Math 322
Homework Problem Set 1

Remark. As we've already briefly discussed in class, the adjacency matrix of a graph allows us to obtain plenty of information about the graph in efficient ways. For example, we discussed some ways of quickly determining the degree of each vertex of the graph by just looking at the adjacency matrix. The following question offers one more such way.

Problem 1. Let $G = (V, E)$ be a finite graph of order n, with $V = \{v_1, v_2, \ldots, v_n\}$. Write A for the adjacency matrix of G. Show that the (j, j)-th entry of the matrix A^2 equals the degree of the vertex v_j.

Problem 2. Consider the following graphs (pictures from the Harris-Hirst-Mossinghoff book).

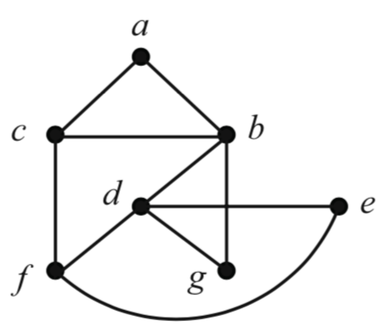

Figure 1: Graph G_1

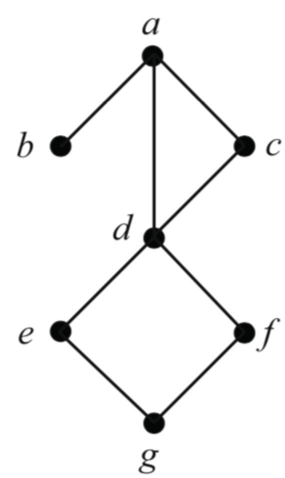

Figure 2: Graph G_2

(i) Write down their adjacency and their incindence matrices.

(ii) For each one of the graphs, determine how many paths in it have c as their terminal vertex *(where you can view the different paths in each graph as oriented, even though the graph itself is not)*. Note that here you are asked to find not only the longest possible paths, but all that end at the vertex c. Justify your answer.

(iii) For each one of the graphs, determine how many different paths avoid the vertex c altogether *(be careful not to count each path twice by considering both possible directions; recall that the given graphs are not directed, so choosing an orientation for any of their subgraphs should only be done if it helps you obtain the correct answer, and should be ignored at the end).*

(iv) Can you find a subgraph of G_1 on 4 vertices and a subgraph of G_2 on 4 vertices that are isomorphic? Justify your answer fully.

(v) What about the analogous question to part (iv) for subgraphs on 5 vertices? Justify your answer fully.

Problem 3. There are 11, essentially different, unlabelled graphs on 4 vertices, or in other words, there are 11 pairwise non-isomorphic graphs of order 4. Draw all of them *(and try to find, if possible, a systematic way to list them so that you avoid drawing the same graph twice or more, and also avoid missing any of the 11 graphs; explain briefly your approach).*

Remark. We will see in class that, for every finite graph G which has at least two vertices, we can find (at least) one pair of vertices which will have the same degree.

The problem below asks you to study a bit further those graphs which have **exactly one** pair of vertices sharing the same degree.

Problem 4. (i) Let $G = (V, E)$ be a finite graph of order n, with $V = \{v_1, v_2, \ldots, v_n\}$, and let

$$\big(\deg(v_1),\, \deg(v_2),\, \ldots,\, \deg(v_n)\big) = (d_1, d_2, \ldots, d_n)$$

be the degree sequence of G (note that we do not rewrite this here in a decreasing way; the order in which the degrees are given corresponds to the order of the vertices).

What is the degree sequence of the complement $\overline{G}$ of the graph G? Determine how it depends on the degree sequence of G.

(ii) Based on part (i), show that if G has exactly one pair of vertices sharing the same degree, then $\overline{G}$ also has this property.

(iii) By relying also on the lists of pairwise non-isomorphic graphs of order 3 and of order 4 that you have been asked to find (please look for the former list in the lecture notes or the Thursday September 10 class), try to find, **for every n between 2 and 8,** two non-isomorphic graphs which have exactly one pair of vertices sharing the same degree, and draw these graphs (briefly explaining why they are not isomorphic).

It might help to look at the lists of pairwise non-isomorphic graphs of order 3 and of order 4 that you have been asked to find (you should be able to find the former list in the lecture notes or from the Thursday September 10 class).

(iv) *(Optional, just for fun, will not be considered at all during grading!)* Does your answer in part (iii) allow you to come up with a description of all non-isomorphic finite graphs which have exactly one pair of vertices sharing the same degree? That is, do you see a pattern which you could generalise to higher orders?

Remark. As we will see in class, for every $n \geqslant 3$, there is a 2-regular graph on n vertices, that is, a graph of order n all of whose vertices have degree 2 (the simplest such graph is, what we will call, the n-cycle).

The two following problems ask an analogous question about the existence of 3-regular graphs and 4-regular graphs on n vertices.

Problem 5. (i) What is the minimum possible order of a 3-regular graph? Justify your answer fully (that is, find what the minimum order $n_{3,\min}$ should be, but also exhibit that this order works by drawing (or describing) a 3-regular graph on $n_{3,\min}$ vertices).

(ii) Is there a maximum order for a finite 3-regular graph? Justify your answer.

(iii) Just as in part (i), write $n_{3,\min}$ for the minimum possible order of a 3-regular graph. Can you find a 3-regular graph of order n for every $n \geqslant n_{3,\min}$? Justify your answer (and if the answer is no, try to determine the integers n which could be orders of 3-regular graphs).

Problem 6. Analogous to Problem 5, but for 4-regular graphs.
(i) What is the minimum possible order of a 4-regular graph? Justify your answer fully (that is, find what the minimum order $n_{4,\min}$ should be, but also exhibit that this order works by drawing (or describing) a 4-regular graph on $n_{4,\min}$ vertices).

(ii) Is there a maximum order for a finite 4-regular graph? Justify your answer.

(iii) Just as in part (i), write $n_{4,\min}$ for the minimum possible order of a 4-regular graph. Can you find a 4-regular graph of order n for every $n \geqslant n_{4,\min}$? Justify your answer (and if the answer is no, try to determine the integers n which could be orders of 4-regular graphs).