Math 322
Suggested solutions to Homework Set 5

Problem 1. (a) Consider two different vertices v_1, v_2 of H. Since H is an orientation of the complete graph on n vertices, either the directed edge (v_1, v_2) or the directed edge (v_2, v_1) is contained in H. Let's assume without loss of generality that $(v_1, v_2) \in E(H)$.

But then we have the directed path $v_1 \to v_2$ from v_1 to v_2.

Since H is strongly connected, we should also be able to find a directed path from v_2 to v_1; say one such path is the following:

$$v_2 \to w_1 \to w_2 \to \cdots \to w_s \to v_1$$

where $w_1, w_2, \ldots, w_s$ are some other vertices of H. Also, given that the edge (v_2, v_1) is **not** contained in H, there must exist at least one other vertex w_i in this path, exactly as we have written above.

Combining the above, we see that H contains the following directed cycle (which contains at least 3 vertices, the vertices v_1, v_2 and at least one more vertex w_i):

$$v_1 \to v_2 \to w_1 \to w_2 \to \cdots \to w_s \to v_1.$$

(b) By Theorem $1''$ we know that H' has a directed Hamilton cycle

$$v_1 \to v_2 \to v_3 \to \cdots \to v_{n'-1} \to v_{n'} \to v_1.$$

Assume towards a contradiction that H' does not contain a directed 3-cycle. This implies that H' cannot contain the directed edge (v_3, v_1) (because otherwise $v_1 \to v_2 \to v_3 \to v_1$ would be a directed 3-cycle).

At the same time, since H' is an orientation of the complete graph on n' vertices, it must contain the directed edge (v_1, v_3).

Next, we make the analogous observation that H' cannot contain the directed edge (v_4, v_1) (because otherwise $v_1 \to v_3 \to v_4 \to v_1$ would be a directed 3-cycle in H').

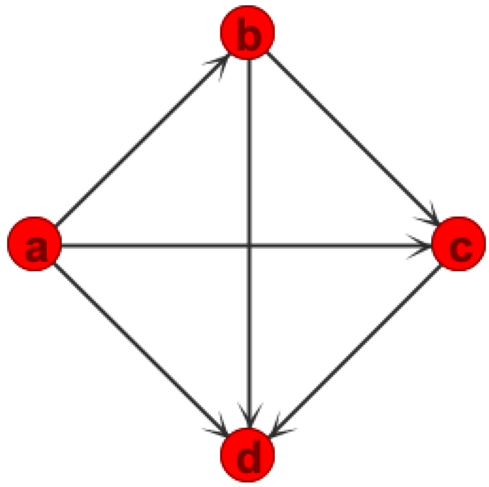
At the same time, since H' is an orientation of the complete graph on n' vertices, it must contain the directed edge (v_1, v_4).

Continuing like this, we see that H' must contain the directed edges (v_1, v_s) for each $s \in \{2, 3, \ldots, n' - 1\}$.

But then H' will contain the directed 3-cycle $v_1 \to v_{n'-1} \to v_{n'} \to v_1$, contradicting our assumption that H' does not contain a directed 3-cycle.

We conclude that this assumption was incorrect, and thus H' does contain at least one directed 3-cycle.

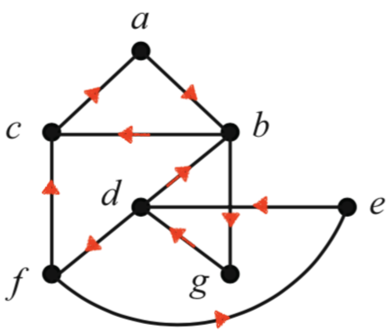
(c) The following graph is a tournament on 4 vertices which does not contain any directed 3-cycles (clearly, it's also not strongly connected):



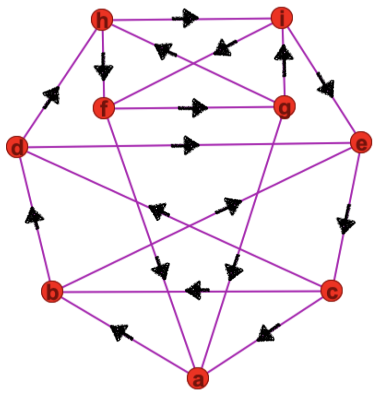
Problem 2. From Theorem $1'$ of Lecture 21 we know that a connected (undirected) graph has a strong orientation if and only if every edge of it belongs to at least one cycle.

Out of the given graphs, H_2 and H_4 don't have this property (e.g. the edge $\{a, b\}$ of H_2 does not belong to any cycle, and similarly the edge $\{0, 11\}$ of H_4 does not belong to any cycle). Thus these two graphs won't have a strong orientation.

On the other hand, H_1 and H_3 should have a strong orientation. Below we give an example for each of the two graphs:



Strong orientation of H_1



Strong orientation of H_3

Note that the first graph is a strong orientation of H_1 because it contains the directed cycles

$$d \to f \to c \to a \to b \to g \to d$$

and

$$e \to d \to f \to e.$$

Given the first cycle, we see that any two vertices of H_1 which are both different from e will be strongly connected (we can get from one vertex to the other by traversing the cycle). Similarly, given the second cycle, e is strongly connected with both d and f.

Finally, to find a directed path from e to a vertex $v \in V(H_1) \setminus \{d, f, e\}$, we can first move to d and then move to the vertex v along the first cycle; analogously, if we are trying to find a directed path from v to e, we can first move from v to f, and then from f directly to e.

Furthermore, the second graph is a strong orientation of H_3 because it contains the directed Hamilton cycle

$$a \to b \to d \to h \to f \to g \to i \to e \to c \to a.$$

Problem 3. Given that the students will essentially be matched with the different positions that the teachers are offering (so, from a graph-theoretical point of view, we are looking for a stable matching of positions and students, since these two sets have the same cardinality), it will be convenient to replace the first table of preferences by the following table:

T_1, Post 1	T_1, Post 2	T_1, Post 3	T_2, Post 1	T_2, Post 2	T_2, Post 3	T_2, Post 4	T_3, Post 1	T_3, Post 2
S_3	S_3	S_3	S_3	S_3	S_3	S_3	S_1	S_1
S_4	S_4	S_4	S_2	S_2	S_2	S_2	S_4	S_4
S_1	S_1	S_1	S_7	S_7	S_7	S_7	S_3	S_3
S_2	S_2	S_2	S_8	S_8	S_8	S_8	S_5	S_5
S_7	S_7	S_7	S_9	S_9	S_9	S_9	S_9	S_9
S_9	S_9	S_9	S_4	S_4	S_4	S_4	S_8	S_8
S_8	S_8	S_8	S_1	S_1	S_1	S_1	S_6	S_6
S_5	S_5	S_5	S_6	S_6	S_6	S_6	S_7	S_7
S_6	S_6	S_6	S_5	S_5	S_5	S_5	S_2	S_2

We now see that in the 1st round of the algorithm, T_1 will make an offer to S_3, T_2 will also make an offer to S_3, while T_3 will make an offer to S_1.

Next, we see that S_1 will provisionally accept the offer by T_3, while S_3 will provisionally accept the offer by T_1 (given that S_3 prefers T_1 over T_2).

Thus, by the end of the 1st round we have the provisional engagements:

$$(T_1, S_3) \quad \text{and} \quad (T_3, S_1).$$

Note that, even though both T_1 and T_3 have already been **provisionally** assigned a student, they will still make offers in the next round since they still have available positions (recall that essentially we are matching the positions with the students, so the above provisional engagements can be read as "S_3 has provisionally accepted the 1st student position offered by T_1, while S_1 has provisionally accepted the 1st student position offered by T_3").

In the 2nd round of the algorithm, T_1 will make an offer to S_4, T_2 will make an offer to S_2, while T_3 will make an offer to S_4.

Next, we see that S_2 will provisionally accept the offer by T_2, while S_4 will provisionally accept the offer by T_3 (given that S_4 prefers T_3 over T_1).

Thus, by the end of the 2nd round we have the provisional engagements:

$$(T_1, S_3), \quad (T_2, S_2), \quad (T_3, S_1) \quad \text{and} \quad (T_3, S_4).$$

We observe that, at the end of this round, all the positions that T_3 is planning to offer have been filled, so T_3 will not be making an offer in the next round (however, T_3 may again make an offer in a later round, if it happens that one of the students who have already chosen T_3 rejects the provisional offer for a more preferred one, and thus that position becomes available again).

In the 3rd round, T_1 makes an offer to S_1, while T_2 makes an offer to S_7.

S_7 has not received any other offers so far, so the student provisionally accepts this offer. On the other hand, S_1 has already accepted an offer made by T_3; given that S_1 prefers T_1, the student rejects the previous offer, and provisionally accepts the offer by T_1.

Thus, by the end of the 3rd round we have the provisional engagements:

$$(T_1, S_1), \quad (T_1, S_3), \quad (T_2, S_2), \quad (T_2, S_7) \quad \text{and} \quad (T_3, S_4).$$

In the 4th round, T_1 makes an offer to S_2, T_2 makes an offer to S_8, while T_3 makes an offer to S_3.

S_8 has not received any other offers so far, so the student provisionally accepts the offer made by T_2. On the other hand, S_2 has provisionally accepted an offer by T_2; since S_2 prefers T_2 over T_1, the student stays with T_2.

Similarly, S_3 has provisionally accepted an offer by T_1; since S_3 prefers T_3 over T_1, the student rejects the previous offer and accepts the new one made by T_3.

Thus, by the end of the 4th round we have the provisional engagements:

$$(T_1, S_1), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

Again, we see that T_3 has no more positions to fill, so this teacher will not be making an offer in the next round.

In the 5th round, T_1 makes an offer to S_7, while T_2 makes an offer to S_9.

S_9 has not received any other offers so far, so the student provisionally accepts this offer. On the other hand, S_7 has already accepted an offer made by T_2; given that S_7 prefers T_2 over T_1, the student rejects the new offer and stays with T_2.

Thus, by the end of the 5th round we have the provisional engagements:

$$(T_1, S_1), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_2, S_9), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

We see that both T_2 and T_3 have no more positions to fill, so in the next round only T_1 will make an offer.

In the 6th round, T_1 makes an offer to S_9. This student has already accepted an offer by T_2, and since S_9 prefers T_2 over T_1, the new offer by T_1 gets rejected.

Thus, by the end of the 6th round we have the same provisional engagements as in the previous round:

$$(T_1, S_1), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_2, S_9), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

In the 7th round, T_1 makes an offer to S_8. This student has already accepted an offer by T_2, and since S_8 prefers T_2 over T_1, this new offer by T_1 also gets rejected.

Again, we have the same provisional engagements as in the previous round:

$$(T_1, S_1), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_2, S_9), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

In the 8th round, T_1 makes an offer to S_5. This student has not received any other offers so far, so this offer gets accepted.

Thus, by the end of the 8th round we have the following provisional engagements:

$$(T_1, S_1), \quad (T_1, S_5), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_2, S_9), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

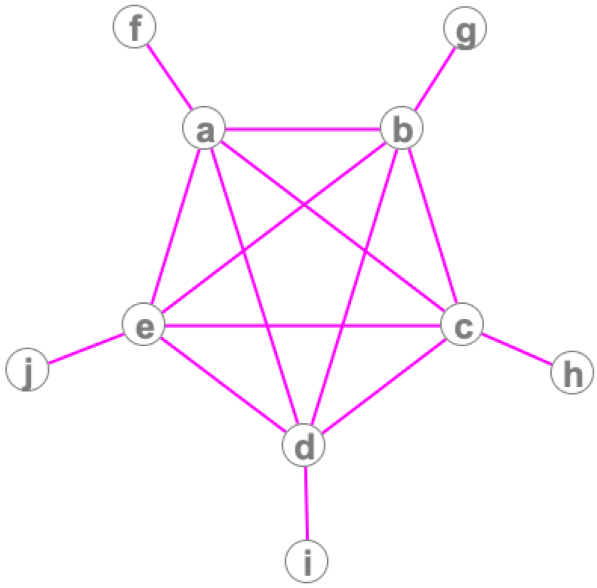
Finally, in the 9th round, T_1 makes an offer to S_6. This student has not received any other offers so far, so this offer gets accepted.

Thus, by the end of the 9th round we have the following provisional engagements:

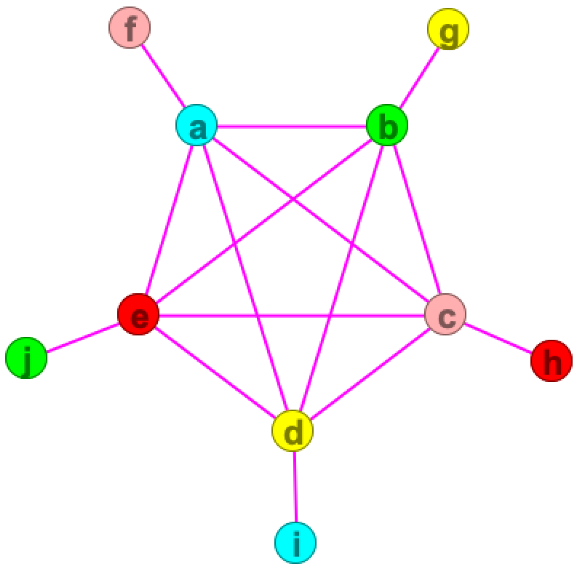
$$(T_1, S_1), \quad (T_1, S_5), \quad (T_1, S_6), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_2, S_9), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

We now observe that each student has accepted a position offered by a teacher, and all positions have been filled, therefore the engagements from the last round become permanent, and the algorithm terminates.

Problem 4. Consider the following graph G_0:



We observe that K_5 is isomorphic to a subgraph of this graph, and hence $\omega(G_0) \geqslant 5$ (in fact, the clique number of G_0 is exactly 5). It follows that $\chi(G_0) \geqslant 5$. Moreover, we can find a 5-colouring of G_0 as follows:



The above show that $\chi(G_0) = 5$.

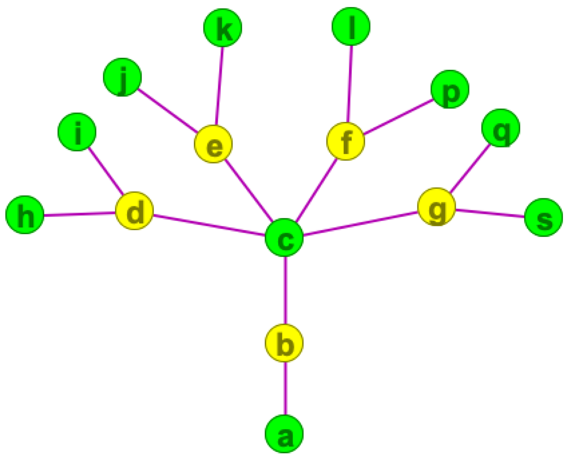
On the other hand, we observe that five vertices of G_0 have degree 5, while the remaining five vertices have degree 1. Thus

$$\operatorname{avgdeg}(G_0) = \frac{1}{10}(5 \cdot 5 + 5 \cdot 1) = 3,$$

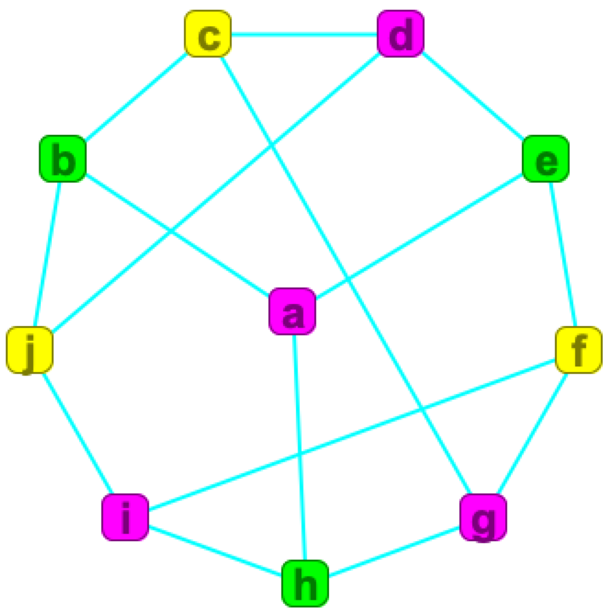
and hence $\operatorname{avgdeg}(G_0) + 1 = 4 < \chi(G_0)$.

Problem 5. We observe that G_1 is a tree (with 15 vertices and 14 edges), and hence G_1 contains no cycles (and in particular, no odd cycles). Thus, as we saw in Lecture 25, $\chi(G_1) = 2$.

We also give a minimal colouring of G_1:

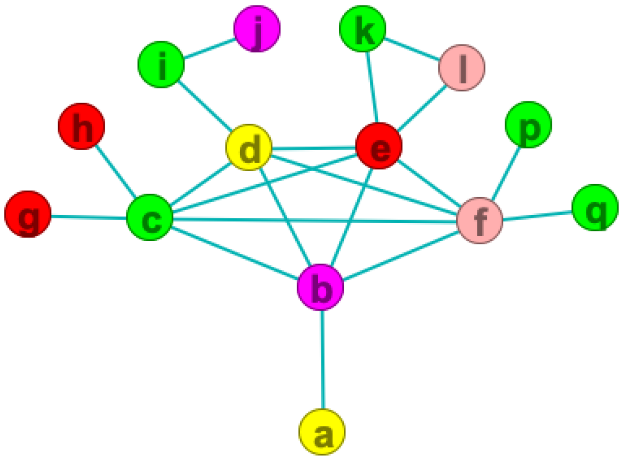


Graph G_2 contains the odd cycle $b\,a\,h\,i\,j\,b$, and hence $\chi(G_2) \geqslant 3$. We will now give a 3-colouring of G_2, which will show that $\chi(G_2) = 3$:



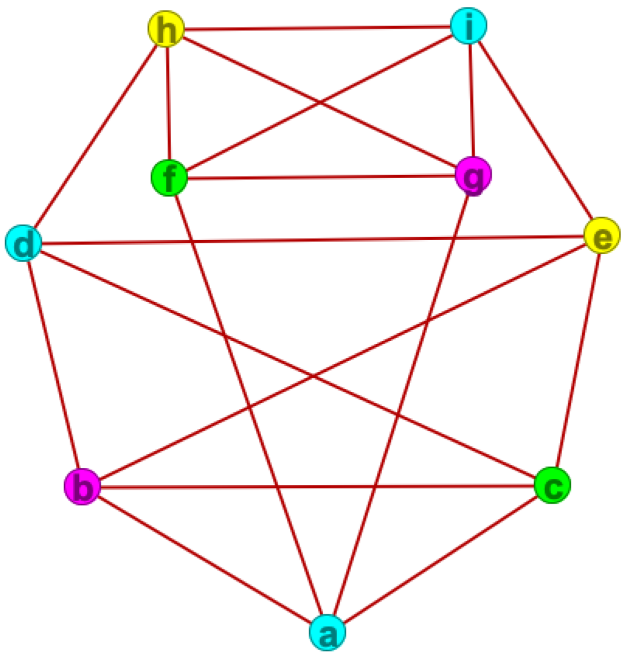
Graph G_3 contains the clique $(\{b, c, d, e, f\}, \{bc, bd, be, bf, cd, ce, cf, de, df, ef\})$, and hence $\chi(G_3) \geqslant \omega(G_3) \geqslant 5$ (we could also check that $\omega(G_3) = 5$).

We will now give a 5-colouring of G_3, which will show that $\chi(G_3) = 5$:



Graph G_4 contains the clique $(\{b, c, d, e\}, \{bc, bd, be, cd, ce, de\})$, and hence $\chi(G_4) \geqslant \omega(G_4) \geqslant 4$ (we could also check that $\omega(G_4) = 4$).

We will now give a 4-colouring of G_4, which will show that $\chi(G_4) = 4$:



Finally, Graph G_5 contains the odd cycle $a\,b\,e\,a$, and hence $\chi(G_5) \geqslant 3$.

We will now give a 3-colouring of G_5, which will show that $\chi(G_5) = 3$: