Problems 2

09/16/2020

1) Let H and K be subgroups of a group G. Show that the union $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

2) Show that the centralizer of a normal subgroup is normal.

3) Let
$$\mathrm{U}_3(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}.$$
Show that $\mathrm{U}_3(\mathbb{R})$ is a subgroup of $\mathrm{SL}_3(\mathbb{R})$, which is not normal, and determine the center of $\mathrm{U}_3(\mathbb{R})$.

4) Let G be a group, such that every element has order 2. Show that G is a vector space over the field with two elements $\mathbb{Z}/2$.

5) (i) Let H be a subgroup of index 2 of a group G. Show that H is normal.
   (ii) Give an example of a group G with a non normal subgroup H of index 3.

6) Compute the center of S_4.

7) Let G be a group of order $p \cdot q$, where $p \neq q$ are prime numbers, which is not cyclic. Show that if G has a normal subgroup of order p then there exist two elements x and y of order p and q, respectively, such that every $g \in G$ can be uniquely written as
$$g = y^j \cdot x^i$$
with $0 \leq i \leq p-1$ and $0 \leq j \leq q-1$.