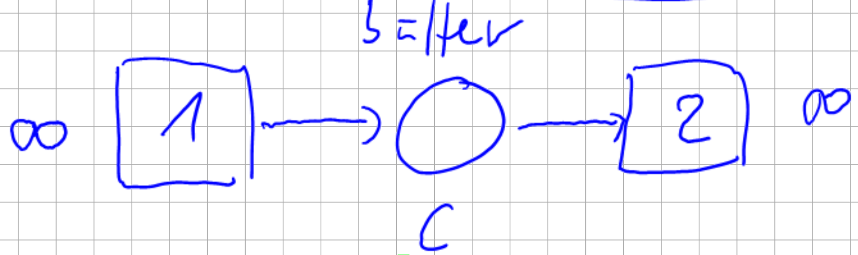
Two machine flow line with reliable machines making discrete parts



first machine never starved

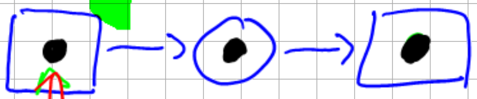
second " " blocked

first machine can be blocked

second " can be starved

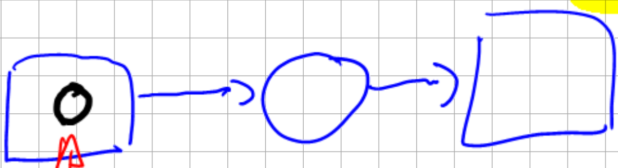
blocking after service (BAS)

C = 1



Part already processed by machine 1

⟹ machine 1 is blocked



Part not yet processed by machine 1

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machine 1 processing time $T_1 \sim exp(\mu_1)$

$$\Rightarrow E[T_1] = \frac{1}{\mu_1}$$

machine 2 processing time $T_2 \sim exp(\mu_2)$

$$\Rightarrow E\{T_2\} = \frac{1}{\mu_2}$$

We have two possible events

Event 1: Machine 1 completes processing of a part, event occurs with rate μ_1

Event 2: Machine 2 completes processing of a part, event occurs with rate μ_2

Analysis: KPI TH, Inventory ? (In expectation)

→ we need state probabilities

→ we need a state description!

→ How can we describe the state of the system at any moment in time???

- We cannot have a state with a non-empty buffer and a starved machine 2!

- We cannot have a state with a non-full buffer and a blocked machine 1!

↑ simultaneously!!

State description

$s = s(n)$ with $n = 0, \ldots, C+2$

number of parts that have already been processed by machine 1, but by machine 2 !

Example $C = 2$ (b = the capacity)

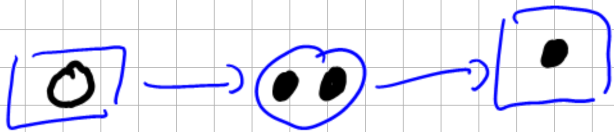
Case 1:

$n = C+2 = 4$

m1 is blocked

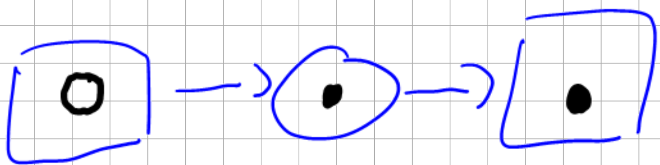
Case 2:

$n = 3$



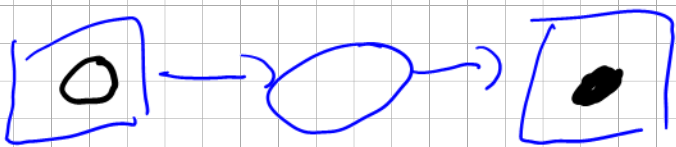
Case 3:

$n = 2$



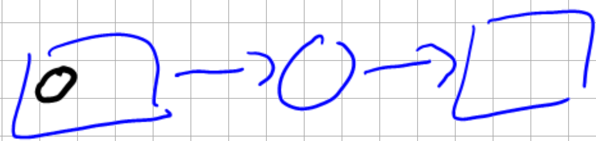
Case 4:

$n = 1$



Case 5:

$n = 0$



machine 2 is starved

Cont. with $C = 2 \Rightarrow n = 0, \dots, C+2 = N$

Transition diagram CTMC

$$\begin{tikzcd} 0 \arrow[r, bend left, "\mu_1"] & 1 \arrow[l, bend left, "\mu_2"] \arrow[r, bend left, "\mu_1"] & 2 \arrow[l, bend left, "\mu_2"] \arrow[r, bend left, "\mu_1"] & 3 \arrow[l, bend left, "\mu_2"] \arrow[r, bend left, "\mu_1"] & 4 \arrow[l, bend left, "\mu_2"] \end{tikzcd}$$

$$\begin{tikzcd} n \arrow[r, bend left, "\mu_1"] & n+1 \end{tikzcd}$$

$$\begin{tikzcd} n-1 & n \arrow[l, bend left, "\mu_2"] \end{tikzcd}$$

1 part in buffer, 1 part on machine 2

second machine is starved

2 parts in buffer, 1 part on machine 2

first machine is blocked

So for $C=2$, we have states $n = 0, \dots, 4$

We are interested in state probabilities

π_0, π_1, π_2, π_3 and π_4

Let us assume that we already know $\pi_0 \dots \pi_4$!!

Can we determine the throughput TH of the line in this case (i.e., knowing $\pi_0 \dots \pi_4$)?

TH_1 throughput through machine 1

TH_2 throughput through machine 2

Throughput is the long-term number of completed parts per time unit

Homework:

a) Determine TH_1 in terms of TH_2, in other words: How does TH_1 relate to TH_2?

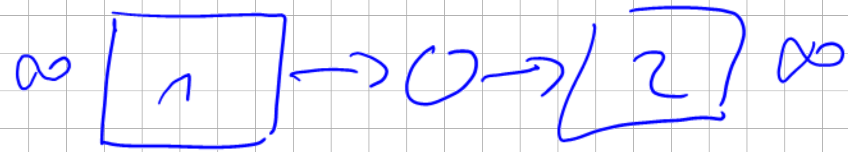
b) Give a formula to compute TH_1 in terms of $\Pi_0, \Pi_1, \dots, \Pi_4, \mu_1, \mu_2$, C as needed!!

c) Give a similar formula for TH_2 ...

d) Give a formula to determine the average number $\bar{n}$ of parts already processed by machine 1 <u>and</u> still in the system, i.e., not yet processed by machine 2!

(average inventory)

Homework



a) TH_1 in terms of TH_2 ?

In steady-state, system probabilities no longer change! Then, <u>in the long run</u>, we have

$$\boxed{TH_1 \overset{!!!}{=} TH_2}$$!

$\Rightarrow$ Conservation of flow COF

5)
$$TH_1 = \mu_1 \cdot \overbrace{\left(1 - \pi_{C+2}\right)}$$

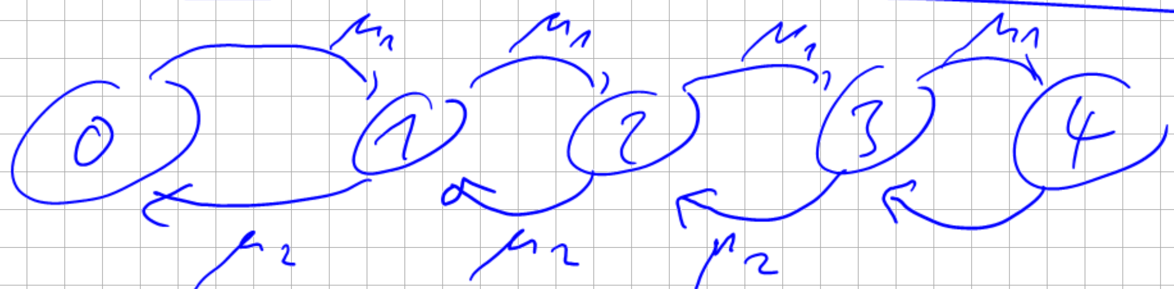
prob. of the extended buffer not being full, i.e., μ_1 not being blocked

$$= \mu_1 \cdot \sum_{n=0}^{C+1} \pi_n$$

✓

$C = 2$

$\Rightarrow N = C + 2 = 4$



c) $TH_2 = \mu_2 \cdot \underline{(1 - \pi_0)}$

L prob. of machine 2 not being starved

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d) Average number $\bar{n}$ (of parts already processed by the first machine, but not the second)

$$\bar{n} = \sum_{i=0}^{N=C+2} i \cdot \pi_i$$

$$\bar{n} = \sum_{n=0}^{N} n \cdot \pi_n$$

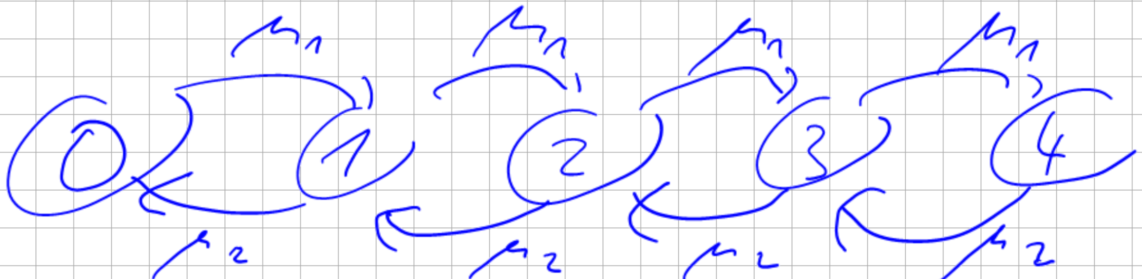
Assume $C = 2 \Rightarrow n = 0, \dots, C+2 = N = 4$

Generator matrix

$$
Q = \begin{array}{c} \\ \text{"0"} \\ \text{"1"} \\ \text{"2"} \\ \text{"3"} \\ \text{"4"} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{"0"} & \text{"1"} & \text{"2"} & \text{"3"} & \text{"4"} \end{array} \\
\begin{pmatrix}
-\mu_1 & \mu_1 & & & \\
\mu_2 & -(\mu_1+\mu_2) & \mu_1 & & \\
 & \mu_2 & -(\mu_1+\mu_2) & \mu_1 & \\
 & & \mu_2 & -(\mu_1+\mu_2) & \mu_1 \\
 & & & \mu_2 & -\mu_2
\end{pmatrix}
\end{array}
$$

We know

$\pi_0 Q = 0$



In addition, we have the normalization constraint

Here: $$\sum_{n=0}^{N=4} \pi_n = 1$$

Modified Q-matrix:

$$Q_{mod} = \begin{pmatrix} -\mu_1 & \mu_1 & & & 1 \\ \mu_2 & -(\mu_1+\mu_2) & \mu_1 & & 1 \\ & \mu_2 & -(\mu_1+\mu_2) & \mu_1 & 1 \\ & & \mu_2 & -(\mu_1+\mu_2) & 1 \\ & & & \mu_2 & 1 \end{pmatrix}$$

Modified right-hand side vector $n_{mod} = (0\ 0\ 0\ 0\ 1)$

$$\pi \cdot Q_{mod} = n_{mod}$$

We now have a system of lin. equations which we can solve for the state prob. vector π

$$\pi = n_{mod} \cdot Q_{mod}^{-1}$$

While Q does not have full rank and is not invertible,

Matrix Q_{mod} does have full rank and is invertible