System Dynamics and Vibrations

Prof. Gustavo Alonso

Chapter 6: Two-degree-of-freedom systems
Part 4

School of General Engineering
Beihang University (BUAA)

Contents

- Introduction
- The equations of motion of two-degree-of-freedom systems.
- Free vibration of undamped systems. Natural modes.
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- Systems admitting rigid-body motions.
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- Introduction to multi-degree-of-freedom systems.

Multi-degree-of-freedom systems

- In the absence of damping, the equations of motion can be decoupled by using a transformation of coordinates, with the modal matrix acting as the transformation matrix.
- To determine the modal matrix, we must solve an algebraic eigenvalue problem, a problem associated with free vibration.
- The free vibration problem leads directly to the eigenvalue problem, the solution of the latter yielding the natural modes of vibration.
- The natural motions, defined as motions in which the system vibrates in any one of the natural modes, can be identified as special cases of free vibration.
- In the general case of free vibration, the motion can be regarded as a linear combination of the natural motions.

Undamped free vibration. The eigenvalue problem

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Free vibration of undamped systems

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{0}$$

Set of *n* simultaneous homogeneous differential equations

Solution (synchronous motion):

$$q_j(t) = u_j f(t), \quad j = 1, 2, \ldots, n$$

$f(t)$ function of time, the same for all the coordinates $q_j(t)$

$u_j \left(j = 1, 2, \ldots, n \right)$ constant amplitudes

Free vibration of undamped systems

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$$

$$\ddot{f}(t)M\mathbf{u} + f(t)K\mathbf{u} = \mathbf{0}$$

$$\ddot{f}(t)\mathbf{u}^T M\mathbf{u} + f(t)\mathbf{u}^T K\mathbf{u} = 0 \longrightarrow$$

escalar equation

$$\lambda = \frac{\mathbf{u}^T K\mathbf{u}}{\mathbf{u}^T M\mathbf{u}}$$

$$\ddot{f}(t) + \lambda f(t) = 0$$

$$\Downarrow$$

(Eigenvalue problem) $$K\mathbf{u} = \lambda M\mathbf{u}$$

Free vibration of undamped systems

$$M\ddot{\mathbf{x}}(t)+K\mathbf{x}(t)=\mathbf{0}$$

proportional to potential energy

$$\lambda=\frac{\mathbf{u}^T K\mathbf{u}}{\mathbf{u}^T M\mathbf{u}}$$

real and positive

$$\lambda=\omega^2$$

proportional to kinetic energy

$$\ddot{f}(t)+\omega^2 f(t)=0$$

Free vibration of undamped systems

$$\ddot{f}(t) + \omega^2 f(t) = 0$$

harmonic solution:

$$f(t) = C\cos\left(\omega t - \phi\right)$$

Same frequency and phase angle for the different degrees of freedom

- The algebraic eigenvalue problem can only be solved numerically, requiring methods of matrix algebra.
- The sole exception is for Two-degrees-of-freedom systems

Free vibration of undamped systems

$$\boldsymbol{K}\mathbf{u} = \omega^2 M\mathbf{u}$$

Eigenvalue problem:

Set of *n* homogeneous algebraic equations
It has non-trivial solutions if:

$$\det\left[K - \omega^2 M\right] = 0$$

characteristic equation

The *n* roots are denoted by $\omega_1^2, \omega_2^2, \ldots \omega_n^2$

and the square roots of those quantities are the system **natural frequencies**

The lowest frequency is referred to as the **fundamental frequency**

Free vibration of undamped systems

- Associated with every one of the frequencies there is a certain nontrivial vector $\boldsymbol{u}_r$ which is the solutuon of the eigenvalue problem:

$$K\mathbf{u}_r = \omega_r^2 M\mathbf{u}_r, \quad r = 1, 2, \ldots, n$$

- The vectors $\boldsymbol{u}_r$ (r = 1, 2,..., n) are called **modal vectors** and represent physically the **natural modes**

- These vectors are unique in the sense that the ratio between any two components u_{ir} and u_{jr} is constant

➔ The shape of the natural modes is unique, but the amplitude is not

Free vibration of undamped systems

- The process of adjusting the magnitude of the natural modes to render them unique is called **normalization**
- The resulting eigenvectors are referred to as **normal modes**

$$\mathbf{u}_r^T M \mathbf{u}_r = 1, \quad r = 1, 2, ..., n$$

- It can be proven that:

$$\mathbf{u}_r^T K \mathbf{u}_r = \omega_r^2, \quad r = 1, 2, ..., n$$

Free vibration of undamped systems

$$M\ddot{\mathbf{q}}(t)+K\mathbf{q}(t)=\mathbf{0}$$

The solutions are therefore:

$$\mathbf{q}_r(t)=\mathbf{u}_r f_r(t), \quad j=1,2,...,n$$

$$f_r(t)=C_r \cos\left(\omega_r t-\phi_r\right), \quad r=1,2,...n$$

The free vibration problem admits special independent solutions in which the system vibrates in any one of the natural modes.
These solutions are referred to as **natural motions**.

Free vibration of undamped systems

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{0}$$

Invoking the superposition principle, the general solution can be written as:

$$\mathbf{q}(t) = \sum_{r=1}^{n} q_r(t) = \sum_{r=1}^{n} \mathbf{u}_r f_r(t) = U\mathbf{f}\left(t\right)$$

$U = \left[u_1 \quad u_2 \quad \ldots \quad u_n\right]$ → modal matrix

$$\mathbf{f}(t) = \left[\, f_1(t) \quad f_2(t) \quad \ldots \quad f_n(t)\right]$$

Orthogonality of modal vectors

- It can be proven, like in the case of Two-degree-of-fredom systems that:

$$\mathbf{u}_s^T M \mathbf{u}_r = 0, \quad r \neq s$$

$$\mathbf{u}_s^T K \mathbf{u}_r = 0, \quad r \neq s$$

as long as *M* and *K* are symmetric

- The orthogonality property plays a crucial role in the vibration of multi-degree-of-freedom systems, as it forms the foundation for modal analysis whereby the response of a system can be represented as a linear combination of the natural modes

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Systems admitting rigid-body motions

- The motion characteristics described so far are typical of positive definite systems, i.e., systems for which the mass and stiffness matrices are real, symmetric and positive definite
- In the case in which the stiffness matrix is only positive semidefinite, there is at least one eigenvector, say $\mathbf{u}_s$, such that $K\mathbf{u}_s = \mathbf{0}$
- In this case, the system is not fully restrained, and $\mathbf{u}_s$ represents a rigid-body mode with the corresponding natural frequency equal to zero, $\omega_s = 0$
- Of course, in this case the function f_s is not harmonic

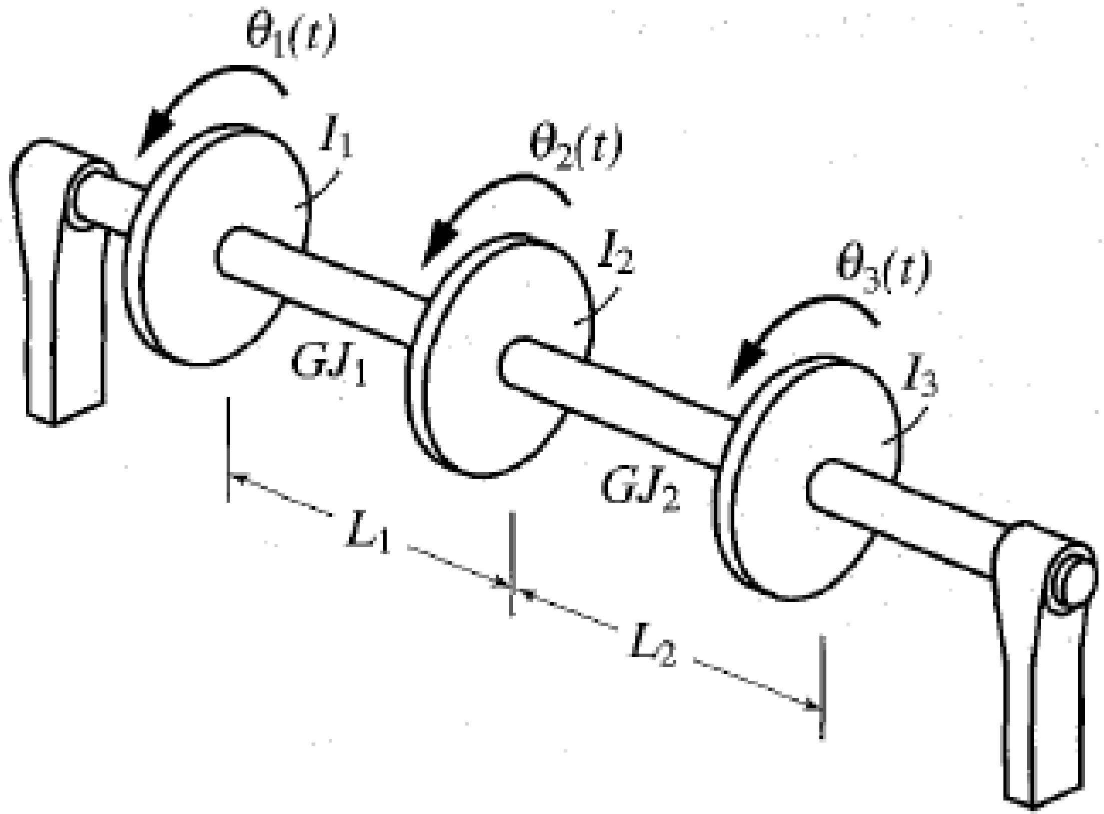
Systems admitting rigid-body motions

- When the mass matrix *M* is positive definite and the stiffness matrix *K* is only positive semidefinite, the system is positive semidefinite
- Physically this implies that the system is supported in such a manner that rigid-body
- motion is possible
- When the potential energy is due to elastic effects alone, if the body undergoes pure rigid-body motion, i.e., without any elastic deformations, then the potential energy is zero without all the coordinates being identically equal to zero

Systems admitting rigid-body motions

- Example:

The system consists of three disks of mass polar moments of inertia I_1 , I_2 and I_3 connected by two segments of a massless shaft of lengths L_1 and L_2 and torsional stiffnesses GJ_1 and GJ_2 respectively. The system is supported at both ends by means of frictionless bearings in such a way that the entire system can rotate freely as a whole. Of course, torsional deformations can also be present, so that in general the motion of the system is a combination of rigid and elastic motions.



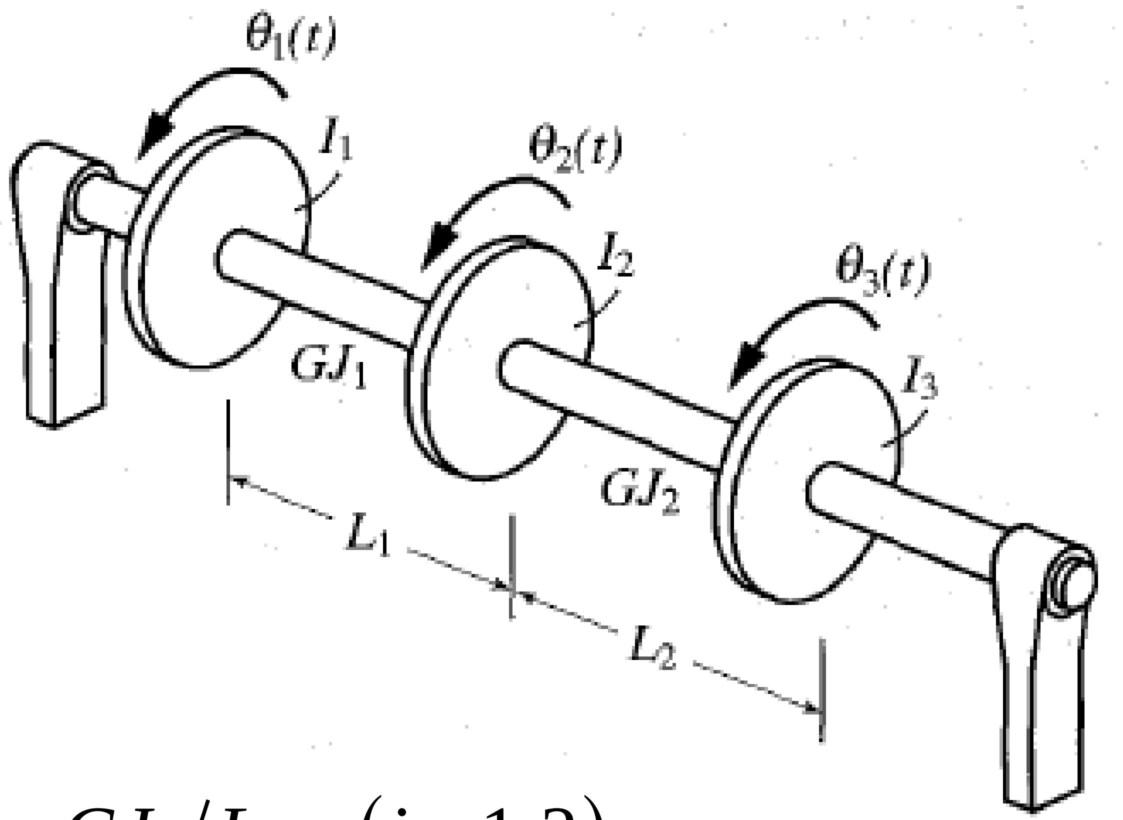
Systems admitting rigid-body motions

$$T = \frac{1}{2}\left(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2 + I_3\dot{\theta}_3^2\right) = \frac{1}{2}\dot{\boldsymbol{\theta}}^T M \dot{\boldsymbol{\theta}}$$

$$M = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$V = \frac{1}{2}\left[k_1\left(\theta_2 - \theta_1\right)^2 + k_1\left(\theta_3 - \theta_2\right)^2\right] = \frac{1}{2}\boldsymbol{\theta}^T K \boldsymbol{\theta}$$

$$K = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$



$$k_i = GJ_i / L_i \quad \left(i = 1, 2\right)$$

Systems admitting rigid-body motions

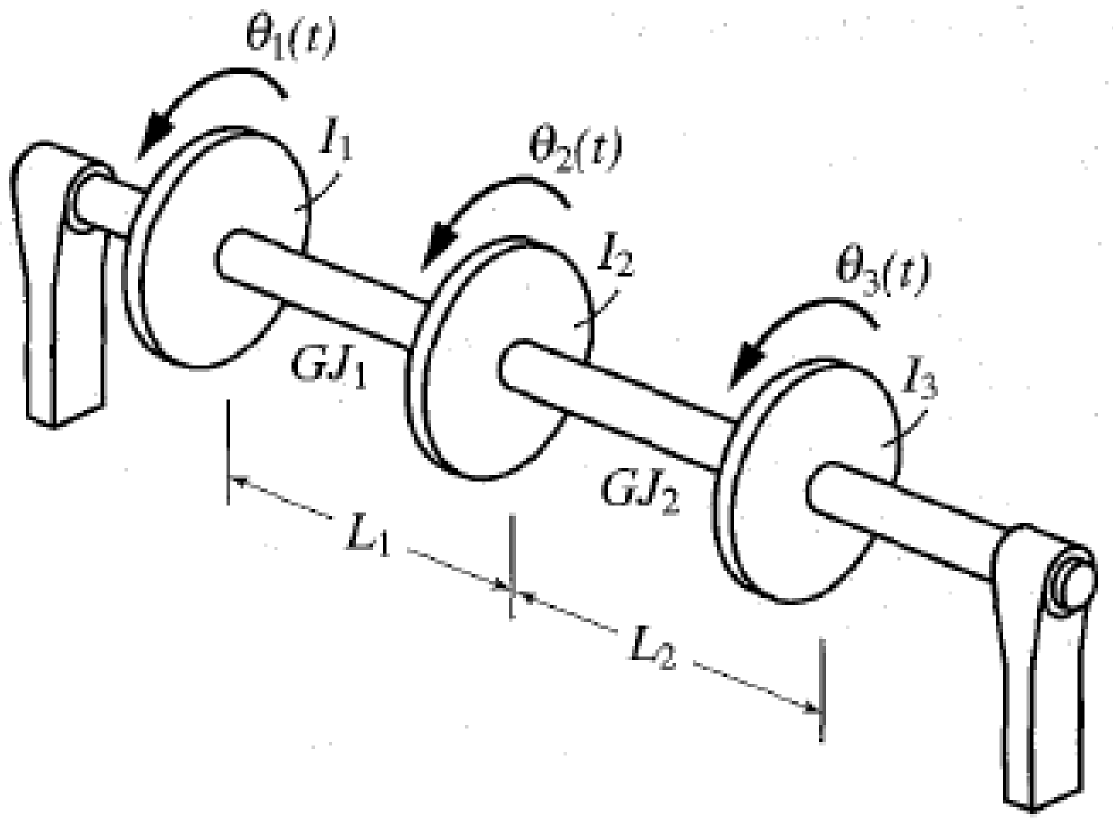
Solution (synchronous motion):

$$\theta_i(t) = \Theta_i \, f(t) \quad i = 1, 2, 3$$

constants harmonic

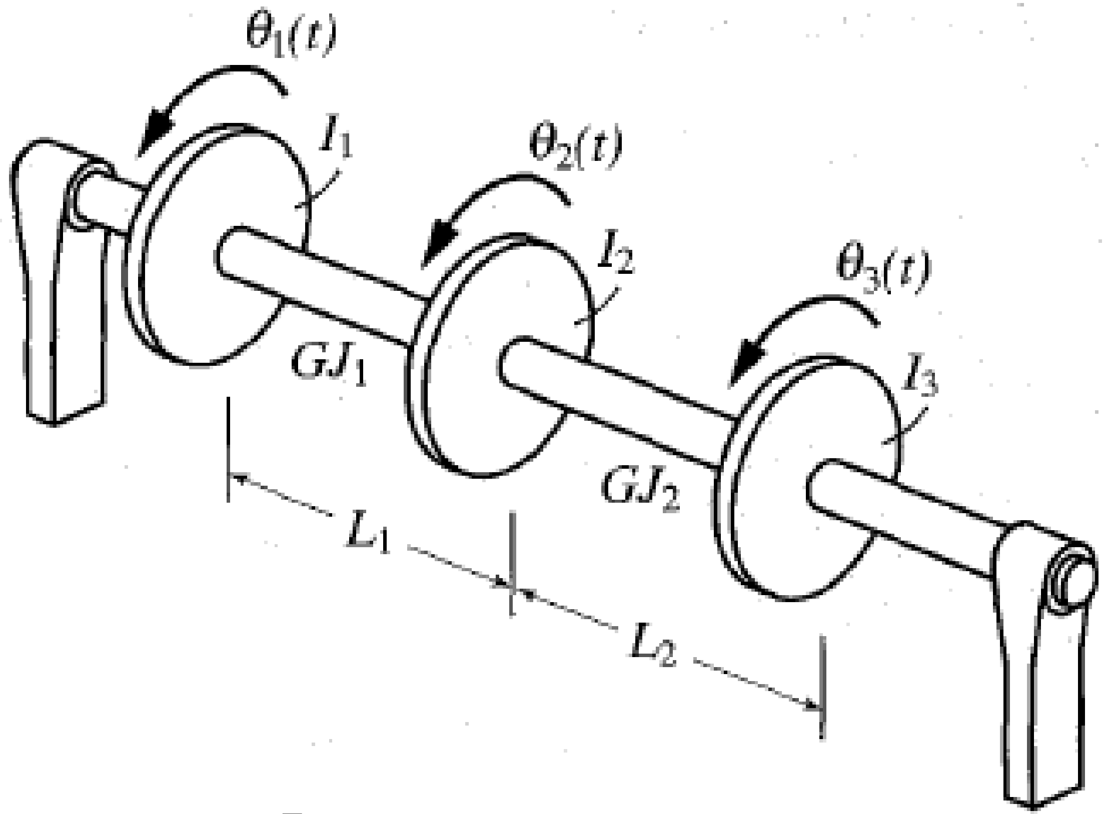
Eigenvalue problem:

$$K\mathbf{\Theta} = \omega^2 M\mathbf{\Theta}$$



Systems admitting rigid-body motions

$$K = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$



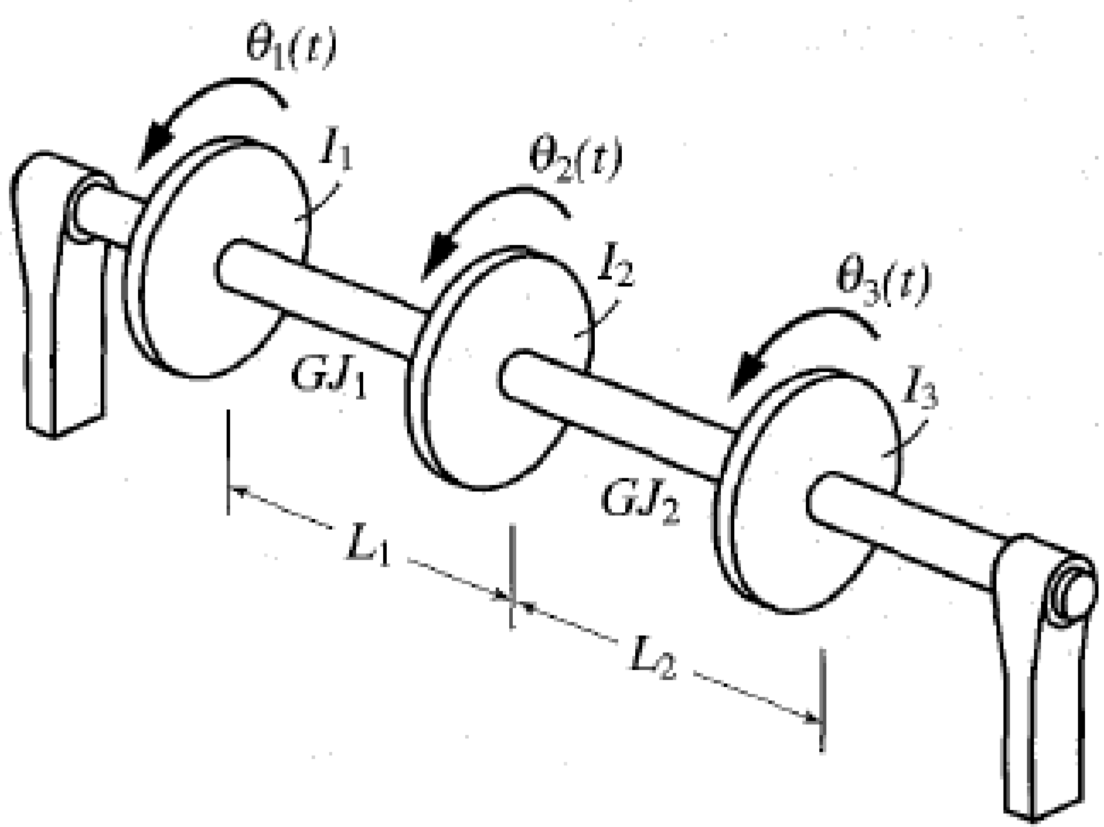
K is singular (the determinant of *K* is equal to zero)

➔ The system admits a rigid-body mode in which the shaft experiences no elastic deformation. The implication is that all three disks undergo the same rotation, so that the rigid-body mode must have the form

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}_0 = \Theta_0 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T = \Theta_0 \mathbf{1}$$

Systems admitting rigid-body motions

$$K\mathbf{\Theta}_0 = \Theta_0 \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$$



We conclude that the eigenvalue problem does indeed admit as a nontrivial solution the rigid-body $\mathbf{\Theta}_0$ = **1** mode with the zero natural frequency, $\omega_0 = 0$.
Note that the rigid-body mode $\mathbf{\Theta}_0$ is possible because both ends of the shaft are free.
It is the only rigid-body mode possible for the system under consideration.

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Response to initial excitations by modal analysis

Equation of motion:

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{0}$$

Initial conditions:

$$\mathbf{q}(0),\quad \dot{\mathbf{q}}(0)$$

The solution can be regarded as a superposition of the normal modes $\mathbf{u}_r$

$$\mathbf{q}(t) = \eta_1(t)\mathbf{u}_1 + \eta_2(t)\mathbf{u}_2 + \ldots + \eta_n(t)\mathbf{u}_n = \sum_{r=1}^{n} \eta_r(t)\mathbf{u}_r$$

The coefficients are defined by

$$\eta_r(t) = \mathbf{u}_r^T M\mathbf{q}(t),\quad \omega_r^2\eta_r(t) = \mathbf{u}_r^T K\mathbf{q}(t),\quad r = 1, 2, , , n$$

Response to initial excitations by modal analysis

In compact matrix form:

$$\mathbf{q}(t) = U\eta(t)$$

$$\eta(t) = U^T M\mathbf{q}(t), \quad \Omega\eta(t) = U^T K\mathbf{q}(t)$$

$$\Longrightarrow \quad \ddot{\eta}(t) + \Omega\eta(t) = \mathbf{0}$$

$\eta_r(t)$ are the modal coordinates

and are subject to the initial conditions: $\eta_r(0), \ \dot{\eta}_r(0) \quad (r = 1, 2, \ldots, n)$

Response to initial excitations by modal analysis

This equation resembles the equation of a harmonic oscillator:

$$\ddot{\eta}(t)+\Omega\eta(t)=\mathbf{0}$$

$$\Rightarrow \quad \eta_r(t)=C_r\cos(\omega_r t-\phi_r)=\eta_r(0)\cos\omega_r t+\frac{\dot{\eta}_r(0)}{\omega_r}\sin\omega_r t, \quad r=1,2,\ldots,n$$

$$\eta_r(0)=\mathbf{u}_r^T M\mathbf{q}(0), \quad \dot{\eta}_r(0)=\mathbf{u}_r^T M\dot{\mathbf{q}}(0), \quad r=1,2,\ldots,n$$

$$\Rightarrow \quad \mathbf{q}(t)=\sum_{r=1}^{n}\left[\mathbf{u}_r^T M\mathbf{q}(0)\cos\omega_r t+\frac{1}{\omega_r}\mathbf{u}_r^T M\dot{\mathbf{q}}(0)\sin\omega_r t\right]\mathbf{u}_r$$

Response to external excitations by modal analysis: undamped systems

Equation of motion:

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{Q}(t)$$

Solve the eigenvalue problem:

$$K\mathbf{u} = \omega^2 M\mathbf{u}$$

To obtain natural frequencies and natural modes (or modal vectors):

$$\omega_r^2, \quad \mathbf{u}_r \quad (r = 1, 2, \dots, n)$$

or in matricial form: $\Omega = \mathrm{diag}\left[\omega_1^2 \quad \omega_2^2 \quad \dots \quad \omega_n^2\right], U = \left[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_n\right]^T$

Response to external excitations by modal analysis: undamped systems

Normalize the modal matrix:

$$U^T M U = I, \quad U^T K U = \Omega$$

Express the solution as a linear combination of the modal vectors:

$$\mathbf{q}(t) = \sum_{r=1}^{n} \eta_r(t) \mathbf{u}_r = U \boldsymbol{\eta}(t)$$

$\eta_r(t)$ are the modal coordinates

Response to external excitations by modal analysis: undamped systems

Replacing in the equations of motion:

$$\ddot{\eta}(t)+\Omega\eta(t)=\mathbf{N}(t)$$

vector of modal forces

$$\mathbf{N}(t)=U^{T}\mathbf{Q}(t)$$

Set of <u>independent</u> modal equations, that resembles the equation of a an undamped single-degree-of-freedom system

$$\ddot{\eta}_r(t)+\omega_r^2\eta_r(t)=N_r(t),\quad r=1,2,\ldots,n$$

$$N_r(t)=\mathbf{u}_r^T\mathbf{Q}(t),\quad r=1,1,\ldots,n$$

Response to external excitations by modal analysis: undamped systems

If the external excitation is **harmonic**: $\mathbf{Q}(t) = \mathbf{Q}_0 \cos \alpha t$

forces amplitudes excitation frequency

Nodal forces are: $N_r(t) = \mathbf{u}_r^T \mathbf{Q}_0 \cos \alpha t, \quad r = 1, 2, \ldots, n$

Using analogy with the response of undamped single-degree-of-freedom systems to harmonic excitations:

$$\eta_r(t) = \frac{\mathbf{u}_r^T \mathbf{Q}_0}{\omega_r^2 - \alpha^2} \cos \alpha t, \quad r = 1, 2, \ldots, n \quad \Longrightarrow \quad \mathbf{q}(t) = \sum_{r=1}^{n} \frac{\mathbf{u}_r^T \mathbf{Q}_0}{\omega_r^2 - \alpha^2} \mathbf{u}_r \cos \alpha t$$

(being a steady-state response, it is not advisable to add to it the effect of initial excitations, which represent a transient response)

Response to external excitations by modal analysis: undamped systems

If the external excitation is **arbitrary**:

Using analogy with the response of undamped single-degree-of-freedom systems to arbitrary excitations:

$$\eta_r(t) = \frac{1}{\omega_r}\int_0^t N_r(t-\tau)\sin\omega_r\tau\, d\tau, \quad r = 1, 2, \ldots, n$$

$$\Longrightarrow \qquad \mathbf{q}(t) = \sum_{r=1}^{n}\left[\frac{\mathbf{u}_r^T}{\omega_r}\int_0^t \mathbf{Q}(t-\tau)\sin\omega_r\tau\, d\tau\right]\mathbf{u}_r$$

The response is the superposition of this response to external excitations and the response to initial excitations previously determined

Response to external excitations by modal analysis: systems with proportional damping

- The modal matrix *U* is able to diagonalize the mass matrix *M* and thestiffness matrix *K* simultaneously, but not the damping matrix *C*
- There is one exception, when *C* can be expressed as a linear combination of *M* and *K* ➔ proportional damping

$$C = \alpha M + \beta K$$

given constant scalars

$$U^T C U = U^T(\alpha M + \beta K)U = \alpha U^T M U + \beta U^T K U = \alpha I + \beta \Omega$$

diagonal matrix

Response to external excitations by modal analysis: systems with proportional damping

The equations of motion in modal form are now:

$$\ddot{\boldsymbol{\eta}}(t) + \left(\alpha I + \beta\Omega\right)\dot{\boldsymbol{\eta}}(t) + \Omega\boldsymbol{\eta}(t) = \mathbf{N}(t)$$

Introducing the notation $\alpha + \beta\omega_r^2 = 2\varsigma_r\omega_r, \quad r = 1,1,\ldots,n$

where $\varsigma_r \;\; (r = 1,2,\ldots,n)$ are modal viscous damping factors

➔We obtain the independent modal equations:

$$\ddot{\eta}_r(t) + 2\varsigma_r\omega_r\dot{\eta}_r(t) + \omega_r^2\eta_r(t) = N_r(t), \quad r = 1,2,\ldots,n$$

that resemble entirely the equation of motion of a viscously damped single-degree-of-freedom system

Response to external excitations by modal analysis: systems with proportional damping

If the system is subjected **to harmonic excitations** $\mathbf{Q}(t) = \mathbf{Q}_0 e^{i\alpha t}$

$$N_r(t) = \mathbf{u}_r^T \mathbf{Q}_0 e^{i\alpha t}, \quad r = 1, 2, \ldots, n$$

Using analogy with viscously damped single-degree-of-freedom systems, the steady state solution is:

$$\eta_r(t) = \frac{\mathbf{u}_r^T \mathbf{Q}_0}{\omega_r^2 - \alpha^2 + i2\varsigma_r \omega_r \alpha} e^{i\alpha t}, \quad r = 1, 2, \ldots, n$$

$$\Rightarrow \quad \mathbf{q}(t) = \sum_{r=1}^{n} \frac{\mathbf{u}_r^T \mathbf{Q}_0}{\omega_r^2 - \alpha^2 + i2\varsigma_r \omega_r \alpha} \mathbf{u}_r e^{i\alpha t}$$

Response to external excitations by modal analysis: systems with proportional damping

$$\eta_r(t) = \frac{\mathbf{u}_r^T \mathbf{Q}_0}{\omega_r^2 - \alpha^2 + i2\varsigma_r\omega_r\alpha} e^{i\alpha t}, \quad r = 1, 2, \ldots, n$$

$$\Rightarrow \quad \mathbf{q}(t) = \sum_{r=1}^{n} \frac{\mathbf{u}_r^T \mathbf{Q}_0}{\omega_r^2 - \alpha^2 + i2\varsigma_r\omega_r\alpha} \mathbf{u}_r e^{i\alpha t}$$

If the excitation is $\mathbf{Q}(t) = \mathbf{Q}_0 \cos \alpha t$

we retain only the real part of $\eta_r(t)$

and if the excitation is $\mathbf{Q}(t) = \mathbf{Q}_0 \sin \alpha t$ we retain the imaginary part

Response to external excitations by modal analysis: systems with proportional damping

The notation can be simplified by introducing the <u>modal frequency responses</u>:

$$G_r(i\alpha) = \frac{1}{1-\left(\alpha/\omega_r\right)^2 + i2\varsigma_r\,\alpha/\omega_r}, \quad r = 1,2,\ldots,n$$

$$\Rightarrow \qquad \mathbf{q}(t) = \sum_{r=1}^{n} \frac{\mathbf{u}_r^T \mathbf{Q}_0}{\omega_r^2} \left|G(i\alpha)\right| \mathbf{u}_r e^{i\left(\alpha t - \phi_r\right)}$$

$$\left|G_r(i\alpha)\right| = \frac{1}{\left\{\left[1-\left(\alpha/\omega_r\right)^2\right]^2 + \left(2\varsigma_r\,\alpha/\omega_r\right)^2\right\}^{1/2}}, \quad r = 1,2,\ldots,n \qquad \phi_r = \tan^{-1}\frac{2\varsigma_r\,\alpha/\omega_r}{1-\left(\alpha/\omega_r\right)^2}, \quad r = 1,2,\ldots,n$$

Response to external excitations by modal analysis: systems with proportional damping

If the system is subjected **to arbitrary excitations**

Using analogy with viscously damped single-degree-of-freedom systems, the steady state solution is:

$$\eta_r(t) = \frac{1}{\omega_{dr}} \int_0^t N_r(t-\tau) e^{-\varsigma_r \omega_r \tau} \sin \omega_{dr} \tau \, d\tau, \quad r = 1, 2, \ldots, n$$

$$\omega_{dr} = \left(1 - \varsigma_r^2\right)^{1/2} \omega_r, \quad r = 1, 2, \ldots, n$$

frequencies of damped oscillations