Reminders from Last Times: We have seen that
$$\det : \mathbb{F}^{n\times n} \longrightarrow \mathbb{F}$$
has the following main properties:

I) it is multilinear in the rows (or columns) of $A\in\mathbb{F}^{n\times n}$, that is, for every $1\leq k\leq n$ and every $r\in\mathbb{F}$, $\bar{u},\bar{v}\in\mathbb{F}^{1\times n}$,

$$\det\begin{pmatrix} -R_1- \\ -R_2- \\ \vdots \\ -r\bar{u}+\bar{v}- \\ \vdots \\ -R_n- \end{pmatrix} = \det\begin{pmatrix} -R_1- \\ -R_2- \\ \vdots \\ -\boxed{r}\bar{u}- \\ \vdots \\ -R_n- \end{pmatrix} + \det\begin{pmatrix} -R_1- \\ -R_2- \\ \vdots \\ -\bar{v}- \\ \vdots \\ -R_n- \end{pmatrix}$$
$$= \boxed{r}\det\begin{pmatrix} -R_1- \\ -R_2- \\ \vdots \\ -\bar{u}- \\ \vdots \\ -R_n- \end{pmatrix} + \det\begin{pmatrix} -R_1- \\ -R_2- \\ \vdots \\ -\bar{v}- \\ \vdots \\ -R_n- \end{pmatrix}.$$

(and similarly if we choose to work with columns).

II) it is alternating in the rows (or columns) of $A\in\mathbb{F}^{n\times n}$, that is, if A has two rows (or two columns) equal, then $\boxed{\det(A)=0.}$

III) $\det(I_n)=1$.

Remark We saw that these properties characterise the determinant.

Moreover, they imply one more very useful property:

Property 7 Let $A \in \mathbb{F}^{n\times n}$ and let $1 \le k < l \le n$.

Then
$$\det \begin{array}{r} \\ \\ k\text{-th row} \rightarrow \\ l\text{-th row} \rightarrow \\ \\ \end{array} \begin{pmatrix} -R_1(A)- \\ -R_2(A)- \\ \vdots \\ -R_l(A)- \\ -R_k(A)- \\ -R_n(A)- \end{pmatrix} = -\det(A).$$

We will now rely on these properties to prove

Theorem 1 Let $A \in \mathbb{F}^{n\times n}$. We have that $\det(A) = 0$ if and only if A is <u>not</u> invertible.

and

Theorem 2 (Multiplication Theorem) Let $A, B \in \mathbb{F}^{n\times n}$. We have that $\det(AB) = \det(A)\cdot\det(B)$.

Both proofs will be based on examining how elementary row operations affect the determinant, or in other words on the following

Theorem 0 Let $A \in \mathbb{F}^{n\times n}$, $\lambda \in \mathbb{F}\setminus\{0\}$, $\mu \in \mathbb{F}$, $1 \le i < j \le n$ or $1 \le j < i \le n$. Then

a) $\det(D_{i;\lambda}A) = \lambda\cdot\det(A) = \det(D_{i;\lambda})\det(A)$.

b) $\det(E_{i,j;\mu}A) = \det(A) = \det(E_{i,j;\mu})\cdot\det(A)$

c) $\det(P_{i,j}A) = -\det(A) = \det(P_{i,j})\cdot\det(A)$.

Recall that $D_{i;\lambda}$ stands for the diagonal elementary matrix corresponding to multiplying the i-th row of A by λ,

$E_{i,j,\mu}$ stands for the elementary matrix corresponding to adding to the i-th row of A the j-th row multiplied by μ,
and $P_{i,j}$ stands for the elementary permutation matrix corresponding to swapping the i-th and the j-th rows of A.

<u>Proof of Theorem 0</u> a) We have already seen that
$$\det(D_{i;\lambda}A) = \lambda\cdot\det(A)$$
(follows from the multilinearity of the determinant).
We have also seen that
$$\det(D_{i;\lambda}) = \det\left(\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & \lambda & \\ 0 & & & & \ddots & \\ & & & & & 1 \end{pmatrix}\right) = \lambda,$$
i-th column

thus $\lambda\cdot\det(A) = \det(D_{i;\lambda})\cdot\det(A)$.

Alternatively, if we only use the main properties here, we can obtain this conclusion by noting that, if $A = I_n$, then $\det(D_{i;\lambda}) = \det(D_{i;\lambda}I_n) = \lambda\cdot\det(I_n) = \lambda$.

c) Similarly, we have seen that
$$\det(P_{i,j}A) = -\det(A)$$
(this is Property 7).
We also note that, if $A = I_n$, then
$$\det(P_{i,j}) = \det(P_{i,j}I_n) = -\det(I_n) = -1,$$
and thus $\det(P_{i,j}A) = \det(P_{i,j})\cdot\det(A)$.

Remains to prove (b):

(b) Let's assume that $1 \le i < j \le n$ (the proof is completely analogous if $j < i$). We have that

$$\det(E_{i,j;\mu} A) = \det\begin{pmatrix} - R_1(A) - \\ - R_2(A) - \\ \vdots \\ - R_i(A) + \mu R_j(A) - \\ \vdots \\ - R_j(A) - \\ \vdots \\ - R_n(A) - \end{pmatrix} \underset{\substack{\text{by linearity in the} \\ i\text{-th row}}}{=}$$

$$= \det\begin{pmatrix} - R_1(A) - \\ - R_2(A) - \\ \vdots \\ - R_i(A) - \\ \vdots \\ - R_j(A) - \\ \vdots \\ - R_n(A) - \end{pmatrix} + \mu \cdot \det\begin{pmatrix} - R_1(A) - \\ - R_2(A) - \\ \vdots \\ - R_j(A) - \\ \vdots \\ - R_j(A) - \\ \vdots \\ - R_n(A) - \end{pmatrix} \underset{\substack{\text{b/c det} \\ \text{is alternating}}}{=} \det(A) + \mu \cdot 0 = \det(A)$$

We can also note that, if $A = I_n$, we get
$$\det(E_{i,j;\mu}) = \det(E_{i,j;\mu} \cdot I_n) = \det(I_n) = 1,$$
thus $\det(E_{i,j;\mu} A) = \det(E_{i,j;\mu}) \cdot \det(A)$.

An important consequence of Thm 0 is the following

Proposition 0. Let $A \in \mathbb{F}^{n \times n}$ and consider elementary matrices $E_1, \ldots, E_m \in \mathbb{F}^{n \times n}$. Then
$$\det(E_1 \cdots E_m \cdot A) = \det(E_1)\det(E_2) \cdots \det(E_m)\det(A).$$

Proof. This follows by induction in m.

Base of induction: $m = 1$

Then the conclusion is that of Thm 0.

Induction Step Assume that for some $m \geq 1$

given any elementary matrices $E_1, \ldots, E_m \in F^{n\times n}$,
$\det(E_1 \cdots E_m A) = \det(E_1) \cdots \det(E_m)\det(A)$. } Inductive Hypothesis

Then, if $\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_m, \tilde{E}_{m+1}$ are $m+1$ elementary matrices in $F^{n\times n}$, we have that

$$\det(\tilde{E}_1 \tilde{E}_2 \cdots \tilde{E}_m \tilde{E}_{m+1} A) = \det(\tilde{E}_1 \cdot (\tilde{E}_2 \cdots \tilde{E}_{m+1} A))$$

$$\underset{\text{by Thm 0}}{=} \det(\tilde{E}_1) \cdot \det(\underbrace{\tilde{E}_2 \cdots \tilde{E}_m \tilde{E}_{m+1}}_{m \text{ elem matrices}} A)$$

$$\underset{\text{by Inductive Hypothesis}}{=} \det(\tilde{E}_1) \cdot \det(\tilde{E}_2) \cdots \det(\tilde{E}_m)\det(\tilde{E}_{m+1}) \cdot \det(A).$$

This completes the proof of the induction step, and thus of Proposition 0.

Important Remark As follows from the proof of Thm 0, the determinant of an elementary matrix is non-zero. (this is immediately clear for elementary matrices of the form $E_{i,j,\mu}$ or $P_{i,j}$, while if the matrix has the form $D_{i,\lambda}$, recall that we need $\lambda \neq 0$ for the matrix to be called elementary, and hence $\det(D_{i,\lambda}) = \lambda \neq 0$).

Proof of Thm 1 We will show that:

- if A is invertible, then $\det(A) \neq 0$.
- if A is not invertible, then $\det(A) = 0$.

This is equivalent to the statement of the theorem. (why?)

Assume first that A is invertible. Then, as we have seen in MATH 127, we can write A as a product of elementary matrices, or in other words there are $m \geq 1$ and elementary matrices $\mathcal{E}_1, \mathcal{E}_2, \dots \mathcal{E}_m$ in $\mathbb{F}^{n\times n}$ such that

$$A = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_m.$$

But then, by Proposition 0,

$$\det(A) = \det(\mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_m) = \det(\mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_m I_n)$$
$$= \underbrace{\det(\mathcal{E}_1)}_{\neq 0} \underbrace{\det(\mathcal{E}_2)}_{\neq 0} \cdots \underbrace{\det(\mathcal{E}_m)}_{\neq 0} \underbrace{\det(I_n)}_{\neq 0}$$

which is $\neq 0$ since it is a product of non-zero elements of the field $\mathbb{F}$.

We conclude that $\det(A) \neq 0$, as we wanted.

Conversely, assume that A is not invertible. Then, again as we have seen in MATH 127, any Row Echelon Form of A has fewer pivots than the number n of its rows, and hence it has at least one zero row. In particular, its last row is definitely zero.

Let's consider a REF A' of A. Then there are $k \geq 1$ and elementary matrices $\widetilde{\mathcal{E}}_1, \widetilde{\mathcal{E}}_2, \dots \widetilde{\mathcal{E}}_k$ in $\mathbb{F}^{n\times n}$ such that $A = \widetilde{\mathcal{E}}_1 \widetilde{\mathcal{E}}_2 \cdots \widetilde{\mathcal{E}}_k A'$.

But then, by Proposition 0,

$$\det(A) = \det(\widetilde{\mathcal{E}}_1)\det(\widetilde{\mathcal{E}}_2) \cdots \det(\widetilde{\mathcal{E}}_k)\det(A')$$
$$= \det(\widetilde{\mathcal{E}}_1)\det(\widetilde{\mathcal{E}}_2) \cdots \det(\widetilde{\mathcal{E}}_k)\det\begin{pmatrix} -R_1(A')- \\ -R_2(A')- \\ \vdots \\ -0 \cdot R_n(A')- \end{pmatrix}$$

since the last row of A' is zero anyway

by linearity in the n-th row

$$= \det(\widetilde{\mathcal{E}}_1)\det(\widetilde{\mathcal{E}}_2) \cdots \det(\widetilde{\mathcal{E}}_k) \cdot 0 \det(A') = 0.$$

Determinants (cont.)

We still have to prove

Theorem 2 (Multiplication Theorem) Let $\mathbb{F}$ be a field, n a positive integer, and let $A, B \in \mathbb{F}^{n\times n}$. Then
$$\det(AB) = \det(A)\cdot\det(B).$$

Proof We will consider two main cases and use a different argument to show the conclusion in each one of them.

Case 1 A is invertible

We have seen then that there are $m \geq 1$ and elementary matrices $E_1, E_2, \ldots E_m \in \mathbb{F}^{n\times n}$ such that
$$A = E_1 E_2 \cdots E_m$$
and we also saw in the proof of Thm 1 last time that
$$\det(A) = \det(E_1)\det(E_2)\cdots\det(E_m).$$
Therefore by Proposition 0
$$\det(AB) = \det((E_1E_2\cdots E_m)B)$$
$$= \det(E_1)\det(E_2)\cdots\det(E_m)\cdot\det(B) = \det(A)\cdot\det(B).$$

Case 2 A is not invertible

Then by Thm 1 we know that $\det(A)=0$, and hence $\det(A)\cdot\det(B) = 0$.

Thus we have to show that $\det(AB)=0$ as well, which is equivalent by Thm 1 to AB being non-invertible.

But $AB = \begin{pmatrix} C_1(AB) & C_2(AB) & \cdots & C_n(AB)\end{pmatrix} = \begin{pmatrix} AC_1(B) & AC_2(B) & \cdots & AC_n(B)\end{pmatrix}$

where, for every $1 \leq j \leq n$,

$$AC_j(B) = A\cdot\begin{pmatrix} B_{1j}\\ B_{2j}\\ \vdots \\ B_{nj}\end{pmatrix} = \begin{pmatrix} A_{11}B_{1j} + A_{12}B_{2j} + \dots + A_{1n}B_{nj} \\ A_{21}B_{1j} + A_{22}B_{2j} + \dots + A_{2n}B_{nj} \\ \vdots \\ A_{n1}B_{1j} + A_{n2}B_{2j} + \dots + A_{nn}B_{nj}\end{pmatrix}$$

$$= \underbrace{B_{1j}C_1(A) + B_{2j}C_2(A) + \dots + B_{nj}C_n(A)}_{\in \text{ Column Space of } A}$$

Thus $CS(AB) \leq CS(A)$, and since A is not invertible, $\operatorname{rank}(AB) = \dim_F CS(AB) \leq \dim_F CS(A) = \operatorname{rank}(A) < n$.
It follows that AB is not invertible either, and hence

$$\det(AB) = 0 = \det(A)\cdot\det(B).$$

Note that in proving Case 1 of this theorem, as well as in proving Thm 1 (which was used here too), we relied on knowing how applications of Gaussian elimination affect the determinant of a matrix.
This is also important in applications:

<u>Example 1</u> Find $\det(A^4)$ if

$$A = \begin{pmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{pmatrix} \in \mathbb{R}^{4\times4}.$$

<u>Solution</u> By multiplicativity of the determinant,

$$\det(A^4) = (\det(A))^4.$$

Also, $A \sim \begin{pmatrix} 1 & 5/2 & -3/2 & -1/2 \\ 3 & 0 & 1 & -3 \\ -3 & 0 & -2 & 9/2 \\ 1 & 5/2 & -1 & -1/4 \end{pmatrix} = D_{4;\frac14} D_{3;\frac12} D_{1;\frac12} A \sim$

$$\begin{pmatrix} 1 & 5/2 & -3/2 & -1/2 \\ 3 & 0 & 1 & -3 \\ 0 & 0 & -1 & 3/2 \\ 0 & 0 & 1/2 & -1/4 \end{pmatrix} = E_{32;1} E_{41;1} D_{4;1/4} D_{3;1/2} D_{1;1/2} A \sim$$

$$\begin{pmatrix} 1 & 5/2 & -3/2 & -1/2 \\ 0 & -15/2 & 11/2 & -3/2 \\ 0 & 0 & -1 & 3/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \underbrace{E_{43;\frac{1}{2}} E_{21;-3} E_{32;1} E_{41;-1} D_{4;1/4} D_{3;1/2} D_{1;1/2} A}_{B}$$

The last matrix is upper triangular, so (as you have to show in HW3) its determinant is equal to the product of its diagonal entries:

$\det(B) = \frac{15}{2}$ and also

$$\det(B) = \det(E_{43;\frac{1}{2}}) \det(E_{21;-3}) \det(E_{32;1}) \det(E_{41;-1}) \cdot \det(D_{4;1/4}) \det(D_{3;1/2}) \det(D_{1;\frac{1}{2}}) \det(A)$$
$$= \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} \det(A) = \frac{1}{16} \det(A)$$

In the end, $\det(A) = 16 \cdot \frac{15}{2} = 120$

and $\det(A^4) = 120^4$

<u>Example 2</u> Find $\det\left(\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & n \end{pmatrix}\right)$ (the entries are in $\mathbb{R}$).

<u>Solution</u> By the main properties of the determinant:

$$\det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 3 & \cdots & 1 \\ \vdots & & & \vdots & \\ 1 & 1 & 1 & & n \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 3 & \cdots & 1 \\ \vdots & & & & \\ 1 & 1 & 1 & \cdots & n \end{pmatrix} + \overset{0}{\cancel{\det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 3 & \cdots & 1 \\ \vdots & \vdots & & \vdots & \\ 1 & 1 & 1 & \cdots & n \end{pmatrix}}}$$

$$= \det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & & & & \vdots \\ 1 & 1 & 1 & \cdots & n \end{pmatrix} + \underbrace{\det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & 1 & \cdots & n \end{pmatrix}}_{0}$$

$$= \cdots = \det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & n-2 & 0 \\ 1 & 1 & 1 & \cdots & 1 & n \end{pmatrix} + \underbrace{\det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & n \end{pmatrix}}_{0}$$

continuing inductively

$$= \det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & & n-2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n-1 \end{pmatrix} + \underbrace{\det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & n-2 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}}_{0} = (n-1)!$$

Applications of Determinants

Characteristic Polynomial of a matrix (or characteristic equation)

Let $\mathbb{F}$ be a field, n a positive integer and $A \in \mathbb{F}^{n\times n}$.

In HW2, Problem 5 we saw that
0 is an eigenvalue of A if and only if
A is not invertible if and only if ← we can now consider this equivalence too b/c of Thm 1
$\det(A) = 0$.

We can come up with similar equivalent conditions for when a non-zero scalar is an eigenvalue of A:
μ is an eigenvalue of A if (by definition)
there exists $\bar{u} \in \mathbb{F}^n\setminus\{\bar{0}\}$ such that $A\bar{u} = \mu\bar{u}$ if and only if
there exists $\bar{u} \in \mathbb{F}^n\setminus\{\bar{0}\}$ such that $A\bar{u} = (\mu I_n)\bar{u}$ if and only if
there exists $\bar{u} \in \mathbb{F}^n\setminus\{\bar{0}\}$ such that $(A-\mu I_n)\bar{u} = \bar{0}$ if and only if
Nullspace$(A-\mu I_n)$ contains non-zero vectors if and only if
$A-\mu I_n$ is not invertible if and only if
$\det(A-\mu I_n) = 0$.

Let us single out the last condition:
μ is an eigenvalue of A if and only if $\det(A-\mu I_n)=0$.

But what is $\det(A-\mu I_n)$? That is, how does it depend on μ?

$$\det(A - \mu I_n) = \det\left(\begin{pmatrix} a_{11}-\mu & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\mu & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\mu & & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\mu \end{pmatrix}\right)$$

only the diagonal entries of A may change

use the Leibniz formula

$$= \sum_{\sigma \text{ } n\text{-permutation}} \operatorname{sgn}(\sigma)\, \tilde{a}_{1\sigma(1)} \tilde{a}_{2\sigma(2)} \tilde{a}_{3\sigma(3)} \cdots \tilde{a}_{n\sigma(n)}$$

$$= (a_{11}-\mu)(a_{22}-\mu)(a_{33}-\mu) \cdots (a_{nn}-\mu) + \sum_{\substack{\sigma \text{ } n\text{-permutation} \\ \sigma \neq id}} \operatorname{sgn}(\sigma)\, \tilde{a}_{1\sigma(1)} \tilde{a}_{2\sigma(2)} \cdots \tilde{a}_{n\sigma(n)}$$

But for summands in the latter sum, how many diagonal entries can appear in the product?
At most $\underline{n-2}$. (why?)

So if view μ for now as a parameter (or variable) varying over $\mathbb{F}$, each such summand is a product of at most $n-2$ linear polynomials in μ of the form $a_{ii}-\mu$ and of some constant polynomials, thus each such summand is a polynomial in μ of degree $\underline{\text{at most}}$ $n-2$.

But then the sum $\sum_{\substack{\sigma \text{ } n\text{-permutation} \\ \sigma \neq id}} \operatorname{sgn}(\sigma)\, \tilde{a}_{1\sigma(1)} \tilde{a}_{2\sigma(2)} \cdots \tilde{a}_{n\sigma(n)}$

is also a polynomial in μ of degree $\underline{\text{at most}}$ $n-2$.

On the other hand, the product
$$(a_{11}-\mu)(a_{22}-\mu) \cdots (a_{nn}-\mu)$$
is a polynomial in μ of degree n with highest order term equal to $(-1)^n \mu^n$.

Very Important Conclusion $p_A(\mu) := \det(A - \mu I_n)$ is a polynomial in μ of degree n with coefficients from $\mathbb{F}$ and highest-order term $(-1)^n \mu^n$.

It is called the characteristic polynomial of A (and the equation $p_A(\mu) = 0$ is called the characteristic equation of A).

It is very useful to us because, as we saw,

$$p_A(\mu) = 0 \Leftrightarrow \det(A - \mu I_n) = 0 \Leftrightarrow \mu \text{ is an eigenvalue of } A,$$

thus the eigenvalues of A are precisely the roots of the characteristic polynomial of A.

This viewpoint facilitates a lot the study of eigenvalues of A:

e.g. we know that every polynomial of degree n with coefficients from $\mathbb{F}$ has at most n roots in $\mathbb{F}$, therefore a matrix $A \in \mathbb{F}^{n \times n}$ can have at most n eigenvalues.

Other useful consequence:

Recall the Fundamental Theorem of Algebra:

every polynomial $q(x)$ with coefficients from $\mathbb{C}$ that has degree $n \geq 1$ has at least one root in $\mathbb{C}$ (in fact, we know that this is equivalent to saying that $q(x)$ has n roots in $\mathbb{C}$ (not necessarily distinct)).

Thus every matrix $A \in \mathbb{C}^{n \times n}$ has at least one eigenvalue.

To appreciate why this is a strong conclusion, let's see examples where it fails when $F=\mathbb{R}$ instead:

Example 1 Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

Prove that it has no eigenvalues in $\mathbb{R}$.

Solution There are two ways here: going by the definition (and understanding the "action" of A better), or working with the characteristic polynomial of A instead.

1st way What does this matrix do when acting on vectors in $\mathbb{R}^2$?

Answer: it rotates each vector $\binom{x}{y}$ about the origin counterclockwise by an angle of 45° (or $\frac{\pi}{4}$ radians) and it also increases its length by a factor of $\sqrt{2}$ (why?)

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix}$$

Note that the fact that A multiplies lengths does not cause an issue with having eigenvalues, but the fact that it rotates each vector $\binom{x}{y}$ to another vector that cannot be a scalar multiple of $\binom{x}{y}$ does.

So no non-zero vector of $\mathbb{R}^2$ can be an eigenvector of A, and hence A cannot have eigenvalues.

<u>2nd way</u> What is $p_A(t)$?

$$p_A(t) = \det(A - tI_2) = \det\left(\begin{pmatrix} 1-t & -1 \\ 1 & 1-t \end{pmatrix}\right) = (1-t)^2 - (-1)\cdot 1$$
$$= 1 + t^2 - 2t + 1 = t^2 - 2t + 2$$

But this polynomial has no real roots (given that we can write it again as
$$p_A(t) = (t^2 - 2t + 1) + 1 = (t-1)^2 + 1$$
which is $\geq 1 > 0$ no matter what real value we give t).

<u>Question</u>: What if we view A as a matrix in $\mathbb{C}^{2\times 2}$? We said that then A must have at least one eigenvalue. Find such an eigenvalue (in fact, you should be able to find all eigenvalues of A) and also find an eigenvector corresponding to it.