\{-m_2\omega^2+\left(k_2+k_3\right)\right\}X_2=0$$

$$\Rightarrow \left\{\begin{aligned} r_1 &= \frac{X_2^{(1)}}{X_1^{(1)}}=\frac{-m_1\omega_1^2+\left(k_1+k_2\right)}{k_2}=\frac{k_2}{-m_2\omega_1^2+\left(k_2+k_3\right)} \\ r_2 &= \frac{X_2^{(2)}}{X_1^{(2)}}=\frac{-m_1\omega_2^2+\left(k_1+k_2\right)}{k_2}=\frac{k_2}{-m_2\omega_2^2+\left(k_2+k_3\right)} \end{aligned}\right.$$

For r_1, $\omega^2=\omega_1^2$

For r_2, $\omega^2=\omega_2^2$

Free vibration of undamped systems. Natural modes

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \\ r_1 X_1^{(1)} \end{Bmatrix}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \\ r_2 X_1^{(2)} \end{Bmatrix}$$

$$\left.\begin{array}{l} \vec{X}^{(1)} \\ \vec{X}^{(2)} \end{array}\right\}$$

modal vectors ➔ normal modes of vibration

Free vibration of undamped systems. Natural modes

Free vibration solution:
$$x_1(t) = X_1 \cos(\omega t + \phi)$$
$$x_2(t) = X_2 \cos(\omega t + \phi)$$

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos\left(\omega_1 t + \phi_1\right) \\ r_1 X_1^{(1)} \cos\left(\omega_1 t + \phi_1\right) \end{Bmatrix} = \text{first mode}$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos\left(\omega_2 t + \phi_2\right) \\ r_2 X_1^{(2)} \cos\left(\omega_2 t + \phi_2\right) \end{Bmatrix} = \text{second mode}$$

Constants $X_1^{(1)}, X_1^{(2)}, \phi_1, \phi_2$ are determined by the initial conditions

Contents

Response to initial excitations

The system can be made to vibrate in its i_{th} normal mode (i = 1, 2) by subjecting it to the specific initial conditions:

$$x_1(t=0) = X_1^{(i)} = \text{some constant}, \qquad \dot{x}_1(t=0) = 0,$$

$$x_2(t=0) = r_i X_1^{(i)}, \qquad \dot{x}_2(t=0) = 0$$

Response to initial excitations

The system can be made to vibrate in its i_{th} normal mode (i = 1, 2) by subjecting it to the specific initial conditions:

$$x_1(t=0) = X_1^{(i)} = \text{some constant}, \qquad \dot{x}_1(t=0) = 0,$$

$$x_2(t=0) = r_i X_1^{(i)}, \qquad \dot{x}_2(t=0) = 0$$

For any other general initial conditions, both modes will be excited:

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t)$$

c_1 and c_2 are constants
we can choose c_1 = c_2 = 1 since the solution already includes the unknown constans $X_1^{(1)}$ and $X_1^{(2)}$

Response to initial excitations

$$\vec{x}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$$

$$\Rightarrow \quad \begin{aligned} x_1(t) &= x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)}\cos\left(\omega_1 t + \phi_1\right) + X_1^{(2)}\cos\left(\omega_2 t + \phi_2\right) \\ x_2(t) &= x_2^{(1)}(t) + x_2^{(2)}(t) = r_1 X_1^{(1)}\cos\left(\omega_1 t + \phi_1\right) + r_2 X_1^{(2)}\cos\left(\omega_2 t + \phi_2\right) \end{aligned}$$

Constants $X_1^{(1)}, X_1^{(2)}, \phi_1, \phi_2$ are determined by the initial conditions

$$\left\{ \begin{aligned} &x_1\left(t=0\right) = x_1(0), \qquad \dot{x}_1\left(t=0\right) = \dot{x}_1\left(0\right), \\ &x_2\left(t=0\right) = x_2(0), \qquad \dot{x}_2\left(t=0\right) = \dot{x}_2\left(0\right) \end{aligned} \right.$$

Response to initial excitations

Introducing the initial conditions in the equations:

$$\left\{\begin{aligned} x_1(t) &= X_1^{(1)}\cos\left(\omega_1 t+\phi_1\right)+X_1^{(2)}\cos\left(\omega_2 t+\phi_2\right) \\ x_2(t) &= r_1 X_1^{(1)}\cos\left(\omega_1 t+\phi_1\right)+r_2 X_1^{(2)}\cos\left(\omega_2 t+\phi_2\right) \end{aligned}\right.$$

$$\left\{\begin{aligned} x_1(0) &= X_1^{(1)}\cos\phi_1+X_1^{(2)}\cos\phi_2 \\ \dot{x}_1(0) &= -\omega_1 X_1^{(1)}\sin\phi_1-\omega_2 X_1^{(2)}\sin\phi_2 \\ x_2(0) &= r_1 X_1^{(1)}\cos\phi_1+r_2 X_1^{(2)}\cos\phi_2 \\ \dot{x}_1(0) &= -\omega_1 r_1 X_1^{(1)}\sin\phi_1-\omega_2 r_2 X_1^{(2)}\sin\phi_2 \end{aligned}\right.$$

four algebraic equations with four unknowns

Response to initial excitations

The four unknowns:
$$
\begin{cases}
X_1^{(1)}\cos\phi_1 = \left\{\dfrac{r_2 x_1(0) - x_2(0)}{r_2 - r_1}\right\}, & X_1^{(2)}\cos\phi_2 = \left\{\dfrac{-r_1 x_1(0) + x_2(0)}{r_2 - r_1}\right\} \\[2ex]
X_1^{(1)}\sin\phi_1 = \left\{\dfrac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1\left(r_2 - r_1\right)}\right\}, & X_1^{(2)}\sin\phi_2 = \left\{\dfrac{r_1 \dot{x}_1(0) - \dot{x}_2(0)}{\omega_2\left(r_2 - r_1\right)}\right\}
\end{cases}
$$

Response to initial excitations

The four unknowns:

$$\begin{cases} X_1^{(1)}\cos\phi_1 = \left\{\dfrac{r_2 x_1(0) - x_2(0)}{r_2 - r_1}\right\}, & X_1^{(2)}\cos\phi_2 = \left\{\dfrac{-r_1 x_1(0) + x_2(0)}{r_2 - r_1}\right\} \\ X_1^{(1)}\sin\phi_1 = \left\{\dfrac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1 (r_2 - r_1)}\right\}, & X_1^{(2)}\sin\phi_2 = \left\{\dfrac{r_1 \dot{x}_1(0) - \dot{x}_2(0)}{\omega_2 (r_2 - r_1)}\right\} \end{cases}$$

The desired solution:

$$\begin{cases} X_1^{(1)} = \left[\left\{X_1^{(1)}\cos\phi_1\right\}^2 + \left\{X_1^{(1)}\sin\phi_1\right\}^2\right]^{1/2} = \dfrac{1}{(r_2 - r_1)}\left[\left\{r_2 x_1(0) - x_2(0)\right\}^2 + \dfrac{\left\{-r_2 \dot{x}_1(0) + \dot{x}_2(0)\right\}^2}{\omega_1^2}\right]^{1/2} \\ X_1^{(2)} = \left[\left\{X_1^{(2)}\cos\phi_2\right\}^2 + \left\{X_1^{(2)}\sin\phi_2\right\}^2\right]^{1/2} = \dfrac{1}{(r_2 - r_1)}\left[\left\{+r_1 x_1(0) - x_2(0)\right\}^2 + \dfrac{\left\{r_1 \dot{x}_1(0) - \dot{x}_2(0)\right\}^2}{\omega_2^2}\right]^{1/2} \\ \phi_1 = \tan^{-1}\left\{\dfrac{X_1^{(1)}\sin\phi_1}{X_1^{(1)}\cos\phi_1}\right\} = \tan^{-1}\left\{\dfrac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1\left[r_2 x_1(0) - x_2(0)\right]}\right\} \\ \phi_2 = \tan^{-1}\left\{\dfrac{X_1^{(2)}\sin\phi_2}{X_1^{(2)}\cos\phi_2}\right\} = \tan^{-1}\left\{\dfrac{r_1 \dot{x}_1(0) - \dot{x}_2(0)}{\omega_2\left[-r_1 x_1(0) + x_2(0)\right]}\right\} \end{cases}$$