1. Prerequisites

In this chapter we briefly review some of the topics you should be familiar with. There are also some that are more of a "nice to know" than "need to know" nature, and not strictly speaking part of this course or required knowledge.

Many of the subtle points raised in the first four sections are merely for the interested. They may be hard to swallow and follow. They won't be required material.

But any mathematician should be aware that mathematics is not as "fool proof" as it is often made out to be. And certainties (or "truths") are always hard to come by regardless of the field of study, even in mathematics.

What you should **not** do is let these sections distract you and make you feel insecure about what is "true" or "false." Follow your instinct but be aware that nothing is ultimately "obvious."

1.1 Logic

Logic is at once a part of mathematics and not part of it. What is meant by that, and is that not a contradiction?

Of course, logic is a fundamental part of the language of mathematics. But on the other hand, if logic was entirely a part of mathematics, it would be subject to the rules of mathematics. What would those rules be, if not logic? This is not a precise statement, of course, but gives you some indication where problems might arise.

Think about the following question:

What does it mean for a statement to be true?

"Truth" is a concept that is very hard if not impossible to define[1].

Of course, we all have intuitive notions of truth. For example, few would argue that the statement "If x is real number then x is not a real number" is true. (As an aside, this **would** be a true statement if there are no real numbers.)

However, to prove logically a statement of the form "If A[2] is true then its negation is not also true," one must make assumptions. Usually the assumption is that for all "statements" A one has that

[1] I would propose it is impossible to define as any sort of definition would require some sort of logical rules, which in themselves probably are meaningless without a concept of "true" or "false." But this is a philosophical and not a mathematical argument.

[2] Here A is a substitute for any "statement."

“A and $\neg A$ is false”

Symbolically, $\neg(A \wedge \neg A)$ is always true. Here $\neg A$ is shorthand for “not A”.

This is obvious (so please prove it[3]). This is commonly referred to as the Law of No Contradiction.

There is ultimately no way around the fact that we need some ground rules that we must treat as obvious and established, and which we cannot prove. The above example is one of them.

In the early 20th century there was a deep and long discussion about what these rules should be, and to this day there is no universally accepted “rule set.”

One of the main controversies was the question whether it is “true” that for any statement A the statement

“A is true or $\neg A$ is true”

is true. Symbolically, is $A \vee \neg A$ always true?

Think about that: the discussions revolved around whether a statement of the form “x is a real number or x is not a real number” is always true.

The postulate (or “axiom”) that for all statements A we always have A holds or $\neg A$ holds is commonly known as the **Law of Excluded Middle** (there is nothing between true or false in a sense) or **Tertium non Datur** (which is Latin for “There is no Third Option”).

The question revolved around whether if $\neg\neg A$ holds, does that always force A to hold. To show must you assume the Law of Excluded Middle.

So called “Intuitionists” reject the LEM. The objections are often grounded in the fact that LEM allows non-constructive proofs (that is, proofs where the existence of an object can be established without giving any concrete recipe of how to obtain the object from given data).

These are profound logical questions, and way above our head (mine included) in this course.

Most of modern mathematics has accepted the Law of Excluded Middle.

No proof by contradiction would be valid without it: A proof by contradiction rests on the fact that, if we show that the negative of a statement is false, then the statement is true. This needs that it is true that for every statement, the statement itself or its negative is true.

In fact, a proof by contradiction of a statement A usually establishes $\neg A \rightarrow A$, which is a contradiction if $\neg A$ is true. Thus, $\neg A$ must be false, that is $\neg\neg A$ is true. The leap of faith is then that this means A is true

[3] Unfortunately, you very likely won’t be able to prove it. This is usually taken as an axiomatic statement that needs no proof.

because “obviously” $\neg\neg A = A$. To prove this latter fact, one needs the Law of Excluded Middle. Adherents of intuitionist logic assert that $\neg\neg A \rightarrow A$ is not a valid “inference” (logical step). They usually do not object to the inference $A \rightarrow \neg\neg A$ (which is essentially the Law of No Contradiction).

If that confuses you, you are not alone. We will of course assume the LEM.

To illustrate the issue somewhat in intuitive terms: Assuming someone would like to prove that aliens exist. Which “proof” would convince you more: a theoretical discussion that the non-existence of aliens leads to a contradiction, and hence aliens don’t “not exist”; or someone showing you an actual alien?

This general problem (to prove existence of something by showing that its non-existence leads to a contradiction) was very controversial at the time.

We also haven’t touched upon the question what a “statement” is. This again seems to be something undefinable as any definition of a “statement” seems to be itself a “statement.”

1.2 Proofs

What is a proof? Again, this is both a mathematical and philosophical question.

Logically, a proof of a statement $A = A_n$ is a sequence of logical statements of the form $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \cdots \rightarrow A_n$ such that if A_i is true this forces A_{i+1} to be true, and A_1 is true.

So $A_i \rightarrow A_{i+1}$ must be what is called a logical implication (that is, if A_i is true then also A_{i+1} is true).

Again, there are problems with this: what are i and $i + 1$? Can we meaningfully talk about these notions without needing “proofs”? Is this definition circular?

There are (philosophical) solutions (or not, depending on where you stand) to these conundrums, and in mathematical day to day business this rarely causes any issues. We all usually know what is meant.

One could regard proofs (and therefore any statements) simply as a trial before a “court of law” where the proponent (prosecutor) tries to establish the fact, and the opponent (defence) tries to argue the particular statement is false. Both parties are subject to certain rules, and a statement is true if the proponent always wins.

Regardless, in modern mathematics, we usually do not worry too much about such foundational questions. We have come to realize that there is likely no solution that satisfies all, and we must live with the uncertainty that comes with it.

1.3 Axioms

Nowadays, in mathematics we usually do not care about what mathematical objects “are” (as that is largely a philosophical question), but rather describe what properties they have. Often these properties are referred to as “axioms” and deemed true by default or definition.

For example, the Peano Axioms are a way to capture fundamental properties of natural numbers (but not all).

For the “user” there are two classes of axioms. The first kind are statements that are treated as self-evident truths, such as the Law of Excluded Middle discussed above.

The second kind are axioms by definition: a function has certain properties by definition, and its properties don't have to have any intrinsic meaning or be particularly self-evident.

This is not a logical distinction, by the way.

Below, we will briefly discuss set theory. In the early 20th century it was also discovered that the "naïve set theory" as proposed and fundamentally furthered by people like Cantor lead to contradictions.

As a result, there was a movement to "axiomatize" mathematics: base everything on a few "first principles."

This applied to the most fundamental mathematical theories of all: set theory. There was hope that one could prove that the axioms of set theory would be consistent (meaning you cannot deduce a logical contradiction from them).

At the heart of this thought is that "truth" is closely related to "provable."

This is a natural thought: if a statement is "true" there should be a way to prove that it actually is.

This approach was supremely successful. However, as Goedel showed in the 1930s, it is also in vain.

Goedel proved in his famous 1st Incompleteness Theorem that any system of axioms that is sufficiently complex (for example complex enough that basic arithmetic of natural numbers can be defined and performed) will allow statements that are neither probable nor disprovable, using only the axioms of the system (and predetermined rules of logic).

Goedel later proved as well that no sufficiently complex system of axioms can prove its own consistency. Here that means that there is no proof using only the axioms of the system (and predermined axioms of logic) that shows that the axioms are not contradictory (meaning that no contradiction could be deduced from them).

For us, these things make little difference. It is mentioned to show that mathematics is far from being without controversies.

Often the first incompleteness theorem is paraphrased as "there are statements that are true but not provable." That is slightly misleading as there is no way to know a statement is "true" without being able to prove it[4]. What is meant is that there are systems that satisfy the axioms where the statement is true, but the axioms are not enough to prove it, and therefore there are other systems where the statement is false.

Think about the Peano Axioms minus the Principle of Induction. There are probably sets that satisfy the modified system, but do not satisfy the POI, and there are sets that are satisfying all of them like $\mathbb{N}$. If that is true, then the POI cannot be proved from the other axioms. It is an "undecidable" statement, but "true" in some instances (like that of $\mathbb{N}$). It can also be shown that the Peano Axioms do not characterize $\mathbb{N}$ uniquely.

[4] Of course, in principle, there could be a proof stating that "one of the following two statements A or B is true" and there might not be a proof of either A or B. But this is not what is meant, as Goedel specified a specific statement that is not provable.

Historically, axioms appeared in Euclidean geometry: "Two (distinct) points define a line." It was later (much later) shown that these axioms do not rely on any intrinsic meaning of the words "line" or "point."

1.4 Sets

Set theory is a foundational topic of modern mathematics. From a purely logical viewpoint **all** mathematical objects (at least those that we encounter in this course) are *sets* (yes, in foundational mathematics, an integer is a set), or at least *classes* (a *class* is a naïve version of a set; see below).

We will use so called *Naïve Set Theory* throughout. That is, for us a **set** is a *collection* of mathematical objects[5]. That means if X is a set and x is a mathematical object, then one and only one of the following two statements is true:

- $x \in X$ ("x is an element of X, or belongs to the collection X"); or
- $x \notin X$ ("x is not an element of X").

For example, ½ ∈ $\mathbb{R}$, but ½ ∉ $\mathbb{Z}$.

If X, Y are sets, we can form $X \cup Y$, the **union** of X, Y, which is the set containing precisely the elements of both X and Y. Similarly, we can form the **intersection** $X \cap Y$ of elements that belong to both.

1.4.1 Naïve set theory

Naïve set theory goes back to Georg Cantor who first established a well-founded theory of cardinality of sets. He showed, for example, that the reals form an uncountable set.

The set theory as used by Cantor works for most practical purposes. However, if taken to the logical extreme, it does lead to contradictions.

As often is true in logic, self-reference is a problem.

So, if sets are collections of arbitrary mathematical objects, then surely there is a set whose elements consists of all sets (as these are undeniably mathematical objects).

Russell's paradox

Let S be the set containing all sets as elements. Let $T \subseteq S$ be the subset formed as follows

$$T = \{A \in S \mid A \notin A\}$$

Is $T \in T$?

T is clearly a set. So if $T \in T$, then by the definition of T, $T \notin T$, a contradiction.

Thus, we must conclude that $T \notin T$. But then, as T is a set, by definition of T, we also find $T \in T$, again a contradiction.

This shows that naïve set theory can become problematic when dealing with large or self-referencing sets.

[5] This is obviously a circular definition. We just chose to define "set" by using a logical synonym "collection," which begs the question, what is a "collection?" This is meant when we said that we do not care what objects "are" rather than what properties they have.

In the case that all sets in question are part or elements of a large (often unmentioned) set, there is no problem.

In modern mathematics this problem has been solved by introducing axiomatic set theory (see below). The role of naïve sets is then played by so called "classes." A class is a set if and only if it is an element of a class.

1.4.2 Axiomatic set theory

To avoid paradoxes such as above, in modern mathematics, set theory is governed by axioms. They are quite formal and beyond the scope of this class. The usually accepted system of axioms is the Zermelo-Fraenkel system of axioms. It is abbreviated ZF, or ZFC if the Axiom of Choice (see below) is included.

One of these axioms stands out, as it was -just like the Law of Excluded Middle- the source of great controversy.

At the heart of the problem with the Law of Excluded Middle is that existence proofs by contradiction don't tell you much about the object that exists.

For example, you might be able to prove that the nonexistence of the solution of an equation $f(x) = 0$ leads to a contradiction. You then know that a solution exists. But that may not help you to find that solution.

Similarly, the Axiom of Choice (AoC) in set theory has led to complaints. The AoC is simple: let X be any set whose elements are *nonempty* sets. Then X admits a **choice function** $C: X \to \bigcup_{A\in X} A$ such that $\forall A \in X: C(A) \in A$. In other words, one can "choose" for each of the sets belonging to X one element.

You have encountered many such applications: For example, when we construct many sequences we argue as follows: "for each $n \in \mathbb{N}$, pick x_n such that …": Consider a nonempty bounded set $A \subseteq \mathbb{R}$ and we want a sequence $a_n \to \sup A$. We then say, "For $n \in \mathbb{N}$ let $a_n \in A$ be any element with $|a_n - \sup A| < \frac{1}{n}$." If $\sup A \notin A$, such an a_n is never unique and the existence of a_n needs the AoC. For that let $A_n \coloneqq \{ x \in A \mid |x - \sup A| < \frac{1}{n}\}$. Then A_n is never empty. Let $X = \{ A_n \mid n \in \mathbb{N}\}$. Then a choice function C has the property that $x_n \coloneqq C(A_n) \in A_n$ as desired. Unfortunately, the AoC tells us nothing about how to explicitly obtain such a C.

To put it into concrete terms, a proof using the AoC might tell you that an equation of the form $f(x) = 0$ has a solution. But it may not tell you anything about how to find x.

1.5 Sequences and limits

1.5.1 Definition and properties

A **sequence** of real numbers is a function[6] $a: \mathbb{N} \to \mathbb{R}$. We usually write a_n instead of $a(n)$, and often simply write a_n for the entire sequence as well. Sometimes it is convenient to have a sequence start at any integer $k \in \mathbb{Z}$, the most often used case being $k = 0$.

A sequence is called **improper** if some (or all) of its elements are $\pm\infty$.

[6] Functions are defined precisely below.

Here $\pm\infty$ should be understood as elements of the **extended real number system** $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with the "obvious" rules for all $c \in \mathbb{R}$:

1. $c + \infty = \infty$
2. $c - \infty = -\infty$
3. $c \cdot \infty = (-c)(-\infty) = \infty$ if $c > 0$
4. $c \cdot \infty = (-c)(-\infty) = -\infty$ if $c < 0$
5. $\infty \cdot \infty == \infty + \infty = \infty$
6. $\infty \cdot (-\infty) = -\infty$

Both $0 \cdot (\pm\infty)$ and $\infty + (-\infty)$ are undefined. There are further rules, involving associativity and distributivity, but really, this is just notation.

By convention, $-\infty < c < \infty$ for all real numbers c.

Definition (Limit)
Let a_n be a sequence. Then $L \in \mathbb{R}$ is called the **limit** of the sequence, $\lim\limits_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon\ \forall n > N$.
This 1obviously also makes sense in case the sequence "starts" at any integer. If L exists we say a_n is **convergent**, and **divergent** otherwise.
We say $\lim\limits_{n\to\infty} a_n = \infty$, if for all $M > 0$ there is $N \in \mathbb{N}$ such that $a_n > M$ for all $n > N$, and $\lim\limits_{n\to\infty} a_n = -\infty$, if for all $M > 0$ there is $N \in \mathbb{N}$ such that M for all $n > N$.
Such a sequence is said to **diverge** to $\pm\infty$, and $\pm\infty$ is then referred to as an **improper** limit of the sequence. EOD.

Limits are unique if they exist.

Recall the main properties of limits:

Proposition (Properties of limits)
In the following let a_n, b_n sequences with limit L, M (allowed to be $\pm\infty$ unless otherwise specified).

1. If $c, d \in \mathbb{R}$, and cL, dM are not both infinite of opposing signs, then $ca_n + db_n \to cL + dM$.
2. If LM is not of the (undefined) type $0 \times \pm\infty$, $a_n b_n \to LM$.
3. If $M \neq 0$, and not both M, N are infinite, then $\frac{a_n}{b_n} \to \frac{L}{M}$ (this sequence is defined for large enough n).
4. If $L = \pm\infty$, then $\frac{1}{a_n} \to 0$ ($\frac{1}{a_n}$ is defined for large enough n).

EOP.

Lemma
If $a_n \leq b_n$ for all n and $a_n \to L, b_n \to M$. Then $L \leq M$. EOL.

Lemma (Squeeze principle)
Let $a_n \leq b_n \leq c_n$(for all n) be sequences and $a_n, c_n \to L$. Then also $b_n \to L$. EOL.

A sequence a_n is called **bounded** if there is $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all n.

Lemma

Any monotone bounded sequence is convergent. EOL.

For any sequence a_n one defines new improper sequences $S_k \coloneqq \sup_{n \geq k}\{a_n\}$ and $I_k = \inf_{n \geq k}\{a_n\}$ (note the sequences may be constant $\pm\infty$).

Both, S_k, I_k are monotone: S_k is decreasing and I_k is increasing.

Exercise
Show a_n is bounded if and only if S_k, I_k converge.
Show a_n is convergent if and only if S_k, I_k converge and have the same limit.

As monotone sequences, S_k, I_k have limits and we define

$$\limsup_{n\to\infty} a_n = \lim_{k\to\infty} S_k$$
$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} I_k$$

The first is called the **limit superior** of a_n, and the second is called the **limit inferior** of a_n. It always exists but may be infinite. EOE.

Lemma
Let a_n be a sequence. Then $\lim_{n\to\infty} a_n = L$ if and only if $\limsup_{n\to\infty} a_n = L = \liminf_{n\to\infty} a_n$. EOL.

Proof. Let S_k, I_k be defined as above. Then $I_k \leq a_k \leq S_k$ for all k. So, if $S_k, I_k \to L$, we have $a_k \to L$ by the Squeeze Principle.

Conversely, if $a_k \to L$, then $S_k \to L$, as for every $\varepsilon > 0$ all but finitely many elements of the sequence satisfy $a_n \leq L + \varepsilon$, which means that $L + \varepsilon \geq S_n$ for all sufficiently large n. We also must have that $L \leq S_k$ for all k: if for some k we have $L > S_k$, then $a_n \leq S_k < L$ for all $n \geq k$ and we cannot have $a_n \to L$. The argument for $I_k \to L$ is similar. QED.

Exercise
Let a_n, b_n be sequences.

1. Show that $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n)$ with equality if b_n is convergent.
2. If $a_n, b_n > 0$ and neither is a zero sequence, show that $\limsup_{n\to\infty} a_n b_n \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right)$.
3. If a_n, b_n are sequences with b_n convergent with limit $L \neq 0$, show that
$$\limsup_{n\to\infty} a_n b_n = L \ \limsup_{n\to\infty} a_n$$

Analogous statements hold for the limit inferior. EOE.

1.5.2 Subsequences

Definition
If a_n is a sequence, then a **subsequence** is a sequence b_k of the form $b_k = a_{n_k}$ such that $n_1 < n_2 < \cdots$. EOD.

Theorem (Bolzano-Weierstrass)
Any bounded sequence has a convergent subsequence. EOT.

Proof. Let a_n be a bounded sequence. Let M be its supremum. Let $n_1 = 1$ and for each $k \in \mathbb{N}$, $k > 1$ let recursively $n_k = \min\left\{m \mid m > n_{k-1};\ a_m > M - \frac{1}{k}\right\}$. Then $a_{n_k} \to M$. QED.

1.5.3 Some topological notions involving limits

When discussing functions defined on subsets of $\mathbb{R}$ most of the time the domain of a function will be an interval. However, often we will take limits of sequences in the domain and then it may become an important question whether this limit is again an element of the domain.

To simplify the discussion, we introduce the following notions:

Definition.
Let $S \subseteq \mathbb{R}$ be a subset.

1. A point $x \in \mathbb{R}$ is an **accumulation point** of S, if there is a sequence $a_n \in S$ such that $a_n \neq x \forall n$ and $a_n \to x$ as $n \to \infty$. If a_n is a sequence, an accumulation point of the sequence is an accumulation point of $\{a_n \mid n \in \mathbb{N}\}$.
2. A point $x \in \mathbb{R}$ is a **boundary point** of S, if for all $\varepsilon > 0$, the sets $(x - \varepsilon, x + \varepsilon) \cap S$ and $(x - \varepsilon, x + \varepsilon) \cap (\mathbb{R} \setminus S)$ are nonempty.
3. An **interior** o **inner** point of S is an element $x \in S$ such that there is $\varepsilon > 0$ for which $(x - \varepsilon, x + \varepsilon) \subseteq S$.
4. The set of all interior points is denoted $\mathring{S}$, and called the **interior** of S; set of all boundary points is denoted ∂S and called the **boundary** of S.
5. S is called **closed** if it contains all its accumulation points.
6. S is called **open** if it is equal to its interior $\mathring{S}$.
7. A subset $T \subseteq S$ is called **relative open**, if it is the intersection of an open set with S.
8. A subset $A \subseteq \mathbb{R}$ is called **discrete**, if for every $a \in A$ there is $\varepsilon > 0$ (maybe depending on a) such that $(a - \varepsilon, a + \varepsilon) \cap A = \{a\}$.

EOD.

Convince yourself that an open interval is open, and a closed interval is closed.

The boundary, or the set of interior points may be empty. For example, $\mathring{\mathbb{N}} = \emptyset = \partial\mathbb{R}$, and $\partial(a, b) = \{a, b\}$. On the other hand, $\partial\mathbb{Q} = \mathbb{R}$.

Exercise.

1. For $a < b$, show that $\partial(a, b) = \partial[a, b] = \{a, b\}$.
2. Show that $\mathbb{N}$ is closed.
3. Show that $\mathbb{Q}$ is neither open nor closed.
4. Show that $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is closed and 0 is its only accumulation point.
5. Show that $[a, b)$ is relative open in any $[a, c]$ where $c \geq b$.
6. Show that any discrete set is closed.
7. Show that $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is discrete.
8. Show that $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ is not discrete.

9. For a nonempty bounded set $A \subseteq \mathbb{R}$, its supremum and infimum always are elements of the boundary.
10. Show that any discrete set of $\mathbb{R}$ is at most countably infinite.
(*Hint*: Let S be a discrete set, and $s \in S$. For s, let I_s be an open interval such that $I_s \cap S = \{s\}$. Now pick a rational point $r_s \in I_s \cap \mathbb{Q}$. Recall that $\mathbb{Q}$ is countably infinite.)

EOE.

1.5.4 Limits of functions

1.5.5 Functions

Definition
A function $f: X \to Y$ is a *rule* that assigns to each $x \in X$ and element $f(x) \in Y$.
The set X is called the **domain** of f and Y is called the **codomain**.
The **range** or **image** of f is the set of its values, i.e. the set $f(X) = \{ f(x) \mid x \in X\}$. EOD.

What is a "rule"? Formally, a function $f: X \to Y$ is a subset $\Gamma \subseteq X \times Y$ such that

1. $\forall x \in X \, \exists y \in Y: (x, y) \in \Gamma$
2. $\forall x \in X$: if $(x, y), (x, y') \in \Gamma$ then $y = y'$

In other words, for each x there is a unique $y = f(x)$ such that $\big(x, f(x)\big) \in \Gamma$. Γ is called the **graph** of f.

The vast majority of functions we will be discussing are functions where the codomain is a subset of $\mathbb{R}$. Such functions are called "real valued."

Note that the domain and codomain are part of the definition of a function. For example, the functions $f: \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by $f(x) = x^2$ is different from the functions $g: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ also defined by $x \mapsto x^2$. Note that f is the **restriction** of g to the set of non-negative real numbers. Unfortunately, we often use the same label for the restricted function. **We often *do* treat f and h interchangeably as they only differ by the codomain**. This is usually only an issue when invertibility of functions is concerned (f is, whereas h is not).

It is a common bad practice to write $f(x)$ for both the *function* and the *value* of the function for some $x \in \mathbb{R}$. You must be careful no know which one is meant. For example, if we say $f(x) = \sqrt{x}$ is continuous, we cannot mean the *value* as that would be nonsensical. But what about "suppose $f(x) > 0$"? Do we mean $f(x) > 0$ for all x or for a given x?

This is occasionally confusing, but rarely poses serious problems. We will follow the bad practice, and often assume both the domain and codomain of a function to be understood implicitly.

For example, we may say things like: "Show $x^2 - \frac{x}{x^3+2}$ is continuous at $x_0 = 0$.", without explicitly stating that the domain of that function is $\mathbb{R} \setminus \{-\sqrt[3]{2}\}$.

What is the domain of $\frac{x^2-1}{x+1}$? You will receive different answers depending on whom you ask. The "correct" (because my) answer is that this is an ill-posed question. A "rule" or "formula" has no domain on its own. However, once can talk about a *maximum subset of* $\mathbb{R}$ where this defines a function. In that case that subset would be $\mathbb{R} \setminus \{-1\}$, even though some would argue that $\frac{x^2-1}{x+1} = x - 1$ and hence is defined everywhere. This is a valid point. It depends on whether your view $\frac{x^2-1}{x+1}$ as a formal, well, "formula", in which case you cannot evaluate it at $x = -1$, or as elements of some set (here what is called a "polynomial ring") where you can say the equation $\frac{x^2-1}{x+1} = x - 1$ is true (but you then need to say what kind of object x is).

1.5.6 Limits of functions

Definition.
Let I be an interval and let x_0 be an accumulation point of I (so $x_0 \in I$ or x_0 e $\in \partial I$).
For any function with domain $I \setminus \{x_0\}$ or I, we say the real number L is the **limit** of f as x approaches x_0, $\lim_{x \to x_0} f(x) = L$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \neq x_0 \in I$ with $|x - x_0| < \delta$, $|f(x) - L| < \varepsilon$. EOD.

The value of f at the place x_0 (in case x_0 is in the domain of f) has no bearing on the value of the limit.

Definition.
Let I be an interval and let x_0 be an accumulation point of I (so $x_0 \in I$ or $x\_0 \in \partial I$).
For any function with domain I, we say the real number ∞ is the **limit** of f as x approaches x_0, $\lim_{x \to x_0} f(x) = \infty$ if for every $M > 0$ there is $\delta > 0$ such that for all $x \neq x_0 \in I$ with $|x - x_0| < \delta$, $f(x) > M$.
Similarly, $\lim_{x \to x_0} f(x) = -\infty$ if for every $M > 0$ there is $\delta > 0$ such that for all $x \neq x_0 \in I$ with $|x - x_0| < \delta$, $f(x) < -M$. EOD.

Exercise.
With the assumptions on I, x_0, f as in the previous definitions, show that:

$$\lim_{x \to x_0} f(x) = L$$

iff for every sequence $x_n \to x_0$ with $x_n \neq x_0$ and $x_n \in I$ for all n, we have $\lim_{n \to \infty} f(x_n) = L$.
Note that this includes the cases $L = \pm\infty$. EOE.

There is also a definition of limit as $x \to \pm\infty$. And this limit may be finite or $\pm\infty$. Rather than repeating all six versions (finite limit and limit $\pm\infty$ for $x \to \pm\infty$), we just define the limit as L (where $L \in \mathbb{R} \cup \{\pm\infty\}$) if for every sequence $x_n \in I$ with $x_n \to \pm\infty$ we have $f(x_n) \to L$.

For this to make sense, I must be unbounded on the side the limit is taken.

1.5.7 One sided limits

In some instances, we want to compute a **one-sided** limit.

Definition
Let I be an interval and $x_0 \in I$, and f a function defined on (at least) $I \setminus \{x_0\}$.
Then we write $L = \lim_{x \to x_0^+} f(x)$ if for every sequence $x_n \in I$ such that $x_n > x$ and $x_n \to x_0$, we have $f(x_n) \to L$.

Similarly, $L = \lim_{x \to x_0^-} f(x)$ if for every sequence $x_n \in I$ with $x_n < x_0$ and $x_n \to x_0$ we have $f(x_n) \to L$.
EOD.

Exercise
Let I be an interval, $a \in I^\circ$, and f a function on I.
Show: $\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$.

1.6 Continuous functions

Definition.
Let I be an interval and let $x_0 \in I$
A function f with domain I is **continuous** at x_0, if $\lim_{x\to x_0} f(x) = f(x_0)$.
f is continuous, if it is continuous at all points of I. EOD.

The set of continuous functions with domain I is often denoted $\mathcal{C}(I)$.

Note that if $f: I \to \mathbb{R}$ is continuous at x_0, then so is the restriction of f to any sub-interval containing x_0.The converse is true, if the subinterval is *relative open* in I.

The functions with domain I that are continuous at x_0 form a vector space. $\mathcal{C}(I)$ is also a vector space, and in fact an *algebra*.

Example

1. $f(x) = x$ is continuous on all of $\mathbb{R}$.
2. $f(x) = |x|$ is continuous on all of $\mathbb{R}$.
3. Any polynomial function is continuous on all of $\mathbb{R}$.
4. Any rational function is continuous on all of $\mathbb{R}$ except the places where the denominator is zero.
5. The trig functions $\sin x$ and $\cos x$ are continuous everywhere.
6. The characterstic function of $\mathbb{Q}$ $\chi_{\mathbb{Q}}$ defined as
$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$
is nowhere continuous.

Lemma
Let f, g be continuous at $x_0 \in I$, and $a, b \in \mathbb{R}$.

1. $af + bg$ is continuous at x_0
2. fg is continuous at x_0
3. If $g(x) \neq 0$ for all x, then $\frac{f}{g}$ is continuous at x_0. EOL.

1.6.1 The Intermediate Value Theorem

One of the most fundamental theorems on continuous functions is the following:

Theorem.
Let f be continuous on $[a, b]$ and suppose $f(a) < 0 < f(b)$.
Then there is $c \in (a, b)$ such that $f(c) = 0$. EOT.

Proof. Let $A = \{x \in [a, b] f(x) < 0\}$. Then A is nonempty ($a \in A$). Also A is bounded. Thus $c :=$ "sup" A exists. Then $f(c) \leq 0$. Indeed, there is a sequence $a_n \in A$ such that $a_n \to c$, and hence $f(a_n) \to f(c)$. But $f(a_n) < 0$ for all n, so $f(c) \leq 0$. In particular $c < b$. Now suppose $f(c) < 0$. Then there is $\delta > 0$ such that $c + \delta < b$ and $f(c + \delta) < 0$, as f is continuous. But that means $c + \delta \in A$, a contradiction. It follows that $f(c) = 0$. It is clear that $c \in (a, b)$. QED.

For the purpose of this section we introduce the following shorthand

$$[\{a,b\}] = \begin{cases} [a,b] \text{if } a < b \\ [b,a] \text{ if } b \geq a \end{cases}$$

Corollary.
Let f be continuous on I, and suppose $y_1 < y_2 \in f(I)$. Then $[y_1, y_2] \subseteq f(I)$.
More precisely, if $a < b \in I$, then $[\{f(a), f(b)\}] \subseteq f([a,b])$. EOC.

Proof. This is an application of the previous theorem: Let $y \in [\{f(a), f(b)\}]$, and suppose $y \neq f(a), f(b)$ and define g on $[a,b]$ as $f(x) - y$, if $f(a) < f(b)$, or $y - f(x)$ if $f(b) < f(a)$.

Then $g(a) < 0 < g(b)$, and there is $c \in [a,b]$ such that $g(c) = 0$. But then $f(c) = y$. QED.

1.6.2 Supremum, infimum, maximum, and minimum

Definition
Let f be real valued function defined on a set $D \subseteq \mathbb{R}$. Let $A \subseteq D$ be any subset.

1. $\sup\limits_A f = \sup\{f(x) \mid x \in A\}$
2. $\inf\limits_A f = \inf\{f(x) \mid x \in A\}$
3. $\max\limits_A f = \max\{f(x) \mid x \in A\}$ if the right hand side exists
4. $\min\limits_A f = \min\{f(x) \mid x \in A\}$ if the right hand side exists

If $A = D$ we often omit the subscript A in the notation above and write for example $\sup f$ instead of $\sup\limits_D f$. We also will often write $f(x)$ instead of f. Finally, we often replace A in the notation by its defining conditions. So instead of $\sup\limits_{(a,b)} f(x)$ we often write $\sup\limits_{a<x<b} f(x)$. EOD.

As with sequences, we can form a function from the supremum:

Let $f: I \to \mathbb{R}$ be a function and let x_0 be an accumulation point of I. x_0 is allowed to be $\pm\infty$. Let $\dot{N}(x_0) = \{J \setminus \{x_0\} \mid J \subseteq I \text{ is an interval with accumulation point } x_0\}$.

We put $\limsup_{x \to x_0} f(x) = \inf\{\sup\limits_J f \mid J \in \dot{N}(x_0)\}$.

If x_0 is a boundary point or $\pm\infty$, then this definition coincides with

$$\limsup_{x \to x_0} f = \inf M(x)$$

where $M(x) = \sup\limits_{[x,x_0)} f$ if x_0 is an upper bound (and $M(x) = \sup\limits_{(x_0,x]} f$ if x_0 is a lower bound for I).

Even if $x_0 \in I^\circ$, it is occasionally useful to define

$$\limsup_{x \to x_0^+} f = \limsup_{x \to x_0} f|_{(x_0,\infty) \cap I}$$

and

$$\limsup_{x \to x_0^-} f = \limsup_{x \to x_0} f|_{(-\infty,x_0) \cap I}$$

(Note that x_0 is always a boundary point of $(x_0, \infty) \cap I$ and $(-\infty, x_0) \cap I$ if $x_0 \in I^\circ$.)

Exercise
Show that if $I = (a, b)$ then $\limsup_{x\to b} f = \lim_{x\to b} M(x)$.

1.6.3 The maximum principle

Theorem.
Let f be continuous on $I = [a, b]$, then f is bounded and there are $c, d \in [a, b]$ such that $f(c) = \sup f$ and $f(d) = \inf f$. EOT.

Proof. Let $L = \sup f$ (where we allow $L = \infty$). There is a sequence $y_n \in f([a, b])$ such that $y_n \to L$.

Then $y_n = f(x_n)$ for some sequence $x_n \in [a, b]$. By Bolzano-Weierstrass, there is a convergent subsequence x_{n_k} with limit c, say. Then $c \in [a, b]$ (as $[a, b]$ is closed), and by continuity $f(x_{n_k}) \to f(c)$. On the other hand, $f(x_{n_k}) = y_{n_k} \to L$. It follows $L = c$. In particular $L < \infty$.

For the infimum apply identical reasoning, where now L is allowed to be $-\infty$. QED.

Corollary
Let f be continuous on $[a, b]$, then $f([a, b]) = [m, M]$ where $m = \min f$ and $M = \max f$. EOC.

Proof. This is a combination of the Maximum Principle and the Intermediate Value Theorem. QED.

1.6.4 Invertible functions

Recall that if X, Y are sets and $f: X \to Y$ is a function, then f is called **invertible** if there is $g: Y \to X$ such that $f(g(y)) = y \forall y \in Y$ and $g\big(f(x)\big) = x\ \forall x \in X$. The function g is unique if it exists. If f is invertible, we write f^{-1} for this function g and call it the **inverse** of f.

Warning
The notation f^{-1} is ambiguous in some cases. If $f: [a, b] \to \mathbb{R}$ is nowhere zero, then there is also a function $h: [a, b] \to \mathbb{R}$ defined by $h(x) = f(x)^{-1}$. This function is also often denoted by f^{-1}. However, in general it is different from the inverse of f (which may not even exist).
To take this point further, let $I = (0,1)$ and consider $f: (0,1) \to (1, \infty)$ defined by $f(x) = x^{-1}$. Then f is invertible, and its inverse f^{-1} is defined on $(0, \infty)$ as $f^{-1}(x) = x^{-1}$. But note that $f \neq f^{-1}$ because the functions have different domains and codomains. On the other hand the function $h: (0,1) \to \mathbb{R}$ defined by $h(x) = f(x)^{-1}$ is just the function $h(x) = x$. EOW.

Remark
We usually identify functions if they only differ by their codomain (otherwise our notation would become very unwieldy).
However, when determining whether a function is invertible, its codomain is important: it must be onto/surjective for having a chance to be invertible. Every function is surjective onto its range by definition. EOR.

Definition
Let $f: X \to Y$ be a function.

1. f is called **injective** (one-one), if $x \neq x'$ implies that $f(x) \neq f(x')$.

2. f is called **surjective** (onto), if $f(X) = Y$, that is, if for each $y \in Y$ there is $x \in X$ such that $f(x) = y$.
3. f is called **bijective** (one-to-one) if f is both, injective and surjective.

EOD.

These notions can be easily understood in terms of *solutions of equations*:

f is injective, if for every $y \in Y$ the equation $f(x) = y$ has **at most one** solution (but may have none).

f is surjective, if for every $y \in Y$ the equation $f(x) = y$ has **at least one** solution (which may not necessarily be unique).

f is bijective, if for every $y \in Y$ the equation $f(x) = y$ has **a unique** solution.

In symbolical language:

f is injective, iff $\forall x, x'\colon f(x') = f(x) \Rightarrow x = x'$.

f is surjective, iff $\forall y\colon \exists x\colon f(x) = y$.

f is bijective, iff $\forall y\colon \exists!\, x\colon f(x) = y$.

Lemma
A function $f\colon X \to Y$ is invertible if and only if it is **bijective**, that is, both, **injective** (one-one) and **surjective** (onto). EOL.

The proof is left as an exercise.

Proposition
Let f be continuous on $[a, b]$ (where $a < b$). Then f is invertible (viewed as function onto its range) iff f is strictly monotone. If that is the case $f([a, b]) = [c, d]$ is again an interval, and $f^{-1}\colon [c, d] \to [a, b]$ is continuous. EOP.

Proof. By the Intermediate Value Theorem (and the Maximum Principle), we know that $f([a, b]) = [c, d]$ where $c = \min f$ and $d = \max f$.

If f is strictly monotone, then it is clearly injective, and bijective onto $[c, d]$.

If f is invertible, then $f(a) \neq f(b)$. Suppose $f(a) < f(b)$. The case $f(a) > f(b)$ is similar.

Let first $x \in (a, b)$. If $f(x) < f(a)$, then, on the interval (x, b) f must attain $f(a)$ again (by the IVT), contradicting that f is invertible. Similarly, if $f(x) > b$, then f attains $f(b)$ on (a, x), again impossible. This $f(a) < f(x) < f(b)$.

Let $x < y \in [a, b]$ and suppose $f(x) \geq f(y)$. Then $f(x) > f(y)$ because f is invertible. Also $f(a) < f(x), f(y)$ by the above, so there is $c \in (a, x)$ such that $f(c) = f(y)$. This is impossible.

It follows there are no $x < y$ with $f(x) \geq f(y)$ and f is strictly monotone.

Now let f be invertible and let $g = f^{-1}$. Suppose g is not continuous at $y_0 \in [c, d] = f([a, b])$. Then there exists a sequence $y_n \in [c, d]$ with $y_n \neq y_0$, $y_n \to y_0$ and $g(y_n)$ does not converge to $g(y_0)$. By definition this means that there is an $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ there is n_k with $\left|g\left(y_{n_k}\right) - g(y_0)\right| > \varepsilon$.

Since $g\left(y_{n_k}\right)$ is bounded, the subsequence y_{n_k} has a subsequence $z_\ell = y_{n_{k_\ell}}$ such that $g(z_\ell)$ converges to $x_0 \in [a, b]$, say. Now f is continuous, so $f\left(g(z_\ell)\right) = z_\ell$ converges to $f(x_0)$. On the other hand, z_ℓ is a subsequence of the original sequence y_n, so it converges to y_0. Thus $f(x_0) = y_0$, which means $x_0 = g(y_0)$. But by construction, $g(y_0)$ is not the limit of $g(z_\ell)$, as $|g(z_\ell) - g(y_0)| > \varepsilon$ for all ℓ. This is a contradiction, and we must have that $g(y_n) \to g(y_0)$. QED.

Corollary
Let I be an interval of any type and f a continuous function on I. If f is injective, then f is strictly monotone on I, $J = f(I)$ is an interval of the same type, and $f^{-1}: J \to I$ is continuous. EOC.

Proof. Exercise. QED.

Example

1. $\sqrt{x}$ is invertible on $[0, \infty)$ as a function to $[0, \infty)$.
2. $\sqrt[3]{x}$ is invertible on $\mathbb{R}$.

2 Differentiation

Differentiation is intended to give a measure of the rate of change of a function.

Suppose you are measuring the position $x(t)$ at time t of a particle moving a long a line. Measuring its position at times t_0 and t say, allows you to define an *average velocity* between t and t_0 as

$$\frac{x(t) - x(t_0)}{t - t_0}$$

If you make the difference $\Delta t = t - t_0$ smaller and smaller the quotient of the distance Δx travelled in Δt

$$\frac{\Delta x}{\Delta t}$$

approximate the “true” velocity at time t_0 better and better. In other words

$$v(t_0) = \lim_{t \to t_0} \frac{x(t) - x(t_0)}{t - t_0}$$

is the velocity at time t_0. It measures the “rate of change” of the distance travelled at a particular time. If you graph the position $x(t)$ over the time t, then $v(t_0)$ is the slope of the *tangent* to the graph of $x(t)$ at the point $(t_0, x(t_0))$.

2.1 Differentiable functions

2.1.1 Definition and examples

Definition.

Let I be an interval and f a function defined on I. We say f is **differentiable** at $x_0 \in I$ if

Equation 1

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. In this case we denote this limit by $f'(x_0)$ and call it the **derivative** of f at x_0.

You will often also find the notation $\frac{df}{dx}(x_0)$ or $\left.\frac{d}{dx}\right|_{x_0} f$ for the derivative of f at x_0.

Examples

1. Any *constant* function is differentiable everywhere it is defined, and its derivative is 0.
2. Any *linear* function is differentiable everywhere: $f(x) = mx + b$, then $f'(x_0) = m$.
3. It is a little more work is to show that *monomials* of the form $f(x) = x^n$ are differentiable everywhere.

   Indeed, consider

$$\frac{x^n - x_0^n}{x - x_0} = x^{n-1} + x^{(n-2)}x_0 + \dots + x x_0^{n-2} + x_0^{n-1} = \sum_{k=0}^{n-1} x^{(n-k-1)}\, x_0^k$$

   The right hand side is a sum of continuous functions, all defined at x_0, and we conclude

$$\frac{dx^n}{dx}(x_0) = n x_0^{n-1}$$

4. The function $f(x) = \sin x$ is differentiable at $x_0 = 0$, and
$$f'(0) = \lim_{x\to 0} \frac{\sin x}{x} = 1$$
5. The function $f(x) = |x|$, defined on all of $\mathbb{R}$, is differentiable everywhere except at $x_0 = 0$. Indeed, if $x_0 \neq 0$, it is easy to see that
$$f'(x_0) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$
But if $x_0 = 0$, then $\lim_{x\to 0} \frac{|x|}{x}$ does not exist (the quotient alternates between ± 1 around 0).
6. The function $f: (-\infty, 0] \to \mathbb{R}$, defined by $f(x) = |x|$ *is* differentiable at 0), as then
$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{|x|}{x} = -1$$
In that sense the *domain* of the function in question is important.

2.1.2 Equivalent definitions

There are several equivalent statements expressing that f is differentiable at x_0:

1. f is differentiable at x_0.
2. $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists and is finite.
3. There is a number $c = f'(x_0)$ such that $f(x_0 + h) = f(x_0) + ch + r(h)$ and $\lim_{h\to 0} \frac{r(h)}{h} = 0$. Here $r(h)$ is the function $f(x_0 + h) - f(x_0) - ch$ (which as a function of h which is always defined on an interval containing 0).

The second statement is often convenient, and the last statement is important as it allows for a direct generalization to functions in more than one variable.

It is a good exercise to show that these three statements are indeed equivalent. For the second, note that if $x_n \neq x_0$ converges to x_0, then $h_n := x_n - x_0 \neq 0$ converges to 0 (and vice versa). This shows that f is differentiable at x_0 with derivative $f'(x_0)$ if and only if $g(h) := f(x_0 + h)$ is differentiable at $h_0 = 0$ with derivative $g'(0) = f'(x_0)$. If f is defined on an interval I containing x_0, g is defined on the interval $I' = -x_0 + I = \{h \in \mathbb{R} \mid h + x_0 \in I\}$.

For the third, note there was a typo in a previous version. $r(h)$ is defined as $f(x_0 + h) - f(x_0) - ch$ which is defined for every $c \in \mathbb{R}$. If and only if $f'(x_0)$ exists and is equal to c does $r(h)$ satisfy that $\frac{r(h)}{h} \to 0$ for $h \to 0$. Again $r(h)$ is defined on an interval containing 0 (the same interval on which $g(h)$ above is defined).

2.1.3 Differentiability and continuity

Lemma.
Let f be defined on an interval I and differentiable at $x_0 \in I$. Then f is continuous at x_0. EOL.

Proof. For the limit of $\frac{f(x) - f(x_0)}{x - x_0}$ for $x \to x_0$ to exist and be finite, it is necessary that $f(x) \to f(x_0)$. But that means f is continuous at x_0. QED.

Note the converse is false. For example, $|x|$ is a continuous function but not differentiable (see Example 5 in 2.1.1 above) at $x_0 = 0$. One can find examples of functions that are continuous everywhere and nowhere differentiable.

Before we study differentiable function in greater detail, we list some immediate consequences of the definition.

2.1.4 Some rules of differentiation

Lemma

Let f, g be functions defined on an interval I and differentiable at some $x_0 \in I$.

1. $f + g$ is differentiable at x_0 and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.
2. fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
3. If $g(x) \neq 0$ for all $x \in I$, then $\frac{f}{g}$ is differentiable and
$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Part 2 is often called the **Product** or **Leibniz Rule**. Part 3 is usually referred to as the **Quotient Rule**.

Proof.

1. This is a consequence of the fact that the limit of a sum is the sum of limits (if they exist).
2. By 2.1.2 above we may write
$$f(x_0+h) = f(x_0) + f'(x_0)h + r(h)$$
$$g(x_0+h) = g(x_0) + g'(x_0)h + s(h)$$
where $\frac{r(h)}{h}, \frac{s(h)}{h} \to 0$ as $h \to 0$. Then
$$f(x_0+h)g(x_0+h) = f(x_0)g(x_0) + (f'(x_0)g(x_0) + f(x_0)g'(x_0))h + t(h)$$
where $t(h) = f'(x_0)g'(x_0)h^2 + (f(x_0) + f'(x_0)h + \ r(h))s(h) + (g(x_0) + g'(x_0)h)r(h)$.
Then $\lim\limits_{h\to 0} \frac{t(h)}{h} = 0$ because $\frac{r(h)}{h}, \frac{s(h)}{h} \to 0$ and the coefficients converge to finite numbers (namely $g(x_0)$ and $f(x_0)$ respectively).
3. We first treat the case $f = 1$. We must show that $\frac{1}{g}$ is differentiable at x_0. To this end consider
$$\frac{g(x)^{-1} - g(x_0)^{-1}}{x - x_0} = \frac{g(x_0) - g(x)}{g(x)g(x_0)(x - x_0)} = \frac{-1}{g(x)g(x_0)}\frac{g(x) - g(x_0)}{x - x_0} \to \frac{-1}{g(x_0)^2}g'(x_0)$$
for $x \to x_0$. This uses the fact that g is continuous at x_0 (2.1.3).
Now the general case follows from 2:
$$\left(\frac{f}{g}\right)'(x_0) = \left(f\left(\frac{1}{g}\right)\right)'(x_0) = \frac{f'(x_0)}{g(x_0)} + f(x_0)\left(-\frac{g'(x_0)}{g(x_0)^2}\right) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

QED.

The second statement of the proposition applies in particular in case one of the functions is constant.

Thus $(cf)'(x_0) = cf'(x_0)$.

This shows that the set $\mathcal{D}(I, x_0)$ of functions defined on I that are differentiable at x_0 form a subspace of the set $\mathcal{F}(I)$ of all functions on I.

So far, we discussed only derivatives at a single point.

Example

1. Any polynomial function is differentiable everywhere.

2. For any $n \in \mathbb{N}$, x^{-n} is differentiable everywhere and $(x^{-n})' = (-n)x^{-n-1}$.
3. Any rational function is differentiable everywhere in its domain.

2.1.5 Linearity of differentiation

Let I be an interval and $x_0 \in I$. If we define

$$\mathcal{D}(I, x_0) = \{ f \in \mathcal{F}(I) \mid \exists f'(x_0)\}$$

then, $\mathcal{D}(I, x_0)$ is a subspace of $\mathcal{F}(I) = \{f: I \to \mathbb{R}\}$, as it is nonempty (it contains for example, the constant functions), and it is closed under addition and scalar multiplication (the multiplication of a function with a constant).

$\mathcal{F}(I)$ is what is called an *algebra* (that is, a vector space with a *compatible* multiplication; it leads too far to formally define the compatibility conditions).

In fact, by 2.1.4, this means that $\mathcal{D}(I, x_0)$ is a *subalgebra* of $\mathcal{F}(I)$ (because it is also closed under the multiplication operation on $\mathcal{F}(I)$.

In addition, the maps $\mathcal{D}(I, x_0) \to \mathbb{R}$, defined by $f \mapsto f'(x_0)$, is a linear transformation (or, a *derivation* of the algebra $\mathcal{D}(I, x_0)$).

2.1.6 Differentiability of some functions

To gain further experience with derivatives we consider the following statement:

Suppose f^2 is differentiable at x_0. Does it follow that f is differentiable at x_0?

The short answer is no: we have seen that $f(x) = |x|$ is not differentiable at 0, yet f^2 is.

So we modify the statement to

Suppose $f^2$is differentiable at x_0 and f is continuous at x_0 with $f(x_0) \neq 0$. Then f is differentiable at x_0, and $f'(x_0) = \frac{(f^2)'(x_0)}{2f(x_0)}$.

Statement 2-1

To see this observe that $f(x)^2 - f(x_0)^2 = \big(f(x) + f(x_0)\big)\big(f(x) - f(x_0)\big)$. Then

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\big(f^2(x) - f^2(x_0)\big)}{(x - x_0)(f(x) + f(x_0))}$$

which is defined if x is close enough to x_0 (such that $f(x) + f(x_0) \neq 0$). This works because f is continuous and $f(x_0) \neq 0$. But the limit for $x \to x_0$ of the right hand side exists (again using the continuity of f and the fact that f^2 is differentiable at x_0).

Example

The function $f(x) = \sqrt{x}$ defined on $[0, \infty)$ is differentiable on $(0, \infty)$ and not differentiable at $x_0 = 0$ (which we will see below).

By the above, as $f^2 = x$ is differentiable, we conclude that whenever $f(x_0) \neq 0$, then $f'(x_0) = \frac{1}{2f(x_0)} = \frac{1}{2\sqrt{x_0}}$. Note this neatly fits into the scheme of computing derivatives of powers of x: $\left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}}$. EOE.

This can be extended to all functions of the form $g(x) = x^{\frac{1}{p}}$ (for $x > 0$).

Indeed, for any function g we have

$$g^p(x) - g^p(x_0) = (g(x) - g(x_0)(g(x)^{p-1} + g(x)^{p-2}g(x_0) + g(x)^{p-3}g(x_0)^2 + \cdots + g(x_0)^{p-1})$$

If $g(x_0) \neq 0$ and g is continuous at x_0, then the second factor on the right is nonzero for x close to x_0, and converges to $pg(x_0)^{p-1}$.

We find that

$$\frac{g(x) - g(x_0)}{x - x_0} = \frac{g^p(x) - g^p(x_0)}{x - x_0} \cdot \frac{1}{\sum_{i=0}^{p-1} g(x)^{p-1-i} g(x_0)^i} \to \frac{1}{pg(x_0)^{p-1}}(g^p)'(x_0)$$

for $x \to x_0$.

So

$$g'(x_0) = \frac{(g^p)'(x_0)}{pg(x_0)^{p-1}}$$

In the case of $g(x) = x^{\frac{1}{p}}$, we get

$$g'(x_0) = \frac{1}{p\sqrt[p]{x_0}^{p-1}} = \frac{1}{px_0^{\frac{p-1}{p}}} = \frac{1}{p}x_0^{\frac{1}{p}-1}$$

Guided exercise
We use the previous example to show that $\cos x$ is differentiable at 0 and $\cos' 0 = 0$.

1. Show that if f, g are functions (defined on an interval I) such that $f^2 + g^2 = c$ is constant on I, then g^2 is differentiable at $x_0 \in I$ if and only if f^2 is.
2. Conclude that $\cos^2 x$ is differentiable at 0 and compute its derivative.
3. Conclude that $\cos x$ is differentiable at 0 with derivative 0.

2.1.7 The derivative as a function

Let f be defined on an interval I. The **derivative** of f, denoted f', is the function$f' \colon D_f \to \mathbb{R}$ defined by $x \mapsto f'(x_0)$ where $D_f = \{x_0 \in I \mid f'(x_0) \text{ exists}\}$.

Of course, the derivative is only interesting if $D_f \neq \emptyset$, and in fact, when we discuss f' as a function, we usually want D_f to comprise a relative open interval (in I) containing a given point in question.

If $x_0 \in D_f$ and D_f contains a relative open interval J[1] with $x_0 \in J$, then it makes sense to ask whether f' is again differentiable at or continuous at x_0. This leads to the following recursive definition:

Definition
Let I be an interval, and f a function defined at x_0.
We say f is **continuously differentiable** at x_0 if $f'(x)$ is defined for x in a relative open interval containing x_0 and f' is continuous at x_0.

For any natural number n we say f is **n-times differentiable at x_0** if:

1. $f^{(n-1)}$ is defined on a relative open interval containing x_0.
2. $f^{(n-1)}$ is differentiable at x_0.

Here for any n the function $f^{(n)}$ is defined as $(f^{(n-1)})'$, where f^{n-1} is defined on a relative open neighbourhood of x_0.

A function is called **smooth** if it is n times differentiable everywhere in its domain for every n. EOD.

These definitions sound more complicated than they actually are.

If f is differentiable at x_0, and also on a set of the form $(x_0 - \varepsilon, x_0 + \varepsilon)$, it makes sense to define $f''(y_0)$ for $y_0 \in (x_0 - \varepsilon, x_0 + \varepsilon)$ as $f''(y_0) = (f')'(y_0)$.

Remark
Suppose f, g are defined and differentiable everywhere on an interval I. Then, as functions on I, we have $(f + g)' = f' + g'$, $(fg)' = f'g + g'f$. If g is nowhere 0 on I, then $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

Example
Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \vdash \begin{cases} -\dfrac{1}{2}x^2 x < 0 \\ \dfrac{1}{2}x^2\ x \geq 0 \end{cases}$$

Then f is differentiable everywhere $f'(x) = |x|$. Thus f is twice differentiable for all $x \neq 0$ (in fact it is n-times differentiable for all n), but not at $x = 0$.

Exercise
Let I be an interval and f be a rational function on I. That is $f = \frac{p}{q}$ where p, q are polynomial functions defined on I and $q(x) \neq 0$ for all $x \in I$. Then f' is defined on I.

Let $f(x_0) = f'(x_0) = 0$. Show that $(x - x_0)^2$ must divide p as a polynomial (that is $p = (x - x_0)^2 g$ where g is a polynomial function on I).

Show that if f is a polynomial such that $f(x_0) = f^{(1)}(x_0) = \cdots = f^{(k)}(x_0) = 0$ then $(x - x_0)^{k+1}$ divides f.

---

[1] This means that J is a proper open interval if I is an open interval. If $I = [a, b]$, then J is open or of the form $[a, c)$, $(c, b]$, or $[a, b]$. And similarly if I is a half-open interval, J is open or contains the boundary point of I.

Example
We have $\sin' x = \cos x$ and $\cos' x = -\sin x$.
Indeed, $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$. Therefore $\sin(x+h) - \sin(x) = \sin(x)\,(\cos(\mathrm{h}) - 1) + \cos(x)\sin(h)$, and

$$\frac{1}{h}(\sin(x+h) - \sin(x)) = \frac{\sin(x)\,(\cos(h)-1)}{h} + \frac{\cos(x)\sin(x)}{h} \to 0 + \cos(x)$$

for $h \to 0$.
Similarly,

$$\begin{aligned}\cos(x+h) - \cos(x) &= \cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)\\ &= \cos(x)\,(\cos(h)-1) - \sin(x)\sin(h)\end{aligned}$$

and the result follows because $\frac{\cos h - 1}{h} \to 0$ and $\frac{\sin h}{h} \to 1$ for $h \to 0$.

This shows that the trig functions are smooth. In fact, they are solutions for the **differential equation**

$$f'' = -f$$

This equation is fundamentally important in classical mechanics (it is essentially describing the *harmonic oscillator*). One can show that *any* twice differentiable function f that satisfies this equation on an interval I is of the form $a\sin x + b\cos x$ for some constants a, b. Of couse any such function satisfies this equation because

$$(a\sin x + b\cos x)'' = a\sin'' x + b\cos'' x = -a\sin x - b\cos x$$

It is possible to *define* the trigonometric functions sine and cosine as **solutions to the differential equation** $f'' = -f$ with the prescribed values $\sin 0 = 0$, $\sin' 0 = 1$ and $\cos 0 = 1$, $\cos' 0 = 0$. This would require that we know such solutions exist and are unique (if enough prescribed values are given). However, it would have the advantage that we would not rely on somewhat murky concepts from Euclidean geometry: the definition of $\sin x$, for example, relies on the definition of the *arc length* of a segment of a unit circle. To define this precisely quite a bit of work is required. We will hopefully see a rigorous definition of the trigonometric functions towards the end of the term. EOE.

2.1.8 The chain rule of differentiation

The chain rule deals with composition of functions.

Proposition
Suppose f is defined on some interval I, g is defined on some interval J, and $g(J) \subseteq I$. If g is differentiable at $x_0 \in J$, and f is differentiable at $y_0 = g(x_0) \in I$, then $h = f \circ g$ is differentiable at x_0 with $h'(x_0) = f'\big(g(x_0)\big)g'(x_0)$. EOP.

Proof. As before let $g(x_0 + h) = g(x_0) + g'(x_0)h + r(h)$ and consider

$$h(x_0 + h) = f\big(g(x_0 + h)\big) = f(g(x_0) + g'(x_0)h + r(h))$$

Then if $f(y_0 + k) = f(y_0) + f'(y_0)k + s(k)$, we find that

$$h(x_0 + h) = f(y_0) + f'(y_0)\big(g'(x_0)h + r(h)\big) + s(g'(x_0)h + r(h))$$

It remains to show that $\lim\limits_{h\to 0}\left(\frac{f'(y_0)r(h) + s\left(g'(x_0)h + r(h)\right)}{h}\right) = 0$.

Note that $\frac{f'(y_0)r(h)}{h} \to 0$, so we must show that $\lim_{h\to 0} \frac{s\left(g'(x_0)h+r(h)\right)}{h} = 0$. Let $\sigma(h) = \frac{s(h)}{h}$ for $h \neq 0$ and $\sigma(0) = 0$. Then σ is continuous at 0, and

$$\frac{s\left(g'(x_0)h + r(h)\right)}{h} = \frac{g'(x_0)h + r(h)}{h}\sigma\left(g'(x_0)h + r(h)\right) \to g'(x_0)\sigma(0) = 0$$

for $h \to 0$. This uses that for all $h \neq 0$, $\frac{s\left(g'(x_0)h+r(h)\right)}{h} = \frac{g'(x_0)h+r(h)}{h}\frac{s\left(g'(x_0)h+r(h)\right)}{g'(x_0)h+r(h)}$ if $g'(x_0)h + r(h) \neq 0$. If $g'(x_0)h + r(h) = 0$, then both $s\left(g'(x_0)h + r(h)\right) = \sigma\left(g'(x_0)h + r(h)\right) = 0$. QED.

Examples

1. $\frac{1}{x}$ has derivative $-\frac{1}{x^2}$. So for any f that is differentiable and nonzero at x_0 we conclude that
$$\left(\frac{1}{f}\right)'(x_0) = \frac{d}{dx}\left(\frac{1}{x} \circ f\right)(x_0) = -\frac{1}{f(x_0)^2}f'(x_0)$$
in line with the quotient rule above.
2. We have seen that $f = \sqrt[p]{x}$ is differentiable for $x > 0$. Then $f^p = x$. So the chain rule would demand that
$$(f^p)' = pf^{p-1}f' = 1$$
It then follows that $f'(x_0) = \frac{1}{pf^{p-1}(x_0)}$, from which we conclude that f cannot be differentiable at $x_0 = 0$. For $x_0 > 0$ this suggests $f'(x_0) = \frac{1}{p\sqrt[p]{x_0^{p-1}}} = \frac{1}{p}x_0^{\frac{1}{p}-1}$ consistent with what we obtained before. But note that this does *not* directly prove that $f'(x_0)$ exists for $x_0 > 0$.
This procedure is known as *implicit differentiation*.
3. Let f be defined on $\mathbb{R}_{>0}$ as $f(x) = \sqrt[p]{x^q}$. Then
$$f'(x_0) = \frac{1}{p}\left(x_0^q\right)^{\frac{1-p}{p}}qx_0^{q-1} = \frac{q}{p}x_0^{\frac{q(1-p)+p(q-1)}{q}} = \frac{q}{p}x_0^{\frac{q}{p}-1}$$
This shows that for any rational number $r > 0$ we have x^r is differentiable at any $x_0 > 0$ and $\left.\frac{d}{dx}\right|_{x_0} x^r = rx^{r-1}$.

EOE.

The second item begs the following question: let f, g be functions such that $g \circ f$ is defined and differentiable. Suppose g is differentiable at $y_0 = f(x_0)$. Does it follow that f is differentiable at x_0?

The chain rule would say $(g \circ f)'(x_0) = g'(y_0)f'(x_0)$ suggesting that $f'(x_0) = \frac{(g\circ f)'(x_0)}{g'(y_0)}$ exists as long as $g'(y_0) \neq 0$.

This is actually true:

Let g be differentiable at $y_0 = f(x_0)$ and let $g \circ f$ be differentiable at x_0. If $g'(y_0) \neq 0$, then f is differentiable at x_0 and $f'(x_0) = \frac{(g\circ f)'(x_0)}{g'(y_0)}$.

To see why this holds, let

$$q(y) = \vdash \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0} & y \neq y_0 \\ g'(y_0) & y = y_0 \end{cases}$$

Then $q(y)$ is continuous at y_0. Moreover, for any $x \neq x_0$ we have

$$\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} = q\big(f(x)\big)\frac{f(x) - f(x_0)}{x - x_0}$$

(because both sides are equal to 0 if $f(x) = f(x_0) = y_0$. As $x \to x_0$, the left hand side converges to $g \circ f'(x_0)$, thus the right hand side converges as well. If $g'(y_0) \neq 0$ this means $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ must exist.

2.2 Local properties of differentiable functions

2.2.1 Positive and negative derivative

Suppose f is differentiable at some $x_0 \in I$. It is a natural question to ask whether we can draw any conclusions from properties of $f'(x_0)$.

For example, does it mean something whether $f'(x_0) > 0$ or $f'(x_0) < 0$?

Lemma
Suppose $f'(x_0) > 0$. Then there is $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0) \cap I$ and all $y \in (x_0, x_0 + \delta) \cap I$ we have

$$f(x) < f(x_0) < f(y)$$

If $f'(x_0) < 0$, there is $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0) \cap I$ and all $y \in (x_0, x_0 + \delta) \cap I$ we have

$$f(x) > f(x_0) > f(y)$$

Proof. We prove the first assertion. The second follows by applying the first to $-f$.

Let $\delta > 0$ be small enough such that $\frac{f(x)-f(x_0)}{x-x_0}$ is close enough to $f'(x_0)$ as to be positive as well whenever $0 < |x - x_0| < \delta$ and $x \in I$.

As $x - x_0 > 0$ if and only if $x > x_0$, this means $f(x) > f(x_0)$ if $x > x_0$ and $f(x) < f(x_0)$ if $x < x_0$. QED.

2.2.2 Local extrema

Definition
Let f be a function defined on an interval I and let $x_0 \in I$. Then f has a

- **local maximum** at x_0, if there is $\delta > 0$ such that $\forall x \in I \cap (x_0 - \delta, x_0 + \delta): f(x) \leq f(x_0)$.
- **local minimum** at x_0, if there is $\delta > 0$ such that $\forall x \in I \cap (x_0 - \delta, x_0 + \delta): f(x) \geq f(x_0)$.

f has a **local extremum** at x_0 if it has a local maximum or minimum at x_0.

A local extremum need not be a global extremum (that is, $f(x_0) = \sup\{ f(x)\}$ or $\inf\{f(x)\}$). Unless I is a closed interval and f is continuous, local or global extreme need not exist.

Lemma
Suppose f has a local extremum at $x_0 \in I^\circ$ and suppose f is differentiable at x_0. Then $f'(x_0) = 0$. EOL.

Proof. This is an immediate consequence from the definition and the lemma in 2.2.1 which shows that the derivative cannot be positive or negative at a local extremum in the interior of I.

For example, if $f'(x_0) > 0$ at the inner point x_0, then there are sequences $x_n < x_0$, $y_n > x_0$ in I such that $x_n, y_n \to x_0$ and $f(x_n) < f(x_0) < f(y_n)$ contradicting the definition of a local maximum or minimum. The proof for negative derivative at x_0 is similar. QED.

This is one of the few cases where it is important that x_0 is an interior point: Consider $f(x) = x$ defined on $[0,1]$. Then f has a local (in fact global) extremum at $x_0 = 1$. But of course $f'(1) = 1 \neq 0$.

Example
Consider $f\colon \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3 - x$. Then $f'(x) = 3x^2 - 1$, and we find candidates for local extrema as $x_{1/2} = \pm\frac{1}{\sqrt{3}}$. Let us focus on $x_2 = \frac{1}{\sqrt{3}}$. Consider the interval $I = [0, b]$. Then f attains its infimum on this interval (Min/Max Principle). That is, there is $x_0 \in I$ such that $f(x_0) = \inf_I f$. There are three possible cases: $x_0 = 0$, $x_0 = b$, or $x_0 \in I^\circ$. In the last case, we must have $f'(x_0) = 0$ and thus, $x_0 = x_1$. $f(0) = 0 > f(x_1) = \frac{1}{3} - \frac{1}{\sqrt{3}}$. So $0 \neq x_0$. Next, for b large enough, $f(b) > 0$ and hence in this case $x_0 \neq b$ either. It follows f has a local minimum at $x_0 = x_1$.
This shows that quite a bit of ad-hoc reasoning is necessary even in the case of a relatively simple function. EOE.

2.2.3 Rolle's Theorem

Theorem
Let f be a continuous function defined on $[a, b]$ and differentiable on at least (a, b) such that $f(a) = f(b)$. Then there is $c \in (a, b)$ such that $f'(c) = 0$. EOT.

The theorem should be intuitively reasonable: if $f(a) = f(b)$ it is not a stretch that this means it must have a horizontal tangent somewhere in (a, b).

Proof. As f is continuous, the maximum principle shows that f must attain its minum and maximum somewhere on $[a, b]$. If f is constant then $f'(c) = 0$ for all c and we are done.

Otherwise $f(a) = f(b)$ cannot be equal to both the maximum and minimum of f. Therefore f has a global extremum at an interior point $x_0 \in I$. It is necessarily also a local extremum, and by the lemma in 2.2.2 this means $f'(x_0) = 0$. QED.

Rolle's Theorem has only one purpose: proving the Mean Value Theorem (below). However it *does* help finding local extrema:

Example
Let f be a continuous function on I, differentiable on at least I°. For any $a < b \in I$ with $f(a) = f(b)$ there must be $c \in (a, b) \subseteq I$ such that $f'(c) = 0$ *and* f has a local extremum at c (as this is what the proof of Rolle's Theorem actually shows).

So in the example above: for $f(x) = x^3 - x$, we know $f(0) = 0$, $f(x_1) < 0$, and $\lim_{x \to \infty} f(x) = \infty$. So there must be $b > 0$ with $f(b) = 0$, and then necessarily x_1 (as the only zero of f' in $(0, b)$ must be the place of a local minimum. EOE.

Note that if f is not differentiable everywhere, there are simple counterexamples to the statement of the theorem: consider $f(x) = |x|$ on $[-1,1]$. Then $f(-1) = f(1)$ but there is no $c \in (-1,1)$ where $f'(c)$ is defined and equal to 0.

2.2.4 The Mean Value Theorem

Rolle's Theorem is a special case of the so called Mean Value Theorem, which relates the derivative to the "average" slope of f.

Example

Suppose you are tasked with enforcing speeding laws along a long stretch of highway (with not much inbetween the two endpoints of the stretches). Rather than bother measuring the instantaneous speed of vehicles you install cameras at both ends of the stretch which record a time coded picture of the licence plate of each passing car. If the stretch of highway covers a distance Δd, and the time difference of the two pictures of a licence plate Δt is so small that $\frac{\Delta d}{\Delta t} > L$ where L is the speed limit (which we assume to be constant), you send the owner of the car a ticket. Now an owner decides to challenge the ticket, and they claim never to have driven faster than L. What is your response in court?
While it is intuitively clear that the owner must be lying, it is a bit technical to prove that this is actually the case, and the proof below relies on the fact that the travelled distance is a continuous function in time and in fact differentiable everywhere except possibly at the two endpoints. EOE.

Theorem

Let $f: [a, b] \to \mathbb{R}$ be continuous and suppose f is differentiable on (a, b).
Then there is $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

EOT.

Proof. Consider the function $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$ defined in $[a, b]$.

Then $g(a) = g(b) = f(a)$. As g is differentiable on (a, b), Rolle's Theorem implies that there is $c \in (a, b)$ where

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

QED.

Now we got the speeding driver: As $\Delta d = d(b) - d(a) > L(b - a)$ we must have that $v(c) > L$ for at least one $c \in (a, b)$ where $v(t) = d'(t)$ is the car's speed at time t.

Applied to this situation the Mean Value Theorem says that you must hit the *average* speed at least once during a trip.

Geometrically this means the slope of the graph of f must be equal to the slope of the line connecting $(a, f(a))$ and $(b, f(b))$ *somewhere*. The slope of this line is in this sense the *average* slope of the graph.

Corollary (Zero derivative means constant)
Let f be continuous on I, and differentiable on at least I°. If $f' = 0$ on I°, then f is constant on all of I. EOC.

Proof. Let $a < b \in I$. We must show that $f(a) = f(b)$. Now $[a,b] \subseteq I$ and f is differentiable on (a,b). By the MVT (applied to f on $[a,b]$) we have $f(b) - f(a) = f'(c)(b-a)$ for some $c \in (a,b)$. Since $f'(c) = 0$ it follows that $f(b) = f(a)$. QED.

In particular, this means if two continuous f, g are defined on an interval I and $f' = g'$ on I°, then $f = g + C$ for some constant C. This is the corollary applied to the difference $f - g$ (which has zero derivative everywhere on I°).

Example
If $f' = m$ is constant on (a,b), then $f = mx + C$ on $[a,b]$. Indeed, $(f - mx)' = 0$ and hence $f - mx = C$ is constant. EOE.

Warning
Note that the fact that the domain here is an *interval* is important:
Let f be defined on $(-1,0)$ as $f(x) = -1$, and on $(0,1)$ as $f(x) = 1$. Then $f'(x) = 0$ everywhere on $D = (-1,0) \cup (0,1) = (-1,1) \setminus \{0\}$ but f is **not** constant.
So, the more general statement would be that f must be constant on any *interval contained in its domain*.
The precise condition for the corollary to hold, which can be generalized to functions in more variables, is that the domain of f must be *connected*. This is a term borrowed from topology and means that the domain cannot be written as a disjoint union of relative open subsets. For $\mathbb{R}$, a subset A is connected if and only if for every $a \neq b \in A$, the line segment connecting them (that is, $[a,b]$) is contained in A. EOW.

2.2.5 Monotone functions and the derivative

Proposition
Let f be continuous on I and differentiable on at least the interior I°. Then
- f is *monotone increasing* iff $f' \geq 0$ on I°.
- f is *monotone decreasing* iff $f' \geq 0$ on I°.

If $f' > 0$ on I°, then f is strictly monotone increasing. If $f' < 0$ on I° then f is strictly monotone decreasing. EOP.

Proof. If f is monotone increasing, we cannot have $f'(c) < 0$ for any $c \in I^\circ$ (see 2.2.1). Conversely, if $f' \geq 0$ on I°, by the MVT 2.2.4 for any $x \leq y \in I$, we have $f(y) - f(x) = f'(c)(y-x) \geq 0$ for some $c \in (x,y)$. Thus, $f(x) \leq f(y)$ and f is monotone increasing.

The proof in the second case is similar.

Finally, suppose $f' > 0$ on I°. Then the argument just given shows that $f(y) - f(x) > 0$ for all $x < y \in I$. Again, the argument is similar if $f' < 0$. QED.

Note that if f is stricyly increasing, it does not follow that $f' > 0$. As an example consider $f(x) = x^3$ defined on $[-1,1]$. Then f is strictly increasing, but $f'(0) = 0$. It is still true however, that $f' \geq 0$.

Example

Suppose I is an interval of any type. Recall that a function $f\colon I \to \mathbb{R}$ is called *invertible* (as a function to its range $J \coloneqq f(I)$) if it is one-one (injective), since it's then *bijective* onto its range J. If f is continuous then J is an interval. We also know that in this case f is invertible if and only if f is strictly monotone on I.

Find a maximal[2] interval I such that $\sin x$ is invertible on I.

Our approach is simple: first identify an interval where $\cos x = \sin' x > 0$. We know that is true on $K = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and can conclude that $\sin x$ is strictly monotone on K, and in fact on $I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We also know that $\cos x < 0$ for any $x > \frac{\pi}{2}$ and close to $\frac{\pi}{2}$. And similarly, $\cos x < 0$ for $x < -\frac{\pi}{2}$ and close to $-\frac{\pi}{2}$. Thus $I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is a maximal interval where $\sin x$ is invertible. Then $J = \sin I = [-1,1]$, and the inverse of $\sin$ is often called $\arcsin$, or, obviously, $\sin^{-1}$.

Similar reasoning shows that $\tan x$ is invertible on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with range $\mathbb{R}$ (and as $\tan x$ is undefined for $x = \pm\frac{\pi}{2}$ this is a maximal interval). The inverse of $\tan$ is denoted $\arctan$; it is defined on all of $\mathbb{R}$ and has range $(-\frac{\pi}{2}, \frac{\pi}{2})$. EOE.

2.2.6 The first derivative test

Using the definition of a local extremum it may be hard to figure out whether any given function has one at a point c.

We have seen that the derivative vanishes at local extrema. The converse is false as again $f(x) = x^3$ shows at $x_0 = 0$. This means to find local extrema, we need a way to decide whether f has a local extremum at its **critical points**. A critical point here is any x_0 where $f'(x_0) = 0$ or where f' is not defined.

Theorem (First Derivative Test)
Let I be an interval, f continuous on I and $c \in I$ a critical point.

- If there is a relative open subinterval $J \subseteq I$ such that $c \in J$ and $f' \le 0$ on $J \cap (-\infty, c)$ and $f' \ge 0$ on $J \cap (c, \infty)$, then f has a **local minimum** at c.
- If there is a relative open subinterval $J \subseteq I$ such that $c \in J$ and $f' \ge 0$ on $J \cap (-\infty, c)$ and $f' \le 0$ on $J \cap (c, \infty)$, then f has a **local maximum** at c.

EOT.

To clarify the "relative open" business: if c is an interior point of I, all this says is that there is $\delta > 0$ such that $J = (c - \delta, c + \delta) \subseteq I$ and, in the first scenario, $f' \le 0$ on $(c - \delta, c)$ and $f' \ge 0$ on $(c, c + \delta)$.

In the second scenario, $f' \ge 0$ on $(c - \delta, c)$ and $f' \le 0$ on $(c, c + \delta)$.

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[2] For the purposes here, a *maximal* I is an interval that is not properly contained in any interval with the same property.

In case $c \in \partial I$, then one of the two (or both if I is just consisting of a point) is vacuous as there are no points in $J \cap (-\infty, c)$ for example if $I = [c, b)$ for some $b > 0$.

Proof. We discuss the first scenario. The proof of the second is similar (or simply apply the first scenario to the function $-f$).

Then f is monotone decreasing on $J \cap (-\infty, c]$ and monotone increasing on $J \cap [c, \infty)$. In particular, there is $\delta > 0$ such $f(x) \geq f(c)$ for all $x \in I \cap (c - \delta, c + \delta)$. But that means f has a local minimum at c.

Example
Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then f is not differentiable at $x = 0$. Does it have a local extremum at $x = 0$?
The answer is no, as there are zero sequences $x_n < 0$ and $y_n > 0$ such that $f(x_n) < 0$ and $f(y_n) > 0$.
EOE.

2.2.7 The second derivative test

Recall that if g is differentiable, then $g'(c) > 0$ tells us that $g(x) < g(c) < g(y)$ for all $x < c < y$ "close enough" to c. Applying this reasoning to $g = f'$ we obtain the second derivative test.

Theorem (Second Derivative Test)
Let f be defined on I and twice differentiable at a critical point x_0. Then

- If $f''(x_0) < 0$, f has a local maximum at x_0.
- If $f''(x_0) > 0$, f has a local minimum at x_0.
- If $f''(x_0) = 0$, f may or may not have a local extremum of either type at x_0.

Proof. We prove the first case. The second is similar, and the last is proven by the examples $x^3, \pm x^4$ at $x_0 = 0$.

By assumption f' is defined on a relative open interval containing x_0. As $f''(x_0) < 0$, the results of 2.2.1 show that $f' > 0$ on some set of the form $(x_0 - \delta, x_0) \cap I$, and $f' < 0$ on $(x_0, x_0 + \delta) \cap I$ (for a suitably small chosen $\delta > 0$).

But then f is strictly monotone decreasing on $(x_0 - \delta, x_0] \cap I$ and strictly monotone decreasing on $[x_0, x_0 + \delta) \cap I$ by 2.2.5.

Thus, f has a local maximum at x_0. QED.

From the proof we see this criterion cannot possibly capture all instances where f has a local extremum, as it forces f to be strictly monotone (of opposing direction) in a neighbourhood of c. For example the theorem says nothing about constant functions, which have local (and global) extrema everywhere.

Example
Find all local extrema of $\sin x$.
Then $\sin' x = \cos x = 0$ means $x = \frac{\pi}{2} + k\pi$ for some $k \in \mathbb{Z}$.
And $\sin'' x = -\sin\, x$. Now $\sin\left(\frac{\pi}{2} + k\pi\right) = \begin{cases} 1 & k \text{ is even} \\ -1 & k \text{ is odd} \end{cases}$

This shows that $\sin x$ has a local maximum at $\frac{\pi}{2} + k\pi$ if k is even, and a local minimum if k is odd. Of course we know that also directly.

Example

Find all local extrema of $f(x) = x^3 - \frac{3}{2}x^2 - 6x - 1$ (defined on $\mathbb{R}$).
First, $f'(x) = 3x^2 - 3x - 6$. Then $f'(x) = 0$ iff $x^2 - x - 2 = 0$. The solution to this equation is $x_1 = \frac{1}{2} + \sqrt{\frac{1}{4} + 2} = 2$ and $x_2 = -1$.
As $f''(x) = 6x - 3$, we find that $f''(x_1) > 0$ and $f''(x_2) < 0$ so f has a local minimum at x_1 and a local maximum at x_2.
Does f have global extrema? EOE.

Example

Let f be a polynomial function of degree 3 such that f' has two distinct roots. Show that f has a local minimum at one and a local maximum at the other. If $f(x) = ax^3 + \cdots$ which is which depending on $a > 0$ or $a < 0$?

We know there are $x_1 < x_2$ such that $f'(x_1) = f'(x_2) = 0$. Rolle's Theorem asserts that then there is $x_3 \in (x_1, x_2)$ with $f''(x_3) = 0$. As f'' is a degree one polynomial, it has exactly one root. This means $f''(x_1) \neq 0$ and $f''(x_2) \neq 0$. Moreover, as f'' is a linear function, the signs of $f''(x_1), f''(x_2)$ are opposed. f has a local maximum at one and a local minimum at the other. If $a > 0$, then f'' is strictly increasing, which means x_1 is a local maximum and x_2 is a local minimum for f. If $a < 0$ the places are interchanged and f has a local minimum at x_1 and a local maximum at x_2.

2.3 Interlude: Exponential functions and logarithms

Definition

A *nonzero* function $f: \mathbb{R} \to \mathbb{R}$ is called an **exponential function** if f is differentiable, and $f(x+y) = f(x)f(y)$ for all x, y. EOD.

Recall that a function f is nonzero, if there is at least one x with $f(x) \neq 0$. At the moment, the only exponential function we know exists is the constant function 1.

Lemma

If f is an exponential function, then $f > 0$. EOL.

Proof. By definition $f \neq 0$, so there is x_0 such that $f(x_0) \neq 0$. But then $f(x_0) = f\left(\frac{1}{2}x_0\right) f\left(\frac{1}{2}x_0\right)$ is a square and hence positive. As this applies to any x_0 with $f(x_0) \neq 0$, we find that $f \geq 0$. Also note $0 < f(x_0) = f(x_0 + 0) = f(x_0)f(0)$, which forces $f(0) = 1$.

Then there is $\varepsilon > 0$ such that $f(x) > 0$ for all $x \in (-\varepsilon, \varepsilon)$. For general x there is $n \in \mathbb{N}$ such that $\frac{x}{n} \in (-\varepsilon, \varepsilon)$. But then $f\left(\frac{x}{n}\right) > 0$ and so $f(x) = f\underbrace{\left(\frac{x}{n} + \frac{x}{n} + \cdots + \frac{x}{n}\right)}_{n \text{ summands}} = f\left(\frac{x}{n}\right)^n > 0$. QED.

A quicker and in fact nicer argument as proposed by some in class: As before conclude that $f \geq 0$. To see that $f > 0$ it suffices to show that $f(x) \neq 0$ for all x. Suppose $f(x_0) = 0$ for some x_0. Then $f(x) = f\big(x_0 + (-x_0 + x)\big) = f(x_0)f(-x_0 + x) = 0$ for all x, contrary to our assumption. QED.

2.3.1 Uniqueness of exponential functions

Definition
If f is an exponential function, then $a := f(1) > 0$ is called the **base** of f.

Exercise
Let f be an exponential function with base $a > 1$.

1. Show that for any rational number r and any real number x we have $f(rx) = f(x)^r$. In particular $f(r) = a^r$.
2. Use the continuity of f and the first part to show that the range of f is all of $\mathbb{R}_{>0}$.
3. Show that the set $\{ a^r \mid r \in \mathbb{Q} \}$ is *dense* in $\mathbb{R}_{>0}$.
4. Use the previous part to show that f is strictly monotone increasing.
5. Discuss what changes in each of the parts when $a < 1$.

EOE.

Note by the first statement we have that for all $x \in \mathbb{R}$ and *any* rational sequence r_n, we have $f(x) = \lim\limits_{n\to\infty} a^{r_n}$, and this does not depend on the sequence r_n. The exercise somewhat motivates the terminology. It is common to write $f(x) = a^x$ for all real numbers, but it is important to realize that this is a definition, and a^x has no intrinsic meaning (for irrational x). There is also potential for ambiguity: a priori it could be that two exponential functions have the same base.

All statements so far do not require that f is differentiable. As a consequence:

Lemma (Uniqueness of Exponential Functions)
For each $a > 0$, there is at most one continuous function $f: \mathbb{R} \to \mathbb{R}$ such that for all x, y, $f(x + y) = f(x)f(y)$ and $f(1) = a$. EOL.

Proof. Let f, g be two such functions. Then for all $r \in \mathbb{Q}$, $f(r) = g(r) = a^r$. As f, g are continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$ this means $f = g$. Indeed, if x is any real number and $r_n \to x$ is a sequence of rational numbers with limit x, then the sequences $f(r_n) = g(r_n)$ have the same limit, namely $f(x) = g(x)$. QED.

Theorem
For every $a > 0$ there is exactly one exponential function with base a. EOT.

Proof. The uniqueness assertion follows from the previous lemma, as all exponential functions are continuous. We have to postpone the existence part for now.

Remark
As mentioned in class one *can* define the exponential function a^x as $a^x = \lim\limits_{n\to\infty} a^{r_n}$ where $r_n \to x$ is a sequence of rational numbers. One then has to show that this limit always exists and is independent of the sequence chosen, and finally that the so defined function is differentiable. All of this is possible.
A more conceptual way is to figure out properties of exponential functions and then in the end show that there is a function satisfying the desired properties. There are essentially two ways to do this, and we will soon see how.

2.3.2 The derivative of an exponential function

Proposition

Let f be the exponential function with base $a > 0$. Then there is a constant c such that $f' = cf$. In particular, $c = f'(0)$. EOP.

Proof. Let $x_0 \in \mathbb{R}$. Then $f(x_0 + h) = f(x_0)f(h)$. Therefore

$$\frac{f(x_0 + h) - f(x_0)}{h} = f(x_0)\frac{f(h) - 1}{h} \to f(x_0)f'(0)$$

for $h \to 0$. This shows that for all x_0, $f'(x_0) = f'(0)f(x_0)$, so the proposition follows with $c = f'(0)$. QED.

Remark
Functions satisfying an equation of the form $f' = cf$ are important in many applications. They describe systems experiencing exponential growth (if $c > 0$) or decay (if $c < 0$) and appear in many contexts. Not every such function is an exponential function. But every such function is a constant multiple of an exponential function, motivating the terminology *exponential* growth or decay. EOR.

Definition
Let f be the (unique) exponential function with base $a > 0$. The number $c = f'(0)$ is called the **natural logarithm** of a and denoted $\ln a$ or $\log a$. EOD.

Lemma
For every $c \in \mathbb{R}$ there exists a unique exponential function f with $f' = cf$. EOL.

Proof. Let g be any exponential function with base $a > 1$, which exists by the Theorem in 2.3.1. We know that g is strictly monotone increasing and that $g > 0$. We also know that $g' = g'(0)g$. Taken together this means $g'(0) > 0$.

Let f be defined as $f(x) = g\left(\frac{c}{g'(0)}x\right)$. Then f is again an exponential function, and $f' = cf$ by the Chain Rule.

As for uniqueness, suppose $f_1' = cf_1$ and $f_2' = cf_2$ for exponential functions f_1, f_2. Then as $f_1 > 0$, f_1^{-1} defined by $f_1^{-1}(x) = \frac{1}{f_1(x)}$ (not to be confused with the *inverse* of f_1, which also exists if $f_1'(0) \neq 0$) is defined and also an exponential function. In fact, $f_1^{-1}(x) = f_1(-x)$. Also, $g = f_1^{-1}f_2$ is easily seen to be an exponential function as well. Then $g'(x) = -f_1'(-x)f_2(x) + f_1(-x)f_2(x) = -cf_1(-x)f_2(x) + cf_1(-x)f_2(x) = 0$ for all x. Therefore g is constant. As $g(0) = 1$, it follows that $f_1 = f_2$. QED.

Definition
The unique exponential function f with $f' = f$ is called **the exponential function** and often denoted exp. Its base is denoted e. EOD.

Note that $f(x) = \exp(cx)$ defines an exponential function with $f'(0) = c$. It is therefore the unique exponential function with base $a = \exp(c)$ where $c = \ln a$.

As the range of exp is all of $\mathbb{R}_{>0}$ and $\exp' = \exp > 0$ everywhere, we know that exp is invertible with some inverse $g: \mathbb{R}_{>0} \to \mathbb{R}$.

It follows that $a = f(1) = \exp(\ln a)$. This in turn also means that $\ln a$ is the unique number y such that $\exp y = a$, which forces $y = g(a) = \ln a$. In other words, the function $a \mapsto \ln(a)$ is the same as the function $a \mapsto g(a)$.

The unique exponential function with base $a > 0$ is $exp(x \ln a)$.

Definition
The inverse function g of exp is called the **(natural) logarithm function**. It is denoted by $\ln$ or $\log$. EOD.

Note that for all $x, y > 0$ we have $\ln(xy) = \ln(x) + \ln(y)$. This is shown simply by applying exp on both sides of the equation. (It also follows by the fact that $\ln$ is the inverse of a group homomorphism and hence itself a group homomorphism.) As a consequence, we also have that $\ln x^n = n \ln x$ for all $n \in \mathbb{N}$.

Moreover, as exp is strictly monotone increasing, so is its inverse $\ln$. Its range must be all of $\mathbb{R}$ and we conclude that $\lim_{x\to 0} \ln x = -\infty$ and $\lim_{x\to\infty} \ln x = \infty$.

As with exponential functions, we can also define logarithms for different bases. If $a > 0$ is any real number except 1, the exponential function with base a has range $\mathbb{R}$ and is invertible. Its inverse function is denoted by $\log_a x$. If $a \neq 1$ then $\ln a \neq 0$.

Note that since $y = a^x = \exp(x \ln a)$, we get $\log_a y = x = \frac{\ln a^x}{\ln a} = \frac{\ln y}{\ln a}$.

Exercise
Show that for any $a > 0$, $a \neq 1$, and any positive x, y, we have

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a x^n = n \log_a x$
3. $\log_a$ is strictly increasing if $a > 1$ and strictly decreasing if $a < 1$.

EOE.

We conclude the section with the following observation on arbitrary differentiable functions satisfying an equation of the form $f' = cf$.

Proposition
Let f be a differentiable function on $\mathbb{R}$ such that $f' = cf$. Then there is a unique constant d such that $f(x) = d \exp(cx)$ for all x. EOP.

Proof. Let $g(x) = \exp(-cx) f(x)$. Then $g'(x) = -c \exp(-cx) f(x) + \exp(-cx)\, cf(x) = 0$. So g is constant, $g = \frac{f}{\exp(cx)} = d$. QED.

This shows that up to multiplication by a constant, the functions of exponential growth or decay are exponential functions.

2.4 Local properties of differentiable functions continued

2.4.1 Invertible functions

Recall that a function $f: I \to J$ is called invertible with inverse g if there is a function $g: J \to I$ such that $f \circ g = \mathrm{id}_J$ and $g \circ f = \mathrm{id}_I$.

In this case, the *inverse* g is denoted f^{-1}.

Remark
For a function f to be invertible it must be both, injective and surjective. However, if f is injective, we often identify it with the function with codomain the range of f, and then it is automatically bijective (as a function to its range). This is not really an issue if f is continuous as then the range is again an interval, if the domain is an interval.

Proposition
Suppose f is continuous on I and invertible and suppose $x_0 \in D_f$ and $f'(x_0) \neq 0$ for some $x_0 \in I$. Then $f^{-1}: f(I) \to I$ is differentiable at $y_0 = f(x_0)$, and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.
If $f'(x_0) = 0$, then f^{-1} is not differentiable at x_0. EOP.

Proof. As f is continuous, $f(I) = J$ is an interval. Let $g = f^{-1}$. If $f'(x_0) \neq 0$, we must show that

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0}$$

exists and is finite. Note that as f is invertible, it is injective, and so is $g = f^{-1}$. This means if $y \neq y_0$, then $x = f^{-1}(y) \neq x_0 = f^{-1}(y_0)$. Let $y_n \in J$ be a sequence with $y_n \neq y_0$ but $y_n \to y_0$. Then $x_n := f^{-1}(y_n) \neq x_0$ is a sequence in I. Moreover, $x_n \to x_0$. This uses the fact that f^{-1} is continuous at y_0. Then, if $f'(x_0) \neq 0$

$$\frac{g(y_n) - g(y_0)}{y_n - y_0} = \frac{x_n - x_0}{f(x_n) - f(x_0)} \to \frac{1}{f'(x_0)}$$

for $n \to \infty$. This shows that g is differentiable if $f'(x_0) \neq 0$.

If $f'(x_0) = 0$, and f^{-1} were differentiable at $y_0 = f(x_0)$, then the chain rule asserts that $1 = (f^{-1} \circ f)'(x_0) = {f^{-1}}'(y_0) f'(x_0) = 0$, which is impossible. QED.

Here is a sometimes useful theorem (which we haven't discussed in class).

Theorem*
Let $f: I \to \mathbb{R}$ be a differentiable function and suppose f' is continuous. Let $x_0 \in I$ and suppose $f'(x_0) \neq 0$. Then there is a relative open neighbourhood J of $x_0 \in J \subseteq I$, and an interval $K \subseteq \mathbb{R}$ such that $f(J) = K$ and $f: J \to K$ is invertible. Moreover, f^{-1} is also continuously differentiable on K.

This is a version of a much more difficult analogous result for functions in several variables (called the Implicit Function Theorem).

Proof. We prove the case $f'(x_0) > 0$ and leave the case $f'(x_0) < 0$ as an exercise.

As f' is continuous, there is a relative open subset $J \subseteq I$ with $x_0 \in J$ such that $f' > 0$ on J. We denote the restriction of f to J by f as well. Then f is strictly monotone on J, and $f(J) = K$ is an interval. Finally, f^{-1} is differentiable everywhere on K by the previous proposition, from which we obtain the formula that ${f^{-1}}'(y) = \frac{1}{f'(f^{-1}(y))}$. As the multiplicative inverse of a composition of two continuous functions, ${f^{-1}}'$ is continuous as well. QED.

The theorem essentially says that any continuously differentiable function is *locally* invertible, as long as its derivative is nonzero.

Warning
As before, it is important to note that $f' > 0$ or $f' < 0$ only guarantees that f is injective (because it is strictly monotone) if the domain of f is an interval.
Consider $D = (0,1) \cup (1,2) = (0,2) \setminus \{1\}$ and $f(x) = x$ on $(0,1)$ and $f(x) = x - 1$ on $(1,2)$. $f' = 1 > 0$ on D, but f is not injective. EOW.

Exercise
Suppose f is a smooth invertible function, and $f'(x) \neq 0$ for all x. Show that f^{-1} is also smooth. EOE.

Example

1. Consider $f(x) = \sin x$. Then f is not invertible unless we restrict its domain to a small enough set, say $I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $f(I) = [-1,1]$ and $f: I \to [-1,1]$ is invertible.
   Its inverse function is often denoted $\sin^{-1} x$ or $\arcsin x$. We find that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos \sin^{-1} x}$, which is defined as long as $x \neq \pm 1$. Here f^{-1} is differentiable on $(-1,1)$. Moreover $\cos t$ is positive on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Now observe that $\cos \sin^{-1} x = \sqrt{1 - (\sin \sin^{-1} x)^2} = \sqrt{1 - x^2}$. The upshot is: $f'(x) = \frac{1}{\sqrt{1-x^2}}$ on $(-1,1)$.
2. Consider $\tan x$ which has domain $\mathbb{R} \setminus \left(\frac{\pi}{2} + \mathbb{Z}\pi\right)$. Recall that $\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cos x + \sin x \sin x}{(\cos x)^2} = \frac{1}{(\cos x)^2}$. This function is positive on $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $\tan x$ is strictly increasing on that interval. As $\sin x$ approaches ± 1 and $\cos x$ approaches 0 towards the boundaries of I, $\tan(I) = (-\infty, \infty)$. And $\tan x$ has an inverse, denoted $\tan^{-1} x$ or $\arctan x$, defined on all of $\mathbb{R}$.
   Then $\frac{d}{dx}\arctan x = (\cos \arctan x)^2$. Now note that $(\tan x)^2 = \frac{(\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2} - 1$ or
   $$(\cos x)^2 = \frac{1}{1 + (\tan x)^2}$$
   Using this we get $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$.
3. Consider the exponential function $f(x) = \exp(x)$. As $f' = f > 0$, f is invertible. We have seen that the range of f is all of $\mathbb{R}_{>0}$. Let $g(x) = \ln x$ be the inverse function. Then for all $y_0 \in \mathbb{R}_{>0}$ $g'(y_0) = \frac{1}{f'(g(y_0))} = \frac{1}{f(g(y_0))} = \frac{1}{y_0}$.
4. Let $y \in \mathbb{R}$ and for $x > 0$ Consider $f(x) = x^y$. Then $f(x) = \exp(y \ln x)$, so the chain rule gives $f'(x) = \frac{\exp(y \ln x) y}{x} = \frac{x^y y}{x} = \frac{\exp(y \ln x - \ln x)}{x} = y\, x^{y-1}$.
5. Let $f(x) = x^x$ defined on $x > 0$. Then $f(x) = \exp(x \log x)$ and $f'(x) = \exp(x \log x)\,(x \log x)' = x^x(\log x + 1) = (\log x + 1)x^x$

2.4.2 L'Hôpital's rule

When computing limits of quotients of functions, we run into trouble when both functions have $\pm\infty$ or 0 as a limit. Here we will find a rule that is surprisingly useful. We have seen examples of this: for example, the limit of $\frac{\sin x}{x}$ for $x \to 0$. Sometimes one can attack these limits directly. But sometimes this is hard.

Consider the following situation: suppose f, g are differentiable on $[a, b]$ and $f(a) = g(a) = 0$. Suppose also that $g(x) \neq 0$ for $x > a$. We want to find $\lim_{x\to a} \frac{f(x)}{g(x)}$ if it exists. If $g'(a) \neq 0$, then this limit is equal to $\frac{f'(a)}{g'(a)}$.

Indeed $\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(x)-f(a)}{x-a}\frac{x-a}{g(x)-g(a)} \to \frac{f'(a)}{g'(a)}$ for $x \to a$.

If f, g are *continuously differentiable* at a, this means $\lim_{a\to x} \frac{f(x)}{g(x)} = \lim_{a\to x} \frac{f'(x)}{g'(x)}$.

This works for example in case $f = \sin x$ and $g = x$. But of course, this is cheating because $\lim_{x\to 0} \frac{\sin x}{x} = \sin' 0$.

Derivatives can be useful when computing some limits of quotients of functions. A different version of a similar rule as the above is

Theorem (L'Hôpital's rule)
Let I be an interval and $a \in I$. Let f, g be functions defined and differentiable on $J = I \setminus \{a\}$. Suppose that $g'(x) \neq 0$ for all $x \in J$. Furthermore, suppose one of the following two statements holds

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.
2. $\lim_{x\to a} g(x) = \pm\infty$.

Suppose $L = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

HISTORY

Many attributions in mathematics are wrong or at least imprecise. This rule is named after Guillaume de l'Hôpital (1661-1704) (a.k.a. Guillaume-François-Antoine Marquis de l'Hôpital, Marquis de Sainte-Mesme, Comte d'Entremont and Seigneur d'Ouques-la-Chaise) who published it in his 1696 book "Analyse des infiniment petits pour l'intelligence des lignes courbes", apparently the first text-book on the differential calculus. It is believed that the rule actually originated with Johann Bernoulli (1667-1748).

Remark
The point of first choosing I and then defining $J = I \setminus \{a\}$ is the following: In the majority of applications I is of the form $[a, b)$ or $(b, a]$, that is a is a boundary point of I. Then J is again an interval ((a, b) or (b, a), respectively. The limit that the l'Hôpital rule then computes is a one-sided limit. But in some cases, we want to compute a two sided limit, and then a is an interior point of I. In that case J is not an interval itself but a union of two intervals: if $I = (c, d)$ and $a \in (c, d)$, then $J = (c, a) \cup (a, d)$. You should not worry too much about this.

The proof will require a stronger version of the Mean Value Theorem.

Lemma (Cauchy Mean Valuey Theorem)
Let f, g be continuous on $[a, b]$ and differentiable on (a, b). Then there is $c \in (a, b)$ such that

$$f'(c)\big(g(b)-g(a)\big)=g'(c)(f(b)-f(a))$$

In particular if $g'(x)\neq 0$ on (a,b), then

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

Proof. Consider h defined on $[a,b]$ as $h(x)=(f(x)-f(a))\big(g(b)-g(a)\big)-(g(x)-g(a))(f(b)-f(a))$. Then h is continuous on $[a,b]$ and differentiable on (a,b). Also $h(a)=h(b)=0$. By Rolle's Theorem there must be $c\in(a,b)$ such that $h'(c)=0$. But hat is equivalent to the first assertion.

Now if $g'(x)\neq 0$ for all $x\in(a,b)$, then $g(b)\neq g(a)$ by the usual Mean Value Theorem. Thus, the second assertion follows. QED.

The usual MVT is the special case where $g(x)=x$.

Proof of l'Hôpital's Rule. The proof follows the one in "Heuser, *Lehrbuch der Analysis*". It has the advantage that it covers most cases with a minimum of case by case considerations. We will show that $\lim_{x\to a^+}\frac{f(x)}{g(x)}=L=\lim_{x\to a^-}\frac{f(x)}{g(x)}$ whenever this makes sense. Both limits exist and are equal if a is an interior point of I, and only one of them exists if a is a boundary point of I.

Suppose $(-\infty,a)\cap J\neq\emptyset$. Then we must show that $\lim_{x\to a^-}\frac{f(x)}{g(x)}=L$.

Suppose first hat $L\in\mathbb{R}\cup\{-\infty\}$, and let $M>L$ arbitrary. We will show that there is $\delta_1>0$ such that $(a-\delta_1,a\subseteq J$ and $\frac{f(x)}{g(x)}<M$ for all $x\in(a-\delta_1,a)$.

To start let N be a number such that $L<N<M$. there is $\delta>0$ such that $(a-\delta,a)\subseteq J$ and we have $\frac{f'(x)}{g'(x)}<N$ for all $x\in(a-\delta,a)$.

Let $y\neq x\in(a-\delta,a)$. Then $g(x)\neq g(y)$ by the MVT applied to $[x,y]$ (if $y>x$) or $[y.x]$ (if $y<x$). The generalized MVT then shows there is z between x,y such that

$$\frac{f(x)-f(y)}{g(x)-g(y)}=\frac{f'(z)}{g'(z)}<N<M$$

(note that $z\in(a-\delta,a)$). This holds for all $y\neq x\in(a-\delta,a)$.

If 1. holds then this means $\lim_{y\to a}\frac{f(x)-f(y)}{g(x)-g(y)}=\frac{f(x)}{g(x)}\leq N<M$. For this we use that $g(x)\neq 0$ for x close to a (because $g(x)=0$ for at most one $x\in(a-\delta,a)$ by the MVT).

If 2. holds, then $g(x),g(y)\neq 0$ for x,y close enough to a as well. If y is close enough to a, then $\frac{g(x)-g(y)}{g(x)}=1-\frac{g(y)}{g(x)}>0$ by 2. The above shows that

$$\frac{f(x)-f(y)}{g(x)-g(y)}\frac{g(x)-g(y)}{g(x)}=\frac{f(x)-f(y)}{g(x)}<N\frac{g(x)-g(y)}{g(x)}=N-N\frac{g(y)}{g(x)}$$

And so

$$\frac{f(x)}{g(x)} < N - N\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

This holds for all $x \neq y$ close enough to a. Now $\lim_{x\to a}\left(N - N\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}\right) = N$. So there is $0 < \delta' < \delta$ such that for all $x \in (a - \delta', a)$, we have $N - \frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} < M$ and then for such x also $\frac{f(x)}{g(x)} < M$.

The upshot is for every $M > L$ we have $\delta_1 = \delta'$ such that for all $x \in (a - \delta_1, a)$

$$\frac{f(x)}{g(x)} < M$$

Now if $L \in \mathbb{R} \cup \{\infty\}$ we can mimic these arguments to construct δ_2 such that for all $M' < L$ and all $x \in (a - \delta_2, a)$, we have

$$\frac{f(x)}{g(x)} > M'$$

Indeed, by the generalized MVT for a chosen $M' < L$ and any N' with $M' < N' < L$

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(z)}{g'(z)} > N' > M'$$

as long as x, y are close enough to a. In Case 1. the limit for $y \to a^-$ of the left hand side

$$\frac{f(x)}{g(x)} \geq N' > M'$$

And in Case 2. this means again

$$\frac{f(x) - f(y)}{g(x)} = \frac{f(x) - f(y)}{g(x) - g(y)}\frac{g(x) - g(y)}{g(x)} > N' - N'\frac{g(y)}{g(x)}$$

and so

$$\frac{f(x)}{g(x)} > N' - N'\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

The limit of the right hand side is N' and so the right hand side is $> M'$ for y close enough to a.

So we find δ_2 as claimed.

Now we are done: if L is infinite, $L = \infty$ say, then we have for any M' a δ_2 such that $\frac{f(x)}{g(x)} > M'$ for all $x \in (a - \delta_2, a)$. If $L = -\infty$ we have for every M a δ_1 such that $\frac{f(x)}{g(x)} < M$ on $(a - \delta_1, a)$.

For finite L we combine the two statements above: for $\varepsilon > 0$ let $M = L + \varepsilon$ and $M' = L - \varepsilon$ and put $\delta = \min\{\delta_1, \delta_2\}$. Then for all $x \in (a - \delta, a)$ we have

$$L - \varepsilon < \frac{f(x)}{g(x)} < L + \varepsilon$$

and it follows that $\lim\limits_{x\to a^-}\frac{f(x)}{g(x)} = L$.

The case that $(a,\infty)\cap J \neq \emptyset$ is similar. Finally, if a is an inner point of I the arguments (applied to $J \cap (a,\infty)$ and $(-\infty,a)\cap J$ separately) given show that $\lim\limits_{x\to a^+}\frac{f(x)}{g(x)} = \lim\limits_{\mathrm{x}\to \mathrm{a}^-}\frac{f(x)}{g(x)}$. QED.

Corollary (L'Hôpital's Rule for $x \to \pm\infty$)
The theorem also holds for intervals $(-\infty, b)$ and (b,∞) and $a = \pm\infty$. EOL.

Poof. We only must modify our argument slightly. We will discuss the case $a = \infty$. As before $g(x) \neq 0$ for x large enough. Indeed, $g'(x) \neq 0$ everywhere, means g is injective. So $g(x) = 0$ for at most one x. We may also assume that $b > 0$.

Consider the functions $F = f \circ \frac{1}{x}$ and $G = g \circ \frac{1}{x}$ on the interval $(0, b^{-1})$.

Then $\lim\limits_{x\to 0} F(x) = \lim\limits_{x\to\infty} f(x) = 0$. Similarly, $\lim\limits_{x\to 0} G(x) = 0$.

Next, $F'(x) = f'\left(\frac{1}{x}\right)\frac{-1}{x^2}$ and $G'(x) = -\frac{g'\left(\frac{1}{x}\right)}{x^2} \neq 0$ on $(0, b^{-1})$. Finally,

$$\lim_{x\to 0}\frac{F'(x)}{G'(x)} = \lim_{x\to 0^+}\frac{f'\left(\frac{1}{x}\right)}{g'\left(\frac{1}{x}\right)} = L$$

We may now apply the theorem to conclude the result. QED.

Examples

1. We have seen $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$. Using l'Hôspital's Rule we can verify this:
$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$
2. Let $f =$ exp be the exponential function. Let $g(x)$ be any polynomial, of degree $n \geq 0$, say. Then $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = \pm\infty$ with the sign equal to the sign of $g(x)$ for large x.
Indeed, by induction on n the result holds for smaller degree polynomials (and the result is trivial for degree 0). Now $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = \lim\limits_{x\to\infty}\frac{f(x)}{g'(x)} = \pm\infty$. As the sign of $g(x)$ is the same as the sign of $g'(x)$ for large x, the result follows.
This is an example where one has to apply the rule repeatedly.
3. With f as in 2., $f(-x) = f(x)^{-1}$. This immediately shows that $f(x) \to 0$ for $x \to -\infty$. Moreover, for every polynomial g, we find $\lim\limits_{x\to-\infty} f(x)g(x) = 0$.
Indeed, this follows as $\lim\limits_{x\to-\infty} f(x)g(x) = \lim\limits_{x\to\infty}\frac{g(-x)}{f(x)} = 0$ by 2.
Similarly, $\lim\limits_{x\to\infty} g(x)f(-x) = 0$ for every polynomial g.
4. Consider $h(x) = x\log\left(1+\frac{1}{x}\right)$ defined for $x > 0$. Then we can write $h(x) = \frac{f(x)}{g(x)}$ with $f(x) = \log\left(1+\frac{1}{x}\right)$ and $g(x) = \frac{1}{x}$. Note that $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = 0$.
We have $f'(x) = \frac{1}{1+\frac{1}{x}}\frac{-1}{x^2}$ and $g'(x) = -\frac{1}{x^2}$. Thus $\frac{f'(x)}{g'(x)} = \frac{1}{1+\frac{1}{x}} \to 1$.

We find that $\lim_{x\to\infty} x\log\left(1+\frac{1}{x}\right) = 1$.

Note, a similar argument shows $\lim_{x\to 0^+}\left(\frac{\log(1+x)}{x}\right) = 1$.

5. Compute $\lim_{x\to\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}$. Here the rule does not help at all. However,

$$\frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{1+e^{-2x}}{1-e^{-2x}} \to 1$$

Application

Part 4. of the example above shows that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

Indeed, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\exp\left(x\log\left(1+\frac{1}{x}\right)\right) = \exp\left(\lim_{x\to\infty}\left(x\log\left(1+\frac{1}{x}\right)\right)\right) = \exp(1) = e$.

We conclude that $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. EOA.

An interesting consequence of l'Hôspital's Rule is the following lemma (which can also be proved using the MVT as in a homework problem):

Lemma

Let I be an interval, $a \in I$, and let $f: I \to \mathbb{R}$ be a function differentiable on (at least) $I \setminus \{a\}$ and continuous on all of I.

Suppose $\lim_{x\to a} f'(x) = L$ is exists and is finite. Then f is differentiable at a and $f'(a) = L$.

In particular, f is continuously differentiable at a. EOL.

Proof. By l'Hôspital's Rule,

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{f'(x)}{1} = L$$

QED.

Example

Let again

$$f(x) = \begin{cases} 0; & x = 0 \\ e^{-\frac{1}{x^2}}; & x \neq 0 \end{cases}$$

Then f is continuous as $\lim_{x\to 0} e^{-\frac{1}{x^2}} = 0$. Moreover for $x \neq 0$, $f'(x) = -\frac{2}{x^3}e^{-\frac{1}{x^2}}$.

We will show that $\lim_{x\to 0} f'(x) = 0$, and conclude by the lemma that $f'(0) = 0$.

For this note that $-\frac{2}{x^3}e^{-\frac{1}{x^2}} = -\frac{2}{x^3 e^{\frac{1}{x^2}}}$. So it suffices to show that $\lim_{x\to 0}\left|x^3 e^{\frac{1}{x^2}}\right| = \infty$.

But $\lim_{x\to 0^+} x^3 e^{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{e^{x^2}}{x^3} = +\infty$, and $\lim_{x\to 0^-} x^3 e^{\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{e^{x^2}}{x^3} = -\infty$. Together $\lim_{x\to 0}\left|x^3 e^{\frac{1}{x^2}}\right| = \infty$ follows. EOE.

2.5 Convex functions

Definition

Let I be an interval. A function f defined on I is called **convex**, if for all $a < b \in I$ we have

Equation 2

$$f(x) \le f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

for all $x \in (a, b)$. It is called **strictly convex** if the inequality above is strict for all $x \in (a, b)$.
f is called **concave** if $-f$ is convex, and strictly concave, if $-f$ is strictly convex. EOD.

Convexity is fundamentally a geometric notion.

Recall that $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ can be identified with the Euclidean plane.

A subset $C \subseteq \mathbb{R}^2$ is called **convex** if for every pair $p, q \in C$, the *line segment* $\{ap + (1 - a)q \mid 0 \le a \le 1\}$ is again a subset of C.

If you are familiar with linear algebra, the line through two points p, q is defined by the vector equation $x = p + t(q - p) = (1 - t)p + tq$ where t ranges over all real numbers.

The right-hand side of Equation 2 is the function whose graph is the line segment joining $(a, f(a))$ and $(b. f(b))$.

We can reparametrize this by putting $s = \frac{x-a}{b-a}$, then $x = (1 - s)a + sb$ and $s \in (0,1)$ whenever $x \in (a, b)$. Equation 2 then becomes

$$f\big(a + s(b - a)\big) \le f(a) + \big(f(b) - f(a)\big)s = (1 - s)f(a) + sf(b)$$

for all $s \in (0,1)$. Rewriting it slightly again, we get

$$f\big((1 - s)a + sb\big) \le (1 - s)f(a) + sf(b)$$

f is convex if this inequality holds for all $s \in (0,1)$ and strictly convex, if it is a strict inequality for all $s \in (0,1)$.

For a concave function the inequalities are simply reversed.

Exercise
How that f is convex on an interval I if and only if for all $n \ge 2$ and all $t_1, t_2, \dots, t_n \in (0,1)$ with $t_1 + t_2 + \cdots + t_n = 1$ we have $f(t_1x_1 + t_2x_2 + \cdots + t_nx_n) \le t_1f(x_1) + t_2f(x_2) + \cdots + t_nf(x_n)$ for all $x_1 \le x_2 \le \cdots \le x_n \in I$.
(*Hint*: The definition gives you the case $n = 2$.) EOE.

In this context, the whole point of this business is to derive inequalities for the values of functions (which are always hard to come by).

Example
For $x, y > 0$ consider the following definitions of a mean value:
There is the **arithmetic mean** defined as $\frac{x+y}{2}$ and there is the **geometric mean** defined as $\sqrt{xy}$. Is there a relationship between them?
In fact, we have $\sqrt{xy} \le \frac{1}{2}(x + y)$.
To prove this, it suffices to show that $\log\sqrt{xy} \le \log\left(\frac{1}{2}(x + y)\right)$.

Note that $\sqrt{xy} = (xy)^{\frac{1}{2}}$. Then $\log\left(\sqrt{xy}\right) = \frac{1}{2}(\log x + \log y)$.
Suppose we knew that $\log x$ was concave (a.k.a concave down). Then we could conclude that

$$\log\left(\frac{1}{2}x + \frac{1}{2}y\right) \geq \frac{1}{2}\log(x) + \frac{1}{2}\log(y) = \frac{1}{2}(\log(x) + \log(y)) = \log\sqrt{xy}$$

as needed.
Using the exercise above, one can modify this argument to show that for $x_1, x_2, \dots, x_n > 0$ we always have

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

EOE.

How could we possibly determine whether or not a function is convex or concave? The general idea is the Horse Race Theorem.

To be convex, the graph of a function must always stay below the secant between two points in the graph:

$$f(t) \leq f(a) + \frac{f(b) - f(a)}{b - a}(t - a)$$

for all $t \in [a, b]$.

Theorem (Convexity Theorem)
Suppose f is differentiable on an interval I. Then f is convex on I iff f' is monotone increasing on I°. f is concave on I iff f' is monotone decreasing on I. EOT.

Proof. We will prove the criterion for convexity. Applied to $-f$ the criterion for concavity follows.

Suppose f is convex, and let $a < b \in I^\circ$. We must show that $f'(a) \leq f'(b)$.

Now $\frac{f(b)-f(a)}{b-a} \geq \frac{f(t)-f(a)}{t-a}$ for all $t \neq a$. This shows that $\lim\limits_{t \to a^+} \frac{f(t)-f(a)}{t-a} = f'(a) \leq \frac{f(b)-f(a)}{b-a}$.

Similarly, $f(t) \leq f(b) + \frac{f(a)-f(b)}{b-a}(b - t)$, and so

$$\frac{f(b) - f(t)}{b - t} \geq \frac{f(b) - f(a)}{b - a}$$

for all $t \in [a, b)$, and so $f'(b) = \lim\limits_{t \to b^-} \frac{f(b)-f(t)}{b-t} \geq \frac{f(b)-f(a)}{b-a} \geq f'(a)$.

Now suppose f' is increasing on I°. We must show that f is convex. Let $a < b \in I$. Then for any $x \in (a, b)$ we must have $\frac{f(x)-f(a)}{x-a} = f'(c_1)$ and $\frac{f(b)-f(x)}{b-x} = f'(c_2)$ for some $c_1 \in (a, x) \subseteq I^\circ$ and some $c_2 \in (x, b) \subseteq I^\circ$. In particular $c_1 < c_2$ so $f'(c_1) \leq f'(c_2)$.

Thus, for all $x \in (a, b)$, $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(x)}{b-x}$

Note $x - a, b - x > 0$, and so

$$(b - x)\big(f(x) - f(a)\big) \leq (x - a)(f(b) - f(x))$$

or

$$(b-a)f(x) \le (b-x)f(a) + (x-a)f(b)$$

Let $s = \frac{x-a}{b-a}$, then $x = (1-s)a + sb$, so $b - x = (1-s)b - (1-s)a$, and $x - a = -sa + sb$.

The above then becomes

$$(b-a)f(x) \le (1-s)(b-a)f(a) + s(b-a)f(b)$$

or, as $b - a > 0$

$$f\big((1-s)a + sb\big) \le (1-s)f(a) + sf(b)$$

for all $s \in (0,1)$. QED.

Note that we have shown that if f' is strictly increasing then f is strictly convex.

Corollary
Suppose f is defined and continuous on an interval I and twice differentiable on I°. Then
1. f is convex iff $f'' \ge 0$ on I°.
2. f is concave iff $f'' \le 0$ on I°.

If $f'' > 0$ then f is strictly convext. If $f'' < 0$ then f is strictly concave.

Proof. $f'' \ge 0$ is equivalent to f' increasing on I°. By the theorem, this is equivalent to f convex. The same reasoning applied to $-f$ gives the second assertion.

We also know that if $f'' > 0$ on I° then f' is strictly increasing on I°, showing that f is strictly convex. Again, the same reasoning applied to $-f$ shows that if $f'' < 0$ then f is strictly concave.

Example
1. e^x is strictly convex.
2. $\log x$ is strictly concave on $I = (0, \infty)$. Indeed the second derivative is $-\frac{1}{x^2} < 0$.
3. If f is a polynomial function of odd degree > 2, then f cannot be concave or convex everywhere. Indeed, f'' is a polynomial of odd degree and hence changes its sign somewhere.

Definition
Let f be defined on an interval I and $c \in I^\circ$ an interior point. Then c is called an **inflection point** if there is $\delta > 0$ such that f is convex on $(c - \delta, c]$ and concave on $[c, c + \delta)$ or vice versa f is concave on $(c - \delta, c]$ and convex on $[c, c + \delta)$. EOD.

To find inflection points, the following lemma is often useful.

Lemma
Let f be twice differentiable at an inflection point c. Then $f''(c) = 0$.

Proof. f is twice differentiable at c, so there is $\delta > 0$ such that both f' is defined on $(c - \delta, c + \delta)$ and f' is increasing on $(c - \delta, c]$ and decreasing on $[c, c + \delta)$ (or vice versa) by the Convexity Theorem. But this means f' has a local maximum at c (or a local minimum). In either case $f''(c) = 0$ by Lemma 2.2.2. QED.

Example

Let $f(x) = (1+x)\sqrt{1-x^2}$ defined on $[-1,1]$. Then f is differentiable (in fact, smooth) on $(-1,1)$ with

$$f'(x) = \frac{-2x^2 - x + 1}{\sqrt{1-x^2}}$$
$$f''(x) = \frac{-2x^3 + x - 1}{(1-x^2)\left(\sqrt{1-x^2}\right)}$$

We find $f'(x) = 0$ for $x_1 = \frac{1}{2}$ (The numerator of f' has a second root namely -1, but the formula given for f' is not defined at -1.)

The numerator of $f''(x)$ has roots -1, $x_2 = \frac{1-\sqrt{3}}{2}$, and $\frac{1+\sqrt{3}}{2}$, of which only x_2 is in $(-1,1)$.
Note that $f''(x) > 0$ for $x < x_2$ and $f'' < 0$ for $x > x_2$.
We conclude the following f is (strictly) convex on $[-1, x_2]$, strictly concave $[x_2, 1]$, and has a local maximum at x_1. x_2 is the only inflection point.
It is elementary to check that $\lim_{x\to 1} f'(x) = -\infty$. This suggests that f is not differentiable at $x = 1$. Indeed, as the limit exists, one can show that f cannot be differentiable at 1 (as the limit would necessarily be equal to that limit, as in the Lemma 2.4.2).
On the other hand L'Hôpital's Rule shows that $\lim_{x\to -1} f'(x) = 0$, and so f is differentiable there. EOE.

2.6 Some complements*

2.6.1 The Horse Race Theorem*

Theorem

Let $I = [a,b]$ and f, g be continuous functions on I, differentiable on (a,b).
Suppose

1. $f(a) \geq f(b)$
2. $f' \geq g'$ on (a,b)

Then $f(b) \geq g(a)$. If $f' > g'$ on (a,b) then $f(b) > g(b)$.

Proof. The function $g = f - g$ is monotone increasing on $[a,b]$ as $h' \geq 0$ on (a,b). Then $h(b) \geq h(a) \geq 0$. If $f' > g'$, then $h' > 0$ and hence h is strictly increasing and $h(b) > 0$. QED.

Example

Consider $f(x) = \exp x$ and $g(x) = 1 + x$ on $[0,b]$ (for any $b > 0$). Then as f is strictly monotone and $f' = f$, we find that $f' > 1 = g'$. As $f(0) = g(0) = 1$, we have $\exp(x) > 1 + x$ for all $x > 0$ and consequently also $x > \log(1+x)$.

3 Power series and Taylor's Theorem

Why are we interested in series? In analysis (or calculus, if you will), they provide a very important and rich class of functions, based on *power series*.

3.1 Review of series

3.1.1 Definition of series

A series is nothing but a very special form of sequence. Given any sequence a_n, we can form the associated series $\sum_{n=1}^{\infty} a_n$. What do we mean by that? Logically the series $\sum_{n=1}^{\infty} a_n$ is just the sequence of numbers $a_1, a_2, \ldots$, but we agree to compute the limit, if it exists, differently namely as

$$\sum_{n=1}^{\infty} a_n \coloneqq \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

We could also think of the series as the sequence $a_1, a_1 + a_2, \ldots$. Note in this sense *any* sequence is a series: the sequence $b_1, b_2, \ldots$ corresponds to to the series $\sum_{n=1}^{\infty} a_n$ where $a_1 = b_1, a_2 = b_2 - b_1, a_3 = b_3 - b_2, \ldots$

So $\sum_{n=1}^{\infty} a_n$ is *not* just the number $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ (if it exists) but also comprises the coefficients $a_1, a_2, \ldots$ In other words, typically two series are considered the same only if all the summands a_n are the same. **It is a slight abuse of notation to denote the series and its limit, if it exists, by the same symbols (namely $\sum_{n=1}^{\infty} a_n$).** But it is very common. If we want to emphasize that we are talking about the value (limit) of a series, we may also write

$\lim \sum_{n=1}^{\infty} a_n$ instead of $\sum_{n=1}^{\infty} a_n$.

Think of a series as the sequence of its partial sums (see below).

Definition
The series $\sum_{n=1}^{\infty} a_n$ is **convergent**, if its limit exists and is finite; otherwise it is **divergent**. Like any sequence, a series can have limit $\pm\infty$.
A **partial sum** of the series $\sum_{n=1}^{\infty} a_n$ is a sum of the form $S_N = \sum_{n=1}^{N} a_n$. The partial sums form a sequence with the same limit (if it exists).

A series can start at any integer (just like a sequence can), but for simplicity we state most results with base 1.

3.1.2 Convergence of series

Lemma
Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Then a_n is a zero sequence. EOL.

Note that a *zero sequence* is a sequence that converges to 0.

Proof. The sequence of partial sums S_N is a Cauchy sequence. Thus, for N sufficiently large $|a_{N+1}| = |S_{N+1} - S_N|$ is smaller than any predetermined $\varepsilon > 0$. QED.

The converse of the lemma is false:

Example

The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞. To see this note that

$$\sum_{n=N+1}^{2N} \frac{1}{n} \geq N \frac{1}{2N} = \frac{1}{2}$$

The partial sums therefore cannot form a Cauchy sequence so must diverge. As it is monotone it must diverge to infinity.

Remark

If a series has only nonnegative terms it is monotone increasing. Then it converges if and only if the sequence of partial sums is bounded. EOR.

One should also keep in mind that the limit of a series is nothing but the limit of the sequence of its partial sums. Thus, all convergence criteria for sequences apply to series just as well.

In particular, this applies to the Cauchy criterion:

Lemma (Cauchy Criterion for Series)

A series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if for every $\varepsilon > 0$ there is $N_0 \in \mathbb{N}$ such that for all $n, m > N_0$, $|\sum_{k=m}^{n} a_k| < \varepsilon$. EOL.

Proof. This is the Cauchy criterion applied to the partial sums. QED.

3.1.3 Absolute convergence

Definition.

A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ converges. EOD.

Note if a series converges absolutely, it also converges by the Cauchy-Criterion:

Indeed, if the series converges absolutely, for any $\varepsilon > 0$ there is n_0 such that $\left|\sum_{n=N}^{M} a_n\right| \leq \sum_{n=N}^{M} |a_n| < \varepsilon$ as long as $N, M > n_0$.

The converse is not true: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges but does not converge absolutely.

Note that a series $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} |a_n| < \infty$, that is, if and only if the sequence of partial sums $S_N = \sum_{n=1}^{N} |a_n|$ is bounded. Indeed, S_N is a monotone increasing sequence and therefore convergent if and only if it is bounded.

3.1.4 The geometric series

One of the most important series is the *geometric series* $\sum_{n=0}^{\infty} a^n$ where a is a real number.

Proposition

1. If $a \neq 1$, $\sum_{n=0}^{N} a^n = \frac{a^{N+1}-1}{a-1}$
2. $\sum_{n=0}^{\infty} a^n$ converges if and only if $|a| < 1$.

EOP.

Proof. The first part is an elementary induction on N. For the second part: if $|a| < 1$, then $\lim_{N \to \infty} a^N = 0$, and the first part shows that in this case

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

If $|a| > 1$, the limit is not finite ($a > 1$) or does not exist ($a < -1$). If $a = 1$, the series clearly diverges to ∞. If $a = -1$, the partial sums alternate between 1 and 0. QED.

3.1.5 Convergence tests for series

The convergence of the geometric series is at the heart of the following convergence criterion:

Theorem (Ratio test for series)

Let $\sum_{n=1}^{\infty} a_n$ be a series for which $a_n \neq 0$ for large enough n. Suppose $R = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists. Then

1. If $|R| < 1$, the series converges.
2. If $|R| = 1$ no statement can be made.
3. If $|R| > 1$, or $R = \pm\infty$, the series does not converge.

EOT.

Proof. This is the consequence of the following more general result. QED.

Theorem (Ratio test, general version)

Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \neq 0$ for $n > N_0$. Let $L = \limsup\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ and $M = \liminf\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$)

1. If $L < 1$, then the series converges absolutely.
2. If $M > 1$ the series does not converge.
3. If $L = 1$ or $M = 1$ no general statement can be made.

EOT.

Remark

To connect the ratio test above to the general ratio test here observe that for a convergent sequence $\limsup$ and $\liminf$ are the same and equal to the limit of the sequence. The only cases not strictly speaking covered are $R = \pm\infty$. But one can adapt the case $M = \infty$ to that (a_n cannot be a zero sequence if $R = \pm\infty$). EOR.

Warning

An earlier version of these notes contained a botched version of the general ratio test. In particular the second statement made no sense. It stated, in the notation of the theorem, that if $L \geq 1$ the series does not converge absolutely. We make no such claim. If $L > 1$ we **cannot** conclude that $\left|\frac{a_{n+1}}{a_n}\right| > L - \varepsilon > 1$ for all large n. For this we would need that $M > 1$. EOW.

Proof. Suppose $L < 1$. Fix $L_0 < 1$ positive such that $L < L_0 < 1$. Then there is $N \in \mathbb{N}$ such that for all $n > N$, we have $\left|\frac{a_{n+1}}{a_n}\right| < L_0$. So $\frac{|a_{N+k}|}{|a_{N+1}|} = \frac{|a_{N+k}|}{|a_{N+k-1}|}\frac{|a_{N+k-1}|}{|a_{N+k-2}|}\cdots\frac{|a_{N+2}|}{|a_{N+1}|} < L_0^{k-1}$. Then $\sum_{n=N+1}^{N+k}|a_n| < \sum_{n=N+1}^{N+k}|a_{N+1}|L_0^{n-(N+1)} = \sum_{i=0}^{k-1}|a_{N+1}|L_0^i$. As $0 < L_0 < 1$, the limit of this for $k \to \infty$ exists and is finite. Thus, $\sum_{n=N+1}^{\infty} |a_n| < \infty$.It follows $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

If $M > 1$, then for n large enough, we have $\left|\frac{a_{n+1}}{a_n}\right| > M - \varepsilon > 1$ (e.g. for $\varepsilon = \frac{M-1}{2}$). In particular this means $|a_{n+1}| \geq |a_n|$ for n large. Then $|a_n|$ for large n is a monotone increasing positive sequence. It therefore cannot be a zero sequence, and $\sum_{n=1}^{\infty} a_n$ cannot be convergent.

If $L = 1$ there are examples of absolutely convergent sequences, and of sequences that do not converge absolutely: Take $a_n = \frac{(-1)^n}{n}$. Then $L = 1$. The series does not converge absolutely, but it does converge. $a_n = \frac{1}{n^2}$ gives a series that its absolutely convergent, and $L = M = 1$. QED. $a_n = \frac{1}{n}$ gives an example of $L = M = 1$ that diverges.

Note that if $a_n \geq 0$ for all n absolute convergence is equivalent to convergence.

There is also a root test:

Theorem (Root test for series)
Let $\sum_{\mathrm{n}=1}^{\infty} a_n$ be a series and suppose $a_n \geq 0$. Let $L = \limsup \sqrt[n]{a_n}$.

1. If $L < 1$, the series converges.
2. If $L > 1$ the series diverges to ∞.
3. If $L = 1$ no general statement can be made.

EOT.

Warning.
An earlier version gave a botched proof for 2. If $L > 1$ it does not follow that for all sufficiently large n, $a_n > L - \varepsilon > 1$. However, it is enough that this holds for infinitely many n. EOW.

Proof. Suppose $L < 1$. Let $S_n = \sup\limits_{m \geq n} \sqrt[n]{a_n}$. Then for large enough n, $n > N_0$, say, we may assume that $S_n \leq L + \varepsilon$ for $\varepsilon = \frac{1-L}{2} > 0$. Then for such n we also have $\sqrt[n]{a_n} \leq L + \varepsilon < 1$, and $a_n \leq (L + \varepsilon)^n$. But then $\sum_{n=N_0}^{\infty} a_n \leq \sum_{n=N_0}^{\infty} (L + \varepsilon)^n < \infty$.

If $L > 1$, then for each $n_0 \in \mathbb{N}$, there is (at least one) $n > n_0$ such that $\sqrt[n]{a_n} > L - \varepsilon > 1$. But then $a_n > 1$ and therefore a_n is not a zero sequence. The series must therefore diverge.

Finally, for $L = 1$, the series is divergent if $a_n = 1$ for all n. But there are examples where it converges. QED.

Exercise

Show that $\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{n}} = 1 = \lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{n^2}}$. EOE.

Exercise
Let $S = \sum_{n=1}^{\infty} a_n$ (here S is the series not its limit) be any series. Let b_n be a subsequence of a_n constructed recursively as follows: Let $n_1 = \min\{n \mid a_n \neq 0\}$, $n_2 = \min\{n > n_1 \mid a_n \neq 0\}$, and so on, that is $n_{k+1} = \min\{\, n > n_k \mid a_n \neq 0\}$; then $b_i = a_{n_i}$ for $i = 1,2, \ldots$
Show that S and $\sum_{n=1}^{\infty} b_n$ have the same limit if either series has a limit. EOE.

This observation allows us to assume for many purposes that $a_n \neq 0$ for all n.

Exercise
Modify the arguments given in this section slightly to obtain the following criteria:
Let $\sum_{n=1}^{\infty} a_n$ be any series.

1. If there is $0 \leq q < 1$ such that $\left|\frac{a_{n+1}}{a_n}\right| \leq q < 1$ for almost all (that is, all but finitely many) n, then the series converges absolutely.

2. If there is $0 \leq q < 1$ such that $\sqrt[n]{|a_n|} \leq q < 1$ for almost all n, then the series is absolutely convergent.

EOE.

Warning

It is tempting to conclude that if $\left|\frac{a_{n+1}}{a_n}\right| < 1$ or $\sqrt[n]{|a_n|} < 1$for *almost all*[1] *n*,then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. However, this need not be the case. Note that if $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$, then $\frac{a_{n+1}}{a_n}, \frac{b_{n+1}}{b_n} < 1$ and $\sqrt[n]{a_n}, \sqrt[n]{b_n} < 1$ for all n. But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
In both cases tough, the ratio and root criteria are inconclusive. EOW.

There are several other criteria for convergence of series.

Compression Test

Suppose $a_n \geq 0$ is monotone decreasing. Then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=0}^{\infty} a_{2^n} 2^n$ converges. EOT.

Proof. Suppose $\sum_{n=0}^{\infty} a_{2^n} 2^n$ converges, to L, say. Since a_n is monotone decreasing,

$$a_1 + (a_2 + a_3) + (a_4 + a_5 + \cdots + a_7) + \cdots + (a_{2^n} + a_{2^n+1} + \cdots + a_{2^{n+1}-1}) \leq$$

$$a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n} \leq L$$

Thus $\sum_{n=1}^{2^{n+1}-1} a_n \leq L$ and therefore the series is bounded and convergent.

Conversely suppose $\sum_{n=1}^{\infty} a_n = M$. Then

$$a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + \cdots + a_8) + \cdots + (a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n})$$

$$\geq a_1 + a_2 + 2a_4 + \cdots + 2^{n-1} a_{2^n}$$

Thus $a_1 + \sum_{k=1}^{n} 2^{k-1} a_{2^k} \leq M$. But then $\sum_{k=1}^{n} 2^k a_{2^k} \leq 2M$ is still bounded, and $\sum_{k=0}^{\infty} a_{2^k} 2^k \leq 2M + a_1 < \infty$.

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called **bounded**, if its sequence of partial sums is bounded. A sequence a_n is called of **bounded variation** if the series $\sum_{n=1}^{\infty} |a_{n+1} - a_n|$ converges. EOD.

Fact

If a sequence a_n is of bounded variation, then it is convergent. EOF.

Proof. The series $\sum_{n=1}^{\infty} (a_{n+1} - a_n)$ is convergent as it is absolutely convergent by definition. Its partial sums S_N satisfy that $S_N = a_{N+1} - a_1$. As S_N is convergent, a_N is. QED.

Fact

If a_n is a monotone bounded sequence, then a_n is of bounded variation. EOF.

[1] "Almost all" natural numbers means "all but finitely many."

Proof. Suppose a_n is monotone increasing. Then $\sum_{n=1}^{N} |a_{n+1} - a_n| = \sum_{n=1}^{N}(a_{n+1} - a_n) = a_N - a_1$. This converges to a finite limit for $N \to \infty$ as a_n is convergent. A similar argument works if a_n is monotone decreasing (or apply this argument to the sequence $-a_n$). QED.

Dirichlet's Rule
Let $\sum_{n=1}^{\infty} a_n$ be a bounded series and let b_n be a monotone sequence with limit 0. Then $\sum_{n=1}^{\infty} a_n b_n$ converges. EOR.

Proof. Let $S_N = \sum_{n=1}^{N} a_n$. There is $B > 0$ such that $|S_N| \leq B$ for all N. Let $N > M > 1$ and $n \geq M$. Then $a_n = S_n - S_{n-1}$ for all $M \leq n \leq N$ and hence

$$\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} b_n (S_n - S_{n-1})$$

We can rewrite the right hand side as

$$S_N b_{N+1} - S_{M-1} b_M + \sum_{n=M}^{N} S_n (b_n - b_{n+1})$$

The absolute value of this can be bounded by $B|b_{N+1}| + B|b_M| + B\sum_{n=M}^{N} |b_n - b_{n+1}|$. For any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that each of these summands is $< \frac{\varepsilon}{3}$ as long as $N, M > n_0$, as b_n is of bounded variation and a zero sequence. QED.

Leibniz Rule
Let $a_n \geq 0$ be a monotone decreasing sequence with limit 0.
Then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges. EOR.

Proof. $\sum_{n=1}^{\infty}(-1)^n$ is a bounded series, so the result follows from Dirichlet's Rule. QED.

Example
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

3.1.6 The exponential series

One of the most important examples of a series is the exponential series.

Definition
For every $x \in \mathbb{R}$ the **exponential series** is defined as $\mathrm{E}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. EOD.

Lemma
The exponential series converges for all $x \in \mathbb{R}$. EOL.

Proof. Let $x \in \mathbb{R}$. We may assume that $x \neq 0$. Then $\frac{\left|\frac{1}{(n+1)!} x^{n+1}\right|}{\left|\frac{1}{n!} x^n\right|} = \frac{|x|}{n+1} \to 0$ for $n \to \infty$. By the ratio test for series, this means the series converges, and in fact absolutely converges. QED.

Because of the lemma, the exponential series gives rise to a function $\mathrm{E}: \mathbb{R} \to \mathbb{R}$

We will see soon that this function is indeed **the** exponential function whose existence we claimed earlier.

3.1.7 Rearranging series

Let $\sum_{n=1}^{\infty} a_n$ be a series. Since addition is commutative, it is tempting to assume one can rearrange the summands and have the same limit.

To make precise what this means, we say a **rearrangement** or **reordering** of the series $S = \sum_{n=1}^{\infty} a_n$ is a series of the form $S_\sigma \coloneqq \sum_{n=1}^{\infty} a_{\sigma(n)}$ where $\sigma: \mathbb{N} \to \mathbb{N}$ is a **bijection**, in this context often also called a **permutation**.of $\mathbb{N}$. Here S, S_σ denote the **series**, and not its limit.

Analogous statements hold for series where the index starts at some integer other than $\mathbb{N}$.

Notation
Let $S \subseteq \mathbb{N}$ be a subset. If a_n is a sequence we want to define the symbols $\sum_{\substack{n=1\\n\in S}}^{N} a_n$ and $\sum_{\substack{n=1\\n\in S}}^{\infty} a_n$.
The first one is defined as the sum of all a_n for which *both*, $n \in S$ and $n \leq N$. In particular, it is equal to 0 if S is empty. The second needs more consideration. The *value* is defined as $\lim_{N\to\infty} \sum_{\substack{n=1\\n\in S}}^{N} a_n$ (if that exists). If we want to think of it as a *series* we must specify a sequence b_n such that $\sum_{\substack{n=1\\n\in S}}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$ as series. We do this as follows

$$b_n = \vdash \begin{cases} a_n & n \in S \\ 0 & n \notin S \end{cases}$$

We also write $\sum_{\substack{n=1\\n\in S}}^{N} a_n$ for the partial sum $\sum_{n=1}^{N} b_n$.
EON.

Example
Let $S = \sum_{n=1}^{\infty} a_n$ be absolutely convergent. Let $P = \{\, n \mid a_n \geq 0\}$ and $Q = \{n \mid a_n < 0\}$. Then both $\sum_{\substack{n=1\\n\in P}}^{\infty} a_n$ and $\sum_{\substack{n=1\\n\in Q}}^{\infty} a_n$ are absolutely convergent.
Indeed, $\sum_{\substack{n=1\\n\in P}}^{N} |a_n|$, $\sum_{\substack{n=1\\n\in Q}}^{N} |a_n|$ are both bounded by $\sum_{n=1}^{N} |a_n|$.

Theorem
If a series S is absolutely convergent, then so is any rearrangement S_σ and the limit is the same. EOT.

Proof. Let $S = \sum_{n=1}^{\infty} a_n$ (the series, not the limit). We first treat the case that $a_n \geq 0$ for all n. Let L be the limit of S.

Let $S_N = \sum_{n=1}^{N} a_n$. Then $S_N \leq L$ for all N. Let σ be any permutation of $\mathbb{N}$. Since S_σ has nonnegative summands we must show that S_σ is bounded to conclude it is convergent.

For any n, let $\Pi_n = \{\, \sigma(1), \sigma(2), \dots, \sigma(n)\} \subseteq \mathbb{N}$. Let $\pi_n = \max \Pi_n$. Then for any $N \in \mathbb{N}$, $\sum_{n=1}^{N} a_{\sigma(n)} \leq \sum_{n=1}^{\pi_N} a_n \leq L$. This uses that $a_n \geq 0$ and that every summand on the left occurs on the right. But this means that S_σ is bounded by L. And consequently $\sum_{n=1}^{\infty} a_{\sigma(n)} \leq L$.

It follows that S_σ has a limit $M \leq L$. We have shown that for every convergent series with nonnegative coefficients, any reordering results in a series with a limit less than or equal to the limit of the original series.

On the other hand, S is a reordering of S_σ. Indeed, $S = (S_\sigma)_\mu$ where μ is the inverse permutation defined by $\mu(n) = m$ if $\sigma(m) = n$. By the argument just given, this shows that $L \le M$. It follows that $L = M$.

Now for the general case. If S is absolutely convergent, we have established that S_σ is also absolutely convergent (and therefore, convergent). It remains to show that the limits are equal.

For $n \in \mathbb{N}$, let $P = \{n \mid a_n \ge 0\}$ and $Q = \{n \mid a_n < 0\}$.

Then $S_N = \sum_{n=1}^{N} a_n = \sum_{\substack{n=1\\ n\in Q}}^{N} a_n + \sum_{\substack{n=1\\ n\in Q}}^{N} a_n = \sum_{\substack{n=1\\ n\in P}}^{N} a_n - \sum_{\substack{n=1\\ n\in Q}}^{N} |a_n|$.

Note that the limits for $N \to \infty$ of both summands on the right exist (as they are both bounded by $\pm \sum_{n=1}^{\infty} |a_n|$).

Thus $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{\substack{n=1\\ n\in P}}^{N} a_n - \lim_{N\to\infty} \sum_{\substack{n=1\\ n\in Q}}^{N} |a_n|$.

The same applies for S_σ. But note that the sets $P_\sigma = \{ n \mid a_{\sigma(n)} \ge 0 \}$ and $Q_\sigma = \{ n \mid a_{\sigma(n)} < 0 \}$ for S_σ are $\sigma^{-1}(P)$ and $\sigma^{-1}(Q)$, respectively. Indeed $n \in P_\sigma$ iff $\sigma(n) \in P$, and similarly for Q_σ.

Therefore the series $\sum_{\substack{n=1\\ n\in P_\sigma}}^{\infty} a_{\sigma(n)}$ is a reordering of the series $\sum_{\substack{n=1\\ n\in P}}^{\infty} a_n$ and same for $\sum_{\substack{n=1\\ n\in Q_\sigma}}^{\infty} |a_{\sigma(n)}|$ and $\sum_{\substack{n=1\\ n\in Q}}^{\infty} |a_n|$. It follows that these two pairs have the same limits, respectively, by the first part. QED.

Corollary
If $a_n \ge 0$ is a sequence such that $\sum_{n=1}^{\infty} a_n = \infty$, then any reordering has the same limit. EOC.

Proof. If there is a rearrangement of the series that has a finite limit, that rearrangement is absolutely convergent, and hence the original series is. QED.

CUTOFF FOR MIDTERM 1

The following proposition shows that this is as good as it gets:

Rearrangement Theorem
Let $S = \sum_{n=1}^{\infty} a_n$ be convergent but not absolutely convergent. For any $L \in \mathbb{R}$ there is a permutation σ such that S_σ has limit L. EOP.

Proof. Let $Q = \{\, i \mid a_i < 0\}$ and $P = \{\, i \mid a_i \ge 0 \,\}$. Note neither P nor N can be finite, otherwise S is absolutely convergent. In fact $\sum_{\substack{n=1\\ n\in P}}^{\infty} a_n = \infty$ and $\sum_{\substack{n=1\\ n\in Q}}^{\infty} a_n = -\infty$. Because if either has a finite limit α say, then the other has also a finite limit β, say, as well, and the limit of S is $\alpha + \beta$, and S is absolutely convergent.

Let now L be given, and we will assume $L \ge 0$. The case $L < 0$ is similar. Then there is n_1 minimal in P such that $A_1 = \; P_1 := \sum_{\substack{n=1\\ n\in P}}^{n_1} a_n > L$. Next there is $m_1 \in N$ minimal such that with $N_1 = \sum_{\substack{n=1\\ n\in Q}}^{m_1} a_n$ we have $U_1 = P_1 + N_1 < L$. Next chose $n_2 > n_1 \in P$ minimal such that with $P_2 = \sum_{\substack{n=n_1+1\\ n\in P}}^{n_2} a_n$ we have $A_2 = P_1 + N_1 + P_2 > L$. Continuing we $n_1, n_2, \ldots,$ and $m_1, m_2, \ldots$ and P_i, N_i such that

$$A_i = P_1 + N_1 + P_2 + N_2 + \cdots + P_{i-1} + N_{i-1} + P_i > L$$

And $U_i = P_1 + N_1 + \cdots + P_i + N_i < L$.

Also $A_i - L \leq a_{n_i}$ and $L - U_i \leq -a_{m_i}$. This follows from the minimal choice of n_i and m_i, respectively.

Indeed, if $A_i - L > a_{n_i}$, then $\left(A_i - a_{n_i}\right) > L$. But then we could choose a smaller number for n_i. Same for $L - U_i$.

We now define $\sigma: \mathbb{N} \to \mathbb{N}$ as follows: Think of P and N as two parallel lanes, and n_i, m_i as the respective points where we change lanes.

Both P, Q are infinite, and therefore we can list them by size $P = \{p_1 < p_2 < \cdots\}$ and $Q = \{q_1 < q_2 < \cdots\}$. For $n \in P$ let $p(n)$ be its position in P. That is $p(p_i) = i$. Similarly, for $n \in Q$, let $q(n)$ be its position in Q. Then $q(q_i) = i$.

For $1 \leq n \leq p(n_1)$ let $\sigma(n) = p_n$. For $p(n_1) < n \leq p(n_1) + q(m_1)$, let $\sigma(n) = q_{n-p(n_1)}$. Then change lanes again, and for $p(n_1) + q(m_1) < n \leq p(n_1) + q(m_1) + p(n_2)$ let $\sigma(n) = p_{n-p(n_1)-q(n_1)}$. Keep doing so. Every natural number appears exactly once as $\sigma(n)$ for some n, and therefore σ is a permutation.

It follows that if S is the original series, then $\lim S_\sigma = L$. Indeed, S is convergent, so a_n is a zero sequence. Let $\varepsilon > 0$. Then there is n_0 such that $|a_n| < \varepsilon$ as long as $n > n_0$.

As long as i is large enough such that $n_i, m_i > n_0$ then the above shows that $A_i - L$ and $L - U_i$ is less than ε. Note that the partial sums of $\sum_{n=1}^{\infty} a_{\sigma(i)}$ are decreasing from A_i to U_i and increasing afterwards from U_i to A_{i+1}

$$\sum_{n=1}^{p(n_1)} a_{\sigma(1)} = A_1$$

$$\sum_{n=1}^{p(n_1)+q(m_1)} a_{\sigma(1)} = A_1 + U_1$$

$$\sum_{n=1}^{p(n_1)+q(m_1)+p(n_2)} a_{\sigma(1)} = A_1 + U_1 + A_2$$

and so on, and in between the partial sums never increase their distance from L. It follows that the limit is L.

If $L < 0$ we apply the same reasoning to $-\sum_{n=1}^{\infty} a_n$ and $-L$. QED.

Note that if we modify the argument as follows, we can also achieve a limit of $\pm\infty$. We will show the case of $L = \infty$.

With P, Q defined as in the proof, let $n_1 \in P$ be minimal such that $A_1 \coloneqq P_1 = \sum_{\substack{n=1 \\ n \in P}}^{n_1} a_n > 1$. Then let $m_1 \in Q$ be minimal such that with $N_1 = \sum_{\substack{n=1 \\ n \in Q}}^{m_1} a_n$ we have $U_1 = P_1 + N_1 < 1$.

If $A_k, U_k, P_k. N_k n_k, m_k$, have been defined we define n_{k+1} as the minimum element $> n_k$ in P such that with $P_{k+1} \coloneqq \sum_{\substack{n=n_k+1 \\ n\in P}}^{n_{k+1}} a_n$,

$$A_{k+1} \coloneqq U_k + P_{k+1} > k$$

Then m_{k+1} is the smallest element $> m_k$ in Q such that with $N_{k+1} = \sum_{\substack{n=m_k+1 \\ n\in Q}}^{m_{k+1}} a_n$ we have

$$U_{k+1} \coloneqq A_{k+1} + N_{k+1} < k$$

Note that both $n_k, m_k \to \infty$ if $k \to \infty$. In particular $|a_{n_k}|$ and $|a_{m_k}|$ are small for large k.

The minimality of n_k, m_k then makes sure that $A_k < k+1$ and $U_k > k-1$.

We then rearrange the series as in the proof above. Then for $N \geq p(n_1) + q(m_1) + p(n_2) + q(m_2) + \cdots + p(n_k) + q(m_k)$ we have $\sum_{n=1}^{N} a_{\sigma(n)} \geq U_k > k-1$. Thus, the limit is ∞.

3.1.8 Products of series

While we haven't yet discussed this in detail, the sum of two series $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} c_n$, where $c_n = a_n + b_n$, is again a series, and if α is any real number and $S = \sum_{n=1}^{\infty} a_n$ is a series, then αS is the series $\sum_{n=1}^{\infty} \alpha a_n$.

The situation is a bit more complicated for products of series. In this case it is useful to start the series at $n = 0$. Consider two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Let S_N, T_M be the partial sums, respectively.

Then $S_N T_M$ is a sum of products of the form $a_i b_j$ where $0 \leq i \leq N$ and $0 \leq j \leq M$. But there is no obvious order. Let $p_0, p_1, \ldots$ be any enumeration of all $a_i b_j$. That is each $a_i b_j$ appears once and only once among the p_k. More precisely, let $f\colon \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ be a bijection, and for $(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0$ let $p_{i,j} = a_i b_j$. Then $p_k \coloneqq p_{f(k)}$. We call the infinite series $\sum_{n=0}^{\infty} p_n$ a *product series* of the original two series.

For example we could use the antidiagonals in the scheme

Equation 3-1

$$\begin{matrix} a_0 b_0 & a_0 b_1 & a_0 b_2 & \ldots \\ a_1 b_0 & a_1 b_1 & a_1 b_2 & \ldots \\ a_2 b_0 & a_2 b_1 & a_2 b_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

$$p_0 = a_0 b_0, p_1 = a_0 b_1, p_2 = a_1 b_0, p_3 = a_0 b_2, \ldots$$

It is then a natural question whether $\sum_{n=0}^{\infty} p_n$ converges, and if so what is the relationship to the the product of limits $(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n)$? Note that the Rearrangement Theorem destroys any hope that we *always* equality even if we have convergence.

Note each *row series* $\sum_{n=1}^{\infty} a_n b_m$ converges (m fixed). Similarly, each *column series* $\sum_{m=1}^{\infty} a_n b_m$ (n fixed) converges.

If both series converge absolutely, however, then $|p_0| + |p_1| + \cdots + |p_k| \leq \sum_{n=1}^{N} \sum_{m=1}^{M} |a_n||b_m|$ for N, M large enough. And then $|p_0| + |p_1| + \cdots + |p_k| \leq (\sum_{n=0}^{\infty} |a_n|)(\sum_{m=0}^{\infty} |b_m|)$, and so *every product* *series converges* absolutely. They all have the same limit as they are rearrangements of each other. Let p be that common limit. We must show that

$$p = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) =: P$$

This follows from looking at top left squares in Equation 3-1 above. Labeling rows and columns starting at 0, and picking $p_0 = a_0 b_0$. $p_1 = a_0 b_1, p_2 = a_1 b_1, p_3 = a_1 b_0, \dots$

Thus we start at the top of column n gow down until we hit the "diagonal" element, and then gow back on row n to the 0th column. Thus, the elements $p_0, p_1, \dots, p_{n^2-1}$ enumerate the n^2 elements in the top left "square" of side length n.

But then $p_1 + p_2 + \cdots + p_{n^2-1} = (a_0 + a_1 + \cdots + a_{n-1})(b_0 + b_2 + \cdots + b_{n-1})$.

This converges to P. But the left hand side is a subsequence of the sequence of partial sums of the series $\sum_{n=0}^{\infty} p_n$ which converges to p. Thus $p = P$.

Theorem
Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent with limit A and B, respectively. Then the *Cauchy product* of these two series

$$\sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$$

converges absolutely and has limit AB. EOT.

Proof. The partial sums of this series form a subsequence of a product series. Any product series converges absolutely to AB. QED.

3.1.9 Final interlude on the exponential function

Example
Let $E(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ be the exponential series. We know it is absolutely convergent since $E(|x|)$ converges for all $x \in \mathbb{R}$.
Then $E(x)E(y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} \frac{1}{(n-k)!} x^k y^{n-k}$. Now observe that $\sum_{k=0}^{n} \frac{1}{k!} \frac{1}{(n-k)!} x^k y^{n-k} = \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k} = \frac{1}{n!}(x+y)^n$.
It follows that for all $x, y \in \mathbb{R}$, we have $E(x)E(y) = E(x+y)$. EOE.

We are more than half-way there to show that $E(x) = \exp(x)$ is actually *the* exponential function, which will close the gap that we haven't yet established that $\exp(x)$ actually exists.

The only thing that is missing is the fact that $E(x)$ defines a differentiable function (it is clearly not zero, since $E(0) = 1$).

We will show this below in greater generality (that every so called *power series* is differentiable in the interior of its domain). But we can deal with $E(x)$ ad hoc.

Note that the example shows that E defines a group homomorphism $\mathbb{R} \to \mathbb{R}_{>0}$. Indeed, $E(x) > 0$ for all x. This follows from $E(x) > 0$ for $x \geq 0$ (simply because $E(x) \geq 1$ for these x). And for $x < 0$, $E(x) = E(-x)^{-1} > 0$ because $1 = E(0) = E(x - x) = E(x)E(-x)$.

Groups "look" everywhere the same, as we can translate properties around an element g to properties around an element h by multiplying everything by hg^{-1}. This is not a precise statement, but it is one of the reasons we are interested in groups.

To illustrate the point: suppose we know that $E(x)$ is differentiable at some point x_0. Then to verify that it is differentiable at y_0 we can observe that $E(y_0 + h) - E(y_0) = E\big((y_0 - x_0) + x_0 + h\big) - E\big((y_0 - x_0) + x_0\big) = E(y_0 - x_0)(E(x_0 + h\,) - E(x_0))$.

Lemma
$E(x)$ is differentiable at $x_0 = 0$, and $E'(0) = 1$.

Proof. $E(h) - E(0) = E(h) - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} h^n$. Then for $h \neq 0$,

$$\frac{E(h) - 1}{h} = \sum_{n=1}^{\infty} \frac{1}{n!} h^{n-1} = F(h)$$

Note the right hand side is a convergent series for $h \neq 0$, because $hF(h)$ is.

We must show that $\lim_{h \to 0} F(h) = 1$. This may seem obvious but it isn't. It says that F as a function of h is continuous at 0. For $h \neq 0$, $F(h) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} h^n$ and $F(h) - 1 = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} h^n$.

Then $|F(h) - 1| \leq \sum_{n=1}^{\infty} \frac{1}{(n+1)!} |h|^n = |h| \sum_{n=0}^{\infty} \frac{1}{(n+1)!} |h|^n \leq |h| E(|h|)$.

Note that if $|h| \leq 1$, then $E(|h|) \leq E(1)$. Thus, for small h, $|F(h) - 1| \leq |h| E(1) \to 0$ for $h \to 0$. QED.

Corollary
The function $x \mapsto E(x)$ defined on $\mathbb{R}$ is differentiable and $E'(x) = E(x)$ for all x.

Proof. $\frac{E(x_0 + h) - E(x_0)}{h} = \frac{E(x_0)(E(h) - 1)}{h} \to E(x_0)E'(0) = E(x_0)$ for $h \to 0$. QED.

We have now closed a gap in our earlier treatment of exponential functions. We have shown that there is a differentiable function $E \neq 0$ such that $E(x + y) = E(x)E(y)$. We have seen that as soon as we have one non-constant exponential function, we have a unique one for every base $a > 0$.

Note that as $E'(x) = E(x)$ it follows that in out earlier notation $\exp(x) = E(x)$.

We also find a new way of computing $e = E(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$. This converges much **much** faster than $\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k$.

For example $\left(1 + \frac{1}{20}\right)^{20} \sim 2.6533$. But $\sum_{n=0}^{20} \frac{1}{n!}$ is indistinguishable from $e \sim 2.718282$ for quite a few digits: indeed, note that $(n + k)! \geq k!\, n!$ (binomial coefficients are natural numbers) Therefore $e -$

$\sum_{n=0}^{20}\frac{1}{n!}=\sum_{n=21}^{\infty}\frac{1}{n!}=\sum_{n=0}^{\infty}\frac{1}{(n+21)!}\leq\frac{1}{21!}\sum_{n=0}^{\infty}\frac{1}{n!}\leq\frac{1}{21!}e$. Of course, now we would need to know that e is small compared to $21!$. It is: $e<2.72$. One can show this using the exponential series:

Let $S_N=\sum_{n=0}^{N}\frac{1}{n!}$. Then $\lim_{N\to\infty}S_N=E(1)=e$.

One can show that $S_N\leq e\leq\frac{(N+1)!}{(N+1)!-1}S_N$. Sor for many practical purposes $e=S_N$ if $N>5$.

3.1.10 Double sequences

Definition
A **double sequence** is a function $a:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$, so that for each $m,n\in\mathbb{N}$, we have a real number $a_{m,n}=a(m,n)$. Sometimes the indices are allowed to be 0 or bounded negative.
A double sequence $a_{m,n}$ is **bounded** if there is $B>0$ such that $|a_{m,n}|<B$ for all m,n. EOD.

As in the case of regular sequences, one can add double sequences and multiply them by constants, both in the obvious ways (so they form a vector space).

There are obvious extensions of that definitions to subsets of $\mathbb{Z}\times\mathbb{Z}$.

Definition (Limits of double sequences)
Let $a_{m,n}$ be a double sequence. A real number L is the **limit** of such a sequence, and the sequence is then called **convergent** to L, if for every $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that for all $m,n>n_0$, $|a_{m,n}-L|<\varepsilon$. In this case we write $\lim_{m,n\to\infty}a_{m,n}=L$. There is an analogous definition for improper limits $\pm\infty$:
We say $\lim_{m,n\to\infty}a_{m,n}=\infty$ if for every M there is n_0 such that $a_{m,n}>M$ for all $m,n>n_0$.
Similarly, $\lim_{m,n\to\infty}a_{m,n}=-\infty$ if for every M there is n_0 such that $a_{m,n}<M$ for all $m,n>n_0$. EOD.

This is a subtle notion. Note that if $a_{m,n}$ converges to, say, L, no statement can be made about the sequences $b_n=a_{n,k}$ or $c_n=a_{k,n}$ (for fixed k). These sequences may not converge, or converge to different numbers, or to $\pm\infty$.

It is useful to think of $a_{m,n}$ as arranged into an "infinite matrix" whose ith row is formed by the sequence $a_{i,n}$ for $n\in\mathbb{N}$. This is just a mental image and doesn't really have a theoretical meaning.

We can then discuss *column* and *row* limits.

Definition
Let $a_{n,m}$ be a double sequence. Its ith **row limit** is defined as $\lim_{m\to\infty}a_{i,m}$ if it exists.
Similarly, its tth **column limit** is defined as $\lim_{n\to\infty}a_{n,i}$, if it exists. EOD.

Example

1. A sequence a_n is a Cauchy sequence if and only if the double sequence $b_{m,n}=a_m-a_n$ converges to 0.
2. Let $a_{m,n}=\frac{(-1)^n}{m}$. Then $\lim_{m\to\infty}a_{m,n}=0$. For fixed n, however, $\lim_{n\to\infty}a_{m,n}$ does not exist.
3. Let $a_{m,n}=\frac{1}{\min\{n,m\}}$. Then $a_{m,n}\to 0$. But for fixed m,n (one at a time) we have $\lim_{n\to\infty}a_{m,n}=\frac{1}{m}$ and $\lim_{m\to\infty}a_{m,n}=\frac{1}{n}$.

4. Let $a_{m,n} = \frac{m}{n}$. Then the row limits are all 0 and the column limits are all ∞. The double sequence has no limit.
5. Let $a_{m,n} = \min\{m, n\}$. Then every row and column converges, but $a_{m,n}$ does not.
6. Let $a_{m,n} = \frac{m}{\max\{m,n\}}$. Then all row limits are 0 and all column limits are 1, and $a_{m,n}$ is not convergent as for any n_0 there are $m, n > n_0$ such that $a_{m,n} = 1$ and there are $m, n > n_0$ such that $a_{m,n} = 2$.
7. Let
$$a_{m,n} = \vdash\begin{cases} 1 & n \neq m \\ 0 & n = m \end{cases}$$
Then all row and column limits are 1, but $a_{m,n}$ does not converge.

In the following a statement about natural numbers is **almost always** true, if the set of natural numbers, where the statement is false, is finite.

As an example, $\frac{n}{n-1}$ is defined for almost all integers. Likewise, for a convergent sequence a_n with limit L and any $\varepsilon > 0$, $|a_n - L| < \varepsilon$ is true for almost all n.

Theorem
Let $a_{m,n}$ be a double sequence with limit $L \in \mathbb{R} \cup \{\pm\infty\}$. Suppose row limits exist and are finite[2] for almost all i. Let L_i be the ith row limit. Then $\lim_{i\to\infty} L_i = L$.
As similar statement holds for column limits. EOT.

Proof. First, let L be finite. Let $n_0 \in \mathbb{N}$ be such that L_i exists for all $i > n_0$. Let $\varepsilon > 0$. There is N_0 such that for all $n, m > N_0$ we have $\left|a_{m,n} - L\right| < \frac{\varepsilon}{2}$. Therefore, if $i > \max(N_0, n_0)$, we have $|L_i - L| \leq \frac{\varepsilon}{2} < \varepsilon$.

The same reasoning works for column limits.

If $L = \infty$, let $M > 1$. Then there exists N_0 such that $a_{m,n} > 2M$ for all $m, n > N_0$. Again, if $i > \max(N_0, n_0)$, then $a_{i,n} > 2M$ for all large enough n and hence $L_i \geq 2M >$M. So $L_i \to \infty$.

The case $L = -\infty$ is similar. QED.

It is crucial that it is a priori known that the double sequence *does have* a limit.

Definition
A double sequence $a_{m,n}$ is called **monotone increasing**, for all m_0, n_0 we have $a_{m_0,n_0} \leq a_{m,n}$ whenever $m > m_0$ *and* $n > n_0$. $a_{m,n}$ is **monotone decreasing**, if the double sequence $-a_{m,n}$ is monotone increasing. EOD.

The definitions of **strictly** monotone increasing/decreasing should be clear now.

Exercise
Prove the usual statements about limits for limits for double sequences (for sums and products of limits of convergent double sequences). EOE.

Lemma

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[2] The restriction that row limits are finite is not necessary. But we want the sequence of row limits to be a proper sequence.

A monotone and bounded double sequence converges. EOL.

Proof. We prove the statement for a bounded monotone increasing double sequence. The case of a monotone decreasing double sequence is similar (and also follows by applying the monotone increasing case to $-a_{m,n}$).

Let $L = \sup\{a_{m,n} \mid m, n \in \mathbb{N}\}$. Then $L \in \mathbb{R}$ because $a_{m,n}$ is bounded. I claim that $\lim_{m,n\to\infty} a_{m,n} = L$.

Indeed, let $\varepsilon > 0$. There is $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ such hat $|a_{m_0,n_0} - L| < \varepsilon$. As the double sequence is monotone increasing this means that for all $m, n > \max\{m_0, n_0\}$, we have $|a_{m,n} - L| < \varepsilon$. QED.

3.1.11 Double series

As in the case of normal sequences we can define series in the case of double sequences.

Definition
Given a sequence $a_{n,m}$, we can form a **double series** as $\sum_{m,n=1}^{\infty} a_{m,n}$, with **partial sums** $S_{M,N} = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n}$. The **limit** or **value** of the double series $\sum_{m,n=1}^{\infty} a_{m,n}$ is $\lim_{M,N\to\infty} S_{M,N}$ if it exists. We call it **convergent** if this limit exists and is finite. A double series is also allowed to start at 0 or any integer.
A double series $\sum_{m,n=1}^{\infty} a_{m,n}$ **convergence absolutely** if the double series $\sum_{m,n=1}^{\infty} |a_{m,n}|$ converges.
The series $\sum_{n=1}^{\infty} a_{m,n}$ is called the mth **row series** of the double series. The series $\sum_{m=1}^{\infty} a_{m,n}$ is called the nth **column series**. If a row or column series converges its limit is called a **row sum** or **column sum**, respectively.
EOD.

If a double series $\sum_{m,n=1}^{\infty} a_{m,n}$ converges absolutely, then it also converges: to see this consider

$$b_{m,n} = \vdash \begin{cases} a_{m,n} & a_{m,n} \geq 0 \\ 0 & a_{m,n} \leq 0 \end{cases} \qquad\qquad c_{m,n} = \vdash \begin{cases} -a_{m,n} & a_{m,n} < 0 \\ 0 & a_{m,n} \geq 0 \end{cases}$$

Then $a_{m,n} = b_{m,n} - c_{m,n}$.

The double series $\sum_{m,n=1}^{\infty} b_{m,n}$ and $\sum_{m,n}^{\infty} c_{m,n}$ converge as they have nonnegative terms and are bounded by $\sum_{m,n=1}^{\infty} |a_{m,n}|$. But then $\sum_{m,n=1}^{\infty} a_{m,n} = \sum_{m,n=1}^{\infty} b_{m,n} - \sum_{m,n=1}^{\infty} c_{m,n}$ converges.

Proposition
Let $\sum_{m,n=1}^{\infty} a_{m,n}$ be a double series. If it converges, and *all* its row series converge, then the iterated series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n}$ converges and for the limits we have

$$\sum_{m,n=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n}$$

A similar statement is true if all column series converge and then

$$\sum_{m,n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}$$

EOP.

Proof. This is an application of the theorem in 3.1.10 to the partial sums of the double series. QED.

We say an *iterated series* $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n}$ converges absolutely, if

$$\lim_{M\to\infty}\lim_{N\to\infty}\sum_{m=1}^{M}\sum_{n=1}^{N}|a_{m,n}| = \lim_{M\to\infty}\sum_{m=1}^{M}\left(\sum_{n=1}^{\infty}|a_{m,n}|\right)$$

is finite (of course the right hand side here makes sense only if $\sum_{n=1}^{\infty}|a_{m,n}|$ converges for all m). This is logically imprecise as $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}$ can also be viewed as a simple series, and then absolute convergence would mean that $\sum_{m=1}^{\infty}\left|\sum_{n=1}^{\infty} a_{m,n}\right|$ converges. Which is not the same, in general.

Cauchy's Double Series Theorem
Let $\sum_{m,n=1}^{\infty} a_{m,n}$ be a double series. Suppose $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{m,n}| = \lim_{M\to\infty}\lim_{N\to\infty}\sum_{m=1}^{M}\sum_{n=1}^{N}|a_{m,n}|$ is finite. Then the series $\sum_{m,n}^{\infty} a_{m.n}$ converges absolutely, every column series converges absolutely, and the iterated series of column series converges absolutely, and we have

$$\sum_{m,n=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n}$$

A similar statement (with the roles of rows and columns interchanged) holds if the iterated series of column series converge absolutely. EOT.

Proof. Under the assumptions of the theorem, the partial sums $A_{M,N} = \sum_{m=1}^{M}\sum_{n=1}^{N}|a_{m,n}| = \sum_{n=1}^{N}\sum_{m=1}^{M}|a_{m,n}|$ are bounded by $A = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{m,n}|$. But then $\sum_{m,n=1}^{\infty} a_{m,n}$ is absolutely convergent and therefore convergent. Likewise all row and column sums are absolutely convergent, as for instance $\sum_{m=1}^{\infty}|a_{m,n}| \le A$ since $\sum_{m=1}^{M}|a_{m,n}| \le A_{M,n} \le A$. The theorem in the last section then shows that

$$\sum_{m,n=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n}$$

QED.

We will see examples where double series are relevant in short order.

3.2 Power series

Definition
A **(formal) power series centered at** $c \in \mathbb{R}$ is a sequence a_n $(n \in \mathbb{N}_0)$, written as $\sum_{n=0}^{\infty} a_n(x-c)$. It **converges** at $x_0 \in \mathbb{R}$ if $\sum_{n=0}^{\infty} a_n(x_0-c)^n$ converges and **diverges** otherwise.

We write $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ for this power series to remind us that we think of something that can be "evaluated" at various $x_0 \in \mathbb{R}$. We often write $f(x_0)$ converges/diverges, and write $f(x_0)$ for the **value** (read *limit*) of the power series at x_0. EOD.

Note that it is possible that a formal power series doesn't converge anywhere except c.

If $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is a formal power series centered at c, and $d \in \mathbb{R}$ we write $f(x+d)$ for the formal power series $\sum_{n=0}^{\infty} a_n\big(x-(c-d)\big)$ centered at $c-d$. In particular, if $d=c$, we obtain a formal power series centred at 0. We will focus our attention mostly on these.

3.2.1 Power series as functions

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a (formal) power series (centred at 0). If $I \subseteq \mathbb{R}$ is a set such that for all $x_0 \in I$, $f(x_0)$ converges, we can define a function $F: I \to \mathbb{R}$ defined by $F(z) = f(z)$, where the right hand side means the limit of the series $\sum_{n=0}^{\infty} a_n z^n$.

By abuse of language we often will identify a formal power series with this function it defines. We will then use the same label (f) for both the power series and the induced function. While this is imprecise, it rarely causes confusion.

3.2.2 The radius of convergence of a power series

Theorem

Let $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ be a power series centered at c. Let $L := \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then $f(x_0)$ converges for all x_0 with $|x_0 - c| < \frac{1}{L}$, and diverges for all x_0 with $|x_0 - c| > \frac{1}{L}$. EOT.

For the purpose of this theorem $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proof. We may assume $c = 0$ (why?). Let first $L < \infty$. We will show that if $|x_0| < \frac{1}{L}$, then $\sum_{n=0}^{\infty} a_n x_0^n$ converges absolutely. We may thus assume that $x_0 > 0$. Then $|a_n x_0^n| < \frac{|a_n|}{L^n}$, and therefore $\sqrt[n]{|a_n| x_0^n} < \sqrt[n]{\frac{|a_n|}{L^n}} = \frac{\sqrt{|a_n|}}{L}$ and so $\limsup_{n\to\infty} \sqrt[n]{|a_n x_0^n|} < 1$.

By the root test in Convergence of series3.1.2, this shows that the series $f(x_0)$ converges absolutely. Similarly, the inequality is reversed if $|x_0| > \frac{1}{L}$ showing that in this case $f(|x_0|)$ diverges. But note that the argument given in proving divergence in the root test was that if for the series $\sum_{n=1}^{\infty} b_n$ with $b_n \geq 0$, we have $\limsup_{n\to\infty} \sqrt{b_n} > 1$, then b_n is not a zero sequence. Thus $|a_n x_0^n|$ is not a zero sequence, and hence $a_n x_0^n$ is not a zero sequence either. Therefore also $f(x_0)$ diverges.

If $L = \infty$, the same argument again shows divergence for all $x_0 \neq 0$. QED.

Definition

For a formal power series f, its **radius of convergence** is defined as $\frac{1}{L}$. It is defined to be ∞ if $L = 0$, and 0 if $L = \infty$. EOD.

Example

1. The radius of convergence of $f(x) = \exp(x)$ is ∞.
2. The radius of convergence of the geometric series $\sum_{n=0}^{\infty} x^n$ is 1.
3. The radius of convergence of the series $\sum_{n=0}^{\infty} n!\, x^n$ is 0.
4. The radius of convergence of $g(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$ is 1 and $g(1)$ diverges, whereas $g(-1)$ converges.
5. The radius of convergence of $h(x) = \sum_{n=0}^{\infty} n x^n$ is 1 and $h(-1)$ and $h(1)$ both diverge ($|(-1)^n n|$ is not a zero sequence).

If a power series $f(x)$ centred at c has radius of convergence $R > 0$ it is in general not clear what happens if $|x_0 - c| = R$. It may converge, diverge, or converge but not absolutely.

Convention
We henceforth identify a power series with the function it defines on the interval defined by its radius of convergence. EOC.

Fact
Let $a_n \geq 0$ be a sequence. For any $k \in \mathbb{N}$, we have
$$\limsup_{n\to\infty} \sqrt[n]{a_{n\pm k}} = \limsup_{n\to\infty} \sqrt[n]{a_n}$$
EOF.

Note that a_{n-k} defines a sequence starting at $n = k+1$.

Proof. Note that $\sqrt[n]{a_{n+k}} = a_{n+k}^{\frac{1}{n}} = \left(a_{n+k}^{\frac{1}{n+k}}\right)^{\frac{n+k}{n}}$. Let $S_n = \sup_{m\geq n}\{\sqrt[m]{a_m}\}$, and $T_n = \sup_{m\geq n}\{\sqrt[m]{a_{m+k}}\}$.

Note that if $m < \ell$, then $x^m \leq x^\ell$ (for nonnegative x). Also $\frac{n+k}{n} > 1$, therefore $\sqrt[m]{a_{m+k}} \geq \sqrt[m+k]{a_{m+k}}$.

We find that $S_{n+k} \leq T_n$. This already shows that if $\lim S_{n+k} = \infty$ then so is $\lim\ T_n$.

On the other hand, $\sqrt[m]{a_{m+k}} = \sqrt[m+k]{a_{m+k}}^{\left(1+\frac{k}{m}\right)}$, and $1 + \frac{k}{m} \leq 1 + \frac{k}{n}$ for all $m \geq n$. Thus $T_n \leq S_{n+k}^{1+\frac{k}{n}}$.

This uses that $x^{\frac{n+k}{n}}$ is monotone increasing. It therefore suffices to show that for any (proper or improper) sequence $b_n \geq 0$ with limit $L \geq 0$, we have $\lim_{n\to\infty} b_n^{\frac{n+k}{n}} = L$. If $L > 0$ (including $L = \infty$) this follows easily for example from the fact that then $b_n^{\frac{n+k}{n}} = \exp\left(\frac{n+k}{n}\log b_n\right) \to \exp(1\cdot\log L) = L$ for $n \to \infty$ if $L < \infty$, and $\exp\left(\frac{n+k}{n}\log b_n\right) \geq b_n$ if $L = \infty$ and n large enough such that $b_n \geq 1$.

In general, (ie. $L = 0$) we still have $b_n^{\frac{n+k}{n}} = \exp\left(\frac{n+k}{n}\log b_n\right) \to \exp\big(1\cdot(-\infty)\big) = 0$, where we define $\exp(-\infty)$ as $\lim_{x\to-\infty}\exp(x) = 0$.

Finally, Let $c_n = 0$ for $n \leq k$ and $c_n = a_{n-k}$ for $n > k$. Then $a_n = c_{n+k}$ and the above shows that $\limsup_{n\to\infty} \sqrt[n]{a_n} = \limsup_{n\to\infty} \sqrt[n]{c_n}$. QED.

Proposition (Radius of convergence for shifted series)
Let $f(x) = \sum_{n=0}^\infty a_n x^n$ be a formal power series (centred at 0) with radius of convergence R.
Let $k \in \mathbb{N}$. Then the *shifted series* $g(x) = \sum_{n=0}^\infty a_{n+k} x^n$ has the same radius of convergence.
It then follows that also the series $h(x) = \sum_{n=k}^\infty a_{n-k} x^n$ has the same radius of convergence. EOP.

Proof. Let $L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. By the above fact $\limsup_{n\to\infty} \sqrt[n]{|a_{n+k}|} = L$. Thus, the shifted series has the same radius of convergence.

If for $n \geq k$ we put $b_n = a_{n-k}$, this shows that $L = \limsup_{n\to\infty} \sqrt[n]{|b_{n+k}|} = \limsup_{n\to\infty} \sqrt[n]{|b_n|}$.

Then $h(x)$ has the same radius of convergence as $f(x)$. QED.

Corollary 1
Let $f(x)$ be a formal power series centred at c with radius of convergence $R > 0$. Then f is continuous at $x_0 = c$. EOC.

Proof. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-c)$. We must show that $\lim\limits_{x \to c} f(x) = a_0 = f(c)$. To do this note that for every x_0, we have

$$|f(x_0) - a_0| = \left|\sum_{n=1}^{\infty} a_n(x_0 - c)^n\right| = |x_0 - c| \left|\sum_{n=0}^{\infty} a_{n+1}(x_0 - c)^n\right|$$

The right hand side involves the shifted series $\sum_{n=0}^{\infty} a_{n+1}(x-c)^n$. This series has the same radius of convergence as $f(x)$.

As $f(x_0)$ is absolutely convergent for $|x_0 - c| < R$, the right hand side is bounded by

$$|x_0 - c| \sum_{n=0}^{\infty} |a_{n+1}||z - c|^n$$

where we choose $z \in (c, c+R)$ such that $|x_0 - c| < |z - c|$ for all x_0 close to c (so typically z closer to $c + R$ than any of the x_0 close to c in question). Thus $\lim\limits_{x_0 \to c} |f(x_0) - a_0| = 0$. QED.

Corollary 2
Let $f(x)$ be a formal power series centred at c with radius of convergence $R > 0$. Then f is differentiable at $x_0 = c$, and $f'(x_0) = a_1$. EOC.

Proof. We must show that $\lim\limits_{x_0 \to c} \frac{f(x_0) - f(c)}{x_0 - c} = a_1$.But note that

$$f(x_0) - f(c) = (x_0 - c) \sum_{n=0}^{\infty} a_{n+1}(x_0 - c)^n$$

We can therefore identify $\frac{f(x)-f(c)}{x-c}$ with the shifted power series $\sum_{n=0}^{\infty} a_{n+1}(x-c)^n$. Corollary 1 tells us that this is a continuous function at $x_0 = c$, and its value at c is a_1. QED.

3.2.3 Formal derivatives of power series

Definition
Let $f = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series. We define $D(f) \coloneqq \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ and call this the **formal derivative** of f. EOD.

Lemma
If f has radius of convergence R then so does $D(f)$. EOL.

Proof. Let $L = \limsup\limits_{n \to \infty} \sqrt[n]{|a_n|}$. We must show that $L = \limsup\limits_{n \to \infty} \sqrt[n]{(n+1)|a_{n+1}|}$. As $\lim\limits_{n \to \infty} \sqrt[n]{n+1} = 1$ this follows from our result for the radius of convergence for shifted series. QED.

Exercise

1. For $k > 0$ show that if we define $D^k(f)$ as applying D k times to f, then
$$D^k(f) = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} a_{n+k} x^n$$
Show that the radius of convergence of $D^k(f)$ is the same as that of f.
2. For a formal power series f as above we define the **formal antiderivative** as $I(f) = \sum_{n=1}^{\infty} \frac{1}{n} a_{n-1} x^n$. Show that $D\big(I(f)\big) = f$.
3. Show that the radius of convergence of $I(f)$ is that of f.

EOE.

Remark
It is tempting to conclude from the preceding results that if f is a power series (centred at 0, say) with radius of convergence $R > 0$, then f is differentiable on $(-R, R)$ and $f' = D(f)$.
This is true, but to prove this is surprisingly involved. Informally speaking it involves to "swap" limits: That means, if $c \in (-R, R)$ to show that $f'(c) = D(f)(c)$, one needs to show that
$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \sum_{n=0}^{\infty} a_n \left(\frac{x^n - c^n}{x - c}\right) = \sum_{n=0}^{\infty} a_n \lim_{x\to c} \left(\frac{x^n - c^n}{x - c}\right) = D(f)(c)$$
It is not entirely obvious why that should be true. EOR.

3.2.4 Analytic functions

A way out of this conundrum is the following strategy: instead of showing directly that a given power series $f(x)$ centered at c with radius of convergence $R > 0$ is differentiable at any $x_0 \in (c - R, c + R)$, we will show that we can rewrite any such power series as a power series *centered at* x_0 again with positive radius of convergence. Then our previous results show that $f(x)$ is differentiable at x_0.

But this idea is more general than just referring to power series, and gives rise to the following idea:

Definition
Let I be an **open** interval, and f a function defined on I. We say that f is **analytic at $\boldsymbol{c} \in \boldsymbol{I}$**, if there is a formal power series g centred at c convergent on an interval $(c - \delta, c + \delta) \subseteq I$ for some $\delta > 0$, such that $f(x) = g(x)$ on $(c - \delta, c + \delta)$. We say f is **analytic** if that holds for every $c \in I$. EOD.

For example, the exponential function is analytic at 0. In fact, every power series with positive radius of convergence is analytic at its centre.

Remark
Analytic functions are differentiable. Indeed, for $c \in I$, we can find $\delta > 0$ such $f(x) = g(x)$ for some power series g centered at c and all $x \in (c - \delta, c + \delta)$. But then f is differentiable at c iff g is. Since g is differentiable at c, f is. EOR

We will now see that power series are analytic.

Theorem (Transformation Theorem)
Let f be a power series centered at c with convergence radius $R > 0$. Then f is analytic on $(c - R, c + R)$. Moreover, if $d \in (c - R, c + R)$, then (as a function) $f(x) = \sum_{k=0}^{\infty} \frac{D^k(f)(d)}{k!} (x - d)^k$ on $(d - S, d + S)$ where $S = \min\{R - (c - d), R - (d - c)\} = R - |d - c|$. EOT.

Proof. We first treat the case $c = 0$. Let $f = \sum_{n=0}^{\infty} a_n x^n$ and let $d \in (-R, R)$. We want to rewrite f as a power series $g = \sum_{n=0}^{\infty} b_n (x-d)^n$ centered at d.

For any $x_0 \in (-R, R)$

$$f(x_0) = f_N\big((x_0 - d) + d\big) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (x_0 - d)^k d^{n-k}$$

We define $\binom{n}{k} = 0$ if $k > n$, then we can write the above as

$$f(x_0) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} (x_0 - d)^k d^{n-k}$$

Note that $f(x_0)$ is absolutely convergent. Now let $|x_0 - d| + |d| < R$.

Then $\sum_{n=0}^{\infty} |a_n| (|x_0 - d| + |d|)^n$ converges, and

$$\sum_{n=0}^{\infty} |a_n| (|x_0 - d| + |d|)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} |x_0 - d|^k |d|^{n-k}$$

Thus, the iterated series for $f(x_0)$ above is absolutely convergent. By Cauchy's Double Series Theorem, we can swap the summation.

$$f(x_0) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} a_n (x_0 - d)^k d^{n-k} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{n}{k} a_n (x_0 - d)^k d^{n-k}$$
$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \binom{n}{k} a_n d^{n-k} \right) (x_0 - d)^k$$

Now observe that $\sum_{n=0}^{\infty} \binom{n}{k} a_n d^{n-k} = \sum_{n=k}^{\infty} \binom{n}{k} a_n d^{n-k} = \frac{1}{k!} \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} a_{n+k} d^n = \frac{1}{k!} D^k(f)(d)$.

Note that $|x_0 - d| + |d| < R$ if and only if $|x_0 - d| < R - |d| = \min\{R - d, R + d\}$.

The case of $c \neq 0$ follows by observing that $f(x + c)$ is a power series centered at 0 and $D^k(f)(x + c) = D^k(f(x + c))$. QED.

Corollary 1
Let $f = \sum_{n=0}^{\infty} a_n (x - c)^n$ be a formal power series centered at c with radius of convergence $R > 0$. Then for any $x_0 \in (c - R, c + R)$, f is differentiable at x_0 and $f'(x_0) = D(f)(x_0)$. EOC.

Proof. We have shown that f is analytic on $I = (c - R, c + R)$. Analytic functions are differentiable.

By the Transformation Theorem $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n(f)(x_0)(x - x_0)^n$ on a small enough interval around $x_0 \in I$. By Corollary 2 in 3.2.2, this means f is differentiable at x_0 with $f'(x_0) = D(f)(x_0)$. In particular, this says that the derivative of f is $D(f)$. QED.

Corollary 2
Let $f = \sum_{n=0}^{\infty} a_n (x - c)^n$ be a formal power series centered at c with radius of convergence $R > 0$.

Then f is smooth on $(c-R, c+R)$, and $f^{(n)}$ is again a power series, namely $D^n(f)$. In particular, $a_n = \frac{1}{n!}D^n(f)(c)$. EOC.

Proof. By Corollary 1, f is differentiable at any point in $(c-R, c+R)$, and its derivative is again a power series, namely $D(f)$, and $a_1 = D(f)(c)$. We now proceed by induction on n. If $n=1$, the assertion is proven. Suppose Corollary 2 holds for a specific natural number n. Then $f^{(n)}$ is a power series, defined by $D^n(f)$. Then applying Corollary 1 to $D^n(f)$, we find that $D^n(f)$ is differentiable with derivative $D\big(D^n(f)\big) = D^{n+1}(f)$. This shows that $f^{(n+1)}$ is again a power series, and in fact equal to $D^{n+1}(f)$.

Applying the Transformation Theorem to $x_0 = c$, we get $f = \sum_{n=0}^{\infty} \frac{1}{n!} D^n(f)(c)(x-c)^n$ and together with the above that means $a_n = \frac{1}{n!} f^{(n)}(c) = \frac{1}{n!} D^n(f)(c)$. QED.

Convention
We henceforth identify a power series with the function it defines on the interval defined by its radius of convergence. For example, if f is a power series, we write f' both for $D(f)$ and the derivative of the function defined by f. EOC.

Corollary 3
Let f, g be two power series centered at c, both convergent on the same nonempty interval I containing c. Then $f = g$ if and only if $f(x_0) = g(x_0)$ for all $x_0 \in I$. EOC.

Remark
This may sound like a tautology, and indeed we are just picking up some loose ends that we have avoided so far, namely the question whether the function defined by a power series determines the power series (that is, its coefficients). To be precise, the issue is, if $f = \sum_{n=0}^{\infty} a_n (x-c)^n$ and $g = \sum_{n=0}^{\infty} b_n (x-c)^n$, then $f = g$ “should” mean $a_n = b_n$ for all n, whereas $f(x_0) = g(x_0)$ for all $x_0 \in I$ only means that the functions defined by the two power series are the same. EOR.

Proof. Let $f = \sum_{n=0}^{\infty} a_n (x-c)^n$ and $g = \sum_{n=0}^{\infty} b_n (x-c)^n$. If $f = g$ then $a_n = b_n$ for all n and therefore $f(x_0) = g(x_0)$ for all $x_0 \in I$.

We must therefore show the converse: if $f(x_0) = g(x_0)$ for all $x_0 \in I$, then $a_n = b_n$ for all n. f, g both define the same function on I. Both functions are smooth by Corollary 1, and $a_n = \frac{1}{n!} f^{(n)}(c)$ and $b_n = \frac{1}{n!} g^{(n)}(c)$. It follows that $a_n = b_n$. QED.

Example
Recall the exponential series $\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. If we take only the “even” or “odd” summands and alternate them, we obtain power series that are also convergent on all of $\mathbb{R}$.

1. Let $S(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1}$. Then the radius of convergence is ∞, because $\sum_{n=1}^{\infty} \frac{1}{(2n-1)!} |x|^{2n-1} \leq E(|x|) < \infty$.
   Then $C(x) := S'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n-1)!} x^{2n-2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ again has radius of convergence ∞.
2. $C'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n}{(2n)!} x^{2n-1} = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1} = -S(x)$.

We find $S'(x) = C(x)$, and $C'(x) = -S(x)$. In particular, $S''(x) = -S(x)$ and $C''(x) = -C(x)$. This should remind you of the sin and cos functions. We will see, that in fact $\sin x = S(x)$ and $\cos x = C(x)$.

Remark

While the concept of an analytic function helped us to show that power series are differentiable (and in fact, smooth), the importance of the concept goes beyond that.
Analytic functions are a large class of functions that we can actually describe effectively (at least in principle), since for every point c in their domain, we are given a power series (again, at least in principle) that agrees with the function around c. In particular, we have something as close as possible to a "formula" for determining the value of the function.
Maybe even more important is the following fact: one can extend most definitions we have encountered so far to complex valued functions defined on subsets of the complex numbers.
Surprisingly, if a function $f: \mathbb{C} \to \mathbb{C}$ is differentiable everywhere (the definition of "differentiable" is the same as the one we have seen, namely that $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists and is a complex number), then it is automatically analytic! In particular, if f is power series on the interval $I = (c - R, c + R)$, then there is a *unique* complex differentiable function $\tilde{f}$ defined on $D = \{z \in \mathbb{C} \mid |z - c| < R\}$ that agrees with f on I.
Consider the following function $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$. We know that $f(x_0)$ converges for every $x_0 > 1$. One can show that this function is analytic, for $x > 1$ and that the series also converges for $x \in \mathbb{C}$ where $\Re(x) > 1$.

Warning

Tying up another loose end, if f is a power series with centre c and *finite* radius of convergence $R > 0$, then we have seen that f is analytic on $I = (c - R, c + R)$. For $d \in I$, we can rewrite f as a power series g with centre d, and we have seen that the radius of convergence of g is *at least* $S = R - |d - c|$. However it *may be strictly larger*. Thus, g may actually converge for points strictly outside I (not just on the boundary of I.
Consider $f(x) = \sum_{n=0}^{\infty} x^n$. For $|x| < 1$ this is equal to $\frac{1}{1-x}$. So let $h(x) = \frac{1}{1-x}$. Not that h is defined at -1, and $h(-1) = \frac{1}{2}$, whereas $f(x)$ does not converge for $x = -1$ (but $f(x)$ is bounded close to $x - 1$ and that is no coincidence).
Consider $d = -\frac{1}{2}$. Then $g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(d)}{n!} (x - d)^n$.
Next, observe that $f'(x) = \frac{1}{(1-x)^2}, f''(x) = \frac{2}{(1-x)^3}, \dots, f^{(n)}(x) = \frac{n}{(1-x)^{n+1}}$. Hence

$$g(x) = \sum_{n=0}^{\infty} n \left(\frac{2}{3}\right)^{n+1} \left(x + \frac{1}{2}\right)^n$$

The inverse of the radius of convergence of this series is $\limsup_{n \to \infty} \sqrt[n]{n \left(\frac{2}{3}\right)^{n+1}} = \frac{2}{3}$, and the radius of convergence is $\frac{3}{2}$. One can show that h is analytic everywhere in $\mathbb{R} \setminus \{1\}$. But there is no single power series convergent in all its domain.
On the other hand if a function is analytic on all of $\mathbb{R}$, then there exists a power series (centred at 0) with convergence radius ∞ agreeing with the function. EOW.

Exercise

For those of you that have some familiarity with (linear) algebra:
Let I be an open interval, and let $\mathcal{O}(I)$ be the set of analytic functions on I.

1. Show that $\mathcal{O}(I)$ is a subspace of $\mathcal{F}(I)$: show that $\mathcal{O}(I)$ is nonempty; show that for all $f, g \in \mathcal{O}(I)$ and any $a, b \in \mathbb{R}$, also $af + bg \in \mathcal{O}(I)$.
2. Show that $\mathcal{O}(I)$ is a *sub-algebra* of $\mathcal{F}(I)$: show that for all $f, g \in \mathcal{O}(I)$, also $fg \in \mathcal{O}(I)$. (*Hint*: the Cauchy product is helpful.)
3. Show that $D: \mathcal{O}(I) \to \mathcal{O}(I)$ defined by $D(f) = f'$ is a linear transformation.
4. Let $a \in I$. Let $\mathfrak{m}_a \subseteq \mathcal{O}(I)$ be defined as $\mathfrak{m}_a = \{f \in \mathcal{O}(I) \mid f(a) = 0\}$. Show that $\mathcal{O}(I) = \mathbb{R} \oplus \mathfrak{m}_a$, where $\mathbb{R}$ is identified with the constant functions on I.

3.3 Identity Theorem for power series*

We have seen that if two formal power series define the same function, then they are equal. However, a much stronger statement is true: if they are centred at the same point, and their functions agree at a countable set with that point an accumulation point, then they are already equal.

Theorem
Let $f(x), g(x)$ be power series centered at c, convergent on the interval $I = (c - R, c + R)$. Let $x_n \neq c \in I$ be any sequence such that $\lim_{n\to\infty} x_n = c$. If $f(x_n) = g(x_n)$ for all n, then $f(x) = g(x)$. EOT.

Proof. We may assume that $c = 0$. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$. We will show that $a_n = b_n$ for all n.

To do this, for $k = 0,1,2, \dots$ we define $f_k = \sum_{n=k}^{\infty} a_n x^{n-k} = \sum_{n=0}^{\infty} a_{n+k} x^n$.

We proceed by induction on n.

First, $a_0 = b_0$ because both f and g define continuous functions, so $a_0 = f(0) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(0) = b_0$.

Suppose that for a given n, $a_m = b_m$ for all $m \leq n$.

Then $f_n - a_n = x \sum_{k=0}^{\infty} a_{k+n+1} x^n = x f_{n+1}$ agrees with $g_n - a_n = x \sum_{k=0}^{\infty} b_{k+n+1} x^n$ on x_m for all m.

Thus, the power series $x f_{n+1} = x g_{n+1}$ on all x_m. In particular,

$$a_{n+1} = f_{n+1}(0) = \lim_{m\to\infty} f_{n+1}(x_m) = \lim_{m\to\infty} g_{n+1}(x_m) = g_{n+1}(0) = b_{n+1}$$

QED.

Exercise
Examine the proof of the previous theorem and see where we used that $x_n \neq c$ for all n. EOE.

3.4 Taylor's Theorem and Taylor series

One of the most interesting theorems in analysis is the fact that one can *approximate* sufficiently differentiable functions just by knowing their multiple derivatives at a given point.

We already saw a glimpse of that by observing that if f is differentiable at x_0, then $f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x)$ with $r(x)$ "small." How can we generalize that to obtain better approximations?

We have also seen that if f is a power series, it may be approximated by polynomials (its partial sums), and the coefficients of those polynomials are related to the derivatives of f at the centre.

This will require some preparation.

3.4.1 General Rolle's Theorem

Recall that Rolle's Theorem states that for f differentiable, if $f(a) = f(b)$ for some $a < b$, there must be $c \in (a, b)$ for which $f'(c) = 0$.

Is there an analogue for higher derivatives? Indeed, there is.

Theoem (General Rolle's Theorem)
Let $I = [a, b]$ be an interval, and f a continuous function on I, n times continuously differentiable on $(a, b]$, where $n \geq 0$ is an integer, and $f(a) = f(b)$. Suppose $f^{(n+1)}$ exists in at least (a, b) and $f^{(k)}(b) = 0$ for $k = 1,2, \dots, n$. Then there is $c \in (a, b)$ such that $f^{(n+1)}(c)$=0.
A similar result holds for intervals $[a, b)$ with the roles of a and b interchanged. EOT.

Proof. We proceed by induction on n. If $n = 0$ this is Rolle's Theorem. Now suppose the theorem holds for a particular n, and let a, b such that $f(a) = f(b)$ and $f^{(k)}(b) = 0$ for all $1 \leq k \leq n + 1$. We must show that there is $c \in (a, b)$ such that $f^{(n+2)}(c) = 0$.

By induction, there exists $c' \in (a, b)$ such hat $f^{(n+1)}(c') = 0$. Then $c' < b$ and applying Rolle's Theorem to $f^{(n+1)}$ on the interval $[c', b]$, there must be $c \in (c', b) \subseteq (a, b)$ such that ${f^{(n+1)}}'(c) = f^{(n+2)}(c) = 0$. QED.

Corollary (of Proof)
Let $I = [a, b]$ be an interval and f a continuous function on I, $n + 1$ times differentiable on $[a, b)$, where $n \geq 0$ is an integer, and $f(a) = f(b)$. Suppose $f^{(k)}(a) = 0$ for $k = 1,2, \dots, n$. Then there is $c \in (a, b)$ such that $f^{(n+1)}(c) = 0$. EOC.

Proof. Exercise. QED.

Original Rolle was used to prove the Mean Value Theorem. General Rolle is then used to prove its natural generalization, known as Taylor's Theorem.

3.4.2 Taylor polynomials

Definition
Let f be a function defined on an interval I. Let $c \in I^{\circ}$. If f is n-times differentiable at c, then the polynomial $P_{f,n,c}(x) = f(c) + f'(c)(x - c) + \frac{1}{2}f''(c)(x - c)^2 + \dots + \frac{1}{n!}f^{(n)}(c)(x - c)^n$ is called the **degree n Taylor polynomial** (or nth Taylor polynomial) of f at c. It is always a polynomial of degree at most n.
The polynomials $P_{f,n.c}$ are often also referred to as **Taylor expansions** of f at c. EOD.

Exercise
Show that if f itself is a polynomial function, then $P_{f,c,n} = f$ for all $c \in \mathbb{R}$ as long as $n \geq \deg f$. EOE.

Example

1. Let $f(x) = \sin x$ defined on $\mathbb{R}$. Then $P_{f,0,n} = x - \frac{1}{6}x^3 + \frac{1}{105}x^5 + \dots = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k-1}$.

2. Let $f(x) = \log(1+x)$ defined on $(-1,1)$. Then $f'(0) = 1$, $f''(x) = \frac{1}{1+x}' = -\frac{1}{(1+x)^2} = -(1+x)^{-2}$, so $f''(0) = -1$. Continuing, $f^{(n)}(0) = (-1)^{n+1}\,(n-1)!$, and $P_{f,n,0} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \pm \cdots + \frac{(-1)^{n+1}}{n}x^n$.

3.4.3 Taylor's Theorem

One of the remarkable facts of analysis is that one often can say a lot of meaningful things about the difference $\left|f(x) - P_{f,n,c}(x)\right|$. If that difference is small, then the Taylor polynomial can serve as a good approximation for f.

Theorem

Suppose f is n times continuously differentiable on an interval $[a,b]$, and suppose $f^{(n+1)}$ exists on at least (a,b). For every $u \in [a,b)$ there is d strictly between u and b such that

$$f(u) - P_{f,n,c}(u) = \frac{(u-c)^{n+1}}{(n+1)!} f^{(n+1)}(d)$$

A similar theorem holds for intervals of the form $[a,b)$ with the roles of a and b interchanged.
EOT.

For $n = 0$, this is essentially the Mean Value Theorem.

Note

In class, I stated the theorem for the half-open interval $(a,b]$. However, it has certain advantages to state it for the closed interval $[a,b]$ (see for example the computation for $\log(2)$ below. EON.

Proof. The idea (not mine) is to apply General Rolle. For this we must find a function h such that $h(u) = h(b)$ and $h^{(k)}(b) = 0$ for all $1 \le k \le n$, and such that $h^{(n+1)}(d) = \frac{(x-c)^{n+1}}{(n+1)!} f^{(n+1)}(d)$ for d strictly between u and b.

Note that $g(x) = f(x) - P_{f,n,b}(x)$ satisfies that $g^{(k)}(b) = 0$ for $0 \le k \le n$. The only thing that is missing is that $g(u) \ne g(b) = 0$ in general.

To fix this define $h(x) = f(x) - P_{f,n,c}(x) - \frac{(x-b)^{n+1}}{(u-b)^{n+1}}\Big(f(u) - P_{f,n,b}(u)\Big)$.

Now $h(u) = 0 = h(b)$. Also $h^{(k)}(b) = 0$ for $1 \le k \le n$. So, there is d between u, c such that $h^{(n+1)}(d) = 0$.

But $h^{(n+1)}(d) = f^{(n+1)}(d) - 0 - \frac{(n+1)!}{(u-b)^{n+1}}\Big(f(u) - P_{f,n,b}(u)\Big)$. Solving this equation for $f^{(n+1)}(d)$ gives the result. QED.

Definition

The "error term" $\frac{(u-b)^{n+1}}{(n+1)!} f^{(n+1)}(d)$ is often called the **Lagrange remainder**. EOD.

Because of the "high" degree in $(u-b)$ the Lagrange remainder may seem large. But note that if u is close to c, then $(u-b)^{n+1}$ is small, made even smaller by the division by $(n+1)!$. Thus, the behaviour of $f^{(n+1)}(x)$ is crucial to analyzing the error term. EOD.

Warning
The d in the Lagrange remainder depends on n and x. Different x means different d in general. EOW.

In particular, if we know that $\left|f^{(n+1)}(x)\right| \le \alpha C^{n+1}$ for some $\alpha, C > 0$ and all n (assuming that f is smooth), this yields an error term that always converges to 0.

Example

1. Let $f(x) = \sin x$. Then for $k = 0,1, \dots$ $f^{(2k)}(x) = (-1)^k \sin x$, and $f^{(2k+1)}(x) = (-1)^k \cos x$. It follows that the Lagrange remainder is always bounded by
$$\frac{|(u-b)^{n+1}|}{(n+1)!}$$
In particular, for fixed $u, b \in \mathbb{R}$, we have $\lim_{n\to\infty} P_{f,n,b}(u) = f(u)$.
2. Let $f(x) = \log(1+x)$ on $(-1,1)$. Recall $f'(x) = \frac{1}{1+x}$, and for $n \ge 1$
$$f^{(n)}(x) = (-1)^{n+1}(n-1)!\,(1+x)^{-n}$$
Then for $u \in (-1,1)$, we have that the Lagrange remainder (with $b = 0$) is
$$R_n(x) = \frac{u^{n+1}}{(n+1)!}(-1)^{n+1} n!\,(1+d)^{-n-1}$$
for some d strictly between u and 0. Note that for the Lagrange remainder we get $\frac{|u|^{n+1}}{(n+1)|1+d|^{n+1}} \le \frac{1}{n+1}$ as long as $x \in [0,1)$, and so $d \ge 0$. Thus for such x, we have $\lim_{n\to\infty} R_n(x) = 0$.
It follows that for $x \in [0,1)$ we have
$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$
What about $x \in (-1,0)$? We know that the right hand side still converges (it is a power series with radius of convergence 1).
Consider $g(x) = \frac{1}{1+x}$. Then on $(-1,1)$ g is a power series, namely $g(x) = \sum_{n=0}^{\infty} (-1)^n x^n$.
Indeed, $g(-x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.
Let $G(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$. Then $D(G) = g$ so G and g have the same radius of convergence. In particular, $G' = g = f'$, so $G = f + C$ for some $C \in \mathbb{R}$. Then $C = G(0) - f(0) = 0$. It follows that $\log(1+x) = G(x)$ on $(-1,1)$.

EOE.

Remark
Let $f(x) = \log(1+x)$. We know that $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ on $(-1,1)$. We also know that the power series diverges for $x = -1$ (harmonic series), and still converges for $x = 1$ (Leibniz Criterion). Could it be that $f(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$? The Lagrange Remainder gives the answer: We know that $R_n(1) = f(1) - P_{f,n,0}(1) = \frac{(-1)^{n+1} 1^{n+1}}{(n+1)(1+d)^{n+1}}$ for some $d \in (0,1)$. Observe here it is crucial that Taylor's Theorem applies for the *closed* interval $[0,1]$. It follows that $R_n(x) \to 0$ for $n \to \infty$. We conclude that
$$f(1) = \log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

EOR.

3.4.4 Taylor series

One of the most important examples of power series are Taylor series associated to smooth functions. Recall that a function f defined on an interval I is called **smooth** if $f^{(n)}$ exists on all of I for all $n \in \mathbb{N}$.

Definition

Let I be an interval, $c \in I^{\circ}$, and f a function defined on I, such that $f^{(n)}(c)$ exists for all $n \in \mathbb{N}$. Then the **Taylor series** of f at c is the formal power series

$$T_{f,c}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

EOD.

Subtlety alert

The existence of $f^{(n)}$ at c for all n implies that for each n there is a $\delta_n > 0$ such that $f, f', f'', \dots, f^{(n)}$ are defined on $(c-\delta_n, c+\delta_n)$. But nothing prevents δ_n from being a zero sequence a priori. So $f^{(n)}$ to exist for all n does not directly imply that there is an open interval containing c where $f^{(n)}$ is defined for all $n \in \mathbb{N}$. EOS.

The convergence radius of T_f may be 0. Even if T_f converges at a point x_0, it may converge to a value other than $f(x_0)$. We identify the Taylor series with the function it induces on the interval centered at c (of lenth twice the convergence radius).

Example

1. Since $\exp'(\mathrm{x}) = \exp(x)$, $T_{\exp,0}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \exp(x)$.
2. More generally, if f is any power series centered at c with positive convergence radius, then f coincides with $T_{f,c}$.
3. For any real number $a \in \mathbb{R}$ and any $n \in \mathbb{N}_0$ we define the **generalized binomial coefficient** $\binom{a}{n}$ as
$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$$
if $n > 0$ and 1, if $n = 0$. The **binomial series** for a is defined as the formal power series $B_a(x) = \sum_{n=0}^{\infty} \binom{a}{n} x^n$. Its radius of convergence is 1.
Indeed, for large n and $x \neq 0$, the ratio test gives $\frac{|\binom{a}{n+1} x^{n+1}|}{|\binom{a}{n} x^n|} = \frac{|x|}{n+1}|a-n|$.
For $n \to \infty$ this has limit $|x|$. Thus $B_a(x)$ converges if $|x| < 1$, and diverges if $|x| > 1$ (see the Ratio Test 3.1.5.).
For $a \in \mathbb{R}$, consider $f(x) = (1+x)^a$. Then $T_{f,0}(x) = B_a(x)$. Note that $f^{(n+1)}(x) = a(a-1)(a-2)\cdots(a-n)(1+x)^{a-n-1}$. Then the Lagrange remainder is $\frac{|x|^{n+1}}{(n+1)!} a(a-1)\cdots(a-n)(1+d)^{a-n-1} = |x|^{n+1}\binom{a}{n+1}(1+d)^{a-n-1} \to 0$ for $n \to \infty$ for as long as $d \in (0,1)$.
It follows that $T_{f,0}$ converges to $f(x)$ for all $x \in [0,1)$. What about $x < 0$?
Then $B_a(x)$ still converges, and $B_a'(x) = \sum_{n=0}^{\infty}(n+1)\binom{a}{n+1} x^n \sum_{n=0}^{\infty} a\binom{a-1}{n} x^n$.

Note $(1+x)B_a'(x) = aB_a(x)$ because $\binom{a-1}{n} + \binom{a-1}{n-1} = \binom{a}{n}$. f also satisfies that $(1+x)f'(x) = af(x)$. Also note that $f(x) > 0$ on $(-1,1)$.

Then $\frac{d}{dx}\left(\frac{B_a(x)}{f(x)}\right) = \frac{B_a'(x)f(x) - B_a(x)f'(x)}{f^2(x)} = \frac{\frac{a}{1+x}\left(B_a(x)f(x) - B_a(x)f(x)\right)}{f^2(x)} = 0$.

Therefore $B_a(x) = Cf(x)$ for some constant C. Now $1 = B_a(0) = f(0)$ forces $C = 1$.

4. Consider the function h defined as

$$h(x) = \vdash\begin{cases} 0 & x \leq 0 \\ \exp\left(-\frac{1}{x}\right) & x > 0 \end{cases}$$

Then $h^{(n)}(0) = 0$ for all n. Thus $T_{h,0} = \sum_{n=0}^{\infty} 0 \cdot x^n$. But for positive x $h(x) > 0$. So the Taylor series with convergence radius ∞ converges nowhere to $h(x)$ for $x > 0$.

EOE.

Proposition

If I is an open interval and f is analytic on I, then for each $c \in I$, the Taylor series $T_{f,c}$ has a positive convergence radius R and $f(x) = T_{f,c}(x)$ for all $x \in (c-R, c+R) \cap I$. EOP.

Exercise

Consider $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-\frac{1}{x^2}}$ if $x \neq 0$ and $f(0) = 0$. Show that f is smooth.
Show that $T_{f,0} = 0$. EOE.

4 Integration

Integration is a procedure somewhat inverse to differentiation. It is also an analogue for an infinite sum.

Consider a particle moving along a line. Suppose you are sitting "on the particle" and you have a speedometer. Your task is to provide the distance travelled in a certain time interval.

Let $v(t)$ denote the velocity measured at time t. To compute an approximation of the distance travelled, you would sample (ie. measure) the velocity at many time spots $0 = t_0 < t_1 < t_2 < \cdots < t_n < T = t_{n+1}$. An approximation for the distance travelled $D(T)$ would then be

$$D(T) \sim \sum_{i=0}^{n} v(t_{i+1})(t_{i+1} - t_i)$$

where $t_0 = 0$ and $t_{n+1} = T$. The smaller the intervals $[t_i, t_{i+1}]$, the more precise this computation will usually be. Assuming the velocity function is continuous for example, the computation will be exact in the limit as the maximum interval size approaches zero.

Of course, there are problems with that. For example, v may not be constant on each of the intervals $[t_i, t_{i+1}]$ but rather vary considerably. Then our approximation may be really bad. However, if v is close to constant on these intervals, our approximation might be very good. For example, if v is continuous, and we choose our interval sizes small enough, this might be a good assumption (as we shall see).

Now let us turn this around, and suppose the distance travelled at time t, $D(t)$ is a differentiable function. Then $D(0) = 0$ and

$$D(T) = D(T) - D(0) = \sum_{i=0}^{n} D(t_{i+1}) - D(t_i) = \sum_{i=0}^{n} D'(c_i)(t_{i+1} - t_i)$$

where $c_i \in (t_i, t_{i+1})$ is suitably chosen (this uses the MVT). Again, if D' is continuous, say, and the intervals are small, then $D'(c_i) \sim D'(t_{i+1})$, which suggests that $D' = v$ (which we know by definition).

In this chapter we will formalize this kind of thinking. It will turn out, that for "reasonable" functions $v(t)$ (e.g. continuous functions) this procedure does indeed give meaningful results.

4.1 Indefinite integrals

4.1.1 Antiderivatives

We have seen how to compute derivatives of functions. We now turn the question around and ask whether we can go back: given f, is there F such that $F' = f$?

Definition
Let f be a function on an interval I. Then a function $F: I \to \mathbb{R}$ is called an **antiderivative** or an **indefinite integral** of f, if $F' = f$ on I. EOD.

Remark
Convince yourself why it makes no sense to talk about the antidervative at a point $x_0 \in I$. EOR.

Antiderivatives may or may not exist. We typically write $F = \int f$ or $F = \int f dx$ (where x is the variable) to indicate that $F' = f$. This notation is somewhat dangerous though, because there are many antiderivatives for a given function f as long as there is one.

Example

1. $\sin x$ is an antiderivative of $\cos x$ on any interval.
2. $\frac{1}{2}x^2$ is an antiderivative of x on any interval.
3. If $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is a formal power series with radius of convergence $R > 0$, $F(x) = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n$ is an antiderviative with the same radius of convergence.

If F is an antiderivative of f, then so is $F + C$ where $C \in \mathbb{R}$ is any constant,

Remark
If the domain of f consists of multiple disconnected intervals, e.g. $(0,1) \cup (2,3)$, then two antiderivatives of f need not differ by a constant (they could differ by different constants on each sub-interval). However, if F, G are two antiderivatives of f on an interval I, then $F - G$ *is* constant because $(F - G)' = 0$ on I (and it is enough that $F' = G'$ on I°). EOR.

We often write $\int f$ or $\int f dx$ for the **set** of all antiderivatives of f.

4.1.2 Integration rules for indefinite integrals

Let $f, g: I \to \mathbb{R}$ be functions defined on an interval I and suppose they each have antiderivatives, F, G respectively.

Linearity
For any $a, b \in \mathbb{R}$ we have $\int (af + bg) = a\int f + b\int g$.

Note, if the antiderivatives F of f and G of g are given, this does not mean, that a *specific* antiderivative of $af + bg$ is of the form $aF + bG$. It does mean that any antiderivative of $af + bg$ is of the form $aF + bG + C$ for some constant C. Or conversely, it means that any $aF + bG$ is an antiderivative of $af + bg$.

Integration by parts
Let $F' = f$ and $G' = g$. Then $\int Fgdx = FG - \int fGdx$.

One should think of this as an inverse to the product rule. It is easily proved by computing the derivative of FG.

Substitution rule
Suppose h is a function with antiderivative H, defined on an interval J such that $G(I) \subseteq J$. Then $\int (h \circ G)gdx = H \circ G$. Here $g = G'$.

Examples

1. $\int ax + bdx = \frac{a}{2}x^2 + bx + C$
2. $\int (ax + b)db = \frac{1}{2}b^2 + C$
3. $\int a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 x dx = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + \frac{a_1}{2}x^2 + a_0 x$
4. $\int \sin(x)\cos(x) = \sin(x)^2 - \int \cos(\mathrm{x})\sin(x) = \frac{1}{2}\sin^2(x)$. ($F = \sin(x)$, $g = \cos(x)$)
5. $\int \sin(x)\cos(x) = \frac{1}{2}\sin^2(x)$ ($h = x$, $G = \sin(x)$)

6. $\int \log x \, dx = \int 1 \cdot \log x \, dx = x \log x - \int x \cdot \frac{1}{x} dx = x \log x - x$
7. $\int xe^x = xe^x - \int e^x = xe^x - e^x$
8. $\int x^2 e^x = x^2 e^x - \int 2xe^x dx = x^2 e^x - 2xe^x + 2e^x$
9. $\int \frac{f'}{f} dx = \int \frac{1}{y} \circ f(x) f'(x) dx = \log |f|$
10. $\int (\arctan x) dx = x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} = x \arctan x - \frac{1}{2} \log(1 + x^2)$

The substitution rule above does not give an antiderivative of f but rather of the expression $(f \circ G)g'$. The antiderivative is then $F \circ G$. To turn it around we must get $F = (F \circ G) \circ G^{-1}$. This is sometimes possible.

Proposition (Substitution Rule)
Suppose the following holds:
1. f is defined on some interval I.
2. g is defined on an interval J and $g(I) = J$.
3. $g'(x) \neq 0$ for all $x \in J$.
4. $(f \circ g)g'$ has an anti-derivative F_0 on J.

Then g^{-1} is defined on I, and $F(x) \coloneqq F_0 \circ g^{-1}(x)$ is an antiderivative for f:
$$\int f(x) dx = \left[\int f\big(g(y)\big) g'(y) dy\right]_{y=g^{-1}(x)}$$

Proof. If $g'(x) \neq 0$ for all $x \in J^\circ$, the Mean Value Theorem guarantees that g is injective. It is therefore strictly increasing or strictly decreasing. Thus, g is invertible as a function $J \to I$. Then $h = g^{-1}: I \to J$ is also bijective. Moreover, h is differentiable and $h'(x) = \frac{1}{g(h(x))}$.

Now consider $F_0 \circ h$: Then $\frac{d}{dx}(F_0 \circ h) = F_0'\big(h(x)\big)h'(x) = f \circ g\big(h(x)\big)g'\big(h(x)\big)h'(x) = f(x)$. QED.

Example
$\int \frac{1}{\sin x} dx$ on $(-\pi, 0)$ or $(0, \pi)$.
Put $x = 2 \arctan t$ on $(-\infty, 0)$ or $(0, \infty)$, respectively. (You should think $x = g(t)$.)
Then $g'(t) = \frac{2}{1+t^2}$.
Then $\arctan t = \frac{x}{2}$, so $t = h(x) = \tan\left(\frac{x}{2}\right)$.
Recall that $\sin(2x) = \sin(x)\cos(x) + \cos(x)\sin(x) = 2\sin(x)\cos(x)$ and hence
$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\left(\sin\left(\frac{x}{2}\right)\right)^2 + \left(\cos\left(\frac{x}{2}\right)\right)^2} = \frac{2\left(\tan\frac{x}{2}\right)}{1 + \left(\tan\frac{x}{2}\right)^2} = \frac{2t}{1+t^2}$$
$(\sin\big(g(t)\big) = \frac{2t}{1+t^2})$.
Let $f = \frac{1}{\sin x}$, then $f \circ g(t) = \frac{1+t^2}{2t}$. $g'(t) = \frac{2}{1+t^2}$, so $(f \circ g)g' = \frac{2t}{2t}$, and $F_0(t) = \log|t|$.
Then $F_0 \circ h(x) = \log\left|\tan\left(\frac{x}{2}\right)\right|$, and we conclude
$$\int \frac{1}{\sin x} dx = \log\left|\tan\left(\frac{x}{2}\right)\right|$$
EOE.

4.2 Definite Riemann integrals

As mentioned in the introduction to the chapter, it is a natural (and important) question whether given a differentiable function F, say, we can recover the value $F(b)$ at b, provided we know $F(a)$, and the *rate of change* (ie. $F'(x)$) on the interval $[a,b]$. It turns out, the answer is yes in many cases (virtually all cases of interest), and the idea is the idea given in the introduction:

If F changes by $F'(x)\Delta x$ on a small interval $[x, x+\Delta x]$, then we should have

$F(b) = F(a) + \sum_{i=0}^{n} F'(c_i)\Delta x$ where $c_0, c_1, \ldots, c_n$ are spaced out in $[a,b]$ in intervals of size Δx. This is absolutely not yet rigorous.

4.2.1 Partitions

In the following we will talk a lot about **partitions** of closed intervals. Let $I = [a,b]$ be a fixed closed interval.

Definition
A **partition** P of $I = [a,b]$ is a finite ordered sequence $a < x_1 < x_2 < \cdots < x_n < b$ of ordered elements of I. So P is just a finite ordered subset of I. P is allowed to be empty. We call n the **size** of P, and denote it by $|P|$ or $\#P$. We always put $x_0 = a$, and $x_{n+1} = b$. The maximum interval size $\max\{\, x_{i+1} - x_i \mid i = 0,1,2,\ldots,n\}$ is called the **mesh size** of P, and denoted $m(P)$. We write $\Pi(I)$ or $\Pi(a,b)$ for the set of all partitions of I.

If P is a partition, its elements are denoted $x_1 < x_2$ … unless otherwise noted. Similarly, of P_n is a sequence of partitions, then its elements are denoted $x_{n,1} < x_{n,2} < \cdots$.

Notice that the partitions of I can be partially ordered by putting $P \leq Q$ if $P \subseteq Q$. We call such a Qa **refinement** of P.

If P, Q are any two paritions of I, then $P \cup Q$ is a **common refinitement**: $P, Q \leq P \cup Q$.

4.2.2 Riemann sums

Definition
Let $I = [a,b]$ and let $P = x_1 < x_2 < \cdots < x_n \in \Pi(I)$ be any partition of size $n \geq 0$. A **tag vector** for P is an element $\boldsymbol{y} = (y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}$ such that

$$a = x_0 \leq y_0 \leq x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_n \leq y_n \leq x_{n+1} = b$$

Or, equivalently, $y_i \in [x_i, x_{i+1}]$. We write $T(P)$ for the set of all tag-vectors. EOD.

For $f: I \to \mathbb{R}$, $P \in \Pi(I)$, and $\boldsymbol{y} \in T(P)$, we define the corresponding **Riemann sum** as the sum

$$S(P, \boldsymbol{y}, f) \coloneqq \sum_{i=0}^{|P|} f(y_i)(x_{i+1} - x_i)$$

A **Riemann sequence** for f is a sequence of the form $S(P_n, \boldsymbol{y}_n, f)$ where $\lim_{\mathrm{n}\to\infty} m(P_n) = 0$. EOD.

Note that the function f in the definition of a Riemann sequence does not vary with n. Technically, a Riemann sequence is more than its value. The partitions, tag-vectors, and function are all part of the definition.

4.2.3 The Riemann integral

Definition
Let $f: [a, b] \to \mathbb{R}$ be a function (where $a \leq b \in \mathbb{R}$). We say f is **Riemann integrable** or **R-integrable** or **integrable**, if every Riemann sequence for f converges. The set of integrable functions on $[a, b]$ is denoted $\mathcal{R}[a, b]$. EOD.

Lemma
Suppose f is integrable on $[a, b]$. Then all Riemann sequences have the same limit. EOL.

Proof. Let $S(P_n, \boldsymbol{y}_n, f)$ and $S(Q_n, \boldsymbol{z}_n, f)$ be two Riemann sequences. Then

$$S(P_1, \boldsymbol{y}_1, f), S(Q_1, \boldsymbol{z}_1, f), S(P_2, \boldsymbol{y}_2, f), S(Q_2, \boldsymbol{z}_2, f), \ldots$$

is again a Riemann sequence.

Formally, we form a new Riemann sequence $S(R_n, \boldsymbol{x}_n, f)$ where

$$(R_n, \boldsymbol{x}_n) = \begin{cases} (P_k, \boldsymbol{y}_k) & n = 2k - 1 \\ (Q_k, \boldsymbol{z}_k) & n = 2k \end{cases}$$

This is again a Riemman sequence and hence convergent. Since the original ones are subsequences, they have the same limit. QED.

Definition
Let f be integrable on $[a, b]$. The **Riemman integral** or **R-integral** or **integral** of f on $[a, b]$ is defined as the common limit of its Riemann sequences. It is usually denoted as
$\int_a^b f$ or $\int_a^b f(x)dx$ or $\int_a^b f dx$. EOD.

Example
Consider $f = \chi_{\mathbb{Q}}$,the characteristic function of $\mathbb{Q}$. Then f is not integrable on any bounded interval $[a, b]$ with $a < b$. Recall that $f(x) = 0$ if $x \notin \mathbb{Q}$ and $f(x) = 1$ if $x \in \mathbb{Q}$.
Consider any Riemann sequence $S(P_n, \boldsymbol{y}_n, f)$ where all the tag points in $\boldsymbol{y}_n$ are rational. Then $S(P_n, \boldsymbol{y}_n, f) = b - a$, and the sequence converges.
If on the other hand all tag points are irrational, then $S(P_n, \boldsymbol{y}_n, f) = 0$ and again the sequence converges. As the limits are not equal, it follows that f cannot be integrable. EOE.

4.2.4 The Fundamental Theorem of Calculus (Part I)

As mentioned in the introduction, if F is a differentiable function (e.g. D above), then it is natural to ask whether we can recover F from $F' = f$. A partial answer to this question is given by the following theorem:

Theorem (First Fundamental Theorem of Calculus, or FTC 1)
Let $F: [a, b] \to \mathbb{R}$ be a continuous function, differentiable on at least (a, b) and suppose $f: [a, b] \to \mathbb{R}$ is a function such that:

1. f is integrable or continuous.
2. f agrees with F' on at least (a, b).

Then $F(b) - F(a) = \int_a^b f(x)dx$, or equivalently

$$F(b) = F(a) + \int_a^b f(x)dx$$

EOT.

Note that the theorem asserts that if F' is continuous then it is integrable. It does not say a priori that every continuous function is integrable: it only asserts that for those that have an indefinite integral.

Proof. Let $P_n = x_{n0} = a < x_{n1} < x_{n2} < \cdots < x_{n|P_n|} < x_{n|P_n|+1} = b$ be a sequence of partitions such that $m(P_n) \to 0$.

Then

$$F(b) - F(a) = \sum_{i=0}^{|P_n|} \Big(F\big(x_{n,i+1}\big) - F(x_{ni}) \Big) = \sum_{i=0}^{|P_n|} F'(c_{ni})\big(x_{n,i+1} - x_{ni}\big)$$

where $c_{ni} \in (x_{ni}, x_{n,i+1})$ is suitably chosen (this uses the MVT).

Put $\boldsymbol{y}_n = \big(c_{n0}, c_{n1}, \dots, c_{n,|P_n|}\big) \in \mathbb{R}^{|P_n|+1}$. Then $\boldsymbol{y}_n$ is a tag vector for P, and the right hand side is equal to $S(P_n, \boldsymbol{y}_n, f)$ (since $f = F'$). This is a Riemann sequence, and since $F(b) - F(a) = S(P_n, \boldsymbol{y}_n, f)$ is a constant sequence we get

$$F(b) - F(a) = \lim_{n\to\infty} S(P_n, \boldsymbol{y}_n, f)$$

If f is integrable, the right hand side is equal to $\int_a^b f(x)dx$, which finishes the proof in this case.

If f is continuous we must show it is integrable. Let $\boldsymbol{z}_n = (z_{n0}, z_{n1}, \dots, z_{n,|P_n|})$ be any tag vector for P_n. Note that as $[a, b]$ is closed and bounded, f is *uniformly continuous* (see below). That is, for any $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in [a, b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

Now $|S(P_n, \boldsymbol{y}_n, f) - S(P_n, \boldsymbol{z}_n, f)| \leq \sum_{i=0}^{|P_n|} |f(c_{ni}) - f(z_{ni})|\big(x_{n,i+1} - x_{ni}\big)$

$$\leq \max_{0\leq i\leq |P_n|} |f(c_{ni}) - f(z_{ni})|(b - a)$$

Let $\varepsilon > 0$ and let $\delta > 0$ be such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ whenever $|x - y| < \delta$. Then there is n_0 such that for all $n > n_0$, $m(P_n) < \delta$. But then

$$|S(P_n, \boldsymbol{y}_n, f) - S(P_n, \boldsymbol{z}_n, f)| \leq \max_{0\leq i\leq |P_n|} |f(c_{ni}) - f(z_{ni})|(b - a) \leq \frac{\varepsilon}{b - a} \cdot (b - a) = \varepsilon$$

It follows that $S(P_n, \boldsymbol{y}_n, f) - S(P_n, \boldsymbol{z}_n, f)$ is a zero sequence, and therefore

$$\lim_{n\to\infty} S(P_n, \boldsymbol{z}_n, f) = \lim_{n\to\infty} S(P_n, \boldsymbol{y}_n, f) = F(b) - F(a)$$

Since we started with an arbitrary sequence of partitions P_n (for which $m(P_n) \to 0$), this shows that for any Riemman sequence for f, we get one (and in fact the same) limit, namely $F(b) - F(a)$. This shows that f is integrable. QED.

Note again, that in this proof, in the case of continuous f, we needed to know hat *a priori* for every sequence P_n with $m(P_n) \to 0$ there is at least one associated sequence of tag-vectors such that the associated Riemann sequence converges. This worked because we knew that f had an anti-derivative.

To complete the proof of the FTC 1, we recall the definition of uniform continuity.

Definition
Let I be an interval. A function f on I is called **uniform continuous** if for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in I$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. EOD.

Uniform continuous functions are obviously continuous. The converse is not always true. But it is true if the interval is closed and bounded, ie. of the form $I = [a, b]$ where $a \leq b \in \mathbb{R}$.

Proposition
Let $I = [a, b]$ be a *closed* bounded interval. If f is a continuous function on I, then it is uniform continuous. EOP.

Proof. Let $\varepsilon > 0$. As f is continuous for every $x_0 \in I$, there exists $\delta(x_0)$ such that for all $x \in I$ with $|x - x_0| < \delta(x_0)$ we have $|f(x) - f(x_0)| < \varepsilon$. We must show that there exists a δ that works for all x_0.

Suppose not. Then for $\delta_n = \frac{1}{n}$ there is a pair x_n, y_n such that $|x_n - y_n| < \delta_n$ but $|f(x_n) - f(y_n)| > \varepsilon$. Note x_n, y_n are each a bounded sequence so contain convergent subsequences. We may therefore replace x_n with a convergent subsequence. Then y_n is automatically also convergent, because we still have $|x_n - y_n| < \frac{1}{n}$. As $[a, b]$ is closed, it must contain the common limit x_0 of x_n, y_n.

Then $0 = |f(x_0) - f(x_0)| = \lim_{n\to\infty} |f(x_n) - f(y_n)| \geq \varepsilon > 0$, which is obviously a contradiction. QED.

Returning to the FTC1:

Warning
The theorem does not make any statement about cases where F' is *not* integrable. There are functions with non-integrable derivative.
Conversely, even if a function is integrable it need not have an antidervative.
This is an unlimited source of mistakes. EOW.

Example
1. Whenever F has a continuous derivative, the FTC1 applies:
2. $\log x = \log 1 + \int_1^x \frac{1}{t} dt = \int_1^x \frac{1}{t} dt$ for all $x > 1$. (This also holds for $x \leq 1$, but we haven't discussed this yet.)
3. $e^x = 1 + \int_0^x e^t dt$ for $x > 0$.
4. $\sin x = \int_0^x \cos t \, dt$ for $x > 0$.
5. Take any *continuous* function f on $[a, b]$ which has an antiderivative F there. Then $F(b) = F(a) + \int_a^b f(x)\, dx$

Notation
If a function f is integrable on $[a, b]$ and has an antiderivative F there, it is common to write to write $[F]_a^b$ for $F(b) - F(a)$.

Thus, for example

$$\int_a^b \cos x\, dx = [\sin x]_a^b$$

EON.

MIDTERM 2 CUTOFF

4.2.5 Linearity of integration

For now, let us record that if f, g are integrable, then so are all linear combinations:

Lemma (Linearity of integration)
Suppose $f, g: [a, b] \to \mathbb{R}$ are integrable. Then $\alpha f + \beta g$ is integrable for all $\alpha, \beta \in \mathbb{R}$ and

$$\int_a^b \alpha f(x) + \beta g(x) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

EOL.

Proof. This immediately follows from the fact that for all Riemann sequences here we have $S(P_n, \boldsymbol{y}_n, cf + dg) = cS(P_n, \boldsymbol{y}_n, f) + dS(P_n, \boldsymbol{y}_n, g)$, together with the linearity of limits of sequences. QED.

If you know linear algebra, the lemma shows that $\mathcal{R}[a, b]$ is a linear subspace of $\mathcal{F}[a, b]$.

4.2.6 Integrable functions are bounded

For an interval I we denote the set of bounded functions on I by $\mathcal{B}I$ or $\mathcal{B}(I)$. So $\mathcal{B}[a, b]$ is the set of bounded functions on $[a, b]$.

Theorem

$$\mathcal{R}[a, b] \subseteq \mathcal{B}[a, b]$$

In other words, every integrable function on $[a, b]$ is bounded. EOT.

Proof. Suppose f is not bounded. Then $\sup f = \infty$ or $\inf f = -\infty$. We will assume the first and show that f is not integrable. The second case then follows by applying the first to $-f$ (which is integrable by the linearity lemma for integration 4.2.3).

We may assume there is a sequence $x_n \in [a, b]$ such that $\lim_{n\to\infty} f(x_n) = \infty$. By Bolzano-Weierstrass such a bounded sequence has a convergent subsequence x_{n_k} with limit $x_0 \in [a, b]$. Then still $\lim_{k\to\infty} f(x_{n_k}) = \infty$, so we may replace the original sequence with this subsequence and assume $x_n \to x_0 \in [a, b]$.

After again restricting to a subsequence, we may assume that $f(x_n) > n^2$. (Indeed, to construct the subsequence x_{n_k} simply recursively put $n_k = \min\{m \mid m > n_{k-1}, f(x_m) > k^2\}$).

Similarly, again choosing a subsequence we may also assume that $|x_n - x_0| < \frac{b-a}{n^3}$.

Let P_n be a partition such that $m(P_n) \to 0$ and such that for each n (large enough), P_n contains an interval of size $\frac{b-a}{n}$, say $[x_{n,i(n)}, x_{n,i(n)+1}]$, which contains x_0 as its mid point.

Let $S(P_n, \boldsymbol{y}_n, f)$ be a Riemann sequence. We may assume that $y_{n,i(n)} = x_0$.

Let $\boldsymbol{y}_n' \in \mathbb{R}^{|P_n|+1}$ be obtained from $\boldsymbol{y}_n$ by replacing $y_{n,i(n)} = x_0$ with x_n which is in the same interval.

Then $S(P_n, \mathbf{y}'_n, f) - S(P_n, \mathbf{y}_n, f) = \big(f(x_n) - f(x_0)\big)\big(x_{n,i(n)+1} - x_{n,i(n)}\big) = \frac{f(x_n)-f(x_0)}{n}(b-a) > n - \frac{f(x_0)}{n} \to \infty$, contradicting the fact that f is integrable. QED.

Example/Exercise

Let $f(x) = \vdash \begin{cases} x\sqrt{x}\sin\frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$

defined on $[0, \infty)$. Then f is differentiable everywhere:

$f'(x) = \frac{3}{2}\sqrt{x}\sin\left(\frac{1}{x}\right) - \frac{x\sqrt{x}}{x^2}\cos\frac{1}{x} = \frac{3}{2}\sqrt{x}\sin\left(\frac{1}{x}\right) - \frac{1}{\sqrt{x}}\cos\frac{1}{x}$ on $(0, \infty)$

and $f'(0) = 0$.

Verify these assertions. Then show that f' is not bounded on any interval $[0, b]$ and hence not integrable on any such interval. EOE.

4.2.7 Darboux integration

The definition of the Riemann integral is an awful way to show that a given function is integrable. The main issue is that we don't have to show convergence of a single Riemann sequence, which is not too bad for reasonable functions, but for *all* of them.

This is important because the convergence of a single, or even infinitely many Riemann sequences is not enough in general.

The notion of Darboux integration is a bit better behaved for functions which have easily determined suprema and infima. But in the end, it will turn out that this is not a new notion.

Let $f \in \mathcal{B}[a, b]$ and let $P \in \Pi(a, b)$. As f is bounded, on each interval $[x_i, x_{i+1}]$ of $[a, b]$, f as a supremum M_i and an infimum m_i.

We put $\mathcal{U}(P, f) \coloneqq \sum_{i=0}^{|P|} M_i(x_{i+1} - x_i)$, and $\mathcal{L}(P, f) \coloneqq \sum_{i=0}^{|P|} m_i(x_{i+1} - x_i)$.

Clearly $\mathcal{U}(P, f) \geq \mathcal{L}(P, f)$. And for every tag vector $\mathbf{y} \in T(P)$ we have

$$\mathcal{L}(P, f) \leq S(P, \mathbf{y}, f) \leq \mathcal{U}(P, f)$$

Note that $\mathcal{U}(P, f) \geq (\inf f)(b - a)$ and $\mathcal{L}(P, f) \leq (\sup f)(b - a)$. We can therefore define:

$$\mathcal{U}(f) \coloneqq \inf_{P \in \Pi(a,b)} \mathcal{U}(P, f)$$

and

$$\mathcal{L}(f) \coloneqq \sup_{P \in \Pi(a,b)} \mathcal{L}(P, f)$$

In the literature you also find the notation

$$\int_{\underline{a}}^{b} f(x)dx = \mathcal{L}(f)$$

And

$$\overline{\int_a^b} f(x)dx = \mathcal{U}(f)$$

Note we will show below that always $\mathcal{L}(f) \leq \mathcal{U}(f)$.

Definition

A function $f \in \mathcal{B}[a,b]$ is called **Darboux-integrable** if $\mathcal{L}(f) = \mathcal{U}(f)$. For such a function this common value is called its **Darboux integral**. We denote it by $\mathcal{D} - \int_a^b f$. EOD.

One of the advantages of this notion is that if $P \leq Q$ is a refinement, then the lower and upper are well behaved:

Fact

If $P \leq Q$, then

$$\mathcal{U}(P,f) \geq \mathcal{U}(Q,f)$$

and

$$\mathcal{L}(P,f) \leq \mathcal{L}(Q,f)$$

EOF.

We show the first inequality, the second follows by applying the first to $-f$, or by similar reasoning.

If $P \leq Q$, then every interval of Q is a subinterval of some interval of P. Therefore, the supremum of f on this subinterval is *smaller* (or equal) than the supremum of f on the interval of P containing it.

Proof. We first assume Q is obtained from P by adding a single element p at position $i+1$, say:

$$P = x_1 < \cdots < x_i < x_{i+1} < \cdots < x_n$$

and

$$Q = x_1 < \cdots < x_i < p < x_{i+1} < \cdots < x_n$$

As before let $M_j = \sup\limits_{[x_j, x_{j+1}]} f(x)$ $(j = 0,1,\ldots,n)$.

Let $A = \sum_{j=0}^{i-1} M_j(x_{j+1} - x_i) + \sum_{j=i+1}^{n} M_j(x_{j+1} - x_j)$.

Then $\mathcal{U}(P,f) = A + M_i(x_{i+1} - x_i)$. Note that the intervals of Q are those of P except the ith and $(i+1)$st one: those are $[x_i, p]$ and $[p, x_{i+1}]$. Let $N_1 = \sup\limits_{[x_i,p]} f(x)$ and $N_2 = \sup\limits_{[p,x_{i+1}]} f(x)$. Then $N_1, N_2 \leq M_i$

and

$$\mathcal{U}(Q,f) = A + N_1(p - x_i) + N_2(x_{i+1} - p) \leq A + M_i\big((p - x_i) + (x_{i+1} - p)\big) = \mathcal{U}(P,f)$$

as claimed. The general case then follows easily by induction on $k \coloneqq |Q| - |P|$. We just dealt with the case $k = 1$. Now suppose we have shown that for a given k, whenever $P \subseteq Q$ and $|Q| = |P| + k$, the claim holds. If now Q contains P and $k+1$ other points, then $P \subseteq Q' \subseteq Q$ where Q' contains k points not in P. By the induction assumption $\mathcal{U}(P,f) \leq \mathcal{U}(Q',f)$ and by the case $k = 1$, $\mathcal{U}(Q',f) \leq \mathcal{U}(Q,f)$. QED.

Corollary
If P, Q are two partitions of $[a, b]$, then always
$$\mathcal{L}(P, f) \leq \mathcal{U}(Q, f)$$
EOC.

Proof. Note that P, Q have a common refinement, namely e.g. $R \coloneqq P \cup Q$. Then
$$\mathcal{L}(P, f) \leq \mathcal{L}(R, f) \leq \mathcal{U}(R, f) \leq \mathcal{U}(Q, f)$$
QED.

It follows that as promised that $\mathcal{L}(f) \leq \mathcal{U}(f)$.

Lemma (Cauchy Criteron for Darboux Integration)
$f \in \mathcal{B}[a, b]$ is Darboux integrable iff for every $\varepsilon > 0$, there is a partition P such that $\mathcal{U}(P, f) - \mathcal{L}(P, f) < \varepsilon$. EOL.

Proof. "If": suppose such a P exists for each $\varepsilon > 0$. Note that $0 \leq \mathcal{U}(f) - \mathcal{L}(f) \leq \mathcal{U}(P, f) - \mathcal{L}(P, f)$ for every P. Therefore, $\mathcal{U}(f) - \mathcal{L}(f) < \varepsilon$ for every $\varepsilon > 0$, and hence $\mathcal{U}(f) = \mathcal{L}(f)$.

"only if": Suppose $\mathcal{U}(f) = \mathcal{L}(f)$ and let $\varepsilon > 0$. Then there is a partition Q such that $\mathcal{U}(Q, f) < \mathcal{U}(f) + \frac{\varepsilon}{2}$ and a partition R such that $\mathcal{L}(R, f) > \mathcal{L}(f) - \frac{\varepsilon}{2}$. By the fact above this still holds if we replace Q, R by a common refinement P. But then $\mathcal{U}(P, f) - \mathcal{L}(P, f) < \mathcal{U}(f) + \frac{\varepsilon}{2} - \mathcal{L}(f) + \frac{\varepsilon}{2} = \varepsilon$. QED.

This criterion is sometimes also referred to as the Riemann Criterion for integration (once one knows that Darboux integrable functions are Riemann integrable).

Corollary
$f \in \mathcal{B}[a, b]$ is Darboux integrable if and only if there is a sequence of partitions P_n with $\lim_{n\to\infty}\big(\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f)\big) = 0$, and if that is the case then
$$\mathcal{D} - \int_a^b f(x)dx = \lim_{n\to\infty} \mathcal{U}(P_n, f) = \lim_{n\to\infty} \mathcal{L}(P_n, f)$$
EOC.

Proof. Suppose P_n exists. Then f is Darboux-integrable by the lemma (for any $\varepsilon > 0$ pick n large enough such that $\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) < \varepsilon$). Conversely, if f is Darboux-integrable, then for each n there is P_n such that $\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) < \frac{1}{n}$. This shows the first part.

Now suppose $\lim_{n\to\infty}\big(\mathcal{U}(P_n. f) - \mathcal{L}(P_n, f)\big) = 0$. Then the result follows from the following lemma (put $c = \mathcal{D} - \int_a^b f(x)dx$, which exists by the first part). QED.

Lemma
Let a_n, b_n be sequences such that $\lim_{n\to\infty}(a_n - b_n) = 0$ and there exists c such that $a_n \geq c \geq b_n$ for all n. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$. EOL.

Proof. If one of the sequences converges, so does the other, and the limits coincide. If $a_n - b_n \to 0$, then $0 \leq a_n - c \leq a_n - b_n \to 0$. QED.

Example
Let $f(x) = x$ defined on $[a, b]$.
Let P_n be the partition $a + \frac{L}{n} < \ a + \frac{2L}{n} < \cdots < a + \frac{(n-1)L}{n}$ where $L = b - a$.
Then

$$\mathcal{U}(P_n, f) = \sum_{i=0}^{n-1} \left(a + \frac{(i+1)L}{n}\right)\frac{L}{n} = \frac{n}{n}aL + \frac{L^2}{n^2}\sum_{i=0}^{n-1} i + 1 = \frac{n}{n}aL + \frac{\frac{L^2}{n^2}(n+1)n}{2}$$

This converges to $aL + \frac{L^2}{2} = a(b-a) + \frac{(b-a)^2}{2} = ab - a^2 + \frac{1}{2}b^2 - ab + \frac{1}{2}a^2 = \frac{1}{2}(b^2 - a^2)$.
Note replacing $i + 1$ in the summation above by i gives $\mathcal{L}(P_n, f)$ but does not change the limit. Hence f is Darboux integrable and

$$\mathcal{D} - \int_a^b x dx = \frac{1}{2}(b^2 - a^2) = \int_a^b x dx$$

by the FTC 1. EOE.

Theorem
Let $f \in \mathcal{B}[a, b]$ and suppose P_n is a sequence of partitions with $m(P_n) \to 0$. Then f is Darboux integrable iff $\lim_{n\to\infty} \mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) = 0$.
In particular, if f is Darboux integrable, for every sequence of partitions P_n with $m(P_n) \to 0$, we have $\lim_{n\to\infty} \mathcal{U}(P_n, f) = \lim_{n\to\infty} \mathcal{L}(P_n, f) = \mathcal{D} - \int_a^b f(x)dx$. EOT.

Proof. Suppose $\lim_{n\to\infty} \mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) = 0$. Then f is Darboux integrable by the Cauchy criterion above. For the converse let f be Darboux integrable, $L = \mathcal{D} - \int_a^b f\,dx$, and let $\varepsilon > 0$. Then there is a partition P such that $\mathcal{U}(P, f) - \mathcal{L}(P, f) < \frac{\varepsilon}{2}$.

We now compare $\mathcal{U}(P_n, f)$ and $\mathcal{U}(P, f)$. For large enough n, so that the mesh size $m(P_n)$ is small compared to the smallest interval length of P, most intervals of P_n must be properly contained in an interval of P. Suppose $m = |P|$, so P has $m + 1$ intervals. Then there are at most $2m$ intervals of P_n that are not properly contained in any interval of P (namely those that contain one of the points in P). Let $P = y_1 < y_2 < \cdots < y_m$.

We get

$$\mathcal{U}(P_n, f) = \sum_{i=0}^{|P_n|} M_{ni}(x_{i+1} - x_i) = \sum_{\substack{i=0 \\ P\cap[x_i, x_{i+1}]=\emptyset}}^{|P_n|} M_{ni}(x_{i+1} - x_i) + \sum_{\substack{i=0 \\ P\cap[x_i, x_{i+1}]\neq\emptyset}}^{|P_n|} M_{ni}(x_{i+1} - x_i)$$

where $M_{ni} = \sup_{[x_{ni}, x_{n,i+1}]} f(x)$.

Similarly, we have

$$\mathcal{L}(P_n, f) = \sum_{\substack{i=0 \\ P\cap[x_{ni}, x_{n,i+1}]\neq\emptyset}}^{|P_n|} m_{ni}(x_{i+1} - x_i) + \sum_{\substack{i=0 \\ P\cap[x_i, x_{i+1}]=\emptyset}}^{|P_n|} m_{ni}(x_{i+1} - x_i)$$

Then

$$\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)\le \sum_{\substack{i=0\\ P\cap[x_i,x_{i+1}]=\emptyset}}^{|P_n|}(M_{ni}-m_{ni})(x_{i+1}-x_i)+\sum_{\substack{i=0\\ P\cap[x_i,x_{i+1}]\neq\emptyset}}^{|P_n|}(M_{ni}-m_{ni})(x_{i+1}-x_i)$$

$$\le \sum_{\substack{i=0\\ P\cap[x_i,x_{i+1}]=\emptyset}}^{|P_n|}(M_{ni}-m_{ni})(x_{i+1}-x_i)+\sum_{\substack{i=0\\ P\cap[x_i,x_{i+1}]\neq\emptyset}}^{|P_n|}(M-I)(x_{i+1}-x_i)$$

$$\sum_{\substack{i=0\\ P\cap[x_i,x_{i+1}]=\emptyset}}^{|P_n|}(M_{ni}-m_{ni})(x_{i+1}-x_i)+(M-I)2mm(P_n)$$

where $M=\sup f$ and $I=\inf f$. (Note all summands are nonnegative, and there at most $2m$ intervals with $P\cap[x_i,x_{i+1}]\neq\emptyset$.)

In the sum on the left, each occurring interval $[x_{ni},x_{n,i+1}]$ is properly contained in an interval of P. If $A_j=\sup\limits_{[y_j,y_{j+1}]} f$ and $a_j=\inf\limits_{[x_j,x_{j+1}]} f$ then

$$\mathcal{U}(P,f)-\mathcal{L}(P,f)=\sum_{j=0}^{m}(A_j-a_j)(y_{j+1}-y_j)$$

It follows that

$$\sum_{\substack{i=0\\ Q\cap[x_i,x_{i+1}]=\emptyset}}^{n}(M_{ni}-m_{ni})(x_{i+1}-x_i)=\sum_{j=0}^{m}\sum_{\substack{i=0\\ [x_{ni},x_{n,i+1}]\subseteq[y_j,y_{j+1}]}}^{|P_n|}(M_{ni}-m_{ni})(x_{i+1}-x_i)\le$$

$$\le\sum_{j=0}^{m}\sum_{\substack{i=0\\ [x_{ni},x_{n,i+1}]\subseteq[y_j,y_{j+1}]}}^{|P_n|}\left(A_j-a_j\right)(x_{i+1}-x_i)\le\sum_{j=0}^{m}(A_j-a_j)(y_{j+1}-y_j)$$

Thus $\mathcal{U}(P_n,f)-\ \mathcal{L}(P_n,f)\le\mathcal{U}(P,f)-\mathcal{L}(P,f)+2m(M-I)m_n(P)<\frac{\varepsilon}{2}+2m(M-I)m_n(P)\to\frac{\varepsilon}{2}$ (for $n\to\infty$).

Thus, for n large, $\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)<\varepsilon$, and so $\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)\to 0$ as needed. QED.

This theorem is important: to test whether a function is Darboux integrable and compute the integral, all we need to do is to check whether the two limits $\lim\limits_{n\to\infty}\mathcal{U}(P_n,f)=\lim\limits_{n\to\infty}\mathcal{L}(P_n,f)$ exist and coincide for *one* choice of sequence P_n with $m(P_n)\to 0$. Contrast that with a Riemann sequence $S(P_n,\boldsymbol{y}_n,f)$, where convergence of a single one does not mean that f is integrable.

4.2.8 Equivalence of Darboux and Riemann integration

Theorem

Let $f \in \mathcal{B}[a,b]$. Then f is Darboux integrable if and only if it is integrable. If f is integrable then

$$\mathcal{D}-\int_a^b f=\int_a^b f$$

Proof. Suppose f is integrable, and let $S(P_n,\boldsymbol{y}_n,f)$ a Riemann sequence as follows: P_n is a sequence of partitions such that $m(P_n)\to 0$. For each i let $y_{n,i}\in[x_{n,i},x_{n,i+1}]$ be chosen such that $f(y_{n,i})>\sup_{[x_{ni},x_{n,i+1}]} f(x)-\frac{1}{n(|P_n|+1)(x_{n,i+1}-x_{n,i})}$.

Let $\boldsymbol{z}_n$ be defined such that for $z_{n,i}\in[x_{ni},x_{n,i+1}]$, $f(z_i)<\inf_{[x_{n,i},x_{n,i+1}]} f(x)+\frac{1}{n|P_n|(x_{n,i+1}-x_{n,i})}$.

Then $\mathcal{L}(P_n,f)\le\ S(P_n,\boldsymbol{y}_n,f), S(P_n,\boldsymbol{z}_n,f)\le\mathcal{U}(P_n,f)$. Note that

$$\mathcal{U}(P_n,f)-S(P_n,\boldsymbol{y}_n,f)<\sum_{i=0}^{|P_n|}\frac{1}{n(|P_n|+1)(x_{n,i+1}-x_{n,i})}(x_{n,i+1}-x_{ni})=\frac{1}{n}$$

We similarly conclude that $S(P_n,\boldsymbol{z}_n,f)-\mathcal{L}(P_n,f)<\frac{1}{n}$.

For n large enough, both Riemann sums are “close” to $I=\ \int_a^b f$, and then necessarily $\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)$ must be “small”. To be precise, let $\varepsilon>0$, and let n be large enough such that $|S(P_n,\boldsymbol{y}_n,f)-I|, |S(P_n,\boldsymbol{z}_n,f)-I|<\frac{\varepsilon}{8}$ and also assume $\frac{1}{n}<\frac{\varepsilon}{4}$. Then

$$\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)\le\mathcal{U}(P_n,f)-S(P_n,\boldsymbol{y}_n,f)+$$
$$|S(P_n,\boldsymbol{y}_n,f)-\ S(P_n,\boldsymbol{z}_n,f)|+S(P_n,\boldsymbol{z}_n,f)-\mathcal{L}(P_n,f)<\varepsilon$$

This shows f is Darboux integrable, and moreover, that $\mathcal{D}-\int_a^b f(x)dx=\int_a^b f(x)dx$.

To show the converse, we know that if P_n is any sequence of partitions such that $m(P_n)\to 0$, then

$$\mathcal{U}(P_n,f),\mathcal{L}(P_n,f)\to\mathcal{D}-\int_a^b f(x)\,dx$$

On the other hand, for every Riemann sequence $S(P_n,\boldsymbol{t}_n,f)$ we have

$$\mathcal{L}(P_n,f)\le S(P_n,\boldsymbol{t}_n,f)\le\mathcal{U}(P_n,f)$$

The squeeze principle shows that every such Riemann sequence converges, and that the common limit is equal to $\mathcal{D}-\int_a^b f$. QED.

We therefore drop the $\mathcal{D}-$ from the notation and treat the two notions of integral as equivalent.

4.2.9 Continuous functions are integrable

Theorem

Let f be continuous on $[a,b]$. Then f is integrable. EOT.

Proof. Let P_n be a sequence of partitions of $[a,b]$ for which $m(P_n)\to 0$.

Note that for each interval $[x_{ni}, x_{n,i+1}]$ of P_n, there are $y_{ni}, z_{ni} \in [x_{ni}, x_{n,i+1}]$ such that $f(y_{ni}) = \sup_{[x_{ni}, x_{n,i+1}]} f(x) =: M_{ni}$ and $f(z_{ni}) = \inf_{[x_{ni}, x_{n,i+1}]} f(x) =: m_{ni}$. This uses that f is continuous on $[a, b]$.

Then $\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) = \sum_{i=0}^{|P_n|}(M_{ni} - m_{ni})\left(x_{n,i+1} - x_{ni}\right)$.

But f is uniformly continuous on $[a, b]$. So, for a given $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$. So, as soon as n is large enough such that $m(P_n) < \delta$, we have

$$\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) < \sum_{i=0}^{|P_n|} \frac{\varepsilon}{b-a}\left(x_{n,i+1} - x_{ni}\right) = \frac{\varepsilon}{b-a}\sum\left(x_{n,i+1} - x_i\right) = \frac{\varepsilon}{b-a}\,(b-a) = \varepsilon$$

The Cauchy Criterion for Darboux integrals does the rest. QED.

This closes a gap in our discussion of the Fundamental Theorem of Calculus Part 1. There, we only proved that continuous functions with an indefinite integral are integrable. Now we know that this was no restriction: all continuous functions are integrable on closed and bounded intervals.

4.2.10 Monotone functions are integrable

As an example of our discussion so far, we show that monotone functions are integrable.

Theorem
Let f be monotone on $[a, b]$. Then f is integrable. EOT.

Proof. We will prove the theorem for the case f is increasing. If f is monotone decreasing, the result follows by applying the monotone increasing result to $-f$ (and observing that f is integrable iff $-f$ is integrable (see Linearity of integration 4.2.5).

We may assume that f is not constant (as we know that constant functions are integrable). Then $f(b) > f(a)$.

For each partition P we conclude that

$$\mathcal{U}(P, f) = \sum_{i=0}^{|P|} f(x_{i+1})(x_{i+1} - x_i)$$

and

$$\mathcal{L}(P, f) = \sum_{i=0}^{|P|} f(x_i)(x_{i+1} - x_i)$$

Now suppose that P is a partition with $m(P) < \delta$. Then

$$\begin{aligned}\mathcal{U}(P, f) - \mathcal{L}(P, f) < \sum_{i=0}^{|P|}\big(f(x_{i+1}) - f(x_i)\big)(x_{i+1} - x_i) &< \delta\sum_{i=0}^{|P|} f(x_{i+1}) - f(x_i)\\ &= \delta(f(b) - f(a))\end{aligned}$$

For $\varepsilon > 0$, if we choose $\delta < \frac{\varepsilon}{f(b)-f(a)}$, then for every partition P with $m(P) < \delta$, we get $\mathcal{U}(P,f) - \mathcal{L}(P,f) < \varepsilon$. QED.

4.2.11 Properties of the definite integral

For a closed interval $[a,b]$, we denote the set of integrable functions on $[a,b]$ by $\mathcal{R}[a,b]$. We have seen that this is a vector space.

Moreover, the map $\mathcal{R}[a,b] \to \mathbb{R}$ defined by $f \mapsto \int_a^b f$ is a linear transformation.

Lemma (First inequality for integrals)
$f, g \in \mathcal{R}[a,b]$ and $f \leq g$. Then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. EOL.

Proof. If $f \leq g$, then for any partition P of $[a,b]$ and any tag-vector $\boldsymbol{y}$ we have

$$S(P,\boldsymbol{y},f) \leq S(P,\boldsymbol{y},g)$$

from which the result follows immediately after considering Riemann sequences. QED.

Corollary
For any $f \in \mathcal{R}[a,b]$, we have

$$\inf f \cdot (b-a) \leq \int_a^b f(x)dx \leq \sup f \cdot (b-a)$$

EOC.

Proof. Observe that constant functions are integrable so put $I(x) = \inf f$ and $S(x) = \sup f$ on $[a,b]$, then $I \leq f \leq S$ and $\int_a^b I(x)dx = \inf f \cdot (b-a) \leq \int_a^b f(x)dx \leq \sup f \cdot (b-a)$. QED.

Lemma (Fundamental inequality for integrals)
Let $f \in \mathcal{R}[a,b]$, then $\left|\int_a^b f(x)dx\right| \leq \|f\|_\infty (b-a)$. EOC.

Here, $\|f\|_\infty \coloneqq \sup|f|$.

Proof. This is immediate from the observation that for every Riemann sum $S(P,\boldsymbol{y},f)$ we have

$$|S(P,\boldsymbol{y},f)| \leq \sup|f| \cdot (b-a)$$

(triangle inequality). As the integral is the limit of a sequence of Riemann sums, the result follows. QED.

Remark
Note we would also like to have a result of the form

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx \leq \sup|f| \cdot (b-a)$$

The problem is, we haven't shown that for an integrable function f, the absolute value function $|f|$ is also integrable. While this is true, this is not immediate. For continuous f, however, $|f|$ is also continuous, and therefore integrable. Then the above inequality is true. EOR.

Lemma
Let $f \in \mathcal{B}[a,b]$ and $c \in (a,b)$. Then $f \in \mathcal{R}[a,b]$ if and only if $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$, and if either is the case then

$$\int_a^b f = \int_a^c f + \int_c^b f$$

EOL.

Note that technically we should use different labels for the restrictions of f to $[a,c]$ and $[c,b]$. However, this would not add anything useful to the discussion.

Proof. We use Darboux integration: Suppose f is integrable on $[a,b]$. We must find sequences of partitions of $[a,c]$ and $[c,b]$ such that the corresponding Darboux integrals converge. To be precise, let P_n be a partition such that $\lim_{n\to\infty}\big(\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)\big)=0$. We may assume that $c \in P_n$, otherwise we just add it without changing the limit (refinements make the difference smaller). Then $Q_n \coloneqq P_n \cap (a,c)$, viewed as a partition of $[a,c]$ and $R_n \coloneqq P_n \cap (c,b)$ viewed as a partition of $[c,b]$, are partitions of $[a,c]$ and $[c,b]$ respectively.

But then

$$\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f) = \big(\mathcal{U}(Q_n,f)-\mathcal{L}(Q_n,f)\big) + \big(\mathcal{U}(R_n,f)-\mathcal{L}(R_n,f)\big)$$

The left hand side, and the two bracketed terms on the right are all nonnegative. Thus the squeeze principle dictates that each bracketed term on the right is a zero sequence.

Conversely, if f is integrable on $[a,c]$ and $[c,b]$, then choosing again sequences of partitions Q_n of $[a,c]$ and R_n of $[c,b]$, we get a sequence of partitions $P_n \coloneqq Q_n \cup \{c\} \cup R_n$ of $[a,b]$, and the above equation still holds. Now the right hand side converges to zero, so the left hand side does.

Regarding the value of the integral: note that both sides of the above equation converge to 0, then we know that

a. $\mathcal{U}(P_n,f) = \mathcal{U}(Q_n,f) + \mathcal{U}(R_n,f)$ for all n
b. $\lim_{n\to\infty}\mathcal{U}(P_n,f) = \int_a^b f(x)\,dx$, $\lim_{n\to\infty}\mathcal{U}(Q_n,f) = \int_a^c f(x)dx$, and $\lim_{n\to\infty}\mathcal{U}(R_n,f) = \int_c^b f(x)dx$.

Our standard results on limits of sequences then tell us that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

A technicality are some degenerate cases (where $c=a$, or $c=d$, or $d=b$). The result still holds if we define for any function f defined in some $\alpha \in \mathbb{R}$ that $\int_\alpha^\alpha f(x)dx = 0$. QED.

Corollary
Let $c < d \in [a,b]$. If $f \in \mathcal{R}[a,b]$, then f is also integrable on $[c,d]$.

Proof. We first apply the lemma to conclude that f is integrable on $[a,c]$ and $[c,b]$. Applying the lemma again to the restriction of f to $[c,b]$, we find that f is integrable on $[c,d]$. QED.

Definition

- Let $[a,b]$ be an interval. If $b=a$, we define $\int_a^b f(x)dx = 0$ for all functions f defined in $a=b$.
- Let $a>b$, we define $\mathcal{R}[a,b] = \mathcal{R}[b,a]$ and $\int_a^b f(x)dx = -\int_b^a f(x)dx$ for all functions integrable on $[b,a]$. EOD.

With these definitions we have for all $c,d,e \in [a,b]$ that

$$\int_c^d f(x)dx + \int_d^e f\,(x)dx = \int_c^e f(x)dx$$

and we do not need to have that $c \leq d \leq e$. (Convince yourself of that.)

For example if $d > c > e$, then $\int_e^c f(x)dx + \int_c^d f(x)dx = \int_e^d f(x)dx$. But then $\int_c^e f = -\int_e^c f(x)dx = \int_c^d f - \int_e^d f = \int_c^d f + \int_d^e f$.

Definition
A function f defined on an interval of the form $[-a, a]$ is called **odd** if for all $x \in [-a, a]$ $f(-x) = -f(x)$. It is called **even** if $f(-x) = f(x)$.

Corollary
Let f be an integrable function on $[-a, a]$.

1. If f is odd, then$\int_{-a}^a f(x)dx = 0$.
2. If f is even, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.

Proof. f is integrable on $[-a, 0]$ and $[0, a]$ and it suffices to show that $\int_{-a}^0 f = -\int_0^a f$ if f is odd, and $\int_{-a}^0 f = \int_0^a f$ if f is even.

Let $P_n = \frac{a}{n} < \frac{2a}{n} < \cdots < \frac{(n-1)a}{n}$, which is a partition of $[0, a]$. We write $-P_n$ for the partition $-\frac{(n-1)a}{n} < -\frac{n-2}{a} < \cdots < -\frac{a}{n}$ of $[-a, 0]$. Any tag-vector $\boldsymbol{y}_n$ for P_n gives rise to the tag vector $-\boldsymbol{y}_n'$ of $-P_n$, where we write $\boldsymbol{y}'$ for the vector $\boldsymbol{y}$ *ordered in reverse*. For example, $(1,2,3)' = (3,2,1)$.

It is then clear that $S(-P_n, -\boldsymbol{y}_n, f) = S(P_n, \boldsymbol{y}_n, f)$ if f is even, and $S(-P_n, -\boldsymbol{y}_n, f) = -S(P_n, \boldsymbol{y}_n, f)$ if f is odd. Since $m(P_n) \to 0$, the claim follows. QED.

4.2.12 Variable integration boundaries

By the above if f is integrable on $[a, b]$, for an fixed $c \in [a, b]$ and any $x \in [a, b]$, we may define

$$F(x) = \int_c^x f(t)dt$$

In other words, we treat the upper boundary as a variable. This defines a function on $[a, b]$.

Note that if f happens to have an anti-derivative on $[a, b]$, F_0, say, then F_0 is also an antiderivative on $[c, x]$ if $x > c$, and on $[x, c]$ if $x < c$.

In the first case $F(x) = F_0(x) - F_0(c)$ by the FTC1. In the second case $F(x) = -\int_x^c f(t)dt = -\big(F_0(c) - F_0(x)\big) = F_0(x) - F_0(c)$.

Finally if $x = c$, then $F(x) = 0$ by definition and also $F_0(x) - F_0(c) = 0$, so in all cases we have

$$F(x) = F_0(x) - F_0(c)$$

This means F and F_0 differ by a constant (namely $\pm F_0(c)$) and so F is differentiable and $F'(x) = f(x)$ on $[a, b]$.

The upshot is that if f has an antiderivative, then F as defined above is one.

We haven't yet precisely discussed which functions do have antiderivatives, and how to find one.

But it seems natural to look among functions of the form $F(x) = \int_c^x f(t)dt$ at least for integrable functions. This is what we will do next.

4.2.13 Second Fundamental Theorem of Calculus

The first observation may be a little bit surprising:

Lemma

Let f be integrable on $[a, b]$ and $F(x) = \int_c^x f(t)dt$ on $[a, b]$. Then F is continuous. EOL.

Proof. First, note that by the discussion in 4.2.11 we have for any fixed $x_0 \in [a, b]$

$$F(x) - F(x_0) = \int_c^x f - \int_c^{x_0} f = \int_c^x f + \int_{x_0}^c f = \int_{x_0}^x f(t)dt$$

To show that F is continuous at x_0 it therefore is enough to show that

$$\lim_{x \to x_0} \int_{x_0}^x f(t)dt = 0$$

Now recall the Fundamental Inequality for Integrals:

$$\left| \int_{x_0}^x f(t)dt \right| \leq S(x - x_0)$$

where $S = \sup_{[x_0, x]} f$. But $0 \leq S \leq \sup_{[a,b]} f$. Therefore $\lim_{x \to x_0} S(x - x_0) = 0$. QED.

Note that this holds even if f itself is not continuous. Integration seems to "smooth" out jumps.

Theorem (Second Fundamental Theorem of Calculus)

Let f be continuous on $[a, b]$. For any $c \in [a, b]$, the function $F(x) = \int_c^x f(t)dt$ is an anti-derivative of f:

$$\frac{d}{dx} \int_c^x f(t)dt = f(x)$$

EOT.

The theorem is an immediate consequence of the following stronger result:

Lemma

Let f be integrable on $[a, b]$, $c \in [a, b]$, and f continuous at $x_0 \in [a, b]$. Let $F(x) = \int_c^x f(t)dt$. Then

$$F'(x_0) = f(x_0)$$

EOL.

Proof. We must show that $\lim_{h \to 0} \frac{1}{h} \left(F(x_0 + h) - F(x_0)\right) = f(x_0)$.

$F(x_0+h)-F(x_0)=\int_{x_0}^{x_0+h} f(t)dt$. Let $\varepsilon>0$. As f is continuous at x_0, there is $\delta>0$ such that $|f(t)-f(x_0)|<\varepsilon$ for all $t\in[a,b]$ with $|t-x_0|<\delta$.

Observe that for all h such that $x_0+h\in[a,b]$ we have $\int_{x_0}^{x_0+h} f(x_0)dt=f(x_0)h$. And thus for such h with $h\neq 0$

$$\frac{1}{h}\big(F(x_0+h)-F(x_0)\big)-f(x_0)=\frac{1}{h}\int_{x_0}^{x_0+h} f(t)-f(x_0)dt$$

Then $\left|\frac{1}{h}\int_{x_0}^{x_0+h}\big(f(t)-f(x_0)\big)dt\right|<\frac{1}{h}\varepsilon\cdot h=\varepsilon$ as long as $|h|<\delta$ by the Fundamental Inequality. QED.

4.2.14 Rules of definite integration

The rules for indefinite integrals have mirror images for definite integrals.

We already have seen that if f,g are integrable in $[a,b]$ and $\alpha,\beta\in\mathbb{R}$, then $\alpha f+\beta g$ is integrable on $[a,b]$ and

$$\int_a^b(\alpha f+\beta g)dx=\alpha\int_a^b fdx+\beta\int_a^b gdx$$

Product Rule

Let f be continuous on $[a,b]$ and let g be continuously differentiable on $[a,b]$. Let F be any antiderivative of f

Then fg is integrable and

$$\int_a^b fgdx=[Fg]_a^b-\int_a^b Fg'dx$$

EOL.

Proof. By the FTC2, we know that F always exists. We also know that Fg' is continuous and therefore integrable.

Now let $U(x)=F(x)g(x)-\int_a^x F(t)g'(t)dt$. Then U is an antiderivative of fg:

$$U'(x)=F'(x)g(x)+F(x)g'(x)-F(x)g'(x)=f(x)g(x)$$

by the FTC2. Thus, by the FTC1, $\int_a^b fgdx=U(b)-U(a)$ wich is what is claimed. QED.

Substitution Rule

Let f be continuous on $[a,b]$. Let g be continuously differentiable on $[\alpha,\beta]$ and suppose $g([\alpha,\beta])\subseteq[a,b]$ and $a=g(\alpha)$ and $b=g(\beta)$. Then

$$\int_a^b f(x)dx=\int_\alpha^\beta f\big(g(t)\big)g'(t)dt$$

EOL.

Proof. By FTC2 f has an antiderivative F on $[a,b]$. Then $F(g(t))$ is an antiderivative of $f\big(g(t)\big)g'(t)$ and hence by the FTC1, we have

$$\int_\alpha^\beta f\big(g(t)\big)g'(t)dt = F\big(g(\beta)\big) - F\big(g(\alpha)\big) = F(b) - F(a) = \int_a^b f(x)dx$$

QED.

The condition that $g(\alpha) = a$ and $g(\beta) = b$ forces $g([\alpha, \beta]) = [a, b]$ by the IVT.

Note that the substitution rule also works if $a > b$ or $\alpha > \beta$:

For example

$$\int_1^0 \sin(t)\cos(t)\,dt = \int_{\sin 1}^0 x dx = \left[\frac{1}{2}x^2\right]_{\sin 1}^0 = -\frac{1}{2}\sin^2(1)$$

4.2.15 Some examples

We have seen that all continuous functions are integrable and that continuous functions have antiderivatives.

However, the converse of both statements has counterexamples:

Let $f(x) = \vdash \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

Then f is integrable on every closed and bounded subinterval. But f has no antiderivative on any interval that contains 0 and also negative points: indeed, if $0 \in [a, b]$ and $a < 0 \leq b$, then f takes both values 0 and 1 on $[a, b]$. By the Intermediate Value Theorem for derivatives (this was a homework assignment) if $F' = f$, then f must take all values between 0,1.

But f is integrable since it is monotone increasing. One could also show this directly by dealing with the one jump using a clever partition.

f is also not continuous at 0. But it may not be surprising that f is integrable (because we know that we can built up an integral by integration over a partition into subintervals). So as long as f has only finitely many jumps we should be fine (even though one has to deal with the jumps).

Here is a more surprising function, defined on $[0, \infty)$

$$g(x) = \vdash \begin{cases} \frac{1}{n} & x = \frac{m}{n} \in \mathbb{Q}, m, n \in \mathbb{N}_0, n \neq 0, \gcd(m, n) = 1 \\ 0 & x \notin \mathbb{Q} \end{cases}$$

(It is possible to extend g to all of $\mathbb{R}$ by allowing negative numerators.) Also note that 1 is the only positive integer with a gcd of 1 with 0, so $g(0) = 1$.

Fact

g is continuous at exactly the irrational positive numbers. EOF.

Proof. If $x_0 \in \mathbb{Q}$ and $x_0 \geq 0$, there is a sequence $x_n \geq 0$ all of its elements are irrational, and such that $x_n \to x_0$ (the irrational numbers are dense). But to be concrete you could use $x_n = x_0 + \frac{\sqrt{2}}{n}$.

Then $g(x_n) = 0$ for all n, so $g(x_n) \to 0 \neq g(x_0)$. So g is not continuous at x_0.

If on the other hand $x_0 \notin \mathbb{Q}$, we must show that for every sequence $x_n \to x_0$ we have $g(x_n) \to 0$.

We discussed this in our online lecture, but admittedly not in a very clear manner. Let me try to be precise here: First off, if $x_n \notin \mathbb{Q}$, then $f(x_n) = 0$ we can restrict our attention to the subsequence of x_n of elements in $\mathbb{Q}$, and hence may assume that $x_n \in \mathbb{Q}$ for all n.

But then, if we denote the subsequence by $x_n = \frac{p_n}{q_n}$, we must have that $\lim\limits_{n\to\infty} q_n = \infty$. (This would show hat $f(x_n) = \frac{1}{q_n} \to 0 = f(x_0)$.)

Indeed, if $\lim x_n = x_0$, then for all but finitely many n, $x_n \in (x_0 - \varepsilon, x_0 + \varepsilon)$ for any given $\varepsilon > 0$.

(I'll phrase the proof below so that it would also work if we extend the definition of f to all of $\mathbb{R}$. For $[0, \infty)$, the absolute value can be omitted.)

Let $y = |x_0| + \varepsilon$. Then there are at most finitely many natural numbers in the interval $\left(0, \frac{1}{\varepsilon}\right)$. So $q < \frac{1}{\varepsilon}$ for only finitely many $q \in \mathbb{N}$. But then there are only finitely many rational numbers $\frac{p}{q}$ in the interval $[0, y)$ with $q < \frac{1}{\varepsilon}$ (as for each q there are only finitely many $p > 0$ such that $\frac{p}{q} < y$). Now all but finitely many $|x_n|$ are contained in $[0, y)$, and as $|x_n| \to |x_0|$, we must have that for all but finitely many n, $|x_n|$ is not equal to one of the finitely many rational numbers with denominator $q < \frac{1}{\varepsilon}$ (each of those rational numbers has a positive distance from $|x_0|$), and therefore, for only finitely many n we can have $q_n < \frac{1}{\varepsilon}$. But that means $q_n \to \infty$. QED.

4.2.16 Oscillation

Definition
Let f be a bounded function on an interval I. We define the **oscillation** of f on I, denoted $\Omega_f(I)$ as

$$\Omega_f(I) \coloneqq \sup_I f - \inf_I f$$

EOD.

Note that if $I \subset J$ and f is defined on J, then $\Omega_f(I) \leq \Omega_f(J)$.

Let now f be defined and bounded on $[a, b]$ and let $x \in [a, b]$. For $\delta > 0$ we define

$$\Omega_{f,x}(\delta) \coloneqq \Omega_f\big([a, b] \cap (x - \delta, x + \delta)\big)$$

(that is, the oscillation of f on the "small" δ interval around x).

$\Omega_{f,x}(\delta)$ is a monotone increasing function on $(0, \infty)$ and bounded below. It follows that

$$\omega_f(x) \coloneqq \lim_{\delta\to 0^+} \Omega_{f,x}(\delta)$$

exists and is ≥ 0. It is called the **oscillation** of f **at** x.

Proposition
$f \in \mathcal{B}[a, b]$ is continuous at $x \in [a, b]$ iff $\omega_f(x) = 0$. EOP.

Proof. Homework exercise.

4.2.17 Sets of measure zero

Definition

A subset $N \subseteq \mathbb{R}$ is called a **zero set**[1] or **set of measure zero**, if for very $\varepsilon > 0$ there are (at most) countably infinitely many open intervals $I_1, I_2, \ldots$ such that $N \subseteq I_1 \cup I_2 \cup \ldots$, and such that $|I_1| + |I_2| + \cdots = \sum_{k=1}^{\infty} |I_k| < \varepsilon$ EOD.

Here for any interval I of the form $(a,b), [a,b], (a,b], [a,b)$ we put $|I| = b - a$.

Terminology

We say a property of points $x \in I$ holds **almost everywhere** in a set I, if the set where the property fails is a set of measure zero.

For example, f defined on I is *almost everywhere continuous* on I if the set $\{ x \in I \mid f \text{ is not continuous at } x\}$ is a zero set.

Exercise

Show that $\mathbb{Q} \subseteq \mathbb{R}$ or more generally any countably infinite set in $\mathbb{R}$ is a zero set. EOE.

Lemma

1. Subsets of a zero set are zero sets.
2. Any finite or countable union of zero sets is again a zero set.

EOL.

Proof. 1. is straight forward: any open cover of a set of of measure zero is also an open cover of any subset.

As for 2. the case of of a finite union is covered by the case of a countably infinite union (why?).

Let Z_k ($k \in \mathbb{N}$) be a countably infinite collection of sets of measure zero, and let $\varepsilon > 0$.

Then for each k, Z_k may be covered by a countably infinite union of open intervals $I_{k\ell}$ such that

$$\sum_{\ell=1}^{\infty} |I_{k\ell}| < \frac{\varepsilon}{2^k}$$

Let $Z = \bigcup_{k=1}^{\infty} Z_k$. Then $Z \subseteq \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} I_{k\ell} = \{x \in \mathbb{R} \mid \exists k, \ell : x \in I_{k,\ell}\}$.

Note that by the Cauchy Double Series Theorem $\sum_{k,\ell=1}^{\infty} |I_{k\ell}| = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} |I_{k\ell}|$ if the right hand side converges (and this then shows that summing up the $|I_{k\ell}|$ in any order is bounded by either of that sum).

Now $\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} |I_{k\ell}| \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \left(\frac{1}{1-\frac{1}{2}} - 1 \right) = \varepsilon$. But note as e.g. $\sum_{\ell=1}^{\infty} |I_{1\ell}| < \frac{\varepsilon}{2}$, we actually have strict inequality. Thus, Z is a set of measure zero. QED.

[1] Not to be confused with the *zero set* of a function f, defined as $\{x \mid f(x) = 0\}$.

4.2.18 Compact sets and the Heine Borel Theorem

If $S \subseteq \mathbb{R}$ is any subset, an **open cover** or **open covering** of S is a family $\{U_i\}_{i\in I}$ of open sets $U_i \subseteq \mathbb{R}$ such that $S \subseteq \bigcup_{i\in I} U_i = \{x \in \mathbb{R} \mid \exists i: x \in U_i\}$.

For example, $U_n := \left(\frac{1}{n}, 1\right)$ defines an open covering of $(0,1)$:

$$(0,1) \subseteq \bigcup_{n=1}^{\infty} U_n$$

In fact, we have equality in this case.

A **subcover** or **subcovering** is then determined by a subset $J \subseteq I$ such that still $S \subseteq \bigcup_{i\in J} U_i$. So $\{U_i\}_{i\in J}$ defines an open cover where each open set also appears in the original cover (with the same index).

A very important case are sets S where we can *always* find a subcover consisting of *finitely many* open sets.

Definition
A subset $K \subseteq \mathbb{R}$ is called **compact**, if every open covering has a finite subcover. In other words, whenever

$$K \subseteq \bigcup_{i\in I} U_i$$

then there are $i_1, i_2, \ldots, i_n \in I$ such that

$$K \subseteq U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_n}$$

EOD.

Note that $(0,1)$ above is *not* compact: the given covering $\{U_n\}_{n\in\mathbb{N}}$ has no finite subcover: indeed, we have $U_1 \subseteq U_2 \subseteq \cdots$. Therefore, for any finite list of indices $n_1 < n_2 < \cdots < n_k$, we have

$$U_{n_1} \cup U_{n_2} \cup \ldots \cup U_{n_k} = U_{n_k} = \left(\frac{1}{n_k}, 1\right)$$

which does not contain $(0,1)$ as a subset.

This leaves the question how we can characterize compact subsets of $\mathbb{R}$. The definition is certainly unwieldy, and it is not at all clear how we could ever determine how a subset might be compact.

Here is a first hint at what the definition means for subsets of $\mathbb{R}$:

Lemma
Let K be a compact subset of $\mathbb{R}$. Then every sequence $x_n \in K$ has a convergent subsequence with limit in K. EOL.

Proof. Let $x_n \in K$ be a sequence. Suppose no $x \in K$ is the limit of a subsequence. Then for each x, there is $\varepsilon_x > 0$ such that there are only finitely many indices n such that $x_n \in (x - \varepsilon_x, x + \varepsilon_x)$. Otherwise, if for every ε, $x_n \in (x - \varepsilon, x + \varepsilon)$ for infinitely many n, then there is a subsequence converging to x.

Let $U_x = (x - \varepsilon_x, x - \varepsilon_x)$. Then $K \subseteq \bigcup_{x\in K} U_x$. This is an open covering. As K is compact, there are finitely many $y_1, y_2, \ldots, y_k \in K$ such that $K \subseteq U_{y_1} \cup U_{y_2} \cup \ldots \cup U_{y_k}$.

But then for at least one i we must have infinitely many n such that $x_n \in U_{y_i}$ (as there are infinitely many $n \in \mathbb{N}$; pigeonhole principle). This is a contradiction to the definition of U_{y_i}. QED.

Corollary
Let K be a compact subset of $\mathbb{R}$. Then K is closed and bounded. EOC.

Proof. First, K must be bounded, otherwise we can easily construct a strictly monotone increasing or decreasing sequence with limit ∞, respectively $-\infty$. Such a sequence has no convergent subsequence, a contradiction to the lemma.

Let $x_n \in K$ be any sequence that has a limit $x_0 \in \mathbb{R}$. Then any subsequence also converges to x_0. The lemma says there is a subsequence converging to an element of K, and hence $x_0 \in K$. This shows that K is closed. QED.

We have already gone one third of the way of the Heine-Borel Theorem. It turns out that "closed and bounded" is the characterization of compact sets we were looking for:

Theorem (Heine-Borel)
A subset K of $\mathbb{R}$ is compact if and only if K is bounded and closed. EOT.

This theorem holds also for subsets of $\mathbb{R}^n$ and is fundamentally important in Analysis. However, it does not hold in arbitrary "topological spaces" where this statement might make sense.

Also, I said "one third" above, because the "if"-direction is significantly more technical to prove that the "only if"-direction.

Proof. We have just seen that a compact set is closed and bounded. So we must show the converse.

Let $K \subseteq \bigcup_{i \in I} U_i$ be an open covering. The U_i are open but need not be open intervals a priori. To get better control over the type of open sets let for any $x \in K$ $i(x)$ be an index in I such that $x \in U_{i(x)}$. Note hat $i(x)$ need not be unique[2]. Then there is[3] $\delta_x > 0$ such that $V_x := (x - \delta_x, x + \delta_x) \subseteq U_{i(x)}$. We define $W_x := (x - \varepsilon_x, x + \varepsilon_x)$ with $\varepsilon_x := \frac{\delta_x}{2}$. It will turn out important that we have this additional buffer. Note that $W_x \subseteq V_x \subseteq U_{i(x)}$.

Then we have $K \subseteq \bigcup_{x \in K} W_x$.

First, we want to get some control on the "size" of the ε_x needed. Ideally, they would be bounded below by a positive ε.

To this end, we put

$$W_\varepsilon = \bigcup_{\substack{x \in K \\ \varepsilon_x \geq \varepsilon}} W_x$$

First, W_ε is open, and second, $W_\varepsilon \subseteq W_\mu$ if $\mu < \varepsilon$.

[2] In fact, the existence of a well-defined $i(x)$ needs the Axiom of Choice.
[3] Again, Axiom of Choice.

Claim. There is $\varepsilon > 0$ such that $K \subseteq W_\varepsilon$.

To prove the claim, suppose it is false. Then for every $n \in \mathbb{N}$ and $\varepsilon = \frac{1}{n}$ there is $x_n \in K$ such that $x_n \notin W_{\frac{1}{n}}$.

As K is bounded and closed, by BW, there is a subsequence x_{n_k} converging to some $x_0 \in K$.

Let $n_0 > \frac{1}{\varepsilon_{x_0}}$. Then if $n_k > n_0$, we have $\frac{1}{n_k} < \varepsilon_{x_0}$, and therefore $x_{n_k} \notin W_{x_0} \subseteq W_{\varepsilon_{x_0}} \subseteq W_{\frac{1}{n_k}}$.

But for $k \to \infty$ we must have $n_k \to \infty$, so for almost all[4] k we have $n_k > n_0$ and therefore $x_{n_k} \notin W_{x_0}$. This contradicts the fact that $x_{n_k} \to x_0$ and proves the claim.

For a nonempty bounded and closed subset K we define its *diameter* $d(K)$ as $d(K) = b - a$, where $a = \min K$ and $b = \max K$ (note that as K is bounded, it has a sup and an inf. Since it is closed, both are elements of K, so $\sup K = \max K$ and $\inf K = \min K$).

For any $x \in \mathbb{R}$ and $\delta > 0$ let $V_x(\delta) = (x - \delta, x + \delta)$.

For fixed $\varepsilon > 0$, we will now show by induction on $n \in \mathbb{N}$ that if a nonempty closed and bounded subset A has $d(A) \leq n\varepsilon$, and $A \subseteq \bigcup_{j \in J} V_{y_j}(\varepsilon_j)$ is an open cover where for each y_j, $\varepsilon_j \geq \varepsilon$ then there are finitely many $j_1, j_2, \dots, j_k \in J$ such that

$$A \subseteq V_{y_{j_1}}(2\varepsilon_{j_1}) \cup V_{y_{j_2}}(2\varepsilon_{j_2}) \cup \dots \cup V_{y_{j_k}}(2\varepsilon_{j_k})$$

Let first $n = 1$. Let $a = \min A$ and $b = \max A$. Then $b - a \leq \varepsilon$. Let $x \in A$ be arbitrary. There is $j \in J$ such that $x \in V_{y_j}(\varepsilon_j)$. Then $|x - y_j| < \varepsilon_j$. And for any $x' \in A$, we have

$$|x' - y_j| \leq |x' - x| + |x - y_j| < \varepsilon + \varepsilon_j \leq 2\varepsilon_j$$

Therefore $x' \in V_{y_j}(2\varepsilon_j)$ and hence $A \subseteq V_{y_j}(2\varepsilon_j)$. This shows the base case.

Now suppose $d(A) \leq (n+1)\varepsilon$, and suppose the assertion is true for any closed and bounded subset of diameter at most $n\varepsilon$.

Let $A \subseteq \bigcup_{j \in J} V_{y_j}(\varepsilon_j)$. Let $a = \min A$, and let $A_1 = A \cap [a, a + n\varepsilon]$. Let $A_2 = A \cap [a + n\varepsilon, a + (n+1)\varepsilon]$. We may assume that both A_1, A_2, are nonempty (why?). Then $d(A_1) \leq n\varepsilon$ and $d(A_2) \leq \varepsilon$.

Both A_1, A_2 are closed and bounded. By the induction hypothesis for A_1 and the case $n = 1$ for A_2, there are finitely many $j_1, j_2, \dots, j_k, j_0 \in J$, such that

$A_1 \subseteq V_{y_{j_1}}(2\varepsilon_{j_1}) \cup V_{y_{j_2}}(2\varepsilon_{j_2}) \cup \dots \cup V_{y_{j_k}}(2\varepsilon_{j_k})$ and $A_2 \subseteq V_{y_{j_0}}(2\varepsilon_{j_0})$.

But then $A = A_1 \cup A_2 \subseteq V_{y_{j_1}}(2\varepsilon_{j_1}) \cup V_{y_{j_2}}(2\varepsilon_{j_2}) \cup \dots \cup V_{y_{j_k}}(2\varepsilon_{j_k}) \cup V_{y_{j_0}}(2\varepsilon_{j_0})$. This proves the claim.

Now returning to our original closed and bounded subset K. Then with $J = K$ we have

[4] All but finitely many.

$$K \subseteq \bigcup_{x \in J} W_x$$

and therefore there are finitely many $x_1, x_2, \ldots, x_k \in K$ such that

$$K \subseteq V_{x_1} \cup V_{x_2} \cup \ldots V_{x_k}$$

(recall $W_x = V_x(\varepsilon_x)$ and $V_x = V_x(2\varepsilon_x)$). But then

$$K \subseteq U_{i(x_1)} \cup U_{i(x_2)} \cup \ldots \cup U_{i(x_k)}$$

QED.

As an application we show:

Lemma
Suppose K is a compact set. Then K is a set of measure zero, if and only if for every $\varepsilon > 0$ there exist finitely many open intervals $I_1, I_2, \ldots, I_n$ with $K \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$ and $\sum |I_k| < \varepsilon$. EOL.

Proof. If K has measure zero, then for each $\varepsilon > 0$ there exists a countable open cover $K \subseteq \bigcup_k I_k$ with $\sum |I_k| < \varepsilon$. As K is compact there is a finite subcover $K \subseteq I_{n_1} \cup I_{n_2} \cup \ldots \cup I_{n_\ell}$, and then $\sum I_{n_i} < \varepsilon$. The if-direction is immediate. QED.

Corollary
If N is a set of measure zero, then N does not contain any interval (a, b) where $a < b$. EOC.

This should be intuitively clear.

Proof. If $(a, b) \subseteq N$, then (a, b) is also a set of measure zero. Let $a' < b' \in (a, b)$. Then $[a', b'] \subseteq (a, b) \subseteq N$. So $K = [a', b']$ is a set of measure zero. It is also compact.

Thus, if $\varepsilon > 0$, there exist n open intervals $I_1, I_2, \ldots, I_n$ such that $K \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$ and such that $\sum_{k=1}^{n} |I_k| < \varepsilon$. Then the same remains true if we replace each open interval $I_k = (a_k, b_k)$ with its closure $\overline{I_k} = [a_k, b_k]$.

If $\varepsilon < b' - a'$, this is impossible: an easy induction on n shows that if $K \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$ is a compact interval for closed intervals I_k with $\sum |I_k| < \varepsilon$ then $d(K) < \varepsilon$. Indeed, this is clear if $n = 1$. Suppose for any union of n closed intervals I_k with $\sum |I_k| < \varepsilon$, any compact subinterval has diameter $< \varepsilon$.

Let $K \subseteq I_1 \cup I_2 \cup \ldots \cup I_{n+1}$ with $\sum |I_k| < \varepsilon$. We may assume that $\max K \in I_{n+1}$.

Then $K = K_1 \cup K_2$, where $K_1 = K \cap (I_1 \cup I_2 \cup \ldots \cup I_n)$ and $K_2 = K \cap I_{n+1}$. Note that K_2 is a compact interval of length at most $|I_{n+1}|$ (because $\max K \in I_{n+1}$). Let $K' = \{x \in K \mid x < \min K_2\}$. Then $K' \subseteq K_1$, and therefore also its closure $K_0 = \overline{K'} \subseteq K_1$. K_0 is a compact interval, and by induction $d(K_0) < \varepsilon - |I_{n+1}|$. Since $d(K) \leq d(K_0) + d(K_2)$ the claim follows. QED.

Corollary
If I is an interval and $N \subseteq I$ is a set of measure zero. Then $I \setminus N$ is dense in I. EOC.

Proof. N does not contain any interval by the previous corollary. Let $x \in I$. Then for each $n \in \mathbb{N}$, there exists $x_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap I$ such that $x_n \notin N$. Then $x_n \to x$. QED.

4.2.19 The Lebesgue Criterion for Integrability

Theorem (Lebesgue Criterion)

A function $f \in \mathcal{B}[a,b]$ is integrable iff f is continuous almost everywhere. EOT.

Proof (following *Heuser*, *Lehrbuch der Analysis I, 1992, p471f)*. Suppose f is continuous almost everywhere. Let N be the set of points in I where f is not continuous. Let $\varepsilon > 0$. As N is a zero set, there exist open intervals $J_1, J_2, \ldots$ such hat $\sum_{i=1}^{\infty} |J_i| < \varepsilon$. We write $I = [a,b]$.

For any $x \in I \setminus N$, f is continuous at x and therefore $\omega_f(x) = 0$. We may therefore find an open interval U_x containing x such that

$$\Omega_f\left(\overline{U}_x \cap I\right) < \varepsilon$$

Then $I \subseteq \bigcup_{\{k=1\}}^{n} J_k \cup \bigcup_{x \notin N} U_x$ is a covering of I by open intervals. Heine Borel below tells us that there must be finitely many of those intervals doing the trick. After relabeling, we may therefore assume that

$$I \subseteq \bigcup_{k=1}^{m} J_k \cup \bigcup_{k=1}^{n} U_k$$

This is still the case if we replace all J_k and all U_k by their closures $\overline{J}_k, \overline{U}_k$.

Let P be a partition of $[a,b]$ with small enough mesh size such that all intervals of P are contained in one of the $\overline{J}_k$ or one of the $\overline{U}_k$. We may then write

$$\mathcal{U}(P,f) - \mathcal{L}(P,f) = S_1 + S_2$$

where S_1 is the sum of all $(M_i - m_i)(x_{i+1} - x_i)$ for which $[x_i, x_{i+1}] \subseteq \overline{J}_k$ for some k, and S_2 is the sum of all others. Thus, if $(M_i - m_i)(x_{i+1} - x_i)$ appears in S_2 then $[x_i, x_{i+1}] \subseteq \overline{U}_k$ for some k. Now f is bounded, so $|S_1| \le (S - I)\sum_{k=1}^{m} \left|\overline{J}_k\right| < (S - I)\varepsilon$ where $S = \sup f$ and $I = \inf f$.

On the other hand if $[x_i, x_{i+1}] \subseteq \overline{U}_k$, then $M_i - m_i \le \Omega_f\left(\overline{U}_k \cap I\right) < \varepsilon$. Therefore $|S_2| < \varepsilon(b-a)$. As we can do that for *every* $\varepsilon > 0$, the Cauchy criterion for Darboux integrals shows that f is integrable.

Now suppose f is integrable, and let N be the set where f is not continuous. Then for all $x \in N$, $\omega_f(x) > 0$.

We may write $N = \bigcup_n \Omega_n$ with

$$\Omega_n = \left\{ x \in N \;\middle|\; \omega_f(x) > \frac{1}{n} \right\}$$

It is then enough that Ω_n is a set of measure zero.

Let $\varepsilon > 0$ and let P be a partition such that $\mathcal{U}(P,f) - \mathcal{L}(P,f) < \frac{\varepsilon}{2n}$. Then

Let $M = \left\{ i \mid N \cap \left(x_i, x_{(i+1)}\right) \neq \emptyset \right\}$. Let $i \in M$ and let $x \in \Omega_n \cap (x_i, x_{i+1})$.

Then there is $\delta > 0$ such that $\Omega_{f,x}(\delta) > \frac{1}{n}$ and $(x - \delta, x + \delta) \subseteq [x_i, x_{i+1}]$. In that case we have for sure that $M_i - m_i > \frac{1}{n}$.

But then $\frac{1}{n}\sum_{i\in M}|(x_i,x_{i+1})| = \frac{1}{n}\sum_{i\in M}(x_{i+1}-x_i) \le \sum_{i\in M}(M_i-m_i)(x_{i+1}-x_i) < \frac{\varepsilon}{2n}$

Note that $N \subseteq \bigcup_{i\in M}(x_i,x_{i+1}) \cup \{a,x_1,x_2,\dots,x_{|P|},b\}$. For $x \in P\cup\{a,b\}$ choose an open interval J_x containing x such that $\sum_{x\in P\cup\{a,b\}}|J_x| < \frac{\varepsilon}{2}$.

Then $N \subseteq \bigcup_{i\in M}(x_i,x_{i+1}) \cup J_a \cup J_{x_1} \cup \dots \cup J_{x_{|P|}} \cup J_b$. The length of all these intervals sums up to something $< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. Therefore, N is a set of measure zero. QED.

4.2.20 Consequences of the Lebesgue Criterion

If f,g are integrable on $[a,b]$, then so are

1. $|f|$
2. fg
3. $\min\{f,g\}$ and $\max\{f,g\}$
4. f^+ and f^-

Here $\min\{f,g\}(x) = \min\{f(x),g(x)\}$ and $\max\{f,g\}(x) = \max\{f(x),g(x)\}$.

$$f^+(x) = \begin{cases} f(x) & f(x)\ge 0 \\ 0 & f(x)<0 \end{cases}$$

Similarly,

$$f^-(x) = \begin{cases} -f(x) & f(x)<0 \\ 0 & f(x)\ge 0 \end{cases}$$

Note that $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

If f,g are almost everywhere continuous, then so are all of the functions listed here, which means they are then integrable.

Note that we need both direction of the Lebesgue Criterion here: if f,g are integrable, then they are almost everywhere continuous, and therefore so are the functions listed. And because of that, these functions are then integrable.

It is a good exercise to convince yourself that if $NC(h)$ denotes the set of points where a function is not continuous, then $NC(fg) \subseteq NC(f)\cup NC(g)$, $NC(f^+) \subseteq NC(f)$ and so on.

To prove directly (using the definition) that any of the functions in 1.,2.,3.,4. is integrable, is rather cumbersome and requires ad-hoc arguments. The admittedly technical Lebesgue Criterion provides a uniform and immediate argument.

It is a good exercise (just to appreciate the point) to try to prove that fg is integrable without using the Lebesgue Criterion.

Triangle inequality for integrals

Let f be integrable on $[a,b]$. Then we have $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx \le \|f\|_\infty(b-a)$. EOL.

Proof. The second inequality is simply the Fundamental Inequality applied to $|f|$, which is integrable by the Lebesgue Criterion.

The first inequality is immediate from the fact that for any partition P and tag vector $\boldsymbol{y}$, we have

$$|S(P,\boldsymbol{y},f)| \leq S(P,\boldsymbol{y},|f|)$$

by the usual triangle inequality for finite sums. Now take limits over a Riemann sequence. QED.

Lemma
Let f be integrable on $[a,b]$ and $F(t) = \int_c^t f(x)dx$ for some $c \in [a,b]$. Then F is almost everywhere differentiable. EOL.

Proof. F is differentiable certainly where f is continuous. As f is almost everywhere continuous, the lemma follows. QED.

Corollary
Let $f \geq 0$ be integrable on $[a,b]$. If there is $x_0 \in [a,b]$ where f is continuous and $f(x_0) > 0$, then $\int_a^b f dx > 0$. EOC.

Proof. Let $F(t) = \int_a^t f(x)dx$. Then F is monotone increasing. It is also differentiable at x_0 and $F'(x_0) = f(x_0) > 0$. But then there is $x > x_0$ such that $F(x) > F(x_0) \geq 0$, and $F(b) \geq F(x) > 0$. QED.

Lemma
Let $f \geq 0$ be integrable on $[a,b]$. Then $\int_a^b f(x)dx = 0$ if and only if $f(x) = 0$ almost everywhere on $[a,b]$. EOL.

Proof. The only if part is clear: let $N = \{x \in [a,b] \mid f(x) > 0\}$. If N is not a set of measure zero, then N is not a subset of $NC(f)$, and there must be $x \in N$ where f is also continuous. By the previous corollary, $\int_a^b f(x)dx > 0$.

Now suppose N is a set of measure zero. And let $U = \{x \in [a,b] \mid x \notin N\}$. Then $f(x) = 0$ for all $x \in U$. Also U is dense in $[a,b]$. In other words, every sub-interval of $[a,b]$ contains elements of U (as N does not contain any interval). Let P_n a sequence of partitions with $m(P_n) \to 0$. We may then choose tagvectors as follows: for each n, $y_{ni} \in \left[x_{ni}, x_{n,i+1}\right] \cap U \neq \emptyset$. Then $S(P_n, \boldsymbol{y}_n, f) = 0$ by construction. It also converges to $\int_a^b f(x)dx$. QED.

Note the assumption that f *is* integrable is crucial: the Dirichlet function $\chi_{\mathbb{Q}}$ is not integrable but equal to zero almost everywhere.

4.3 Improper integrals

So far, we discussed integration on closed and bounded intervals. This was possible for *bounded* functions. What about unbounded functions? And what about unbounded intervals?

Consider the following example from classical mechanics:

Two masses m_1 and m_2 enact a gravitational force of $F(r) = \frac{Gm_1m_2}{r^2}$ on each other, where G is some constant, and r is the distance between the centre of masses.

For example, if the first body is Earth, and the second body is Enterprise, then to take Enterprise out of Earth's gravity field you must increase r to infinity (or reasonably close to that).

To do so, you must spend work (energy) (force times distance). However, the force is not constant.

So going form R_0 to R, you could compute $S(P, y, F) = \sum F(y_i)(r_{i+1} - r_i)$ as an approximation of the energy necessary to increase the distance from r to R where P is a partition of $[R_0, R]$ and y is a tag vector, so that $F(y)$ is an approximation of the average force on the interval $[r_i, r_{i+1}]$. It is reasonable to assume that the finer the mesh size of P the better the approximation of the actual energy needed.

Then $E(R) = \int_{R_0}^{R} \frac{Gm_1m_2}{r^2} dr = \left[-\frac{Gm_1m_21}{r}\right]_{R_0}^{R} = Gm_1m_2\left(\frac{1}{R_0} - \frac{1}{R}\right)$.

To completely escape Earth's gravity (which is of course impossible in this model), you therefore need an amount of energy equal to

$$E = \lim_{R\to\infty}\left(Gm_1m_2\left(\frac{1}{R_0} - \frac{1}{R}\right)\right) = \frac{Gm_1m_2}{R_0}$$

It is clearly natural to define

$$E = \int_{R_0}^{\infty} F(x)dx$$

4.3.1 Definition of improper integrals

Suppose f is integrable in an interval $[a, b]$. Then we have seen that

$$G(t) = \int_a^t f dx$$

is a continuous function on $[a, b]$, and in particular,

$$\lim_{t\to b^-} G(t) = G(b) = \int_a^b f dx$$

We may use the left hand side to define the right hand side, if f is not integrable on $[a, b]$. More precisely consider the following definition.

Definition
Let f be a function defined on an interval $[a, b)$ (and $b = \infty$ is allowed) and suppose f is integrable on all intervals $[a, t]$ where $a < t < b$. Then the **improper integral** $\int_a^b f dx$ is defined as

$$\int_a^b f dx \coloneqq \lim_{t\to b^-} \int_a^t f dx$$

if that limit exists (finite or infinite). Likewise, if f is defined on $(a, b]$ (and $a = -\infty$ is allowed) and integrable on all $[t, b]$ for $a < t < b$, then

$$\int_a^b f dx \coloneqq \lim_{t\to a+} \int_t^b f dx$$

if that limit exists. I
If the limt is finite, we say the integral **converges**, and **diverges** otherwise. EOD.

Example

$$\int_0^1 \frac{1}{t} dt = \infty$$

Indeed, for any $t \in (0,1]$, we have $\int_t^1 \frac{1}{x} dx = \log(1) - \log(t) = -\log t \to \infty$ for $t \to 0$. EOE.

Remark

If f is defined on all of $[a, b]$, and is integrable, then this is not a new definition and the improper integral $\int_a^b f dx$ is the Riemann integral $\int_a^b f dx$. In fact, if f is bounded on $(a, b]$ say, and the improper integral $\int_a^b f\, dx$ exists, then, if we define $f(a)$ to be any number, f is bounded on $[a, b]$, and integrable, and its integral agrees with the improper integral.

This is a consequence of the Lebesgue criterion: Let f_n be the restriction of f to $[a + \frac{1}{n}, b]$. Then $NC(f) \subseteq \{a\} \cup \bigcup_{n \in \mathbb{N}} NC(f_n)$ is a set of measure zero.

The argument is similar if f is defined and bounded on $[a, b)$ and we choose any value for $f(b)$.

Therefore, improper integrals over bounded intervals are only interesting for unbounded functions.

EOR.

We can combine the two types of improper integrals above:

Definition

Let f be defined on (a, b) (and$a = -\infty$, or $b = \infty$ is allowed), and suppose for every $a < s \leq t < b$ f is integrable on $[s, t]$.

If for some $c \in (a, b)$ both improper integrals $\int_a^c f dx$ and $\int_c^b f dx$ exist, we put

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

EOD.

Note the existence is independent of the particular choice of c: for example, if $c_1 < c_2 \in (a, b)$ then

$$\int_a^{c_1} f dx + \int_{c_1}^{c_2} f dx = \int_a^{c_2} f dx$$

if one of the two sides exists. This is an elementary computation involving sums of limits. The same holds for the improper integrals $\int_{c_1}^b f dx$ and $\int_{c_2}^b f dx$ (if one exists so does the other, and $\int_{c_1}^b f dx = \int_{c_1}^{c_2} f dx + \int_{c_2}^b f dx$.

In particular, in the situation of the definition we have

$$\lim_{s \to a^+} \lim_{t \to b^-} \int_s^t f dx = \lim_{t \to b^-} \lim_{s \to a^+} \int_s^t f dx = \int_a^b f dx$$

Why?

4.3.2 Integration over unbounded intervals

Example

$$\int_0^\infty e^{-x} dx = 1$$

Indeed, $\int_0^t e^{-x} dx = [-e^{-x}]_0^t = e^0 - e^{-t} \to 1$ for $t \to \infty$. EOE.

Such integrals behave in a similar way as infinite series as far as convergence is concerned.

In the following we will assume throughout that f is defined on $[a, \infty)$ and integrable on all closed subintervals $[a, t]$.

Example
$\int_1^\infty \frac{1}{x^a} dx$ converges if and only if $a > 1$.
Indeed, if $a \neq 1$, then $\int_1^t \frac{1}{x^a} dx = \left[\frac{x^{1-a}}{1-a}\right]_1^t = \frac{t^{1-a}-1}{1-a}$ which has a finite limit only if $a > 1$, and this limit is then $\frac{1}{a-1}$.
If $a = 1$, then $\int_1^t \frac{1}{x} dx = \log t$, which diverges. EOE.

Cauchy Criterion for Integrals
$\int_a^\infty f dx$ converges if and only if for every ε, there is $t_0 > a$ such that for all $s > r > t_0$ we have

$$\left|\int_r^s f dx\right| < \varepsilon$$

EOL.

Compare this with the Cauchy criterion for infinite series: $\sum a_n$ converges if for every $\varepsilon > 0$ there is n_0 such hat for all $\ell > k > m_0$, $\left|\sum_{n=k}^{\ell} a_k\right| < \varepsilon$.

Proof. To simplify notation let $F(t) = \int_a^t f dx$. Then $\int_r^s f dx = F(s) - F(r)$. Suppose $L = \lim_{t\to\infty} F(t)$ exists and is finite. Let $\varepsilon > 0$. Then there is t_0 such that for all $t > t_0$, we have $|F(t) - L| < \frac{\varepsilon}{2}$. For $s > r > t_0$, we then have

$$|F(s) - F(r)| \leq |F(s) - L| + |L - F(r)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

For the converse suppose that t_0 exists for each $\varepsilon > 0$.

We must show that $\lim_{t\to\infty} F(t)$ exists and is finite. Equivalently, it suffices to show that for every sequence $x_n \in [a, \infty)$ with $\lim_{n\to\infty} x_n = \infty$, we have $F(x_n)$ is a Cauchy sequence. For then each $F(x_n)$ converges to a finite limit. If x_n, y_n are two such sequences then z_n forms as $x_1, y_1, x_2, y_2, \ldots$ is a third sequence and $F(z_n)$ has a finite limit. But $F(x_n)$ and $F(y_n)$ are subsequences and hence must have the same limit.

So let $\varepsilon > 0$ and let t_0 as in the statement. There is n_0 such that $x_n > t_0$ for all $n > t_0$, and then $|F(x_m) - F(x_n)| < \varepsilon$ for all $m, n > n_0$. (If $x_m > x_n$ put $s = x_m, r = x_n$, if $x_m < x_n$, put $s = x_n, r = x_m$, and if $x_m = x_n$ this is clear.) QED.

Example
Consider $\int_0^\infty \frac{\sin x}{x} dx$. Technically this is doubly improper integral as $\frac{\sin x}{x}$ is not defined at 0. But $\lim_{x\to 0} \frac{\sin x}{x} = 1$, so what we mean here is the integral of f defined as $f(0) = 1$ and $f(x) = \frac{\sin x}{x}$ for $x > 0$. (As f is bounded around 0, there the value of f at 0 is irrelevant.)

Then for $s > r > 0$, $\int_r^s \frac{\sin x}{x} dx = \left[-\frac{\cos x}{x}\right]_r^s - \int_r^s \frac{\cos x}{x^2} dx$ by the product rule.
Taking absolute values we find

$$\left|\int_r^s \frac{\sin x}{x} dx\right| \leq \frac{1}{s} + \frac{1}{r} + \int_r^s \frac{1}{x^2} dx = \frac{1}{s} + \frac{1}{r} + \left[-\frac{1}{x}\right]_r^s = \frac{2}{r}$$

Where we used the First Inequality and the Triangle Inequality to conclude

$$\left|\int_r^s \frac{\cos x}{x^2} dx\right| \leq \int_r^s \frac{|\cos x|}{x^2} dx \leq \int_r^s \frac{1}{x^2} dx$$

EOE.

Monotone criterion
Let $f \geq 0$. Then $\int_a^\infty f dx$ converges if and only if there is B such that for all $t > a$, $\int_a^t f dx < B$. EOL.

Proof. The function $F(t) = \int_a^t f dx$ is monotone increasing as $f \geq 0$, and therefore $F(t) - F(s) = \int_s^t f dx \geq 0$ if $t > s$ (this uses the First Inequality). As F is monotone increasing, it has a finite limit for $t \to \infty$ iff it is bounded above, and then $\lim_{t\to\infty} F(t) = \sup F$. QED.

Example
$\int_0^\infty x^x e^{-x^2} dx$ converges.
Note that here we put $0^0 = 1$ as we know that $\lim_{x\to 0^+} x^x = 1$. Also, $x^x e^{-x^2} = e^{x\log x - x^2} = e^{x(\log x - x)}$.
Now $\lim_{x\to\infty} (\log x - x) = -\infty$, from which we conclude that there is $t_0 > 0$ such that for $x > t_0$, $\log x - x < -1$.

$$\int_{t_0}^t x^x e^{-x^2} dx \leq \int_{t_0}^t e^{-x} dx \leq \int_0^\infty e^{-x} dx$$

The monotone criterion asserts that $\int_{t_0}^\infty x^x e^{-x^2} dx$ converges and therefore also

$$\int_0^\infty x^x e^{-x^2} dx = \int_0^{t_0} x^x e^{-x^2} dx + \int_{t_0}^\infty x^x e^{-x^2} dx$$

converges. EOE.

Definition
We say $\int_a^\infty f dx$ is absolutely convergent, if $\int_a^\infty |f| dx$ converges. EOD.

Lemma (Triangle Inequality)
If $\int_a^\infty f dx$ is absolutely convergent, then it is convergent and

$$\left|\int_a^\infty f dx\right| \leq \int_a^\infty |f|\, dx$$

EOL.

Proof. If the integral is absolutely convergent, the Cauchy Criterion applies to $\int_a^\infty |f| dx$. Given $\varepsilon > 0$, there is t_0 such that for all $s > r > t_0$ $\int_r^s |f| dx < \varepsilon$ (we do not need to take the absolute value of the integral as $|f| \geq 0$ and hence the integral is always nonnegative).

But then also

$$\left|\int_r^s f dx\right| \le \int_r^s |f| dx < \varepsilon$$

Thus, $\int_a^\infty f dx$ converges. The triangle inequality then follows from taking limits on both sides of the inequality

$$\left|\int_a^t f dx\right| \le \int_a^t |f| dx$$

QED.

Example
$\int_0^\infty \frac{\sin x}{x} dx$ does not converge absolutely.
For $n \in \mathbb{N}$, consider $\int_0^{n\pi} \frac{\sin x}{x} dx$. Then

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx = \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \ge \sum_{k=0}^{n-1} \frac{1}{(k+1)\pi} \cdot \int_0^\pi \sin x \, dx = \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1} \cdot [-\cos x]_0^\pi$$

Limit Criterion
Let $f, g > 0$ be defined on $[a, \infty)$ and suppose $L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is finite.

1. If $L > 0$, then $\int_a^\infty f dx$ converges iff $\int_a^\infty g dx$ does.
2. If $L = 0$, then $\int_a^\infty f dx$ converges if $\int_a^\infty g dx$ does.

EOL.

Proof. Suppose $\int_a^\infty g dx$ converges. There is x_0 such that for all $x \ge x_0 \ge a$, we have $\frac{f(x)}{g(x)} < L + 1$. Then $f(x) < (L+1)g(x)$ and $\int_{x_0}^t f(x)dx \le (L+1)\int_{x_0}^t g(x)dx \le (L+1)\int_{x_0}^\infty g(x)dx$. By the Monotone Criterion we then have $\int_a^\infty f(x)dx$ converges.

If $L > 0$, then $\frac{g(x)}{f(x)} \to L^{-1}$ for $x \to \infty$ and we can repeat the argument if with the roles of f and g interchanged if $\int_a^\infty f dx$ converges to conclude that $\int_a^\infty g dx$ converges. QED.

Remark
The similarity between infinite series and such integrals is not absolute: if $\int_a^\infty f(x)dx$ converges it does not necessarily mean that f is bounded or that $\lim_{x\to\infty} f(x) = 0$.
As an example define f on $[2, \infty)$ as follows:
For a natural number $n \ge 2$, on $[n, n+1)$ f is defined as $f(x) = n$ if $n \le x \le n + \frac{1}{n^3}$, and $f(x) = 0$ otherwise.
Then f is integrable on any bounded interval of the form $[2, t]$. By the monotone criterion we must show $\int_2^t f(x)dx$ is bounded.
Let $N > t$ be a natural number, then f is a step function on $[n, n+1]$ and

$$\int_2^t f(x)dx \le \int_2^N f(x)dx = \sum_{n=2}^{N-1} \int_n^{n+1} f(x)dx = \sum_{n=2}^{N-1} \frac{n}{n^3} \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

EOR.

4.3.3 The integral criterion

Theorem (Integral criterion)
Let $f \geq 0$ be defined and monotone descending on $[m, \infty)$ where $m \in \mathbb{N}$.
Then $\int_m^\infty f(x)dx$ converges if and only if the infinite series $\sum_{n=m}^\infty f(n)$ converges. EOT.

Proof. Since $f \geq 0$ and f is monotone descending we have for all $n \geq m$ in $\mathbb{N}$,

$$f(n) \geq \int_n^{n+1} f(x)dx \geq f(n+1)$$

Since $\int_m^{N+1} f(x)dx = \sum_{n=m}^N \int_n^{n+1} f(x)dx$ we conclude

$$\sum_{n=m}^N f(n) \geq \int_m^{N+1} f(x)dx \geq \sum_{n=m}^N f(n+1)$$

The monotone criterion then shows that the integral converges if the series converges. And if the integral converges the series is bounded and hence convergent. QED.

Note we do not need to assume that f is integrable as this is automatic as it is monotone.

Example
$\sum_{n=1}^\infty \frac{1}{n^a}$ converges iff $a > 1$.
Indeed, $\sum_{n=1}^\infty \frac{1}{n^a}$ converges iff $\int_1^\infty \frac{1}{x^a} dx$ converges. EOE.

4.3.4 Improper integrals

Finally, all criteria for convergence of integrals over unbounded intervals apply essentially unchanged to improper integrals of unbounded functions on bounded intervals.

Rather than reproving everything, we illustrate this by means of a few examples.

$\int_a^b \frac{1}{(x-a)^c} dx$ converges iff $c < 1$.

If $c \neq 1$, then $\int \frac{1}{(x-a)^c} = \frac{(x-a)^{1-c}}{1-c}$. Hence the integral becomes $\lim\limits_{t\to a} \frac{(b-a)^{1-c}-(t-a)^{1-c}}{1-c}$ which is finite only if $1 - c > 0$. And in this case it is equal to $\frac{(b-a)^{1-c}}{1-c}$.

If $c = 1$, the integral is $\log(b - a) - \lim\limits_{t\to a} \log(t - a)$ which is not finite.

$\int_0^1 \frac{\log x}{\sqrt{x}} dx$ converges (absolutely).

We use the limit criterion: let $f(x) = \frac{|\log x|}{\sqrt{x}}$ and $g(x) = \frac{1}{x^{\frac{3}{4}}}$ defined and positive on $(0,1]$.

Then $\frac{f(x)}{g(x)} = x^{\frac{1}{4}}|\log x|$. By LH we know that $\lim\limits_{x\to 0} x^a \log x = 0$ whenever $a > 0$. (Indeed, $\frac{\log x}{\frac{1}{x^a}} \to 0$ because $\frac{\frac{1}{x}}{(-a)x^{-a-1}} = \frac{x^a}{-a} \to 0$).

Therefore $\int_0^1 f(x)dx$ converges if $\int_0^1 g(x)\,dx$ converges. The latter converges by the previous example.

4.3.5 Step functions*

It is possible to phrase Darboux-integration entirely in terms of step functions. This is relevant as the more general notion of Lebesgue integration can also be based on step functions.

The idea is to take a function f and "approximate" it by functions that are constant on some subintervals.

Definition
Let $[a,b]$ be a closed interval, and let I be one of $[a,b],(a,b),[a,b),(a,b]$. A function $\varphi: I \to \mathbb{R}$ is called a **step function** if there is a partition $P = x_1 < x_2 < \cdots < x_n \in \Pi(a,b)$, such that φ Is constant on each of the intervals (x_i, x_{i+1}) $(i = 0,1,\dots,n)$. If φ is a step function, and P is a partition with this property then φ, P are said to be **compatible**. For any P, we denote by $\Phi(P)$ the set of all compatible step functions on $[a,b]$. For any step function φ we denote by $\Pi(\varphi)$ the set of all compatible partitions in $\Pi(a,b)$. EOD.

We do not care what the values of φ are at the boundaries of the intervals. Any piecewise constant function is a step function.

Lemma
Step functions on $[a,b]$ are integrable. In fact, if φ is a step function and P is a compatible partition, then

$$\int_a^b \varphi dx = \sum_{i=0}^{|P|} \alpha_i (x_{i+1} - x_i)$$

where $\varphi(x) = \alpha_i$ on the open interval (x_i, x_{i+1}). EOL.

Proof. One can prove this directly using Darboux integration (see the homework problem solution). However, now that we have more theory at our hands, we give a quick and dirty proof:

Note φ is integrable because it is bounded and almost everywhere continuous. Therefore,

$$\int_a^b \varphi dx = \sum_{i=0}^{|P|} \int_{x_i}^{x_{i+1}} \varphi dx$$

It therefore suffices to show that $\int_{x_i}^{x_{i+1}} \varphi dx = \alpha_i(x_{i+1} - x_i)$. But this is clear: as φ is integrable, its integral coincides with the *improper* integral on (x_i, x_{i+1}). Therefore

$$\int_{x_i}^{x_{i+1}} \varphi dx = \lim_{r \to x_i^+} \lim_{s \to x_{i+1}^-} \int_r^s \varphi dx = \lim_{r \to x_i^+} \lim_{s \to x_{i+1}^-} \alpha_i (s - r) = \alpha_i (x_{i+1} - x_i)$$

QED.

6. Multivariable calculus

In this chapter we briefly consider functions of more than one variable. We will focus on the case of two variables, but the case of more than two variables is very similar.

6.1 Topology of $\mathbb{R}^2$

$\mathbb{R}^2$ is what is called a **vector space**: we can add two elements (componentwise) and0 we can scale any element by multiplying its components with a fixed real number. We leave the details of this definition to linear algebra.

There is a **zero vector**, denoted 0 or $\mathbf{0}$ such that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ and of course $\mathbf{0} = (0,0)$.

Let $\boldsymbol{x} \in \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. A **norm** on $\mathbb{R}^2$ is a function

$$\|\cdot\|: \mathbb{R}^2 \to \mathbb{R}$$

such that

1. $\|\boldsymbol{x}\| \geq 0$ for all $\boldsymbol{x}$
2. $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x} = (0,0)$
3. $\|c\boldsymbol{x}\| = |c|\|\boldsymbol{x}\|$
4. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

The usual consequences similar to the absolute value for real numbers follow: For example, for all $\boldsymbol{x}, \boldsymbol{y}$ we have

$$\|\boldsymbol{x} + \boldsymbol{y}\| \geq |\|\boldsymbol{x}\| - \|\boldsymbol{y}\||$$

Indeed, 3. and 4. imply that $\|\boldsymbol{x} - \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$. Then

$$\|\boldsymbol{x}\| \geq \|\boldsymbol{x} - \boldsymbol{y}\| - \|\boldsymbol{y}\|$$

This holds for all $\boldsymbol{x}, \boldsymbol{y}$, so we may replace $\boldsymbol{x}$ with $\boldsymbol{x} + \boldsymbol{y}$ get

$$\|\boldsymbol{x} + \boldsymbol{y}\| \geq \|\boldsymbol{x}\| - \|\boldsymbol{y}\|$$

Exchanging $\boldsymbol{x}, \boldsymbol{y}$, we also get $\|\boldsymbol{x} + \boldsymbol{y}\| \geq \|\boldsymbol{y}\| - \|\boldsymbol{x}\|$, and therefore the claim.

Example
The following three examples are the ones used most often:

1. The maximum norm $\|\boldsymbol{x}\|_\infty \coloneqq \max\{|x|, |y|\}$
2. $\|\boldsymbol{x}\|_1 \coloneqq |x| + |y|$
3. The Euclidean norm $|\boldsymbol{x}| \coloneqq \|\boldsymbol{x}\|_2 := \sqrt{x^2 + y^2}$

Proposition
All norms on $\mathbb{R}^2$ are **equivalent**. That is, if $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are two norms, there exist positive constants A, B such that for all $\boldsymbol{x}$ we have

$$A\|\boldsymbol{x}\|_\alpha \leq \|\boldsymbol{x}\|_\beta \leq B\|\boldsymbol{x}\|_\alpha$$

EOP.

Proof. It is enough to show hat A, B exist in case $\|\cdot\|_\alpha = \|\cdot\|_\infty$. Let $\|\cdot\|$ be any norm. Then $\|\boldsymbol{x}\| = \|xe_1 + ye_2\|$ where $e_1 = (1{,}0)$ and $e_2 = (0{,}1)$.

$$\|xe_1 + ye_2\| \leq |x|\|e_1\| + |y|\|e_2\| \leq B \max\{|x|, |y|\} = B\|\boldsymbol{x}\|_\infty$$

where $B = 2\max\{\|e_1\|, \|e_2\|\}$.

On the other hand let $A = \inf\{\ \|\boldsymbol{x}\| \mid \|\boldsymbol{x}\|_\infty = 1\}$. Note A exists since $\|\boldsymbol{x}\| > 0$ for all $\boldsymbol{x}$ with $\|\boldsymbol{x}\|_\infty = 1$.

If $A > 0$, then for any $\boldsymbol{x} \neq \boldsymbol{0}$ we have $\|c\boldsymbol{x}\|_\infty = 1$ where $c = \|\boldsymbol{x}\|_\infty^{-1}$, and therefore $\|c\boldsymbol{x}\| \geq A$ and thus $\|\boldsymbol{x}\| \geq Ac^{-1} = A\|\boldsymbol{x}\|_\infty$.

Let $S = \{\ \boldsymbol{x} \mid \|\boldsymbol{x}\|_\infty = 1\ \}$, and let $\boldsymbol{x}_n \in S$ be a sequence such that $\|\boldsymbol{x}_{\boldsymbol{n}}\| \to A$.

The entries of $\boldsymbol{x}_n = (x_n, y_n)$ are all bounded because $\|\boldsymbol{x}_n\|_\infty = 1$. Thus, we may choose a subsequence such that both x_n, y_n converge (see the last part of the proof of the Fundamental Theorem of Algebra), to x_0, y_0 respectively. Let $\boldsymbol{x}_0 = (x_0, y_0)$. Then $\max\{|x_n - x_0|, |y_n - y_0|\}$ is a zero sequence, that is, $\|\boldsymbol{x}_n - \boldsymbol{x}_0\|_\infty \to 0$.

Then $\|\boldsymbol{x}_n - \boldsymbol{x}_0\| \leq B\|\boldsymbol{x}_n - \boldsymbol{x}_0\|_\infty \to 0$. But then by the discussion above

$$|\|\boldsymbol{x}_n\| - \|\boldsymbol{x}_0\|| \leq \|\boldsymbol{x}_n - \boldsymbol{x}_0\| \to 0$$

On the other hand $\|\boldsymbol{x}_n\| \to A$. Therefore, $\|\boldsymbol{x}_0\| = A$.

But $\|\boldsymbol{x}_0\|_\infty = 1$ (because $\|\boldsymbol{x}_n\| = 1$ for all n) and therefore $\boldsymbol{x}_0 \neq \boldsymbol{0}$. Thus $A > 0$. QED.

Definition
Let $\boldsymbol{x}_n$ be a sequence of elements of $\mathbb{R}^2$. We say $\boldsymbol{x}_n$ converges to $\boldsymbol{x}_0$ if
$$\lim_{n\to\infty} |\boldsymbol{x}_n - \boldsymbol{x}_0| = 0$$
EOD.

We have chosen the Euclidean norm for this definition. But as all norms are equivalent, convergence (and limits) do not depend on the norm chosen.

Lemma
$\boldsymbol{x}_n = (x_n, y_n) \to \boldsymbol{x}_0 = (a, b)$ iff $x_n \to a$ and $x_n \to b$. EOL.

Proof. Suppose $\boldsymbol{x}_n \to \boldsymbol{x}_0$. Then $\|\boldsymbol{x}_n - \boldsymbol{x}_0\|_\infty \to 0$. Therefore $\max\{|x_n - a|, |y_n - b|\} \to 0$ and therefore both $x_n \to a$ and $y_n \to b$.

Conversely suppose $x_n \to a$ and $y_n \to b$, then $(x_n - a)^2 + (b_n - a)^2 \to 0$ (sum and products of limits) and as $\sqrt{x}$ is continuous we get $\sqrt{(x_n - a)^2 + (y_n - b)^2} \to 0$. QED.

Definition
A subset A of $\mathbb{R}^2$ is called **closed** if A contains the limit of any convergent sequence in A.
EOD.

Definition
As subset U of $\mathbb{R}^2$ is called **open** if for every $\boldsymbol{x} \in U$ there is $\varepsilon > 0$ such that
$$B_\varepsilon(\boldsymbol{x}) = \{\ \boldsymbol{y} \in \mathbb{R}^2 \mid |\boldsymbol{y} - \boldsymbol{x}| < \varepsilon\ \} \subseteq U$$

EOD.

Exercis
Show that U is open if and only if $\mathbb{R}^2 \setminus U$ is closed.

Since convergence (and hence being closed) does not depend on the norm, "open" does also not depend on the norm chosen.

If $a < b$ and $c < d$, then $(a, b) \times (c, d)$ is open.

6.1.1 Continuous functions

Let $f: D \to \mathbb{R}$ be a function, where D is open.

Definition
Let $\boldsymbol{x}_0$ be an accumulation point of D. That is, there exists a sequence $\boldsymbol{x}_n \in D$ with $\boldsymbol{x}_n \neq \boldsymbol{x}_0$ for all n such that $\boldsymbol{x}_n \to \boldsymbol{x}_0$.
If $\alpha \in \mathbb{R}$, we write $\lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0} f(\boldsymbol{x}) = \alpha$ if $\lim\limits_{n\to\infty} f(\boldsymbol{x}_n) = \alpha$ for every sequence $\boldsymbol{x}_n \in D$ with limit $\boldsymbol{x}_0$.
Equivalently, for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $\boldsymbol{x} \in B_\delta(\boldsymbol{x}_0) \cap D$ we have $|f(\boldsymbol{x}) - \alpha| < \varepsilon$.
EOD.

Definition
Let $\boldsymbol{x}_0 \in D$. f is called **continuous** at $\boldsymbol{x}_0$ if $\lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0} f(\boldsymbol{x}) = f(\boldsymbol{x}_0)$. EOD.

For example every norm is continuous.

Example

$$f(x,y) = \vdash \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is *not* continuous at 0.
While for every zero sequences x_n, y_n we have $f(x_n, 0) \to 0$ and $f(0, y_n) \to 0$ in general we do not have $f(x_n, y_n) \to 0$. Consider $x_n = y_n = \frac{1}{n}$, then

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \frac{1}{2} \to \frac{1}{2} \neq 0$$

This is an example where the limit does not exist (even though $\lim\limits_{x\to0}\lim\limits_{y\to0} f(x,y) = \lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y) = 0$).
EOE.

6.2 Differentiation

6.2.1 Directional derivatives

In the following let f be a function defined on an open subset $D \subseteq \mathbb{R}^2$.

Let $\boldsymbol{x}_0 \in D$ and $\boldsymbol{v} \in \mathbb{R}^2$. There is $\varepsilon > 0$ such that $\boldsymbol{x}_0 + t\boldsymbol{v} \in D$ for all $t \in (-\varepsilon, \varepsilon)$. (This uses that D is open.) Consider the function $g(t) = f(\boldsymbol{x}_0 + t\boldsymbol{v})$ defined on $(-\varepsilon, \varepsilon)$.

Definition

The **directional derivative** of f at $\boldsymbol{x}_0$ with respect to $\boldsymbol{v}$ is defined as $g'(0)$ if g is differentiable at 0, and denoted by $\partial_{\boldsymbol{v}} f(\boldsymbol{x}_0)$. EOD.

You will also find notation such as $\frac{\partial}{\partial v}\Big|_{\boldsymbol{x}=\boldsymbol{x_0}} f(\boldsymbol{x})$ or similar. Often it is also assumed that $\|\boldsymbol{v}\|_2 = 1$.

$$\partial_{\boldsymbol{v}} f(\boldsymbol{x}_0) = \lim_{t\to 0} \frac{f(\boldsymbol{x_0} + t\boldsymbol{v}) - f(\boldsymbol{x}_0)}{t}$$

Example

Let $\boldsymbol{v} = (1,2)$ and $f(x,y) = e^{x-y^2}$.
Then $g(t) = e^{x_0+t-(y_0+2t)^2} = e^{x_0-y_0^2}e^{t+4y_0t-4t^2}$ and $g'(t) = e^{x_0-y_0^2}e^{(1+4y_0)t+4t^2}(1 + 4y_0 + 8t)$
Thus, $\partial_{(1,2)} f(x_0, y_0) = e^{x_0-y_0^2}(1 + 4y_0)$. EOE.

6.2.2 Partial derivatives

Consider $e_1 = (1,0)$ and $e_2 = (0,1)$. Then the **partial derivatives** of f are defined as the directional derivatives $\partial_{e_1} f$ and $\partial_{e_2} f$.

Definition

We define $\frac{\partial f}{\partial x}(\boldsymbol{x}_0) \coloneqq \partial_{e_1} f(\boldsymbol{x}_0)$ and $\frac{\partial f}{\partial y}(\boldsymbol{x}_0) \coloneqq \partial_{e_2} f(\boldsymbol{x}_0)$. EOD.

For example,

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{d}{dt}\Big|_{t=0} f(x_0 + t, y_0) = \lim_{t\to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t}$$

If the partial derivatives exist everywhere on D, then we obtain functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

6.2.3 $\mathcal{C}^1$-functions

Definition

$f: D \to \mathbb{R}$ is called a $\mathcal{C}^1$-function at $\boldsymbol{x}_0 \in D$, if both partial derivatives of f exist in an open neighborhood U of $\boldsymbol{x}_0$ and are continuous at $\boldsymbol{x}_0$.
f is a $\mathcal{C}^1$-function if it is $\mathcal{C}^1$ at all points of D. EOD.

We can iteratively define $\mathcal{C}^m$ functions for $m > 1$: we say f is $\mathcal{C}^m$ at $\boldsymbol{x}_0$ if f is $\mathcal{C}^1$ at $\boldsymbol{x}_0$ and the partial derivatives of f are $\mathcal{C}^{m-1}$ at $\boldsymbol{x}_0$. And $\mathcal{C}^m(D)$ is then the set of functions defined on D that are $\mathcal{C}^m$.

We say a function is $\mathcal{C}^0$ if it is continuous at $\boldsymbol{x}_0$, and consequently $\mathcal{C}^0(D)$ denotes the set of continuous functions on D.

The existence of partial derivatives may be useful in some cases, but in general, there is nothing that connects $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and we cannot draw a lot of meaningful conclusions about the function f. For instance, suppose f is a function where the partial derivatives exist at $\boldsymbol{0}$. This may tell us something about f along the x- and y-axes, but tells us certainly nothing about anything else.

This changes completely for $\mathcal{C}^1$-functions. The point of derivatives is to give meaningful insights into the change of value of a function.

Suppose f is $\mathcal{C}^1$ at $\boldsymbol{x}_0$. In the following all vectors $\boldsymbol{h} = (h, k)$ are "small" enough such that $\boldsymbol{x}_0 + \boldsymbol{h} \in D$, but also $\boldsymbol{x}_0 + (r, s) \in D$ for all (r, s) with $|r| \leq |h|$ and $|s| \leq |k|$.

We assume that both h, k are nonzero, otherwise the discussion reduces to a one-variable scenario.

$$f(\boldsymbol{x}_0 + \boldsymbol{h}) - f(\boldsymbol{x}_0) = \big(f(\boldsymbol{x}_0 + \boldsymbol{h}) - f(\boldsymbol{x_0} + ke_2)\big) + \big(f(\boldsymbol{x}_0 + ke_2) - f(\boldsymbol{x}_0)\big)$$

We treat each of the brackets on the right separatedly.

Let $\boldsymbol{x}_0 = (x_0, y_0)$. Then

$$f(\boldsymbol{x}_0 + \boldsymbol{h}) - f(\boldsymbol{x}_0 + ke_2) = f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)$$

Viewing this as a function of the first variable (and here then h), the MVT tells us that there is h' with $|h'| < |h|$ such that

$$f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) = \frac{\partial f}{\partial x}(x_0 + h', y_0 + k)h$$

Therefore

$$f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) = \frac{\partial f}{\partial x}(x_0, y_0)h + \left(\frac{\partial f}{\partial x}(x_0 + h', y_0 + k)h - \frac{\partial f}{\partial x}(x_0, y_0)h\right)$$
$$= \frac{\partial f}{\partial x}(x_0, y_0)h + \varrho_1(h, k)h$$

where $\varrho_1(h, k) = \frac{\partial f}{\partial x}(x_0 + h', y_0 + k) - \frac{\partial f}{\partial x}(x_0, y_0)$ (keeping in mind that $h' = h'(h)$ depends on h).

Note this equation still holds, if $h = 0$ and then $\varrho_1(h, k) = 0$.

Similarly,

$$f(x_0, y_0 + k) - f(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)k + \left(\frac{\partial f}{\partial y}(x_0, y_0 + k')k - \frac{\partial f}{\partial y}(x_0, y_0)k\right)$$
$$= \frac{\partial f}{\partial y}(x_0, y_0)k + \varrho_2(k)k$$

with $\varrho_2(k) = \frac{\partial f}{\partial y}(x_0, y_0 + k') - \frac{\partial f}{\partial y}(x_0, y_0)$ and again keeping in mind that $k' = k'(k)$ depends on k. Again, this holds if $k = 0$ and then $\varrho_2(k) = 0$.

Notice that because f is $\mathcal{C}^1$ (and this is where we use this hypothesis), we can conclude that

$$\lim_{(h,k) \to 0} \varrho_1(h, k) = 0$$

Similarly, $\lim_{(h,k) \to 0} \varrho_2(k) = 0$. If we define $R(h, k) \coloneqq \varrho_1(h, k)h + \varrho_2(k)k$ we find

$$\frac{R(h, k)}{|h| + |k|} = \varrho_1(h, k)\left(\frac{h}{|h| + |k|}\right) + \varrho_2(k)\left(\frac{k}{|h| + |k|}\right) \to 0$$

for $(h, k) \to 0$. As all norms are equivalent, we have shown that

$$f(\boldsymbol{x}_0+\boldsymbol{h})-f(\boldsymbol{x}_0)=\frac{\partial f}{\partial x}(\boldsymbol{x}_0)h+\frac{\partial f}{\partial y}(\boldsymbol{x}_0)k+R(\boldsymbol{h})$$

with $\lim_{\boldsymbol{h}\to\boldsymbol{0}}\frac{R(\boldsymbol{h})}{\|\boldsymbol{h}\|}=0$.

This should remind you of the definition of differentiability in one variable.

6.2.4 Differentiation

Definition

f is **differentiable** at $\boldsymbol{x}_0$ if there is a linear transformation[1] $T:\mathbb{R}^2\to\mathbb{R}$ such that

$$f(\boldsymbol{x}_0+\boldsymbol{h})=f(\boldsymbol{x}_0)+T(\boldsymbol{h})+R(\boldsymbol{h})$$

where $\lim_{\boldsymbol{h}\to\boldsymbol{0}}\frac{R(\boldsymbol{h})}{\|\boldsymbol{h}\|}=0$. f is differentiable if it is differentiable at every point in its domain.
If f is differentiable at $\boldsymbol{x}_0$ we call $f'(\boldsymbol{x}_0)\coloneqq T$ its derivative. EOD.

As an immediate consequence of the definition we obtain:

Lemma

If f is differentiable at $\boldsymbol{x}_0$, it is continuous at $\boldsymbol{x}_0$. EOL.

Proof. $\lim_{\boldsymbol{h}\to\boldsymbol{0}}(f(\boldsymbol{x}_0+\boldsymbol{h})-f(\boldsymbol{x}_0))=\lim_{\boldsymbol{h}\to\boldsymbol{0}}(f'(\boldsymbol{x}_0)(\boldsymbol{h})+R(\boldsymbol{h}))=0$. QED.

Remark

A linear transformation on $\mathbb{R}^2$ can be described by its standard matrix. Here, this means T is determined by two numbers a,b and we then write[2] $T=[a,b]$ such that

$$T\begin{pmatrix}x\\y\end{pmatrix}=[a,b]\begin{bmatrix}x\\y\end{bmatrix}=ax+by$$

It is common to write the elements of $\mathbb{R}^2$ here as **column vectors**. EOR.

This begs the question how to determine a and b.

Proposition

If f is differentiable at $\boldsymbol{x}_0$ then all partial derivatives at $\boldsymbol{x}_0$ exist, and

$$f'(\boldsymbol{x}_0)=\left[\frac{\partial f}{\partial x}(\boldsymbol{x}_0),\frac{\partial f}{\partial y}(\boldsymbol{x}_0)\right]$$

More generally, all directional derivatives at $\boldsymbol{x}_0$ exist and $\partial_{\boldsymbol{v}}f(\boldsymbol{x}_0)=f'(\boldsymbol{x}_0)(\boldsymbol{v})$.
EOP.

Proof. Let $\boldsymbol{v}\in\mathbb{R}^2$. Then

$$g(t)\coloneqq f(\boldsymbol{x}_0+t\boldsymbol{v})=f(\boldsymbol{x}_0)+f'(\boldsymbol{x}_0)(t\boldsymbol{v})+R(t\boldsymbol{v})=f(\boldsymbol{x}_0)+tf'(\boldsymbol{x}_0)(\boldsymbol{v})+R(t\boldsymbol{v})$$

And

$$\lim_{t\to 0}\frac{R(t\boldsymbol{v})}{t}=0$$

[1] A linear transformation is a function such that $T(\boldsymbol{x}+\boldsymbol{y})=T(\boldsymbol{x})+T(\boldsymbol{y})$ and $T(c\boldsymbol{x})=cT(\boldsymbol{x})$.
[2] This is very sloppy: it depends on the choice of a basis for $\mathbb{R}^2$. We usually choose (e_1,e_2).

Because $\lim_{t\to 0}\frac{R(t\boldsymbol{v})}{\|t\boldsymbol{v}\|} = 0$ and $\frac{R(t\boldsymbol{v})}{\|t\boldsymbol{v}\|} = \frac{1}{\|\boldsymbol{v}\|}\frac{R(t\boldsymbol{v})}{|t|}$.

But this means $g'(0)$ exists and is equal to $f'(\boldsymbol{x}_0)(\boldsymbol{v})$. (This was one of the three equivalent definitions for a function to be differentiable.)

But this means $\partial_{e_1} f(\boldsymbol{x}_0) = f'(\boldsymbol{x}_0)(e_1)$ and $\partial_{e_2} f(\boldsymbol{x}_0) = f'(\boldsymbol{x}_0)(e_2)$. Together

$$f'(\boldsymbol{x}_0)(\boldsymbol{h}) = \frac{\partial f}{\partial x}(\boldsymbol{x}_0)h_1 + \frac{\partial f}{\partial y}(\boldsymbol{x}_0)h_2 = \left[\frac{\partial f}{\partial x}(\boldsymbol{x}_0), \frac{\partial f}{\partial y}(\boldsymbol{x}_0)\right]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

QED.

By the previous section, we conclude that if f is C^1 at $\boldsymbol{x}_0$, then f is differentiable at $\boldsymbol{x}_0$.

The converse is not true.

Exercise
Consider

$$f(x,y) = \begin{cases} (x^2+y^2)\sin(x^2+y^2)^{-\frac{1}{2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that f is differentiable at $(0,0)$ but not C^1. EOE.

On the other hand, the existence of all directional derivatives is not enough for a function to be differentiable:

Exercise
Consider

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that all directional derivatives exist, but that f is not differentiable at $(0,0)$. (Check continuity.) EOE.

6.3 Differentiation in $\mathbb{R}^n$

The situation does not change significantly if we pass to functions on open subsets (defined in the same way) of $\mathbb{R}^n$.

But what if we also change the co-domain from $\mathbb{R}$ to $\mathbb{R}^m$ (say)?

Let $f: D \to \mathbb{R}^m$ be a function where $D \subseteq \mathbb{R}^n$. We write elements of $\mathbb{R}^m$ as column vectors so that

$$f(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{bmatrix}$$

Then f is *continuous, differentiable, C^m at $\boldsymbol{x}_0$* if each f_i is.

In particular $f'(\boldsymbol{x}_0)$ is then a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$. It corresponds to an $m \times n$ matrix, namely the matrix

$$\begin{bmatrix} f_1'(\boldsymbol{x}_0) \\ f_2'(\boldsymbol{x}_0) \\ \vdots \\ f_m'(\boldsymbol{x}_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

This matrix is often called the **Jacobi** matrix of f.

Then $f(\boldsymbol{x}_0 + \boldsymbol{h}) = f(\boldsymbol{x}_0) + f'(\boldsymbol{x}_0)(\boldsymbol{h}) + R(\boldsymbol{h})$

where $R(\boldsymbol{h})$ is now a vector-valued function and

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{R(\boldsymbol{h})}{\|\boldsymbol{h}\|} = \boldsymbol{0}$$

as before.

7. Some Applications

7.1 Infectious diseases

This discussion follows *Heuser, Lehrbuch der Analysis I, p315ff, 1992.*

Please note that the discussion is likely to be inaccurate for any actual infectious disease. It is meant as an indication on how one could use mathematics to describe complicated developments in the "real world." It is not meant to draw any conclusions from in any real event (this disclaimer seems warranted given our current situation).

Consider the situation of a disease outbreak in a closed system (that is, the total population is constant).

At each time t (starting at 0), the total population is then partitioned into three classes:

$s(t)$, the number of *susceptible* individuals. That is, individuals that have not yet been but could potentially become infected.

$v(t)$, the number of *infected* individuals.

$r(t)$ the number of *removed* individuals: this contains all individuals who recovered from the infection (and therefore became immune), and all deceased.

Then $N = s(t) + v(t) + r(t)$ is the total number of individuals.

In a small time interval Δt, it is reasonable to assume, that the number of infected individuals changes at a rate proportional to the time interval and the numbers of infected and susceptible individuals. Assuming transmission from infected to susceptible, the change in infections should be roughly proportional to the number of contacts between infected and susceptible individuals in a given time interval. The number of contacts is itself proportional (at least approximately) to the number $s(t)v(t)$.

Thus, we would guess

$$\Delta s(t) = -\alpha s(t)v(t)\Delta t$$

for some positive constant α. Similarly, it is reasonable to assume that a proportion $\beta v(t)\Delta t$ of infected individuals recover or die (thus enter the pool of removed individuals) for some $\beta > 0$. Assuming that recovered individuals are no longer susceptible, we would get

$$\Delta v(t) = \alpha s(t)v(t)\Delta t - \beta v(t)\Delta t$$

Finally

$$\Delta r(t) = \beta v(t)\Delta t$$

In the limit ($\Delta t \to 0$), this becomes a system of equations of the form

$$\begin{aligned}
s'(t) &= -\alpha s(t)v(t) \\
v'(t) &= \alpha s(t)v(t) - \beta v(t) \\
r'(t) &= -\beta v(t)
\end{aligned}$$

We will call α, β the *rate of infection*, and *removal rate*, respectively. We will assume that we start with $s(0) = s_0 > 0, v(0) = v_0 > 0, r(0) = r_0 > 0$. (Therefore, this model does not cover the immediate start of an epidemic (where $r, v = 0$)).

Of course, the real world assumptions mean $\alpha, \beta > 0$, and $s(t), v(t) > 0$. It follows that $s' < 0$, so the number of susceptible individuals decreases strictly over time.

But this also means that s is invertible, so we can write $t = g(s)$ for some function g. This is slightly imprecise as s is a function and not really a variable. What this means is that $s(t)$ is invertible, so there is a function $g(y)$ such that $g\big(s(t)\big) = t$ and $s\big(g(y)\big) = y$. Because of this, it is common to label y also by s. So $s = s(t)$ and $t = t(s)$ with the understanding that $t(s) = g(y)$ (and $y = s(t)$).

Finally, as $r(t)$ is determined by $s(t)$ and $v(t)$, and neither s', nor v' depend on r above, we ignore it. Now consider $A(t) = \big(s(t), v(t)\big)$. Over time, $A(t)$ describes a *curve* in $\mathbb{R}^2$. As $t = g(s)$, this curve is the graph $\big(s, V(s)\big)$ for a function $V(s) = v(g(s))$. Then

$$\frac{d}{ds}V(s) = \frac{dv}{dt}\big(g(s)\big)\frac{dg}{ds}$$

Now $\frac{dg}{ds}(s) = \frac{1}{\left(\frac{ds}{dt}\right)(g(s))} = \frac{1}{s'(g(s))}$ and

$$\frac{dV}{ds}(s) = \frac{v'\big(g(s)\big)}{s'\big(g(s)\big)}$$

But $s'\big(g(s)\big) = -\alpha s\big(g(s)\big)v\big(g(s)\big) = -\alpha sV(s)$. $v'\big(g(s)\big) = \alpha s\big(g(s)\big)v\big(g(s)\big) - \beta v\big(g(s)\big) = \alpha sV(s) - \beta V(s)$. Together,

$$\frac{dV}{ds}(s) = \frac{\alpha sV(s) - \beta V(s)}{-\alpha sV(s)} = -1 + \frac{\beta}{\alpha}\frac{1}{s} = -1 + \frac{\gamma}{s}$$

where $\gamma = \frac{\beta}{\alpha}$, whenever $s, V(s) \neq 0$. But notice that meaningful in reality are only situations where $s > 0$ and $V(s) > 0$, as cases where $s = 0$ mean that there will be no further infections, and cases where $V(s) = 0$ mean that there are no infected (and so also no further infection occurs). This shows that this is not a good model to describe the *beginning* of an epidemic, as any epidemic probably starts with $V(s) = 0$.

The equation $V'(s) = -1 + \frac{\gamma}{s}$ means $V(s) = -s + \gamma \log s + C$ for some constant C.

Now $s_0 = s(0)$ means $V(s_0) = v(0) = v_0$. Thus, $v_0 = V(s_0) = -s_0 + \gamma \log s_0 + C$ means

$$C = v_0 + s_0 - \gamma \log s_0$$

Since $\gamma \log s - \gamma \log s_0 = \gamma \log \frac{s}{s_0}$ we obtain the formula

$$V(s) = v_0 + s_0 - s + \gamma \log\left(\frac{s}{s_0}\right)$$

This is not a complicated function. Only values $V(s) \geq 0$ are meaningful.

It is a good exercise to verify the following facts:

1. There is s_1 such that $V(s_1) = 0$, and $0 < s_1 < \gamma$.
2. There is s_2 such that $V(s_2) = 0$, and $s_2 > \gamma$.
3. V is positive precisely on (s_1, s_2).
4. V has a (unique) local (and global) maximum at $s = \gamma$.

There are two cases:

1. $s = s(t)$ reaches s_1 in finite time; or
2. $\lim_{t \to \infty} s(t) = s_1$

Suppose there is no t_0 such that $s(t_0) = s_1 > 0$. Then $v(t) > 0$ for all *relevant* t (that is time points t that correspond to meaningful values of the function s, v). Consequently $s(t)$ is strictly decreasing for all such t and bounded below and $\lim_{t \to \infty} s(t) = \inf s(t)$. Note $\inf s(t) \geq s_1$ as otherwise there is t_0 such that $s(t_0) = s_1$.

If $S = \inf s(t) > s_1$, then $V(S) = \lim_{t \to \infty} V\big(s(t)\big) > 0$.

But note hat this means that $s'(t) < C$ for some $C < 0$ and that is impossible (why?).

Therefore eventually, we reach s_1 (or get close to it).

This shows the importance of the value of γ. If γ is very large (for example, larger than our starting value of s_0, then because s is a strictly decreasing function of time, we move towards s_1 over time without ever hitting the maximum $V(\gamma)$ of infected, so V is also strictly decreasing over time, and an epidemic can be avoided.

If on the other hand $s_0 > \gamma$, then s is still decreasing, but V is initially *increasing*, and will eventually reach its maximum $V(\gamma)$.

More reassuring is maybe the fact hat never *all* individuals will be infected, as $s_1 > 0$ (so when there are no longer any infected ($V(s) = 0$), then s is still positive).

Of course, reality is usually more complex than such simple models. But this indicates that the first order of business is to try to increase the value of γ.

γ can be increased by *decreasing* α or *increasing* β. α can be decreased by limiting contact between infected and susceptible individuals (e.g. *social distancing*).

β can be increased, for example, by improving the health care system, developing a vaccine or cure, and generally trying for a higher ratio of recovery. It also shows that if the disease has a high mortality rate, this keeps β high and γ down. The Ebola virus may be an example of a large β value (but I am no expert, and that is really just a guess).

But for a mathematician it should be interesting to note that this relatively simple qualitative analysis can yield concrete insights into the dynamics of an epidemic (think about it, no deep understanding of the disease itself or its existing or potential cure is needed). As with any model, there are of course limitations:

1. We assumed that the infection and recovery coefficients remained constant over time. This may not be realistic, for example if measures are enacted over time to increase γ.
2. We assumed a rather simple relation between the number of susceptible and infected individuals to get a measure of the rate of growth of the number of infected. The real world may be (and likely is) more complicated. Over large time spans any model becomes less and less reliable (for example, this model does not allow or predict recurring epidemics of the same disease; but we have the annual flu season.).
3. We assumed that once recovered or deceased, an individual is neither susceptible nor infectious. This is certainly not true for all diseases. If someone who died from the disease still is infectious, we could adapt by not removing them from the pool of infected, but maybe the rate of infection is different for a dead individual compared to a live individual. I believe Ebola is an example where deceased individuals are still infectious. But in any event, even if recovered individuals are no longer infectious, they may become susceptible again after a certain time span (which is for example unknown in the current COVID-19 pandemic).
4. We assumed our model is closed and the total number of individuals is constant. This is not realistic, as people are born and die (for unrelated reasons) all the time. This introduces variance into our model. For example, if many infected individuals die of unrelated causes, it may impact our model; likewise, if the disease can be transmitted from mothers to newborns (but maybe at a different rate), this also will change our model. And finally, as health care resources are usually limited, if an epidemic strains the resources of a health care system, some people with unrelated illnesses might die because they don't receive treatment for their otherwise treatable conditions in time.

None of this is exact. And I am not an epidemiologist. Please don't use this analysis to draw any conclusions about the currently ongoing COVID-19 pandemic. The model/analysis presented here is way too simplified to be of significant use. It is merely meant as an example of how one could start to discuss a situation such as the one we are currently in with mathematical tools.

7.2 Differential equations

Loosely speaking, a differential equation relates the rate of growth (that is the derivative) or higher derivatives of one or more quantities to the values of these and other quantities.

You may be familiar with Newton's laws of motion in classical mechanics: if $v(t)$ denotes the velocity of an object at time t, then $v'(t) = \frac{1}{m} f(t)$ where $f(t)$ is the magnitude of the force applied to the object and m is its mass. Note v itself is a derivative, namely $v(t) = x'(t)$ where $x(t)$ is the position of the object (assuming the object travels along a one-dimensional real line; this can be generalized to space).

In some important instances the force working on an object is proportional to the position $x(t)$ (think of an ideal spring attached to the object where we measure $x(t)$ as the elongation of the spring. Then $f(t) = -kx(t)$ where k is some (positive) constant.

Thus $x''(t) = -kx(t)$, which means $x(t) = a\cos\left(\sqrt{k}t\right) + b\sin(\sqrt{k}\,t)$ for some constants a, b. This is known as the *harmonic oscillator* in physics. Note a, b are determined by the initial conditions (a by the position $x(0)$ and b by $x'(0) = b$.

Differential equations come in many shapes and sizes, and vary greatly in complexity.

A reasonably general concept of a differential equation is the following:

Definition
A nth order **ordinary differential equation** (ODE) is a function f defined on an open subset D of $\mathbb{R}^{n+2}$ of the following form: $D = (r,s) \times (a_0, b_0) \times (a_1, b_1) \times \ldots \times (a_n, b_n)$, the "hyper cube" defined by real numbers $r < s, a_0 < b_0, a_1 < b_1, \ldots, a_n < b_n$). EOD.

This seams like a very odd definition. After all what is the equation here?

It becomes clear when we talk about **solutions**:

Definition
A **solution** to a differential equation f as above, is a function $x(t)$ defined on (r,s) such that

$$f\left(t, x(t), x'(t), \ldots, x^{(n)}(t)\right) = 0$$

Of course this necessitates that the range of x is contained in (a_0, b_0), and more generally the range of $x^{(k)}$ is in (a_k, b_k).
There are modifications of that for more general domains D of f. EOD.

As stated, this is far too general a concept. Certainly, for us.

We will focus on a very special case:

Consider the case that f is a function in $n+1$ variables of the form

$$f(t, x_0, x_1, \ldots, x_n) = b(t) + a_0(t)x_0 + a_1(t)x_1 + \cdots + a_n(t)x_n$$

Then we are looking for functions $x(t)$ that satisfy

$$0 = b(t) + a_0(t)x(t) + a_1(t)x'(t) + \cdots + a_n(t)x^{(n)}(t)$$

This is known as a **linear ODE**. If $b = 0$, it is customary to associate the function

$$g(t, y) = a_0(t) + a_1(t)y + \cdots + a_n(t)y^n$$

to this equation. Further specializing, a **linear ODE with constant coefficients** corresponds to a **polynomial** $g(y) = a_0 + a_1 y + \cdots + a_n y^n$, and we are looking for functions $x(t)$ that satisfy $g = 0$ if y^k is replaced by $x^{(k)}(t)$.

We may assume $a_n \neq 0$ and then $a_n = 1$ (by scaling everything by a_n^{-1}).

The harmonic oscillator corresponds to the polynomial $y^2 + k$.

Let $g(y) = y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0$. By the Fundamental Theorem of Algebra g has (possibly complex) roots. Let λ be a root of g. That is, $g(\lambda) = 0$.

Then $f(x) = e^{\lambda x}$ is a solution.

Indeed, $f^{(k)}(x) = \lambda^k e^{\lambda x} = \lambda^k f(x)$. It follows that

$$\begin{aligned} f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \cdots + a_1 f'(x) + a_0 f(x) &= (\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0)f(x) \\ &= g(\lambda)f(x) = 0 \end{aligned}$$

It turns out that this holds even if λ is a complex number.

If g is a polynomial of degree n with n distinct roots $\lambda_1, \lambda_2, \dots, \lambda_n$, then the $e^{\lambda_i x}$ are linearly independent solutions and any other solution is of the form

$$a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + \cdots + a_n e^{\lambda_n x}$$

where $a_1, a_2, \dots, a_n \in \mathbb{C}$.

Example

Let $g(y) = y^2 + k$ (for some $k > 0$). Then g has the complex roots $\pm\sqrt{k}$i. And $f_1 = e^{\sqrt{k}ix}$ and $f_2 =$ e^$(-\sqrt{k}ix)$ are two solutions.

Note that $\Re(f)$ is also a solution if f is. Same for $\Im(f)$. In this case $\Re(f) = \cos(\sqrt{k}\,x)$. EOE.

5. The complex numbers

5.1 The complex plane

5.1.1 Introduction

Our plan is to extend the set of real numbers by adding solutions to the equation $x^2 + 1 = 0$. To motivate the definitions that are about to come, let us work backwards: let us assume that there exists a field C, say, that contains the real numbers as a subfield such that in C we can solve the above equation: there is an element $\boldsymbol{i}$ for which $\boldsymbol{i}^2 = -1$. Since C contains $\mathbb{R}$ we can form the set $C' = \{a + b\boldsymbol{i} \mid a, b \in \mathbb{R}\}$. Indeed, as $a, b, \boldsymbol{i}$ are all elements of C, so is $a + b\boldsymbol{i}$ for each choice of $a, b \in \mathbb{R}$.

Note $C'$is closed under addition

$$(a + b\boldsymbol{i}) + (c + d\boldsymbol{i}) = (a + c) + (b + d)\boldsymbol{i} \in C'$$

What about the multiplication?

$$(a + b\boldsymbol{i})(c + d\boldsymbol{i}) = ac + bc\boldsymbol{i} + ad\boldsymbol{i} + bd\boldsymbol{i}^2 = (ac - bd) + (ad + bc)\boldsymbol{i} \in C'$$

It will turn out that C' is actually a subfield of C (all that is left to show would be that the multiplicative inverses of elements in C' are again in C').

Note that if $a + b\boldsymbol{i} = c + d\boldsymbol{i}$, then $a - c = (d - b)\boldsymbol{i}$ and after squaring both sides we have

$$(a - c)^2 = (d - b)^2(-1)$$

But this is an equation among real numbers. The left hand side is always nonnegative, and the right hand side is never positive, which forces both sides to be equal to 0. Thus, $a = c$ and $b = d$.

5.1.2 The definition

We will now use the above to *define* a field that behaves much like C' above.

Let $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$ be the set of ordered pairs of real numbers. Two such pairs (a, b), (c, d) are equal iff $a = c$ and $b = d$.

We define an addition $+$ on $\mathbb{R}^2$ by

$$(a, b) + (c, d) \coloneqq (a + c, b + d)$$

and a multiplication by

$$(a, b)(c, d) \coloneqq (ac - bd, ad + bc)$$

One then checks (by uninspired direct computation) that both operations are associative and commutative.

Moreover, there is an identity element for the addition $0 \coloneqq (0,0)$ and any (a, b) has an additive inverse (a', b') such that $(a, b) + (a', b') = 0$. Indeed, $(a', b') = (-a, -b)$.

The element $1 \coloneqq (1,0)$ is a multiplicative identity, that is, $1(a, b) = (a, b)$ for all (a, b).

It is also elementary to verify the usual distributive law that connects addition and multiplication in a field.

Every nonzero element also has a multiplicative inverse (as we see soon).

Definition
$\mathbb{R}^2$ together with addition and multiplication introduced above is called the **field of complex numbers** and denoted $\mathbb{C}$.

Before we discuss multiplicative inverses, we simplify the notation:

Let $a, a' \in \mathbb{R}$. Then

$(a,0)(a',0) = (aa',0)$ and $(a,0) + (a',0) = (a + a',0)$.

In other words, if we identify the real number a with $(a,0)$, then it does not matter whether we perform computations in $\mathbb{R}$ or in $\mathbb{C}$. We therefore *identify* $a \in \mathbb{R}$ with $(a,0) \in \mathbb{C}$ and often write a also for $(a,0) \in \mathbb{C}$. Obviously, this only applies if we think of $\mathbb{R}^2$ as the field $\mathbb{C}$. If we think of $\mathbb{R}^2$ as $\mathbb{R}^2$, the cartesian product of $\mathbb{R}$ with itself, we do *not* identify a and $(a,0)$.

But then

$$(a,b) = a(1,0) + b(0,1) = a + bi$$

where we put $i \coloneqq (0,1)$.

Then $i^2 = (0,1)(0,1) = (-1,0) = -1$.

Every complex number z can be written uniquely as $z = a + bi$ where $a, b \in \mathbb{R}$. a is called the **real part** of z and often denoted by $\mathfrak{R}(z)$ or $\mathrm{Re}(z)$, and b is called the **imaginary part** of z, denoted $\mathfrak{I}(z)$ or $\mathrm{Im}(z)$.

Let $z = a + bi$ and consider

$$z(a - bi) = a^2 + b^2 \in \mathbb{R}_{\geq 0}$$

Also $a^2 + b^2 > 0$ if and only if $z \neq 0$. But then

$$z \cdot \frac{1}{a^2 + b^2}(a - bi) = 1$$

so $z^{-1} = \frac{1}{a^2+b^2}(a - bi)$ is a multiplicative inverse.

Because of the fact that $z(a - bi) \in \mathbb{R}$, we define the **complex conjugate** of z as

$$\bar{z} \coloneqq a - bi$$

Properties of the conjugate

1. $\overline{z + w} = \bar{z} + \overline{w}$
2. $\overline{zw} = \bar{z}\overline{w}$
3. $\bar{z} = z$ iff $z \in \mathbb{R}$

It is natural to think of $\mathbb{C}$ as a plane (it is after all $\mathbb{R}^2$) with coordinate axis given by the real number and purely imaginary numbers (the complex numbers of the form bi with $b \in \mathbb{R}$). Therefore $\mathbb{C}$ is often referred to as the *complex plane*, or the *Gaussian (number) plane*.

5.2 Analysis in the complex plane

5.2.1 Absolute value and distance

In order to do analysis (that is, discuss differential calculus functions of a complex variable) one needs to have a solid notion of convergence, which ultimately rests on a notion of "distance" between points.

Since $z\bar{z} \in \mathbb{R}_{\geq 0}$ and $z\bar{z} > 0$ if $z \neq 0$, it makes sense to define

$$|z| := \sqrt{z\bar{z}} = \sqrt{\Re(z)^2 + \Im(z)^2}$$

This is called the **absolute value** of z.

It also coincides with the *Euclidean length* of z when viewed as an element of $\mathbb{R}^2$.

Properties of the absolute value

1. $|z| \in \mathbb{R}_{\geq 0}$
2. $|z| = 0$ iff $z = 0$
3. $|zw| = |z||w|$
4. $|z + w| \leq |z| + |w|$ (triangle inequality)
5. If $z \in \mathbb{C}$ is a real number then $|z| = \sqrt{z^2}$ is just its usual absolute value.

All of these properties are straight forward. We give a proof of 4.

Let a, b, c, d be real numbers then

$$(a + c)^2 + (b + d)^2 \leq a^2 + b^2 + 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} + c^2 + d^2$$

Indeed $a^2 + b^2 + c^2 + d^2$ appear on both sides, so this inequality is equivalent to

$$2ac + 2db \leq 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$$

Or

$$ac + bd \leq \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$$

Since $ac + bd \leq |a||c| + |b||d|$ it is enough to prove this for $a, b, c, d \geq 0$.

Squaring both sides again, this is equivalent to

$$(ac)^2 + 2abcd + (bd)^2 \leq (a^2 + b^2)(c^2 + d^2) = (ac)^2 + (ad)^2 + (bc)^2 + (bd)^2$$

It thus boils down to $2abcd \leq (ad)^2 + (bc)^2$. But this is true since $(ad - bc)^2 \geq 0$.

The triangle inequality now follows if we put $z = a + bi$ and $w = c + di$. QED.

Remark
Note this is not a very elegant or efficient proof of the triangle inequality.
A more general result, known as *Minkowski's Inequality,* says

$$\left(\sum_{i=1}^{n}(x_i+y_i)^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n}x_i^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n}y_i^2\right)^{\frac{1}{2}}$$

where $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ are real numbers. This inequality, in turn, can be obtained by a use of Hölder's Inequality (cf. Homework 3)

$$\sum_{i=1}^{n}|x_i y_i| \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}|y_i|^q\right)^{\frac{1}{q}}$$

whenever $\frac{1}{p}+\frac{1}{q}=1$. EOR.

Using the absolute value, we can define the **distance** between two complex numbers z, w as $d(z,w) \coloneqq |z-w|$.

It has the usual properties we would require of a distance, namely:

1. $d(z,w) \in \mathbb{R}_{\geq 0}$
2. $d(z,w) = 0$ iff $z = w$
3. $d(z,w) = d(w,z)$
4. $d(z,w) \leq d(z,u) + d(u,w)$ for all $u \in \mathbb{C}$

These are immediate translations of the properties of the absolute value.

Remark
The field of complex numbers cannot be ordered[1] in the sense of field theory: suppose $P \subseteq \mathbb{C}$ defines an order (that is P is the set of "positive" elements). Then for each $z, w \in P$, $zw \in P$. And for each nonzero $z \in \mathbb{C}$, $z^2 \in P$. But $-1 \notin P$, because 1 is always positive in any ordered field. But $i^2 = -1$. So we cannot compare two arbitrary complex number. Nevertheless it is common to talk about "large" or "small" complex numbers in the sense that they have a "large" or "small" absolute value.

5.2.2 Sequences and limits

Now that we have a concept of the distance between complex numbers, we can make pronouncements what it means to "approach" a complex number. In other words we can define limits of sequences:

A **sequence** of complex numbers is, not surprisingly, a sequence $z_1, z_2, \dots, z_n, \dots$ of complex numbers, that is, a function $f: \mathbb{N} \to \mathbb{C}$, and we write $z_i = f(i)$. Sometimes such sequences, just as in the real case, may start at a different natural number, or even at an integer.

Definition
Let z_n be a sequence of complex numbers. The complex number z_0 is called the **limit** of the sequence z_n if $\lim_{n\to\infty}|z_n - z_0| = 0$.
If that is the case, we write $z_0 = \lim_{n\to\infty} z_n$. A sequence with a limit is said to be **convergent** and divergent otherwise EOD.

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[1] In the sense of field theory. Of course, like any set, the underlying allows for even a well-ordering by the Axiom of Choice. However such an ordering is not compatible with the field structure. So $\mathbb{C}$ cannot be turned into an *ordered field*.

$|z_n - z_0|$ is a sequence of real numbers, and therefore we know what it means to have limit 0.

Limits are unique if they exist (and the proof is similar to the proof for sequences of real numbers).

Lemma
Let z_n be a sequence of complex numbers. Then $z_0 = a_0 + b_0 i$ $(a_0, b_0 \in \mathbb{R})$ is the limit of z_n if and only if $\lim_{n\to\infty} \Re(z_n) = a_0$ and $\lim_{n\to\infty} \Im(z_n) = b_0$. EOL.

Proof. Suppose $z_0 = \lim_{n\to\infty} z_n$. Then $|z_n - z_0| \to 0$ and thus

$$\sqrt{(\Re(z_n) - a_0)^2 + (\Im(z_n) - b_0)^2} \to 0$$

It follows that also $|\Re(z_n) - a_0| = \sqrt{(\Re(z_n) - a\_0)^2} \le \sqrt{(\Re(z_n) - a_0)^2 + (\Im(z_n) - b_0)^2}$ converges to 0. Likewise, $|\Im(z_n) - b_0| \to 0$. It follows that $\lim_{n\to\infty} \Re(z_n) = a_0$ and $\lim_{n\to\infty} \Im(z_n) = b_0$.

For the converse suppose $\Re(z_n) \to a_0$ and $\Im(z_n) \to b_0$.

Then also $(\Re(z_n) - a_0)^2 + (\Im(z_n)^2 - b_0)^2 \to 0$. As $\sqrt{x}$ is continuous at 0, it follows that

$$\sqrt{(\Re(z_n) - a_0)^2 + (\Im(z_n) - b_0)^2} \to 0$$

and hence $z_n \to z_0$. QED.

5.2.3 Open and closed sets

Now that we know what convergence means, we can find analogues for some of the related concepts we developed for subsets of real numbers.

Definition
A subset $A \subseteq \mathbb{C}$ is **closed** if for every sequence $z_n \in A$ that has a limit $z_0 \in \mathbb{C}$, that limit is an element of A. EOD.

Immediate examples are the empty set $\emptyset$ and the set $\mathbb{C}$ of all complex numbers.

Example
Let $D = \{z \in \mathbb{C} \mid |z| \le R\}$. Then D is closed.
Indeed, if $z_n \in D$ is a convergent sequence with limit z_0, we have $|z_n - z_0| \to 0$.
By the above if $z_0 = a_0 + b_0 i$ and $z_n = a_n + b_n i$ $(a_n, b_n \in \mathbb{R})$ then $a_n \to a_0$ and $b_n \to b_0$. Therefore

$$|z_n| = \sqrt{a_n^2 + b_n^2} \to \sqrt{a_0^2 + b_0^2} = |z_0|$$

The left hand side is $\le R$ for all n and therefore $|z_0| \le R$ as well. EOE.

Definition
A set $U \subseteq \mathbb{C}$ is **open** if for every $z \in U$ there is $\varepsilon > 0$ (in $\mathbb{R}$) such that

$$B_\varepsilon(z) := \{ w \in \mathbb{C} \mid |w - z| < \varepsilon \} \subseteq U$$

EOD.

Exercise
1. Show that $B_\varepsilon(z)$ is open.
2. Show that U is open iff $\mathbb{C} \setminus U$ is closed.

EOE.

5.2.4 Continuous functions

A *function* is the same as before. But now the domain and codomain will be subsets of the complex numbers.

Mostly we will discuss functions whose domains are open sets. One of the reasons is that while open "discs" of the form $B_R(z)$ are good analogues of open intervals in $\mathbb{R}$, open sets (even so called "connected" ones) can be substantially more complicated and the boundary of those (essentially defined in the same way as for subsets of the real line) can be really "complex."

Definition

Let $D \subseteq \mathbb{C}$ be an open set and $f: D \to \mathbb{C}$ be a function. Let $z_0 \in \mathbb{C}$ be a complex number for which there exists at least one sequence in D with limit z_0. Then $L \in \mathbb{C}$ is called the **limit** of $f(z)$ as $z \to z_0$ if for **every sequence** $z_n \in D$ with $z_n \to z_0$ we have $\lim_{n\to\infty} f(z_n) = L$.

If that is the case, we write $\lim_{z\to z_0} f(z) = L$.

Equivalently, $\lim_{z\to z_0} f(z) = L$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $z \in D$ with $z \in B_\delta(z)$, we have $|f(z) - L| < \varepsilon$. EOD.

Exercise

Show that the two equivalent definitions of $\lim_{z\to z_0} f(z)$ are indeed equivalent. EOE.

Definition

Let $D \subseteq \mathbb{C}$ be an open subset and $f: D \to \mathbb{C}$ a function. Let $z_0 \in D$. Then f is **continuous** at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$.

Equivalently, f is continuous at z_0 if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $z \in B_\delta(z_0) \cap D$ we have $|f(z) - f(z_0)| < \varepsilon$.

f is **continuous**, if it is continuous at all $z_0 \in D$. EOD.

Example

Let $f(z) = \bar{z}$, defined on $D = \mathbb{C}$. Then f is continuous.

Indeed, let $z_n \to z_0$ be any convergent sequence. We have seen that $z_n \to z_0$ means $\Re(z_n) \to \Re(z_0)$ and $\Im(z_n) \to \Im(z_0)$. But then also $\overline{z_n} = \Re(z_n) - \Im(z_n)i \to \Re(z_0) - \Im(z_0)i = \overline{z_0}$. EOE.

The rules of continuity apply unchanged from the real case:

Sums, products, and quotients (where defined) of continuous functions are continuous. We will not repeat the proofs for each of those statements as they are essentially the same as in the real case.

Example

Let $f(z) = |z|$. Then f is continuous.

Again let $z_n \to z_0$ be a convergent sequence. Then $\sqrt{a_n^2 + b_n^2} \to \sqrt{a_0^2 + b_0^2}$ where $a_n = \Re(z_n), b_n = \Im(z_n)$, and $a_0 = \Re(z_0)$ and $b_0 = \Im(z_0)$, respectively. EOE.

Exercise

Let f, g be continuous functions defined on real or complex domains. Suppose $f \circ g$ is defined (that is the range of g is contained in the domain of f). Show that $f \circ g$ is continuous.

This is a special case of the more genera fact that compositions of continuous functions between topological spaces are continuous.
Here, it is much more simple. EOE.

Using the exercise, the previous example is immediate: $f(z) = |z|$ is the composition of the continuous functions $f(x) = \sqrt{x}$ and $g(z) = z\bar{z}$.

Example
If $f(z)$ is a polynomial function (a.k.a a polynomial), then f is continuous on $\mathbb{C}$.
So, if $f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$, then $\lim_{z \to z_0} f(z) = f(z_0)$.
Indeed, $\lim_{z \to z_0} z^n = z_0^n$. This follows by induction from the formula

$$z^n - z_0^n = (z - z_0)(z^{n-1} + z^{n-2} z_0 + \cdots + z z_0^{n-2} + z_0^{n-1})$$

The right hand side is equal to something of the form $(z - z_0)g(z)$ with $g(z) \to n z_0^{n-1}$ by induction.
Therefore $z^n - z_0^n \to 0$. EOE.

5.2.5 Differentiation

Not surprisingly differentiation of functions on the complex plane is defined in the same way as it was for functions in intervals on the real line.

Definition
Let $D \subseteq \mathbb{C}$ be an open subset, $f: D \to \mathbb{C}$ be a function, and $z_0 \in D$.
We say f is **complex differentiable** at z_0 if

$$\lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If that is the case, this limit is denoted $f'(z_0)$ and called the **(complex) derivative** of f at z_0. EOD.

Note that technically, we haven't defined the "limit $z \to z_0$ and $z \neq z_0$". What is meant is the following: define $g(z)$ on $D^* \coloneqq D \setminus \{z_0\}$ as $g(z) = \frac{f(z) - f(z_0)}{z - z_0}$. Then $f'(z_0) = \lim_{z \to z_0} g(z)$ if that limit exists.

(Since D is open and $z_0 \in D$, there exists a sequence $z_n \in D^*$ with $z_n \to z_0$ (why?).)

Example
Like in the real case, monomials of the form z^n where ($n \in \mathbb{N}$) are differentiable everywhere:

$$z^n - z_0^n = (z - z_0)(z^{n-1} + z^{n-2} z_0 + \cdots + z z_0^{n-2} + z_0^{n-1})$$

Then

$$\frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z^{n-2} z_0 + \cdots + z z_0^{n-2} + z_0^{n-1} \to n z_0^{n-1}$$

EOE.

The usual rules for differentiation still apply. More precisely, suppose f, g are defined on D and differentiable at $z_0 \in D$. Then:

1. $f + g$ is differentiable at z_0 and $(f + g)'(z_0) = f'(z_0) + g'(z_0)$
2. fg is differentiable at z_0 and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$
3. It follows that $\alpha f + \beta g$ is differentiable at z_0 with $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$ where $\alpha, \beta \in \mathbb{C}$.

Suppose f is defined on an open subset $E \subseteq \mathbb{C}$ and g is defined on an open subset D and $g(D) \subset E$ (so $f \circ g$ is a well-defined function). If f is differentiable at w_0 and g is differentiable at z_0 and $g(z_0) = w_0$ then

$$(f \circ g)'(z_0) = f'(w_0)g'(z_0)$$

(So the Chain Rule still holds.)

5.2.6 Holomorphic functions

Definition

Suppose f is defined on an open $D \subseteq \mathbb{C}$. f is called **holomorphic**, if f is complex differentiable everywhere on D. EOD.

Example

$$f(z) = \bar{z}$$

is not complex differentiable anywhere, and is therefore not holomorphic.
Let's first treat the case $z_0 = 0$.
Then

$$\frac{\bar{z}}{z} = \frac{a - bi}{a + bi}$$

does not converge for $z \to 0$.
Indeed, choose $z_n = \frac{1}{n} i$ and $w_n = \frac{1}{n}$.
Then $\frac{\overline{z_n}}{z_n} = -1$ whereas $\frac{\overline{w_n}}{w_n} = 1$. So $\frac{\bar{z}}{z}$ has no limit for $z \to 0$.
But similarly then $z_n = z_0 + \frac{1}{n}$, $w_n = z_0 + \frac{1}{n} i$ gives sequences for which $\frac{f(z_n) - f(z_0)}{z_n - z_0}$ and $\frac{f(w_n) - f(z_0)}{z_n - z_0}$ have distinct limits. EOE.

Polynomials are holomorphic, as are convergent power series (defined as in the real setting). For the latter one needs a version of the Transformation Theorem.

In fact, every holomorphic function can locally be written as a power series:

Theorem

Holomorphic functions are analytic. EOT.

This is beyond our means to prove.

5.2.7 The exponential function

Among the most interesting holomorphic functions is the *Riemann Zeta function* defines as

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Note this is as written not a power series, and it is defined on the set $D = \{z \in \mathbb{C} \mid \Re(z) > 1\}$.

But what is $\frac{1}{n^z}$ supposed to mean? If z is a real number, then $n^z = e^{z \log n}$. Does this make sense also if z is complex?

Definition

For $z \in \mathbb{C}$ we define

$$e^z := \exp(z) := \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

EOD.

e^z converges for all z: indeed, just as in the real case one shows that for $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $M, N > n_0$,

$$\left| \sum_{n=M}^{N} \frac{1}{n!} z^n \right| \leq \sum_{n=M}^{N} \frac{1}{n!} |z|^n < \varepsilon$$

Likewise, the proof we have for the formula $e^{z+w} = e^z e^w$ carries over essentially unchanged.

It then follows that $(e^z)' = e^z$.

We obtain a holomorphic function, called the **exponential function**, defined on all of $\mathbb{C}$. One can show that its range is all of $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Contrary to the real case, however, the exponential function is no longer injective, as we shall see.

5.2.8 Complex numbers of absolute value 1

Let $U = \{z \in \mathbb{C} \mid |z| = 1\}$. Geometrically, in the Gaussian plane, this is the circle of radius 1 around the origin.

Indeed, $|z| = 1$ iff $\Re(z)^2 + \Im(z)^2 = 1$.

Another interesting property is that $|z| = 1$ iff $z\bar{z} = |z|^2 = 1$, and thus

$$z \in U \Leftrightarrow \bar{z} = z^{-1}$$

Note that if $z \neq 0$, then $\frac{z}{|z|} \in U$.

Finally, $z, w \in U$, then $|zw| = |z||w| = 1$, so $zw \in U$. As $1 \in U$, and $|z^{-1}| = \frac{1}{|z|} = 1$ for $z \in U$, this show that U is a subgroup of the (multiplicative) group $\mathbb{C}^*$. Note this is the *unitary group* of rank 1.

What does multiplication of elements in U mean geometrically?

To get into that, note that for any complex number w we have

$$\overline{e^w} = e^{\overline{w}}$$

Indeed, this follows from the continuity of $w \mapsto \overline{w}$ and e.g. the following reasoning:

$$\overline{e^w} = \overline{\lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} w^n} = \lim_{N\to\infty} \overline{\sum_{n=0}^{N} \frac{1}{n!} w^n} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} \overline{w}^n = e^{\overline{w}}$$

On the other hand, for each $w \in \mathbb{C}$, $(e^w)^{-1} = e^{-w}$.

It follows that e^w has absolute value 1, if and only if $\overline{e^w} = e^{\overline{w}} = (e^w)^{-1} = e^{-w}$, or $\overline{w} = -w$.

But $\overline{w} = -w$ iff $w = i\alpha$ for some $\alpha \in \mathbb{R}$. The upshot is:

$e^{i\alpha} \in U$ for all $\alpha \in \mathbb{R}$.

Let $f(\alpha) = e^{i\alpha}$. What can we say about f? It is a function $f: \mathbb{R} \to U \subseteq \mathbb{C}$.

Moreover $f(\alpha + \beta) = e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} = f(\alpha)f(\beta)$.

Let us write $f(\alpha) = C(\alpha) + iS(\alpha)$, that is $C(\alpha) = \Re(f(\alpha))$ and $S(\alpha) = \Im(f(\alpha))$.

Lemma
C, S are smooth functions on $\mathbb{R}$, in fact, both are power series:

$$C(\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \alpha^{2n}$$

and

$$S(\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \alpha^{2n+1}$$

EOL.

Proof. Left as an exercise. However, use $e^{i\alpha} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\alpha)^n$ together with the fact, that for every sequence $\Re(\lim z_n) = \lim\ R(z_n)$, $\Im(\lim z_n) = \lim\ \Im(z_n)$. Finally

$$i^n = \vdash \begin{cases} 1 & n \equiv 0\ (4) \\ i & n \equiv 1\ (4) \\ -1 & n \equiv 3\ (4) \\ -i & n \equiv 3\ (4) \end{cases}$$

QED.

Note from the lemma we immediately gather that C is an even function and S is an odd function, with $S(0) = 0$ and $C(0) = 1$.

Moreover,

$$1 = \left|e^{i\alpha}\right| = \sqrt{C(\alpha)^2 + S(\alpha)^2}$$

and thus $C(\alpha)^2 + S(\alpha)^2 = 1$ for all $\alpha \in \mathbb{R}$.

This immediately also shows that the range of both C, S is a subset of $[-1,1]$. Also, from the power series description it follows easily that $S'(\alpha) = C(\alpha)$ and $C'(\alpha) = -S(\alpha)$.

Finally, because $f(\alpha + \beta) = f(\alpha)f(\beta)$ we have

$$C(\alpha + \beta) = C(\alpha)C(\beta) - S(\alpha)S(\beta)$$

and

$$S(\alpha + \beta) = C(\alpha)S(\beta) + S(\alpha)C(\beta)$$

Definition
The function $C(\alpha)$ is called the **cosine** of α, and we often write $\cos\alpha$ instead of $C(\alpha)$.

The function $S(\alpha)$ is called the **sine** of α, and we often write $\sin\alpha$ instead of $S(\alpha)$.

Remark
You may wonder why this definition makes sense. Have we not defined $\sin x$ and $\cos x$ already? The short answer is no, we only think we did. The original "definition" involved angles, triangles, and arc-lengths. All not very well defined concepts if we are to build analysis from the ground up.
For example, the concept of arc-length typically involves integration. EOR.

Theorem
The functions S, C as above have the following properties:

1. The range of both functions are equal to $[-1,1]$.
2. C has a smallest positive root x_0.
3. Let $P = 2x_0$. Then S, C are $2P$-periodic. That is for all x, $S(x+2P) = S(x)$ and $C(x+2P) = C(x)$.
4. The roots of S are of the form kP, were $k \in \mathbb{Z}$. The roots of C are of the form $x_0 + kP$ where $k \in \mathbb{Z}$.
5. S is strictly increasing on $[-x_0, x_0]$, and strictly decreasing on $[x_0, x_0 + P]$.
6. For each $z \in U$ there is a unique $\alpha \in [0,2P)$ such that $f(\alpha) = z$.

EOT.

This was essentially a homework problem.

5.3 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra asserts that any nonconstant complex polynomial has t least one complex root. It was first rigorously proven by Carl Friedrich Gauss in his dissertation in 1799.

Theorem (Fundamentl Theorem of Algebra)
Let $f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$ be a polynomial function on $\mathbb{C}$, with $n > 0$ and $c_n \neq 0$. Then there is $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.

Proof (Sketch). The proof outlined here follows the one given in Michael Artin, *Algebra*, 1997.

We will proceed in several steps. First the general idea: to show that there is z_0 with $f(z_0) = 0$ we consider the function $g(z) = |f(z)|$. First, we will consider the question whether g (which is real-valued) can have a global minimum > 0, and we will see it cannot. So first we will show that **if** $g(z_0)$ is minimal, then $g(z_0) = 0$, which of course means $f(z_0) = 0$. The second question is then, does $g(z)$ have a global minimum, and we will answer that in the affirmative. Taken together, we must have that z_0 exists.

So suppose first that there is $z_0 \in \mathbb{C}$ with $g(z_0)$ minimal, but $g(z_0) > 0$. We will show that is impossible.

Step 1
For any $c \in \mathbb{C}$ and any fixed $k \in \mathbb{N}$, the equation $z^k = c$ has a solution.
If $c = 0$, then $z = 0$ is a solution. Otherwise we may write $c = re^{i\alpha}$ for $r = |c| > 0$ and some $\alpha \in \mathbb{R}$. We know that the range of x^k is all of $\mathbb{R}_{>0}$ so there is $\sqrt[k]{r} \in \mathbb{R}_{>0}$. Then

$$z = \sqrt[k]{r} e^{\frac{i\alpha}{k}}$$

is a solution. Indeed, $z^k = \left(\sqrt[k]{r}^k\right)\left(e^{\frac{i\alpha}{k}}\right)^k = re^{i\alpha} = c$.

Step 2

We may assume that $z_0 = 0$ and $c_0 = 1$.

Consider $\bar{f}(z) = f(z + z_0)$. Then $\bar{f}$ is again a polynomial, and $|\bar{f}(0)|$ is minimal if $g(z_0)$ is. We may replace f with $\bar{f}$. Then $c_0 = f(0) \neq 0$ because we assumed $g(0) > 0$.

Note that $|c_0^{-1}f(z)|$ is minimal iff $|f(z)|$ is. So we may replace f with c_0^{-1}.

Step 3

We may assume $f(z) = 1 - z^k + z^{k+1}h(z)$ for some $k \in \mathbb{N}$ and some polynomial h.

Let $k = \min\{m > 0 \mid c_m \neq 0\}$. Then $f(z) = 1 + c_k z^k + z^{k+1}H(z)$ for some polynomial H by Step 2.

By Step 1, there is $t \neq 0$ such that $t^k = -c_k$.

Let $f^*(z) = f(tz) = 1 - z^k + z^{k+1}t^{k+1}H(tz)$ and put $h(z) = t^{k+1}H(tz)$. Then h is a polynomial. Als $|f^*(z)|$ is minimal iff $g(tz)$ is. For example this is true if $tz = 0$, i.e. $z = 0$.

We replace f^* with f.

Now we can show that $g(0)$ cannot be minimal. First let us restrict g to $\mathbb{R}_{\geq 0}$.

Then $g(x) = \left|1 - x^k + x^{k+1}h(x)\right| \leq \left|1 - x^k\right| + |x^{k+1}h(x)|$

For "small" $x \geq 0$ we have $1 - x^k > 0$. Then

$$g(x) \leq 1 - x^k + x^{k+1}|h(x)| = 1 - x^k(1 - x|h(x)|)$$

h is a polynomial, so it is continuous at 0. Therefore $|h(x)|$ is bounded around 0, and $\lim_{x\to 0} x|h(x)| = 0$.

For small enough positive x, we therefore have $1 - x|h(x)| > 0$, and therefore

$$1 - x^k(1 - x|h(x)|) < 1 = g(0)$$

This contradicts that $g(0)$ was a minimum of g.

What is left is to show that there is indeed z_0 such that $g(z_0)$ is minimal.

The crucial ingredient here is the fact that for all $R > 0$, there is $S > 0$ such that $|f(z)| > R$ whenever $|z| > S$. We informally write $\lim_{|z|\to\infty} |f(z)| = \infty$.

To prove this, proceed by induction on $\deg f$. If $\deg f = 1$, then $f = c_1 z + c_0$ and $|f(z)| \geq |c_1||z| - |c_0|$, which clearly satisfies the assertion.

If the assertion is true for every nonconstant polynomial of degree at most n, let $\deg f = n + 1$, then

$f(z) = zh(z) + c_0$ where $\deg h = n$.

Then $|f(z)| \geq |z||h(z)| - |c_0|$ and as $\lim_{|z|\to\infty} |h(z)| = \infty$, the same follows for $\lim_{|z|\to\infty} |f(z)|$.

Now pick $z_1 \in \mathbb{C}$. If $f(z_1) = 0$, then $|g(z_1)|$ is minimal and we are done.

Otherwise $R \coloneqq g(z_1) > 0$. There is S as above such that $|f(z)| > R$ for all z with $|z| > S$.

Let $D \coloneqq \{z \in \mathbb{C} \mid |z| \leq S\}$.

Then D is closed and bounded. "bounded" here means $|z|$ is bounded on D which follows by definition.

As for closed, if $z_n \in D$ is a sequence with limit $z_0 \in \mathbb{C}$, then $|z_n| \to |z_0|$ as $|z|$ is a continuous function.

But then $|z_0| \leq S$ because all $|z_n|$ are. Therefore D is closed.

Finally, g is continuous on D. Let $x_0 = \inf g(D)$, which exists because g is bounded below.

There is a sequence $z_n \in D$ such that $g(z_n) \to x_0$.

Let us write $z_n = a_n + b_n i$ with $a_n, b_n \in \mathbb{R}$. Since D is bounded, so is a_n, b_n. There is a convergent subsequence a_{n_k}. Then b_{n_k} is also bounded and has a convergent subsequence $b_{n_{k_\ell}}$. As a subsequence of a convergent sequence $a_{n_{k_\ell}}$ is still convergent, which means $z_{n_{k_\ell}}$ is convergent with limit z_0, say. Then $g\left(z_{n_{k_\ell}}\right) \to g(z_0)$ because g is continuous; and by definition of z_n, $g\left(z_{n_{k_\ell}}\right) \to x_0$. Together $g(z_0) = x_0$ and $x_0 = \min g(D)$.

$g(x_0)$ is a global minimum of g, because $g(x_0)$ is the minimal value of g on D, and $g(z) > R \geq x_0$ for all $z \notin D$. QED.