Math 227 – Review for Final Exam: Problems from past exams (or similar to such problems)

Problem 1. Let A be a subset of a vector space. Prove that $\mathrm{span}(A)$ is the smallest subspace containing A.

Problem 2. (I) Each of the following examples gives a linear map f from a vector space V_1 to another vector space V_2, where V_1 and V_2 are vector spaces over the same field $\mathbb{F}$. In addition, each of these linear maps is either onto or 1-1 or both. Determine which properties each of the following linear maps possesses (and justify your answer).

(a) The linear map $f : \mathbb{Z}_7^3 \to \mathbb{Z}_7^3$ given by

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{Z}_7^3 \quad \mapsto \quad f(\bar{x}) := \begin{pmatrix} 6 & 0 & 3 \\ 2 & 2 & 5 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{Z}_7^3.$$

(b) The trace operator from $\mathbb{Z}_5^{4\times 4}$ to $\mathbb{Z}_5$, defined by

$$A = (a_{ij})_{1\leqslant i,j\leqslant 4} \in \mathbb{Z}_5^{4\times 4} \quad \mapsto \quad \mathrm{tr}(A) := \sum_{i=1}^{4} a_{ii}.$$

(c) The linear map $f : \mathbb{C}^m \to \mathbb{C}^n$ given by

$$\bar{w} \in \mathbb{C}^m \quad \mapsto \quad f(\bar{w}) := \begin{pmatrix} \sqrt{2} & 1-\boldsymbol{i} \\ 3 & 3e \\ 1 & e \end{pmatrix} \bar{w} \in \mathbb{C}^n$$

(determine also the values of m and n here).

(d) The operator $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ which we call the "left shift" and is defined by

$$(x_n)_{n\in\mathbb{N}} = (x_1, x_2, x_3, x_4, \dots) \in \mathbb{R}^{\mathbb{N}} \mapsto T\big((x_1, x_2, x_3, x_4, \dots)\big) := (x_2, x_3, x_4, x_5, \dots).$$

(e) The operator $S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ which we call the "right shift" and is defined by

$$(x_n)_{n\in\mathbb{N}} = (x_1, x_2, x_3, x_4, \dots) \in \mathbb{R}^{\mathbb{N}} \mapsto S\big((x_1, x_2, x_3, x_4, \dots)\big) := (0, x_1, x_2, x_3, x_4, \dots).$$

(f) The linear extension f from the space of real polynomials $\mathcal{P}$ to itself of the function

$$\phi(x^{2i}) = x^{2i+1} \quad \text{and} \quad \phi(x^{2i+1}) = x^{2i} \quad \text{for every } i \in \mathbb{Z}_{\geqslant 0}$$

(observe that ϕ is indeed a function from the standard basis $\{1,\, x,\, x^2,\, x^3, \ldots\}$ of $\mathcal{P}$ into $\mathcal{P}$, so we know that it has a unique linear extension $f : \mathcal{P} \to \mathcal{P}$ (recall the relevant theorem here); to better understand this linear extension f, consider also some specific examples of polynomials in $\mathcal{P}$, and determine their images under f).

(II) Under what conditions for the vector spaces V_1 and V_2 are we allowed to say that, if we already know that f is onto, then we also have that f is 1-1? Or similarly, if we already know that f is 1-1, then we also have that f is onto? (You may wish to review Main Theorem E here.)

Problem 3. Consider the operators T and S from $\mathbb{R}^{\mathbb{N}}$ to itself that were defined in the previous problem.

(i) Show that **every** real number λ is an eigenvalue of the left shift T, and for each such λ find an eigenvector of T corresponding to λ.

(ii) Show that the right shift S has **no** eigenvalues.

(iii) If Id is the identity operator on $\mathbb{R}^{\mathbb{N}}$, show that $T \circ S = \mathrm{Id}$ but $S \circ T \neq \mathrm{Id}$.

Problem 4. Let $Q : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be given by

$$\begin{aligned}(x_n)_{n\in\mathbb{N}} = (x_1, x_2, x_3, x_4, x_5 \ldots) &\in \mathbb{R}^{\mathbb{N}} \\ &\mapsto \quad Q\big((x_1, x_2, x_3, x_4, x_5, \ldots)\big) = (2x_1, 0, 2x_3, 0, 2x_5, 0, \ldots).\end{aligned}$$

Show that Q is linear. Is Q onto? One-to-one? Find $\mathrm{Ker}(Q)$ and $\mathrm{Range}(Q)$. Identify all the eigenvalues of Q and the eigenspaces corresponding to them.

Problem 5. Let A be a 5×3 matrix with entries from a field $\mathbb{F}$, and let B be the 3×3 matrix made up of the first three rows of A. Show that, if $\det B \neq 0$, then the columns of A are $\mathbb{F}$-linearly independent.

Problem 6. Find a 2×2 *real* matrix with eigenvalues $1 \pm 2i$. (Partial credit will be given for a *complex* matrix.)

Problem 7. Let A be a matrix in $\mathbb{C}^{4\times 4}$ which has **only real** entries, and suppose that $\frac{4}{5} + \frac{3}{5}i$ and $\frac{3}{5} + \frac{4}{5}i$ are eigenvalues of A.

(i) Show that A is diagonalisable (over $\mathbb{C}$) (<u>*hint:*</u> observe that $p_A(t)$ will be a polynomial with real coefficients (why?); but then, if $z \in \mathbb{C}$ is a root of $p_A(t)$, what can you say about the value $p_A(\bar{z})$, where $\bar{z}$ is the conjugate of z?).

(ii) Show that every eigenvalue λ of A^9 satisfies $|\lambda| = 1$.

Problem 8. Let $\mathbb{F}$ be a field, let A, B be matrices in $\mathbb{F}^{n\times n}$, and suppose that A and B are similar. Prove that, for every $k \geqslant 2$, A^k and B^k are also similar.

Problem 9. Give the definition of an inner product space (distinguish between the real and the complex case).

Recall the following

Definitions. (I) A matrix $A \in \mathbb{R}^{n\times n}$ is called *symmetric* if $A = A^T$.
(II) A matrix $B \in \mathbb{C}^{n\times n}$ is called *Hermitian* if $B = B^*$.

Alternative Terminology. A symmetric or Hermitian matrix is sometimes also called *self-adjoint.*

Problem 10. (i) Let A be a matrix in $\mathbb{R}^{n\times n}$. Show that, for all $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, we have $\langle A\tilde{x}, \tilde{y}\rangle = \langle \tilde{x}, A^T\tilde{y}\rangle$ (where $\langle \cdot, \cdot\rangle$ is the standard inner product on $\mathbb{R}^n$).

(ii) Let B be a matrix in $\mathbb{C}^{n\times n}$. Show that, for all $\tilde{v}, \tilde{w} \in \mathbb{C}^n$, we have $\langle B\tilde{v}, \tilde{w}\rangle = \langle \tilde{v}, B^*\tilde{w}\rangle$ (where $\langle \cdot, \cdot\rangle$ is the standard inner product on $\mathbb{C}^n$; *be careful about how this is defined, in particular, that its definition is slightly different from that of the standard inner product on* $\mathbb{R}^n$).

(iii) Consider now a self-adjoint matrix $E \in \mathbb{C}^{n\times n}$. Prove that $\langle E\tilde{u}, \tilde{u}\rangle$ is a real number for every $\tilde{u} \in \mathbb{C}^n$.

Recall the following

Definition. Let $\big(V, \langle \cdot, \cdot, \rangle\big)$ be a (real or complex) inner product space, and let S be a subspace of V. We can define the *orthogonal complement* of S by

$$S^\perp := \{\bar{w} \in V : \text{for every } \bar{y} \in S,\ \bar{w} \perp \bar{y}\} = \{\bar{w} \in V : \text{for every } \bar{y} \in S,\ \langle \bar{w}, \bar{y}\rangle = 0\}.$$

Recall that you are asked in HW6, Problem 2 to prove that $S^\perp$ will be a subspace of V too, and that we will have $V = S \oplus S^\perp$.

Problem 11. Let $V = \mathbb{R}^{4\times 4}$.
(i) Verify that we can turn V into a real inner product space by setting

$$A, B \in \mathbb{R}^{4\times 4} \quad \mapsto \quad \langle A, B\rangle := \mathrm{tr}(AB^T),$$

that is, verify that this is an inner product on V *(you may also wish to recall HW5, Problem 1 here, where you were asked to verify a useful formula for this inner product in terms of the matrix entries).*

(ii) Let S be the subspace of **upper triangular** matrices in V which have **zero trace**. Find a basis for it, as well as a basis for $S^{\perp}$.

(iii) Recall that S should be isomorphic to a vector space of the form $\mathbb{R}^n$ for some n. Determine n here, and define a linear isomorphism from S to $\mathbb{R}^n$ (explain also why the linear map you will define is indeed an isomorphism).

Problem 12. Let Y be the solution set of the linear system

$$\begin{cases} x_1 \qquad\quad - x_3 - 6x_4 = 0 \\ \qquad x_2 - x_3 \qquad\quad = 0 \\ x_1 - x_2 \qquad\quad - 6x_4 = 0 \end{cases},$$

where the coefficients are taken from $\mathbb{R}$. Consider also the following vectors from $\mathbb{R}^4$:

$$\bar{u} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ -19 \end{pmatrix}, \; \bar{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 14 \end{pmatrix}, \quad \text{and} \quad \bar{w} = \begin{pmatrix} 8 \\ 6 \\ 1 \\ -4 \end{pmatrix}.$$

(a) Show that Y is a subspace of $\mathbb{R}^4$.

(b) Determine the dimensions of Y and of $Y^{\perp}$.

(c) Find a basis of Y.

(d) Find an orthonormal basis of Y, and extend it to an orthonormal basis of $\mathbb{R}^4$.

(e) Find the point of Y that is the closest to $\bar{u}$ (that is, find the element $\bar{y}_0$ of Y such that $\|\bar{y}_0 - \bar{u}\|$ is smallest possible among elements of Y; *hint: recall how we can find $\|\bar{x}\|$, for any given $\bar{x} \in \mathbb{R}^4$, when we have an orthonormal basis of the space*).

(f) True or False: $[\bar{u}]_Y + [\bar{v}]_Y = [\bar{w}]_Y$ in $\mathbb{R}^4/Y$ *(justify your answer).*

Problem 13. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(1) = 1$ and $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
True or False: we necessarily have $f(x) = x$ for every $x \in \mathbb{R}$ *(justify your answer, that is, prove the statement if true, or give a counterexample if false).*