MATH 117 Assignment #1: Sets and logic

Due 11:59 pm Saturday September 14, 2019

1. [3 pts] **(a)** Give the set of all subsets of $\{a, b, c\}$.

(b) Give the set of all subsets of $\{\{\}\}$.

2. [2 pts]

(a) True or false: $\{a, b, \{a, b\}\} = \{a, b\}$

(b) True or false: $\{1, 2, 3, 2\} = \{2, 3, 1\}$

3. [2 pts] For this question let $A = \{n \in \mathbb{Z} \,|\, -3 \leq n < 5\}$ and $B = \{k \in \mathbb{N} \,|\, k \leq 10\}$. Find $A \cup B$ and $A \cap B$. (NOTE: '$<$' means "less than", and '$\leq$' means "less than or equal". So for example, $2 < 3$, $2 \leq 3$, $2 \leq 2$).

4. [1 pt] Find two different elements of the set $\{n \in \mathbb{N} \,|\, n = k^2 \text{ for some } k \in \mathbb{Z}\}$.

5. [6 pts] Say whether each of these is **true** or **false**. Justify your answer.

(a) $2 \in \mathbb{N}$ AND $-2 \notin \mathbb{Z}$

(b) $2 \in \mathbb{N}$ OR $-10 \in \mathbb{Z}$

(c) IF $1 > 2$ THEN $1 + 1 = 2$

(d) $1 > 2$ IFF $1 + 1 = 2$

(e) IF $1 > 2$ THEN $-2 \in \mathbb{N}$

(f) IF $1 < 2$ THEN $1 + 1 = 2$

6. [4 pts] For any sets A and B, prove:

$$\text{If } A \subseteq B, \text{ then } B = A \cup B.$$

MATH 117. Solutions to Assignment #1

1(a) [2 pts] The subsets of $\{a, b, c\}$ are

$$\{\}\,,\quad \{a\}\,,\quad \{b\}\,,\quad \{c\}\,,\quad \{a,b\}\,,\quad \{a,c\}\,,\quad \{b,c\}\,,\quad \{a,b,c\}$$

Therefore, the *set* of all subsets is just the above list, but with brackets around it:

$$\big\{\{\}\,,\ \{a\}\,,\ \{b\}\,,\ \{c\}\,,\ \{a,b\}\,,\ \{a,c\}\,,\ \{b,c\}\,,\ \{a,b,c\}\big\}$$

As always, the empty set $\{\}$ is a subset, as is the full set (here, the full set is $\{a, b, c\}$).

For a set with n elements, there will be exactly 2^n subsets. Here, $n = 3$ so we should have $2^3 = 8$ subsets. If you know about factorials and binomial coefficients (we'll discuss them later in the course), the number of subsets with exactly k elements is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. So there should be exactly $\binom{3}{0} = 1$ subset with 0 elements (namely $\{\}$), exactly $\binom{3}{1} = 3$ subsets with 1 elements (namely $\{a\}$, $\{b\}$, $\{c\}$), exactly $\binom{3}{2} = 3$ subsets with 2 elements (namely $\{a, b\}$, $\{a, c\}$, $\{b, c\}$), and exactly $\binom{3}{3} = 1$ subset with 3 elements (namely $\{a, b, c\}$).

1(b) [1 pt] The subsets of $\big\{\{\}\big\}$ are

$$\{\}\,,\quad \big\{\{\}\big\}$$

So the set of all subsets is

$$\big\{\{\}\,,\ \{\{\}\}\big\}$$

Two subsets are always easy to write down: the empty set, and the full set. Since $\big\{\{\}\big\}$ has exactly 1 element, namely $\{\}$, it will have exactly $2^1 = 2$ subsets.

It's kind of weird that $\{\}$ is *both* an element *and* a subset of $\big\{\{\}\big\}$. But to be fair, $\big\{\{\}\big\}$ is a bit weird itself!

2(a) [1 pt] It is **FALSE** that $\{a, b, \{a, b\}\} = \{a, b\}$. The set on the left has 3 elements, while the set on the right has only 2.

2(b) [1 pt] It is **TRUE** that $\{1, 2, 3, 2\} = \{2, 3, 1\}$. The order you write the elements doesn't matter, nor does it matter if you write some elements more than once. (It's a bit silly to write an element more than once, but if you really want to, you can and it won't change anything.)

3. [2 pts] We have that $A = \{-3,-2,-1,0,1,2,3,4\}$ and $B = \{1,2,\ldots,10\}$. (Notice that the elements of A are integers, but those of B are natural numbers so start with 1.) So

$$A \cup B = \{-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10\}$$

$$A \cap B = \{1,2,3,4\}$$

4. [1 pt] In words, the set $\{n \in \mathbb{N} \,|\, n = k^2 \text{ for some } k \in \mathbb{Z}\}$ is the list of all natural numbers which are perfect squares. So the first several elements in it are $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, etc. Just choose two of these, e.g. 1 and 4.

But note that 0 is not in this set. This is because we require $n \in \mathbb{N}$, so even though $k = 0$ is possible, $n = 0$ is not.

5(a) [1 pt] This is **false**. First, "$2 \in \mathbb{N}$" is **true**. Since "$-2 \in \mathbb{Z}$" is **true**, "$-2 \notin \mathbb{Z}$" is **false**. Now use the Truth table for AND, as given in Quiz2 solutions or Week1 notes.

5(b) [1 pt] This is **true**. Both "$2 \in \mathbb{N}$" and "$-10 \in \mathbb{Z}$" are **true**. Now use the OR truth table, as given in Quiz2 or the Week1 notes.

5(c) [1 pt] This is **true**. The premise "$1 > 2$" is **false**. So the conclusion doesn't matter: the IF...THEN is automatically **true**.

5(d) [1 pt] This is **false**. "$1 > 2$" is **false** and "$1 + 1 = 2$" is **true**, so look at the IFF truth table as given in Quiz2 or the Week2 notes.

5(e) [1 pt] This is **true**, for the same reason as in 5(c).

5(f) [1 pt] This is **true**. The premise "$1 < 2$" is **true**, and the conclusion "$1 + 1 = 2$" is **true**, so the IF...THEN is **true** as explained in Quiz2 or the Week2 notes.

6. [4 pts] Assume $A \subseteq B$. We want to prove that $B = A \cup B$. To prove an equality of sets, we also need to prove two directions: (i) that $B \subseteq A \cup B$, and (ii) that $A \cup B \subseteq B$. (This is explained in the Week2 notes, "Logic and sets: proof strategies".)

The proof of (i) is easy. We proved this in class (Lemma 1 in the Week2 notes). So there is no need to do anything here, except reference that Lemma.

To prove (ii), choose any $x \in A \cup B$. That means that either $x \in A$, or $x \in B$. We want to prove that $x \in B$. So the only potential problem is when $x \in A$. But we are assuming here (see the first paragraph of this "$\Longrightarrow$" proof) that $A \subseteq B$, so $x \in A$ implies $x \in B$. Thus no matter what, $x \in B$, and we have shown that $A \cup B \subseteq B$, as desired. (This proof also follows the proof strategies outlined in the Week2 notes, "Logic and sets: proof strategies".)

Together, (i) and (ii) tell us $B = A \cup B$, as desired.

MATH 117 Assignment #2: Induction and Peano axioms

Due 11:59 pm Sunday September 22, 2019

1. [2 pts] Let P, Q be any two statements. Prove that

$$\text{NOT}(P \Rightarrow Q) \ \text{ IFF } \ (P \text{ AND NOT}(Q))$$

(HINT: One way is to consider all possible truth values of P and Q)

2. [6 pts] For any sets A, B, C, prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

3. Recall Peano's axioms of the natural numbers, given in Class (or see our Week 2 summary). Define multiplication recursively by: $m \cdot 1 = m$ and $m \cdot S(n) = m \cdot n + m$.

(a) [2 pts] Compute $3 \cdot 2$ (recall that by definition, $3 = S(S(1))$, $4 = S(3)$, etc). (WARNING: you'll have to compute $3 + 3$ using the definition of addition.)

(b) [5 pts] Prove distributivity: $(k + m) \cdot n = k \cdot n + m \cdot n$ for all $k, m, n \in \mathbb{N}$.

4. [5 pts] Prove using induction that, for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$$

(Recall that $\sum_{k=1}^{n} k2^k$ is shorthand for $1 \cdot 2^k + 2 \cdot 2^2 + \cdots + n \cdot 2^n$. For this question and the next, you can assume the usual basic properties of natural numbers – you don't have to reduce everything to Peano's axioms!)

5. [5 pts] Prove using induction that, for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

MATH 117. Solutions to Assignment #2

1. We have to show that for every possible truth value of P and Q, the truth value of NOT($P \Rightarrow Q$) equals that of (P AND NOT(Q)).

Consider first P **false**. Then ($P \Rightarrow Q$) is **true** (no matter what Q is), so NOT($P \Rightarrow Q$) is **false**. Also, in this case (P AND NOT(Q)) will be **false** (no matter what Q is). So when P is **false**, the truth value of NOT($P \Rightarrow Q$) equals that of (P AND NOT(Q)).

So it suffices to consider P **true**. If now Q is **true**, then NOT($P \Rightarrow Q$) is **false** and (P AND NOT(Q)) is also **false**. And if Q is **false**, then NOT($P \Rightarrow Q$) is **true** and (P AND NOT(Q)) is also **true**. So in both cases, we again have that the truth value of NOT($P \Rightarrow Q$) equals that of (P AND NOT(Q)).

2. To prove any two sets S, T are equal, we need to show $S \subseteq T$ and $S \supseteq T$. So to prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, there are two directions to show: **(i)** $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and **(ii)** $A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$

First, **(i)**. Choose any $x \in A \cup (B \cap C)$. That means either $x \in A$ or $x \in B \cap C$, i.e. either $x \in A$, or both $x \in B$ and $x \in C$. We want to show $x \in (A \cup B) \cap (A \cup C)$. So we want to show both $x \in A \cup B$ and $x \in A \cup C$, i.e. both ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$).

If $x \in A$, then it is certainly true that both $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$ and we're done.

Otherwise, we know that both $x \in B$ and $x \in C$. Because $x \in B$, we know $x \in A \cup B$. And because $x \in C$, we know $x \in A \cup C$. Together, these again tell us $x \in (A \cup B) \cap (A \cup C)$.

Those last two paragraphs take care of **(i)**: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now let's show **(ii)**. Choose any $x \in (A \cup B) \cap (A \cup C)$. That means both ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$). We want to show $x \in A \cup (B \cap C)$. So we want to show that either $x \in A$, or both $x \in B$ and $x \in C$.

If $x \in A$, then there is nothing to prove: certainly $x \in A \cup (B \cap C)$. So it suffices to consider $x \notin A$. Then $x \in (A \cup B) \cap (A \cup C)$ would mean both $x \in B$ and $x \in C$. And again we're done.

This concludes the proof of **(ii)**.

3(a)

$$3 \cdot 2 = 3 \cdot S(1) = 3 \cdot 1 + 3 = 3 + 3 = 3 + S(2) = S(3+2) = S(3+S(1)) = S(S(3+1)) = S(S(S(3))) = 6$$

3(b) Let $K = \{n \in \mathbb{N} \mid (k+m) \cdot n = k \cdot n + m \cdot n \text{ for all } k, m \in \mathbb{N}\}$. We want to use Axiom A3 to show that $K = \mathbb{N}$. That means we must show **(i)** $1 \in K$ and **(ii)** if $n \in K$, then $S(n) \in K$.

(i) $(k+m) \cdot 1 = k + m = k \cdot 1 + k \cdot m$, so $1 \in K$.

(ii) $(k+m) \cdot S(n) = (k+m) \cdot n + (k+m) = (k \cdot n + m \cdot n) + (k+m)$. Using associativity and commutativity of addition, we can rearrange $(k \cdot n + m \cdot n) + (k+m)$ to $(k \cdot n + k) + (m \cdot n + m)$.

Now, $k \cdot n + k = k \cdot S(n)$ and $m \cdot n + m = m \cdot S(n)$. Thus $(k+m) \cdot S(n) = k \cdot S(n) + m \cdot S(n)$. This says that $S(n) \in K$.

Therefore $K = \mathbb{N}$, and we're done.

I'm using here the usual hierarchy of arithmetic operations. So e.g. $k \cdot n + m \cdot n = (k \cdot n) + (m \cdot n)$.

Incidentally, if you want to explicitly show how to rearrange $(k \cdot n + m \cdot n) + (k + m)$ to $(k \cdot n + k) + (m \cdot n + m)$ using associativity and commutativity, the argument in full detail would be

$$(k \cdot n + m \cdot n) + (k + m) = ((k \cdot n + m \cdot n) + k) + m = (k \cdot n + (m \cdot n + k)) + m = (k \cdot n + (k + m \cdot n)) + m$$

$$= ((k \cdot n + k) + m \cdot n) + m = (k \cdot n + k) + (m \cdot n + m)$$

using associativity, associativity, commutativity, associativity in that order.

4. Here, the statement $P(n)$ is "$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$".

First, we must show $P(1)$, i.e. that $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$ is true for $n = 1$. As usual, this is easy: LHS is $\sum_{k=1}^{1} k2^k = 1 \cdot 2^1 = 2$, and RHS is $(1-1)2^{1+1} + 2 = 2$.

Now suppose that $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$ holds for some n (this is the induction hypothesis). Is it true that $\sum_{k=1}^{n+1} k2^k = n2^{n+2} + 2$? (that is $P(n+1)$). We compute

$$\sum_{k=1}^{n+1} k2^k = \sum_{k=1}^{n} k2^k + (n+1)2^{n+1} = ((n-1)2^{n+1} + 2) + (n+1)2^{n+1} = 2n2^{n+1} + 2 = n2^{n+2} + 2$$

as desired ($2n2^{n+1} = n2^1 2^{n+1} = n2^{n+2}$).

By induction, together these tell us $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$ is true for all n.

5. First, we must show that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ holds for $n = 1$. This is easy: LHS is $\sum_{k=1}^{1} k^3 = 1^3 = 1$ and RHS is $\frac{1^2(1+1)^2}{4} = \frac{2^2}{4} = 1$.

Now suppose that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ holds for some n. Is it true that $\sum_{k=1}^{n+1} k^3 = \frac{(n+1)^2(n+2)^2}{4}$? We compute

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + n^3 + 3n^2 + 3n + 1 = \frac{n^4 + 2n^3 + n^2 + 4(n^3 + 3n^2 + 3n + 1)}{4}$$

$$= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4}$$

which equals

$$\frac{(n+1)^2(n+2)^2}{4} = \frac{(n^2 + 2n + 1)(n^2 + 4n + 4)}{4} = \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4}$$

Again, by induction, we're done.

MATH 117 Assignment #3: fields

Due 11:59 pm Wednesday October 2, 2019

1. [3 pts] Let $\mathbb{F}$ be any field, with operations $+$ and $\cdot$ as usual. Define $2 = 1 + 1$, where 1 is the multiplicative unit as usual. Prove that $x + x = 2 \cdot x$ for any $x \in \mathbb{F}$.

2. [9 pts] Here is a possible theorem and a possible proof. For each step in the 'proof', either explain why it is valid (i.e. why it follows from the previous steps and/or previous theorems and/or the axioms of a field), or explain why it is not valid.

Theorem(?). Let $\mathbb{F}$ be any field, and $a \in \mathbb{F}$. If $a = -a$, then $a = 0$.
Proof(?).
(a) Assume $a = -a$. *Can we do this? Is this a good way to start the proof?*
(b) Then $a + a = 2 \cdot a$, where $2 = 1 + 1$. *Is this a valid step? Why or why not?*
(c) But also $a + a = a + (-a)$. *Valid? Why or why not?*
(d) So also $a + a = 0$. *Valid? Why or why not?*
(e) Therefore $2 \cdot a = 0$. *Valid? Why or why not?*
(f) Multiply each side of $2 \cdot a = 0$ by 2^{-1}: we get $2^{-1} \cdot (2 \cdot a) = 2^{-1} \cdot 0$. *Valid? Why or why not?*
(g) But $2^{-1} \cdot (2 \cdot a) = (2^{-1} \cdot 2) \cdot a = 1 \cdot a = a$. *Valid? Why or why not?*
(h) And $2^{-1} \cdot 0 = 0$. *Valid? Why or why not?*
(i) Therefore $a = 0$. Q.E.D. *Valid? Why or why not?*

3. [6 pts] Let $\mathbb{F}$ be any field, and a, b be any elements in $\mathbb{F}$. Prove each of the following. You may assume anything we proved in class.

(a) $a - b = 0$ iff $a = b$.

(b) If $ab = 0$ then either $a = 0$ or $b = 0$.

(c) $a^2 - b^2 = (a + b)(a - b)$

4. In this question we're going to do a slight notation change. Define new operations $\oplus$ and $\odot$ on the real numbers $\mathbb{R}$ through the formulas $x \oplus y = x + y + 1$ and $x \odot y = 2xy + 2x + 2y + 1$ (in these formulas, the expressions on the right are evaluated using usual real number addition and multiplication, so e.g. $1 \oplus 1 = 3$ and $1 \odot 1 = 7$). It turns out that $\mathbb{R}$ with these weird operations is a field (but don't show that).

(a) [2 pts] Verify that, for $\mathbb{R}$ with these weird operations, the additive identity is $0' = -1$, by showing that $0' = -1$ satisfies axiom **(AN)** of a field.

(b) [3 pts] Find the multiplicative identity $1'$, by finding a number $1'$ satisfying axiom **(MN)** of a field. (HINT: $1' \neq 1$.)

(c) [3 pts] Find the additive inverse $\ominus 2$ of 2, i.e. the number x such that $x \oplus 2 = 0'$. (HINT: it is not -2.) Here, 2 is the usual real number, not $2' = 1' \oplus 1'$ as in Qu.1.

(d) [3 pts] Find the multiplicative inverse of 2, i.e. the number x such that $x \odot 2 = 1'$. (Hint: it is not 0.5.)

5. [4 pts] Simplify these expressions (so don't use the word 'NOT').

(a) NOT(for all $x \in A$ there exists a $y \in B$ such that $x > y$)

(b) NOT(there exists a $y \in B$ such that for all $x \in A$, $x > y$)

6. [3 pts] Prove that no rational number squared equals 3.

MATH 117. Solutions to Assignment #3

1. For any $x \in \mathbb{F}$, $x + x = (1 \cdot x) + (1 \cdot x)$ (by Axiom MN). Then Distributivity (Axiom D) gives $(1 + 1) \cdot x$, and then the definition $2 = 1 + 1$ gives us $2 \cdot x$.

2(a) [1 pt] **This is a valid way to start the proof.** You can always assume the premise.

2(b) [1 pt] **This is valid:** We proved this in Question 3 of this Assignment.

2(c) [1 pt] **This is valid:** We are using Step (a) of this proof, substituting a for '$-a$'.

2(d) [1 pt] **This is valid: Axiom AI** says $a \oplus$ '$(-a)$' $= 0'$.

2(e) [1 pt] **This is valid:** We're combining steps (b) and (d) of the proof.

2(f) [1 pt] **THIS IS NOT VALID for all fields**. '$2'^{-1}$' exists only if $2' \neq 0'$. In some fields (e.g. the **even,odd** field), $2' = 0'$.

2(g) [1 pt] Assuming step (f), i.e. assuming that '$2'^{-1}$' $\odot (2' \odot a)$ is sensible, **this is valid**. It uses **Axioms AA, AI, AN**.

2(h) [1 pt] Assuming step (f), i.e. that '$2'^{-1}$' $\odot 0'$ is sensible, **this is valid**: it is Lemma F.2(b) in the Week3 notes.

2(i) [1 pt] **This is valid:** it combines steps (g) and (h).

By the way, $\mathbb{F} = \{\textbf{even}, \textbf{odd}\}$ is a counterexample to this so-called 'Theorem': $-\textbf{odd} = \textbf{odd}$ even though $\textbf{odd} \neq 0'$. So tracing this example through the so-called 'proof', you will find a place where the proof does something illegal.

3(a) This is easy. $a - b = 0$ iff $(a + -b) + b = 0 + b$, where we substitute what subtraction means, and we add the same thing to both sides. This holds iff $a + (-b + b) = b$ using associativity and the definition (Axiom AN) of 0. And that holds iff $a + 0 = b$, using the definition (Axiom AI) of additive inverse. And Axiom AN again tells us that holds iff $a = b$.

We can use 'iff' throughout these steps, because these steps are reversible.

3(b) Assume $ab = 0$. a either equals 0 or it doesn't. If $a = 0$, then we're done, because then it is certainly true that either $a = 0$ or $b = 0$. So it suffices to consider the case where $a \neq 0$. Then a^{-1} exists, so we can multiply both sides of $ab = 0$ by a^{-1}: the left-side simplifies to b (after using Axioms MA, MI, and MN) and the right-side simplifies to 0 (using Lemma F.2(b)). Thus we get $b = 0$, and we're done.

3(c) This is easy. By the definition of subtraction, $(a + b)(a - b) = (a + b)(a + -b)$. By Axiom D, $(a + b)(a + -b) = (a + b)a + (a + b)(-b)$. By Lemma F.1(e), this equals $\big(aa + ba\big) + \big(a(-b) + b(-b)\big)$. By Lemma F.2(c), we can replace $-b$ with $(-1) \cdot b$, so we get $(a^2 + ba) + \big(a((-1)b) + b((-1)b)\big)$. Using associativity (Axiom MA) and commutativity (Axiom MC) multiple times, we can write $a((-1)b) = (a(-1))b = ((-1)a)b = (-1)(ab) = -(ba)$, where we also snuck in Lemma F.2(c). Likewise, $b((-1)b)$ simplifies to $-b^2$. So we get $(a^2 + ba) + (-ba - b^2)$. Using associativity AA a couple times, this becomes $a^2 + (ba + (-ba - b^2)) = a^2 + ((ba + -ba) - b^2)$, which collapses to $a^2 + (0 - b^2) = a^2 - b^2$, using Axioms AI and AN.

It's tedious to show all these steps. If you mention *all* axioms and lemmas you need, and show the calculation $(a + b)(a - b) = aa - ab + ba - bb = a^2 - b^2$, then you'd get full marks.

4(a) [2 pts] The neutral element $0'$ is defined by **Axiom AN**: it is required to satisfy $0' \oplus y = y$ for all $y \in \mathbb{R}$. So we need to verify that $-1 \oplus y = y$ for all y. Substituting in the formula for $\oplus$, $-1 \oplus y$ becomes $-1 + y + 1$, where now we use the familiar operations of $\mathbb{R}$. Of course this simplifies to y, as desired. So we have verified that **AN** holds, and we're done.

4(b) [3 pts] **Axiom MN** says that $1' \odot y = y$ holds for all $y \in \mathbb{R}$. Substituting in the formula for $\odot$, this becomes $2 \cdot 1' \cdot y + 2 \cdot 1' + 2y + 1 = y$, where now we use the familiar operations of $\mathbb{R}$ ($\cdot$ here is the usual multiplication of $\mathbb{R}$). So we can manipulate this equation in the usual way. We want to solve it for $1'$, and it must hold for all $y \in \mathbb{R}$. We get $2 \cdot 1'y + 2 \cdot 1' = -1 - y$. The only way this can hold for all $y \in \mathbb{R}$, is if that $1' = -1/2$.

4(c) [3 pts] Write x for the additive inverse of 2: so $x \oplus 2 = 0'$. Substituting $0' = -1$ (from 4(a)) and evaluating $x \oplus 2$ using the given formula for '$\oplus$', we rewrite $x \oplus 2 = 0'$ as $x + 2 + 1 = -1$, where now all the operations are the usual ones. So solve $x + 2 + 1 = -1$ using the usual operations, we obtain $x = -4$. So the additive inverse $\ominus 2$ of 2 in this field is -4.

Note that $-1 \odot 2 = -4$. Coincidence?

4(d) [3 pts] Write x for the multiplicative inverse of 2: so $x \odot 2 = 1'$. Substituting $1' = -1/2$ (from 4(b)) and evaluating $x \odot 2$ using the given formula for '$\odot$', we rewrite $x \odot 2 = 1'$ as $4x + 2x + 4 + 1 = -1/2$, where now all the operations are the usual ones. So solve $6x + 5 = -1/2$ using the usual operations, we obtain $x = -11/12$. So the multiplicative inverse of 2 in this field is $-11/12$!

5(a) Move 'NOT' through the expression. NOT changes "for all $x \in A$" to "there exists an $x \in A$". NOT changes "there exists a $y \in B$" to "for all $y \in B$". NOT changes "$x > y$" to "$x \leq y$". Putting it all together, we see that "NOT(for all $x \in A$ there exists a $y \in B$ such that $x > y$)" simplifies to "there exists an $x \in A$ such that for all $y \in B$, $x \leq y$".

You can't change the order of "there exists an $x \in A$" and "for all $y \in B$" without changing the meaning.

(b) By the same reasoning in (a), "NOT(there exists a $y \in B$ such that for all $x \in A$, $x > y$)" is the same as "for all $y \in B$, there exists an $x \in A$ such that $x \leq y$".

6(a) [3 pts] Suppose for contradiction that $\sqrt{3}$ is rational. That means $\sqrt{3} = m/n$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. As before, we can assume that m and n don't have any common divisors > 1. In particular, we can require that at least one of m, n is not divisible by 3.

Squaring both sides, we get $m^2 = 3n^2$. Hence m^2 is divisible by 3, which forces m to be divisible by 3 (if m has a remainder of 1 or 2 when dividing by 3, then m^2 will have a remainder of 1). So $m = 3m'$ where $m' \in \mathbb{Z}$. Substituting in, we get $(3m')^2 = 2n^2$, i.e. $3m'^2 = n^2$. Hence also n is divisible by 3, which contradicts that m, n don't share a common divisor > 1. This contradiction means $\sqrt{3}$ is irrational.

There is a valid proof here using even,odd arguments. But that proof isn't as good, because it won't work for many other sqareroots. E.g. try to get it to work for $\sqrt{17}$.

MATH 117 Assignment #4

Due 11:59 pm Sunday October 13, 2019

1. [3 pts] Choose any $n \in \mathbb{N}$. Prove however you like that $\sqrt{2}+\sqrt{n}$ is irrational. (Hint: one way, for $n \neq 2$, is to consider $\sqrt{2}-\sqrt{n}$.)

2. [4 pts] Let $\mathbb{F}$ be any ordered field, and $a, b, c, d \in \mathbb{F}$. Prove each of the following. You may assume anything we proved in class.

(a) If $a \geq b$ and $b \geq c$, then $a \geq c$.

(b) If $a > b > 0$ and $c > d > 0$ then $ac > bd$.

3. [3 pts] Let a be any real number satisfying $a \geq -1$. Prove that $(1+a)^n \geq 1 + na$ whenever $n \in \mathbb{N}$. (HINT: use induction)

4. [6 pts] On $\mathbb{R}$ define operations $x \oplus y = x + y - 2$ and $x \odot y = -xy + 2x + 2y - 2$, where the addition and multiplication on the right side of those formulas is usual real number arithmetic. So e.g. $0 \oplus 1 = -1$ and $0 \odot 1 = 0$. You can assume that the set $\mathbb{R}$ with these operations $\oplus, \odot$ is a field. (We had a similar question on Assignment 3.) Choose $P = \{x \in \mathbb{R} \mid x < 2\}$. Prove that $\mathbb{R}$ with these operations and this P, satisfies axioms **OA**, **OM**, and **OT** of an ordered field.

5. [2 pts] Let $\mathbb{F}$ be any ordered field. For this question, assume $a, b, c, d \in \mathbb{F}$ and they obey $a < b \leq c < d$. (This means $a < b$ and $b \leq c$ and $c < d$.)

Prove that $(a, c) \cap (b, d) = (b, c)$.

6(a) [2 pts] Find all $x \in \mathbb{R}$ satisfying $\left|\frac{2}{x-13}\right| > \frac{8}{9}$. (HINT: it'll be an interval)

6(b) [3 pts] Find all $x \in \mathbb{R}$ satisfying $\big||x-1|-8\big| < 2$.

7. [4 pts] Let our field be $\mathbb{R}$. Suppose all we know is $|x| \leq 2$. How big could

$$|x^5 - 3x^4 + x^2 - 2x - 3|$$

get? (Don't use calculus to help you with this question. Just use absolute value properties to find an upper bound. Your upper bound doesn't have to be very good, it just has to work)

MATH 117 Assignment #4 Solutions

1. [3 pts] Assume for contradiction that $\sqrt{2}+\sqrt{n}$ is rational, say $\sqrt{2}+\sqrt{n}=a/b$ where $a\in\mathbb{Z}$ and $b\in\mathbb{N}$. Note that $(\sqrt{2}+\sqrt{n})(\sqrt{2}-\sqrt{n})=2-n$, so $\sqrt{2}-\sqrt{n}=(2-n)b/a$ is also rational. So $2\sqrt{2}=(\sqrt{2}+\sqrt{n})+(\sqrt{2}-\sqrt{n})=a/b+(2-n)b/a$ is also rational. But this contradicts the fact that $\sqrt{2}$ is irrational. Hence $\sqrt{2}+\sqrt{n}$ is irrational.

You can also prove this in other ways, e.g. by squaring.

2(a) [2 pts] If $a=b$ then $a\geq c$ is the same as $b\geq c$ so there is nothing to prove. Likewise, if $b=c$ then $a\geq b$ is the same as $a\geq c$, and again there is nothing to prove. So it suffices to consider the case where $a>b$ and $b>c$. But this is Lemma F.1(a), and we obtain $a>c$, which implies the weaker $a\geq c$.

2(b) [2 pts] Assume $a>b>0$ and $c>d>0$. Then $a-b\in P$ and $c-d\in P$. We compute $ac-bd=(ac-bc)+(bc-bd)$ by associativity of addition and Axiom AI. Using distributivity twice, we get $(a-b)c+b(c-d)$. But we know $b>0$ and $c>0$, so Axioms OM and OA tell us that $(a-b)c+b(c-d)>0$. Hence $ac>bd$.

3. [3 pts] *Base case:* For $n=1$, $(a+1)^1=a+1=1+1a$ for all $a\geq -1$ (in fact for all a).
Induction hypothesis: Suppose $(a+1)^n\geq 1+na$ for all $a\geq -1$.

Then

$$(a+1)^{n+1}=(a+1)^n(a+1)\geq(1+na)(a+1)=a+na^2+1+na=1+(n+1)a+na^2\geq 1+(n+1)a$$

since $na^2\geq 0$.

4. [2 pts] First, let's prove **OA**. Choose any $x,y\in P$, so $x<2$ and $y<2$. We want to show $x\oplus y\in P$, which is the same as proving $x+y-2<2$, which is the same as proving $x+y<4$. But this is clear, from Lemma OF.1(d), since $x<2$ and $y<2$.

Some of you might be uneasy about working backwards like this from the conclusion. You can do this as long as all of those steps can be reversed. So let's try to do that. As above, we know $x<2$ and $y<2$. This implies $x+y<4$ by Lemma OF.1(d), so $x+y-2<2$, so $x\oplus y\in P$, as desired. Working backward from the conclusion, to see what you need to prove, is a standard tactic in proving theorems. But make sure each step is reversible! Just make sure that at each step you can say "which is the same as proving" or "which we know is true if we can prove..."

[2 pts] Next, let's prove **OM**. Choose any $x,y\in P$, so $x<2$ and $y<2$. We want to show $x\odot y\in P$, which is the same as proving $-xy+2x+2y-2<2$, which is the same as proving $0<xy-2x-2y+4$. But we know $0<2-x$ and $0<2-y$, so $0<(2-x)(2-y)=4-2x-2y+xy$, which is what we needed to show.

[2 pts] Finally, let's prove **OT**. First note that the additive identity $0'$ here is 2: $x\oplus 2=x$. Also, the additive inverse $\ominus x=4-x$: $x\oplus(4-x)=2=0'$.

Choose any $x\in\mathbb{R}$, and suppose $x\notin P$ and $x\neq 0'$. So $x\geq 2$ and $x\neq 2$, so $x>2$. We need to show $\ominus x\in P$. But $\ominus x=4-x$, so we need to show $4-x<2$. Since we know $x>2$, it is certainly true that $4-x<2$.

5. [2 pts] There are two directions to this: we must prove $(a,c) \cap (b,d) \subseteq (b,c)$, and also $(a,c) \cap (b,d) \supseteq (b,c)$.

First, suppose $x \in (a,c) \cap (b,d)$. Then $a < x < c$ and $b < x < d$. So in particular $b < x$ and $x < c$, i.e. $b < x < c$, i.e. $x \in (b,c)$

Now, suppose $x \in (b,c)$, i.e. $b < x < c$. But we're told $a < b$. So $b < x$ and $a < b$ implies $a < x$ (see Lemma OF.1). Similarly, we're told $c < d$, so that and $x < c$ implies $x < d$. Thus $x \in (a,c)$ and also $x \in (b,d)$, i.e. $x \in (a,c) \cap (b,d)$, and we're done.

6(a) Since $|x-13| > 0$ and $9/8 > 0$, we can rewrite $\left|\frac{2}{x-13}\right| > \frac{8}{9}$ as $\frac{9}{4} > \left|x-13\right|$. But we know $\left|x-r\right| < r$ is the same as $x \in (c-r, c+r)$. So the set of all x satisfying $\frac{9}{4} > \left|x-13\right|$, is the interval $(13-\frac{9}{4}, 13+\frac{9}{4})$, i.e. $(\frac{43}{4}, \frac{61}{4})$.

6(b) *case 1.* $x-1 \geq 0$, i.e. $x \geq 1$. Then the inequality $\left||x-1|-8\right| < 2$ reduces to $\left|(x-1)-8\right| < 2$, i.e. $\left|x-9\right| < 2$. This corresponds to x in the interval $(7,11)$. Taking the intersection of this with $[1,\infty)$ gives $(7,11)$.

case 2. $x-1 < 0$, i.e. $x < 1$. Then the inequality $\left||x-1|-8\right| < 2$ reduces to $\left|-(x-1)-8\right| < 2$, i.e. $\left|x+7\right| < 2$. This corresponds to x in the interval $(-9,-5)$. Taking the intersection of this with $(-\infty,1)$ gives $(-9,-5)$.

So the final answer is the union of the x in cases 1 and 2, namely the x in the union $(-9,-5) \cup (7,11)$.

7. By the triangle inequality,

$$|x^5 - 3x^4 + x^2 - 2x - 3| \leq \left|x^5\right| + \left|-3x^4\right| + \left|x^2\right| + \left|-2x\right| + \left|-3\right| = |x|^5 + 3|x|^4 + |x|^2 + 2|x| + 3$$

Because everything is now $+$ signs, the right side gets bigger the bigger $|x|$ gets. So we can replace $|x|$ with its upper bound 2:

$$|x|^5 + 3|x|^4 + |x|^2 + 2|x| + 3 \leq 2^5 + 3 \cdot 2^4 + 2^2 + 2 \cdot 2 + 3 = 91$$

So 91 is an upper bound for $|x^5 - 3x^4 + x^2 - 2x - 3|$.

For this question, you don't need to find the supremum, or even a really good bound. Any upper bound is fine.

MATH 117 Assignment #5: sup, inf, limits

Due 11:59 pm Wednesday October 23, 2019

1. For this question, use the ordered field $\mathbb{R}$.

(a) [3 pts] Let $A = \{x^{-2} \mid x \in (0,1)\}$. Find $\max A$, $\min A$, $\sup A$, $\inf A$, if they exist. Is A bounded?

(b) [3 pts] Let $B = \{-n^{-2} \mid n \in \mathbb{N}\}$. Find $\max B$, $\min B$, $\sup B$, $\inf B$, if they exist. Is B bounded?

(c) [3 pts] Let $C = (3,4] \cup (7,9]$. Find $\max C$, $\min C$, $\sup C$, $\inf C$, if they exist. Is C bounded?

2. [3 pts] Let $\mathbb{F}$ be any ordered field. Choose any $a, b \in \mathbb{F}$ and let S be the 'interval' $S = [a,b)$, i.e. the set $\{x \in \mathbb{F} \mid a \leq x < b\}$. Identify $\inf S$ and $\sup S$, if they exist. Prove your answer is correct.

3. [3 pts] For this question, use the ordered field $\mathbb{R}$. For $|x - y| < 0.01$ and $x, y \in (0,2)$, show that $|x^2 - y^2| < 0.04$. (Don't use calculus; just use properties of absolute values.)

4(a) [3 pts] Using the definition of limit, show that $\lim_{n\to\infty} \frac{(n+1)^2}{n^2+1} = 1$. So for each $\epsilon > 0$ find an N such that $|1 - \frac{(n+1)^2}{n^2+1}| < \epsilon$ for all $n > N$.

(b) [3 pts] Using the definition of limit, show that $\lim_{n\to\infty} a_n = 0$, where

$$a_n = \begin{cases} \frac{1}{\sqrt{n}} & \text{if } n \text{ is odd} \\ \frac{10}{n^2} & \text{if } n \text{ is even} \end{cases}$$

So for each $\epsilon > 0$ find an N such that $|a_n| < \epsilon$ for all $n > N$.

5. [4 pts] Let a_n be a sequence converging to L, and suppose each $a_n > 0$. Prove that the sequence $\sqrt{a_n}$ converges to $\sqrt{L}$.

MATH 117 Assignment #5 Solutions

1(a) As x gets closer to 0, x^{-2} tends to ∞. So $\max A$ does not exist, and $\sup A = \infty$.

For $x < 1$, $x^{-1} > 1$ so $x^{-2} > 1$. But as x gets closer to 1, x^{-2} tends to 1. So $\min A$ does not exist, but $\inf A = 1$.

Because $\sup A$ is infinite, A is unbounded.

1(b) As n increases, n^{-1} decreases, so n^{-2} also decreases, and $-n^{-2}$ increases. So the smallest value of $-n^{-2}$ will correspond to the smallest n (i.e. $n = 1$), and the largest value of $-n^{-2}$ will correspond to the largest n (i.e. to the limit as $n \to \infty$).

At $n = 1$, $-n^{-2} = -1$, so $\min B = \inf B = -1$.

$\lim_{n\to\infty} - n^{-2} = 0$, so $\sup B = 0$. But $-n^{-2} < 0$ for any n, so $\max B$ does not exist.

Because inf and sup of B are both finite, B is bounded.

1(c) The max is 9, so the sup is also 9. The inf is 3, but the minimum doesn't exist (if it too existed, it would have to equal 3, but $3 \notin C$). C is bounded (below by 3, above by 9).

2. [3 pts] First, let's prove that $\inf [a, b)$ is a. We need to show 2 things: **(i)** a is a lower bound of $[a, b)$; **(ii)** if ℓ is any lower bound of $[a, b)$, then $\ell \le a$. The proof of **(i)** is easy: $[a, b) = \{x \in \mathbb{F} \,|\, a \le x < b\}$. In particular, for any $x \in [a, b)$, $a \le x$ (this is half of the definition of that interval). This is what it means for a to be a lower bound of $[a, b)$. To see **(ii)**, first note that $a \in [a, b)$. So if ℓ is a lower bound of $[a, b)$, then in particular $\ell \le a$ (since a is in that interval) which is what we had to prove. So we have proven that a equals $\inf [a, b)$.

Next, we want to prove that $b = \sup [a, b)$. This means we need to prove that: **(i)** b is an upper bound of $[a, b)$; and **(ii)** if u is any other upper bound of $[a, b)$, then $b \le u$. Again, the proof of **(i)** is easy: $[a, b) = \{x \in \mathbb{F} \,|\, a \le x < b\}$ so in particular $x \le b$ for all $x \in [a, b)$, which is what it means for b to be an upper bound of $[a, b)$. The proof of **(ii)** is harder: suppose for contradiction that u is an upper bound of $[a, b)$ and $u < b$. We proved in class that any open interval like (u, b) is nonempty, in any ordered field $\mathbb{F}$. In other words, there is an $x \in \mathbb{F}$ such that $u < x < b$. Now, $u \ge a$ (since u, being an upper bound, must be at least as big as anything in $[a, b)$). So $x \in [a, b)$, but $x > u$. This contradicts u being an upper bound of $[a, b)$. Hence $b \le u$, and $b = \sup [a, b)$.

3. First note that $|x^2 - y^2| = |(x - y)(x + y)| = |x - y|\,|x + y|$. We are told that $|x - y| < 0.01$. What is the biggest $|x + y|$ can be? Well, we're told that $x, y \in (0, 2)$, so $|x + y| = x + y < 2 + 2 = 4$. Thus

$$|x^2 - y^2| = |x - y|\,|x + y| < 0.01 \cdot 4 = 0.04$$

4(a) [3 pts] First note that

$$\left|1 - \frac{(n+1)^2}{n^2+1}\right| = \left|\frac{(n^2+1) - (n^2+2n+1)}{n^2+1}\right| = \left|\frac{-2n}{n^2+1}\right| = \frac{2n}{n^2+1}$$

We want to find a N so that, for any $\epsilon > 0$, $\frac{2n}{n^2+1} < \epsilon$ for all $n > N$.

What makes $\frac{2n}{n^2+1}$ hard to deal with is the bottom n^2+1. So the plan is to replace it with a bigger fraction, which is simpler. Now, to make a fraction $\frac{a}{b}$ bigger, when a and b are both positive, you can either make the top bigger and keep the bottom, or keep the top the same and make the bottom smaller (but still positive). In other words: If $0 < a < c$ and $0 < d < b$, then both

$$\frac{a}{b} < \frac{c}{b} \text{ and } \frac{a}{b} < \frac{a}{d}$$

It's the bottom of $\frac{2n}{n^2+1}$ which is complicated, so its the bottom we want to change. What is something both smaller and simpler than n^2+1? Obviously n^2. So $\frac{2n}{n^2+1} < \frac{2n}{n^2} = \frac{2}{n}$. To make that fraction $< \epsilon$, n has to be $> \frac{2}{\epsilon}$. So this is the way the proof should go:

Choose any $\epsilon > 0$. Take $\frac{N=2}{\epsilon}$. Then for any $n > N$,

$$|L-a_n| = \left|1 - \frac{(n+1)^2}{n^2+1}\right| = \left|\frac{(n^2+1)-(n^2+2n+1)}{n^2+1}\right| = \left|\frac{-2n}{n^2+1}\right| = \frac{2n}{n^2+1} < \frac{2n}{n^2} = \frac{2}{n} < \frac{2}{N} = \epsilon$$

and we're done.

4(b) [3 pts] Suppose we just had to show $1/\sqrt{n} \to 0$. Then $N = 1/\epsilon^2$ would work (we did this example in class).

Suppose we just had to show $10/n^2 \to 0$. Then $10/N^2 \leq \epsilon$ means $10/\epsilon \leq N^2$ which means $\sqrt{10/\epsilon} \leq N$, so $N = \sqrt{10/\epsilon}$ works.

Our sequence is a combination of these two. So this is how the proof goes.

Choose any $\epsilon > 0$. Let $N = \max\{1/\epsilon^2, \sqrt{10/\epsilon}\}$. Choose any $n > N$. We want to show that for $n > N$, $|0 - a_n| < \epsilon$. If n is odd then, since $n > 1/\epsilon^2$, we have

$$|0 - a_n| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{1/\epsilon^2}} = \epsilon$$

as desired. And if n is odd then, since $n > \sqrt{10/\epsilon}$, we have

$$|0 - a_n| = \frac{10}{n^2} < \frac{10}{(\sqrt{10/\epsilon})^2} = \frac{10}{10/\epsilon} = \epsilon$$

In all cases, $|0 - a_n| < \epsilon$, and we're done.

5. [4 pts] Since all $a_n > 0$, L cannot be negative, so $\sqrt{L}$ exists (i.e. is a real number).

Suppose first that $L > 0$. Choose any $\epsilon > 0$. Since $a_n \to L$, there is an N such that $|L - a_n| < \sqrt{L}\epsilon$ for all $n > N$. Choose any $n > N$. Then $|\sqrt{L} - \sqrt{a_n}| = \frac{|L-a_n|}{|\sqrt{L}+\sqrt{a_n}|}$. But $|L - a_n| < \sqrt{L}\epsilon$ and $|\sqrt{L} + \sqrt{a_n}| = \sqrt{L} + a_n > \sqrt{L}$, so putting these together we get that $\frac{|L-a_n|}{|\sqrt{L}+\sqrt{a_n}|} < \frac{\sqrt{L}\epsilon}{\sqrt{L}} = \epsilon$. So this means $\sqrt{a_n} \to \sqrt{L}$ when $L > 0$.

Now suppose $L = 0$. Choose any $\epsilon > 0$. Then there exists an $N > 0$ such that, for all $n > N$, $|0 - a_n| < \epsilon^2$. Choose any $n > N$. Then $a_n < \epsilon^2$, so $\sqrt{a_n} < \epsilon$. This tells us $\sqrt{a_n} \to 0$, as desired.

MATH 117 Assignment #6: limits

Due 11:59 pm Wednesday October 30, 2019

1. [3 pts] Using the definition, show that the sequence $\frac{n-2}{\sqrt{n}+1}$ tends to ∞.

2. [12 pts] Compute the limits of the following sequences (if they exist). Show your work (as always, you can use the Theorems and Lemmas and Corollaries proved in class).

(a) $\frac{8n^4-3n^3+2n^2+17n-11}{-2n^4+3n^3+5n^2-12}$

(b) $\frac{-n^3+5n^2-11}{n^4+n^3+n^2+n+1}$

(c) $\frac{2n+(-1)^n}{(-1)^n n-\sqrt{n}}$

(d) $\frac{2n+(-1)^n}{n-(-1)^n\sqrt{n}}$

(e) $\sqrt{n^4+2n+1}-\sqrt{n^4-3n+2}$

(f) $\frac{n^2+n\cos(n^3+5n)}{\frac{1}{n}+\sqrt{n}-n^2}$

3(a) [3 pts] Let a_n be a sequence, and assume that $\sup a_n = \infty$. Prove that there is a subsequence $b_k = a_{n_k}$ such that b_k tends to ∞.

3(b) [3 pts] Let a_n be a sequence, and assume that there is a subsequence $b_k = a_{n_k}$ such that b_k tends to ∞. Prove that $\sup a_n = \infty$.

4. [3 pts] For any $a > 0$, prove $a^{1/n}$ converges to 1. ($a^{1/n}$ means the unique positive number x such that $x^n = a$; you can assume it always exists.) Hint: write $a^{1/n} = 1 + x_n$ and use the inequality $(x+1)^n \geq 1 + nx$ valid for all $x \geq -1$ and $n \in \mathbb{N}$ which we proved on Assignment 4.

5. [4 pts] Explain why each of the following statements are not equivalent to the definition that $a_n \to L$. In order to do this, for each of these statements find an example of a sequence which either satisfies the statement but fails to converge to L, or which converges to L but fails the statement.

(a) $\forall \epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $|a_n - L| < \epsilon$

(b) there is an N such that, for all $\epsilon > 0$, and all $n > N$, $|a_n - L| < \epsilon$.

MATH 117 Assignment #6 Solutions

1. [3 pts] We want to show that, for any M, there is an N such that for all $n > N$, $\frac{n-2}{\sqrt{n}+1} > M$. So the idea is to find a simpler and smaller fraction than $\frac{n-2}{\sqrt{n}+1}$, and we find how big n has to be to make the simpler fraction bigger than M.

Given a fraction a/b where both a, b are positive, the way to make a smaller fraction is to make the top smaller and keep the bottom, or make the bottom bigger and keep the top. Put another way, if $0 < c < a$ and $0 < b < d$, then both

$$\frac{a}{b} > \frac{c}{b} \quad \text{and} \quad \frac{a}{b} > \frac{a}{d}$$

Both $n-2$ and $\sqrt{n}+1$ are complicated. We need to make $\sqrt{n}+1$ bigger, so make it $2\sqrt{n}$: $\sqrt{n}+1 \leq 2\sqrt{n}$ for all $n \geq 1$. We need to make $n-2$ smaller: a good choice is $n/2$: $n-2 \geq n/2$ for all $n \geq 4$. So as long as $n \geq 4$, $\frac{n-2}{\sqrt{n}+1} \geq \frac{n-2}{2\sqrt{n}} \geq \frac{n/2}{2\sqrt{n}} = \sqrt{n}/4$, which looks much much simpler! We want $\sqrt{n}/4 > M$, which means $\sqrt{n} > 4M$, which means $n > 16M^2$.

That is the secret work which goes into the proof. This is the way the proof should go:

Choose any M. Let $N = \max\{16M^2, 4\}$. Choose any $n > N$. Then, because $n > 4$

$$\frac{n-2}{\sqrt{n}+1} \geq \frac{n-2}{2\sqrt{n}} \geq \frac{n/2}{2\sqrt{n}} = \frac{\sqrt{n}}{4} > \frac{\sqrt{N}}{4} = \frac{4M}{4} = M$$

and we're done!

This was meant to be a hard question.

2(a) [2 pts] First note $\frac{8n^4-3n^3+2n^2+17n-11}{-2n^4+3n^3+5n^2-12} = \frac{8-3/n+2/n^2+17/n^3-11/n^4}{-2+3/n+5/n^2-12/n^4}$. Using the theorem about sums, products and quotients of limits, and the fact that $1/n^k \to 0$ for any $k > 0$, we get that

$$\frac{8-3/n+2/n^2+17/n^3-11/n^4}{-2+3/n+5/n^2-12/n^4} \to \frac{8-3\cdot 0+2\cdot 0+17\cdot 0-11\cdot 0}{-2+3\cdot 0+5\cdot 0-12\cdot 0} = \frac{8}{-2} = -4$$

So the limit is -4.

2(b) [2 pts] By the same argument, note that

$$\frac{-n^3+5n^2-11}{n^4+n^3+n^2+n+1} = \frac{-1/n+5/n^2-11/n^4}{1+1/n+1/n^2+1/n^3+1/n^4} \to \frac{-0+5\cdot 0-11\cdot 0}{1+0+0+0+0} = 0$$

so the limit is 0.

2(c) [2 pts] First note that $\frac{2n+(-1)^n}{(-1)^n n-\sqrt{n}} = \frac{2+(-1)^n/n}{(-1)^n-1/\sqrt{n}}$. Let a_k be the subsequence with $n_k = 2k$: then

$$a_k = \frac{2+1/(2k)}{1-\sqrt{2k}} \to \frac{2+0}{1-(\sqrt{2}\cdot 0)} = \frac{2}{1} = 2$$

Compare it to the subsequence b_k with $n_k = 2k-1$: $b_k = \frac{2-1/(2k-1)}{-1-1/\sqrt{2k-1}} \to \frac{2-0}{-1-0} = -1$ (We know $1/(2k-1) \to 0$ because it is a subsequence of $1/n$, which certainly converges to 0; we know $1/\sqrt{2k-1} \to 0$ because it is a subsequence of $1/\sqrt{n}$, which we know converges to 0. By the Subsequence Theorem proved in class, subsequences of convergent sequences themselves converge.) So the sequence diverges, because it has subsequences converging to different numbers.

2(d) [2 pts] We know $(-1)^n/n \to 0$, by the Squeeze Theorem (it is squeezed between $-1/n$ and $1/n$). Similarly, $(-1)^n/\sqrt{n} \to 0$, by the Squeeze Theorem (it is squeezed between $-1/\sqrt{n}$ and $1/\sqrt{n}$). So

$$\frac{2n+(-1)^n}{n-(-1)^n\sqrt{n}} = \frac{2+(-1)^n/n}{1-(-1)^n/\sqrt{n}} \to \frac{2+0}{1-0} = 2$$

So the limit is 2.

2(e) [2 pts] We did a question like this in Week8 notes.

$$\sqrt{n^4+2n+1}-\sqrt{n^4-3n+2} = (\sqrt{n^4+2n+1}-\sqrt{n^4-3n+2})\frac{\sqrt{n^4+2n+1}+\sqrt{n^4-3n+2}}{\sqrt{n^4+2n+1}+\sqrt{n^4-3n+2}}$$

$$= \frac{(\sqrt{n^4+2n+1}-\sqrt{n^4-3n+2})(\sqrt{n^4+2n+1}+\sqrt{n^4-3n+2})}{\sqrt{n^4+2n+1}+\sqrt{n^4-3n+2}}$$

$$= \frac{n^4+2n+1-(n^4-3n+2)}{\sqrt{n^4+2n+1}+\sqrt{n^4-3n+2}} = \frac{5n-1}{\sqrt{n^4+2n+1}+\sqrt{n^4-3n+2}}$$

How does that fraction behave as $n \to \infty$? The complicated part is the bottom. n^4+2n+1 and n^4-3n+2 both grow like n^4 (the dominant term for both), so $\sqrt{n^4+2n+1}$ and $\sqrt{n^4-3n+2}$ both grow like $\sqrt{n^4} = n^2$, so the bottom should grow like $2n^2$, and the top only grows like $5n$, so it looks like this fraction has limit 0. Let's prove this. Both the top and the bottom are positive. We want to find a bigger simpler fraction which still converges to 0 (so we can then apply the Squeeze Theorem). So we want a smaller simpler bottom. Certainly, $n^4+2n+1 > n^4$. Also, $n^4-3n+2 > n^4/4$ when $n > 2$: to see this, note that $n^3 > 4$ when $n > 2$ (cube both sides), so $3n^3/4 > 3$ hence $3n^4/4 > 3n$ hence $n^4-3n > n^4/4$ hence $n^4-3n+2 > n^4/4$. So for $n > 2$,

$$\frac{5n-1}{\sqrt{n^4+2n+1}+\sqrt{n^4-3n+2}} < \frac{5n-1}{\sqrt{n^4}+\sqrt{n^4/4}} = \frac{5n-1}{n^2+n^2/2} = \frac{5n-1}{3n^2/2}$$

which tends to 0 by the usual argument.

So the proof that we get a limit of 0, is the Squeeze Theorem. One sequence is $a_n = 0$. The other is $c_n = \frac{5n-1}{3n^2/2}$. Both of these have limit 0, and $a_n < \sqrt{n^4+2n+1} - \sqrt{n^4-3n+2} < c_n$.

2(f) [2 pts] $\frac{n^2+n\cos(n^3+5n)}{\frac{1}{n}+\sqrt{n}-n^2} = \frac{1+\cos(n^3+5n)/n}{\frac{1}{n^3}+1/(n\sqrt{n})-1}$. Now, $1/n^3 \to 0$ and $1/(n\sqrt{n}) \to 0$. To prove $\cos(n^3+5n)/n \to 0$, use the Squeeze Thm: $-1 \le \cos(\text{anything}) \le 1$, so $-1 \le$

$\cos(n^3+5n) \leq 1$, so $-1/n \leq \cos(n^3+5n)/n \leq 1/n$, so $\lim_{n\to\infty} \cos(n^3+5n)/n = 0$. So putting this all together, we get $\lim_{n\to\infty} \frac{n^2+n\cos(n^3+5n)}{1/n+\sqrt{n}-n^2} = \frac{1+0}{0+0-1} = -1$.

3(a) [3 pts] Let a_n be a sequence, and $\sup a_n = \infty$. Then for any M we choose, there must be infinitely many n such that $a_n > M$ (proof by contradiction: if there were only finitely many, then there would only be finitely many values of a_n which are $> M$, so we could take the maximum of that finite set, and that number — not ∞ — would be the supremum).

In particular, there must be infinitely many n with $a_n > 1$. Choose one of these at random and call it n_1. Next, there must be infinitely many n with $a_n > \max\{2, a_{n_1}\}$; take one of them bigger than n_1 and call it n_2. Next, there must be infinitely many n with $a_n > \max\{3, a_{n_2}\}$; take one of them bigger than n_2 and call it n_3. Etc etc. The result is a sequence $n_1 < n_2 < n_3 < \cdots$ such that $a_{n_1} < a_{n_2} < a_{n_3} < \cdots$ and such that $k < a_{n_k}$. So the subsequence $b_k = a_{n_k}$ monotonically increases to ∞.

3(b) [3 pts] Let a_n be a sequence, which has a subsequence $b_k = a_{n_k}$ such that b_k tends to ∞. Suppose for contradiction that $\sup a_n$ is not infinity. That means a_n is bounded from above, so there is some M such that $a_n < M$ for all n.

Now, b_k tends to ∞ so there is some N such that $b_k > M$ for all $k > N$. So there is an a_n, e.g. $a_{n_{N+1}} = b_{N+1}$, which is bigger than M. But this contradicts how we chose M. This contradiction means $\sup a_n = \infty$.

4. [3 pts] Consider first $a \geq 1$. Write $a^{1/n} = 1 + x_n$, so $a \geq 1$ implies $a^{1/n} \geq 1$, which implies $x_n \geq 0$. Then $a = (1+x_n)^n$, which is $\geq 1+nx_n$ by Question 5(a), so $(a-1)/n \geq x_n$. Thus $(a-1)/n \geq x_n \geq 0$, so by the Squeeze Theorem $x_n \to 0$, so $a^{1/n} = 1+x_n \to 1+0 = 1$.

Now consider $0 < a < 1$, so $1/a > 1$. From the previous paragraph, we know $1/a^{1/n} = (1/a)^{1/n} \to 1$, so $a^{1/n} \to 1$ by the Limit of Quotients.

5(a) The given statement works with the sequence $a_n = (-1)^n$ and $L = 1$ (or $L = -1$), but L is not a limit of that sequence. On the other hand, when $a_n \to L$, then the given statement is true.

5(b) The sequence $a_n = 1/n$ converges to $L = 0$, but doesn't satisfy statement 5(b).

MATH 117 Assignment #7: Sequences again

Due 11:59 pm Sunday October 28, 2018

1. [10 pts] Find the following limits. Show your work.

(a) $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$

(b) $(0.99)^n n^{1000}$

(c) $\frac{n!}{100^n}$ where $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ (e.g. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$).

(d) $\frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}}$

(e) $\frac{(n!)^2}{(2n)!}$

2. [3 pts] Find a bounded sequence a_n and three subsequences $b_k = a_{n_k}$, $c_k = a_{n'_k}$, $d_k = a_{n''_k}$ such that b_k, c_k, d_k each converge to a different values. So $b_k \to L$, $c_k \to L'$, $d_k \to L''$, and L, L', L'' are all different. Justify your answer.

3. [5 pts] Consider the sequence a_n given by

$$-1, 0, 1,$$
$$-2, -1\tfrac{1}{2}, -1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, 1\tfrac{1}{2}, 2,$$
$$-3, -2\tfrac{2}{3}, 2\tfrac{1}{3}, -2, -1\tfrac{2}{3}, -1\tfrac{1}{3}, -1, -\tfrac{2}{3}, -\tfrac{1}{3}, 0, \tfrac{1}{3}, \tfrac{2}{3}, 1, 1\tfrac{1}{3}, 1\tfrac{2}{3}, 2, 2\tfrac{1}{3}, 2\tfrac{2}{3}, 3,$$
$$-4, -3\tfrac{3}{4}, \ldots$$

So we first run through the 3 fractions from -1 to $+1$ with denominators of 1, then we run through the 9 fractions from -2 to $+2$ with denominators of 2, then ..., then run through the $2k^2 + 1$ fractions from $-k$ to k with denominators of k, etc etc etc

(a) Show that, for any $r \in \mathbb{Q}$, there are infinitely many n such that $a_n = r$.

(b) Show that every real number x is the limit of a subsequence of a_n.

4. [6 pts] For this question, let a_n be any sequence.

(a) Show that $\lim_{n\to\infty} a_n$ exists iff $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

(b) Consider any convergent subsequence a_{n_k} of a_n. Show that $\liminf_{n\to\infty} a_n \leq \lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n$.

(c) Show that a_n has subsequences which converge to $\liminf_{n\to\infty} a_n$ and to $\limsup_{n\to\infty} a_n$.

5. [4 pts] Find the limits of all convergent subsequences of $\sqrt{2 + 2(-1)^n \cos(\pi/(n^2+1))}$.

6. [4 pts] Let $a \in \mathbb{R}$. Show that there is a sequence $n_1 < n_2 < n_3 < \cdots$ of natural numbers such that $\lim_{k\to\infty} \sin(a^{n_k})$ exists. (The sequence will depend on a.)

MATH 117 Assignment #7 Solutions

1(a) [2 pts] The easiest way to do this is to note that a_n is monotone decreasing. It is also bounded below by 0 (i.e. all a_n are positive). So by the Bounded Monotone Theorem, a_n must converge, and it will converge to the infimum L of the sequence. Certainly the limit L will be ≥ 0, since 0 is a lower bound of all a_n. Can it converge to some $L > 0$? No of course not, since for any $L > 0$ I can find an $N \in \mathbb{N}$ such that $1/N < L$, and some a_n's will equal $1/N$. So the limit has to be 0.

This can also be proved the long way. Choose any $\epsilon > 0$, and let $K > 1/\epsilon$, so $1/K < \epsilon$. How many terms a_n in the sequence are $\geq 1/K$? Well, $1 + 2 + \cdots + K$, since exactly one will equal 1, exactly 2 will equal 1/2, ..., exactly K will equal $1/K$. Let $N = 1+2+\cdots+K$. If you choose any $n > N$, then $|a_n| \leq \frac{1}{K+1} < \frac{1}{K} < \epsilon$, and we're done.

1(b) [2 pts] The ratio test is a good way to do this. Look at the sequence

$$\frac{(0.99)^{n+1}(n+1)^{1000}}{(0.99)^n n^{1000}} = 0.99\,(1 + \frac{1}{n})^{1000}$$

What does that sequence tend to? Well, certainly $1 + \frac{1}{n} \to 1$, so $(1 + \frac{1}{n})^{1000} \to 1^{1000} = 1$ so the ratio $0.99\,(1 + \frac{1}{n})^{1000}$ tends to 0.99. Because this is positive and less than 1, we know by the Ratio Test that the sequence $(0.99)^n n^{1000}$ must tend to 0.

1(c) [2 pts] Again, use the ratio test:

$$\left(\frac{(n+1)!}{100^{n+1}}\right) / \left(\frac{n!}{100^n}\right) = \frac{n+1}{100}$$

As $n \to \infty$, $(n+1)/100 \to \infty$. By the ratio test, this means the sequence $\frac{n!}{100^n}$ will tend to either $+\infty$ or $-\infty$. But it must be $+\infty$, because all terms in the sequence are positive.

1(d) [2 pts] The dominant term on the top say is 3^n, so divide top and bottom by it:

$$\frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}} = \frac{(-2/3)^n + 1}{-2(-2/3)^n + 3} \to \frac{0+1}{-2 \cdot 0 + 3} = \frac{1}{3}$$

1(e) [2 pts] Use ratio test:

$$\frac{((n+1)!)^2(2n)!}{(2n+2)!(n!^2)} = \frac{((n+1)!/n!)^2}{(2n+2)!/(2n)!} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{2(2n+1)} \to \frac{1}{4}$$

2. [3 pts] There are lots of such sequences. The easiest is 0,1,2,0,1,2,0,1,2,....:
The subsequence $a_1, a_4, a_7, ... = 0, 0, 0, ...$ has limit 0.
The subsequence $a_2, a_5, a_8, ... = 1, 1, 1, ...$ has limit 1.
The subsequence $a_3, a_6, a_9, ... = 2, 2, 2, ...$ has limit 2.

3(a) [2 pts] To clarify things, let's call "level 1" the first 3 terms $-1, 0, 1$ in our sequence; "level 2" are the next 9 terms $-2, -1\frac{1}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$, etc. So "level k" means the $2k^2 + 1$ fractions from $-k$ to k with denominators of k.

Choose any rational number $r = a/b$, where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Let m be the smallest integer greater than or equal to $|r|$; then $r = a/b$ will appear in level k, where k is the smallest multiple of b which is greater than m: if $k = db$, then $r = da/(db)$ is a fraction between $-k$ and k (in fact it's between $-m$ and m) with denominator k.

But it will also appear in the $2k$th level, since we can write $r = (2da)/(2db)$, and in the $3k$th level, and $4k$th level, ...

3(b) [3 pts] Choose any $x \in \mathbb{R}$. We can approximate x with a sequence $r_1, r_2, r_3, ...$ of rational numbers tending to x.

My subsequence is: find any n_1 such that $a_{n_1} = r_1$ (by part (a), we know there is such an n_1, in fact infinitely many). Now choose n_2 so that $a_{n_2} = r_2$, and the "level" of n_2 is greater than that of n_1 (since there are infinitely many n's with $a_n = r_2$, we know there are such n_2 — in fact there will be infinitely many). Next choose n_3 so that $a_{n_3} = r_3$, and the "level" of n_3 is greater than that of n_2. etc etc etc

This is a subsequence, because $n_1 < n_2 < n_3 < \cdots$. And $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ converges to x (since $r_1, r_2, r_3, ...$ do).

4. This question was meant to assume that a_n is any **BOUNDED** sequence, but i forgot to include that word. The marker won't take off any marks if you only deal with the case that both $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are finite.

(a) [2 pts] There are 2 directions to show.

$\Longrightarrow$ Suppose first that $\lim_{n\to\infty} a_n$ exists and equals L. We want to show that $\liminf_{n\to\infty} a_n = L = \limsup_{n\to\infty} a_n$. Without loss of generality we'll just show that $L = \limsup_{n\to\infty} a_n$. The proof that $L = \liminf_{n\to\infty} a_n$ is identical. Let's also assume L is finite. (I'll explain after that how to do L infinite). Write $s_n = \sup\{a_n, a_{n+1}, ...\}$. Then $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n$.

Choose any $\epsilon > 0$. Then there exists an N such that for all $n > N$, $|a_n - L| < \epsilon$. That implies that, for all $n > N$, $a_n < L + \epsilon$, so $s_n \leq L + \epsilon$. Also, $a_n > L - \epsilon$, so $s_n > L - \epsilon$. Hence $|s_n - L| \leq \epsilon$, and we're done: the limit of s_n must also be L.

(If $L = \infty$, then for any M, all sufficiently big n will have $a_n > M$ and hence $s_n > M$, and we get that $s_n \to \infty$. $L = -\infty$ is done similarly.)

$\Longrightarrow$ Now suppose that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. Call this common value L. We want to prove that $\lim_{n\to\infty} a_n = L$. Assume L is finite (I'll explain at the end how to do L infinite). Write $s_n = \sup\{a_n, a_{n+1}, ...\}$ and $i_n = \inf\{a_n, a_{n+1}, ...\}$. Then $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n$ and $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} i_n$.

Choose any $\epsilon > 0$. Because $\lim_{n\to\infty} s_n = L$, there exists an N' such that for all $n > N'$, $|s_n - L| < \epsilon$. That means that, for all $n > N'$, $s_n < L + \epsilon$, so all a_n satisfy $a_n < L + \epsilon$ (since s_n is their sup).

Similarly, because $\lim_{n\to\infty} i_n = L$, there exists an N'' such that for all $n > N''$, $|i_n - L| < \epsilon$. That means that, for all $n > N''$, $i_n > L - \epsilon$, so all a_n satisfy $a_n > L - \epsilon$ (since i_n is their inf).

So if we take $N = \max\{N', N''\}$, then we get that, for all $n > N$, we have both $a_n < L + \epsilon$ (since $n > N'$) and $a_n > L - \epsilon$ (since $n > N''$). In other words, $|a_n - L| < \epsilon$, and we're done!

(If $L = \infty$, then for all M, all sufficiently big n will have $i_n > M$, so all $a_n > M$, so $\lim_{n\to\infty} a_n = \infty$. Similarly if $L = -\infty$.)

(b) [2 pts] Assume $b_k = a_{n_k}$ is a subsequence of a_n with $\lim_{k\to\infty} b_k = L$ for some L. Consider first that L is finite (we'll do L infinite later). We want to show that $L \le \limsup_{n\to\infty} a_n$. The proof that $\liminf_{n\to\infty} a_n \le L$ is identical.

Suppose for contradiction that this is wrong. That it, suppose $L > \limsup_{n\to\infty} a_n$. Choose $\epsilon = (L - \limsup_{n\to\infty} a_n)/2$ (which is positive by our hypothesis). Then there exists an N such that for all $k > N$, $b_k > L - \epsilon$. That means that for all n, $s_n > L - \epsilon$, since s_n is the sup of the a_m for $m \ge n > N$, but this will include some of the n_k since $n_k \to \infty$. Hence $\limsup_{m\to\infty} a_m = \lim_{m\to\infty} s_n \ge L - \epsilon$. But $L - \epsilon = (L + \limsup_{n\to\infty} a_n)/2 > \limsup_{n\to\infty} a_n$, a contradiction. We're done.

(Strictly speaking, L must be finite, because we used the word 'converges', but that is just a technicality. If $L = \infty > \limsup_{n\to\infty} a_n$, then choose $M > \limsup_{n\to\infty} a_n$. All sufficiently big k have $b_k > M$. By the above reason, this means all $s_n > M$, so $\limsup_{m\to\infty} a_m \ge M$. This contradicts $M > \limsup_{n\to\infty} a_n$.)

(c) We will show that a_n has a subsequence which converge to $\limsup_{n\to\infty} a_n$. The proof for $\liminf_{n\to\infty} a_n$ is similar.

Let $L = \limsup_{n\to\infty} a_n$. Choose any $\epsilon > 0$ and N. We will show that there is an $n > N$ such that $|a_n - L| < \epsilon$. Suppose not. Then for all $n > N$, $|a_n - L| \ge \epsilon$. If there were infinitely many n with $a_n - L \ge \epsilon$, then for all m, $s_m \ge L + \epsilon$, so $L = \limsup_{n\to\infty} a_n \ge L + \epsilon$, a contradiction. Therefore choosing N a little bigger (so we get past the finitely many n with $a_n - L \ge \epsilon$), all $n > N$ have $L - a_n \ge \epsilon$. Therefore all $n > N$ have $L - \epsilon > a_n$, hence for all $m > N$, $L - \epsilon \ge s_m$, which taking limits gives $L - \epsilon \ge L$, a contradiction.

Now apply Midterm 2 question 11: we get that there is a subsequence converging to $L = \limsup_{n\to\infty} a_n$.

(If $L = \infty$, a similar (but slightly easier) argument shows that for any M and N, there is an $n > N$ such that $a_n > M$. The same argument used in the proof of the Midterm 2 Qu.11 then gives us a subsequence $b_k \to \infty = L$.)

5. [4 pts] You can assume for this question that $\lim_{n\to\infty} \cos(\pi/(n^2+1)) = \cos(0) = 1$. (In language we will soon learn, this is because $\cos(x)$ is continuous).

So for n even, the sequence tends to $\sqrt{2+2} = 2$ and for n odd, the sequence tends to $\sqrt{2-2} = 0$.

More generally, if the subsequence n_k has infinitely many odd n's but only finitely many even n's, then it has the same limit as the purely odd subsequence. And if instead the subsequence n_k has infinitely many even n's but only finitely many odd n's, then it has the same limit as the purely even subsequence. Those are the only convergent subsequences: if some subsequence n_k has infinitely many odds *and* infinitely many evens, then the odd subsequence of the subsequence(!) will have a different limit from the even subsequence of the subsequence, so that subsequence a_{n_k} won't converge.

6. [4 pts] Let $a_n = \sin(a^n)$. Then $-1 \leq a_n \leq 1$ for all n, so a_n is bounded. By Bolzano-Weierstrass, there is a subsequence a_{n_k} which is convergent. In other words, $\lim_{k\to\infty} \sin(a^{n_k})$ exists.

MATH 117 Assignment #8: Limits for functions

Due 11:59 pm Wednesday November 27, 2019

1. [3 pts] Use Cauchy's Theorem to prove that $\sum_{k=1}^{n} \frac{1}{k^2}$ converges to a finite number as $n \to \infty$. (Hint: use the fact that $\frac{1}{k^2} < \frac{1}{k\,(k-1)} = \frac{1}{k-1} - \frac{1}{k}$.)

2. [2 pts] Use Cauchy's theorem to prove that the sequence $x_n = (-1)^n\left(1 - \frac{1}{n}\right)$ diverges. (No credit will be given for using any other method.)

3. [1 pt] What is the natural domain of

$$f(x) = 5\sqrt{x} - \sin(1/x) + 27\frac{x^2 - 16}{x - 10}$$

(i.e. find the largest subset of $\mathbb{R}$ where $f(x)$ as stated is defined).

4. [8 pt] Using the definition of the limit of a function, show that

(a) $\lim_{x\to\infty} \frac{3x+4}{5x+7} = \frac{3}{5}$

(b) $\lim_{x\to-\infty} \sqrt{-x} = \infty$

(c) $\lim_{x\to 3} 5x^2 = 45$

(d) $\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty$

5. [3 pts] Suppose that $f : \mathbb{R} \to \mathbb{R}$. Assume that $\lim_{x\to\infty} f(x) = \infty$. Prove that, if x_n is any sequence tending to ∞, $\lim_{n\to\infty} f(x_n) = \infty$.

MATH 117 Assignment #8: Solutions

1. [3 pts] Write $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$ as usual. We want to prove s_n is Cauchy. Choose any $\epsilon > 0$. Let $N = 1/\epsilon$. Then for any $m > n > N$,

$$|s_m - s_n| = \sum_{k=n+1}^{m} \frac{1}{k^2} < \sum_{k=n+1}^{m} \frac{1}{k\,(k-1)} = \sum_{k=n+1}^{m} \left(\frac{1}{k-1} - \frac{1}{k} \right)$$

$$= \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m-1} - \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m} \right) = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \frac{1}{N} = \epsilon$$

Therefore $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is Cauchy.

(Of course this means it converges. It can be shown that the limit is $\pi^2/6$)

2. [2 pts] Choose $\epsilon = 1$. Then for any $N \geq 1$, take $n = 2N$ and $m = 2N+1$. Therefore $m > n > N$ and

$$|x_m - x_n| = \left|(-1)\big(1 - \frac{1}{2N+1}\big) - \big(1 - \frac{1}{2N}\big)\right| = \left|-1 + \frac{1}{2N+1} - 1 + \frac{1}{2N}\right| = 2 - \frac{1}{2N} - \frac{1}{2N+1}$$

$$\geq 2 - \frac{1}{2} - \frac{1}{3} > 1 = \epsilon$$

So x_n is not Cauchy. Therefore x_n diverges.

(No credit will be given for using any other method.)

3. [1 pt] $\sqrt{x}$ requires that $x \geq 0$. $\sin(1/x)$ requires that $x \neq 0$. $\frac{x^2-16}{x-10}$ requires that $x \neq 10$. Putting all this together, we get that the natural domain is $(0, 10) \cup (10, \infty)$.

4(a) [2 pts] This is an infinite-finite limit. Take an arbitrary $\varepsilon > 0$. Put $N = \frac{1}{25\epsilon}$. If $x > N$ then $x > \frac{1}{25\epsilon}$, so that

$$\left| \frac{3x+4}{5x+7} - \frac{3}{5} \right| = \frac{1}{5(5x+7)} \leq \frac{1}{25x} < \epsilon$$

(b) [2 pts] This is an infinite-infinite limit. Choose any M. (We're interested in M very large.) Take $N = -M^2$. Then for any $x < N$,

$$\sqrt{-x} > \sqrt{-N} = \sqrt{M^2} = |M| \geq M$$

Here we're using that if $x < y$, then $-x > -y$, and if $a > b > 0$, then $\sqrt{a} > \sqrt{b}$.

(c) [2 pts] This is a finite-finite limit. Choose any $\epsilon > 0$. Let $\delta = \min\{\frac{\epsilon}{35}, 1\}$. Then for any $0 < |x - 3| < \delta$,

$$|5x^2 - 45| = 5|x^2 - 9| = 5|(x-3)(x+3)| = 5|x+3|\,|x-3|$$

We want to show that this is $< \epsilon$, so we But because $\delta \geq 1$, $|x-3| < \delta$ implies that $x \in (2,4)$, so $5|x+3|$ ranges from 25 to 35. In particular, $5|x+3| < 35$, so $|5x^2-45| < 35|x-3| < 35\delta$. Now, $\delta \leq \frac{\epsilon}{35}$ tells us $|5x^2-45| < 35|x-3| < \epsilon$ and we're done.

(d) [2 pts] This is a finite-infinite limit. Choose any N. Let $\delta = \frac{1}{\sqrt{|N|}}$. Then $0 < |x-2| < \delta$ implies

$$\frac{1}{(x-2)^2} = \frac{1}{|x-2|^2} > \frac{1}{\delta^2} = \frac{1}{1/|N|} = |N| \geq N$$

where we use the Golden Rule of fraction inequalities, as well as the fact that if $a > b > 0$, then $a^2 > b^2$.

5. [3 pts] Assume $\lim_{x\to\infty} f(x) = \infty$. Also, assume that $x_n \to \infty$. We want to prove that $f(x_n) \to \infty$.

So choose any M. Then, because $\lim_{x\to\infty} f(x) = \infty$, there is some N' such that, whenever $x > N'$, $f(x) > M$. And, because $x_n \to \infty$, there exists some N such that, whenever $n > N$, $x_n > N'$. Putting these together, whenever $n > N$, $x_n > N'$ and hence $f(x_n) > M$. We're done!

MATH 117 Assignment #9: Solutions

1. [2 pts] NOT($\lim_{x\to 0} \sin(1/x) = 0$) means that there exists a $\epsilon_0 > 0$ such that $\forall \delta > 0$, there is an x_δ such that $|x_\delta - 0| < \delta$ and $\sin(1/x_\delta) - 0| \geq \epsilon_0$.

The reason the limit can't be 0, is because for any $L \in [-1, 1]$, there is a sequence $x_n \to 0$ such that $\sin(1/x_n) \to L$. So if we choose ϵ_0 to be anything less than 1, and we choose x_δ sufficiently close to 0 so that $\sin(1/x)$ is close to $L = 1$, then $|\sin(1/x) - 0|$ will be $\geq \epsilon_0$. Note that $\sin(1/x) = 1$ whenever $1/x = \frac{\pi}{2} + 2n\pi$ for any $n \in \mathbb{Z}$. So take

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$$

Then $\sin(1/x_n) = 1$ for all n. Also, $1/(\frac{\pi}{2} + 2\pi n) \to 0$ as $n \to \infty$. We will use x_n to get the argument.

Let $\epsilon_0 = 1/2$. Choose any $\delta > 0$. Because $x_n \to 0$, there is an N such that for any $n > N$, $|x_n - 0| < \delta$. So take e.g. $x_\delta = x_{N+1}$. Then $|x_\delta - 0| < \delta$ but $|\sin(1/x_\delta) - 0| = 1 > \epsilon_0$. We are done.

2(a) [1 pt] $x \to 0^-$ means we're interested only in *negative* x close to 0. But this doesn't matter much here. We know there are lots of rational, negative (or positive) $x \in \mathbb{Q}$ as close as you want to 0. To be precise, take $x_n = -1/n$: then each x_n is rational (and negative), so $f(x_n) = 0$, and $x_n \to 0$. And there are lots of irrational, negative x as close as you want to 0. To be precise, we can take $x'_n = -\pi/n$: then each x'_n is irrational (and negative), so $f(x'_n) = 1$, and $x'_n \to 0$. We can use these two sequences to get our argument.

So suppose for contradiction that $\lim_{x\to 0^-} f(x) = L$ for some number $L \in \mathbb{R}$. Take $\epsilon_0 = 1/2$. Choose any $\delta > 0$. We want to find an x_δ such that $-\delta < x_\delta < 0$ and $|f(x_\delta) - L| \geq \epsilon_0$.

Now, either $|L - 0| \geq \epsilon_0$ or $|L - 1| \geq \epsilon_0$ (or both). Suppose first that $|L - 0| \geq \epsilon_0$. Then since $x_n \to 0$, there exists an N such that for all $n > N$, $|x_n - 0| < \delta$. So let $x_\delta = x_{N+1}$. Then $-\delta < x_\delta < 0$ and $f(x_\delta) = 0$, so $|f(x_\delta) - L| \geq \epsilon_0$. This contradicts $\lim_{x\to 0^-} f(x) = L$.

Suppose instead that $|L - 1| \geq \epsilon_0$. Then since $x'_n \to 0$, there exists an N such that for all $n > N$, $|x'_n - 0| < \delta$. So let $x_\delta = x'_{N+1}$. Then $-\delta < x'_\delta < 0$ and $f(x'_\delta) = 1$, so $|f(x'_\delta) - L| \geq \epsilon_0$. This means that $\lim_{x\to 0^-} f(x)$ *does not exist.* This also contradicts $\lim_{x\to 0^-} f(x) = L$, and we're done.

For what it's worth, $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0} f(x)$ also do not exist.

2(b) [1 pt] $x \to 1^+$ means we're interested only in $x > 1$ close to 1. For such x, $f(x) = 0$, so $\lim_{x\to 1^+} f(x) = 0$.

For what it's worth, $\lim_{x\to 1^-} f(x) = 3$.

3. [3 pts] $\Longrightarrow$ Assume that $\lim_{x\to a} f(x) = L$. We want to prove that $\lim_{x\to a^+} f(x) = L$ (the proof that $\lim_{x\to a^-} f(x) = L$ is similar).

Choose any $\epsilon > 0$. Because $\lim_{x \to a} f(x) = L$, there exists a $\delta > 0$ such that whenever $0 < |x - a| < \delta$, we have $|f(x) - L| < \epsilon$. Take any x satisfying $0 < x - a < \delta$. Then $0 < |x - a| < \delta$, so $|f(x) - L| < \epsilon$. We've just proved that $\lim_{x \to a^+} f(x) = L$ (it's as easy as that!)

$\Longleftarrow$ Assume that $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$. We want to prove that $\lim_{x \to a} f(x) = L$.

Choose any $\epsilon > 0$. Because $\lim_{x \to a^+} f(x) = L$, we know that there exists a $\delta' > 0$ such that for any x satisfying $0 < x - a < \delta'$, $|f(x) - L| < \epsilon$. And because $\lim_{x \to a^-} f(x) = L$, we know that there exists a $\delta'' > 0$ such that for any x satisfying $0 > x - a > -\delta''$, $|f(x) - L| < \epsilon$.

Take $\delta = \min\{\delta', \delta''\}$. Then $\delta > 0$. Choose any x satisfying $0 < |x - a| < \delta$. If $x > a$, then $0 < x - a < \delta \leq \delta'$ so $|f(x) - L| < \epsilon$. Otherwise, $x < a$, in which case $0 > x - a > -\delta \geq -\delta''$ so again $|f(x) - L| < \epsilon$. Thus in either case, $|f(x) - L| < \epsilon$. Hence $\lim_{x \to a} f(x) = L$.

4. [1 pt] $f(x)$ is continuous at $x = 2$ if $\lim_{x \to 2} f(x) = f(2)$. So we first need to compute the limit $\lim_{x \to 2} f(x)$, and then define $f(2)$ to be that value. We know from Qu.3 above, that to show that $\lim_{x \to 2} f(x)$ exists, and to compute its value, we need to compute both $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$ and see those 2 limits are equal.

Compute first $\lim_{x \to 2^+} f(x)$. For $x > 2$, $f(x) = \frac{3}{1-x}$. The function $\frac{3}{1-x}$, being a ratio of polynomials, is continuous wherever it is defined, i.e. at any $x \neq 1$. In particular, it is continuous at $x = 2$. So $\lim_{x \to 2^+} f(x) = \lim_{x \to 2} \frac{3}{1-x} = \frac{3}{1-2} = -3$.

Compute next $\lim_{x \to 2^-} f(x)$. For $x < 2$, $f(x) = 1 - x^2$. The function $1 - x^2$ is a polynomial, so is continuous everywhere. In particular, it is continuous at $x = 2$. So $\lim_{x \to 2^-} f(x) = \lim_{x \to 2} 1 - x^2 = 1 - 2^2 = -3$.

So $\lim_{x \to 2} f(x)$ exists and equals -3. So if we define $f(2)$ to be -3, $f(x)$ will be continuous there.

5. [4 pts] The function $\frac{x^2-1}{x-1}$ is a quotient of continuous functions (polynomials), so is continuous everywhere it is defined (i.e. anywhere the denominator doesn't vanish, so it is continuous when $x \neq 1$). Choose any $a \in \mathbb{R}$. Then for all sufficiently small $\delta > 0$, any x satisfying $0 < |x - a| < \delta$ will be a non-integer (take $\delta < 1$ if $a \in \mathbb{Z}$, otherwise take δ to be less than the distance from a to the nearest integer). What this means is that, in the definition of $\lim_{x \to a} f(x)$, we can replace $f(x)$ with the continuous function $\frac{x^2-1}{x-1} = x + 1$, and we obtain $\lim_{x \to a} f(x) = a + 1$.

When $a \notin \mathbb{Z}$, then $f(a) = \frac{a^2-1}{a-1} = a + 1 = \lim_{x \to a} f(x)$. Hence $f(x)$ is continuous at any $a \notin \mathbb{Z}$.

When $a \in \mathbb{Z}$, then $f(a) = a^2 + 2a - 1$, which equals $\lim_{x \to a} f(x) = a + 1$ iff $a^2 + 2a - 1 = a + 1$, i.e. iff $a^2 + a - 2 = 0$, i.e. iff $a = 1$ or $a = -2$. Thus $f(x)$ is also continuous at $x = 1$ and

$x = -2$. At all other $x \in \mathbb{Z}$, it fails to be continuous.

6. [3 pts] The problem is that $\sqrt{2}$ is not rational, and we are only told what $f(x)$ equals, when x is rational. But we're also told that $f(x)$ is continuous. By the Bridge Thm for Continuity, since $f(x)$ is continuous at $x = \sqrt{2}$, we can find any sequence x_n converging to $\sqrt{2}$, and $f(\sqrt{2}) = \lim_{n\to\infty} f(x_n)$.

So choose x_n to be any sequence of rational numbers tending to $\sqrt{2} = 1.4142135623....$ We know from class that there are lots of these, one is $x_1 = 1, x_2 = 1.4, x_3 = 1.41, x_4 = 1.414, x_5 = 1.4142, x_6 = 1.41421, ...$ Since each $x_n \in \mathbb{Q}$, $f(x_n) = x_n^4 - 3$. As $n \to \infty$, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^4 - 3 = (\sqrt{2})^4 - 3$ (since $x^4 - 3$, being a polynomial, is continuous), which in turn equals $4 - 3 = 1$. So $\lim_{n\to\infty} f(x_n) = 1$. By continuity of f, $f(\sqrt{2}) = 1$.

More generally, continuity of f, and the fact that there are sequences of rationals tending to any $x \in \mathbb{R}$, forces $f(x) = x^4 - 3$ for all x.

7(a) [2 pts] Let x_n be any sequence tending to 0. Since $-1 \le \cos(\theta) \le 1$ for all $\theta \in \mathbb{R}$, so $-x_n^2 \le x_n^2 \cos(1/x_n^2) \le x_n^2$. By the Squeeze Thm, $\lim_{n\to\infty} x_n^2 \cos(1/x_n^2) = 0$. Because x_n is arbitrary, the Bridge Thm concludes that $\lim_{x\to 0} x^2 \cos(1/x^2) = 0$.

(We don't have a Squeeze Thm for function limits, so you should prove it first before you use it. You can prove it exactly as I did here, by combining it with the Bridge Thm.)

(b) [1 pt] $\lim_{x\to 0} \frac{x^2}{\sin(x)} = \lim_{x\to 0} x \frac{x}{\sin(x)} = \lim_{x\to 0} x \lim_{x\to 0} \frac{x}{\sin(x)} = 0 \cdot 1 = 0$, using the Pretty Useful Thm and the fact proved in class that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.