Math 227
Suggested solutions to Homework Set 4

Problem 1. First we find the cofactor matrix C of A:

$$C = \big(C_{ij}\big)_{1\leqslant i,j\leqslant 4} = \big((-1)^{i+j}\det M_{ij}\big)_{1\leqslant i,j\leqslant 4}$$

where

$$(-1)^{1+1}\det M_{11} = \det\left(\begin{pmatrix} 1 & 0 & 6 \\ 0 & 7 & 0 \\ 0 & 10 & 0 \end{pmatrix}\right) = 0,$$

$$(-1)^{1+2}\det M_{12} = -\det\left(\begin{pmatrix} 3 & 0 & 6 \\ 4 & 7 & 0 \\ 8 & 10 & 0 \end{pmatrix}\right) = 8,$$

$$(-1)^{1+3}\det M_{13} = \det\left(\begin{pmatrix} 3 & 1 & 6 \\ 4 & 0 & 0 \\ 8 & 0 & 0 \end{pmatrix}\right) = 0,$$

$$(-1)^{1+4}\det M_{14} = -\det\left(\begin{pmatrix} 3 & 1 & 0 \\ 4 & 0 & 7 \\ 8 & 0 & 10 \end{pmatrix}\right) = -5 = 6,$$

$$(-1)^{2+1}\det M_{21} = -\det\left(\begin{pmatrix} 2 & 0 & 5 \\ 0 & 7 & 0 \\ 0 & 10 & 0 \end{pmatrix}\right) = 0,$$

$$(-1)^{2+2}\det M_{22} = \det\left(\begin{pmatrix} 0 & 0 & 5 \\ 4 & 7 & 0 \\ 8 & 10 & 0 \end{pmatrix}\right) = -3 = 8,$$

$$(-1)^{2+3}\det M_{23} = -\det\left(\begin{pmatrix} 0 & 2 & 5 \\ 4 & 0 & 0 \\ 8 & 0 & 0 \end{pmatrix}\right) = 0,$$

$$(-1)^{2+4}\det M_{24} = \det\left(\begin{pmatrix} 0 & 2 & 0 \\ 4 & 0 & 7 \\ 8 & 0 & 10 \end{pmatrix}\right) = 10,$$

$$(-1)^{3+1}\det M_{31} = \det\left(\begin{pmatrix} 2 & 0 & 5 \\ 1 & 0 & 6 \\ 0 & 10 & 0 \end{pmatrix}\right) = 7,$$

$$(-1)^{3+2}\det M_{32} = -\det\left(\begin{pmatrix} 0 & 0 & 5 \\ 3 & 0 & 6 \\ 8 & 10 & 0 \end{pmatrix}\right) = -7 = 4,$$

$$(-1)^{3+3}\det M_{33} = \det\left(\begin{pmatrix} 0 & 2 & 5 \\ 3 & 1 & 6 \\ 8 & 0 & 0 \end{pmatrix}\right) = 1,$$

$$(-1)^{3+4}\det M_{34} = -\det\left(\begin{pmatrix} 0 & 2 & 0 \\ 3 & 1 & 0 \\ 8 & 0 & 10 \end{pmatrix}\right) = 5,$$

$$(-1)^{4+1}\det M_{41} = -\det\left(\begin{pmatrix} 2 & 0 & 5 \\ 1 & 0 & 6 \\ 0 & 7 & 0 \end{pmatrix}\right) = 5,$$

$$(-1)^{4+2}\det M_{42} = \det\left(\begin{pmatrix} 0 & 0 & 5 \\ 3 & 0 & 6 \\ 4 & 7 & 0 \end{pmatrix}\right) = 6,$$

$$(-1)^{4+3}\det M_{43} = -\det\left(\begin{pmatrix} 0 & 2 & 5 \\ 3 & 1 & 6 \\ 4 & 0 & 0 \end{pmatrix}\right) = -6 = 5,$$

and finally $(-1)^{4+4}\det M_{44} = \det\left(\begin{pmatrix} 0 & 2 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 7 \end{pmatrix}\right) = -9 = 2.$

Thus

$$C = \begin{pmatrix} 0 & 8 & 0 & 6 \\ 0 & 8 & 0 & 10 \\ 7 & 4 & 1 & 5 \\ 5 & 6 & 5 & 2 \end{pmatrix}.$$

We can now also find $\det(A)$ (by using e.g. the 1st row of C, thus in other words using the Laplace expansion of $\det(A)$ along the 1st row):

$$\det(A) = 2 \cdot 8 + 5 \cdot 6 = 5 + 8 = 2.$$

We conclude that

$$A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{2} \begin{pmatrix} 0 & 0 & 7 & 5 \\ 8 & 8 & 4 & 6 \\ 0 & 0 & 1 & 5 \\ 6 & 10 & 5 & 2 \end{pmatrix}$$

$$= 6 \begin{pmatrix} 0 & 0 & 7 & 5 \\ 8 & 8 & 4 & 6 \\ 0 & 0 & 1 & 5 \\ 6 & 10 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 9 & 8 \\ 4 & 4 & 2 & 3 \\ 0 & 0 & 6 & 8 \\ 3 & 5 & 8 & 1 \end{pmatrix}.$$

We now use Gauss-Jordan elimination instead to find A^{-1}:

$$(A \mid I_4) = \left(\begin{array}{cccc|cccc} 0 & 2 & 0 & 5 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 6 & 0 & 1 & 0 & 0 \\ 4 & 0 & 7 & 0 & 0 & 0 & 1 & 0 \\ 8 & 0 & 10 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \underset{R_1 \leftrightarrow R_3}{\sim} \left(\begin{array}{cccc|cccc} 3 & 1 & 0 & 6 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 5 & 1 & 0 & 0 & 0 \\ 4 & 0 & 7 & 0 & 0 & 0 & 1 & 0 \\ 8 & 0 & 10 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{4R_1,\ 3R_3,\ 7R_4}{\sim} \left(\begin{array}{cccc|cccc} 1 & 4 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 5 & 1 & 0 & 0 & 0 \\ 1 & 0 & 10 & 0 & 0 & 0 & 3 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 & 0 & 7 \end{array}\right) \underset{\substack{R_3-R_1 \\ R_4-R_1}}{\sim} \left(\begin{array}{cccc|cccc} 1 & 4 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 7 & 10 & 9 & 0 & 7 & 3 & 0 \\ 0 & 7 & 4 & 9 & 0 & 7 & 0 & 7 \end{array}\right)$$

$$\underset{6R_2,\ 8R_3,\ 8R_4}{\sim} \left(\begin{array}{cccc|cccc} 1 & 4 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 8 & 6 & 0 & 0 & 0 \\ 0 & 1 & 3 & 6 & 0 & 1 & 2 & 0 \\ 0 & 1 & 10 & 6 & 0 & 1 & 0 & 1 \end{array}\right) \underset{\substack{R_3-R_2 \\ R_4-R_2}}{\sim} \left(\begin{array}{cccc|cccc} 1 & 4 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 8 & 6 & 0 & 0 & 0 \\ 0 & 0 & 3 & 9 & 5 & 1 & 2 & 0 \\ 0 & 0 & 10 & 9 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\underset{4R_3,\ 10R_4}{\sim} \left(\begin{array}{cccc|cccc} 1 & 4 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 8 & 6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 9 & 4 & 8 & 0 \\ 0 & 0 & 1 & 2 & 6 & 10 & 0 & 10 \end{array}\right) \underset{R_4 - R_3}{\sim} \left(\begin{array}{cccc|cccc} 1 & 4 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 8 & 6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 9 & 4 & 8 & 0 \\ 0 & 0 & 0 & 10 & 8 & 6 & 3 & 10 \end{array}\right)$$

$$\underset{10R_4}{\sim} \left(\begin{array}{cccc|cccc} 1 & 4 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 8 & 6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 9 & 4 & 8 & 0 \\ 0 & 0 & 0 & 1 & 3 & 5 & 8 & 1 \end{array}\right) \underset{\substack{R_1-R_4,\ R_2-R_4 \\ R_3-R_4}}{\sim} \left(\begin{array}{cccc|cccc} 1 & 4 & 0 & 0 & 5 & 5 & 6 & 9 \\ 0 & 1 & 0 & 0 & 4 & 4 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 1 & 3 & 5 & 8 & 1 \end{array}\right)$$

$$\underset{R_1 - R_2}{\sim} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 9 & 8 \\ 0 & 1 & 0 & 0 & 4 & 4 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 1 & 3 & 5 & 8 & 1 \end{array}\right) = (I_4 \mid A^{-1}).$$

We remark that our answer here agrees with the answer above *(and each method can serve as a check for the other one; of course the latter method will in general require much fewer computations, especially in similar problems of much larger size).*

Problem 2. (i) We have that

$$\begin{aligned}
p_{A_1}(t) &= \det\left(\begin{pmatrix} 3-t & 0 & 1 \\ -2.5 & -2-t & -1.5 \\ 2 & 4 & -2-t \end{pmatrix}\right) \\
&= (3-t)\cdot\det\left(\begin{pmatrix} -2-t & -1.5 \\ 4 & -2-t \end{pmatrix}\right) \;+\; 1\cdot\det\left(\begin{pmatrix} -2.5 & -2-t \\ 2 & 4 \end{pmatrix}\right) \\
&= (3-t)((2+t)^2 + 1.5\cdot 4) + (-10 + 2(2+t)) \\
&= (3-t)(4 + 4t + t^2 + 6) - 6 + 2t \\
&= 30 + 12t + 3t^2 - 10t - 4t^2 - t^3 - 6 + 2t \\
&= -t^3 - t^2 + 4t + 24.
\end{aligned}$$

We now use the hint, that one of the roots of p_{A_1} is an integer k. Then we will have that $t-k$ divides $p_{A_1}(t)$, and hence, because $p_{A_1}(t)$ has only integer coefficients, k will need to divide the constant term of $p_{A_1}(t)$. Indeed, assume that $q(t) = b_2t^2 + b_1t + b_0$ is such that

$$(t-k)(b_2t^2 + b_1t + b_0) = (t-k)q(t) = p_{A_1}(t) = -t^3 - t^2 + 4t + 24.$$

We can then conclude that $b_2 = -1$, which furthermore implies that $-1 = b_1 - b_2k = b_1 + k \;\Rightarrow\; b_1 = -k-1 \in \mathbb{Z}$; this in turn gives us that $4 = b_0 - b_1k = b_0 + k(k+1) \;\Rightarrow\; b_0 = 4 - k(k+1) \in \mathbb{Z}$, which finally allows us to conclude that $24 = -b_0k \;\Rightarrow\; k$ divides 24.

We now observe that the divisors of 24 are the integers $1, 2, 3, 4, 6, 8, 12, 24$ and their additive inverses. Trying out each one of them, we see that 3 is definitely a root of $p_{A_1}(t)$. We can then also reverse the above process to find that the polynomial $q_1(t) = -t^2 - 4t - 8$ is such that

$$p_{A_1}(t) = (t-3)q_1(t) = (t-3)(-t^2 - 4t - 8).$$

Finally, we find the roots of the quadratic polynomial $q_1(t)$ using standard methods:

$$\mathrm{root}_{1,2} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\cdot(-1)\cdot(-8)}}{2\cdot(-1)} = \frac{4 \pm 4\boldsymbol{i}}{-2} = -2 \mp 2\boldsymbol{i}.$$

We conclude that the eigenvalues of A_1 are the numbers $3, -2+2\boldsymbol{i}$ and $-2-2\boldsymbol{i}$.

We now find the eigenspace for each of these eigenvalues.

We want to find $N(A_1 - 3I_3)$, where

$$A_1 - 3I_3 = \begin{pmatrix} 0 & 0 & 1 \\ -2.5 & -5 & -1.5 \\ 2 & 4 & -5 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0.6 \\ 1 & 2 & -2.5 \end{pmatrix} \sim$$
$$\sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0.6 \\ 0 & 0 & -3.1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0.6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is a Row Echelon Form of $A_1 - 3I_3$, so it has the same nullspace. Given that its only free variable is the variable x_2, the dimension of this nullspace is 1; moreover, if we set $x_2 = 1$, we can find the non-zero solution $\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -2 \\ 1 \\ 0 \end{smallmatrix}\right)$, which spans $N(A_1 - 3I_3)$.

Next we want to find $N(A_1 - (-2 + 2\boldsymbol{i})I_3)$, where

$$A_1 - (-2 + 2\boldsymbol{i})I_3 = \begin{pmatrix} 5 - 2\boldsymbol{i} & 0 & 1 \\ -2.5 & -2\boldsymbol{i} & -1.5 \\ 2 & 4 & -2\boldsymbol{i} \end{pmatrix} \xrightarrow{R_1 + 2R_2 + \boldsymbol{i}R_3 \to R_1'}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ -2.5 & -2\boldsymbol{i} & -1.5 \\ 2 & 4 & -2\boldsymbol{i} \end{pmatrix} \sim \begin{pmatrix} 1 & 0.8\boldsymbol{i} & 3.75 \\ 1 & 2 & -\boldsymbol{i} \\ 0 & 0 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0.8\boldsymbol{i} & 3.75 \\ 0 & 2 - 0.8\boldsymbol{i} & -3.75 - \boldsymbol{i} \\ 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is a Row Echelon Form of $A_1 - (-2+2\boldsymbol{i})I_3$, so it has the same nullspace. Given that its only free variable is the variable x_3, the dimension of this nullspace is 1; moreover, if we set $x_3 = 1$, we can find the non-zero solution $\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{0.8-3i}{2-0.8i} - 3.75 \\ \frac{3.75+i}{2-0.8i} \\ 1 \end{smallmatrix}\right)$, which spans $N(A_1 - (-2 + 2\boldsymbol{i})I_3)$.

Finally we find $N(A_1 - (-2 - 2\boldsymbol{i})I_3)$, where

$$A_1 - (-2 - 2\boldsymbol{i})I_3 = \begin{pmatrix} 5 + 2\boldsymbol{i} & 0 & 1 \\ -2.5 & 2\boldsymbol{i} & -1.5 \\ 2 & 4 & 2\boldsymbol{i} \end{pmatrix} \xrightarrow{R_1 + 2R_2 - \boldsymbol{i}R_3 \to R_1'}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ -2.5 & 2\boldsymbol{i} & -1.5 \\ 2 & 4 & 2\boldsymbol{i} \end{pmatrix} \sim \begin{pmatrix} 1 & -0.8\boldsymbol{i} & 3.75 \\ 1 & 2 & \boldsymbol{i} \\ 0 & 0 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & -0.8\boldsymbol{i} & 3.75 \\ 0 & 2 + 0.8\boldsymbol{i} & -3.75 + \boldsymbol{i} \\ 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is a Row Echelon Form of $A_1-(-2-2\boldsymbol{i})I_3$, so it has the same nullspace. Given that its only free variable is the variable x_3, the dimension of this nullspace is 1; moreover, if we set $x_3 = 1$, we can find the non-zero solution $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{0.8+3\boldsymbol{i}}{2+0.8\boldsymbol{i}} - 3.75 \\ \frac{3.75-\boldsymbol{i}}{2+0.8\boldsymbol{i}} \\ 1 \end{pmatrix}$, which spans $N(A_1 - (-2-2\boldsymbol{i})I_3)$.

(ii) We have that

$$\begin{aligned}
p_{A_2}(t) &= \det\left(\begin{pmatrix} -t & 1 & 2 \\ 1 & 4-t & 3 \\ 3 & 2 & 4-t \end{pmatrix}\right) \\
&= -t\cdot\det\left(\begin{pmatrix} 4-t & 3 \\ 2 & 4-t \end{pmatrix}\right) - 1\cdot\det\left(\begin{pmatrix} 1 & 3 \\ 3 & 4-t \end{pmatrix}\right) + 2\cdot\det\left(\begin{pmatrix} 1 & 4-t \\ 3 & 2 \end{pmatrix}\right) \\
&= -t\cdot\big((4-t)^2-1\big) - (4-t-4) + 2\big(2-3(4-t)\big) \\
&= -t(1-3t+t^2-1)+t+2(2-2+3t) \\
&= -t^3+3t^2+2t = t(-t^2+3t+2).
\end{aligned}$$

We thus see that 0 is an eigenvalue of A_2. Next, by inspection, that is, by plugging every element of $\mathbb{Z}_5$ into $q_2(t) = -t^2+3t+2$, we see that $p_{A_2}(t)$ has no other roots in $\mathbb{Z}_5$.

We conclude that A_2 has only one eigenvalue, equal to 0. We now find the corresponding eigenspace, which is the nullspace of $A_2 - 0I_3 = A_2$. We have

$$A_2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 4 & 3 \\ 3 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 2 \\ 1 & 4 & 3 \\ 1 & 4 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 2 \\ 1 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is a Row Echelon Form of A_2, so it has the same nullspace. Given that its only free variable is the variable x_3, the dimension of this nullspace is 1; moreover, if we set $x_3 = 1$, we can find the non-zero solution $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, which spans $N(A_2)$.

(iii) We have that

$$
\begin{aligned}
p_{A_3}(t) = \det\left(\begin{pmatrix} 2-t & 0 & 6 \\ 0 & 1-t & 2 \\ 3 & 0 & 5-t \end{pmatrix}\right) \\
&= (2-t)\cdot\det\left(\begin{pmatrix} 1-t & 2 \\ 0 & 5-t \end{pmatrix}\right) + 6\cdot\det\left(\begin{pmatrix} 0 & 1-t \\ 3 & 0 \end{pmatrix}\right) \\
&= (2-t)\cdot(1-t)\cdot(5-t) - 6\cdot 3\cdot(1-t) \\
&= (1-t)\big((2-t)(5-t)-4\big) \\
&= (1-t)\big(3+t^2-4\big) = (1-t)(t^2-1) = -(t-1)^2(t+1).
\end{aligned}
$$

We conclude that the eigenvalues of A_3 are the elements 1 and $-1 = 6$ of $\mathbb{Z}_7$ (and also that the eigenvalue 1 has algebraic multiplicity 2).

We now find the corresponding eigenspace for each of these eigenvalues. We want to find $N(A_3 - I_3)$, where

$$
A_3 - I_3 = \begin{pmatrix} 1 & 0 & 6 \\ 0 & 0 & 2 \\ 3 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 6 \\ 0 & 0 & 2 \\ 1 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.
$$

The last matrix is a Row Echelon Form of $A_3 - I_3$, so it has the same nullspace. Given that its only free variable is the variable x_2, the dimension of this nullspace is 1 **(and thus we see that the geometric multiplicity of the eigenvalue 1 is equal to 1 and hence smaller than its algebraic multiplicity).** Moreover, if we set $x_2 = 1$, we can find the non-zero solution $\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, which spans $N(A_3 - I_3)$.

Next, we also need to find $N(A_3 + I_3) = N(A_3 - (-1)I_3)$, where

$$
A_3 + I_3 = \begin{pmatrix} 3 & 0 & 6 \\ 0 & 2 & 2 \\ 3 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.
$$

The last matrix is a Row Echelon Form of A_3+I_3, so it has the same nullspace. Given that its only free variable is the variable x_3, the dimension of this nullspace is 1; moreover, if we set $x_3 = 1$, we can find the non-zero solution $\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 5 \\ -1 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 5 \\ 6 \\ 1 \end{smallmatrix}\right)$, which spans $N(A_3 + I_3)$.

Problem 3. (I) We first show that, if f is injective, then the vectors $\phi(\bar{u}_1),\ \phi(\bar{u}_2), \dots, \phi(\bar{u}_k)$ are linearly independent. Note that, since f extends ϕ, these vectors coincide with the vectors $f(\bar{u}_1),\ f(\bar{u}_2), \dots, f(\bar{u}_k)$ respectively.

Also, if we assume that f is injective, any two of these vectors will be different. Thus, showing that the vectors $f(\bar{u}_1), f(\bar{u}_2), \dots, f(\bar{u}_k)$ are linearly independent will be equivalent to showing that the set $\big\{f(\bar{u}_1),\ f(\bar{u}_2), \dots, f(\bar{u}_k)\big\}$ is a linearly independent subset of V_2.

Given that the set $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k\}$ is a basis of V_1 by our assumptions, and thus a linearly independent subset of it, what we need to show reduces by the above to HW3, Problem 7.

Next we show that, if the vectors $\phi(\bar{u}_1),\ \phi(\bar{u}_2), \dots, \phi(\bar{u}_k)$ are linearly independent, then f will be injective.

Assume that the vectors $\phi(\bar{u}_1),\ \phi(\bar{u}_2), \dots, \phi(\bar{u}_k)$ are linearly independent, and consider $\bar{x}, \bar{y} \in V_1$ such that $f(\bar{x}) = f(\bar{y})$. We need to show that $\bar{x} = \bar{y}$.

Given that the set $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k\}$ is a basis of V_1, we can find scalars $\lambda_1, \lambda_2, \dots, \lambda_k,\ \mu_1, \mu_2, \dots, \mu_k$ such that

$$\begin{aligned} \bar{x} &= \lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \cdots + \lambda_k\bar{u}_k \\ \text{and}\ \ \bar{y} &= \mu_1\bar{u}_1 + \mu_2\bar{u}_2 + \cdots + \mu_k\bar{u}_k \end{aligned}$$

(we also recall that these are the only ways we can write $\bar{x}$ and $\bar{y}$ as linear combinations of the vectors $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k$).

But then, by the way we define the linear extension f of ϕ, we know that

$$\begin{aligned} f(\bar{x}) &= \lambda_1\phi(\bar{u}_1) + \lambda_2\phi(\bar{u}_2) + \cdots + \lambda_k\phi(\bar{u}_k) \\ \text{and}\ \ f(\bar{y}) &= \mu_1\phi(\bar{u}_1) + \mu_2\phi(\bar{u}_2) + \cdots + \mu_k\phi(\bar{u}_k). \end{aligned}$$

We can then write

$$\begin{aligned} \bar{0}_{V_2} &= f(\bar{x}) - f(\bar{y}) \\ &= \big(\lambda_1\phi(\bar{u}_1) + \lambda_2\phi(\bar{u}_2) + \cdots + \lambda_k\phi(\bar{u}_k)\big) - \big(\mu_1\phi(\bar{u}_1) + \mu_2\phi(\bar{u}_2) + \cdots + \mu_k\phi(\bar{u}_k) \\ &= (\lambda_1 - \mu_1)\phi(\bar{u}_1) + (\lambda_2 - \mu_2)\phi(\bar{u}_2) + \cdots + (\lambda_k - \mu_k)\phi(\bar{u}_k) \end{aligned}$$

Given that the vectors $\phi(\bar{u}_1),\ \phi(\bar{u}_2), \dots, \phi(\bar{u}_k)$ have been assumed linearly independent, this implies that all the coefficients $\lambda_i - \mu_i$ are equal to $0_{\mathbb{F}}$. In other words, for every $i \in \{1, 2, \dots, k\}$, $\lambda_i = \mu_i$, and hence

$$\bar{x} = \lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \cdots + \lambda_k\bar{u}_k = \mu_1\bar{u}_1 + \mu_2\bar{u}_2 + \cdots + \mu_k\bar{u}_k = \bar{y}.$$

Given that we showed that $f(\bar{x}) = f(\bar{y})$ implies $\bar{x} = \bar{y}$ for an arbitrary pair of vectors from V_1, we can conclude that f is injective.

(II) We first show that, if g is surjective, then the $\{\psi(\bar{u}) : \bar{u} \in \mathcal{B}\}$ is a spanning set of V_2. Note that, since g extends ψ, this set coincides with the set $\{g(\bar{u}) : \bar{u} \in \mathcal{B}\}$.

If we assume that g is surjective, then we will have that $V_2 = \mathrm{Range}(g)$, so it will suffice to show that $\{g(\bar{u}) : \bar{u} \in \mathcal{B}\}$ is a spanning set of $\mathrm{Range}(g)$.

By our assumptions, $\mathcal{B}$ is a basis of V_1, and hence a spanning set of V_1. Therefore, what we need to show reduces by the above to Problem 4, part (b) of the 1st Midterm Exam.

Next we show that, if the set $\{\psi(\bar{u}) : \bar{u} \in \mathcal{B}\}$ is a spanning set of V_2, then g is surjective.

Assume that the set $\{\psi(\bar{u}) : \bar{u} \in \mathcal{B}\}$ is a spanning set of V_2, and consider $\bar{w} \in V_2$. We need to show that there exists $\bar{z} \in V_1$ such that $\bar{w} = g(\bar{z})$.

Since the set $\{\psi(\bar{u}) : \bar{u} \in \mathcal{B}\}$ spans V_2, we can find scalars $\mu_1, \mu_2, \ldots, \mu_k$ such that

$$\bar{w} = \mu_1\psi(\bar{u}_1) + \mu_2\psi(\bar{u}_2) + \cdots + \mu_k\psi(\bar{u}_k).$$

Also, since g extends ψ, we can rewrite the latter as follows:

$$\bar{w} = \mu_1 g(\bar{u}_1) + \mu_2 g(\bar{u}_2) + \cdots + \mu_k g(\bar{u}_k),$$

which by the linearity of g implies that,

$$\begin{aligned} \bar{w} &= g(\mu_1\bar{u}_1) + g(\mu_2\bar{u}_2) + \cdots + g(\mu_k\bar{u}_k) \\ &= g\big(\mu_1\bar{u}_1 + \mu_2\bar{u}_2 + \cdots + \mu_k\bar{u}_k\big) \\ &= g(\bar{z}) \end{aligned}$$

with $\bar{z} = \mu_1\bar{u}_1 + \mu_2\bar{u}_2 + \cdots + \mu_k\bar{u}_k \in V_1$.

Thus we have shown that the arbitrary element in V_2 is the image under g of some element in V_1. This shows that g is surjective.

Problem 4. By Main Theorem C, we know that

$$\dim_{\mathbb{F}} \mathrm{RS}(A) = \dim_{\mathbb{F}} \mathrm{CS}(A).$$

Thus, by the results we discussed in class, about Bases and Linear Functions (see the March 9 Lecture Notes), we know that the two spaces, viewed as vector spaces over $\mathbb{F}$, are isomorphic.

In other words, we can find an isomorphism $f : \mathrm{RS}(A) \to \mathrm{CS}(A)$, or more specifically

a bijective function $f : \mathrm{RS}(A) \to \mathrm{CS}(A)$ that also satisfies

- for every two $\bar{x}, \bar{y} \in \mathrm{RS}(A)$, $f(\bar{x} + \bar{y}) = f(\bar{x}) + f(\bar{y})$;
- for every $r \in \mathbb{F}$ and for every $\bar{z} \in \mathrm{RS}(A)$, $f(r \cdot \bar{z}) = r \cdot f(\bar{z})$.

Next, we observe that, when we view $\mathrm{RS}(A)$ and $\mathrm{CS}(A)$ as vector spaces over the subfield $\mathbb{K}$, the corresponding operations of vector addition remain the same, while the operations of scalar multiplication are simply restrictions of the corresponding operations of scalar multiplication that we had before (the latter were maps from $\mathbb{F} \times \mathrm{RS}(A)$ to $\mathrm{RS}(A)$, or from $\mathbb{F} \times \mathrm{CS}(A)$ to $\mathrm{CS}(A)$ respectively, while we now only need to consider their respective restrictions which have domain $\mathbb{K} \times \mathrm{RS}(A)$ or $\mathbb{K} \times \mathrm{CS}(A)$).

Thus, if we view $\mathrm{RS}(A)$ and $\mathrm{CS}(A)$ as vector spaces over the subfield $\mathbb{K}$, the above function f is still a bijective function from $\mathrm{RS}(A)$ to $\mathrm{CS}(A)$ (note that $\mathrm{RS}(A)$ and $\mathrm{CS}(A)$ still consist of the same elements as before), and it also still satisfies:

$$\text{for every two } \bar{x}, \bar{y} \in \mathrm{RS}(A), \;\; f(\bar{x} + \bar{y}) = f(\bar{x}) + f(\bar{y}).$$

Finally, if $s \in \mathbb{K}$ and $\bar{z} \in \mathrm{RS}(A)$, then we also have that $s \in \mathbb{F}$ and $\bar{z} \in \mathrm{RS}(A)$, and thus, by what we stated previously,

$$f(s \cdot \bar{z}) = s \cdot f(\bar{z}).$$

Thus the function $f : \mathrm{RS}(A) \to \mathrm{CS}(A)$ is still a linear function when we view $\mathrm{RS}(A)$ and $\mathrm{CS}(A)$ as vector spaces over $\mathbb{K}$. Since it is also bijective, we can conclude that it is an isomorphism from the $\mathbb{K}$-vector space $\mathrm{RS}(A)$ to the $\mathbb{K}$-vector space $\mathrm{CS}(A)$.

But then this implies that the two vector spaces have the same dimension over $\mathbb{K}$, which is what we wanted to show.

Problem 5. (i) Let us write f_1 for the rotation by angle $\frac{\pi}{6}$ about the origin, f_2 for the reflection across the line $x = y$, and finally f_3 for the rotation by angle $-\frac{\pi}{6}$ about the origin. We observe that the desired linear transformation f is equal to $f_3 \circ f_2 \circ f_1$, and thus if A_{f_i} is the standard matrix representation of the function f_i for $i = 1, 2, 3$, we should have

$$A_f = A_{f_3} \cdot A_{f_2} \cdot A_{f_1}$$

(given that composition of such linear transformations (that have a matrix representation) corresponds to multiplication of the corresponding matrices; how can we justify this?).

Thus it suffices to find A_{f_i} for each $i = 1, 2, 3$, and then consider the above product. We recall that

$$A_{f_i} = \begin{pmatrix} | & | \\ f_i(\bar{e}_1) & f_i(\bar{e}_2) \\ | & | \end{pmatrix}.$$

But f_1 maps the vector $\bar{e}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ to the vector $\begin{pmatrix} \cos(\pi/6) \\ \sin(\pi/6) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$, while it maps the vector $\bar{e}_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ to the vector $\begin{pmatrix} \cos(\pi/2+\pi/6) \\ \sin(\pi/2+\pi/6) \end{pmatrix} = \begin{pmatrix} -\sin(\pi/6) \\ \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$. We thus see that

$$A_{f_1} = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}.$$

Very similarly, we find that

$$A_{f_3} = \begin{pmatrix} \cos(-\pi/6) & -\sin(-\pi/6) \\ \sin(-\pi/6) & \cos(-\pi/6) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}.$$

Finally, concerning the reflection f_2, we can see that it maps the vector $\bar{e}_1$ to the vector $\bar{e}_2$, and the vector $\bar{e}_2$ to the vector $\bar{e}_1$. It follows that

$$A_{f_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can conclude that a matrix representation for the desired linear transformation f is

$$\begin{aligned} A_f = A_{f_3} \cdot A_{f_2} \cdot A_{f_1} &= \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}. \end{aligned}$$

(ii) Similarly to part (i), let us write g_1 for the rotation around the x-axis by angle $\frac{\pi}{3}$, and g_2 for the rotation around the z-axis by angle $\frac{\pi}{4}$. We observe that the desired linear transformation g is equal to $g_2 \circ g_1$, and thus if A_{g_i} is the standard matrix representation of the function g_i for $i = 1, 2$, we should have

$$A_g = A_{g_2} \cdot A_{g_1}.$$

Again we recall that

$$A_{g_i} = \begin{pmatrix} | & | & | \\ g_i(\bar{e}_1) & g_i(\bar{e}_2) & g_i(\bar{e}_3) \\ | & | & | \end{pmatrix}.$$

But the function g_1 leaves the vector $\bar{e}_1$ unchanged (since the x-axis is the axis of the rotation g_1), while it maps the vector $\bar{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ to the vector $\begin{pmatrix} 0 \\ \cos(\pi/3) \\ \sin(\pi/3) \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{pmatrix}$, and the vector $\bar{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to the vector $\begin{pmatrix} 0 \\ -\sin(\pi/3) \\ \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{3}/2 \\ 1/2 \end{pmatrix}$. In other words,

$$A_{g_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/3) & -\sin(\pi/3) \\ 0 & \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

Similarly, we see that the function g_2 leaves the vector $\bar{e}_3$ unchanged, while it maps the vector $\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ to the vector $\begin{pmatrix} \cos(\pi/4) \\ \sin(\pi/4) \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$, and the vector $\bar{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ to the vector $\begin{pmatrix} -\sin(\pi/4) \\ \cos(\pi/4) \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$. In other words,

$$A_{g_2} = \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can conclude that

$$\begin{aligned} A_g = A_{g_2} \cdot A_{g_1} &= \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/4 & \sqrt{6}/4 \\ \sqrt{2}/2 & \sqrt{2}/4 & -\sqrt{6}/4 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}. \end{aligned}$$