System Dynamics and Vibrations

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Chapter 3: Single degree-of-freedom systems
Part 3

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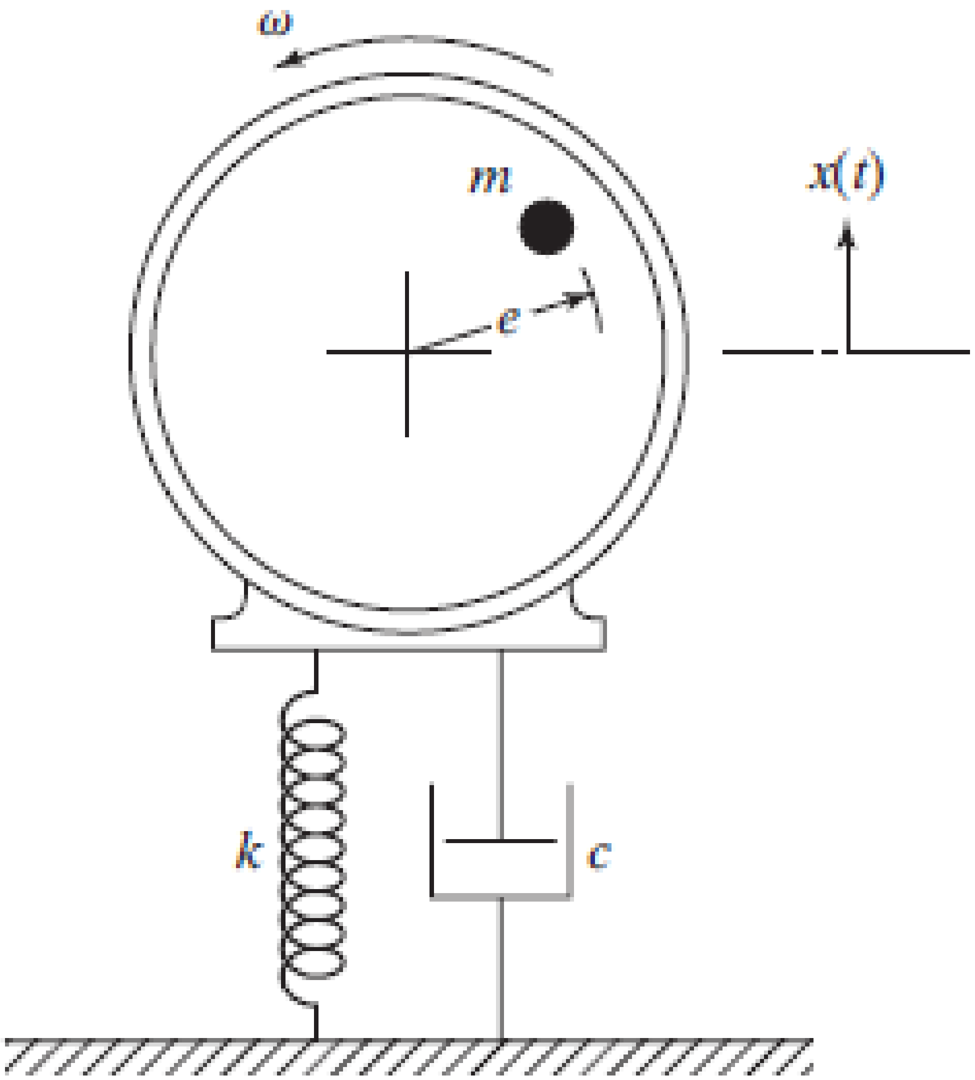


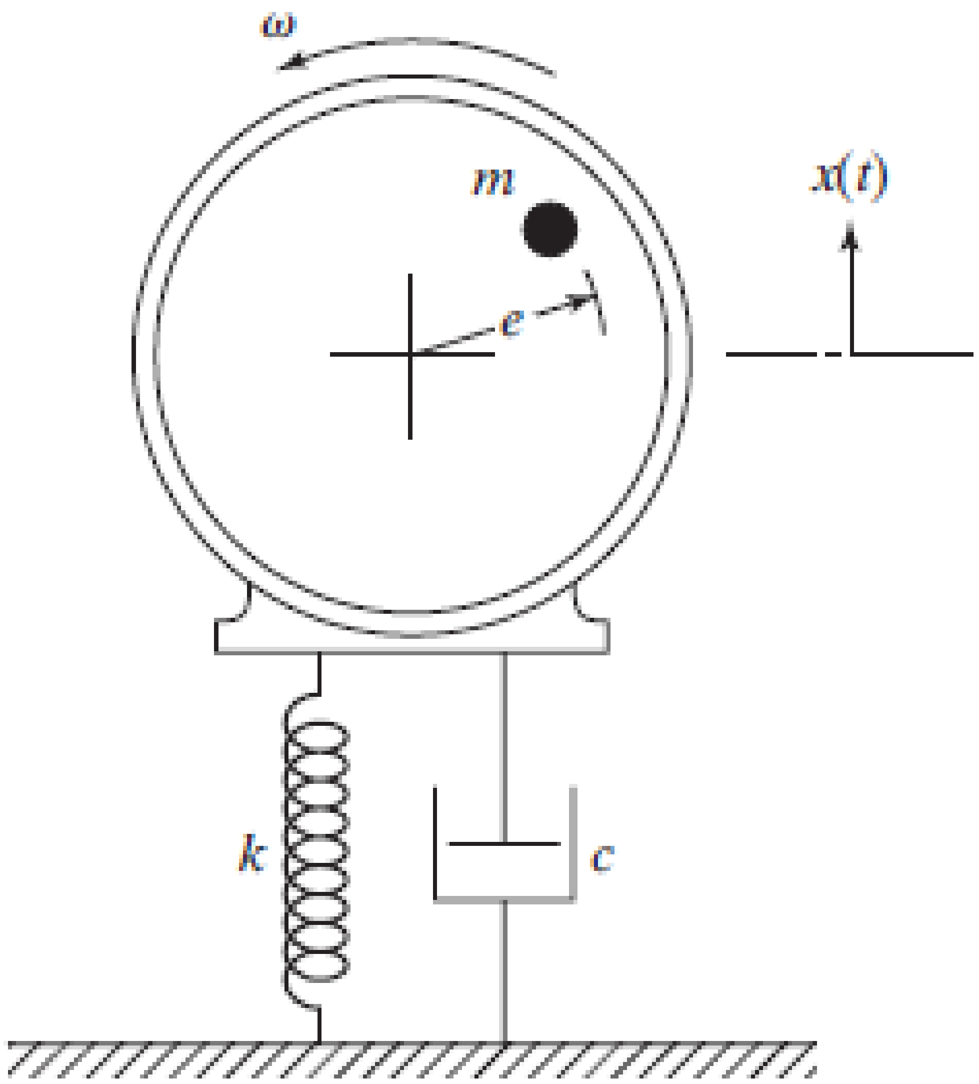
Systems with rotating unbalanced masses

$$M\ddot{x}(t)+c\dot{x}(t)+kx(t)=me\omega^2\sin\omega t$$

$$\ddot{x}(t)+2\varsigma\omega_n\dot{x}(t)+\omega_n^2x(t)=\frac{m}{M}e\omega^2\sin\omega t$$

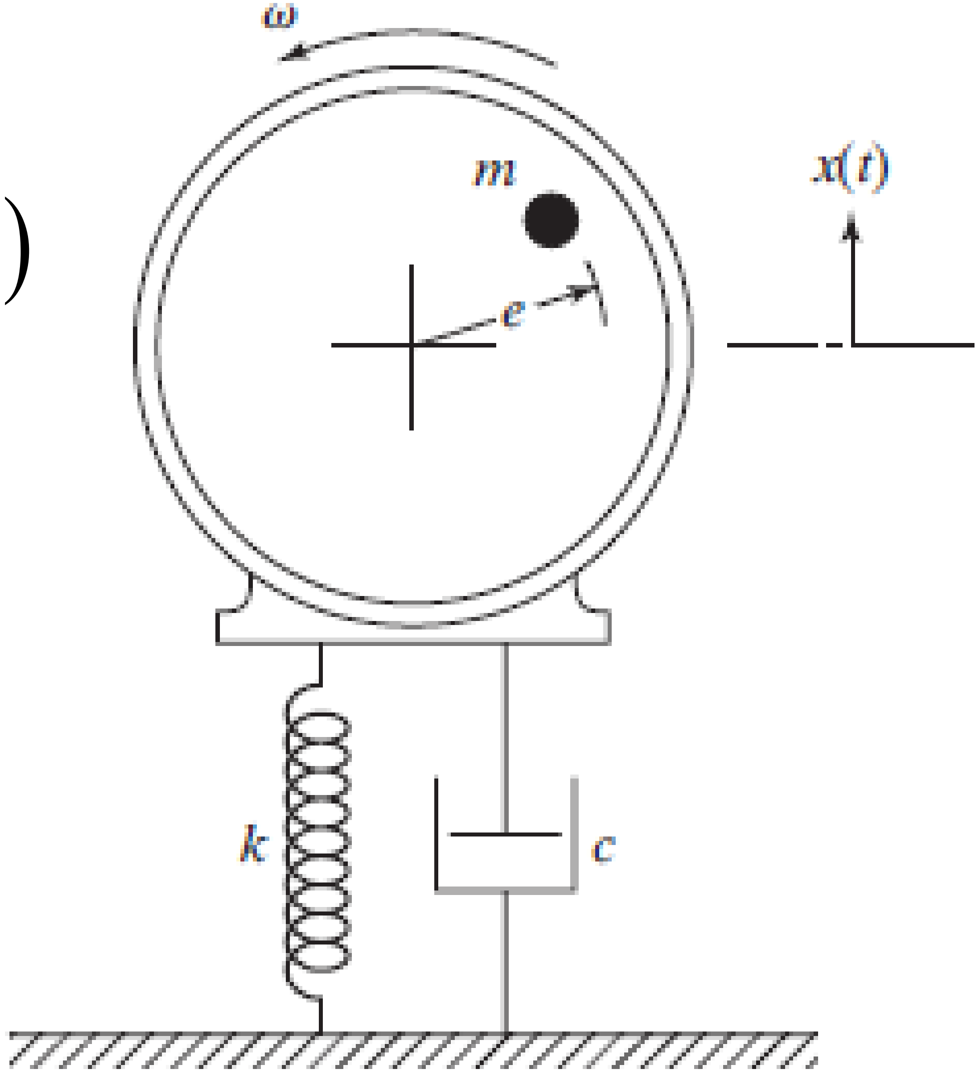
$$\omega_n=\sqrt{k/M}$$ natural frequency of undamped oscillations

$$\varsigma=\frac{c}{2M\omega_n}$$ viscous damping factor



Systems with rotating unbalanced masses

$$x(t) = \frac{m}{M} e \left(\frac{\omega}{\omega_n} \right)^2 \left| G(i\omega) \right| \sin\left(\omega t - \phi \right)$$



$$\left| G(i\omega) \right| = \frac{1}{\left\{ \left[1 - \left(\omega/\omega_n \right)^2 \right]^2 + \left(2\varsigma \left(\omega/\omega_n \right)^2 \right) \right\}^{1/2}}$$

$$\phi(\omega) = \tan^{-1} \frac{2\varsigma\, \omega/\omega_n}{1 - \left(\omega/\omega_n \right)^2}$$

Systems with rotating unbalanced masses

$$x(t) = \frac{m}{M} e \left(\frac{\omega}{\omega_n} \right)^2 \left| G(i\omega) \right| \sin\left(\omega t - \phi \right)$$

$$x(t) = \left| X \right| \sin\left(\omega t - \phi \right)$$

$$\frac{M \left| X \right|}{me} = \left(\frac{\omega}{\omega_n} \right)^2 \left| G(i\omega) \right|$$

Systems with rotating unbalanced masses

$$x(t) = \left|X\right| \sin\left(\omega t - \phi\right)$$

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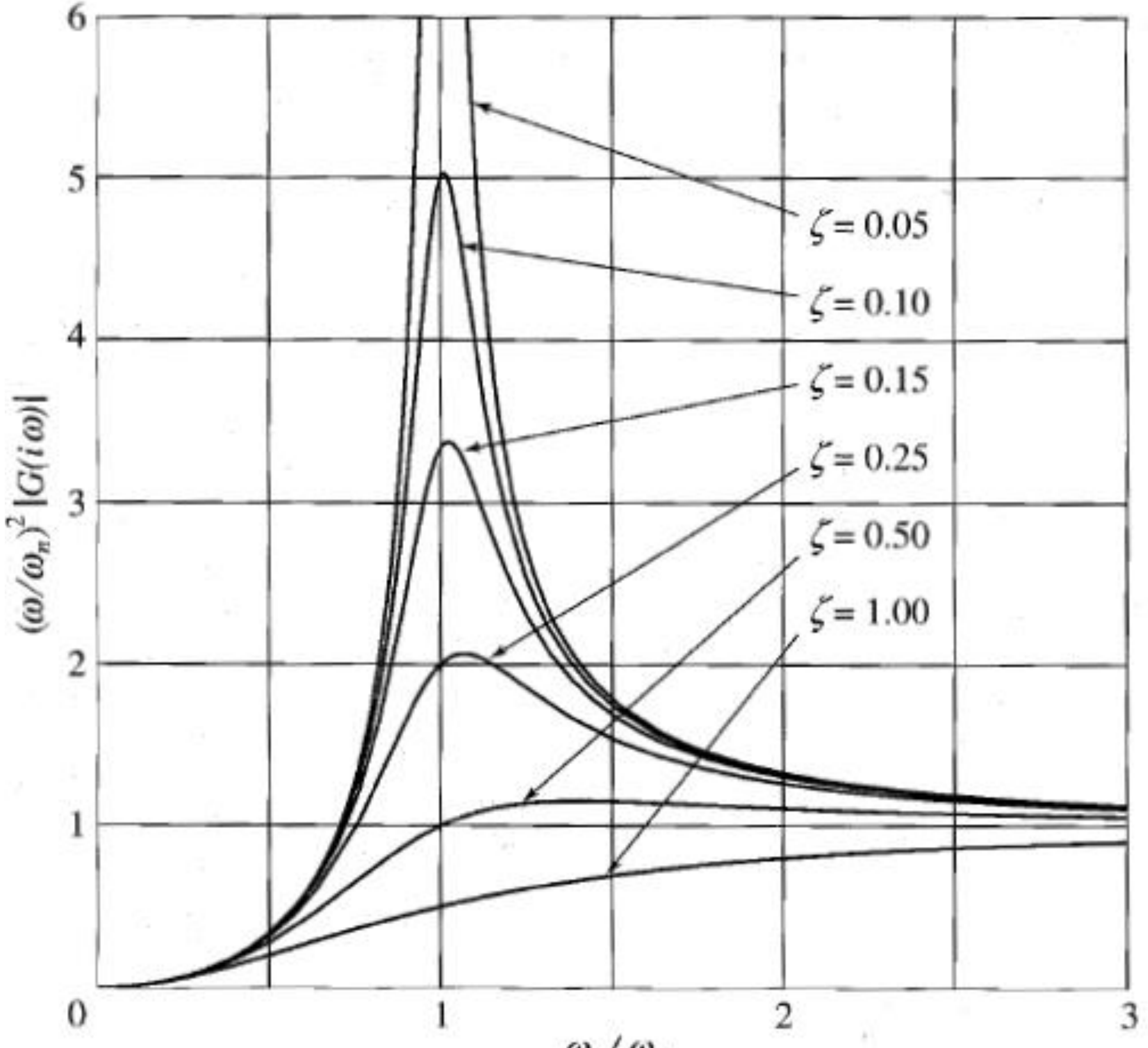

Systems with rotating unbalanced masses

$$x(t) = \left|X\right| \sin\left(\omega t - \phi\right)$$

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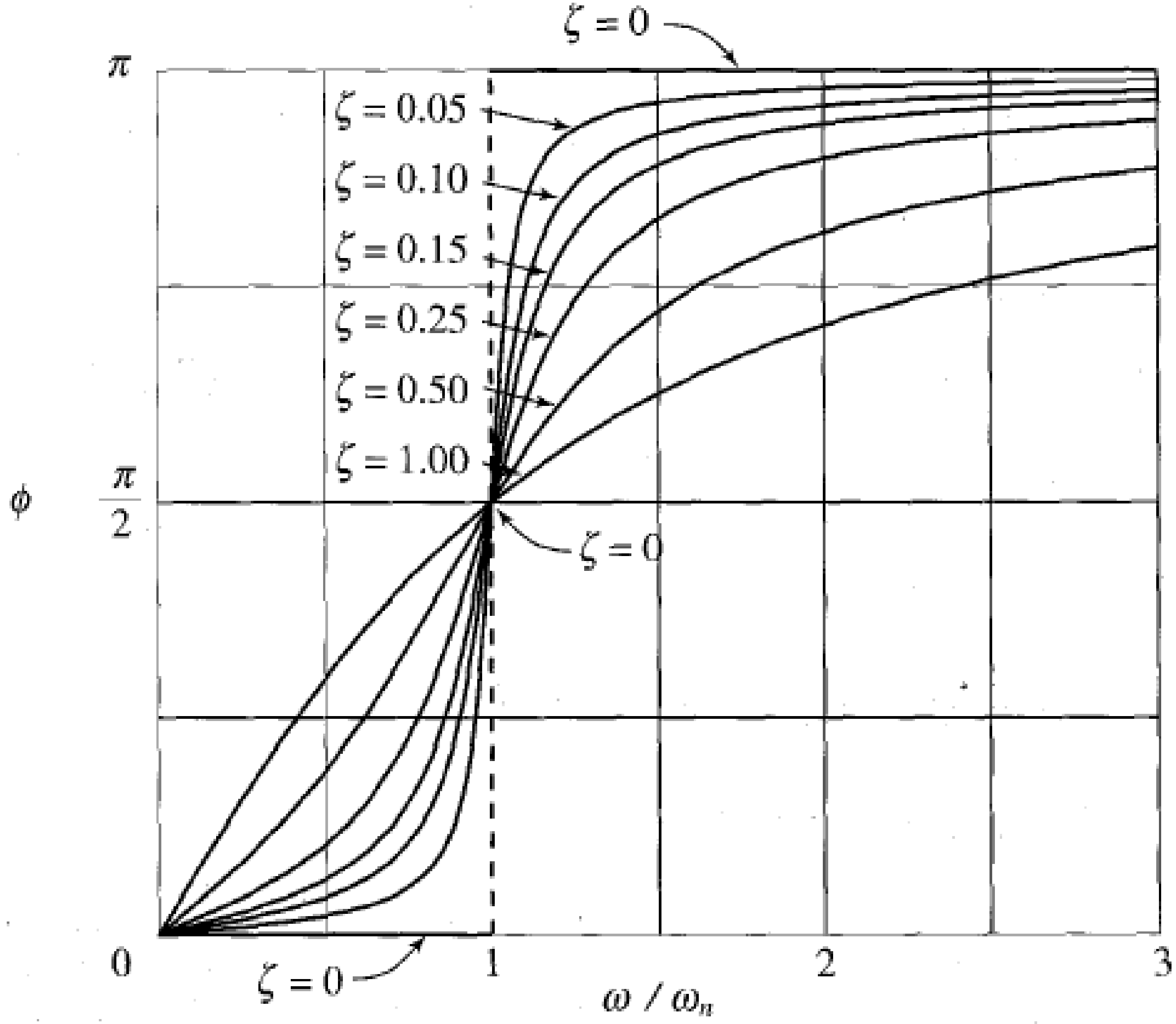
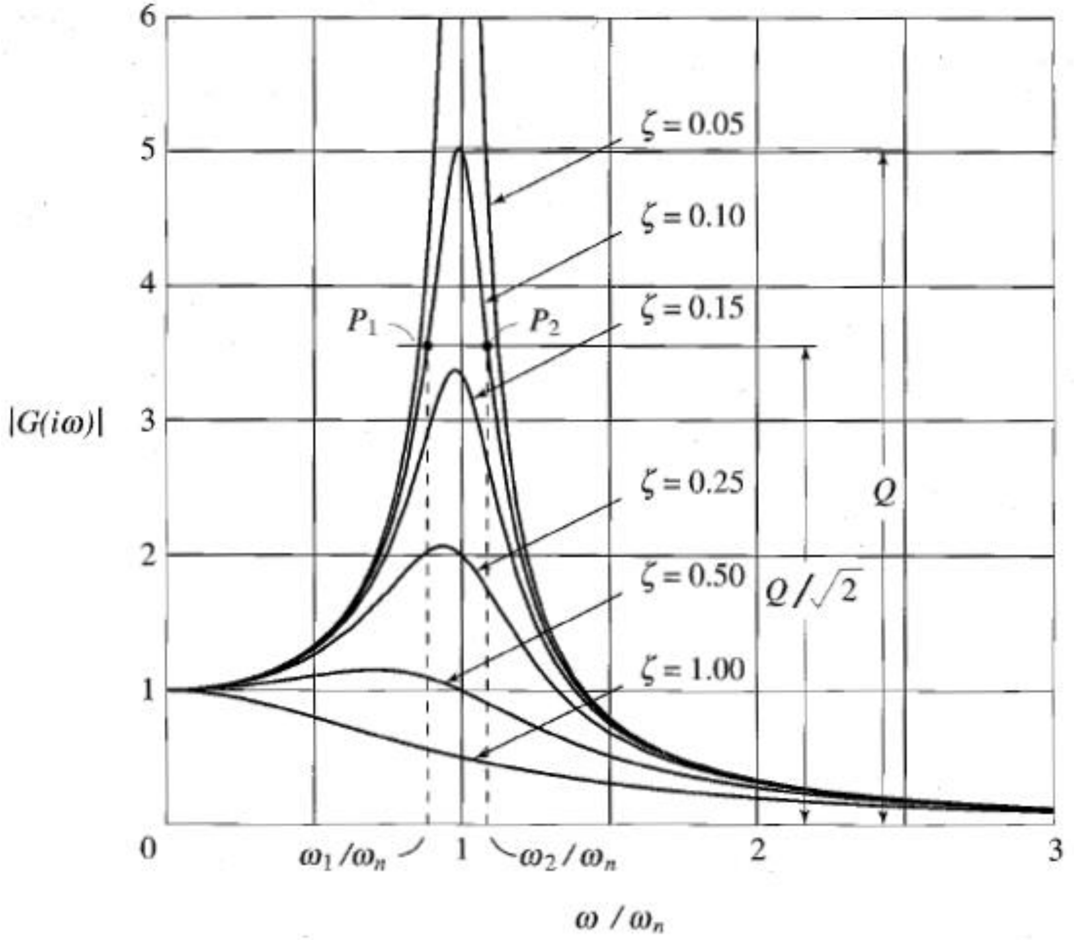
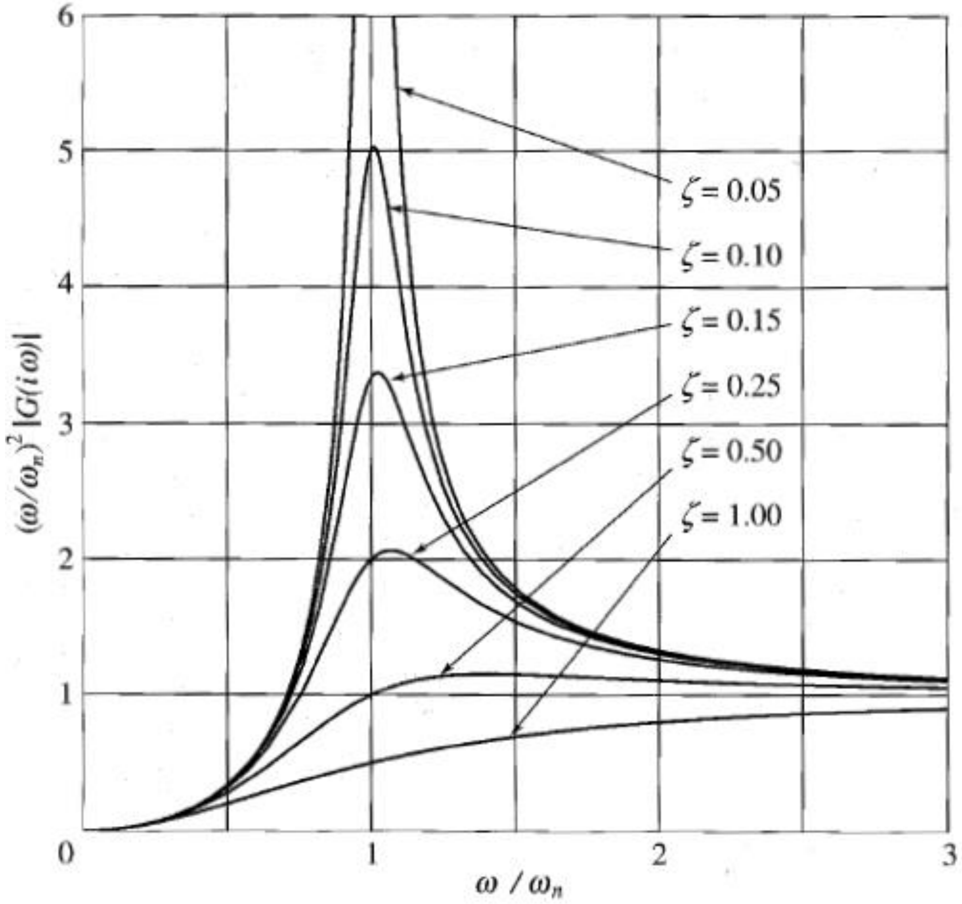


Systems with rotating unbalanced masses



harmonic excitation



rotating unbalanced mass

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- Coulomb damping. Dry friction.
- Response of SDOF systems to harmonic excitations.
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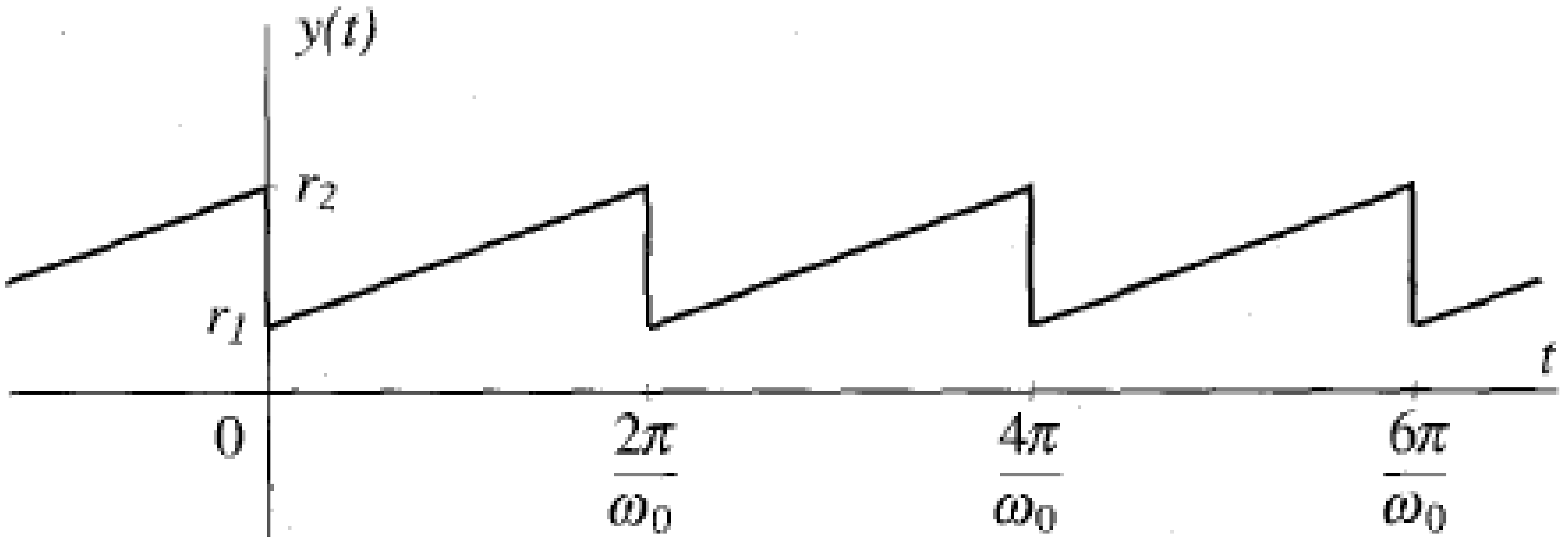
- Response to periodic excitation. Fourier series.
- Response to non-periodic excitation: step, ramp and impulse.

Response to periodic excitation

- The excitation repeat itself every time interval *T* (period)



Response to periodic excitation

- Periodic functions can be expressed as linear combinations of harmonic functions known as Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{p=1}^{\infty}\left(a_p \cos p\omega_0 t + b_p \sin p\omega_0 t\right)$$

$$\omega_0 = \frac{2\pi}{T}$$ fundamental frequency

p = 1, 2, ...   are integers

$p\omega_0$ ➔ <u>harmonics</u>

Response to periodic excitation

- Periodic functions can be expressed as linear combinations of harmonic functions known as Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{p=1}^{\infty}\left(a_p \cos p\omega_0 t + b_p \sin p\omega_0 t\right)$$

$$\omega_0 = \frac{2\pi}{T}$$

fundamental frequency

$$a_p = \frac{2}{T}\int_0^T f(t)\cos p\omega_0 t dt a_p$$

$$b_p = \frac{2}{T}\int_0^T f(t)\sin p\omega_0 t dt$$

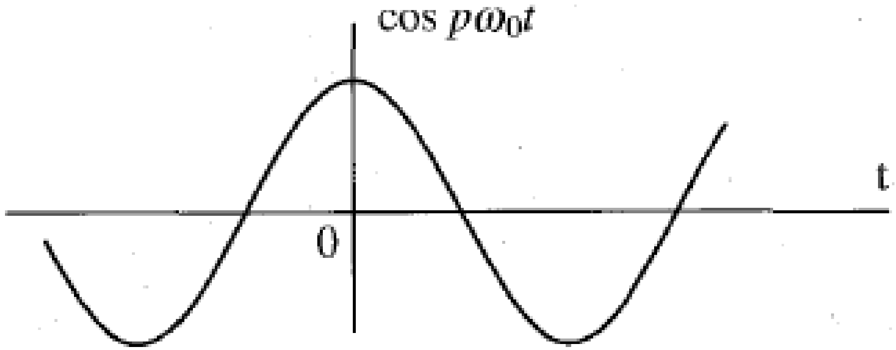
p = 1, 2, ...   are integers

$p\omega_0$ ➔ <u>harmonics</u>

Response to periodic excitation

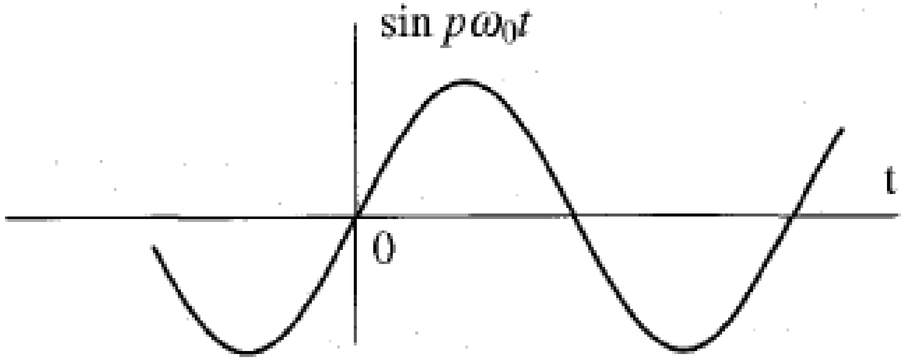
- even functions $f(t) = f(-t)$

$$f(t) = \frac{1}{2}a_0 + \sum_{p=1}^{\infty} a_p \cos p\omega_0 t$$



- odd functions $f(t) = -f(-t)$

$$f(t) = \sum_{p=1}^{\infty} b_p \sin p\omega_0 t$$



Response to periodic excitation

- Periodic functions can be expressed as linear combinations of harmonic functions known as Fourier series

$$f(t) = \sum_{p=-\infty}^{\infty} C_p e^{ip\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$C_p = \frac{1}{T}\int_0^T f(t) e^{-ip\omega_0 t} dt$$

$$p = 0, \pm 1, \pm 2, \ldots$$

Response to periodic excitation

- Periodic functions can be expressed as linear combinations of harmonic functions known as Fourier series

$$f(t) = \frac{1}{2} A_0 + \mathrm{Re}\left(\sum_{p=1}^{\infty} A_p e^{ip\omega_0 t} \right)$$

$$A_p = \frac{2}{T} \int_0^T f(t) e^{-ip\omega_0 t} dt$$

$$p = 0, 1, 2, \ldots$$

Response to periodic excitation

- To obtain the response we apply the principle of superposition

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)=kf(t)$$

$$f(t)=\mathrm{Re}(Ae^{i\omega t})$$

$$x(t)=\mathrm{Re}\left[AG(i\omega)e^{i\omega t}\right]=\mathrm{Re}\left[A\left|G(i\omega)\right|e^{i(\omega t-\phi}\right]$$

$$x(t)=\frac{1}{2}A_0+\mathrm{Re}\left[\sum_{p=1}^{\infty}A_pG_pe^{ip\omega_0 t}\right]=\frac{1}{2}A_0+\mathrm{Re}\left[\sum_{p=1}^{\infty}A_p\left|G_p\right|e^{i(p\omega_0 t-\phi_p)}\right]$$

Response to periodic excitation

$$x(t) = \frac{1}{2}A_0 + \mathrm{Re}\left[\sum_{p=1}^{\infty} A_p G_p e^{ip\omega_0 t}\right] = \frac{1}{2}A_0 + \mathrm{Re}\left[\sum_{p=1}^{\infty} A_p \left|G_p\right| e^{i(p\omega_0 t - \phi_p)}\right]$$

$$G_p = \frac{1}{1-\left(p\,\omega_0/\omega\right)^2 + i2\varsigma\,p\,\omega_0/\omega_n}$$

$$\left|G_p\right| = \frac{1}{\left\{\left[1-\left(p\,\omega_0/\omega_n\right)^2\right]^2 + \left(2\varsigma\,p\,\omega_0/\omega_n\right)^2\right\}^{1/2}}$$

$$\phi_p$$

$$\phi_p = \tan^{-1}\frac{2\varsigma\,p\,\omega_0/\omega_n}{1-\left(p\,\omega_0/\omega_n\right)^2}$$

Response to periodic excitation

$$x(t) = \frac{1}{2}A_0 + \mathrm{Re}\left[\sum_{p=1}^{\infty} A_p G_p e^{ip\omega_0 t}\right] = \frac{1}{2}A_0 + \mathrm{Re}\left[\sum_{p=1}^{\infty} A_p \left|G_p\right| e^{i(p\omega_0 t - \phi_p)}\right]$$

$$G_p = \frac{1}{1-\left(p\,\omega_0/\omega\right)^2 + i2\varsigma\, p\,\omega_0/\omega_n}$$

$$\left|G_p\right| = \frac{1}{\left\{\left[1-\left(p\,\omega_0/\omega_n\right)^2\right]^2 + \left(2\varsigma\, p\,\omega_0/\omega_n\right)^2\right\}^{1/2}}$$

$$\phi_p = \tan^{-1}\frac{2\varsigma\, p\,\omega_0/\omega_n}{1-\left(p\,\omega_0/\omega_n\right)^2}$$

- Each harmonic component in *x(t)* is shifted by the phase angle ϕ_p
- The response remains periodic and with the same period as the excitation.
- As *p* increases, ϕ_p tends to a value inversely proportional to p^2, so that the participation of the higher harmonics in the response decreases rapidly.