Reminder: We are trying to establish the fact that

$\boxed{\text{Subspaces coincide with Linear Spans}}$

Last time we proved

Lemma 1 Let V be a vector space over a field $\mathbb{F}$, and let S_1 be a subspace of V. Then S_1 is the linear span of some subset of V.

We also want the reverse direction.

Proposition 1 Let V be a vector space over a field $\mathbb{F}$, and let $T \subseteq V$. Then $\text{span}(T)$ is a subspace of V.

Proof We have already seen that, regardless of what T is, $\text{span}(T)$ is nonempty since it contains $\bar{0}_V$.

We also need to check that

span(T) is closed under addition

Let $w_1, w_2 \in \text{span}(T)$ and assume that both of them are non-zero (because otherwise we will clearly have $w_1+w_2 \in \text{span}(T)$)

Then $\exists\, n_1, n_2 \in \mathbb{N}$ and $u_1, \ldots u_{n_1}, v_1, \ldots v_{n_2} \in T$ and $\lambda_1, \ldots \lambda_{n_1}, \mu_1, \ldots \mu_{n_2} \in \mathbb{F}$ such that

$$w_1 = \lambda_1 u_1 + \cdots + \lambda_{n_1} u_{n_1}, \quad w_2 = \mu_1 v_1 + \cdots + \mu_{n_2} v_{n_2}.$$

Thus $w_1+w_2 = \lambda_1 u_1 + \cdots + \lambda_{n_1} u_{n_1} + \mu_1 v_1 + \cdots + \mu_{n_2} v_{n_2}$, which shows that w_1+w_2 is in $\text{span}(T)$. (indeed, we've just observed that w_1+w_2 can be written as a linear combination of n_1+n_2 vectors from T (not necessarily distinct here)).

span(T) is closed under scalar multiplication Let $w_1 \in \text{span}(T)$

be as before, and let $r \in \mathbb{F}$. Then

$$r \cdot w_1 = r(\lambda_1 u_1 + \dots + \lambda_{n_1} u_{n_1}) = r \cdot (\lambda_1 u_1) + \dots + r(\lambda_{n_1} u_{n_1})$$
$$= (r\lambda_1) u_1 + \dots + (r\lambda_{n_1}) u_{n_1}$$

which shows that $r \cdot w_1$ is in span(T).

Combining the above, we conclude that span(T) is a subspace of V.

<u>Proposition 2</u> Let $\{S_i : i \in I\}$ be a family of subspaces of V (not necessarily finite).

We define $\bigcap_{i \in I} S_i$ to be the set $\{w \in V : w \in S_i \text{ for every } i \in I\}$.

We have that $\bigcap_{i \in I} S_i$ is a subspace of V too.

<u>Proof</u> We recall that every subspace of V contains the zero vector $\bar{0}_V$, thus $\{\bar{0}_V\} \subseteq \bigcap_{i \in I} S_i$.

We now need to check that:

<u>$\bigcap_{i \in I} S_i$ is closed under addition</u> Let $w_1, w_2 \in \bigcap_{i \in I} S_i$.

Then, by definition, $w_1, w_2 \in S_i$ for every $i \in I$.
Fix some $i_0 \in I$. Then $w_1, w_2 \in S_{i_0}$, and given that S_{i_0} is a subspace of V, we also get that $w_1 + w_2 \in S_{i_0}$.
Since i_0 is arbitrary, we can conclude that

$$w_1 + w_2 \in S_i \text{ for every } i \in I.$$

But this means that $w_1 + w_2 \in \bigcap_{i \in I} S_i$, as we wanted.

<u>$\bigcap_{i \in I} S_i$ is closed under scalar multiplication</u>
Very similar, left as practice.
Combining the above, we get the conclusion.

<u>Corollary of Propositions 1 and 2</u> Let V be a vector space over a field $\mathbb{F}$, and let $T \subseteq V$. Then

$$\mathrm{span}(T) = \bigcap \{ S \text{ subspace of } V : T \subseteq S \}.$$

Equivalently, span(T) is the <u>smallest subspace of V</u> containing T.

<u>Proof</u> It suffices to show that

$$\mathrm{span}(T) \subseteq \bigcap \{ S \leq V : T \subseteq S \} \text{ and } \bigcap \{ S \leq V : T \subseteq S \} \subseteq \mathrm{span}(T).$$

For the first inclusion, consider any subspace S of V that contains T.

Since S is a subspace, we have seen that we have $S = \mathrm{span}(S)$. Also, since $T \subseteq S$, we get that $\mathrm{span}(T) \subseteq \mathrm{span}(S)$.

Thus $\mathrm{span}(T) \subseteq \mathrm{span}(S) = S$.

We conclude that $\mathrm{span}(T) \subseteq \bigcap \{ S \leq V : T \subseteq S \}$.

On the other hand, we have seen that span(T) is a subspace of V and clearly $T \subseteq \mathrm{span}(T)$.

Thus $\mathrm{span}(T) \in \{ S \leq V : T \subseteq S \}$, which implies that every element $w \in \bigcap \{ S \leq V : T \subseteq S \}$ must be an element of span(T) too. In other words,

$$\bigcap \{ S \leq V : T \subseteq S \} \subseteq \mathrm{span}(T).$$

Last time: we finished discussing that
Subspaces coincide with Linear Spans.

This leads to more interesting examples of subspaces:

Definition 1 Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{m\times n}$.
i) We set $RS(A)$ to be the Row Space of A, that is, the linear span of all rows of A.
$RS(A)$ is a subspace of $\mathbb{F}^n$.
ii) We set $CS(A)$ to be the Column Space of A, that is, the linear span of all columns of A.
$CS(A)$ is a subspace of $\mathbb{F}^m$.
iii) We set $N(A)$ to be the Nullspace of A, that is, the set of vectors $\bar{x} \in \mathbb{F}^n$ such that $A\bar{x} = \bar{0} \in \mathbb{F}^m$.

Remark We can check that $N(A)$ is a subspace of $\mathbb{F}^n$ (practice)

Definition 2 The rank of A is set to be the dimension of $CS(A)$.
(Note that, since $CS(A)$ is a subspace of $\mathbb{F}^m$, we must have $\text{rank}(A) \leq m$.)
Equivalently, the rank of A is set to be the dimension of $RS(A)$.
(Note that, since $RS(A)$ is a subspace of $\mathbb{F}^n$, we must have $\text{rank}(A) \leq n$.)

The two versions of Definition 2 are equivalent because of one of the most important theorems in Linear Algebra.

Main Theorem C Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{m\times n}$. Then $\dim_{\mathbb{F}} CS(A) = \dim_{\mathbb{F}} RS(A)$.

Remark Combining the observations we included in Definition 2, we can write

$$\operatorname{rank}(A) \leq \min\{m,n\}.$$

Definition 3 The nullity of A is set to be the dimension of $N(A)$.

Main Theorem D Let $A \in \mathbb{F}^{m\times n}$. Then

$$\dim_{\mathbb{F}} CS(A) + \dim_{\mathbb{F}} N(A) = n.$$

In other words, $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$.

Very related to Main Thm D, is another very important theorem in Linear Algebra:

Main Theorem E Let $\mathbb{F}$ be a field, and let V_1, V_2 be vector spaces over $\mathbb{F}$. Consider also a linear map

$$\ell : V_1 \longrightarrow V_2.$$

Then $\dim_{\mathbb{F}} \operatorname{Range}(\ell) + \dim_{\mathbb{F}} \operatorname{Ker}(\ell) = \dim_{\mathbb{F}} V_1$.

Our purpose now is to prove these theorems.

Subspaces and bases of subspaces

1) Let V be a vector space over a field $\mathbb{F}$. We saw that $\{\bar{0}_V\}$ is a subspace of V. Basis of $\{\bar{0}_V\}$?

Should be among the subsets of $\{\bar{0}_V\}$, so it is either $\emptyset$ or $\{\bar{0}_V\}$. But $\{\bar{0}_V\}$ is not a linearly independent set since e.g. we can have $1_{\mathbb{F}} \cdot \bar{0}_V = \bar{0}_V$

linear combination that gives us the zero vector where the scalar is non-zero

On the other hand, ϕ (the empty set) is linearly independent (why?) and $\mathrm{span}(\phi) = \{\bar{0}_V\}$, as we saw.

2) Recall the set $S_1 = \{p(x) \in \mathcal{P}_4 : p(x) \text{ is divided by } x-1\}$
$$= \{p(x) \in \mathcal{P}_4 : \exists\, q(x) \in \mathcal{P}_3 \text{ s.t. } p(x) = (x-1)q(x)\}$$

We've seen this set is the kernel of the linear map
$\mathcal{E}: \mathcal{P}_4 \to \mathbb{R}$, $p(x) \in \mathcal{P}_4 \longmapsto \mathcal{E}(p(x)) := p(1)$,
thus S_1 is a subspace of $\mathcal{P}_4$.

Basis for S_1? Note that the arbitrary element in S_1 can be written in the form $(x-1)\,q(x)$ for some $q(x) \in \mathcal{P}_3$, so we could first recall how to write the arbitrary polynomial in $\mathcal{P}_3$ in terms of a basis of $\mathcal{P}_3$.

Standard basis of $\mathcal{P}_3$ is the set $\mathcal{B}_0 = \{1, x, x^2, x^3\}$, and if $q(x) = c_0 + c_1x + c_2x^2 + c_3x^3$, then
$$(x-1)\cdot q(x) = c_0(x-1) + c_1(x-1)x + c_2(x-1)x^2 + c_3(x-1)x^3$$
$$= c_0(x-1) + c_1(x^2-x) + c_2(x^3-x^2) + c_3(x^4-x^3).$$
This shows that the set
$$\mathcal{B}_1 = \{(x-1)\cdot r(x) : r(x) \in \mathcal{B}_0\} = \{x-1, x^2-x, x^3-x^2, x^4-x^3\},$$
which is clearly a subset of S_1, is a <u>spanning set</u> of S_1.

Is $\mathcal{B}_1$ linearly independent? We check the definition:
let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ with
$$\lambda_1(x-1) + \lambda_2(x^2-x) + \lambda_3(x^3-x^2) + \lambda_4(x^4-x^3) = 0$$

$\Rightarrow -\lambda_1 + (\lambda_1 - \lambda_2)x + (\lambda_2 - \lambda_3)x^2 + (\lambda_3 - \lambda_4)x^3 + \lambda_4 x^4 = 0$

$\Rightarrow \lambda_1 = 0 = \lambda_4$ and $\lambda_1 - \lambda_2 = 0 = \lambda_2 - \lambda_3$
which further imply $\lambda_3 = \lambda_2 = \lambda_1 = 0$.

We conclude that B_1 is linearly independent.

Thus B_1 is a basis of S_1 (and hence $\dim_{\mathbb{R}} S_1 = 4$)

Subspaces and bases of subspaces (cont.)

3) Let $\mathbb{F}$ be a field. Consider any vector space V over $\mathbb{F}$.

There is only one subspace of V that has dimension 0. This is the zero subspace $\{\bar{0}_V\}$.

4) If moreover the dimension of V is finite, $\dim_{\mathbb{F}} V < \infty$, then there is only one subspace of V that has dimension equal to the dimension of the whole space. In other words,

$\boxed{\text{if } \dim_{\mathbb{F}} V < \infty \text{ and } S \leq V \text{ with } \dim_{\mathbb{F}} S = \dim_{\mathbb{F}} V, \text{ then } S = V.}$

Indeed, let B be a basis of the subspace S.

We know that B is a linearly independent subset of S with size $|B| = \dim_{\mathbb{F}} S$.

But then S is a linearly independent subset of V too, with size $|B| = \dim_{\mathbb{F}} V$, thus B must be a maximal linearly independent subset of V (since otherwise we should be able to find a subset T of V which is linearly independent and satisfies $B \subsetneq T$, but then, since B is finite, the latter implies that $|T| > |B| = \dim_{\mathbb{F}} V$, which contradicts the fact that we should have $|T| \leq \dim_{\mathbb{F}} V$).

Thus B is a basis of V too, and hence

$S = \mathrm{span}(B) = V$, as claimed.

5) Recall that on the Math 127 Final Exam you had to show that

$$S_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 1.5 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\}$$

is a linearly independent subset of $\mathbb{R}^{2\times 2}$ (and hence it is a basis of $\mathbb{R}^{2\times 2}$ given that $\dim_{\mathbb{R}} \mathbb{R}^{2\times 2} = 4$).

What can we say about

$\mathrm{span}\left(\left\{\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\right\}\right)$? It is a subspace of $\mathbb{R}^{2\times 2}$ with dimension 1.

What about $\mathrm{span}\left(\left\{\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 1.5 & 4 \end{pmatrix}\right\}\right)$?
It is a subspace of $\mathbb{R}^{2\times 2}$ of dimension 2.

Terminology Let V be a vector space over a field $\mathbb{F}$.
If S is a subspace of V of dimension 1, we sometimes call S a line of V. (or vector line)
If T is a subspace of V of dimension 2, we sometimes call T a plane of V (or vector plane).

If B is a basis of V, and u is a vector of B, then the subspace R of V that is spanned by $B \setminus \{u\}$ (all the vectors in the basis except one) is called a hyperplane of V.

If we know that $\dim_{\mathbb{F}} V$ is finite, then equivalently a <u>hyperplane</u> of V is any subspace R of V with dimension $\dim_{\mathbb{F}} V - 1$.

6) Let $A_1 = \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{5\times 4}$. What is $\dim_{\mathbb{R}} RS(A_1)$? A basis for $RS(A_1)$?

<u>Remark</u> $\dim_{\mathbb{R}} RS(A_1) \leq$ # of non-zero rows of $A_1 = 3$

But here the non-zero rows of A_1 are linearly independent: if $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ are such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 3 \\ 8 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 3\lambda_1 + 2\lambda_2 \\ 8\lambda_1 + 4\lambda_2 + \lambda_3 \end{pmatrix}$$

then we can conclude that $\lambda_1 = 0 = \lambda_2$ and $0 = 8\lambda_1 + 4\lambda_2 + \lambda_3 = \lambda_3$.

What helps us reach this conclusion? The fact that A_1 is in Row Echelon Form, and so we can do <u>forward substitution</u>.

7) Let $A_2 = \begin{pmatrix} 4 & 0 & 1 & 10 \\ 0 & 1 & 2 & 4 \\ 3 & 0 & 9 & 3 \\ 1 & 1 & 5 & 1 \\ 3 & 5 & 8 & 6 \end{pmatrix} \in \mathbb{Z}_{11}^{5\times 4}$. A basis for $RS(A_2)$?

This matrix is not in REF, so the above reasoning does not apply immediately. But recall

Fact we discussed in MATH 127: Elementary row operations (that is, applications of Gaussian elimination on the rows of a matrix A) preserve the Row Space of A. In other words, if $A \sim B$ then $RS(A) = RS(B)$.

Special cases: • if B is a REF of A, then $RS(A) = RS(B)$

• $RS(A) = RS(RREF(A))$.

Consequence of this: if B is a REF of A, then

$\dim RS(A) = \dim RS(B) = \#$ of non-zero rows of B $= \#$ of pivots of B.

apply now the reasoning from part (6)

In the case of the given matrix A_2,

$$A_2 = \begin{pmatrix} 4 & 0 & 1 & 10 \\ 0 & 1 & 2 & 4 \\ 3 & 0 & 9 & 3 \\ 1 & 1 & 5 & 1 \\ 3 & 5 & 8 & 6 \end{pmatrix} \overset{\substack{3R_1 \\ 4R_3 \\ 4R_5}}{\sim} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 4 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 9 & 10 & 2 \end{pmatrix} \overset{\substack{R_3-R_1 \\ R_4-R_1 \\ R_5-R_1}}{\sim} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 9 & 7 & 5 \end{pmatrix}$$

$$\overset{\substack{R_4-R_2 \\ R_3 \leftrightarrow R_4 \\ R_3 \leftrightarrow R_5}}{\sim} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 9 & 7 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{5R_3}{\sim} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\substack{R_3-R_2 \\ R_4-7R_3}}{\sim} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B_2$$

thus $\dim_{\mathbb{Z}_{11}} RS(A_2) = 3$ and a basis of $RS(A_2) = RS(B_2)$ is the set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 10 \end{pmatrix} \right\}.$$

8) Let $A_3 = \begin{pmatrix} 1 & 0 & 3 & 8 & 6 \\ 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 10 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 5}$.

What is $\dim_{\mathbb{R}} CS(A_3)$? A basis for $CS(A_3)$?