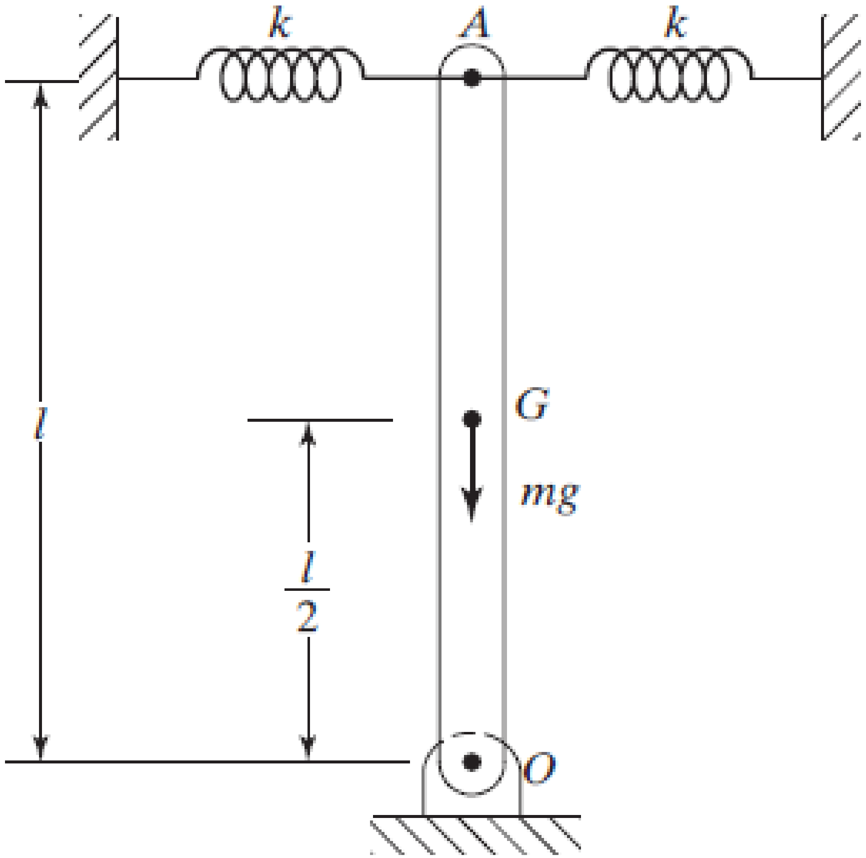
System Dynamics and Vibrations

Prof. Gustavo Alonso

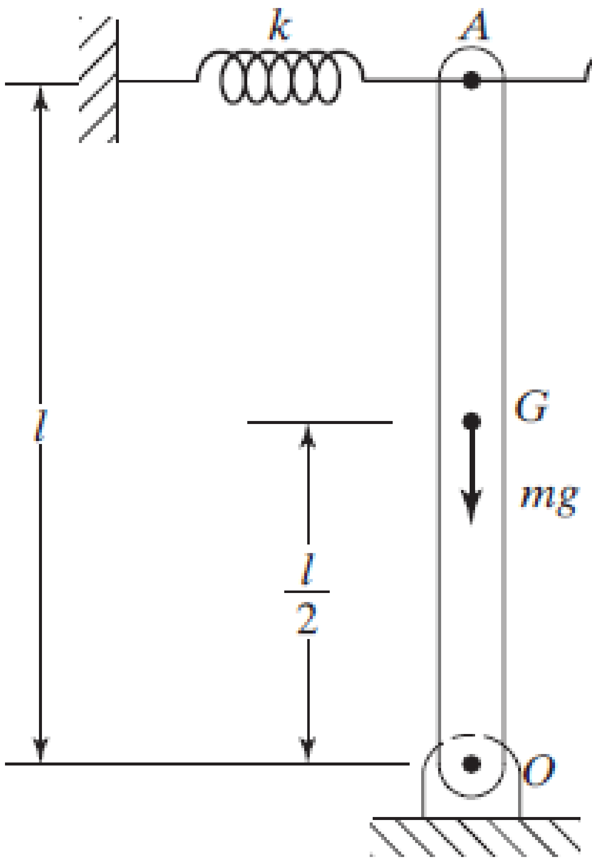
Chapter 5: Dynamic stability. Part I. Exercise

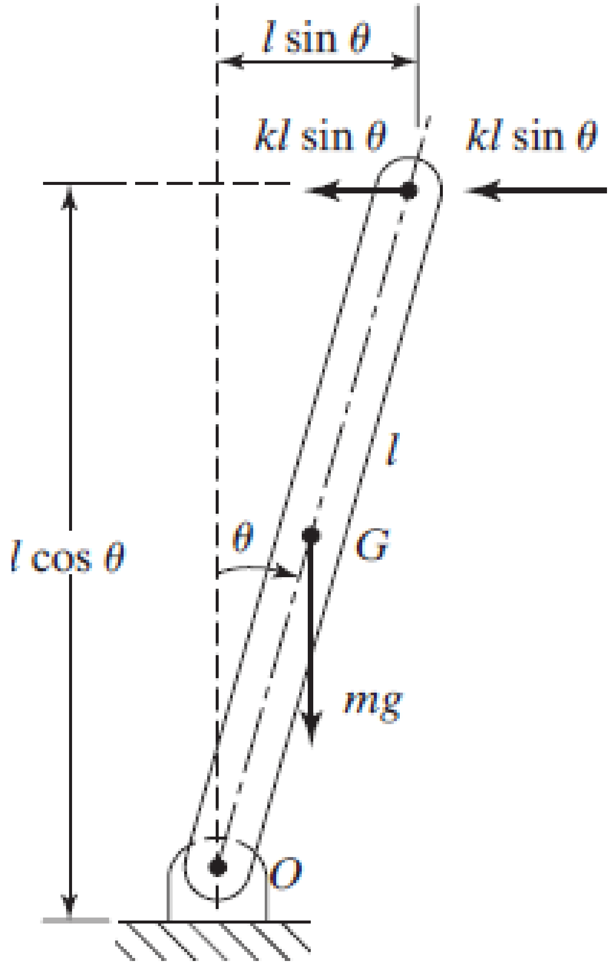
School of General Engineering
Beihang University (BUAA)

Consider a uniform rigid bar, of mass m and legth l, pivoted at one end and connected symmetrically by two springs at the other end. Assuming that the springs are unstretched when the bar is vertical, derive the equation of motion of the system for small angular displacements (θ) of the bar about the pivot point, and investigate the stability of the system.



Consider a uniform rigid bar, of mass *m* and legth *l*, pivoted at one end and connected symmetrically by two springs at the other end. Assuming that the springs are unstretched when the bar is vertical, derive the equation of motion of the system for small angular displacements (θ) of the bar about the pivot point, and investigate the stability of the system.

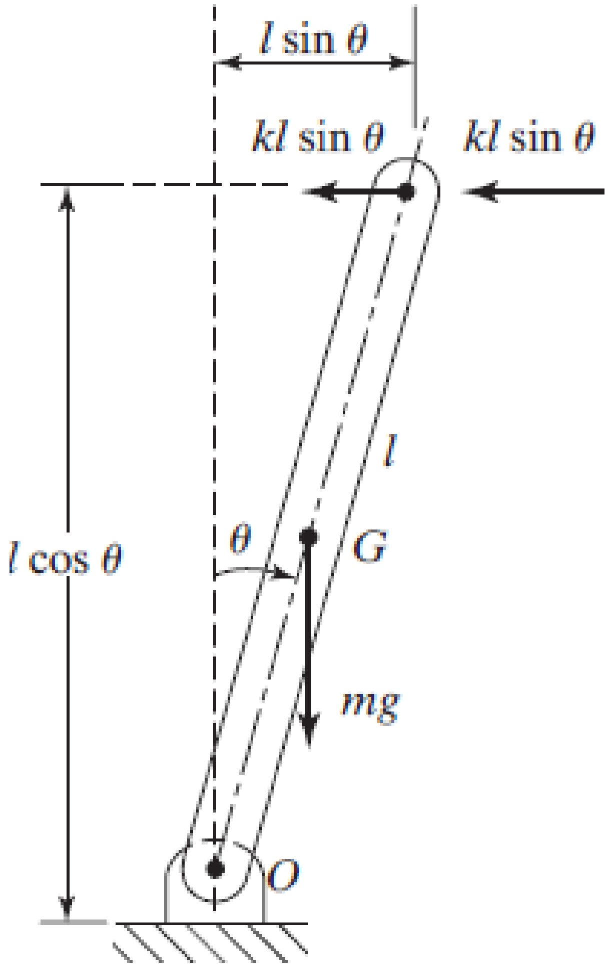




Equation of motion of the bar: rotation about point O

➔ Moment due to angular acceleration:

$$J_0\ddot{\theta} = \left(\frac{ml^2}{3}\right)\ddot{\theta}$$

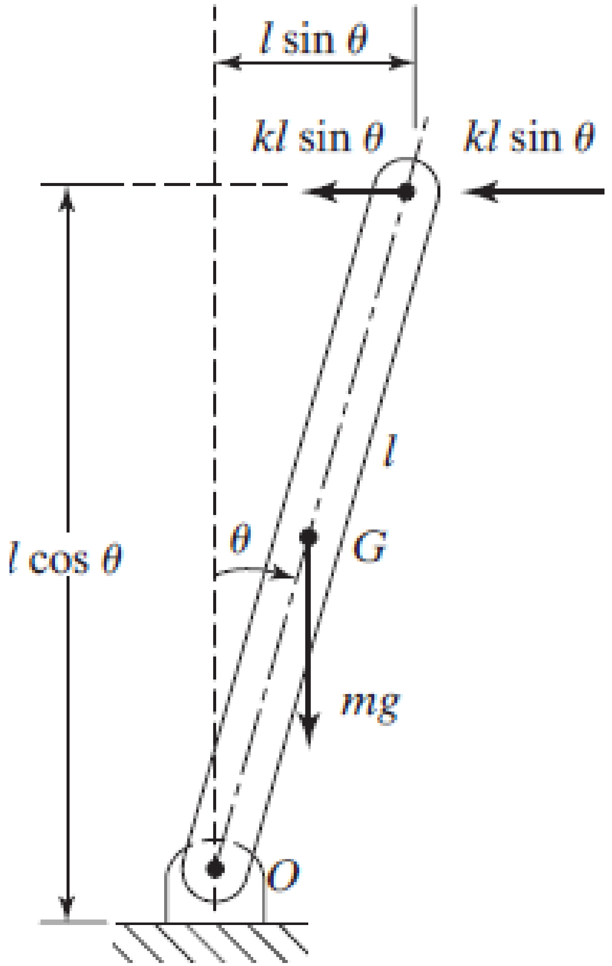


$$J_0\ddot{\theta} = \left(\frac{ml^2}{3}\right)\ddot{\theta}$$

$$\frac{ml^2}{3}\ddot{\theta} + (2kl\sin\theta)l\cos\theta - W\frac{l}{2}\sin\theta = 0$$

$$\frac{ml^2}{3}\ddot{\theta} + 2kl^2\theta - \frac{Wl}{2}\theta = 0$$ (small rotations)

$$\ddot{\theta} + \alpha^2\theta = 0 \qquad \alpha^2 = \left(\frac{12kl^2 - 3Wl}{2ml^2}\right)$$

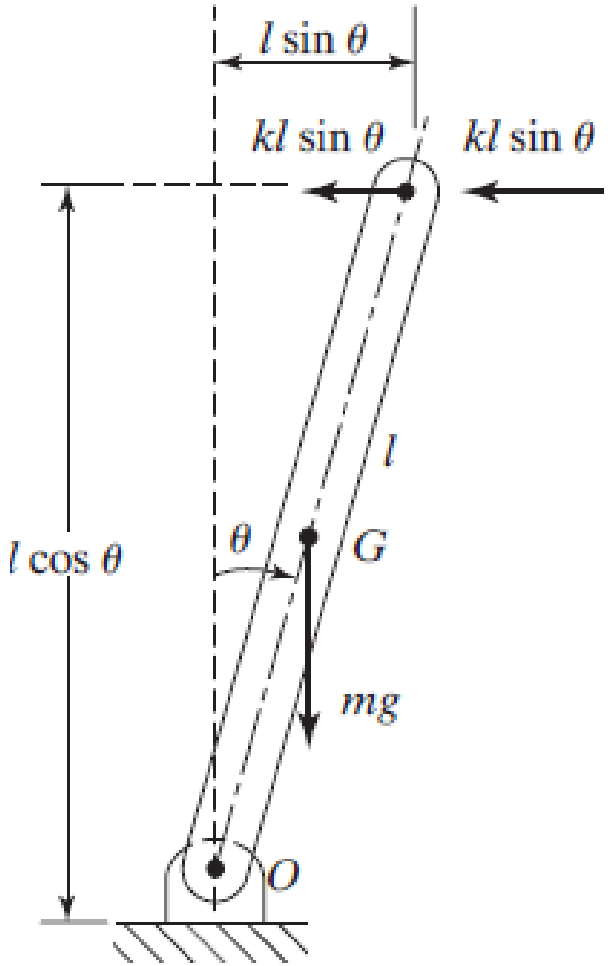


$$\ddot{\theta} + \alpha^2 \theta = 0$$

characteristic equation:

$$s^2 + \alpha^2 = 0$$

The solution depends on the sign of α^2:



Linearization about equilibrium points

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- The nature of the motion (around equilibrium points) depends on the values of the roots *s* (complex numbers, in general):
  - In all cases in which s_1 and s_2 are both real and negative or complex conjugates with negative real part the motion in the neighborhood of an equilibrium point is asymptotically stable
  - In all cases in which s_1 and s_2 are pure imaginary the motion is merely stable
  - If either s_1 or s_2 is real and positive, or both s_1 and s_2 are real and positive, or s_1 and s_2 are complex conjugates with positive real part, the motion is unstable

The solution depends on the sign of α^2:

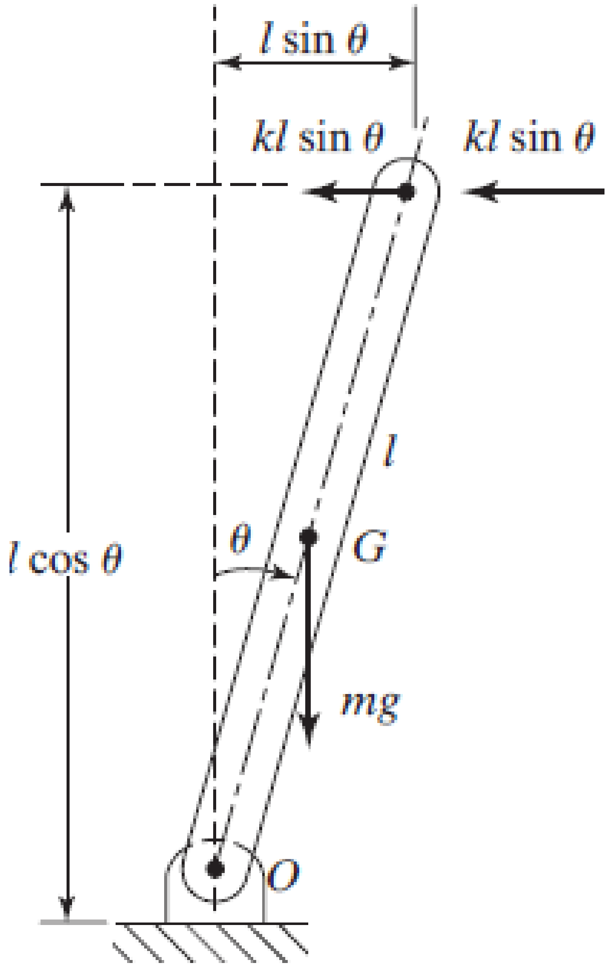
Case 1:

$$(12kl^2 - 3Wl)/2ml^2 > 0,$$

Stable oscillations:

$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

$$\omega_n = \left(\frac{(12kl^2 - 3Wl}{2ml^2} \right)^{1/2}$$



The solution depends on the sign of α^2:

Case 2:

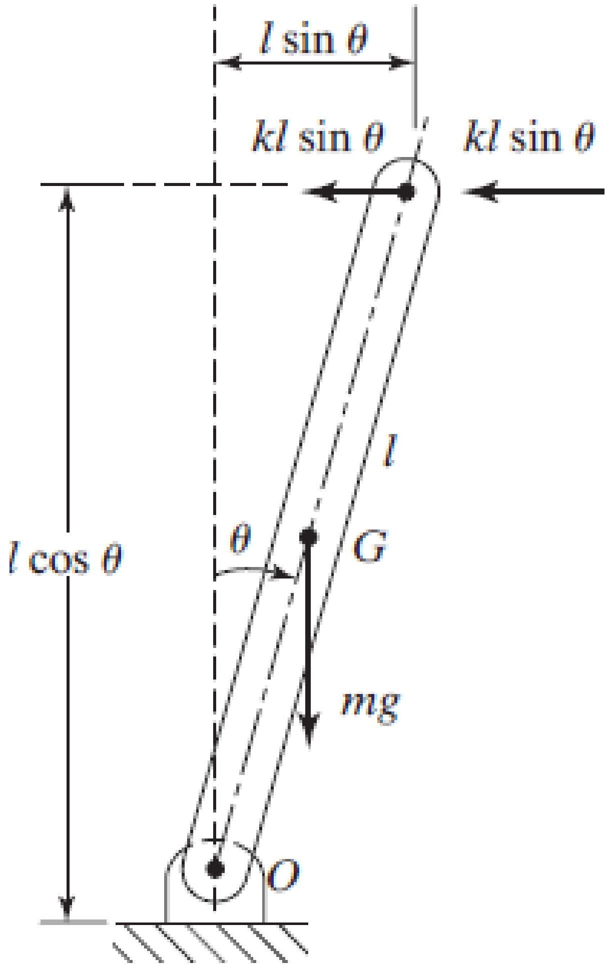
$$(12kl^2 - 3Wl)/2ml^2 = 0,$$

Equation of motion reduces to: $\ddot{\theta} = 0$

$$\theta(t) = C_1 t + C_2$$

For initial conditions:

$$\left.\begin{aligned} \theta(t=0) &= \theta_0 \\ \dot{\theta}(t=0) &= \dot{\theta}_0 \end{aligned}\right\} \quad \theta(t) = \dot{\theta}_0 t + \theta_0$$



➔ instable system

The solution depends on the sign of α^2:

Case 3:

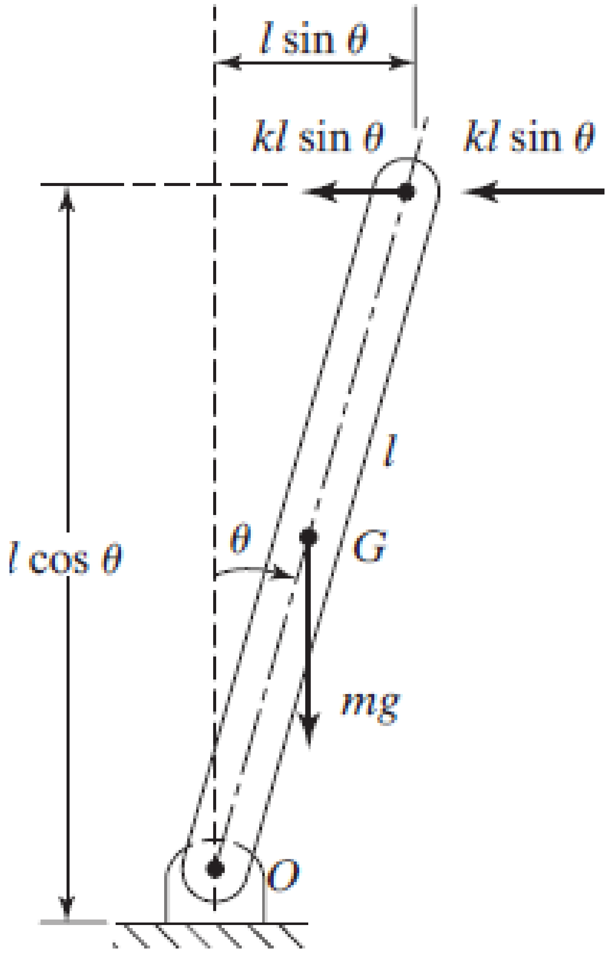
$$(12kl^2 - 3Wl)/2ml^2 < 0,$$

The solution is:

$$\theta(t) = B_1 e^{\alpha t} + B_2 e^{-\alpha t}$$

For initial conditions:

$$\left.\begin{aligned} \theta(t = 0) &= \theta_0 \\ \dot{\theta}(t = 0) &= \dot{\theta}_0 \end{aligned}\right\} \quad \theta(t) = \frac{1}{2\alpha}[(\alpha\theta_0 + \dot{\theta}_0)e^{\alpha t} + (\alpha\theta_0 - \dot{\theta}_0)e^{-\alpha t}]$$



➔ instable system