System Dynamics and Vibrations

Prof. Gustavo Alonso

Chapter 5: Dynamic stability. Part II.

School of General Engineering
Beihang University (BUAA)

Contents

- Lyapunov’s second method for stability.
- Stability of two-degree-of-freedom systems from equilibrium

Lyapunov’s second method for stability

- Lyapunov stability theory was developed by Lyapunov, a Russian mathematician in 1892, and came from his doctoral dissertation.
- The Lyapunov stability theory is used to describe the stability of a dynamic system

The concept of stability. Introduction to Lyapunov's stability

- Autonomous system: no input external to the system
- Described by differential equations

$$\dot{x} = f(x,t), \quad x(t_0) = x_0 \qquad t \in \left[t_0, \infty\right)$$

where *x* is the <u>system state</u>

Positive definite function

V(x) is a positive definite function if:

V(x) is continuous and derivable for any value of *x*

V(0) = 0

V(x) > 0, for *x* ≠ 0

Then:

$$\lim_{\|x\| \to \infty} V\left(\|x\|\right) = \infty$$

Lyapunov function

The core idea of Lyapunov second method is to find a function similar to “energy” function, transform the problem of system stability into analyzing the positive definite problem of the “energy” function and its first derivative, such “energy” function is Lyapunov function.

For instance, a quadratic function:

$$V(x) = \mathbf{x}^T P \mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{12} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

x is a real vector and *P* is real symmetric matrix

Lyapunov stability theorem (second method)

For a continuous time-varying, nonlinear, autonomous system described by the differential equation: $\dot{x} = f(x,t), \quad t \in [t_0, \infty)$

If we can find a scalar function $V(x,t), \quad V(0,t) = 0$

which is:

Positive definite and $\dot{V}(x,t) \leq 0, \quad x \neq 0$

Then *V(x)* is called a Lyapunov function and the system is stable in the sense of Lyapunov

If $\dot{V}(x,t) < 0, \quad x \neq 0$

then the system is asymptotically stable

Lyapunov stability theorem

- The Lyapunov function has an analogy with the potential energy of dynamic systems: if the system loses energy over time and the energy is never restored then eventually the system must grind to a stop and reach some final resting state.
- However, finding a function that gives the precise energy of a physical system can be difficult
- Also, for abstract mathematical systems, economic systems or biological systems, the concept of energy may not be applicable.
- Lyapunov's realization was that stability can be proven without requiring knowledge of the true physical energy, provided a Lyapunov function can be found to satisfy the above constraints.

Contents

Stability of two-degree-of-freedom systems from equilibrium

When the system is subjected to self-exciting forces, the force terms can be combined with the damping/stiffness terms, and the resulting equations of motion can be expressed as

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By substituting the solution:

$$x_j(t) = X_j e^{st}, \quad j = 1, 2$$

and setting the determinant of the coefficient matrix to zero, we obtain the characteristic equation:

Stability of two-degree-of-freedom systems from equilibrium

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

The coefficients a_i are real numbers, since they are derived from the physical parameters of the system

Being s_i the roots of the characteristic equation, we have

$$(s - s_1)(s - s_2)(s - s_3)(s - s_4) = 0$$

Performing the multiplications and identifying terms with the previous equation, we obtain:

Stability of two-degree-of-freedom systems from equilibrium

$$a_0 = 1$$

$$a_1 = -\left(s_1 + s_2 + s_3 + s_4\right)$$

$$a_2 = s_1 s_2 + s_1 s_3 + s_1 s_4 + s_2 s_3 + s_2 s_4 + s_3 s_4$$

$$a_3 = -\left(s_1 s_2 s_3 + s_1 s_2 s_4 + s_1 s_3 s_4 + s_2 s_3 s_4\right)$$

$$a_4 = s_1 s_2 s_3 s_4$$

The criterion for stability is that the real parts of s_i must be negative to avoid increasing exponentials in the solution

$$x_j(t) = X_j e^{st}, \quad j = 1, 2$$

Stability of two-degree-of-freedom systems from equilibrium

Using the properties of a quartic equation, it can be derived that a <u>necessary and sufficient condition for stability </u>is that all the coefficients of the equation (a_0, a_1, a_2, a_3 and a_4) be positive and that the condition

$$a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2$$

be fulfilled

Stability of two-degree-of-freedom systems from equilibrium

A more general technique, suitable to investigate the stability of multi-degree-of-freedom systems, is the Routh-Hurwitz criterion:

The system will be stable if all the coefficients of the equation (a_0, a_1, a_2, a_3 and a_4) be positive and the following determinant are positive:

$$T_1 = |a_1| > 0$$

$$T_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = a_1 a_2 - a_0 a_3 > 0$$

$$T_3 = \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 > 0$$