System Dynamics and Vibrations

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Chapter 6: Two-degree-of-freedom systems
Exercises - 4

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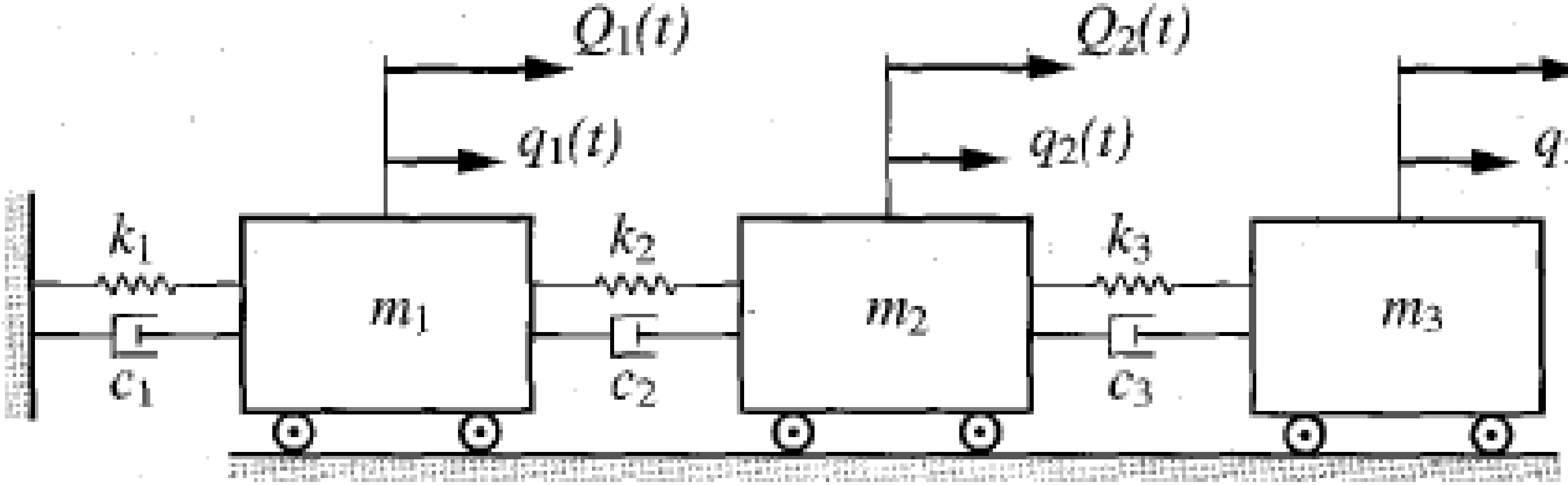
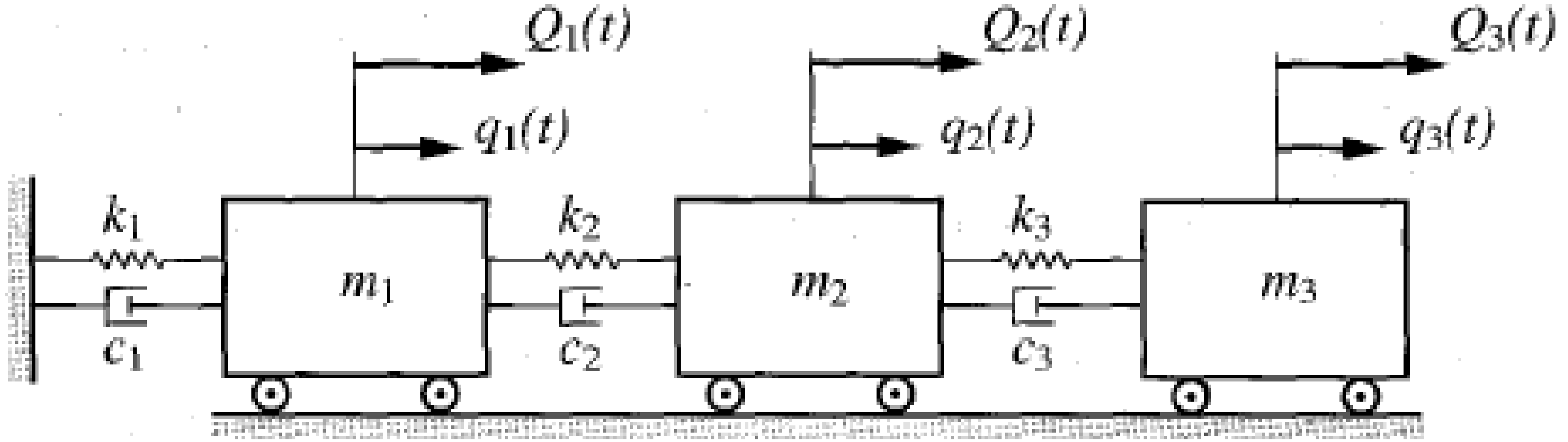
Exercise 7

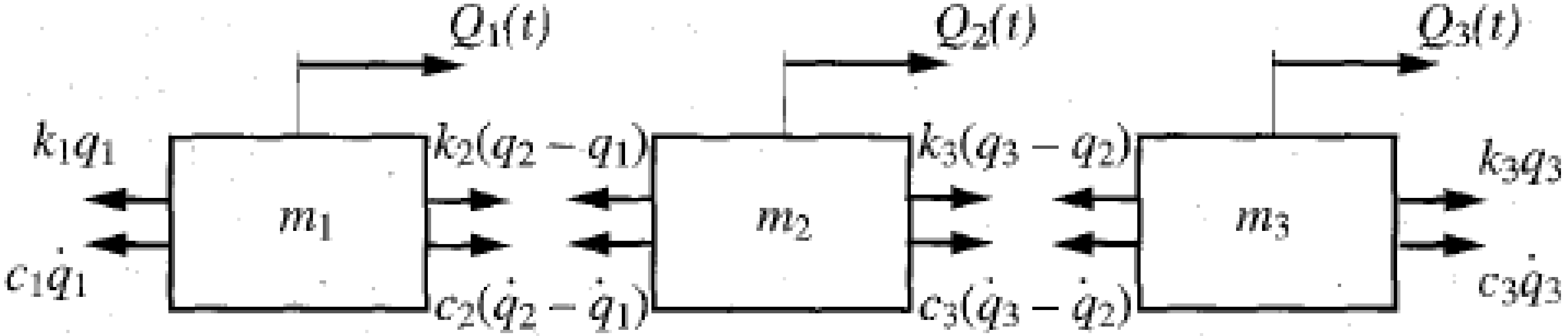
Consider the three-degree-of-freedom system of the figure and derive the system differential equations of motion by Newton's second law. The springs exhibit linear behavior and the dampers are viscous.



Exercise 7

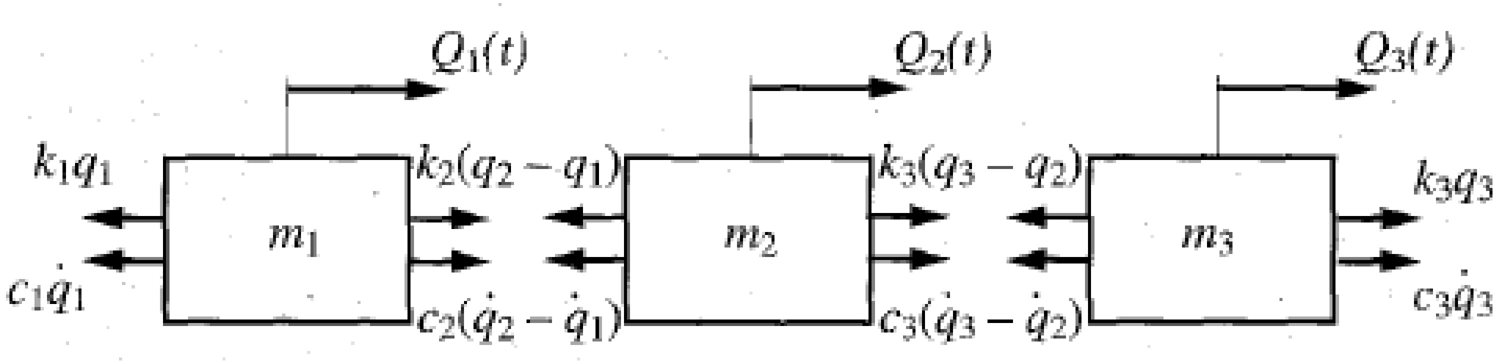


$Q_1(t)$
$Q_2(t)$
$Q_3(t)$
$q_1(t)$
$q_2(t)$
$q_3(t)$
k_1
k_2
k_3
m_1
m_2
m_3
c_1
c_2
c_3



$Q_1(t)$
$Q_2(t)$
$Q_3(t)$
$k_1 q_1$
$c_1 \dot{q}_1$
m_1
$k_2(q_2 - q_1)$
$c_2(\dot{q}_2 - \dot{q}_1)$
m_2
$k_3(q_3 - q_2)$
$c_3(\dot{q}_3 - \dot{q}_2)$
m_3
$k_3 q_3$
$c_3 \dot{q}_3$

Exercise 7



$$Q_1 + c_2\left(\dot{q}_2 - \dot{q}_1\right) + k_2\left(q_2 - q_1\right) - c_1\dot{q}_1 - k_1 q_1 = m_1\ddot{q}_1$$
$$Q_2 + c_3\left(\dot{q}_3 - \dot{q}_2\right) + k_3\left(q_3 - q_2\right) - c_2\left(\dot{q}_2 - \dot{q}_1\right) - k_2\left(q_2 - q_1\right) = m_2\ddot{q}_2$$
$$Q_3 - c_3\left(\dot{q}_3 - \dot{q}_2\right) - k_3\left(q_3 - q_2\right) = m_3\ddot{q}_3$$

$$m_1\ddot{q}_1 + \left(c_1 + c_2\right)\dot{q}_1 - c_2\dot{q}_2 + \left(k_1 + k_2\right)q_1 - k_2 q_2 = Q_1$$
$$m_2\ddot{q}_2 - c_2\dot{q}_1 + \left(c_2 + c_3\right)\dot{q}_2 - k_2 q_1 + \left(k_2 + k_3\right)q_2 - k_3 q_3 = Q_2$$
$$m_3\ddot{q}_3 - c_3\dot{q}_2 + c_3\dot{q}_3 - k_3 q_2 + k_3 q_3 = Q_3$$

Exercise 7

$$m_1\ddot{q}_1 + (c_1 + c_2)\dot{q}_1 - c_2\dot{q}_2 + (k_1 + k_2)q_1 - k_2 q_2 = Q_1$$

$$m_2\ddot{q}_2 - c_2\dot{q}_1 + (c_2 + c_3)\dot{q}_2 - k_2 q_1 + (k_2 + k_3)q_2 - k_3 q_3 = Q_2$$

$$m_3\ddot{q}_3 - c_3\dot{q}_2 + c_3\dot{q}_3 - k_3 q_2 + k_3 q_3 = Q_3$$

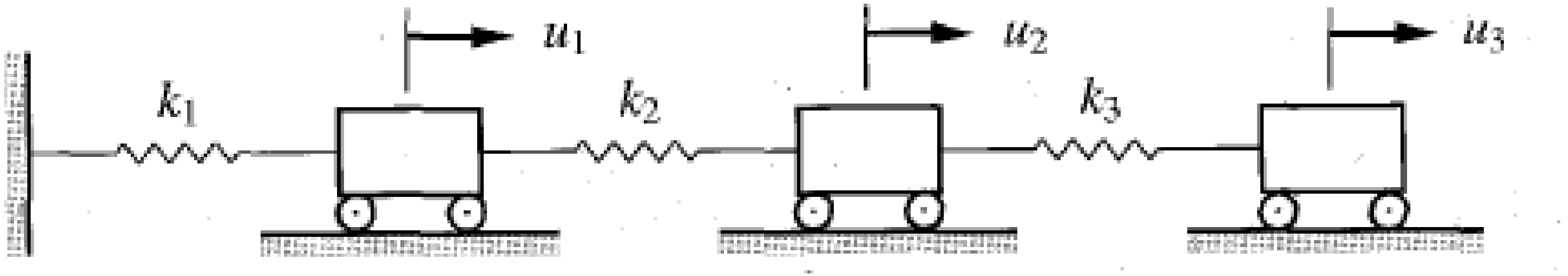
$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

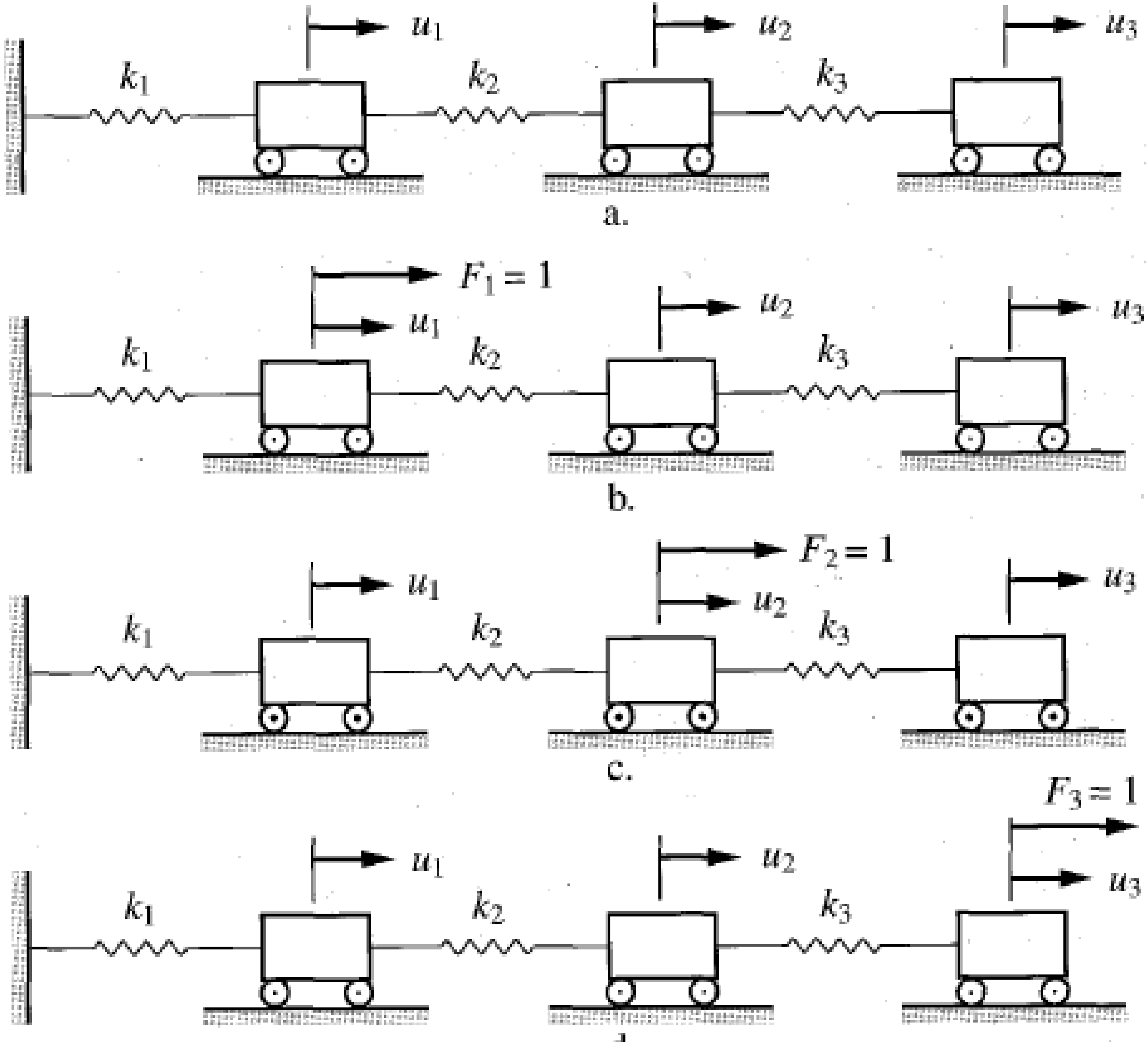
Exercise 8

Consider the three-degree-of-freedom system of the figure and use the definitions to calculate the flexibility and stiffness matrices.



Exercise 8

To calculate the flexibility influence coefficients:



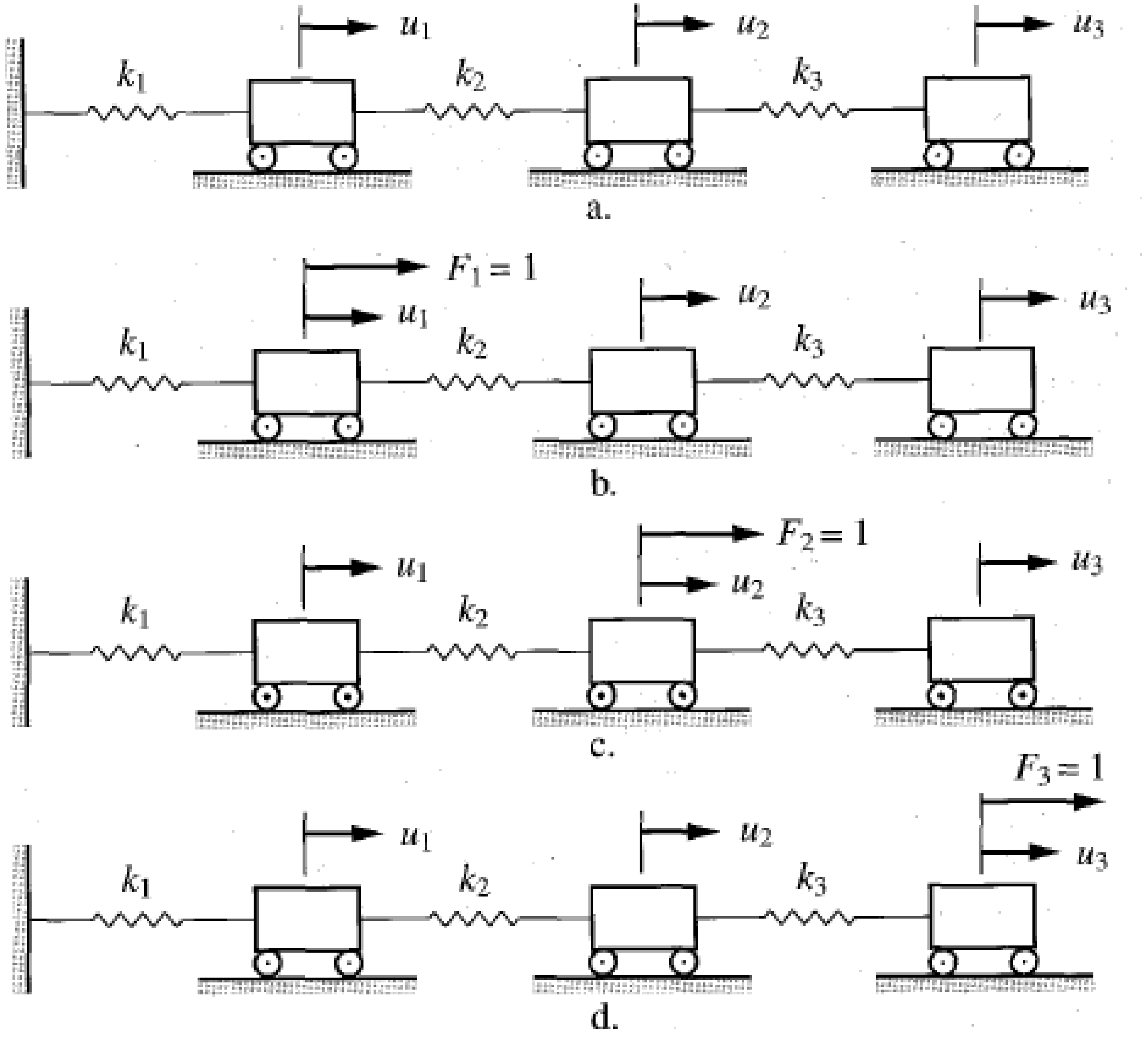
Exercise 8

$$a_{11} = u_1 = \frac{1}{k_1}, \quad a_{21} = u_2 = u_1 = \frac{1}{k_1}, \quad a_{31} = u_3 = u_2 = u_1 = \frac{1}{k_1}$$

$$a_{12} = u_1 = \frac{1}{k_1}, \quad a_{22} = u_2 = \frac{1}{k_1} + \frac{1}{k_2}, \quad a_{32} = u_3 = u_2 = \frac{1}{k_1} + \frac{1}{k_2}$$

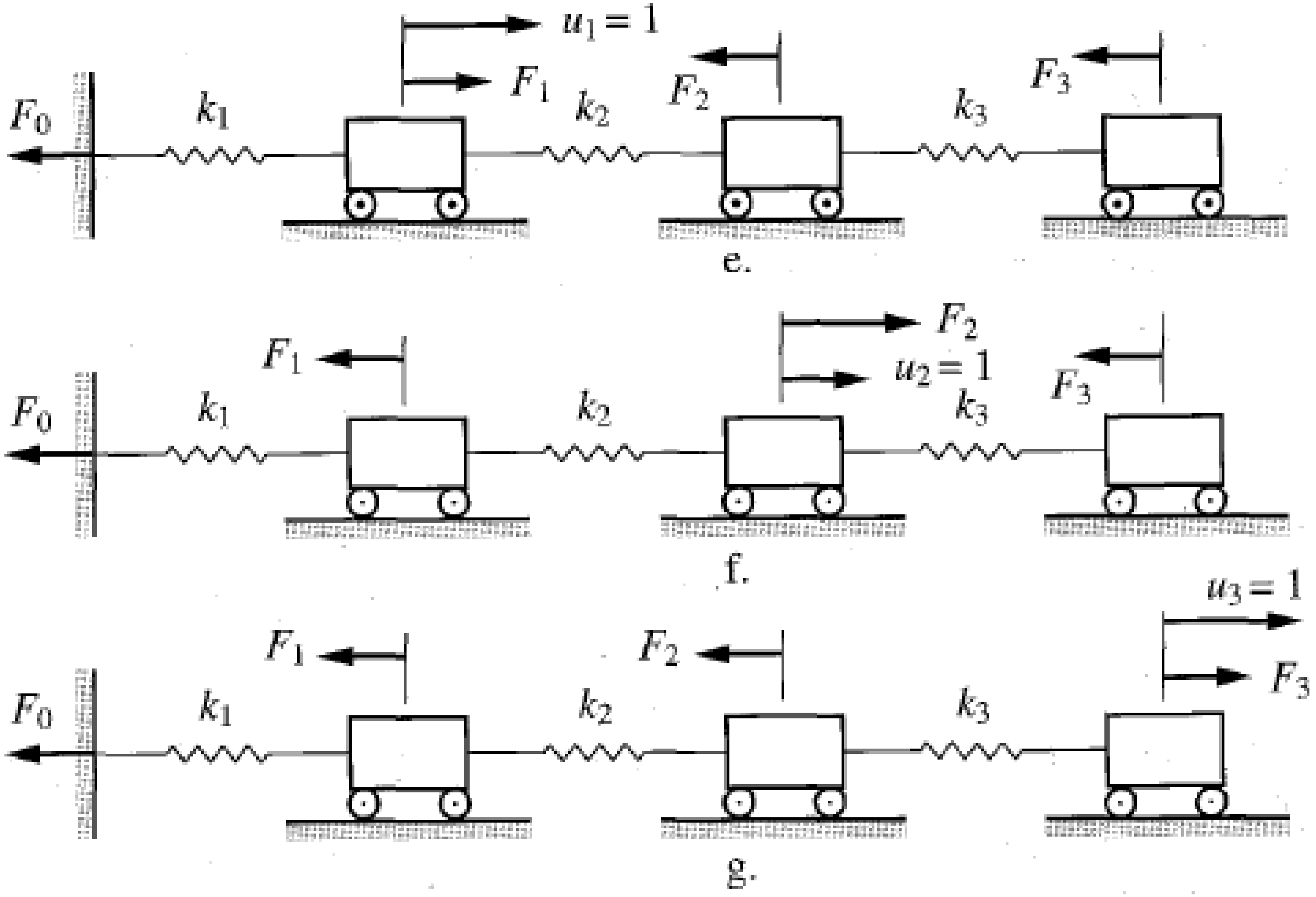
$$a_{13} = u_1 = \frac{1}{k_1}, \quad a_{23} = u_2 = \frac{1}{k_1} + \frac{1}{k_2}, \quad a_{33} = u_3 = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

$$A = \begin{bmatrix} \frac{1}{k_1} & \frac{1}{k_1} & \frac{1}{k_1} \\ \frac{1}{k_1} & \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} \\ \frac{1}{k_1} & \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \end{bmatrix}$$



Exercise 8

To calculate the stiffness influence coefficients:

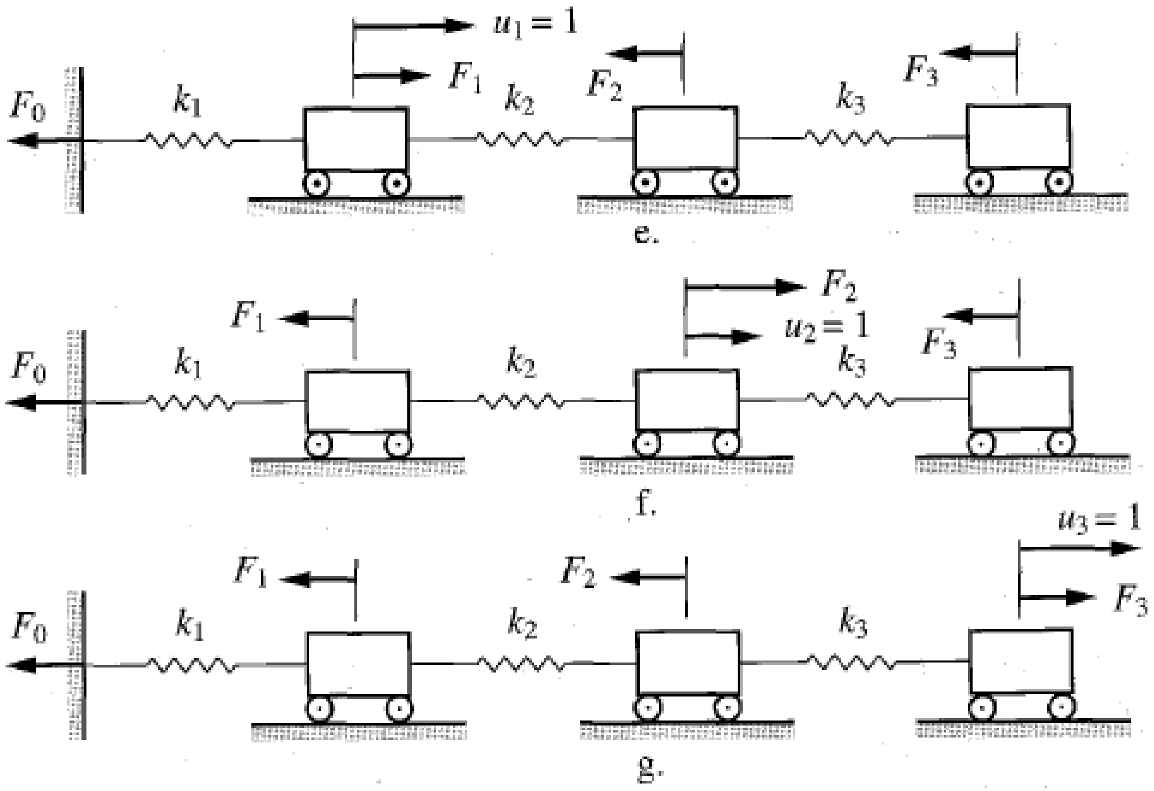


Exercise 8

$$F_0 + F_1 + F_2 = 0$$

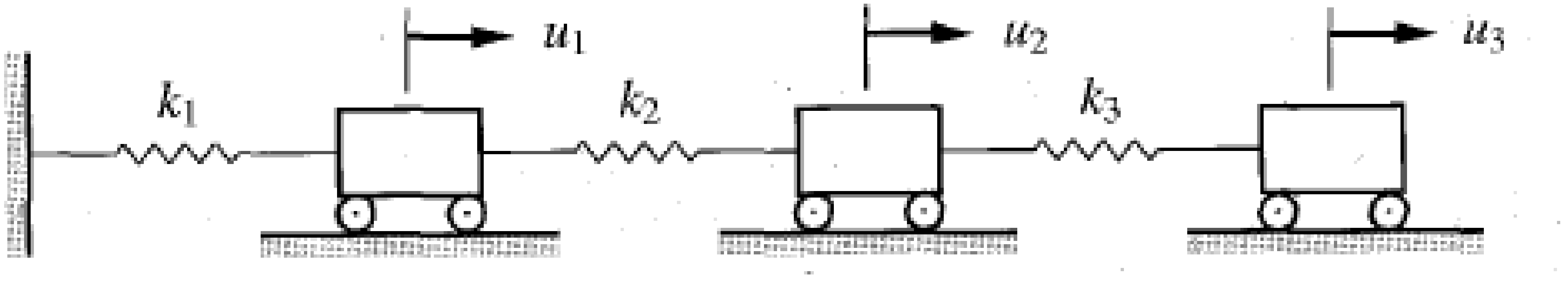
$$k_{11} = F_1 = k_1 + k_2,\quad k_{21} = F_2 = -k_2,\quad k_{31} = F_3 = 0$$
$$k_{12} = F_1 = -k_2,\quad k_{22} = F_2 = k_2 + k_3,\quad k_{32} = F_3 = -k_3$$
$$k_{13} = F_1 = 0,\quad k_{23} = F_2 = -k_3,\quad k_{33} = F_3 = k_3$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

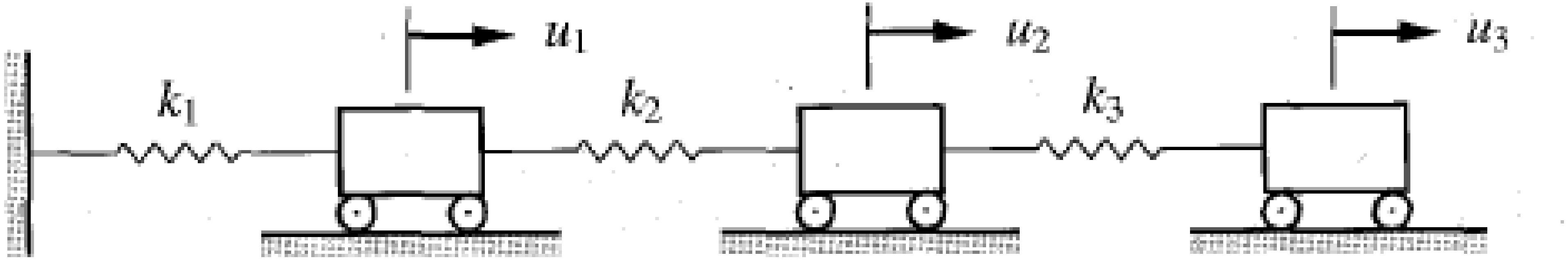


Exercise 9

Consider the three-degree-of-freedom system of the figure and derive the stiffness matrix <u>by means of the potential energy</u>.



Exercise 9



The elongations of the springs k_1, k_2 and k_3 are u_1, $u_2 - u_1$ and $u_3 - u_2$, respectively

$$V = \frac{1}{2}\left[k_1 u_1^2 + k_2\left(u_2 - u_1\right)^2 + k_3\left(u_3 - u_2\right)^2\right]$$

$$= \frac{1}{2}\left[\left(k_1 + k_2\right)u_1^2 + \left(k_2 + k_3\right)u_2^2 + k_3 u_3^2 - 2k_2 u_1 u_2 - 2k_3 u_2 u_3\right]$$

Exercise 9

$$V = \frac{1}{2}\left[\left(k_1 + k_2\right)u_1^2 + \left(k_2 + k_3\right)u_2^2 + k_3 u_3^2 - 2k_2 u_1 u_2 - 2k_3 u_2 u_3\right]$$

$$V = \frac{1}{2}\mathbf{u}^T K \mathbf{u}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

Exercise 10

The two-degree-of-freedom system of the figure consists of two masses on a string of tension *T* vibrating in the vertical plane. Let

$m_1 = m$

$m_2 = 2m$

$L_1 = L_2 = L$

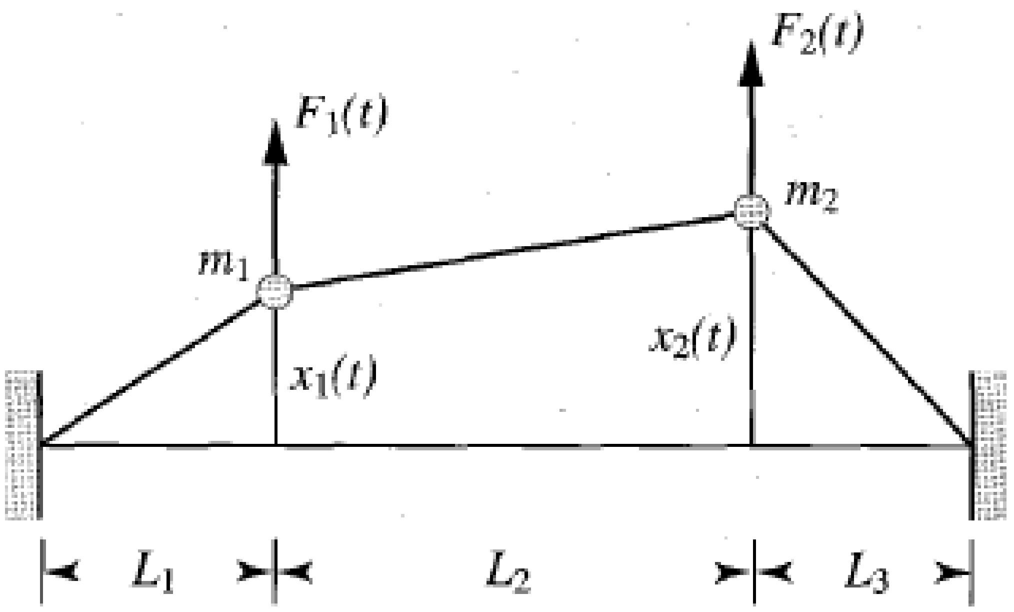
$L_3 = 0.5L$

<u>Show that the modal matrix diagonalizes the mass and stiffness matrices simultaneously</u>.

Then, normalize the modal matrix so as to satisfy

$U^T M U = I$ where I is the identity matrix, and show that the associated diagonal matrix

$U^T K U$ has the natural frequencies squared as the diagonal entries



Exercise 10

Show that the modal matrix diagonalizes the mass and stiffness matrices simultaneously.

Then, normalize the modal matrix so as to satisfy $U^T M U = I$ where I is the identity matrix, and show that the associated diagonal matrix $U^T K U$ has the natural frequencies squared as the diagonal entries

$$M = m\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$K = \frac{T}{L}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix}$$

From Exercise 3 we have already determined:

Exercise 10

<u>Show that the modal matrix diagonalizes the mass and stiffness matrices simultaneously</u>.

$$M' = U^T M U = m\begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix} = m\begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix}$$

$$K' = U^T K U = \frac{T}{L}\begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix} = \frac{T}{L}\begin{bmatrix} 3 & 0 \\ 0 & 3.75 \end{bmatrix}$$

Exercise 10

Then, normalize the modal matrix so as to satisfy
$U^T MU = I$ where I is the identity matrix, and
show that the associated diagonal matrix
$U^T KU$ has the natural frequencies squared
as the diagonal entries

$$\mathbf{u}_1 = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_1^T M \mathbf{u}_1 = \alpha_1^2 m \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\alpha_1^2 m = 1$$

$$\Rightarrow \quad \alpha_1 = 1/\sqrt{3m} \quad \Longrightarrow \quad \mathbf{u}_1 = \frac{1}{\sqrt{3m}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Exercise 10

$$\mathbf{u}_1 = \frac{1}{\sqrt{3m}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$U = \frac{1}{\sqrt{3m}}\begin{bmatrix} 1 & \sqrt{2} \\ 1 & -0.5\sqrt{2} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{1.5m}}\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$M' = U^T M U = \left(\frac{1}{\sqrt{3m}}\right)^2 m \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -0.5\sqrt{2} \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -0.5\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K' = U^T k U = \left(\frac{1}{\sqrt{3m}}\right)^2 \frac{T}{L} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -0.5\sqrt{2} \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -0.5\sqrt{2} \end{bmatrix} = \frac{T}{mL}\begin{bmatrix} 1 & 0 \\ 0 & 2.5 \end{bmatrix}$$

Exercise 10

$$K' = U^T k U = \left(\frac{1}{\sqrt{3m}}\right)^2 \frac{T}{L} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -0.5\sqrt{2} \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -0.5\sqrt{2} \end{bmatrix} = \frac{T}{mL} \begin{bmatrix} 1 & 0 \\ 0 & 2.5 \end{bmatrix}$$

$$\omega_1 = \sqrt{\frac{T}{mL}}$$

$$\omega_2 = \sqrt{\frac{5T}{2mL}}$$

$$K' = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$