Constructing new vector spaces from already known ones

1) Sum of two subspaces

Let $\mathbb{F}$ be a field, V a vector space over $\mathbb{F}$, and S_1, S_2 subspaces of V.

We have seen that $S_1 \cap S_2$ is a subspace of V too, and that in fact

$S_1 \cap S_2$ is the largest subspace of V contained in both S_1 and S_2.

What about the smallest subspace of V containing both S_1 and S_2?

This would be the sum $S_1 + S_2$ of S_1 and S_2 (why?)

Definition Let $\mathbb{F}$ be a field, V a vector space over $\mathbb{F}$ and S_1, S_2 subspaces of V.

The sum $S_1 + S_2$ of S_1 and S_2 is defined as follows:

$$S_1 + S_2 := \{\bar{x} \in V : \exists\, \bar{u} \in S_1 \text{ and } \bar{w} \in S_2 \text{ s.t. } \bar{x} = \bar{u} + \bar{w}\}.$$

Lemma $S_1 + S_2$ is a subspace of V.

In fact it is the smallest subspace of V that contains both S_1 and S_2.

Proof We need to show that $S_1 + S_2$ is nonempty (and moreover that it contains S_1 and S_2), and also that $S_1 + S_2$ is closed under vector addition and under scalar multiplication.

Let $\bar{u} \in S_1$. Then

$$\bar{u} = \bar{u} + \bar{0}_V \in S_1 + S_2 \text{ since } \bar{0}_V \in S_2.$$

Thus $S_1 \subseteq S_1 + S_2$.

Similarly $S_2 \subseteq S_1 + S_2$.

Consider now $\bar{x}_1, \bar{x}_2 \in S_1 + S_2$ and $r \in \mathbb{F}$. The definition tells us that we can find

$$\bar{u}_1, \bar{u}_2 \in S_1 \quad \text{and} \quad \bar{w}_1, \bar{w}_2 \in S_2$$

such that $\bar{x}_1 = \bar{u}_1 + \bar{w}_1$ and $\bar{x}_2 = \bar{u}_2 + \bar{w}_2$.

Then

$$\bar{x}_1 + \bar{x}_2 = (\bar{u}_1 + \bar{w}_1) + (\bar{u}_2 + \bar{w}_2) = \underbrace{\underbrace{(\bar{u}_1 + \bar{u}_2)}_{\in S_1 \text{ since } S_1 \text{ is a subspace}} + \underbrace{(\bar{w}_1 + \bar{w}_2)}_{\in S_2}}_{\in S_1 + S_2}$$

and $r \cdot \bar{x}_1 = r(\bar{u}_1 + \bar{w}_1) = \underbrace{(r\bar{u}_1)}_{\in S_1} + \underbrace{(r\bar{w}_1)}_{\in S_2} \in S_1 + S_2$

Thus $\bar{x}_1 + \bar{x}_2 \in S_1 + S_2$ and $r \cdot \bar{x}_1 \in S_1 + S_2$.
We conclude that $S_1 + S_2$ is a subspace of V, and that it contains S_1 and S_2.

Practice: Why is $S_1 + S_2$ the smallest subspace of V containing both S_1 and S_2?

Examples 1) Let $V = \mathbb{Q}^3$ (viewed as a vector space over $\mathbb{Q}$) and let
$S_1 = \left\{ \begin{pmatrix} p \\ 0 \\ p \end{pmatrix} : p \in \mathbb{Q} \right\}$, $S_2 = \left\{ \begin{pmatrix} q \\ 0 \\ -q \end{pmatrix} : q \in \mathbb{Q} \right\}$.

What is $S_1 + S_2$?

2) Let $\mathcal{P}$ be the vector space of all polynomials with real coefficients in the variable x,
$$\mathcal{P} = \{ p : \mathbb{R} \to \mathbb{R} \mid \exists\, m \in \mathbb{Z}_{\geq 0} \text{ and } a_0, a_1, \ldots a_m \in \mathbb{R} \text{ s.t. } p(x) = a_0 + a_1 x + \cdots + a_m x^m \},$$
viewed as a vector space over $\mathbb{R}$,
and let
$S_1 = \{ p \in \mathcal{P} : p(1) = p(2) = p(5) = 0 \}$, $S_2 = \mathcal{P}_2 = \operatorname{span}(\{1, x, x^2\})$.
What is $S_1 + S_2$?

3) Let $\mathbb{F}$ be a field, and let $V = \mathbb{F}^{4\times 4}$, the vector space of all 4×4 matrices with entries from $\mathbb{F}$.
Moreover, let $\mathcal{U}$ be the subspace of upper triangular matrices in $\mathbb{F}^{4\times 4}$, and let L be the subspace of lower triangular matrices in $\mathbb{F}^{4\times 4}$.
What is $\mathcal{U} + L$?

Important Remark: We do not always have
$$\dim_{\mathbb{F}}(S_1 + S_2) = \dim_{\mathbb{F}} S_1 + \dim_{\mathbb{F}} S_2.$$
Does any of the examples above provide an example where equality here fails?

It's useful to know in which cases we have equality.

<u>Definition</u> Let $\mathbb{F}$ be a field, V a vector space over $\mathbb{F}$, and let S_1, S_2 be subspaces of V.
$\boxed{\text{If } S_1 \cap S_2 = \{\bar{0}_V\}}$ then the sum of S_1 and S_2

is called the <u>direct sum</u> of S_1 and S_2 and we write $S_1 \oplus S_2$.

<u>Remark</u> In this case we have
$$\dim_{\mathbb{F}}(S_1 \oplus S_2) = \dim_{\mathbb{F}} S_1 + \dim_{\mathbb{F}} S_2.$$

In fact this follows because a basis for $S_1 \oplus S_2$ can be found as below:
if B_1 is a basis of S_1, and B_2 is a basis of S_2, then $B_1 \cup B_2$ is a basis of $S_1 \oplus S_2$ (practice).
Moreover, $B_1 \cap B_2 = \phi$ (why?), and thus
$$|B_1 \cup B_2| = |B_1| + |B_2|.$$

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2) <u>Cartesian Product of two Vector Spaces</u>

<u>Definition</u> Let $\mathbb{F}$ be a field, and let V and W be vector spaces over $\mathbb{F}$.
The Cartesian product $V \times W$ of V and W is the set of all ordered pairs with first component from V and second component from W:
$$V \times W := \{ (\bar{v}, \bar{w}) : \bar{v} \in V \text{ and } \bar{w} \in W \}.$$

Given two "vectors"/ordered pairs $(\bar{v}_1, \bar{w}_1)$ and $(\bar{v}_2, \bar{w}_2)$ in $V \times W$, and scalar $r \in \mathbb{F}$, we define

$$(\bar{v}_1, \bar{w}_1) + (\bar{v}_2, \bar{w}_2) = (\bar{v}_1 + \bar{v}_2, \bar{w}_1 + \bar{w}_2)$$
$$r \cdot (\bar{v}_1, \bar{w}_1) = (r \cdot \bar{v}_1, r \cdot \bar{w}_1).$$

<u>Lemma 1</u> $V \times W$ with these operations, that is,

$$\Big(V \times W, \text{component-wise vector addition}, \text{component-wise scalar multiplication}\Big),$$

is a vector space over $\mathbb{F}$. (practice)
Also the zero vector of $V \times W$ is the ordered pair $(\bar{0}_V, \bar{0}_W)$.

<u>Lemma 2</u> The subspace $V \times \{\bar{0}_W\} = \{(\bar{v}, \bar{0}_W) : \bar{v} \in V\}$ of $V \times W$ is isomorphic to V.
Similarly, the subspace $\{\bar{0}_V\} \times W$ of $V \times W$ is isomorphic to W.
Finally, $V \times W = (V \times \{\bar{0}_W\}) \oplus (\{\bar{0}_V\} \times W)$.

<u>Remark.</u> Because of Lemma 2, we often call $V \times W$ the (external) direct sum of V and W as well, and also use the notation $V \oplus W$.

<u>Proposition 1</u> Let V be a vector space over a field $\mathbb{F}$, and let S be a subspace of V.
Then S is an (algebraic) direct summand of V, which

means that there exists another subspace T of V such that $V = S \oplus T$

(in other words, such that

$$V = S + T \quad \text{and} \quad S \cap T = \{\bar{0}_V\}).$$

<u>Proof</u> Let $\mathcal{B}$ be a basis of S. Then $\mathcal{B}$ is also a linearly independent subset of V, so we can extend it to a basis $\mathcal{C}$ of V.

Set $T = \mathrm{span}(\mathcal{C} \setminus \mathcal{B}) = \mathrm{span}(\{\bar{u} \in \mathcal{C} : \bar{u} \notin \mathcal{B}\})$.

Then T is a subspace of V.

Also $S + T = \mathrm{span}(\mathcal{B}) + \mathrm{span}(\mathcal{C} \setminus \mathcal{B}) = \mathrm{span}(\mathcal{C}) = V$.

Finally, let $\bar{x} \in S \cap T$. Then $\bar{x}$ can be written as a linear combination of vectors in $\mathcal{B}$, but also it can be written as a linear combination of vectors in $\mathcal{C} \setminus \mathcal{B}$.

More specifically, we can find $k \geq 0$, $m \geq 0$ and distinct $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k \in \mathcal{B}$, $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m \in \mathcal{C} \setminus \mathcal{B}$ and $\lambda_1, \lambda_2, \ldots, \lambda_k, \mu_1, \mu_2, \ldots, \mu_m \in \mathbb{F}$ so that

$$\bar{x} = \lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \cdots + \lambda_k \bar{u}_k = \mu_1 \bar{w}_1 + \mu_2 \bar{w}_2 + \cdots + \mu_m \bar{w}_m.$$

But then $\lambda_1 \bar{u}_1 + \cdots + \lambda_k \bar{u}_k + (-\mu_1) \bar{w}_1 + (-\mu_2) \bar{w}_2 + \cdots + (-\mu_m) \bar{w}_m = \bar{0}_V$

and since $\overbrace{\{\bar{u}_1, \ldots, \bar{u}_k, \bar{w}_1, \ldots, \bar{w}_m\}}^{\text{all these vectors are distinct}}$ is a linearly independent subset of V (given that it is a subset of the basis $\mathcal{C}$), we can conclude that $\lambda_1 = \cdots = \lambda_k = \mu_1 = \cdots = \mu_m = 0_{\mathbb{F}}$.

This gives $\bar{x} = \lambda_1 \bar{u}_1 + \cdots + \lambda_k \bar{u}_k = \bar{0}_V$, and hence $S \cap T = \{\bar{0}_V\}$.

Constructing new vector spaces from already known ones (cont.)

3) Quotient Space (in more detail, quotient of a vector space by a subspace)

Let $\mathbb{F}$ be a field, V a vector space over $\mathbb{F}$ and S a subspace of V.

Based on S we can define a relation on V as follows (or in other words we can consider a subset of $V \times V$ as described below):

given $\bar{x}, \bar{y}$ in V

$\bar{x} \sim_S \bar{y}$ (or equivalently $(\bar{x}, \bar{y}) \in R_S$)

if and only if $\bar{x} - \bar{y} \in S$.

Very Useful Proposition. The relation R_S defined above is an equivalence relation.

Proof Need to show that R_S is reflexive, symmetric and transitive.

Reflexive Recall that, since S is a subspace, $\bar{0}_V \in S$. Therefore, for every $\bar{x} \in V$, $\bar{x} - \bar{x} \in S$ and hence $\bar{x} \sim_S \bar{x}$.

Symmetric Consider $\bar{x}, \bar{y} \in V$ such that $\bar{x} \sim_S \bar{y}$. We need to show that $\bar{y} \sim_S \bar{x}$ as well.

But by definition $\bar{x} \sim_S \bar{y}$ implies $\bar{x} - \bar{y} \in S$. Given that S is a subspace, we can then write

$$\bar{y} - \bar{x} = -(\bar{x} - \bar{y}) = (-1) \cdot (\bar{x} - \bar{y}) \in S,$$

and hence $\bar{y} \sim_S \bar{x}$ as desired.

<u>Transitive</u> Consider $\bar{x}, \bar{y}, \bar{z} \in V$ which satisfy

$$\bar{x} \sim_S \bar{y} \quad \text{and} \quad \bar{y} \sim_S \bar{z}.$$

We need to show that $\bar{x} \sim_S \bar{z}$ as well.
By definition of R_S we get

$\bar{x} - \bar{y} \in S$ and $\bar{y} - \bar{z} \in S$. Thus, since S is a subspace,

$$\bar{x} - \bar{z} = \bar{x} - \bar{y} + \bar{y} - \bar{z} = \underbrace{(\bar{x} - \bar{y})}_{\in S} + \underbrace{(\bar{y} - \bar{z})}_{\in S} \in S \text{ too.}$$

In other words, $\bar{x} \sim_S \bar{z}$ as we wanted.

<u>Corollary</u> Let $\mathbb{F}$, V, $S \leq V$ as above, and let R_S be the equivalence relation just defined.
For every $\bar{x}$ we write $[\bar{x}]_S$ or $\bar{x} + S$ for the equivalence class of $\bar{x}$: this is the subset of V which is defined by

$$[\bar{x}]_S \,(= \bar{x} + S) := \{ \bar{y} \in V : \bar{x} \sim_S \bar{y} \}.$$

Let V/S (read as V mod S or V by S) be the collection of all these equivalence classes:

$$V/S = \{ [\bar{x}]_S : \bar{x} \in V \}$$ (note that this is a subset of the powerset of V.

Then V/S is a partition of V (that is, a "splitting" of

V into <u>non-empty</u> subsets of it such that each element of V is contained in <u>exactly one</u> of these subsets).

<u>Remark</u> To quickly explain why each equivalence class is nonempty, note that $\bar{x} \in [\bar{x}]_S = \bar{x} + S$.
In fact, by using the Cancellation Law for vector addition, we can check that each equivalence class has the same size as the subspace S.

Also we just saw that each element $\bar{x}$ of V is contained in at least one of these equivalence classes: $\bar{x} \in [\bar{x}]_S$. (note that this is a consequence of the reflexivity property).
But why do we get that $\bar{x}$ is contained in <u>exactly one</u> of these equivalence classes?
Because, whenever we have $\bar{x} \in [\bar{z}]_S$ for some $\bar{z} \in V$, we get $[\bar{x}]_S = [\bar{z}]_S$ (due to the symmetry property and transitivity property)
Indeed, if $\bar{x} \in [\bar{z}]_S$, then by definition

$$\bar{z} \sim_S \bar{x} \implies \bar{x} \sim_S \bar{z} \implies \bar{z} \in [\bar{x}]_S.$$

Also, if $\bar{y} \in [\bar{x}]_S$, then $\bar{x} \sim_S \bar{y}$, which combined with $\bar{z} \sim_S \bar{x}$ (and the transitivity property) gives $\bar{z} \sim_S \bar{y}$, or in other words $\bar{y} \in [\bar{z}]_S$.
Thus $[\bar{x}]_S \subseteq [\bar{z}]_S$, and similarly we can show the reverse inclusion too.

Can we define a structure on V/S? YES

Let $[\bar{x}]_S, [\bar{y}]_S$ in V/S. Define

$$[\bar{x}]_S + [\bar{y}]_S := [\bar{x}+\bar{y}]_S.$$

Also, if $r \in \mathbb{F}$, define

$$r.[\bar{x}]_S := [r.\bar{x}]_S.$$

<u>Proposition 2</u> These operations are well-defined. That is, if $[\bar{x}_1]_S = [\bar{x}_2]_S$ and $[\bar{y}_1]_S = [\bar{y}_2]_S$, then

$$[\bar{x}_1+\bar{y}_1]_S = [\bar{x}_2+\bar{y}_2]_S,$$

and also, for every $r \in \mathbb{F}$,

$$[r\cdot\bar{x}_1]_S = [r\cdot\bar{x}_2]_S.$$

<u>Proof</u> We have already seen that

$$[\bar{x}_1]_S = [\bar{x}_2]_S \text{ if and only if } \bar{x}_1 \sim_S \bar{x}_2.$$

Thus by our assumptions we have $\bar{x}_1 \sim_S \bar{x}_2$ and similarly $\bar{y}_1 \sim_S \bar{y}_2$. But then

$\bar{x}_1 - \bar{x}_2 \in S$, $\bar{y}_1 - \bar{y}_2 \in S$, and thus

$$(\bar{x}_1+\bar{y}_1) - (\bar{x}_2+\bar{y}_2) = (\bar{x}_1-\bar{x}_2) + (\bar{y}_1-\bar{y}_2) \in S.$$

$$\Rightarrow \bar{x}_1+\bar{y}_1 \sim_S \bar{x}_2+\bar{y}_2.$$

Similarly $r\cdot\bar{x}_1 - r\cdot\bar{x}_2 = r\cdot(\bar{x}_1-\bar{x}_2) \in S$

$$\Rightarrow r\cdot\bar{x}_1 \sim_S r\cdot\bar{x}_2.$$

Proposition 3 Viewed together with these operations of vector addition and scalar multiplication, $V/_S$ becomes a vector space over $\mathbb{F}$, called the quotient of V by the subspace S, or more simply the quotient space (associated to S).

Proof (practice)

Question What is the zero vector of $V/_S$?

Answer It is the equivalence class of $\bar{0}_V$, which coincides with the subspace S.

Next very important question How do we find a basis for $V/_S$?

Theorem Let $\mathbb{F}$, V, $S \leq V$ be as before. Consider another subspace T of V such that

$$V = S \oplus T,$$

and moreover consider a basis B_T of T,

$$B_T = \{\bar{u}_i : i \in I\} \text{ where } I \text{ is some index set.}$$

Then, whenever we have two distinct indices i_1, i_2 from I (namely $i_1 \neq i_2$), we get that

$$[\bar{u}_{i_1}]_S \neq [\bar{u}_{i_2}]_S.$$

Moreover, the set $\{[\bar{u}_i]_S : i \in I\} = \{\bar{u}_i + S : i \in I\}$ is a basis of $V/_S$.