System Dynamics and Vibrations

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Chapter 3: Single degree-of-freedom systems
Part 2

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Contents

Introduction: types of excitations

- Initial excitations: Initial displacements, initial velocities, both

  ➔ Free vibration (free response): no further external factors affecting the system ➔ homogeneous equation

- Applied forces / moments ➔ forced vibration / response

  ➔The response depends on the type of applied (external) forces / moments

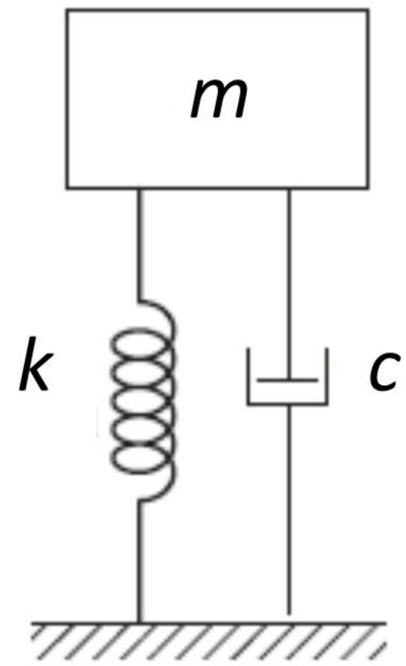
Response of SDOF systems to harmonic excitations

- Steady-state excitations:
  - Harmonic
  - Periodic
- Steady-state excitations occur frequently in various areas of engineering
- The response is analysed in the frequency domain (rather than in time domain)

Response of SDOF systems to harmonic excitations

$$m\ddot{x} + c\dot{x} + kx = F$$

$$\left.\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= v_0 \end{aligned}\right\} \quad \text{initial conditions}$$



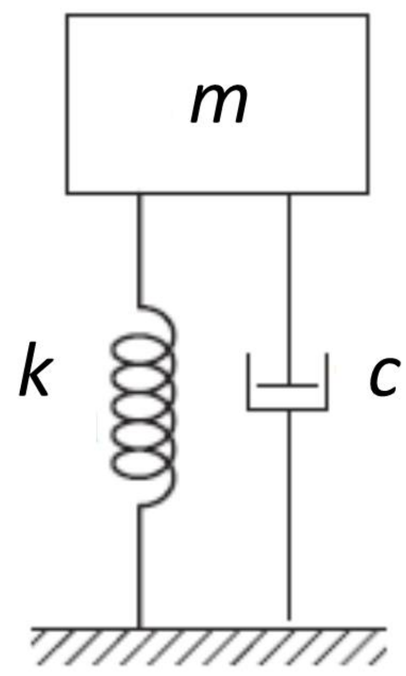
being *x(t)* the response and *F(t)* the excitation

Response of SDOF systems to harmonic excitations

$$m\ddot{x} + c\dot{x} + kx = F$$

$$x(0) = x_0$$

$$\dot{x}(0) = v_0$$



➔ <u>Principle of superposition</u>:

- Response to initial excitations (already solved)
- **Response to the applied force *F(t)***

(obtained separately and combined linearly)

Response of SDOF systems to harmonic excitations

- Harmonic force

$$F(t) = kf(t) = kA\cos\omega t$$

$$f(t) = A\cos\omega t$$

ω is the excitation frequency, or driving freqeuncy

Response of SDOF systems to harmonic excitations

$$m\ddot{x} + c\dot{x} + kx = F = kA\cos\omega t$$

$$\ddot{x}(t) + 2\varsigma\omega_n\dot{x}(t) + \omega_n^2 x(t) = \omega_n^2 A\cos\omega t$$

$$\omega_n = \sqrt{k/m}$$ natural frequency of undamped oscillations

$$\varsigma = \frac{c}{2m\omega_n}$$ viscous damping factor

Response of SDOF systems to harmonic excitations

- The response to harmonic excitation:
  - is also harmonic
  - has the same frequency as the excitation frequency

$$x(t) = C_1 \sin \omega t + C_2 \cos \omega t$$

C_1 and C_2 are constants yet to be determined

Response of SDOF systems to harmonic excitations

$$\left.\begin{aligned}&\ddot{x}(t)+2\varsigma\omega_n\dot{x}(t)+\omega_n^2x(t)=\omega_n^2A\cos\omega t\\&x(t)=C_1\sin\omega t+C_2\cos\omega t\end{aligned}\right\}\Rightarrow$$

$$\left[\left(\omega_n^2-\omega^2\right)C_1-2\varsigma\omega\omega_nC_2\right]\sin\omega t+\left[2\varsigma\omega\omega_nC_1+\left(\omega_n^2-\omega^2\right)C_2\right]\cos\omega t$$

$$=\omega_n^2A\cos\omega t$$

➔ coefficients of $\sin\omega t$ and $\cos\omega t$ must be equal

Response of SDOF systems to harmonic excitations

$$\left.\begin{aligned}\left(\omega_n^2-\omega^2\right)C_1-2\varsigma\omega\omega_n C_2&=0\\2\varsigma\omega\omega_n C_1+\left(\omega_n^2-\omega^2\right)C_2&=\omega_n^2 A\end{aligned}\right\}$$

$$\Rightarrow$$

$$C_1=\frac{2\varsigma\,{}^{\omega}\!/_{\omega_n}}{\left[1-\left({}^{\omega}\!/_{\omega_n}\right)^2\right]^2+\left(2\varsigma\,{}^{\omega}\!/_{\omega_n}\right)^2}A$$

$$C_2=\frac{1-\left({}^{\omega}\!/_{\omega_n}\right)^2}{\left[1-\left({}^{\omega}\!/_{\omega_n}\right)^2\right]^2+\left(2\varsigma\,{}^{\omega}\!/_{\omega_n}\right)^2}A$$

Response of SDOF systems to harmonic excitations

- Steady-state solution:

$$x(t)=\frac{A}{\left[1-\left(\omega/\omega_n\right)^2\right]^2+\left(2\varsigma\,\omega/\omega_n\right)^2}\left\{\frac{2\varsigma\omega}{\omega_n}\sin\omega t+\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]\cos\omega t\right\}$$

Response of SDOF systems to harmonic excitations

- Steady-state solution:

$$x(t)=\frac{A}{\left[1-\left(\omega/\omega_n\right)^2\right]^2+\left(2\varsigma\,\omega/\omega_n\right)^2}\left\{\frac{2\varsigma\omega}{\omega_n}\sin\omega t+\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]\cos\omega t\right\}$$

- To find the physical interpretation we change the notation:

Response of SDOF systems to harmonic excitations

$$\sin\phi = \frac{2\varsigma\, \omega/\omega_n}{\left\{\left[1-\left(\omega/\omega_n\right)^2\right]^2 + \left(2\varsigma\, \omega/\omega_n\right)^2\right\}^{1/2}}$$

$$\cos\phi = \frac{1-\left(\omega/\omega_n\right)^2}{\left\{\left[1-\left(\omega/\omega_n\right)^2\right]^2 + \left(2\varsigma\, \omega/\omega_n\right)^2\right\}^{1/2}}$$

$$x(t) = X\cos\left(\omega t - \phi\right)$$

$$X = X(\omega) = \frac{A}{\left\{\left[1-\left(\omega/\omega_n\right)^2\right]^2 + \left(2\varsigma\, \omega/\omega_n\right)^2\right\}^{1/2}}$$

$$\phi = \phi(\omega) = \tan^{-1}\frac{2\varsigma\, \omega/\omega_n}{1-\left(\omega/\omega_n\right)^2}$$

Response of SDOF systems to harmonic excitations

$$x(t) = X\cos\left(\omega t - \phi\right)$$

$$X = X\left(\omega\right) = \frac{A}{\left\{\left[1-\left(\omega/\omega_n\right)^2\right]^2 + \left(2\varsigma\,\omega/\omega_n\right)^2\right\}^{1/2}}$$

amplitude

$$\phi = \phi\left(\omega\right) = \tan^{-1}\frac{2\varsigma\,\omega/\omega_n}{1-\left(\omega/\omega_n\right)^2}$$

phase angle

Contents

- Introduction
- Undamped SDOF systems. Harmonic oscillator.
- Viscously damped SDOF.
- Coulomb damping. Dry friction.
- Response of SDOF systems to harmonic excitations.
- Frequency response plots.
- Systems with rotating unbalanced masses.
- Response to periodic excitation. Fourier series.
- Response to non-periodic excitation: step, ramp and impulse.

Frequency response plots

$$F(t) = kf(t) = kA\cos\omega t$$

$$f(t) = Ae^{i\omega t}$$

$\Rightarrow$ if $\begin{cases} f(t) = A\cos\omega t & \Rightarrow \quad \mathrm{Re}\, x(t) \\ f(t) = A\sin\omega t & \Rightarrow \quad \mathrm{Im}\, x(t) \end{cases}$

$$\ddot{x}(t) + 2\varsigma\omega_n \dot{x}(t) + \omega_n^2 x(t) = \omega_n^2 A e^{i\omega t}$$

$$x(t) = X(i\omega)e^{i\omega t}$$

$$\Rightarrow \quad X(i\omega) = \frac{A}{1-\left(\omega/\omega_n^2\right)^2 + i2\varsigma\,\omega/\omega_n}$$

Frequency response plots

$$G(i\omega) = \frac{X(i\omega)}{A} = \frac{1}{1-\left(\omega/\omega_n^2\right)^2 + i2\varsigma\,\omega/\omega_n}$$

$\Rightarrow$ frequency response

$$x(t) = AG(i\omega)e^{i\omega t}$$

$$\left.\begin{aligned}
G(i\omega) &= \left|G(i\omega)\right| e^{-i\phi(\omega)} \\
\left|G(i\omega)\right| &= \left\{\left[\operatorname{Re} G(i\omega)\right]^2 + \left[\operatorname{Im} G(i\omega)\right]^2\right\}^{1/2} \\
\phi(\omega) &= \tan^{-1}\left[\frac{-\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}\right]
\end{aligned}\right\} \quad x(t) = A\left|G(i\omega)\right| e^{i(\omega t-\phi)}$$

Frequency response plots

$$x(t) = A\left|G(i\omega)\right| e^{i(\omega t - \phi)}$$

if

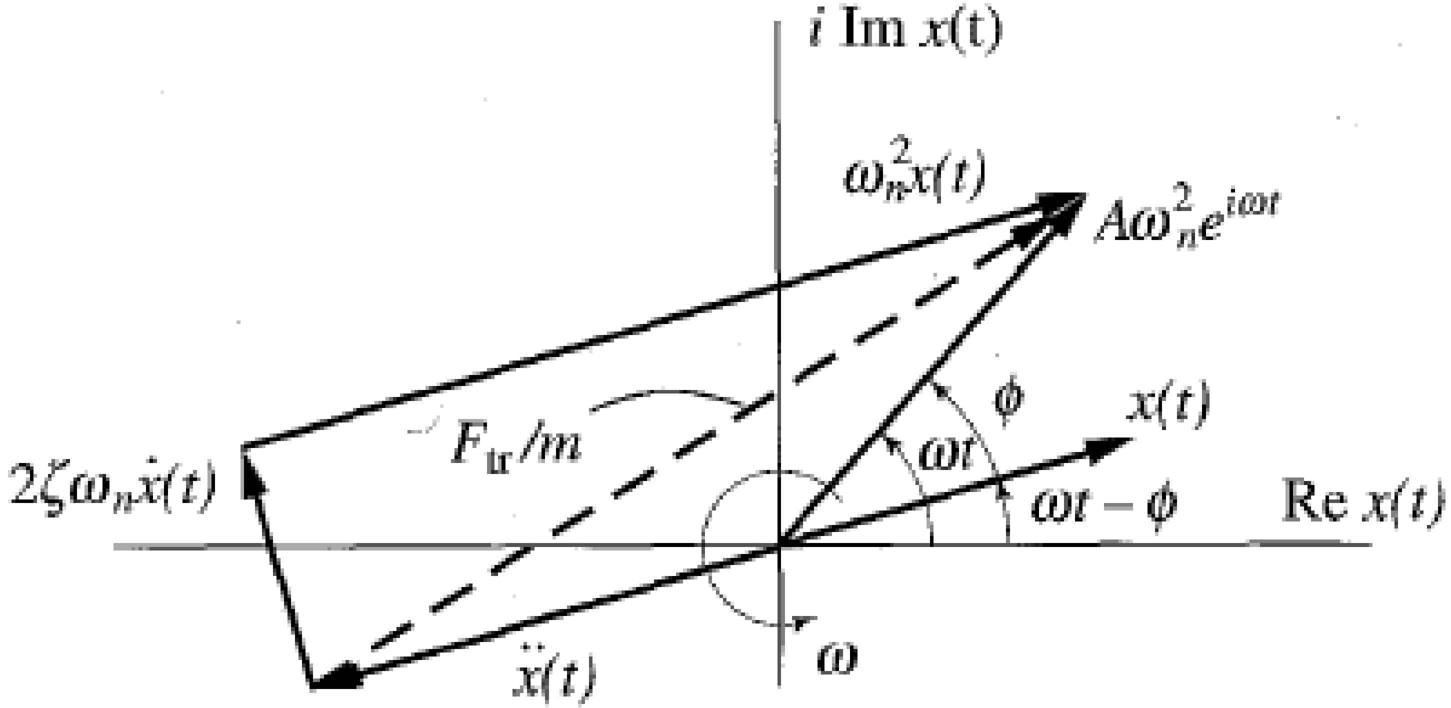
$$\begin{cases} f(t) = A\cos\omega t \;\Rightarrow\; x(t) = \mathrm{Re}\, A\left|G(i\omega)\right| e^{i(\omega t-\phi)} = A\left|G(i\omega)\right| \cos(\omega t - \phi) \\[2ex] f(t) = A\sin\omega t \;\Rightarrow\; x(t) = \mathrm{Im}\, A\left|G(i\omega)\right| e^{i(\omega t-\phi)} = A\left|G(i\omega)\right| \sin(\omega t - \phi) \end{cases}$$

Frequency response plots

$$x(t) = A\left|G(i\omega)\right|e^{i(\omega t-\phi)}$$

$$\dot{x}(t) = i\omega x(t)$$

$$\ddot{x}(t) = -\omega^2 x(t)$$



$$\ddot{x}(t) + 2\varsigma\omega_n \dot{x}(t) + \omega_n^2 x(t) = \omega_n^2 A e^{i\omega t}$$

Frequency response plots

$$\left|G(i\omega)\right| = \left[G(i\omega)\bar{G}(i\omega)\right]^{1/2} = \frac{1}{\left\{\left[1-\left(\omega/\omega_n\right)^2\right]^2 + \left(2\varsigma\left(\omega/\omega_n\right)^2\right)\right\}^{1/2}}$$

$$\phi(\omega) = \tan^{-1}\left[\frac{-\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}\right] = \tan^{-1}\frac{2\varsigma\,\omega/\omega_n}{1-\left(\omega/\omega_n\right)^2}$$

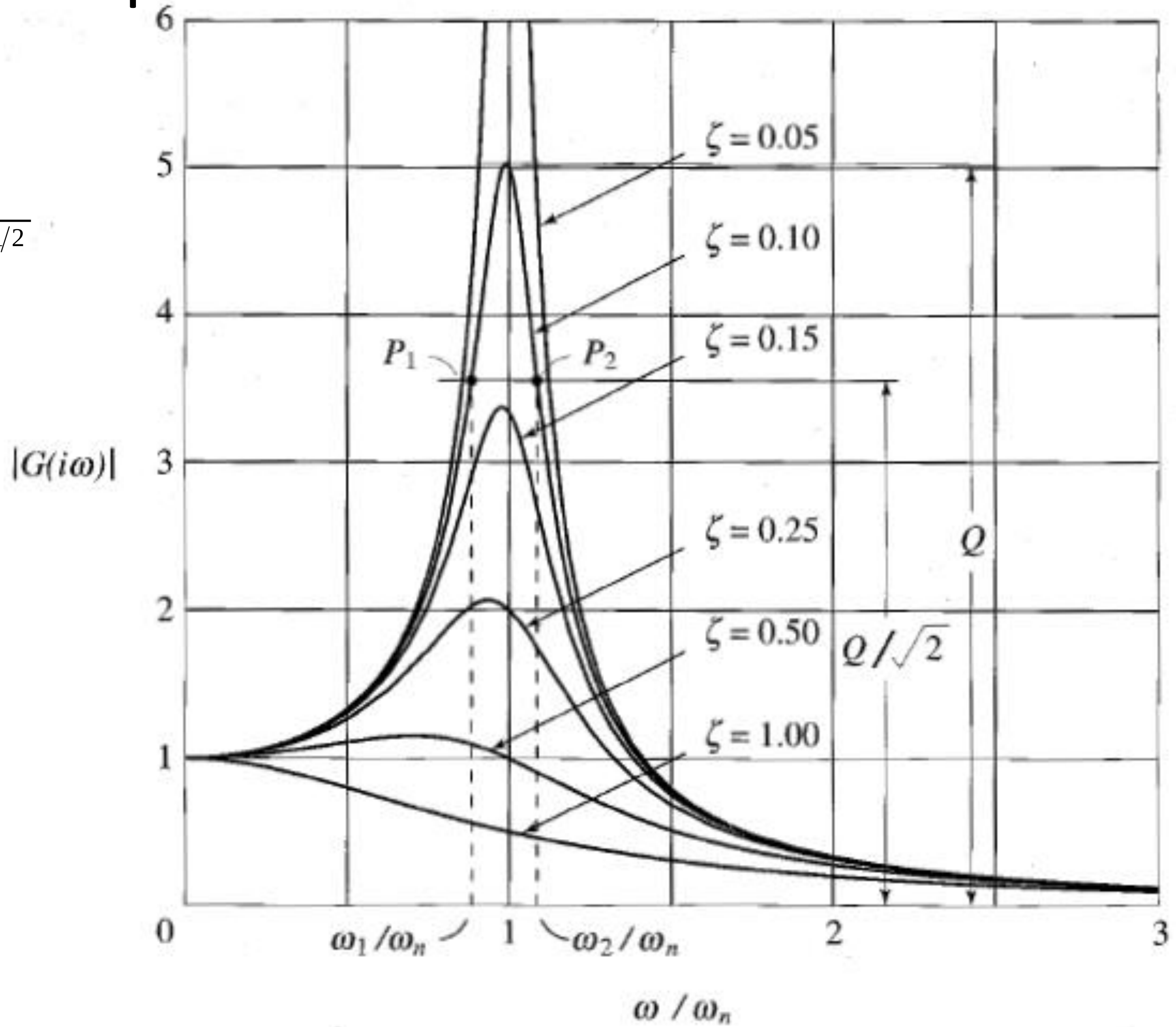
Frequency response plots

$$|G(i\omega)| = \frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + \left(2\varsigma(\omega/\omega_n)^2\right)\right\}^{1/2}}$$

response peaks:

$$\frac{d\,|G(i\omega)|}{d(\omega/\omega_n)} = 0 \;\Rightarrow\; \frac{\omega}{\omega_n} = \sqrt{1-2\varsigma^2}$$

$\varsigma > 1/\sqrt{2}$ ⇒ no peaks



Frequency response plots

response peaks:

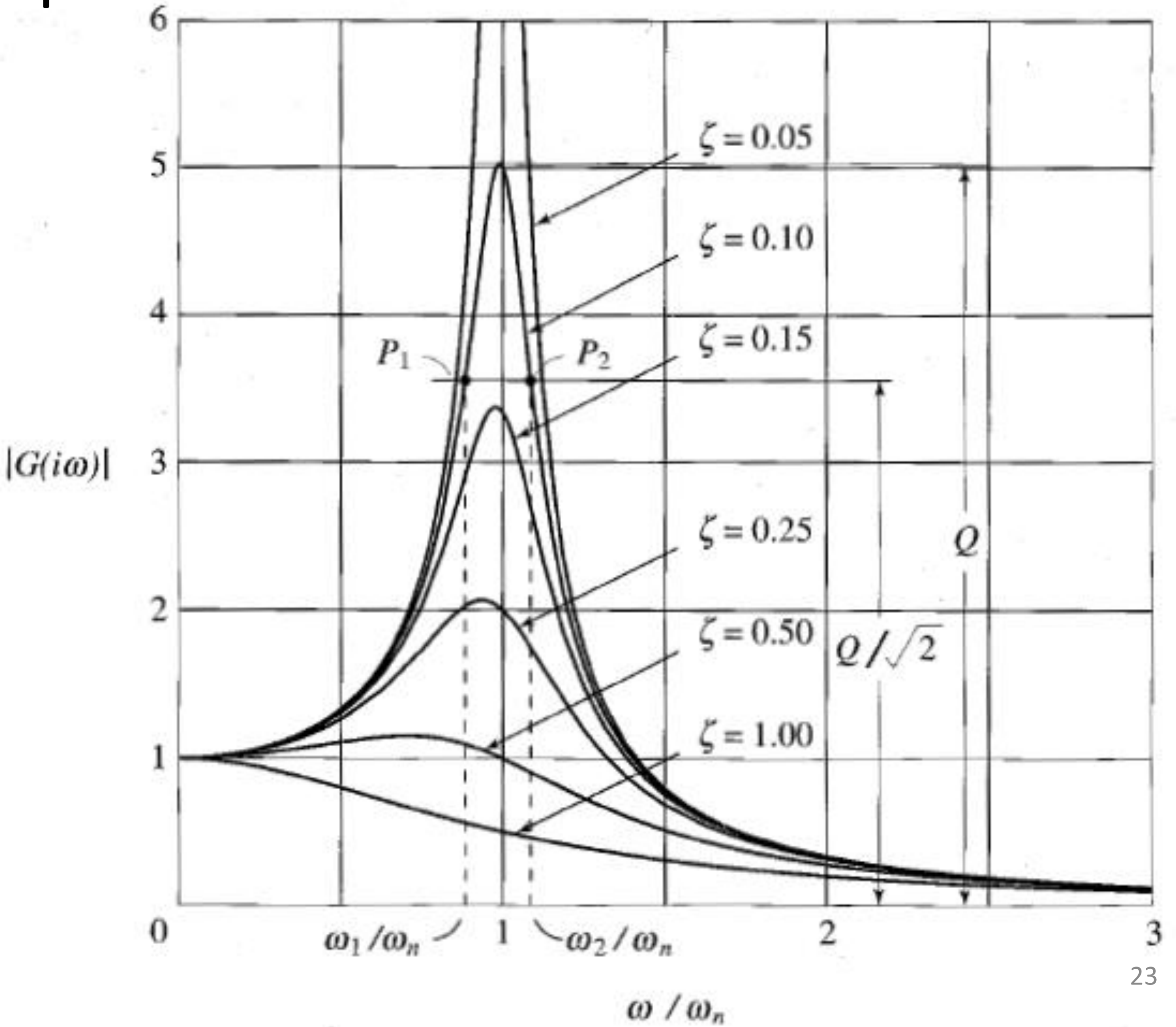
$$\left|G(i\omega)\right|_{\max} = \frac{1}{2\varsigma\sqrt{1-\varsigma^2}}$$

light damping:

$$\varsigma \ll 1 \Rightarrow \left|G(i\omega)\right|_{\max} = Q \cong \frac{1}{2\varsigma}$$

experimental way to estimate viscous damping:

$$\varsigma \cong \frac{1}{2Q}$$



Frequency response plots

P_1, P_2: half-power points
(the power absorbed by the damper is proportional to the square of the amplitude)

$$\left|G(i\omega)\right|_{\max} = \frac{1}{\sqrt{2}Q} = \frac{1}{2\sqrt{2}\varsigma}$$

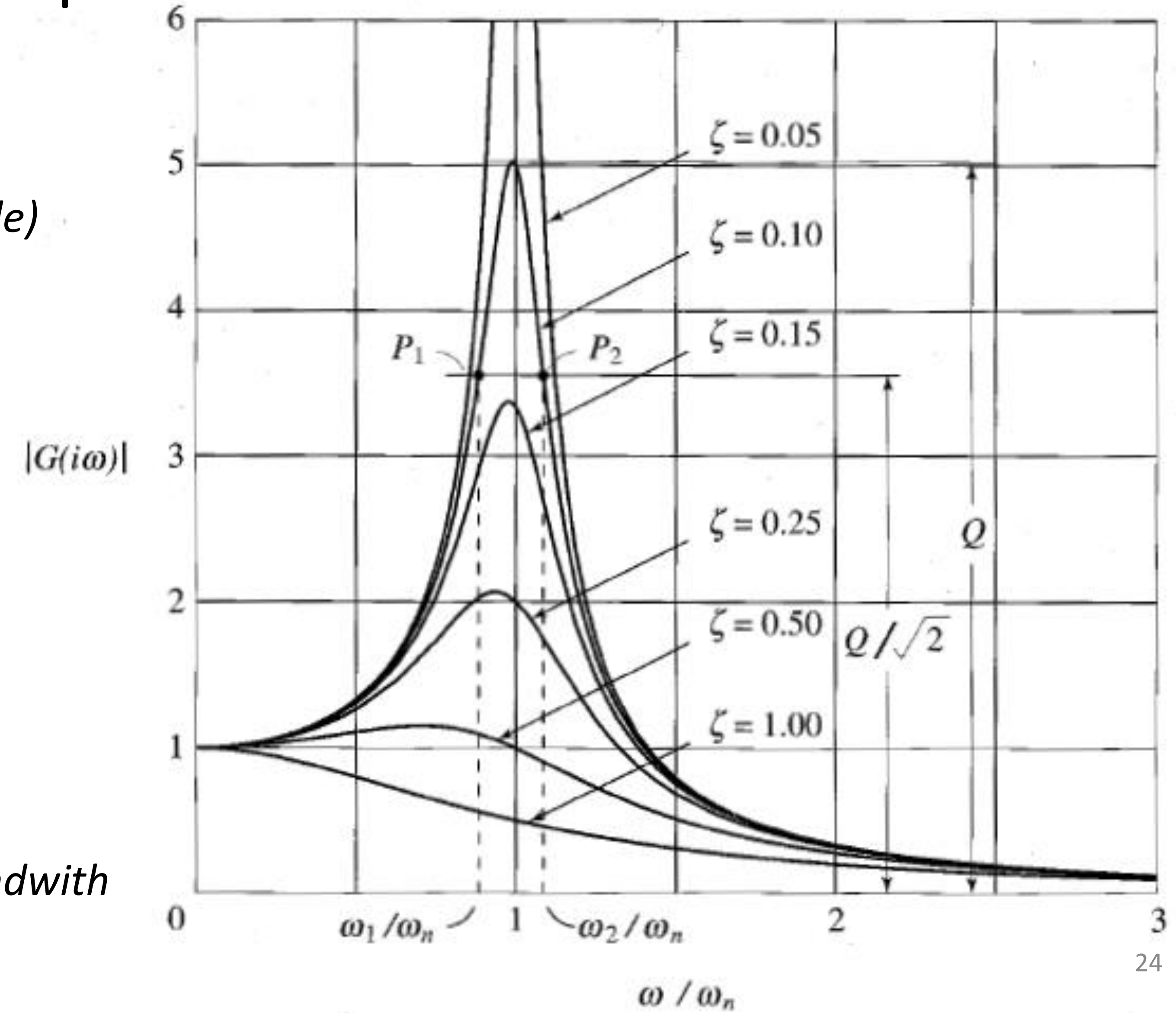
$$\left.\begin{array}{l}\left(\omega_1/\omega_n\right)^2 \\ \left(\omega_2/\omega_n\right)^2\end{array}\right. \cong 1-2\varsigma^2 \mp 2\varsigma$$

$$\varsigma \ll 1 \quad \Rightarrow \quad \begin{array}{l}\omega_1+\omega_2 \cong 2\omega_n \\ \Delta\omega = \omega_2-\omega_1 \cong 2\varsigma\omega_n\end{array}$$

bandwith

$$Q \cong \frac{1}{2\varsigma} \cong \frac{\omega_n}{\Delta\omega}$$

high Q implies small bandwith

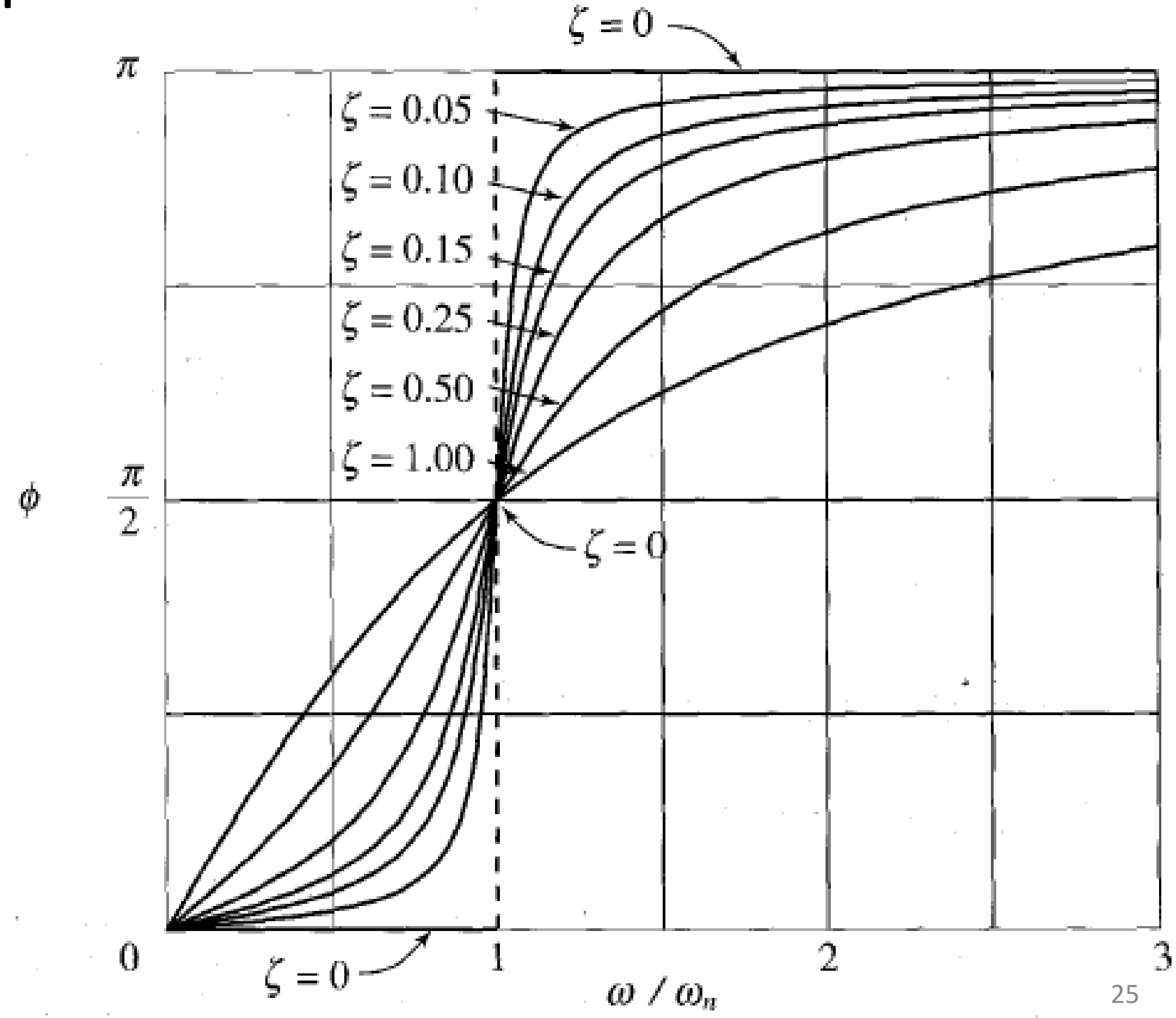


Frequency response plots

$$\phi(\omega) = \tan^{-1} \frac{2\varsigma\, \omega/\omega_n}{1-\left(\omega/\omega_n\right)^2}$$

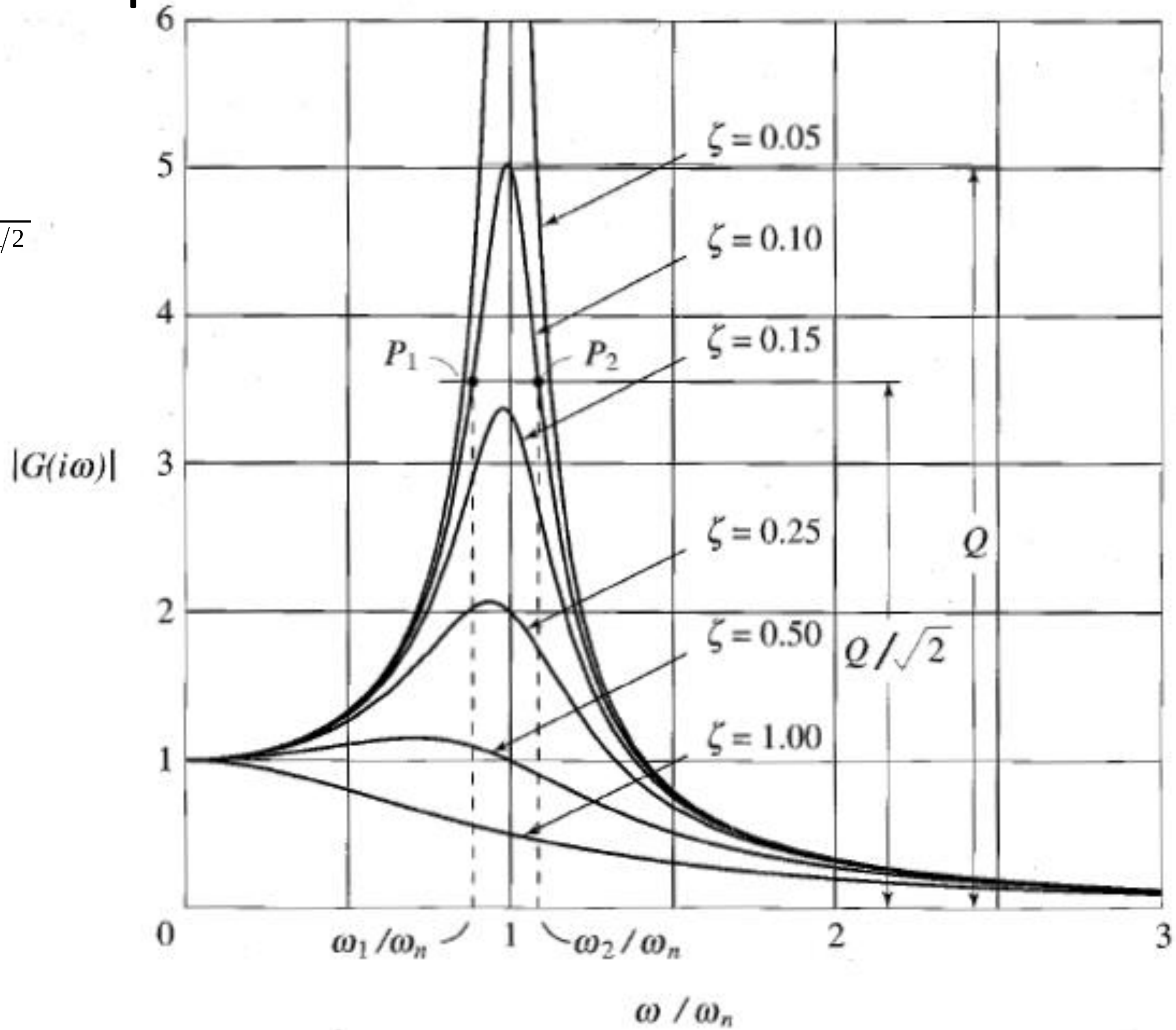
$$\omega/\omega_n = 1 \quad \Rightarrow \quad \phi = \frac{\pi}{2}$$

for any value of ς



Frequency response plots

$$|G(i\omega)| = \frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + \left(2\varsigma(\omega/\omega_n)^2\right)\right\}^{1/2}}$$



undamped case:

$\varsigma = 0$ ➡ $|G(i\omega)| \to \infty$

at $\omega = \omega_n$

➡ **resonance**

Frequency response plots

$$\phi(\omega) = \tan^{-1} \frac{2\varsigma\, \omega/\omega_n}{1-(\omega/\omega_n)^2}$$

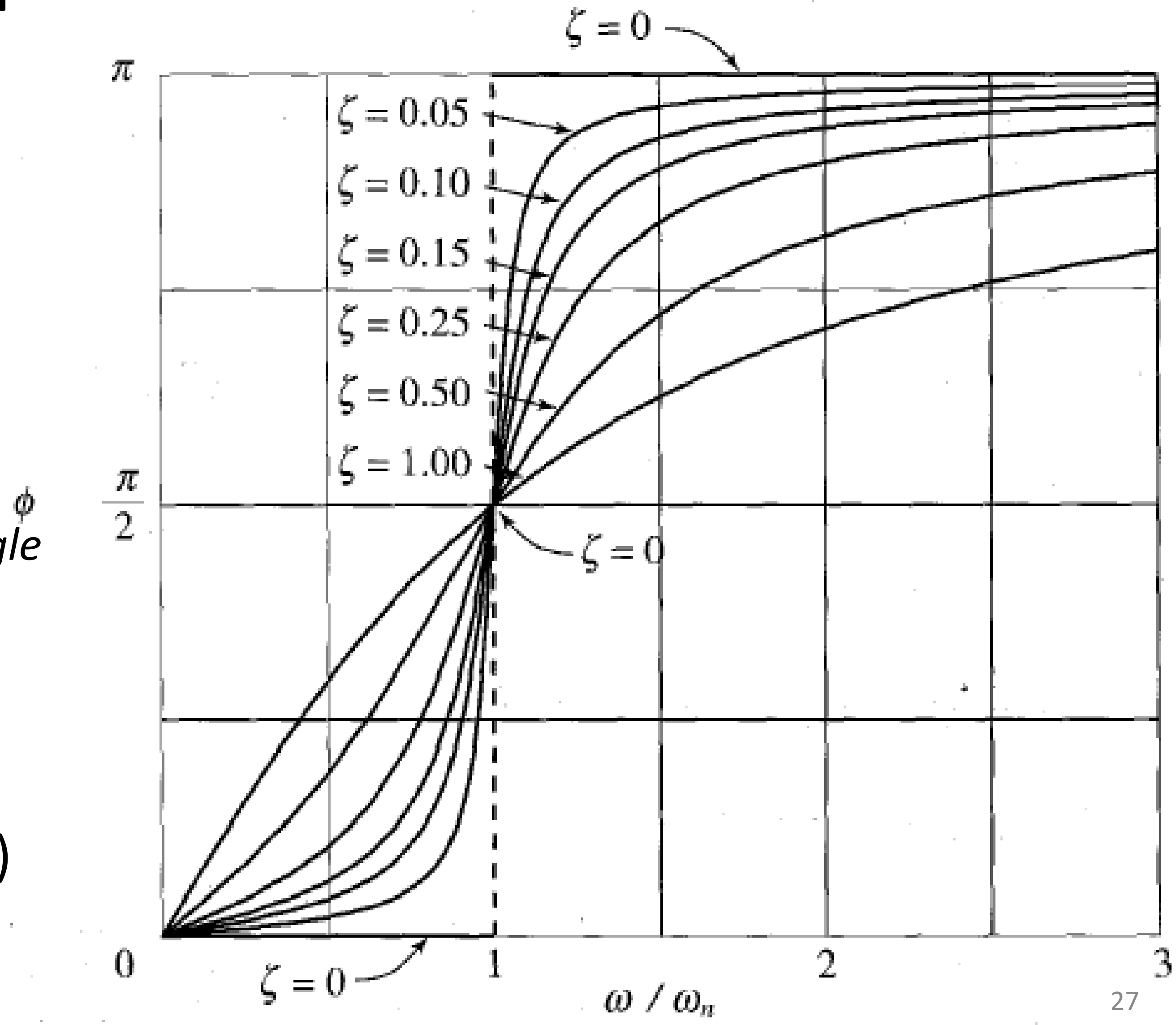
undamped case:

$\varsigma = 0$ ➡ *discontinuity in the phase angle*

$\omega/\omega_n < 1$ ➡ $\phi = 0$

$\omega/\omega_n = 1$ ➡ $\phi = \dfrac{\pi}{2}$ (resonance)

$\omega/\omega_n > 1$ ➡ $\phi = \pi$



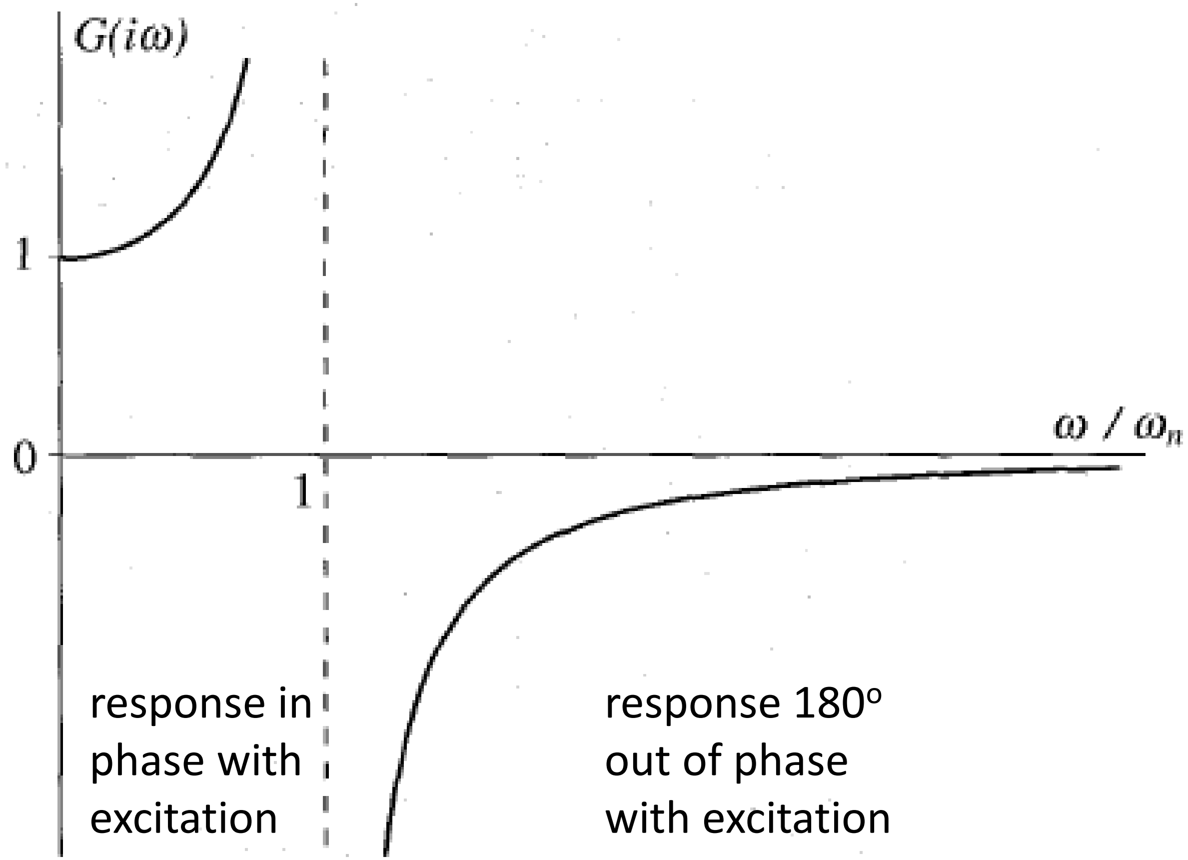
Response of SDOF systems to harmonic excitations: undamped case

$$\ddot{x}(t) + \cancel{2\varsigma}\omega_n \dot{x}(t) + \omega_n^2 x(t) = \omega_n^2 A\cos\omega t$$

$$x(t) = AG(\omega)\cos\omega t$$

$$G(\omega) = \frac{1}{1-\left(\omega/\omega_n\right)^2}$$

real function (different than in the damped case (complex function)



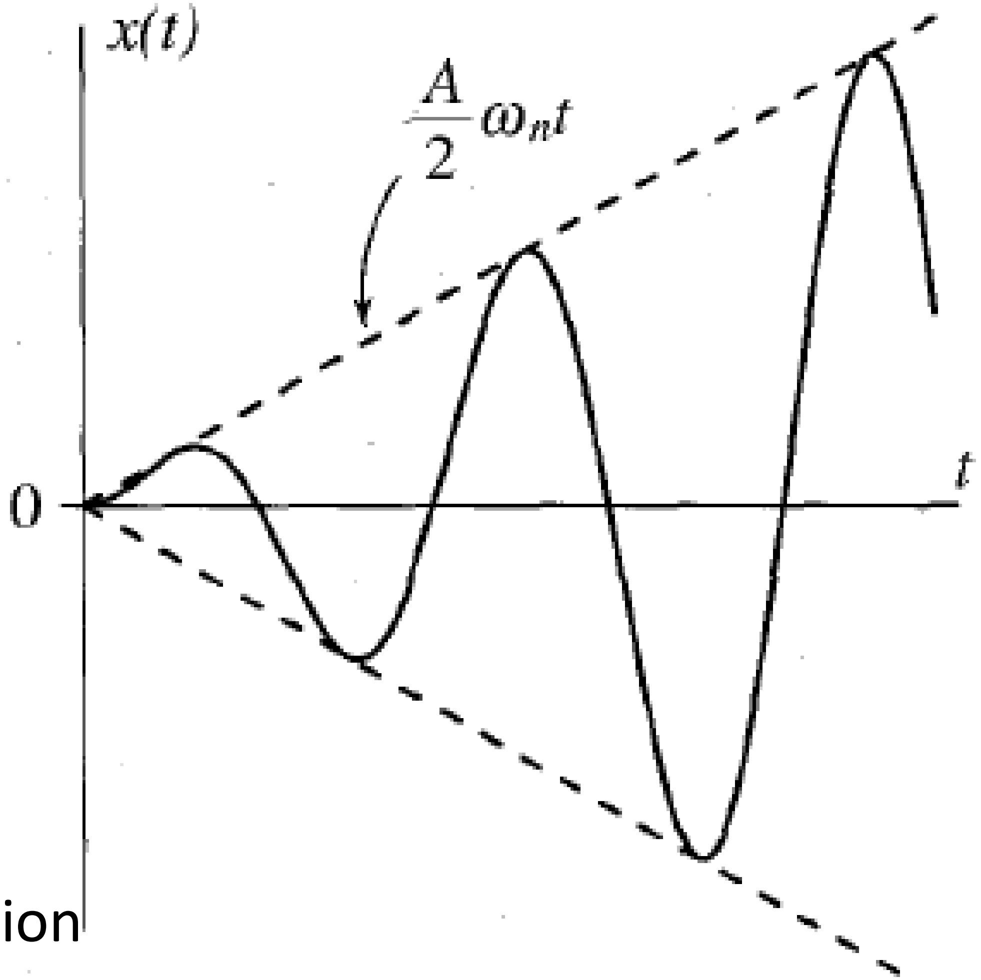
Response of SDOF systems to harmonic excitations: undamped case

At resonance: $\omega = \omega_n$

$$\ddot{x}(t) + \omega_n^2 x(t) = \omega_n^2 A \cos \omega_n t$$

$$x(t) = \frac{A}{2} \omega_n t \sin \omega_n t$$

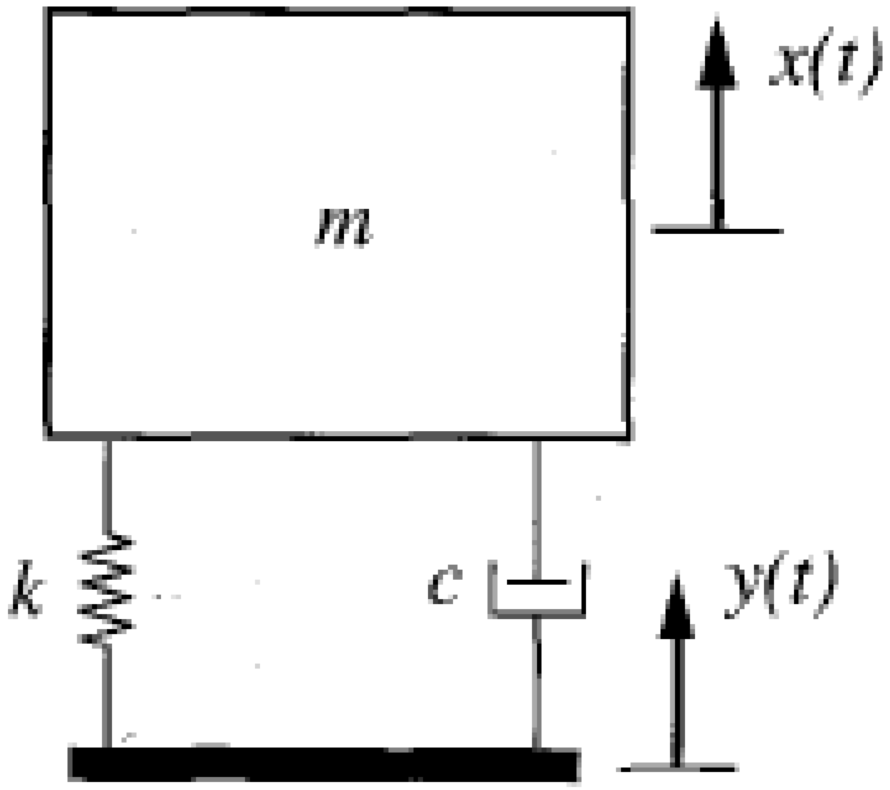
is a particular solution



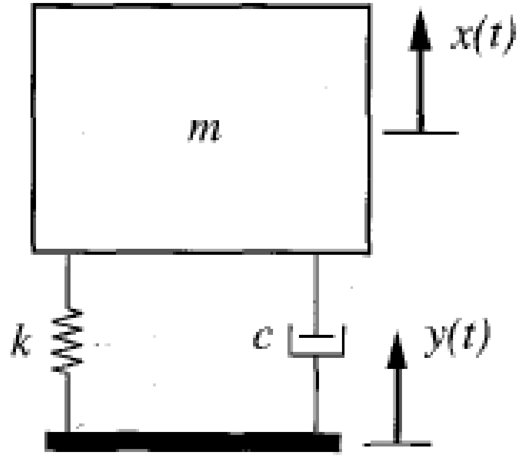
➔ response 90^o out of phase with excitation

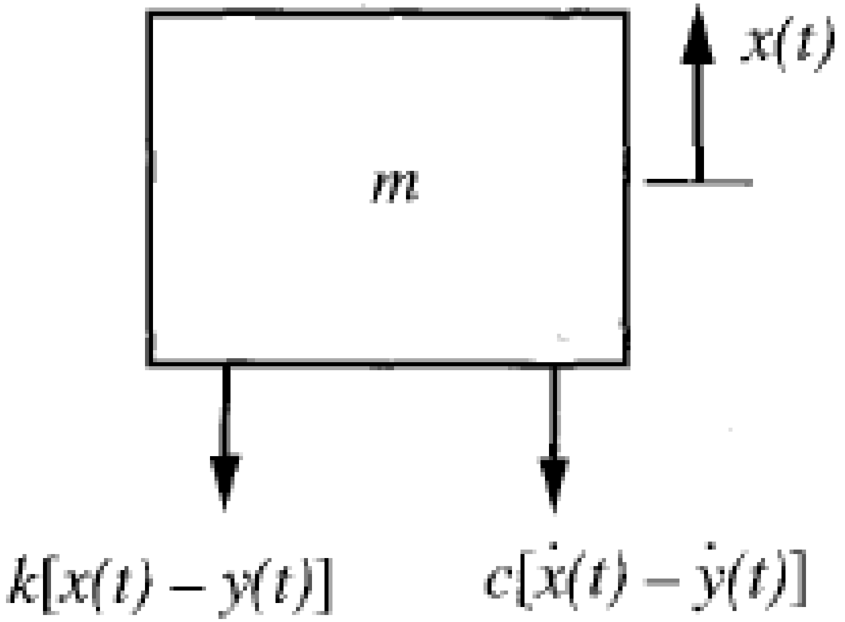
Harmonic motion of the base

- Equipment placed on a vibrating foundation
- Vehicle traveling on a bumpy road
- Engine mounted on an aircraft wing
- ...



Harmonic motion of the base





$$-c\left(\dot{x}-\dot{y}\right)-k\left(x-y\right)=m\ddot{x}$$

$$\ddot{x}+2\varsigma\omega_n\dot{x}+\omega_n^2x=2\varsigma\omega_n\dot{y}+\omega_n^2y$$

$$y(t)=\mathrm{Re}\,Ae^{i\omega t}$$

$$x(t)=X(i\omega)e^{i\omega t}$$

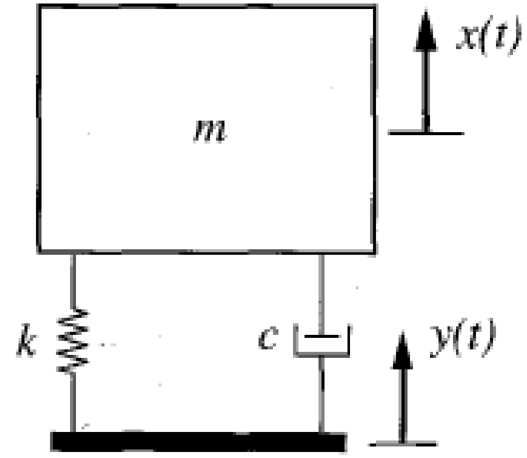
If the excitation is $A\cos\omega t$ the response is $\mathrm{Re}\,x(t)$

If the excitation is $A\sin\omega t$ the response is $\mathrm{Im}\,x(t)$

Harmonic motion of the base

$$X(i\omega) = \left(1 + i2\varsigma\,\omega/\omega_n\right)G(i\omega)A \qquad G(i\omega) = \frac{1}{1-\left(\omega/\omega_n^2\right)^2 + i2\varsigma\,\omega/\omega_n}$$

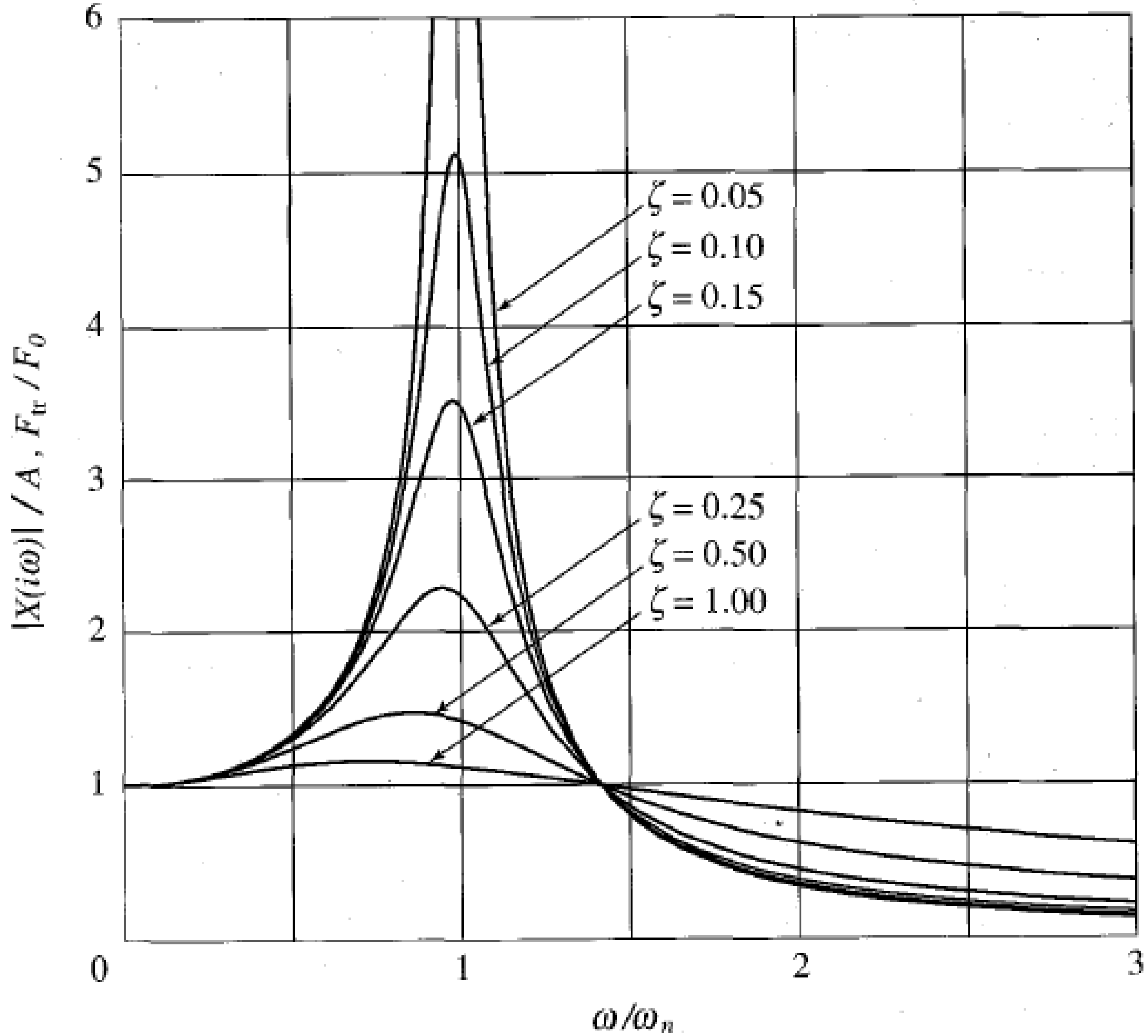
$$x(t) = \left|X(i\omega)\right| e^{i(\omega t - \phi)}$$

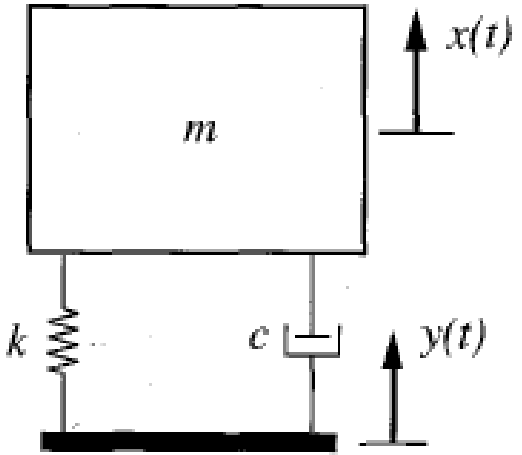


$$\left|X(i\omega)\right| = \left[1 + \left(\frac{2\varsigma\omega}{\omega_n}\right)^2\right]^{1/2} \left|G(i\omega)\right| A$$

$$\phi(\omega) = \tan^{-1}\left[\frac{-\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}\right] = \tan^{-1}\frac{2\varsigma\left(\omega/\omega_n\right)^3}{1-\left(\omega/\omega_n\right)^2 + \left(2\varsigma\,\omega/\omega_n\right)^2}$$

Harmonic motion of the base





Harmonic motion of the base