Hints and solutions to Problems 1

1) Let $g \in G$. As G is a finite group the set
$$\left\{ g^l \,\middle|\, l \in \mathbb{N} \right\}$$
is finite and so there exist natural numbers $n > m$, such that $g^m = g^n$. Multiplying both sides by g^{-m} gives $g^{n-m} = e$.

2) Let $\{a, b, c\} \subseteq X$ be a subset of three different elements of X. Define maps $f, g : X \longrightarrow X$ by $f(a) = b$, $f(b) = c$, $f(c) = a$, and $f(x) = x$ for all $x \in X \setminus \{a, b, c\}$; and by $g(a) = b$, $g(b) = a$, and $g(x) = x$ for all $x \in X \setminus \{a, b\}$. Then both f and g are bijections (check this!), and we have $g(f(a)) = a$ but $f(g(a)) = c$, *i.e.* $f \circ g \neq g \circ f$.

3) One direction is clear. For the other, since X is finite l_x and r_x injective implies that both maps are also surjective and so bijections. Hence given $x \in X$ there exists an element $e_x \in X$, such that $e_x \cdot x = x$. Multiplying this equation by $z \in X$ on the right gives (using the associative law)
$$e_x \cdot (x \cdot z) \,=\, (e_x \cdot x) \cdot z \,=\, x \cdot z$$
for all $z \in X$. Since l_x is onto as well the set of all $x \cdot z$, $z \in X$, is equal X, and so we get $e_x \cdot y = y$ for all $y \in X$, *i.e.* e_x is a (left) neutral element. Since r_y is onto for all $y \in X$ there exists for all $y \in X$ an element $u_y \in X$, such that $u_y \cdot y = e_x$, *i.e.* every $y \in X$ has an inverse with respect to the neutral element e_x. It follows that X is a group.

   The example of $\mathbb{N}$ with the usual addition as operation shows that the claim is wrong if X is not finite.

4) Verification of the axioms.

5) This is clear if G is a group with one or two elements.

   To study the cases that G has 3 or 4 elements we note first that for $g \neq e$, where e is the neutral element, we have $g \cdot h \neq h \neq h \cdot g$ for all $h \in G$. This follows from Lemma 1.5 (i) of the Lecture Notes as the neutral element G solves $X \cdot h = h$ and $h \cdot X = h$.

   If now $G = \{e, a, b\}$ is a group with three elements (e denotes the neutral element) this implies $a \cdot b = e$ and so $b = a^{-1}$, which commutes with a, *i.e.* G is commutative.

   If $G = \{e, a, b, c\}$ is a group with 4 elements this implies $a \cdot b = e$ or $a \cdot b = c$. In the former case $b = a^{-1}$, and so a and b commute, and in the latter $b \cdot a$ can not be equal e as then $b = a^{-1}$ and so $a \cdot b = e$ as well. Since $b \cdot a$ can also not be equal a or b we get $b \cdot a = c$ and so $a \cdot b = b \cdot a$. The proof that a and c, and also b and c commute is analogous.