System Dynamics and Vibrations

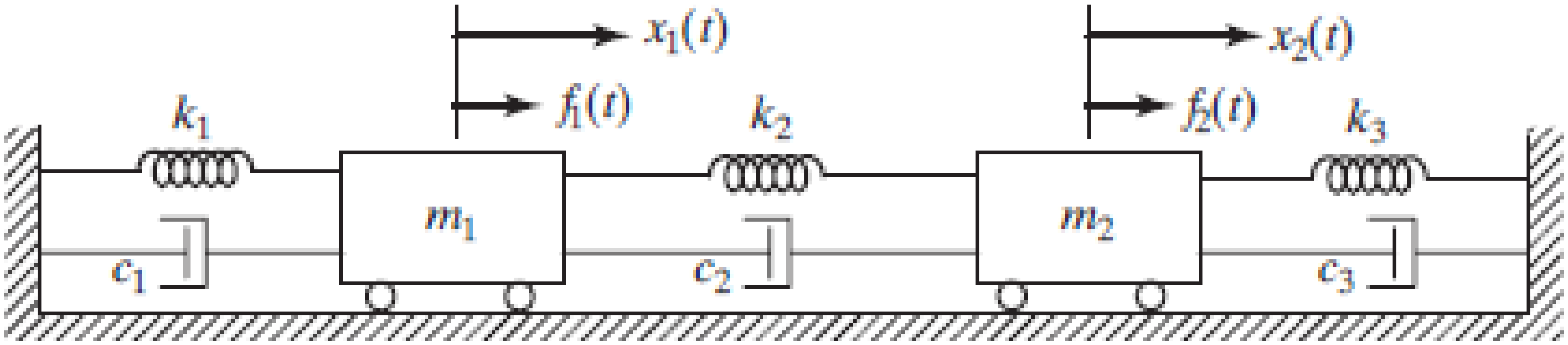
Prof. Gustavo Alonso

Chapter 6: Two-degree-of-freedom systems
Part 2

School of General Engineering
Beihang University (BUAA)

Contents

Equations of motion of 2-DOF systems



$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2 x_2 = f_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = f_2$$

Equations of motion of 2-DOF systems

$$m_1\ddot{x}_1 + \left(c_1 + c_2\right)\dot{x}_1 - c_2\dot{x}_2 + \left(k_1 + k_2\right)x_1 - k_2x_2 = f_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + \left(c_2 + c_3\right)\dot{x}_2 - k_2x_1 + \left(k_2 + k_3\right)x_2 = f_2$$

system of two <u>coupled</u> second order differential equations

$$\left[M\right]\ddot{\vec{x}}(t) + \left[C\right]\dot{\vec{x}}(t) + \left[K\right]\vec{x}(t) = \vec{F}(t)$$

Equations of motion of 2-DOF systems

$$\left[M\right]\ddot{\vec{x}}(t)+\left[C\right]\dot{\vec{x}}(t)+\left[K\right]\vec{x}(t)=\vec{F}(t)$$

mass, damping and stiffness matrices:

$$\left[M\right]=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\left[C\right]=\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}$$

$$\left[K\right]=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

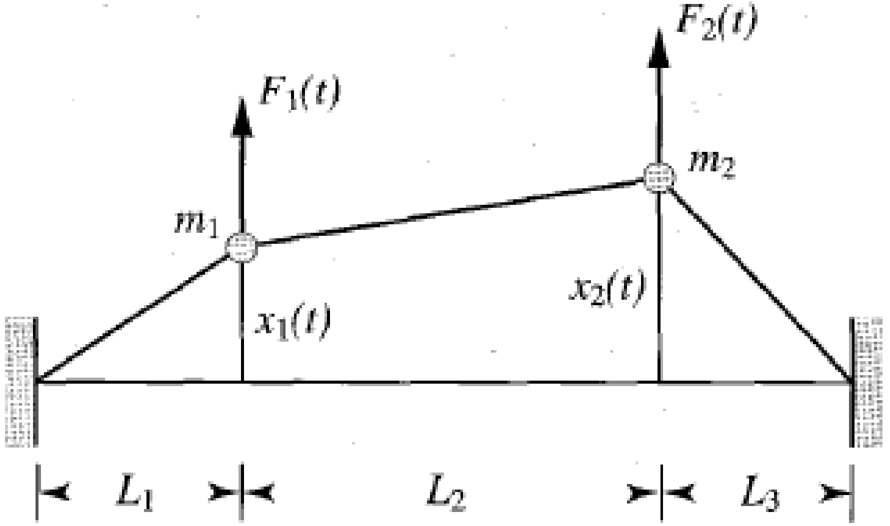
displacementand force vectors:

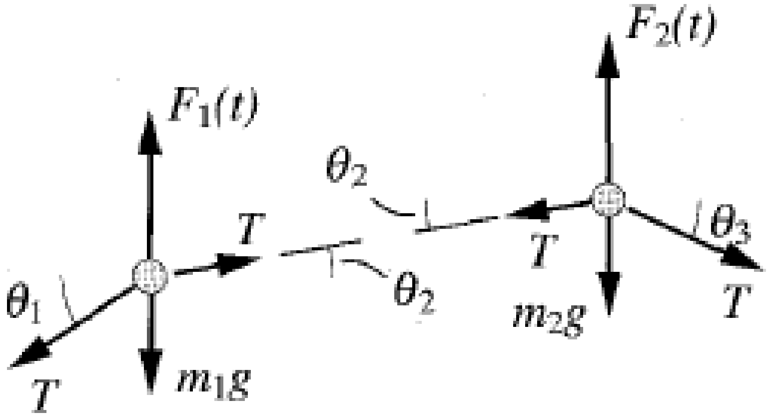
$$\vec{x}(t)=\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

$$\vec{F}(t)=\begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

Equations of motion of 2-DOF systems

Two masses suspended on a string with a tension *T*





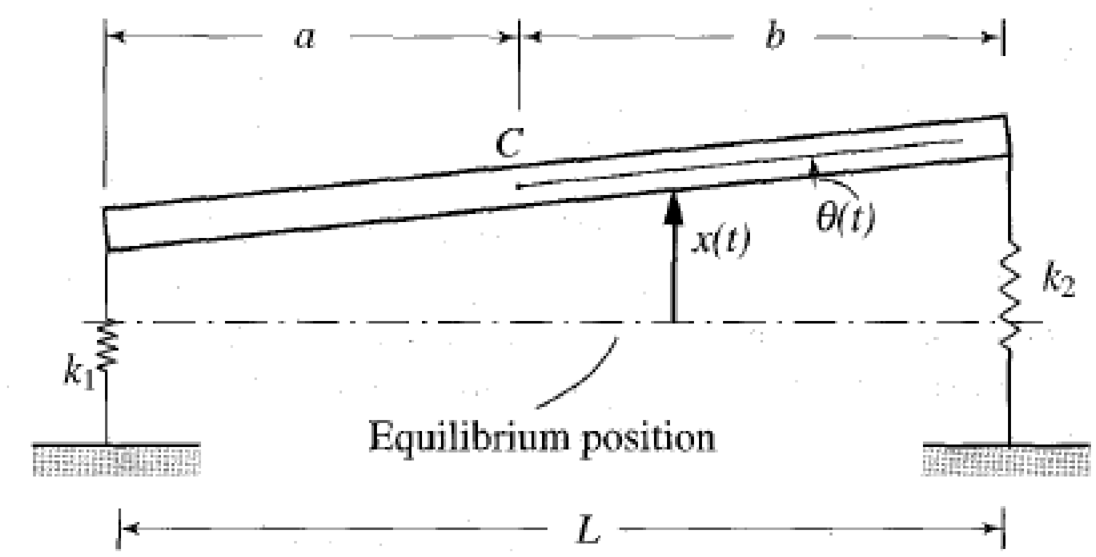
$$m_1 \frac{d^2 x_1}{dt^2} + \left(\frac{T}{L_1} + \frac{T}{L_2} \right) x_1 - \frac{T}{L_2} x_2 = F_1$$

$$m_2 \frac{d^2 x_2}{dt^2} - \frac{T}{L_2} x_1 + \left(\frac{T}{L_2} + \frac{T}{L_3} \right) x_2 = F_2$$

(with the assumption that displacements are small and being x_i the vibration about the equilibrium position)

Equations of motion of 2-DOF systems

Slab of mass *m* supported on two springs



$$m\ddot{x}+\left(k_1+k_2\right)x-\left(k_1a-k_2b\right)\theta=F$$

$$I_C\ddot{\theta}-\left(k_1a-k_2b\right)x+\left(k_1a^2-k_2b^2\right)=Fc$$

Free vibration of undamped systems

$$[M]\ddot{\vec{x}}(t)+[K]\vec{x}(t)=\cancel{\vec{F}(t)}$$

Free-vibration

$$[M]=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$[K]=\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$$

Free vibration of undamped systems

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$$

Solution:

$$\mathbf{x}(t) = f(t)\mathbf{u}$$

$f(t)$	time-dependent amplitude
$\mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$	constant vector representing the displacement pattern

Free vibration of undamped systems

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$$

$$\ddot{f}(t)M\mathbf{u} + f(t)K\mathbf{u} = \mathbf{0}$$

$$\ddot{f}(t)\mathbf{u}^T M\mathbf{u} + f(t)\mathbf{u}^T K\mathbf{u} = 0$$

escalar equation

$$\lambda = \frac{\mathbf{u}^T K\mathbf{u}}{\mathbf{u}^T M\mathbf{u}}$$

$$\ddot{f}(t) + \lambda f(t) = 0$$

(Eigenvalue problem) $$K\mathbf{u} = \lambda M\mathbf{u}$$

Free vibration of undamped systems

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$$

proportional to potential energy

$$\lambda = \frac{\mathbf{u}^T K\mathbf{u}}{\mathbf{u}^T M\mathbf{u}}$$

real and positive

$$\lambda = \omega^2$$

proportional to kinetic energy

$$\ddot{f}(t) + \omega^2 f(t) = 0$$

Free vibration of undamped systems

$$\ddot{f}(t) + \omega^2 f(t) = 0$$

harmonic solution:

$$f(t) = C\cos(\omega t - \phi)$$

Same frequency and phase angle but different amplitudes for the different degrees of freedom

- The algebraic eigenvalue problem can only be solved numerically, requiring methods of matrix algebra.
- The sole exception is for Two-degrees-of-freedom systems

Free vibration of undamped systems

$$\boldsymbol{K}\mathbf{u} = \lambda \boldsymbol{M}\mathbf{u}$$

Eigenvalue problem:

$$\left(k_{11} - \omega^2 m_1\right)u_1 + k_{12}u_2 = 0$$

$$k_{12}u_1 + \left(k_{12} - \omega^2 m_2\right)u_2 = 0$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$$

Two homogeneous algebraic equations
It has non-trivial solutions if:

$$\det\begin{bmatrix} k_{11} - \omega^2 m_1 & k_{12} \\ k_{12} & k_{12} - \omega^2 m_2 \end{bmatrix} = 0$$

Free vibration of undamped systems

$$\det\begin{bmatrix} k_{11}-\omega^2 m_1 & k_{12} \\ k_{12} & k_{12}-\omega^2 m_2 \end{bmatrix} = 0 \longrightarrow \text{characteristic equation} \Longrightarrow$$

$$\underbrace{\begin{matrix}\omega_1^2 \\ \omega_2^2\end{matrix} = \frac{1}{2}\left(\frac{k_{11}}{m_{11}}+\frac{k_{22}}{m_{22}}\right) \mp \frac{1}{2}\sqrt{\left(\frac{k_{11}}{m_{11}}+\frac{k_{22}}{m_{22}}\right)^2 - 4\,\frac{k_{11}k_{22}-k_{12}^2}{m_1 m_2}}}_{\text{eigenvalues}}$$

Free vibration of undamped systems

- Synchronous harmonic motion can take place in only two ways, one with the frequency ω_1 and the other with the frequency ω_2
- The natural frequencies ω_1 and ω_2 play a role for two-degree-of-freedom systems similar to that played by the natural frequency ω_n for single-degree-of-freedom systems

$$f_1(t) = C_1 \cos\left(\omega_1 t - \phi_1\right)$$

$$f_2(t) = C_2 \cos\left(\omega_2 t - \phi_2\right)$$

Free vibration of undamped systems

- To obtain the shape of the displacement configuration for each case:

$$\left(k_{11} - \omega^2 m_1\right)u_1 + k_{12}u_2 = 0$$

$$k_{12}u_1 + \left(k_{12} - \omega^2 m_2\right)u_2 = 0$$

- Let $\omega^2 = \omega_i^2, \quad u_1 = u_{1i}, \quad u_2 = u_{2i} \qquad (i = 1, 2)$

$$\left.\begin{aligned} \left(k_{11} - \omega_i^2 m_1\right)u_{1i} + k_{12}u_{2i} &= 0 \\ k_{12}u_{1i} + \left(k_{12} - \omega_i^2 m_2\right)u_{2i} &= 0 \end{aligned}\right\} \quad i = 1,\ 2$$

Two sets of homogeneous algebraic equations

Free vibration of undamped systems

$$\left.\begin{aligned}\left(k_{11}-\omega_i^2 m_1\right)u_{1i}+k_{12}u_{2i}&=0\\ k_{12}u_{1i}+\left(k_{12}-\omega_i^2 m_2\right)u_{2i}&=0\end{aligned}\right\}\quad i=1,\ 2$$

Two sets of homogeneous algebraic equations

It is not possible to solve for both u_{1i} and u_{2i} uniquely, but only for the ratios u_{2i}/u_{1i}, (i = 1,2)

$$\frac{u_{21}}{u_{11}}=-\frac{k_{11}-\omega_1^2 m_1}{k_{12}}=-\frac{k_{12}}{k_{22}-\omega_1^2 m_2}$$

$$\frac{u_{22}}{u_{12}}=-\frac{k_{11}-\omega_2^2 m_1}{k_{12}}=-\frac{k_{12}}{k_{22}-\omega_2^2 m_2}$$

The ratios u_{2i}/u_{1i}, (i = 1,2) determine uniquely the shape of the displacement profile assumed by the system while it oscillates with the frequency ω_1 and ω_2, respectively

Free vibration of undamped systems

$$\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} \Rightarrow$$

Eigenvectors, or modal vectors

Natural frequency + modal vector = mode of vibration

ω_1 $\mathbf{u}_1$ → first mode of vibration

ω_2 $\mathbf{u}_2$ → second mode of vibration

Free vibration of undamped systems

- The natural modes of vibrations represent a characteristic property of the system
- Because the problem is homogeneous, a modal vector multiplied by a
- constant represents the same modal vector
- We can render a modal vector unique by means of **normalization**
- One normalization scheme is to assign a given value to one of the components of the modal vector, typically to assign the value 1 to the component largest in magnitude ➔ divide all components of the vector by the value largest in magnitude
- Another normalization scheme is to assign the value 1 to the magnitude of the vector, which implies division of all the vector components by the magnitude of the vector. Vectors of unit magnitude are called unit vectors.
- Following normalization, the natural modes are referred to as **normal modes**.
- Normalization is arbitrary and does not affect the mode shape

Free vibration of undamped systems

- The complete synchronous motions are therefore (natural motions):

$$\mathbf{x}_1(t) = f_1(t)\mathbf{u}_1 = C_1\mathbf{u}_1 \cos\left(\omega_1 t - \phi_1\right)$$

$$\mathbf{x}_2(t) = f_2(t)\mathbf{u}_2 = C_2\mathbf{u}_2 \cos\left(\omega_2 t - \phi_2\right)$$

- Natural motions represent harmonic oscillations at the natural frequencies with the system configuration in the shape of the modal vectors, i.e., they represent vibration in the natural modes.
- Each of these natural motions can be excited independently of the other.
- In general, however, the free vibration of a conservative system is a superposition of the natural motions:

$$\mathbf{x}(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t) = C_1 \cos\left(\omega_1 t - \phi_1\right)\mathbf{u}_1 + C_2 \cos\left(\omega_2 t - \phi_2\right)\mathbf{u}_2$$

Response to initial excitations

$$\mathbf{x}(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t) = C_1 \cos\left(\omega_1 t - \phi_1\right)\mathbf{u}_1 + C_2 \cos\left(\omega_2 t - \phi_2\right)\mathbf{u}_2$$

Constants C_1, C_2, ϕ_1, ϕ_2 are determined by the initial conditions

$$\mathbf{x}\left(0\right) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad \dot{\mathbf{x}}\left(0\right) = \mathbf{v}\left(0\right) = \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix}$$

$$\left.\begin{aligned} x_{10} &= u_{11} C_1 \cos\phi_1 + u_{12} C_2 \cos\phi_2 \\ x_{20} &= u_{21} C_1 \cos\phi_1 + u_{22} C_2 \cos\phi_2 \\ v_{10} &= \omega_1 u_{11} C_1 \sin\phi_1 + \omega_2 u_{12} C_2 \sin\phi_2 \\ v_{20} &= \omega_1 u_{21} C_1 \sin\phi_1 + \omega_2 u_{22} C_2 \sin\phi_2 \end{aligned}\right\} \qquad U = \left[\mathbf{u}_1 \quad \mathbf{u}_2\right] = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

modal matrix U

Response to initial excitations

$$C_1 \cos\phi_1 = \frac{u_{22}x_{10} - u_{12}x_{20}}{|U|}, \quad C_2 \cos\phi_2 = \frac{u_{11}x_{20} - u_{21}x_{10}}{|U|}$$

$$C_1 \sin\phi_1 = \frac{u_{22}v_{10} - u_{12}v_{20}}{\omega_1 |U|}, \quad C_2 \sin\phi_2 = \frac{u_{11}v_{20} - u_{21}v_{10}}{\omega_2 |U|}$$

$$\mathbf{x}(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t) = C_1 \cos\left(\omega_1 t - \phi_1\right)\mathbf{u}_1 + C_2 \cos\left(\omega_2 t - \phi_2\right)\mathbf{u}_2$$

$$= \frac{1}{|U|}\left\{\left[\left(u_{22}x_{10} - u_{12}x_{20}\right)\cos\omega_1 t + \frac{u_{22}v_{10} - u_{12}v_{20}}{\omega_1}\sin\omega_1 t\right]\mathbf{u}_1 + \left[\left(u_{11}x_{20} - u_{21}x_{10}\right)\cos\omega_2 t + \frac{u_{11}v_{20} - u_{21}v_{10}}{\omega_2}\sin\omega_2 t\right]\mathbf{u}_2\right\}$$

Coordinate transformations. Coupling

- In many dynamical systems, the mass matrix is diagonal, but the stiffness matrix is not ➔ the differential equations of motion are **coupled**

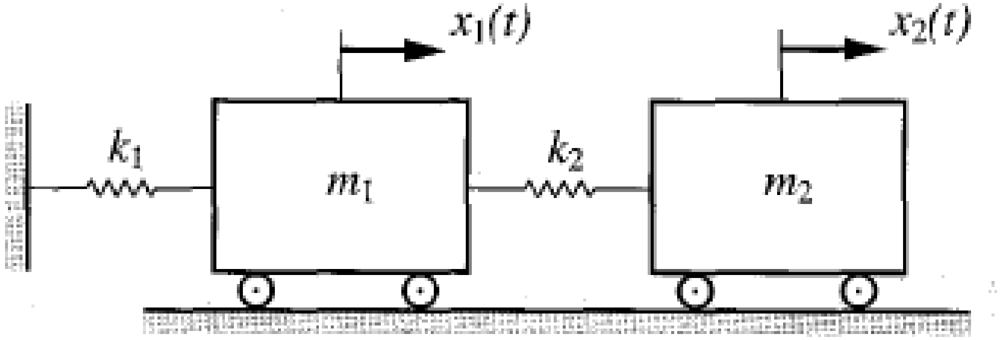
$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$$

Coordinate transformations. Coupling

- In many dynamical systems, the mass matrix is diagonal, but the stiffness matrix is not ➔ the differential equations of motion are **coupled**



$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

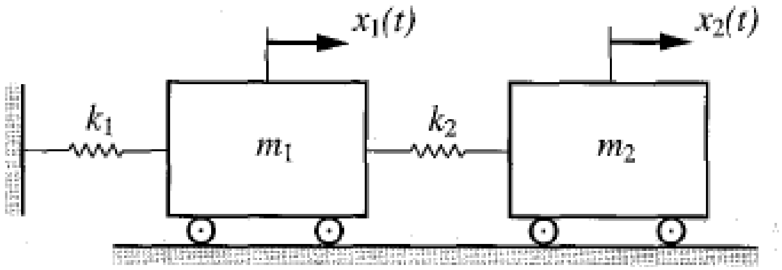
Coordinate transformations. Coupling

Next, we wish to describe the motion of the system by means of a different set of coordinates, namely, the elongations of the springs $z_1(t),\ z_1(t)$

$$x_1(t) = z_1(t),\quad x_2(t) = z_1(t) + z_2(t)$$ → **coordinate transformation**

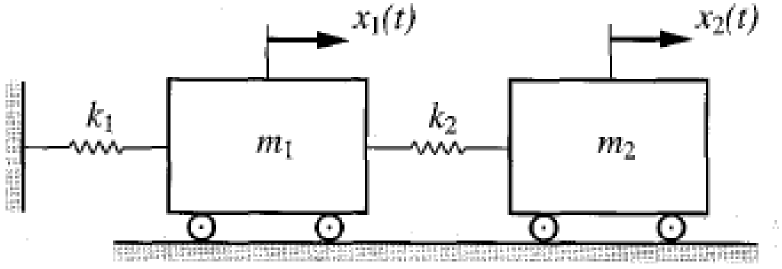
$$\mathbf{x}(t) = T\mathbf{z}(t)$$

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$ → **transformation matrix**



Coordinate transformations. Coupling

The equations of motion in the new coordinates is:



$$M'\ddot{\mathbf{z}}(t) + K'\mathbf{z}(t) = \mathbf{0}$$

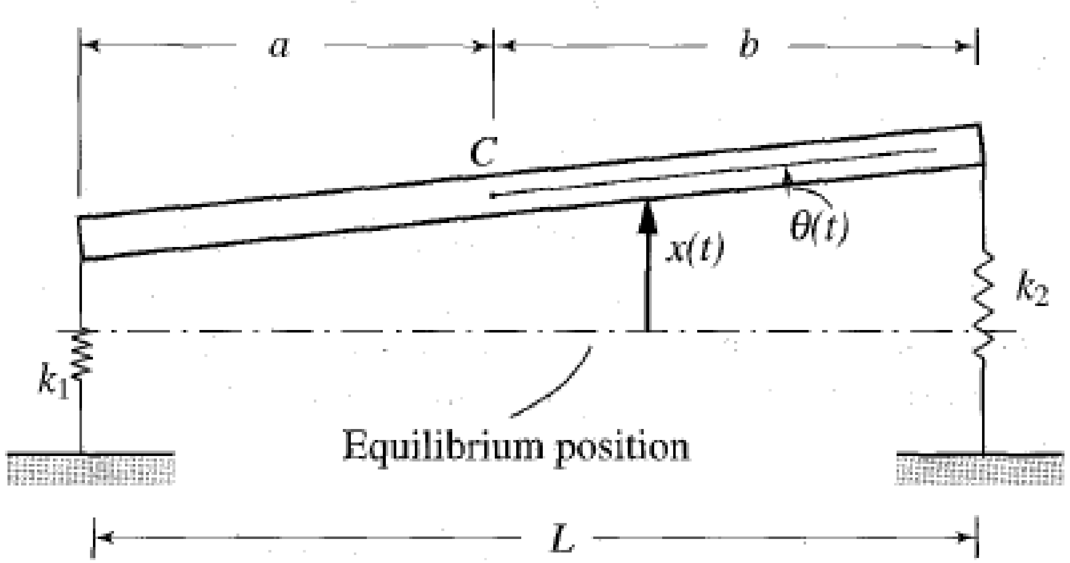
$$M' = T^T M T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}$$

$$K' = T^T K T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

➔ **the stiffness matrix is diagonal, but the mass matrix is not**

Coordinate transformations. Coupling

$$m\ddot{x}+\left(k_1+k_2\right)x-\left(k_1a-k_2b\right)\theta=F$$
$$I_C\ddot{\theta}-\left(k_1a-k_2b\right)x+\left(k_1a^2-k_2b^2\right)=Fc$$



$$M=\begin{bmatrix} m & 0 \\ 0 & I_C \end{bmatrix},\quad K=\begin{bmatrix} k_1+k_2 & -\left(k_1a-k_2b\right) \\ -\left(k_1a-k_2b\right) & k_1a^2+k_2b^2 \end{bmatrix}$$

Coordinate transformations. Coupling

- Next, we define the motion in terms of the vertical translation $x_1(t)$ of point *0* on the slab and the rotation $\theta(t)$, where *0* lies at distances a_1 and b_1 from the springs k_1 and k_2, respectively.
- Point *0* is not arbitrary but chosen so that a vertical force acting at 0 causes the slab to undergo pure translation.
- For this to happen, the moment about *0* must be zero, which implies that a_1 and b_1 must satisfy the condition:

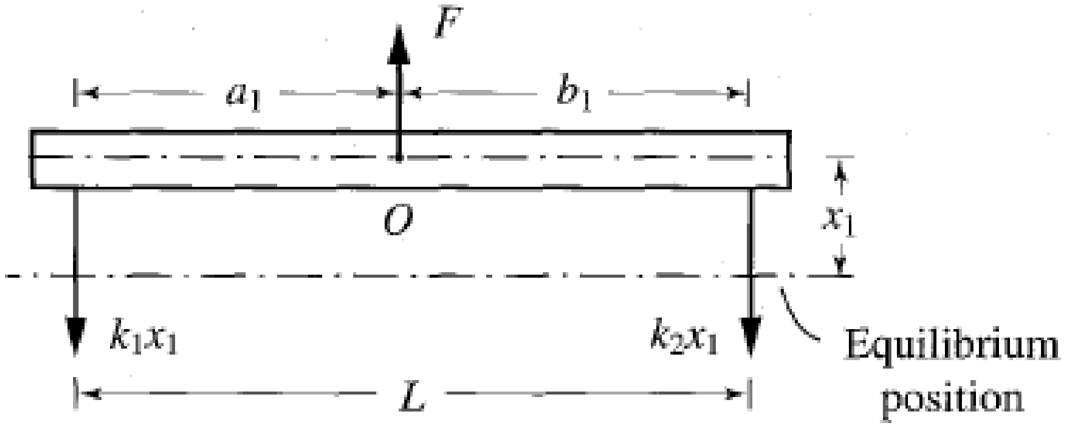
$$k_1 x_1 a_1 = k_2 x_1 b_1$$

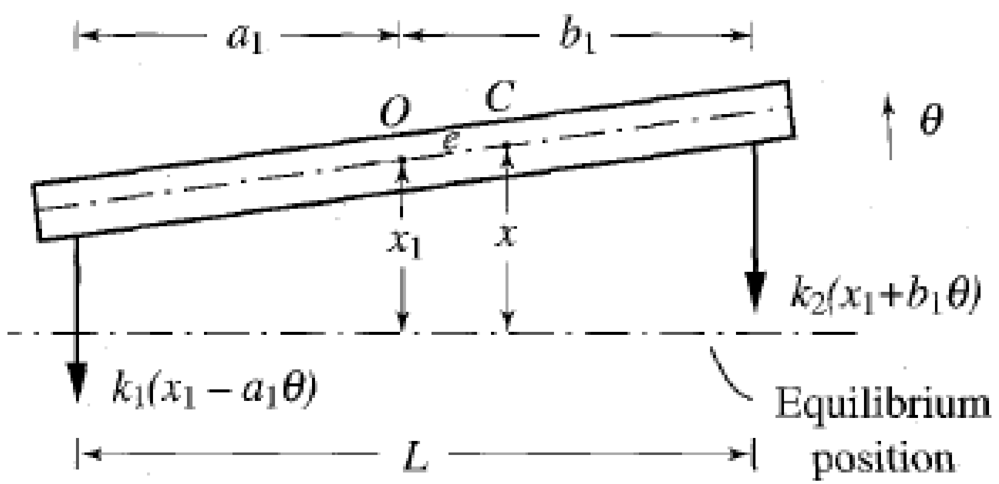
- Coordinate transformation:

$$x(t) = x_1(t) + e\theta(t)$$

$$\mathbf{x}(t) = T\mathbf{x}_1(t)$$

$$\mathbf{x} = \begin{bmatrix} x & \theta \end{bmatrix}^T, \ \mathbf{x}_1 = \begin{bmatrix} x_1 & \theta \end{bmatrix}^T \qquad T = \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix}$$





Coordinate transformations. Coupling

The equations of motion in the new coordinates is:

$$M_1\ddot{\mathbf{x}}_1(t) + K_1\mathbf{x}_1(t) = \mathbf{0}$$

$$M_1 = T^T M T = \begin{bmatrix} 1 & 0 \\ e & 1 \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & I_C \end{bmatrix}\begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m & em \\ em & I_O \end{bmatrix} \qquad I_O = I_C + me^2$$

$$K_1 = T^T K T = \begin{bmatrix} 1 & 0 \\ e & 1 \end{bmatrix}\begin{bmatrix} k_1 + k_2 & -\left(k_1 a - k_2 b\right) \\ -\left(k_1 a - k_2 b\right) & k_1 a^2 + k_2 b^2 \end{bmatrix}\begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & 0 \\ 0 & k_1 a_1^2 + k_2 b_1^2 \end{bmatrix}$$

➔ **the stiffness matrix is diagonal, but the mass matrix is not**

Coordinate transformations. Coupling

Conclusion:

- Coupling is not an inherent characteristic property of the system, but of the coordinates used to describe the motion of the system
- For a coordinate transformation to justify the effort it must facilitate the solution of the equations of motion ➔it must remove both the dynamic and the elastic coupling from the system at the same time

  ➔ The coordinate transformation must diagonalize the mass and stiffness matrices simultaneously
- Such a coordinate transformation does indeed exist and that the transformation matrix is the modal matrix
- The coordinates corresponding to the independent equations of motion are known as natural coordinates, or principal coordinates, andare unique for a given system.

Orthogonality of modes. Natural coordinates

- Modal vectors:
  - Represent configuration vectors experienced by conservative systems vibrating freely in synchronous motion
  - Help to solve the problem of forced vibrations for conservative systems
  - Help to solve the free-vibration problem for systems with more than two degrees of freedom

  ➔ <u>Modal vectors are orthogonal with respect to both the mass matrix and the stiffness matrix</u>

$$\frac{u_{21}}{u_{11}} = -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} = -\frac{k_{12}}{k_{22} - \omega_1^2 m_2}$$

$$\frac{u_{22}}{u_{12}} = -\frac{k_{11} - \omega_2^2 m_1}{k_{12}} = -\frac{k_{12}}{k_{22} - \omega_2^2 m_2}$$

$$\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}$$

Orthogonality of modes. Natural coordinates

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$$

$$\mathbf{u}_1 = u_{11} \begin{bmatrix} 1 \\ -\dfrac{k_{11} - \omega_1^2 m_1}{k_{12}} \end{bmatrix}, \quad \mathbf{u}_2 = u_{12} \begin{bmatrix} 1 \\ -\dfrac{k_{11} - \omega_2^2 m_1}{k_{12}} \end{bmatrix}$$

Orthogonality of modes. Natural coordinates

$$\mathbf{u}_1 = u_{11}\begin{bmatrix} 1 \\ -\dfrac{k_{11}-\omega_1^2 m_1}{k_{12}} \end{bmatrix}, \quad \mathbf{u}_2 = u_{12}\begin{bmatrix} 1 \\ -\dfrac{k_{11}-\omega_2^2 m_1}{k_{12}} \end{bmatrix}$$

$$\begin{matrix}\omega_1^2 \\ \omega_2^2\end{matrix} = \frac{1}{2}\left(\frac{k_{11}}{m_{11}}+\frac{k_{22}}{m_{22}}\right) \mp \frac{1}{2}\sqrt{\left(\frac{k_{11}}{m_{11}}+\frac{k_{22}}{m_{22}}\right)^2 - 4\frac{k_{11}k_{22}-k_{12}^2}{m_1 m_2}}$$

$$\mathbf{u}_2^T M \mathbf{u}_1 = u_{11}u_{12}\begin{bmatrix} 1 \\ -\dfrac{k_{11}-\omega_2^2 m_1}{k_{12}} \end{bmatrix}^T \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} 1 \\ -\dfrac{k_{11}-\omega_1^2 m_1}{k_{12}} \end{bmatrix} = \frac{u_{11}u_{12}}{k_{12}^2}\left[m_1^2 m_2 \omega_1^2\omega_2^2 - m_1 m_2 k_{11}\left(\omega_1^2+\omega_2^2\right) + m_1 k_{12}^2 + m_2 k_{11}^2\right]$$

Orthogonality of modes. Natural coordinates

$$\mathbf{u}_2^T M \mathbf{u}_1 = u_{11}u_{12} \begin{bmatrix} 1 \\ -\dfrac{k_{11} - \omega_2^2 m_1}{k_{12}} \end{bmatrix}^T \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} 1 \\ -\dfrac{k_{11} - \omega_1^2 m_1}{k_{12}} \end{bmatrix} = \frac{u_{11}u_{12}}{k_{12}^2}\left[m_1^2 m_2 \omega_1^2 \omega_2^2 - m_1 m_2 k_{11}\left(\omega_1^2 + \omega_2^2\right) + m_1 k_{12}^2 + m_2 k_{11}^2 \right]$$

$$\omega_1^2 \omega_2^2 = \frac{k_{11}k_{22} - k_{12}^2}{m_1 m_2}, \omega_1^2 + \omega_2^2 = \frac{m_1 k_{22} + m_2 k_{11}}{m_1 m_2} \quad \Rightarrow \quad \mathbf{u}_2^T M \mathbf{u}_1 = \mathbf{u}_1^T M \mathbf{u}_2 = 0$$

The modal vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal with respect to the mass matrix M

Orthogonality of modes. Natural coordinates

$$\mathbf{u}_2^T K \mathbf{u}_1 = u_{11}u_{12} \begin{bmatrix} 1 \\ -\dfrac{k_{11}-\omega_2^2 m_1}{k_{12}} \end{bmatrix}^T \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -\dfrac{k_{11}-\omega_1^2 m_1}{k_{12}} \end{bmatrix} = \frac{u_{11}u_{12}}{k_{12}^2}\left[m_1^2 k_{22}\omega_1^2\omega_2^2 - m_1\left(k_{11}k_{22}-k_{12}^2\right)\left(\omega_1^2+\omega_2^2\right) + k_{11}\left(k_{11}k_{22}-k_{12}^2\right)\right]$$

$$\omega_1^2\omega_2^2 = \frac{k_{11}k_{22}-k_{12}^2}{m_1 m_2}, \omega_1^2+\omega_2^2 = \frac{m_1 k_{22}+m_2 k_{11}}{m_1 m_2} \quad \Rightarrow \quad \mathbf{u}_2^T K \mathbf{u}_1 = \mathbf{u}_1^T K \mathbf{u}_2 = 0$$

The modal vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal with respect to the stiffness matrix K

Orthogonality of modes. Natural coordinates

$$\mathbf{K}\mathbf{u} = \lambda \mathbf{M}\mathbf{u} \qquad \lambda = \omega^2$$

$$\mathbf{K}\mathbf{u}_1 = \omega_1^2 \mathbf{M}\mathbf{u}_1 \quad \Rightarrow \quad \mathbf{u}_1^T \mathbf{K}\mathbf{u}_1 = \omega_1^2 \mathbf{u}_1^T \mathbf{M}\mathbf{u}_1$$

$$\mathbf{K}\mathbf{u}_2 = \omega_2^2 \mathbf{M}\mathbf{u}_2 \qquad \mathbf{u}_2^T \mathbf{K}\mathbf{u}_2 = \omega_2^2 \mathbf{u}_2^T \mathbf{M}\mathbf{u}_2$$

$$\left.\begin{aligned} \mathbf{u}_1^T \mathbf{M}\mathbf{u}_1 &= m'_{11}, \quad \mathbf{u}_2^T \mathbf{M}\mathbf{u}_2 = m'_{22} \\ \mathbf{u}_1^T \mathbf{K}\mathbf{u}_1 &= k'_{11}, \quad \mathbf{u}_2^T \mathbf{K}\mathbf{u}_2 = k'_{22} \end{aligned}\right\} \quad \omega_1^2 = \frac{k'_{11}}{m'_{11}}, \quad \omega_2^2 = \frac{k'_{22}}{m'_{22}}$$

Orthogonality of modes. Natural coordinates

$$\left.\begin{aligned} &M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0} \\ &\mathbf{x}(t) = q_1(t)\mathbf{u}_1 + q_2(t)\mathbf{u}_2 \end{aligned}\right\} \Rightarrow$$

$$M\left[\ddot{q}_1(t)\mathbf{u}_1 + \ddot{q}_2(t)\mathbf{u}_2\right] + K\left[q_1(t)\mathbf{u}_1 + q_2(t)\mathbf{u}_2\right] = 0$$

$$\Rightarrow \left.\begin{aligned} &\ddot{q}_1(t) + \omega_1^2 q_1(t) = 0 \\ &\ddot{q}_2(t) + \omega_2^2 q_2(t) = 0 \end{aligned}\right\}$$

modal equations (independent)

Orthogonality of modes. Natural coordinates

$$\ddot{q}_1(t) + \omega_1^2 q_1(t) = 0$$

$$\ddot{q}_2(t) + \omega_2^2 q_2(t) = 0$$

independent equations

$$\Rightarrow \quad q_1(t) = C_1 \cos\left(\omega_1 t - \phi_1\right)$$

$$q_2(t) = C_2 \cos\left(\omega_2 t - \phi_2\right)$$

by analogy with the harmonic oscillator

$$\mathbf{x}(t) = C_1 \cos\left(\omega_1 t - \phi_1\right)\mathbf{u}_1 + C_2 \cos\left(\omega_2 t - \phi_2\right)\mathbf{u}_2$$

The free response of conservative systems is a superposition of the natural modes multiplied by the natural coordinates

Orthogonality of modes. Natural coordinates

- The free response of conservative systems is a superposition of the natural modes multiplied by the natural coordinates
- The same modal approach can be used to solve for the response of undamped systems to applied forces ➔ modal analysis
- Coordinate transformation capable of decoupling the equations of motion both inertially and elastically (diagonalize the mass matrix M and the stiffness matrix K simultaneously):

$$T = U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$$

- The real power of modal analysis becomes evident in the case of multi-degree-of-freedom systems