Cheat sheet

You may assume anything on this. If you want to assume something else proven in class, mention it explicitly.

Squeeze Theorem. If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $b_n \to L$

Pretty useful Theorem. If $a_n \to L, b_n \to L'$ (where $L, L' \in \mathbb{R} \cup \{\pm\infty\}$), then:
(a) $a_n + b_n \to L + L'$ (unless $L = -L' = \infty$ or $L = -L' = -\infty$). Also, for any constant $c \in \mathbb{R}$, $c\, a_n \to cL$ (unless both $c = 0$ and $L = \pm\infty$).
(b) $a_n b_n \to LL'$ (unless $L = 0, L' = \pm\infty$, or $L = \pm\infty, L' = 0$). If no $b_n = 0$, and also $L' \neq 0$, then $a_n/b_n \to L/L'$ (unless both L, L' are infinite).

Bolzano-Weierstrass Theorem. a_n bounded $\implies$ convergent subsequence.

Cauchy sequence Theorem. a_n converges $\iff$ it is Cauchy.

Power sequence. $r^n \to 0$ if $|x| < 1$; $r^n \to \infty$ if $r > 1$.
$n^r \to 0$ if $r < 0$; $n^r \to \infty$ if $r > 0$.

Ratio test. If $a_{n+1}/a_n \to R$ and $R > 1$, then $a_n \to \infty$ or $-\infty$. If $|R| < 1$, then $a_n \to 0$. If $R < -1$, then a_n diverges.

Geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r| < 1$.

Harmonic series $\sum_{n=1}^{\infty} n^{-1}$ diverges.

$\cos(-\theta) = \cos(\theta + 2\pi) = \cos(\theta)\,, \quad -\sin(-\theta) = \sin(\theta + 2\pi) = \sin(\theta), \quad \tan(\theta) = \sin(\theta)/\cos(\theta)$
$\cos^2(\theta) + \sin^2(\theta) = 1\,, \quad -1 \le \sin(\theta) \le 1\,, \quad -1 \le \cos(\theta) \le 1\,, \quad \theta\cos(\theta) \le \sin(\theta) \le \theta \ \forall |\theta| < \pi/2,$
$\cos(\theta + \phi) = \cos(\theta)\, \cos(\phi) - \sin(\theta)\, \sin(\phi)\,, \quad \sin(\theta + \phi) = \sin(\theta)\, \cos(\phi) + \cos(\theta)\, \sin(\phi)$

Bridge Theorem. $\lim_{x\to a} f(x) = L$ iff for all sequences $a_n \to a$ ($a_n \neq a$), $f(a_n) \to L$.

Pretty useful Theorem for functions. If $\lim_{x\to a} f(x) = L, \lim_{x\to a} g(x) = L'$ then:
(a) $\lim_{x\to a} cf(x) + dg(x) = cL + dL'$, for any constants $c, d \in \mathbb{R}$.
(b) $\lim_{x\to a} f(x)g(x) = LL'$. If no $g(x) = 0$ in interval about a, and also $L' \neq 0$, then $\lim_{x\to a} f(x)/g(x) = L/L'$.

Bridge Corollary. f continuous at $x = a$ iff for all $a_n \to a$, $f(a_n) \to f(a)$.

Continuous Theorem (a) f, g continuous at $x = a$ implies $\alpha f(x) + \beta g(x)$ and $f(x)g(x)$ continuous at $x = a$. If $g(a) \neq 0$, then $f(x)/g(x)$ also continuous at $x = a$.
(b) $g(x)$ continuous at $x = a$ and $f(y)$ continuous at $y = g(a)$, $\implies f(g(x))$ continuous at $x = a$.

Examples. All polys are continuous. All rational functions $\frac{poly(x)}{poly(x)}$ are rational wherever their denominators don't vanish. $\sin(x)$ and $\cos(x)$ are continuous everywhere. For any $r \in \mathbb{Q}$, x^r is continuous on $[0, \infty)$. $f(x) = |x|$ is continuous everywhere.
Intermediate Value Theorem. $f(x)$ be continuous on $[a, b]$. Choose any L between $f(a)$ and $f(b)$. Then there is some c between a and b such that $f(c) = L$.

Maximum Principle. Every continuous function $f(x)$ on a closed interval $[a, b]$ is bounded. $f(x)$ will attain both its supremum and infimum on S.

Fact. f is invertible iff it is one-to-one and onto.

Continuity of inverse Theorem. Suppose $f : [a, b] \to [c, d]$ is continuous and invertible. Then $f^{-1} : [c, d] \to [a, b]$ is continuous. $f(x)$ will be strictly increasing or strictly decreasing on $[a, b]$.

MATH 117 Midterm #1 Cheat sheet

This sheet will be available to you during your midterm 1.
You may assume anything on this. Anything else should be justified.

Peano's Axioms

A0. $1 \in \mathbb{N}$
A1. If $n \in \mathbb{N}$, then $S(n) \neq 1$
A2. If $m, n \in \mathbb{N}$ and $m \neq n$, then $S(m) \neq S(n)$
A3. If $K \subseteq \mathbb{N}$, and $1 \in K$, and $n \in K \Rightarrow S(n) \in K$, then $K = \mathbb{N}$

Field Axioms

(AC) $x + y = y + x \ \forall x, y \in F$.
(AA) $(x + y) + z = x + (y + z) \ \forall x, y, z \in F$.
(AN) There is some $0 \in F$ such that $0 + x = x + 0 = x \ \forall x \in F$.
(AI) For each $x \in F$, there is some $-x \in F$ such that $x + (-x) = (-x) + x = 0$.
(MC) $xy = yx \ \forall x, y \in F$.
(MA) $(xy)z = x(yz) \ \forall x, y, z \in F$.
(MN) There is a $1 \in F$ such that $1x = x1 = x \ \forall x \in F$.
(MI) For each $x \in F$, except for $x = 0$, there is some $x^{-1} \in F$ such that $xx^{-1} = x^{-1}x = 1$.
(D) $x(y + z) = (y + z)x = xy + xz \ \forall x, y, z \in F$.

Lemmas

Lemma S. $A \subseteq A \cup B$. $A \supseteq A \cap B$. $A \cup B = B \cup A$. $A \cap B = B \cap A$.

Lemma P. Addition in $\mathbb{N}$ is associative and commutative. $k, m, n \in \mathbb{N}$ implies $(k+m)n = kn + mn$

Generalized Associativity and Commutativity. If we know both associativity and commutativity holds, then we can reorder and rearrange brackets in any (finite) expression.

Lemma F. Let $\mathbb{F}$ be any field. Then:

(a) $\forall x \in \mathbb{F}$, $-(-x) = x$.
(b) $\forall x \in \mathbb{F}$, $0x = 0$.
(c) $\forall x \in \mathbb{F}$, $-x = (-1)x$.
(d) $(-1)^2 = 1$.

Lemma. $\sqrt{2}$ is irrational.

Lemma. $\mathbb{R}$ and $\mathbb{Q}$ are fields. $\{even, odd\}$ is a field.

Midterm 1 Review Sheet

The course until now fits under 3 themes. **Theme 1** is "Language and grammar of math", namely sets and elementary logic. **Theme 2** is "Elementary proof strategies". **Theme 3** is "Numbers and axioms". All 3 will be tested on midterm, roughly equally.

Sets and set notation (Theme 1)

You have to be comfortable with the basics of sets and set notation. This is explained in the Week1 summary, as well as Section 1.A of Bowman's notes. Quiz 1 was on this, as was most of Assignment 1. The notation you need to know:

$a \in A$, $A \subseteq B$, $A \cup B$, $A \cap B$, $A = B$. The empty set $\{\}$. Set-builder notation.

The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ of numbers.

You don't need to know Russell's Paradox.

Also useful is $\sum$ notation for sums (Week3). We often see that in induction proofs.

Elementary logic (Theme 1)

You have to be comfortable with elementary logic: AND, OR, NOT, IF...THEN, IFF. This is discussed in the Week1 and Week2 summaries, and was the subject of Quiz 2 and Assn.1 and Assn.2.

Know the difference between Theorem, Definition, Axiom. (Week2)

Know what "for all" (abbreviated $\forall$) and "there exists" (abbreviated $\exists$) mean. Know how to negate an expression involving them (see Quiz4, and Week4).

Proof strategies (Theme 2)

You should know the basic proof strategies for sets, i.e. for showing $A \subseteq B$, $A = B$, etc. Assn.1 and 2. This is discussed in the Week2 notes.

Know what "counterexample" means. (see e.g. Quiz 4, and Week4 notes)

You should be comfortable with **Induction**, an important proof strategy. It is discussed in Week3, Assn.2, and Quiz 3, as well as Bowman's notes 1.E.

Be comfortable with the **Proof-by-contradiction** proof strategy, it is a very useful friend! It is discussed in the Week4 notes (Thursday), and Assn.3. See also Section 1.B of Bowman's notes.

Peano's axioms (Theme 3)

This axiomatic system is discussed on Assn.2 and in the Week2 and Week3 notes. You don't need to memorise the axioms of $\mathbb{N}$ — if needed, they will be listed in your midterm. You won't be asked to regurgitate any Peano axiom proofs.

Know what *commutativity* and *associativity* mean. Know that together they imply "generalised associativity and commutativity". See the Week3 notes, and Quiz 3 Question 1.

Fields (Theme 3)

This is the subject of Assn.3. In Week3 summary you'll find listed all 9 axioms of a field. Be comfortable with them, but don't memorize these: they will be given to you on your Midterm Cheatsheet. Simple proofs involving fields are on Assn.3, and some are given in the Week4 summary. Bowman's notes Section 1.C should also be looked at, though he includes here also the axioms for an ordered field. When in doubt, follow the treatment of field in our notes rather than those in Bowman's notes.

Know what division and subtraction mean in a field.

You don't need to memorize the theorems of a field given in the Week4 summary. These will be listed on your Midterm cheat sheet, and you will be able to assume them when you prove things on your midterm. You also don't need to memorize their proofs, but you should understand their proofs. You may find it helpful to use similar ideas in your midterm proofs.

Know the real field $\mathbb{R}$, the rational field $\mathbb{Q}$, and the$\{\mathbf{even}, \mathbf{odd}\}$ field. You don't need to know the rational function field, nor tropical numbers. You don't need to know the field of complex numbers.

You can use the weird $\oplus$, $\odot$ notation if you like, but you are also free to use the usual notation (as long as it is unambiguous).

MATH 117 Midterm #1 solutions

1(a) [1 pt] True or False: $(3 \geq 2)$ OR $(7 \geq 5)$ (justify your answer)

Solution: '$3 \geq 2$' is **true**. '$7 \geq 5$' is **true**. true OR true = **TRUE**

1(b) [1 pt] True or False: If $\sqrt{2}$ is rational then $\sqrt{2}$ is irrational. (justify your answer)

Solution: premise is false so **TRUE**

1(c) [1 pt] True or False: If $\sqrt{2}$ is irrational, then $\sqrt{2}$ is rational. (justify your answer)

Solution: premise true, conclusion false so **FALSE**

1(d) [1 pt] True or False: for all $x \in \mathbb{R}$, there exists a $y \in \mathbb{R}$ such that $x^2 < y$. (justify your answer)

Solution: **True:** no matter what x is, take $y = x^2 + 1$
(this question wasn't done so well)

1(e) [1 pt] Simplify: NOT(for all $x \in A$, there exists a $y \in B$ such that $x^2 < y$)

Solution: there exists an $x \in A$ such that for all $y \in B$, $x^2 \geq y$

2. [4 pts] Prove for all $n \in \mathbb{N}$ that the sum of the first n odd numbers is n^2: e.g. $1+3 = 2^2$ and $1+3+5 = 3^2$.

Solution: The first odd number is 1, the second is 3, the third is 5,... The formula for the nth odd number is $2n-1$ (if you want, you can prove this from induction, but it is easy enough that you could just state it). So the statement that we need to prove is $\sum_{k=1}^{n}(2k-1) = n^2$.

The base case is $n = 1$: we need to show that $1 = 1^1$. This is obvious.

Now for the inductive step. Assume that $\sum_{k=1}^{n}(2k-1) = n^2$ for some $n \in \mathbb{N}$ (this is the inductive hypothesis). We need to show it is true for $n+1$: i.e. we need to show $\sum_{k=1}^{n+1}(2k-1) \stackrel{?}{=} (n+1)^2$

The LHS is

$$\sum_{k=1}^{n+1}(2k-1) = \sum_{k=1}^{n}(2k-1) + (2(n+1)-1) = n^2 + 2n + 2 - 1 = n^2 + 2n + 1$$

using the inductive hypothesis.

The RHS is $(n+1)^2 = n^2 + 2n + 1$.

LHS=RHS, so we're done!

3. [2 pts] Prove that if $r \neq 0$ is rational and x is irrational, then rx must be irrational. (You may assume $\mathbb{Q}$ is a field).

Solution: Assume $r \neq 0$ is rational, and x is irrational. Suppose for contradiction that rx is rational. Since $\mathbb{Q}$ is a field, r^{-1} exists and is in $\mathbb{Q}$. Moreover, rational times rational must be rational. So $r^{-1}(rx) = x$ must be rational. This contradicts the hypothesis that x is irrational. This contradiction means rx must be irrational.

4. [2 pts] Is it always true that $A \cup (B \cap C) = (A \cup B) \cap C$? Prove your answer.

Solution: No, it is not always true. A counterexample is $A = \{1\}$, $B = \{2\}$, $C = \{\}$: then LHS is $A \cup (B \cap C) = \{1\} \cup (\{2\} \cap \{\}) = \{1\} \cup \{\} = \{1\}$ and RHS is $(A \cup B) \cap C = (\{1\} \cup \{2\}) \cap \{\} = \{1, 2\} \cap \{\} = \{\}$, so LHS $\neq$ RHS here.

There are lots of other counterexamples — if ever there is an $x \in A$ which is not in C, then x will be in LHS but not in RHS.

5. [2 pts] Prove that in any field, if $a \neq 0$, then $(-a)^{-1} = -(a^{-1})$.

Solution: We need to show that $(-a)\cdot(-(a^{-1})) \stackrel{?}{=} 1$. LHS: $(-a)\cdot(-(a^{-1})) = (-1)a(-1)(a^{-1}) = (-1)^2 aa^{-1} = 1 \cdot 1 = 1$, as desired. (I'm freely using here generalized associativity and commutativity)

6. Let $\mathbb{F}$ be a field. As always, define $2 = 1 + 1$, $3 = 2 + 1$, $4 = 3 + 1$, etc.

(a) [2 pts] Let $x, y \in \mathbb{F}$. If both $x \neq 0$ and $y \neq 0$, prove that $x \cdot y \neq 0$.

Solution: $x \neq 0$ implies x^{-1} exists. Suppose for contradiction that $x \cdot y = 0$. Then $y = x^{-1}(xy) = 0$ by Lemma F(b), which contradicts $y \neq 0$.

(b) [2 pts] Prove that, in any field $\mathbb{F}$, $2 \cdot 2 = 4$ (HINT: don't use Peano's Axioms).

Solution: $2 \cdot 2 = (1 + 1) \cdot 2 = 1 \cdot 2 + 1 \cdot 2$ by distributivity, which equals $2 + 2$ by MN, which equals $2 + (1 + 1) = (2 + 1) + 1 = 3 + 1 = 4$.

(c) [1 pt] Assume that $2 \neq 0$ in $\mathbb{F}$. Prove that $4 \neq 0$ in $\mathbb{F}$.

Solution: By (a) and (b), $2 \neq 0$ implies $4 = 2 \cdot 2 \neq 0$

7. [1 pt] True or False: Any field must be infinite. Justify your answer.

Solution: **False.** A counterexample: $\{even, odd\}$ is a field with only 2 elements. There are lots of other finite fields.

8. [1 pt] Define the natural number 5.

Solution: The natural numbers are defined using Peano's axioms. 5 is $S(S(S(S(1))))$.

9. [2 pts] Let $P = \{x \in \mathbb{R} \mid x > 0\}$. Prove that NOT(for all $x, y, z \in P$, $x/(y/z) = (x/y)/z$).

Solution: A counterexample: Take $x = 2 = y = z$. Then $x/(y/z) = 2$ and $(x/y)/z = 1/2$.

More generally, $x/(y/z) = xz/y$ and $(x/y)/z = x/(yz)$, so as long as $z \neq 1$ you'll get a counterexample.

10. [1 pt] Using only associativity, show that $(a + b) + (c + d) = a + ((b + c) + d)$. (Don't use generalized associativity)

Solution: $(a + b) + (c + d) = a + (b + (c + d)) = a + ((b + c) + d)$

Midterm 2 Review Sheet

Our exam is Friday Nov 1, in usual place.

The course until now fits under 6 themes. **Theme 1** is “Language and grammar of math”. **Theme 2** is “Elementary proof strategies”. **Theme 3** is “Numbers and axioms”. **Theme 4** is “Inequalities and absolute value”. **Theme 5** is “sup, inf, and completeness”. **Theme 6** is “Sequences and limits”. Themes 1-3 were tested on Midterm 1. Midterm 2 will focus on Themes 4-6, roughly equally.

Ordered Field (Theme 4)

The 3 axioms of an ordered field are given in the Week5 summary. You can also see Section 1.C of Bowman’s notes. Don’t need to memorize them; they’ll be on the Cheat-sheet. Know the examples of $\mathbb{Q}$ and $\mathbb{R}$ (you don’t need to know the rational function example). Lemma OF.1 and Lemma OF.2 in Week5 are useful, but will be put on Cheat-sheet. Don’t memorize their proofs, but understand their proofs. Question 2 on Assn.4 is good to look over, as is Qu.1 on PilesOfQuestions2.

Know when x is positive and negative, in terms of the set P. Know what $x > y$, $x < y$ etc mean in terms of P.

Intervals: Theme 4

Be comfortable with the interval notation, described in Week6 summary. It is also covered in Section 1.G of Bowman’s notes. Lemma OF.3 says these are never empty. See Qu.5 of Assn.4 and Quiz 5.

Absolute value: Theme 4

This is discussed in Section 1.D of Bowman’s notes, as well as Week 6. Know the definition and the basic properties. Triangle inequality is the most important of these. See Qu.6,7 of Assn.4 and Qu.3 of Assn.5, Qu.1.5, 1.7, 1.8 on PilesOfQuestions2. See also Quiz5.

Bounds, max, min, sup, inf: Theme 5

This is discussed in Week6 and Week7, as well as Sections 1.H and 1.I of Bowman’s notes. See Qu.1,2 of Assn.5, and Qu.2-4, 6-7 of PilesOfQuestions2. You have to know the differences between these words:

upper bound, lower bound, bounded above, bounded below, bounded, unbounded.
minimum, maximum, supremum, infimum

Completeness: Theme 5

Know this axiom. Know that $\mathbb{Q}$ doesn’t obey it, but $\mathbb{R}$ does. This is discussed in Week7, as well as Bowman’s notes 1.J.

The Archimedian property can be useful.

Sequences and limit definitions: Theme 6

Know the definition of $\lim_{n\to\infty} a_n = L$ (Week 7; p.33 Bowman), including $L = \pm\infty$ (Week8). See p.52 Bowman; Qu.4-5 Assn.5; Qu.1,5 Assn.6; Qu.8,11 PilesOfQuestions2.

The Golden Rule of Fraction Inequalities (Week8) is quite useful in simplifying fractions

Subsequences: Theme 6

Know what subsequences mean; know Subsequence Theorem (Week 8; Section 2.C Bowman); know how to use it to prove sequences don't converge (Week 8)

Subsequences also appear on Qu.5 Assn.6 and Quiz6.

Squeeze Theorem: Theme 6

See Week 8; p.34 Bowman.

Know how to use it to prove sequences converge

For this purpose, the inequalities $-1 \leq \cos(\theta) \leq 1$ and $-1 \leq \sin(\theta) \leq 1$ (Week8) are useful

Bounded sequences: Theme 6

Know what bounded sequences are; know Bounded Lemma (Week8; p.35 Bowman)

Limits of sums, products etc: Theme 6

Know limits of sums, products, quotients (Week8; p.36 Bowman), including infinite limit cases (Fri Week 6). See Qu.9,13 PilesOfQuestions2 and Quiz6.

Know $\lim n^r$ (Week8)

Monotone Convergence Theorem: both Theme 5 and 6

Know Monotone Convergence Theorem (Week9; Section 2.B Bowman). It needs Completeness axiom. See also Qu.10,12,14 of PilesOfQuestions2.

Be good at evaluating limits. Lots of limit questions, solved by various means, are collected in Qu.2,4 Assn.6; Qu.9 PilesOfQuestions2

You don't need to know the Power Corollary (i.e. how c^n converges) for your midterm. But feel free to use it if it is helpful. You don't need lim inf and lim sup for your midterm.

<u>**MATH 117**</u> midterm 2

1. [1 pt] Find max, min, sup and inf of the set $\{|x| \mid x \in (-2,1]\}$ of all absolute values of numbers in the interval $(-2,1]$. Just state your results (no justification is needed.)

   max=1, **min** does not exist, **sup**=1, **inf**$= -2$

2. [2 pts] Which of the following (there may be more than one) must hold in any complete ordered field $\mathbb{F}$? (no justification is needed)

   (a) For any nonempty set $S \subset \mathbb{F}$, the minimum, maximum, infimum and supremum of S all exist, and $\min S = \inf S$ and $\max S = \sup S$ **NO**

   (b) Whenever $a < b$ we have $|a| < |b|$ **NO**

   (c) Any bounded sequence converges. **NO**

   (d) For all $a, b \in \mathbb{F}$, $|a+b| \geq |a| - |b|$ **YES**

   You didn't need to give a reason, but: **(a)** $(-2,1]$ does not have a minimum; **(b)** $-2 < -1$ but $|-2| > |-1|$; **(c)** $(-1)^n$ is bounded but divergent; **(d)** this is part of triangle inequality

3. [2 pts] Suppose a_n is bounded monotone increasing with limit A and b_n is bounded monotone decreasing with limit B. Find $\sup\{a_n - 2b_n\}$ and $\inf\{a_n - 2b_n\}$, i.e. find the supremum and infimum of the sequence $c_n = a_n - 2b_n$. Justify your answer.

   solution: a_n is monotone increasing and b_n is monotone decreasing, so $a_n - 2b_n$ is monotone increasing. So $\sup\{a_1 - 2b_1, a_2 - 2b_2, ...\}$ is the limit, which by Pretty Useful Thm is $A - 2B$. And $\inf\{a_1 - 2b_1, a_2 - 2b_2, ...\}$ is the first term, which is $a_1 - 2b_1$.

4. [1 pt] Suppose $a_n \to 1$, $b_n \to -1$, and $c_n \to 3$. Compute

$$\lim_{n\to\infty} \frac{a_n + 2}{\frac{c_n}{n} + b_n}$$

   solution: This is pretty easy, and follows immediately from Pretty Useful Thm. First, $\frac{c_n}{n} \to 0$ (you could just write this down and i'd believe you, but a formal proof would use the Bounded sequence Lemma and the Squeeze Lemma). So the limit is $\frac{1+2}{0-1} = -3$.

5. [3 pts] Let $\mathbb{F}$ be an ordered field. Let $k \in \mathbb{N}$. Let $a \in \mathbb{F}$. Prove there can be at most one $b \in \mathbb{F}$ with $b > 0$ and $b^k = a$.

   solution: Suppose for contradiction that there are more than one: so $b, b' > 0$ and $b^k = a = b'^k$. Since $b \neq b'$, one of them is bigger – without loss of generality say $b > b'$. Then $bb > bb' > b'b' > 0$, i.e. $b^2 > b'^2 > 0$. And $bb^2 > bb'^2 > b'b'^2$, i.e. $b^3 > b'^3 > 0$. etc etc until we get $b^k > b'^k > 0$. But this contradicts $b^k = a = b'^k$.

   (you could formally prove $b^k > b'^k > 0$ by an induction on k, but that is easy enough that you could just skip it as I did.)

6. [2 pts] Using whatever method you like, evaluate the limit

$$\lim_{n\to\infty} \frac{n + \cos(\frac{1}{n})}{(-1)^n + \frac{5}{\sqrt{n}}}$$

   or show the limit doesn't exist.

solution: This does not have a limit. To see this, consider first the subsequence consisting of even n: $n = 2k$. The sequence becomes

$$\frac{2k + \cos(\frac{1}{2k})}{1 + \frac{5}{\sqrt{2k}}} > \frac{2k-1}{2} > \frac{k}{2}$$

by Golden Rule, at least for $k > 25/2$. So this subsequence tends to $+\infty$. Next, consider the subsequence consisting of odd n: $n = 2k+1$. The sequence becomes

$$\frac{2k + 1 + \cos(\frac{1}{2k+1})}{-1 + \frac{5}{\sqrt{2k+1}}} < \frac{2k}{-1} < -2k$$

by Golden Rule. So this subsequence tends to $-\infty$. So the full sequence has no limit.

(To handle Golden Rule when numerator or denominator is negative, a safe idea is to first multiply both sides by -1 (changing the direction of inequality).)

7. [3 pts] Find
$$\lim_{n\to\infty} \frac{n^2}{5-n}$$
and prove your result, **using definition of limit**.

solution: I claim this diverges to $-\infty$. To see this, choose any $M < 0$. Take $N = -M$. Then for any $n > N$,
$$\frac{n^2}{5-n} = \frac{n}{\frac{5}{n} - 1} < \frac{n}{-1} = -n < -N = M$$
by Golden Rule. Therefore I'm done.

(to see whether $\frac{n^2}{5-n}$ goes to $+\infty$ or $-\infty$, just look at what happens for large n: the top grows much faster than the bottom, and the top is positive while the bottom is negative. So it grows to $-\infty$.)

8. [3 pts] Let $p(x) = 2x^2 - 2x - 1$. Prove that $|p(a) - p(b)| < 1$ whenever $a, b \in (-2, 1) \subseteq \mathbb{R}$ and $|a - b| < 0.1$.

solution: $|p(a) - p(b)| = \big|(2a^2 - 2a - 1) - (2b^2 - 2b - 1)\big| = \big|(2a^2 - 2b^2) - (2a - 2b)\big| \le \big|2a^2 - 2b^2\big| + \big|2a - 2b\big|$ by triangle inequality. This can be factored into $2\big|a-b\big|\,\big|a+b\big| + 2\big|a-b\big|$. Now substitute in $|a-b| < 0.1$ to get $< 0.2|a+b| + 0.2$. But $|a+b| \le |a| + |b| < |-2| + |-2| = 4$ since $a, b \in (-2, 1]$. Putting it all together, we get $|p(a) - p(b)| < 0.2 \cdot 4 + .2 = 1$.

(you want to overestimate $|p(a) - p(b)|$, and show your overestimate is still < 1. Because one of your conditions is $|a - b| < .1$, you want to use triangle inequality but in a way that keeps a and b together.)

9. [3 pts] Let $\mathbb{F}$ be an ordered field. Let $a, b \in \mathbb{F}$, with $a < b$. Prove that there are infinitely many elements in the interval (a, b). (HINT: divide something by $1 < 2 < 3 < \cdots$)

One solution: We know from class that there is an element in (a, b) – call it x_1. (We have a formula for such an element, namely $\frac{a+b}{2}$, but that's not important.) So $a < x_1 < b$. The same lemma tells us there is an element, call it x_2, in (a, x_2). Continuing in this way, we get a sequence x_n of elements in $\mathbb{F}$ such that $a < \cdots < x_3 < x_2 < x_1 < b$. Those x_n are our infinitely many elements.

Second solution: We know $b - a > 0$ and $0 < 1 < 2 <$ ..., so $0 < \cdots < \frac{1}{3} > \frac{1}{2} < \frac{1}{1}$ and hence $0 < \cdots < \frac{b-a}{3} < \frac{b-a}{2} < \frac{b-a}{1}$. Consider $x_n = a + \frac{b-a}{n}$. Then $a < \cdots < x_3 < x_2 < x_1 = b$. Those x_n are our infinitely many elements.

10. [2 pts] Must a bounded monotone increasing sequence of rational numbers converge to a rational number? Explain your answer.

solution: No. A simple example is $1, 1.4, 1.41, 1.414, 1.4142, ...$, which converges to $\sqrt{2}$: it is a bounded monotone increasing sequence of rational numbers which converges to an irrational number.

(The Monotone Convergence Theorem tells us that a bounded monotone increasing sequence of *real numbers* must converge to a *real number*. But that proof uses completeness, and $\mathbb{Q}$ is not complete.)

11. [3 pts] Choose any $L \in \mathbb{R}$, and let a_n be any sequence in $\mathbb{R}$. Suppose that for all $\epsilon > 0$ and all N, there exists an $n > N$ such that $|a_n - L| < \epsilon$. Show that there is a subsequence $b_k = a_{n_k}$ of a_n which converges to L. (HINT: For each $k \in \mathbb{N}$ define n_k recursively by taking $\epsilon = 1/k$ and $N = n_{k-1}$.)

solution: Start with $k = 1$. Let $\epsilon_1 = 1/1$ and $N_1 = 1$, then by hypothesis there is some $n > N_1$, call it n_1, such that $|a_{n_1} - L| < \epsilon_1$. Now choose $k = 2$ and pick $\epsilon_2 = 1/2$ and $N_2 = n_1$: then by hypothesis there is some $n > N_2$, call it n_2, such that $|a_{n_2} - L| < \epsilon_2$. Continue inductively: if we know what n_k is, then take $\epsilon_{k+1} = 1/(k+1)$ and $N_{k+1} = n_k$, so by induction we get an $n_{k+1} > N_{k+1}$ such that $|a_{n_{k+1}} - L| < \epsilon_{k+1}$.

Let $b_k = a_{n_k}$. I claim that this subsequence b_k converges to L. To see that, choose any $\epsilon > 0$. Take $N = 1/\epsilon$. Then for any $k < N$, $|b_k - L| = |a_{n_k} - L| < \epsilon_k = 1/k < \epsilon$, and we're done.

(Many of you claimed that $a_n \to L$. But the hypothesis says that SOME $n > N$ satisfies $|a_n - L| < \epsilon$, not ALL $n > N$ satisfies $|a_n - L| < \epsilon$,.)

Additional questions for Midterm 1 practise

1. Let A be the set of all even numbers in $\mathbb{N}$, B the set of all odd numbers in $\mathbb{N}$, C the set of all numbers in $\mathbb{N}$ divisible by 3, and $D = \{n \in \mathbb{N} \,|\, 3n < 10\}$. Identify $A \cap B$, $A \cup B$, $A \cap C$, $A \cap D$, and $\mathbb{N} \cup A$.

2. Given three logical statements P, Q, and R. By comparing truth tables, verify that

(a) NOT(P OR Q) = (NOT P) AND (NOT Q)
(b) P AND (Q AND R) = (P AND Q) AND R
(c) P OR (Q AND R) = (P OR Q) AND (P OR R)
(d) P AND (Q OR R) = (P AND Q) OR (P AND R)

3. Simplify: NOT($\forall x \in \mathbb{R}$ there exists a $y \in \mathbb{N}$ such that $\forall z \in \mathbb{Z}$, $x + y > z$). Is that true or false?

4(a) Let r and s be rational numbers. For each of the following expressions, determine whether it is necessarily a rational number: $r + s$, $r - s$, rs, r/s? (Explain why or provide counterexamples).

(b) Let r be rational, and s and t be irrational numbers. For each of the following expressions, determine whether it is necessarily an irrational number: $r + s$, rs, $s + t$, st? (Explain why or provide counterexamples).

5(a) Prove that no rational number cubed equals 2.

(b) Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

6. Prove that for all $n \in \mathbb{N}$, 3 divides $5^n - 2^n$.

7. Prove that, for all $n \in \mathbb{N}$,

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

8. Prove, using Peano's axioms, that $k \cdot (m + n) = k \cdot m + k \cdot n$.

9. Prove the Cancellation Laws: Let $\mathbb{F}$ be any field. Then:

(a) $\forall x, y, z \in \mathbb{F}$, $x + z = y + z \implies x = y$.

(b) $\forall x, y, z \in \mathbb{F}$, $z + x = z + y \implies x = y$.

(c) $\forall x, y, z \in \mathbb{F}$ where $z \neq 0$, $xz = yz \implies x = y$.

(d) $\forall x, y, z \in \mathbb{F}$ where $z \neq 0$, $zx = zy \implies x = y$.

10. Let $\mathbb{F}$ be any field. Prove:

(a) $\forall x \in F$ where $x \neq 0$, $(x^{-1})^{-1} = x$.

(b) $\forall x, y \in \mathbb{F}$, $-(x + y) = -x + (-y)$.

(c) $\forall x, y, z \in \mathbb{F}$, $x(y - z) = xy - xz$.

11. Let $\mathbb{F}$ be any field. Define $x^1 = x$ and $x^{n+1} = x^n x$. Prove $x^n x^m = x^{m+n}$ for all $m, n \in \mathbb{N}$. (HINT: use induction, or Peano's Axiom A3)

(Sketches of) solutions to the additional questions for Midterm 1 practise

1. $A \cap B = \{\}$ because no natural number is both even and odd.
$A \cup B = \mathbb{N}$ because every natural number is either even or odd.
$A \cap C$ is the set of all natural numbers divisible by 6.
$A \cap D = \{2\}$. $\mathbb{N} \cup A = \mathbb{N}$

2. I'll just do (a).

P	Q	P OR Q	NOT(P OR Q)	NOT P	NOT Q	(NOT P) AND (NOT Q)
t	t	t	f	f	f	f
t	f	t	f	f	t	f
f	t	t	f	t	f	f
f	f	f	t	t	t	t

3. "NOT($\forall x \in \mathbb{R}$ there exists a $y \in \mathbb{N}$ such that $\forall z \in \mathbb{Z}$, $x + y > z$)" is the same as "there is an $x \in \mathbb{R}$ such that, for all $y \in \mathbb{N}$, there is a $z \in \mathbb{Z}$ such that $x + y \leq z$".

That statement is true: take $x = 0$, then for all y, take $z = y$: we get $x + y = z$.

4(a) $r + s \in \mathbb{Q}$ by closure of addition: write $r = \frac{a}{b}$ and $s = \frac{c}{d}$, then $\frac{a}{b} + \frac{c}{d} = \frac{ad+bd}{bd} \in \mathbb{Q}$. A similar argument shows $r - s \in \mathbb{Q}$. $rs = \frac{ac}{bd} \in \mathbb{Q}$. But r/s won't be rational, if $s = 0$.

(b) Prove $r + s$ is irrational, using proof by contradiction: Suppose for contradiction that $r+s$ is rational. Then $(r+s)-r = s$ will be rational, by 4(a), but this contradicts s being irrational.

$r = 0$ and $s = \sqrt{2}$ is a counterexample to rs being irrational.
$s = \sqrt{2}$ and $t = -\sqrt{2}$ is a counterexample to $s + t$ being irrational: $s + t = 0 \in \mathbb{Q}$.
$s = \sqrt{2} = t$ is a counterexample to st being irrational: $st = 2 \in \mathbb{Q}$.

5(a) Suppose for contradiction that the cube-root of 2 is rational. Then it equals m/n where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. We can assume that at least one of m, n is odd (otherwise we keep dividing 2 from m and n). So $2 = m^3/n^3$, i.e. $m^3 = 2n^3$. So m must be even: say $m = 2\ell$. So $(2\ell)^3 = 2n^3$, i.e. $8\ell^3 = 2n^3$, i.e. $4\ell^3 = n^3$, so n is even. That contradicts that at least one of m, n must be odd.

(b) Here is one way to do it. Suppose for contradiction that $\sqrt{2}+\sqrt{3}$ is rational. Notice that $(\sqrt{3}-\sqrt{2})(\sqrt{2}+\sqrt{3}) = \sqrt{3}^2 - \sqrt{2}^2 = 1$ by difference of squares. So $\sqrt{3}-\sqrt{2} = 1/(\sqrt{2}+\sqrt{3})$ will also be rational (the inverse of a rational is rational). So $(\sqrt{2}+\sqrt{3})-(\sqrt{3}-\sqrt{2}) = 2\sqrt{2}$ is rational. Then $2^{-1}(2\sqrt{2}) = \sqrt{2}$ is rational (by 4(a) above). But this contradicts what we know about $\sqrt{2}$.

Another proof: If $\sqrt{2}+\sqrt{3}$ is rational, then so would be $(\sqrt{2}+\sqrt{3})^2 = 5 + 2\sqrt{6}$, so so would $\sqrt{6}$. You can prove $\sqrt{6}$ is irrational by the usual even,odd argument.

6. Prove this by induction. Certainly true for $n = 1$. Now suppose it is true for n: so 3 divides $5^n - 2^n$. Say $5^n - 2^n = 3L$. Is it true for $n+1$? I.e., does 3 divide $5^{n+1} - 2^{n+1}$? Well, note that $5^n = 3L + 2^n$, by our inductive hypothesis. OK, $5^{n+1} = 5 \cdot 5^n = (3+2) \cdot (3L + 2^n) =$

$9L + 3 \cdot 2^n + 6L + 2 \cdot 2^n = 15L + 3 \cdot 2^n + 2^{n+1}$. So $5^{n+1} - 2^{n+1} = 15L + 3 \cdot 2^n$, which is indeed a multiple of 3: the quotient is $5L + 2^n$.

7. We'll prove this by induction. True for $n = 1$. Assume true for n: so $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$. Is it true for $n+1$, i.e. is $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\cdots\left(1 - \frac{1}{(n+1)^2}\right) \stackrel{?}{=} \frac{n+2}{2(n+1)}$? Well,

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\cdots\left(1 - \frac{1}{(n+1)^2}\right) = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\cdots\left(1 - \frac{1}{n^2}\right)\left(1 - \frac{1}{(n+1)^2}\right)$$
$$= \frac{n+1}{2n}\left(1 - \frac{1}{(n+1)^2}\right)$$

using the inductive hypothesis. And

$$\frac{n+1}{2n}\left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+1}{2n}\left(\frac{n^2+2n+1-1}{(n+1)^2}\right) = \frac{n^2+2n}{2n(n+1)} = \frac{n+2}{2(n+1)}$$

as desired.

8. The easiest way to prove this is using commutativity of addition and the fact that on a previous assignment we proved the other distributivity. But for fun, let's try to prove this distributivity directly.

Let $K = \{n \in \mathbb{N} \mid k \cdot (m+n) = k \cdot m + k \cdot n \text{ for all } k, m \in \mathbb{N}\}$. First we want to prove that $1 \in K$, i.e. that $k \cdot (m+1) \stackrel{?}{=} k \cdot m + k \cdot 1$ for all $k, m \in \mathbb{N}$. Simplifying the LHS, we get $k \cdot (m+1) = k \cdot S(m) = k \cdot m + k$ by the definition of multiplication. And simplifying the RHS, we know $k \cdot 1 = k$, by definition of multiplication. So LHS=RHS, and $1 \in K$.

Now suppose $n \in K$, i.e. that $k \cdot (m+n) = k \cdot m + k \cdot n$ $\forall k, m \in \mathbb{N}$. We want to show that $S(n) \in K$, i.e. that $k \cdot (m + S(n)) \stackrel{?}{=} k \cdot m + k \cdot S(n)$ $\forall k, m \in \mathbb{N}$. LHS: $k \cdot (m+S(n)) = k \cdot S(m+n)$ by definition of addition, which equals $k \cdot (m+n) + k$ by definition of multiplication. But the inductive hypothesis says $k \cdot (m+n) + k = (k \cdot m + k \cdot n) + k$. Now simplify the RHS: $k \cdot m + k \cdot S(n) = k \cdot m + (k \cdot n + k)$ by definition of multiplication. Then LHS=RHS, by associativity of addition.

9. These are all easy. Let's prove (d) (the rest are similar). We can assume $zx = zy$ and also $z \neq 0$. That means z^{-1} exists. Multiply both sides of $zx = zy$ by z^{-1}: we get $z^{-1}(zx) = z^{-1}(zy)$. Use associativity of multiplication: $(z^{-1}z)x = (z^{-1}z)y$. Use definition of inverse: we get $1x = 1y$. Now use definition of multiplicative identity: $x = y$, and we're done.

10(a) To prove that the inverse of x^{-1} is x, we need to show $(x^{-1})x \stackrel{?}{=} 1$. But this is just the definition of inverse, and commutativity of multiplication.

(b) From a theorem proven in class, $-(x+y) = (-1) \cdot (x+y)$, which by distributivity equals $(-1) \cdot x + (-1) \cdot y$, which by the same theorem equals $(-x) + (-y)$, and we're done.

(c) $x(y-z) = x(y + (-1)z) = xy + x((-1)z) = xy + (x(-1))z = xy + ((-1)x)z = xy + (-1)(xz) = xy + (-xz) = xy - xz$.

11. We proved this in class. Let $K = \{n \in \mathbb{N} \,|\, x^m x^n = x^{m+n} \ \forall m \in \mathbb{N}\}$.

First, we need to show $1 \in K$, i.e. $x^m x^1 = x^{m+1} \ \forall m \in \mathbb{N}$. This is easy: x^{m+1} is defined to be $x^m x$, and $x^1 = x$.

Next, suppose $n \in K$, i.e. that $x^m x^n = x^{m+n} \ \forall m \in \mathbb{N}$. We need to show that $n + 1 \in K$, i.e. that $x^{m+n+1} \stackrel{?}{=} x^m x^{n+1}$. LHS: $x^{m+n+1} = x^{m+n} x$ by definition of exponential, and $x^{m+n} x = x^m x^n x$ by our inductive hypothesis. And $x^m x^n x = x^m x^{n+1}$ by definition of exponential. And that is RHS!

Additional questions for Midterm 2 practise

1. Let F be any ordered field. For this question, assume $a, b, c, d \in F$ and they obey $a < b < c < d$. (This means $a < b$ and $b < c$ and $c < d$.)What should $(a, c) \cup (b, d)$ equal? Prove your answer.

1.5. Suppose a and b are two real numbers such that $|a - b| \leq \frac{1}{2}$. Show that $|a^2 - b^2| \leq |b| + \frac{1}{4}$.

1.7. Find all real numbers x satisfying $|x+1|-|x-1|+|x|-|2x-6| \geq 2x-4$.

1.8. Show (using induction) that for every natural number n and any real numbers $a_1, \ldots, a_n$ the following is true:

$$|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$$

2. Let F be an ordered complete field, and A, B be subsets of F. Prove: If $A \subseteq B$, then $\sup A \leq \sup B$.

3. Prove that in $\mathbb{Q}$, the sup of $S = \{x \in \mathbb{Q} \,|\, x^2 < 2\}$ does not exist.

4. Let $\mathbb{F}$ be any ordered field, and $S \subseteq \mathbb{F}$ be any *finite* set. Prove that $\min S$ and $\max S$ both exist. (HINT: use induction)

5. Prove that any ordered field has infinitely many distinct numbers in it (except for the trivial field $\mathbb{F} = \{0\}$).

6. True or false. Determine whether the following statements are true or false. Prove your answers.

(a) If $x_n \leq y_n$ for all n then $\sup x_n \leq \sup y_n$.

(b) If $x_n < y_n$ for all n then $\sup x_n < \sup y_n$.

7. Suppose that x_n and y_n are two sequences such that $\inf x_n = -7$, $\sup x_n = 5$, $\inf y_n = 1$ and $\sup y_n = 3$.

(a) Find $\sup(x_n + 7)$, $\sup(3x_n)$, and $\sup(6 - x_n)$.

(b) Find $\sup(x_n^2)$.

(c) Show that $\sup(x_n + y_n) \leq 8$ and $\inf(x_n + y_n) \geq -6$.

(d) Show (by finding examples of appropriate sequences x_n and y_n) that the inequalities in (c) need not be equalities.

8. Prove the following limits using the definition of convergence.

(a) $\frac{10000n}{3n^2+1} \to 0$

(b) $\frac{n^2-n}{n^2+1} \to 1$

9. Compute the limits of the following sequences (however you like):

(a) $\frac{3n^4-2n^2+7}{5+3n-3n^3-2n^4}$

(b) $\frac{243+3n^5-n^{10}}{n^{17}-156n^{16}}$

(c) $\frac{(n+1)(n+2)(n+3)}{n^3}$

(d) $\frac{n+(-1)^n}{n-(-1)^n}$

(e) $\sqrt{n^2+1} - n$

(f) $\frac{3n-n^5}{1+n+n^4\sqrt{n}}$

(g) $\frac{1}{\sqrt{n}} \cos(n^2+n)$

10. Which of these sequences are monotone increasing? monotone decreasing? For those which are monotonic, compute the limit.

(a) $a_n = 1/\sqrt{n}$

(b) $a_n = \cos(n)/n$

(c) $a_n = -n^2$

11. Suppose $a_n \to L$. Let b_n be a new sequence defined by $b_1 = 0$ and $b_n = a_{n-1}$. Let c_n be a new sequence defined by $c_n = a_{n+1}$. Prove that $\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} c_n$.

12. Let $a_1 = 1$ and $a_{n+1} = (a_n + 2)/2$ for all $n \in \mathbb{N}$. Show that a_n is strictly increasing and bounded above by 2. Find the limit.

13. Let $a_n = 1/n$. Find a sequence b_n such that $b_n \to \infty$ and

(a) $a_n b_n \to 0$

(b) $a_n b_n \to \infty$

(c) $a_n b_n \to 7$

(d) $\lim_{n\to\infty} a_n b_n$ does not exist (not even ∞).

14. Suppose that a_n is increasing and has a convergent subsequence. Show that a_n is convergent.

(Sketches of) solutions to Midterm 2 practise questions

1. It should equal (a, d). To show this, we need to show that $(a, c) \cup (b, d) \subseteq (a, d)$ and $(a, c) \cup (b, d) \supseteq (a, d)$.

To show $(a, c) \cup (b, d) \subseteq (a, d)$, let $x \in (a, c) \cup (b, d)$. Then either $x \in (a, c)$ (i.e. $a < x < c$) or $x \in (b, d)$ (i.e. $b < x < d$). If $a < x < c$, then $a < x < d$ (since $c < d$). If $b < x < d$, then $a < x < d$ (since $a < b$). In both cases we get $x \in (a, d)$.

To show $(a, c) \cup (b, d) \supseteq (a, d)$, let $x \in (a, d)$, i.e. $a < x < d$. If $x < c$, then $x \in (a, c)$. If $x \geq c$, then $x \in (b, d)$. In either case we get $x \in (a, c) \cup (b, d)$.

1.5. $|a^2 - b^2| = |a + b|\,|a - b|$, so we need to bound $|a + b|$. But the triangle inequality tells us $|a| - |b| \leq |a - b| \leq \frac{1}{2}$, so $|a| \leq |b| + \frac{1}{2}$. And the triangle inequality gives $|a + b| \leq |a| + |b| \leq 2|b| + \frac{1}{2}$. Putting all this together gives $|a^2 - b^2| \leq (2|b| + \frac{1}{2})\frac{1}{2} = |b| + \frac{1}{4}$, as desired.

1.7. You could do this by $2^4 = 16$ cases, but it is simpler than that. Note that $x + 1 = 0$ iff $x = -1$; $x - 1 = 0$ iff $x = 1$; $x = 0$; $2x - 6 = 0$ iff $x = 3$. So there are 4 values of x where one of these terms change sign. We can order the cases based on how x compares to 3, 1, 0 and -1.

Case 1. When $x \geq 3$, then $2x - 6 \geq 0$, $x > 0$, $x - 1 > 0$ and $x + 1 > 0$, so for these x the inequality becomes $2x - 4 \leq (x + 1) - (x - 1) + (x) - (2x - 6) = -x + 8$, i.e. $3x \leq 12$, i.e. $x \leq 4$. So the first case has solutions $x \in [3, 4]$.

Case 2. When $1 \leq x < 3$, then $2x - 6 < 0$, $x > 0$, $x - 1 \geq 0$ and $x + 1 > 0$, so for these x the inequality becomes $2x - 4 \leq (x + 1) - (x - 1) + (x) + (2x - 6) = 3x - 4$, i.e. $0 \leq x$. So the 2nd case has solutions $x \in [1, 3)$.

Case 3. When $0 \leq x < 1$, then $2x - 6 < 0$, $x \geq 0$, $x - 1 < 0$ and $x + 1 > 0$, so for these x the inequality becomes $2x - 4 \leq (x + 1) + (x - 1) + (x) + (2x - 6) = 5x - 6$, i.e. $2 \leq 3x$, i.e. $\frac{2}{3} \leq x$. So the 3rd case has solutions $x \in [\frac{2}{3}, 1)$.

Case 4. When $-1 \leq x < 0$, then $2x - 6 < 0$, $x < 0$, $x - 1 < 0$ and $x + 1 \geq 0$, so for these x the inequality becomes $2x - 4 \leq (x + 1) + (x - 1) - (x) + (2x - 6) = 3x - 6$, i.e. $2 \leq x$. So the 4th case has no solutions.

Case 5. When $x < -1$, then $2x - 6 < 0$, $x < 0$, $x - 1 < 0$ and $x + 1 < 0$, so for these x the inequality becomes $2x - 4 \leq -(x + 1) + (x - 1) - (x) + (2x - 6) = x - 8$, i.e. $x \leq -4$. So the 5th case has solutions $x \in (-\infty, -4]$.

Thus the complete list of solutions is $x \in [3,4] \cup [1,3) \cup [\frac{2}{3},1) \cup (-\infty,-4] = [\frac{2}{3},4] \cup (-\infty,-4]$.

1.8. Certainly true for $n=1$: $\left|\sum_{i=1}^{1} a_i\right| = |a_1| = \sum_{i=1}^{1} |a_i|$.

Inductive hypothesis: suppose for some n that $\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|$. Now consider $n+1$ real numbers $a_1,\ldots,a_{n+1}$. We want to show $\left|\sum_{i=1}^{n+1} a_i\right| \leq \sum_{i=1}^{n+1} |a_i|$.

Look at LHS:

$$\Big|\sum_{i=1}^{n+1} a_i\Big| = \Big|\left(\sum_{i=1}^{n} a_i\right) + a_{n+1}\Big| \leq \Big|\sum_{i=1}^{n} a_i\Big| + |a_{n+1}| \leq \Big(\sum_{i=1}^{n} |a_i|\Big) + |a_{n+1}|$$

where the second last inequality is the usual triangle inequality, and the last inequality is the inductive hypothesis. The RHS of this new inequality is the same as the RHS of what we want to prove.

2. This is easy. If the field is ordered and complete, then we know that supremums always exist. Write $\alpha = \sup A$ and $\beta = \sup B$. Note that β will certainly be an upper bound for A: for any $a \in A$, we know a is also in B, so $a \leq \beta$, and hence β is an upper bound for A. But α is the *least* upper bound for A, so α is $\leq$ any other upper bound. So $\alpha \leq \beta$ and we're done.

3. Suppose for contradiction that $\sup S$ exists in $\mathbb{Q}$. That means that there exists some $s \in \mathbb{Q}$ such that s is an upper bound of S, and if u is any other upper bound, then $s \leq u$.

Suppose first that $s^2 > 2$. We know that the interval $(0, \frac{s^2-2}{2s})$ is nonempty in $\mathbb{Q}$ (why?). So choose any rational number $\epsilon \in (0, \frac{s^2-2}{2s})$. Define $s' = s - \epsilon$; then $s' \in \mathbb{Q}$ because s and ϵ are. Also, $s' < s$ (since $\epsilon > 0$) and $0 < s'$ (since $\epsilon < s$ (why?)), and

$$(s')^2 = (s-\epsilon)^2 = s^2 - 2s\epsilon + \epsilon^2 > s^2 - 2s\epsilon > s^2 - 2s\frac{s^2-2}{2s} = 2$$

. Hence s' is an upper bound of S (why?) but $s' < s$, so we get a contradiction to s being the least upper bound.

Suppose next that $s^2 < 2$. Then $s \geq 1.4$ since $1.4 \in S$. We know that the interval $(0, \frac{2-s^2}{3s})$ is nonempty in $\mathbb{Q}$ (why?). So choose any rational number $\epsilon \in (0, \frac{2-s^2}{3s})$. Then $\epsilon < \frac{2-1.4^2}{3\cdot 1.4} < 0.01$. Define $r = s + \epsilon$; then $r \in \mathbb{Q}$ because

s and ϵ are. Also, $r > s$ (since $\epsilon > 0$), and

$$r^2 = (s+\epsilon)^2 = s^2 + 2s\epsilon + \epsilon^2 < s^2 + 2s\epsilon + \epsilon s < s^2 + 3s\frac{2-s^2}{3s} = 2$$

since $\epsilon < 0.01 < s$. Hence $r \in S$, so s is not an upper bound of S, a contradiction.

The only remaining possibility is that $s^2 = 2$. But in a previous class we ruled out the possibility that $\sqrt{2}$ is rational. Therefore in $\mathbb{Q}$, the sup of $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ does not exist.

4. "Without loss of generality" we will consider only maximum. Minimum is done in exactly the same way.

Suppose S has just 1 element, say $S = \{s\}$. Then the maximum of S is just s.

Inductive hypothesis: suppose that any $S \subseteq \mathbb{F}$ with exactly n elements in it, so $S = \{s_1, s_2, \ldots, s_n\}$ has a maximum. Now consider a set S with $n+1$ elements, say $S = \{s_1, s_2, \ldots, s_{n+1}\}$. We want to show that it too must have a maximum.

Consider the set $S' = \{s_1, s_2, \ldots, s_n\}$ obtained from S by throwing away the element s_{n+1}. Then S' has exactly n elements, so by inductive hypothesis it has a maximum, say s_i. Then if $s_i > s_{n+1}$, we know $\max S = s_i$. And if instead $s_i < s_{n+1}$, we know $\max S = s_{n+1}$. In either case, we get that S has a maximum.

5. This is easy. We know that in any ordered field (except for the stupid field $\mathbb{F} = \{0\}$), there is a copy of $\mathbb{N}$: $1 < 2 < 3 < \cdots$. These are all distinct, so the field has infinitely many numbers in it. (Of course, there are lots of finite fields, but they can't be ordered.)

6(a) This is **True**. If $\sup y_n = \infty$, there is nothing to prove. If $s = \sup y_n$ is finite, then (amongst other things) s is an upper bound for the y_n, which means s is an upper bound for the x_n (which are smaller than the y_n), which means $\sup x_n \leq s$ (since $\sup x_n$ is the least of the upper bounds of the x_n).

(b) This is **False**. A simple counterexample is $x_n = -1/n$ for all n, while $y_n = 0$. Then $x_n < y_n$ for all n, but $\sup x_n = 0 = \sup y_n$.

7(a) $\sup(x_n + 7) = 7 + \sup x_n = 12$ and $\sup(3x_n) = 3\sup x_n = 15$, and $\sup(6 - x_n) = 6 - \inf x_n = 13$ (since $\sup(-S) = -\inf S$ (why?)).

(b) The function $f(x) = x^2$ increases for $x > 0$ and decreases for $x < 0$. In other words, for positive x, x^2 is biggest for the biggest x, while for negative x, x^2 is biggest for the smallest ('most negative') x. Therefore $\sup(x_n^2) = \max\{(\sup x_n)^2, (\inf x_n)^2\} = \max\{25, 49\} = 49$.

(c) For each k, $x_k + y_k \leq \sup x_n + \sup y_n = 5 + 3 = 8$, so $\sup(x_n + y_n) \leq 8$. Similarly, $\inf(x_n + y_n) \geq -7 + 1 = -6$.

(d) Choose $x_1 = -7, x_2 = 5$, and $x_n = 0$ for all $n \geq 3$. Choose $y_1 = 3, y_2 = 1, y_n = 2$ for all $n \geq 3$. Then $\inf x_n = -7$, $\sup x_n = 5$, $\inf y_n = 1$ and $\sup y_n = 3$. Moreover, $x_n + y_n = -4, 6, 2, 2, 2, 2, \ldots$ so $\sup(x_n + y_n) = 6 < 8$ and $\inf(x_n + y_n) = -4 > -6$.

8(a) Choose any $\epsilon > 0$. Let $N = \frac{10000}{3\epsilon}$. Then for any $n > N$,

$$\frac{10000n}{3n^2+1} < \frac{10000n}{3n^2} = \frac{10000}{3n} < \frac{10000}{3N} = \frac{10000}{3\frac{10000}{3\epsilon}} = \epsilon$$

and we're done!

How did i do this? I did this secret work: I need to know how big n has to get, so that $\frac{10000n}{3n^2+1} < \epsilon$. It is hard to solve for n in terms of ϵ directly, so i simplify the inequality. There are lots of ways to do this, but the simplest is to write $\frac{10000n}{3n^2+1} < \frac{10000n}{3n^2}$.

GOLDEN RULE of FRACTION INEQUALITIES: Assuming the numerator(=top) and denominator(=bottom) of a fraction are both positive, a fraction gets bigger if you make the numerator bigger and/or make the denominator smaller.

Here we made the denominator smaller: $3n^2 + 1 > 3n^2$. The point is, if i can make sure $\frac{10000n}{3n^2} < \epsilon$, then i know for sure that $\frac{10000n}{3n^2+1} < \epsilon$. And it is easy to make sure $\frac{10000n}{3n^2} < \epsilon$: cancelling an n from top and bottom, and moving n and ϵ to other sides, i get $\frac{10000}{3\epsilon} < n$. So if i choose $N = \frac{10000}{3\epsilon}$, and i require that $n > N$, then (by reversing the logic) i know for sure that $\frac{10000n}{3n^2} < \epsilon$ and hence that $\frac{10000n}{3n^2+1} < \epsilon$.

(b) Choose any $\epsilon > 0$. Let $N = \frac{2}{\epsilon}$. Then for any $n > N$,

$$\left|\frac{n^2-n}{n^2+1} - 1\right| = \left|\frac{n^2-n-n^2-1}{n^2+1}\right| = \left|\frac{-n-1}{n^2+1}\right| = \frac{n+1}{n^2+1} < \frac{2n}{n^2} = \frac{2}{n} < \frac{2}{N} = \epsilon$$

by making both numerator bigger and denominator smaller of $\frac{n+1}{n^2+1}$. And we're done!

How did i do this? In other words, how did i know what to choose for N? I did the error calculation

$$|a_n - L| = \left|\frac{n^2-n}{n^2+1} - 1\right| = \cdots = \frac{n+1}{n^2+1}$$

as above. I want to know how big n has to be, so that the error is less than ϵ. To do this, i want to simplify the error $\frac{n+1}{n^2+1}$ so that it is a little bigger and much simpler: i chose $\frac{n+1}{n^2+1} < \frac{2n}{n^2}$ (there are other simplifications). This simplifies to $\frac{2}{n}$. So the error between $L = 1$ and $a_n = \frac{n^2-n}{n^2+1}$ is less than $\frac{2}{n}$. So if i make sure $\frac{2}{n} < \epsilon$, then i know that $\left|\frac{n^2-n}{n^2+1} - 1\right| < \epsilon$. That's why i chose $N = \frac{2}{\epsilon}$.

9(a)

$$\frac{3n^4 - 2n^2 + 7}{5 + 3n - 3n^3 - 2n^4} = \frac{3 - \frac{2}{n^2} + \frac{7}{n^4}}{\frac{5}{n^4} + \frac{3}{n^3} - \frac{3}{n} - 2} \to \frac{3 - 0 + 0}{0 + 0 - 0 - 2} = -\frac{3}{2}$$

9(b)

$$\frac{243 + 3n^5 - n^{10}}{n^{17} - 156n^{16}} = \frac{\frac{243}{n^{17}} + \frac{3}{n^{12}} - \frac{1}{n^7}}{1 - \frac{156}{n}} \to \frac{0 + 0 - 0}{1 - 0} = 0$$

9(c)

$$\frac{(n+1)(n+2)(n+3)}{n^3} = \frac{n+1}{n} \cdot \frac{n+2}{n} \cdot \frac{n+3}{n} = \left(1+\tfrac{1}{n}\right)\left(1+\tfrac{2}{n}\right)\left(1+\tfrac{3}{n}\right) \to (1+0)(1+0)(1+0) = 1$$

9(d)

$$\frac{n + (-1)^n}{n - (-1)^n} = \frac{1 + \frac{(-1)^n}{n}}{1 - \frac{(-1)^n}{n}} \to \frac{1+0}{1-0} = 1$$

9(e)

$$\begin{aligned} \lim\left(\sqrt{n^2+1} - \sqrt{n^2}\right) &= \lim\left(\sqrt{n^2+1} - \sqrt{n^2}\right)\left(\frac{\sqrt{n^2+1} + \sqrt{n^2}}{\sqrt{n^2+1} + \sqrt{n^2}}\right) \\ &= \lim \frac{n^2 + 1 - n^2}{\sqrt{n^2+1} + \sqrt{n^2}} = \lim \frac{1}{\sqrt{n^2+1} + \sqrt{n^2}} \to 0. \end{aligned}$$

Hence

$$\lim(\sqrt{n^2+1}-n)=0.$$

9(f) Note that we can write the given sequence in the form $x_n y_n$, where $x_n = \sqrt{n}$ and $y_n = \frac{\frac{3}{n^4\sqrt{n}}-1}{\frac{1}{n^4\sqrt{n}}+\frac{1}{n^3\sqrt{n}}+1}$. Note also that $x_n \to +\infty$ while $y_n \to \frac{0-1}{0+0+1} = -1$, so that (y_n) converges to a negative number. It follows that $x_n y_n \to -\infty$.

9(g)

$$\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}} = \frac{(-1)^n \cdot \left(\frac{2}{3}\right)^n \cdot \frac{1}{3}+\frac{1}{3}}{(-1)^{n+1}\left(\frac{2}{3}\right)^{n+1}+1} \to \frac{0+\frac{1}{3}}{0+1} = \frac{1}{3}$$

9(h) Let $x_n = \frac{1}{\sqrt{n}}\cos(n^2+n)$. Then $|x_n| = \frac{\left|\cos(n^2+n)\right|}{\sqrt{n}} \leq |1\sqrt{n}| \to 0$. It follows that $|x_n| \to 0$, hence $x_n \to 0$.

10(a) The sequence $a_n = -1/\sqrt{n}$ is monotone increasing: $\sqrt{n} < \sqrt{n+1}$, so $1/\sqrt{n+1} < 1/\sqrt{n}$, so $-1/\sqrt{n+1} > -1/\sqrt{n}$. The sequence converges to 0.

(b) $a_n = \cos(n)/n$ is not monotonic, because about half of the a_n are negative and the other have are positive.

(c) $a_n = -n^2$ is monotone decreasing: $n^2 < (n+1)^2$ so $-n^2 > -(n+1)^2$. The limit is $-\infty$.

11. I'll do the proof for L finite; you can change this so that it works for L infinite.

Choose any $\epsilon > 0$. Because $a_n \to L$, there is an N' such that $|a_n - L| < \epsilon$ for all $n > N'$. Let $N = N'+1$. Then for $n > N$, $n-1 > N-1 = N'$ so $|b_n - L| = |a_{n-1} - L| < \epsilon$. Thus $b_n \to L$.

A similar argument shows $c_n \to L$. For any $\epsilon > 0$, choose $N = N'-1$. Then $n > N$ implies $n+1 > N+1 = N'$ so $|c_n - L| = |a_{n+1} - L| < \epsilon$.

12. If $x < 2$, then $x+2 < 2+2 = 4$ so $(x+2)/2 < 4/2 = 2$. This means that $a_n < 2$ for all n (prove this by induction on n). Hence a_n is bounded above by 2.

Suppose $x \geq (x+2)/2$. Then $x/2 \geq 1$, i.e. $x \geq 2$. Hence if $x < 2$, then $x < (x+2)/2$. That means that $a_n < a_{n+1}$ for all n. So a_n is monotone increasing.

By the monotone Convergence theorem, a_n converges to some finite number L. Now take the limit as $n \to \infty$ of both sides

13(a) Take $b_n = \sqrt{n}$. Then $a_n b_n = \frac{1}{n}\sqrt{n} = \frac{1}{\sqrt{n}} \to 0$

(b) Take $b_n = n^2$. Then $a_n b_n = \frac{1}{n}n^2 = n \to \infty$

(c) Take $b_n = 7n$. Then $a_n b_n = 7 \to 7$

(d) Take $b_n = \sqrt{n}$ when n is even, and $b_n = 7n$ when n is odd. Then $b_n \to \infty$ as desired. Also, $a_n b_n$ does not have a limit: the subsequence $c_k = a_{2k}b_{2k} = \frac{1}{2k}\sqrt{2k} = \frac{1}{\sqrt{2k}}$ of $a_n b_n$ converges to 0 as $k \to \infty$, while the subsequence $d_k = a_{2k+1}b_{2k+1} = 7$ converges to 7.

14. Let a_{n_k} be the convergent subsequence. By the Monotone Convergence Theorem, $\lim a_n = \sup a_n$ (which could be a number or infinity); also $\lim a_n = \lim a_{n_k}$. Since the latter is a real number (not infinity), hence $\lim a_n$ is a number.

Additional questions for Final Exam practise

(some of these involve material from the final week)

Question 0. Calculate $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ for the following examples:

(a) $a_n = \frac{(-1)^n n+8}{n+1}$
(b) $a_n = 3 + \cos(\pi n/4)$
(c) $a_n = n - 3(-1)^n\sqrt{n}$

Question 1.1. Prove the following limits using the definition of convergence.

(a) $\frac{10000n}{3n^2+1} \to 0$

(b) Show that $x_n \to 0$ where $x_n = \begin{cases} 1/\sqrt{n} & \text{if } n \text{ is even and} \\ 100/n^2 & \text{if } n \text{ is odd} \end{cases}$

(c) $\frac{n^2-1}{n^2+1} \to 1$

(d) $\lim_{n\to\infty} \frac{1}{\sqrt[m]{n^2+1}+\sqrt[m]{n^2}} = 0$, where $m \in \mathbb{N}$.

Question 1.2. Compute the limits of the following sequences (however you like):

(a) $\frac{3n^4-2n^2+7}{5+3n-3n^3-2n^4}$

(b) $\frac{243+3n^5-n^{10}}{n^{17}-156n^{16}}$

(c) $\frac{(n+1)(n+2)(n+3)}{n^3}$

(d) $\frac{n+(-1)^n}{n-(-1)^n}$

(e) $\frac{n^2}{n!}$

(f) $\sqrt[8]{n^2+1} - \sqrt[4]{n}$

(g) $\frac{3n-n^5}{1+n+n^4\sqrt{n}}$

(h) $\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}$

(i) $\frac{1}{\sqrt{n}}\cos(n^2+n)$

(j) $\frac{(n!)^2}{(2n)!}$

Question 1.3. Use Cauchy's criterion to prove that the sequence $x_n = (-1)^n\left(1-\frac{1}{n}\right)$ diverges.

Question 1.4. Find the limit if it exists (use any method).

(a) $\lim\limits_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$

(b) $\lim\limits_{x\to 0} x^3\sin\frac{1}{x^2}$

(c) $\lim\limits_{x\to 0}\sin^2 x\cos^3 x$

(d) $\lim\limits_{x\to\sqrt{2}\pi}\frac{\sin\sqrt{x^2+2\pi^2}}{1+\cos x}$

Question 1.5. If $f(x)$ is continuous at $x=a$, and $f(a)\neq 0$, prove that there is an interval (c,d) such that $a\in(c,d)$ and $f(x)\neq 0$ for all $x\in(c,d)$.

Question 2.1. Use the definition to show that the function $f(x)=\sqrt{x}$ is continuous on $(0,\infty)$.

Question 2.2. Prove that the following are equivalent: (i.e. (a) is true iff (b) is true iff(c) is true).

(a) $\lim\limits_{x\to c^-} f(x)=L$

(b) For every sequence x_n, if $x_n\to c$ and $x_n<c$ for all n then $f(x_n)\to L$.

(c) For every strictly increasing sequence x_n, if $x_n\to c$ then $f(x_n)\to L$.

Question 3. Find the limit if it exists (show your work).

(a) $\lim_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$

(b) $\lim_{x\to 0} x^3 \sin\frac{1}{x^2}$

(c) $\lim_{x\to 0^+} \frac{1}{\sqrt{x}}$

(d) $\lim_{x\to 3^-} \frac{x+3}{x-3}$

(e) $\lim_{x\to 0} \sin^2 x \cos^3 x$

(f) $\lim_{x\to \sqrt{2}\pi} \frac{\sin\sqrt{x^2+2\pi^2}}{1+\cos x}$

(g) $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$

(h) $\lim_{x\to 0} \frac{\tan x}{\sin 2x}$

(i) $\lim_{x\to 0} \frac{1-\cos 2x}{\sin^2 x}$

(j) $\lim_{x\to 0} \frac{\cos x-\cos 3x}{x^2}$

(k) $\lim_{x\to \pi} (\pi - x)\text{cotan}(2x)$

(l) $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} x^2+3 & \text{if } x > 0, \\ \frac{3}{1-x} & \text{if } x < 0 \end{cases}$$

(m) $\lim_{n\to\infty} x_n$

where $x_n = \cos\frac{1}{n^2+1}$

(n) $\lim_{x\to 0} \sqrt{\frac{\sin x}{x}}$

4. [3 pts] Find a function $f : \mathbb{R} \to \mathbb{R}$ which is not continuous at some point, but which satisfies the conclusion of the Intermediate Value Theorem: i.e., for any points $a < b$, and any y between $f(a)$ and $f(b)$, there is some x, $a \leq x \leq b$, such that $f(x) = y$. (Hint: one such function can be built from sine)

5. [3 pts] Use the Intermediate Value Theorem to prove that the equation $x^5 - 3x + 1 = 0$ has at least three real roots.

6. [3 pts] Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ n & \text{if } x = m/n \text{ is reduced, } m \in \mathbb{Z},\ n \in \mathbb{N} \end{cases}$$

(m/n 'reduced' means m, n don't have any common divisors). For example, $f(-1/2) = f(3/2) = 2$ and $f(\sqrt{2}) = 0$. Prove that f is unbounded on any interval $[a, b]$, where $a < b$.

7. (a) [3 pts] Suppose $f(x)$ is continuous on the interval $[a, b]$, and $f(x) > 0$ whenever $a \leq x \leq b$. Prove that the supremum $\sup\{1/f(x) : a \leq x \leq b\}$ of all values of $1/f(x)$ for $x \in [a, b]$, is finite.

(b) [1 pt] Show that if we drop the requirement that $f(x)$ be continuous, but keep everything else, then the conclusion fails: i.e. find a function $f(x)$ defined and positive everywhere in $[a, b]$, but $\sup\{1/f(x) : a \leq x \leq b\} = \infty$.

(c) [1 pt] Show that if we drop the requirement that the interval be closed, but keep everything else, then the conclusion fails: i.e. i.e. find a function $f(x)$ defined, continuous and positive everywhere in $(a, b]$, but $\sup\{1/f(x) : a \leq x \leq b\} = \infty$.

8.1. [5 pts] Determine which of the following functions are invertible. For those which are invertible, compute the inverse function.

(a) $f\colon [0, 2) \to [0, 2)$ given by

$$f(x) = \begin{cases} x + 1 & \text{if } 0 \leq x < 1 \\ x - 1 & \text{if } 1 \leq x < 2 \end{cases}$$

(b) $f(x) = x(1 - x)$ on $\mathbb{R}$.

8.2. [4 pts] Prove that every invertible continuous function on an interval (a, b) is either strictly increasing or strictly decreasing there.

Solutions to PilesofQuestions3

Question 0(a) Noice that $a_n \leq \frac{n+8}{n+1}$ and $\frac{n+8}{n+1} \to 1$. So no subsequence of a_n can ever converge to something larger than 1. But there is a subsequence converging to 1: namely the subsequence of even n. So $\limsup_{n\to\infty} a_n = 1$ (since it equals the largest limit of any convergent subsequence).

Likewise, $a_n \geq \frac{-n+8}{n+1}$ and $\frac{-n+8}{n+1} \to -1$. So no subsequence of a_n can ever converge to something smaller than -1. But there is a subsequence converging to -1: namely the subsequence of odd n. So $\liminf_{n\to\infty} a_n = -1$ (since it equals the smallest limit of any convergent subsequence).

Qu.0(b) $-1 \leq \cos(x) \leq 1$, so $2 = 3 - 1 \leq a_n \leq 3 + 1 = 4$. But there is a subsequence of a_n converging to 4: take $n_k = 8, 16, 24, 32, ...$ So $\limsup_{n\to\infty} a_n = 4$. And there is a subsequence of a_n converging to 2: take $n_k = 4, 12, 20, 28, ...$ So $\liminf_{n\to\infty} a_n = 2$.

Qu.0(c) Again we note that $n - 3\sqrt{n} \leq a_n \leq n + 3\sqrt{n}$. But both $n - 3\sqrt{n} \to \infty$ and $n + 3\sqrt{n} \to \infty$. Therefore $a_n \to \infty$, so $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \infty$.

1.1(a) Given an arbitrary $\epsilon > 0$. Take N be any natural number greater that $\frac{1000}{3\epsilon}$. Suppose that $n \geq N$. Then $n > \frac{1000}{3\epsilon}$. It follows that

$$|x_n - c| = \left|\frac{10000n}{3n^2+1} - 0\right| = \frac{10000n}{3n^2+1} \leq \frac{10000n}{3n^2} \leq \frac{10000}{3n} < \epsilon.$$

1.1(b) Given an arbitrary $\epsilon > 0$. Let N be any integer greater than both $\frac{1}{\epsilon^2}$ and $\frac{10}{\sqrt{\epsilon}}$. If $n \geq N$ then $n \geq \frac{1}{\epsilon^2}$ and $n \geq \frac{10}{\sqrt{\epsilon}}$. Now,

- if n is even, then $|x_n - 0| = \frac{1}{\sqrt{n}} < \epsilon$ because $n \geq \frac{1}{\epsilon^2}$; while
- if n is even, then $|x_n - 0| = \frac{100}{n^2} < \epsilon$ because $n \geq \frac{10}{\sqrt{\epsilon}}$.

In either case, $|x_n - 0| < \epsilon$.

1.1(c) Given an arbitrary $\epsilon > 0$. Let N be any integer greater than $\sqrt{\frac{2}{\epsilon}}$. If $n \geq N$ then $n > \sqrt{\frac{2}{\epsilon}}$, so that

$$|x_n - 1| = \left|\frac{n^2-1}{n^2+1} - 1\right| = \left|\frac{n^2-1+(n^2+1)}{n^2+1}\right| = \left|\frac{-2}{n^1+1}\right| = \frac{2}{n^2+1} < \frac{2}{n^2} < \epsilon.$$

1.1(d) We claim that the sequence converges to zero. Indeed, given and $\epsilon > 0$, we have

$$\left| \frac{1}{\sqrt[m]{n^2+1} + \sqrt[m]{n^2}} - 0 \right| \le \frac{1}{2\sqrt[m]{n^2}}.$$

The latter expression is less than ϵ provided that

$$n > \frac{1}{\left(\sqrt{2\epsilon}\right)^m}.$$

1.2(a)

$$\frac{3n^4 - 2n^2 + 7}{5 + 3n - 3n^3 - 2n^4} = \frac{3 - \frac{2}{n^2} + \frac{7}{n^4}}{\frac{5}{n^4} + \frac{3}{n^3} - \frac{3}{n} - 2} \to \frac{3 - 0 + 0}{0 + 0 - 0 - 2} = -\frac{3}{2}$$

1.2(b)

$$\frac{243 + 3n^5 - n^{10}}{n^{17} - 156n^{16}} = \frac{\frac{243}{n^{17}} + \frac{3}{n^{12}} - \frac{1}{n^7}}{1 - \frac{156}{n}} \to \frac{0 + 0 - 0}{1 - 0} = 0$$

1.2(c)

$$\frac{(n+1)(n+2)(n+3)}{n^3} = \frac{n+1}{n} \cdot \frac{n+2}{n} \cdot \frac{n+3}{n} = \left(1+\tfrac{1}{n}\right)\left(1+\tfrac{2}{n}\right)\left(1+\tfrac{3}{n}\right) \to (1+0)(1+0)(1+0) = 1$$

1.2(d)

$$\frac{n + (-1)^n}{n - (-1)^n} = \frac{1 + \frac{(-1)^n}{n}}{1 - \frac{(-1)^n}{n}} \to \frac{1+0}{1-0} = 1$$

1.2(e) Use ratio test: $((n+1)^2/(n+1)!)/(n^2/n!) = (1+1/n)/n \to 0$ so $\frac{n^2}{n!} \to 0$

1.2(f)

$$\lim\left(\sqrt[8]{n^2+1} - \sqrt[8]{n^2}\right) = \lim\left(\sqrt[8]{n^2+1} - \sqrt[8]{n^2}\right)\left(\frac{\sqrt[8]{n^2+1} + \sqrt[8]{n^2}}{\sqrt[8]{n^2+1} + \sqrt[8]{n^2}}\right)$$

$$= \lim \frac{\sqrt[4]{n^2+1} - \sqrt[4]{n^2}}{\sqrt[8]{n^2+1} + \sqrt[8]{n^2}} = \lim \frac{\sqrt{n^2+1} - \sqrt{n^2}}{\left(\sqrt[8]{n^2+1} + \sqrt[8]{n^2}\right)\left(\sqrt[4]{n^2+1} + \sqrt[4]{n^2}\right)}$$

$$= \lim \frac{1}{\left(\sqrt[8]{n^2+1} + \sqrt[8]{n^2}\right)\left(\sqrt[4]{n^2+1} + \sqrt[4]{n^2}\right)\left(\sqrt{n^2+1} + \sqrt{n^2}\right)}$$

$$= \lim\left(\frac{1}{\sqrt[8]{n^2+1} + \sqrt[8]{n^2}}\right) \lim\left(\frac{1}{\sqrt[4]{n^2+1} + \sqrt[4]{n^2}}\right) \lim\left(\frac{1}{\sqrt{n^2+1} + \sqrt{n^2}}\right).$$

Each of the limits in the last line tends to 0. Hence

$$\lim(\sqrt[8]{n^2+1}-\sqrt[4]{n})=0.$$

1.2(g) Note that we can write the given sequence in the form $x_n y_n$, where $x_n = \sqrt{n}$ and $y_n = \frac{\frac{3}{n^4\sqrt{n}}-1}{\frac{1}{n^4\sqrt{n}}+\frac{1}{n^3\sqrt{n}}+1}$. Note also that $x_n \to +\infty$ while $y_n \to \frac{0-1}{0+0+1} = -1$, so that (y_n) converges to a negative number. It follows that $x_n y_n \to -\infty$.

1.2(h)

$$\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}} = \frac{(-1)^n\cdot\left(\frac{2}{3}\right)^n\cdot\frac{1}{3}+\frac{1}{3}}{(-1)^{n+1}\left(\frac{2}{3}\right)^{n+1}+1} \to \frac{0+\frac{1}{3}}{0+1} = \frac{1}{3}$$

1.2(i) Let $x_n = \frac{1}{\sqrt{n}}\cos(n^2+n)$. Then $|x_n| = \frac{\left|\cos(n^2+n)\right|}{\sqrt{n}} \leq |1\sqrt{n}| \to 0$. It follows that $|x_n| \to 0$, hence $x_n \to 0$.

1.2(j)

$$\frac{(n!)^2}{(2n)!} = \frac{1\cdot 2\cdot\cdots\cdot n\cdot 1\cdot 2\cdot\cdots\cdot n}{1\cdot 2\cdot\cdots\cdot n\cdot(n+1)\cdot\cdots\cdot n} = \frac{1\cdot 2\cdot\cdots\cdot n}{(n+1)\cdot\cdots\cdot n} \leq \left(\tfrac{1}{2}\right)^n \to 0,$$

hence $x_n \to 0$.

1.3. For every $n > 10$, $x_{2n} - x_{2n-1} > 1$. Hence, every tail of (x_n) contains terms that are more that 1 away from each other. By Cauchy's criterion, the sequence diverges.

1.4(a) For $x \neq 2$, after multiplying the numerator and the denominator by the same terms, we get

$$\begin{aligned}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2} &= \frac{\left(\sqrt{1+2x}-3\right)\left(\sqrt{1+2x}+3\right)\left(\sqrt{x}+2\right)}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)\left(\sqrt{1+2x}+3\right)} \\ &= \frac{\left((1+2x)-3^2\right)\left(\sqrt{x}+2\right)}{(x-2^2)\left(\sqrt{1+2x}+3\right)} = 2\frac{\sqrt{x}+2}{\sqrt{1+2x}+3}.\end{aligned}$$

Since this is a continuous function, we get

$$\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2} = \lim_{x\to 4} 2\frac{\sqrt{x}+2}{\sqrt{1+2x}+3} = 2\frac{\sqrt{4}+2}{\sqrt{1+2\cdot 4}+3} = \frac{4}{3}.$$

1.4(b) Since $\left|x^3 \sin \frac{1}{x^2}\right| \leq |x^3|$ for all $x \neq 0$ and $\lim_{x\to 0} x^3 = 0$, it follows by the Squeeze Theorem that $\lim_{x\to 0} x^3 \sin \frac{1}{x^2} = 0$.

1.4(c) The function is continuous (being the product of continuous functions), hence the limit equals the value: $\lim_{x\to 0} \sin^2 x \cos^3 x = \sin^2 0 \cos^3 0 = 0$.

1.4(d) The function is continuous because it is built out of continuous functions using only the algebraic operations and compositions. Hence, the limit equals the value

$$\lim_{x\to\sqrt{2}\pi} \frac{\sin\sqrt{x^2+2\pi^2}}{1+\cos x} = \frac{\sin\sqrt{\left(\sqrt{2}\pi\right)^2+2\pi^2}}{1+\cos\left(\sqrt{2}\pi\right)} = \frac{\sin 2\pi}{1+\cos\left(\sqrt{2}\pi\right)} = 0$$

1.5. Take $\epsilon = |g(a)|$, then there is some $\delta > 0$ such that, for all $|x-a| < \delta$, $|f(x) - f(c)| < \epsilon$. Take $c = a - \delta, d = a + \delta$; then for all $x \in (c, d)$, $f(c) - \epsilon < f(x) < f(c) + \epsilon$, i.e. $f(x) \neq 0$.

Qu. 2.1. Choose any $a > 0$. We want to show $f(x) = \sqrt{x}$ is continuous at a. Choose any $\epsilon > 0$. Let $\delta = \min\{a/2, \epsilon\sqrt{a}\}$. Then for any $|x-a| < \delta$,

$$|f(x) - f(a)| = |\sqrt{x} - \sqrt{a}| = |x-a|/(\sqrt{x}+\sqrt{a}) < \delta/\sqrt{a} = \epsilon$$

Qu. 2.2. We will prove that all three are equivalent to each other, by proving **(a)** $\Rightarrow$ **(b)**, then proving **(b)** $\Rightarrow$ **(c)**, and finally proving **(c)** $\Rightarrow$ **(a)**. This automatically implies the remaining 3 directions: e.g. to see that **(b)** $\Rightarrow$ **(a)**, we know that **(b)** implies **(c)**, which in turn implies **(a)**.

(a) $\Longrightarrow$ **(b)**: Assume $\lim_{x\to c^-} f(x) = L$. Let x_n be any sequence tending to c, with all $x_n < c$. We want to show that $f(x_n) \to L$.

Choose any $\epsilon > 0$. Because $\lim_{x\to c^-} f(x) = L$, it means that there exists a $\delta > 0$ such that for all x with $c - \delta < x < c$, we have $|f(x) - L| < \epsilon$. Because $x_n \to c$, there is an N such that for all $n > N$, $|x_n - c| < \delta$. We also know that $x < c$, so in fact $c - \delta < x < c$ for all $n > N$. But this implies $|f(x_n) - L| < \epsilon$ for all $n > N$, as desired.

(b) $\Longrightarrow$ **(c)**: This is trivial to show. A strictly increasing sequence $x_n \to c$ necessarily satisfies $x_n < c$ for all n.

(c) $\Longrightarrow$ **(a)**: Assume that for every strictly increasing sequence $x_n \to c$, then $f(x_n) \to L$. Suppose for contradiction that $\lim_{x\to c^-} f(x) \neq L$. That means there is some $\epsilon_0 > 0$ such that, for all $\delta > 0$, there is an x_δ such that $c - \delta < x_\delta < c$ and $|f(x_\delta) - L| \geq \epsilon_0$. The strategy is to take δ smaller and smaller to construct a strictly increasing sequence $c_n \to c$ with $|f(c_n) - L| \geq \epsilon_0$. The existence of such a sequence contradicts the hypothesis that **(c)** is true, a contradiction.

Start with $\delta = 1$. Let $c_1 = x_\delta$: so $c - \delta < c_1 < c$ and $|f(c_1) - L| \geq \epsilon_0$. Now take $\delta = (c - c_1)/2$. Let $c_2 = x_\delta$ for this new δ: so $c_1 < c - \delta < c_2 < c$ and $|f(c_2) - L| \geq \epsilon_0$. Repeat ad infinitum: take the new $\delta = (c - c_2)/2$ etc. The result is a sequence $c_1 < c_2 < \cdots$ with $c_n \to c$, but $|f(c_n) - L| \geq \epsilon_0$. As mentioned above, this contradicts **(b)**, so we're done.

Qu. 3(a) For $x \neq 2$, after multiplying the numerator and the denominator by the same terms, we get

$$\frac{\sqrt{1+2x}-3}{\sqrt{x}-2} = \frac{\left(\sqrt{1+2x}-3\right)\left(\sqrt{1+2x}+3\right)\left(\sqrt{x}+2\right)}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)\left(\sqrt{1+2x}+3\right)}$$
$$= \frac{\left((1+2x)-3^2\right)\left(\sqrt{x}+2\right)}{(x-2^2)\left(\sqrt{1+2x}+3\right)} = 2\frac{\sqrt{x}+2}{\sqrt{1+2x}+3}.$$

Since this is a continuous function, we get

$$\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2} = \lim_{x\to 4} 2\frac{\sqrt{x}+2}{\sqrt{1+2x}+3} = 2\frac{\sqrt{4}+2}{\sqrt{1+2\cdot 4}+3} = \frac{4}{3}.$$

(b) Since $\left|x^3 \sin\frac{1}{x^2}\right| \leq \left|x^3\right|$ for all $x \neq 0$ and $\lim\limits_{x\to 0} x^3 = 0$, it follows by the Squeeze Thm for function limits that $\lim\limits_{x\to 0} x^3 \sin\frac{1}{x^2} = 0$. (the squeeze thm for function limits is proved on Assn.9, using Bridge thm and Squeeze thm for sequences).

(c) Given any sequence x_n such that $x_n > 0$ for all n and $x_n \to 0$, we have $\sqrt{x_n} \to 0$ by continuity of $\sqrt{x}$. Since the sequence $\sqrt{x_n})$ is positive, it follows that $\frac{1}{\sqrt{x_n}} \to +\infty$. By the $\lim\limits_{x\to c^+}$ analogue of Qu.2.2, we then know that $\lim\limits_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty$.

(d) Consider an arbitrary sequence x_n such that $x_n < 3$ for all n and $x_n \to 3$. We have $3 - x_n > 0$ for all n and $3 - x_n \to 0$; hence $\frac{1}{3-x_n} \to +\infty$. Also, $-(x_n + 3) \to -6$, which is a negative number. Hence, $\frac{x_n+3}{x_n-3} = -(x_n+3)\cdot\frac{1}{3-x_n} \to -\infty$. It follows that $\lim\limits_{x\to 3^-} \frac{x+3}{x-3} = -\infty$.

(e) The function is continuous (being the product of continuous functions), hence the limit equals the value: $\lim\limits_{x\to 0} \sin^2 x \cos^3 x = \sin^2 0 \cos^3 0 = 0$.

(f) The function is continuous because it is built out of continuous functions using only the algebraic operations and compositions. Hence, the limit equals the value

$$\lim_{x\to\sqrt{2}\pi} \frac{\sin\sqrt{x^2+2\pi^2}}{1+\cos x} = \frac{\sin\sqrt{\left(\sqrt{2}\pi\right)^2+2\pi^2}}{1+\cos\left(\sqrt{2}\pi\right)} = \frac{\sin 2\pi}{1+\cos\left(\sqrt{2}\pi\right)} = 0$$

(g) Using the substitution $t = 3x$, we get $\lim\limits_{x\to 0} \frac{\sin 3x}{3x} = \lim\limits_{t\to 0} \frac{\sin t}{t} = 1$. Similarly, we get $\lim\limits_{x\to 0} \frac{\sin 5x}{5x} = 1$. Combining these, we have

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \frac{3}{5} \lim_{x\to 0} \frac{\left(\frac{\sin 3x}{3x}\right)}{\left(\frac{\sin 5x}{5x}\right)} = \frac{3}{5} \frac{\lim\limits_{x\to 0} \frac{\sin 3x}{3x}}{\lim\limits_{x\to 0} \frac{\sin 5x}{5x}} = \frac{3}{5}\cdot\frac{1}{1} = \frac{3}{5}.$$

(h) Since $\sin 2x = 2\sin x\cos x$, so that

$$\frac{\tan x}{\sin 2x} = \frac{\sin x/\cos x}{2\sin x\cos x} = \frac{1}{2\cos^2 x} \to \frac{1}{2\cdot 1^2} = \frac{1}{2} \text{ as } x \to 0.$$

(i) Note that $\frac{1-\cos 2x}{\sin^2 x} = 4\cdot\frac{1-\cos 2x}{(2x)^2}/\left(\frac{\sin x}{x}\right)^2$ Using the substitution $t = 2x$, we get $\lim\limits_{x\to 0} \frac{1-\cos 2x}{(2x)^2} = \lim\limits_{t\to 0} \frac{1-\cos t}{t^2} = \frac{1}{2}$. Hence, $\lim\limits_{x\to 0} \frac{1-\cos 2x}{\sin^2 x} = 4\cdot\frac{1}{2}/1^2 = 2$.

(j)

$$\frac{\cos x - \cos 3x}{x^2} = \frac{\cos x - 1}{x^2} + \frac{1-\cos 3x}{x^2} = -\frac{1-\cos x}{x^2} + 9\cdot\frac{1-\cos 3x}{(3x)^2} \to -\frac{1}{2} + 9\cdot\frac{1}{2} = 4.$$

(k)

$$\lim_{x\to\pi}(\pi - x)\mathrm{cotan}(2x) = \lim_{x\to\pi}\frac{\pi - x}{\sin(2x)}\cos(2x) = \lim_{y\to 0}\frac{y}{\sin(y)}\frac{\cos(2\pi)}{2\cos(\pi)} = -1/2$$

using continuity of cosine and the substitution $y = \pi - x$.

(l) $\lim\limits_{x\to 0^+} f(x) = 3$ and $\lim\limits_{x\to 0^-} f(x) = 3$. Since the one-sided limits are equal, we have $\lim\limits_{x\to 0} f(x) = 3$

(m) Since $\lim_{x\to 0} \cos x = 1$ and $t_n := \frac{1}{n^2+1} \to 0$, the Bridge Theorem yields that $\cos t_n \to 1$. Thus, $\cos \frac{1}{n^2+1} \to 1$.

(n) Note that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $g(x) = \sqrt{x}$ is continuous at 1. Since continuous functions respect limits, we have $\lim_{x\to 0} \sqrt{\frac{\sin x}{x}} = \sqrt{\lim_{x\to 0} \frac{\sin x}{x}} = \sqrt{1} = 1$.

4. There are lots of possible answers. One is $f(x) = \sin(1/x)$ when $x \neq 0$, $f(x) = 0$. This is continuous at all $x \neq 0$, but it is not continuous at $x = 0$. In fact, for any $y \in [-1, 1]$, there is a sequence $0 < x_n \to 0$ and a sequence $0 > x'_n \to 0$ such that $f(x_n) \to y$ and $f(x'_n) \to y$: write $\sin(\theta) = y$, and take $x_n = 1/(2n\pi + \theta)$ and $x'_n = 1/(\theta - 2\pi n)$. So certainly $\lim_{x\to 0} f(x)$ does not exist.

Choose any $a < b$, and any y between $f(a)$ and $f(b)$. We want to show there is a c, $a \leq c \leq b$, such that $f(c) = y$. If $0 < a$ or $b < 0$, then $f(x)$ is continuous in $[a, b]$, so the existence of such a c is guaranteed by the Intermediate value Theorem. So it suffices to consider the case where $0 \in [a, b]$. Choose θ so that $\sin(\theta) = y$. Then for all sufficiently big n, either $x_n \in [a, b]$ (if $b > 0$) or $x'_n \in [a, b]$ (if $a < 0$) or both; moreover $f(x_n)$ and/or $f(x'_n)$ equals y; so we'll be able to find a $c \in [a, b]$ with $f(c) = y$ (in fact there will be infinitely mant such c).

5. We find $f(-2) = -25$, $f(0) = 1$, $f(1) = -2$, and $f(2) = 30$, so $f(x)$ has a root in $(-2, 0)$, a root in $(0, 1)$, and a root in (1,2).

6. Choose any rational number $r = m/n$ in (a, b). For any $M \in \mathbb{N}, M > n$, such that $1/M < b - r$, note that $x_M = r + 1/M$ will be a rational number in the interval (a, b), and with denominator $\geq M$ (because $1/M$ has a bigger denominator than $r = m/n$). Then $f(x_M)$ will be $\geq M$. Taking M bigger and bigger, we get that $\sup f(x) = \infty$ on (a, b).

7(a) By the Maximum principle, the inf of $f(x)$ on $[a, b]$ will equal $f(c)$ for some $c \in [a, b]$, so will be > 0. Then $\sup 1/f(x)$ on $[a, b]$ will equal $1/f(c)$, so will be finite.

(b) Take $[a, b] = [0, 1]$, $f(x) = x$ for all $x > 0$, and $f(0) = 1$. Then $f(x)$ isn't continuous at 0, but it is positive everywhere, and $\sup 1/f(x) = \infty$ on that interval.

(c) Take $(a, b] = (0, 1]$, $f(x) = x$. Then $f(x)$ is continuous everywhere, and is positive everywhere, but $\sup 1/f(x) = \infty$ on that interval.

8.1(a) Note that $f(x)$ for $x \in [0,1)$ is onto the interval [1,2), while for $x \in [1,2)$, $f(x)$ is onto the interval [0,1). So $f\colon [0,2) \to [0,2)$ is onto.

To show $f(x)$ is one-to-one on [0,2), suppose $f(a) = f(b)$. Then a and b must either both be in [0,1) (if their images $f(a) = f(b)$ are in [1,2)), or both in [1,2) (if their images are in [0,1)). So we either have $a + 1 = f(a) = f(b) = b + 1$, or $a - 1 = f(a) = f(b) = b - 1$; in either case we have $a = b$. Therefore $f(x)$ is one-to-one on [0,2).

Since $f\colon [0,2) \to [0,2)$ is both onto and one-to-one, it is invertible. The value of $f^{-1}(y)$ depends on whether $y \in [0,1)$ (in which case we use $f(x) = x - 1$) or $y \in [0,1)$ (in which case we use $f(x) = x + 1$). The result is:

$$f^{-1}(y) = \begin{cases} y+1 & \text{if} \quad 0 \le y < 1 \\ y-1 & \text{if} \quad 1 \le y < 2 \end{cases}$$

In other words, $f^{-1}(y) = f(y)$!

8.1(b) $f(x) = x(1-x)$ is not one-to-one on $\mathbb{R}$. For example, $f(0) = f(1) = 0$. So that means it cannot be invertible on $\mathbb{R}$.

It is also not onto, so also for that reason isn't invertible. For example, $f(x) = 1$ would imply $x(1 - x) = 1$, i.e. $x^2 - x + 1 = 0$, which doesn't have a solution in $\mathbb{R}$ (using the quadratic formula). In fact $f(x)$ never gets larger than 1/4

8.2. Suppose for contradiction that $f(x)$ is continuous and invertible on (a, b), but neither strictly increasing not strictly decreasing. This means there are numbers $c < c'$ and $d < d'$ in (a, b), such that $f(c) \ge f(c')$ and $f(d) \le f(d')$. In both cases we must have strict inequalities (i.e. $f(c) > f(c')$ and $f(d) < f(d')$, as otherwise $f(x)$ wouldn't be one-to-one.

If $f(c) \ge f(d')$, then $f(c) \ge f(d') > f(d)$, which by the Intermediate Value Theorem implies there is some d'' between c and d with $f(d'') = f(d')$. But $f(x)$ is one-to-one, so $d'' = d'$. Then $d < d'$ implies $d < d' \le c < c'$ (because $d'' = d'$ must be between c and d).

On the other hand, if $f(c) < f(d')$ then $f(d') > f(c) > f(c')$, which by the Intermediate Value Theorem and one-to-one-ness implies c is between d' and c'. And now $c < c'$ implies $d < d' < c < c'$.

Thus in either case we learn that $d < c$.

In the same way, if $f(c') \ge f(d)$ then $f(c) > f(c') \ge f(d)$ so $c < c' \le d < d'$. On the other hand, if $f(c') < f(d)$ then $f(c') < f(d) < f(d')$ so $c < c' < d < d'$. Thus in either case we learn that $c < d$.

So we have that both $d < c$ and $c < d$, a contradiction.