Math 322
Suggested solutions to Homework Set 1

Problem 1. Let us write $a_{i,j}$ for the (i,j)-th entry of the adjacency matrix A of G, and $b_{i,j}$ for the (i,j)-th entry of the matrix $A^2 = A \cdot A$; in other words, we write $A = \big(a_{i,j}\big)_{1\leqslant i,j\leqslant n}$ and $A^2 = \big(b_{i,j}\big)_{1\leqslant i,j\leqslant n}$.

We recall that

$$a_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E(G) \\ 0 & \text{otherwise} \end{cases}.$$

We also observe that the (i,j)-th entry of the matrix A^2 is equal to the dot product of the i-th row and the j-th column of A:

$$b_{i,j} = \langle \mathrm{Row}_i(A),\, \mathrm{Col}_j(A)\rangle.$$

In particular, for every $1 \leqslant j \leqslant n$,

$$\begin{aligned} b_{j,j} &= \langle \mathrm{Row}_j(A),\, \mathrm{Col}_j(A)\rangle \\ &= \sum_{s=1}^{n} a_{j,s} \cdot a_{s,j} \\ &= \sum_{s=1}^{n} a_{j,s}^2 \end{aligned}$$

where the last equality follows because the matrix A is symmetric.

But the last sum has summands equal only to 0 or 1 (depending on whether $a_{j,s}$ is equal to 0 or 1), so it is equal to the total number of indices $s \in \{1, 2, \ldots, n\}$ such that $a_{j,s} = 1$. But each such index corresponds to a vertex v_s of G which is adjacent to the vertex v_j, while at the same time the remaining indices s' (for which we have $a_{j,s'} = 0$) correspond to vertices of G which are **not** neighbours of v_j.

Based on all the above, we see that

$$\begin{aligned} b_{j,j} &= \sum_{s=1}^{n} a_{j,s}^2 \\ &= \big|\{s \in \{1, 2, \ldots, n\} : a_{j,s} = 1\}\big| \\ &= \big|\{s \in \{1, 2, \ldots, n\} : v_s \text{ is adjacent to the vertex } v_j\} \\ &= \deg(v_j)\,, \end{aligned}$$

which is what we needed to show.

Problem 2. (i) For graph G_1 we have

$$A_{G_1} = \begin{array}{cc} & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

and

$$I_{G_1} = \begin{array}{cc} & \begin{array}{cccccccccc} ab & ac & bc & bd & bg & cf & de & df & dg & ef \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

Similarly, for graph G_2 we have

$$A_{G_2} = \begin{array}{cc} & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

and

$$I_{G_2} = \begin{array}{cc} & \begin{array}{cccccccc} ab & ac & ad & cd & de & df & eg & fg \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

(ii) We will list the paths in G_1 according to their length.

Length 1: Then the path has only two vertices, an initial vertex and the vertex c (viewed as its terminal vertex), so the initial vertex here can only be a neighbour of c in G_1. Thus, we get three paths of length 1:

$$a\,c, \qquad b\,c, \qquad f\,c\,.$$

Length 2: Then the second vertex in the path has to be a neighbour of c, while the initial vertex has to be a neighbour of a neighbour of c. We get the following paths of length 2:

$$b\,a\,c, \qquad a\,b\,c, \qquad d\,b\,c, \qquad g\,b\,c, \qquad d\,f\,c, \qquad e\,f\,c.$$

Length 3: We have the following paths (again building on the previous paths we found):

$$\begin{array}{ccccc} d\,b\,a\,c, & g\,b\,a\,c, & e\,d\,b\,c, & f\,d\,b\,c, & g\,d\,b\,c, \\ d\,g\,b\,c, & b\,d\,f\,c, & e\,d\,f\,c, & g\,d\,f\,c, & d\,e\,f\,c. \end{array}$$

Length 4: We have the following paths:

$$\begin{array}{c} e\,d\,b\,a\,c, \qquad f\,d\,b\,a\,c, \qquad g\,d\,b\,a\,c, \qquad d\,g\,b\,a\,c, \\ f\,e\,d\,b\,c, \qquad e\,f\,d\,b\,c, \qquad e\,d\,g\,b\,c, \qquad f\,d\,g\,b\,c, \\ a\,b\,d\,f\,c, \qquad g\,b\,d\,f\,c, \qquad b\,g\,d\,f\,c, \qquad b\,d\,e\,f\,c, \qquad g\,d\,e\,f\,c. \end{array}$$

Length 5: We have the following paths:

$$\begin{array}{cccc} f\,e\,d\,b\,a\,c, & e\,f\,d\,b\,a\,c, & e\,d\,g\,b\,a\,c, & f\,d\,g\,b\,a\,c, \\ & f\,e\,d\,g\,b\,c, & e\,f\,d\,g\,b\,c, & \\ a\,b\,g\,d\,f\,c, & a\,b\,d\,e\,f\,c, & g\,b\,d\,e\,f\,c, & b\,g\,d\,e\,f\,c. \end{array}$$

Length 6: We have the following paths:

$$f\,e\,d\,g\,b\,a\,c, \qquad e\,f\,d\,g\,b\,a\,c, \qquad a\,b\,g\,d\,e\,f\,c.$$

Note also that the paths in the last case are the longest possible we can have, since they pass by every vertex of G_1. We conclude that there are 45 different paths in G_1 with terminal vertex c.

We now list the paths in G_2 which end at c.

Length 1: We have the following paths (where necessarily the initial vertex is a neighbour of c):

$$a\,c, \qquad d\,c.$$

Length 2: We have the following paths:

$$b\,a\,c, \qquad d\,a\,c, \qquad a\,d\,c, \qquad e\,d\,c, \qquad f\,d\,c.$$

Length 3: We have the following paths:

$$e\,d\,a\,c, \qquad f\,d\,a\,c, \qquad b\,a\,d\,c, \qquad g\,e\,d\,c, \qquad g\,f\,d\,c.$$

Length 4: We have the following paths:

$$g\,e\,d\,a\,c, \qquad g\,f\,d\,a\,c, \qquad f\,g\,e\,d\,c, \qquad e\,g\,f\,d\,c.$$

Length 5: We have the following paths:

$$f\,g\,e\,d\,a\,c, \qquad e\,g\,f\,d\,a\,c.$$

Note that in G_2 we do not have any paths of length 6. We conclude that there are 18 different paths in G_2 with terminal vertex c.

(iii) Again we list the paths in G_1 that avoid the vertex c completely according to their length, but also according to their pair of endvertices **(and also, to avoid counting each path twice, we write the endvertices of the path in alphabetical order)**.

Length 1: These paths will essentially be sequences of only two vertices which are adjacent:

$$a\,b, \qquad b\,d, \qquad b\,g, \qquad d\,e, \qquad d\,f, \qquad d\,g, \qquad e\,f.$$

Length 2: We find these paths by adding one more vertex to each of the above paths which is either adjacent to the initial vertex of that path or to the terminal vertex of that path, whenever that's possible:

$$\begin{array}{ccccc} a\,b\,d, & a\,b\,g, & b\,d\,e, & b\,d\,f, & b\,d\,g, \\ d\,b\,g, & b\,g\,d, & d\,e\,f, & e\,d\,f, & d\,f\,e, \end{array}$$
$$e\,d\,g, \qquad f\,d\,g.$$

Length 3: We have the following paths:

$$\begin{array}{cccc} a\,b\,d\,e, & a\,b\,d\,f, & a\,b\,d\,g, & a\,b\,g\,d, \\ b\,d\,e\,f, & b\,d\,f\,e, & e\,d\,b\,g, & f\,d\,b\,g, \\ b\,g\,d\,e, & b\,g\,d\,f, & f\,e\,d\,g, & e\,f\,d\,g. \end{array}$$

Length 4: We have the following paths:

$$\begin{array}{cccc} a\,b\,d\,e\,f, & a\,b\,d\,f\,e, & a\,b\,g\,d\,e, & a\,b\,g\,d\,f, \\ f\,e\,d\,b\,g, & e\,f\,d\,b\,g, & b\,g\,d\,e\,f, & b\,g\,d\,f\,e. \end{array}$$

Length 5: We have the following paths:

$$a\,b\,g\,d\,e\,f, \qquad a\,b\,g\,d\,f\,e.$$

Note also that the paths in the last case are the longest possible we can have, since they pass by every vertex of G_1 except c. We conclude that there are 41 different paths in G_1 that avoid the vertex c.

Similarly, we list the paths in G_2 that avoid the vertex c:

Length 1: Again, these are essentially sequences of two adjacent vertices:

$$a\,b, \qquad a\,d, \qquad d\,e, \qquad d\,f, \qquad e\,g, \qquad f\,g.$$

Length 2: We find these paths by adding one more vertex to each of the above paths which is either adjacent to the initial vertex of that path or to the terminal vertex of that path, whenever that's possible:

$$b\,a\,d, \qquad a\,d\,e, \qquad a\,d\,f, \qquad d\,e\,g, \qquad e\,d\,f, \qquad d\,f\,g, \qquad e\,g\,f.$$

Length 3: We have the following paths:

$$\begin{array}{ccc} b\,a\,d\,e, & b\,a\,d\,f, & a\,d\,e\,g, \\ a\,d\,f\,g, & f\,d\,e\,g, & d\,e\,g\,f, \end{array}$$
$$e\,d\,f\,g, \qquad d\,f\,g\,e.$$

Length 4: We have the following paths:

$$b\,a\,d\,e\,g, \qquad b\,a\,d\,f\,g, \qquad a\,d\,e\,g\,f, \qquad a\,d\,f\,g\,e.$$

Length 5: We have the following paths:

$$b\,a\,d\,e\,g\,f, \qquad b\,a\,d\,f\,g\,e.$$

Again the paths in the last case are the longest possible we can have, since they pass by every vertex of G_2 except c. We conclude that there are 27 different paths in G_2 that avoid the vertex c.

(iv) The answer is yes. For clarity here, we will write the vertices of G_2 as $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and so on.

Consider the following subgraph of G_1:

$$H_1 = \big(\{b, c, d, f\},\, \{bc,\, bd,\, cf,\, df\}\big)$$

as well as the following subgraph of G_2:

$$H_2 = \big(\{\tilde{d}, \tilde{e}, \tilde{f}, \tilde{g}\},\, \big\{\tilde{d}\tilde{e},\, \tilde{d}\tilde{f},\, \tilde{e}\tilde{g},\, \tilde{f}\tilde{g}\big\}\big).$$

We claim that $H_1 \cong H_2$.

Indeed, consider the following bijection τ from $V(H_1)$ to $V(H_2)$:

$$\tau(b) = \tilde{d}, \quad \tau(c) = \tilde{e}, \quad \tau(d) = \tilde{f} \ \text{ and } \ \tau(f) = \tilde{g}.$$

Then we have that

- there does exist an edge $\tau(b)\,\tau(c)$ in H_2 (it is the edge $\tilde{d}\tilde{e}$ of H_2);
- there does exist an edge $\tau(b)\,\tau(d)$ in H_2 (it is the edge $\tilde{d}\tilde{f}$ of H_2);
- there does exist an edge $\tau(c)\,\tau(f)$ in H_2 (it is the edge $\tilde{e}\tilde{g}$ of H_2);
- and there does exist an edge $\tau(d)\,\tau(f)$ in H_2 (it is the edge $\tilde{f}\tilde{g}$ of H_2).
- Also, there is no edge of H_2 that does not correspond to an edge of H_1 under this rule.

Thus τ is indeed an isomorphism from H_1 to H_2.

(v) We want to answer the analogous question to part (iv), but for subgraphs on 5 vertices now. Again we will write $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and so on for the vertices of G_2 to avoid any confusion.

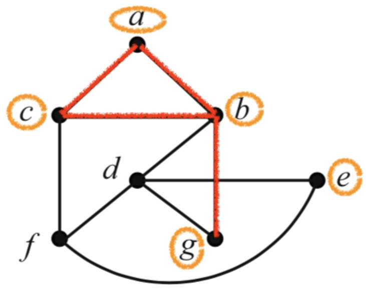
Note that none of the two parts requires the subgraphs of G_1 and G_2 to be induced subgraphs, even though in part (iv) we came up with a pair of induced subgraphs.

On the other hand, we could consider very simple examples for this part:

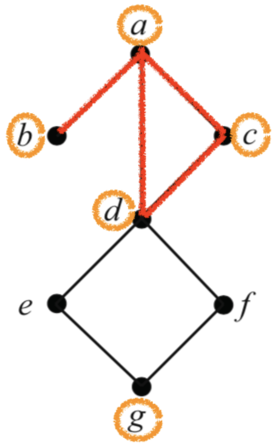
- say, the subgraph of G_1 which consists of the 3-cycle formed by the vertices a, b, c together with the vertices e and g viewed as isolated vertices now,
- and similarly the subgraph of G_2 which consists of the 3-cycle formed by the vertices $\tilde{a}, \tilde{c}, \tilde{d}$ together with the vertices $\tilde{b}$ and $\tilde{g}$ viewed as isolated vertices now.

It is not hard to check that these two subgraphs would be isomorphic (and that any bijection that maps the vertices of the first 3-cycle to the vertices of the second 3-cycle, and hence also maps the first pair of isolated vertices to the second pair of isolated vertices, would give a graph isomorphism).

That said, in this case we also have that the corresponding induced subgraphs are isomorphic. The subgraphs are highlighted in the pictures below



Subgraph K_1 induced by the vertices a, b, c, e, g



Subgraph K_2 induced by the vertices $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{g}$

while in (V, E)-notation they can be described as follows:

$$K_1 = \big(\{a, b, c, e, g\},\ \{ab,\ ac,\ bc,\ bg\}\big) \subseteq G_1,$$
$$K_2 = \big(\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{g}\},\ \{\tilde{a}\tilde{b},\ \tilde{a}\tilde{c},\ \tilde{a}\tilde{d}, \tilde{c}\tilde{d}\}\big) \subseteq G_2.$$

Note that the bijection σ from $V(K_1)$ to $V(K_2)$ given by

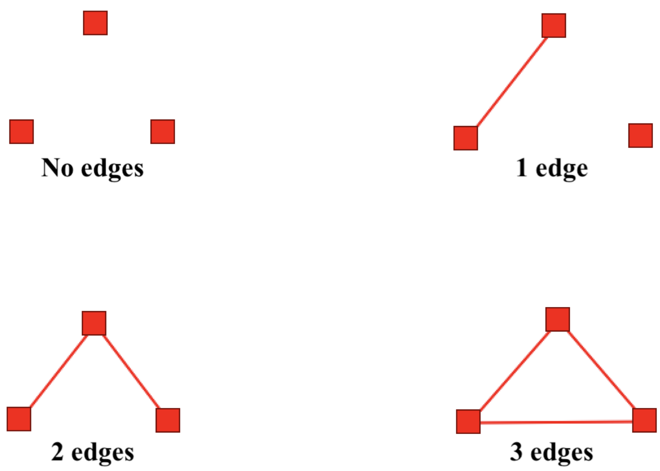
$$\sigma(a) = \tilde{d}, \quad \sigma(b) = \tilde{a}, \quad \sigma(c) = \tilde{c}, \quad \sigma(e) = \tilde{g} \ \text{ and } \ \sigma(g) = \tilde{b}$$

leads to a graph isomorphism from K_1 to K_2. Indeed,

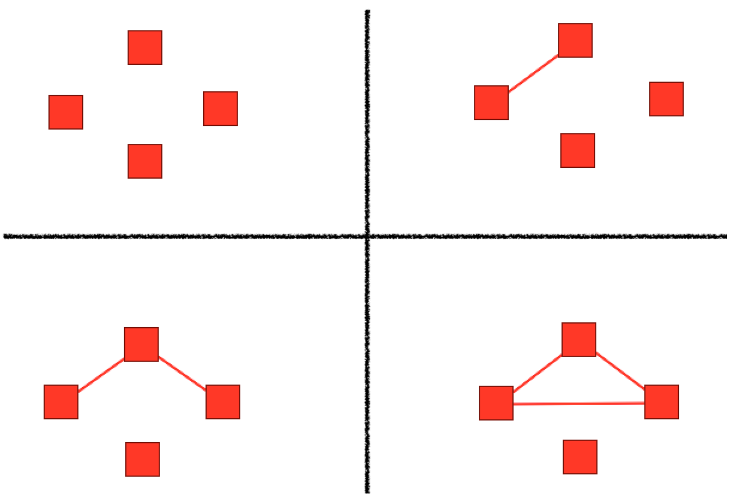
- there does exist an edge $\sigma(a)\,\sigma(b)$ in K_2 (it is the edge $\tilde{d}\tilde{a}$ of K_2);
- there does exist an edge $\sigma(a)\,\sigma(c)$ in K_2 (it is the edge $\tilde{d}\tilde{c}$ of K_2);
- there does exist an edge $\sigma(b)\,\sigma(c)$ in K_2 (it is the edge $\tilde{a}\tilde{c}$ of K_2);
- and there does exist an edge $\sigma(b)\,\sigma(g)$ in K_2 (it is the edge $\tilde{a}\tilde{b}$ of K_2).
- Also, there is no edge of K_2 that does not correspond to an edge of K_1 under this rule.

Thus the answer to part (v) is affirmative.

Problem 3. Recall that there are four unlabelled graphs on 3 vertices, which can be seen in the following picture:

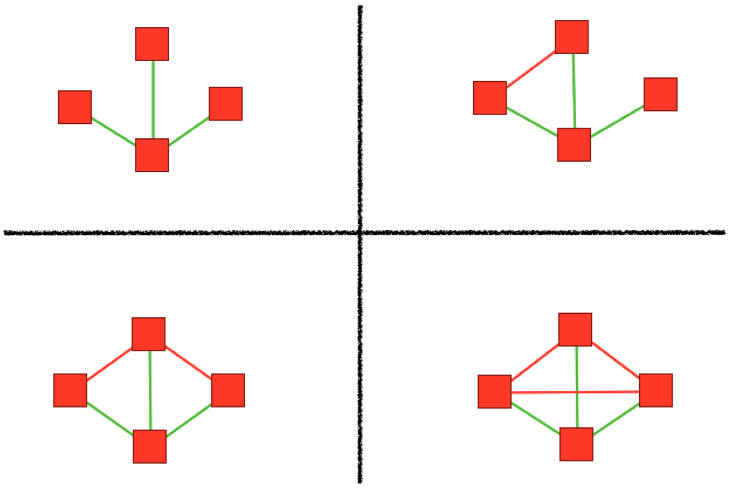


We can immediately construct some of the graphs on 4 vertices that we are asked to find by adding one more vertex to each of these graphs (without joining it to any of the vertices we have already):



Note that all four graphs here are different, given that each has a different degree sequence. In particular, the first one has degree sequence $(0, 0, 0, 0)$, the second one has degree sequence $(1, 1, 0, 0)$, the third one has degree sequence $(2, 1, 1, 0)$, and the fourth one has degree sequence $(2, 2, 2, 0)$.

Moreover, we could construct four more graphs on 4 vertices by adding a vertex to the graphs on 3 vertices that we have found, and by joining this time the new vertex with any other vertex:



Again, these are four different graphs since each of them has a different degree sequence. In particular, the first graph here has degree sequence $(3, 1, 1, 1)$, the second one has degree sequence $(3, 2, 2, 1)$, the third one has degree sequence $(3, 3, 2, 2)$, and the fourth one has degree sequence $(3, 3, 3, 3)$. In addition, these are different from the previous four graphs we found (why?).

We thus see that we have already found eight of the eleven graphs we are looking for. To find the remaining three, we examine which degree sequences we could still consider: by the Corollary of the Handshaking Lemma, which states that V_{odd} must have even cardinality, we can conclude that there are no other graphical sequences with 4 terms in which at least one term would be equal to 3. Indeed, if we have at least one term equal to 3, then an odd number of the remaining terms must be odd numbers as well, whose value must be $\leqslant 4 - 1 = 3$; thus an odd number of the remaining terms must be equal to 1 or 3, which leads to only four possibilities:

- either the sequence consists of four terms equal to 3;
- or the sequence has three terms equal to 3 and one term equal to 1 (it is not hard to check now that this would not be a graphical sequence);
- or the sequence has two terms equal to 3 and two terms equal to 1 (again, it is not hard to check that this would not be a graphical sequence);

- or the sequence has two terms equal to 3 and no terms equal to 1 (and then, since we cannot have a term equal to 0 if we want the sequence to be graphical, both remaining terms will be equal to 2);

- or the sequence has exactly one term equal to 3 and three terms equal to 1;

- or the sequence has exactly one term equal to 3 and one term equal to 1 (and then, since we cannot have a term equal to 0 if we want the sequence to be graphical, both remaining terms will be equal to 2).

Similarly, we can see that there are no other graphical sequences with 4 terms in which at least one term is equal to 0. Indeed, if we consider such a graphical sequence, and a graph H that realises it, then by removing one isolated vertex of H (which we are guaranteed to have in the setting we are considering right now), we would end up with a graph on 3 vertices whose degree sequence has to be one of the four sequences we can find in the first picture. But the degrees of the vertices of this subgraph of H would be the same as their degrees within H (why?), and thus the degree sequence of H would be one of the sequences we can find in the second picture.

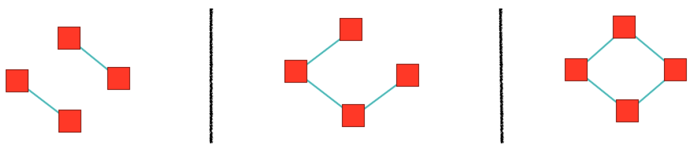
It follows that, if we are to find three additional graphical sequences with 4 terms, different from all the above, then the values we can choose for the terms are only 1 or 2 (and we also need to have an even number of 1s). This leads to three possibilities:

- the sequence contains four terms equal to 1, and thus it is of the form $(1, 1, 1, 1)$;

- the sequence contains two terms equal to 1, which implies that the remaining two terms equal 2, so the sequence is of the form $(2, 2, 1, 1)$;

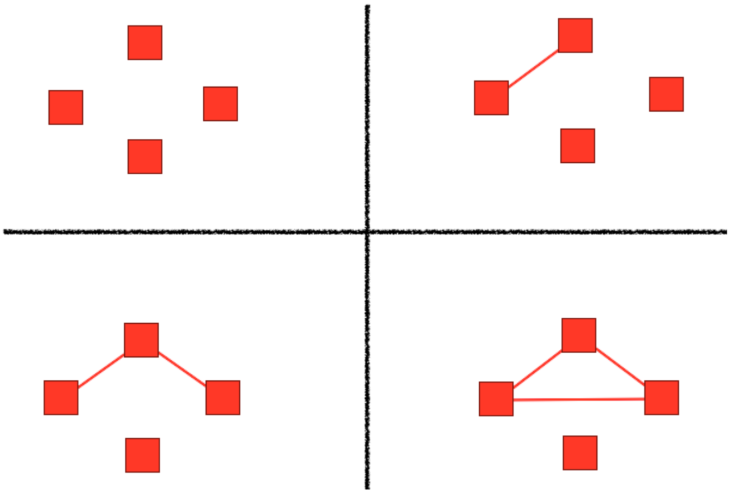
- the sequence does not contain terms equal to 1, so it consists of four terms equal to $2:$ it is the sequence $(2, 2, 2, 2)$.

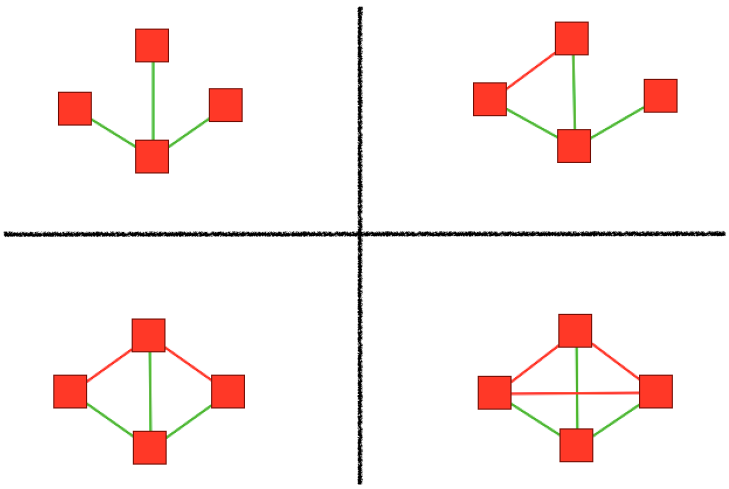
We can now check that all these three sequences are graphical and realised by the following graphs respectively:

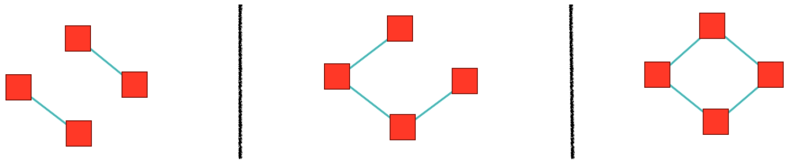


Given also that they are all different, including being different from the previous eight sequences we found, we can conclude that these last graphs are the remaining graphs on 4 vertices that we needed to find.

We list all eleven graphs again:







Problem 4. (i) Let us consider an index $j \in \{1, 2, \ldots, n\}$. Then the degree $\deg_G(v_j)$ of the vertex v_j in G is equal to the cardinality of the (open) neighbourhood $N_G(v_j)$ of v_j in G, that is, of that subset of the vertex set $\{v_1, v_2, \ldots, v_n\} \setminus \{v_j\}$ which contains only the neighbours of v_j.

Clearly the number of the remaining vertices in $\{v_1, v_2, \ldots, v_n\} \setminus \{v_j\}$, that is, the vertices which are not neighbours of v_j in G, is

$$\big|\{v_1, v_2, \ldots, v_n\} \setminus \{v_j\}\big| - \deg_G(v_j) = (n-1) - \deg_G(v_j).$$

But these vertices are exactly the vertices that will be neighbours of v_j in the complement $\overline{G}$ of G.

We can conclude that

$$\deg_{\overline{G}}(v_j) = (n-1) - \deg_G(v_j).$$

This gives us that the degree sequence of $\overline{G}$ is the sequence

$$\begin{aligned} &\big((n-1) - \deg_G(v_1),\, (n-1) - \deg_G(v_2),\, \ldots,\, (n-1) - \deg_G(v_n)\big) \\ = &\big((n-1) - d_1,\; (n-1) - d_2,\; \ldots,\; (n-1) - d_n\big)\,. \end{aligned}$$

(ii) As we just saw in part (i), we have that, if the degree sequence of G is the sequence

$$(d_1, d_2, \ldots, d_n)$$

(where the order here is according to the labelling of the vertices of G, and not necessarily monotonic), then the degree sequence of $\overline{G}$ is the sequence

$$\big((n-1) - d_1,\; (n-1) - d_2,\; \ldots,\; (n-1) - d_n\big)\,.$$

But then, for any two indices i, j in $\{1, 2, \ldots, n\}$ such that $d_i = d_j$, we will also have that $(n-1) - d_i = (n-1) - d_j$, and similarly, if we know that $(n-1) - d_i = (n-1) - d_j$, we will get that $d_i = d_j$.

In other words, the two degree sequences above have repeated terms in the same positions.

This implies that, if there is exactly one pair of indices i, j from $\{1, 2, \ldots, n\}$ such that $d_i = d_j$, then the sequence

$$\big((n-1) - d_1,\; (n-1) - d_2,\; \ldots,\; (n-1) - d_n\big)\,.$$

will also have the same property: any two of its terms will be different, except for the i-th and the j-term, which will be equal.

(iii) When the order n of the graph is 2, there are only two unlabelled graphs:

(these clearly are non-isomorphic). Both of these graphs satisfy the required property: the degree sequence of the first one is $(0, 0)$, while the degree sequence of the second one is $(1, 1)$, and thus the only two terms in either sequence have equal values.

Similarly, by looking at the list of unlabelled graphs of order 3, we see that only two of them have the required property:

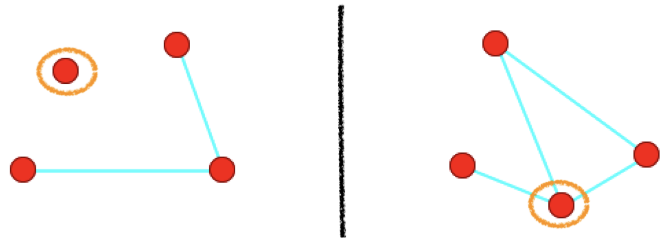
[note that here we have drawn them in such a way that each of them corresponds to one of the previous two graphs, and can be viewed as constructed based on that graph:

- the graph on the left is constructed by adding one vertex to the previous graph on the left and then joining this new vertex with every other vertex,

- while the graph on the right is constructed by adding an isolated vertex to the previous graph on the right;

the vertices that have been added are enclosed in an orange circle, while the choice of which scheme of construction to use (that is, whether to add a vertex and connect it to every other vertex, or whether to add an isolated vertex) follows from trying to preserve the required property for the degree sequences (e.g. if the previous degree sequence already had a term equal to 0, then we don't add an isolated vertex, because if we did so, our new sequence would have more than one pair of equal terms)].

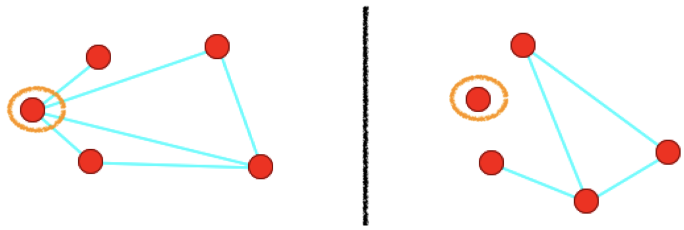
We can continue like this. Again, by looking at the list of unlabelled graphs of order 4 that we found in the previous problem (and at their degree sequences), we see that only two of them have the required property:



(and again we can view these as constructed from the previous graphs we found according to one of the schemes we described above, that is, by either adding one vertex to the graph we already have, and then joining this new vertex with every other vertex, or by adding an isolated vertex to the graph we already have).

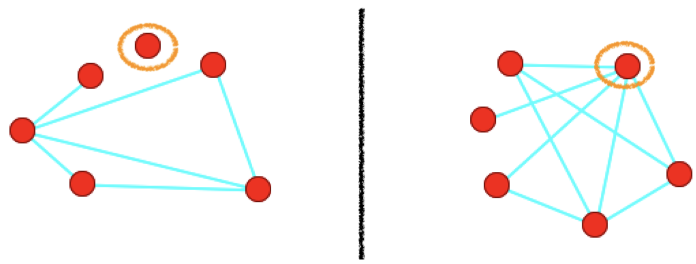
Just to double check, the degree sequences of these two graphs are $(2, 1, 1, 0)$ and $(3, 2, 2, 1)$ respectively, so they have the desired property. Moreover, one graph is the complement of the other.

We continue analogously: two graphs of order 5 that have the required property are



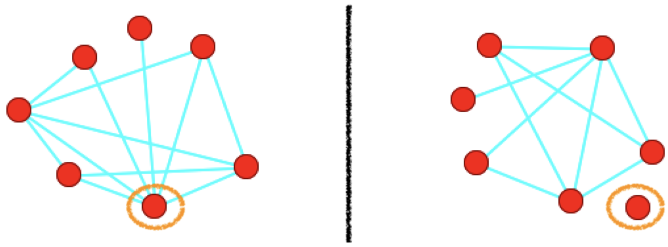
(their degree sequences are $(4, 3, 2, 2, 1)$ and $(3, 2, 2, 1, 0)$ respectively, and we can check that one is the complement of the other; also these two graphs are definitely non-isomorphic since they have different degree sequences).

Two graphs of order 6 that have the required property are



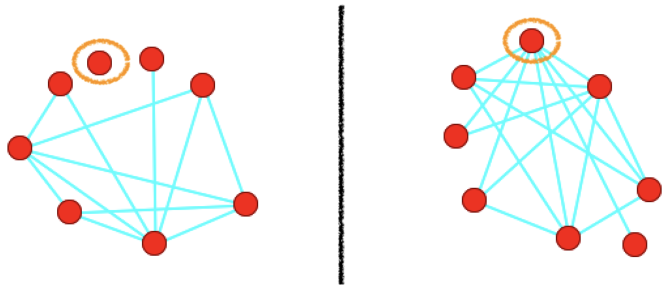
(their degree sequences are $(4, 3, 2, 2, 1, 0)$ and $(5, 4, 3, 3, 2, 1)$ respectively, and we can check that one is the complement of the other).

Two graphs of order 7 that have the required property are



(their degree sequences are $(6, 5, 4, 3, 3, 2, 1)$ and $(5, 4, 3, 3, 2, 1, 0)$ respectively, and we can check that one is the complement of the other).

Finally, two graphs of order 8 that have the required property are



(their degree sequences are $(6, 5, 4, 3, 3, 2, 1, 0)$ and $(7, 6, 5, 4, 4, 3, 2, 1)$ respectively, and we can check that one is the complement of the other).

In all the above pictures we have circled in orange the vertex that we add to the corresponding graph found in the picture right before (where this vertex then either becomes an isolated vertex, or is joined with every other vertex, depending on what the degree sequence of the previous graph was).

Problem 4 (cont.) Part (iv) *(Optional question)* Based on the two schemes that we described, of constructing graphs with the required property from graphs with one vertex less, which we already know have the required property, we can now determine all finite graphs which have the required property. We begin with three observations/claims.

Claim (A) If a finite graph G has the required property, then its complement $\overline{G}$ has the required property too; this was precisely part (ii) of this problem.

Claim (B) By the Pigeonhole Principle it follows that such a graph must either have an isolated vertex or a vertex which is joined with every other vertex (and then its complement will have the opposite property); this is because we require the degree sequence of such a graph to contain $n-1$ different numbers from the set $\{0, 1, 2, \ldots, n-2, n-1\}$, and at the same time, for any fixed graph of order n, we cannot have both 0 and $n-1$ to be terms of its degree sequence (why?).

Claim (C) For each order n, there are exactly two non-isomorphic graphs with the required property, and one is the complement of the other.

Note that Claim (B) shows that any graph that has the required property cannot be isomorphic to its complement (since they will have different degree sequences). Therefore, given both Claims (A) and (B), we can now confirm Claim (C) by simply checking that there are **exactly two graphs of order n** with the required property.

We use induction in n to prove Claim (C).

Base Case: We can check that the claim is true when $n = 2$ or $= 3$ or $= 4$ (simply because we have listed all unlabelled graphs of order 2, 3 and 4, so we can verify that the ones we drew in part (iii) are the only ones with the required property).

Inductive Hypothesis: We assume that we have already confirmed the claim when $n = n_0$ for some $n_0 \geqslant 4$: that is, we assume that there are exactly two non-isomorphic graphs of order n_0 which have exactly one pair of vertices sharing the same degree. Let us denote these two graphs by $H_{n_0,1}$ and $H_{n_0,2}$.

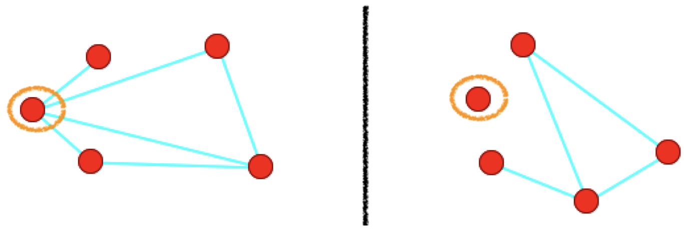
Induction Step: We show that the claim is true when $n = n_0 + 1$.

First of all, we show that there are graphs of order $n_0 + 1$ with the required property. By Claim (B), which we explained above, one of $H_{n_0,1}$ and $H_{n_0,2}$ has an isolated vertex, while the other one has a vertex of degree $n_0 - 1$ (note that we end up having both cases because one graph is the complement of the other).

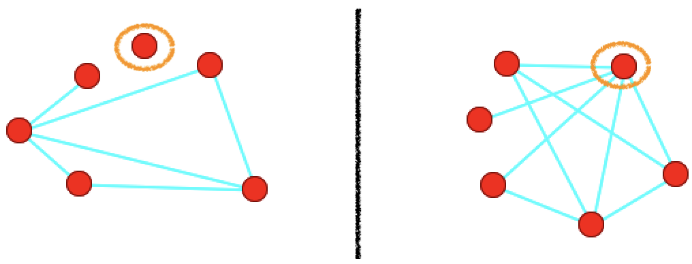
Suppose that $H_{n_0,1}$ is the graph that has a vertex of degree $n_0 - 1$ (and hence does not have an isolated vertex). Add a new vertex w_0 to $H_{n_0,1}$, and do not join it with any other vertex, thus constructing a new graph G_1 on $n_0 + 1$ vertices, in which w_0 is an isolated vertex; in other words, G_1 is the disjoint union of $H_{n_0,1}$ and the null graph on one vertex:

$$G_1 = H_{n_0,1} \oplus N_1.$$

For instance, if $H_{n_0,1}$ is the graph on the left here:



then G_1 will be the graph on the left in the following picture (and w_0 will be the vertex within the orange circle):



Observe now that the degree sequence of G_1 is

$$\begin{aligned}&\big(\text{degree sequence of } H_{n_0,1},\ \ \deg_{G_1}(w_0)\big)\\ &\qquad\qquad\qquad\qquad = \big(\text{degree sequence of } H_{n_0,1},\ \ 0\big).\end{aligned}$$

Therefore, given that the degree sequence of $H_{n_0,1}$ does not have any terms equal to 0, but does have exactly one pair of terms which are equal, we obtain that the degree sequence of G_1 also has exactly one pair of terms which are equal (while all other terms are distinct).

We conclude that G_1 is a graph on n_0+1 vertices which has the required property, and then its complement $\overline{G_1}$ will also be such a graph.

Finally, we show that G_1 and $\overline{G_1}$ are the only graphs on n_0+1 vertices that have the required property. Assume towards a contradiction that K_1 is another graph on n_0+1 vertices, which is not isomorphic to G_1 or to $\overline{G_1}$, but still has the required property.

Then, by Claim (A) both K_1 and $\overline{K_1}$ have the required property, while by our assumption none of them can be isomorphic to G_1 or to $\overline{G_1}$. Also, by Claim (B) one of K_1 or $\overline{K_1}$ must have an isolated vertex. Let us suppose that K_1 has the isolated vertex, say vertex u_0 (otherwise we work with $\overline{K_1}$ instead).

But then the degree sequence of the graph $K_1 - u_0$ (that is, the graph we get by removing the vertex u_0 of K_1) coincides with the first n_0 terms of the degree sequence of K_1, so it must also have exactly one pair of terms which are equal. In other words,

- $K_1 - u_0$ is a graph on n_0 vertices,
- which has exactly one pair of vertices sharing the same degree,
- and also does not have an isolated vertex (note that it would not be possible for K_1 to have two isolated vertices, given that we need K_1 to have a vertex of degree $(n_0+1)-2 = n_0-1$).

But according to the inductive hypothesis, only the graph $H_{n_0,1}$ (and any graph isomorphic to it) has these properties, therefore $K_1 - u_0$ must be isomorphic to $H_{n_0,1}$.

This would imply that K_1 and G_1 are also isomorphic (since we can construct each of them from either $K_1 - u_0$ or $H_{n_0,1}$ by just adding an isolated vertex), and it would contradict the assumption that K_1 is different from G_1.

We conclude that any graph on n_0+1 vertices with the required property must be isomorphic either to G_1 or to $\overline{G_1}$ (depending on whether it has an isolated vertex or not).

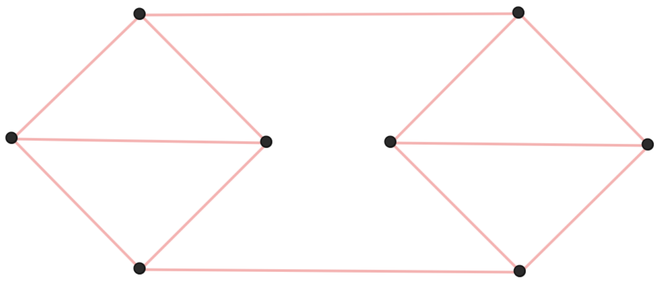
Problem 5. (i) For a graph G to have vertices with degree 3, we need G to contain at least 4 vertices; thus $n_{3,\min} \geqslant 4$.

At the same time, the complete graph K_4 on 4 vertices is a 3-regular graph, so $n_{3,\min} \leqslant |K_4| = 4$.

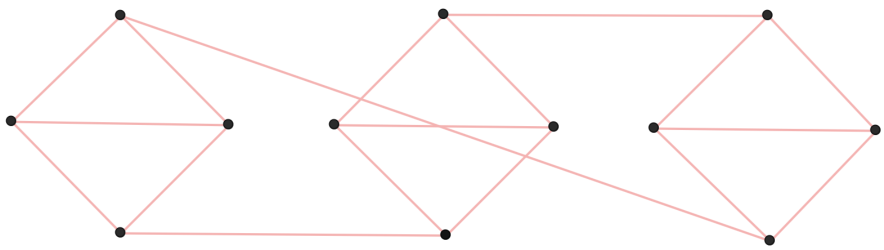
We conclude that $n_{3,\min} = 4$.

(ii) Observe that, for every $k \geqslant 2$, we can take the disjoint union of k copies of K_4, and thus get a 3-regular graph on $4k$ vertices (which has k connected components). This shows that there are 3-regular graphs with order $4k$ for every $k \geqslant 2$, and thus that there is not a maximum possible order of a finite 3-regular graph.

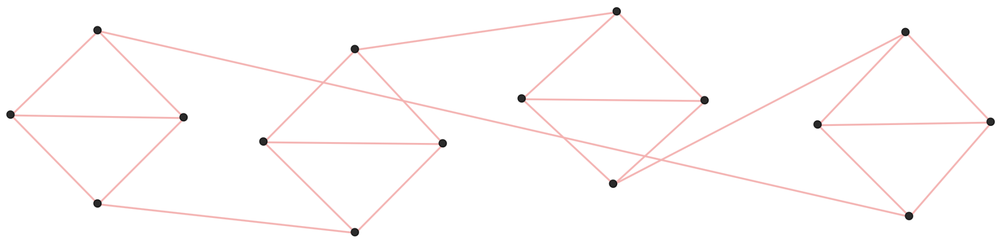
(iii) We have already seen in part (ii) that, for every $k \geqslant 1$, there exists a 3-regular graph with $4k$ vertices. In fact, the graphs we found are disconnected, except when $k = 1$, but we could have also come up with connected 3-regular graphs with $4k$ vertices: see some initial examples in the pictures below.



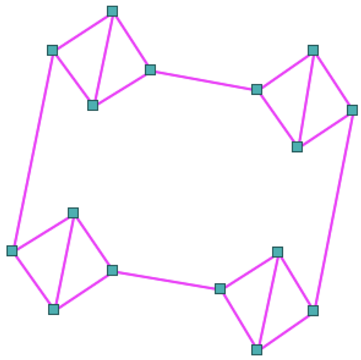
Connected 3-regular graph of order 8



Connected 3-regular graph of order 12



Connected 3-regular graph of order 16



One more representation of the above 3-regular graph of order 16

Next we note that there are no 3-regular graphs of order n when n is odd. This is because of the Corollary of the Handshaking Lemma, which tells us that V_{odd} (the subset of the vertices of the graph that have odd degree) must have even cardinality. But here V_{odd} will coincide with the entire vertex set V of the graph (since we want every vertex to have degree 3), so the cardinality of V, or equivalently the order of the graph, must be even.

We finally examine whether there are 3-regular graphs of order $4k+2$ for $k \geqslant 1$. We start with the smallest order we could have in this case, which is 6; in other words, we want to determine whether the sequence

$$(3, 3, 3, 3, 3, 3)$$

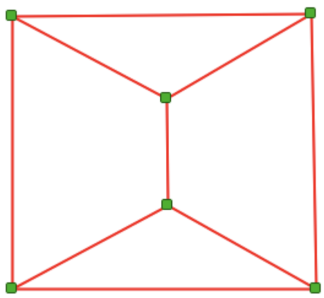
is graphical. By the Havel-Hakimi theorem, we have that

$$(3, 3, 3, 3, 3, 3) \text{ is graphical}$$
$$\text{if and only if}$$

$$(2,2,2,3,3) \text{ is graphical,} \quad \text{or in other words } (3,3,2,2,2) \text{ is graphical,}$$
$$\text{if and only if}$$
$$(2,1,1,2) \text{ is graphical,} \quad \text{or in other words } (2,2,1,1) \text{ is graphical.}$$

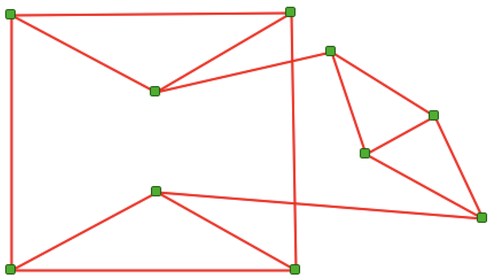
We now note that the last sequence is graphical (it coincides with the degree sequence of a path on 4 vertices), and thus all the previous sequences will be graphical too.

In fact, if we reverse the process (and recall how we were showing in the proof of the theorem that, if the shorter sequence is graphical, then the longer one is graphical too), we can come up with a 3-regular graph of order 6:



3-regular graph of order 6

Finally, for any other $k > 1$, we can construct a 3-regular graph of order $4k + 2$ if we take the disjoint union of this graph of order 6 and of $k - 1$ copies of K_4. Or, analogously to above, we can also come up with connected constructions: e.g.



Connected 3-regular graph of order 10

We conclude that there are 3-regular graphs of order $n \geqslant 4$ for every $n = 4k$ or $= 4k + 2$ for some $k \geqslant 1$, and there are no 3-regular graphs of any other order.

Problem 6. (i) For a graph G to have vertices with degree 4, we need G to contain at least 5 vertices; thus $n_{4,\min} \geqslant 5$.

At the same time, the complete graph K_5 on 5 vertices is a 4-regular graph, so $n_{4,\min} \leqslant |K_5| = 5$.

We conclude that $n_{4,\min} = 5$.

(ii) Observe that, for every $k \geqslant 2$, we can take the disjoint union of k copies of K_5, and thus get a 4-regular graph on $5k$ vertices (which has k connected components). This shows that there are 4-regular graphs with order $5k$ for every $k \geqslant 2$, and thus that there is not a maximum possible order of a finite 4-regular graph.

(iii) We have already seen in part (ii) that, for every $k \geqslant 1$, there exists a 4-regular graph with $5k$ vertices. We now examine whether we can find 4-regular graphs of order n whenever (a) $n = 5k + 1$, or (b) $n = 5k + 2$, or (c) $n = 5k + 3$, or finally (d) $n = 5k + 4$ (where $k \geqslant 1$ is a positive integer).

Case of $5k+1$: We begin with the smallest order we could have in this case, which is 6; in other words, we want to determine whether the sequence

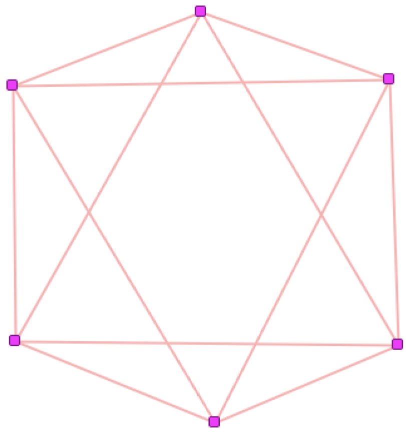
$$(4, 4, 4, 4, 4, 4)$$

is graphical. By the Havel-Hakimi theorem, we have that

$$\begin{gathered}(4, 4, 4, 4, 4, 4) \text{ is graphical}\\ \text{if and only if}\\ (3, 3, 3, 3, 4) \text{ is graphical, } \text{ or in other words } (4, 3, 3, 3, 3) \text{ is graphical,}\\ \text{if and only if}\\ (2, 2, 2, 2) \text{ is graphical.}\end{gathered}$$

We now note that the last sequence is graphical (it coincides with the degree sequence of a cycle on 4 vertices), and thus all the previous sequences will be graphical too.

Again, if we reverse the process (and recall how we were showing in the proof of the theorem that, if the shorter sequence is graphical, then the longer one is graphical too), we can come up with a 4-regular graph of order 6:



4-regular graph of order 6

Next we note that, for any other $k > 1$, we can construct a graph of order $5k+1$ by taking the disjoint union of this graph and of $k-1$ copies of K_5.

Case of $5k+2$: We begin with the smallest order we could have in this case, which is 7; in other words, we want to determine whether the sequence

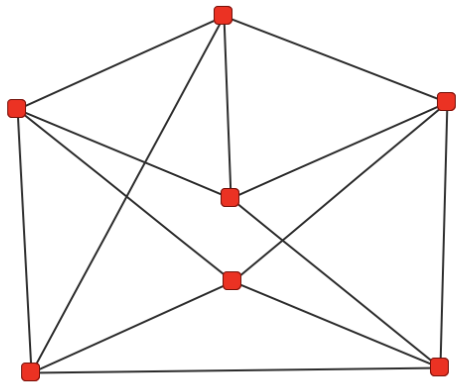
$$(4,4,4,4,4,4,4)$$

is graphical. By the Havel-Hakimi theorem, we have that

$$\begin{gathered}(4,4,4,4,4,4,4) \text{ is graphical}\\ \text{if and only if}\\ (3,3,3,3,4,4) \text{ is graphical, or in other words } (4,4,3,3,3,3) \text{ is graphical,}\\ \text{if and only if}\\ (3,2,2,2,3) \text{ is graphical, or in other words } (3,3,2,2,2) \text{ is graphical,}\\ \text{if and only if}\\ (2,1,1,2) \text{ is graphical, or in other words } (2,2,1,1) \text{ is graphical.}\end{gathered}$$

We now note that the last sequence is graphical (it coincides with the degree sequence of a path on 4 vertices), and thus all the previous sequences will be graphical too.

We can also come up with an instance of a 4-regular graph on 7 vertices:



4-regular graph of order 7

Finally, observe that, for any other $k > 1$, we can construct a graph of order $5k + 2$ by taking the disjoint union of this graph and of $k - 1$ copies of K_5.

Case of $5k + 3$: We begin with the smallest order we could have in this case, which is 8; in other words, we want to determine whether the sequence

$$(4, 4, 4, 4, 4, 4, 4, 4)$$

is graphical. By the Havel-Hakimi theorem, we have that

$$(4, 4, 4, 4, 4, 4, 4, 4) \text{ is graphical}$$

$$\text{if and only if}$$

$$(3, 3, 3, 3, 4, 4, 4) \text{ is graphical, or in other words } (4, 4, 4, 3, 3, 3, 3) \text{ is graphical,}$$

$$\text{if and only if}$$

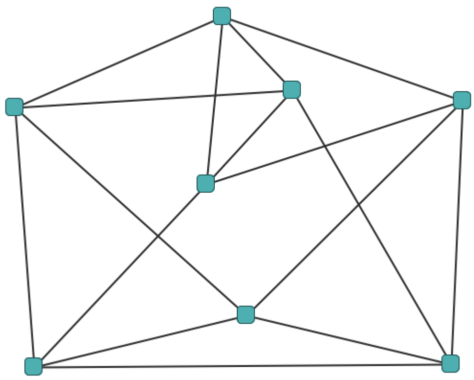
$$(3, 3, 2, 2, 3, 3) \text{ is graphical, or in other words } (3, 3, 3, 3, 2, 2) \text{ is graphical,}$$

$$\text{if and only if}$$

$$(2, 2, 2, 2, 2) \text{ is graphical.}$$

We now note that the last sequence is graphical (it coincides with the degree sequence of a cycle on 5 vertices), and thus all the previous sequences will be graphical too.

We can also come up with an instance of a 4-regular graph on 8 vertices:



4-regular graph of order 8

Finally, observe that, for any other $k > 1$, we can construct a graph of order $5k+3$ by taking the disjoint union of this graph and of $k-1$ copies of K_5.

Case of $5k+4$: We begin with the smallest order we could have in this case, which is 9; in other words, we want to determine whether the sequence

$$(4, 4, 4, 4, 4, 4, 4, 4, 4)$$

is graphical. By the Havel-Hakimi theorem, we have that

$$(4,4,4,4,4,4,4,4,4) \text{ is graphical}$$

$$\text{if and only if}$$

$$(3,3,3,3,4,4,4,4) \text{ is graphical, \ or in other words } (4,4,4,4,3,3,3,3) \text{ is graphical,}$$

$$\text{if and only if}$$

$$(3,3,3,2,3,3,3) \text{ is graphical, \ or in other words } (3,3,3,3,3,3,2) \text{ is graphical,}$$

$$\text{if and only if}$$

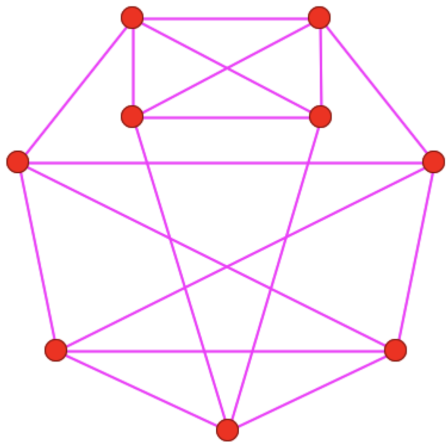
$$(2,2,2,3,3,2) \text{ is graphical, \ or in other words } (3,3,2,2,2,2) \text{ is graphical,}$$

$$\text{if and only if}$$

$$(2,1,1,2,2) \text{ is graphical, \ or in other words } (2,2,2,1,1) \text{ is graphical.}$$

We now note that the last sequence is graphical (it coincides with the degree sequence of a path on 5 vertices), and thus all the previous sequences will be graphical too.

We can also come up with an instance of a 4-regular graph on 9 vertices:



4-regular graph of order 9

Finally, observe that, for any other $k > 1$, we can construct a graph of order $5k + 4$ by taking the disjoint union of this graph and of $k - 1$ copies of K_5.

We conclude that there are 4-regular graphs of every order $n \geqslant 5$.

Natural follow-up question; For practice/fun: Can you also find **connected** 4-regular graphs of every order $n \geqslant 5$? Could you 'manipulate' the examples mentioned above (that is, the different disjoint unions we came up with in either case) to construct connected examples instead?