Problems 5

10/26/2020

1) Let F be a field.
   (i) Let G be a finite group of order $n \geq 1$. Show that G is isomorphic to a subgroup of $\mathrm{GL}_n(F)$.
   (ii) Show that A_n is isomorphic to a subgroup of $\mathrm{SL}_n(F)$ for all $n \geq 1$.

2) Show that a non trivial abelian group is simple if and only if it is cyclic of prime order.

3) Let $\sigma_1, \ldots, \sigma_l \in S_n$ be permutations, such that
$$X_{\sigma_i} \cap X_{\sigma_j} = \emptyset$$
for all $1 \leq i \neq j \leq l$. Prove:
$$\mathrm{ord}(\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_l) = \mathrm{lcm}\left(\mathrm{ord}(\sigma_1), \mathrm{ord}(\sigma_2), \ldots, \mathrm{ord}(\sigma_l)\right).$$
(Here lcm denotes the *least common multiple*.)

4) Show that $\mathrm{Aut}(A_4) \simeq S_4$.

5) Show that the alternating group A_4 has no subgroup of order 6.

6) Let G be a group (finite or infinite). Show that if G has a subgroup H of index n then G has a normal subgroup K, such that n divides the index $[G : K]$, and this index divides $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

7) Show that A_5 has no subgroups of order 15, 20, or 30.

8) A subgroup G of the symmetric group S_n is called *l-transitive* for some $l \geq 1$ if given two ordered sets
$$\{i_1, i_2, \ldots, i_l\},\ \{j_1, j_2, \ldots, j_l\} \subseteq \{1, 2, \ldots, n\}$$
of l different integers then there exists $g \in G$, such that $g(i_r) = j_r$ for all $1 \leq r \leq l$.

   Show that the alternating group A_n is $(n-2)$-transitive but not $(n-1)$-transitive.