Math 227
Suggested solutions to Homework Set 3

Problem 1. (i) Since A_1 is a 3×3 matrix, and since its last row has a zero entry, it should be efficient to just compute its determinant by expanding over the third row:

$$\begin{aligned}
\det(A_1) &= (-1)^{3+1} a_{3,1} \det(M_{3,1}) + (-1)^{3+3} a_{3,3} \det(M_{3,3}) \\
&= 2 \cdot \big(4\boldsymbol{i} - 3(3+\boldsymbol{i})\big) + 5\boldsymbol{i} \cdot \big(2 \cdot (3 + \boldsymbol{i} - 7\boldsymbol{i}\big) \\
&= 2 \cdot (-9 + \boldsymbol{i}) + 5\boldsymbol{i} \cdot (6 - 5\boldsymbol{i}) = 7 + 32\boldsymbol{i}.
\end{aligned}$$

(ii) We use Gaussian elimination to find a Row Echelon Form of A_2:

$$A_2 = \begin{pmatrix} 1 & 2 & 6 & 4 \\ 2 & 3 & 2 & 5 \\ 3 & 4 & 2 & 2 \\ 4 & 5 & 5 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 6 & 4 \\ 1 & 5 & 1 & 6 \\ 1 & 6 & 3 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix} = D_{2;4} D_{3;5} D_{4;2} A_2$$

$$\sim \begin{pmatrix} 1 & 2 & 6 & 4 \\ 0 & 3 & 2 & 2 \\ 0 & 4 & 4 & 6 \\ 0 & 1 & 4 & 4 \end{pmatrix} = E_{21;-1} E_{31;-1} E_{41;1} D_{2;4} D_{3;5} D_{4;2} A_2$$

$$\sim \begin{pmatrix} 1 & 2 & 6 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 4 & 4 \end{pmatrix} = D_{2;5} D_{3;2} E_{21;-1} E_{31;-1} E_{41;1} D_{2;4} D_{3;5} D_{4;2} A_2$$

$$\sim \begin{pmatrix} 1 & 2 & 6 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} = E_{32;-1} E_{42;-1} D_{2;5} D_{3;2} E_{21;-1} E_{31;-1} E_{41;1} D_{2;4} D_{3;5} D_{4;2} A_2$$

$$\begin{pmatrix} 1 & 2 & 6 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 1 & 1 \end{pmatrix} = D_{3;3} E_{32;-1} E_{42;-1} D_{2;5} D_{3;2} E_{21;-1} E_{31;-1} E_{41;1} D_{2;4} D_{3;5} D_{4;2} A_2$$

$$\sim \begin{pmatrix} 1 & 2 & 6 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix} = E_{43;-1} D_{3;3} E_{32;-1} E_{42;-1} D_{2;5} D_{3;2} E_{21;-1} E_{31;-1} E_{41;1} D_{2;4} D_{3;5} D_{4;2} A_2.$$

We can finally conclude by the Multiplication Theorem, and because the determinant of Type 2 elementary matrices is always equal to 1, that

$$\det\left[\begin{pmatrix} 1 & 2 & 6 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}\right] = \det(D_{3;3})\det(D_{2;5})\det(D_{3;2})\det(D_{2;4})\det(D_{3;5})\det(D_{4;2})\det(A_2)$$

$$= 3\cdot 5\cdot 2\cdot 4\cdot 5\cdot 2\cdot \det(A_2) = 3\det(A_2),$$

which implies that

$$\det(A_2) = 3^{-1}\det\left[\begin{pmatrix} 1 & 2 & 6 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}\right] = 5\cdot 1^3\cdot 2 = 3.$$

if we also use Problem 3 of this homework assignment.

(iii) Similarly to part (ii), we recall that if we use Type 2 elementary row operations on A_3, its determinant won't change. Therefore, the matrix we get from A_3 if we subtract the last row from each one of the other rows has the same determinant as A_3. But the latter matrix is

$$B_3 = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}.$$

Moreover, if we consecutively add each of the first $n-1$ rows of B_3 to the last row, the determinant of the new matrices we get at each step will again remain the same:

$$\det(A_3) = \det(B_3) = \det\left[\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}\right] =$$

$$= \det\left[\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 2 \end{pmatrix}\right] = \cdots\cdots = \det\left[\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & n-1 \end{pmatrix}\right].$$

Finally, the last matrix is upper triangular, so its determinant is equal to the product of its diagonal entries (as we see in Problem 3 of this homework). We can conclude that $\det(A_3) = (-1)^{n-1}(n-1)$.

Problem 2. (i) For any choice of values for a, b, c, d for which the corresponding matrix will be invertible, we will also have that its determinant will be non-zero.

This immediately shows that both a, b have to be non-zero. Indeed, if a were zero, then the 3rd and 4th rows of the matrix would be the same, so the determinant would be zero, whereas if b were zero, then the 2nd row would be equal to minus the 1st row, and again by multilinearity and the alternating property of the determinant, the determinant of the matrix would be zero.

We will now show that, even if a, b are non-zero, the determinant of the corresponding matrix is zero. Indeed, in such a case, we will have

$$\det\left[\begin{pmatrix} a & b & c & d \\ -a+b & 0 & -c+b & -d+b \\ 0 & 1 & 2 & 3 \\ a & a+1 & a+2 & a+3 \end{pmatrix}\right]$$

$$= \det\left[\begin{pmatrix} a & b & c & d \\ -a & -b & -c & -d \\ 0 & 1 & 2 & 3 \\ a & a+1 & a+2 & a+3 \end{pmatrix}\right] + \det\left[\begin{pmatrix} a & b & c & d \\ b & b & b & b \\ 0 & 1 & 2 & 3 \\ a & a+1 & a+2 & a+3 \end{pmatrix}\right]$$

$$= \det\left[\begin{pmatrix} a & b & c & d \\ b & b & b & b \\ 0 & 1 & 2 & 3 \\ a & a+1 & a+2 & a+3 \end{pmatrix}\right]$$

$$= \det\left[\begin{pmatrix} a & b & c & d \\ b & b & b & b \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}\right] + \det\left[\begin{pmatrix} a & b & c & d \\ b & b & b & b \\ 0 & 1 & 2 & 3 \\ a & a & a & a \end{pmatrix}\right]$$

$$= \det\left[\begin{pmatrix} a & b & c & d \\ b & b & b & b \\ 0 & 1 & 2 & 3 \\ a & a & a & a \end{pmatrix}\right] = \frac{b}{a} \cdot \det\left[\begin{pmatrix} a & b & c & d \\ a & a & a & a \\ 0 & 1 & 2 & 3 \\ a & a & a & a \end{pmatrix}\right] = 0.$$

Thus the determinant of such a matrix is always zero, which is equivalent to the matrix being non-invertible. In other words, there isn't any choice of values for a, b, c, d for which the corresponding matrix will be invertible.

(ii) We can note that $w = \exp\left(\frac{2\pi \boldsymbol{i}}{3}\right)$ satisfies $w^3 = \exp\left(2\pi \boldsymbol{i}\right) = 1$, and similarly $w^2 = \exp\left(\frac{4\pi \boldsymbol{i}}{3}\right)$ satisfies $(w^2)^3 = 1$. Thus, the numbers $w, w^2, 1$ (which are all different) are the three roots of the polynomial $x^3 - 1$, which is a special case of the polynomials we are asked to work with in part (iii). This shows that part (ii) could also follow by what we prove in part (iii).

Alternatively, we can note that

$$\det(B) = \det\left[\begin{pmatrix} 1 & w & w^2 \\ w^2 & 1 & w \\ w & w^2 & 1 \end{pmatrix}\right] = \det\left[\begin{pmatrix} 1 & w & w^2 \\ w^2 & 1 & w \\ {\color{red}w}\cdot 1 & {\color{red}w}\cdot w & {\color{red}w}\cdot w^2 \end{pmatrix}\right]$$
$$= w \cdot \det\left[\begin{pmatrix} 1 & w & w^2 \\ w^2 & 1 & w \\ 1 & w & w^2 \end{pmatrix}\right] = 0,$$

where in the second equality we are using linearity of the determinant in the 3rd row, and in the last equality we are using the alternating property.

(iii) We have

$$\det(C) = z_1 \cdot \det\left(\begin{pmatrix} z_3 & z_1 \\ z_1 & z_2 \end{pmatrix}\right) - z_2 \cdot \det\left(\begin{pmatrix} z_2 & z_1 \\ z_3 & z_2 \end{pmatrix}\right) + z_3 \cdot \det\left(\begin{pmatrix} z_2 & z_3 \\ z_3 & z_1 \end{pmatrix}\right)$$
$$= z_1 z_3 z_2 - z_1^3 - z_2^3 + z_2 z_1 z_3 + z_3 z_2 z_1 - z_3^3 = 3 z_1 z_2 z_3 - z_1^3 - z_2^3 - z_3^3.$$

We now recall that z_1, z_2, z_3 are the roots of the polynomial $x^3 + ax + b$, therefore we have

$$x^3 + ax + b = (x - z_1)(x - z_2)(x - z_3)$$
$$= x^3 - (z_1 + z_2 + z_3)x^2 + (z_1 z_2 + z_1 z_3 + z_2 z_3)x - z_1 z_2 z_3.$$

By equating coefficients, we can conclude that $z_1 + z_2 + z_3 = 0$, $a = z_1 z_2 + z_1 z_3 + z_2 z_3$ and $b = -z_1 z_2 z_3$.

Moreover, again because the z_i are roots of the polynomial $x^3 + ax + b$, we get for every $i = 1, 2, 3$ that $z_i^3 + a z_i + b = 0 \Leftrightarrow z_i^3 = -a z_i - b$.

Therefore, we can write

$$\det(C) = 3 z_1 z_2 z_3 - z_1^3 - z_2^3 - z_3^3 = -3b + (a z_1 + b) + (a z_2 + b) + (a z_3 + b)$$
$$= -3b + a(z_1 + z_2 + z_3) + 3b = a(z_1 + z_2 + z_3) = 0.$$

Problem 3. Let $U = (u_{ij})_{1 \leqslant i,j \leqslant n} \in \mathbb{F}^{n \times n}$ be an upper triangular matrix. By the Leibniz formula, we have that

$$\det(U) = \sum_{\sigma\ n\text{-permutation}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} u_{i,\sigma(i)} = \prod_{i=1}^{n} u_{i,i} + \sum_{\substack{\sigma\ n\text{-permutation} \\ \sigma \neq \mathrm{id}}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} u_{i,\sigma(i)}.$$

We will now show that all the products in the latter sum are equal to 0. This would follow if each such product contained an entry of U below the diagonal, that is, if in each such product there were some $i \in \{1, 2, \ldots, n\}$ such that $\sigma(i) < i$.

Consider an n-permutation σ_0 that is different from the identity permutation. Then there must exist $i_0 \in \{1, 2, \ldots, n\}$ such that $\sigma_0(i_0) \neq i_0$. Assume also towards a contradiction that the product corresponding to σ_0 is non-zero, which therefore implies that for every $i \in \{1, 2, \ldots, n\}$ we have $\sigma_0(i) \geqslant i$.

Let us then focus on the images under σ_0 of the set $\{i_0, i_0 + 1, \ldots, n\}$, which has size $n - i_0 + 1$. We have:

- $\sigma_0(i_0) \neq i_0$ and $\sigma_0(i_0) \geqslant i_0$, therefore we must have $\sigma_0(i_0) > i_0$, or in other words $\sigma_0(i_0) \in \{i_0 + 1, \ldots, n\}$;
- also, for every $j \in \{i_0 + 1, \ldots, n\}$ we have $\sigma_0(j) \geqslant j$, and thus $\sigma_0(j) \in \{i_0 + 1, \ldots, n\}$.

Combining the above, we see that the set $\{\sigma_0(j) : i_0 \leqslant j \leqslant n\}$ is a subset of the set $\{i_0 + 1, \ldots, n\}$. But since σ_0 is an injective function from $\{1, 2, \ldots, n\}$ to itself, no two images can be equal, and thus the set $\{\sigma_0(j) : i_0 \leqslant j \leqslant n\}$ contains $n - i_0 + 1$ different images. This contradicts the conclusion that it is a subset of a set with size $n - i_0$.

We conclude that the assumption we made that we could find an n-permutation σ_0 which is different from the identity permutation and for which $\sigma_0(i) \geqslant i$ for all $i \in \{1, 2, \ldots, n\}$ was incorrect. Thus, for every permutation σ that is not the identity permutation, there must exist $i \in \{1, 2, \ldots, n\}$ such that $\sigma(i) < i$. This implies that the corresponding product of matrix entries in the Leibniz formula for $\det(U)$ will contain the zero factor $u_{i,\sigma(i)}$, and therefore it will be zero itself. So in the end, $\det(U) = \prod_{i=1}^{n} u_{i,i}$, as claimed.

Next, given a lower triangular matrix $L = (l_{ij})_{1 \leqslant i,j \leqslant n} \in \mathbb{F}^{n \times n}$, we can argue similarly that for every n-permutation σ that is not the identity permutation, there must exist $j \in \{1, 2, \ldots, n\}$ such that $\sigma(j) > j$. This would immediately give that the corresponding product of matrix entries in the

Leibniz formula for $\det(L)$ would contain the zero factor $l_{j,\sigma(j)}$, and thus it would be zero itself, which would finally imply that $\det(L) = \prod_{j=1}^{n} l_{j,j}$.

Alternatively, we could simply note that L^T is upper triangular in this case, so from the previous argument we have $\det(L) = \det(L^T) = \prod_{j=1}^{n} l_{j,j}$.

Problem 4. Let us write λ for a parameter that can take values in $\mathbb{F}$. We recall that $p_A(\lambda) = \det(A - \lambda I_n)$, while $p_{A^T}(\lambda) = \det(A^T - \lambda I_n)$.

At the same time, $A^T - \lambda I_n = (A - \lambda I_n)^T$, and therefore, as we have seen in class,

$$\det(A^T - \lambda I_n) = \det\big((A - \lambda I_n)^T\big) = \det(A - \lambda I_n).$$

We conclude that the two polynomials coincide, so they have the same roots, or in other words A and A^T have the same eigenvalues (moreover, the algebraic multiplicity of each of these eigenvalues λ_0 is the same when we view λ_0 as an eigenvalue of A as when we view it as an eigenvalue of A^T; *see HW4 for the definition of this notion, as well as for the notion of "geometric multiplicity of an eigenvalue"*).

Question. Does each of the common eigenvalues of A and A^T have the same geometric multiplicity too when viewed as an eigenvalue of A as when viewed as an eigenvalue of A^T?

Problem 5. (i) We show that similarity of matrices is an equivalence relation, or in other words that it is *reflexive*, *symmetric* and *transitive*.

Reflexivity: Let $A \in \mathbb{F}^{n\times n}$. Then $A = I_n A I_n = I_n^{-1} A I_n$.

Symmetry: Let $A, B \in \mathbb{F}^{n\times n}$, and assume B is similar to A, or in other words that there is an invertible matrix $E \in \mathbb{F}^{n\times n}$ such that $B = E^{-1}AE$. Then $A = E(E^{-1}AE)E^{-1} = EBE^{-1} = (E^{-1})^{-1}BE^{-1}$, which shows that A is similar to B as well.

Transitivity: Let $A, B, C \in \mathbb{F}^{n\times n}$, and assume that A is similar to B and B is similar to C. In other words, we can find invertible matrices E_1, E_2 in $\mathbb{F}^{n\times n}$ such that $A = E_1^{-1}BE_1$ and $B = E_2^{-1}CE_2$.

But then, $A = E_1^{-1}BE_1 = E_1^{-1}(E_2^{-1}CE_2)E_1 = (E_1^{-1}E_2^{-1})C(E_2E_1)$, which shows that A is similar to C given that the product E_2E_1 is an invertible matrix too, and $E_1^{-1}E_2^{-1} = (E_2E_1)^{-1}$.

(ii) Let us consider two matrices $A, B \in \mathbb{F}^{n\times n}$ that are similar. Then we can find an invertible matrix $E \in \mathbb{F}^{n\times n}$ such that $B = E^{-1}AE$. But then, by the Multiplication Theorem,

$$\begin{aligned}\det(B) = \det(E^{-1}AE) &= \det(E^{-1})\det(A)\det(E)\\ &= \det(E^{-1})\det(E)\det(A) = \det(E^{-1}E)\det(A)\\ &= \det(I_n)\det(A) = \det(A).\end{aligned}$$

Important Note. In deriving the desired conclusion here, we also showed that $\det(E^{-1}) = \frac{1}{\det(E)}$.

Let us now write λ for a parameter that can take values in $\mathbb{F}$. We recall that $p_A(\lambda) = \det(A - \lambda I_n)$, and similarly $p_B(\lambda) = \det(B - \lambda I_n)$. But we also have

$$\lambda I_n = \lambda(E^{-1}E) = \lambda(E^{-1}I_nE) = E^{-1}(\lambda I_n)E,$$

therefore

$$B - \lambda I_n = E^{-1}AE - E^{-1}(\lambda I_n)E = E^{-1}(A - \lambda I_n)E$$

by combining the left and the right distributive property of matrix multiplication.

We can thus use the Multiplication Theorem again to write

$$\det(B-\lambda I_n) = \det\big(E^{-1}(A-\lambda I_n)E\big) = \det(E^{-1})\det(A-\lambda I_n)\det(E) = \det(A-\lambda I_n),$$

which shows that the two characteristic polynomials coincide.

Problem 6. We will show that the set $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\}$ of eigenvectors of A corresponding to k different eigenvalues is linearly independent by using induction in k.

Base case: $k = 1$ Then the set $\{\bar{u}_1\}$ contains only one eigenvector of A, which by definition is a non-zero vector, and therefore the set is linearly independent.

Induction Step Assume that for some $k \geqslant 1$, $k < n$, we have already shown that,

if $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\}$ is a set of k eigenvectors of A
corresponding to k different eigenvalues,
then this set is linearly independent.

Assume now that we have one more eigenvector of A, the vector $\bar{u}_{k+1}$, corresponding to an eigenvalue λ_{k+1} of A which is different from the previously mentioned. We will show that the set $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k, \bar{u}_{k+1}\}$ is linearly independent too.

We note that it suffices to show that $\bar{u}_{k+1}$ cannot be written as a linear combination of the vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k$. Indeed, in this case $\bar{u}_{k+1}$ will not belong to the linear span of the latter vectors, and since the latter vectors are linearly independent by the Inductive Hypothesis, we will be able to conclude that the set $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k, \bar{u}_{k+1}\}$ is also linearly independent (this is precisely one of the results on the topic of linear independence that we discussed in MATH 127).

Assume towards a contradiction that $\bar{u}_{k+1}$ can be written as a linear combination of the vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k$. Then, it should be possible to find $\mu_1, \mu_2, \ldots, \mu_k \in \mathbb{F}$ such that

$$\bar{u}_{k+1} = \mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k \tag{1}$$

$$\begin{aligned} \Rightarrow \qquad \lambda_{k+1}\bar{u}_{k+1} = A\bar{u}_{k+1} &= A\big(\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k\big) \\ &= \mu_1 A\bar{u}_1 + \mu_2 A\bar{u}_2 + \cdots + \mu_k A\bar{u}_k \\ &= \mu_1 \lambda_1 \bar{u}_1 + \mu_2 \lambda_2 \bar{u}_2 + \cdots + \mu_k \lambda_k \bar{u}_k. \end{aligned}$$

At the same time, (1) would imply that

$$\lambda_{k+1}\bar{u}_{k+1} = \lambda_{k+1}\big(\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k\big) = \mu_1 \lambda_{k+1} \bar{u}_1 + \mu_2 \lambda_{k+1} \bar{u}_2 + \cdots + \mu_k \lambda_{k+1} \bar{u}_k.$$

Combining the above, we get that

$$\begin{aligned}
&\mu_1(\lambda_1 - \lambda_{k+1})\bar{u}_1 + \mu_2(\lambda_2 - \lambda_{k+1})\bar{u}_2 + \cdots + \mu_k(\lambda_k - \lambda_{k+1})\bar{u}_k \\
&= (\mu_1\lambda_1\bar{u}_1 + \mu_2\lambda_2\bar{u}_2 + \cdots + \mu_k\lambda_k\bar{u}_k) - (\mu_1\lambda_{k+1}\bar{u}_1 + \mu_2\lambda_{k+1}\bar{u}_2 + \cdots + \mu_k\lambda_{k+1}\bar{u}_k) \\
&= \lambda_{k+1}\bar{u}_{k+1} - \lambda_{k+1}\bar{u}_{k+1} = \bar{0}.
\end{aligned}$$

But since by the Inductive Hypothesis we have that the set $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\}$ is linearly independent,

$$\mu_1(\lambda_1 - \lambda_{k+1})\bar{u}_1 + \mu_2(\lambda_2 - \lambda_{k+1})\bar{u}_2 + \cdots + \mu_k(\lambda_k - \lambda_{k+1})\bar{u}_k = \bar{0}$$

implies that $\mu_i(\lambda_i - \lambda_{k+1}) = 0$ for every $i \in \{1, 2, \ldots, k\}$. Given also that $\lambda_i - \lambda_{k+1} \neq 0$ for every such i (since λ_{k+1} is a different eigenvalue from the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of A by assumption), we conclude that $\mu_1 = \mu_2 = \cdots = \mu_k = 0$.

But then (1) gives

$$\bar{u}_{k+1} = \mu_1\bar{u}_1 + \mu_2\bar{u}_2 + \cdots + \mu_k\bar{u}_k = \bar{0},$$

which contradicts the assumption that $\bar{u}_{k+1}$ is an eigenvector of A.

We conclude that the assumption we made, that $\bar{u}_{k+1}$ can be written as a linear combination of the vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k$, was incorrect, and therefore the set $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k, \bar{u}_{k+1}\}$ is also linearly independent.

Problem 7. Note that, since f is injective, the set $\{f(\bar{u}_1), f(\bar{u}_2), \dots, f(\bar{u}_k)\}$ contains k different images of f, and thus k different elements of V_2 (indexed by $i \in \{1, 2, \dots, k\}$).

Consider $\mu_1, \mu_2, \dots, \mu_k \in \mathbb{F}$ that satisfy

$$\mu_1 f(\bar{u}_1) + \mu_2 f(\bar{u}_2) + \cdots + \mu_k f(\bar{u}_k) = \bar{0}_{V_2}.$$

We need to show that $\mu_1 = \mu_2 = \cdots = \mu_k = 0_{\mathbb{F}}$.

By the linearity of f, we get

$$\begin{aligned}\bar{0}_{V_2} &= \mu_1 f(\bar{u}_1) + \mu_2 f(\bar{u}_2) + \cdots + \mu_k f(\bar{u}_k)\\ &= f(\mu_1 \bar{u}_1) + f(\mu_2 \bar{u}_2) + \cdots + f(\mu_k \bar{u}_k)\\ &= f(\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k).\end{aligned}$$

This shows that $\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k \in \mathrm{Ker}(f)$.

But since f is injective, we must have $\mathrm{Ker}(f) = \{\bar{0}_{V_1}\}$, or in other words $\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k = \bar{0}_{V_1}$. Since by assumption the set $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k\}$ is linearly independent, we can conclude that $\mu_1 = \mu_2 = \cdots = \mu_k = 0_{\mathbb{F}}$, as we wanted.