1) $\displaystyle\int_0^\infty \frac{\cos(x)}{1+x}\,dx$

Let $c > 0$ and $R > c$. Then integration by parts yields

$$\int_c^R \frac{\cos(x)}{1+x}\,dx = \frac{\sin x}{x+1}\Big|_c^R + \int_c^R \frac{\sin x}{(x+1)^2}\,dx$$

Now clearly $\displaystyle\frac{\sin x}{x+1}\Big|_c^R = \frac{\sin R}{R+1} - \frac{\sin c}{c+1} \xrightarrow{R\to\infty} -\frac{\sin c}{c+1} \xrightarrow{c=0} \frac{0}{1} = 0$

now clearly $\displaystyle\int_c^R \frac{1}{(x+1)^2}\,dx = -\frac{1}{(x+1)}\Big|_c^R = -\frac{1}{(R+1)} + \frac{1}{(c+1)} \xrightarrow{R\to\infty} \frac{1}{(c+1)} \xrightarrow{c=0} \frac{1}{1}$

$\therefore \displaystyle\int_0^R \frac{1}{(x+1)^2}\,dx$ exists. Since $\displaystyle\left|\frac{\sin x}{(x+1)^2}\right| \le \frac{1}{(x+1)^2} \;\forall x>0$

$\therefore$ by the comparision test shows that $\displaystyle\int_0^\infty \frac{\cos(x)}{x+1}\,dx$ exists.

However $\displaystyle\int_0^\infty \frac{\cos(x)}{1+x}\,dx$ does not converge absolutely. Let $n \in \mathbb{N}$

and then note

$$\int_0^{n\pi} \frac{|\cos(x)|}{x+1}\,dx = \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\cos x|}{x+1}\,dx$$

$$\ge \sum_{k=1}^{n} \frac{1}{(k\pi+1)} \int_{(k-1)\pi}^{k\pi} |\cos x|\,dx$$

ie $\therefore \int_0^{n\pi} \frac{|\cos(x)|}{x+1} dx \geq \sum_{k=1}^{n} \frac{1}{(k\pi+1)} \int_{(k-1)\pi}^{k\pi} |\cos x| \, dx = \sum_{k=1}^{n} \frac{2}{(k\pi+1)}$ 1)

now let $b_k = \frac{1}{k} \therefore \sum_{k=1}^{\infty} b_k = \infty \quad a_k = \frac{2}{k\pi+1}$

Then $\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{2}{k\pi+1} \cdot k = \frac{2}{\pi + 1/k} \xrightarrow{k\to\infty} \boxed{\frac{2}{\pi}} > 0$

$\therefore$ by limit comparison test $\sum_{k=1}^{\infty} a_k$ diverges

$\therefore$ since it diverges it follows that the improper integral $\int_0^{\infty} \frac{|\cos(x)|}{x+1} dx$ does not exist.

$\boxed{\text{QED}}$

2) $\emptyset \neq U \subset \mathbb{R}$, $r, s \in \mathbb{N}_0$ $\omega \in \Lambda^r(C^1(U))$ $\phi \in \Lambda^s(C^1(U))$ is closed.

By definition $d\omega = 0$ and $d\phi = 0$. (from HW)

Then to show $\omega \wedge \phi$ is closed.

Then by proposition 7.2.5 $d(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^r \omega \wedge d\phi$
$= 0 \wedge \phi + (-1)^r \omega \wedge 0$
$= \underline{\underline{0}}$

$\therefore \omega \wedge \phi$ is closed too.

$\boxed{\text{QED}}$

$f \in PC_{2\pi}(\mathbb{R})$ $\quad f(t) = \begin{cases} -1 & t \in (-\pi, 0] \\ 2 & t \in (0, \pi] \end{cases}$

Extending f to a function in $PC_{2\pi}(x)$ using definition 9.5.2(a)
For $n \in \mathbb{N}_0$ we obtain.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

$$= \frac{1}{\pi}\left(-\int_{-\pi}^{0}\cos(nt)\,dt + 2\int_{0}^{\pi}\cos(nt)\,dt\right)$$

$$= \frac{1}{\pi}\left(-\int_{0}^{\pi}\cos(nt)\,dt + 2\int_{0}^{\pi}\cos(nt)\,dt\right)$$

$$= \frac{1}{\pi}\left(\int_{0}^{\pi}\cos(nt)\,dt\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n}\sin nt\Big|_{0}^{\pi}\right) = \frac{1}{\pi}\left(\frac{\sin \pi t}{n}\right) = \underline{\underline{0}}$$

$\therefore a_n = 0 \quad \forall n \in \mathbb{N}_0 - \{0\}$ but $a_0 = 1$ as $\frac{1}{\pi}\int_0^{\pi} dt = \frac{\pi}{\pi} = 1$

$$\text{now } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left(-\int_{-\pi}^{0}\sin(nt)\,dt + 2\int_{0}^{\pi}\sin(nt)\,dt\right)$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}\int_{-\pi n}^{0}\sin(t)\,dt + \frac{2}{n}\int_{0}^{\pi n}\sin(nt)\,dt\right)$$

$$= \frac{1}{\pi n}\left(\cos t \Big|_{-\pi n}^{0} - 2\cos t \Big|_{0}^{\pi n} \right)$$

$$= \frac{1}{\pi n}\left(1 - \cos(\pi n) - 2\cos(\pi n) + 2 \right)$$

$$= \frac{1}{\pi n}\left(3 - 3\cos(\pi n)\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ 6/\pi n & n \text{ is odd} \end{cases}$$

$\therefore$ It follows that $f(x) \sim \frac{6}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin((2n+1)x) + \frac{1}{2}$

~~Now from theorem 9.1.3 and~~

~~Let us assume that $f(x)$~~

now note that for integer multiples of π $f(x\pi) = 1/2$

By theorem 9.3.13 then $f(x)$ converges pointwise $\forall x$ that are not integer multiples of π.

$\therefore$ It converges pointwise to $1/2$

Let $g(x) = \begin{cases} f(x) & x \neq k\pi \quad \forall k \in \mathbb{Z} \\ 1/2 & x = k\pi \quad \forall k \in \mathbb{Z} \end{cases}$

and $g(x)$ is clearly discontinuous.

~~Now assume that~~ Let $f_n = \sum_{k=0}^{n} \frac{1}{(2k+1)} \sin((2k+1)x)$

and since each f_n is continuous. Let us assume that $f_n \to g$ uniformly on $\mathbb{R}$. Then by theorem 9.1.3

we get g is continuous which is a contradiction.

$\therefore$ The fourier series of f does not converge uniformly on $\mathbb{R}$. $\boxed{QED}$

4) $U := \{(x,y) \in \mathbb{R}^2 : x > 0\}$ $\quad f = (P, Q) : U \to \mathbb{R}^2$

$$P(x,y) = \frac{\tan^{-1}(y)}{x} \qquad Q(x,y) = \frac{\log x}{1+y^2} \qquad \forall (x,y) \in U$$

$Y = Y_1 \oplus Y_2$, $\quad Y_1 = [(1,7), (\sqrt{3},\pi)]$ $\quad Y_2 = [(\sqrt{3},\pi), (e,1)]$

To find $\displaystyle\int_Y P\,dx + Q\,dy$

First to find $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ ~~and $\frac{\partial Q}{\partial x}$~~

$$= \frac{1}{(1+y^2)\cdot x} = \frac{1}{x \cdot (1+y^2)}$$

$\therefore$ by proposition 6.2.11 we have a conservative vector field.

Now to find the potential function

Let $F: \mathbb{R}^2 \to \mathbb{R}$ be a potential function for f.

Then $\displaystyle F(x,y) = \int \frac{\tan^{-1}(y)}{x}\,dx$

$$= \tan^{-1}(y) \int \frac{1}{x}\,dx = \tan^{-1}(y)\log(x) + G(y)$$

Then $\dfrac{\partial F}{\partial y}(x,y) = \dfrac{\partial G}{\partial y} + \dfrac{1}{1+y^2}\log(x)$

Now $\dfrac{\partial G}{\partial y} = \dfrac{1}{1+y^2}\log(x) - \left(\dfrac{\log x}{1+y^2}\right) = 0$

$\therefore \dfrac{\partial G}{\partial y} = 0$ and G is a constant

$\therefore F(x,y) = \tan^{-1}(y)\log(x) + C$

Now $\gamma = \gamma_1 \oplus \gamma_2$ $\quad \gamma = \begin{cases} \gamma_1(t) & t \in [(1,7),(3,\pi)] \\ \gamma_2(t) & t \in [(3,\pi),(e,1)] \end{cases}$

now clearly $\gamma(a) = (1,7)$

$\gamma(b) = (e,1)$

and γ is clearly a piecewise C^1-curve.

$\therefore$ by Fundamental theorem for Curve integrals

$$\int_\gamma P\,dx + Q\,dy = \int_\gamma J \cdot dx = F(\gamma(b)) - F(\gamma(a))$$

$$= -\left(\tan^{-1}(7)\log(1) + C - \left(\tan^{-1}(1)\log(e) + C\right)\right)$$

$$= -\left(0 - \frac{\pi}{4}\right) \quad \left(\tan^{-1}(1) = \frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\pi}{4}$$

$$\therefore \int_\gamma P\,dx + Q\,dy = \frac{\pi}{4}$$

$\boxed{QED}$

5) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

a) Radius of convergence

Using the ratio test let $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ $x_0 = 0$

then $a_n = \frac{1}{n}$ $\therefore R = \lim_{n\to\infty} \frac{1/(n)}{1/(n+1)} = \frac{n+1}{n} \Rightarrow 1 + \frac{1}{n} \xrightarrow{n\to\infty} \underline{\underline{1}}$

$\therefore$ Radius of convergence is 1.

b) Since by definition 9.2.3 $\sum_{n=1} \frac{x^n}{n}$ has a radius of convergence 1 and converges uniformly on $[-r, r]$ $\forall r \in [0,1)$

and diverges $\forall x \in \mathbb{R}$ with $|x| > 1$. At $x = 1$ we clearly get $\sum_{n=1} \frac{1}{n}$ which diverges, $x = -1$ we get $\sum_{n=0} \frac{(-1)^n}{n}$ which converges.

c). Now by corollary 9.2.4 we have $\int : (-1, 1) \to \mathbb{R}$

$$\int'(x) = \sum_{n=1}^{\infty} n \cdot \frac{1}{n} (x)^{n-1} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n$$

Now since we already know by (b) it converges uniformly on $[-r, r]$ $\forall r \in [0,1)$ we can use the geometric series theorem to get $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $|x| < 1$

$\therefore \int'(x) = \frac{1}{1-x}$

now $f'(x) = \frac{1}{1-x}$

then $\int f'(x)dx = \int \frac{1}{1-x} dx$

$= -\ln(|x-1|) + C$

$\therefore \int f'(x)dx = \int \sum_{n=0}^{\infty} x^n dx = \sum_{n=0}^{\infty} \int x^n dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}$

$\therefore \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(|x-1|) + C$

Let $x = 0 \Rightarrow C = 0$

$\therefore \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(|x-1|) = -\ln(1-x)$ as $|x| < 1$

$\forall x \in [-r, r]\ \forall r \in [0, 1)$

At $x = -1$ we get by example in 9.3

we have $\log(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$

Then $-\log(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

Then at $x = 1$ we get $-\log(2) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = f(-1)$

$\therefore$ at $x = -1$ we get $-\log(2)$

as $f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \underline{\underline{-\log(2)}}$.

$\boxed{QED}$

6) C be the counterclockwise oriented circle $x^2 - 2x + y^2 = 0$
$\Rightarrow (x-1)^2 + y^2 = 1$

$$\int_C 2xy\,dx + (x+1)^2\,dy$$

Now $(x-1)^2 + y^2 = 1$ is clearly a normal domain.
Let D be the ~~circle~~ region enclosed in C.

Then by green theorem we get

$$\int_C 2xy\,dx + (x+1)^2\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left(2(x+1) - 2x\right) dA$$

$$= 2\iint_D dA$$

$$= 2\cdot\underbrace{\pi r^2}_{\text{area of circle}} \quad (r = 1) \text{ [from Note]}$$

$$= 2\pi$$

$$\therefore \int_C 2xy\,dx + (x+1)^2\,dy = 2\pi$$

Note $\iint_D dA$ = By polar coordinate conversion we get

$$\iint_D dA = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^{2\pi} \left.\frac{r^2}{2}\right]_0^1 d\theta$$

$$= 2\pi \cdot \frac{1}{2} = \pi$$

$\boxed{\text{QED}}$

*) $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$ $\quad f, g : [a, b) \to \mathbb{R}$

f, g are riemann integrable on $[a, c]$ $\forall c \in [a, b)$

Since $\int_a^b f(x)\,dx$ converges absolutely $\Rightarrow \int_a^b |f(x)|\,dx$ exists

and g is bounded $\exists\, C > 0$ s.t. $|g(x)| \leq C$ $\forall x \in [a, c]$ $\forall c \in [a, b)$

Then $\quad |f(x)g(x)| \leq |f(x)|C$

Then note that $\int_a^b C|f(x)|\,dx = C\int_a^b |f(x)|\,dx$ exists as

$\int_a^b |f(x)|\,dx$ exists as $\int_a^b f(x)\,dx$ converges absolutely.

By comparison test (8.2.6) $\int_a^b f(x)g(x)\,dx$ converges absolutely.

and $\tan^{-1}(x)$ is defined on all $x \in \mathbb{R}$.

Counterexample:

Let $f = \frac{\sin x}{x}$ then by example 1 in theorem 8.2.6 : $\int_1^\infty \frac{\sin x}{x} dx$

we know it converges but not absolutely convergent.

Let $g = \sin x$ which is clearly bounded by 1 above.

Then $\int^\infty g(x)\,dx$

Now set $c' > 1$ then $\int_1^{c'} \sin x\,dx = -\cos x \Big|_1^{c'} \Rightarrow \cos 1 - \cos c' < \infty$

which exists.

∴ Both f and g are also riemann integrable on $[a, c]$ $\forall$ $c \in [a,b]$

Now
$$\int_1^\infty fg\,dx = \int_1^\infty \frac{\sin^2 x}{x}\,dx$$

$g(x)$ is continuous and $[1, c']$ is a closed interval so riemann integrable.

Integration by parts yield

6)

$\int_1^\infty \frac{\sin^2 x}{x} dx$ Now fix $R>0$

Then $\int_1^R \frac{\sin^2 x}{x} = -\frac{1}{x}\left(\frac{\sin 2x + 2x}{4}\right)\Big|_1^R + \int_1^R \frac{\sin 2x + 2x}{4 \cdot x^2} dx$

$= \frac{-\sin 2x}{x \cdot 4}\Big|_1^R - \frac{2}{4}\Big|_1^R + \int_1^R \frac{\sin 2x\, dx}{4x^2} + \int_1^R \frac{2x\, dx}{4x^2}$

$= \frac{\sin 2R}{4R} \xrightarrow{R \to \infty} 0$

$\therefore -\frac{1}{x}\left(\frac{\sin 2x + 2x}{4}\right)\Big|_1^R = \frac{-1}{2} + \frac{-1}{2} - \sin 2$

$= \underline{\underline{-1 - \sin 2}}$ (exists)

now $\int_1^R \frac{2dx}{2x} = \frac{1}{2}\ln(x)\Big|_1^R = \frac{1}{2}\ln R \xrightarrow{R\to\infty} \infty$

$\therefore$ diverges.

$\therefore \int_1^\infty \frac{\sin^2 x}{x} dx$ does not exist.

$\boxed{\text{QED}}$ $\therefore$ It is necessary for $\int_a^b f(x)dx$ to be absolutely convergent too.

8) $\emptyset \neq K \subset \mathbb{R}^N$ $\emptyset \neq L \subset \mathbb{R}^M$ $F: K \times L \to \mathbb{R}$ be continuous
$(y_n)_{n=1}^{\infty}$ be a sequence in L. $\forall n \in \mathbb{N}$

$$f_n : K \to \mathbb{R} \quad x \to F(x, y_n)$$

To show $(f_n)_{n=1}^{\infty}$ has a subsequence that converges uniformly to K.

we know that clearly $K \times L \subset \mathbb{R}^{N+M}$ is compact as proved in the HW of Q17. Then and since F is continuous by theorem 2.3.4 we get $F(K \times L)$ is compact and by corollary 2.3.5 we get $F(K \times L)$ is bounded. Then since $(y_n)_{n=1}^{\infty}$ is defined in L by theorem 2.1.12 we get

that $(y_n)_{n=1}^{\infty}$ has a subsequence that converges to a point in L. — 2

Let this subsequence be $(y_{n_k})_{k=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ that

converges to $y \in L$. Now by Theorem 2.4.2 we also F is uniformly continuous. $\to$ 1
Now to show that $(f_{n_k})_{k=1}^{\infty}$ subsequence of $(f_n)_{n=1}^{\infty}$

converges uniformly to $F(x, y)$ on K.

Let $\varepsilon > 0$ and $\delta > 0$ then by 1 $\forall (x, y_{n_k}) \in K \times L$
Let $f_y(x) = F(x, y)$
we get $\|(x, y_{n_k}) - (x, y)\| < \delta \Rightarrow |f_{n_k}(x) - f_y(x)|$

$$= |F(x, y_{n_k}) - F(x, y)| < \varepsilon$$

$\hookrightarrow$ 3

~~det~~

Now $\|(x, y_{n_k}) - (x,y)\| = \|y_{n_k} - y\| < \delta$

Then by ② we get

$\lim_{k\to\infty} y_{n_k} = y$.

$\Rightarrow \exists n_\varepsilon \in \mathbb{N}$ s.t $\|y_{n_k} - y\| < \delta \ \forall\ n_k \geq n_\varepsilon$.

$\therefore$ Then by choosing $n_\varepsilon \in \mathbb{N}$ s.t $\|y_{n_k} - y\| < \delta \ \forall\ n_k \geq n_\varepsilon$

and hence $\|(x, y_{n_k}) - (x,y)\| = \|y_{n_k} - y\| < \delta \ \forall x \in K$

and hence from ③ we get

$|f_{n_k}(x) - f_y(x)| = |F(x, y_{n_k}) - F(x,y)| < \varepsilon$.

$\therefore$ All in all $(f_{n_k})_{k=1}^{\infty}$ uniformly converges also to f_y on K which is a subsequence of $(f_n)_{n=1}^{\infty}$.

$\boxed{\text{QED}}$