Assign 1

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Thought Question 1

Chapter 2
Subsection 2.1

Question

I was thinking about Random variables in different Probability Spaces and thought of this question. Let's say suppose we have X which is a random variable defined on (Ω_0, A_0, P_0).Do there exist random variables

$$\{U_i | i \in \mathbb{N}\}$$

in specified probability spaces

$$T_i = \{\Omega_i, A_i, P_i\} \ i \in \mathbb{N}$$

which have the same distribution as X ie

$$U_i \text{ is a random variable defined on } T_i$$

.

Solution

My question deals with only the existence of such random variables and not actually finding it. When I say they are equal in distribution I mean X and each U_i have the same probability distribution.

$$P_X := P_0 \circ X^{-1} \wedge \forall U_i : P_{U_i} := P_i \circ U_i^{-1} \text{on } (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ are the same}$$

But now this generalizes this question which I am not sure. The thing is I know the answer should only depend on the distribution of X and the specified T_i and not on (Ω_0, A_0, P_0).I feel that larger the T_i increases the chance of this existence or maybe consider

$$T_i = ([0, 1], \mathcal{B}([0, 1]), \text{Leb}) \exists i \in \mathbb{N}$$

This definitely should be able to generate all distributions or even realize them on $(\mathbb{R}, \mathcal{B}(\mathbb{R})$. So probably this could work.

Thought Question 2

Chapter 2
Subsection 2.4

Question

When do we have the fact that $E[X] = (E[X])^{-1}$? What significance does the inverse expectation indicate?

Solution

First of all let us deal with this simple case

$$\mathbb{E}[f(X)] = \begin{cases} \sum_{x\in\mathcal{X}} f(x)p(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} f(x)p(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

Now let $f(X) = \frac{1}{X}$, Then

$$\mathbb{E}[\frac{1}{X}] = \begin{cases} \sum_{x\in\mathcal{X}} \frac{p(x)}{f(x)} & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} \frac{p(x)}{f(x)}dx & \text{if } X \text{ is continuous} \end{cases}$$

and let us put aside the cases when $E(X) = 0$

$$(\mathbb{E}[X])^{-1} = \begin{cases} \frac{1}{\sum_{x\in\mathcal{X}} xp(x)} & \text{if } X \text{ is discrete} \\ \frac{1}{\int_{\mathcal{X}} xp(x)dx} & \text{if } X \text{ is continuous} \end{cases}$$

We need to compare both these equations which yield in

$$(\mathbb{E}[X])^{-1} = \mathbb{E}[\frac{1}{X}]$$

$$\begin{cases} \frac{1}{\sum_{x\in\mathcal{X}} xp(x)} & \text{if } X \text{ is discrete} \\ \frac{1}{\int_{\mathcal{X}} f(x)xdx} & \text{if } X \text{ is continuous} \end{cases} = \begin{cases} \sum_{x\in\mathcal{X}} \frac{p(x)}{f(x)} & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} \frac{p(x)}{f(x)}dx & \text{if } X \text{ is continuous} \end{cases}$$

Which leads to say that, basically, $E(1/X) \neq \frac{1}{E(X)}$ since the inverse of the integral/sum is not the same as integral/sum of inverses Unless this is a constant equation and equates to 1.

Another reason why this usually wont happen is due to the fact ; let us consider the covariance when (assuming we are dealing with a positive variable and vice versa inequalities change when it is negative) $\text{Cov}[X,Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}(Y)] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

Then

$$\text{Cov}[X, \frac{1}{X}] = \mathbb{E}[X\frac{1}{X}] - \mathbb{E}[X]\mathbb{E}[\frac{1}{X}]$$

Now since these both are inverses of each other ie X and $1/X$ therefore the covariance should be less than or equal to 0.

This implies

=

$$E(X \cdot 1/X) - E(X)E(1/X) \leq 0$$

=

$$1 - E(X)E(1/X) \leq 0$$

=

$$E(X)E(1/X) \geq 1$$

$$E(1/X) \geq (E(X))^{-1}$$

So it probably dominates and hence usually cant happen unless its almost constant. I am not sure now where to proceed but I feel like this should be correct unless we take simple examples and construct it way by way.

About the significance I am not really sure to be honest and dont really see much of a significance of the inverse expectation.