System Dynamics and Vibrations

Prof. Gustavo Alonso

Chapter 1: Elements of analytical dynamics
Part 1

School of General Engineering
Beihang University (BUAA)

Introduction

- ➢ Newtonian mechanics (vectorial mechanics)
  - Equations of motion for **individual particles** expressed in terms of **physical coordinates and forces** (represented both by vectors)
  - Free-body diagram for each of the masses in the system
  - Involve **reaction forces and interacting forces** (kinematical constraints) ➔ many equations and unknowns

Introduction

- ➢ Analytical mechanics (Lagrange, variational approach)
  - Considers the **systems as a whole** ➔ reaction and constraint forces are excluded
  - Dynamics problems formulated in terms of: **kinetic energy, potential energy, virtual work of non-conservative forces**
  - Equations of motion formulated in terms of generalized coordinates and generalized forces ➔ broader and more abstract approach
  - The mathematical formulation is independent of any special system of coordinates

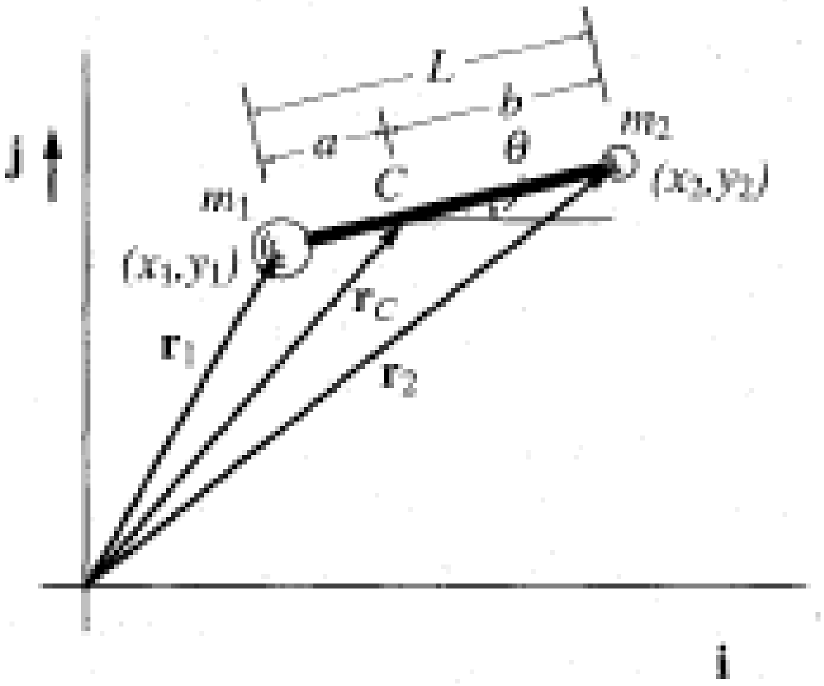
Contents

- Introduction
- Degrees of freedom and generalized coordinates
- The principle of virtual work
- The principle of D’Alembert
- Lagrange’s equations

Degrees of freedom and generalized coordinates

- Equations of motion for a system by the Newtonian approach:
  - Isolate the masses
  - Draw one free-body diagram for each of the masses, including all forces acting upon it:
    - Applied forces
    - Reaction forces
    - Internal forces
  - We obtain normally more equations and unknowns than necessary
  - The motion is described in terms of physical coordinates: may not always be independent ➔ equations of motion + constraint equations

Degrees of freedom and generalized coordinates



- Movement in *x*, *y* – plane
- Position vectors:

$$\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j},\ \ \mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j},$$

- x_1, x_2, y_1, y_2 are not independent

$$\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 = L^2$$

➔ constraint equation (+ the equations of motion)

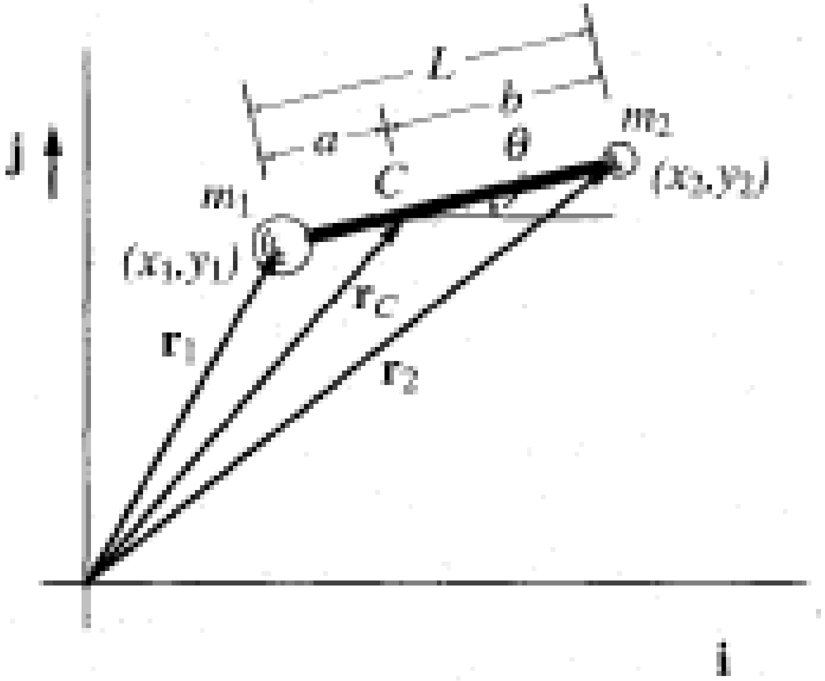
Degrees of freedom and generalized coordinates

- A better choice of coordinates obviates the difficulties of working with too may coordinates and constraint equations
- For a system of *N* mass particles with positions defined by the radius vectors $\mathbf{r}_i$ (x_i, y_i, z_i) in a three-dimensional space:

$$n = 3N - c$$

  - c number of constraints
  - n number of **degrees of freedom of the system**

Degrees of freedom and generalized coordinates



- Much simpler in generalized coordinates:

$$\mathbf{r}_C = \mathbf{r}_C(x_C, y_C)$$
$$\theta$$

- Three independent coordinates:

$$x_C, y_C, \theta$$

- No constraint equation is needed

$$a = \frac{m_2 L}{m_1 + m_2},$$

$$\mathbf{r}_1 = \mathbf{r}_C - a(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}),\ \mathbf{r}_2 = \mathbf{r}_C + b(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}),$$

$$b = \frac{m_1 L}{m_1 + m_2},$$

Degrees of freedom and generalized coordinates

- Independent coordinates are normally called generalized coordinates: q_1, q_2, … , q_n
- Coordinate transformation:

$$x_1 = x_1(q_1, q_2, ..., q_n)$$

$$y_1 = y_1(q_1, q_2, ..., q_n)$$

$$z_1 = z_1(q_1, q_2, ..., q_n)$$

$$x_2 = x_2(q_1, q_2, ..., q_n)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$z_N = z_N(q_1, q_2, ..., q_n)$$

Degrees of freedom and generalized coordinates

- Generalized coordinates $q_1, q_2, \ldots, q_n$ are not unique for a given system
- Normally only one or two sets may represent a suitable choice ➔ the equations of motion in terms of these coordinates have the simplest form
- In vibrations, this choice is obvious for most of the cases

Contents

The principle of virtual work

- Basically a statement of the **static equilibrium** of a mechanical system
- The first variational principle of mechanics
- Tool to transition from Newtonian mechanics to Lagrangian mechanics
- New concepts:
    - Virtual displacements
    - Constraint forces

The principle of virtual work

- System of *N* particles in a three-dimensional space
- We define <u>**virtual displacements**</u> as

infinitesimal changes in the coordinates $x_1, y_1, z_1, x_2, \ldots, z_n$

$$\delta x_1, \delta y_1, \delta z_1, \delta x_2, \ldots, \delta z_N$$

- Virtual displacements must be:
  - <u>Consistent with the system constraints, but</u>
  - <u>Arbitrary</u>

The principle of virtual work

- Virtual displacements represent small variations in the coordinates resulting from imagining the system in a slightly displaced position:
  - Virtual displacements take place instantaneously ➔ $\delta t = 0$
- Symbol δ emphasizes the virtual character of the instantaneous variations
- Symbol *d* represents actual differentials of position coordinates taking place in the time interval *dt* (forces can change in that time interval)
  - Virtual displacements, being infinitesimal, obey the rules of differential calculus

The principle of virtual work

- We assume that every one of the N particles in the system is acted upon by the resultant force

$$\mathbf{R}_i = \mathbf{F}_i + \mathbf{f}_i\,,\ i = 1, 2, \ldots, N$$

$\mathbf{F}_i$ is an applied force (gravitational force, aerodynamic lift and drag, magnetic forces, etc.)

$\mathbf{f}_i$ is a constraint force (for example the force that keeps a particle confined to a given surface)

The principle of virtual work

- For a system in equilibrium every particle must be at rest
- Then:

$$\mathbf{R}_i = \mathbf{F}_i + \mathbf{f}_i = \mathbf{0}, \qquad i = 1, 2, \ldots N$$

$$\overline{\delta W_i} = \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0, \quad i = 1, 2, \ldots N$$

$\overline{\delta W_i}$ represents the **<u>virtual work</u>** perfomed by the resultant force vector $\mathbf{R}_i$ over the virtual displacement vector $\delta \mathbf{r}_i$ of particle *i*

The principle of virtual work

- Summing up over *i*:

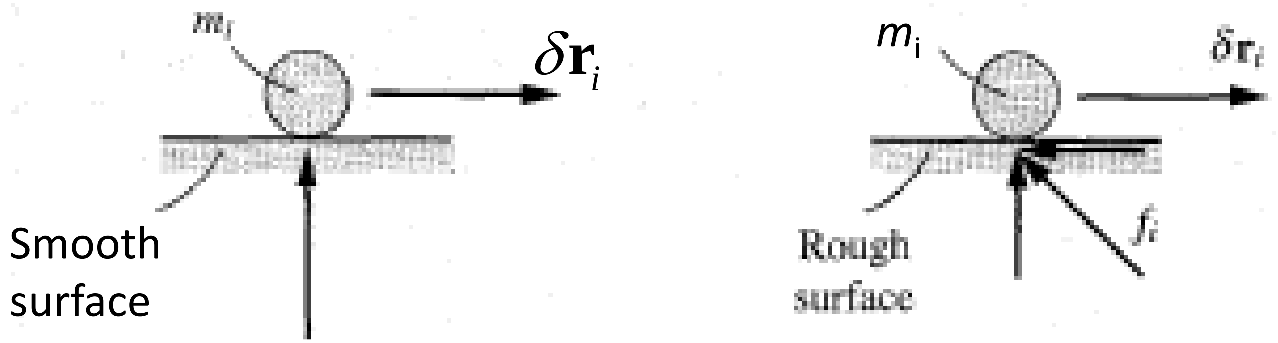
$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0$$

➔ the virtual work for the entire system must vanish

$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i + \sum_{i=1}^{N} \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

The principle of virtual work

- We limit ourselves to systems for which the virtual work performed by the constraint forces is zero:



The principle of virtual work

- We limit ourselves to systems for which the virtual work performed by the constraint forces is zero:
- Particle or body on a smooth surface (no friction)
- Other examples:
  - Articulated bodies
  - Bodies forced to be in contact
  - Bodies rolling and pivoting (without sliding) one on each other

➔ solid mechanics

The principle of virtual work

- We limit ourselves to systems for which the virtual work performed by the constraint forces is zero:

$$\sum_{i=1}^{N} \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

- Therefore:

$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$$

➔ **<u>Principle of virtual work</u>**: the work performed by the applied forces through infinitesimal virtual displacements compatible with the system constraints is zero

The principle of virtual work

- When the **virtual displacements are all independent**:
- We can invoke the arbitrariness of the virtual displacements

➔ equation $\overline{\delta W} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$

can be satisfied for all possible values of $\delta \mathbf{r}_i$

only if $\mathbf{F}_i = \mathbf{0}, \quad i = 1, 2, \ldots N$ ➔ **Equilibrium conditions**

The principle of virtual work

- If the coordinates $\mathbf{r}_i$ (i = 1, 2, ..., N) are not **<u>independent</u>** (but related by constraint equations):

➔It is more convenient to switch to generalized coordinates q_1, q_2, ..., q_n

We can express $\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, ..., q_n) \quad i = 1, 2, ...N$

➔ generalized coordinates are independent by definition

The principle of virtual work

- Virtual displacements are:

$$\delta \mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial q_1}\delta q_1 + \frac{\partial \mathbf{r}_i}{\partial q_2}\delta q_2 + \ldots + \frac{\partial \mathbf{r}_i}{\partial q_n}\delta q_n = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k, \quad i = 1, 2, \ldots N$$

$$\delta q_k \qquad (k = 1, 2, \ldots, n)$$

are virtual generalized displacements, all independent

The principle of virtual work

$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} \mathbf{F}_i \cdot \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k = \sum_{k=1}^{n} \left(\sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) \delta q_k = \sum_{k=1}^{n} Q_k \delta q_k = 0$$

where $$Q_k = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}, \qquad (k = 1, 2, \ldots, n)$$

are known as **generalized forces**

The principle of virtual work

- Being $\delta q_k \quad (k = 1, 2, \ldots, n)$

all independent, and therefore entirely arbitrary

- Letting $\delta q_1 = 1, \quad \delta q_2 = \delta q_3 = \ldots = \delta q_n = 0$

$$\overline{\delta W} = \sum_{k=1}^{n} Q_k \delta q_k = 0 \Rightarrow Q_1 = 0$$

- Repeating the argument with $k = 2, 3, \ldots, n$ in sequence

➔ we obtain the equilibrium conditions:

$$Q_k = 0, \quad (k = 1, 2, \ldots, n)$$