Hydrodynamics

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1 Introduction

The equations of one-dimensional hydrodynamics in the conservation form

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}}{\partial x} = 0 \tag{1}$$

where $\vec{U} = (\rho, \rho v, E)$ are the conserved variables for mass, momentum, and energy, and $\vec{F} = (\rho v, \rho v^2 + P, (E+P)v)$ are the fluxes of those quantities. ρ is the density, v is the velocity, P is the pressure, and $E = \rho e + \frac{\rho v^2}{2}$ is the total energy density. The quantity is the specific internal energy. For an ideal gas:

$$P = (\gamma - 1)\rho e \tag{2}$$

where γ is the adiabatic index. Therefore, we can get the following equations based on equation 2 and other conditions.

$$\vec{U} = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} = \begin{pmatrix} \rho \\ \rho v \\ \rho e + \frac{\rho v^2}{2} \end{pmatrix} \tag{3}$$

$$\vec{F} = \begin{pmatrix} \rho v \\ \rho v^2 + P \\ (E+P)v \end{pmatrix} = \begin{pmatrix} \rho v \\ \rho(v^2 + e(\gamma - 1)) \\ \rho v(\gamma e + \frac{v^2}{2}) \end{pmatrix} \tag{4}$$

By observation, we can get $\vec{U}$ and $\vec{F}$ just using the values of ρ, v, e and γ.

2 Sod Shock Tube Problem

2.1 Overview

The first part of our project focuses on the application of Low-order 1D method to solve Sod Shock Tube Problem. First, we calculate the Harten-Lax-van Leer approximation flux $\vec{F}^{HLL}$ based on the formula and $\vec{U}^{t_n}$ and $\vec{F}^{t_n}$ for time at t_n

$$\vec{F}^{\mathrm{HLL}} = \frac{\alpha^+ \vec{F}_L^{t_n} + \alpha^- \vec{F}_R^{t_n} - \alpha^+\alpha^- \left(\vec{U}_R^{t_n} - \vec{U}_L^{t_n}\right)}{\alpha^+ + \alpha^-}. \tag{5}$$

where L, R can be $n, n+1$ represents the spacial index, and the formula will calculate $\vec{F}^{\text{HLL}}$ between $n, n+1$, which is $n+\frac{1}{2}$. $\alpha^{\pm}$ can be calculated by

$$\alpha^{\pm}_{i+\frac{1}{2}} = \max\left\{0, \pm\lambda^{\pm}\left(\vec{U}_i^{t_n}\right), \pm\lambda^{\pm}\left(\vec{U}_{i+1}^{t_n}\right)\right\} \tag{6}$$

where $\lambda^{\pm} = v \pm \sqrt{\frac{\partial P}{\rho}}$, v is the velocity. Next, We can calculate the derivative of $\vec{U}$ using following equation

$$\frac{\partial \vec{U}_i}{\partial t} = -\frac{\vec{F}_{i+\frac{1}{2}} - \vec{F}_{i-\frac{1}{2}}}{\Delta x} \tag{7}$$

Then, we are able to obtain $\vec{U}^{t_{n+1}}$, which is $\vec{U}$ at next time step $n+1$, given $\vec{U}^{t_n}$ and $\frac{\partial \vec{U}_i}{\partial t}$,

$$\begin{aligned} \vec{U}^{t_{n+1}} &= \vec{U}^{t_n} + \Delta t \cdot \frac{\partial \vec{U}_i^{t_n}}{\partial t} \\ &= \vec{U}^{t_n} + \Delta t \cdot -\frac{\vec{F}_{i+\frac{1}{2}} - \vec{F}_{i-\frac{1}{2}}}{\Delta x} \end{aligned} \tag{8}$$

Finally, we can calculate $\vec{F}^{t_{n+1}}$ based on $\vec{U}^{t_{n+1}}$.

2.2 Derivation And Code Implementation

The idea is to create matrix with nx columns and nt rows, each column represents a discrete position of the simulation, and each row represents the simulation at different time step. Each element in the matrix is an object we defined as cell, which includes $\vec{U}$, $\vec{F}$, $F^{\vec{HLL}}$, $\lambda^{\pm}$ at that location and time. Just as the table shows below for the $\vec{U}$ arrangement in matrix:

...	$U_{i-2}^{t_n}$	$U_{i-1}^{t_n}$	$U_i^{t_n}$	$U_{i+1}^{t_n}$	$U_{i+2}^{t_n}$	...
...	$U_{i-2}^{t_{n+1}}$	$U_{i-1}^{t_{n+1}}$	$U_i^{t_{n+1}}$	$U_{i+1}^{t_{n+1}}$	$U_{i+2}^{t_{n+1}}$	...

and then for this particular question, we assigned $\rho_L, P_L, e_L = U_i$ for all position at left half, and $\rho_R, P_R, e_R = U_i$ on the right half. Some preliminary conditions are given as:

$$\begin{aligned} \frac{P_L}{P_R} &= 8 \\ \frac{\rho_L}{\rho_R} &= 10 \\ \gamma &= 1.4 \end{aligned} \tag{9}$$

Using equation 3 and 4, we are able to get the following expressions for velocity v and pressure P:

$$\begin{cases} v = \frac{\rho v}{\rho} = \frac{U(0)}{U(1)} \\ P = (\gamma-1)\rho \cdot \left(\frac{E}{\rho} - \frac{1}{2}v^2\right) = (\gamma-1)U(0)(\frac{U(2)}{U(0)} - \frac{1}{2}(\frac{U(1)}{U(0)})^2) \end{cases} \tag{10}$$

where $U(i)$ denotes the value in the ith row of matrix $\vec{U}$. Then we can calculate $\vec{F}$ using 10.

2.3 Result

current results:

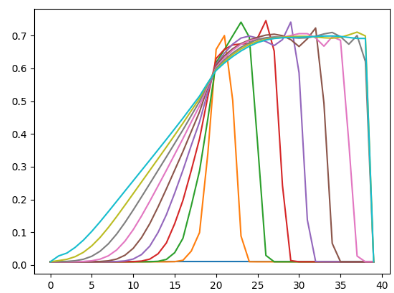

Figure 1: velocity distribution, horizontal axis position, vertical axis velocity. different line represents velocity distribution at different time. nt = 100, nx = 40

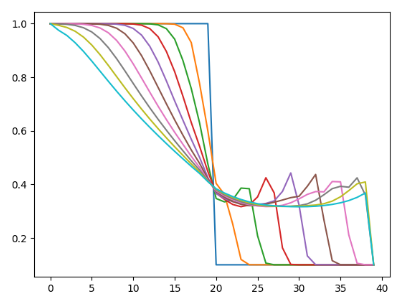

Figure 2: density distribution, horizontal axis position, vertical axis density. different line represents density distribution at different time. nt = 100, nx = 40

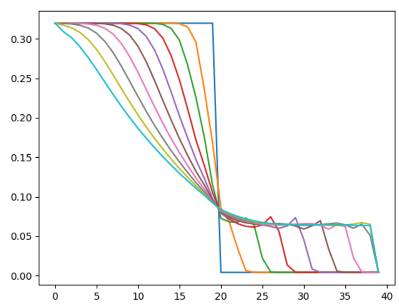

Figure 3: pressure distribution, horizontal axis position, vertical axis pressure. different line represents pressure distribution at different time. nt = 100, nx = 40