System Dynamics and Vibrations

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Chapter 3: Single degree-of-freedom systems
Part 1

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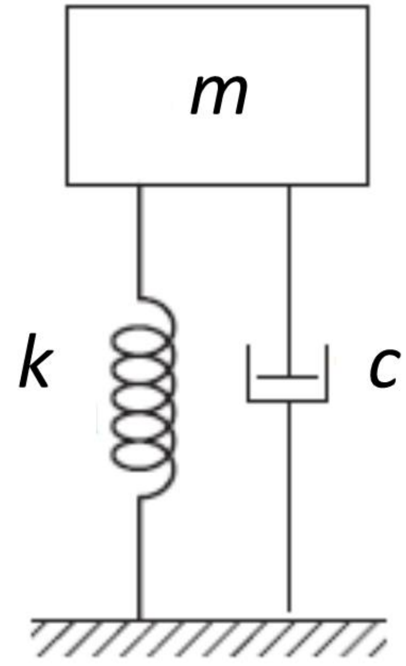
Contents

- Introduction
- Undamped SDOF systems. Harmonic oscillator.
- Viscously damped SDOF.
- Coulomb damping. Dry friction.
- Response of SDOF systems to harmonic excitations.
- Frequency response plots.
- Systems with rotating unbalanced masses.
- Response to periodic excitation. Fourier series.
- Response to non-periodic excitation: step, ramp and impulse.

Introduction: the SDOF

- Single degree-of-freedom (SDOF) system: motion described by a single variable (or coordinate)

$$m\ddot{x} + c\dot{x} + kx = F$$



being *x(t)* the response and *F(t)* the excitation

Introduction: types of excitations

- Initial excitations: Initial displacements, initial velocities, both

➔ Free vibration (free response): no further external factors affecting the system ➔ homogeneous equation

- Applied forces / moments ➔ forced vibration / response

➔The response depends on the type of applied (external) forces / moments

Contents

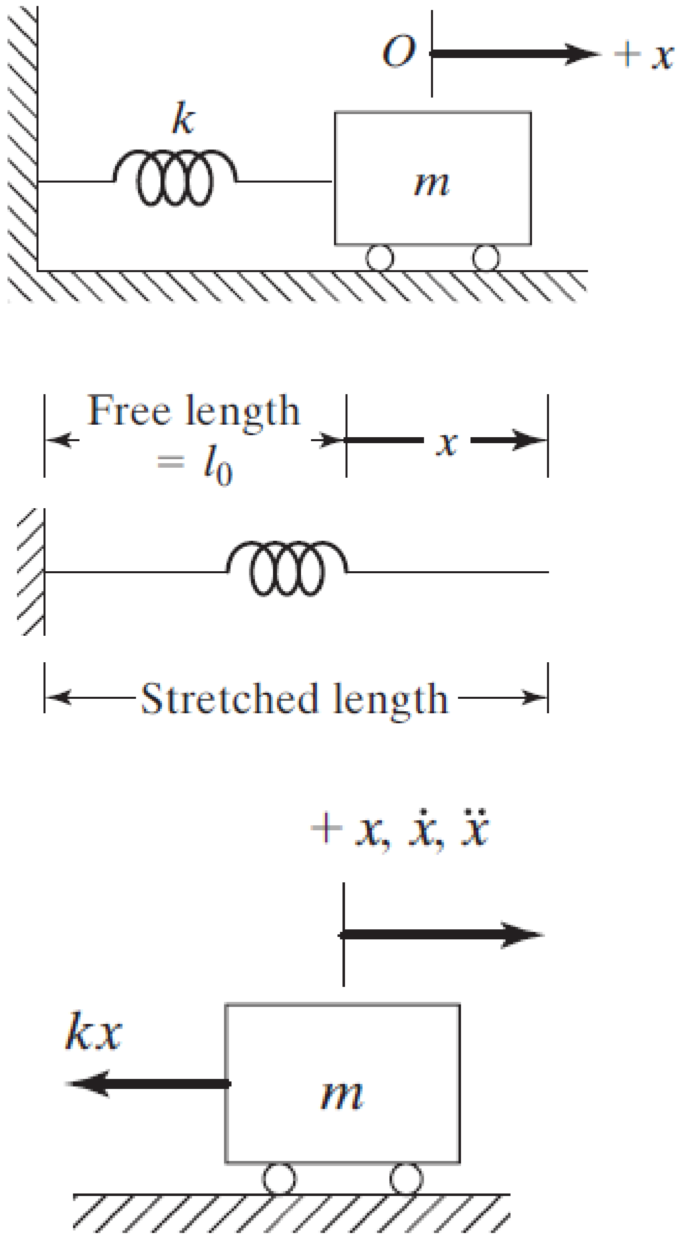
Undamped SDOF systems

$$m\ddot{x} + \cancel{c\dot{x}} + kx = \cancel{F}$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega_n^2 x = 0$$

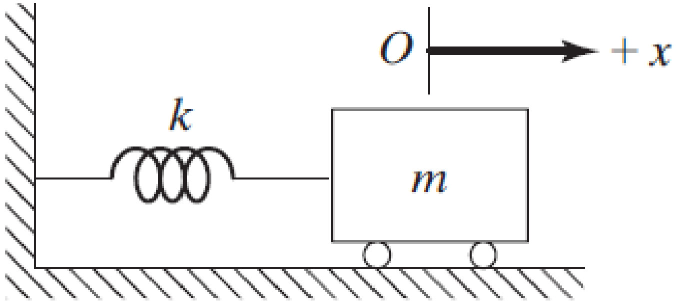
$$\Rightarrow \omega_n = \sqrt{k/m} \qquad \text{(rad/s)}$$

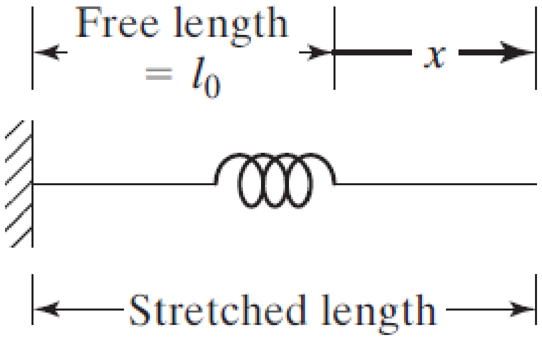


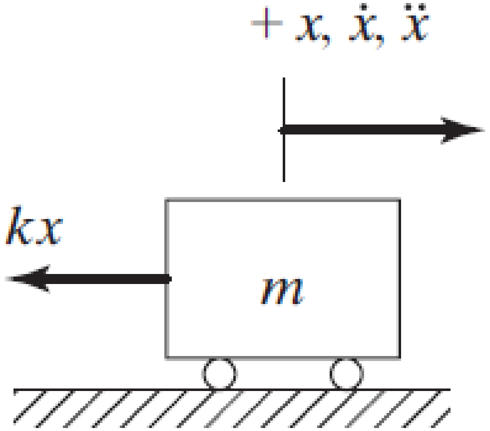
Undamped SDOF systems

$$\ddot{x}+\omega_n^2 x=0$$

$$\left.\begin{aligned} x(0)&=x_0\\ \dot{x}(0)&=v_0 \end{aligned}\right\}\ \text{initial conditions}$$







x(t) are small displacements about the equilibrium position

Undamped SDOF systems

solution: $$\ddot{x} + \omega_n^2 x = 0$$

$$x(t) = Ae^{st}$$

$$s^2 + \omega_n^2 = 0$$

$$\left.\begin{matrix} s_1 \\ s_2 \end{matrix}\right. = \pm i\omega_n$$

$$x(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$$

Undamped SDOF systems

$$x(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$$

x(t) must be real:

$$A_1 = \frac{C}{2} e^{-i\phi}, A_2 = \overline{A_1} = \frac{C}{2} e^{i\phi}$$

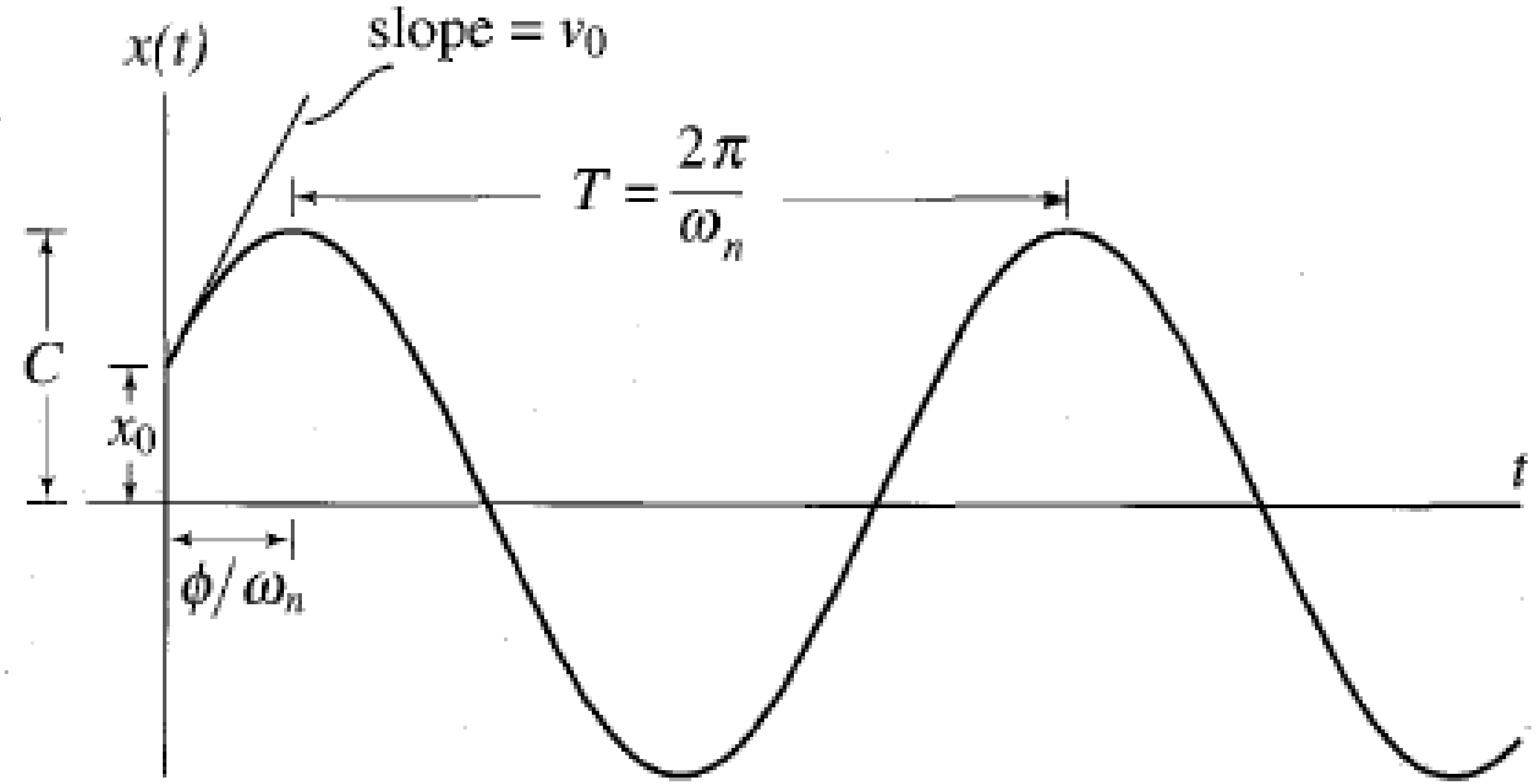
$$\Rightarrow x(t) = \frac{C}{2}\left[e^{i(\omega_n t - \phi)} + e^{-i(\omega_n t - \phi)} \right] = C\cos\left(\omega_n t - \phi\right)$$

C and ϕ are constants of integration

<u>harmonic oscillation</u>

Undamped SDOF systems: harmonic oscillator

$$x(t) = C\cos\left(\omega_n t - \phi\right)$$



<u>initial excitations</u>:

C: amplitude

ϕ: phase angle

<u>system parameters</u>:

ω_n: natural frequency

$$\omega_n = \sqrt{k/m}$$

Undamped SDOF systems: harmonic oscillator

$$x(t) = C\cos\left(\omega_n t - \phi\right)$$

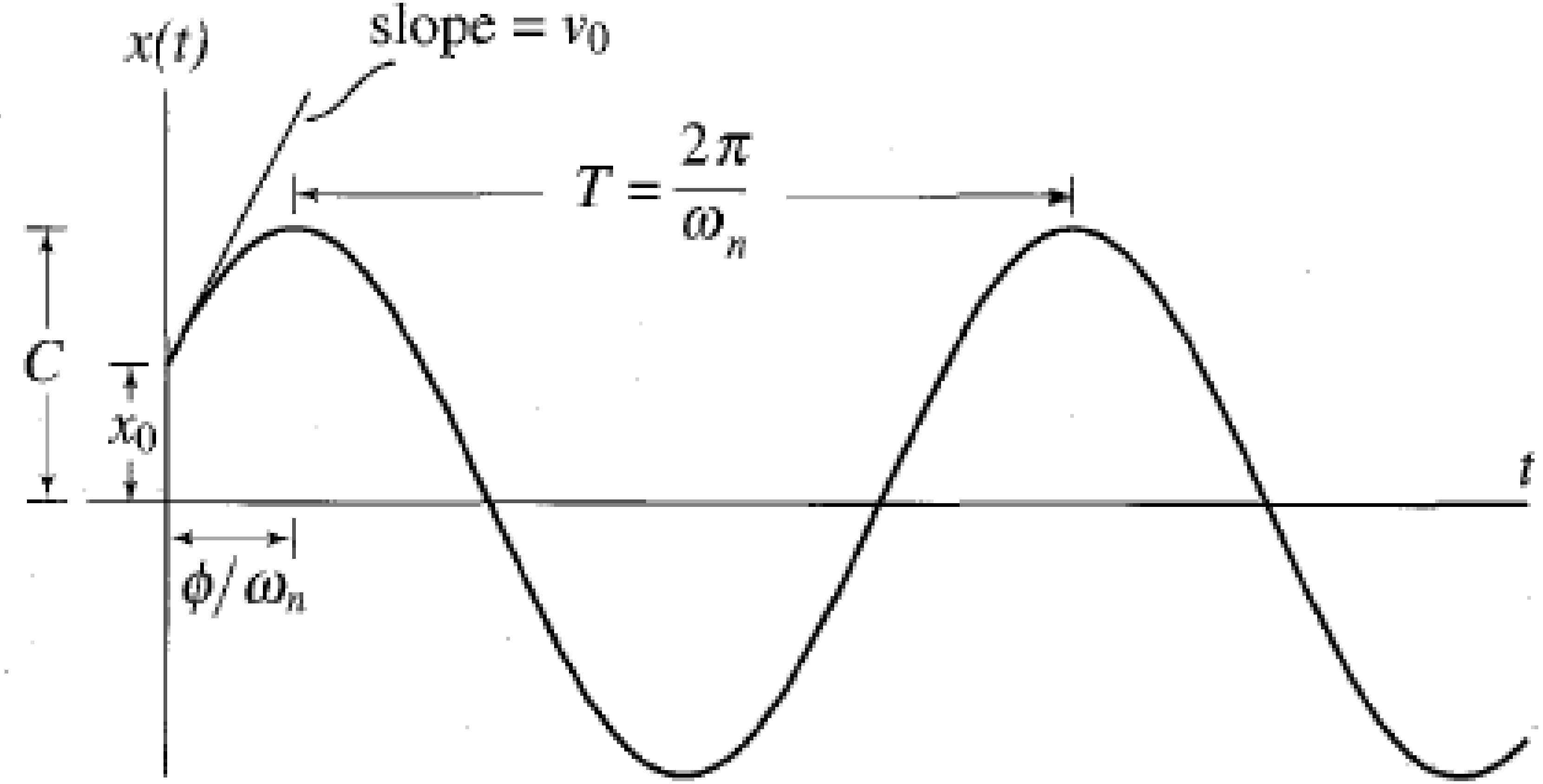
Initial conditions:

$$\left.\begin{aligned} x(0) &= x_0 = C\cos\phi \\ \dot{x}(0) &= v_0 = \omega_n C\sin\phi \end{aligned}\right\} \qquad \begin{aligned} C &= \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \\ \phi &= \tan^{-1}\frac{v_0}{x_0\omega_n} \end{aligned} \qquad \Rightarrow$$

$$x(t) = C\cos\left(\omega_n t - \phi\right) = C\left(\cos\omega_n t\cos\phi + \sin\omega_n t\sin\phi\right) = x_0\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$$

Undamped SDOF systems: harmonic oscillator

$$x(t) = C\cos\left(\omega_n t - \phi\right)$$



$$\omega_n = \sqrt{k/m}$$ (rad/s)

$$T = \frac{2\pi}{\omega_n}$$ (s)

$$f_n = \frac{1}{T}$$ (Hz)

Undamped SDOF systems: harmonic oscillator

- Harmonic oscillator ➔ conservative system

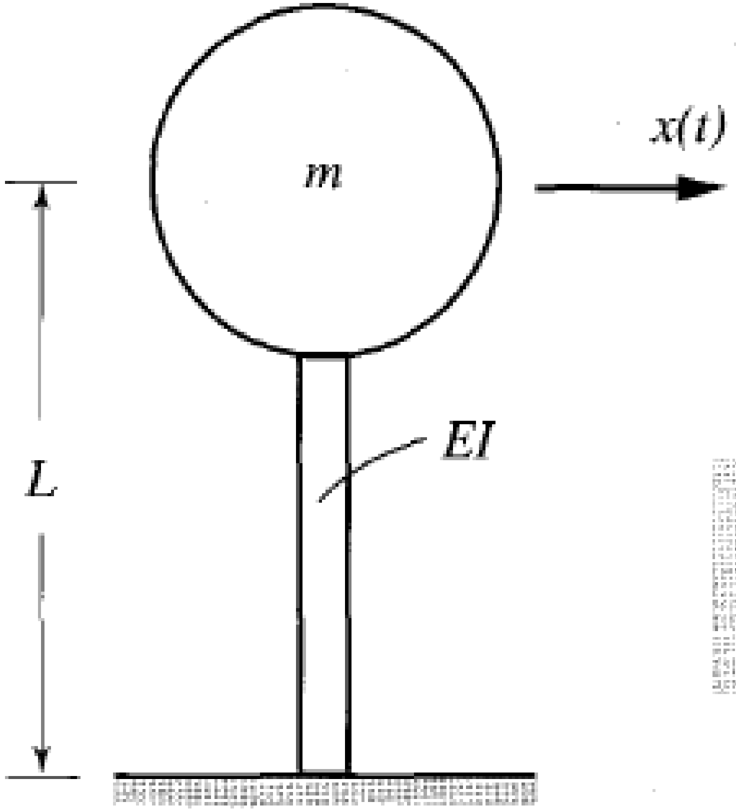
(no damping, no dissipation of energy)

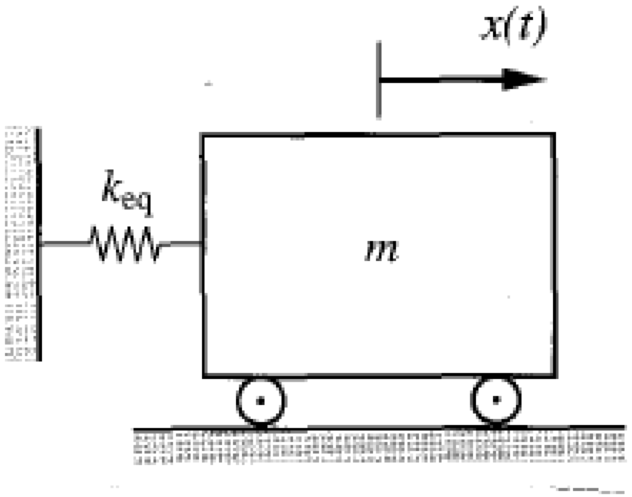
- Stable motion, pure oscillation
- Good approximation for physical systems when the rate of energy dissipation is so small that it takes many oscillation cycles before a reduction in the amplitude can be discerned

Undamped SDOF systems: examples

- Water tank

Assumption: the tank and water act as a rigid body and the support column is a massless uniform cantilever beam of bending stiffness EI.



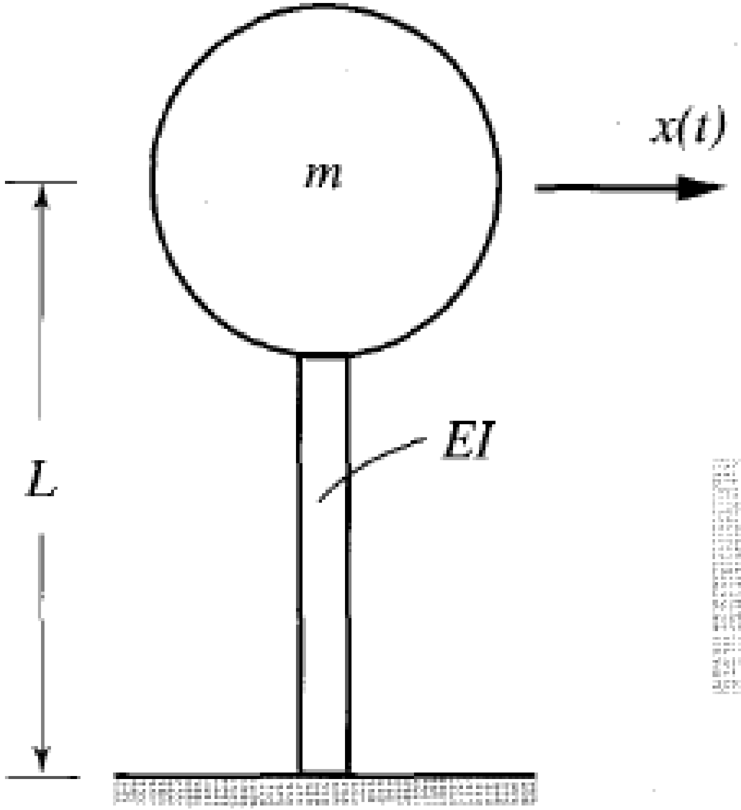


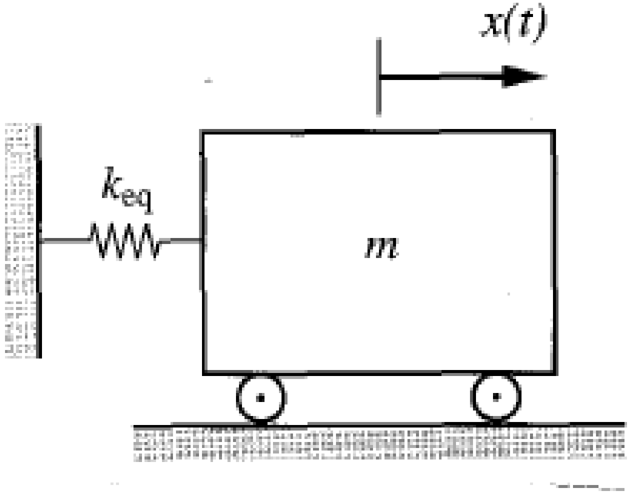
Undamped SDOF systems: examples

- Water tank

$$k_{eq} = \frac{3EI}{L^3}$$

$$\omega_n = \sqrt{k_{eq}/m}$$

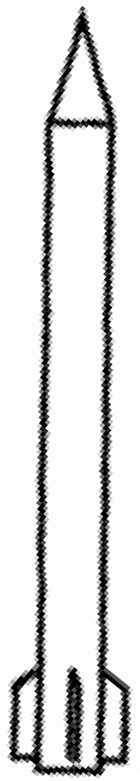


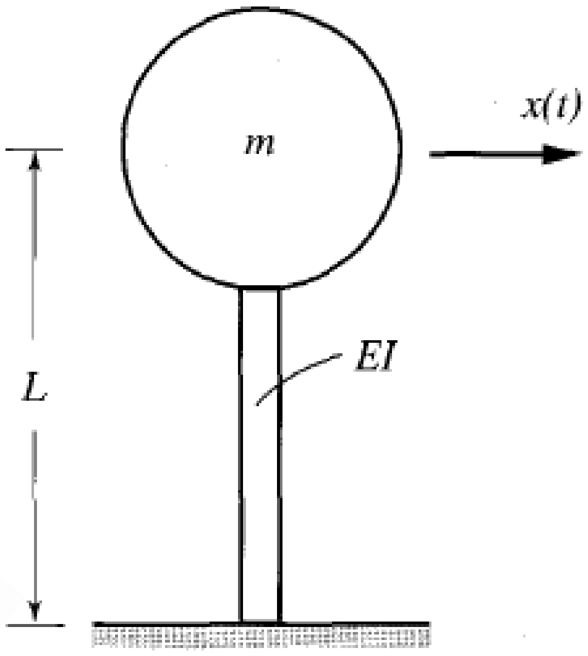


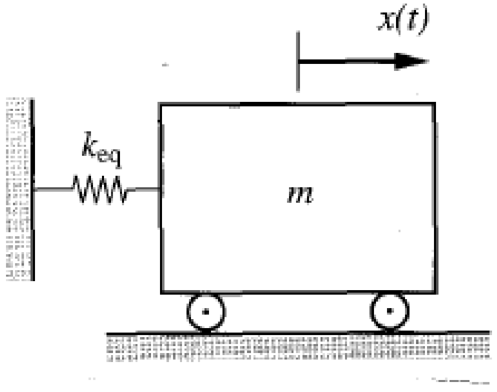
Undamped SDOF systems: examples

- Rocket launcher / missile

Assumption: all the mass is located at the center of gravity: L = COG







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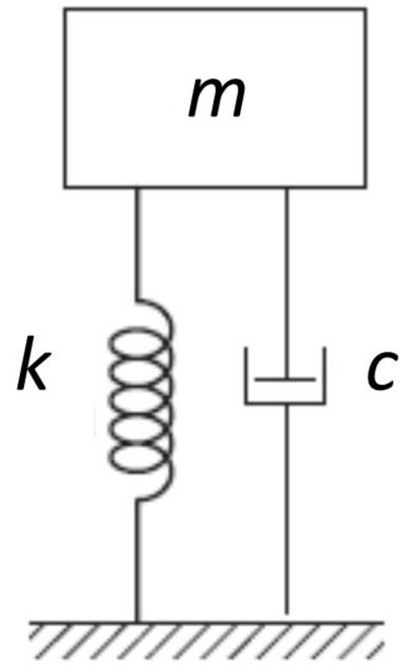
Viscously damped SDOF

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = 0$$

$$\omega_n = \sqrt{k/m} \qquad \varsigma = \frac{c}{2m\omega_n}$$

(viscous damping factor)



Viscously damped SDOF

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = 0$$

$$x(0) = x_0$$

$$\dot{x}(0) = v_0$$

$$x(t) = Ae^{st}$$

$$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$$

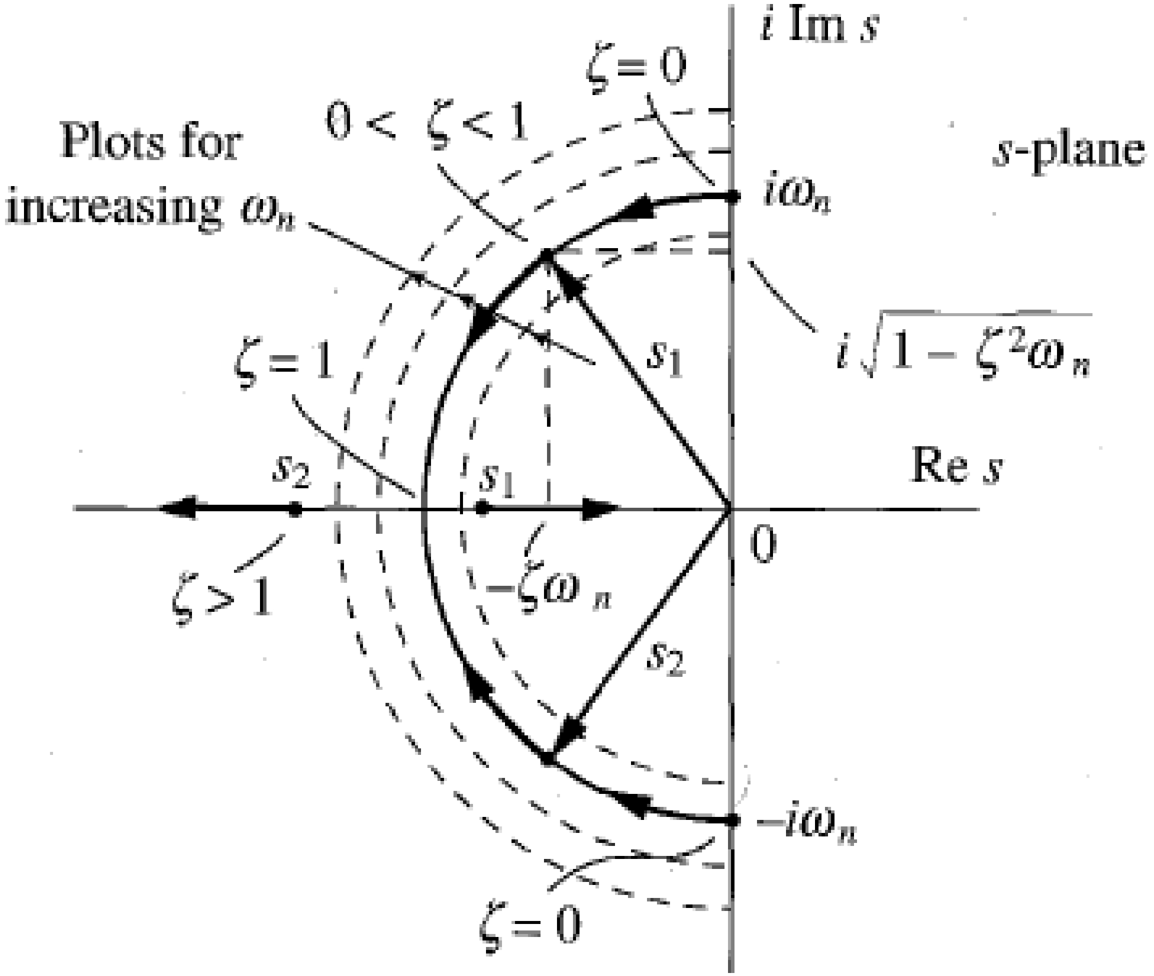
$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\varsigma\omega_n \pm \sqrt{\varsigma^2 - 1}\omega_n$$

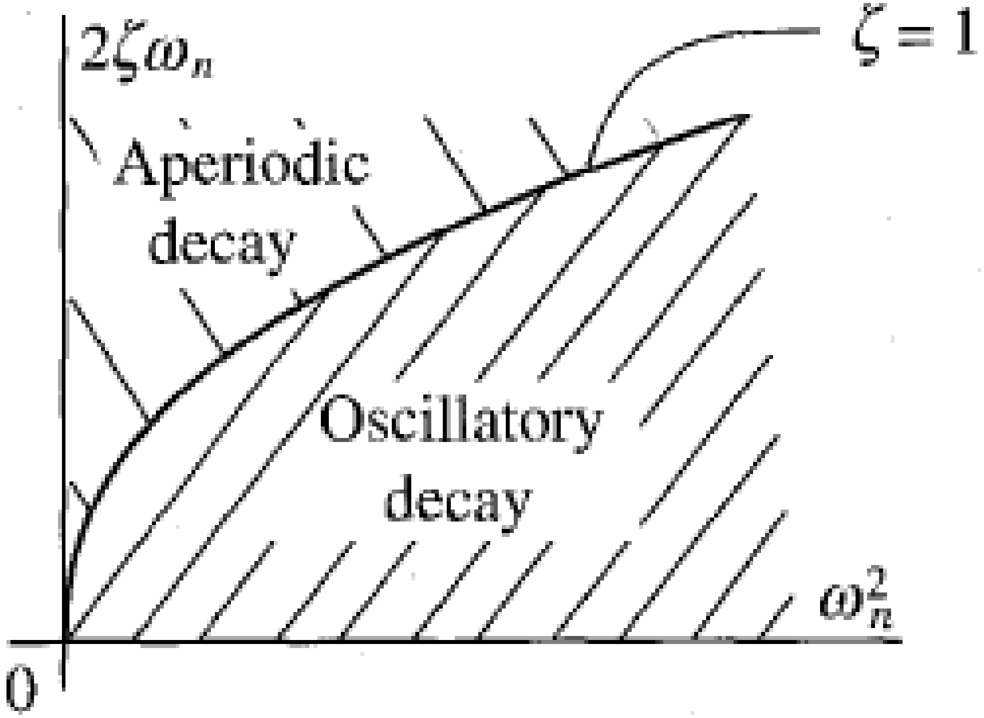
The motion depends of s_1, s_2 ➔ ζ

Viscously damped SDOF

$$x(t) = Ae^{st}$$

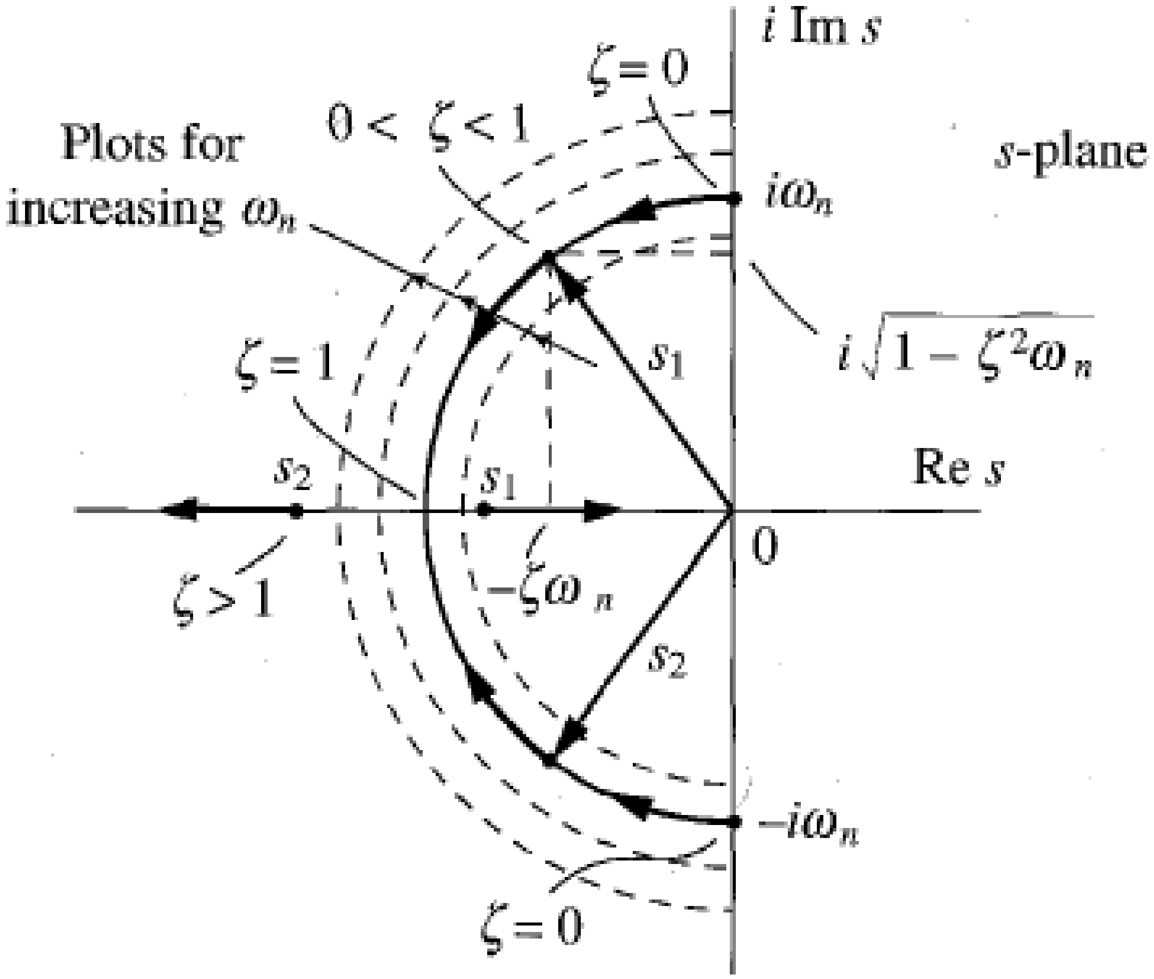
$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\varsigma\omega_n \pm \sqrt{\varsigma^2 - 1}\omega_n$$





Viscously damped SDOF

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\varsigma\omega_n \pm \sqrt{\varsigma^2 - 1}\omega_n$$



ζ = 0 ➔ harmonic oscillator

$0 < \zeta < 1$ ➔ oscllatory decay (underdamping)

ζ = 1 ➔ aperiodic decay (critical damping)

ζ > 1 ➔ aperiodic decay (overdamping)

Viscously damped SDOF

solution: $$\ddot{x} + 2\varsigma\omega_n \dot{x} + \omega_n^2 x = 0$$

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\left.\begin{aligned} x(0) &= A_1 + A_2 = x_0 \\ \dot{x}(0) &= s_1 A_1 + s_2 A_2 = v_0 \end{aligned}\right\} \qquad \begin{aligned} A_1 &= \frac{-s_2 x_0 + v_0}{s_1 - s_2} \\ A_2 &= \frac{s_1 x_0 - v_0}{s_1 - s_2} \end{aligned} \qquad \Rightarrow$$

$$\Rightarrow \qquad x(t) = \frac{-s_2 x_0 + v_0}{s_1 - s_2} e^{s_1 t} + \frac{s_1 x_0 - v_0}{s_1 - s_2} e^{s_2 t}$$

Viscously damped SDOF

$$x(t)=\frac{-s_2x_0+v_0}{s_1-s_2}e^{s_1t}+\frac{s_1x_0-v_0}{s_1-s_2}e^{s_2t}$$

$0 \quad < \zeta < 1$ ➔ oscllatory decay (<u>underdamping</u>)

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\varsigma\omega_n \pm i\omega_d$$

$$\omega_d = \sqrt{1-\varsigma^2}\,\omega_n$$

frequency of damped vibration

➔

$$x(t) = Ce^{-\varsigma\omega_n t}\cos\left(\omega_d t-\phi\right)$$

$$C = \sqrt{x_0^2+\left(\frac{\varsigma\omega_n x_0+v_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\frac{\varsigma\omega_n x_0+v_0}{\omega_d x_0}$$

Viscously damped SDOF

$0 \quad < \zeta < 1$ ➔ oscllatory decay (<u>underdamping</u>)

exponentially decaying function

$$x(t) = Ce^{-\varsigma\omega_n t}\cos\left(\omega_d t - \phi\right)$$

$$C = \sqrt{x_0^2 + \left(\frac{\varsigma\omega_n x_0 + v_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\frac{\varsigma\omega_n x_0 + v_0}{\omega_d x_0}$$

Viscously damped SDOF

$0 \quad < \zeta < 1$ ➔ oscllatory decay (<u>underdamping</u>)

harmonic function

$$x(t) = Ce^{-\varsigma\omega_n t}\cos\left(\omega_d t - \phi\right)$$

$$C = \sqrt{x_0^2 + \left(\frac{\varsigma\omega_n x_0 + v_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\frac{\varsigma\omega_n x_0 + v_0}{\omega_d x_0}$$

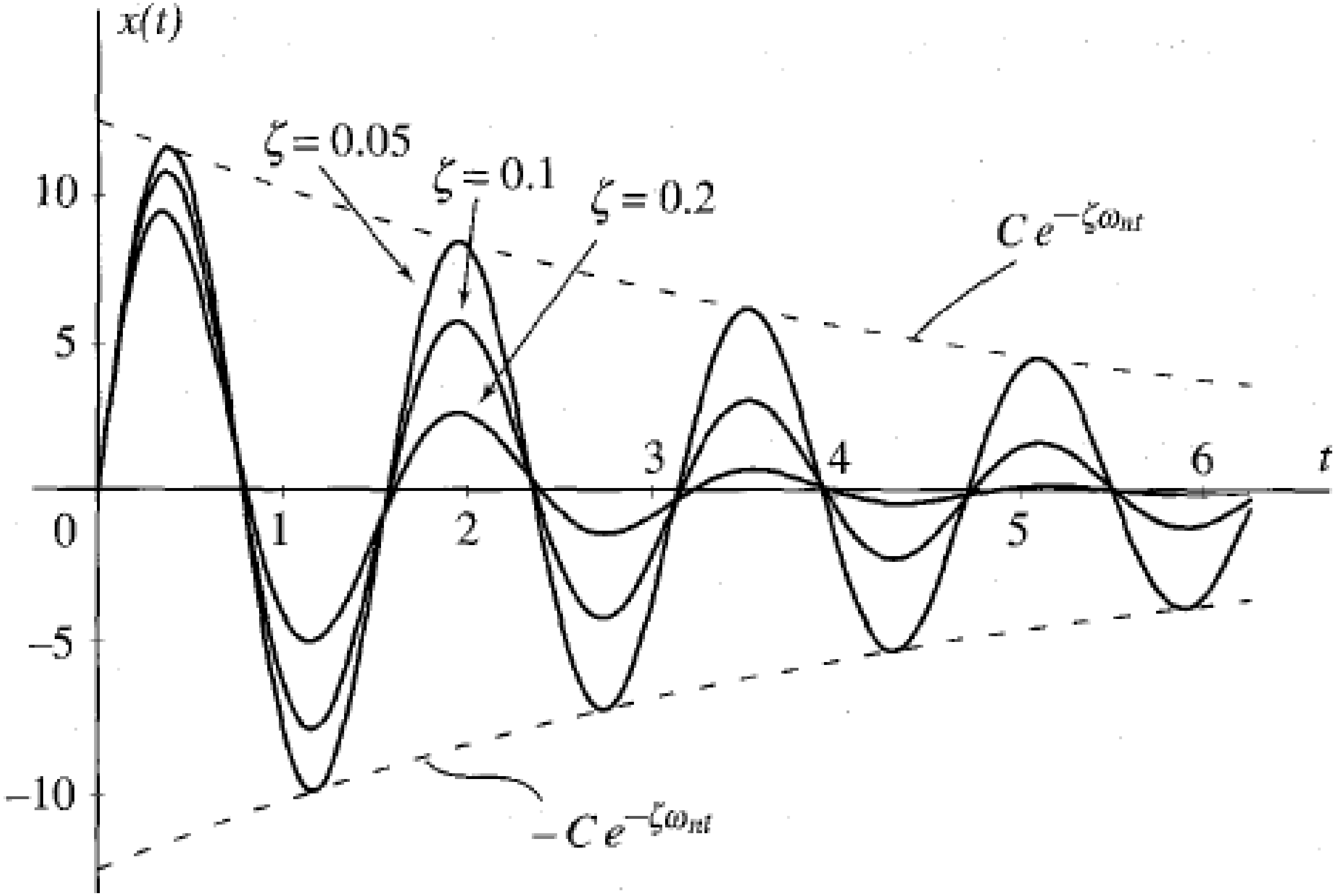
Viscously damped SDOF

$0 < \zeta < 1$ ➔ oscllatory decay (underdamping)

$$x(t) = Ce^{-\varsigma\omega_n t}\cos\left(\omega_d t - \phi\right)$$

$$C = \sqrt{x_0^2 + \left(\frac{\varsigma\omega_n x_0 + v_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\frac{\varsigma\omega_n x_0 + v_0}{\omega_d x_0}$$



Viscously damped SDOF

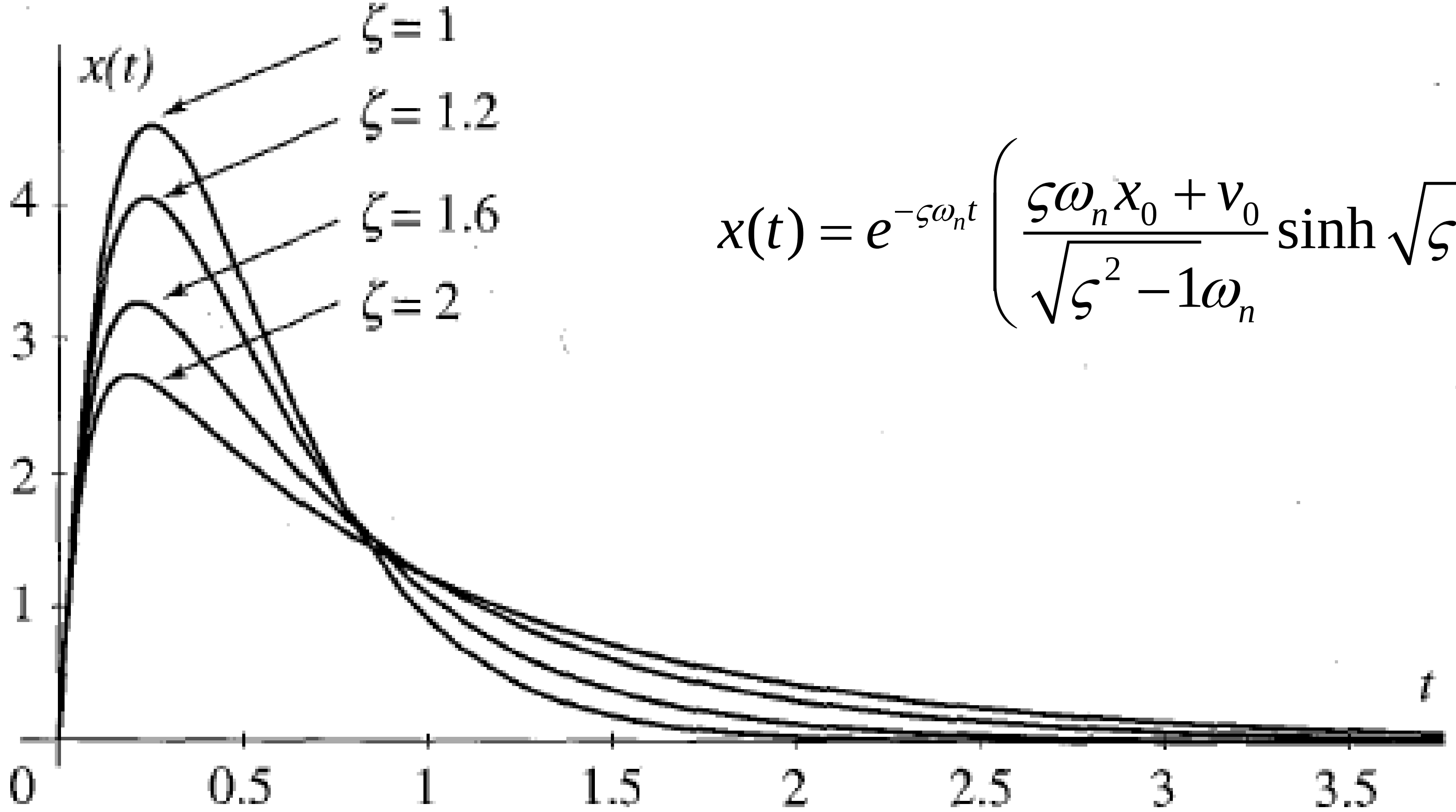
$$x(t) = \frac{-s_2 x_0 + v_0}{s_1 - s_2} e^{s_1 t} + \frac{s_1 x_0 - v_0}{s_1 - s_2} e^{s_2 t}$$

$\zeta > 1$ ➔ aperiodic decay (over<u>damping)</u>

$$x(t) = e^{-\varsigma \omega_n t} \left(\frac{\varsigma \omega_n x_0 + v_0}{\sqrt{\varsigma^2 - 1}\omega_n} \sinh \sqrt{\varsigma^2 - 1}\omega_n t + x_0 \cosh \sqrt{\varsigma^2 - 1}\omega_n t \right)$$

Viscously damped SDOF

$\zeta > 1$ ➔ aperiodic decay (overdamping)



$$x(t) = e^{-\varsigma\omega_n t}\left(\frac{\varsigma\omega_n x_0 + v_0}{\sqrt{\varsigma^2 - 1}\omega_n}\sinh\sqrt{\varsigma^2 - 1}\omega_n t + x_0\cosh\sqrt{\varsigma^2 - 1}\omega_n t\right)$$

Viscously damped SDOF

$\zeta = 1$ ➔ aperiodic decay (critically damped systems)

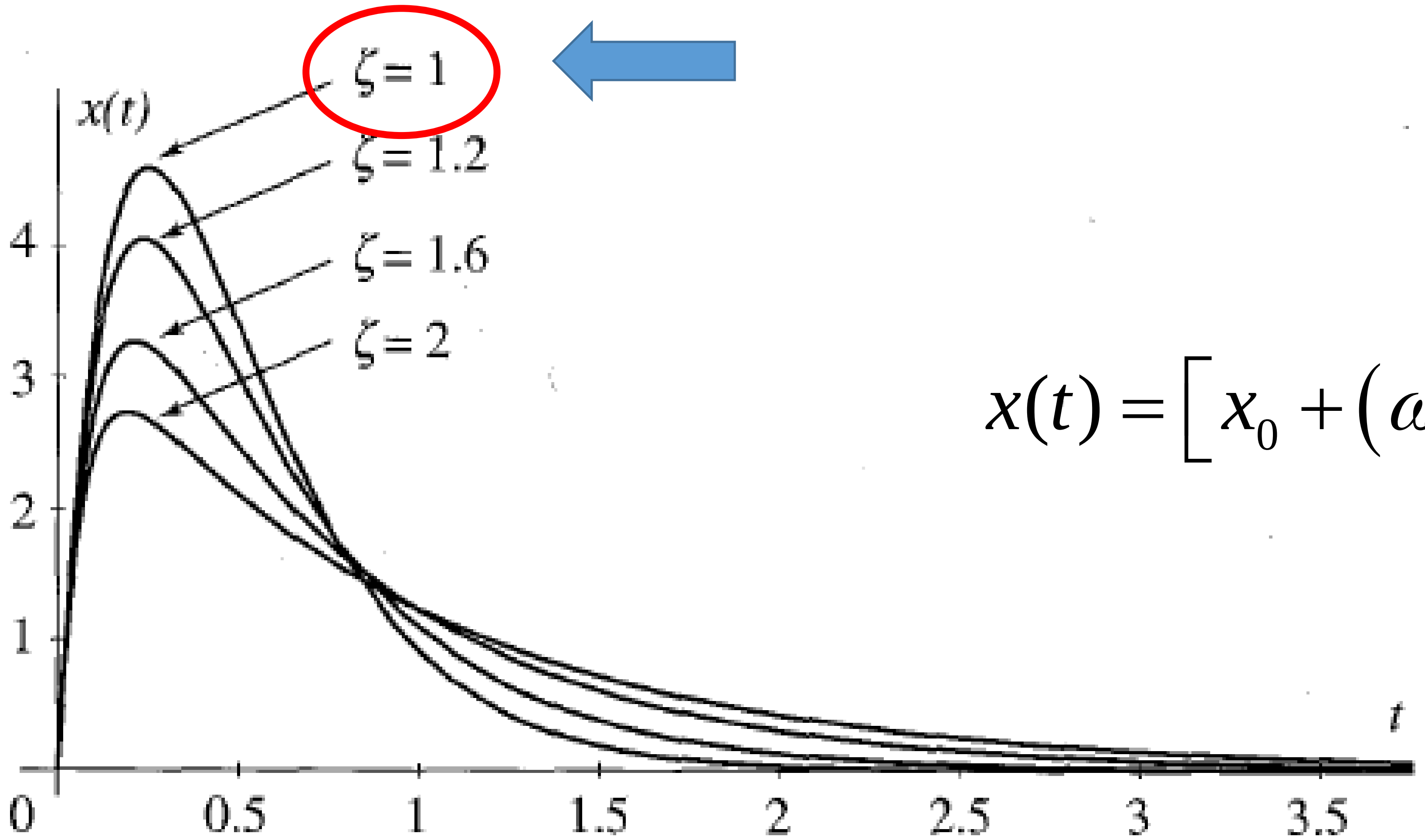
$$s_1 = s_2 = -\omega_n$$

$$x(t) = e^{-\varsigma\omega_n t}\left(\frac{\varsigma\omega_n x_0 + v_0}{\sqrt{\varsigma^2 - 1}\omega_n}\sinh\sqrt{\varsigma^2 - 1}\omega_n t + x_0\cosh\sqrt{\varsigma^2 - 1}\omega_n t\right)$$

$$\left.\begin{array}{l} \displaystyle\lim_{\varsigma\to 1}\frac{\sinh\sqrt{\varsigma^2 - 1}\omega_n t}{\sqrt{\varsigma^2 - 1}\omega_n} = t \\ \displaystyle\lim_{\varsigma\to 1}\cosh\sqrt{\varsigma^2 - 1}\omega_n t = 1 \end{array}\right\} \qquad x(t) = \left[x_0 + \left(\omega_n x_0 + v_0\right)t\right]e^{-\omega_n t}$$

Viscously damped SDOF

$\zeta = 1$ ➔ aperiodic decay (critically damped systems)



$$x(t) = \left[x_0 + \left(\omega_n x_0 + v_0 \right) t \right] e^{-\omega_n t}$$

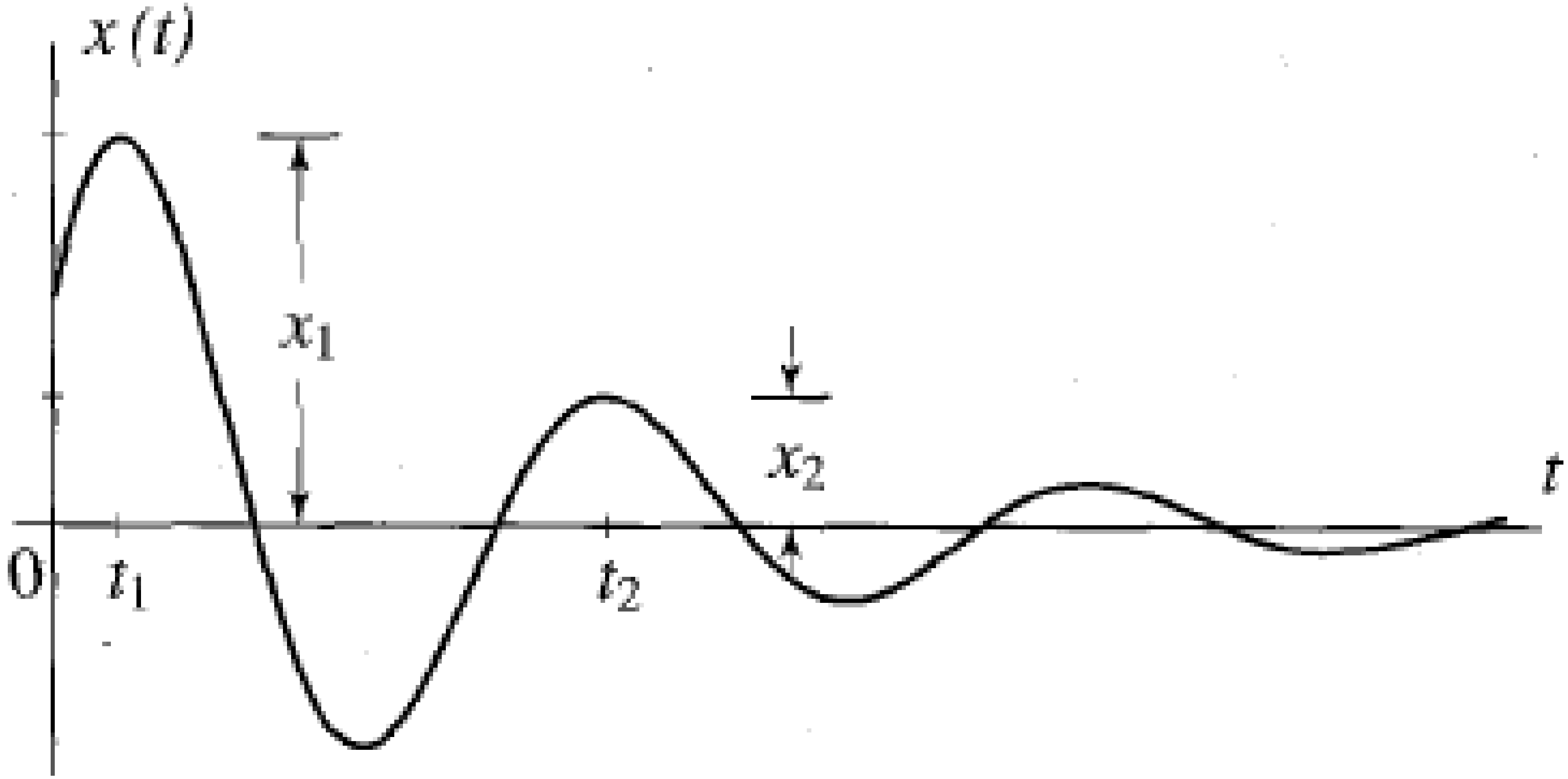
Viscously damped SDOF: damping measurement

- Damping is generally difficult to characterize
- It depends on many factors: the nature of connectin between individual components, air resistance, etc.
- Experimental procedure to measure ζ

➔ *measuring the drop in amplitude at the completion of one cycle of vibration*

Viscously damped SDOF: damping measurement

- Experimental procedure to measure ζ

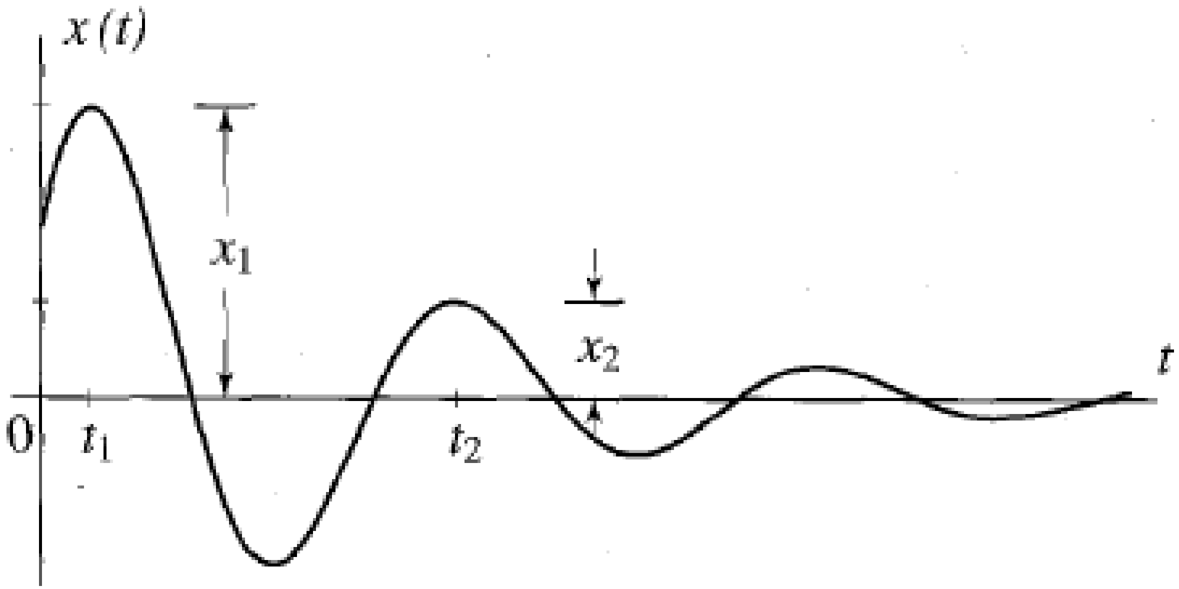


Viscously damped SDOF: damping measurement

$$x(t) = Ce^{-\varsigma\omega_n t}\cos\left(\omega_d t - \phi\right)$$

$$t_2 = t_1 + T$$

$$T = {2\pi}/{\omega_d}$$



$$\frac{x_1}{x_2} = \frac{x(t_1)}{x(t_2)} = e^{\varsigma\omega_n T} = e^{2\pi\varsigma\omega_n/\omega_d} = e^{2\pi\varsigma/\sqrt{1-\varsigma^2}}$$

Viscously damped SDOF: damping measurement

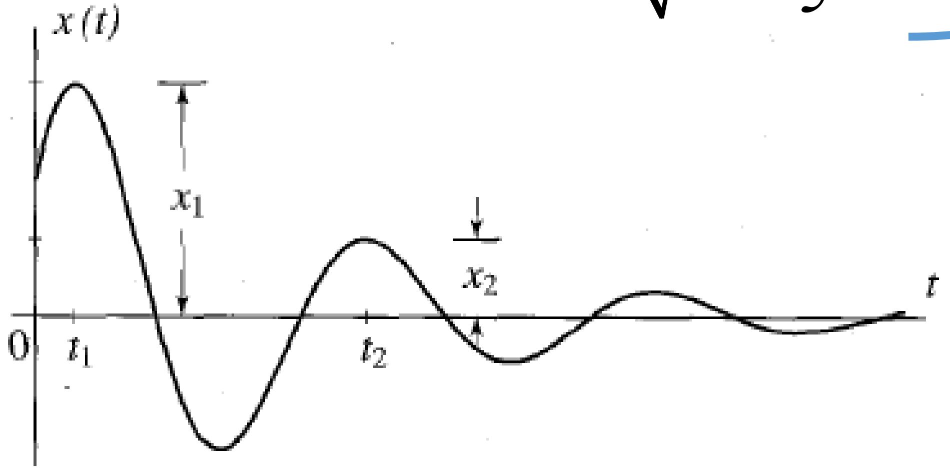
$$\frac{x_1}{x_2} = e^{2\pi\varsigma / \sqrt{1-\varsigma^2}}$$

$$\delta = \ln\frac{x_1}{x_2} = \frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}}$$

δ ➔ logarithmic decrement

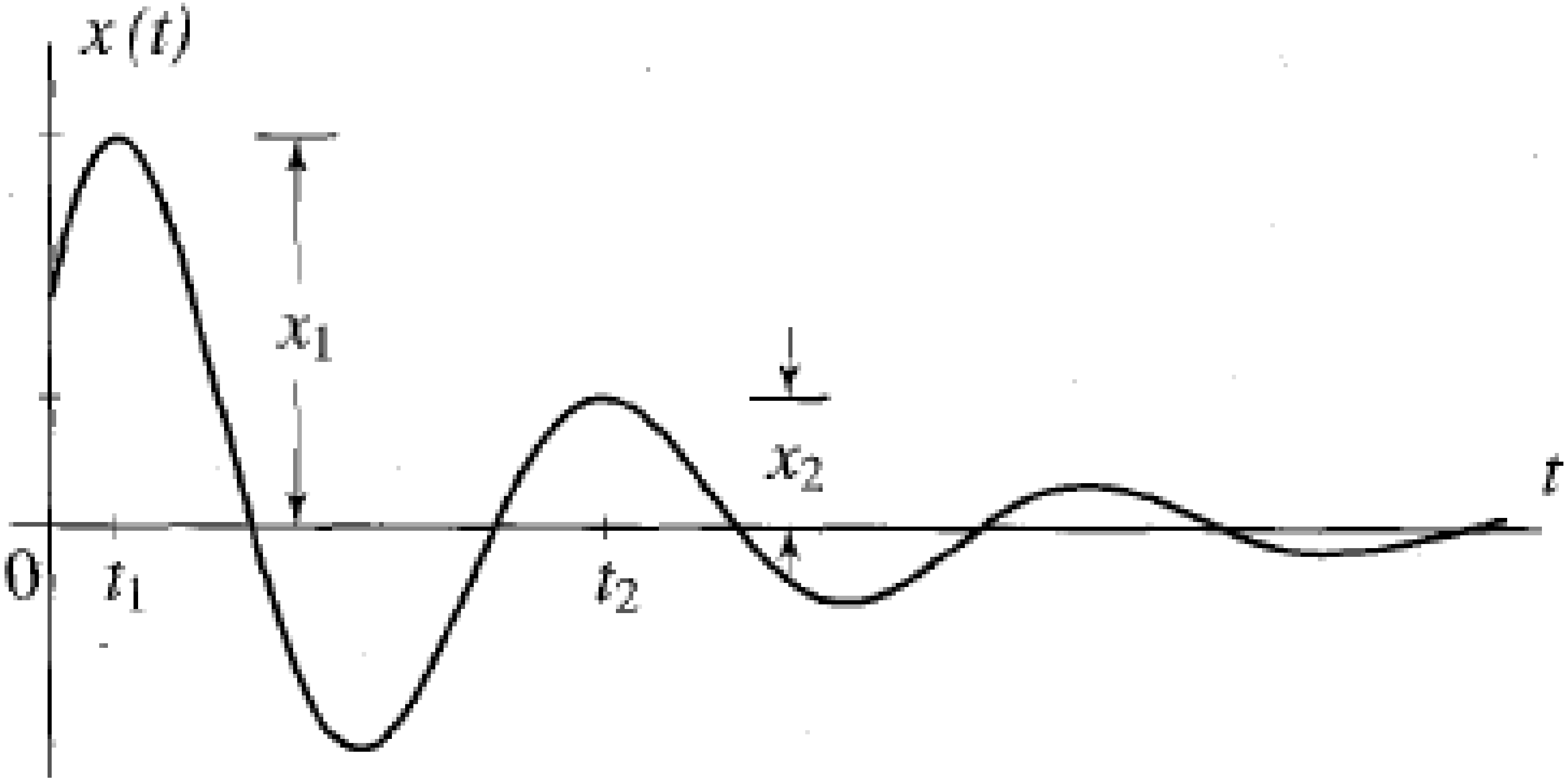
$$\varsigma = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \cong \frac{\delta}{2\pi}$$

$$\varsigma \ll 1$$



Viscously damped SDOF: damping measurement

- More accurate procedure to measure ζ

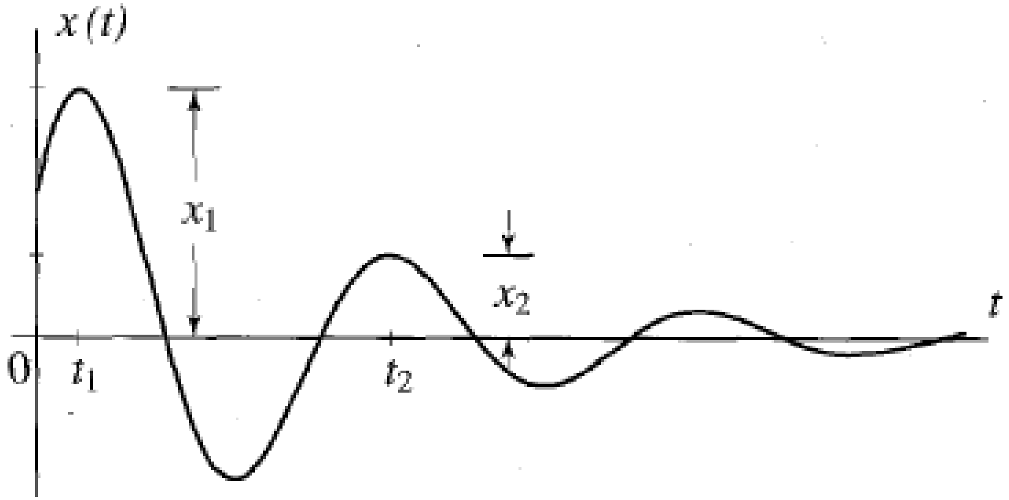


Viscously damped SDOF: damping measurement

$$x(t) = Ce^{-\varsigma\omega_n t}\cos\left(\omega_d t - \phi\right)$$

$$x_1, x_{j+1}$$

$$t_{j+1} = t_1 + jT$$

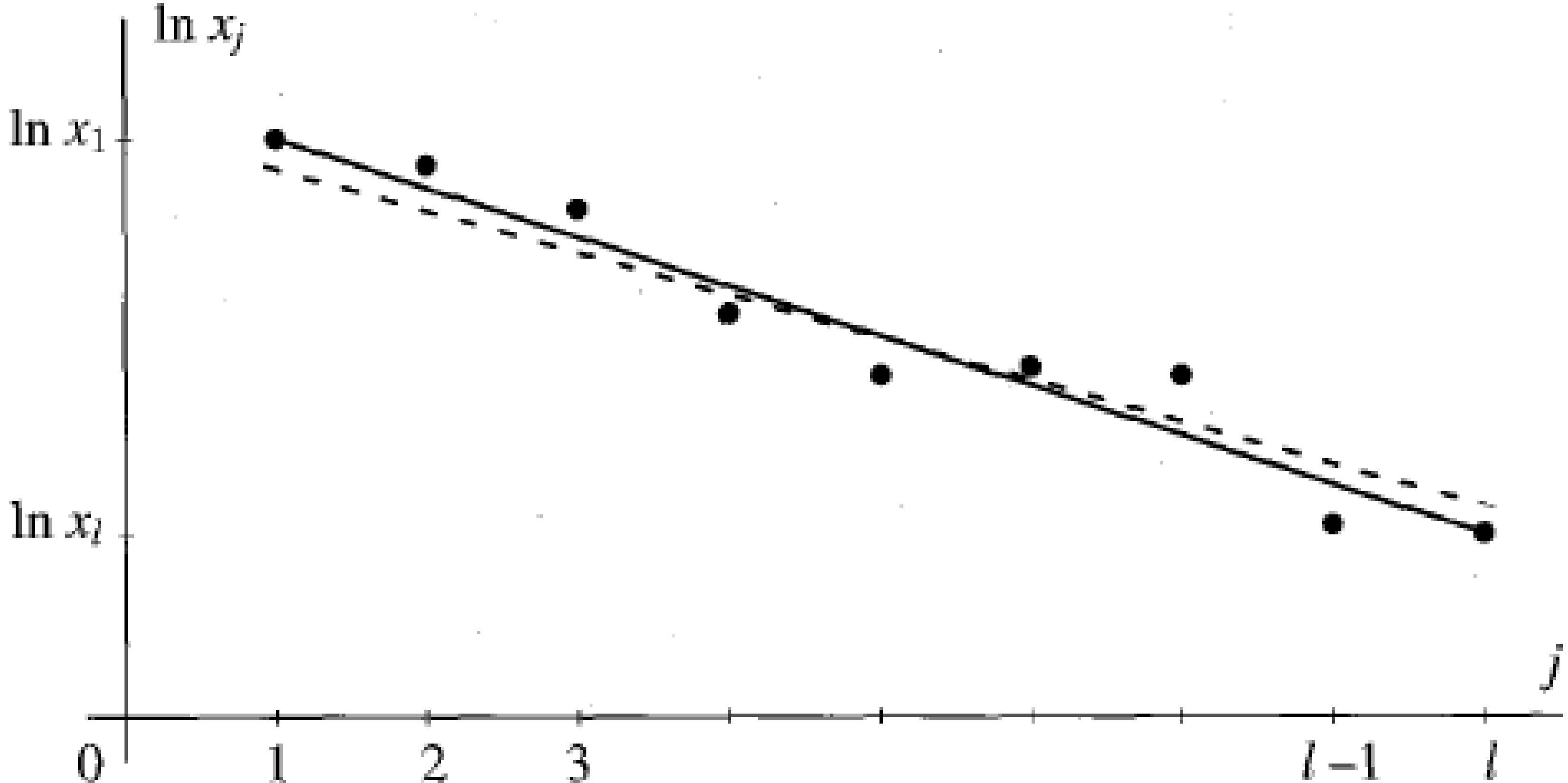


$$\frac{x_1}{x_{j+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\ldots\frac{x_j}{x_{j+1}} = e^{\varsigma\omega_n T} = \left(e^{2\pi\varsigma/\sqrt{1-\varsigma^2}}\right)^j = e^{j2\pi\varsigma/\sqrt{1-\varsigma^2}}$$

$$\Rightarrow \quad \delta = \frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}} = \frac{1}{j}\ln\frac{x_1}{x_{j+1}}$$

Viscously damped SDOF: damping measurement

$$\delta = \frac{1}{j}\ln\frac{x_1}{x_{j+1}} \Rightarrow \ln x_j = \ln x_1 - \delta\left(j-1\right), \quad j = 1, 2, \ldots, l$$



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Damping elements

- Damping is the mechanism by which the vibrational energy is gradually converted into heat or sound
- Dampers are assumed to have neither mass nor elasticity
- Damping forces exists only if there is relative velocity between the two ends of the damper
- **Viscous damping**:
  - Related to the interaction of the body with the surrounding fluid
  - Depends on: the size and shape of the body, the viscosity of the fluid, the frequency of vibration, the velocity of the body, etc.
  - The damping force is proportional to the velocity of the vibrating body

Damping elements

- **Coulomb or dry-friction damping**:
  - The damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body
  - It is caused by friction between rubbing surfaces that either are dry or have insufficient lubrication
- **Material or Solid or Hysteretic Damping:**
  - When a material is deformed, energy is absorbed and dissipated by the material
  - The effect is due to friction between the internal planes, which slip or slide as the deformations take place.
  - When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop as indicated
  - The area of this loop denotes the energy lost per unit volume of the body per cycle due to damping

Coulomb damping. Dry friction

- Static friction coefficient μ_s

  $0 < \mu_s < 1$ is a material property

- Kinetic friction coefficient μ_d

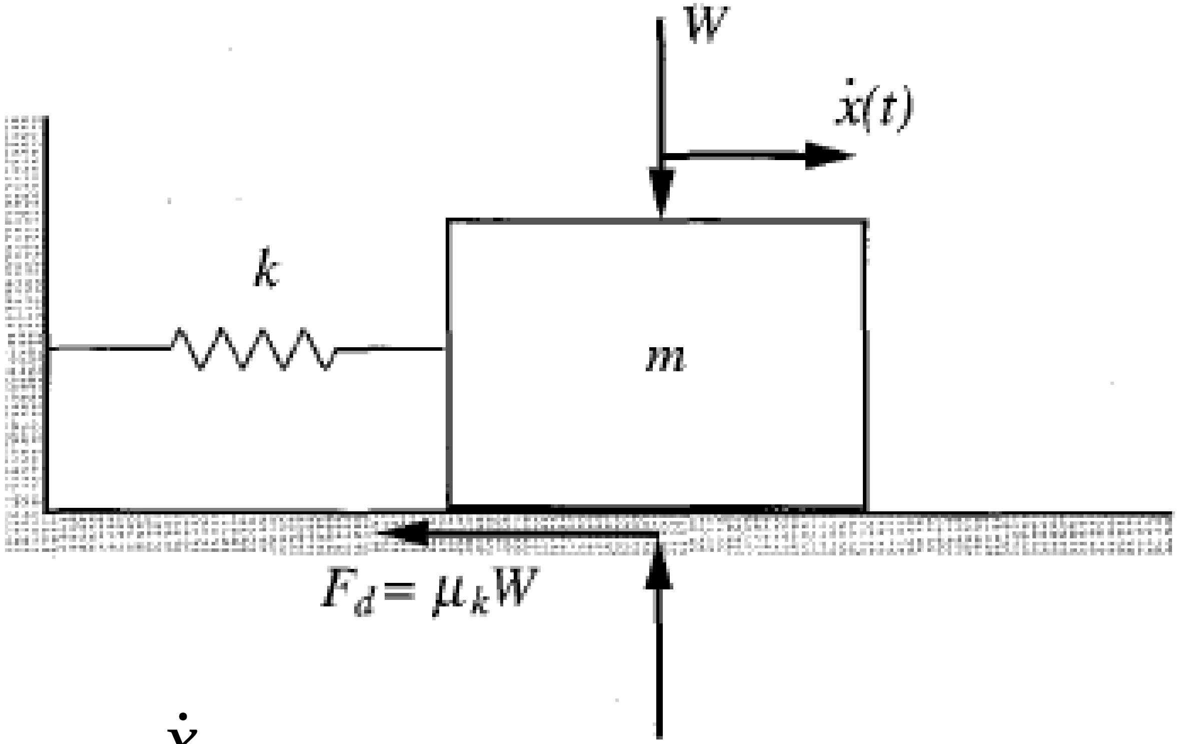
  $\mu_k < \mu_s$

- Damping force

  $$F_d = \mu_k W$$

- Equation of motion

  $$m\ddot{x} + F_d \operatorname{sgn}(\dot{x}) + kx = 0$$

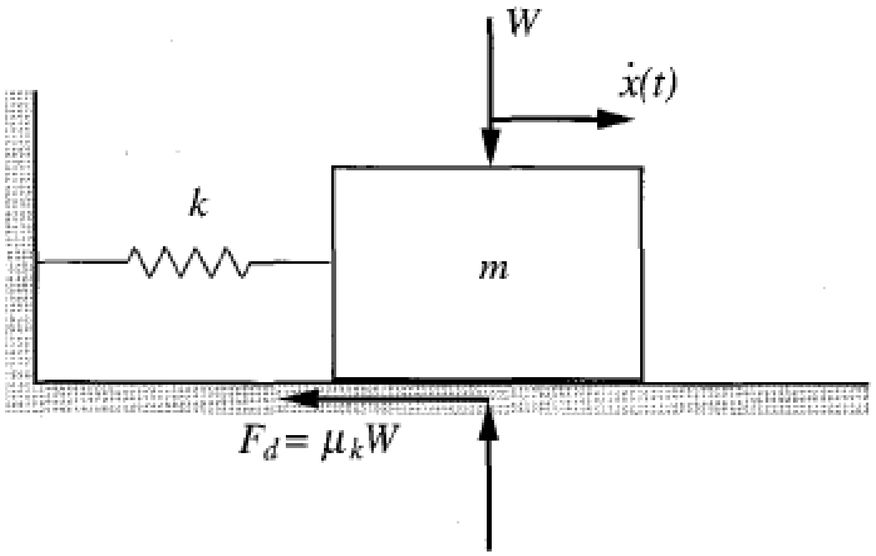


$$\operatorname{sgn}(\dot{x}) = \frac{\dot{x}}{|\dot{x}|}$$

Coulomb damping. Dry friction

$$m\ddot{x} + F_d \operatorname{sgn}(\dot{x}) + kx = 0$$

$$\begin{cases} m\ddot{x} + kx = -F_d & \text{for} \quad \dot{x} > 0 \\ m\ddot{x} + kx = F_d & \text{for} \quad \dot{x} < 0 \end{cases}$$



Solution can be obtained for one time interval at a time, depending on the sign of $\dot{x}$

Let's asume motion starts from rest and $x(0) = x_0$

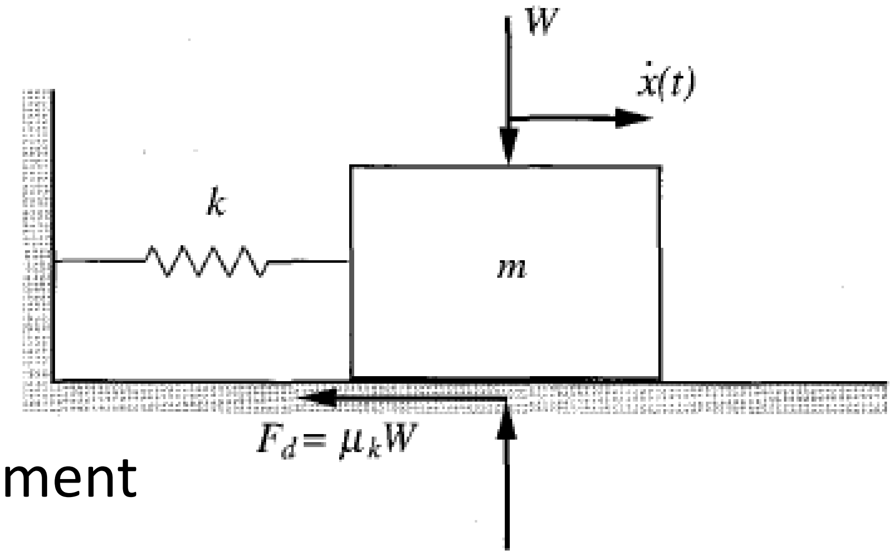
x(0) is sufficiently large so the restoring force in the spring excedes the static friction force

➔ $\dot{x} < 0$

Coulomb damping. Dry friction

$$m\ddot{x} + kx = F_d$$

$$\ddot{x} + \omega_n^2 x = \omega_n^2 f_d \qquad \omega_n^2 = \frac{k}{m}$$



$$f_d = \frac{F_d}{k}$$ represents an equivalent displacement

Initial conditions:

$$x(0) = x_0$$

$$\dot{x}(0) = 0$$

Solution:

$$x(t) = \left(x_0 - f_d\right)\cos\omega_n t + f_d$$

harmonic oscillation superposed on the average response f_d

valid for $0 \le t \le t_1$ (at t_1 velocity reduces to 0)

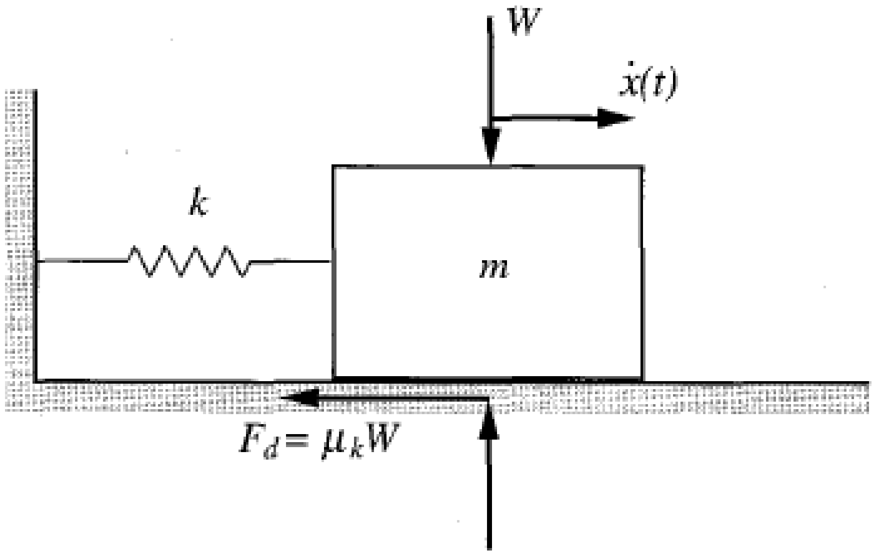
Coulomb damping. Dry friction

$$x(t) = \left(x_0 - f_d\right)\cos\omega_n t + f_d$$

$$\dot{x}(t) = -\omega_n\left(x_0 - f_d\right)\sin\omega_n t$$

$$\dot{x}(t_1) = 0 \Rightarrow t_1 = \frac{\pi}{\omega_n}$$

$$x(t_1) = -\left(x_0 - 2f_d\right)$$

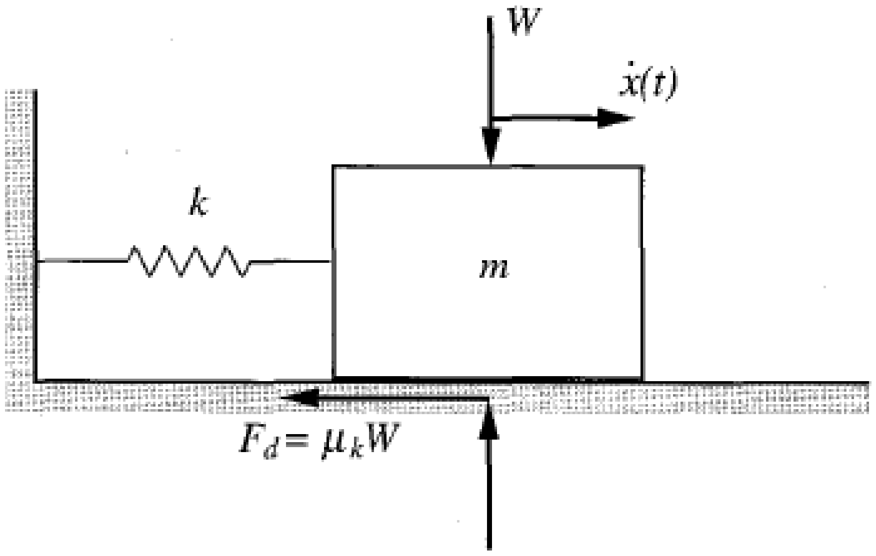


If *x(t₁)* is sufficiently large to overcome the static friction, the velocity becomes positive

➔ $$\ddot{x} + \omega_n^2 x = -\omega_n^2 f_d$$

Coulomb damping. Dry friction

$$\ddot{x} + \omega_n^2 x = -\omega_n^2 f_d$$



Initial conditions:

$$x(t_1) = -\left(x_0 - 2 f_d\right)$$

$$\dot{x}(t_1) = 0$$

$$t_2 = \frac{2\pi}{\omega_n}$$

$$x(t_2) = -\left(x_0 - 4 f_d\right)$$

Solution:

$$x(t) = \left(x_0 - 3 f_d\right)\cos\omega_n t - f_d$$

the amplitude of the harmonic oscillation is smaller by $2f_d$

valid for $t_1 \le t \le t_2$ (at t_2 velocity reduces to 0)

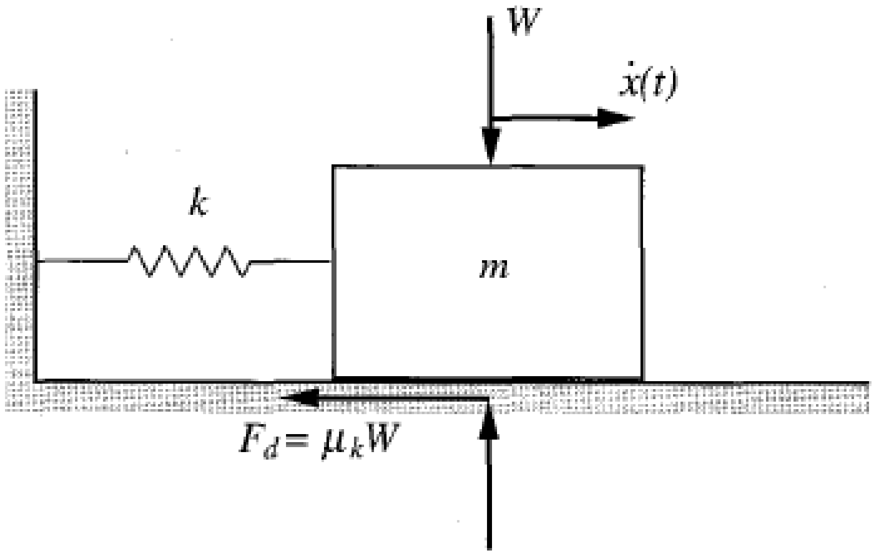
Coulomb damping. Dry friction

$$x(t) = \left(x_0 - 3f_d\right)\cos\omega_n t - f_d$$

$$t_1 \le t \le t_2$$

$$t_2 = \frac{2\pi}{\omega_n}$$

$$x(t_2) = -\left(x_0 - 4f_d\right)$$



Coulomb damping. Dry friction

- The procedure can be repeated for $t > t_2$

every time switching back and forth the sign of the damping force

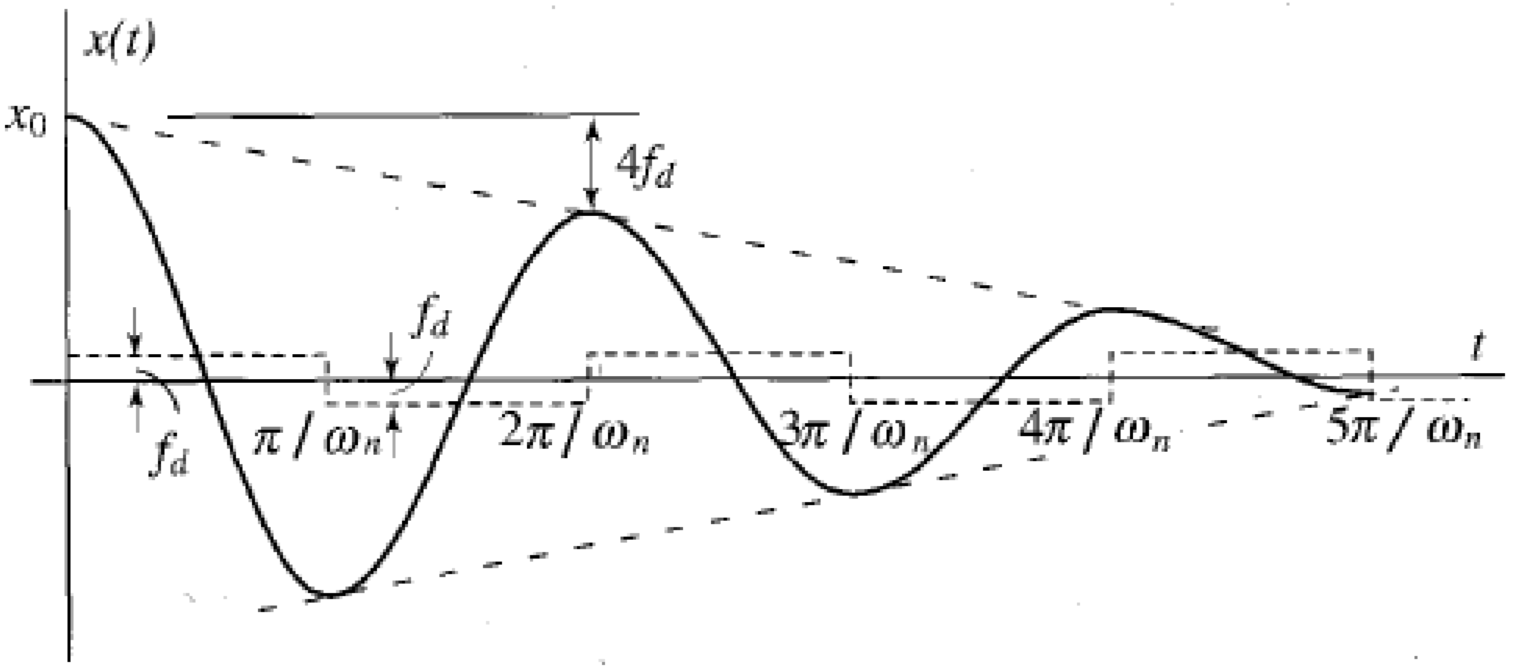
- A pattern can be identified:
  - The average value of the solution alternates between f_d and $-f_d$
  - At the end of each half-cycle the displacement magnitude is reduced by $2f_d$

➔for Coulomb damping the decay is linear with time, as opposed to the exponential decay for viscous damping

- The motion stops abruptly when the displacement at the end of a given half-cycle is not sufficiently large for the restoring force in the spring to overcome the static friction

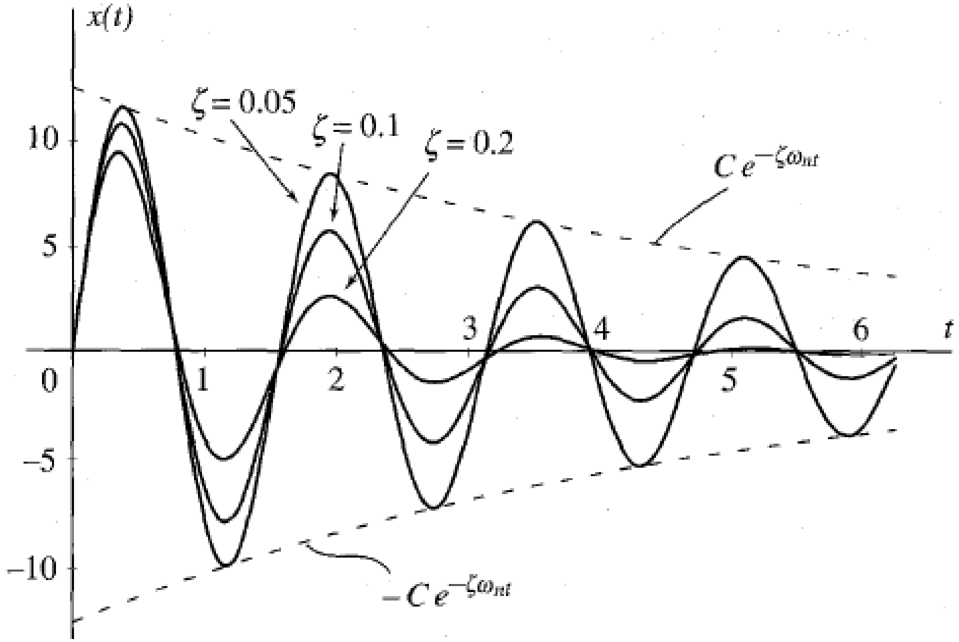
Coulomb damping. Dry friction

- The motion stops at the end of the half-cycle for which the amplitude of the harmonic component is smaller than $2f_d$
- Denoting by *n* the half-cycle just prior to the cessation of motion, we conclude that *n* is the smallest integer satisfying the inequality

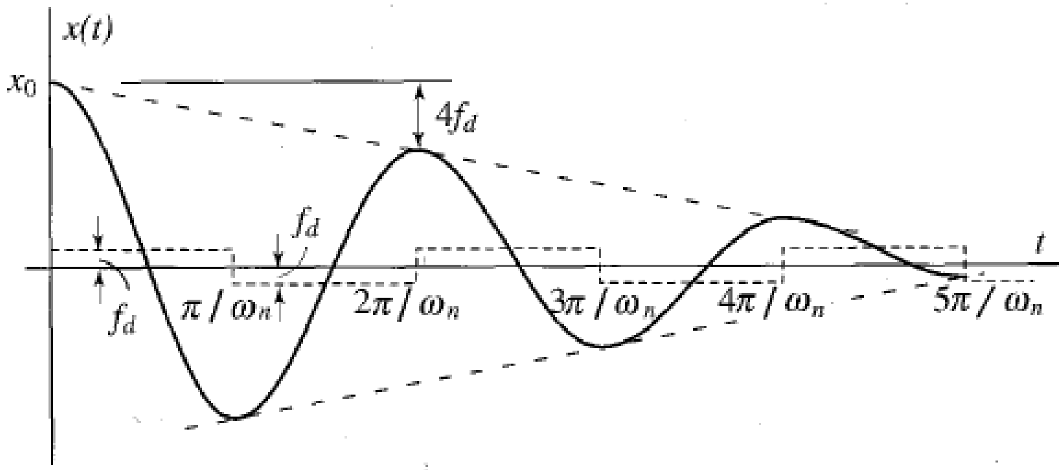


$$x_0 - (2n-1) f_d < \left(1 + \frac{\mu_s}{\mu_k}\right) f_d$$

Coulomb damping. Dry friction



Viscous damping



Coulomb damping