System Dynamics and Vibrations

Prof. Gustavo Alonso

Chapter 6: Two-degree-of-freedom systems
Exercises - 5

School of General Engineering
Beihang University (BUAA)

Exercise 11

Consider the three-degree-of-freedom system of Exercise 7 and solve the associated eigenvalue problem for the parameters:

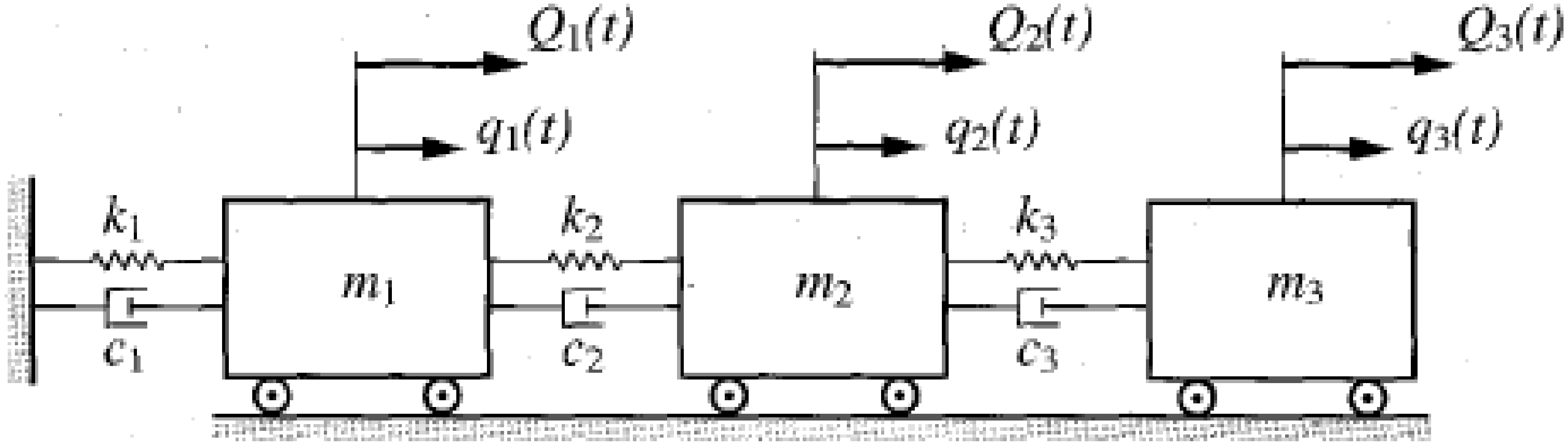
$m_1 = m_2 = m$; $m_3 = 2m$

$c_1 = c_2 = c_3 = 0$

$k_1 = k_2 = k$; $k_3 = 2k$

Then, derive the solution to the **free vibration problem for the initial excitations**

$$\mathbf{q}(0) = q_0 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T, \ \dot{\mathbf{q}}(0) = 0$$



Exercise 11

In Exercise 7 we derived the mass, damping and stiffness matrices:

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

For the given parameters, the matrices become:

$$M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$K = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

Exercise 11

Eigenvalue problem: $$\det\left[K - \omega^2 M \right] = 0$$

$$\det\begin{bmatrix} 2k-\omega^2 m & -k & 0 \\ -k & 3k-\omega^2 m & -2k \\ & -2k & 2k-2\omega^2 m \end{bmatrix} = -2m^3\left[\omega^6 - 6\frac{k}{m}\omega^4 + 8\left(\frac{k}{m}\right)^2\omega^2 - \left(\frac{k}{m}\right)^3 \right] = 0$$

$$\omega_1^2 = 0.1392\frac{k}{m}$$

$$\omega_2^2 = 1.7459\frac{k}{m}$$

$$\omega_3^2 = 4.1149\frac{k}{m}$$

Exercise 11

Modal vectors:

➔insert each of the natural frequencies ω_r, in sequence, in

$$K\mathbf{u}_r = \omega_r^2 M\mathbf{u}_r, \quad r = 1, 2, \ldots, n$$

and solve the corresponding algebraic equations:

For r = 1

$$\left(k_{11} - \omega_1^2 m_1\right)u_1 + k_{12}u_2 + k_{13}u_3 = 0$$

$$k_{12}u_1 + \left(k_{22} - \omega_1^2 m_2\right)u_2 + k_{23}u_3 = 0$$

$$k_{13}u_1 + k_{23}u_2 + \left(k_{33} - \omega_1^2 m_3\right)u_3 = 0$$

homogeneous system: no unique solution
➔ one of the three components can be assigned an arbitray value (one of the three equations is redundant)

Exercise 11

Choosing arbitrarily u_3 = 1
and retaining the first two equations of the system:

$$k\left(2-0.1392\right)u_1 - ku_2 = 0$$
$$-ku_1 + k\left(3-0.1392\right)u_2 = 2k$$

$$u_1 = 0.4626, \quad u_2 = 0.8608$$

The first modal vector is then:

$$\mathbf{u}_1 = \begin{bmatrix}0.4626 & 0.8608 & 1.000\end{bmatrix}^T$$

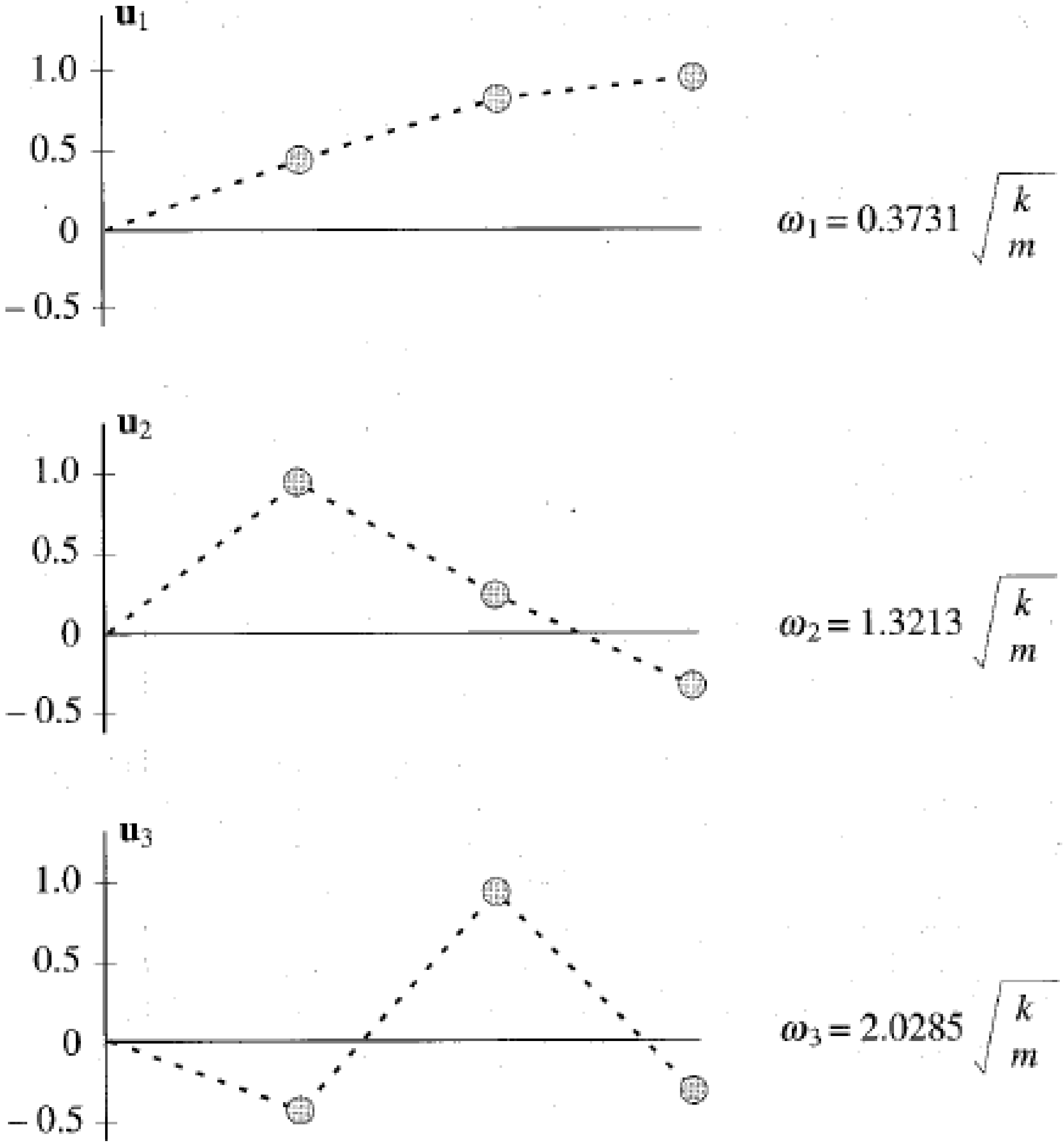
Following the same procedure the other two modal vectors are:

$$\mathbf{u}_2 = \begin{bmatrix}1.0000 & 0.2541 & -0.3407\end{bmatrix}^T$$
$$\mathbf{u}_3 = \begin{bmatrix}-0.4728 & 1.000 & -0.3210\end{bmatrix}^T$$

Exercise 11

All modal vectors have been normalized so that the largest component is equal to 1

The modes can be represented:



Exercise 11

The solution of the free-vibration problem has the general form:

$$\mathbf{q}(t) = \sum_{r=1}^{n} q_r(t) = \sum_{r=1}^{n} \mathbf{u}_r f_r(t) = \sum_{r=1}^{n} C_r \mathbf{u}_r \cos\left(\omega_r t - \phi_r\right)$$

$$= C_1 \begin{bmatrix} 0.4626 \\ 0.8606 \\ 1.0000 \end{bmatrix} \cos\left(0.3731\sqrt{\frac{k}{m}}t - \phi_1\right) + C_2 \begin{bmatrix} 1.0000 \\ 0.2541 \\ -0.3407 \end{bmatrix} \cos\left(1.3213\sqrt{\frac{k}{m}}t - \phi_2\right) + C_3 \begin{bmatrix} -0.4728 \\ 1.0000 \\ -0.3210 \end{bmatrix} \cos\left(2.0285\sqrt{\frac{k}{m}}t - \phi_3\right)$$

C_r amplitudes and phase angles are constants of integration, determined
ϕ_r with the initial conditions

Exercise 11

Initial conditions:

$$\mathbf{q}(0) = q_0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = C_1 \begin{bmatrix} 0.4626 \\ 0.8606 \\ 1.0000 \end{bmatrix} \cos\phi_1 + C_2 \begin{bmatrix} 1.0000 \\ 0.2541 \\ -0.3407 \end{bmatrix} \cos\phi_2 + C_3 \begin{bmatrix} -0.4728 \\ 1.0000 \\ -0.3210 \end{bmatrix} \cos\phi_3$$

Differentiating **q**(0) and letting *t* = 0

$$\dot{\mathbf{q}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.3731\sqrt{\frac{k}{m}} C_1 \begin{bmatrix} 0.4626 \\ 0.8606 \\ 1.0000 \end{bmatrix} \sin\phi_1 + \sqrt{\frac{k}{m}} 1.3213 C_2 \begin{bmatrix} 1.0000 \\ 0.2541 \\ -0.3407 \end{bmatrix} \sin\phi_2 + 2.0285\sqrt{\frac{k}{m}} C_3 \begin{bmatrix} -0.4728 \\ 1.0000 \\ -0.3210 \end{bmatrix} \sin\phi_3$$

Exercise 11

$$\mathbf{q}(0) = q_0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = C_1 \begin{bmatrix} 0.4626 \\ 0.8606 \\ 1.0000 \end{bmatrix} \cos\phi_1 + C_2 \begin{bmatrix} 1.0000 \\ 0.2541 \\ -0.3407 \end{bmatrix} \cos\phi_2 + C_3 \begin{bmatrix} -0.4728 \\ 1.0000 \\ -0.3210 \end{bmatrix} \cos\phi_3$$

$$\dot{\mathbf{q}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.3731\sqrt{\frac{k}{m}} C_1 \begin{bmatrix} 0.4626 \\ 0.8606 \\ 1.0000 \end{bmatrix} \sin\phi_1 + \sqrt{\frac{k}{m}} 1.3213 C_2 \begin{bmatrix} 1.0000 \\ 0.2541 \\ -0.3407 \end{bmatrix} \sin\phi_2 + 2.0285\sqrt{\frac{k}{m}} C_3 \begin{bmatrix} -0.4728 \\ 1.0000 \\ -0.3210 \end{bmatrix} \sin\phi_3$$

The second system represents three homogeneous algebraic equations in the unknowns $C_1\sin\phi_1$, $C_2\sin\phi_2$ and $C_3\sin\phi_3$
For a nontrivial solution to exist, the determinant of the coefficients must be equal to zero.
But, the columns of the determinant represent the modal vectors, which are orthogonal, and hence independent by definition.
It follows that the determinant cannot be zero, so that the system can only be satisfied trivially, or

$$C_1\sin\phi_1 = C_2\sin\phi_2 = C_3\sin\phi_3 = 0$$

Because the case $C_1 = C_2 = C_3 = 0$ must be ruled out, we conclude that

$$\phi_1 = \phi_2 = \phi_3 = 0$$

Exercise 11

$$\mathbf{q}(0) = q_0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = C_1 \begin{bmatrix} 0.4626 \\ 0.8606 \\ 1.0000 \end{bmatrix} \cos\phi_1 + C_2 \begin{bmatrix} 1.0000 \\ 0.2541 \\ -0.3407 \end{bmatrix} \cos\phi_2 + C_3 \begin{bmatrix} -0.4728 \\ 1.0000 \\ -0.3210 \end{bmatrix} \cos\phi_3$$

$$\dot{\mathbf{q}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.3731\sqrt{\frac{k}{m}} C_1 \begin{bmatrix} 0.4626 \\ 0.8606 \\ 1.0000 \end{bmatrix} \sin\phi_1 + \sqrt{\frac{k}{m}} 1.3213 C_2 \begin{bmatrix} 1.0000 \\ 0.2541 \\ -0.3407 \end{bmatrix} \sin\phi_2 + 2.0285\sqrt{\frac{k}{m}} C_3 \begin{bmatrix} -0.4728 \\ 1.0000 \\ -0.3210 \end{bmatrix} \sin\phi_3$$

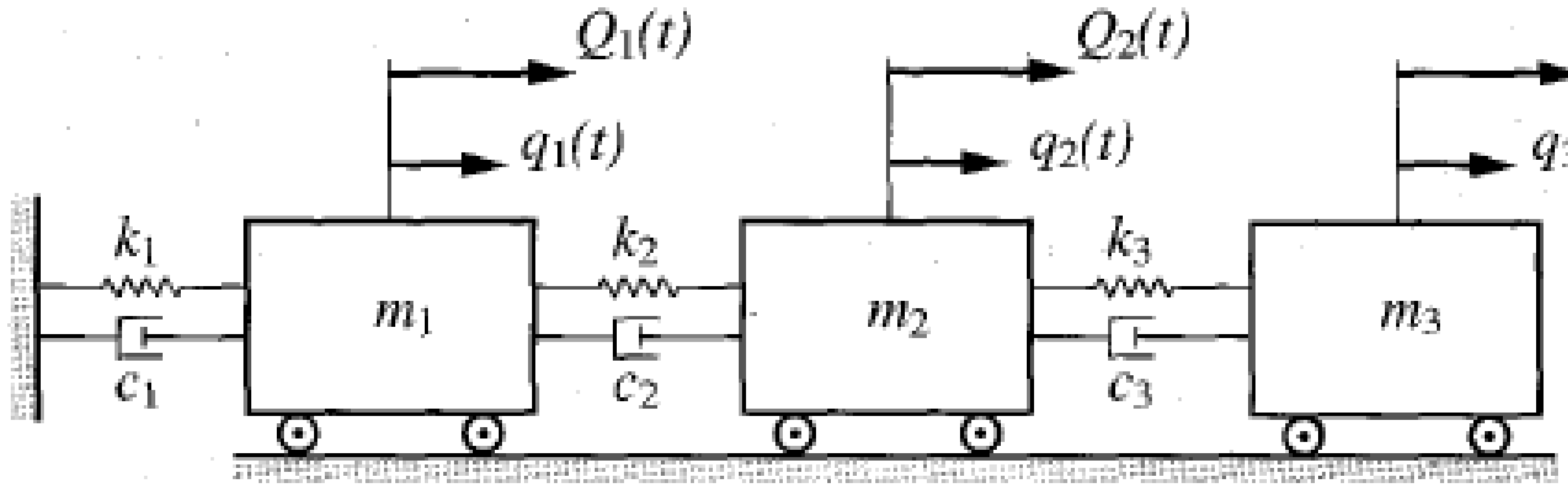
Then, solving the first system we obtain: $C_1 = 2.7696 q_0,\quad C_2 = -0.4132 q_0,\quad C_3 = -0.2791 q_0,$

The solution of the free vibration problem is then:

$$\mathbf{q}(t) = q_0 \left\{ \begin{bmatrix} 1.2812 \\ 2.3841 \\ 2.7696 \end{bmatrix} \cos 0.3731\sqrt{\frac{k}{m}} t + \begin{bmatrix} -0.4132 \\ -0.1050 \\ 0.1408 \end{bmatrix} \cos 1.3213\sqrt{\frac{k}{m}} t + \begin{bmatrix} 0.1320 \\ -0.2791 \\ 0.0896 \end{bmatrix} \cos 2.0285\sqrt{\frac{k}{m}} t \right\}$$

Exercise 12

Obtain the solution of the free vibration problem for the three-degree-of-freedom system of Exercise 11 by means of modal analysis



Exercise 12

The free vibration response derived by means of modal analysis is given by:

$$\mathbf{q}(t)=\sum_{r=1}^{n}\left[\mathbf{u}_r^T M\mathbf{q}\left(0\right)\cos\omega_r t+\frac{1}{\omega_r}\mathbf{u}_r^T M\dot{\mathbf{q}}\left(0\right)\sin\omega_r t\right]\mathbf{u}_r$$

in which the modal vectors are normalized according to:

$$\mathbf{u}_r^T M\mathbf{u}_r=1,\quad r=1,2,\ldots,n$$

From Exercise 11 we have

$$\mathbf{u}_1=\left[0.4626 \quad 0.8608 \quad 1.000\right]^T$$

$$\mathbf{u}_2=\left[1.0000 \quad 0.2541 \quad -0.3407\right]^T$$

$$\mathbf{u}_3=\left[-0.4728 \quad 1.000 \quad -0.3210\right]^T$$

Exercise 12

Using the procedure of Exercise 10, the re-normalized normal vectors are:

$$\mathbf{u}_1 = m^{-1/2}\begin{bmatrix} 0.2691 \\ 0.5008 \\ 0.5817 \end{bmatrix}, \quad \mathbf{u}_2 = m^{-1/2}\begin{bmatrix} 0.8782 \\ 0.2231 \\ -0.2992 \end{bmatrix}, \quad \mathbf{u}_3 = m^{-1/2}\begin{bmatrix} -0.3954 \\ 0.8363 \\ -0.2685 \end{bmatrix}$$

$$q(t) = q_0\left\{\begin{bmatrix} 0.2691 \\ 0.5008 \\ 0.5817 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cos 0.3731\sqrt{\frac{k}{m}}t \begin{bmatrix} 0.2691 \\ 0.5008 \\ 0.5817 \end{bmatrix} + \begin{bmatrix} 0.8782 \\ 0.2231 \\ -0.2992 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cos 0.3731\sqrt{\frac{k}{m}}t \begin{bmatrix} 0.8782 \\ 0.2231 \\ -0.2992 \end{bmatrix} + \begin{bmatrix} -0.3954 \\ 0.8363 \\ -0.2685 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cos 0.3731\sqrt{\frac{k}{m}}t \begin{bmatrix} -0.3954 \\ 0.8363 \\ -0.2685 \end{bmatrix}\right\}$$

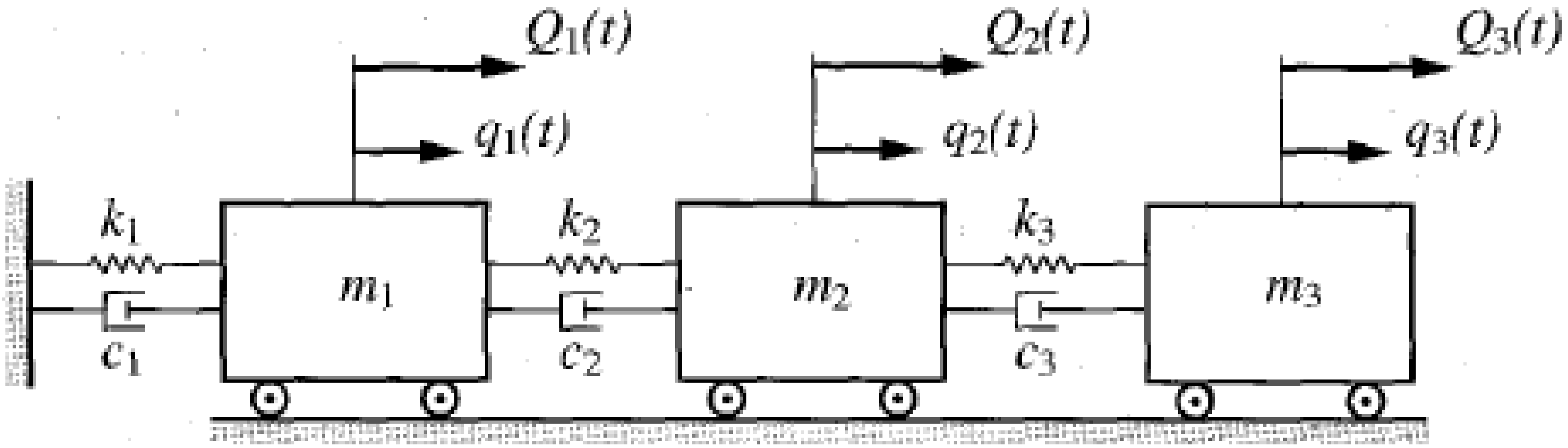
$$\mathbf{q}(t) = q_0\left\{\begin{bmatrix} 1.2812 \\ 2.3841 \\ 2.7696 \end{bmatrix} \cos 0.3731\sqrt{\frac{k}{m}}t + \begin{bmatrix} -0.4132 \\ -0.1050 \\ 0.1408 \end{bmatrix} \cos 1.3213\sqrt{\frac{k}{m}}t + \begin{bmatrix} 0.1320 \\ -0.2791 \\ 0.0896 \end{bmatrix} \cos 2.0285\sqrt{\frac{k}{m}}t\right\}$$

Exercise 13

Use modal analysis to derive the response of the **undamped** three-degree-of-freedom system of Exercise 11 to the **excitation**:

$$Q_1(t) = Q_2(t) = 0, \quad Q_3(t) = Q_0 \boldsymbol{U}(t)$$

$\boldsymbol{U}(t) \longrightarrow$ unit step function



Exercise 13

The equations of motion for the system are given in matrix form by:

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{Q}(t)$$

From Exercise 11 we know:

$$M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$K = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

Exercise 13

The response is given by: $$\mathbf{q}(t) = \sum_{r=1}^{n} \eta_r(t)\mathbf{u}_r = U\boldsymbol{\eta}(t)$$

The modal matrix *U* has been calculated in Exercise 12:

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.2691 & 0.8782 & -0.3954 \\ 0.5008 & 0.2231 & 0.8363 \\ 0.5817 & -0.2992 & -0.2685 \end{bmatrix}$$

U has been normalized to satisfy $U^T M U = I$

The modal coordinates are given by equations:

$$\eta_r(t) = \frac{1}{\omega_r} \int_0^t N_r(t-\tau) \sin \omega_r \tau d\tau, \quad r = 1, 2, \ldots, n$$

Exercise 13

The modal forces are: $N_r(t) = \mathbf{u}_r^T \mathbf{Q}(t), \quad r = 1,1,\ldots,n$

$$N_1(t) = \mathbf{u}_1^T Q(t) = m^{-1/2} \begin{bmatrix} 0.2691 \\ 0.5008 \\ 0.5817 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ Q_0 \boldsymbol{U}(t) \end{bmatrix} = 0.581731 \frac{Q_0}{\sqrt{m}} \boldsymbol{U}(t)$$

$$N_2(t) = \mathbf{u}_2^T Q(t) = m^{-1/2} \begin{bmatrix} 0.8782 \\ 0.2231 \\ -0.2992 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ Q_0 \boldsymbol{U}(t) \end{bmatrix} = 0.299166 \frac{Q_0}{\sqrt{m}} \boldsymbol{U}(t)$$

$$N_3(t) = \mathbf{u}_3^T Q(t) = m^{-1/2} \begin{bmatrix} -0.3954 \\ 0.8363 \\ -0.2685 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ Q_0 \boldsymbol{U}(t) \end{bmatrix} = 0.268493 \frac{Q_0}{\sqrt{m}} \boldsymbol{U}(t)$$

Exercise 13

From Exercise 11 the natural frequencies were determined:

$$\omega_1 = 0.373087\sqrt{\frac{k}{m}}, \quad \omega_2 = 1.321325\sqrt{\frac{k}{m}}, \quad \omega_3 = 2.028523\sqrt{\frac{k}{m}}$$

Now the modal coordinates can be calculated:

$$\eta_1(t) = \frac{0.581731Q_0}{\sqrt{m}\omega_1}\int_0^t \boldsymbol{U}(t-\tau)\sin\omega_1\tau d\tau = \frac{0.581731Q_0}{\sqrt{m}\omega_1^2}\left(1-\cos\omega_1 t\right) = \frac{4.179285Q_0\sqrt{m}}{k}\left(1-\cos 0.373087\sqrt{\frac{k}{m}}t\right)$$

$$\eta_2(t) = \frac{0.299166Q_0}{\sqrt{m}\omega_2}\int_0^t \boldsymbol{U}(t-\tau)\sin\omega_2\tau d\tau = \frac{0.299166Q_0}{\sqrt{m}\omega_2^2}\left(1-\cos\omega_2 t\right) = \frac{0.171353Q_0\sqrt{m}}{k}\left(1-\cos 1.321325\sqrt{\frac{k}{m}}t\right)$$

$$\eta_3(t) = \frac{0.268493Q_0}{\sqrt{m}\omega_3}\int_0^t \boldsymbol{U}(t-\tau)\sin\omega_3\tau d\tau = \frac{0.268493Q_0}{\sqrt{m}\omega_3^2}\left(1-\cos\omega_3 t\right) = \frac{0.065249Q_0\sqrt{m}}{k}\left(1-\cos 2.028523\sqrt{\frac{k}{m}}t\right)$$

Exercise 13

Finally, the response is:

$$\mathbf{q}(t)=\frac{Q_0}{k}\left\{4.179285\left(1-\cos 0.373087\sqrt{\frac{k}{m}}t\right)\begin{bmatrix}0.2691\\0.5008\\0.5817\end{bmatrix}+0.171353\left(1-\cos 1.321325\sqrt{\frac{k}{m}}t\right)\begin{bmatrix}0.8782\\0.2231\\-0.2992\end{bmatrix}+0.065249\left(1-\cos 2.028523\sqrt{\frac{k}{m}}t\right)\begin{bmatrix}-0.3954\\0.8363\\-0.2685\end{bmatrix}\right\}$$

Exercise 14

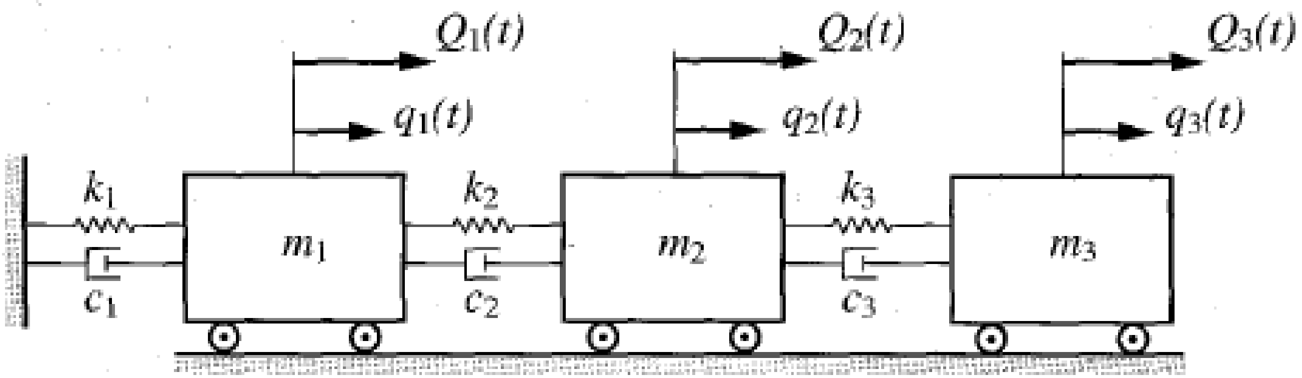
Use modal analysis to derive the response of the three-degree-of-freedom system of Exercise 11 to the excitation:

$$Q_1(t) = Q_2(t) = 0, \quad Q_3(t) = Q_0 \boldsymbol{\mathcal{U}}(t)$$

$\boldsymbol{\mathcal{U}}(t) \longrightarrow$ unit step function

The system possesses **<u>proportional damping</u>** with the proportionality constants:

$$\alpha = 0.2\sqrt{k/m}, \beta = 0.01\sqrt{m/k}$$



Exercise 14

The modal coordinates depend now on the frequencies of damped oscillations

$$\eta_r(t) = \frac{1}{\omega_{dr}} \int_0^t N_r(t-\tau) e^{-\varsigma_r \omega_r \tau} \sin \omega_{dr} \tau d\tau, \quad r = 1, 2, ..., n$$

$$\omega_{dr} = \left(1 - \varsigma_r^2\right)^{1/2} \omega_r, \quad r = 1, 2, ..., n$$

We first need to calculate the modal damping factors: $\varsigma_r \ (r = 1, 2, ..., n)$

$$\alpha + \beta \omega_r^2 = 2 \varsigma_r \omega_r, \quad r = 1, 1, ..., n$$

Exercise 14

$$\varsigma_1 = \frac{\alpha + \beta\omega_1^2}{2\omega_1} = \frac{0.2 + 0.01 \times 0.373087^2}{2 \times 0.373087} = 0.269908$$

$$\varsigma_2 = \frac{\alpha + \beta\omega_2^2}{2\omega_2} = \frac{0.2 + 0.01 \times 1.321325^2}{2 \times 1.321325} = 0.082288$$

$$\varsigma_3 = \frac{\alpha + \beta\omega_3^2}{2\omega_3} = \frac{0.2 + 0.01 \times 2.028523^2}{2 \times 2.028523} = 0.059440$$

$$\omega_{d1} = \left(1 - \varsigma_1^2\right)^{1/2} \omega_1 = 0.359240\sqrt{k/m}$$

$$\omega_{d2} = \left(1 - \varsigma_2^2\right)^{1/2} \omega_2 = 1.316844\sqrt{k/m}$$

$$\omega_{d3} = \left(1 - \varsigma_3^2\right)^{1/2} \omega_3 = 2.021356\sqrt{k/m}$$

Exercise 14

The modal forces remain the same as in Exercise 13. Then:

$$\eta_1(t) = \frac{0.581731Q_0}{\sqrt{m}\omega_{d1}}\int_0^t \boldsymbol{U}(t-\tau)e^{-\varsigma_1\omega_1\tau}\sin\omega_{d1}\tau d\tau = \frac{0.581731Q_0}{\sqrt{m}\omega_{d1}}\left[1-e^{-\varsigma_1\omega_1 t}\left(\cos\omega_{d1}t+\frac{\varsigma_1\omega_1}{\omega_{d1}}\sin\omega_{d1}t\right)\right]$$

$$= \frac{4.179285Q_0\sqrt{m}}{k}\left[1-e^{-0.100699t}\left(\cos 0.359204\sqrt{\frac{k}{m}}t+0.289312\sin 0.359204\sqrt{\frac{k}{m}}t\right)\right]$$

$$\eta_2(t) = \frac{0.299166Q_0}{\sqrt{m}\omega_{d2}}\int_0^t \boldsymbol{U}(t-\tau)e^{-\varsigma_2\omega_2\tau}\sin\omega_{d2}\tau d\tau = \frac{0.299166Q_0}{\sqrt{m}\omega_{d2}}\left[1-e^{-\varsigma_2\omega_2 t}\left(\cos\omega_{d2}t+\frac{\varsigma_2\omega_2}{\omega_{d2}}\sin\omega_{d2}t\right)\right]$$

$$= \frac{0.171353Q_0\sqrt{m}}{k}\left[1-e^{-0.108729t}\left(\cos 1.316844\sqrt{\frac{k}{m}}t+0.082568\sin 1.316844\sqrt{\frac{k}{m}}t\right)\right]$$

$$\eta_3(t) = \frac{0.268493Q_0}{\sqrt{m}\omega_{d3}}\int_0^t \boldsymbol{U}(t-\tau)e^{-\varsigma_3\omega_3\tau}\sin\omega_{d3}\tau d\tau = \frac{0.268493Q_0}{\sqrt{m}\omega_{d3}}\left[1-e^{-\varsigma_3\omega_3 t}\left(\cos\omega_{d3}t+\frac{\varsigma_3\omega_3}{\omega_{d3}}\sin\omega_{d3}t\right)\right]$$

$$= \frac{0.065249Q_0\sqrt{m}}{k}\left[1-e^{-0.120575t}\left(\cos 2.021356\sqrt{\frac{k}{m}}t+0.059651\sin 2.021356\sqrt{\frac{k}{m}}t\right)\right]$$

Exercise 14

And the response is:

$$\mathbf{q}(t) = \frac{Q_0}{k} 4.179285 \left[1 - e^{-0.100699t} \left(\cos 0.359240 \sqrt{\frac{k}{m}} t + 0.289312 \sin 0.359240 \sqrt{\frac{k}{m}} t \right) \right] \begin{bmatrix} 0.2691 \\ 0.5008 \\ 0.5817 \end{bmatrix}$$

$$+ \frac{Q_0}{k} 0.171353 \left[1 - e^{-0.108729t} \left(\cos 1.316844 \sqrt{\frac{k}{m}} t + 0.082568 \sin 1.316844 \sqrt{\frac{k}{m}} t \right) \right] \begin{bmatrix} -0.8782 \\ -0.2231 \\ 0.2992 \end{bmatrix}$$

$$+ \frac{Q_0}{k} 0.065249 \left[1 - e^{-0.120575t} \left(\cos 2.021356 \sqrt{\frac{k}{m}} t + 0.059651 \sin 2.021356 \sqrt{\frac{k}{m}} t \right) \right] \begin{bmatrix} 0.3954 \\ -0.8363 \\ 0.2685 \end{bmatrix}$$