System Dynamics and Vibrations

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Chapter 6: Two-degree-of-freedom systems
Exercises - 1

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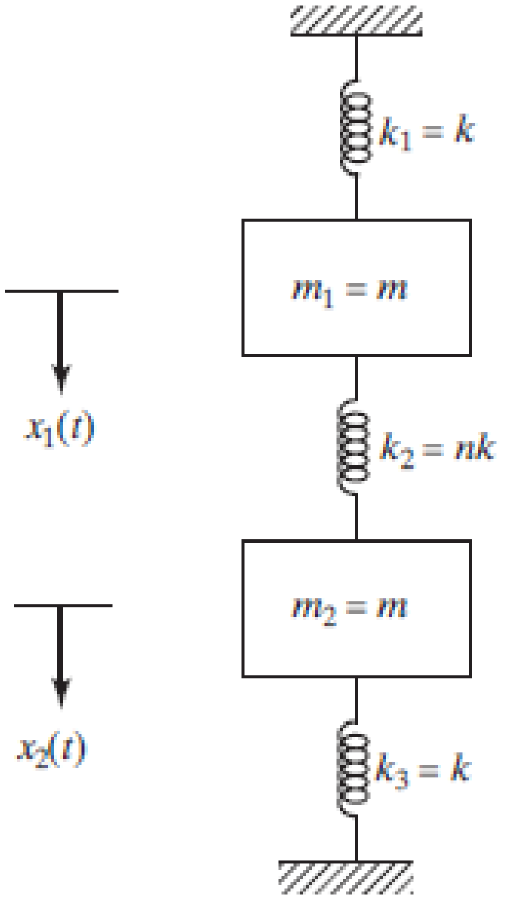
Exercise 1

Find the natural frequencies and mode shapes of a spring-mass system which is constrained to move in the vertical direction only.
Take $n = 1$

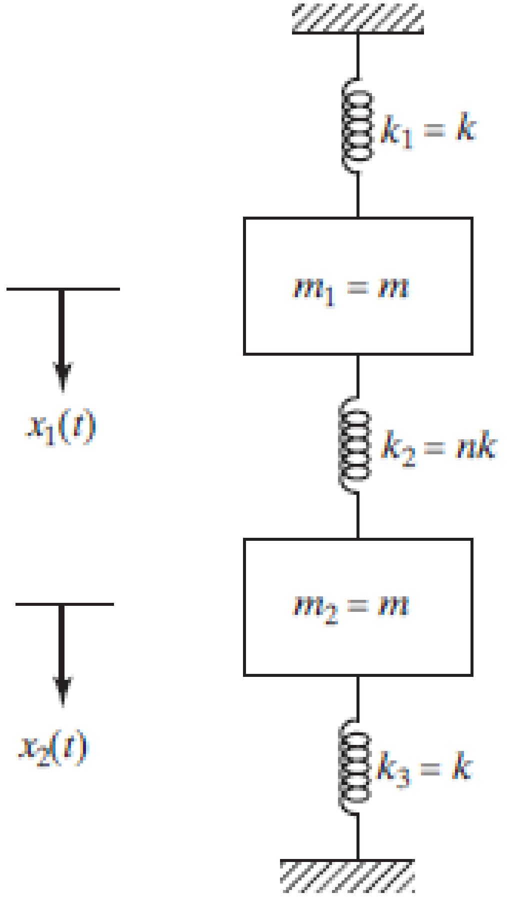
Approach: measure x_1 and x_2 from the static equilibrium position of the masses m_1 and m_2 respectively



Exercise 1

$$m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 - k_2 x_1 + \left(k_2 + k_3\right)x_2 = 0$$

$$m_1\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m_2\ddot{x}_2 - kx_1 + 2kx_2 = 0$$



Exercise 1

harmonic solutions:

$$x_1(t) = X_1 \cos(\omega t + \phi)$$
$$x_2(t) = X_2 \cos(\omega t + \phi)$$

$$m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 - k_2x_1 + \left(k_2 + k_3\right)x_2 = 0$$

Exercise 1

$$\det\begin{bmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + 2k \end{bmatrix} = 0$$

$$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$

characteristic equation

Exercise 1

$$\omega_1 = \left\{ \frac{4km - \left[16k^2m^2 - 12k^2m^2\right]1/2}{2m^2} \right\}^{1/2} = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \left\{ \frac{4km + \left[16k^2m^2 - 12k^2m^2\right]1/2}{2m^2} \right\}^{1/2} = \sqrt{\frac{3k}{m}}$$

Exercise 1

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + 2k}{k} = \frac{k}{-m\omega_1^2 + 2k} = 1$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m\omega_2^2 + 2k}{k} = \frac{k}{-m\omega_1^2 + 2k} = -1$$

Exercise 1

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}} t + \phi_1 \right) \\ X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}} t + \phi_1 \right) \end{Bmatrix} = \text{first mode}$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}} t + \phi_2 \right) \\ -X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}} t + \phi_2 \right) \end{Bmatrix} = \text{second mode}$$

Exercise 1

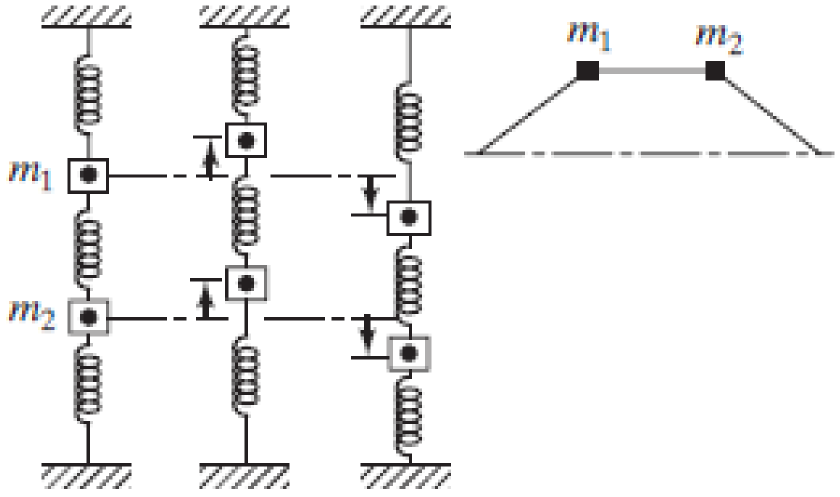
$$x_1(t) = x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) + X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$

$$x_2(t) = x_2^{(1)}(t) + x_2^{(2)}(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) - X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$

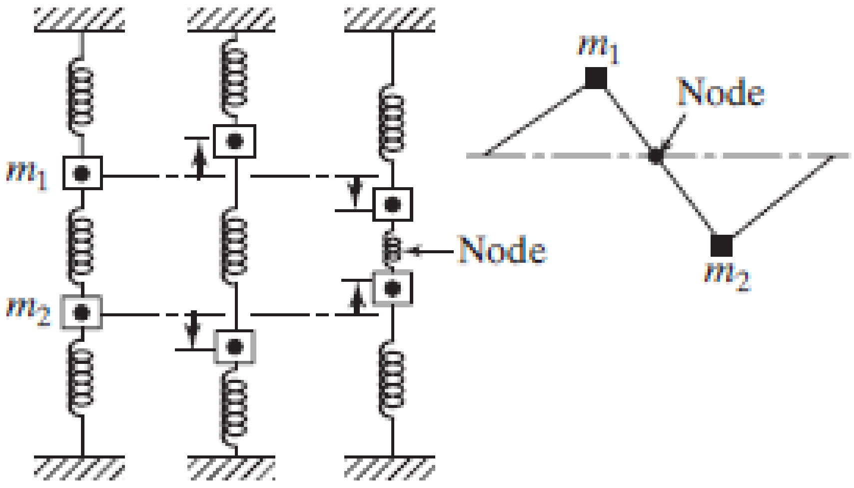
Exercise 1

$$x_1(t) = x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) + X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$

$$x_2(t) = x_2^{(1)}(t) + x_2^{(2)}(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) - X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$



(a) First mode



(b) Second mode

Exercise 2

Consider a torsional system consisting of two discs mounted on a shaft. The three segments of the shaft have rotational spring constants k_{t1}, k_{t2}, and k_{t3}

The discs have mass moments of inertia J_1 and J_2 respectively.

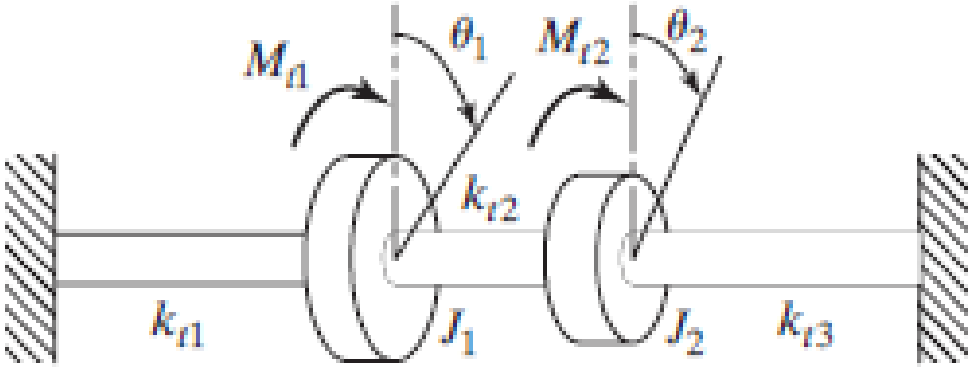
Find the natural frequencies and mode shapes for the system for:

$J_1 = J_0$

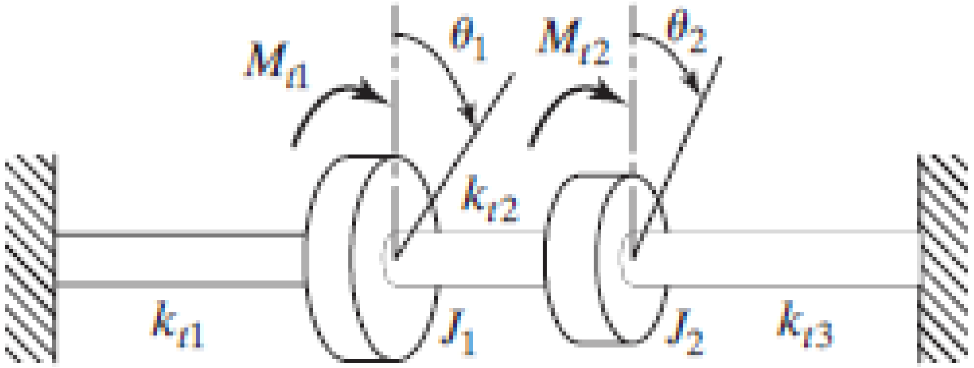
$J_2 = 2\ J_0$

$k_{t1}, = k_{t2} = k_t$

$k_{t3} = 0$



Exercise 2



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$J_1 = J_0$

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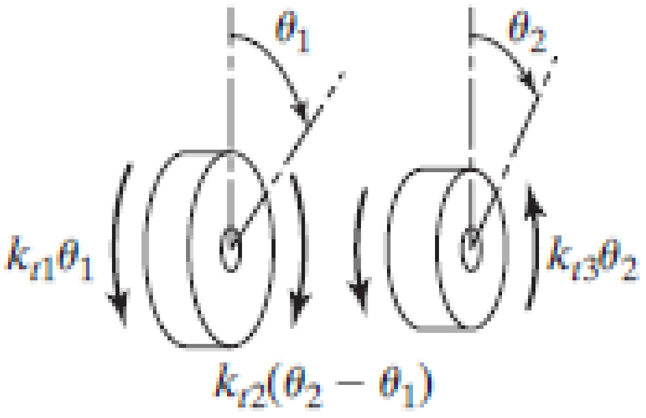
$k_{t1}, = k_{t2} = k_t$

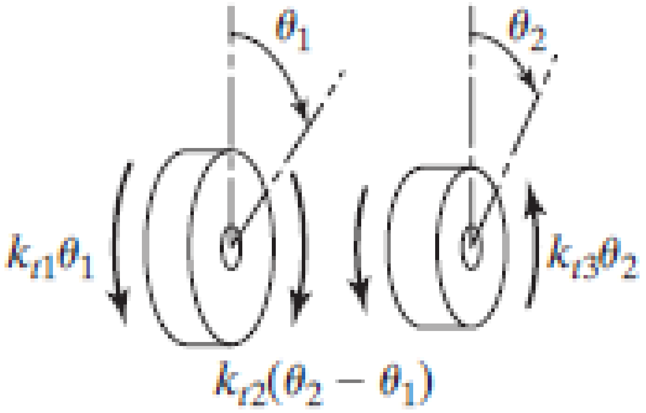
$k_{t3} = 0$

<u>Approach: write the equations of motion</u>:



Exercise 2

Equations of motion:

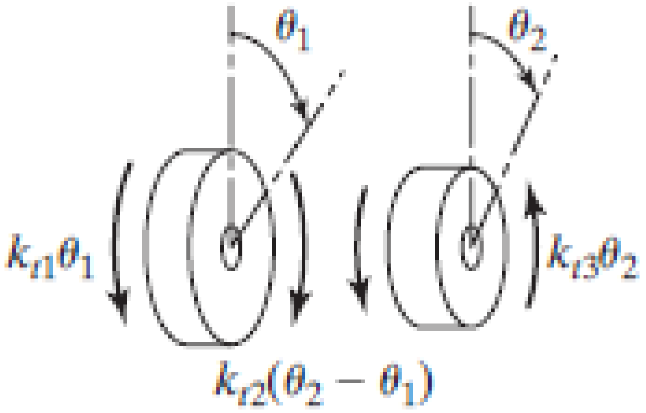


$$J_1\ddot{\theta}_1 = -k_{t1}\theta_1 + k_{t2}\left(\theta_2 - \theta_1\right) + M_{t1}$$

$$J_2\ddot{\theta}_2 = -k_{t2}\left(\theta_2 - \theta_1\right) - k_{t3}\theta_2 + M_{t2}$$

Exercise 2

Equations of motion:



$$J_1\ddot{\theta}_1 = -k_{t1}\theta_1 + k_{t2}\left(\theta_2 - \theta_1\right) + M_{t1}$$

$$J_2\ddot{\theta}_2 = -k_{t2}\left(\theta_2 - \theta_1\right) - k_{t3}\theta_2 + M_{t2}$$

Free vibration:

$$J_1\ddot{\theta}_1 + \left(k_{t1} + k_{t2}\right)\theta_1 - k_{t2}\theta_2 = 0$$

$$J_2\ddot{\theta}_2 - k_{t2}\theta_1 + \left(k_{t2} + k_{t3}\right)\theta_2 = 0$$

Exercise 2

For:

$J_1 = J_0$

$J_2 = 2\,J_0$

$k_{t1}, = k_{t2} = k_t$

$k_{t3} = 0$

$$J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - k_t\theta_1 + k_t\theta_2 = 0$$

Exercise 2

$$J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - k_t\theta_1 + k_t\theta_2 = 0$$

$$\theta_1(t) = \Theta_1 \cos\left(\omega t + \phi\right)$$

$$\theta_2(t) = \Theta_2 \cos\left(\omega t + \phi\right)$$

Exercise 2

$$J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - k_t\theta_1 + k_t\theta_2 = 0$$

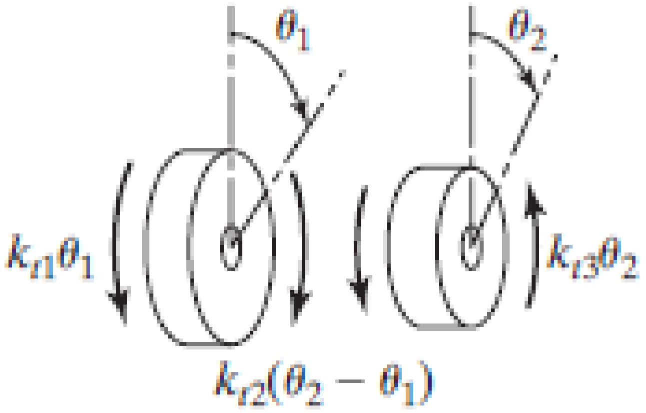
$$\theta_1(t) = \Theta_1 \cos\left(\omega t + \phi\right)$$

$$\theta_2(t) = \Theta_2 \cos\left(\omega t + \phi\right)$$

$$2\omega^4 J_0^2 - 5\omega^2 J_0 k_t + k_t^2 = 0$$

Exercise 2

$$2\omega^4 J_0^2 - 5\omega^2 J_0 k_t + k_t^2 = 0$$



$$\omega_1 = \sqrt{\frac{k_t}{4J_0}\left(5 - \sqrt{17}\right)}$$

$$\omega_2 = \sqrt{\frac{k_t}{4J_0}\left(5 + \sqrt{17}\right)}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = 2 - \frac{\left(5 - \sqrt{17}\right)}{4}$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = 2 - \frac{\left(5 + \sqrt{17}\right)}{4}$$