北京航空航天大學
BEIHANG UNIVERSITY

飞行力学
Flight Mechanics

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Introduction

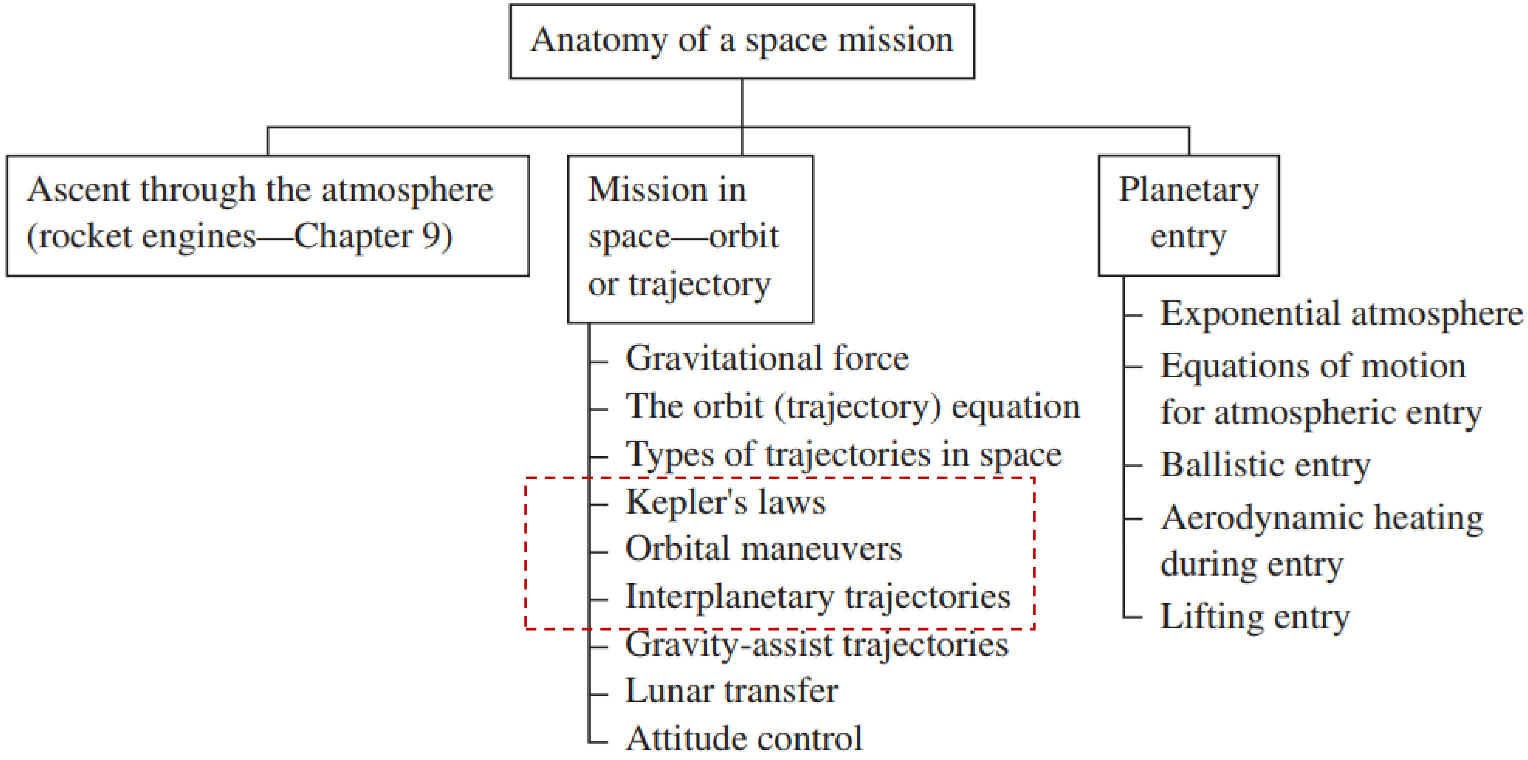


Figure 8.7 Road map for Ch. 8.

Contents

Elliptical orbit

Terminology

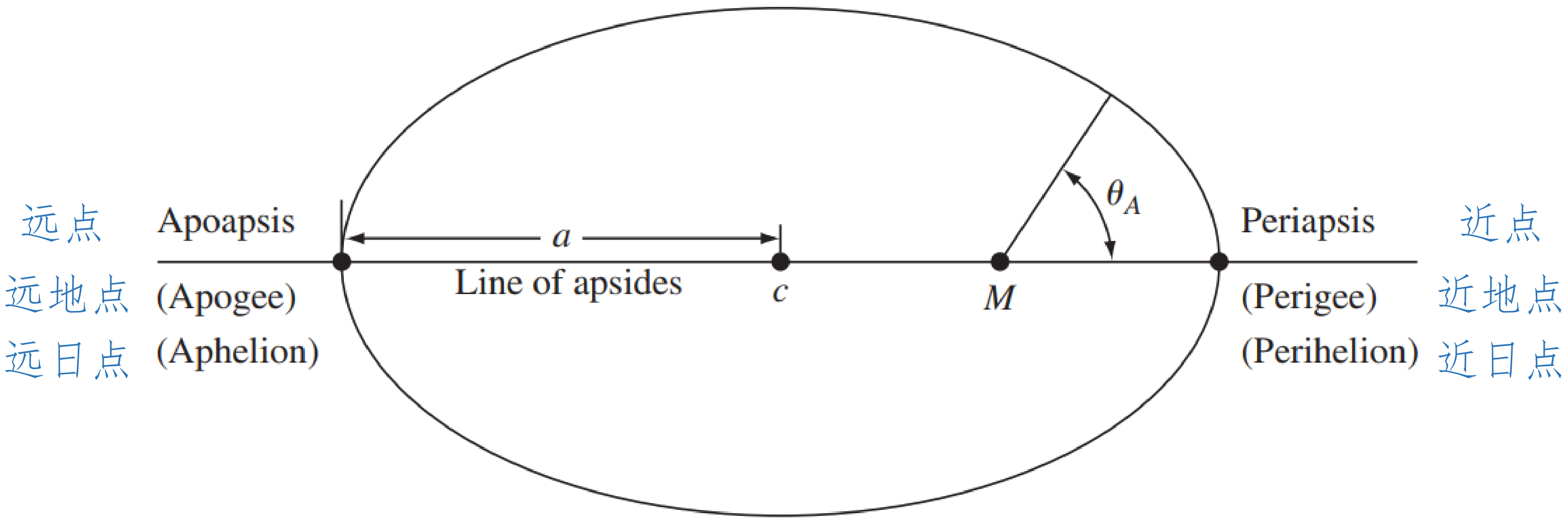


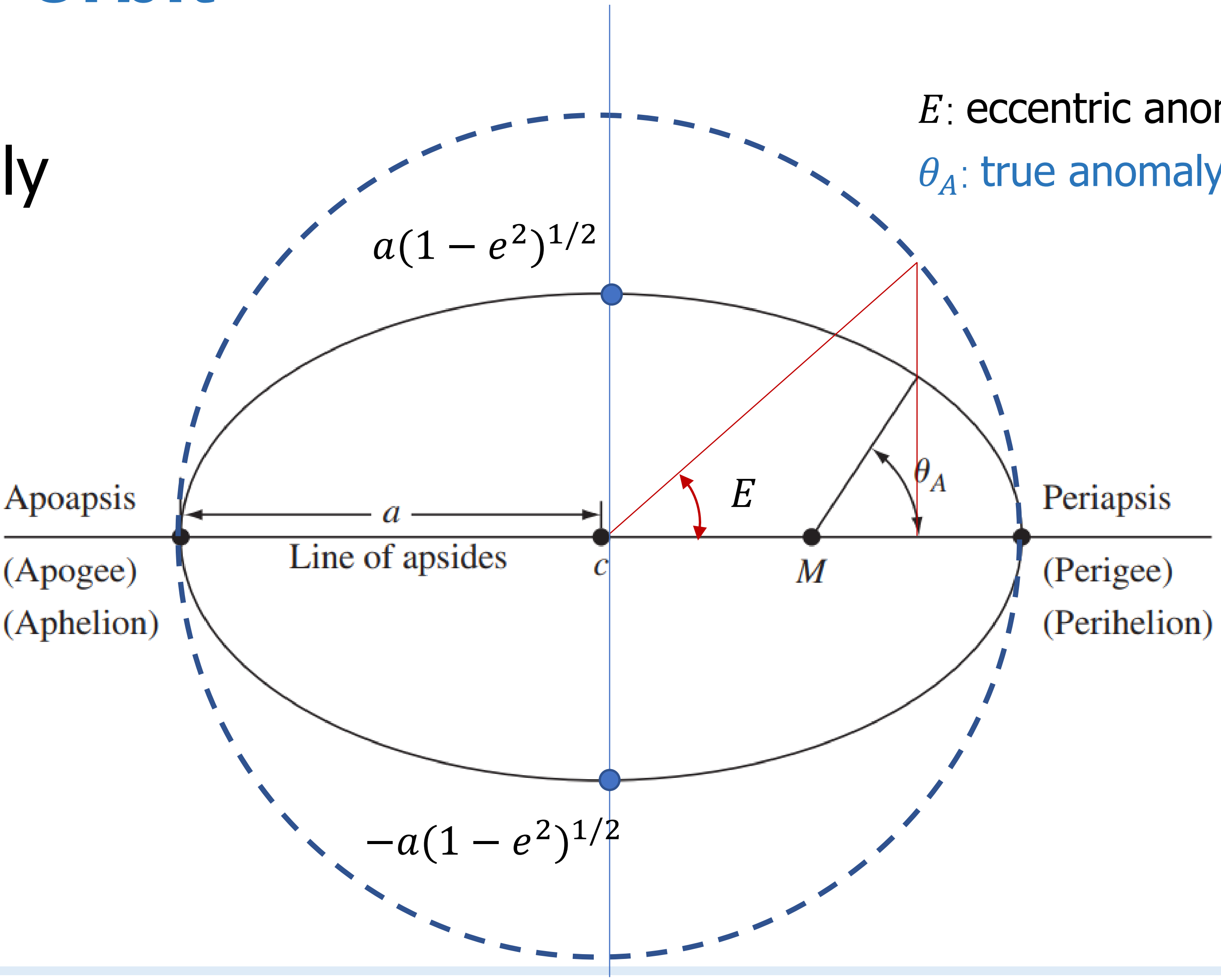
Figure 8.19 Terminology for an elliptical orbit.

Elliptical orbit

True anomaly

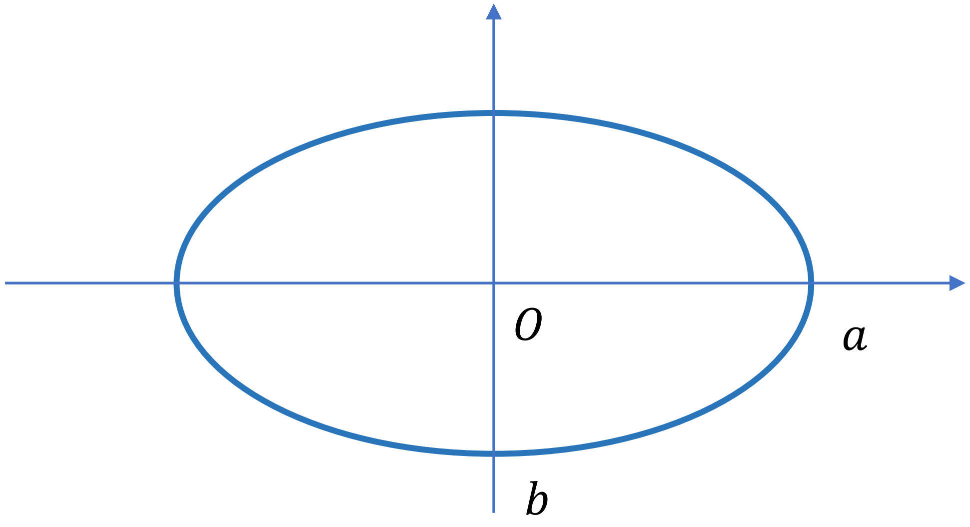
E: eccentric anomaly (偏近点角)

θ_A: true anomaly (真近点角)



Elliptical orbit

Eccentricity (偏心率)



Consider the ellipse with equation given by:

$$\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$$

Eccentricity in OXY coordinate is defined as:

$$\boldsymbol{e} = \sqrt{1 - \left(\frac{b}{a}\right)^2} \quad \Leftrightarrow b = a\sqrt{1 - e^2}$$

Energy equation

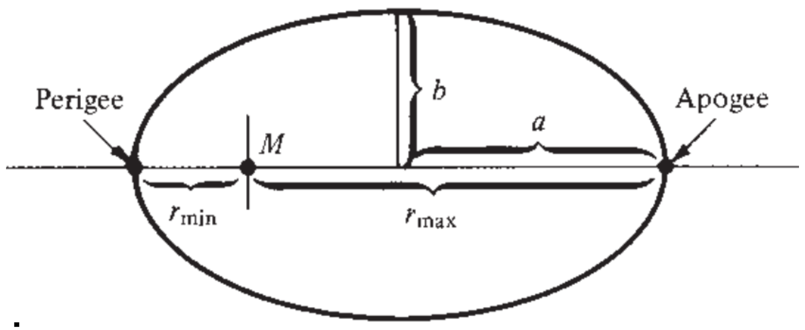
Total energy

From the definition of eccentricity e in previous lecture:

$$\boxed{e = \sqrt{1 + \frac{2h^2 H}{mk^4}}}$$

$$\Rightarrow H = -\frac{1}{2} m \frac{k^4}{h^2} (1 - e^2)$$

$$\Rightarrow H = -\frac{1}{2} mk^2 \frac{1 - e^2}{h^2/k^2} = -m\frac{k^2}{2a}$$



The semimajor axis a:

$$\boxed{a = \frac{1}{2}(r_{max} + r_{min}) = \frac{1}{2}\frac{h^2}{k^2}\left(\frac{1}{1-e} + \frac{1}{1+e}\right) = \frac{h^2/k^2}{1-e^2}}$$

Semimajor axis is an exclusive measure of the total energy of the spacecraft!

Energy equation

Total energy

$$H = -m\frac{k^2}{2a}$$

From the definition of total energy

$$H \equiv \frac{1}{2}mV^2 - \frac{k^2 m}{r}$$

(kinetic energy + potential energy)

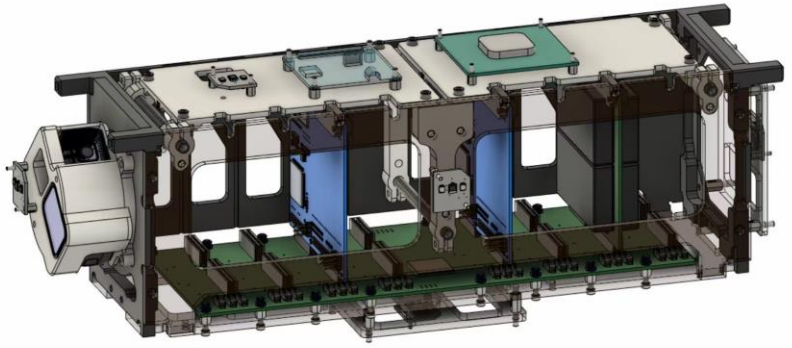
$$\Rightarrow V = \sqrt{\frac{2k^2}{r} - \frac{k^2}{a}}$$

The magnitude of the spacecraft velocity is a function of position specified by r.

Orbit calculation

Question

- How you calculate an orbit for your satellite?
- What you should tell the rocket launch company, such as SpaceX, to bring your CubeSat to that orbit?



TUM CubeSat model

Orbit calculation

Review – example 8.1

At the end of a rocket launch of a space vehicle, the burnout velocity is 9 km/s in a direction due north and 3° above the local horizontal. The altitude above sea level is 806 km. The burnout point is located at the 27° above the equator. Calculate and plot the trajectory of the space vehicle.

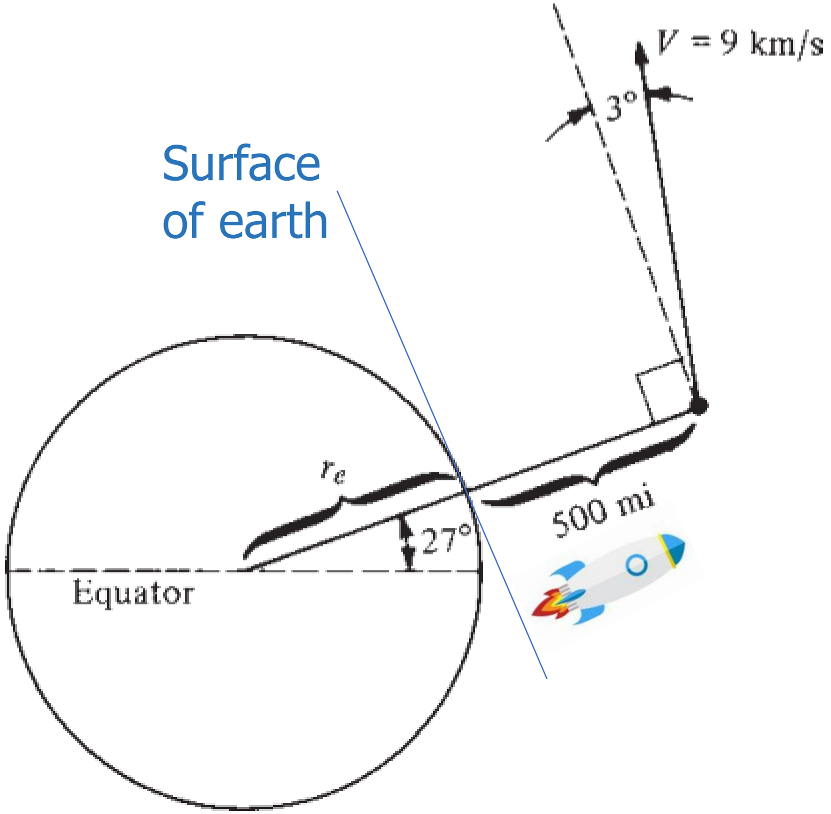


Figure 8.14 Burnout conditions for Example 8.1.

Orbit calculation

Review – example 8.1

The orbit equation:

$$r = \frac{p}{1 + e\cos(\theta - C)}$$

where $p = h^2/k^2$, and $k^2 = GM = 3.986 \times 10^{14}\ m^3/s^2$

$$h = r^2\dot{\theta} = r(r\dot{\theta}) = rV_\theta$$

$$e = \sqrt{1 + \frac{2h^2H}{mk^4}}$$

Unknowns: 1. angular momentum per unit mass h.
2. eccentricity e.
3. constant C.

Orbit calculation

$$h = r^2\dot{\theta} = r(r\dot{\theta}) = rV_\theta$$

Review – example 8.1

1) Calculate h:

$$h = rV_\theta = r_b V \cos\beta_b = (7.2 \times 10^6)(9 \times 10^3)\cos 3° = 6.47 \times 10^{10}\,\mathrm{m^2/s}$$

$$h^2 = 4.188 \times 10^{21}\,\mathrm{m^4/s^2}$$

$$p \equiv \frac{h^2}{k^2} = \frac{4.188 \times 10^{21}}{3.986 \times 10^{14}} = 1.0506 \times 10^7\ \mathrm{m}$$

Orbit calculation

Review – example 8.1

$$e = \sqrt{1 + \frac{2h^2 H}{mk^4}}$$

2) Calculate eccentricity e:

where $H/m = (T - |\Phi|)/m$:

$$\frac{T}{m} = \frac{V^2}{2} = \frac{(9 \times 10^3)^2}{2} = 4.05 \times 10^7 \text{ m}^2/\text{s}^2$$

$$\left|\frac{\Phi}{m}\right| = \frac{GM}{r_b} = \frac{k^2}{r_b} = \frac{3.986 \times 10^{14}}{7.2 \times 10^6} = 5.536 \times 10^7 \text{ m}^2/\text{s}^2$$

$$\frac{H}{m} = (4.05 - 5.536) \times 10^7 = -1.486 \times 10^7 \text{ m}^2/\text{s}^2$$

$$e = \left[1 + \frac{2h^2}{k^4}\left(\frac{H}{m}\right)\right]^{1/2}$$

$$= \left[1 + \frac{2(4.188 \times 10^{21})(-1.486 \times 10^7)}{(3.986 \times 10^{14})^2}\right]^{1/2}$$

$$= \sqrt{0.2166} = 0.4654$$

Orbit calculation

Review – example 8.1

$$r_b = r_e + h_G = 6.4 \times 10^6 + 0.805 \times 10^6 = 7.2 \times 10^6 \text{ m}$$

3) Calculate the constant C:

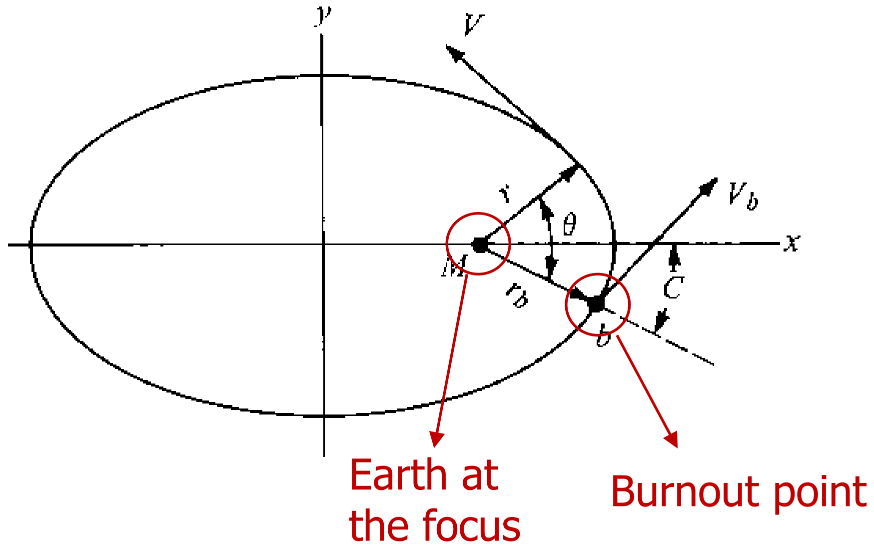
$\theta = 0$ at burnout point

$$r_b = \frac{p}{1 + e\cos(-C)}$$

$$7.2 \times 10^6 = \frac{1.0506 \times 10^7}{1 + 0.4654\cos(-C)}$$

$$\Rightarrow \quad \cos(-C) = 0.9878$$

$$C = -8.96°$$



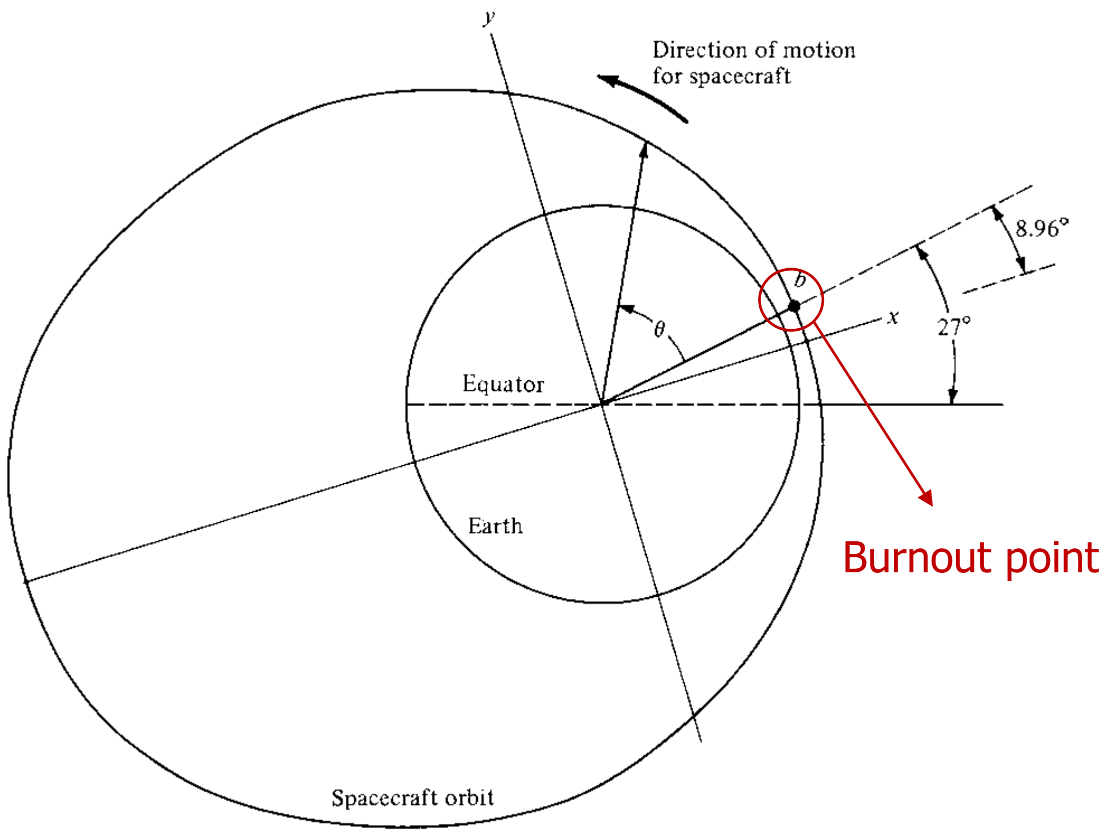
Ellipse $e < 1$

Orbit calculation

Review – example 8.1

The complete orbit equation is:

$$r = \frac{1.0506 \times 10^7}{1 + 0.4654\cos(\theta + 8.96°)}$$



Orbit calculation

Example 8.3

Consider the spacecraft orbit calculated in Example 8.1 . Calculate the spacecraft's velocity at (a) the perigee (近地点), (b) the apogee (远地点), and (c) a true anomaly (真近点角) of 120°.

Orbit calculation

Example 8.3

Consider the spacecraft orbit calculated in Example 8.1 . Calculate the spacecraft's velocity at (a) the perigee (近地点), (b) the apogee (远地点), and (c) a true anomaly (真近点角) of 120°.

From the energy equation, we have:

$$\Rightarrow V = \sqrt{\frac{2k^2}{r} - \frac{k^2}{a}}$$

Unknowns:

1. semimajor a.
2. radius r.

Orbit calculation

Calculate the r and a

$$r = \frac{1.0506 \times 10^7}{1 + 0.4654\cos(\theta + 8.96°)}$$

The true anomaly, θ_A, is measured from the axis of symmetry; hence $\theta_A = \theta + 8.96°$, and the orbit equation can be written as

$$r = \frac{1.0506 \times 10^7}{1 + 0.4654\cos\theta_A}$$

At perigee $\theta_A = 0$, $r_p = 7.169 \times 10^6\ m$

At apogee $\theta_A = 180°$, $r_a = 1.965 \times 10^7\ m$

At $\theta_A = 120°$, $r_a = 1.369 \times 10^7\ m$

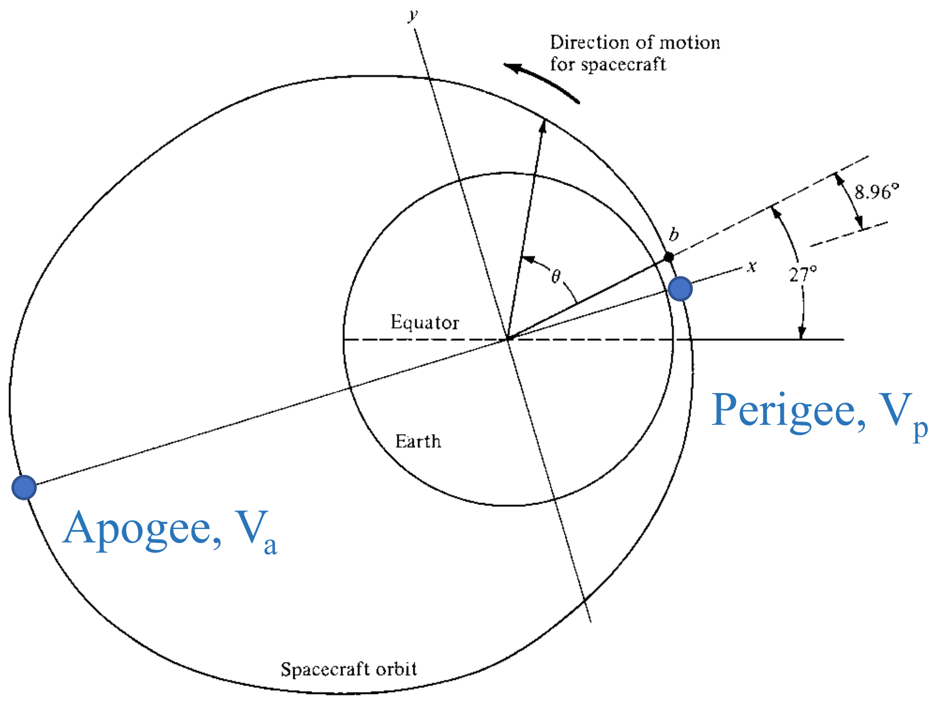
The semimajor, $a = \frac{r_p + r_a}{2} = 1.341 \times 10^7\ m$

The problem is solved.

Orbit calculation

Discussion

- The velocity is maximum at perigee and minimum at apogee.
- The burnout position is near perigee, thus the velocity 9 km/s is close to V_p
- At $\theta_A = 120°$, the calculated velocity should larger than V_a, but smaller than V_p



Orbit Maneuver

Plane changes

- The plane of orbit has a specific inclination angle relative to the equatorial plane
- How to change this inclination angle?

Orbit Maneuver

Plane changes

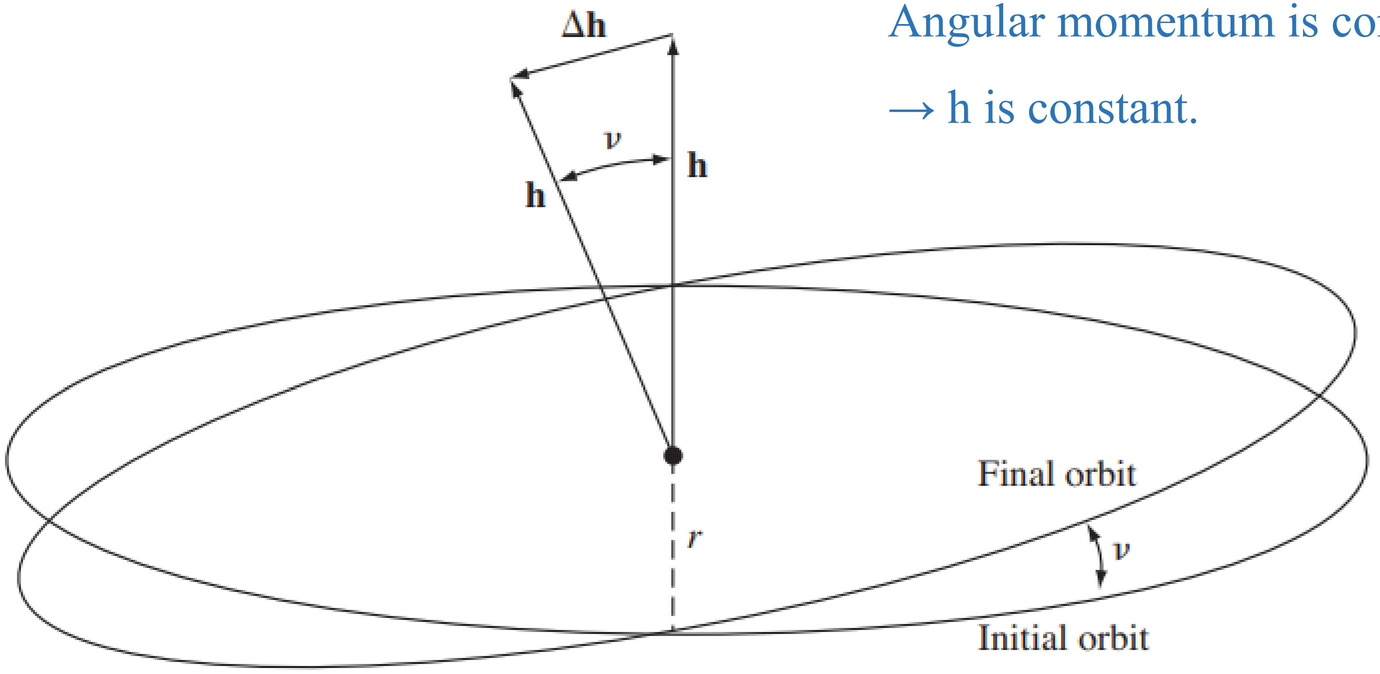


Figure 8.21 Schematic for an orbital plane change.

Angular momentum is conserved:

→ h is constant.

Orbit Maneuver

Plane changes

$$(\Delta h)^2 = h^2 + h^2 - 2h^2 \cos v$$

geometric relationship

$$(\Delta h)^2 = h^2[2(1-\cos v)]$$

$$\Rightarrow \quad \Delta h = 2rV_\theta \sin\left(\frac{v}{2}\right)$$

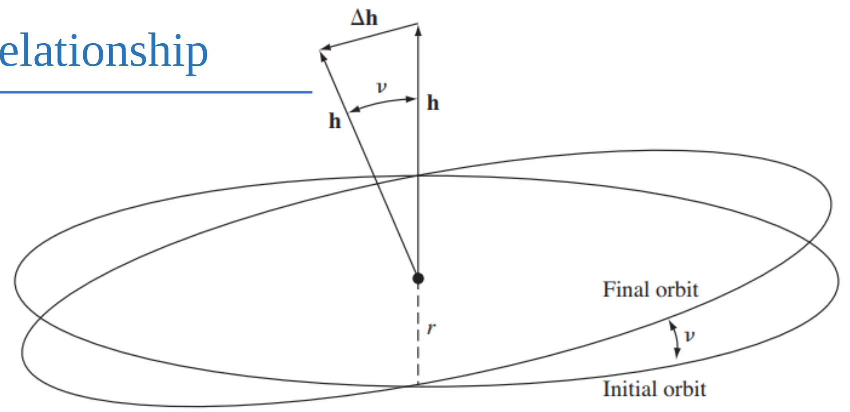
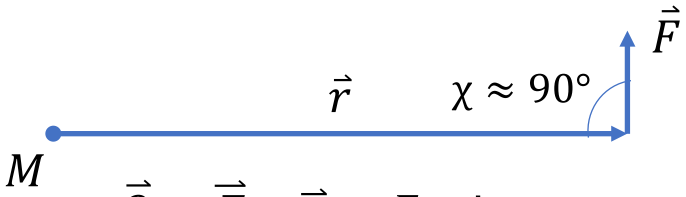


Figure 8.21 Schematic for an orbital plane change.

Orbit Maneuver

Plane changes



$$\vec{Q} = \vec{\boldsymbol{F}} \times \vec{\boldsymbol{r}} = Fr\sin\chi$$

$$\boxed{\Delta h = 2rV_\theta \sin\left(\frac{\nu}{2}\right)}$$

Torque = Time rate of change of angular momentum $\Rightarrow \vec{Q} = d\vec{\boldsymbol{h}}/dt$

$$\Delta h = Q\Delta t = Fr\Delta t$$

$$\Delta h = rF\Delta t = r\Delta V$$

Both h and F are defined as per unit mass

$$\Rightarrow \quad \Delta V = 2V_\theta \sin\left(\frac{\nu}{2}\right)$$

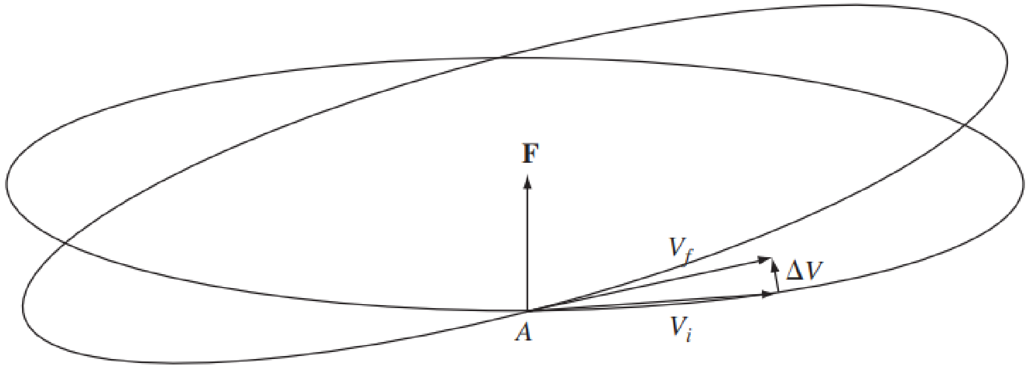


Figure 8.22 Application of the impulse for a simple orbital plane change.

Orbit Maneuver

Discussion

$$\Delta V = 2V_\theta \sin\left(\frac{v}{2}\right)$$

- Orbit maneuver cost money
- The smallest $\boldsymbol{\Delta V}$ correspond to the point on the orbit where V_θ is minimum
- The best efficiency is achieved by executing the plane change at **the apogee**

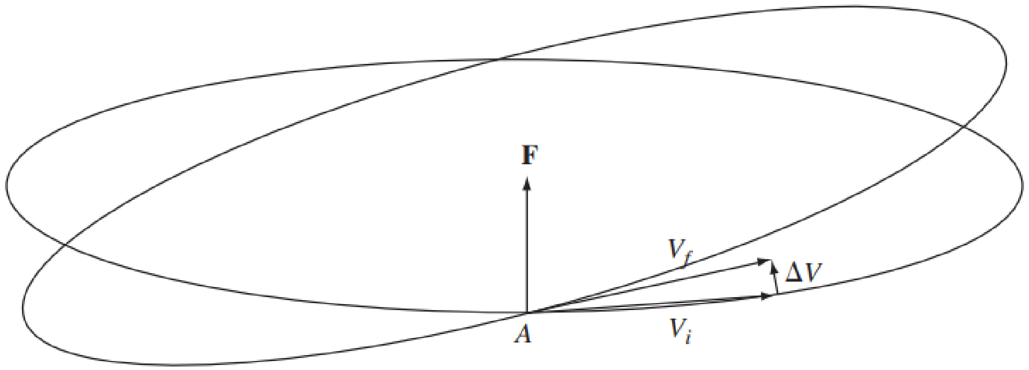


Figure 8.22 Application of the impulse for a simple orbital plane change.

Orbit Maneuver

Example 8.5

Consider the orbit determined in Example 8.1 and drawn in Fig. 8.15 . The given burnout conditions stated that the burnout velocity direction was due north. Therefore the plane of the orbit shown in Fig. 8.15 is perpendicular to the equatorial plane and contains both the north and south poles. An edge view of this orbital plane is shown in Fig. 8.23 , perpendicular to the equatorial plane. An impulse is applied to the spacecraft to change the inclination angle of the orbit by 10°, as shown in Fig. 8.23 . Note that the planes of the initial and final orbits and the equatorial plane all include the focus F of the elliptical orbits, which is the center of the earth (the assumed origin of the central gravitational force field). The impulse is applied at the ascending node of the original orbit. Calculate the value of the impulse $\boldsymbol{\Delta V}$ required to perform this plane change maneuver.

$$\Delta V = 2V_\theta \sin\left(\frac{v}{2}\right)$$

Unknowns?

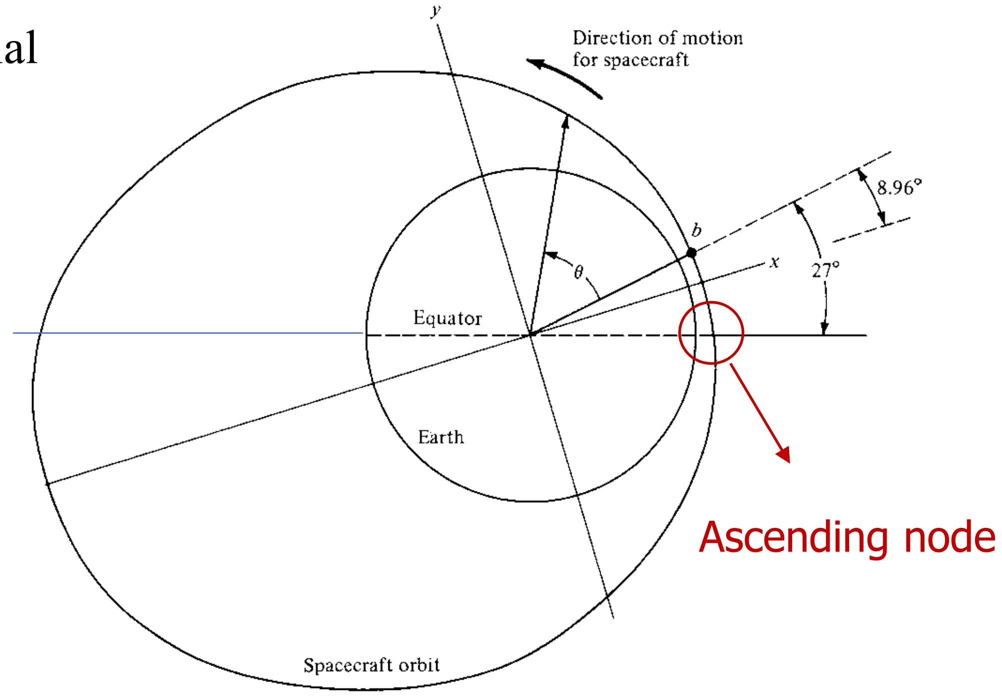
Orbit Maneuver

Example 8.5

The ascending node is where the orbit crosses the equatorial plane, as shown in the figure.

The true anomaly of at the ascending node is

$$\theta_A = 8.96° - 27° = -18.04°$$



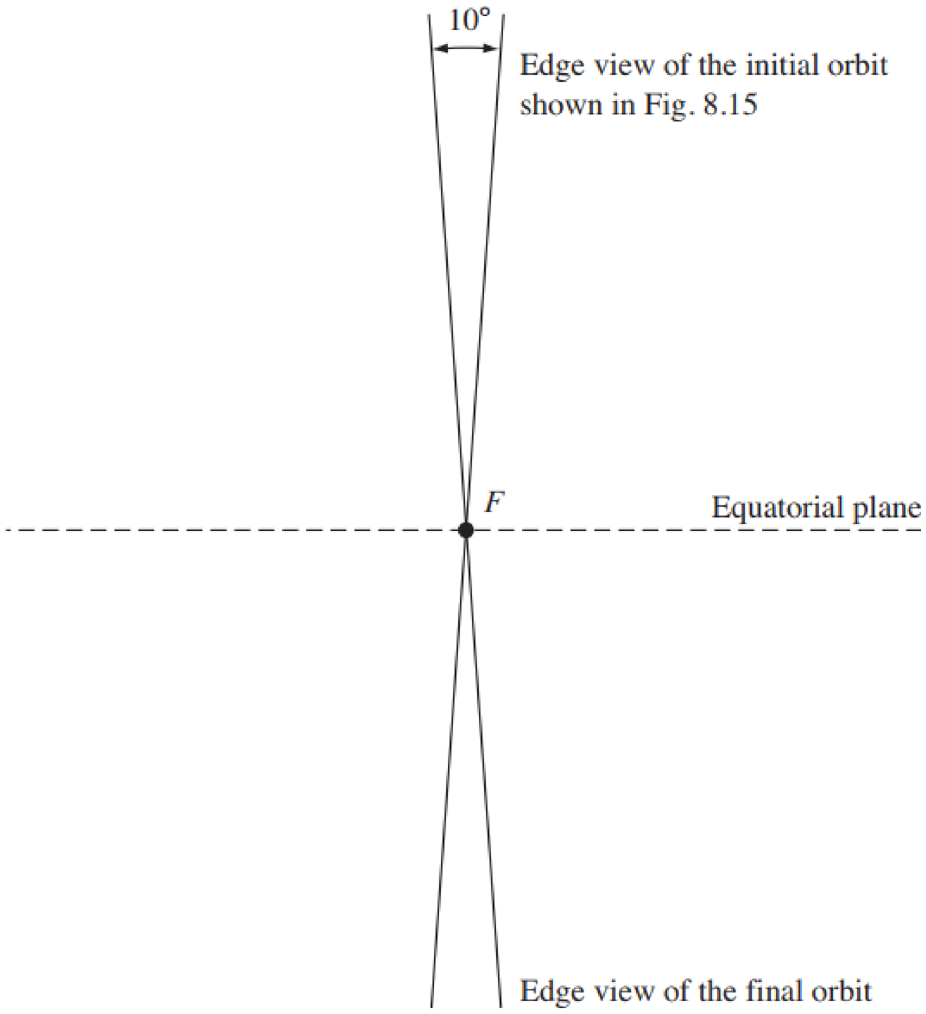
Orbit Maneuver

Example 8.5

$$r = \frac{1.0506\times10^7}{1+0.4654\cos\theta_A} = \frac{1.0506\times10^7}{1+0.4654\cos(-18.04°)} = 7.283\times10^6\ \text{m}$$

$$V_\theta = \frac{h}{r} = \frac{6.47\times10^{10}}{7.283\times10^6} = 8884\ \text{m/s}$$

$$\Delta \mathbf{V} = 2V_\theta \sin\left(\frac{v}{2}\right) = 2(8884)\sin(5°) = \boxed{1549\ \text{m/s}}$$



Orbit Maneuver

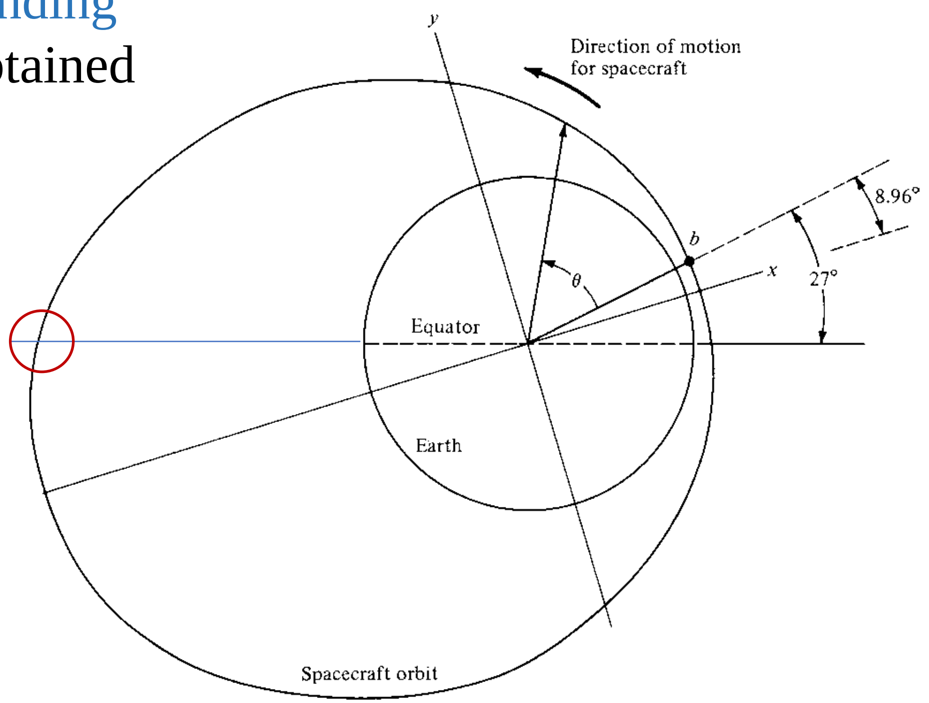
Example 8.6

Repeat Example 8.5 with the impulse applied at the descending node. Compare the impulse for this case with the result obtained for the ascending node in Example 8.5 .

At the descending node:

$\theta_A = 180° - 18.04° = 161.96°$

Descending node



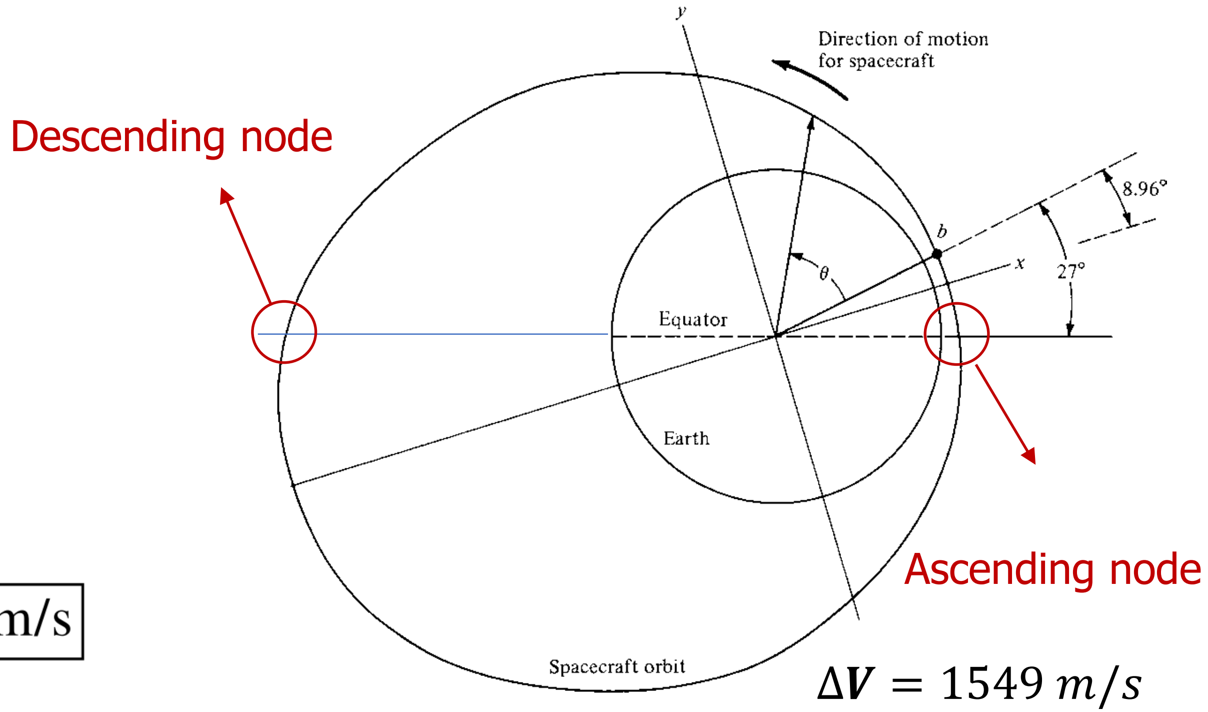
Orbit Maneuver

Example 8.6

$$r = \frac{1.0506\times10^{7}}{1+0.4654\cos(161.96°)} = 1.885\times10^{7}\ \text{m}$$

$$V_\theta = \frac{h}{r} = \frac{6.47\times10^{10}}{1.885\times10^{7}} = 3432\ \text{m/s}$$

$$\mathbf{\Delta V} = 2V_\theta \sin\left(\frac{v}{2}\right) = 2(3432)\sin 5° = \boxed{598.2\ \text{m/s}}$$

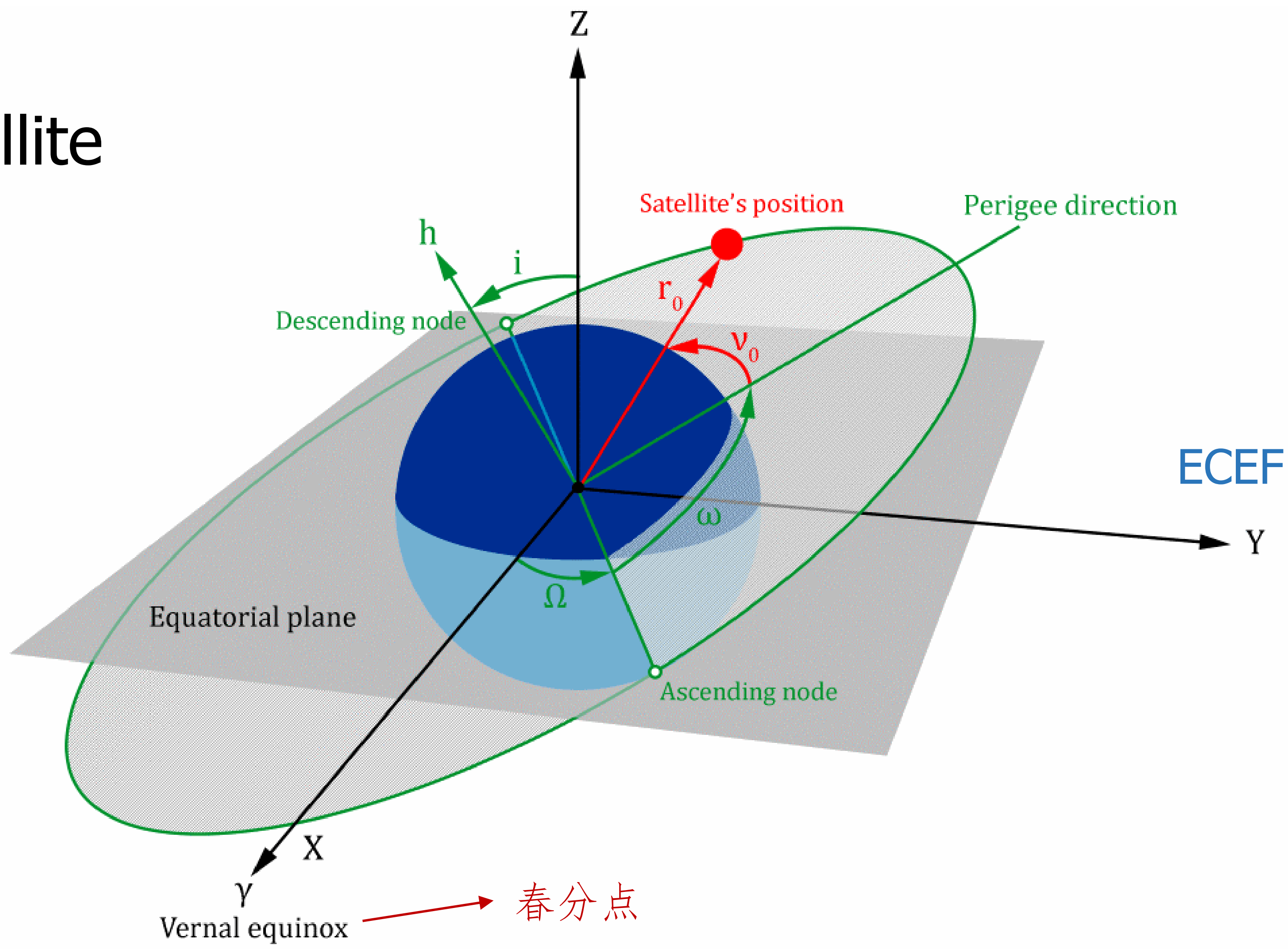


$\Delta \boldsymbol{V} = 1549\ m/s$

More efficient at descending node!

A real example

skCUBE satellite

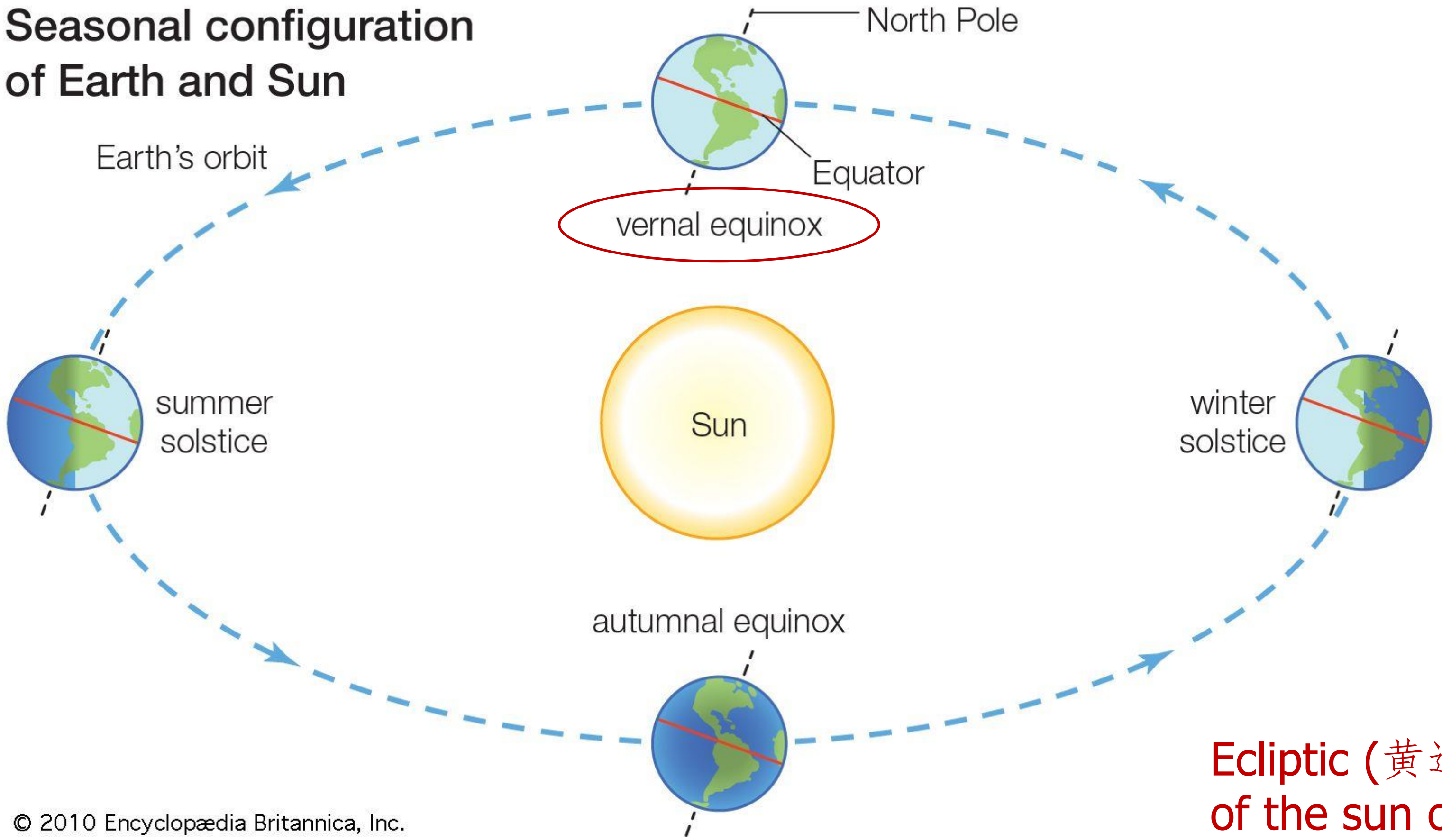


ECEF frame

The vernal equinox

(spring equinox / march equinox)

The vernal equinox



Ecliptic (黄道) is the path of the sun on the star sky

Orbit Maneuver

Maneuver requirements

- change to a new orbit with a different eccentricity, a different semimajor axis, and a new direction of the line of apsides
- but in **the same plane** as the original orbit

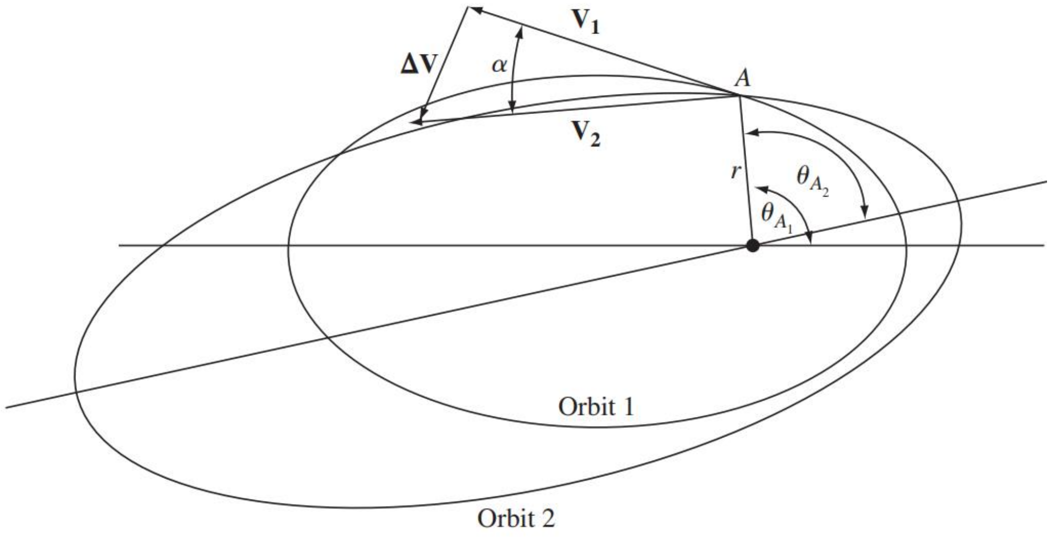


Figure 8.24 Schematic of a coplanar orbital transfer for two intersecting orbits (not to scale).

Orbit Maneuver

Calculate the impulse

$$(\Delta V)^2 = V_1^2 + V_2^2 - 2V_1V_2\cos\alpha$$

Unknowns: V_1, V_2, α

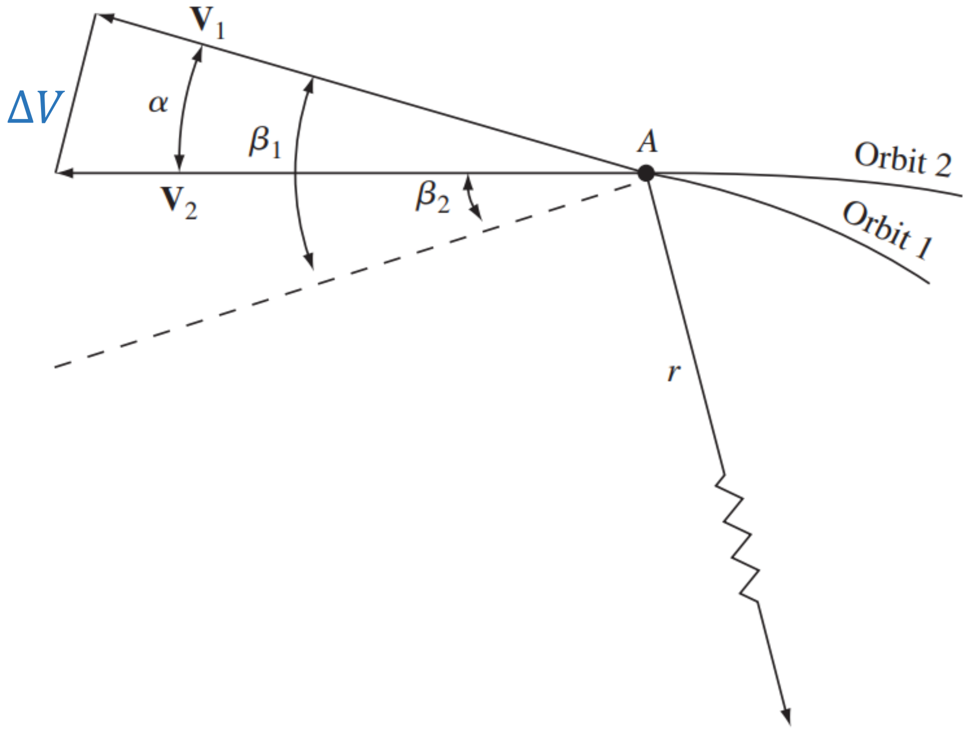


Figure 8.25 Detail at point *A* as seen in Fig. 8.24.

smallest impulse (smallest energy) required to make the orbital transfer occurs when $\alpha = 0$ (point where the two orbits are tangent to each other)

Orbit Maneuver

Example 8.7

Consider a spacecraft moving in the orbit calculated in Example 8.1. For this orbit, from Example 8.1 , the eccentricity is $e_1 = 0.4654$, and the periapsis and apoapsis are $r_{p,1} = 7.169 \times 10^6$ and $r_{a,1} = 1.965 \times 10^7$ m, respectively. At the point on the orbit given by the true anomaly $\theta_A = 90°$, a single impulse is applied to the spacecraft that transfers the spacecraft to a new orbit with $e_2 = 0.6$ and $r_{p,2} = 8000$ km. Calculate the value $\Delta \boldsymbol{V}$ of this impulse.

Orbit Maneuver

Example 8.7

$$r = \frac{1.0506\times10^{7}}{1+0.4654\ \cos\theta_A}$$

Consider a spacecraft moving in the orbit calculated in Example 8.1. For this orbit, from Example 8.1 , the eccentricity is $e_1 = 0.4654$, and the periapsis and apoapsis are $r_{p,1} = 7.169\times 10^6$ and $r_{a,1} = 1.965 \times 10^7$ m, respectively. At the point on the orbit given by the true anomaly $\theta_A = 90°$, a single impulse is applied to the spacecraft that transfers the spacecraft to a new orbit with $e_2 = 0.6$ and $r_{p,2} = 8000$ km. Calculate the value $\boldsymbol{\Delta V}$ of this impulse.

$$(\Delta V)^2 = V_1^2 + V_2^2 - 2\,V_1 V_2 \cos\alpha$$

Unknowns: V_1, V_2, α

$$V = \sqrt{\frac{2k^2}{r} - \frac{k^2}{a}}$$

Orbit Maneuver

Example 8.7

$$r_1 = \frac{h^2/k^2}{1+e\cos\theta_A} = \frac{1.0506\times10^7}{1+0.4654\cos 90°} = \frac{1.0506\times10^7}{1+0} = 1.0506\times10^7\ \mathrm{m}$$

$$a_1 = \frac{r_{p,1}+r_{a,1}}{2} = \frac{7.169\times10^6+1.965\times10^7}{2} = 1.341\times10^7\ \mathrm{m}$$

(Given conditions)

$$V_1 = \sqrt{\frac{2k^2}{r_1}-\frac{k^2}{a_1}} = \sqrt{\frac{2(3.986\times10^{14})}{1.0506\times10^7}-\frac{3.986\times10^{14}}{1.341\times10^7}}$$

$$= \sqrt{7.588\times10^7-2.972\times10^7} = 6794\ \mathrm{m/s}$$

Similar for V_2

Orbit Maneuver

The relation between flight path angle and true anomaly

$$\tan\beta = \frac{e\sin\theta_A}{1+e\cos\theta_A}$$

Eq. (8.96) of Textbook

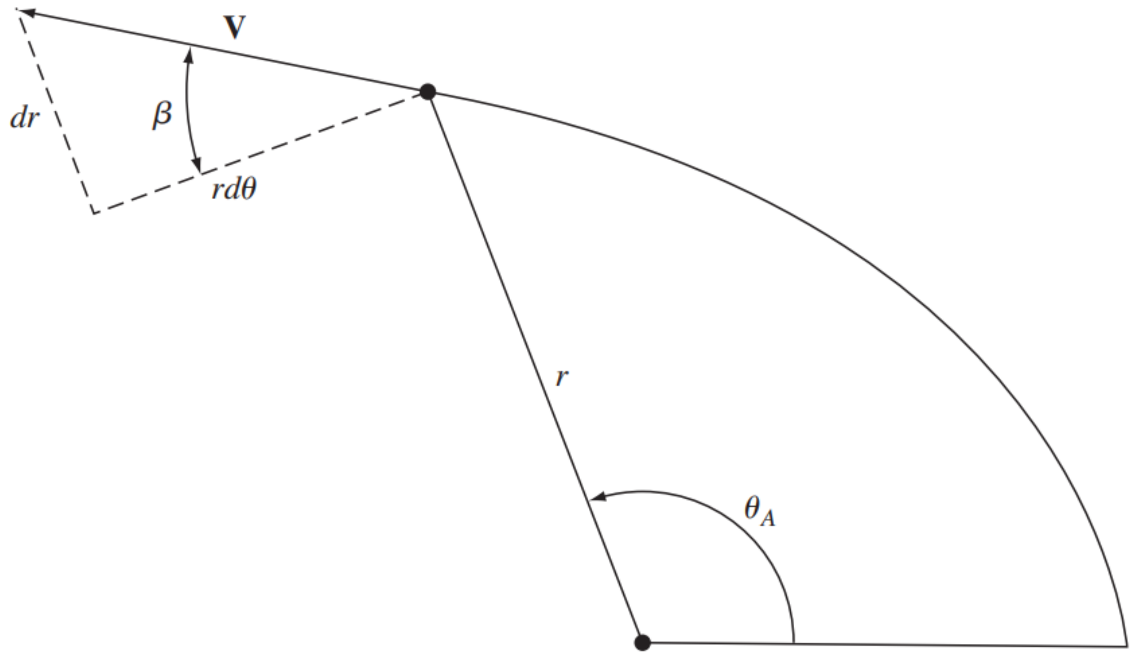
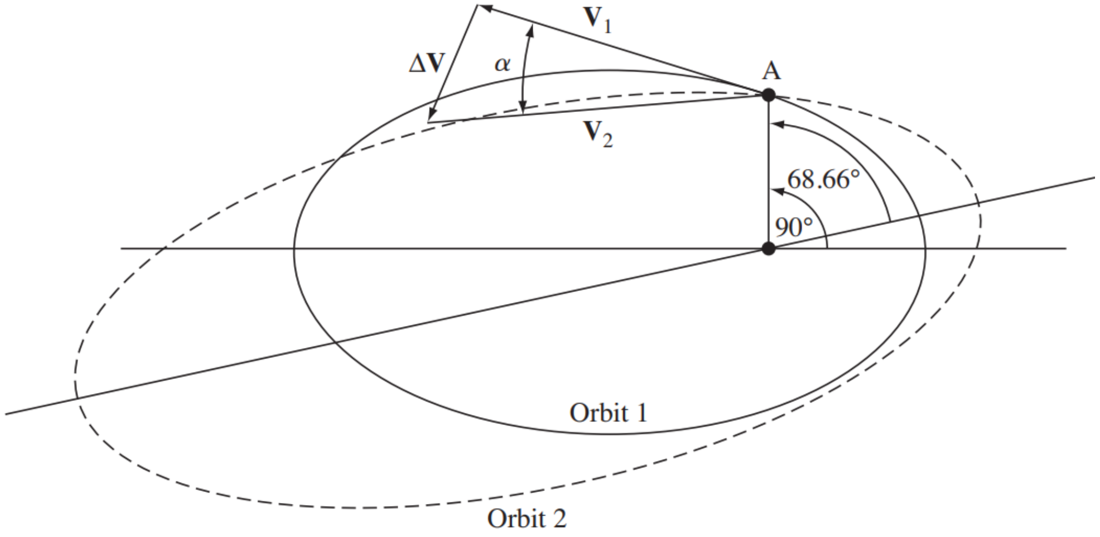


Figure 8.26 Diagram for the calculation of the flight path angle.

Orbit Maneuver

Example 8.7



$$\tan\beta_1 = \frac{e_1 \sin\theta_A}{1+e_1\cos\theta_A} = \frac{0.4654 \sin 90°}{1+0.4654\cos 90°} = 0.4654$$

$$\tan\beta_2 = \frac{e_2 \sin\theta_A}{1+e_2\cos\theta_A} = \frac{(0.6)\sin(68.66°)}{1+(0.6)\cos(68.66°)} = \frac{0.5589}{1.21834} = 0.4587$$

$$\Rightarrow \quad \alpha = \beta_1 - \beta_2 = 24.957 - 24.64 = 0.317°$$

The problem is solved.

Orbit Maneuver

Scenario 1 – Two impulses

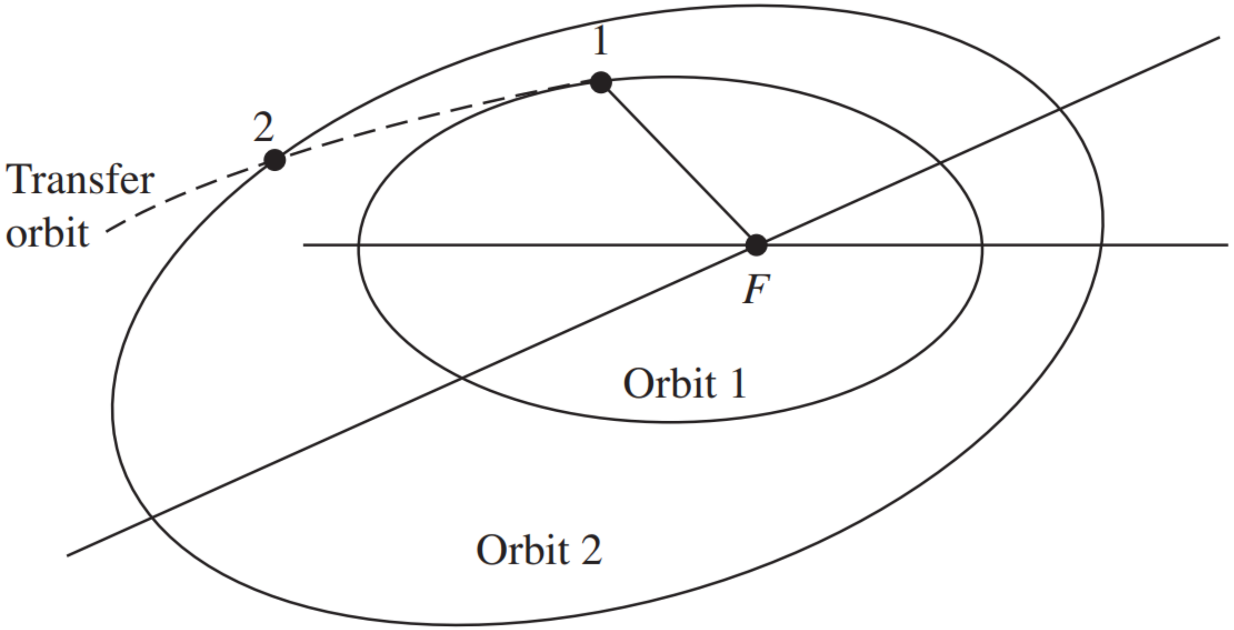


Figure 8.28 Generic sketch of a transfer orbit.

Orbit Maneuver

Scenario 2 – Hohmann transfer orbit

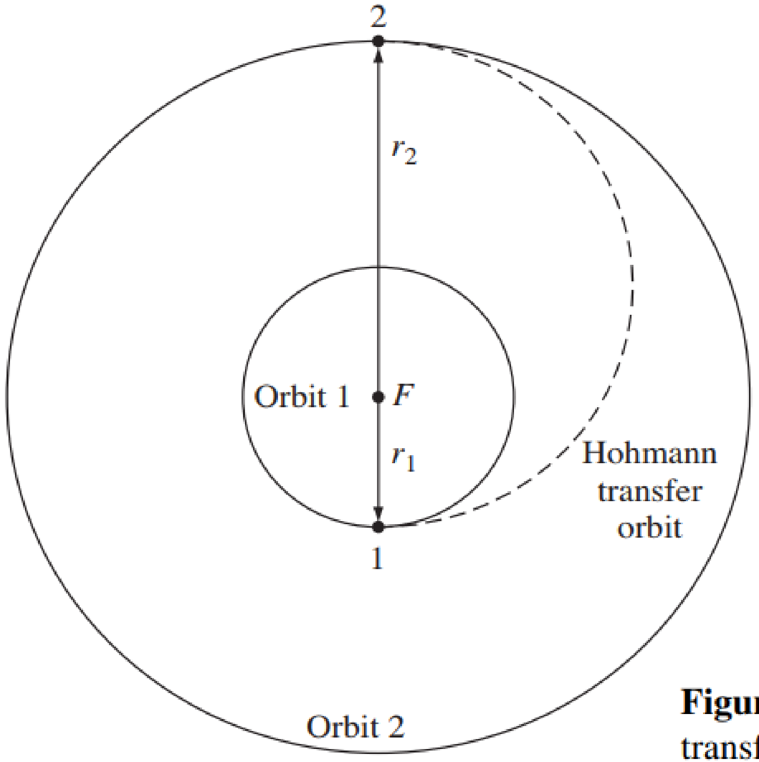


Figure 8.29 Illustration of the Hohmann transfer orbit.

Fun during quarantine

Some recommendations

- Kerbal space program
- www.heavens-above.com
- Star Walk (App)
- 实地观测点

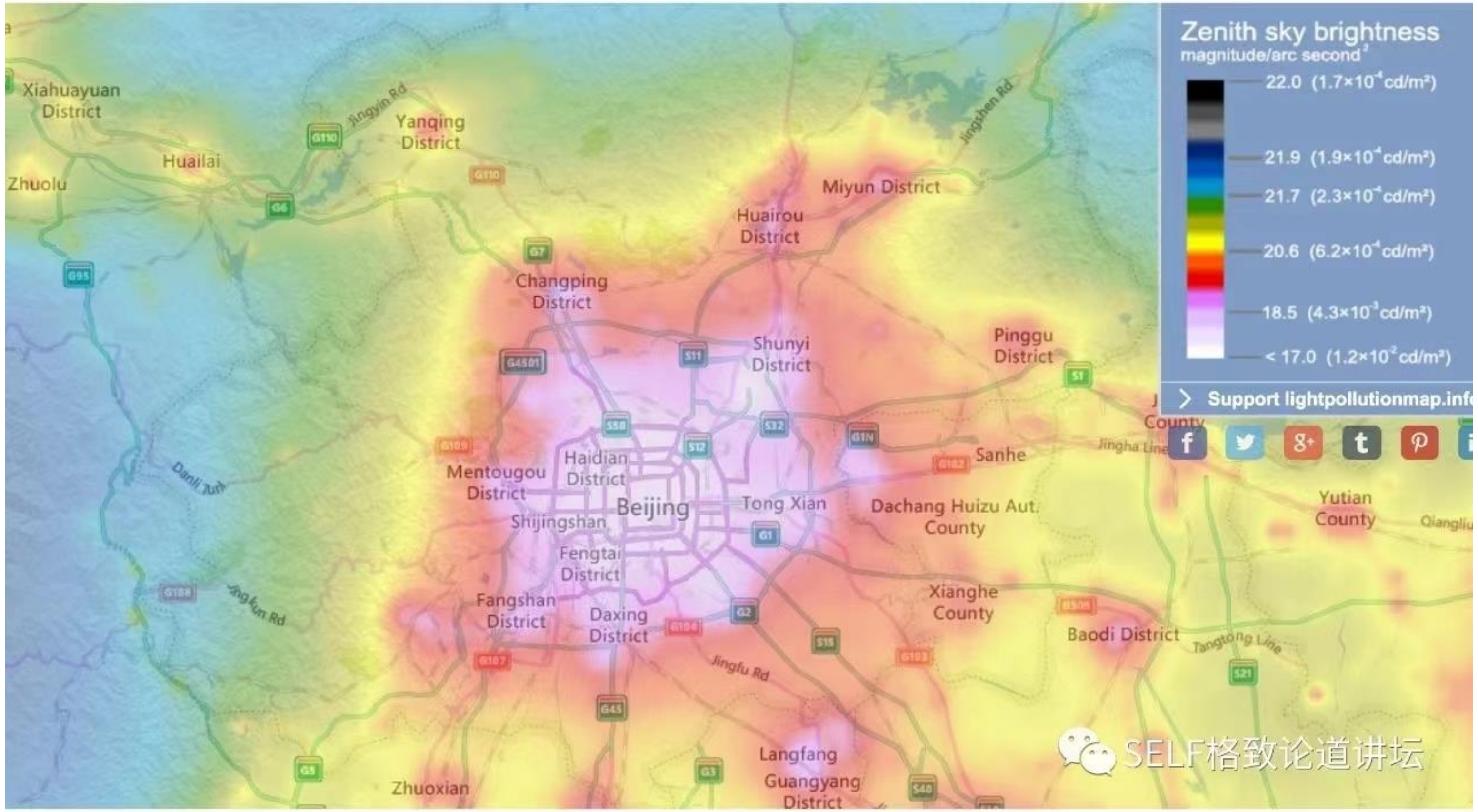
Watch the sky

Source: 中科院国家天文台

Watch the sky

National Astronomical Observatory – Miyun（密云观测站）

Watch the sky



[Source: Light pollution map]

Watch the sky

Source: 格致论道讲坛

Watch the sky

Source: 叶梓颐

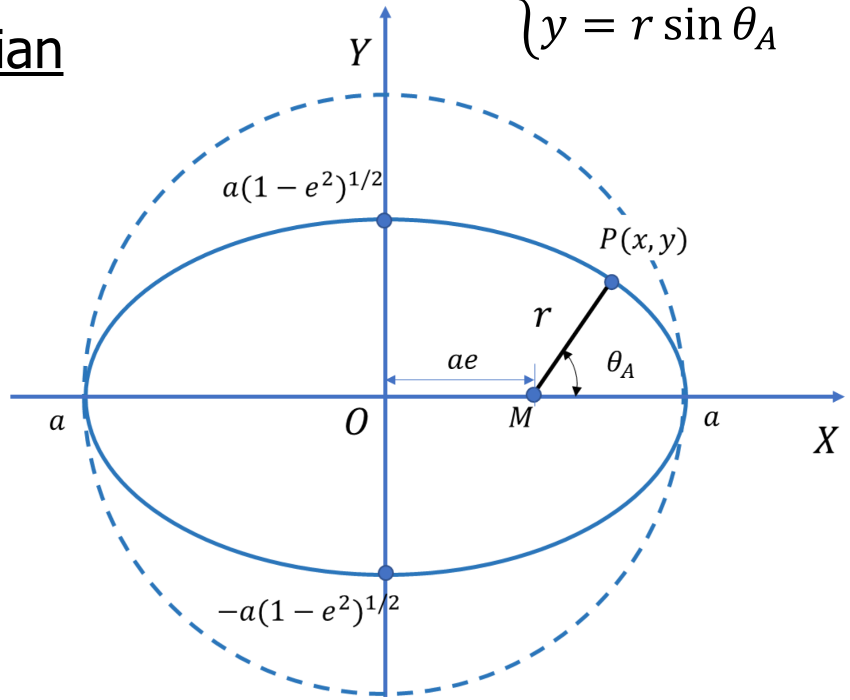
Homework

Question 1.

The two different expressions for the ellipse in Cartesian coordinate system

$$\begin{cases} x = r\cos\theta_A + ae \\ y = r\sin\theta_A \end{cases} \qquad \frac{x^2}{a^2} + \frac{x^2}{a^2(1-e^2)} = 1$$

$$\begin{cases} x = r\cos\theta_A + ae \\ y = r\sin\theta_A \end{cases}$$



a) Derive the following equation:

$$r = \frac{a(1-e^2)}{1 + e\cos\theta_A} \qquad \text{(Eq. 1.1)}$$

Homework

b) Based on the definition of true anomaly θ_A (真近点角), eccentric anomaly E (偏近点角) and mean anomaly M (平近点角), prove that

$$\cos\theta_A = \frac{\cos E - e}{1 - e\cos E} \qquad \sin\theta_A = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E} \qquad \text{(Eq. 1.2)}$$

c) Derive the following equations:

$$M = E - e \cdot \sin E \qquad \text{(Eq. 1.3)}$$

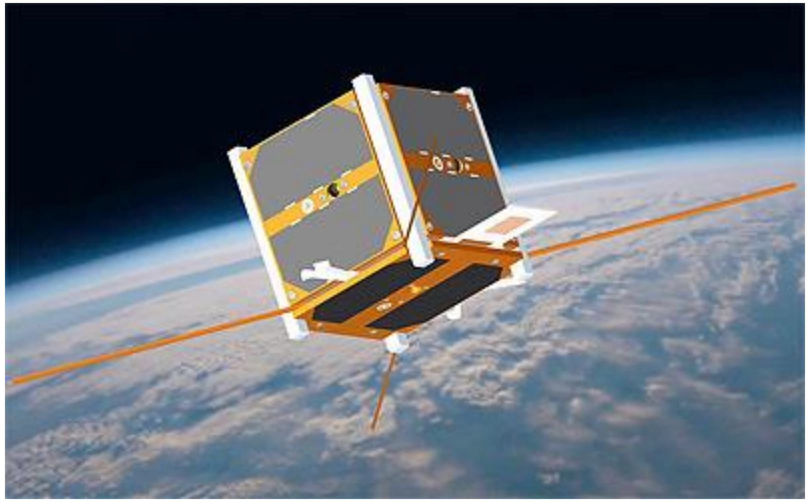
Homework

Question 2.

skCUBE satellite is the first Slovak satellite launched into the Earth orbit on 23.6.2017. Its orbit has the following parameters: (not all parameters will be used)

Orbit Inclination (degrees)	97.3621	i = 97,3621°
Right Ascension of Ascending Node (degrees)	144.5852	Ω = 144,582
Eccentricity (decimal point assumed)	0012165	e = 0,0012165
Argument of Perigee (degrees)	160.6847	ω = 160,6847°
Mean Anomaly (degrees)	199.4853	M = 199,4853°
Mean Motion (revolutions/day)	15.22573301	n = 15,22573301

initial mean anomaly M_0

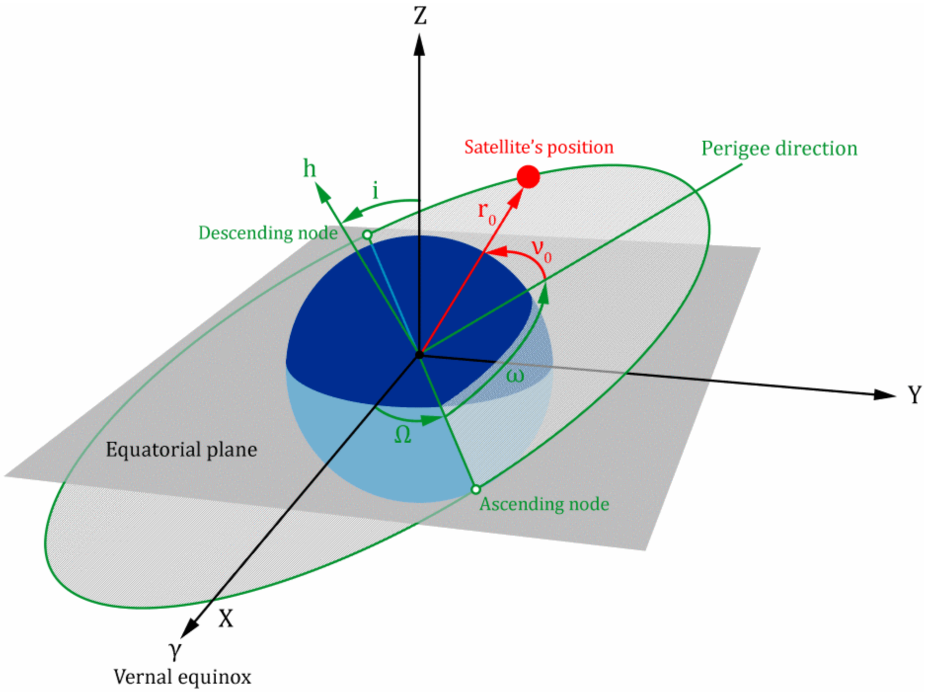


Homework

Question 2.

The geocentric gravitational constant and equatorial radius of the earth are given as

$$GM = 3.986 \times 10^{14}\ m^3/s^2,\ r = 6.378 \times 10^6 m$$



a) Calculate the period of the satellite τ.
b) Calculate the semimajor axis a.
c) Calculate the radius of perigee r_p and apogee r_a.

Homework

Question 2.

d) The mean anomaly M can be written as $M = M_0 + n(t - t_0)$, where t_0 is zero. Divide the period τ with 1 minute interval and plot M, E, and θ_A as a function of time.
(hints: E can be solved from Eq.1.3 of question 1. by iterative method)

e) Plot the orbit of the satellite according to Eq.1.1.

f) Assume you want let this satellite fly directly over Mariupol (longitude 37.55, latitude 47.09) to observe what happened near Ukraine - Russia border. You were allowed to apply one velocity impulse. What orbit maneuver will you make?

Due on 19th May. (Please submit the electronic version or scanned answer sheets.)