System Dynamics and Vibrations

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Chapter 6: Two-degree-of-freedom systems
Exercises - 3

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Exercise 5

The two-degree-of-freedom system of the figure consists of two masses on a string of tension *T* vibrating in the vertical plane. Let

$m_1 = m$

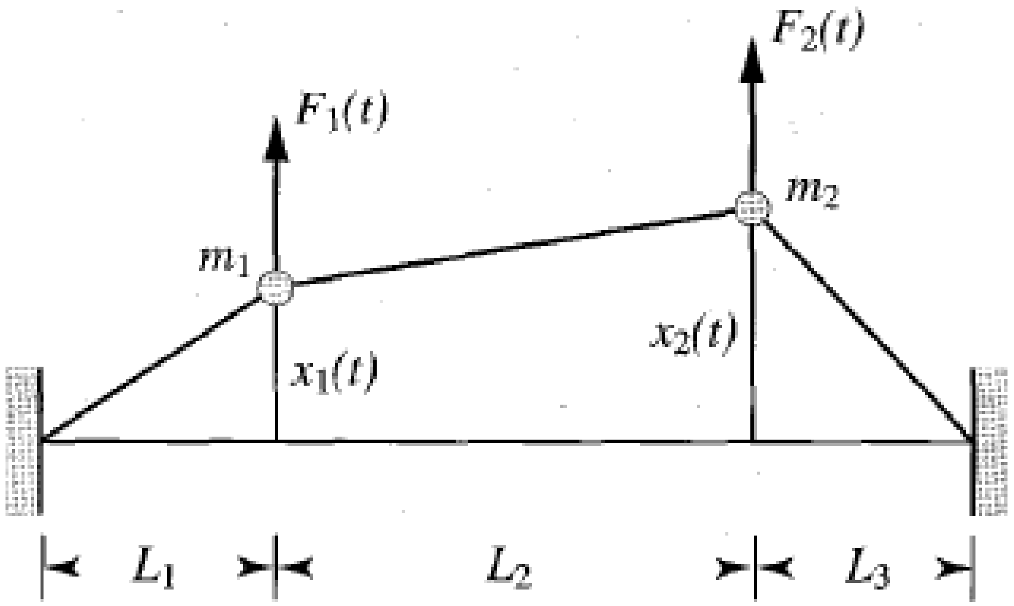
$m_2 = 2m$

$L_1 = L_2 = L$

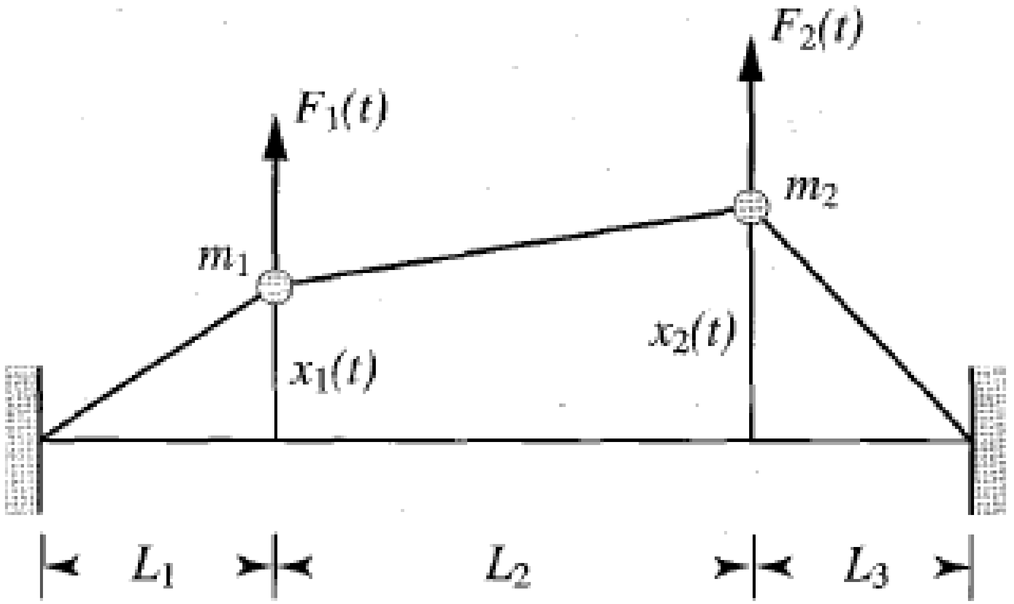
$L_3 = 0.5L$

Determine the response to an <u>initial displacement</u>:

$$x_{10} = 1.2 \text{ cm}$$



Exercise 5



From Exercise 3 we have already obtained:

- The natural frequencies
- The modal matrix

$$\omega_1 = \sqrt{\frac{T}{mL}}$$

$$\omega_2 = \sqrt{\frac{5T}{2mL}}$$

$$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix}$$

Exercise 5

$$\mathbf{x}(t)=\mathbf{x}_1(t)+\mathbf{x}_2(t)=C_1\cos\left(\omega_1 t-\phi_1\right)\mathbf{u}_1+C_2\cos\left(\omega_2 t-\phi_2\right)\mathbf{u}_2$$

$$=\frac{1}{|U|}\left\{\left[\left(u_{22}x_{10}-u_{12}x_{20}\right)\cos\omega_1 t+\frac{u_{22}v_{10}-u_{12}v_{20}}{\omega_1}\sin\omega_1 t\right]\mathbf{u}_1+\left[\left(u_{11}x_{20}-u_{21}x_{10}\right)\cos\omega_2 t+\frac{u_{11}v_{20}-u_{21}v_{10}}{\omega_2}\sin\omega_2 t\right]\mathbf{u}_2\right\}$$

$$|U|=u_{11}u_{22}-u_{12}u_{21}=-1.5$$

$$\mathbf{x}(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=\frac{1}{-1.5}\left\{\left(-0.5\right)\times 1.2\cos\sqrt{\frac{T}{mL}}t\begin{bmatrix}1\\1\end{bmatrix}-1\times 1.2\cos\sqrt{\frac{5T}{2mL}}t\begin{bmatrix}1\\-0.5\end{bmatrix}\right\}$$

$$=\begin{bmatrix}0.4\cos\sqrt{\frac{T}{mL}}t+0.8\cos\sqrt{\frac{5T}{2mL}}t\\0.4\cos\sqrt{\frac{T}{mL}}t-0.4\cos\sqrt{\frac{5T}{2mL}}t\end{bmatrix}(\mathrm{cm})$$

Exercise 6

The two-degree-of-freedom system of the figure consists of two masses on a string of tension *T* vibrating in the vertical plane. Let

$m_1 = m$

$m_2 = 2m$

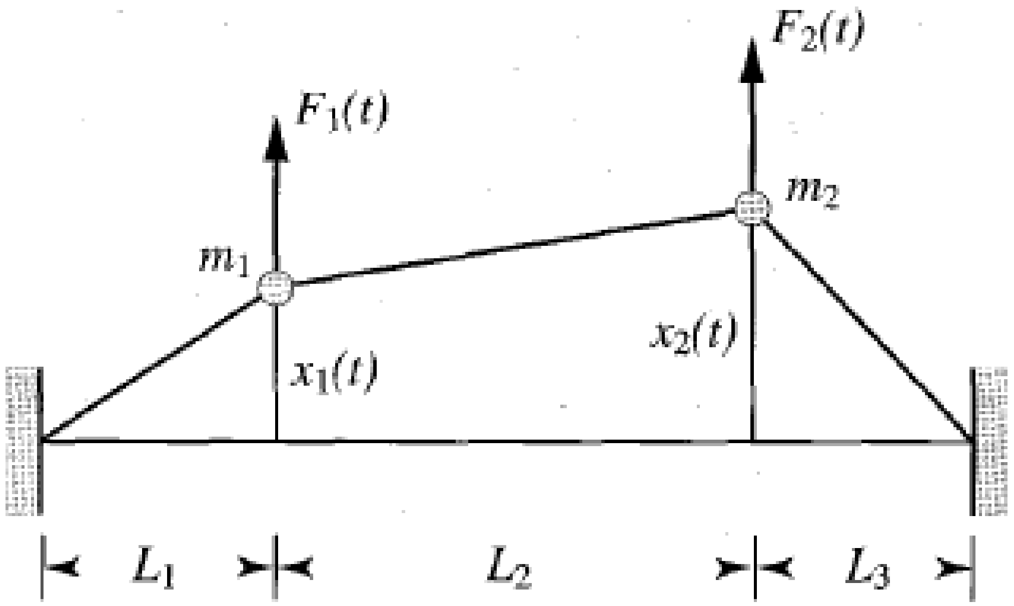
$L_1 = L_2 = L$

$L_3 = 0.5L$

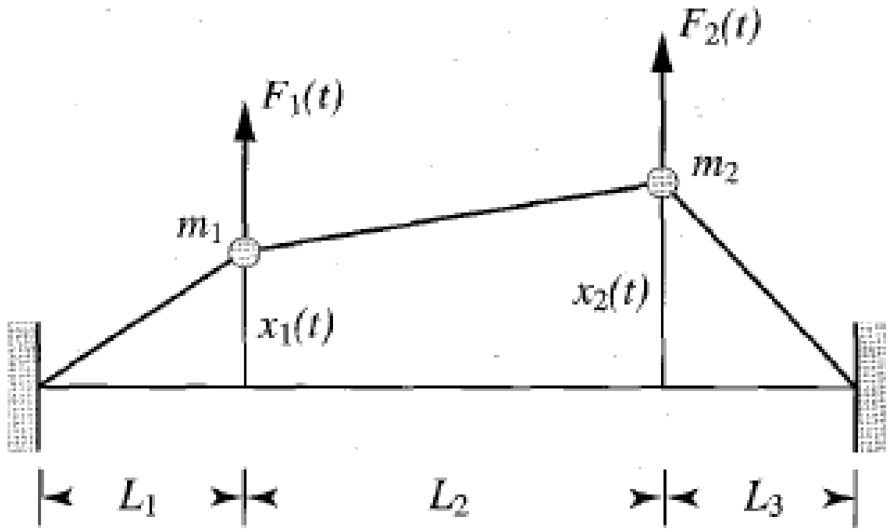
Determine the response to an <u>initial displacement</u>:

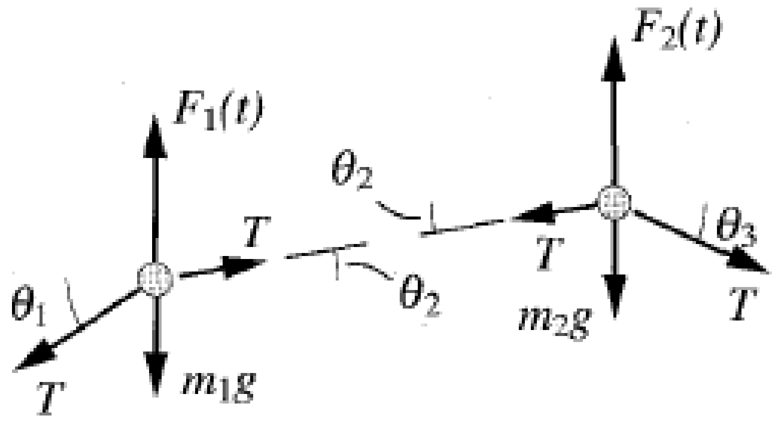
$$x_{10} = 1.2 \text{ cm}$$

<u>**using modal analysis**</u>



Exercise 6





$$m_1 \frac{d^2 x_1}{dt^2} + \left(\frac{T}{L_1} + \frac{T}{L_2} \right) x_1 - \frac{T}{L_2} x_2 = F_1$$

$$m_2 \frac{d^2 x_2}{dt^2} - \frac{T}{L_2} x_1 + \left(\frac{T}{L_2} + \frac{T}{L_3} \right) x_2 = F_2$$

(with the assumption that displacements are small and being x_i the vibration about the equilibrium position)

Exercise 6

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = 0$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = m\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\mathbf{x}(0) = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix}, \ \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \frac{T}{L}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\omega_1 = \sqrt{\frac{T}{mL}}$$

$$\omega_2 = \sqrt{\frac{5T}{2mL}}$$

$$\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

Exercise 6

Modal analysis:

- Linear transformation $\mathbf{x}(t) = q_1(t)\mathbf{u}_1 + q_2(t)\mathbf{u}_2 = q_1(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + q_2(t)\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

- Modal equations

$$\ddot{q}_1(t) + \omega_1^2 q_1(t) = 0$$
$$\ddot{q}_2(t) + \omega_2^2 q_2(t) = 0$$

$$\Rightarrow$$

$$\ddot{q}_1(t) + \sqrt{\frac{T}{mL}} q_1(t) = 0$$
$$\ddot{q}_2(t) + \sqrt{\frac{5T}{2mL}} q_2(t) = 0$$

Exercise 6

Modal analysis:

- Initial modal displacements

$$\mathbf{x}(t) = q_1(t)\mathbf{u}_1 + q_2(t)\mathbf{u}_2 = q_1(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + q_2(t)\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = q_1(0)\mathbf{u}_1 + q_2(0)\mathbf{u}_2 = q_1(0)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + q_2(0)\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$\Rightarrow \quad q_1(0) = 0.4, \; q_2(0) = 0.8$$

- Initial modal velocities $\quad \dot{\mathbf{x}}(0) = 0 \quad \Rightarrow \quad \dot{q}_1(0) = \dot{q}_2(0) = 0$

Exercise 6

Modal analysis:

$$\ddot{q}_1(t) + \sqrt{\frac{T}{mL}} q_1(t) = 0$$

$$\ddot{q}_2(t) + \sqrt{\frac{5T}{2mL}} q_2(t) = 0$$

$$q_1(0) = 0.4, \ q_2(0) = 0.8$$

$$\dot{q}_1(0) = \dot{q}_2(0) = 0$$

$$q_1(t) = q_1(0)\cos\omega_1 t = 0.4\cos\sqrt{\frac{T}{mL}}t$$

$$q_2(t) = q_2(0)\cos\omega_2 t = 0.8\cos\sqrt{\frac{5T}{2mL}}t$$

$$\mathbf{x}(t) = 0.4\cos\sqrt{\frac{T}{mL}}t\begin{bmatrix}1\\1\end{bmatrix} + 0.8\cos\sqrt{\frac{5T}{2mL}}t\begin{bmatrix}1\\-0.5\end{bmatrix}$$