Math 322
Suggested solutions to Homework Set 2

Problem 1. (i) Let us write $a_{s,t}$ for the (s,t)-th entry of the adjacency matrix A of G, and $b_{s,t}$ for the (s,t)-th entry of the matrix $A^2 = A \cdot A$; in other words, we write $A = \big(a_{s,t}\big)_{1\leqslant s,t\leqslant n}$ and $A^2 = \big(b_{s,t}\big)_{1\leqslant s,t\leqslant n}$.

We recall that

$$a_{s,t} = \begin{cases} 1 & \text{if } \{v_s, v_t\} \in E(G) \\ 0 & \text{otherwise} \end{cases}.$$

We also have that

$$\begin{aligned} b_{i,j} &= \langle \mathrm{Row}_i(A),\, \mathrm{Col}_j(A)\rangle \\ &= \sum_{s=1}^{n} a_{i,s} \cdot a_{s,j} \\ &= \Big|\Big\{s \in \{1,2,\ldots,n\} : a_{i,s} = a_{s,j} = 1\Big\}\Big| \\ &= \Big|\Big\{s \in \{1,2,\ldots,n\} : \{v_i, v_s\} \in E(G) \text{ and } \{v_s, v_j\} \in E(G)\Big\}\Big|. \end{aligned}$$

In other words, $b_{i,j}$ equals the number of vertices v_s which are adjacent both to v_i and to v_j, which is what we wanted to show.

As the subsequent remark also states, this is the same as the total number of $v_i - v_j$ walks of length 2 in the graph G.

(ii) We will use induction in k to verify the following

Claim 1. For every $s, t \in \{1,2,\ldots,n\}$ (with s not necessarily different from t), the (s,t)-th entry of A^k is equal to the total number of $v_s - v_t$ walks of length k in G (that is, walks in G which have length k and endvertices the vertex v_s and the vertex v_t).

Note that this includes the desired conclusion of part (ii), as well as the corresponding conclusion for the cases where $s = t$ (which part (ii) did not require us to consider); however it is easier to prove the entire claim using induction. Observe also that, when $s = t$ (or in other words, v_s and v_t stand for the same vertex of G), $b_{s,s}$ counts closed walks of length k.

Base Case: $k = 2$. We observe that we have verified the claim in this case in part (i) as well as in HW1, Problem 1 (where we showed that the (i,i)-th entry of A^2 equals the number of neighbours of the vertex v_i; note that, as we argue in the Remark after part (i), we can check that the number of neighbours of v_i coincides with the number of closed walks of length 2 which start (and end) at v_i).

Induction Step: Assume that the claim has been verified for some $k \geqslant 2$, and we now want to prove it for $k+1$. For convenience, let us write $w_{s,t}$ for the (s,t)-th entry of the matrix A^k (that is, $A^k = \big(w_{s,t}\big)_{1\leqslant s,t\leqslant n}$). We also keep writing $A = \big(a_{s,t}\big)_{1\leqslant s,t\leqslant n}$.

Consider $i, j \in \{1, 2, \dots, n\}$ (not necessarily different). Then

$$\begin{aligned}(i,j)\text{-th entry of } A^{k+1} &= (i,j)\text{-th entry of } A^k \cdot A \\ &= \langle \mathrm{Row}_i(A^k),\, \mathrm{Col}_j(A)\rangle \\ &= \sum_{s=1}^{n} w_{i,s}\cdot a_{s,j} \\ &= \sum_{s:a_{s,j}=1} w_{i,s}.\end{aligned}$$

In other words, if $v_{l_1}, v_{l_2}, \dots, v_{l_{d_j}}$ are the neighbours of v_j in G where $d_j = \deg(v_j)$ (that is, $l_1, l_2, \dots, l_{d_j}$ are those indices s for which we have $a_{s,j} = 1$), then

$$\begin{gathered}\text{the } (i,j)\text{-th entry of } A^{k+1} \text{ equals} \\ \text{the number of walks in } G \text{ of length } k \text{ which start at } v_i \text{ and end at } v_{l_1} \\ +\ \text{the number of walks in } G \text{ of length } k \text{ which start at } v_i \text{ and end at } v_{l_2} \\ +\ \text{the number of walks in } G \text{ of length } k \text{ which start at } v_i \text{ and end at } v_{l_3} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\ \text{the number of walks in } G \text{ of length } k \text{ which start at } v_i \text{ and end at } v_{l_{d_j}}.\end{gathered}$$

This final sum equals the total number of walks in G of length $k+1$ which start at v_i and end at v_j: indeed,

- note that, if $v_i\, u_1\, u_2\, \cdots\, u_{k-1}\, u_k\, v_j$ is such a walk, then the vertex u_k must be a neighbour of v_j. Say it is the vertex v_{l_r} for some $1 \leqslant r \leqslant d_j$, in which case the initial part $v_i\, u_1\, u_2\, \cdots\, u_{k-1}\, u_k$ of this walk, which is itself a walk of length k from v_i to $u_k = v_{l_r}$, is counted above by the term w_{i,l_r}.

- Conversely, for any walk $v_i\, z_1\, z_2\, \cdots\, z_{k-1}\, v_{l_r}$ of length k which starts at v_i and ends at a neighbour v_{l_r} of v_j, we can get a walk of length $k+1$ which ends at v_j by just adding the vertex v_j at the end of the sequence $v_i\, z_1\, z_2\, \cdots\, z_{k-1}\, v_{l_r}$; thus each such walk of length k corresponds uniquely to a v_i–v_j walk of length $k+1$.

We conclude that the (i, j)-th entry of A^{k+1} equals the total number of $v_i - v_j$ walks of length $k+1$ in G. This completes the proof of the Induction Step too.

(iii) *(Practice Question)* Note that, by proving the entire Claim 1 in part (ii), we have also verified that each diagonal entry $w_{j,j}$ of A^k counts the total number of closed walks of length k which start (and end) at the vertex v_j.

Problem 2. (i) The graph G_1 below has order 6 and size 12. Thus, its complement has size 3, and is clearly disconnected (we can also verify this from the pictures).

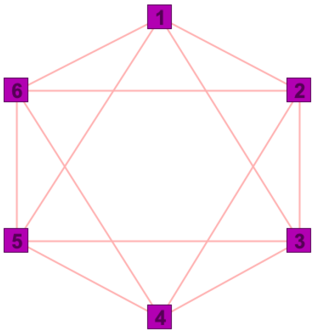


Figure 1: Graph G_1

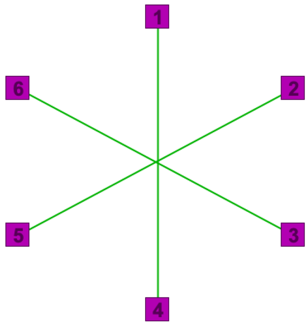


Figure 2: Complement of G_1

(ii) The 7-cycle below has order 7 and is connected, with a connected complement too.

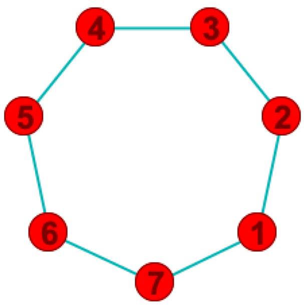


Figure 3: Graph C_7

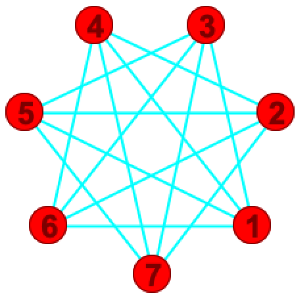


Figure 4: Complement of C_7

(iii) Let us write $u_1, u_2, \ldots, u_n$ for the vertices of H. Then $v = u_{j_0}$ for some $j_0 \in \{1, 2, \ldots, n\}$.

- We have that $V(H - v) = V(H - u_{j_0}) = \{u_i : 1 \leqslant i \leqslant n,\, i \neq j_0\}$. Also, $E(H - v) = E(H - u_{j_0})$ no longer contains any edges of H that v was incident with, but it contains all other edges, that is, all edges of H whose endvertices are in the set $U' = \{u_1, u_2, \ldots, u_n\} \setminus \{u_{j_0}\}$.

  But then the vertex set of $\overline{H - v} = \overline{H - u_{j_0}}$ is again

$$U' = \{u_1, u_2, \ldots, u_n\} \setminus \{u_{j_0}\},$$

while its edge set is

$$[U']^2 \setminus E(H - u_{j_0}),$$

that is, it contains an edge joining two vertices in U' if and only if those two vertices were not joined in $H - u_{j_0}$ (equivalently, if and only if those two vertices were not joined in H).

- We have that $V(\overline{H}) = V(H) = \{u_1, u_2, \ldots, u_n\}$, while

$$E(\overline{H}) = [V(H)]^2 \setminus E(H),$$

  that is, it contains an edge joining two vertices in $\{u_1, u_2, \ldots, u_n\}$ if and only if those two vertices are not joined in H.

  In particular, for any two vertices in $U' = \{u_1, u_2, \ldots, u_n\} \setminus \{u_{j_0}\}$, we have that $\overline{H}$ contains an edge if and only if these two vertices are not joined in H. But these edges of $\overline{H}$ are exactly the edges that will not be removed when we delete the vertex $v = u_{j_0}$, and coincide with the edge set of $\overline{H - v} = \overline{H - u_{j_0}}$, as we saw above.

  Finally, $V(\overline{H} - v) = U'$.

We conclude that $\overline{H - v}$ and $\overline{H} - v$ have the same vertex set, and also the same edge set. Thus they are the same graph.

Problem 3. Let us write $v_1, v_2, \ldots, v_n$ for the vertices of G (recall that $n \geqslant 2$) and $e_1, e_2, \ldots, e_m$ for the edges of G, which are also the vertices of $L(G)$.

Consider two different edges e_i, e_j of G. We need to show that there is a $e_i - e_j$ path in $L(G)$. Let v_{i_1}, v_{i_2} be the two endvertices of e_i, and v_{j_1}, v_{j_2} be the two endvertices of e_j (note that these aren't necessarily four distinct vertices, because the edges e_i and e_j that we have considered might have a common endvertex).

We can consider two cases:

Case 1: e_i and e_j have a common endvertex; say $v_{i_1} = v_{j_1}$. Then e_i and e_j are neighbours in $L(G)$, and therefore the sequence $e_i \, e_j$ is a path from e_i to e_j.

Case 2: e_i and e_j don't have a common endvertex. Then $v_{i_1}, v_{i_2}, v_{j_1}$ and v_{j_2} are four different vertices of G. Given that G is connected, we can find a $v_{i_1} - v_{j_1}$ path in G. We write this path here, including the edges of G that it traverses:

$$v_{i_1} - e_{s_1} - w_1 - e_{s_2} - w_2 - e_{s_3} - w_3 - \cdots\cdots - w_{l-2} - e_{s_{l-1}} - w_{l-1} - e_{s_l} - v_{j_1} \, .$$

Observe that $w_1, w_2, \ldots, w_{l-1}$ are $l-1$ distinct vertices of G which are also all different from v_{i_1} and v_{j_1}. It also follows that $e_{s_1}, e_{s_2}, \ldots, e_{s_l}$ are l distinct edges of G. Moreover, consecutive edges here are adjacent (since they share one endvertex), and hence the sequence

$$e_{s_1} \, e_{s_2} \, e_{s_3} \, \cdots \, e_{s_{l-1}} \, e_{s_l}$$

is a path in $L(G)$.

We now check that this path can give us the $e_i - e_j$ path in $L(G)$ that we want. We consider a few cases:

- $e_{s_1} = e_i$ and $e_{s_l} = e_j$. This happens if we already have that $w_1 = v_{i_2}$ and $w_{l-1} = v_{j_2}$. Then the path we already found is the path we wanted.
- $e_{s_1} = e_i$ but $e_{s_l} \neq e_j$. In this case, e_{s_l} is a neighbour of e_j in $L(G)$ (given that these two edges of G both have v_{j_1} as an endvertex). Therefore, the sequence

$$e_i = e_{s_1} \, e_{s_2} \, e_{s_3} \, \cdots \, e_{s_{l-1}} \, e_{s_l} \, e_j$$

  is also a path in $L(G)$.

– $e_{s_l} = e_j$ but $e_{s_1} \neq e_i$. We can deal with this case completely analogously to the previous one: the sequence

$$e_i \, e_{s_1} \, e_{s_2} \, e_{s_3} \, \cdots \, e_{s_{l-1}} \, e_{s_l} = e_j$$

is a path in $L(G)$.

– Finally, in the case that both $e_{s_1} \neq e_i$ and $e_{s_l} \neq e_j$, the e_i–e_j path we want is the path

$$e_i \, e_{s_1} \, e_{s_2} \, e_{s_3} \, \cdots \, e_{s_{l-1}} \, e_{s_l} \, e_j$$

(note that, in this case, e_{s_1} and e_i are neighbours in $L(G)$ since they share the vertex v_{i_1} when viewed as edges of G, and similarly e_{s_l} and e_j are neighbours in $L(G)$ since they share the vertex v_{j_1} when viewed as edges of G).

Problem 4. (i) We start with the assumption that $G_1 \cong G_2$. Then we know that, if $V(G_1) = \{v_1, v_2, \ldots, v_n\}$, while $V(G_2) = \{u_1, u_2, \ldots, u_n\}$, we can find a bijection

$$f : V(G_1) \to V(G_2)$$

such that, for every $i, j \in \{1, 2, \ldots, n\}$, we will have that

$$\{v_i, v_j\} \in E(G_1) \quad \textbf{if and only if} \quad \{f(v_i), f(v_j)\} \in E(G_2).$$

This also implies that, if $E(G_1) = \{e_1, e_2, \ldots, e_{m-1}, e_m\}$ is the edge set of G_1 and $E(G_2) = \{d_1, d_2, \ldots, d_{m-1}, d_m\}$ is the edge set of G_2, then we also get a bijection

$$g : E(G_1) \to E(G_2).$$

Indeed, we can set $g(e_s) = d_t$ if the following holds true: the endvertices of e_s are the vertices v_i, v_j of G_1 while the endvertices of d_t are the vertices $u_k = f(v_i)$ and $u_l = f(v_j)$ of G_2 (note that this rule gives us a function from $E(G_1)$ to $E(G_2)$ which is injective, given that f is injective, and also surjective, since $\{f(v_i), f(v_j)\} \in E(G_2)$ only if $\{v_i, v_j\} \in E(G_1)$).

We now recall that $V(L(G_1)) = E(G_1) = \{e_1, e_2, \ldots, e_{m-1}, e_m\}$, while $V(L(G_2)) = E(G_2) = \{d_1, d_2, \ldots, d_{m-1}, d_m\}$. We already have the bijection

$$g : V(L(G_1)) \to V(L(G_2)).$$

We check that this is a graph isomorphism. We have that

$$\{e_i, e_j\} \in E(L(G_1))$$

if and only if

the edges e_i and e_j of G_1 have a common endvertex, say vertex $v_s \in V(G_1)$

if and only if

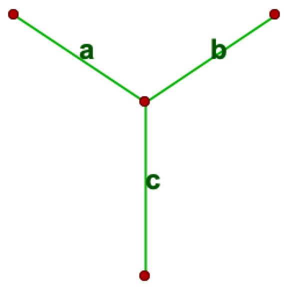
the edges $g(e_i)$ and $g(e_j)$ of G_2 have a common endvertex, vertex $f(v_s) \in V(G_2)$

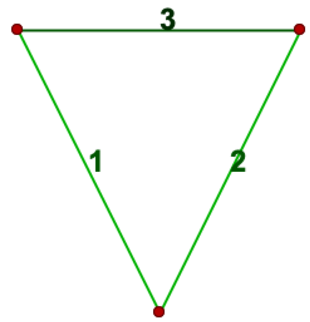
if and only if

$$\{g(e_i), g(e_j)\} \in E(L(G_2)).$$

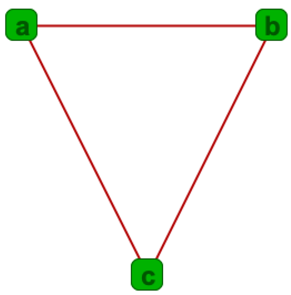
Thus g preserves adjacencies. Given that we have found a graph isomorphism from $L(G_1)$ to $L(G_2)$, we conclude that $L(G_1) \cong L(G_2)$.

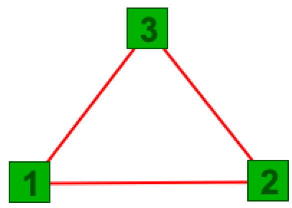
(ii) In the picture below we have labelled representations of the graphs $K_{1,3}$ and K_3 (note that we have only labelled the edges of the graphs):



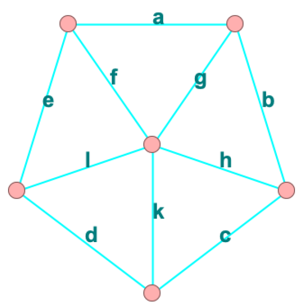


We observe that each of these graphs has 3 edges, any two of which are adjacent. Therefore, the line graph of either of these graphs is isomorphic to K_3, the complete graph on 3 vertices:

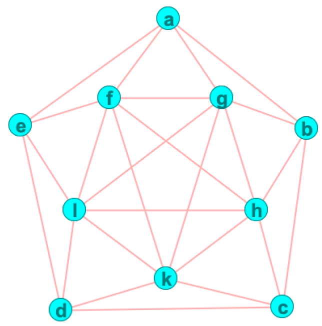




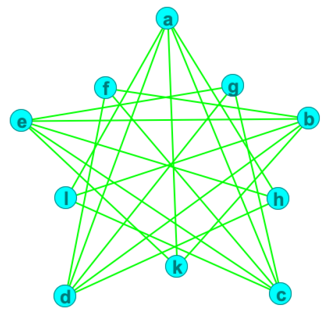
Problem 5. (i) Here is a labelled representation of W_6:



Based on this labelling, the line graph of W_6 is given by



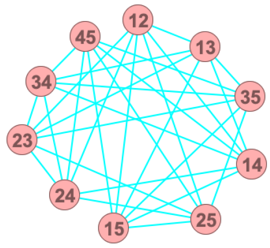
Moreover, the complement $\overline{L(W_6)}$ of $L(W_6)$ is



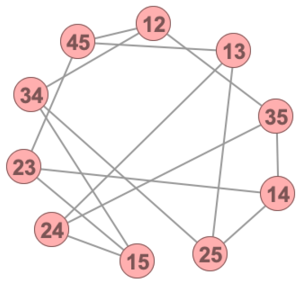
We now turn to the complete graph K_5. We know that it contains $\binom{5}{2} = 10$ edges, which we can label as follows: suppose that the vertices of K_5 are denoted by 1, 2, 3, 4 and 5; then the edge joining e.g. vertices 1 and 4 can be referred to as edge 14.

Observe also that each edge of K_5 is adjacent to exactly six other edges of K_5 (why?), so $L(K_5)$ will be a 6-regular graph on 10 vertices.

Below is a drawing of $L(K_5)$ based on the labelling just described:

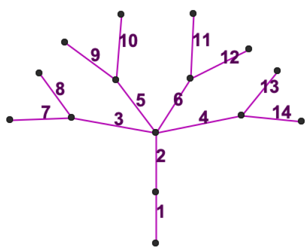


Also, here is the complement $\overline{L(K_5)}$ of $L(K_5)$ (which contains 15 edges, given that $L(K_5)$ contains $\frac{10\cdot 6}{2} = 30$ edges, and given that the complete graph on 10 vertices has $\binom{10}{2} = 45$ vertices):

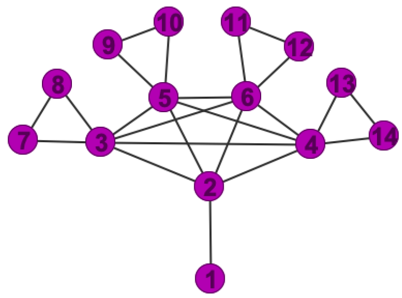


(ii) In the given picture, we have that only the graphs A, B and E are trees.

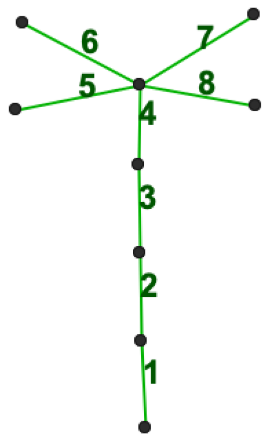
- Regarding the graph A, we note that it contains 14 edges, and thus $L(A)$ will contain 14 vertices. More specifically, we can label the edges of A as follows:



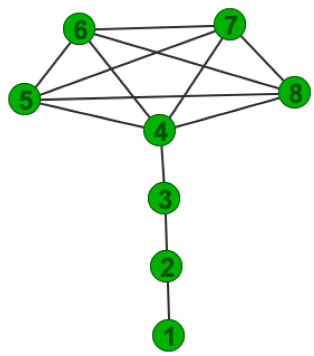
in which case its line graph will be



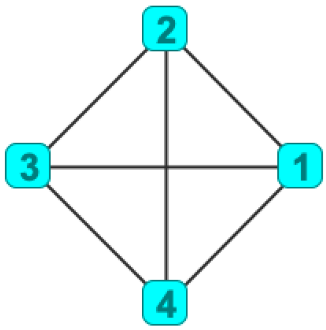
- Regarding the graph B, we note that it contains 8 edges, and thus $L(A)$ will contain 8 vertices. More specifically, we can label the edges of B as follows:



in which case its line graph will be



- Finally, we note that graph E coincides with $K_{1,4}$. It has 4 edges, any two of which are adjacent, and thus its line graph is K_4:



Problem 6. We recall that, in any bipartite graph $K_{s,t}$, the vertices in the first partite set all have degree t, while the vertices in the second partite set all have degree s.

Therefore, $\delta(K_{s,t}) = \min\{s, t\}$. If we now had that both s and t were different from 1 (and thus larger than 1, since they are both positive integers), it would follow that the minimum degree of the graph is $\geqslant 2$. But then, by Problem 7, part (ii) of this homework set, we would be able to find a cycle in $K_{s,t}$.

This contradicts the assumption that $K_{s,t}$ is a tree. Therefore, we must have that at least one of s and t is equal to 1, and hence $K_{s,t}$ turns out to be a star.

Problem 7. (i) Let s_0 be the largest possible length of a path in G (clearly we can find a maximum value here, since a path in G can contain at most all the vertices of G, so it can have length at most $|G|-1$). We need to show that $s_0 \geqslant k$.

Assume towards a contradiction that $s_0 < k$, and consider a path P_0 in G which has length s_0. Let us write $x_0, x_1, x_2, \ldots, x_{s_0-1}, x_{s_0}$ for the vertices which P_0 passes through (in the order that these vertices appear in the path, with x_0 being one of the endvertices of P_0, viewed here as its initial vertex).

Since $\delta(G) \geqslant k$, we have that $\deg(x_{s_0}) \geqslant k > s_0$, and thus we can find at least one neighbour of x_{s_0} which is not among the vertices $\{x_0, x_1, \ldots, x_{s_0-1}\}$. Say this is the vertex y_0 of G.

But then the walk

$$x_0\, x_1\, x_2\, \cdots\, x_{s_0-1}\, x_{s_0}\, y_0$$

is also a path of G (given that all the vertices are distinct), and it has length $s_0+1 > s_0$, which contradicts the assumption that P_0 had the largest possible length among paths in G.

We conclude that the assumption that s_0, the largest possible length of a path in G, is $< k$ was incorrect.

(ii) Again, let us consider a path P_1 of G of largest possible length. Then P_1 has the form

$$z_0\, z_1\, z_2\, \ldots\, z_{l-1}\, z_l$$

for some distinct vertices $z_0, z_1, z_2, \ldots, z_{l-1}$ and z_l of G, with $l \geqslant k \geqslant 2$.

Claim. All the neighbours of z_0 will be among the vertices $z_1, \ldots, z_{l-1}, z_l$. This is because if this were not the case, then we could find a neighbour y_1 of z_0 which would not be among the vertices already in the path P_1, and then the walk

$$y_1\, z_0\, z_1\, z_2\, \ldots\, z_{l-1}\, z_l$$

would be a longer path in G, contradicting the assumption that P_1 has longest possible length.

We look for the last vertex in P_1 which is a neighbour of z_0; say this is vertex z_{t_1}. Based on the claim, we know that $t_1 \geqslant \deg(z_0) \geqslant \delta(G) \geqslant k \geqslant 2$. Then the walk

$$z_0\, z_1\, z_2\, \ldots\, z_{t_1-1}\, z_{t_1}\, z_0$$

is a cycle of G which passes through $t_1 + 1 \geqslant k + 1$ vertices.