System Dynamics and Vibrations

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Chapter 2: Concepts from vibrations
Part 2

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Contents

- Introduction
- Modeling of mechanical systems
- System differential equations of motion
- **Linearization about equilibrium points**
- System and response characteristics. The superposition principle.

Linearization about equilibrium points

$$m\ddot{y} = F(y, \dot{y})$$

- Let's consider a solution having the form:

$$y(t) = y_e + x(t)$$

- being *x(t)* a relatively small displacement from equilibrium
- then:

$$\dot{y}(t) = \dot{x}(t)$$

$$\ddot{y}(t) = \ddot{x}(t)$$

Linearization about equilibrium points

- Expandig $F(y,\dot{y})$ in a Taylor series about an equlibrium point y_e:

$$F(y,\dot{y}) = F(y_e,0) + \left.\frac{\partial F(y,\dot{y})}{\partial y}\right|_{\substack{y=y_e\\ \dot{y}=0}} x + \left.\frac{\partial F(y,\dot{y})}{\partial \dot{y}}\right|_{\substack{y=y_e\\ \dot{y}=0}} \dot{x} + O(x^2)$$

$$m\ddot{y} = F(y,\dot{y}) \Rightarrow \left\{ \begin{array}{l} \displaystyle \frac{1}{m}\left.\frac{\partial F(y,\dot{y})}{\partial y}\right|_{\substack{y=y_e\\ \dot{y}=0}} = -b \\ \displaystyle \frac{1}{m}\left.\frac{\partial F(y,\dot{y})}{\partial \dot{y}}\right|_{\substack{y=y_e\\ \dot{y}=0}} = -a \end{array} \right\} \Rightarrow m\ddot{x} + a\dot{x} + bx = 0$$

Linearization about equilibrium points

- We have assumed that displacements from equilibrium are sufficiently small that the nonlinear terms can be ignored

$$m\ddot{y} = F(y, \dot{y}) \quad \Rightarrow \quad \ddot{x} + a\dot{x} + bx = 0$$

➔*linearized equation of motion about equilibrium*

(small motions assumption)

- The motion characteristics in the neighborhoud of equilibrium depend on parameters *a*, *b*

Linearization about equilibrium points

$$\ddot{x} + a\dot{x} + bx = 0$$

- Linear equation with constant coefficients:

$$x(t) = Ae^{st}$$

A: amplitude

s: constant exponent

- Combining

$$\left.\begin{aligned} m\ddot{x} + a\dot{x} + bx = 0 \\ x(t) = Ae^{st} \end{aligned}\right\} \quad s^2 + as + b = 0$$

Linearization about equilibrium points

$$s^2 + as + b = 0$$

➔ Characteristic equation (algebraic equation)

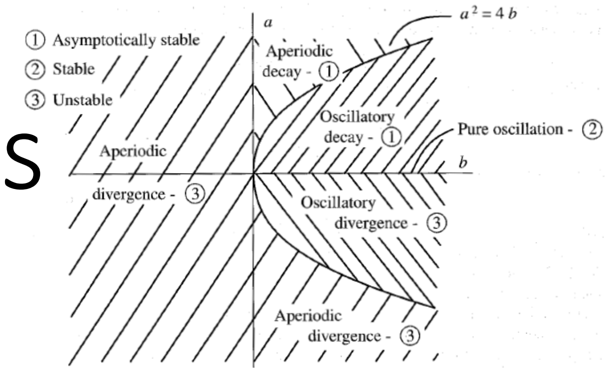
- The roots are:

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

- So the solution to $m\ddot{x} + a\dot{x} + bx = 0$ is:

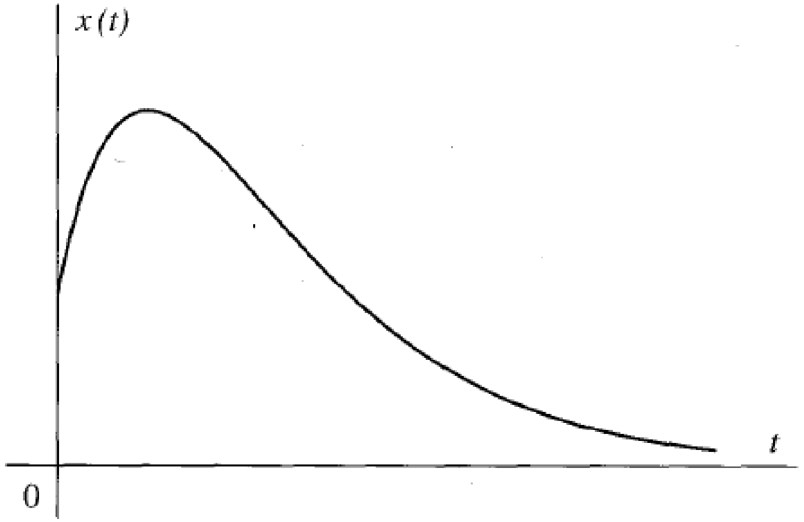
$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Linearization about equilibrium points

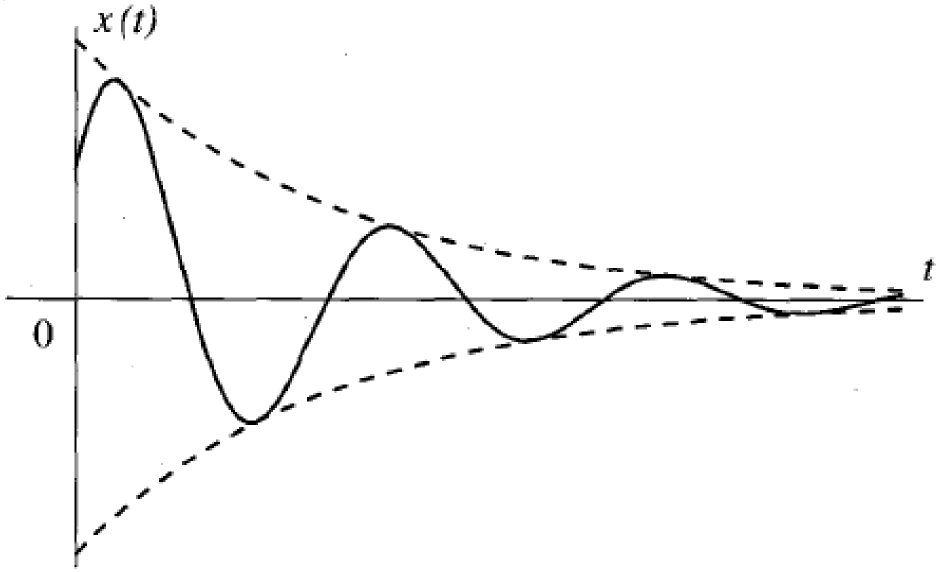


1. Asymptotically stable solution (a>0, b>0)

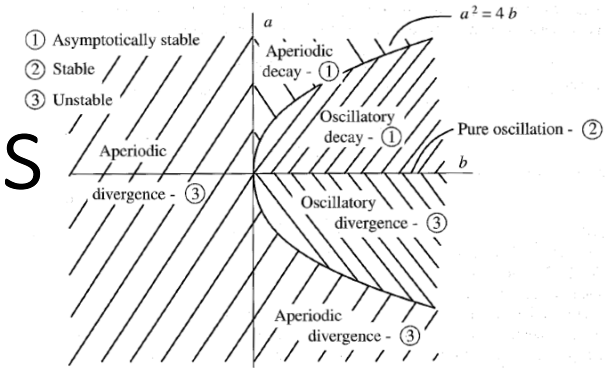
Aperiodically decay



Decaying oscillation

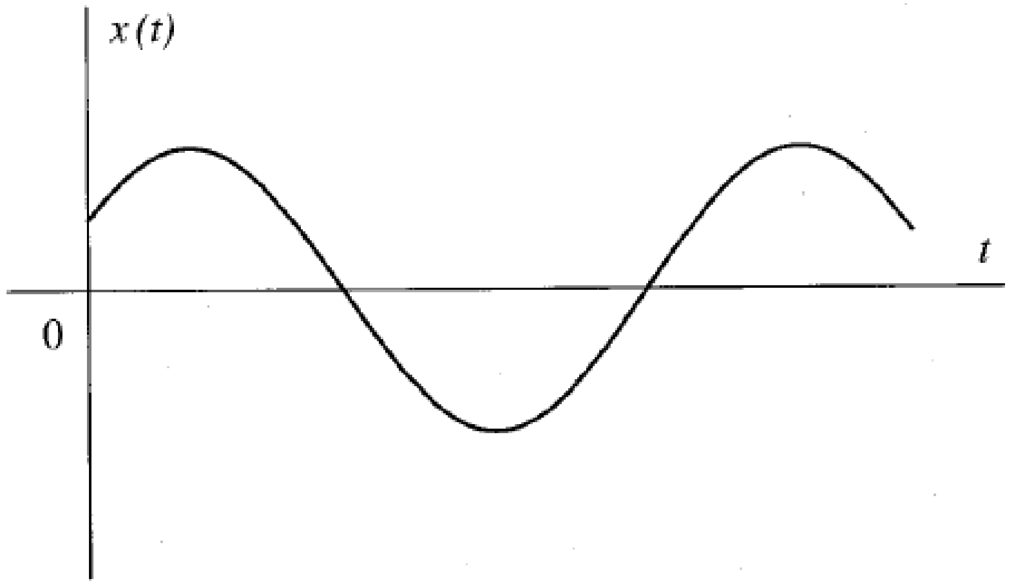


Linearization about equilibrium points

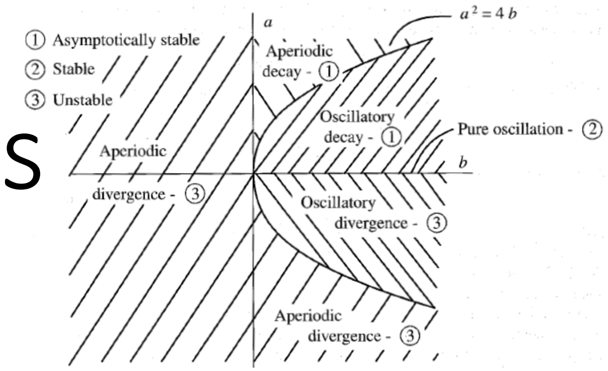


2. Stable motion (a=0, b>0)

Harmonic oscillation

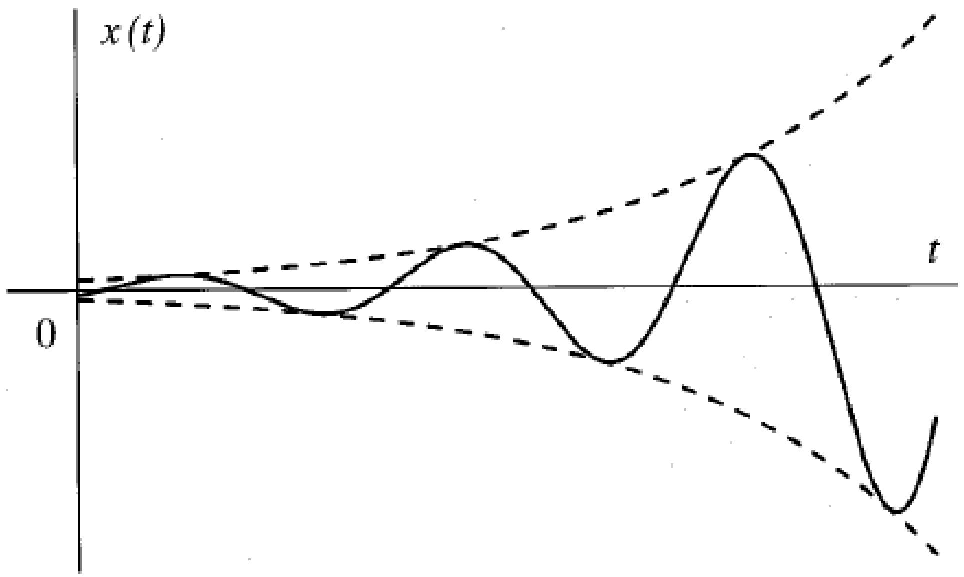


Linearization about equilibrium points

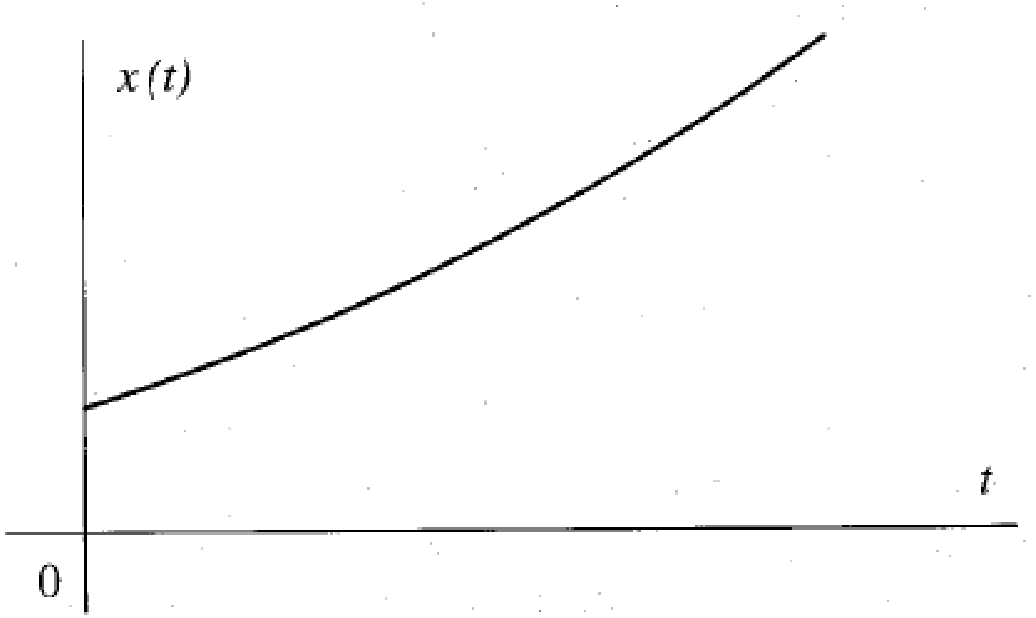


3. Unstable motion (b<0, b>0 & a<0)

Diverging oscillation



Aperiodically diverging motion

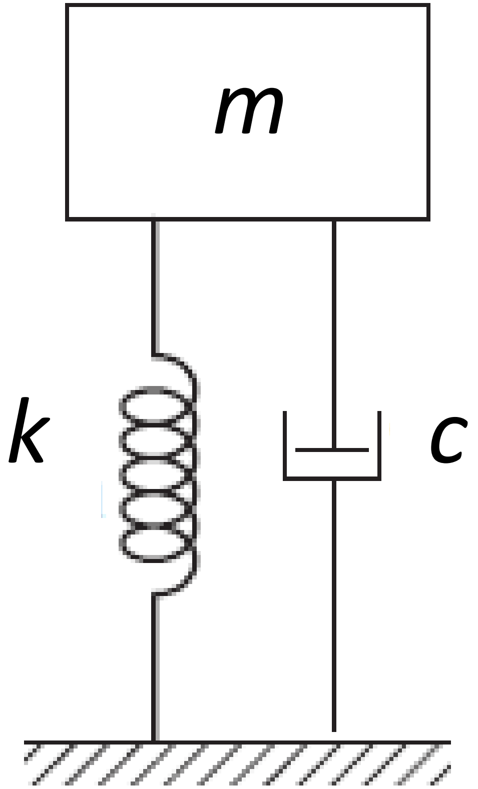


Exercises

$$m\ddot{y} = F(y, \dot{y})$$

1. Mass-spring-damper system

$$m\ddot{y} = -c\dot{y} - ky - mg$$



Equilibrium points:

$$F(y, \dot{y}) = --c\dot{y} - ky - mg$$

$$F(y_e, 0) = -ky_e - mg = 0$$

$$y_e = -\frac{mg}{k}$$

Exercises

$$m\ddot{y} = F(y, \dot{y})$$

1. Mass-spring-damper system

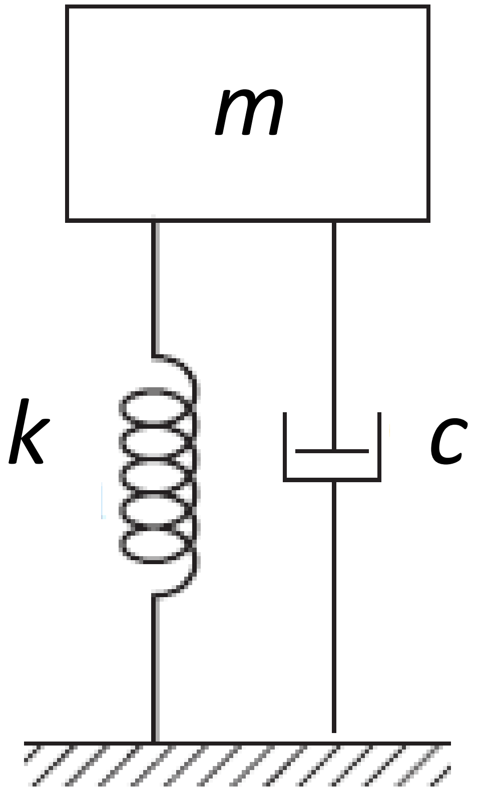
Stability of equilibrium point:

$$y(t) = y_e + x(t) = -\frac{mg}{k} + x(t)$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$a = \frac{c}{m}$$

$$b = \frac{k}{m}$$



Exercises

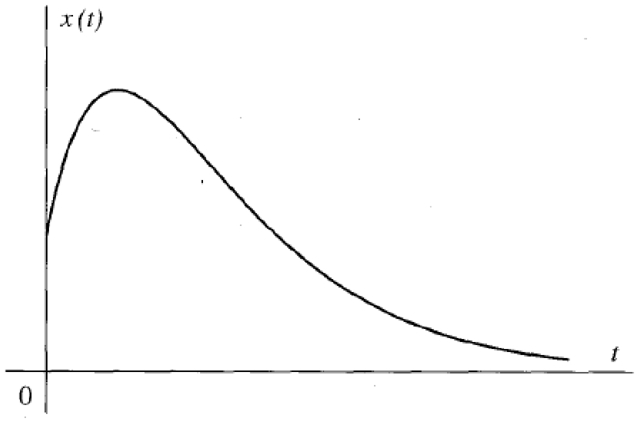
1. Mass-spring-damper system

Stability of equilibrium point:

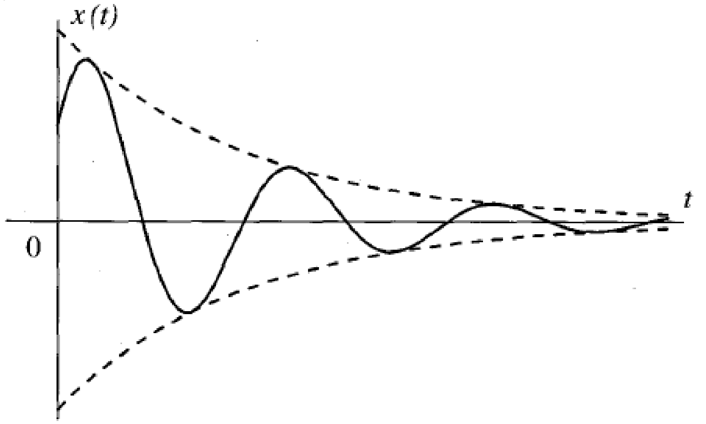
asymptotically stable

$$a = \frac{c}{m} > 0$$
$$b = \frac{k}{m} > 0$$

$$c^2 \geq 4km$$

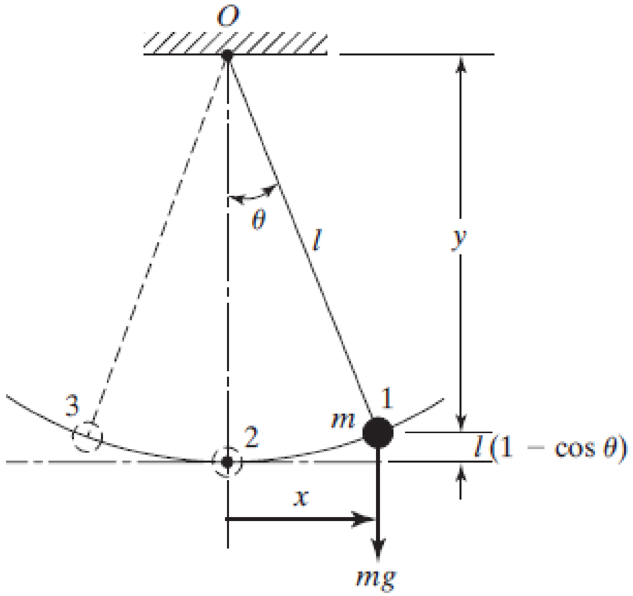


$$c^2 < 4km$$



Exercises

2. Simple pendulum $\ddot{\theta} + \frac{g}{l} \sin\theta = 0$



Equilibrium points:

$$m\ddot{y} = F(y, \dot{y}) \qquad F(y, \dot{y}) = F(\theta, \dot{\theta}) = F(\theta, 0) = -\frac{g}{l} \sin\theta$$

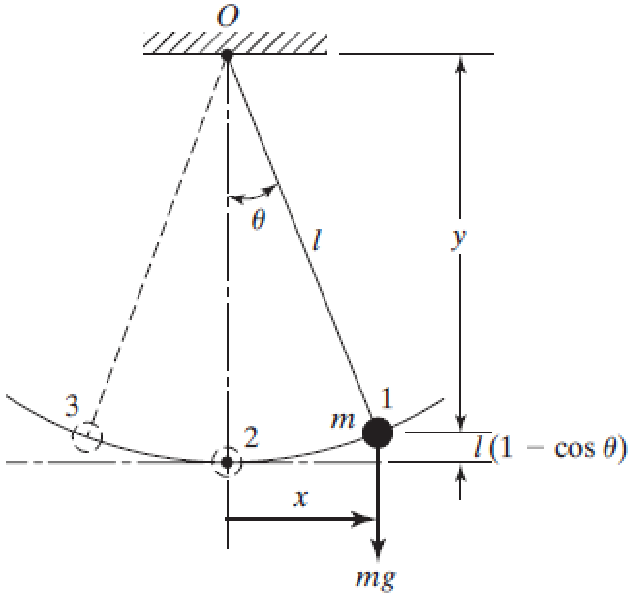
$$m = 1$$

$$y = \theta$$

$$F(\theta, 0) = -\frac{g}{l} \sin\theta = 0$$

equation of equilibrium

Exercises



2. Simple pendulum

Equilibrium points:

$$F(\theta, 0) = -\frac{g}{l}\sin\theta = 0$$

equation of equilibrium

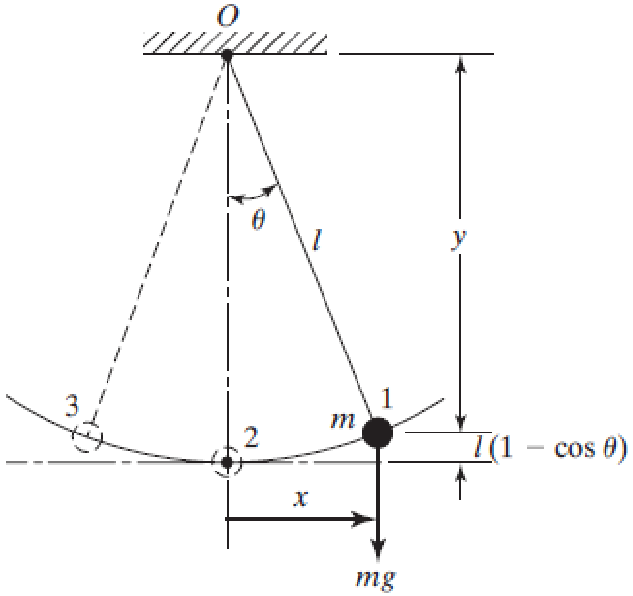
solutions: $\theta_e = 0, \pm\pi, \pm 2\pi, \ldots$

$$\theta_{e1} = 0$$

$$\theta_{e2} = \pi$$

Exercises

2. Simple pendulum



Stability of equilibrium points:

$$\theta_{e1} = 0$$

$$\theta_{e2} = \pi$$

Linearization:

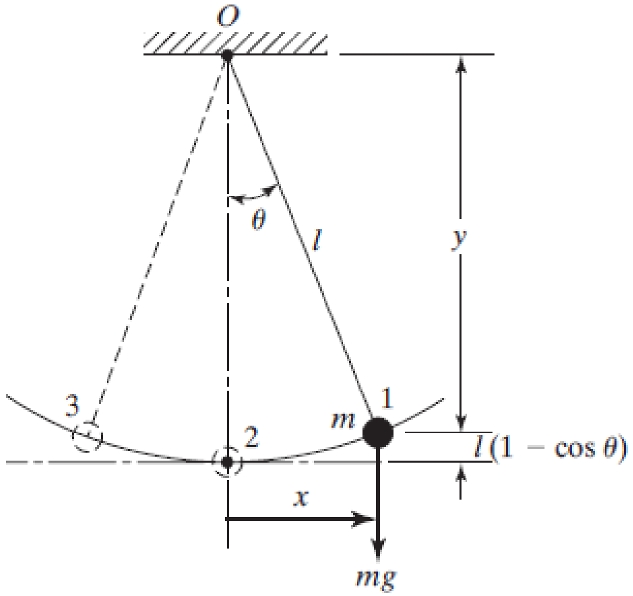
$$\theta(t) = \theta_e + \phi(t)$$

$$\ddot{\phi} + b\phi = 0$$

$$b = -\left.\frac{\partial F(\theta, 0)}{\partial \theta}\right|_{\theta=\theta_e} = \frac{g}{l}\cos\theta_e$$

Exercises

2. Simple pendulum



Stability of equilibrium points:

$$\theta_{e1} = 0 \qquad b = \frac{g}{l} > 0 \qquad \text{stable}$$

$$\theta_{e2} = \pi \qquad b = -\frac{g}{l} < 0 \qquad \text{unstable}$$

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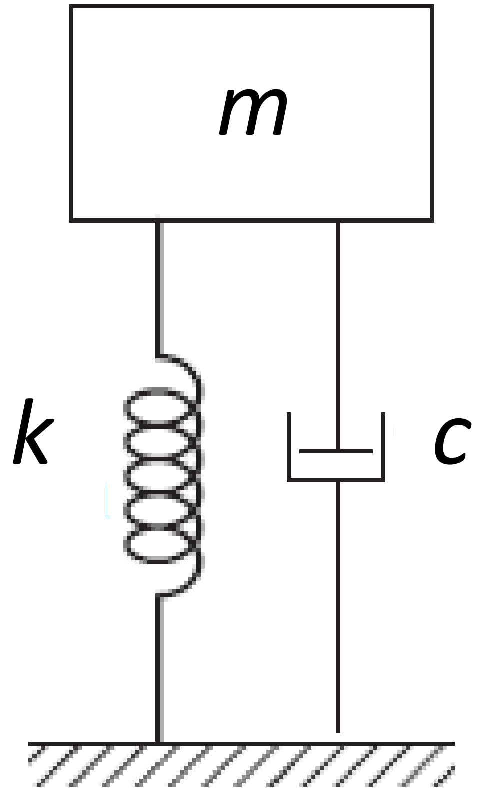
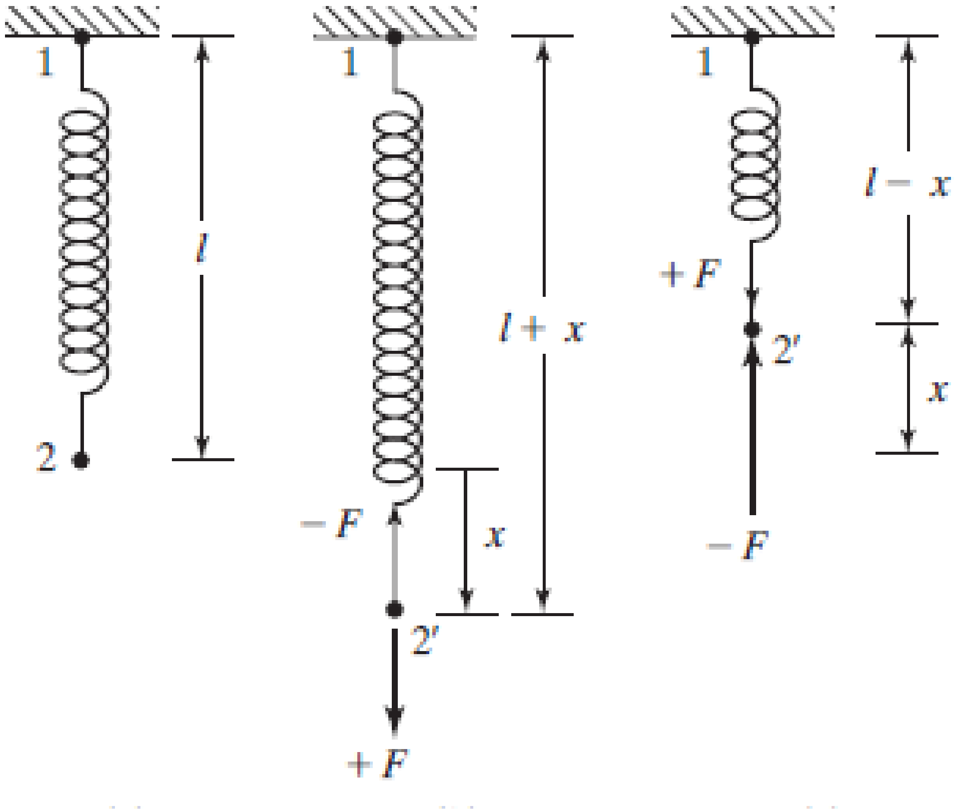
Mass-spring-damper system



Spring elements

- Type of mechanical link
- Characterized by stiffness, *k*
- Normally negligible mass and damping
- Any elastic or deformable body or member, such as a cable, bar, beam, shaft or plate, can be considered as a spring.



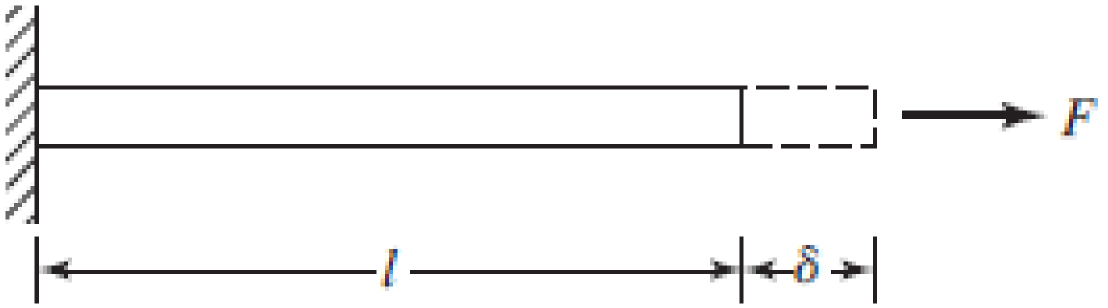
$$F = kx$$

$$V = \frac{1}{2}kx^2$$

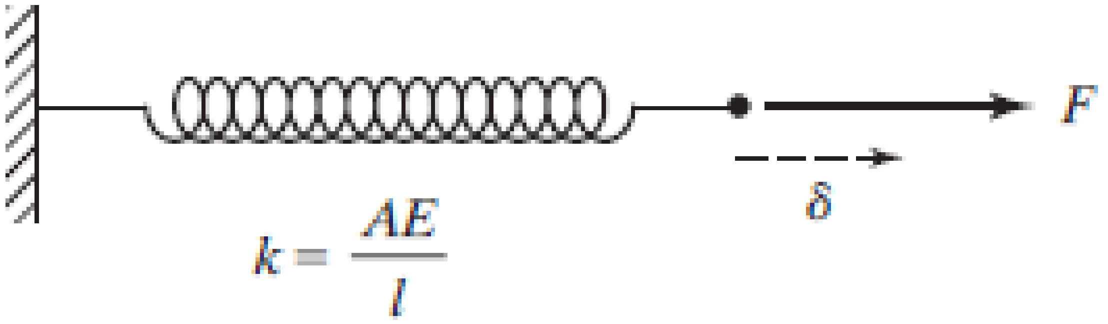
Spring elements: spring constant of a rod

$$\delta = \frac{\delta}{l} l = \varepsilon l = \frac{\sigma}{E} l = \frac{Fl}{AE}$$

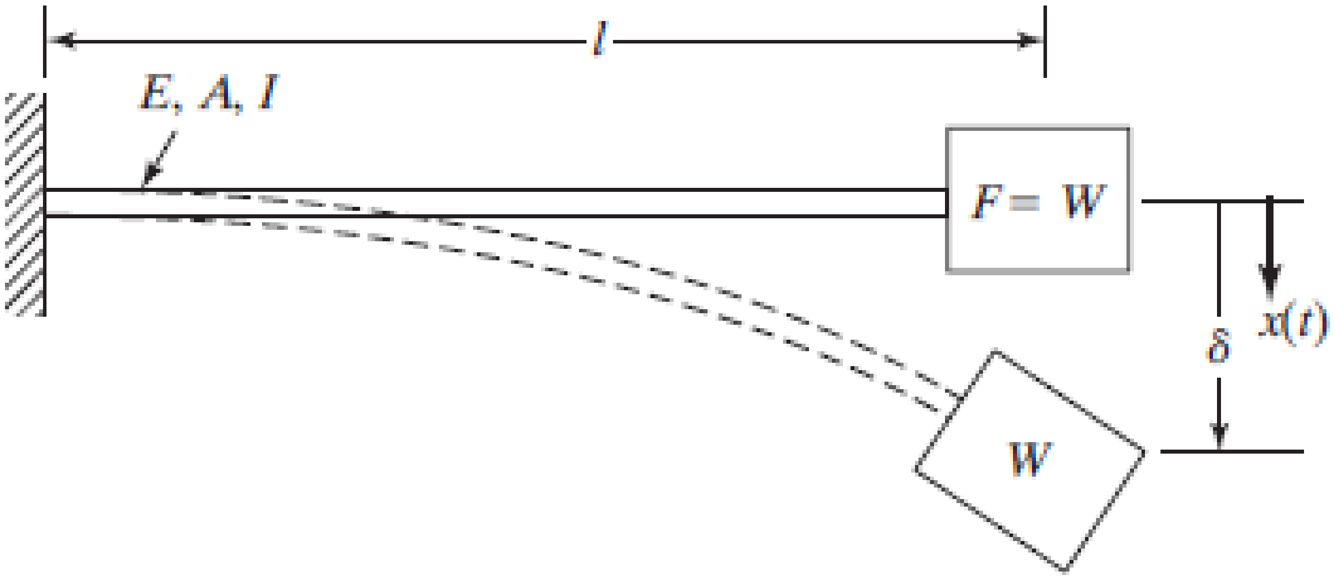
$$k = \frac{F}{\delta} = \frac{AE}{l}$$



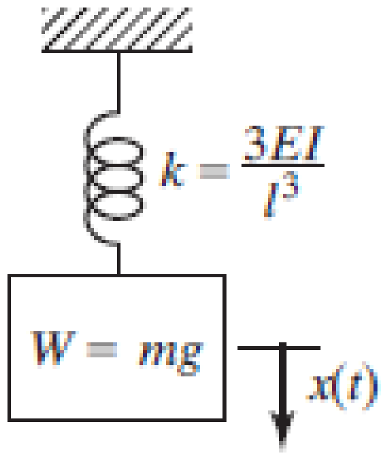
(a)



Spring elements: spring constant of a cantilever beam



(a) Cantilever with end force

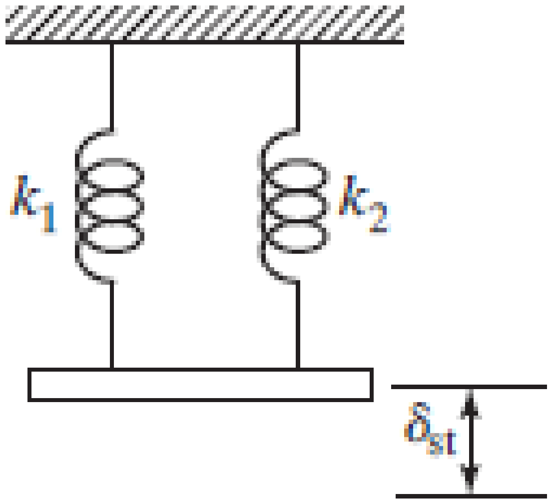


(b) Equivalent spring

$$\delta = \frac{Wl^3}{3EI} \Rightarrow k = \frac{W}{\delta} = \frac{3EI}{l^3}$$

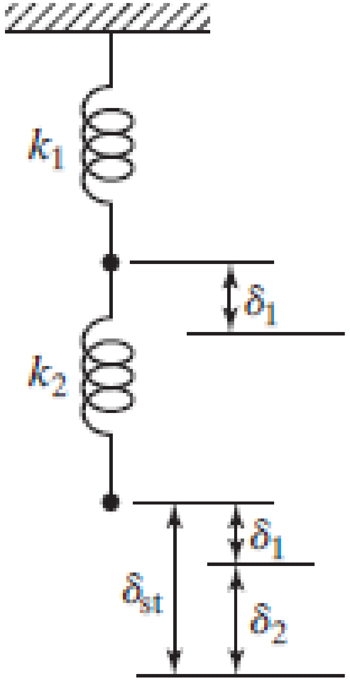
Spring elements: combination of springs

- Springs in parallel



$$k_{eq} = k_1 + k_2 + \ldots + k_n$$

- Springs in series

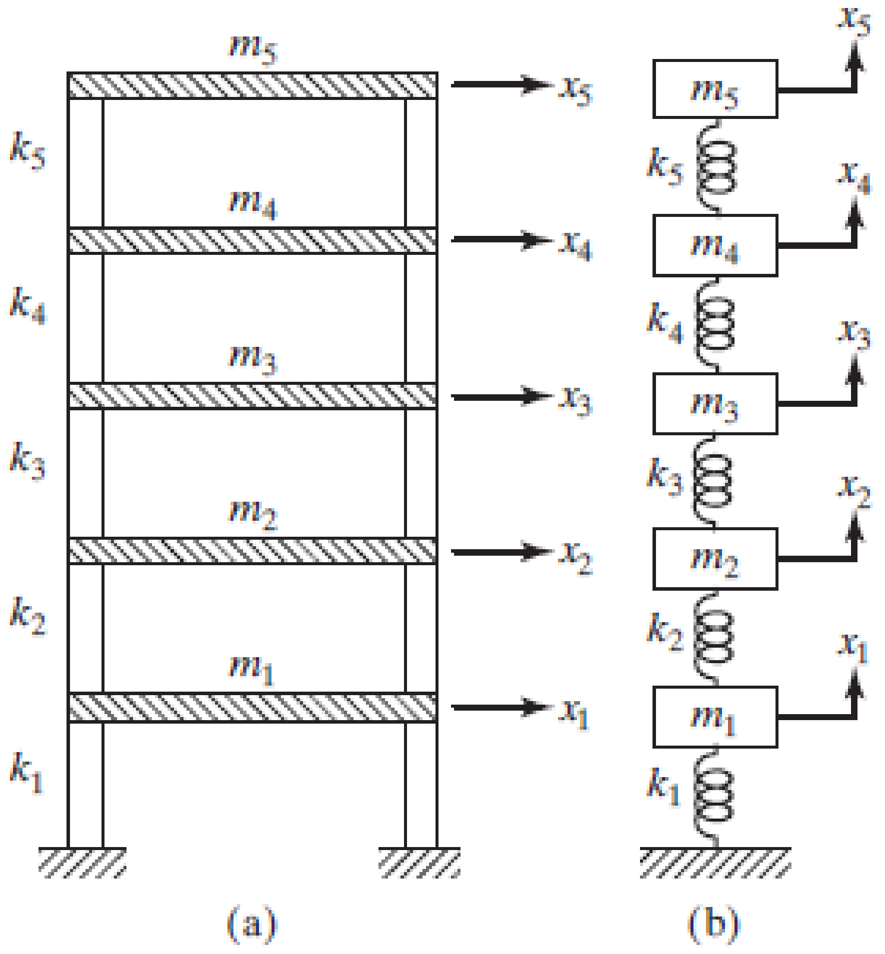


$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n}$$

Mass or Inertia elements

- Assumed to be a rigid body
- It can gain or lose kinetic energy if the velocityof the body changes
- In many practical cases, several masses appear in combination

➔ For a simpe analysis, we can replace these masses by a single equivalent mass



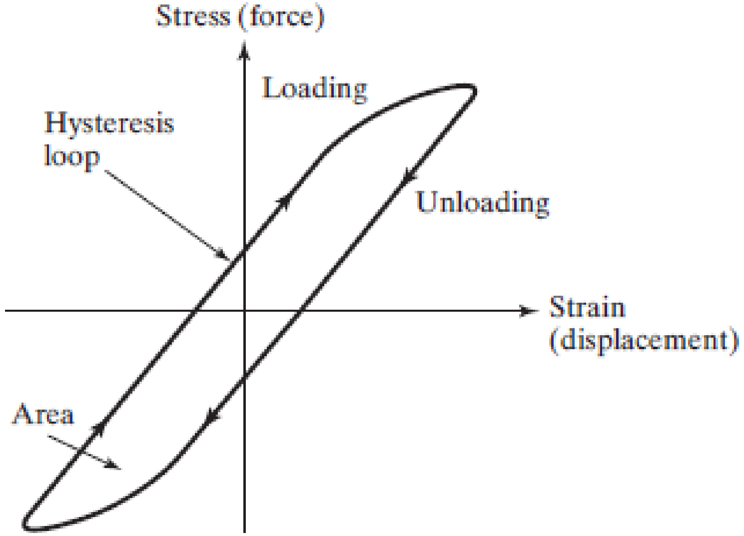
Damping elements

- Damping is the mechanism by which the vibrational energy is gradually converted into heat or sound
- Dampers are assumed to have neither mass nor elasticity
- Damping forces exists only if there is relative velocity between the two ends of the damper
- **Viscous damping**:
  - Related to the interaction of the body with the surrounding fluid
  - Depends on: the size and shape of the body, the viscosity of the fluid, the frequency of vibration, the velocity of the body, etc.
  - The damping force is proportional to the velocity of the vibrating body

Damping elements

- **Coulomb or dry-friction damping**:
  - The damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body
  - It is caused by friction between rubbing surfaces that either are dry or have insufficient lubrication
- **Material or Solid or Hysteretic Damping:**
  - When a material is deformed, energy is absorbed and dissipated by the material
  - The effect is due to friction between the internal planes, which slip or slide as the deformations take place.
  - When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop as indicated
  - The area of this loop denotes the energy lost per unit volume of the body per cycle due to damping

Damping elements



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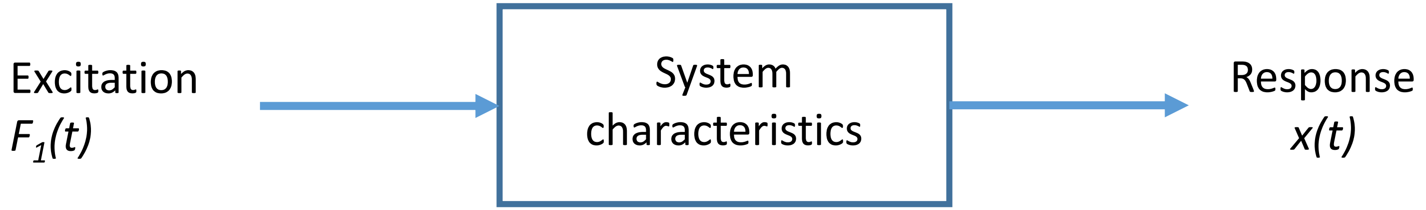
- Linearization about equilibrium points
- **System and response characteristics. The superposition principle**.

The superposition principle

- How do the system characteristics affect the response of the system?
- A systemis defned as an aggregation of components working together as a single unit.
- System charscteristics depend on:
  - Individual components
  - The manner in which these components are conected to one another

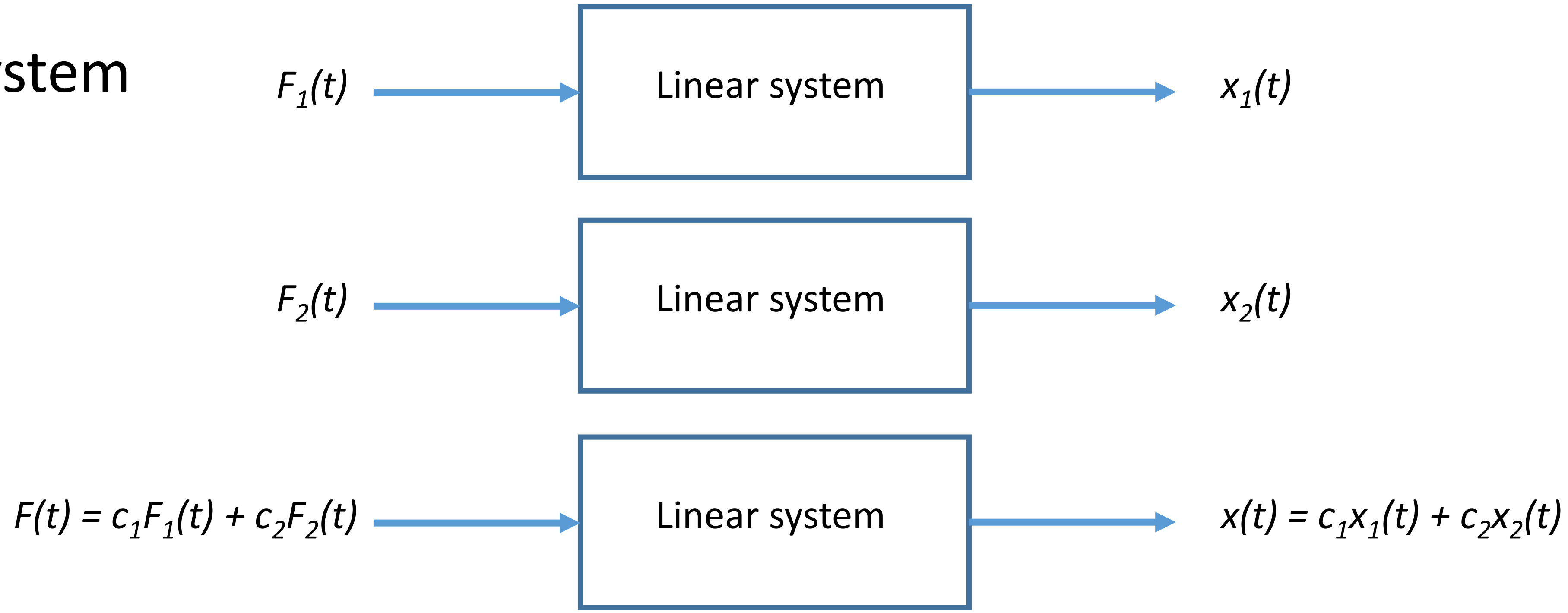


The superposition principle

- Systems can be:
  - Linear
  - Nonlinear

The superposition principle

- Linear system



- Nonlinear system: $x(t) \neq c_1 x_1(t) + c_2 x_2(t)$

The superposition principle

- Principle of superposition (linear system):

  ➔If a linear system is acted upon by a linear combination of individual excitations, the individual responses can be first obtained separately and then combined linearly to obtain the total response

- A system is linear if the dependent variable *x(t)* and all its time derivatives appear in the equation of motion to the first power or zero power only

$$m\ddot{x} + c\dot{x} + kx = F$$ linear

$$m\ddot{x} + c\dot{x} + k(x + \varepsilon x^3) = F$$ nonlinear

The superposition principle

- Time dependency:
  - Linear time-invariant systems (linear systems with constant coefficients)
    - For example: $m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$
    - If the excitation *F(t)* is delayed by an amount of time τ, the resonse *x(t)* is delayed by the same amount t
  - Time-varying systems (systems with time-dependent coefficients)
    - For example: $m\ddot{x}(t) + k(1 + a\cos\omega t)x(t) = F(t)$

The superposition principle

- Response to <u>**initial excitations**</u>

➔simplest problem

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) = 0$$

- Homogeneous equation ➔ solution has exponential form
- Characteristic equation to determine the exponents
- Coefficients of the exponential terms obtained by letting $x(t), \dot{x}(t)$ evaluated at *t* = 0 match the initial displacement and velocity

The superposition principle

- Response to **<u>harmonic excitations</u>**

➔it is also harmonic

- The response has the same frequency as the excitation frequency, but it differs in **magnitude** and possesses a **phase angle** relative to the excitation
- Magnitude and phase angle depend on the driving frequency
- The response to harmonic excitations is a steady-state response

Nature of excitation: harmonic motion

- Oscillatory motion may repeat itself regularly (like in the case of a simple pendulum), or
- It may display considerable irregularity (as in the case of ground motion during an earthquake)
- **Periodic motion**: the motion is repeated after equal intervals of time
- The simplest type of periodic motion is ***harmonic motion***

Nature of excitation: harmonic motion

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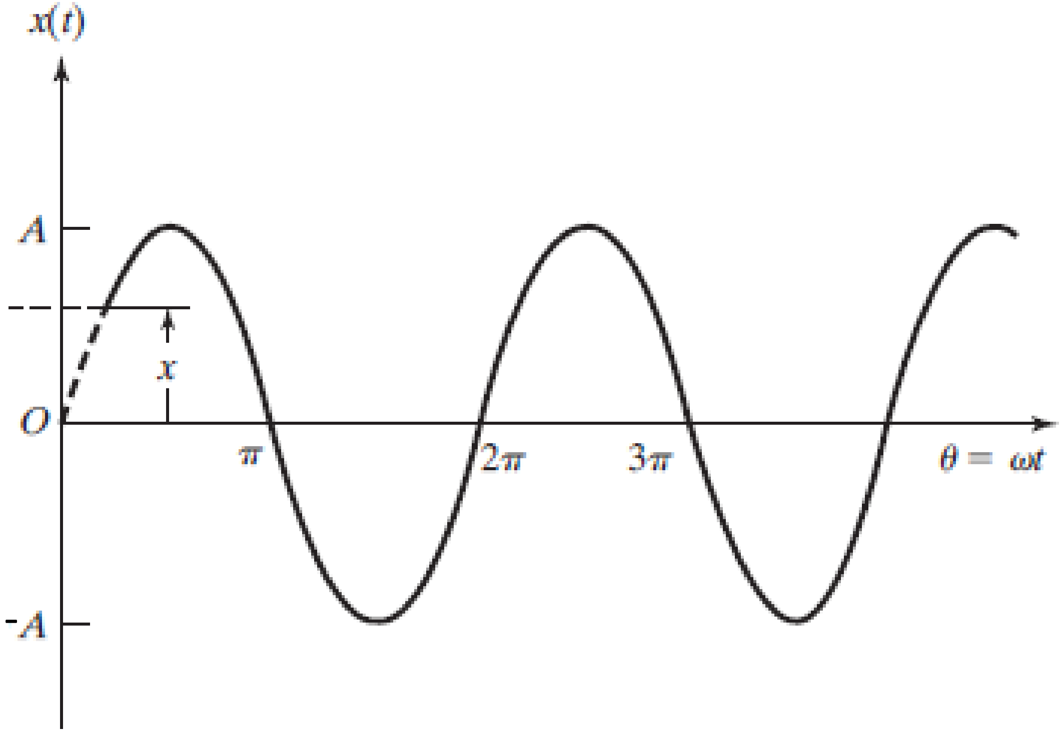


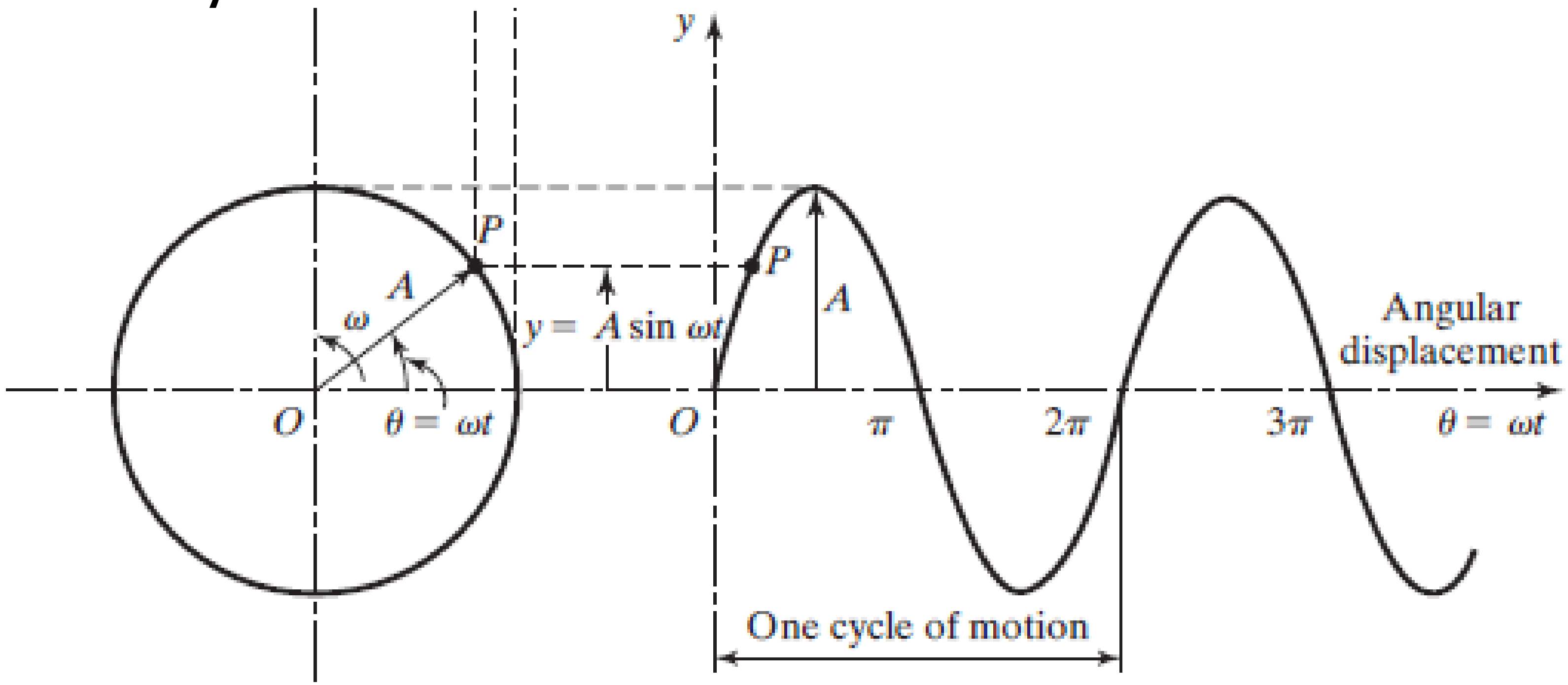
Nature of excitation: harmonic motion

- Harmonic motion can be represented conveniently by means of a vector $\overrightarrow{OP}$ of magnitude A rotating at a constant angular velocity

$$x(t) = A\sin\theta = A\sin\omega t$$

$$\dot{x}(t) = \omega A\cos\omega t$$

$$\ddot{x}(t) = -\omega^2 A\sin\omega t = -\omega^2 x(t)$$



Nature of excitation: harmonic motion

- Complex-number representation of harmonic motion

$$\vec{X} = a + ib$$

real and imaginary parts

$$\vec{X} = A\cos\theta + iA\sin\theta = Ae^{i\theta}$$

-

$$A = \left(a^2 + b^2\right)^{1/2}$$

modulus

$$\theta = \tan^{-1}\frac{b}{a}$$

argument

The superposition principle

- Response to periodic excitations

➔ Periodic excitations can be represented by Fourier series, i.e. series of harmonic functions

- Applying the principle of superposition, the response to periodic excitations can be expressed in the form of a series of harmonic responses ➔ it is a steady-state response

Nature of excitation: periodic motion

- Any periodic function of time can be represented by Fourier series as an infinite sum of sine and cosine terms

$$x(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t \right)$$

-

$$a_0 = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t)dt = \frac{2}{\tau}\int_0^{\tau} x(t)dt$$

$$\omega = \frac{2\pi}{\tau}$$

$$a_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t)\cos n\omega t dt = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t dt$$

$$b_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t)\sin n\omega t dt = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t dt$$

The superposition principle

- Response to an arbitrary excitation:

➔ Superposition of impulse forces of different magnitde and applied at different times

- The impulse response is defined by the response to a unit impulse applied at t= 0
- Assuming that the impulse response is known, the response of a linear system with constant coefficients can be expressed as a superposition of impulse responses of different magnitudes and applied at different times.
- This superposition is called the convolution integral, or the superposition integral.

The superposition principle

- Response to random excitations:

➔ the response is also a random function

- Random functions are characterized by their mean value, mean square value, autocorrelation function, power spectral density function, ...
- Solution obtained by Fourier transfors ➔ working in the frequency domain
- Results are defined in terms of probability distributions