1) a) $\partial_{xx}u + \partial_x u = 0$

Let $v = \partial_x u = 0$

$\Rightarrow \partial_x v + v = 0$

$= e^x \partial_x v + e^x v = 0$

$= \partial_x(e^x v) = 0 \quad \Rightarrow \quad e^x v = c + f(y)$

$= v = ce^{-x} + e^{-x} f(y) \Rightarrow \partial_x u = ce^{-x} + e^{-x} f(y)$

$= u = \int ce^{-x} + e^{-x} f(y)\, dx$

$$\boxed{u(x,y) = -ce^{-x} - ce^{-x} f(y) + g(y)}$$

b) $\partial_{xy}u + \partial_y u = y$

Let $v = \partial_y u$

Then $\Rightarrow \partial_x v + v = y$

$= e^x \partial_x v + e^x v = ye^x$

$= \partial_x (e^x v) = ye^x$

$\Rightarrow e^x v = ye^x + f(y)$

$\Rightarrow v = y + e^{-x} f(y)$

$= \partial_y u = y + e^{-x} f(y) = u(x,y) = \int y + e^{-x} f(y)\, dy$

$= \frac{y^2}{2} + e^{-x} g(y) + P($

$\{ g(y) = \int f(y)\, dy$

$$\boxed{\therefore u(x,y) = \frac{y^2}{2} + e^{-x} g(y) + P(x)}$$

2) a) $\partial_x u = u$

$u|_{x^3} = x^2$

Then the characteristic system is

$$\begin{cases} \frac{dy}{dx} = 0 \Rightarrow y = c \\ \frac{du}{dx} = u \Rightarrow u = p_1 e^x \end{cases}$$

$\therefore$ The general soln is then along the characteristic line $y = c$, where $p_1 = h(y)$ that gives

$$u(x,y) = h(y)e^x$$

new initial condition $u = h(x^3)e^x = x^2$

$$\Rightarrow h(x^3) = x^2 e^{-x}$$

$$= h(x) = x^{(2/3)} e^{-(x^{1/3})}$$

$\therefore$ The soln is $\boxed{u(x,y) = y^{2/3} e^{-y^{1/3}} \cdot e^{+x}}$

b) $\partial_x u = u^2$ and $u = 1$ on $y = x$

If $u = u(x)$ then $\frac{du}{dx} = u^2 \Rightarrow \frac{-1}{u} = x + c$

$$\Rightarrow u = \frac{-1}{x+c}$$

$\therefore$ The PDE can be solved $u(x,y) = \frac{-1}{x + c(y)}$ where $c = c(y)$

Since $u\big|_{x} = 1 \Rightarrow 1 = \frac{-1}{x + c(x)} \Rightarrow c(x) = -(1+x)$

$$\therefore\ u(x,y) = \frac{-1}{x + (-1 - y)} = \frac{-1}{x - y - 1}$$

3) $\partial_{tt} u = c^2 \partial_{xx} u$

a) $u = f(x-ct)$ solves

$\partial_x u = f'(x-ct)$

$\partial_{xx} u = f''(x-ct)$

$\partial_t u = -c f'(x-ct)$

$\partial_{tt} u = c^2 f''(x-ct)$

$\therefore \partial_{tt} u = c^2 \partial_{xx} u$

$u = g(x+ct)$

$\partial_x u = g'(x+ct)$

$\partial_{xx} u = g''(x+ct)$

$\partial_t u = c g'(x+ct)$

$\partial_{tt} u = c^2 g''(x+ct)$

$\therefore \partial_{tt} u = c^2 \partial_{xx} u$

$\therefore u = f(x-ct) + g(x+ct)$

$\partial_x u = f'(x-ct) + g'(x+ct)$

$\partial_{xx} u = f''(x-ct) + g''(x+ct)$

$\partial_t u = -c f'(x-ct) + c g'(x+ct)$

$\partial_{tt} u = c^2 f''(x-ct) + c^2 g''(x+ct)$

$\Rightarrow \therefore \boxed{\partial_{tt} u = c^2 \partial_{xx} u}$

b) To show $\partial_{tt} - c^2 \partial_{xx} = 0$ can be written as

$(\partial_t - c\partial_x)(\partial_t + c\partial_x) = 0$

$\Rightarrow \partial_t(\partial_t) + \partial_t \cdot c\partial_x - c\partial_x \partial_t - c^2 \partial_x(\partial_x)$

Since both ∂_{tx} and ∂_{xt} are defined and continuous
$\therefore$ by Clairaut's theorem

$\partial_{tx} u = \partial_{xt} u$

$\Rightarrow \partial_{tt} - c^2 \partial_{xx} = 0$

$\Rightarrow \partial_{tt} = c^2 \partial_{xx}$

c) $\partial_t u + c\partial_x u = 0$

The characteristic eqn is

$$\begin{cases} \frac{dx}{dt} = c \Rightarrow x = ct + c_2 \Rightarrow \Rightarrow c_2 = x - ct \\ \frac{du}{dt} = 0 \Rightarrow u = c_1 \end{cases}$$

∴ The general soln along the characteristic line $x = ct + c_2$ and since c_1 is another arbitrary function $c_1 = f(x - ct)$ $(c_2 = x - ct)$

∴ $u(x,y) = c_1 = f(x - ct)$

d) $\partial_t u - c\partial_x u = 0$

The characteristic equation is

$$\begin{cases} \frac{dx}{dt} = -c \Rightarrow x = -ct + c_2 \Rightarrow c_2 = x + ct \\ \frac{du}{dt} = 0 \Rightarrow u = c_1 \end{cases}$$

The general soln along the characteristic line $x = c_2 - ct$ and since c_1 is another arbitrary function

$c_1 = f(x + ct)$

∴ $u(x,y) = c_1 = f(x + ct)$

e) $\xi = x - ct \qquad \eta = x + ct$

$u = u(\xi, \eta)$

now that $\partial_x \xi = 1 \qquad \partial_x \eta = 1$

$\partial_t \xi = -c \qquad \partial_t \eta = c$

now $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial t} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial t}$

$$= -c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = c\left[\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial \xi\partial \eta}\frac{\partial \xi}{\partial t} - \frac{\partial u}{\partial \eta\partial \xi}\frac{\partial \eta}{\partial t} - \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial t}\right]$$

$$= c^2\left[\partial_{\eta\eta}u + \partial_{\xi\xi}u - 2\partial_{\xi\eta}u\right]$$

clairaut's theorem

$\Rightarrow \frac{1}{c^2}\partial_{tt}u = \partial_{\eta\eta}u + \partial_{\xi\xi}u - 2\partial_{\xi\eta}u$

Similarly $\partial_x u = \partial_\eta u + \partial_\xi u$

$$\partial_{xx}u = \partial_{\eta\eta}u\,\partial_x\eta + \partial_{\xi\eta}u\,\partial_x\xi + \partial_{\eta\xi}\,\partial_x\eta + \partial_{\xi\xi}u\,\partial_x\xi$$

$\Rightarrow \partial_{xx}u = \partial_{\eta\eta}u + \partial_{\xi\xi}u + 2\partial_{\xi\eta}u$

$\Rightarrow \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = \partial_{xx}u$

$\Rightarrow \partial_{\eta\eta}u + \cancel{\partial_{\xi\xi}u} - 2\partial_{\xi\eta}u = \cancel{\partial_{\eta\eta}u} + \cancel{\partial_{\xi\xi}u} + 2\partial_{\xi\eta}u$

$\Rightarrow 4\, \partial_{\xi N} u = 0$

$\Rightarrow \partial_{\xi N} u = 0$

Then 11ly by the notes

$$\partial_\xi [\partial_N u] = 0$$

Let $v = \partial_N u$

$$\Rightarrow u = \int g(N)\, d_N + h(\xi)$$

$$\therefore \underline{\underline{u(\xi, N) = h(\xi) + g(N)}}$$

4) $\partial_t u + v \partial_x u = 0$

$u(x,0) = u_0(x)$

Now the characteristic equation is

$$\begin{cases} \frac{dx}{dt} = v \Rightarrow x = vt + c_1 \Rightarrow c_1 = x - vt \\ \frac{du}{dt} = 0 \Rightarrow u = c_2 \end{cases}$$

$\therefore u(x,t) = f(x - vt)$

now $u(x,0) = u_0(x) = f(x)$

and since $E_0 = \int_{-\infty}^{\infty} |u_0(x)| dx < \infty$

$= \int_{-\infty}^{\infty} |f(x)| dx < \infty$ — ①

now $E(t) = \int_{-\infty}^{\infty} u(x,t) = \int_{-\infty}^{\infty} f(x - vt) dx$

let $p = x - vt \Rightarrow dp = dx$

$= \int_{-\infty}^{\infty} f(p) dp$

~~since absolute convergence is stronger also.~~

$= \int_{-\infty}^{\infty} f(x) dx \overset{\text{from ①}}{<} \infty = E_0$

5) a) $\vec{n} = (-\partial_x u(x,y), -\partial_y u(x,y), 1)$

$F = (u, u, 1)$

$\lambda \vec{n} = F$

$\lambda \cdot -\partial_x u(x,y) = u$

$\lambda \cdot -\partial_y u(x,y) = u$

$\lambda \cdot 1 = 1$

$\lambda = 1 \Rightarrow$ ~~aa~~ $\quad \lambda 1 = 1 \Rightarrow \lambda = 1$

$\cancel{-\partial_x u}$ $\quad -\partial_x u = u$ $\quad \Longrightarrow \quad \frac{-\partial_x u}{u} = \frac{-\partial_y u}{u} = 1$

$-\partial_y u = u$

We have $-\partial_x u = +u \Rightarrow \frac{du}{dx} = -u \Rightarrow u = e^{-x} c_1(y)$

$-\partial_y u = u \Rightarrow \frac{du}{dy} = -u \Rightarrow u = e^{-y} c_2(x)$

~~$e^{-y} c_2(x) = e^{-x} c_1(y)$~~

Then $e^{-y} c_2(x) = e^{-x} c_1(y)$

$c_2(x) = h_1(x) e^{-x}$
$c_1(y) = h_2(y) e^{-y}$

$\Rightarrow \cancel{e^{-y}} \cdot \cancel{e^{-x}} h_1(x) = \cancel{e^{-x}} \cancel{e^{-y}} \cdot h_2(y)$
$h_1(x) = h_2(y)$
$\Rightarrow h_1(x) = h_2(y) = K$ (a constant)

$\Rightarrow u(x,y) = e^{-y}(e^{-x} \cdot K)$
$u(x,y) = \underline{\underline{e^{-(x+y)} \cdot K}}$

b) $\bar{F} = (x, y, -1)$
$\vec{n} = (-\partial_x u, -\partial_y u, 1)$

~~is tangent~~

$\bar{F} \cdot \hat{n}$ is the desired pde: $\Rightarrow \bar{F} \cdot \hat{n} = (-\partial_x u) \cdot x - (\partial_y u)(y) - 1 = 0$

$\Rightarrow x \partial_x u + y \partial_y u = -1$

$\therefore$ Characteristic eqn

$$\begin{cases} \frac{dy}{dx} = \frac{y}{x} \Rightarrow y = c_1 x & \Rightarrow c_1 = y/x \\ \frac{du}{dx} = \frac{-1}{x} \Rightarrow u = -\ln|x| + c_2 \end{cases}$$

Then $c_2 = h(c_1)$

$$\Rightarrow c_2 = h(y/x)$$

$$\therefore \quad u(x,y) = -\ln(x) + h(y/x)$$

$$\boxed{\text{QED}}$$