Reminder Examples of complex inner product spaces

0′) $W = \mathbb{C}^n$. For $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n$ we set

$$\langle \bar{x}, \bar{y} \rangle := \sum_{i=1}^{n} x_i \overline{y_i}.$$

1′) $W = \mathbb{C}^{n\times n}$. First define for every $B = (b_{ij}) \in \mathbb{C}^{n\times n}$, the <u>conjugate transpose</u> B^* of B as follows:

$$B^* := \left(\overline{b_{ji}}\right)_{1\le i,j\le n}$$

Then define for $A, B \in \mathbb{C}^{n\times n}$

$$\langle A, B \rangle := tr(AB^*)$$

Important example for Fourier Analysis

2′) Let $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ (the unit circle in $\mathbb{C} \equiv \mathbb{R}^2$). Set $W = \mathbf{C}(\mathbb{T})$ (the space of all continuous functions from $\mathbb{T}$ to $\mathbb{C}$, that is, complex-valued functions; given that every point z_0 on $\mathbb{T}$ can be uniquely identified with an angle $\theta_0 \in [0, 2\pi)$ because we have

$$z_0 = \underbrace{Re(z_0) + i\,Im(z_0)}_{\text{their squares should add to } 1} =$$

$$= \cos(\theta_0) + i\sin(\theta_0) = e^{i\theta_0},$$

functions in $\mathbf{C}(\mathbb{T})$ can be identified with continuous functions $\tilde{f} : [0, 2\pi] \to \mathbb{C}$ which satisfy $\tilde{f}(0) = \tilde{f}(2\pi)$).

We then define for $f, g \in \mathbf{C}(\mathbb{T})$

$$\langle f, g\rangle := \frac{1}{2\pi}\oint_{\mathbb{T}} f\cdot\bar{g} = \frac{1}{2\pi}\int_0^{2\pi}\left(\tilde{f}\cdot\overline{\tilde{g}}\right)(t)\,dt.$$

<u>Practice</u> check that indeed all the above are examples of complex inner product spaces.

<u>Important Remark</u> An inner product defined on a complex vector space W is a function

$$\langle\cdot,\cdot\rangle : W\times W \longrightarrow \mathbb{C}.$$

Recall that in $\mathbb{C}$ we cannot have an ordering that is <u>compatible</u> with the field structure, therefore we cannot ask whether we have $z \geq 0$ for a complex number z.

However, we still can require positive-definiteness of the inner product, exactly as stated above: $\langle\cdot,\cdot\rangle$ should satisfy

$$\boxed{\text{for every } \bar{x}\in W \quad \langle\bar{x},\bar{x}\rangle\geq 0 \quad \text{and } \langle\bar{x},\bar{x}\rangle = 0 \text{ iff } \bar{x}=\bar{0}_W.}$$

This makes sense to ask because if $\langle\bar{x},\bar{x}\rangle = z_0\in\mathbb{C}$, then

conjugate symmetry $\Rightarrow z_0 = \langle\bar{x},\bar{x}\rangle = \overline{\langle\bar{x},\bar{x}\rangle} = \overline{z_0}$

$\Rightarrow \mathrm{Re}(z_0) + i\,\mathrm{Im}(z_0) = \mathrm{Re}(z_0) - i\,\mathrm{Im}(z_0) \Rightarrow \mathrm{Im}(z_0) = 0.$

Thus we always have

$$\langle\bar{x},\bar{x}\rangle\in\mathbb{R},$$

and then it makes sense to ask whether positive-definiteness is also satisfied.

Given the previous remark, it also makes sense to define the <u>length</u> (or <u>norm</u>) of vectors $\bar{x}$ in a complex inner product space W exactly as for real inner product spaces.

<u>Definition</u> Let $(W, <\cdot,\cdot>)$ be a complex inner product space. For every $\bar{x} \in W$ we define

$$\|\bar{x}\| := \sqrt{<\bar{x}, \bar{x}>}.$$

This is the <u>length</u> (or <u>norm</u>) of $\bar{x}$.

<u>Definition</u> Let $(V, <\cdot,\cdot>)$ be an inner product space (real or complex). Once we have defined a notion of "length" of vectors in V, we can define a notion of "distance" between any two vectors too (here we will rely on the vector space structure too):
for $\bar{x}, \bar{y} \in V$

$$\text{dist}(\bar{x}, \bar{y}) := \|\bar{x} - \bar{y}\| = \sqrt{<\bar{x}-\bar{y}, \bar{x}-\bar{y}>}.$$

<u>Remark</u> This is truly a notion of distance:

1) $\text{dist}(\bar{x}, \bar{y}) \geq 0$ always, and $\text{dist}(\bar{x}, \bar{y}) = 0$ iff $\bar{x} = \bar{y}$.
2) $\text{dist}(\bar{x}, \bar{y}) = \text{dist}(\bar{y}, \bar{x})$ (symmetric)
3) $\text{dist}(\bar{x}, \bar{z}) \leq \text{dist}(\bar{x}, \bar{y}) + \text{dist}(\bar{y}, \bar{z})$ (triangle inequality)

<u>Proof</u> Recall that $\text{dist}(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\|$ as we defined above, thus properties (1) and (2) follow immediately.

Indeed for property (1)

$$\text{dist}(\bar{x},\bar{y})=0 \text{ iff } \|\bar{x}-\bar{y}\|=0 \text{ iff } \bar{x}-\bar{y}=\bar{0}_V \text{ iff } \bar{x}=\bar{y},$$

while for property (2)

$$\text{dist}(\bar{x},\bar{y}) = \sqrt{<\bar{x}-\bar{y}, \bar{x}-\bar{y}>} = \sqrt{<-(\bar{y}-\bar{x}), -(\bar{y}-\bar{x})>}$$

$$\underset{\substack{\uparrow\\ \text{linearity}\\ \text{in the 1st}\\ \text{argument}}}{=} \sqrt{(-1)\cdot <\bar{y}-\bar{x}, -(\bar{y}-\bar{x})>} \underset{\substack{\uparrow\\ \text{linearity (or conjugate linearity)}\\ \text{in the 2nd argument}}}{=} \sqrt{(-1)\cdot(-1)\cdot <\bar{y}-\bar{x}, \bar{y}-\bar{x}>}$$

$$= \sqrt{<\bar{y}-\bar{x}, \bar{y}-\bar{x}>} = \text{dist}(\bar{y},\bar{x}).$$

Finally, to prove property (3) we need the following

<u>Lemma</u> (Cauchy-Schwartz inequality) If $(V, <\cdot,\cdot>)$ is an inner product space (real or complex), and $\bar{x},\bar{y}\in V$, then

$$|<\bar{x},\bar{y}>| \leq \|\bar{x}\|\cdot\|\bar{y}\|.$$

Also $|<\bar{x},\bar{y}>| = \|\bar{x}\|\cdot\|\bar{y}\|$ iff $\bar{x}$ and $\bar{y}$ are parallel vectors (one is a scalar multiple of the other).

<u>Proof of CS-inequality</u> We begin with the following remark which is useful in general:

For every $\bar{x}\in V$ and every $\lambda\in\mathbb{F}$ (where $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$)

$$\|\lambda\cdot\bar{x}\| = |\lambda|\cdot\|\bar{x}\|.$$

Indeed, $\|\lambda\cdot\bar{x}\| = \sqrt{<\lambda\bar{x},\lambda\bar{x}>} \underset{\substack{\uparrow\\ \text{linearity in the 1st argument}}}{=} \sqrt{\lambda\cdot<\bar{x},\lambda\bar{x}>}$

$$\underset{\substack{\uparrow\\ \text{linearity (or conjugate}\\ \text{linearity) in the 2nd}\\ \text{argument}}}{=} \sqrt{\lambda\cdot\bar{\lambda}<\bar{x},\bar{x}>} = \sqrt{|\lambda|^2\cdot<\bar{x},\bar{x}>} = |\lambda|\cdot\sqrt{<\bar{x},\bar{x}>}$$

We also need the following:

<u>Note 2</u> Given $\bar{x}, \bar{y} \in V$, we can find $\lambda_0 \in \mathbb{F}$ s.t. $\bar{y} \perp \bar{x} - \lambda_0 \bar{y}$ (which is read as "$\bar{y}$ is orthogonal to $\bar{x} - \lambda_0 \bar{y}$" and means

$$\langle \bar{y}, \bar{x} - \lambda_0 \bar{y} \rangle = 0_{\mathbb{F}} \;)$$

Indeed, either $\langle \bar{y}, \bar{x} \rangle = 0_{\mathbb{F}}$ already,

or $\langle \bar{y}, \bar{x} \rangle \neq 0_{\mathbb{F}}$ in which case we immediately know that $\bar{x} \neq \bar{0}_V$, $\bar{y} \neq \bar{0}_V$ (or equivalently $\|\bar{x}\| \neq 0$, $\|\bar{y}\| \neq 0$) (why?)

But then
$$\langle \bar{y}, \bar{x} - \lambda \bar{y} \rangle = \langle \bar{y}, \bar{x} \rangle - \bar{\lambda} \langle \bar{y}, \bar{y} \rangle$$
$$= \langle \bar{y}, \bar{x} \rangle - \bar{\lambda} \|\bar{y}\|^2$$
and to make this expression equal to 0, we just need to choose
$$\bar{\lambda}_0 = \frac{\langle \bar{y}, \bar{x} \rangle}{\|\bar{y}\|^2} \Rightarrow \lambda_0 = \frac{\overline{\langle \bar{y}, \bar{x} \rangle}}{\|\bar{y}\|^2} = \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{y}\|^2}.$$

We can now prove the C-S inequality:

if $\langle \bar{x}, \bar{y} \rangle = 0$, the conclusion follows immediately.

Otherwise, if $\langle \bar{x}, \bar{y} \rangle \neq 0$, then we already saw that $\|\bar{y}\| \neq 0$ and that for $\lambda_0 = \dfrac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{y}\|}$, $\bar{y} \perp \bar{x} - \lambda_0 \bar{y}$.

In other words,

$0 = \langle \bar{y}, \bar{x} - \lambda_0 \bar{y} \rangle$ which implies

$$\|\bar{x}\|^2 = \langle \bar{x}, \bar{x} \rangle = \langle (\bar{x} - \lambda_0 \bar{y}) + \lambda_0 \bar{y}, (\bar{x} - \lambda_0 \bar{y}) + \lambda_0 \bar{y} \rangle \overset{\substack{\text{consecutive applications} \\ \text{of linearity}}}{=}$$
$$\langle \bar{x} - \lambda_0 \bar{y}, \bar{x} - \lambda_0 \bar{y} \rangle + \cancel{\langle \lambda_0 \bar{y}, \bar{x} - \lambda_0 \bar{y} \rangle} + \cancel{\langle \bar{x} - \lambda_0 \bar{y}, \lambda_0 \bar{y} \rangle} + \langle \lambda_0 \bar{y}, \lambda_0 \bar{y} \rangle$$
$$= \|\bar{x} - \lambda_0 \bar{y}\|^2 + |\lambda_0|^2 \cdot \langle \bar{y}, \bar{y} \rangle = \|\bar{x} - \lambda_0 \bar{y}\|^2 + |\lambda_0|^2 \cdot \|y\|^2$$
$$\geq |\lambda_0|^2 \cdot \|y\|^2 = \frac{|\langle \bar{x}, \bar{y} \rangle|^2}{\|\bar{y}\|^2} \Rightarrow |\langle \bar{x}, \bar{y} \rangle| \leq \|\bar{x}\| \cdot \|\bar{y}\|.$$

Finally note that we get equality only if
$\|\bar{x}-\lambda_0\bar{y}\|^2 = 0 \Leftrightarrow \bar{x} = \lambda_0\bar{y} \Rightarrow \bar{x} \,//\, \bar{y}$

The converse is also true (and can be seen more quickly): if $\bar{x} \,//\, \bar{y}$, then

$$|\langle \bar{x}, \bar{y} \rangle| = \|\bar{x}\| \cdot \|\bar{y}\|$$

(follows by linearity and the definition of norm, as well as the remark $\|\lambda\bar{x}\| = |\lambda| \cdot \|\bar{x}\|$ $\forall \bar{x} \in V$ and $\lambda \in \mathbb{F}$).

We can finally verify the triangle inequality for the distance:
note that $\mathrm{dist}(\bar{x}, \bar{z}) = \|\bar{x}-\bar{z}\|$, $\mathrm{dist}(\bar{x},\bar{y}) = \|\bar{x}-\bar{y}\|$ and $\mathrm{dist}(\bar{y},\bar{z}) = \|\bar{y}-\bar{z}\|$.
Thus equivalently we need to prove

$$\|\bar{x}-\bar{z}\| \leq \|\bar{x}-\bar{y}\| + \|\bar{y}-\bar{z}\|$$

But

$$\|\bar{x}-\bar{z}\|^2 = \langle \bar{x}-\bar{z}, \bar{x}-\bar{z} \rangle = \langle (\bar{x}-\bar{y})+(\bar{y}-\bar{z}), (\bar{x}-\bar{y})+(\bar{y}-\bar{z}) \rangle$$

$$\underset{\text{consecutive applications of linearity}}{=} \langle \bar{x}-\bar{y}, \bar{x}-\bar{y} \rangle + \langle \bar{y}-\bar{z}, \bar{x}-\bar{y} \rangle + \langle \bar{x}-\bar{y}, \bar{y}-\bar{z} \rangle + \langle \bar{y}-\bar{z}, \bar{y}-\bar{z} \rangle$$

$$= \|\bar{x}-\bar{y}\|^2 + \|\bar{y}-\bar{z}\|^2 + \langle \bar{x}-\bar{y}, \bar{y}-\bar{z} \rangle + \overline{\langle \bar{x}-\bar{y}, \bar{y}-\bar{z} \rangle}$$

$$= \|\bar{x}-\bar{y}\|^2 + \|\bar{y}-\bar{z}\|^2 + 2\,\mathrm{Re}(\langle \bar{x}-\bar{y}, \bar{y}-\bar{z} \rangle)$$

$$\leq \|\bar{x}-\bar{y}\|^2 + \|\bar{y}-\bar{z}\|^2 + 2\,|\langle \bar{x}-\bar{y}, \bar{y}-\bar{z} \rangle|$$

$$\underset{\text{CS-inequality}}{\leq} \|\bar{x}-\bar{y}\|^2 + \|\bar{y}-\bar{z}\|^2 + 2\|\bar{x}-\bar{y}\| \cdot \|\bar{y}-\bar{z}\|$$

$$= (\|\bar{x}-\bar{y}\| + \|\bar{y}-\bar{z}\|)^2$$

$$\Rightarrow \mathrm{dist}(\bar{x}, \bar{z}) = \|\bar{x}-\bar{z}\| \leq \|\bar{x}-\bar{y}\| + \|\bar{y}-\bar{z}\| = \mathrm{dist}(\bar{x},\bar{y}) + \mathrm{dist}(\bar{y},\bar{z}).$$