MATH 217 (Fall 2020)
Honors Advanced Calculus, I

Solutions #1

1. Let $+$ and $\cdot$ be defined on $\{\spadesuit, \dagger, \bigcirc, \mathrm{A}\}$ through:

$+$	$\spadesuit$	$\dagger$	$\bigcirc$	A
$\spadesuit$	$\spadesuit$	$\dagger$	$\bigcirc$	A
$\dagger$	$\dagger$	$\bigcirc$	A	$\spadesuit$
$\bigcirc$	$\bigcirc$	A	$\spadesuit$	$\dagger$
A	A	$\spadesuit$	$\dagger$	$\bigcirc$

$\cdot$	$\spadesuit$	$\dagger$	$\bigcirc$	A
$\spadesuit$	$\spadesuit$	$\spadesuit$	$\spadesuit$	$\spadesuit$
$\dagger$	$\spadesuit$	$\dagger$	$\bigcirc$	A
$\bigcirc$	$\spadesuit$	$\bigcirc$	$\spadesuit$	$\bigcirc$
A	$\spadesuit$	A	$\bigcirc$	$\dagger$

.

Do these turn $\{\spadesuit, \dagger, \bigcirc, \mathrm{A}\}$ into a field?

Solution: The neutral element of $\{\spadesuit, \dagger, \bigcirc, \mathrm{A}\}$ with respect to $+$, i.e., the zero, is $\spadesuit$. According to the second table, $\bigcirc \cdot \bigcirc = \spadesuit$ holds, which is impossible in a field.

2. Show that

$$\mathbb{Q}[i] := \{p + i\,q : p, q \in \mathbb{Q}\} \subset \mathbb{C}$$

with $+$ and $\cdot$ inherited from $\mathbb{C}$, is a field. Is there a way to turn $\mathbb{Q}[i]$ into an ordered field?

(*Hint*: Many of the field axioms are true for $\mathbb{Q}[i]$ simply because they are true for $\mathbb{C}$; in this case, just point it out and don't verify the axiom in detail.)

Solution: Let $p, q, r, s \in \mathbb{Q}$. Then

$$(p + i\,q) + (r + i\,s) = (p + r) + i\,(q + s) \in \mathbb{Q}[i]$$

and

$$(p + i\,q)(r + i\,s) = \underbrace{(pr - qs)}_{\in\mathbb{Q}} + i\,\underbrace{(qr + ps)}_{\in\mathbb{Q}} \in \mathbb{Q}[i]$$

hold, so that (F 1) is satisfied.

Since (F 2), (F 3), and (F 4) hold for $\mathbb{C}$, they also hold for $\mathbb{Q}[i]$.

Since $0 = 0 + i\,0, 1 = 1 + i\,0 \in \mathbb{Q}[i]$, (F 5) is satisfied as well.

Let $p, q \in \mathbb{Q}$, and let $x = p + i\,q$. Then $-x = -p + i\,(-q) \in \mathbb{Q}[i]$ as well. Suppose that $x \neq 0$, so that $p^2 + q^2 \neq 0$. Set

$$y := \frac{p}{p^2 + q^2} - i\,\frac{q}{p^2 + q^2} \in \mathbb{Q}[i].$$

It is immediate that $xy = 1$. Hence, (F 6) is also satisfied.

Assume that there is $P \subset \mathbb{Q}[i]$ as in the definition of an ordered field. Then either $i \in P$ or $-i \in P$ holds, so that in either case $-1 = i^2 = (-i)^2 \in P$, which contradicts the fact that $1 \in P$.

3. Let $\varnothing \neq S \subset \mathbb{R}$ be bounded below, and let $-S := \{-x : x \in S\}$. Show that:

   (a) $-S$ is bounded above;

   (b) S has an infimum, namely $\inf S = -\sup(-S)$.

   Solution:

   (a) Let L be a lower bound for S, i.e., $L \leq x$ for all $x \in S$. It follows that $-x \leq -L$ for each $x \in S$ and thus $x \leq -L$ for each $x \in -S$. Hence, $-L$ is an upper bound for $-S$.

   (b) Let $C := \sup(-S)$, so that $x \leq C$ for all $x \in -S$. It follows that $-x \geq -C$ for all $x \in -S$, i.e., $x \geq -C$ for all $x \in S$. Hence, $-C$ is a lower bound for S. Let C' be another other lower bound for S. In the solution to (a), we have seen that $-C'$ is an upper bound for $-S$, and thus $-C' \geq C$ by the definition of a supremum. It follows that $C' \leq -C$. Hence, $-C = \inf S$ holds.

4. Find $\sup S$ and $\inf S$ in $\mathbb{R}$ for

$$S := \left\{ (-1)^n \left(1 - \frac{1}{n}\right) : n \in \mathbb{N} \right\}.$$

   Justify, i.e., *prove*, your findings.

   Solution: For odd $n \in \mathbb{N}$, $(-1)^n \left(1 - \frac{1}{n}\right)$ is negative, and for even n, we have

$$(-1)^n \left(1 - \frac{1}{n}\right) = 1 - \frac{1}{n} \leq 1.$$

   Hence, S is bounded above by 1. Assume that $\sup S < 1$, and let $\epsilon := 1 - \sup S$. In class, we saw that there is $n \in \mathbb{N}$ with $0 < \frac{1}{n} < \epsilon$, so that

$$\underbrace{1 - \frac{1}{2n}}_{\in S} > 1 - \frac{1}{n} > 1 - \epsilon = \sup S,$$

   which is impossible.

   Similarly, one sees that $\inf S = -1$.

5. Let $S, T \subset \mathbb{R}$ be non-empty and bounded above. Show that

$$S + T := \{x + y : x \in S,\ y \in T\}$$

is also bounded above with

$$\sup(S+T) = \sup S + \sup T.$$

Solution: Let $x \in S$ and $y \in T$. Then $x \leq \sup S$ and $y \leq \sup T$. It follows that

$$x + y \leq \sup S + \sup T,$$

so that $\sup S + \sup T$ is an upper bound for $S + T$. Consequently,

$$\sup(S+T) \leq \sup S + \sup T$$

holds.

Assume that $\sup(S+T) < \sup S + \sup T$. Let $\epsilon := \frac{1}{2}(\sup S + \sup T - \sup(S+T))$. Choose $x \in S$ and $y \in T$ such that

$$x > \sup S - \epsilon \qquad \text{and} \qquad y > \sup T - \epsilon.$$

It follows that

$$x + y > \sup S + \sup T - 2\epsilon = \sup(S+T),$$

which is a contradiction.

6*. An ordered field $\mathbb{O}$ is said to have the *nested interval property* if $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$ for each decreasing sequence $I_1 \supset I_2 \supset I_3 \supset \cdots$ of closed intervals in $\mathbb{O}$.

Show that an Archimedean ordered field with the nested interval property is complete.

Solution: Let $\varnothing \neq S \subset \mathbb{O}$ be bounded above. Choose $a_1 \in S$ and let $b_1 > a_1$ be an upper bound for S. Let $I_1 := [a_1, b_1]$, and let $c_1 := \frac{1}{2}(b_1 - a_1)$. There are two possibilities:

Case 1: c_1 is an upper bound for S. In this case, let $a_2 := a_1$, $b_2 := c_1$, and $I_2 := [a_2, b_2]$.

Case 2: c_1 is not an upper bound for S. In this case, there is $a_2 \in S$ with $a_2 > c_1$. Let $b_2 := b_1$, and define $I_2 := [a_2, b_2]$.

Let $c_2 := \frac{1}{2}(b_2 - a_2)$. Depending on whether c_2 is an upper bound for S or not, we find a_3 and b_3 as we found a_2 and b_2 and define $I_3 := [a_3, b_3]$.

Continuing in this fashion, we obtain a decreasing sequence $I_1 \supset I_2 \supset I_3 \supset \cdots$ of closed intervals in $\mathbb{O}$ with the following properties for all $n \in \mathbb{N}$:

- $I_n = [a_n, b_n]$, where $a_n \in S$ and $b_n \in \mathbb{O}$ is an upper bound for S;
- $(b_{n+1} - a_{n+1}) \leq \frac{1}{2}(b_n - a_n)$.

This second fact yields that

$$(b_{n+1} - a_{n+1}) \leq \frac{1}{2^n}(b_1 - a_1) \leq \frac{1}{n}(b_1 - a_1)$$

for all $n \in \mathbb{N}$ by induction on n.

Since $\mathbb{O}$ has the nested interval property, there is $x \in \bigcap_{n=1}^{\infty} I_n$. We claim that x is the supremum of S in $\mathbb{O}$.

Assume that x is not an upper bound for S, i.e., there is $y \in S$ such that $y > x$. Use the fact that $\mathbb{O}$ is Archimedean to find $n \in \mathbb{N}$ such that

$$(b_{n+1} - a_{n+1}) \leq \frac{1}{n}(b_1 - a_2) < y - x.$$

Since $x \geq a_{n+1}$, we obtain

$$y - x > b_{n+1} - a_{n+1} \geq b_{n+1} - x,$$

and adding x on both sides yields $y > b_{n+1}$, which contradicts b_{n+1} being an upper bound for S.

Hence, x is an upper bound for S.

Assume that there is an upper bound $y \in \mathbb{O}$ with $y < x$. Again use the fact that $\mathbb{O}$ is Archimedean to find $n \in \mathbb{N}$ such that

$$(b_{n+1} - a_{n+1}) \leq \frac{1}{n}(b_1 - a_2) < x - y.$$

Since $b_{n+1} \geq x$, we obtain

$$x - y > b_{n+1} - a_{n+1} \geq x - a_{n+1},$$

and subtracting x and multiplying with -1 on both sides yields that $a_{n+1} > y$ which contradicts y being an upper bound for S.

Hence, x is the least upper bound for S, i.e., $x = \sup S$.

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Solutions #10

1. Let $a, b > 0$. Determine the area of the ellipse

$$E := \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

Solution: Use the following coordinate transformation:

$$\phi\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (r, \theta) \mapsto (ra\cos\theta, rb\sin\theta),$$

so that $E = \phi([0,1] \times [0, 2\pi])$. Since

$$J_\phi(r, \theta) = \begin{bmatrix} a\cos\theta & -ra\sin\theta \\ b\sin\theta & rb\cos\theta \end{bmatrix}$$

and thus

$$\det J_\phi(r, \theta) = abr,$$

change of variables yields

$$\begin{aligned} \mu(E) &= \int_E 1 \\ &= \int_{[0,1]\times[0,2\pi]} abr \\ &= ab \int_0^1 \left(\int_0^{2\pi} r \, d\theta \right) dr \\ &= 2\pi ab \int_0^1 r \, dr \\ &= \pi ab. \end{aligned}$$

2. Let D in spherical coordinates be given as the solid lying between the spheres given by $r = 2$ and $r = 4$, above the xy-plane and below the cone given by the angle $\theta = \frac{\pi}{3}$. Evaluate the integral $\int_D xyz$.

Solution: In spherical coordinates, D is described as

$$\left\{ (r, \theta, \sigma) \in \mathbb{R}^3 : r \in [2, 4],\ \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2} \right], \sigma \in [0, 2\pi] \right\},$$

so that

$$\begin{aligned} \int_D xyz &= \int_2^4 \left(\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\int_0^{2\pi} (r\cos\theta\cos\sigma)(r\cos\theta\sin\sigma)(r\sin\theta) r^2 \cos\theta \, d\theta \right) d\sigma \right) dr \\ &= \left(\int_2^4 r^5 \, dr \right) \left(\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^3\theta \sin\theta \, d\theta \right) \left(\int_0^{2\pi} \cos\sigma \sin\sigma \, d\sigma \right). \end{aligned}$$

Since (substitute $u = \sin\sigma$)

$$\int_0^{2\pi} \sin\sigma \cos\sigma \, d\sigma = \int_0^0 u \, du = 0,$$

we have $\int_D xyz = 0$.

3. Let $K \subset \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(1,3)$, and $(0,3)$. Evaluate the line integral

$$\int_{\partial K} x^2 y^2 \, dx + 4xy^3 \, dy$$

where ∂K is oriented counterclockwise.

Solution: Note that

$$K = \{(x,y) \in \mathbb{R}^2 : x \in [0,1], \; y \in [3x, 3]\}.$$

Green's Theorem then yields

$$\begin{aligned}
\int_{\partial K} x^2 y^2 \, dx + 4xy^3 \, dy &= \int_K \frac{\partial}{\partial x} 4xy^3 - \frac{\partial}{\partial y} x^2 y^2 \\
&= \int_K 4y^3 - 2x^2 y \\
&= \int_0^1 \left(\int_{3x}^3 4y^3 - 2x^2 y \, dy \right) dx \\
&= \int_0^1 y^4 - x^2 y^2 \Big|_{3x}^3 \, dx \\
&= \int_0^1 81 - 9x^2 - 72x^4 \, dx \\
&= 81x - 3x^3 - \frac{72x^5}{5} \Big|_0^1 \\
&= \frac{318}{5}.
\end{aligned}$$

4. Let $\varnothing \neq U \subset \mathbb{R}^3$ be open, and let $f, g : U \to \mathbb{R}$ be twice continuously partially differentiable. Show that $\operatorname{div}(\nabla f \times \nabla g) = 0$ on U, where $\times$ denotes the cross product in $\mathbb{R}^3$.

Solution: First, note that

$$\nabla f \times \nabla g = \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}, -\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}, \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \right).$$

It follows that

$$\begin{aligned}
&\operatorname{div}(\nabla f \times \nabla g)\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\\
&= \frac{\partial^2 f}{\partial x\partial y}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial x\partial z} - \frac{\partial^2 f}{\partial x\partial z}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial x\partial y}\\
&\qquad - \frac{\partial^2 f}{\partial y\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial y\partial z} + \frac{\partial^2 f}{\partial y\partial z}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial y\partial x}\\
&\qquad\qquad + \frac{\partial^2 f}{\partial z\partial x}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial z\partial y} - \frac{\partial^2 f}{\partial z\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial z\partial x}\\
&= \frac{\partial f}{\partial x}\left(-\frac{\partial^2 g}{\partial y\partial z} + \frac{\partial^2 g}{\partial z\partial y}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial^2 g}{\partial x\partial z} - \frac{\partial^2 g}{\partial z\partial x}\right) + \frac{\partial f}{\partial z}\left(-\frac{\partial^2 g}{\partial x\partial y} + \frac{\partial^2 g}{\partial y\partial x}\right)\\
&\qquad + \frac{\partial g}{\partial x}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right) + \frac{\partial g}{\partial y}\left(-\frac{\partial^2 f}{\partial x\partial z} + \frac{\partial^2 f}{\partial z\partial x}\right) + \frac{\partial g}{\partial z}\left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)\\
&= 0
\end{aligned}$$

by Clairaut's Theorem.

5. Let

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad (x,y,z) \mapsto (x\cos^2 y + \arctan(yz)), (y+e^z), z\sin^2 y).$$

Evaluate $\int_S f\cdot n\,d\sigma$ where S is the sphere with radius $r>0$ centered at the origin, and n is the outward pointing normal unit vector.

Solution: Let V denote the solid ball with radius $r>0$ centered at the origin, so that $S = \partial V$. Gauß' Theorem asserts that

$$\int_S f\cdot n\,d\sigma = \int_V \operatorname{div} f.$$

As

$$\operatorname{div} f = \frac{\partial}{\partial x}(x\cos^2 y + \arctan(yz)) + \frac{\partial}{\partial y}(y+e^z) + \frac{\partial}{\partial z} z\sin^2 y = \cos^2 y + 1 + \sin^2 y = 2,$$

this means that

$$\int_S f\cdot n\,d\sigma = 2\,\mu(V) = \frac{8}{3}r^3\pi.$$

6*. Let $D \subset \mathbb{R}^2$ be the trapeze with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$. Evaluate $\int_D \exp\left(\frac{x+y}{x-y}\right)$. (*Hint*: Consider

$$\phi\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (u,v) \mapsto \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)$$

and apply Change of Variables.)

Solution: Let

$$K := \{(u,v) \in \mathbb{R}^2 : 1 \leq v \leq 2, \quad -v \leq u \leq v\}.$$

Then K is compact with content such that $\phi(K) = D$. Obviously, ϕ is injective, and as

$$\det J_\phi(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2},$$

the Change of Variables Theorem applies and yields

$$\begin{aligned}
\int_D \exp\left(\frac{x+y}{x-y}\right) &= \frac{1}{2}\int_D \exp\left(\frac{u}{v}\right) \\
&= \frac{1}{2}\int_1^2 \left(\int_{-v}^{v} \exp\left(\frac{u}{v}\right)\, du\right) dv \\
&= \frac{1}{2}\int_1^2 \left(v\exp\left(\frac{u}{v}\right)\Big|_{u=-v}^{u=v}\right) dv \\
&= \frac{1}{2}\int_1^2 \left(e - \frac{1}{e}\right) v\, dv \\
&= \frac{3}{4}\left(e - \frac{1}{e}\right).
\end{aligned}$$

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Solutions #2

1. For any set S, its *power set* $\mathfrak{P}(S)$ is defined to be the set consisting of all subsets of S. Show that there is *no* surjective map from S to $\mathfrak{P}(S)$. (*Hint*: Assume that there is a surjective map $f\colon S \to \mathfrak{P}(S)$ and consider the set $\{x \in S : x \notin f(x)\}$.)

   Solution: Assume there is a surjective map $f\colon S \to \mathfrak{P}(S)$, and let

   $$T := \{s \in S : s \notin f(s)\} \in \mathfrak{P}(S).$$

   Since f is surjective, there must be $s \in S$ such that $T = f(s)$. By the definition of T, we have

   $$s \in T \quad \Longleftrightarrow \quad s \notin f(s) = T,$$

   which is nonsense. Hence, there can be no surjective map $f\colon S \to \mathfrak{P}(S)$.

2. Which of the following sets are convex:

   (i) $\{(x, y) \in \mathbb{R}^2 : x > y\}$;

   (ii) $\{x \in \mathbb{R}^N : \|x\| > 2\}$;

   (iii) $\mathbb{R} \setminus \mathbb{Q}$;

   (iv) $\{(x, y, z) \in \mathbb{R}^3 : x + y + z \geq 2020\}$?

   Justify your answers.

   Solution: In each of the following, let C be the set under consideration.

   (a) Let $(x_1, y_1), (x_2, y_2) \in C$, and let $t \in [0, 1]$. It is clear that $t(x_1, y_1) + (1 - t)(x_2, y_2) \in C$ if $t = 0$ or $t = 1$. We may thus suppose without loss of generality that $t \in (0, 1)$. We have

   $$x_1 > y_1 \qquad \text{and} \qquad x_2 > y_2.$$

   Multiplying these inequalities with t and $1 - t$, respectively, we obtain

   $$tx_1 > ty_2 \qquad \text{and} \qquad (1 - t)x_2 > (1 - t)y_2.$$

   Adding these two inequalities, eventually yields

   $$tx_1 + (1 - t)x_2 > ty_1 + (1 - t)y_2,$$

   so that $t(x_1, y_1) + (1 - t)(x_2, y_2) \in C$. Hence, C is convex.

(b) Let $x \in C$. Then $\|-x\| = \|x\| > 2$, so that $-x \in C$ as well. Since

$$0 = \frac{1}{2}x + \frac{1}{2}(-x) \notin C,$$

the set C cannot be convex.

(c) Let $x, y \in C$, and suppose, without loss of generality, that $x < y$. As we have seen in class, there is $q \in (x, y) \cap \mathbb{Q}$. Set $t := \frac{y-q}{y-x}$, so that $t \in [0,1]$ and $q = tx + (1-t)y$. Hence, C is not convex.

(d) Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in C$, and let $t \in [0,1]$. Then

$$x_j + y_j + z_j \geq 2020$$

folds for $j = 1, 2$ and therefore

$$t(x_1 + y_1 + z_1) \geq t\,2020 \qquad \text{and} \qquad (1-t)(x_2 + y_2 + z_2) \geq (1-t)2020.$$

Adding these two inequalities yields

$$t(x_1 + y_1 + z_1) + (1-t)(x_2 + y_2 + z_2) \geq 2020.$$

Hence, C is convex.

3. Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^N$. Show that $\bigcap_{C \in \mathcal{C}} C$ is again convex. Is $\bigcup_{C \in \mathcal{C}} C$ necessarily convex?

Solution: Let $x, y \in \bigcap_{C \in \mathcal{C}} C$, i.e., $x, y \in C$ for each $C \in \mathcal{C}$. Let $t \in [0,1]$. Since each $C \in \mathcal{C}$ is convex, we have $tx + (1-t)y \in C$ for each $C \in \mathcal{C}$. Hence, $tx + (1-t)y \in \bigcap_{C \in \mathcal{C}} C$. Consequently, $\bigcap_{C \in \mathcal{C}} C$ is convex.

Let $x, y \in \mathbb{R}^N$ be such that $x \neq y$, and set $\mathcal{C} = \{\{x\}, \{y\}\}$. Then $\{x\}$ and $\{y\}$ are convex, but $\frac{1}{2}x + \frac{1}{2}y \notin \{x\} \cup \{y\}$.

4. Show that $\mathbb{Z}$ is closed in $\mathbb{R}$, but not open, and that $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed.

Solution: Let $x \in \mathbb{R} \setminus \mathbb{Z}$, and let $\lfloor x \rfloor$ be the largest integer less than or equal to x, e.g., $\lfloor 2 \rfloor = 2$, $\lfloor \pi \rfloor = 3$, or $\left\lfloor -\frac{9}{5} \right\rfloor = -5$. It follows that $\lfloor x \rfloor < x < \lfloor x \rfloor + 1$ (as $x \notin \mathbb{Z}$, the equalities must be strict). Set

$$\epsilon := \min\{x - \lfloor x \rfloor, \lfloor x \rfloor + 1 - x\},$$

so that

$$(x - \epsilon, x + \epsilon) \subset (\lfloor x \rfloor, \lfloor x \rfloor + 1).$$

It follows that $(x - \epsilon, x + \epsilon) \cap \mathbb{Z} = \varnothing$. Hence, $\mathbb{R} \setminus \mathbb{Z}$ is open, and $\mathbb{Z}$ is closed.

Assume that $\mathbb{Q}$ is open. Then, for any $q \in \mathbb{Q}$, there is $\epsilon > 0$ such that $(q-\epsilon, q+\epsilon) \subset \mathbb{Q}$. Choose $n \in \mathbb{N}$ so large that $\frac{\sqrt{13}}{n} < \epsilon$; it follows that $q + \frac{\sqrt{13}}{n} \in (q-\epsilon, q+\epsilon)$, but $q + \frac{\sqrt{13}}{n} \notin \mathbb{Q}$, which is a contradiction.

Assume that $\mathbb{Q}$ is closed, i.e., $\mathbb{R} \setminus \mathbb{Q}$ is open. Then, for any $x \in \mathbb{R} \setminus \mathbb{Q}$, there is $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset \mathbb{R} \setminus \mathbb{Q}$. In class, however, it was shown that there is a rational number between $x - \epsilon$ and $x + \epsilon$. Hence, $\mathbb{R} \setminus \mathbb{Q}$ cannot be open, so that $\mathbb{Q}$ is not closed.

5. Let $\varnothing \neq S \subset \mathbb{R}^N$ be arbitrary, and let $\varnothing \neq U \subset \mathbb{R}^N$ be open. Show that

$$S + U := \{x + y : x \in S,\, y \in U\}$$

is open.

Solution: Let $x \in S$, and define

$$x + U := \{x + y : y \in U\}.$$

We claim that $x + U$ is open. Let $\tilde{x} \in x + U$, so that $\tilde{x} - x \in U$. Let $\epsilon > 0$ be such that $B_\epsilon(\tilde{x} - x) \subset U$, and let $\tilde{y} \in \mathbb{R}^N$ be such that $\|\tilde{x} - \tilde{y}\| < \epsilon$. It follows that

$$\|(\tilde{y} - x) - (\tilde{x} - x)\| = \|\tilde{y} - \tilde{x}\| < \epsilon,$$

i.e., $\tilde{y} - x \in B_\epsilon(\tilde{x} - x) \subset U$ and thus $\tilde{y} \in x + U$. Hence, $x + U$ is open.

Since

$$S + U := \bigcup_{x \in S} (x + U),$$

it is clear that $S + U$ is also open.

6* For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, set

$$\|x\|_1 := |x_1| + \cdots + |x_N| \qquad \text{and} \qquad \|x\|_\infty := \max\{|x_1|, \ldots, |x_N|\}.$$

(a) Show that the following are true for $j = 1, \infty$, $x, y \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$:

(i) $\|x\|_j \geq 0$ and $\|x\|_j = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\|_j = |\lambda| \|x\|_j$;

(iii) $\|x + y\|_j \leq \|x\|_j + \|y\|_j$.

(b) For $N = 2$, sketch the sets of those x for which $\|x\|_1 \leq 1$, $\|x\| \leq 1$, and $\|x\|_\infty \leq 1$.

(c) Show that

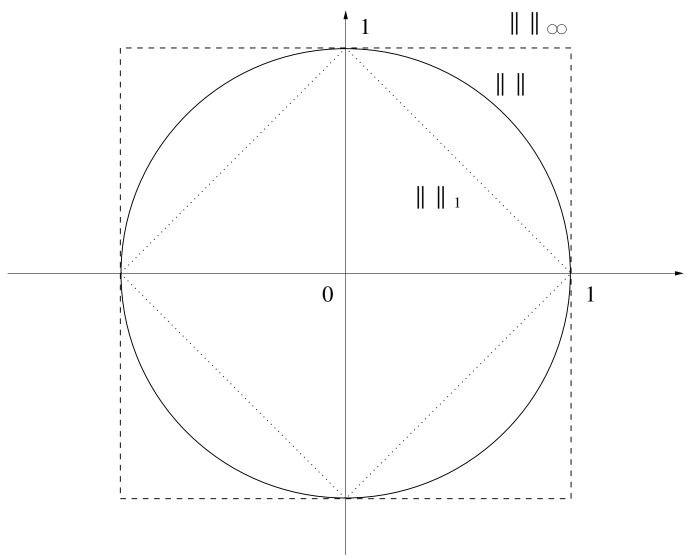
$$\|x\|_1 \leq \sqrt{N} \|x\| \leq N \, \|x\|_\infty$$

for all $x \in \mathbb{R}^N$.

Solution:

(a) The verification of (a) is routine (just use the corresponding properties of the absolute value on $\mathbb{R}$).

(b) Your sketch should look like this:



(c) Let $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, and let $y = (1, \ldots, 1)$. The Cauchy–Schwarz Inequality then yields that

$$\|x\|_1 = \sum_{j=1}^{N} |x_j y_j| \leq \|x\| \|y\| = \sqrt{N}\, \|x\|.$$

Moreover, we have

$$\|x\| = \sqrt{\sum_{j=1}^{N} x_j^2} \leq \sqrt{\sum_{j=1}^{N} \|x\|_\infty^2} = \sqrt{N}\, \|x\|_\infty.$$

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Honors Advanced Calculus, I

Solutions #3

1. Let $S \subset \mathbb{R}^N$. Show that $x \in \mathbb{R}^N$ is a cluster point of S if and only if each neighbourhood of x contains an infinite number of points in S.

   Solution: Let $x \in \mathbb{R}^N$ be a cluster point of S, and assume that there is a neighborhood U of x such that $U \cap S$ contains only finitely many. If $U \cap S = \{x\}$, then x cannot be a cluster point by definition, so suppose that $(U \cap S) \setminus \{x\}$ is a non-empty finite set. Define
   $$\epsilon := \min\{\|x - y\| : y \in (U \cap S) \setminus \{x\}\}.$$
   Then $\epsilon > 0$, and $U \cap B_\epsilon(x)$ is a neighborhood of x of which the intersection with S contains at most x. Hence, x cannot be a cluster point of S.

   For the converse, let U be any neighborhood of x. Then $U \cap S$ is infinite and therefore has to contain at least one point from $S \setminus \{x\}$.

2. Let $S \subset \mathbb{R}^N$ be any set. Show that ∂S is closed.

   Solution: Let $x \in \mathbb{R}^N \setminus \partial S$. Then there is $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \cap S = \varnothing$ or $B_{\epsilon_0}(x) \cap S^c = \varnothing$.

   Suppose that $B_{\epsilon_0}(x) \cap S = \varnothing$, and let $y \in B_{\epsilon_0}(x)$. Since $B_{\epsilon_0}(x)$ is open, there is $\epsilon > 0$ such that $B_\epsilon(y) \subset B_{\epsilon_0}(x)$; it follows that $B_\epsilon(y) \cap S = \varnothing$ as well, so that $y \notin \partial S$.

   The case where $B_{\epsilon_0}(x) \cap S^c = \varnothing$ is treated analogously.

3. Which of the following sets are compact:

   (a) $\{x \in \mathbb{R}^N : r \leq \|x\| \leq R\}$ with $0 < r < R$;

   (b) $\{(x, y) \in \mathbb{R}^2 : x - y \in [0, 1]\}$;

   (c) $\{(t \cos t, t \sin t) : t \in (0, \infty)\}$?

   Justify your answers.

   Solution: In each of the solutions let the set under consideration be denoted by K.

   (a) It is clear that K is bounded. Since
   $$K = B_R[x_0] \cap R_r(x_0)^c$$
   it is also closed and therefore compact by the Heine–Borel Theorem.

   (b) Since $(x, x - 1) \in K$ for each $x \in \mathbb{R}$, K is not bounded and thus not compact.

(c) We claim that K is not closed by showing that $(0,0)$ is a cluster point of K, but not in K. Since

$$\|(t\cos t, t\sin t)\| = \sqrt{t^2((\cos t)^2 + (\sin t)^2)} = t$$

for $t \in (0,\infty)$, it is clear that $(0,0) \notin K$. Let $\epsilon > 0$, and choose $t_0 \in (0,\epsilon)$; then we have

$$\|(0,0) - (t_0\cos t_0, t_0\sin t_0)\| = t_0 < \epsilon,$$

so that $(t_0\cos t_0, t_0\sin t_0) \in B_\epsilon((0,0)) \cap K$. Hence, $(0,0)$ is a cluster point of K.

Alternatively, one can observe that K is not bounded and thus not compact.

4. Show that:

(a) if $U_1 \subset \mathbb{R}^N$ and $U_2 \subset \mathbb{R}^M$ are open, then so is $U_1 \times U_2 \subset \mathbb{R}^{N+M}$;

(b) if $F_1 \subset \mathbb{R}^N$ and $F_2 \subset \mathbb{R}^M$ are closed, then so is $F_1 \times F_2 \subset \mathbb{R}^{N+M}$;

(c) if $K_1 \subset \mathbb{R}^N$ and $K_2 \subset \mathbb{R}^M$ are compact, then so is $K_1 \times K_2 \subset \mathbb{R}^{N+M}$.

Solution:

(a) Let $(x_0, y_0) \in U_1 \times U_2$. As U_1 and U_2 are open, there are $\epsilon_1, \epsilon_2 > 0$ such that $B_{\epsilon_1}(x_0) \subset U_1$ and $B_{\epsilon_2}(y_0) \subset U_2$. Set $\epsilon := \min\{\epsilon_1, \epsilon_2\}$. Let $(x,y) \in B_\epsilon((x_0,y_0))$. Then we have

$$\|x - x_0\| \leq \|(x,y)-(x_0,y_0)\| < \epsilon_1 \qquad \text{and} \qquad \|y-y_0\| \leq \|(x,y)-(x_0,y_0)\| < \epsilon_2$$

so that $(x,y) \subset B_{\epsilon_1}(x_0) \times B_{\epsilon_2}(y_0) \subset U_1 \times U_2$. Hence, $U_1 \times U_2$ is open.

(b) Note that

$$(F_1 \times F_2)^c = (\mathbb{R}^N \times F_2^c) \cup (F_1^c \times \mathbb{R}^M)$$

is open by (a), so that $F_1 \times F_2$ has to be closed.

(c) By (b), $K_1 \times K_2$ is closed. Let $r_1, r_2 > 0$ be such that $K_j \subset B_{r_j}[0]$ for $j = 1, 2$. For $(x,y) \in K_1 \times K_2$, it follows that

$$\|(x,y)\| = \left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}} \leq \sqrt{2}\max\{\|x\|, \|y\|\} \leq \sqrt{2}\max\{r_1, r_2\}.$$

so that $K_1 \times K_2 \subset B_{\sqrt{2}\max\{r_1,r_2\}}[0]$. Hence, $K_1 \times K_2$ is also bounded and thus compact by the Heine–Borel Theorem.

5. Show that a subset K of $\mathbb{R}^N$ is compact if and only if it has the *finite intersection property*, i.e., if $\{F_i : i \in \mathbb{I}\}$ is a family of closed sets in $\mathbb{R}^N$ such that $K \cap \bigcap_{i\in\mathbb{I}} F_i = \varnothing$, then there are $i_1, \ldots, i_n \in \mathbb{I}$ such that $K \cap F_{i_1} \cap \cdots \cap F_{i_n} = \varnothing$.

Solution: Suppose that K is compact and that $\{F_i : i \in \mathbb{I}\}$ is a family of closed sets in $\mathbb{R}^N$ such that $K \cap \bigcap_{i\in\mathbb{I}} F_i = \varnothing$. It follows that

$$K \subset \left(\bigcap_{i\in\mathbb{I}} F_i\right)^c = \bigcup_{i\in\mathbb{I}} F_i^c,$$

so that $\{F_i^c : i \in \mathbb{I}\}$ is an open cover for K. Since K is compact, there are $i_1, \ldots, i_n \in \mathbb{I}$ such that

$$K \subset F_{i_1}^c \cup \cdots \cup F_{i_n}^c = (F_{i_1} \cap \cdots \cap F_{i_n})^c,$$

and thus

$$K \cap F_{i_1} \cap \cdots \cap F_{i_n} = \varnothing.$$

Conversely, suppose that K has the finite intersection property, and let $\{U_i : i \in \mathbb{I}\}$ be an open cover for K, so that

$$K \cap \bigcap_{i\in\mathbb{I}} U_i^c = \varnothing.$$

It follows that there are $i_1, \ldots, i_n \in \mathbb{I}$ such that

$$K \cap U_{i_1}^c \cap \cdots \cap U_{i_n}^c = \varnothing$$

and thus

$$K \subset U_{i_1} \cup \cdots \cup U_{i_n}.$$

Hence, K is compact.

6*. For $j = 1, \ldots, N$, let $I_j = [a_j, b_j]$ with $a_j < b_j$, and let $I := I_1 \times \cdots \times I_N$. Determine ∂I. (*Hint*: Draw a sketch for $N = 2$ or $N = 3$.)

Solution: Since I is closed by part (b) of Problem 4, it is clear that $\partial I \subset I$.

For $j = 1, \ldots, N$ let

$$J_j := I_1 \times \cdots \times I_{j-1} \times \{a_j, b_j\} \times I_{j+1} \times \cdots \times I_N.$$

and let $J := J_1 \cup \cdots \cup J_N$.

We claim that $\partial I = J$.

It is immediate from this definition that

$$I \setminus J = (a_1, b_1) \times \cdots \times (a_N, b_N),$$

which is open by part (a) of Problem 4. Hence, for any $x \in I \setminus J$, there is $\epsilon > 0$ such that $B_\epsilon(x) \subset I \setminus J \subset I$. It follows that $B_\epsilon(x) \cap I^c = \varnothing$, so that x cannot be a boundary point. It follows that $\partial I \subset J$.

For the converse inclusion, let $x \in J$. Without loss of generality, suppose that $x \in J_1$, i.e., $x_1 = a_1$ or $x_1 = b_1$. Without loss of generality also suppose that $x_1 = a_1$. Let $\epsilon > 0$, and let $\delta < \min\{\epsilon, b_1 - a_1\}$. Define

$$y := (x_1 + \delta, x_2, \ldots, x_N) \qquad \text{and} \qquad z := (x_1 - \delta, x_2, \ldots, x_N).$$

Then $y, z \in B_\epsilon(x)$, but $y \in I$, whereas $z \notin I$. Hence, x is a boundary point of I.

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Honors Advanced Calculus, I

Solutions #4

1. For $0 \leq r \leq R$ and $\epsilon \in (0,1)$, determine whether or not the set

$$\left\{(x,y,z) \in \mathbb{R}^3 : r^2 \leq x^2 + y^2 \leq R^2,\ z^2 \in [\epsilon, 1]\right\}$$

is (a) open, (b) closed, (c) compact, or (d) connected.

Solution: Let the set under consideration be called S.

Let $((x_n, y_n, z_n))_{n=1}^\infty$ be a sequence in S converging to $(x,y,z) \in \mathbb{R}^3$. It follows that

$$r^2 \leq x_n^2 + y_n^2 \leq R^2 \qquad \text{and} \qquad z_n^2 \in [\epsilon, 1]$$

for all $n \in \mathbb{N}$. Since $x_n \to x$, $y_n \to y$, and $z_n \to z$, the properties of the limit in $\mathbb{R}$ and the fact that $[\epsilon, 1]$ is closed in $\mathbb{R}$ yield that

$$r^2 \leq x^2 + y^2 \leq R^2 \qquad \text{and} \qquad z^2 \in [\epsilon, 1],$$

so that $(x,y,z) \in S$. Consequently, S is closed.

Note that

$$x^2 + y^2 + z^2 \leq R^2 + 1,$$

for $(x,y,z) \in S$, so that $S \subset B_{\sqrt{R^2+1}}[(0,0,0)]$, i.e., S is bounded. Hence, S is compact by the Heine Borel Theorem.

As $\varnothing \neq S \neq \mathbb{R}^3$, it is clear that S cannot be open.

Finally, S is not connected because $\{U, V\}$ with

$$U := \left\{(x,y,z) \in \mathbb{R}^3 : z < 0\right\} \qquad \text{and} \qquad V := \left\{(x,y,z) \in \mathbb{R}^3 : z > 0\right\}$$

is a disconnection for S as one checks easily.

2. A set $S \subset \mathbb{R}^N$ is called *star shaped* if there is $x_0 \in S$ such that $tx_0 + (1-t)x \in S$ for all $x \in S$ and $t \in [0,1]$. Show that every star shaped set is connected, and give an example of a star shaped set that fails to be convex.

Solution: Let S be star shaped, and let $x_0 \in S$ be as in the definition. Assume that there is a disconnection $\{U, V\}$ of S. Without loss of generality suppose that $x_0 \in U$. Let $x \in V \cap S$, and set

$$\tilde{U} := \{t \in \mathbb{R} : tx_0 + (1-t)x \in U\} \qquad \text{and} \qquad \tilde{V} := \{t \in \mathbb{R} : tx_0 + (1-1)t \in V\}.$$

As in the proof for the connectedness of convex sets, one sees that $\left\{\tilde{U}, \tilde{V}\right\}$ is a disconnection for $[0,1]$, which is impossible.

Set, for instance,

$$S := \left\{(x,y) \in \mathbb{R}^2 : y \leq |x|\right\}.$$

For $(x,y) \in S$, i.e., such that $y \leq |x|$, and $t \in [0,1]$, we have $(1-t)y \leq |(1-t)x|$, so that $((1-t)x, (1-t)y) = t(0,0) + (1-t)(x,y) \in S$. Hence, S is star shaped. Clearly, $(1,1), (-1,1) \in S$ whereas

$$(0,1) = \frac{1}{2}(1,1) + \frac{1}{2}(-1,1) \notin S.$$

Hence, S is not convex.

3. Let $C \subset \mathbb{R}^N$ be connected. Show that $\overline{C}$ is also connected.

   Solution: Assume that there is a disconnection $\{U,V\}$ for $\overline{C}$. It is then obvious that $(C \cap U) \cap (C \cap V) = \varnothing$ and $(C \cap U) \cup (C \cup V) = C$. Assume that $C \cap U = \varnothing$, i.e., $C \subset U^c$. As U is open, U^c is closed, so that $\overline{C} \subset U^c$ as well, i.e., $\overline{C} \cap U = \varnothing$. But this is impossible because $\{U,V\}$ is a disconnection for $\overline{C}$. Similarly, one sees that $C \cap V \neq \varnothing$.

   All in all, $\{U,V\}$ is a disconnection for C, which is impossible because C is connected.

4. Let $S \subset \mathbb{R}^N$, and let $x \in \mathbb{R}^N$. Show that $x \in \overline{S}$ if and only if there is a sequence $(x_n)_{n=1}^\infty$ in S such that $x = \lim_{n\to\infty} x_n$.

   Solution: Suppose that there is a sequence $(x_n)_{n=1}^\infty$ in S such that $x = \lim_{n\to\infty} x_n$. As $(x_n)_{n=1}^\infty$ is also contained in $\overline{S}$ and since $\overline{S}$ is closed, it follows that $x \in \overline{S}$.

   Conversely, let $x \in \overline{S}$. If $x \in S$, there certainly is a sequence $(x_n)_{n=1}^\infty$ converging to x: just set $x_n := x$ for $n \in \mathbb{N}$. If $x \notin S$, then x must be a cluster point of S by the definition of $\overline{S}$, i.e., for each $n \in \mathbb{N}$, there is $x_n \in B_{\frac{1}{n}}(x) \cap S$, so that $x_n \to x$.

5. Let $(x_n)_{n=1}^\infty$ be a convergent sequence in $\mathbb{R}^N$ with limit x. Show that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact.

   Solution: Let $\{U_i : i \in \mathbb{I}\}$ be an open cover for $K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$. Choose $i_0 \in \mathbb{I}$ such that $x \in U_{i_0}$. Since U_{i_0} is open, it is a neighborhood of x. Hence, there is $n_0 \in \mathbb{N}$ such that $x_n \in U_{i_0}$ for all $n \geq n_0$. For $j = 1, \ldots, n_0 - 1$, choose $i_j \in \mathbb{I}$ such that $x_j \in U_{i_j}$. It follows that

   $$K \subset U_{i_0} \cup U_{i_1} \cup \cdots \cup U_{i_{n_0-1}},$$

   so that K is compact as claimed.

6*. Show that $\mathbb{R}^N \setminus \{0\}$ is disconnected if and only if $N = 1$.

Solution: If $N = 1$, then $\{(-\infty, 0), (0, \infty)\}$ is a disconnection for $S := \{x \in \mathbb{R}^N : x \neq 0\}$.

Let $N \geq 2$ and assume that there is a disconnection $\{U, V\}$ for S. Fix $x \in U \cap S$ and $y \in V \cap S$.

Suppose first that $x + t(y - x) \neq 0$ for all $t \in \mathbb{R}$. Define

$$\tilde{U} := \{t \in \mathbb{R} : x + t(y - x) \in U \cap S\}$$

and

$$\tilde{V} := \{t \in \mathbb{R} : x + t(y - x) \in V \cap S\}.$$

As in the proof of the connecteness of convex sets, one sees that $\left\{\tilde{U}, \tilde{V}\right\}$ is a disconnection for $\mathbb{R}$, which is not possible.

Suppose now that there is $t_0 \in \mathbb{R}^N$ such that $x + t_0(y - x) = 0$. Since $y \neq 0$, we have $t_0 \neq 1$ and thus $x = -\frac{t_0}{1-t_0}y$. Let $j \in \{1, \ldots, N\}$ be such that $y_j \neq 0$; then we have $-\frac{t_0}{1-t_0} = \frac{x_j}{y_j}$ and thus $x = \frac{x_j}{y_j}y$. Let $\epsilon > 0$ be such that $B_\epsilon(x) \subset U \cap S$. Fix $k \in \{1, \ldots, N\} \setminus \{j\}$, and define $\tilde{x} \in \mathbb{R}^N$ by letting

$$\tilde{x}_l := \begin{cases} x_l, & l \neq k, \\ x_k + \epsilon, & k = l, \end{cases}$$

for $l = 1, \ldots, N$. It follows that $\tilde{x} \in B_\epsilon(x) \subset U \cap S$. Assume that there is $\tilde{t}_0 \in \mathbb{R}$ such that $\tilde{x} + \tilde{t}_0(y - \tilde{x}) = 0$. Then—as before—it follows that

$$\tilde{x} = \frac{\tilde{x}_j}{y_j}y = \frac{x_j}{y_j}y = x,$$

which is impossible by the definition of $\tilde{x}$. Hence, $\tilde{x} + t(y - \tilde{x}) \neq 0$ must hold for all $t \in \mathbb{R}$, which is impossible as we just saw.

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Solutions #5

1. (a) Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}^N$ such that there is $\theta \in (0,1)$ with

$$\|x_{n+2} - x_{n+1}\| \leq \theta \|x_{n+1} - x_n\|$$

for $n \in \mathbb{N}$. Show that $(x_n)_{n=1}^\infty$ converges.

(*Hint*: Show first that

$$\|x_{n+1} - x_n\| \leq \theta^{n-1} \|x_2 - x_1\|$$

for $n \in \mathbb{N}$, and then use this and the fact that $\sum_{n=0}^\infty \theta^n$ converges to show that $(x_n)_{n=1}^\infty$ is a Cauchy sequence.)

(b) (*Banach's Fixed Point Theorem.*) Let $\varnothing \neq F \subset \mathbb{R}^N$ be closed, and let $f \colon F \to \mathbb{R}^N$ be such that $f(F) \subset F$ and that there is $\theta \in (0,1)$ with

$$\|f(x) - f(y)\| \leq \theta \|x - y\|$$

for $x, y \in F$. Show that there is a unique $x_0 \in F$ such that $f(x_0) = x_0$.

Solution:

(a) We use induction to prove that

$$\|x_{n+1} - x_n\| \leq \theta^{n-1} \|x_2 - x_1\|$$

for $n \in \mathbb{N}$. The claim is trivially true for $n = 1$. Suppose now that the claim has been proven for a particular $n \in \mathbb{N}$. Then

$$\|x_{n+2} - x_{n+1}\| \leq \theta \|x_{n+1} - x_n\| \leq \theta\, \theta^{n-1} \|x_2 - x_1\| = \theta^n \|x_2 - x_1\|$$

holds, which proves the claim for $n + 1$.

Let $m > n \geq 2$. We obtain:

$$\begin{aligned}
\|x_m - x_n\| &\leq \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\| \\
&= \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| \\
&\leq \sum_{k=n}^{m-1} \theta^{k-1} \|x_2 - x_1\| \\
&= \sum_{k=n-1}^{m-2} \theta^k \|x_2 - x_1\| \\
&= \|x_2 - x_1\| \left(\sum_{k=0}^{m-2} \theta^k - \sum_{k=0}^{n-2} \theta^k \right)
\end{aligned}$$

Let $\epsilon > 0$. Since $\sum_{n=0}^{\infty} \theta^n$ converges, $\left(\sum_{k=0}^{n} \theta^k\right)_{n=1}^{\infty}$ is a Cauchy sequence. Hence, there is $n_\epsilon \in \mathbb{N}$ such that

$$\left| \sum_{k=0}^{m-2} \theta^k - \sum_{k=0}^{n-2} \theta^k \right| < \frac{\epsilon}{\|x_2 - x_1\| + 1}$$

for $n, m \in \geq n_\epsilon$.

Let $n, m \geq n_\epsilon$. If $n = m$, we have $\|x_m - x_n\| = 0 < \epsilon$. If $n > m$, note that $\|x_m - x_n\| = \|x_n - x_m\|$ and switch the roles of n and m. Hence, we may suppose that $m > n$. We thus have

$$\|x_m - x_n\| \leq \|x_2 - x_1\| \left(\sum_{k=0}^{m-2} \theta^k - \sum_{k=0}^{n-2} \theta^k \right) < \frac{\epsilon \|x_2 - x_1\|}{\|x_2 - x_1\| + 1} < \epsilon.$$

Hence, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and therefore converges.

(b) Define $(x_n)_{n=1}^{\infty}$ inductively as follows. Let $x_1 \in F$ be arbitrary, and for $n \in \mathbb{N}$, set $x_{n+1} := f(x_n)$. It follows that

$$\|x_{n+2} - x_{n+1}\| = \|f(x_{n+1}) - f(x_n)\| \leq \theta \|x_{n+1} - x_n\|$$

for $n \in \mathbb{N}$. By Problem 4 on Assignment #4, $(x_n)_{n=1}^{\infty}$ converges to some $x_0 \in \mathbb{R}^N$, and as F is closed we have $x \in F$. Since f is continuous, we have

$$f(x_0) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} x_n = x_0.$$

This proves the existence of x_0.

To see that x_0 is unique, let $\tilde{x}_0 \in F$ be such that $f(\tilde{x}_0) = \tilde{x}_0$. It follows that

$$\|x_0 - \tilde{x}_0\| = \|f(x_0) - f(\tilde{x}_0)\| \leq \theta \|x_0 - \tilde{x}_0\|.$$

As $\theta \in (0, 1)$, this means that $\|x_0 - \tilde{x}_0\| = 0$ and thus $x_0 = \tilde{x}_0$.

2. Let $D := \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$, and let

$$f : D \to \mathbb{R}, \quad (x, y) \mapsto \frac{x^2}{y}$$

Show that:

(a) $\lim_{\substack{t \to 0 \\ t \neq 0}} f(tx_0, ty_0) = 0$ for all $(x_0, y_0) \in D$;

(b) $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.

Solution:

(a) Let $(x_0, y_0) \in D$. For $t \in \mathbb{R} \setminus \{0\}$, we then have that $(tx_0, ty_0) \in D$ as well such that

$$f(tx_0, ty_0) = \frac{t^2\, x_0^2}{t\, y_0} = t\frac{x_0^2}{y_0}$$

It follows that $\lim_{\substack{t \to 0 \\ t \neq 0}} f(tx_0, ty_0) = 0$.

(b) For $n \in \mathbb{N}$, set $(x_n, y_n) := \left(\frac{1}{n}, \frac{1}{n^2}\right)$, so that

$$f(x_n, y_n) = \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1.$$

It follows that $\lim_{n\to\infty} f(x_n, y_n) = 1$. Since by (a), $\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = 0$, we conclude that that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

3. Let $\varnothing \neq D \subset \mathbb{R}^N$ have the property that every continuous function $f : D \to \mathbb{R}$ is bounded. Show that D is compact.

Solution: Assume that D is not compact. By Heine–Borel, there are two possibilities.

Case 1: D is unbounded. Then

$$f \colon D \to \mathbb{R}, \quad x \mapsto \|x\|$$

is an unbounded continuous function.

Case 2: D is not closed, i.e., there is $x_0 \in \overline{D} \setminus D$. Then

$$f \colon D \to \mathbb{R}, \quad x \mapsto \frac{1}{\|x - x_0\|}$$

is an unbounded continuous function.

Both cases lead to contractions, so that D must be both closed and bounded, i.e., compact.

4. Let $\varnothing \neq D \subset \mathbb{R}^N$. A function $f \colon D \to \mathbb{R}^M$ is called *Lipschitz continuous* if there is $C \geq 0$ such that

$$\|f(x) - f(y)\| \leq C\|x - y\|$$

for all $x, y \in D$.

Show that:

(a) each Lipschitz continuous function is uniformly continuous;

(b) if $f : [a, b] \to \mathbb{R}$ is continuous such that f is differentiable on (a, b) with f' bounded on (a, b), then f is Lipschitz continuous;

(c) the function

$$f \colon [0, 1] \to \mathbb{R}, \quad x \mapsto \sqrt{x}$$

is uniformly continuous, but not Lipschitz continuous.

Solution:

(a) Suppose that, for $f\colon D \to \mathbb{R}^M$, there is $C \geq 0$ such that

$$\|f(x) - f(y)\| \leq C\|x - y\|$$

for all $x, y \in D$. Let $\epsilon > 0$, and choose $\delta := \frac{\epsilon}{C+1}$. For $x, y \in D$ with $\|x-y\| < \delta$, it follows that

$$\|f(x) - f(y)\| \leq C\|x - y\| < C\frac{\epsilon}{C+1} < \epsilon.$$

Hence, f is uniformly continuous.

(b) Set $C := \sup_{\xi \in (a,b)} |f'(\xi)|$. Let $x, y \in [a, b]$, and suppose without loss of generality that $x < y$. By the Mean Value Theorem, there is $\xi \in (x, y)$ such that

$$f'(\xi) = \frac{f(y) - f(x)}{y - x},$$

so that

$$|f(x) - f(y)| = |f'(\xi)||x - y| \leq C|x - y|.$$

(c) As f is continuous and as $[0, 1]$ is compact, it follows that f is uniformly continuous. Assume that there is $C \geq 0$ as in the definition of Lipschitz continuity. It then follows that

$$\frac{1}{2\sqrt{x}} = f'(x) \leq C$$

for $x \in (0, 1]$, which is impossible.

5. Let $C \subset \mathbb{R}^N$. We say that $x_0, x_1 \in C$ can be *joined by a path* if there is a continuous function $\gamma\colon [0, 1] \to \mathbb{R}^N$ with $\gamma([0, 1]) \subset C$, $\gamma(0) = x_0$, and $\gamma(1) = x_1$. We call C *path connected* if any two points in C can be joined by a path.

   Show that any path connected set is connected.

   Solution: Assume that C is not connected, i.e., there is a disconnection $\{U, V\}$ for C. Choose $x_0 \in U \cap C$ and $x_1 \in V \cap C$. Since C is path connected, there is a continuous function $\gamma\colon [0, 1] \to \mathbb{R}^N$ with $\gamma([0, 1]) \subset C$, $\gamma(0) = x_0$, and $\gamma(1) = x_1$. Since γ is continuous, there are open sets $\tilde{U}, \tilde{V} \subset \mathbb{R}$ such that

$$\tilde{U} \cap [0, 1] = \gamma^{-1}(U) \qquad \text{and} \qquad \tilde{V} \cap [0, 1] = \gamma^{-1}(V).$$

   It is easy to see that $\left\{\tilde{U}, \tilde{V}\right\}$ is a disconnection for $[0, 1]$, which is impossible.

6*. Let

$$C := \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x > 0\right\} \subset \mathbb{R}^2.$$

Show that $\overline{C}$ is connected, but not path connected. (*Hint*: Show that $\{0\} \times [-1,1] \in \overline{C}$ and that any point in $\{0\} \times [-1,1]$ cannot be joined by a path with any point of the form $\left(x, \sin\left(\frac{1}{x}\right)\right)$ with $x > 0$.)

Solution: The map

$$(0,\infty) \to \mathbb{R}^2, \quad t \mapsto \left(t, \sin\left(\frac{1}{t}\right)\right)$$

is continuous and has C as its range. As $(0,\infty)$ is connected, C is connected as is $\overline{C}$ by Solution 3 to Assignment #4.

Let $y \in [-1,1]$, and let $x_y > 0$ be such that $\sin x_y = y$. For $n \in \mathbb{N}$, let $x_n := \frac{1}{2n\pi + x_y}$. It follows that

$$\left(x_n, \sin\left(\frac{1}{x_n}\right)\right) = (x_n, \sin x_y) = (x_n, y) \to (0, y),$$

so that $(0,y) \in \overline{C}$.

Let $y \in [-1,1]$, let $t_0 > 0$, and suppose that there is a continuous function $\gamma = (\gamma_1, \gamma_2) : [0,1] \to \overline{C}$ such that $\gamma(0) = (0,y)$ and $\gamma(1) = \left(t_0, \sin\left(\frac{1}{t_0}\right)\right)$. Let $a := \sup\{t \in [0,1] : \gamma_1(t) = 0\}$. It follows that $\gamma_1(a) = 0$, $a \in [0,1)$, and $\gamma_2(t) = \sin\left(\frac{1}{\gamma_1(t)}\right)$ for $t \in (a,1]$. Consider

$$\tau : [0,1] \to [a,1], \quad t \mapsto a + t(1-a)$$

Then $\gamma \circ \tau$ is a path joining $(0, \gamma_2(a))$ with $\left(t_0, \sin\left(\frac{1}{t_0}\right)\right)$. Replacing γ by $\gamma \circ \tau$, we can thus suppose without loss of generality that $\gamma_1(t) > 0$ for all $t \in (0,1]$.

Let $n \in \mathbb{N}$, and note that $\lim_{t \to 0} \gamma_1(t) = 0 < \gamma_1\left(\frac{1}{n}\right)$. Choose $m_n \in \mathbb{N}$ such that:

- if n is even, then so is m_n, and if n is odd, so is m_n;
- $\frac{1}{m_n\pi + \frac{\pi}{2}} \leq \gamma_1\left(\frac{1}{n}\right)$.

Then use the Intermediate Value Theorem to find $t_n \in \left(0, \frac{1}{n}\right]$ such that $\gamma_1(t_n) = \frac{1}{m_n\pi + \frac{\pi}{2}}$

It follows that $t_n \to 0$, so that $\gamma(t_n) \to (0,y)$. However, we have

$$\gamma_2(t_n) = \sin\left(m_n\pi + \frac{\pi}{2}\right) = (-1)^n$$

for $n \in \mathbb{N}$, which does not converge as $n \to \infty$.

MATH 217 (Fall 2020)

Honors Advanced Calculus, I

Solutions #6

1. Determine the Jacobians of

$$\mathbb{R}^3 \to \mathbb{R}^3, \quad (r, \theta, \phi) \mapsto (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$$

and

$$\mathbb{R}^3 \to \mathbb{R}^3, \quad (r, \theta, z) \mapsto (r \cos\theta, r \sin\theta, z).$$

Solution: The first Jacobian is

$$\begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

and the second one

$$\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. An $N \times N$ matrix X is *invertible* if there is $X^{-1} \in M_N(\mathbb{R})$ such that $XX^{-1} = X^{-1}X = I_N$ where I_N denotes the unit matrix.

   (a) Show that $U := \{X \in M_N(\mathbb{R}) : X \text{ is invertible}\}$ is open. (*Hint*: $X \in M_N(\mathbb{R})$ is invertible if and only if $\det X \neq 0$.)

   (b) Show that the map

   $$f \colon U \to M_N(\mathbb{R}), \quad X \mapsto X^{-1}$$

   is totally differentiable on U, and calculate $Df(X_0)$ for each $X_0 \in U$. (*Hint*: You may use that, by Cramer's Rule, f is continuous.)

Solution:

(a) Since $\det \colon M_N(\mathbb{R}) \to \mathbb{R}$ is continuous and $\mathbb{R} \setminus \{0\}$ is open, $U = \det^{-1}(\mathbb{R} \setminus \{0\})$ is open.

(b) Let $X_0 \in U$. Since U is open by (i), $X_0 + H \in U$ for $\|H\|$ sufficiently small. Note that

$$\begin{aligned} (X_0 + H)^{-1} - X_0^{-1} &= -X_0^{-1}((X_0 + H) - X_0)(X_0 + H)^{-1} \\ &= -X_0^{-1} H (X_0 + H)^{-1}. \end{aligned}$$

Define

$$T\colon M_N(\mathbb{R}) \to M_N(\mathbb{R}), \quad X \mapsto -X_0^{-1} X X_0^{-1}.$$

For $\|H\|$ sufficiently small, we have

$$\begin{aligned} \frac{\|f(X_0+H) - f(X_0) - TH\|}{\|H\|} &= \frac{1}{\|H\|} \left\| X_0^{-1} H (X_0+H)^{-1} - X_0^{-1} H X_0^{-1} \right\| \\ &= \left\| X_0^{-1} \frac{H}{\|H\|} \left((X_0+H)^{-1} - X_0^{-1} \right) \right\|. \end{aligned}$$

As $\|H\| \to 0$, the term $X_0^{-1}\frac{H}{\|H\|}$ stays bounded whereas $(X_0+H)^{-1} - X_0^{-1} \to 0$ by the continuity of f. It follows that

$$\lim_{\|H\|\to 0} \frac{\|f(X_0+H) - f(X_0) - TH\|}{\|H\|} = 0.$$

Hence, f is differentiable at X_0 and $Df(X_0) = T$.

3. Let

$$p\colon (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \to \mathbb{R}^2, \quad (r,\theta) \mapsto (r\cos\theta, r\sin\theta),$$

let $\varnothing \neq U \subset \mathbb{R}^2$ be open, and let $f : U \to \mathbb{R}$ be twice continuously partially differentiable. Show that

$$(\Delta f) \circ p = \frac{\partial^2 (f\circ p)}{\partial r^2} + \frac{1}{r}\frac{\partial (f\circ p)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 (f\circ p)}{\partial \theta^2}$$

on $p^{-1}(U)$. (*Hint*: Apply the chain rule twice.)

Solution: First, note tht

$$J_p(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

The chain rule implies that

$$\begin{aligned} &\left(\frac{\partial (f\circ p)}{\partial r}(r,\theta), \frac{\partial (f\circ p)}{\partial \theta}(r,\theta) \right) \\ &= J_{f\circ p}(r,\theta) \\ &= J_f(p(r,\theta)) J_p(r,\theta) \\ &= \left(\cos\theta \frac{\partial f}{\partial x}(p(r,\theta)) + \sin\theta \frac{\partial f}{\partial y}(p(r,\theta)), -r\sin\theta\frac{\partial f}{\partial x}(p(r,\theta)) + r\cos\theta \frac{\partial f}{\partial y}(p(r,\theta)) \right), \end{aligned}$$

so that

$$\frac{\partial (f\circ p)}{\partial r}(r,\theta) = \cos\theta \frac{\partial f}{\partial x}(p(r,\theta)) + \sin\theta \frac{\partial f}{\partial y}(p(r,\theta))$$

and

$$\frac{\partial (f\circ p)}{\partial \theta}(r,\theta) = -r\sin\theta \frac{\partial f}{\partial x}(p(r,\theta)) + r\cos\theta \frac{\partial f}{\partial y}(p(r,\theta)).$$

It follows that

$$
\begin{aligned}
&\frac{\partial^2(f\circ p)}{\partial r^2}(r,\theta)\\
&=\cos\theta\frac{\partial}{\partial r}\frac{\partial f}{\partial x}(p(r,\theta))+\sin\theta\frac{\partial}{\partial r}\frac{\partial f}{\partial y}(p(r,\theta))\\
&=(\cos\theta)^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta))+\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))\\
&\qquad+\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))+(\sin\theta)^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&=(\cos\theta)^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta))+2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))+(\sin\theta)^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta))
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial^2(f\circ p)}{\partial \theta^2}(r,\theta)\\
&=\frac{\partial}{\partial\theta}\left(-r\sin\theta\frac{\partial f}{\partial x}(p(r,\theta))+r\cos\theta\frac{\partial f}{\partial y}(p(r,\theta))\right)\\
&=-r\cos\theta\frac{\partial f}{\partial x}(p(r,\theta))-r\sin\theta\frac{\partial}{\partial\theta}\frac{\partial f}{\partial x}(p(r,\theta))\\
&\qquad-r\sin\theta\frac{\partial f}{\partial y}(p(r,\theta))+r\cos\theta\frac{\partial}{\partial\theta}\frac{\partial f}{\partial y}(p(r,\theta))\\
&=\ -r\cos\theta\frac{\partial f}{\partial x}(p(r,\theta))+r^2(\sin\theta)^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta))-r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))\\
&\qquad-r\sin\theta\frac{\partial f}{\partial y}(p(r,\theta))-r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))+r^2(\cos\theta)^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&=\ -r\frac{\partial(f\circ p)}{\partial r}(r,\theta)+r^2(\sin\theta)^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta))-2r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))\\
&\qquad+r^2(\cos\theta)^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta)).
\end{aligned}
$$

This means that

$$
\begin{aligned}
&r^2\frac{\partial^2(f\circ p)}{\partial r^2}(r,\theta)+r\frac{\partial(f\circ p)}{\partial r}(r,\theta)+\frac{\partial^2(f\circ p)}{\partial \theta^2}(r,\theta)\\
&=r^2(\cos\theta)^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta))+2r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))+r^2(\sin\theta)^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&\qquad+r^2(\sin\theta)^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta))-2r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))\\
&\qquad+r^2(\cos\theta)^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&=r^2((\cos\theta)^2+(\sin\theta)^2)\frac{\partial^2 f}{\partial x^2}(p(r,\theta))+r^2((\cos\theta)^2+(\sin\theta)^2)\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&=r^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta))+r^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&=r^2(\Delta f)(p(r,\theta)).
\end{aligned}
$$

Division by r^2 then yields the claim.

4. Let

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \begin{cases} \frac{xy^3}{x^2+y^4}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Show that:

(a) f is continuous at $(0,0)$;

(b) for each $v \in \mathbb{R}^2$ with $\|v\| = 1$, the directional derivative $D_v f(0,0)$ exists and equals 0;

(c) f is not totally differentiable at $(0,0)$.

(*Hint for* (c): Assume towards a contradiction that f is totally differentiable at $(0,0)$, and compute the first derivative of $\mathbb{R} \ni t \mapsto f(t^2,t)$ at 0 first directly and then using the chain rule. What do you observe?)

Solution:

(a) Note that, for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have

$$|f(x,y)| = |y| \frac{\sqrt{x^2y^4}}{x^2+y^4} \leq |y| \frac{1}{2} \frac{x^2+y^4}{x^2+y^4} = \frac{|y|}{2}.$$

Hence, if $(x_n, y_n) \to 0$, it follows that $|f(x_n,y_n)| \leq \frac{|y_n|}{2} \to 0 = f(0,0)$.

(b) Let $v = (v_1, v_2)$ have norm one. For $t \neq 0$, we have

$$f(tv_1, tv_2) = \frac{t^4 v_1 v_2^3}{t^2(v_1^2 + t^2 v_2^4)} = t^2 \frac{v_1 v_2^3}{v_1^2 + t^2 v_2^4},$$

so that

$$\frac{f((0,0)+tv) - f(0,0)}{t} = t \frac{v_1 v_2^3}{v_1^2 + t^2 v_2^4}.$$

It follows that

$$D_v f(0,0) = \lim_{\substack{t \to 0 \\ t \neq 0}} \frac{f((0,0)+tv) - f(0,0)}{t} = 0.$$

(c) Let

$$g\colon \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto (t^2, t),$$

so that

$$(f \circ g)(t) = \frac{t^2 t^3}{t^4 + t^4} = \frac{t}{2}$$

for $t \in \mathbb{R}$ and thus $\left.\frac{d(f \circ g)}{dt}(t)\right|_{t=0} = \frac{1}{2}$.

Assume that f is totally differentiable at $(0,0)$. From (b), it is clear that $Df(0,0) = (0,0)$. The chain rule then yields that

$$\frac{d(f \circ g)}{dt}(t)\bigg|_{t=0} = Df(g(0))Dg(0) = (0,0)Dg(0) = 0,$$

which is a contradiction.

5. Let $x, y \in \mathbb{R}$. Show that there is $\theta \in [0,1]$ such that

$$\sin(x+y) = x + y - \frac{1}{2}(x^2 + 2xy + y^2)\sin(\theta(x+y)).$$

Solution: Let

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \sin(x+y).$$

By Taylor's Theorem, there is $\theta \in [0,1]$, such that

$$f(x,y) = f(0,0) + (\text{grad } f)(0,0) \cdot (x,y) + \frac{1}{2}(\text{Hess } f)(\theta x, \theta y)(x,y) \cdot (x,y).$$

Clearly, $f(0,0) = 0$ holds. Since

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = \cos(x+y),$$

we have

$$(\text{grad } f)(0,0) \cdot (x,y) = (1,1) \cdot (x,y) = x + y.$$

Moreover, since

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = -\sin(x+y)$$

we also have

$$\begin{aligned}
&(\text{Hess } f)(\theta x, \theta y)(x,y) \cdot (x,y) \\
&= \left(\begin{bmatrix} -\sin(\theta(x+y)) & -\sin(\theta(x+y)) \\ -\sin(\theta(x+y)) & -\sin(\theta(x+y)) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) \cdot \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -\sin(\theta(x+y))(x+y) \\ -\sin(\theta(x+y))(x+y) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \\
&= -(x^2 + 2xy + y^2)\sin(\theta(x+y)).
\end{aligned}$$

Hence,

$$\sin(x+y) = x + y - \frac{1}{2}(x^2 + 2xy + y^2)\sin(\theta(x+y))$$

holds.

6*. Let $\varnothing \neq C \subset \mathbb{R}^N$ be open and connected, and let $f \colon C \to \mathbb{R}$ be differentiable such that $\nabla f \equiv 0$. Show that f is constant. (*Hint*: First, treat the case where C is convex using the chain rule; then, for general C, assume that f is not constant, let $x, y \in C$ such that $f(x) \neq f(y)$, and show that $\{U, V\}$ with $U := \{z \in C : f(z) = f(x)\}$ and $V := \{z \in C : f(z) \neq f(x)\}$ is a disconnection for C.)

Solution: First, suppose that C is convex, and assume that f is not constant, i.e., there are $x, y \in C$ with $f(x) \neq f(x)$. Since C is convex, $\{x + t(y - x) : t \in [0, 1]\}$ is contained in C. Define

$$g \colon [0,1] \to \mathbb{R}, \quad t \mapsto f(x + t(y - x)).$$

Then g is continuous and differentiable on $(0, 1)$. The chain rule yields

$$g'(t) = (\nabla f(x + t(y - x))) \cdot (y - x) = 0$$

for $t \in (0, 1)$. From one variable calculus, we know that this means that g is constant. However, we have $g(0) = f(x) \neq f(y) = g(1)$, which is a contradiction.

For the general case, assume that f is not constant, and let $x, y \in C$ such that $f(x) \neq f(y)$. Define

$$U := \{z \in C : f(z) = f(x)\} \qquad \text{and} \qquad V := \{z \in C : f(z) \neq f(x)\}.$$

As f is continuous, there is an open set $\tilde{V} \subset \mathbb{R}^N$ such that $V = C \cap \tilde{V}$. Since C is also open, this means that V is open.

We claim that U is open as well. Let $z \in U$, and choose $\epsilon > 0$ such that $B_\epsilon(z) \subset C$. As $B_\epsilon(z)$ is convex, it follows from the convex case that f is constant on $B_\epsilon(z)$, i.e., $f(z') = f(x)$ for all $z' \in B_\epsilon(z)$, so that $B_\epsilon(x) \subset U$. As $z \in U$ is arbitrary, this proves the claim.

By definition, $U \neq \varnothing \neq V$, $U \cap V = \varnothing$, and $U \cup V = C$. Hence, $\{U, V\}$ is a disconnection for C, which is a contradiction.

MATH 217 (Fall 2020)

Honors Advanced Calculus, I

Solutions #7

1. Determine and classify all stationary points of

$$f\colon (-\pi,\pi)\times(-3,4)\to\mathbb{R},\quad (x,y)\mapsto(3+2\cos x)\cos y.$$

If f attains a local minimum or maximum at one of its stationary points, evaluate it there.

Solution: The first order partial derivatives of f are computed as

$$\frac{\partial f}{\partial x}=-2(\sin x)\cos y \qquad\text{and}\qquad \frac{\partial f}{\partial y}=-(3+2\cos x)\sin y.$$

Since $3+2\cos x\neq 0$ for all $x\in\mathbb{R}$, a necessary and sufficient condition for $\frac{\partial f}{\partial y}(x,y)=0$ is that $\sin y=0$, i.e., $y\in\pi\mathbb{Z}$. Since $y\in(-3,4)$, this means that $y\in\{0,\pi\}$. Since $\cos y\neq 0$ for those y, we require that $\sin x=0$ in order to have $\frac{\partial f}{\partial x}(x,y)=0$, i.e., $x=0$ (because $x\in\pi\mathbb{Z}\cap(-\pi,\pi)$).

Hence, $(0,0)$ and $(0,\pi)$ are the only stationary points of f.

The next step is to compute Hess f. We have

$$\frac{\partial^2 f}{\partial x^2}=-2(\cos x)\cos y,\qquad \frac{\partial^2 f}{\partial y^2}=-(3+2\cos x)\cos y,$$

and

$$\frac{\partial^2 f}{\partial x\,\partial y}=\frac{\partial^2 f}{\partial x\,\partial y}=2(\sin x)(\sin y),$$

so that

$$(\operatorname{Hess} f)(0,0)=\begin{bmatrix}-2 & 0\\ 0 & -5\end{bmatrix}$$

and

$$(\operatorname{Hess} f)(0,\pi)=\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}.$$

Hence, $(\operatorname{Hess} f)(0,0)$ is negative definite, so that f attains a local maximum at $(0,0)$, namely 5, whereas $(\operatorname{Hess} f)(0,\pi)$ is positive definite, so that f attains a local minimum at $(0,\pi)$, namely -5.

2. Determine and classify the stationary points of

$$f\colon\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R},\quad (x,y)\mapsto\frac{1}{y}-\frac{1}{x}-4x+y.$$

If f attains a local minimum or maximum at a stationary point, evaluate the function there.

Solution: We have

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{x^2} - 4 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{1}{y^2} + 1.$$

Hence, the set of stationary points of f is

$$\left\{ \left(\frac{1}{2}, 1\right), \left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, -1\right), \left(-\frac{1}{2}, -1\right) \right\}$$

Since

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\frac{2}{x^3}, \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{2}{y^3},$$

and

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = 0,$$

we have

$$(\text{Hess } f)(x,y) = \begin{bmatrix} -\frac{2}{x^3} & 0 \\ 0 & \frac{2}{y^3} \end{bmatrix}.$$

It follows that $(\text{Hess } f)(x,y)$ is indefinite at $\left(\frac{1}{2}, 1\right)$ and $\left(-\frac{1}{2}, -1\right)$—so that f has saddles at those points—, positive definite at $\left(-\frac{1}{2}, 1\right)$, and negative definite at $\left(\frac{1}{2}, -1\right)$. Hence, f has a local minimum at $\left(-\frac{1}{2}, 1\right)$, namely $f\left(-\frac{1}{2}, 1\right) = 6$, and a local maximum at $\left(\frac{1}{2}, -1\right)$, namely $f\left(\frac{1}{2}, -1\right) = -6$.

3. Determine the minimum and the maximum of

$$f \colon D \to \mathbb{R}, \quad (x,y) \mapsto \sin x + \sin y + \sin(x+y),$$

where $D := \left\{(x,y) \in \mathbb{R}^2 : 0 \le x, y \le \frac{\pi}{2}\right\}$, and all points of D where they are attained.

Solution: By the compactness of D and the continuity of f, the function attains both a minimum and a maximum on D.

Note that $\operatorname{int} D = \left\{(x,y) \in \mathbb{R}^2 : 0 < x, y < \frac{\pi}{2}\right\}$. We start with classifying the stationary points of f on $\operatorname{int} D$.

First, determine the gradient of f:

$$\frac{\partial f}{\partial x}(x,y) = \cos x + \cos(x+y) \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = \cos y + \cos(x+y).$$

For $(x,y) \in \operatorname{int} D$ to be a stationary point, it is thus necessary and sufficient that

$$\cos x + \cos(x+y) = 0 = \cos y + \cos(x+y)$$

or, equivalently, that

$$\cos x = \cos y = -\cos(x+y).$$

Since cos in injective on $\left(0, \frac{\pi}{2}\right)$, this means that $x = y$ and thus $\cos x = -\cos(2x)$. For $x \in \left(0, \frac{\pi}{2}\right)$, this is possible only if $x = \frac{\pi}{3}$. Hence, $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is the only stationary point of f.

Next, we calculate the Hessian:

$$(\text{Hess } f)(x,y) = \begin{bmatrix} -\sin x - \sin(x+y) & -\sin(x+y) \\ -\sin(x+y) & -\sin y - \sin(x+y) \end{bmatrix}.$$

Since

$$-\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

and

$$\left(\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)\right)^2 - \left(\sin\left(\frac{2\pi}{3}\right)\right)^2 = 3 - \frac{3}{4} > 0$$

the Hessian matrix is negative definite at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$. Hence, f has a local maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, namely $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3}{2}\sqrt{3}$.

Therefore, we know that f attains a local maximum in int D, which is the only local extremum there. We thus have to check the behaviour of f on ∂D.

Let

$$\begin{aligned}
f_1 &: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}, \quad x \mapsto f(x,0) = 2\sin x; \\
f_2 &: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}, \quad y \mapsto f\left(\frac{\pi}{2}, y\right) = 1 + \sin y + \cos y; \\
f_3 &: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}, \quad y \mapsto f\left(x, \frac{\pi}{2}\right) = 1 + \sin x + \cos x; \\
f_4 &: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}, \quad x \mapsto f(0,y) = 2\sin y.
\end{aligned}$$

It is immediate that f_1 and f_4 attain their respective minimum—0—at 0 and their respective maximum—2—at $\frac{\pi}{2}$.

Since

$$f_2'(y) = \cos y - \sin y,$$

there is only one candidate for a local extremum of f_2 on $\left(0, \frac{\pi}{2}\right)$, namely $y = \frac{\pi}{4}$. We have

$$f_2(0) = f_3(0) = f_2\left(\frac{\pi}{4}\right) = f_3\left(\frac{\pi}{4}\right) = 2 \quad \text{and} \qquad f_2\left(\frac{\pi}{4}\right) = f_3\left(\frac{\pi}{4}\right) = 1 + \sqrt{2}.$$

Any extremal point of f which is not in int D, must lie on the boundary and hence be either one of $\left\{(0,0), \left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$ or a local extremal point of f_1, f_2, f_3, or f_4. Comparing the values of f at those possible values, we obtain that

- f attains its minimum—0—at $(0,0)$;
- f attains its maximum—$\frac{3}{2}\sqrt{3}$—at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

4. Let $(x_n)_{n=1}^\infty$ be a convergent sequence in $\mathbb{R}^N$ with limit x. Show that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ has content zero.

   Solution: Let $\epsilon > 0$, and choose $a_1, b_1, \ldots, a_N, b_N$ with $a_j < b_j$ for $j = 1, \ldots, N$ such that
   $$x \in (a_1, b_1) \times \cdots (a_N, b_N) =: J_0 \qquad \text{and} \qquad \prod_{j=1}^N b_j - a_j < \frac{\epsilon}{2}.$$
   As $\lim_{n\to\infty} x_n = x$, and since J_0 is a neighborhood of x, there is $n_0 \in \mathbb{N}$ such that $x_n \in J_0$ for all $n \geq n_0$. Set
   $$I_0 := [a_1, b_1] \times \cdots [a_N, b_N]$$
   Then I_0 is a compact interval in $\mathbb{R}^N$ with
   $$\{x_n : n \geq n_0\} \cup \{x\} \subset I_0 \qquad \text{and} \qquad \mu(I_0) < \frac{\epsilon}{2}.$$
   As a finite set, $\{x_1, \ldots, x_{n_0-1}\}$ has content zero, i.e., there are compact intervals $I_1, \ldots, I_m \subset \mathbb{R}^N$ such that
   $$\{x_1, \ldots, x_{n_0-1}\} \subset \bigcup_{j=1}^m I_j \qquad \text{and} \qquad \sum_{j=1}^m \mu(I_j) < \frac{\epsilon}{2}.$$
   It follows that
   $$\{x_n : n \in \mathbb{N}\} \cup \{x\} \subset \bigcup_{j=0}^m I_j \qquad \text{and} \qquad \sum_{j=0}^m \mu(I_j) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
   As $\epsilon > 0$ is arbitrary, we conclude that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ has content zero.

5. Let $I \subset \mathbb{R}^N$ be a compact interval. Show that ∂I has content zero.

   Solution: Let
   $$I = [a_1, b_1] \times \cdots \times [a_N, b_N].$$
   For $j = 1, \ldots, N$ and $x \in \mathbb{R}^N$, set
   $$S_{j,x} := [a_1, b_1] \times \cdots \times [a_{j-1}, b_{j-1}] \times \{x\} \times [a_{j+1}, b_{j+1}] \times \cdots \times [a_N, b_N]$$
   In Problem 6* on Assignment #3, you showed that
   $$\partial I = \bigcup_{j=1}^N S_{j,a_j} \cup S_{j,b_j}$$

It is therefore sufficient to show that, $\mu(S_{j,x}) = 0$ for any $j = 1, \ldots, N$ and $x \in \mathbb{R}$. Let $\epsilon > 0$, and let

$$J := [a_1, b_1] \times \cdots \times [a_{j-1}, b_{j-1}] \times [x - \delta, x + \delta] \times [a_{j+1}, b_{j+1}] \times \cdots \times [a_N, b_N],$$

where

$$\delta < \frac{1}{2} \prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{\epsilon}{b_k - a_k}.$$

We then have

$$S_{j,x} \subset J \qquad \text{and} \qquad \mu(J) = 2\delta \prod_{\substack{k=1 \\ k \neq j}}^{N} (b_k - a_k) < \epsilon,$$

so that $\mu(S_{j,x}) = 0$.

6*. Let $I_1, \ldots, I_n \subset \mathbb{R}$ be compact intervals such that $\mathbb{Q} \cap [0,1] \subset I_1 \cup \cdots \cup I_n$. Show that $\sum_{j=1}^{n} \mu(I_j) \geq 1$.

Solution: Let $\epsilon > 0$. For $j = 1, \ldots, n$ and $I_j = [a_j, b_j]$ with $0 \leq a_j$ and $b_j \leq 1$, set $J_j := (a_j - \epsilon, b_j + \epsilon)$. We claim that $[0,1] \subset J_1 \cup \cdots \cup J_n$. To see this, let $x \in [0,1]$. Then there is $q \in \mathbb{Q} \cap [0,1]$ such that $|x - q| < \epsilon$, i.e., $q - \epsilon < x < q + \epsilon$. Let $j_q \in \{1, \ldots, n\}$ be such that $q \in I_{j_q}$, i.e., $a_{j_q} \leq q \leq b_{j_q}$. It follows that

$$a_{j_q} - \epsilon \leq q - \epsilon < x < q + \epsilon \leq b_{j_q} + \epsilon,$$

i.e., $x \in J_{j_q}$.

Let $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $\{t_0, t_1, \ldots, t_m\}$ consists precisely of 0 and 1 and those boundary points of $J_1, \ldots, J_n$ that lie in $[0,1]$. Then we obtain that

$$\begin{aligned}
1 = \sum_{k=1}^{m} t_k - t_{k-1} &\leq \sum_{j=1}^{n} \sum_{(t_{k-1}, t_k) \subset J_j} t_k - t_{k-1} \\
&\leq \sum_{k=1}^{n} (b_n + \epsilon) - (a_n - \epsilon) = 2n\epsilon + \sum_{k=1}^{n} b_k - a_k = 2n\epsilon + \sum_{k=1}^{n} \mu(I_k).
\end{aligned}$$

As $\epsilon > 0$ is arbitrary, this yields the claim.

MATH 217 (Fall 2020)
Honors Advanced Calculus, I

Solutions #8

1. Let I be a compact interval, and let $f = (f_1, \ldots, f_M) : I \to \mathbb{R}^M$. Show that f is Riemann integrable if and only if $f_j : I \to \mathbb{R}$ is Riemann integrable for each $j = 1, \ldots, M$ and that, in this case,

$$\int_I f = \left(\int_I f_1, \ldots, \int_I f_M \right)$$

holds.

Solution: Suppose that f is Riemann integrable. Fix $k \in \{1, \ldots, M\}$, and let $y = (y_1, \ldots, y_M)$ be the Riemann integral of f over I. Let $\epsilon > 0$. Then there is a partition $\mathcal{P}_\epsilon$ of I such that, for each refinement $\mathcal{P}$ of $\mathcal{P}_\epsilon$ and each associated Riemann sum $S(f, \mathcal{P})$, we have

$$|S(f_k, \mathcal{P}) - y_k| \leq \|S(f, \mathcal{P}) - y\| < \epsilon.$$

This means that f_k is Riemann integrable with $\int_I f_k = y_k$.

Conversely, suppose that f_j is Riemann integrable with integral y_j for $j = 1, \ldots, M$. Set $y := (y_1, \ldots, y_M)$. Let $\epsilon > 0$. For each $j = 1, \ldots, M$, there is a partition $\mathcal{P}_j$ of I such that, for each refinement $\mathcal{P}$ of $\mathcal{P}_j$, we have

$$|S(f_j, \mathcal{P}) - y_j| < \frac{\epsilon}{\sqrt{M}}$$

for each Riemann sum $S(f_j, \mathcal{P})$. Let $\mathcal{P}_\epsilon$ be a common refinement of $\mathcal{P}_1, \ldots, \mathcal{P}_M$. Then for every refinement $\mathcal{P}$ of $\mathcal{P}_\epsilon$ and each Riemann sum $S(f, \mathcal{P})$, we obtain

$$\|S(f, \mathcal{P}) - y\| \leq \sqrt{M} \max_{j=1,\ldots,M} |S(f_j, \mathcal{P}) - y_j| < \sqrt{M} \frac{\epsilon}{\sqrt{M}} = \epsilon.$$

Consequently, f is Riemann integrable with $\int_I f = y$.

2. Let $I \subset \mathbb{R}^N$ be a compact interval, and let $f : I \to \mathbb{R}^M$ be Riemann integrable. Show that f is bounded.

Solution: Assume towards a contradiction that f is not bounded.

Let $\mathcal{P}$ be a partition of I—with corresponding subdivision $(I_\nu)_\nu$ of I—such that

$$\left\| S(f, \mathcal{P}) - \int_I f \right\| < 1$$

for each Riemann sum $S(f, \mathcal{P})$ of f corresponding to $\mathcal{P}$. In particular, this means that

$$\|S(f, \mathcal{P})\| \leq 1 + \left\| \int_I f \right\| =: C$$

for each such Riemann sum $S(f,\mathcal{P})$. Since f is assumed to be unbounded and since $I = \bigcup_\nu I_\nu$, there is at least one ν_0 such that f is unbounded on I_{ν_0}. Choose $x_{\nu_0} \in I_{\nu_0}$ such that

$$\|f(x_{\nu_0})\| > \frac{1}{\mu(I_{\nu_0})}\left(C + \left\|\sum_{\nu \neq \nu_0} f(x_\nu)\mu(I_\nu)\right\|\right).$$

For the Riemann sum

$$S_0(f,\mathcal{P}) := \sum_\nu f(x_\nu)\mu(I_\nu),$$

we thus obtain

$$\begin{aligned}
\|S_0(f,\mathcal{P})\| &= \left\|\sum_\nu f(x_\nu)\mu(I_\nu)\right\| \\
&\geq \left|\|f(x_{\nu_0})\|\mu(I_{\nu_0}) - \left\|\sum_{\nu \neq \nu_0} f(x_\nu)\mu(I_\nu)\right\|\right| \\
&= \|f(x_{\nu_0})\|\mu(I_{\nu_0}) - \left\|\sum_{\nu \neq \nu_0} f(x_\nu)\mu(I_\nu)\right\| \\
&> C.
\end{aligned}$$

which is impossible.

3. Let $\varnothing \neq D \subset \mathbb{R}^N$ be bounded, and let $f, g \colon D \to \mathbb{R}$ be Riemann-integrable. Show that $fg \colon D \to \mathbb{R}$ is Riemann-integrable.

   Do we necessarily have
   $$\int_D fg = \left(\int_D f\right)\left(\int_D g\right)?$$

   (*Hint*: First, treat the case where $f = g$ and then the general case by observing that $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2.)$

   Solution: Without loss of generality suppose that D is a compact interval I.

   Let $C \geq 0$ such that $|f(x)| \leq C$ for $x \in I$. Let $\epsilon > 0$ and let $\mathcal{P}_\epsilon$ be a partition of I such that
   $$|S_1(f,\mathcal{P}_\epsilon) - S_2(f,\mathcal{P}_\epsilon)| < \frac{\epsilon}{2(C+1)}$$
   for all Riemann sums $S_1(f,\mathcal{P}_\epsilon)$ and $S_2(f,\mathcal{P}_\epsilon)$ corresponding to $\mathcal{P}_\epsilon$. Let $(I_\nu)_\nu$ the subdivision of I induced by $\mathcal{P}_\epsilon$, and let $x_\nu, y_\nu \in I_\nu$ be support points. As in the proof of Proposition 4.2.12(iii), one sees that
   $$\sum_\nu |f(x_\nu) - f(y_\nu)|\mu(I_\nu) < \frac{\epsilon}{2(C+1)}.$$

It follows that

$$\begin{aligned}\sum_\nu |f(x_\nu)^2 - f(y_\nu)^2|\mu(I_\nu) &= \sum_\nu |f(x_\nu)+f(y_\nu)||f(x_\nu)-f(y_\nu)|\mu(I_\nu)\\ &\leq \sum_\nu 2C|f(x_\nu)-f(y_\nu)|\mu(I_\nu)\\ &< 2C\frac{\epsilon}{2(C+1)}\\ &< \epsilon.\end{aligned}$$

Hence, f^2 is Riemann-integrable by Corollary 4.2.6.

For Riemann-integrable $f, g\colon I \to \mathbb{R}$, we have

$$fg = \frac{1}{2}((f+g)^2 - f^2 - g^2),$$

so that fg is also Riemann-integrable.

However, we have, for instance,

$$\int_0^1 x^2\,dx = \frac{1}{3} \neq \frac{1}{4} = \left(\int_0^1 x\,dx\right)^2.$$

4. Let $\varnothing \neq D \subset \mathbb{R}^N$ have content zero, and let $f\colon D \to \mathbb{R}^M$ be bounded. Show that f is Riemann-integrable on D such that

$$\int_D f = 0.$$

Solution: Let $C \geq 0$ be such that $\|f(x)\| \leq C$ for $x \in D$.

Let $I \subset \mathbb{R}^N$ be a compact interval such that $D \subset I$, and extend f to $\tilde{f}\colon I \to \mathbb{R}^M$ as pointed out in class. Let $\epsilon > 0$, and choose a partition $\mathcal{P}$ of I with corresponding subdivision $(I_\nu)_\nu$ of I such that

$$\sum_{I_\nu \cap D \neq \varnothing} \mu(I_\nu) < \frac{\epsilon}{C+1}.$$

Let $\mathcal{Q}$ be a refinement of $\mathcal{P}$ with corresponding subdivision $(J_\lambda)_\lambda$. It follows that

$$\sum_{J_\lambda \cap D \neq \varnothing} \mu(J_\lambda) < \frac{\epsilon}{C+1}.$$

For each λ, pick a support point $y_\lambda \in J_\lambda$. Then we have

$$\left\|\sum_\lambda \tilde{f}(y_\lambda)\mu(J_\lambda)\right\| = \left\|\sum_{J_\lambda \cap D \neq \varnothing} f(y_\lambda)\mu(J_\lambda)\right\| \leq C \sum_{J_\lambda \cap D \neq \varnothing} \mu(J_\lambda) < \epsilon.$$

It follows that $\int_D f = 0$.

5. Let $\varnothing \neq U \subset \mathbb{R}^N$ be open with content, and let $f : U \to [0, \infty)$ be bounded and continuous such that $\int_U f = 0$. Show that $f \equiv 0$ on U.

Solution: Assume that there is $x_0 \in U$ such that $f(x_0) \neq 0$, i.e., $f(x_0) > 0$. By the continuity of f, there is $\delta > 0$, such that $B_\delta(x_0) \subset U$ and $f(x) > \frac{f(x_0)}{2}$ for all $x \in B_\delta(x_0)$. Let

$$J := \left[x_{0,1} - \frac{\delta}{3\sqrt{N}}, x_{0,1} + \frac{\delta}{3\sqrt{N}}\right] \times \cdots \times \left[x_{0,N} - \frac{\delta}{3\sqrt{N}}, x_{0,N} + \frac{\delta}{3\sqrt{N}}\right],$$

so that $J \subset B_\delta(x_0)$. We thus obtain

$$\int_I f \geq \int_I f\chi_J = \int_J f \geq \int_J \frac{f(x_0)}{2} = \frac{f(x_0)}{2}\mu(J) > 0,$$

which is a contradiction.

6*. The function

$$f\colon [0,1] \times [0,1] \to \mathbb{R}, \quad (x,y) \mapsto xy$$

is continuous and thus Riemann integrable. Evaluate $\int_{[0,1]\times[0,1]} f$ using only the definition of the Riemann integral, i.e., in particular, without use of Fubini's Theorem.

Solution: For $n \in \mathbb{N}$, let

$$\mathcal{P}_n := \left\{\frac{j}{n} : j = 0, \ldots, n\right\} \times \left\{\frac{k}{n} : k = 0, \ldots, n\right\}.$$

For $(j,k) \in \{0, \ldots, n\}$, let $x_{j,k} := \left(\frac{j}{n}, \frac{k}{n}\right)$. The corresponding Riemann sum is then

$$\begin{aligned} S_n(f, \mathcal{P}_n) &= \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{jk}{n^2}\frac{1}{n^2} \\ &= \frac{1}{n^4}\left(\sum_{j=1}^{n} j\right)\left(\sum_{k=1}^{n} k\right) \\ &= \frac{1}{n^4}\frac{n^2(n+1)^2}{4} \\ &\to \frac{1}{4}. \end{aligned}$$

We claim that $\int_{[0,1]^2} f = \frac{1}{4}$.

Let $\epsilon > 0$, and choose $\delta > 0$ such that $|(f(x,y) - f(x',y')| < \frac{\epsilon}{3}$ for all $(x,y),(x',y') \in [0,1]^2$ such that $\|(x,y) - (x',y')\| < \delta$. Choose a partition $\mathcal{P}_0$ of I such that the following are true for the corresponding subdivision $(I_\nu)_\nu$ of $[0,1]^2$:

- if $(x,y),(x',y') \in I_\nu$ for some ν, then $\|(x,y) - (x',y')\| < \delta$;

- if $\mathcal{P}$ is any refinement of $\mathcal{P}_0$, then $\left|S(f,\mathcal{P}) - \int_I f\right| < \frac{\epsilon}{3}$ for any Riemann sum $S(f,\mathcal{P})$ corresponding to $\mathcal{P}$).

Choose $n_0 \in \mathbb{N}$ be such that the following are true for the corresponding subdivision $(J_\mu)_\mu$ of $[0,1]^2$:

- if $(x,y),(x',y') \in J_\mu$ for some μ, then $\|(x,y)-(x',y')\| < \delta$;
- for any $n \geq n_0$, we have $\left|\frac{1}{4} - S_n(f,\mathcal{P}_n)\right| < \frac{\epsilon}{3}$.

Let $\mathcal{Q}$ be any common refinement of $\mathcal{P}_0$ and $\mathcal{P}_{n_0}$, and let $(K_\lambda)_\lambda$ be the corresponding partition of $[0,1]^2$, and let $S(f,\mathcal{Q})$ be a corresponding Riemann sum. Then we have

$$\begin{aligned}\left|\frac{1}{4} - \int_{[0,1]^2} f\right| &\leq \underbrace{\left|\frac{1}{4} - S_{n_0}(f,\mathcal{P}_{n_0})\right|}_{<\frac{\epsilon}{3}} - |S_{n_0}(f,\mathcal{P}_{n_0}) - S(f,\mathcal{Q})| + \underbrace{\left|S(f,\mathcal{Q}) - \int_{[0,1]^2 f}\right|}_{<\frac{\epsilon}{3}} \\ &< \frac{2}{3}\epsilon + |S_{n_0}(f,\mathcal{P}_{n_0}) - S(f\mathcal{Q})|\end{aligned}$$

Let $S(f,\mathcal{Q}) = \sum_\lambda f(x_\lambda)\mu(K_\lambda)$ with $x_\lambda \in K_\lambda$), and $S_{n_0}(f,\mathcal{P}_{n_0}) = \sum_\nu f(y_\nu)\mu(I_\nu)$. It follows that

$$\begin{aligned}|S_{n_0}(f,\mathcal{P}_{n_0}) - S(f\mathcal{Q})| &= \left|\sum_\nu f(y_\nu)\mu(I_\nu) - \sum_\lambda f(x_\lambda)\mu(K_\lambda)\right| \\ &\leq \sum_\nu \sum_{K_\lambda \subset I_\nu} \underbrace{|f(y_\nu) - f(x_\lambda)|}_{<\frac{\epsilon}{3}} \mu(K_\lambda) \\ &< \frac{\epsilon}{3},\end{aligned}$$

so that, all in all, $\left|\frac{1}{4} - \int_{[0,1]^2} f\right| < \epsilon$. As $\epsilon > 0$ was arbitrary, this means that $\int_{[0,1]^2} f = \frac{1}{4}$ as claimed.

MATH 217 (Fall 2020)

Honors Advanced Calculus, I

Solutions #9

1. Let $I \subset \mathbb{R}^N$ be a compact interval. Show that

$$\mathcal{A} := \{A \subset I : A \text{ has content}\}$$

is an *algebra* over I, i.e.,

(a) $\varnothing, I \in \mathcal{A}$,

(b) if $A \in \mathcal{A}$, then $I \setminus A \in \mathcal{A}$, and

(c) if $A_1, \ldots, A_n \in \mathcal{A}$, then $A_1 \cup \cdots \cup A_n \in \mathcal{A}$.

Solution: As the constant functions $0 = \chi_\varnothing$ and $1 = \chi_I$ are trivially Riemann integrable on I, (a) is clear.

Let $A \in \mathcal{A}$, i.e., χ_A is Riemann integrable on I. Consequently, $\chi_{I \setminus A} = \chi_I - \chi_A$ is Riemann integrable, so that $I \setminus A \in \mathcal{A}$.

For (c), we may suppose that $n = 2$. So, let $A, B \in \mathcal{A}$. By (b), this means that $I \setminus A, I \setminus B \in \mathcal{A}$. As $\chi_{(I \setminus A) \cap \chi(I \setminus B)} = \chi_{I \setminus A}\chi_{I \setminus B}$, it follows from Problem 3 on Assignment #8, that $(I \setminus A) \cap (I \setminus B) \in \mathcal{B}$ and—by (b) again—$A \cup B = I \setminus ((I \setminus A) \cap (I \setminus B)) \in \mathcal{A}$.

2. Define

$$f\colon [0,1]^3 \to \mathbb{R}, \quad (x,y,z) \mapsto \begin{cases} xy, & z \leq xy, \\ z, & z \geq xy. \end{cases}$$

Evaluate $\int_{[0,1]^3} f$.

Solution: By Fubini's Theorem, we have

$$\int_{[0,1]^3} f = \int_0^1 \left(\int_0^1 \left(\int_0^1 f(x,y,z)\, dz \right) dy \right) dx.$$

Let $(x,y) \in [0,1]^2$, so that $xy \in [0,1]$. Consequently, we obtain for the innermost integral that

$$\int_0^1 f(x,y,z)\, dz = \int_0^{xy} xy\, dz + \int_{xy}^1 z\, dz = x^2y^2 + \left[\frac{z^2}{2}\right]_{z=xy}^{z=1} = \frac{1}{2}(x^2y^2 + 1)$$

It follows that

$$\begin{aligned}
\int_{[0,1]^3} f &= \int_0^1 \left(\int_0^1 \frac{1}{2} x^2 y^2 + 1 \, dy \right) dx \\
&= \frac{1}{2} \int_0^1 \left(\int_0^1 x^2 y^2 \, dy \right) dx + \frac{1}{2} \\
&= \frac{1}{2} \left(\int_0^1 x^2 \, dx \right) \left(\int_0^1 y^2 \, dy \right) + \frac{1}{2} \\
&= \frac{1}{18} + \frac{1}{2} \\
&= \frac{5}{9}.
\end{aligned}$$

3. Let

$$D := \{(x, y) \in \mathbb{R} : x, y \geq 0, \ x^2 + y^2 \leq 1\},$$

and let

$$f \colon D \to \mathbb{R}, \quad (x, y) \mapsto \frac{4y^3}{(x+1)^2}$$

Evaluate $\int_D f$.

Solution: Define $\phi, \psi \colon [0,1] \to \mathbb{R}$ through

$$\phi(x) = 0 \qquad \text{and} \qquad \psi(x) = \sqrt{1 - x^2}$$

for $x \in [0,1]$, so that

$$D = \{(x, y) \in\in \mathbb{R} : x \in [0,1], \ \phi(x) \leq y \leq \psi(x)\}.$$

It follows that

$$\begin{aligned}
\int_D f &= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \frac{4y^3}{(x+1)^2} \, dy \right) dx \\
&= \int_0^1 \left(\frac{y^4}{(x+1)^2} \bigg|_{y=0}^{y=\sqrt{1-x^2}} \right) dx \\
&= \int_0^1 \frac{(1-x^2)^2}{(x+1)^2} \, dx \\
&= \int_0^1 (1-x)^2 \, dx \\
&= -\frac{(1-x)^3}{3} \bigg|_{x=0}^{x=1} \\
&= \frac{1}{3}.
\end{aligned}$$

4. Let $a < b$, let $f\colon [a,b] \to [0,\infty)$ be continuous, and let

$$D := \{(x,y) : x \in [a,b],\ y \in [0, f(x)]\}.$$

Show that D has content and that

$$\mu(D) = \int_a^b f(x)\,dx.$$

Solution: Note that

$$\begin{aligned}\partial D = \{(a,y) : y \in [0, f(a)]\} \\ \cup \{(x, f(x)) : x \in [a,b]\} \cup \{(b,y) : y \in [0, f(b)]\} \cup \{(x,0) : x \in [a,b]\}.\end{aligned}$$

Each of the sets on the right hand side of this equality has content zero, so that ∂D has content zero, and D has content.

From Fubini's Theorem, we obtain that

$$\begin{aligned}\mu(D) &= \int_D 1 \\ &= \int_a^b \left(\int_0^{f(x)} dy \right) dx \\ &= \int_a^b f(x)\,dx.\end{aligned}$$

5. Let $R > 0$, and define, for $0 < \rho < R$,

$$A_{\rho,R} := \{(x,y,z) \in \mathbb{R}^3 : \rho^2 \le x^2 + y^2 + z^2 \le R^2\}.$$

Determine

$$\lim_{\rho \to 0} \int_{A_{\rho,R}} \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

Solution: Use spherical coordinates. This means that, for $0 < \rho < R$, we have $A_{\rho,R} = \phi(K)$ where

$$K := \left\{ (r,\theta,\sigma) \in \mathbb{R}^3 : r \in [\rho, R],\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \theta \in [0, 2\pi] \right\}.$$

It follows that

$$
\begin{aligned}
\int_{A_{\rho,R}} \frac{1}{\sqrt{x^2+y^2+z^2}} &= \int_K \frac{r^2\cos\theta}{r} \\
&= \int_K r\cos\theta \\
&= \int_\rho^R \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2\pi} r\cos\theta \, d\sigma \right) d\theta \right) dr \\
&= 2\pi \int_\rho^R \left(r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\sigma d\theta \right) dr \\
&= 4\pi \int_\rho^R r \, dr \\
&= 2\pi(R^2-\rho^2) \\
&\overset{\rho\to 0}{\to} 2\pi R^2.
\end{aligned}
$$

6*. Define $f\colon [0,1]\times[0,1] \to \mathbb{R}$ by letting

$$
f(x,y) = \begin{cases} 2^{2n}, & \text{if } (x,y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n}, 2^{-n+1}) \text{ for some } n \in \mathbb{N}, \\ -2^{2n+1}, & \text{if } (x,y) \in [2^{-n-1}, 2^{-n}) \times [2^{-n}, 2^{-n+1}) \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}
$$

Show that the iterated integrals

$$
\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx \qquad \text{and} \qquad \int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy
$$

both exist, but that

$$
\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx \neq \int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy.
$$

Why doesn't this contradict Fubini's Theorem?

Solution: Fix $y_0 \in [0,1)$; let $n \in \mathbb{N}$ be such that $y_0 \in [2^{-n}, 2^{-n+1})$. We then have that

$$
f(x,y_0) = \begin{cases} 2^{2n}, & \text{if } x \in [2^{-n}, 2^{-n+1}), \\ -2^{2n+1}, & \text{if } x \in [2^{-n-1}, 2^{-n}), \\ 0, & \text{otherwise} \end{cases}
$$

and therefore

$$
\int_0^1 f(x,y_0)\, dx = \int_{2^{-n}}^{2^{-n+1}} 2^{2n}\, dx - \int_{2^{-n-1}}^{2^{-n}} 2^{2n+1}\, dx = 2^n - 2^n = 0.
$$

All in all,

$$
\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy = 0
$$

holds. Similarly, if $x_0 \in \left[0, \frac{1}{2}\right)$, we obtain

$$\int_0^1 f(x_0, y)\, dy = 0.$$

If, however, $x_0 \in \left[\frac{1}{2}, 1\right)$, we get

$$f(x_0, y) = \begin{cases} 4, & \text{if } y \in \left[\frac{1}{2}, 1\right) \\ 0, & \text{otherwise} \end{cases}$$

Therefore, we have

$$\int_0^1 \left(\int_0^1 f(x, y)\, dy \right) dx = \int_{\frac{1}{2}}^1 \left(\int_{\frac{1}{2}}^1 4\, dy \right) dx = 1.$$

As f is unbounded, it cannot be Riemann integral. Hence, Fubini's Theorem does not apply.