System Dynamics and Vibrations

Prof. Gustavo Alonso

Chapter 3: Single degree-of-freedom systems
Part 4

School of General Engineering
Beihang University (BUAA)

Contents

Response to nonperiodic excitations

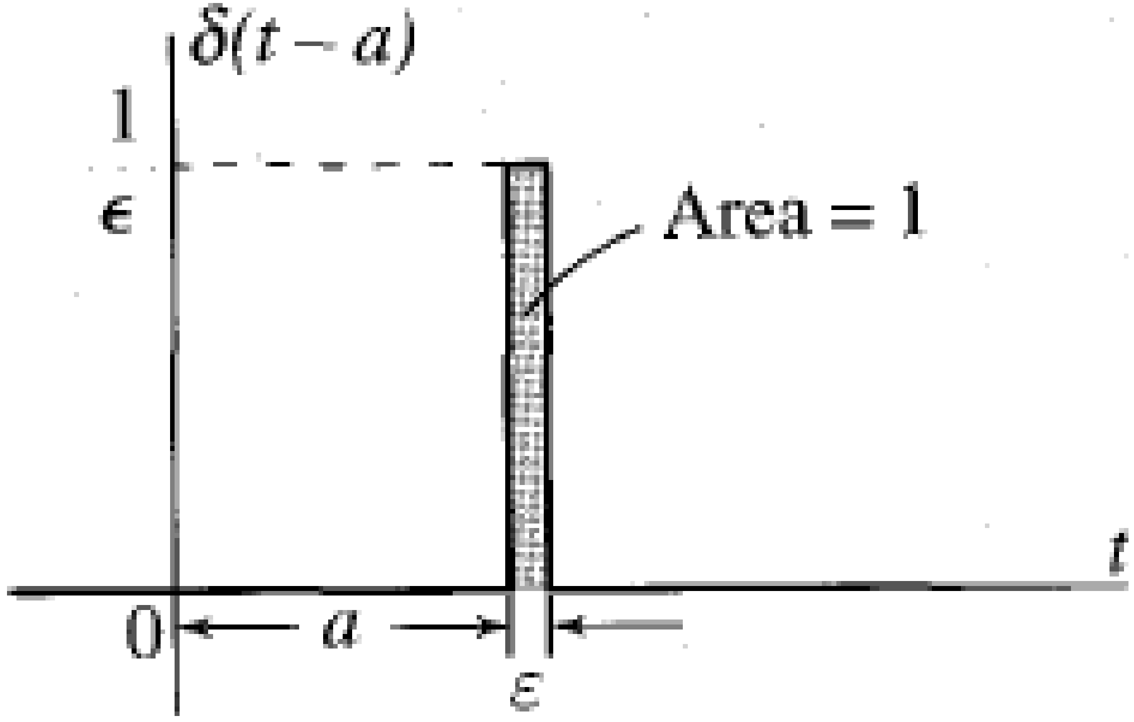
- Nonperiodic excitations are often referred to as <u>transient</u>
- The term transient is to be interpreted in the sense that nonperiodic excitations are <u>not steady state</u>.
- By virtue of the superposition principle, the response of linear systems to nonperiodic excitations can be combined with the response to initial excitations to obtain the total response

The unit impulse. Impulse response.

- Unit impulse (Dirac delta function):

$$\delta\left(t-a\right)=0 \quad \text{for} \quad t \neq a$$

$$\int_{-\infty}^{\infty} \delta\left(t-a\right) dt = 1$$



The unit impulse. Impulse response.

- **Impulsive force**:

  ➔ Very large force acting over a very short time interval at $t = a$

$$F(t) = \hat{F}\delta\left(t - a\right)$$

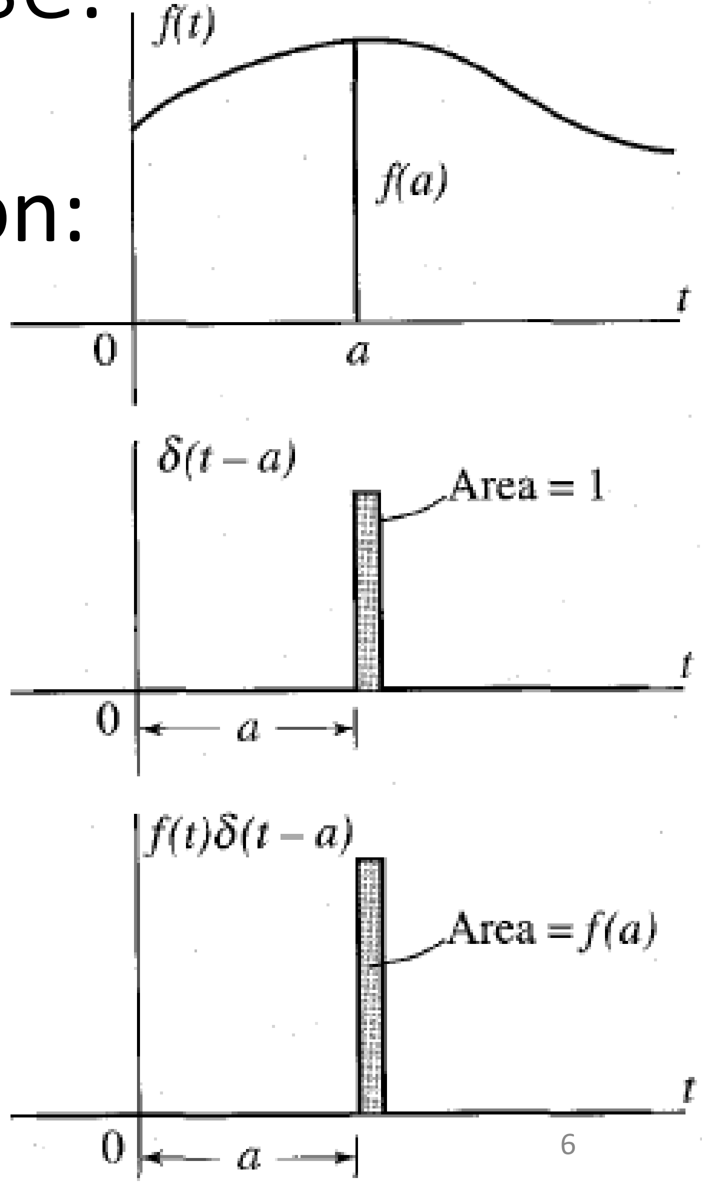
$\hat{F}$ is the magnitude of the impulse [Ns]

The unit impulse. Impulse response.

- Sampling property of Dirac delta function:

$$\int_{-\infty}^{\infty} f(t)\delta\left(t-a\right)dt \cong f(a)\int_{-\infty}^{\infty}\delta\left(t-a\right)dt = f(a)$$

➔simple way of evaluating integrals involving delta function

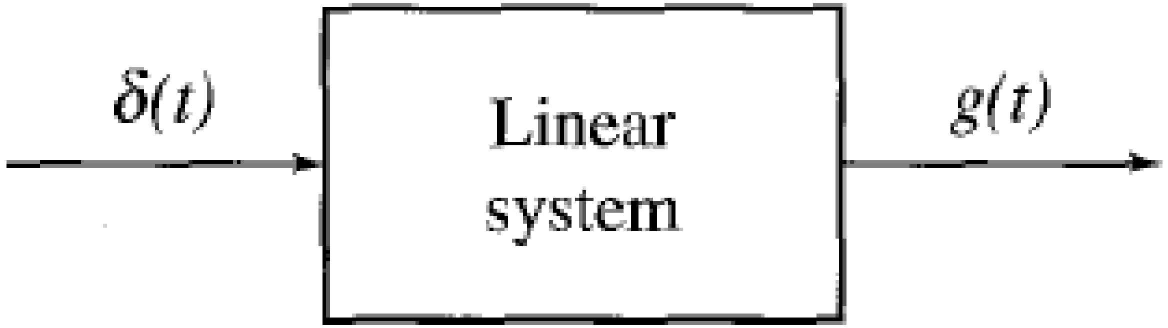


The unit impulse. Impulse response.

- **<u>Impulsive response *g(t)*</u>**:

  ➔ response to delta function $\delta(t)$

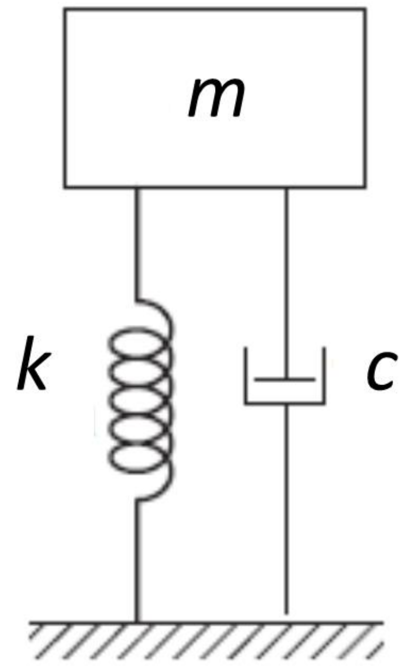
  - *applied at t=0*
  - *initial excitations equal to zero*



The unit impulse. Impulse response.

- **Impulsive response of a SDOF**:

$$m\ddot{g}(t) + c\dot{g}(t) + kg(t) = \delta(t)$$
$$g(0) = 0$$
$$\dot{g}(0) = 0$$



The unit impulse. Impulse response.

$$\int_0^{\varepsilon}\left(m\ddot{g}(t)+c\dot{g}(t)+kg(t)\right)dt=\int_0^{\varepsilon}\delta(t)dt=1$$

$\varepsilon<<1$ is the duration of the impulse

$$\lim_{\varepsilon\to 0}\int_0^{\varepsilon} m\ddot{g}(t)dt=\lim_{\varepsilon\to 0} m\dot{g}(t)\Big|_0^{\varepsilon}=\lim_{\varepsilon\to 0} m\left[\dot{g}(\varepsilon)-\dot{g}(0)\right]=m\dot{g}(0+)$$

$$\lim_{\varepsilon\to 0}\int_0^{\varepsilon} c\dot{g}(t)dt=\lim_{\varepsilon\to 0} cg(t)\Big|_0^{\varepsilon}=\lim_{\varepsilon\to 0} c\left[g(\varepsilon)-g(0)\right]=0$$

$$\lim_{\varepsilon\to 0}\int_0^{\varepsilon} kg(t)dt=\lim_{\varepsilon\to 0} kg(0)t\Big|_0^{\varepsilon}=\lim_{\varepsilon\to 0} kg(0)\varepsilon=0$$

$\dot{g}(0+)$

is the slope of the impulse response curve at the termination of the impulse

The unit impulse. Impulse response.

$$\int_0^{\varepsilon}\left(m\ddot{g}(t)+c\dot{g}(t)+kg(t)\right)dt=\int_0^{\varepsilon}\delta(t)dt=1$$

$$\lim_{\varepsilon\to 0}\int_0^{\varepsilon}\left(m\ddot{g}(t)+c\dot{g}(t)+kg(t)\right)dt=m\dot{g}(0+)=1$$

➔ the effect of a unit impulse at t=0 is to produce an equivalent initial velocity:

$$\dot{g}(0+)=\frac{1}{m}$$

The unit impulse. Impulse response.

- **<u>Impulsive response of a SDOF</u>**:

➔ homogeneous system subjected to an equivalent initial velocity

$$x(t) = Ce^{-\varsigma\omega_n t}\cos\left(\omega_d t - \phi\right)$$

$$C = \sqrt{x_0^2 + \left(\frac{\varsigma\omega_n x_0 + v_0}{\omega_d}\right)^2}$$

$$\omega_d = \sqrt{1-\varsigma^2}\,\omega_n$$

frequency of the damped oscillation

$$\phi = \tan^{-1}\frac{\varsigma\omega_n x_0 + v_0}{\omega_d x_0}$$

The unit impulse. Impulse response.

- **<u>Impulsive response of a SDOF</u>**:

➔ homogeneous system subjected to an equivalent initial velocity

$$C = \sqrt{x_0^2 + \left(\frac{\varsigma \omega_n x_0 + v_0}{\omega_d} \right)^2} = \frac{1}{m\omega_d}$$

$$\phi = \tan^{-1} \frac{\varsigma \omega_n x_0 + v_0}{\omega_d x_0} = \tan^{-1} \infty = \pi/2$$

The unit impulse. Impulse response.

- **Impulsive response of a SDOF**:

  ➔ homogeneous system subjected to an equivalent initial velocity

$$C=\sqrt{x_0^2+\left(\frac{\varsigma\omega_n x_0+v_0}{\omega_d}\right)^2}=\frac{1}{m\omega_d}$$

$$\phi=\tan^{-1}\frac{\varsigma\omega_n x_0+v_0}{\omega_d x_0}=\tan^{-1}\infty=\pi/2$$

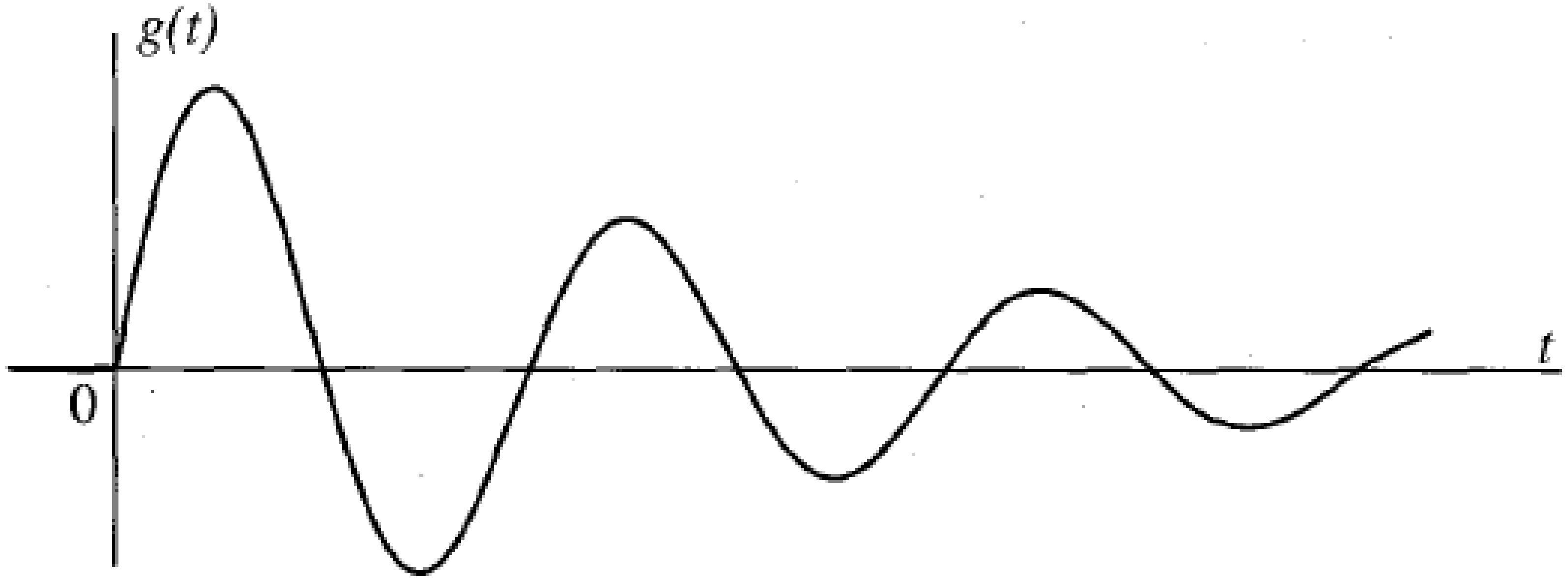
$$g(t)=Ce^{-\varsigma\omega_n t}\cos\left(\omega_d t-\phi\right)$$

$$g(t)=\begin{cases}\dfrac{1}{m\omega_d}e^{-\varsigma\omega_n t}\sin\omega_d t & t>0\\ 0 & t<0\end{cases}$$

The unit impulse. Impulse response.

- **<u>Impulsive response of a SDOF</u>**:

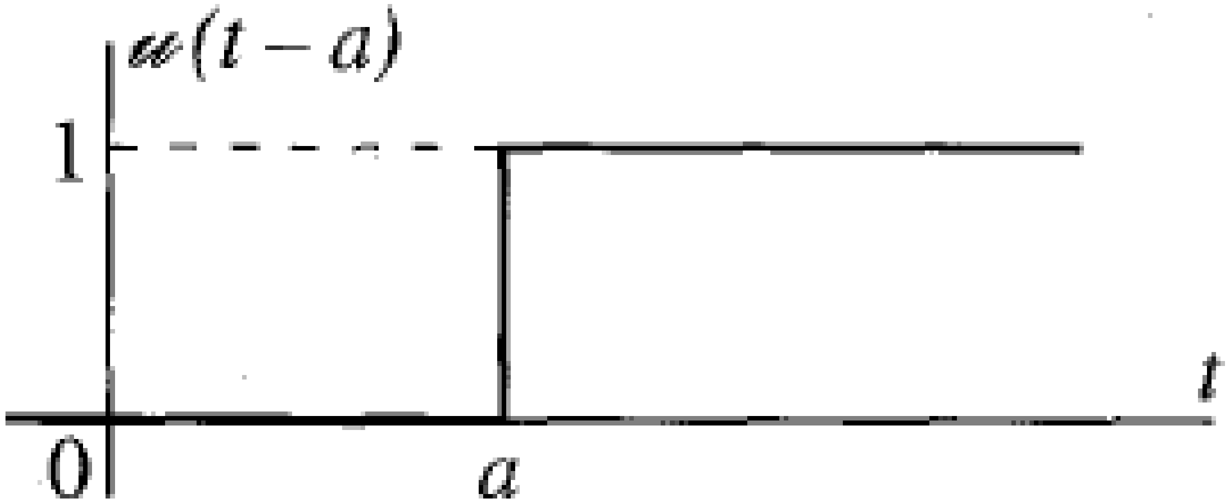
➔ underdamped system $\zeta < 1$ $$g(t) = Ce^{-\varsigma\omega_n t}\cos\left(\omega_d t - \phi\right)$$



The unit step function. Step response.

- The unit step function:

$$u(t-a)=\begin{cases}0 & \text{for} \quad t<a \\ 1 & \text{for} \quad t>a\end{cases}$$



The unit step function. Step response.

- The unit step function:

$$u(t-a)=\int_{-\infty}^{t}\delta(\tau-a)d\tau$$

$$\delta(\tau-a)=\frac{du(t-a)}{dt}$$

The unit step function. Step response.

- The step response *s(t)*:

➔response of a system to a unit step function applied at $t = 0$, with the initial conditions being equal to zero

- It can be proved that:

$$\mathrm{s}(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

The unit impulse. Impulse response.

- **<u>Step response of a SDOF</u>**:

$$\mathrm{s}(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

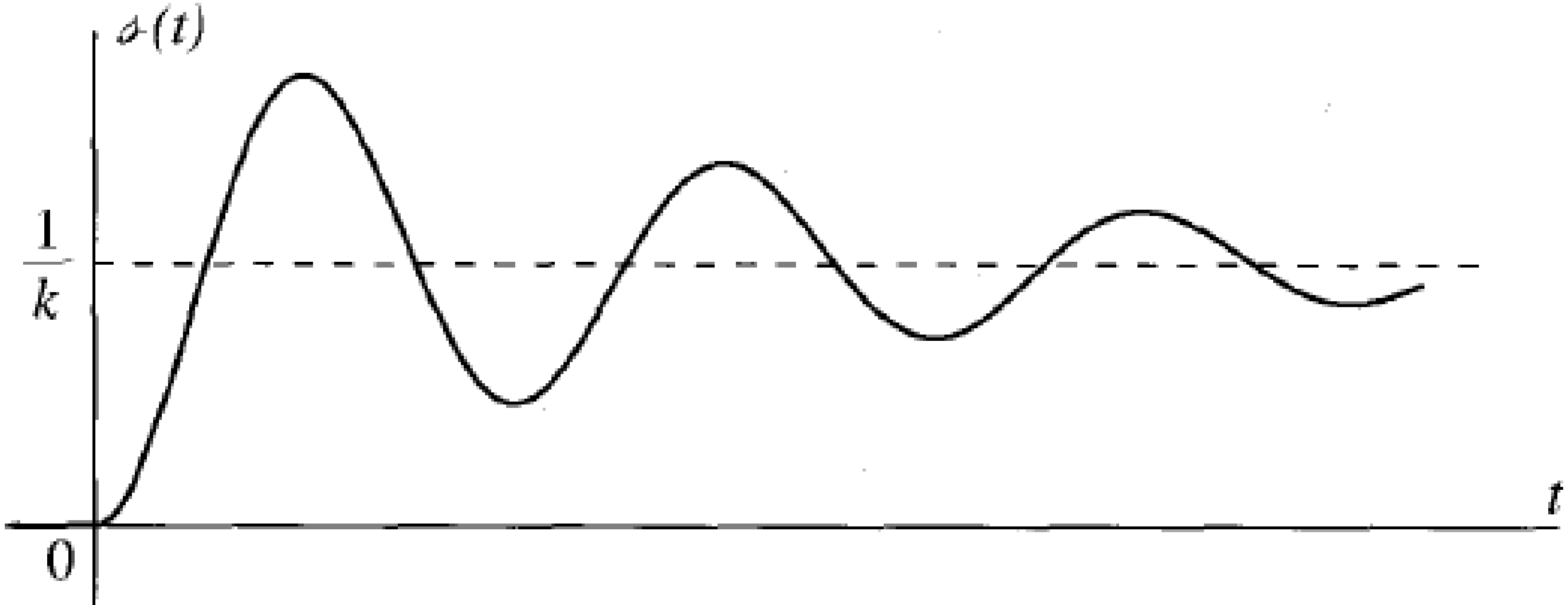
$$g(t) = \frac{1}{m\omega_d} e^{-\varsigma\omega_n t} \sin \omega_d t u(t)$$

$$\Rightarrow \quad \mathrm{s}(t) = \frac{1}{k}\left[1 - e^{-\varsigma\omega_n t}\left(\cos \omega_d t + \frac{\xi\omega_n}{\omega_d} \sin \omega_d t\right)\right]u(t)$$

The unit impulse. Impulse response.

- **<u>Step response of a SDOF</u>**:

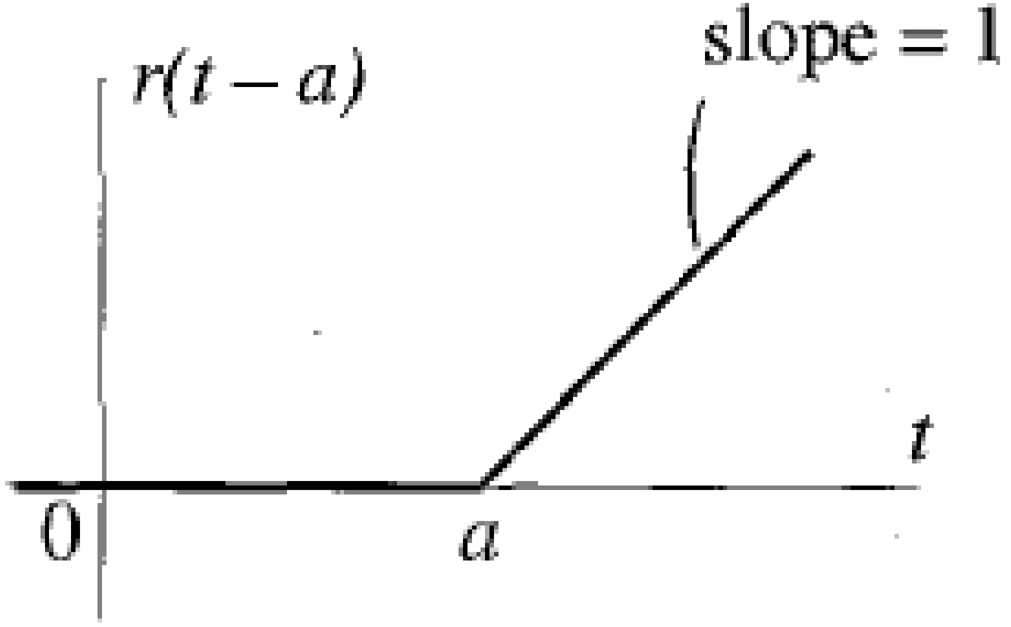
$$s(t)=\frac{1}{k}\left[1-e^{-\varsigma\omega_n t}\left(\cos\omega_d t+\frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right)\right]u(t)$$



The unit ramp function. Ramp response.

- The unit ramp function:

$$r(t-a) = (t-a)u(t-a)$$

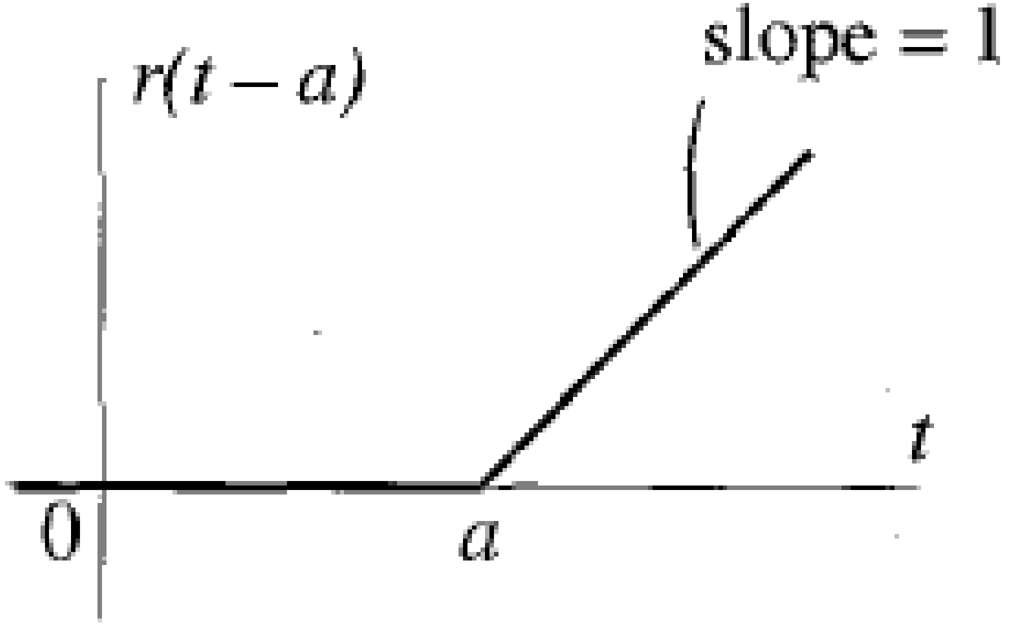


The unit ramp function. Ramp response.

- The unit ramp function:

$$r(t-a)=\int_{-\infty}^{t} u(\tau-a)d\tau$$

$$u(\tau-a)=\frac{dr(t-a)}{dt}$$



The unit ramp function. Ramp response.

- The ramp response: $r(t)$

➔ response of a system to a unit ramp function beginning at t = 0, with the initial conditions being equal to zero

$$r(t) = \int_{-\infty}^{t} s(\tau)d\tau$$