Math 127, A1
1st Midterm Exam – September 27, 2019

Name: ______________________________

This exam has 5 problems with a total worth of 65 points. To earn maximum credit, you need to accumulate 45 points or more.

General instructions (important, read back of the page too).

- Notes, formula sheets, calculators, or electronic aids are **not** allowed.
- All cell phones should be turned off and left in your bags.
- You must show your work and justify your answers to receive full credit. A correct answer without any justification will receive little or no credit.
- Wherever a numerical answer is expected (especially one involving several numerical operations), it is advisable to give your answer in a form that indicates by which process you get it (besides the justifications/explanations you should provide); it is even fine if you set up computations without doing them all in such cases.
- You may not leave the exam room until at least 30 minutes have elapsed.

- If you need extra space for a problem, use the reverse side of the page of the corresponding problem and indicate this clearly.
- The last double-sided page, as well as the rest of this page, can be freely used as scratch paper.

Do not write any part of your final answers there

because these parts will not be considered during grading.

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Problem 1 (*max. 10 points*) Consider $\mathbb{R}^2$ with operations of addition and multiplication defined as follows: for every two ordered pairs $(a, b), (c, d) \in \mathbb{R}^2$,

$$(a,b) + (c,d) \stackrel{\text{def}}{=} (a+c, b+d) \in \mathbb{R}^2$$
$$\text{and} \quad (a,b)(c,d) \stackrel{\text{def}}{=} (ad+bc, bd) \in \mathbb{R}^2.$$

Show that multiplication is commutative and associative.

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Problem 2 (*max. 15 points*) Consider the following structures:

1. $\mathbb{R}^2$ with operations of addition and multiplication defined as in Problem 1;
2. the set of numbers of the form $a + b\sqrt{11}$ where $a, b \in \mathbb{Q}$, with the usual addition and multiplication of real numbers;
3. $\mathbb{Z}_3^2$ with coordinate-wise addition and multiplication $\big($that is, for every two ordered pairs $(a, b), (c, d) \in \mathbb{Z}_3^2$, $(a, b) + (c, d) \stackrel{\text{def}}{=} (a + c, b + d)$ and $(a, b)(c, d) \stackrel{\text{def}}{=} (ac, bd)\big)$.

Only one of them is a field. Find which one and confirm that it is a field (you can either check this directly for the structure in question, or proceed by elimination, that is, by proving that the other two are not fields).

Take for granted that, in all the above structures, addition and multiplication are commutative and associative (that is, you don't need to check these properties; regarding Structure 1 and multiplication, this is precisely the content of Problem 1).

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Problem 3 (a) (*max. 5 points*) Let $\mathbb{F}$ be a field and let V be a vector space over $\mathbb{F}$. State what the distributive laws in V say.

(b) (*max. 5 points*) Recall that we can turn $\mathbb{Z}_5^6$ into a vector space over $\mathbb{Z}_5$ by defining vector addition and scalar multiplication as follows:

$$(\bar{x},\bar{y}) \in \mathbb{Z}_5^6 \times \mathbb{Z}_5^6 \quad \mapsto \quad \bar{x}+\bar{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \\ x_4+y_4 \\ x_5+y_5 \\ x_6+y_6 \end{pmatrix} \in \mathbb{Z}_5^6,$$

$$\bar{x} \in \mathbb{Z}_5^6,\ \lambda \in \mathbb{Z}_5 \quad \mapsto \quad \lambda \cdot \bar{x} = \lambda \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \lambda x_4 \\ \lambda x_5 \\ \lambda x_6 \end{pmatrix} \in \mathbb{Z}_5^6.$$

Show that indeed the distributive laws in $\mathbb{Z}_5^6$ are satisfied.

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Problem 4 (a) (*max. 5 points*) Consider the following vectors in $\mathbb{R}^5$:

$$\bar{a} = \begin{pmatrix} 3 \\ 0.5 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \ \bar{b} = \begin{pmatrix} -6 \\ 4 \\ 5 \\ 3 \\ -1 \end{pmatrix}, \ \bar{c} = \begin{pmatrix} -7 \\ 4 \\ 5 \\ -9 \\ -1 \end{pmatrix}.$$

Is $\bar{c}$ a linear combination of $\bar{a}$ and $\bar{b}$ (where the scalars come from $\mathbb{R}$)? Justify your answer.

(b) (*max. 5 points*) Consider the following vectors in $\mathbb{R}^4$:

$$\bar{d} = \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix}, \ \bar{p} = \begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix}, \ \bar{q} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix}.$$

Is $\bar{d}$ a linear combination of $\bar{p}$ and $\bar{q}$ (where the scalars come from $\mathbb{R}$)? Justify your answer.

(c) (*max. 5 points*) Consider the following vectors in $\mathbb{Z}_5^4$:

$$\bar{u} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \ \bar{v} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 3 \end{pmatrix}, \ \bar{w} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 3 \end{pmatrix}.$$

Is $\bar{u}$ a linear combination of $\bar{v}$ and $\bar{w}$ (where the scalars come from $\mathbb{Z}_5$)? Justify your answer.

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Problem 5 (*max. 15 points*) Consider the following vectors which have real coordinates:

$$\bar{a} = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 4 \end{pmatrix}, \ \bar{b} = \begin{pmatrix} 2 \\ -2 \\ 3 \\ -5 \end{pmatrix}, \ \bar{c} = \begin{pmatrix} 6 \\ 3 \\ 9 \\ -1 \\ 8 \\ -1 \end{pmatrix}, \ \bar{d} = \begin{pmatrix} -7 \\ 4 \\ 5 \\ 4 \\ -1 \end{pmatrix},$$

$$\bar{u} = \begin{pmatrix} -4 \\ 3 \\ 9 \end{pmatrix}, \ \bar{v} = \begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix}, \ \bar{w} = \begin{pmatrix} 0 \\ 3 \\ 5 \\ -2 \end{pmatrix}.$$

(a) (*max. 9 points*) Find the sum of any two of the given vectors, if it makes sense.

(b) (*max. 6 points*) Compute the dot product of any two different vectors out of the ones you found in part (a), whenever it makes sense.

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Scratch paper.

Scratch paper.

Math 127
Suggested solutions to 1st Midterm

Problem 1. *Multiplication is commutative:* We need to show that, for every $(a,b),(c,d) \in \mathbb{R}^2$,

$$(a,b)(c,d) = (c,d)(a,b).$$

Fix two ordered pairs $(a,b),(c,d) \in \mathbb{R}^2$: by definition we have

$$(a,b)(c,d) = (ad+bc, bd) \quad \text{and} \quad (c,d)(a,b) = (cb+da, db).$$

We now recall that addition and multiplication in $\mathbb{R}$ are commutative, therefore $ad+bc = da+cb = cb+da$ and $bd = db$. We thus see that the corresponding components of $(a,b)(c,d)$ and $(c,d)(a,b)$ are equal, which implies that the two products are equal.

Since (a,b) and (c,d) were arbitrary pairs in $\mathbb{R}^2$, the proof is complete.

Multiplication is associative: We show that, for every $(a,b),(c,d),(f,g) \in \mathbb{R}^2$,

$$\big((a,b)(c,d)\big)(f,g) = (a,b)\big((c,d)(f,g)\big).$$

Fix three ordered pairs $(a,b),(c,d),(f,g) \in \mathbb{R}^2$: by definition we have

$$\begin{aligned} \big((a,b)(c,d)\big)(f,g) &= (ad+bc,bd)(f,g) = \big((ad+bc)g+(bd)f,(bd)g\big) \\ \text{and} \quad (a,b)\big((c,d)(f,g)\big) &= (a,b)(cg+df,dg) = \big(a(dg)+b(cg+df),b(dg)\big). \end{aligned}$$

We now recall that addition and multiplication in $\mathbb{R}$ are associative and that multiplication distributes over addition, therefore

$$\begin{aligned} (ad+bc)g+(bd)f &= \big((ad)g+(bc)g\big)+(bd)f && \text{(distributive law)} \\ &= (ad)g+\big((bc)g+(bd)f\big) && \text{(addition is associative)} \\ &= a(dg)+\big(b(cg)+b(df)\big) && \text{(multiplication is associative)} \\ &= a(dg)+b(cg+df) && \text{(distributive law).} \end{aligned}$$

Similarly, because multiplication is associative, $(bd)g = b(dg)$. We conclude that the corresponding components of $\big((a,b)(c,d)\big)(f,g)$ and $(a,b)\big((c,d)(f,g)\big)$ are equal, which implies that $\big((a,b)(c,d)\big)(f,g) = (a,b)\big((c,d)(f,g)\big)$.

Since $(a,b),(c,d)$ and (f,g) were arbitrary pairs in $\mathbb{R}^2$, the proof is complete.

Problem 2. *1st way:* We show directly that Structure 2 is a field (*note:* the proof will be completely analogous to how we proved that Structure 3 in HW1, Pb2 is a field). Let us denote this structure by $\mathbb{Q}(\sqrt{11})$.

We first check that, for any two elements $x, y \in \mathbb{Q}(\sqrt{11})$, $x+y \in \mathbb{Q}(\sqrt{11})$ and $xy \in \mathbb{Q}(\sqrt{11})$.

Indeed, if $x, y \in \mathbb{Q}(\sqrt{11})$, then by definition of the set there are some $a, b, c, d \in \mathbb{Q}$ such that $x = a + b\sqrt{11}$ and $y = c + d\sqrt{11}$. But then

$$x + y = (a + b\sqrt{11}\,) + (c + d\sqrt{11}\,) = (a + c) + (b + d)\sqrt{11} \in \mathbb{Q}(\sqrt{11})$$

given that $a + c, b + d \in \mathbb{Q}$. Similarly,

$$\begin{aligned} xy = (a + b\sqrt{11}\,)(c + d\sqrt{11}\,) &= ac + bc\sqrt{11} + ad\sqrt{11} + 11\,bd \\ &= (ac + 11bd) + (ad + bc)\sqrt{11} \in \mathbb{Q}(\sqrt{11}) \end{aligned}$$

given that $ac + 11bd, ad + bc \in \mathbb{Q}$.

It follows that addition and multiplication in $\mathbb{Q}(\sqrt{11})$ are well-defined, and furthermore that they are commutative and associative (*as we already knew from the statement of the problem*), and also that they satisfy the distributive law (given that they already have these properties when we consider the operations in the entire $\mathbb{R}$).

Moreover, $0 \in \mathbb{Q}(\sqrt{11})$ (since $0 \in \mathbb{Q}$ and we can write $0 = 0 + 0 \cdot \sqrt{11}$), therefore there is a neutral element of addition in $\mathbb{Q}(\sqrt{11})$. Similarly, $1 \in \mathbb{Q}(\sqrt{11})$ (since $0, 1 \in \mathbb{Q}$ and we can write $1 = 1 + 0 \cdot \sqrt{11}$), therefore there is an identity element in $\mathbb{Q}(\sqrt{11})$.

Finally, for every $x = a + b\,\sqrt{11} \in \mathbb{Q}(\sqrt{11})$ the element $-x$ (the additive inverse of x in $\mathbb{R}$) is also in $\mathbb{Q}(\sqrt{11})$, given that

$$-x = -(a + b\,\sqrt{11}) = -a - b\,\sqrt{11} = (-a) + (-b)\,\sqrt{11}$$

and $-a, -b \in \mathbb{Q}$ when a, b are rationals.

Similarly, for every non-zero $x = a + b\,\sqrt{11} \in \mathbb{Q}(\sqrt{11})$ the element $1/x$ (the multiplicative inverse of x in $\mathbb{R}$) is also in $\mathbb{Q}(\sqrt{11})$, given that

$$\begin{aligned} 1/x = 1/(a + b\sqrt{11}\,) = \frac{a - b\,\sqrt{11}}{(a + b\,\sqrt{11}\,)(a - b\,\sqrt{11}\,)} &= \frac{a - b\,\sqrt{11}}{a^2 - 11\,b^2} \\ &= \frac{a}{a^2 - 11\,b^2} + \frac{-b}{a^2 - 11\,b^2}\,\sqrt{11} \end{aligned}$$

and both $a/(a^2 - 11\,b^2)$ and $(-b)/(a^2 - 11\,b^2)$ are in $\mathbb{Q}$ when a, b are rationals (we observe that here we could multiply and divide by $a - b\,\sqrt{11}$ because this number is non-zero (why?)).

We conclude that $\mathbb{Q}(\sqrt{11})$ is a field.

2nd way: We prove that Structure 1 and Structure 3 are not fields. Our approach will be as follows. We recall that we have proven in class the following: assume $\mathbb{F}$ is a field; if $x, y \in \mathbb{F}$ and $xy = 0_{\mathbb{F}}$ (where $0_{\mathbb{F}}$ is the neutral element of addition in $\mathbb{F}$), then either $x = 0_{\mathbb{F}}$ or $y = 0_{\mathbb{F}}$ (or both).

Therefore, for each of the Structures 1 and 3, if we find non-zero elements x, y such that $xy = 0$, we will be done (because the above fact will not hold in that structure, so that structure cannot be a field).

<u>$\mathbb{R}^2$ *with the operations from Problem 1 is not a field.*</u> Observe first that $(0, 0)$ is the neutral element of addition (where each coordinate here is the 0 element in $\mathbb{R}$). Indeed, for every $(a, b) \in \mathbb{R}^2$, we have

$$(0, 0) + (a, b) = (0 + a, 0 + b) = (a, b).$$

It follows that $(1, 0)$ is a non-zero element of $\mathbb{R}^2$. We now check that

$$(1, 0)(1, 0) = (1 \cdot 0 + 0 \cdot 1, 0 \cdot 0) = (0, 0).$$

This concludes the proof that Structure 1 is not a field.

<u>$\mathbb{Z}_3^2$ *with coordinate-wise operations is not a field.*</u> Observe first that $(0, 0)$ is the neutral element of addition (where each coordinate here is the 0 element in $\mathbb{Z}_3$). Indeed, for every $(a, b) \in \mathbb{Z}_3^2$, we have

$$(0, 0) + (a, b) = (0 + a, 0 + b) = (a, b).$$

It follows that $(1, 0)$ and $(0, 2)$ are non-zero elements of $\mathbb{Z}_3^2$. We now check that

$$(1, 0)(0, 2) = (1 \cdot 0, 0 \cdot 2) = (0, 0).$$

This concludes the proof that Structure 3 is not a field.

Problem 3. (a) *1st Distributive Law:*

$$\text{for all } \lambda \in \mathbb{F} \text{ and all } \bar{x}, \bar{y} \in V, \quad \lambda \cdot (\bar{x} + \bar{y}) = \lambda \cdot \bar{x} + \lambda \cdot \bar{y}.$$

2nd Distributive Law:

$$\text{for all } \lambda, \mu \in \mathbb{F} \text{ and all } \bar{x} \in V, \quad (\lambda + \mu) \cdot \bar{x} = \lambda \cdot \bar{x} + \mu \cdot \bar{x}.$$

(b) We check the 1st distributive law: consider $\bar{x}, \bar{y} \in \mathbb{Z}_5^6$ and $\lambda \in \mathbb{Z}_5$.
Then

$$\lambda \cdot (\bar{x} + \bar{y}) = \lambda \cdot \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \\ x_4+y_4 \\ x_5+y_5 \\ x_6+y_6 \end{pmatrix} = \begin{pmatrix} \lambda(x_1+y_1) \\ \lambda(x_2+y_2) \\ \lambda(x_3+y_3) \\ \lambda(x_4+y_4) \\ \lambda(x_5+y_5) \\ \lambda(x_6+y_6) \end{pmatrix},$$

$$\text{while} \quad \lambda \cdot \bar{x} + \lambda \cdot \bar{y} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \lambda x_4 \\ \lambda x_5 \\ \lambda x_6 \end{pmatrix} + \begin{pmatrix} \lambda y_1 \\ \lambda y_2 \\ \lambda y_3 \\ \lambda y_4 \\ \lambda y_5 \\ \lambda y_6 \end{pmatrix} = \begin{pmatrix} \lambda x_1+\lambda y_1 \\ \lambda x_2+\lambda y_2 \\ \lambda x_3+\lambda y_3 \\ \lambda x_4+\lambda y_4 \\ \lambda x_5+\lambda y_5 \\ \lambda x_6+\lambda y_6 \end{pmatrix}.$$

By the distributive law in $\mathbb{Z}_5$, we see that, for every index i, $\lambda(x_i + y_i) = \lambda x_i + \lambda y_i$. Therefore the corresponding components of $\lambda \cdot (\bar{x} + \bar{y})$ and of $\lambda \cdot \bar{x} + \lambda \cdot \bar{y}$ are equal, which shows that the two expressions are equal.

Given that $\lambda \in \mathbb{Z}_5$ and $\bar{x}, \bar{y} \in \mathbb{Z}_5^6$ were arbitrary, the proof is complete.

We now check the 2nd distributive law: consider $\lambda, \mu \in \mathbb{Z}_5$ and $\bar{x} \in \mathbb{Z}_5^6$.
Then

$$(\lambda + \mu) \cdot \bar{x} = \begin{pmatrix} (\lambda+\mu)x_1 \\ (\lambda+\mu)x_2 \\ (\lambda+\mu)x_3 \\ (\lambda+\mu)x_4 \\ (\lambda+\mu)x_5 \\ (\lambda+\mu)x_6 \end{pmatrix},$$

$$\text{while} \quad \lambda \cdot \bar{x} + \mu \cdot \bar{x} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \lambda x_4 \\ \lambda x_5 \\ \lambda x_6 \end{pmatrix} + \begin{pmatrix} \mu x_1 \\ \mu x_2 \\ \mu x_3 \\ \mu x_4 \\ \mu x_5 \\ \mu x_6 \end{pmatrix} = \begin{pmatrix} \lambda x_1 + \mu x_1 \\ \lambda x_2 + \mu x_2 \\ \lambda x_3 + \mu x_3 \\ \lambda x_4 + \mu x_4 \\ \lambda x_5 + \mu x_5 \\ \lambda x_6 + \mu x_6 \end{pmatrix}.$$

By the distributive law in $\mathbb{Z}_5$, we see that, for every index i, $(\lambda + \mu)x_i = \lambda x_i + \mu x_i$. Therefore the corresponding components of $(\lambda + \mu) \cdot \bar{x}$ and of $\lambda \cdot \bar{x} + \mu \cdot \bar{x}$ are equal, which shows that the two expressions are equal.

Given that $\lambda, \mu \in \mathbb{Z}_5$ and $\bar{x} \in \mathbb{Z}_5^6$ were arbitrary, the proof is complete.

Problem 4. (a) The question is asking whether there are $t, s \in \mathbb{R}$ such that

$$\begin{pmatrix} -7 \\ 4 \\ 5 \\ -9 \\ -1 \end{pmatrix} = t \cdot \begin{pmatrix} 3 \\ 0.5 \\ 1 \\ 0 \\ -2 \end{pmatrix} + s \cdot \begin{pmatrix} -6 \\ 4 \\ 5 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3t - 6s \\ 0.5t + 4s \\ t + 5s \\ 3s \\ -2t - s \end{pmatrix}.$$

This is equivalent to finding a solution to the following system of linear equations:

$$\left\{ \begin{array}{rcr} 3t - 6s & = & -7 \\ 0.5t + 4s & = & 4 \\ t + 5s & = & 5 \\ 3s & = & -9 \\ -2t - s & = & -1 \end{array} \right\}.$$

Looking at the 4th equation, we note that we should have $s = -3$. Combining this with the 3rd equation, we note that we should have $t + 5 \cdot (-3) = 5 \Leftrightarrow t = 5 - 5 \cdot (-3) = 5 + 15 = 20$. Therefore, there is only one choice of values for t and s that would satisfy the 3rd and the 4th equation at the same time, $t = 20$ and $s = -3$.

We now check whether this choice of values for t and s solves the system: we note that

$$3 \cdot 20 - 6 \cdot (-3) = 60 + 18 = 78 \neq -7,$$

therefore the 1st, 3rd and 4th equations cannot be satisfied at the same time, which shows that the system cannot have a solution.

We conclude that $\bar{c}$ cannot be written as a linear combination of $\bar{a}$ and $\bar{b}$.

(b) The question is asking whether there are $t, s \in \mathbb{R}$ such that

$$\begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} = t \cdot \begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix} + s \cdot \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 4t + 2s \\ 2t + 2s \\ -2t \\ -7t - 2s \end{pmatrix}.$$

This is equivalent to finding a solution to the following system of linear equations:

$$\left\{ \begin{array}{rcrcl} 4t & + & 2s & = & 3 \\ 2t & + & 2s & = & 5 \\ -2t & & & = & 2 \\ -7t - 2s & & & = & 0 \end{array} \right\}.$$

Looking at the 3rd equation, we note that we should have $t = -1$. Combining this with the 2nd equation, we note that we should have $2 \cdot (-1) + 2s = 5 \Leftrightarrow$

$2s = 5 - 2\cdot(-1) = 7 \Leftrightarrow s = 7/2$. Therefore, there is only one choice of values for t and s that would satisfy the 2nd and the 3rd equation at the same time, $t = -1$ and $s = 7/2$.

We now check whether this choice of values for t and s solves the system: we note that

$$4\cdot(-1) + 2\cdot(7/2) = -4 + 7 = 3 \quad \text{and} \quad (-7)\cdot(-1) - 2\cdot(7/2) = 7 - 7 = 0.$$

We conclude that we can write $\bar{d}$ as a linear combination of $\bar{p}$ and $\bar{q}$:

$$\begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} = (-1)\cdot \begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix} + \frac{7}{2}\cdot \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix}.$$

(c) The question is asking whether there are $t, s \in \mathbb{Z}_5$ such that

$$\begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} = t\cdot \begin{pmatrix} 4 \\ 2 \\ 3 \\ 3 \end{pmatrix} + s\cdot \begin{pmatrix} 2 \\ 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4t + 2s \\ 2t + 2s \\ 3t \\ 3t + 3s \end{pmatrix}.$$

This is equivalent to finding a solution to the following system of linear equations:

$$\left\{ \begin{array}{lllll} 4t & + & 2s & = & 3 \\ 2t & + & 2s & = & 0 \\ 3t & & & = & 2 \\ 3t & + & 3s & = & 0 \end{array} \right\}.$$

Looking at the 3rd equation, we note that we should have $3t = 2 \Leftrightarrow t = 2\cdot 3^{-1} = 2\cdot 2 = 4$ (recall that in part (c) we are working with scalars from $\mathbb{Z}_5$).

Combining this with the 2nd equation, we note that we should have $2\cdot 4 + 2s = 0 \Leftrightarrow 2s = -2\cdot 4 = -3 = 2 \Leftrightarrow s = 2\cdot 2^{-1} = 1$.

Therefore, there is only one choice of values for t and s that would satisfy the 2nd and the 3rd equation at the same time, $t = 4$ and $s = 1$.

We now check whether this choice of values for t and s solves the system: we note that

$$4\cdot 4 + 2\cdot 1 = 1 + 2 = 3 \quad \text{and} \quad 3\cdot 4 + 3\cdot 1 = 2 + 3 = 0.$$

We conclude that we can write $\bar{u}$ as a linear combination of $\bar{v}$ and $\bar{w}$:

$$\begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} = 4\cdot \begin{pmatrix} 4 \\ 2 \\ 3 \\ 3 \end{pmatrix} + 1\cdot \begin{pmatrix} 2 \\ 2 \\ 0 \\ 3 \end{pmatrix} = 4\cdot \begin{pmatrix} 4 \\ 2 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \\ 3 \end{pmatrix}.$$

Problem 5. (a) By commutativity of vector addition, we have

$$\bar{a}+\bar{b}=\bar{b}+\bar{a}=\begin{pmatrix}3\\-5\\3\\-1\end{pmatrix}, \qquad \bar{a}+\bar{w}=\bar{w}+\bar{a}=\begin{pmatrix}1\\0\\5\\2\end{pmatrix},$$

$$\bar{b}+\bar{w}=\bar{w}+\bar{b}=\begin{pmatrix}2\\1\\8\\-7\end{pmatrix}, \quad \text{and} \;\; \bar{u}+\bar{v}=\bar{v}+\bar{u}=\begin{pmatrix}1\\7\\7\end{pmatrix}.$$

Every other pair of the given vectors will be formed by vectors of different dimension, so it won't make sense to add these.

(b) Out of the vectors we found in part (a), there is only one 3-dimensional vector, and three 4-dimensional ones, therefore we compute the dot product only for pairs of these 4-dimensional vectors. By commutativity/symmetry of the dot product, we have

$$\left\langle \left(\begin{smallmatrix}3\\-5\\3\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\5\\2\end{smallmatrix}\right) \right\rangle = \left\langle \left(\begin{smallmatrix}1\\0\\5\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}3\\-5\\3\\-1\end{smallmatrix}\right) \right\rangle = 1\cdot 3+0\cdot(-5)+5\cdot 3+2\cdot(-1)=16,$$

$$\left\langle \left(\begin{smallmatrix}3\\-5\\3\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\1\\8\\-7\end{smallmatrix}\right) \right\rangle = \left\langle \left(\begin{smallmatrix}2\\1\\8\\-7\end{smallmatrix}\right), \left(\begin{smallmatrix}3\\-5\\3\\-1\end{smallmatrix}\right) \right\rangle = 2\cdot 3+1\cdot(-5)+8\cdot 3+(-7)\cdot(-1)=32,$$

$$\text{and} \quad \left\langle \left(\begin{smallmatrix}1\\0\\5\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\1\\8\\-7\end{smallmatrix}\right) \right\rangle = \left\langle \left(\begin{smallmatrix}2\\1\\8\\-7\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\5\\2\end{smallmatrix}\right) \right\rangle = 2\cdot 1+1\cdot 0+8\cdot 5+(-7)\cdot 2=28.$$

Math 127, A1
2nd Midterm Exam – November 4, 2019

Name: ____

This exam has 4 problems with a total worth of 65 points. To earn maximum credit, you need to accumulate 45 points or more.

General instructions (important, read back of the page too).

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- The last double-sided page can be freely used as scratch paper.

Do not write any part of your final answers there

because these parts will not be considered during grading.

Notation/Clarifications.

1. Excluding Problem 4 where you are specifically asked to use matrices, in all other instances where you have to analyse or show something about a linear system and/or a matrix, you can choose to work not in the setting that is given but in the other setting, as long as you "translate" the problem correctly and explain your approach.

2. Recall the following notation that we can use for elementary matrices in $\mathbb{F}^{n\times n}$, where $\mathbb{F}$ is a given field and n a positive integer.

- Given a non-zero element $\lambda \in \mathbb{F}$ and an index i_0 with $1 \leqslant i_0 \leqslant n$, $\boldsymbol{D_{i_0;\lambda}}$ is the diagonal matrix $\big(d_{ij}\big)_{1\leqslant i,j\leqslant n}$ in $\mathbb{F}^{n\times n}$ which satisfies $d_{i_0,i_0} = \lambda$, $d_{ii} = 1$ if $i \neq i_0$ (and $d_{ij} = 0$ if $i \neq j$).
- Given indices i_0, j_0 with $1 \leqslant i_0, j_0 \leqslant n$ and $i_0 \neq j_0$, $\boldsymbol{P_{i_0,j_0}}$ is the matrix in $\mathbb{F}^{n\times n}$ which satisfies the following:
  - its i_0-th column is the vector $\bar{e}_{j_0}$ in $\mathbb{F}^n$,
  - its j_0-th column is the vector $\bar{e}_{i_0}$ in $\mathbb{F}^n$,
  - and if $r \notin \{i_0, j_0\}$, then its r-th column is the vector $\bar{e}_r$ in $\mathbb{F}^n$.

  Recall that $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ are the standard basis vectors in $\mathbb{F}^n$.

  Equivalently, P_{i_0,j_0} is the matrix that we get if we swap the i_0-th and the j_0-th row of the identity matrix $I_n \in \mathbb{F}^{n\times n}$.
- Given indices i_0, j_0 with $1 \leqslant i_0, j_0 \leqslant n$ and $i_0 \neq j_0$, and given an element $\mu \in \mathbb{F}$, $\boldsymbol{E_{i_0,j_0;\mu}}$ is the matrix $\big(e_{ij}\big)_{1\leqslant i,j\leqslant n}$ in $\mathbb{F}^{n\times n}$ which satisfies the following:
  - its diagonal elements are all 1,
  - $e_{i_0,j_0} = \mu$,
  - and if $i \neq j$ and $(i,j) \neq (i_0, j_0)$, then $e_{ij} = 0$.

Name and Student ID: ______________________________

Problem 1 (a) (*max. 5 points*) Let $LS1$ be a staircase linear system with m equations in n unknowns. Show that, if the system has more than n pivots, it is inconsistent.

(b) (*max. 10 points*) Let κ, λ, μ be unknown constants/parameters allowed to take values in $\mathbb{Z}_5$. Consider the following linear system with coefficients from $\mathbb{Z}_5$:

$$\left\{ \begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 4 \\ 2x_1 & + & \kappa x_2 & + & 2x_3 & = & 1 \\ x_1 & + & x_2 & + & (\mu^2 - 3)x_3 & = & \lambda \end{array} \right\}.$$

Find all combinations of κ, λ, μ, if any exist, for which the corresponding system has a unique solution. Justify your answer. (You do not need to find the solution to any of these systems.)

Name and Student ID: ______________________________

Problem 2 (*max. 10 points*) Consider the following sets of vectors from $\mathbb{Z}_3^4$:

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad S_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Exactly two of them have the same linear span: find which two, and also justify why the span of the remaining set is different (the scalars are taken from $\mathbb{Z}_3$).

Name and Student ID: ____

Problem 3 (a) (*max. 10 points*) Let $n > 1$ and let A, B be square matrices in $\mathbb{Z}_{11}^{n\times n}$. Suppose that A and B are both invertible. Prove that A, B are row equivalent.

(b) (*max. 10 points*) <u>TRUE OR FALSE?</u> Determine if the following statement is correct:

"For every $n > 1$ and $A, B \in \mathbb{R}^{n\times n}$, if A, B have the same number of non-zero rows, the same number of non-zero columns, and the same number of non-zero entries, then they are row equivalent."

and justify your answer fully.

Name and Student ID: ______________________________

Problem 4 (a) (*max. 10 points*) Consider the following matrices from $\mathbb{R}^{3\times 3}$:

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix}.$$

Find elementary matrices $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_k$ for some $k \geqslant 1$ so that

$$B = \mathcal{E}_k \cdots \mathcal{E}_2 \mathcal{E}_1 A.$$

(b) (*max. 10 points*) Determine whether AB^3AB is invertible and justify your answer. (It is not required to find its inverse if it exists.)

Name and Student ID: ____________________

Scratch paper.

Scratch paper.

Math 127
Suggested solutions to 2nd Midterm

Problem 1. (a) By the definition of staircase system, each column (containing the coefficients of one specific variable or the constant terms) can contain at most one pivot coefficient. Therefore, since there are more than n pivots, it is not possible that all of them are found in columns corresponding to the variables. In other words, there is a pivot in the column of constant terms, and the equation containing this pivot must have the form

$$0x_1 + 0x_2 + \cdots + 0x_n = c,$$

where c is the pivot. But by definition again, the pivot c is a non-zero element, so this equation cannot be satisfied, which makes the system inconsistent as well.

(b) We use Gaussian elimination on the augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & \kappa & 2 & 1 \\ 1 & 1 & \mu^2-3 & \lambda \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & \kappa-4 & 0 & 3 \\ 0 & 4 & \mu^2-4 & \lambda-4 \end{array}\right).$$

We observe that, if $\kappa - 4 = 0 \Leftrightarrow \kappa = 4$, then the new equivalent system will have an equation of the form

$$0x_1 + 0x_2 + 0x_3 = 3,$$

and so it will be inconsistent.

Therefore, from now on we only consider combinations with $\kappa \neq 4$. In such a case, $\kappa - 4$ in the second row of the last matrix is a pivot, and we can continue using Gaussian elimination to get to a matrix in REF:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & \kappa-4 & 0 & 3 \\ 0 & 4 & \mu^2-4 & \lambda-4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & \kappa-4 & 0 & 3 \\ 0 & 0 & \mu^2-4 & \lambda-4+3\cdot(\kappa-4)^{-1} \end{array}\right).$$

The last matrix is now in REF, regardless of what μ^2-4 and $\lambda-4+3\cdot(\kappa-4)^{-1}$ are. However, if $\mu^2 - 4 = 0$, then the system will either be inconsistent (if $\lambda - 4 + 3 \cdot (\kappa - 4)^{-1} \neq 0$), or it will have more than one solution (if $\lambda - 4 + 3 \cdot (\kappa - 4)^{-1} = 0$).

On the other hand, if $\mu^2 - 4 \neq 0$, then the system will have 3 pivots (as many as the variables), and no pivot in the last column, therefore it will have a unique solusion.

Therefore, we conclude that the only "good" combinations are those with $\kappa \neq 4$ and $\mu^2 - 4 \neq 0$. Since we work in $\mathbb{Z}_5$, this is equivalent to having $\kappa \neq 4$ and $\mu \in \{0, 1, 4\}$.

Problem 2. As we have seen in homework, for every $i, j \in \{1, 2, 3\}$ we have that

$$\begin{aligned} \operatorname{span}(S_i) = \operatorname{span}(S_j) & \qquad \text{if and only if} \\ \operatorname{span}(S_i) \subseteq \operatorname{span}(S_j) \ \text{ and } \ \operatorname{span}(S_j) \subseteq \operatorname{span}(S_i) & \qquad \text{if and only if} \\ S_i \subseteq \operatorname{span}(S_j) \ \text{ and } \ S_j \subseteq \operatorname{span}(S_i). & \end{aligned}$$

We show that $\operatorname{span}(S_1) = \operatorname{span}(S_2)$ by checking the third statement.

We observe that $S_2 \subseteq S_1$, and therefore we immediately get that $S_2 \subseteq \operatorname{span}(S_1)$.

Similarly, we note that the first two vectors of S_1 are also in S_2, and hence in $\operatorname{span}(S_2)$. Thus it remains to check that the third vector of S_1 is also in $\operatorname{span}(S_2)$: in other words, we need to check that there are $\lambda, \mu \in \mathbb{Z}_3$ so that

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda + \mu \\ 2\lambda \\ \mu \\ \lambda + \mu \end{pmatrix}.$$

Observe that here we do not have to find specific λ and μ that satisfy this vector equation, but simply to show that such exist. Equivalently, we have to check that the following matrix corresponds to a consistent system:

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right), \quad \text{which is true, since it is } \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

and the last matrix is in REF and has no pivot in the last column.

We can conclude that $S_1 \subseteq \operatorname{span}(S_2)$, and hence that $\operatorname{span}(S_1) = \operatorname{span}(S_2)$ as we wanted.

We now check that $\operatorname{span}(S_3) \neq \operatorname{span}(S_1) = \operatorname{span}(S_2)$. For this, it would suffice to show that $S_3 \not\subseteq \operatorname{span}(S_2)$. But the first vector of S_3 is in S_2 and hence also in $\operatorname{span}(S_2)$, thus we would need to show that one (or both) of the systems with augmented matrices

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right), \qquad \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right)$$

is inconsistent. In fact, we could try to solve both systems at the same time (given that they have the same coefficient matrix):

$$\left(\begin{array}{cc|c|c} 1 & 1 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This shows that both systems are inconsistent, and hence that $S_3 \not\subseteq \operatorname{span}(S_2)$ as we wanted.

Problem 3. (a) By a Theorem we stated in class, the RREF of A is the identity matrix I_n, and the same is true for the RREF of B. Therefore we can find elementary matrices $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k$, $\widetilde{\mathcal{E}}_1, \widetilde{\mathcal{E}}_2, \ldots, \widetilde{\mathcal{E}}_s$, for some positive integers $k, s \geqslant 1$, so that

$$I_n = \mathcal{E}_k \cdots \mathcal{E}_2 \mathcal{E}_1 A \quad \text{and} \quad I_n = \widetilde{\mathcal{E}}_s \cdots \widetilde{\mathcal{E}}_2 \widetilde{\mathcal{E}}_1 B.$$

Given that elementary matrices are invertible, we can also write

$$B = \widetilde{\mathcal{E}}_1^{-1} \widetilde{\mathcal{E}}_2^{-1} \cdots \widetilde{\mathcal{E}}_s^{-1} I_n.$$

Therefore

$$B = \widetilde{\mathcal{E}}_1^{-1} \widetilde{\mathcal{E}}_2^{-1} \cdots \widetilde{\mathcal{E}}_s^{-1} I_n = \widetilde{\mathcal{E}}_1^{-1} \widetilde{\mathcal{E}}_2^{-1} \cdots \widetilde{\mathcal{E}}_s^{-1} \big(\mathcal{E}_k \cdots \mathcal{E}_2 \mathcal{E}_1 A\big).$$

Since inverses of elementary matrices are also elementary matrices, this shows that $A \sim B$.

(b) We show that the statement

> "For every $n > 1$ and $A, B \in \mathbb{R}^{n \times n}$, if A, B have the same number of non-zero rows, the same number of non-zero columns, and the same number of non-zero entries, then they are row equivalent"

is false.

Proof Strategy: Since the statement claims something is true **for every** $n > 1$, and **every** two matrices in $\mathbb{R}^{n \times n}$ which have certain properties (regarding their rows, columns and entries), we disprove the statement by finding a counterexample, that is, by finding some $n > 1$ and two matrices in $\mathbb{R}^{n \times n}$ which have these properties but do not satisfy the desired conclusion.

We can choose $n = 3$ and A and B to be the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that each of these matrices has 2 non-zero rows, 3 non-zero columns and 3 non-zero entries.

Moreover, both matrices are in RREF: indeed, only the last row of each matrix is zero, while the first non-zero entry of each non-zero row is 1, and is the only non-zero entry in its column; finally, in each of these matrices, the first non-zero entry of the second row is to the right of the first non-zero entry of the first row.

But now we recall that, according to a Theorem we stated in class, every matrix has a unique (row equivalent) Reduced Row Echelon Form. Therefore, since A is in RREF, and since trivially $A \sim A$, A cannot be row equivalent to any other matrix in RREF. In other words, since A and B are different and are both in RREF, they are not row equivalent.

Alternative solution: Consider the following linear systems with coefficients from $\mathbb{R}$:

$$LS_1 : \left\{ \begin{array}{rcrcl} x_1 & + & 2x_2 & = & 0 \\ x_1 & + & 3x_2 & = & 0 \end{array} \right\} \quad \text{and} \quad LS2 : \left\{ \begin{array}{rcrcl} x_1 & + & 2x_2 & = & 0 \\ 2x_1 & + & 4x_2 & = & 0 \end{array} \right\}.$$

By finding equivalent staircase systems, we can check that LS_1 has a unique solution, while LS_2 has infinitely many solutions. Indeed,

$$\left\{ \begin{array}{rcrcl} x_1 & + & 2x_2 & = & 0 \\ x_1 & + & 3x_2 & = & 0 \end{array} \right\} \quad \Leftrightarrow \quad \left\{ \begin{array}{rcrcl} x_1 & + & 2x_2 & = & 0 \\ & & x_2 & = & 0 \end{array} \right\},$$

which has only pivot variables and no pivot in the last column, while

$$\left\{ \begin{array}{rcrcl} x_1 & + & 2x_2 & = & 0 \\ 2x_1 & + & 4x_2 & = & 0 \end{array} \right\} \quad \Leftrightarrow \quad \left\{ \begin{array}{rcrcl} x_1 & + & 2x_2 & = & 0 \\ 0x_1 & + & 0x_2 & = & 0 \end{array} \right\},$$

which has one free variable and no pivot in the last column.

Therefore, by recalling a Proposition we proved in class, we can conclude that the augmented matrices of the two systems,

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{pmatrix},$$

are not row equivalent (because if they were, the systems would be equivalent).

In fact, even the coefficient matrices of these two systems,

$$\widetilde{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

are not row equivalent (given that in this case $A \sim B$ if and only if $\widetilde{A} \sim \widetilde{B}$, because the last columns in A and B are zero).

We finally note that A and B (or $\widetilde{A}$ and $\widetilde{B}$) have the same number of non-zero rows, the same number of non-zero columns, and the same number of non-zero entries, as we wanted (and we can use the square matrices $\widetilde{A}$ and $\widetilde{B}$ for the desired counterexample).

Conclusion/Side Remark: By what we showed above, it follows that the negation of the given statement is true:

"There is $n > 1$ and there are $A, B \in \mathbb{R}^{n \times n}$ such that

- A, B have the same number of non-zero rows, the same number of non-zero columns, and the same number of non-zero entries,
- and A, B are **not** row equivalent".

In fact, an even stronger statement is true:

"**For every $n > 1$**, there are $A, B \in \mathbb{R}^{n \times n}$ such that A, B have the same number of non-zero rows, the same number of non-zero columns, and the same number of non-zero entries, and A, B are **not** row equivalent",

but part (b) does not ask to show this.

Problem 4. (a) We observe that

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 3 \end{pmatrix} \sim D_{3;\frac{1}{3}}A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sim P_{23}D_{3;\frac{1}{3}}A = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}.$$

It remains to check whether/how we can write the row vector $(0,3,0) \in \mathbb{R}^{1\times 3}$ as a linear combination of the third row of the last matrix and one or both of the remaining rows: in other words, we want to find $\kappa, \lambda, \mu \in \mathbb{R}$, with $\kappa \neq 0$, so that

$$(0,3,0) = \kappa(2,-1,0) + \lambda(1,4,3) + \mu(1,0,1),$$

or equivalently

$$\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \kappa \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\kappa + \lambda + \mu \\ -\kappa + 4\lambda \\ 3\lambda + \mu \end{pmatrix}.$$

For this reason, we check whether the linear system with augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ -1 & 4 & 0 & 3 \\ 0 & 3 & 1 & 0 \end{array}\right)$$

is consistent. But

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ -1 & 4 & 0 & 3 \\ 0 & 3 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 4.5 & 0.5 & 3 \\ 0 & 3 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 4.5 & 0.5 & 3 \\ 0 & 0 & \frac{2}{3} & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 0 \\ 0 & 1 & \frac{1}{9} & \frac{2}{3} \\ 0 & 0 & 1 & -3 \end{array}\right).$$

We therefore see that the system has a solution (in fact, a unique solution), which is given by $\mu = -3$, $\lambda + \frac{1}{9}(-3) = \frac{2}{3} \Rightarrow \lambda = 1$ and $\kappa + \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-3) = 0 \Rightarrow \kappa = 1$.

We can thus continue writing:

$$A \sim P_{23}D_{3;\frac{1}{3}}A = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \sim E_{31;1}P_{23}D_{3;\frac{1}{3}}A = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 0 & 1 \\ 3 & 3 & 3 \end{pmatrix}$$

$$\sim E_{32;(-3)}E_{31;1}P_{23}D_{3;\frac{1}{3}}A = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix}.$$

We conclude that

$$B = E_{32;(-3)} E_{31;1} P_{23} D_{3;\frac{1}{3}} A.$$

(b) It suffices to check whether A and B are invertible. Indeed, if A^{-1} and B^{-1} exist, then AB^3AB will also be invertible as a product of invertible matrices, and $\left(AB^3AB\right)^{-1}$ will be equal to $B^{-1}A^{-1}\left(B^{-1}\right)^3 A^{-1}$.

We can check using Gaussian elimination whether A and B are invertible.

In fact, we could first observe that, because A and B are row equivalent, it suffices to check that one of them is invertible. Indeed, if we already knew that A is invertible, then $B = E_{32;(-3)} E_{31;1} P_{23} D_{3;\frac{1}{3}} A$ would be invertible as a product of invertible matrices.

Conversely, we see that

$$A = D^{-1}_{3;\frac{1}{3}} P^{-1}_{23} E^{-1}_{31;1} E^{-1}_{32;(-3)} B = D_{3;3} P_{23} E_{31;(-1)} E_{32;3} B,$$

and therefore, in the same way as above, we can conclude that A is invertible if we already know that B is invertible.

We now check whether B is invertible by finding a Row Echelon Form of it:

$$B = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 \\ 0 & -4 & -2 \\ 0 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 \\ 0 & -4 & -2 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}.$$

Since the last matrix, which is a REF of B, has 3 pivots, B is invertible. By what we showed above, A is also invertible, and so is AB^3AB.

Math 127 – Recitation of October 1

During this recitation hour we discussed HW2, Problem 3, as well as a similar question to HW2, Problem 6. An outline of the discussion is given below.

HW2, Problem 3. We need to fill out the following tables of addition and multiplication so that the operations will satisfy the axioms of a field:

$+$	0	1	c	d
0				
1				
c				
d				

$\cdot$	0	1	c	d
0				
1				
c				
d				

.

We deal with the table of multiplication here. First of all we note that the first and the second row have to be filled out in the following way to reflect the axiom about the multiplicative identity as well as the property "$0 \cdot x = 0$ for every x in the field":

$\cdot$	0	1	c	d
0	0	0	0	0
1	0	1	c	d
c				
d				

.

Also, in order to ensure that commutativity will hold, we have to fill out analogously the corresponding columns:

$\cdot$	0	1	c	d
0	0	0	0	0
1	0	1	c	d
c	0	c		
d	0	d		

.

We now recall the cancellation law for multiplication, which we should make holds true if we want this table of multiplication to be a table of a field: the law implies that the coloured part of the table below should have the property that different cells in the same row contain different elements, and analogously (because of commutativity as well) different cells in the same column contain different elements:

$\cdot$	0	1	c	d
0	0	0	0	0
1	0	1	c	d
c	0	c		
d	0	d		

.

Indeed, if we look at a row corresponding to a non-zero element x and at different cells within this row, one of them should contain the product $x \cdot y$ for some $y \in \mathbb{F}_4$ and the other cell should contain the product $x \cdot z$ for some $z \in \mathbb{F}_4$, $z \neq y$. But the cancellation law gives

$$\text{if } x \neq 0, \quad \text{then } x \cdot y = x \cdot z \quad \Rightarrow \quad y = z,$$
$$\text{or equivalently:} \quad \text{if } x \neq 0, \quad \text{then } y \neq z \quad \Rightarrow \quad x \cdot y \neq x \cdot z.$$

This shows that we can only use the elements 1 and d to fill out the rest of the third row in the above table: $c \cdot c$ should be equal to either 1 or d, and similarly $c \cdot d$ should be equal to either 1 or d and different from $c \cdot c$.

But $c \cdot d$ cannot be equal to d either, given that we have $d \neq 1$, so if we set $c \cdot d = d = 1 \cdot d$, the cancellation law will not hold for this particular table of multiplication (more simply we can note that d is already in the fourth column and second row, so we cannot fill out the cell corresponding to $c \cdot d$ using d again).

This leaves only one possibility for $c \cdot d$: $c \cdot d = 1$, and it also implies that we must have $c \cdot c = d$.

Furthermore, by commutativity we must have $d \cdot c = c \cdot d = 1$. Finally, there is one remaining cell to fill out, and there is only one choice left for it: $d \cdot d = c$. Therefore, the table of multiplication must be filled out as follows:

$\cdot$	0	1	c	d
0	0	0	0	0
1	0	1	c	d
c	0	c	d	1
d	0	d	1	c

.

We now check the axioms concerning only multiplication (since we only filled out the table of multiplication).

We first observe that, simply by the way we filled out the table, there is an identity element, the element 1, and also multiplication is commutative.

We also note that every row corresponding to a non-zero element has a cell containing the element 1, which shows that every non-zero element has a multiplicative inverse.

Therefore, it remains to check that multiplication is associative, that is, to check that, for all $x, y, z \in \mathbb{F}_4$, we have

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

Here we need to check that this identity is true for all combinations of x, y, z from $\mathbb{F}_4$, however we can group these combinations into a few different main cases.

Case 1: one of $\boldsymbol{x, y, z}$ is 0. For convenience, we break this case into 3 smaller cases:

- $\underline{x = 0}$ Then we have $(x \cdot y) \cdot z = (0 \cdot y) \cdot z = 0 \cdot z = 0$, while $x \cdot (y \cdot z) = 0 \cdot (y \cdot z) = 0$, therefore $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in this case.
- $\underline{y = 0}$ Then we have $(x \cdot 0) \cdot z = (0 \cdot x) \cdot z = 0 \cdot z = 0$ using commutativity as well, while $x \cdot (0 \cdot z) = x \cdot 0 = 0$ too, as we wanted.
- $\underline{z = 0}$ Then we have $(x \cdot y) \cdot 0 = 0 \cdot (x \cdot y) = 0$ using commutativity as well, while $x \cdot (y \cdot 0) = x \cdot 0 = 0$, as we wanted.

Case 2: one of $\boldsymbol{x, y, z}$ is 1. We break this case into 3 smaller cases:

- $\underline{x = 1}$ Then we have $(x \cdot y) \cdot z = (1 \cdot y) \cdot z = y \cdot z = 1 \cdot (y \cdot z) = x \cdot (y \cdot z)$.
- $\underline{y = 1}$ Then we have $(x \cdot 1) \cdot z = x \cdot z = x \cdot (1 \cdot z)$.
- $\underline{z = 1}$ Then we have $(x \cdot y) \cdot 1 = x \cdot y = x \cdot (y \cdot 1)$.

It remains to check the cases where $x, y, z \in \{c, d\}$. There are two main cases here:

Case 3: $\boldsymbol{x = y = z}$. Then $(x \cdot x) \cdot x = x \cdot (x \cdot x)$ simply by commutativity.

Case 4: $\boldsymbol{\{x, y, z\} = \{c, d\}}$. In other words, in this case we have two of x, y, z being equal to each other and equal either to c or d, and the third element of x, y, z being equal to the remaining element of $\{c, d\}$.

We will break this case into 3 smaller cases and we will use the fact that $c \cdot c = d$, $d \cdot d = c$ and $c \cdot d = d \cdot c = 1$.

$x = y \neq z$. Given the products above, we can see that, regardless of whether $x = c$ or $x = d$, we have that $x \cdot y = x \cdot x = z$, $z \cdot z = x$, and $x \cdot z = y \cdot z = 1$. Therefore, we have

$$(x \cdot y) \cdot z = z \cdot z = x, \qquad \text{while} \quad x \cdot (y \cdot z) = x \cdot 1 = x,$$

which shows what we wanted in this case.

$x \neq y = z$. Similarly here we have $x \cdot y = x \cdot z = 1$, while $x \cdot x = y = z$, $y \cdot z = z \cdot z = x$. Therefore, we have

$$(x \cdot y) \cdot z = 1 \cdot z = z, \qquad \text{while } \; x \cdot (y \cdot z) = x \cdot x = z,$$

which shows what we wanted.

$x = z \neq y$. Here we have $x \cdot y = 1 = y \cdot z$. Therefore,

$$(x \cdot y) \cdot z = 1 \cdot z = z = x = x \cdot 1 = x \cdot (y \cdot z),$$

which is what we wanted.

We have now checked all cases regarding associativity, so we can conclude that multiplication in $\mathbb{F}_4$, in the way that we defined it, is associative.

Question 2; similar to HW2, Problem 6. In each of the parts below, you are given three sets of vectors from the specified vector space; two of them have the same linear span, while the span of the third one is different. Find in each case which two sets have the same linear span.
(i) S_1, S_2, S_3 are subsets of $\mathbb{R}^4$.

$$S_1 = \left\{ \begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 6 \\ 5 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ -6 \\ 0 \\ 6 \end{pmatrix} \right\}, \quad S_3 = \left\{ \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix} \right\}.$$

(ii) T_1, T_2, T_3 are subsets of $\mathbb{Z}_5^3$.

$$T_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}, T_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}, T_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Solution. (i) We begin with two important remarks.

Remark 1. When we want to show that two sets A, B are equal, in most cases, instead of showing directly that $A = B$, it's more convenient to find a way to show that $A \subseteq B$ and $B \subseteq A$.

In this problem we will show that $\text{span}(S_1) \subseteq \text{span}(S_3)$ and $\text{span}(S_3) \subseteq \text{span}(S_1)$.

To show that $\text{span}(S_1) \subseteq \text{span}(S_3)$, we need to show that every vector $v \in \text{span}(S_1)$ is also contained in $\text{span}(S_3)$. Consider a vector $v \in \text{span}(S_1)$; then there are scalars λ_v, μ_v in $\mathbb{R}$ so that

$$v = \lambda_v \cdot \begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix} + \mu_v \cdot \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix}.$$

We need to show that we can find scalars $t, s \in \mathbb{R}$ (that will depend on these λ_v, μ_v) so that

$$v = \lambda_v \cdot \begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix} + \mu_v \cdot \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} \quad = \quad t \cdot \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix}.$$

The following remark helps do this in a more efficient way.

Remark/Claim 2. To show that $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_3)$, it suffices to check that

$$\begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix} \in \mathrm{span}(S_3) \qquad \text{and} \qquad \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} \in \mathrm{span}(S_3) \tag{1}$$

(note that anyway we would need to check that these two vectors are contained in $\mathrm{span}(S_3)$, what the claim tells us is that this is also the only thing we need to do).

Proof of Claim 2. Suppose we have shown (1), that is, we have found $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{R}$ so that

$$\begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix} = \lambda_1 \cdot \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} + \mu_1 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} = \lambda_2 \cdot \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} + \mu_2 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix}.$$

Then for an arbitrary vector

$$v = \lambda_v \cdot \begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix} + \mu_v \cdot \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix}$$

in $\mathrm{span}(S_1)$, we will be able to write

$$\begin{aligned} v &= \lambda_v \cdot \begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix} + \mu_v \cdot \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} \\ &= \lambda_v \cdot \left[\lambda_1 \cdot \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} + \mu_1 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix} \right] + \mu_v \cdot \left[\lambda_2 \cdot \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} + \mu_2 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix} \right] \\ &= (\lambda_v \cdot \lambda_1) \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} + (\lambda_v \cdot \mu_1) \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix} + (\mu_v \cdot \lambda_2) \cdot \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} + (\mu_v \cdot \mu_2) \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix} \\ &= \big(\lambda_v \cdot \lambda_1 + \mu_v \cdot \lambda_2\big) \cdot \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} \quad + \quad \big(\lambda_v \cdot \mu_1 + \mu_v \cdot \mu_2\big) \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix}. \end{aligned}$$

But this shows that v is also contained in $\operatorname{span}(S_3)$, and since v was arbitrary we can conclude that $\operatorname{span}(S_1) \subseteq \operatorname{span}(S_3)$ as we wanted.

It remains to check (1). To show that

$$\begin{pmatrix} 4 \\ 2 \\ -2 \\ -7 \end{pmatrix} = \lambda_1 \cdot \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} + \mu_1 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix}, \tag{2}$$

we need to solve the linear system

$$\left\{ \begin{array}{rcrcr} 3\lambda_1 & - & 3\mu_1 & = & 4 \\ 5\lambda_1 & - & \mu_1 & = & 2 \\ 2\lambda_1 & + & 2\mu_1 & = & -2 \\ & & 6\mu_1 & = & -7 \end{array} \right\}.$$

This is left as an exercise here: check that $\lambda_1 = 1/6$ and $\mu_1 = -7/6$.

Similarly, to show that

$$\begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} = \lambda_2 \cdot \begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} + \mu_2 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix},$$

we need to solve the linear system

$$\left\{ \begin{array}{rcrcr} 3\lambda_1 & - & 3\mu_1 & = & 2 \\ 5\lambda_1 & - & \mu_1 & = & 2 \\ 2\lambda_1 & + & 2\mu_1 & = & 0 \\ & & 6\mu_1 & = & -2 \end{array} \right\}.$$

We must have $\lambda_2 = 1/3$, $\mu_2 = -1/3$.

We conclude that (1) is true, and therefore, recalling Remark/Claim 2 as well, that $\operatorname{span}(S_1) \subseteq \operatorname{span}(S_3)$.

Next, we need to also show that $\operatorname{span}(S_3) \subseteq \operatorname{span}(S_1)$. Again it suffices to check that

$$\begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \end{pmatrix} \in \operatorname{span}(S_3) \qquad \text{and} \qquad \begin{pmatrix} -3 \\ -1 \\ 2 \\ 6 \end{pmatrix} \in \operatorname{span}(S_3).$$

Show that

$$\begin{pmatrix}3\\5\\2\\0\end{pmatrix} = (-1)\cdot\begin{pmatrix}4\\2\\-2\\-7\end{pmatrix} + \frac{7}{2}\cdot\begin{pmatrix}2\\2\\0\\-2\end{pmatrix} \tag{3}$$

(this was Problem 4(b) on the 1st Midterm Exam), and that

$$\begin{pmatrix}-3\\-1\\2\\6\end{pmatrix} = (-1)\cdot\begin{pmatrix}4\\2\\-2\\-7\end{pmatrix} + \frac{1}{2}\cdot\begin{pmatrix}2\\2\\0\\-2\end{pmatrix}$$

(note that we can find the last linear combination more quickly by combining (2) (where $\lambda_1 = 1/6$ and $\mu_1 = -7/6$) and (3); why?).

Having shown that $\text{span}(S_1) \subseteq \text{span}(S_3)$ and $\text{span}(S_3) \subseteq \text{span}(S_1)$, we conclude that $\text{span}(S_1) = \text{span}(S_3)$.

Finally, we check that $\text{span}(S_2) \nsubseteq \text{span}(S_1)$. This follows immediately if we show that

$$\begin{pmatrix}6\\5\\2\\0\end{pmatrix} \notin \text{span}(S_1).$$

To show this, check that the system

$$\left\{\begin{array}{rcrcr} 4\lambda & + & 2\mu & = & 6\\ 2\lambda & + & 2\mu & = & 5\\ -2\lambda & & & = & 2\\ -7\lambda & - & 2\mu & = & 0 \end{array}\right\}$$

is inconsistent.

(ii) Left as a practice exercise: show that $\text{span}(T_1) = \text{span}(T_2) \neq \text{span}(T_3)$.

Math 127 – Recitation of October 8

During this recitation hour we discussed the following problem, taken from a homework assignment given by Professor Gannon in a past term of the course.

Left as a practice exercise for now for anyone who wants to try to work on it.

Problem. Let $k \in \mathbb{Z}_{17}$ be some (unknown) constant. Consider the system

$$\left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & = & 4 \\ 3x_1 & - & x_2 & + & 5x_3 & = & -2 \\ 2x_1 & - & 3x_2 & + & (k^2-8)x_3 & = & k-2 \end{array} \right\}$$

where the coefficients come from $\mathbb{Z}_{17}$.

(a) For which value(s) of k, if any, does the system have

(i) a unique solution?

(ii) no solution?

(iii) more than one solutions? (And specify how many solutions the system has in each subcase here.)

(b) Solve the system where possible. That is, for those k in (i) above, find the unique solution $(x_1, x_2, x_3) \in \mathbb{Z}_{17} \times \mathbb{Z}_{17} \times \mathbb{Z}_{17}$ (this solution will depend on k), and for each of the k in (iii) above, parametrise the solutions (x_1, x_2, x_3) appropriately.

[*Hint/Clarification.* In part (b), when you try to find the unique solution of the system for each k in (i) above, do not plug in different values of k from (i), but rather try to find a formula for the unique solution in terms of k.]

Math 127 – Recitation of October 15

During this recitation hour we discussed the following problem, which is related to Problem 5 of HW3.

Problem. Let k, λ, μ, u, v, w be unknown constants/parameters allowed to take values in $\mathbb{Z}_5$. Consider the following linear system with coefficients from $\mathbb{Z}_5$:

$$\left\{ \begin{array}{rcrcrcrcrcrcrcl} \kappa x_1 & + & \lambda x_2 & + & 3x_3 & - & 4x_4 & + & 3x_5 & + & x_6 & - & x_7 & = & 1 \\ 0x_1 & + & \mu x_2 & + & x_3 & + & 0x_4 & + & 2x_5 & - & x_6 & + & 2x_7 & = & 2 \\ 0x_1 & + & 0x_2 & + & 2x_3 & - & 4x_4 & + & x_5 & + & 4x_6 & + & 0x_7 & = & 1 \\ 0x_1 & + & 0x_2 & + & 0x_3 & + & x_4 & - & 2x_5 & + & 0x_6 & + & ux_7 & = & 0 \\ 0x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & + & x_5 & + & x_6 & + & x_7 & = & 3 \\ 0x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & + & vx_5 & + & 0x_6 & + & 2x_7 & = & 0 \\ 0x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & + & 0x_5 & + & 0x_6 & + & (u^2-1)x_7 & = & w \end{array} \right\}.$$

(a) For which combinations of $\kappa, \lambda, \mu, u, v$ and w, if any, do we get an upper triangular/staircase system?

(b) For which of the "good" combinations you found in part (a), does the corresponding system have

(i) no solution?

(ii) a unique solution?

(iii) or more than one solutions? (For each of the combinations in this case determine also how many solutions the corresponding system has.)

See suggested solution on next page; good idea to come up with your own approach to this problem first, before checking out the suggested solution.

Solution. (a) We first observe that the first six rows are non-zero rows for any combination of $\kappa, \lambda, \mu, u, v$ and w, so regardless of whether the last row is zero or non-zero, the condition that all non-zero rows should be above any non-zero row will be satisfied.

We will now examine what the leading non-zero coefficient of every non-zero row is for the different combinations of the parameters.

To make sure we do not forget/omit to check any combinations, we consider two main cases which cover all possible combinations (not just the "good" ones that we're looking for in part (a)).

Case 1: *combinations with* $\kappa \neq 0$. For any "good" combinations here, the first pivot of the corresponding system is κ. Therefore, the value of λ will not matter, because λ could not be a pivot coefficient anyway.

For the other rows we have:

- the leading non-zero coefficient of the second row is either μ if $\mu \neq 0$, or otherwise the coefficient 1 of x_3;
- the leading non-zero coefficient of the third row is the coefficient 2 of x_3;
- the leading non-zero coefficient of the fourth row is the coefficient 1 of x_4;
- the leading non-zero coefficient of the fifth row is the coefficient 1 of x_5;
- the leading non-zero coefficient of the sixth row is either v if $v \neq 0$, or otherwise the coefficient 2 of x_7;
- the seventh and last row is either zero, or its leading non-zero coefficient is $u^2 - 1$ if $u^2 - 1 \neq 0$, or otherwise it is w.

We now note that, if $\mu = 0$, then the leading non-zero coefficients of the second and of the third row would be in the same column. Thus, the corresponding system would not satisfy the definition of an upper triangular/staircase system, and hence such a combination would not be "good".

Similarly, if $v \neq 0$, then the leading non-zero coefficients of the fifth and of the sixth row would be in the same column, and again the corresponding system would not be upper triangular.

The above show that any "good" combination in this case necessarily satisfies $\mu \neq 0$ and $v = 0$.

The latter also implies that the leading non-zero coefficient of the sixth row is the coefficient 2 of x_7, and hence the coefficient $u^2 - 1$ of x_7 in the next row should be 0 for the system to be upper triangular: given that $u \in \mathbb{Z}_5$, we see that we must have $u = 1$ or $u = 4$ for any "good" combinations.

Finally, we note that if $w \neq 0$, then there is a leading non-zero coefficient in the last row, the coefficient w, which is to the right of all previous leading non-zero coefficients. On the other hand, if $w = 0$, then the last row is zero, which still allows for an upper triangular system.

We conclude that for all combinations for which $\kappa \neq 0$, if we also have

$$\mu \neq 0, \quad \text{and} \quad v = 0, \quad \text{and} \quad u \in \{1, 4\},$$

then the leading non-zero coefficient of every non-zero row is to the right of the leading non-zero coefficient of any previous row, and thus the corresponding system is upper triangular.

On the other hand, if at least one of the above conditions is not met, then the corresponding system is not upper triangular.

Case 2: *combinations with* $\kappa = 0$. We show that there are no "good" combinations in this case. Indeed, there are three subcases we could consider here:

Subcase 1: $\mu = 0$. As before, we can see that in this subcase the leading non-zero coefficients of the second and of the third row would be in the same column, thus any system here would not be upper triangular.

Subcase 2: $\mu \neq 0$ and $\lambda \neq 0$. Then the leading non-zero coefficients of the first and of the second row would both be in the second column, which again would violate the definition of an upper triangular system.

Subcase 3: $\mu \neq 0$ and $\lambda = 0$. Then the leading non-zero coefficient of the first row would be to the right of μ, the leading non-zero coefficient of the second row, and again we wouldn't get an upper triangular system.

We conclude that the only combinations giving an upper triangular system come from the first main case, and are the combinations satisfying the conditions

$$\kappa \neq 0, \quad \text{and} \quad \mu \neq 0, \quad \text{and} \quad v = 0, \quad \text{and} \quad u \in \{1, 4\}. \tag{1}$$

(b) We consider two main subcases that the "good" combinations we found in part (a) belong to:

$w \neq 0$. For any combination here, the corresponding system has the following pivots: the coefficient κ of x_1 in the first row, the coefficient μ of x_2 in the second row, the coefficient 2 of x_3 in the third row, the coefficient 1 of x_4 in the fourth row, the coefficient 1 of x_5 in the fifth row, the coefficient 2 of x_7 in the sixth row, and the constant term coefficient w in the seventh row.

We thus see that there is a pivot in the last column, and hence the system is inconsistent (in other words it has no solution).

$w = 0$. For any combination here, the corresponding system has pivots in the first six rows, while the last row is zero. The pivots are: the coefficient κ of x_1 in the first row, the coefficient μ of x_2 in the second row, the coefficient 2 of x_3 in the third row, the coefficient 1 of x_4 in the fourth row, the coefficient 1 of x_5 in the fifth row, and the coefficient 2 of x_7 in the sixth row.

We thus see that there is no pivot in the last column, so each system in this case is consistent. We also note that none of the pivots is a coefficient of x_6, therefore x_6 is a free variable. On the other hand, each of the remaining variables has a pivot coefficient. This shows that each system in this case has 5 solutions (as many as the elements of $\mathbb{Z}_5$, or, in other words, as many as the choices we can make for the value of the one free variable x_6).

Finally, we observe that there is no "good" combination that gives a system with a unique solution (since all "good" combinations give a system where x_6 is a free variable).