Final Exam

12/14/2020, 0.00am – 23.59pm

1) (i) Let G be a cyclic group and $f \in \mathrm{Aut}(G)$. Show that $f(H) = H$ for all subgroups H of G, where $f(H) = \{f(h)|h \in H\}$.
   (ii) Let $n \geq 2$ be an integer. Show that there are exactly two homomorphisms of groups $S_n \longrightarrow \mathbb{Z}/2$.
   (iii) Is the group $\mathrm{Aut}(\mathbb{Z}/16)$ cyclic?

   `(1+2+1 credits)`

2) Let H be a normal subgroup of a group G and K another subgroup of G.
   (i) Is K a normal subgroup of G if $K \subseteq H$ and K is normal in H?
   (ii) Show that $|H \cap K| \cdot |HK| = |H| \cdot |K|$ if H and K are finite groups.

   `(1+1 credits)`

3) Let G be a finite group with only two Sylow subgroups. Show that G is isomorphic to the direct product of its two Sylow subgroups. `(2 credits)`

4) Let p be a prime number. Determine the number of p-Sylow subgroups of the symmetric group S_p and of the alternating group A_p. `(3 credits)`

5) For which of the following integers n exists a simple group of order n?

   (a) $n = 60$, (b) $n = 330$, and (c) $n = 360$.

   `(3 credits)`

6) Denote by $\Gamma(n,k)$, where $2 \leq k \leq n$ are integers, the set of all cycles of length k in S_n. Show that:
   (i) $\Gamma(n,k) \subseteq A_n$ if and only if k is odd, and if k is even then $\Gamma(n,k) \cap A_n = \emptyset$.
   (ii) If k is odd then there exists $\sigma, \tau \in \Gamma(n,k)$, such that

   $$\Gamma(n,k) \;=\; \mathrm{Conj}_{A_n}(\sigma) \,\cup\, \mathrm{Conj}_{A_n}(\tau)\,.$$

   `(1+2 credits)`

7) Let G be a finite group. Show that G is isomorphic to a subgroup of a finite simple group. `(3 credits)`

ALL ANSWERS HAVE TO BE JUSTIFIED.