University of Alberta
CMPUT 272 Winter 2020
Final Examination

Time allowed: **180 minutes**
(plus 15 minutes extra for uploading your files)

There are **11 questions** and a total of **86 marks**.

This is an OPEN BOOK exam.

You may refer to the textbook, course notes, eClass page, and the internet.

You must write the exam entirely on your own,
without any communication, hints, copying, or assistance from anyone else.

The work you submit must be your own.

Do not copy an answer from any source.

Do not share this document with anyone.

In addition to the usual properties of numbers, rules of inference, logical equivalences, definitions, and methods of proof, you may use the following facts in your proofs.
You might not need any of them.

$\sqrt{2}$ is irrational
the product of two odd numbers is odd
positive integer powers of odd numbers are odd
the product of two even numbers is even
positive integer powers of even numbers are even
the product of an even number and an odd number is even
the sum of two odd numbers is even
the sum of two even numbers is even
every number is even or odd but not both
the 7 properties of integer division listed on page 5 of Lecture 8 notes

Good luck!

Question 1 (8 marks) This question is about the following argument form.

$$\begin{array}{l} f \rightarrow z \\ z \rightarrow \sim o \\ a \rightarrow s \\ s \rightarrow \sim z \wedge o \\ \sim f \rightarrow a \wedge \sim o \\ \therefore \sim o \wedge f \end{array}$$

a) (3 marks) Explain how the truth table below shows that the argument form is valid. Be specific and point out the relevant rows and columns in the truth table.

	z	f	a	s	o	$f \rightarrow z$	$z \rightarrow \sim o$	$a \rightarrow s$	$s \rightarrow \sim z \wedge o$	$\sim f \rightarrow a \wedge \sim o$	$\sim o \wedge f$
1	F	F	F	F	F	T	T	T	T	F	F
2	F	F	F	F	T	T	T	T	T	F	F
3	F	F	F	T	F	T	T	T	T	F	F
4	F	F	F	T	T	T	T	T	T	F	F
5	F	F	T	F	F	T	T	F	T	T	F
6	F	F	T	F	T	T	T	F	T	F	F
7	F	F	T	T	F	T	T	T	F	T	F
8	F	F	T	T	T	T	T	T	T	F	F
9	F	T	F	F	F	F	T	T	T	T	T
10	F	T	F	F	T	F	T	T	T	T	F
11	F	T	F	T	F	F	T	T	F	T	T
12	F	T	F	T	T	F	T	T	T	T	F
13	F	T	T	F	F	F	T	F	T	T	T
14	F	T	T	F	T	F	T	F	T	T	F
15	F	T	T	T	F	F	T	T	F	T	T
16	F	T	T	T	T	F	T	T	T	T	F
17	T	F	F	F	F	T	T	T	T	F	F
18	T	F	F	F	T	T	F	T	T	F	F
19	T	F	F	T	F	T	T	T	F	F	F
20	T	F	F	T	T	T	F	T	F	F	F
21	T	F	T	F	F	T	T	F	T	T	F
22	T	F	T	F	T	T	F	F	T	F	F
23	T	F	T	T	F	T	T	T	F	T	F
24	T	F	T	T	T	T	F	T	F	F	F
25	T	T	F	F	F	T	T	T	T	T	T
26	T	T	F	F	T	T	F	T	T	T	F
27	T	T	F	T	F	T	T	T	F	T	T
28	T	T	F	T	T	T	F	T	F	T	F
29	T	T	T	F	F	T	T	F	T	T	T
30	T	T	T	F	T	T	F	F	T	T	F
31	T	T	T	T	F	T	T	T	F	T	T
32	T	T	T	T	T	T	F	T	F	T	F

b) (1 mark) Give a statement form containing all of the variables z, f, a, s, and o, that is logically implied by the premises.

c) (4 marks) Complete the formal proof below by filling in the empty space and the conclusion's justification.. Give a reason for each step: a logical equivalence or a rule of inference as well as line numbers of previous statements used in the step. Be careful to indent correctly.

1.	$f \to z$	Premise
2.	$z \to \sim o$	Premise
3.	$a \to s$	Premise
4.	$s \to \sim z \wedge o$	Premise
5.	$\sim f \to a \wedge \sim o$	Premise
6.	$\quad \sim f$	Assume for Hypothetical Reasoning
7.	$\quad a \wedge \sim o$	5, 6, Modus Ponens
8.	$\quad a$	7, Specialization
9.	$\quad s$	3, 8, Modus Ponens
10.	$\quad \sim z \wedge o$	4, 9, Modus Ponens
11.	$\quad o$	10, Specialization
12.	$\quad \sim o$	7, Specialization
13.	$\quad o \wedge \sim o$	11, 12, Conjunction
14.	$\quad \mathbf{c}$	13, Negation law

$\therefore \sim o \wedge f$ ________________________

Question 2 (8 marks) In the statements below,

the domain of all variables is $\mathbb{R}$, the set of real numbers.

For each statement, indicate whether it is true or false. If you can justify your answer with an example or a counterexample, then do so. Otherwise, write a sentence of explanation. Note that, in an example or counterexample, you must identify specific elements of the domain.

a) (2 marks) $\forall x, \forall y, \exists z, (x \leq z \leq y)$

b) (2 marks) $\exists x, \forall y, (y \neq 0 \rightarrow xy = 1)$

c) (2 marks) $\exists x, \exists y, ((y \neq 0) \wedge (\sqrt{2} = x/y))$

d) (2 marks) $\forall x, \exists y, ((x + y = 2) \wedge (2x - y = 1))$

Question 3 (14 marks) Let $P(n)$ be a predicate on integers. In each part below, indicate (Yes or No) whether the information provided guarantees that $P(n)$ is true for all integers $n \geq 1$. Explain. Be specific. Mention rules of inference and give counterexamples.

a) (2 marks) $P(n)$ is true for an arbitrary positive integer n.

b) (2 marks) $P(1), P(2), \ldots, P(10^9)$ are all true.

c) (2 marks) $P(1)$ is true and, for all integers $k \geq 2$, if $P(k)$ is true then $P(k+1)$ is true.

d) (2 marks) $P(1), P(2), \ldots, P(9)$ are true and, for an arbitrary integer $k \geq 9$, if $P(i)$ is true for all i from 1 through k then $P(k+1)$ is true.

e) (2 marks) $P(1), P(2), P(3)$ are true and, for all integers $k \geq 1$, if $P(k)$ is true then $P(3k)$ is true.

f) (2 marks) For all $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true.

g) (2 marks) For all integers $k \geq 0$, if $P(i)$ is true for all i from 1 through k then $P(k+1)$ is true.

Question 4 (6 marks) Prove the following statement.

$$\text{For all } n \in \mathbb{Z}, \quad \text{if } n \text{ is odd then } 8 \mid (n^2 - 1).$$

Question 5 (10 marks) Suppose that $a_2, a_3, a_4, \ldots$ is a sequence defined as follows:

$$\begin{aligned} a_2 &= \frac{1}{2}, \\ a_n &= a_{n-1} + \frac{1}{n} \quad \text{for all integers } n \geq 3. \end{aligned}$$

Use mathematical induction to prove:

$$\text{for all integers } n \geq 2, \quad \sum_{i=2}^{n} a_i \; = \; (n+1)\, a_n - (n-1).$$

- Clearly label the Basis step and the Inductive Step.
- Identify the inductive hypothesis and label it as such.
- Indicate each place where you use the inductive hypothesis.

Question 6 (4 marks) Prove the following statement, where sets A, B, and C are subsets of a universal set U.

For all sets A, B, and C, if $A \subseteq B$ and $B \cap C = \emptyset$ then $A \cap C = \emptyset$.

Question 7 (10 marks) Let U be a nonempty set. For $A \subseteq U$, define the function

$$f_A : U \mapsto \{0, 1\}$$

as follows: For all $u \in U$,

$$f_A(u) = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{if } u \notin A \end{cases}$$

For example, if $U = \{x, y\}$ and $A = \{x\}$ then $f_A(x) = 1$ and $f_A(y) = 0$.

a) (2 marks) Is f_A one-to-one for all nonempty U and all $A \subseteq U$? Prove or give a counterexample.

b) (2 marks) Is f_A onto for all nonempty U and all $A \subseteq U$? Prove or give a counterexample.

c) (2 marks) Describe how A is related to the inverse image of each element of the codomain of f_A.

d) (4 marks) Show that the following holds for all nonempty sets U, all subsets A and B of U, and all $u \in U$:

$$f_{A \cap B}(u) = f_A(u) \cdot f_B(u)$$

Question 8 (10 marks) Let R be a relation on a set $A = \{1, 2, 3, 4, 5, 6, 7\}$ where $|A| = 7$. The following facts are known about R:

$$(5, 2) \in R \qquad (6, 7) \in R \qquad (3, 2) \in R \qquad (7, 5) \in R \qquad (3, 2) \in R \qquad (2, 6) \in R$$

There may be other ordered pairs in R as well.

a) (6 marks) Prove that R is not a partial order relation.

b) (4 marks) Suppose that R is an equivalence relation that contains as a subset the ordered pairs listed above. There are a few possibilities for R, depending upon which ordered pairs appear in R (in addition to those listed above). List the partitions of A that correspond to all such equivalence relations. Justify your answer.

Question 9 (4 marks) How many of the numbers from 1 through 99,999 contain

exactly two 2's and exactly one 7?

Examples: 07202 and 12372 contain exactly two 2's and exactly one 7;
22770, 56789, and 31415 do not.

Your answer should be an expression involving combinations and/or factorials. You do not have to compute the numerical answer. Justify your answer.

Question 10 (6 marks) You have b identical blue marbles and w identical white marbles, where $b < w$. Using combinations and/or factorials, give expressions for the number of linear arrangements of the $w + b$ marbles under each of the following conditions. Justify your answers.

a) (2 marks) The total number (with no restrictions).

b) (2 marks) No two blue marbles are next to each other.

c) (2 marks) No two blue marbles are next to each other, and if two white marbles are next to each other then they are both before the first blue marble or both after the last blue marble. For example, if $b = 5$ and $w = 10$, then two such arrangements would be:

$$wwbwbwbwbwbwwww \qquad \text{and} \qquad wwwwwwbwbwbwbwb$$

Question 11 (6 marks) Let A be a set of nine positive integers, each of which is less than or equal to 30. That is, $A \subseteq \{1, 2, 3, \ldots, 30\}$ with $|A| = 9$.

Use the pigeonhole principle to prove that A contains two different subsets of size 3 whose elements when added up give the same sum.

For example, if $A = \{4, 8, 9, 10, 17, 21, 22, 25, 30\}$, then the following 3-element subsets of A have the same sum: $\{4, 10, 22\}$, $\{9, 10, 17\}$.