Reminder from Last Time

Definition Let V_0 be a vector space over a field $\mathbb{F}$, and let $g : V_0 \to V_0$ be a linear map. A non-zero vector $\bar{u}$ of V_0 is called an eigenvector of g if there exists some $\lambda \in \mathbb{F}$ such that $g(\bar{u}) = \lambda \bar{u}$.

In this case λ is called an eigenvalue of g.

Another setting in which we study eigenvectors and eigenvalues: If $V_0 = \mathbb{F}^n$ for some n, then we have seen that any linear map $g : \mathbb{F}^n \to \mathbb{F}^n$ has a matrix representation A_g.

Conversely, any matrix $B \in \mathbb{F}^{n\times n}$ gives us a linear map by setting $\bar{x} \in \mathbb{F}^n \longmapsto B\bar{x} \in \mathbb{F}^n$.

Thus it makes sense to also define eigenvectors and eigenvalues of a square matrix A:

Definition Let $A \in \mathbb{F}^{n\times n}$. A non-zero vector $\bar{v} \in \mathbb{F}^n$ is called an eigenvector of A if there exists some $\mu \in \mathbb{F}$ such that $\boxed{A\bar{v} = \mu \bar{v}.}$

In this case μ is called an eigenvalue of A.

Very Important Remark Let $g : V_0 \to V_0$ be a linear map, and let $\bar{u} \in V_0$ be an eigenvector of g corresponding to eigenvalue λ.

By definition, $\bar{u}$ is non-zero and satisfies

$$g(\bar{u}) = \lambda \bar{u} = \lambda \underbrace{id_{V_0}}_{\text{this is a linear map too, see HW2, Pb2}}(\bar{u})$$

$$\iff g(\bar{u}) - \lambda\, id_{V_0}(\bar{u}) = \bar{0}_{V_0} \iff (g - \lambda\, id_{V_0})(\bar{u}) = \bar{0}_{V_0}$$

In other words,

$\bar{u}$ is an eigenvector of g corresponding to eigenvalue λ

if and only if

$\bar{u} \in \ker(g - \lambda\, id_{V_0})$ and $\bar{u}$ is non-zero.

Moreover, focusing only on the eigenvalue λ

λ is an eigenvalue of g

if and only if

$\ker(g - \lambda\, id_{V_0}) \neq \{\bar{0}_{V_0}\}$ (in fact, $\ker(g - \lambda\, id_{V_0})$ will be a larger subspace)

if and only if

$g - \lambda\, id_{V_0}$ is not injective.

We have the analogous remarks in the matrix setting: as practice, try to state these remarks. See also the related Problems 5 and 6 from HW2.

One example: Consider the matrix $A = \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 4}$

Does A have eigenvalues in $\mathbb{R}$? If yes, find them.

Solution A real number λ would be an eigenvalue of A if and only if the Nullspace of $A - \lambda I_4$ contained more vectors than just the zero vector (this is one of the analogous remarks we have in the matrix setting).

But the latter can be equivalently restated as: the linear system $(A - \lambda I_4)\bar{x} = \bar{0}$ has more than just the trivial solution.

So let's study its coefficient matrix:

$$A-\lambda I_4 = \begin{pmatrix} 1-\lambda & 0 & 3 & 8 \\ 0 & 1-\lambda & 2 & 4 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{pmatrix}$$ where λ should be treated as a parameter.

Let's consider cases for the value of λ:

Case 1 If we set $\lambda=1$, then two columns of $A-\lambda I_4$ become zero, so certainly $A-\lambda I_4$ cannot have 4 pivots, and thus $(A-\lambda I_4)\bar{x}=\bar{0}$ has more than one solutions.

$\rightarrow$ 1 is an eigenvalue of A.

Case 2 If we set $\lambda=-1$, then two rows of $A-\lambda I_4$ will be equal to each other, and therefore # of pivots of $A-\lambda I_4 = \dim RS(A-\lambda I_4) \leq 3$, which shows again that

-1 is an eigenvalue of A.

Case 3 If $\lambda=0$, then

$$A-\lambda I_4 = \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\rightarrow$ 0 is NOT an eigenvalue of A.

Finally

Case 4 If $\lambda \notin \{-1, 0, 1\}$, then $1-\lambda \neq 0$, $\lambda \neq 0$ and $\lambda^2 \neq 1$, thus

$$\begin{pmatrix} 1-\lambda & 0 & 3 & 8 \\ 0 & 1-\lambda & 2 & 4 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{pmatrix} \xrightarrow{\lambda R_4 + R_3 \to R_4} \begin{pmatrix} 1-\lambda & 0 & 3 & 8 \\ 0 & 1-\lambda & 2 & 4 \\ 0 & 0 & \underset{\neq 0}{-\lambda} & 1 \\ 0 & 0 & 0 & \underset{\neq 0}{-\lambda^2+1} \end{pmatrix}$$

pivot

allowed row operation since $\lambda \neq 0$

pivots

$\rightsquigarrow$ A has no eigenvalues in $\mathbb{R} \setminus \{-1, 0, 1\}$.

We conclude that A has only two eigenvalues in $\mathbb{R}$, the eigenvalues 1 and -1.

The above analysis was not too complicated because A was a "nice enough" matrix, with many zero entries in its first two columns.

In general, a much more efficient method for studying eigenvalues is by using the notion of

Determinants

Let $\mathbb{F}$ be a field. For any $n \geq 2$, we will define a function

$$\det : \mathbb{F}^{n \times n} \longrightarrow \mathbb{F}$$

that we will eventually show satisfies properties such as

(i) $\det(A) = 0_{\mathbb{F}} \iff A$ is <u>not</u> invertible

(ii) $\det(AB) = \det(A) \cdot \det(B)$

(iii) $\det(A^T) = \det(A)$

One way to define these functions is by giving a recursive definition:

Base case: $n=2$

Examples $A_1 = \begin{pmatrix} 2 & 3 \\ -1 & 7 \end{pmatrix} \in \mathbb{R}^{2\times 2}$

then $\det(A_1) = 2\cdot 7 - 3\cdot(-1) = 17 \in \mathbb{R}$

$A_2 = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \in \mathbb{Z}_7^{2\times 2}$, then $\det(A_2) = 3\cdot 6 - 4\cdot 5 = 5 \in \mathbb{Z}_7$

General definition If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{F}^{2\times 2}$, then we set

$$\det(A) = ad - bc.$$

Inductive clause $n>2$

Assume that we have defined already the function det on $\mathbb{F}^{(n-1)\times(n-1)}$, and consider $A \in \mathbb{F}^{n\times n}$.

To define $\det(A)$, we will look at certain "submatrices" of A (matrices we get if we remove some row and column of A), and will take the determinants of those (which we already know how to do).

Definition If $A \in \mathbb{F}^{n\times n}$, and $1 \le i \le n$, $1 \le j \le n$, then we write M_{ij} for the matrix $\in \mathbb{F}^{(n-1)\times(n-1)}$ that we get by removing the i-th row of A and the j-th column of A, and leaving the remaining rows and columns (which will be slightly "reduced") in the positions that they were. We call M_{ij} the ij-th minor of A.

<u>Examples</u> If $A = \begin{pmatrix} 3 & 1 & 5 \\ 2 & -1 & e \\ \sqrt{3} & 4 & -1 \end{pmatrix} \in \mathbb{R}^{3\times 3}$,

then M_{12} of A is the matrix $\begin{pmatrix} 2 & e \\ \sqrt{3} & -1 \end{pmatrix}$,

while M_{33} of A is the matrix $\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$.

Reminder One way to define determinants of square matrices is by giving a recursive definition:

Base case: $n=2$

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{F}^{2\times 2}$, then we set

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Inductive clause: $n>2$

Assume now that we have already defined

$$\det : \mathbb{F}^{(n-1)\times(n-1)} \longrightarrow \mathbb{F}$$

Recall If $B \in \mathbb{F}^{n\times n}$, and $1\le i\le n$, $1\le j\le n$ are given, then M_{ij} is the matrix $\in \mathbb{F}^{(n-1)\times(n-1)}$ that we get by removing the i-th row and the j-th column of B.

M_{ij} is the ij-th minor of B.

Definition If $B\in\mathbb{F}^{n\times n}$, then we define $\det(B)$ in any of the following ways:

i) pick a row of B, say the k-th row; then

$$\det(B) \underline{\qquad} \sum_{j=1}^{n} (-1)^{k+j}\, b_{kj} \det(\underset{\mathbb{F}^{(n-1)\times(n-1)}}{M_{kj}})$$

a suitable linear combination of the determinants of the minors M_{kj}.

so by assumption we know how to compute this determinant

We call this the expansion of $\det(B)$ over the k-th row.

ii) pick a column of B, say the l-th column; then

$$\det(B) \underset{\substack{\uparrow}}{=\!=} \sum_{i=1}^{n} (-1)^{i+l}\, b_{il} \det(M_{il})$$

suitable
linear combination
of the determinants
of the minors $M_{-,l}$

This is the expansion of $\det(B)$ over the l-th column.

Important Remark All these different expansions give the same result (we'll see why soon), thus this definition is good.

Example 1 What is $\det(B)$ if

$$B = \begin{pmatrix} 3 & 1 & 5 \\ 2 & -1 & e \\ \sqrt{3} & 4 & -1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \ ?$$

Use expansion over the 2nd row. Use also expansion over the 3rd column, and compare.

Solution: Using expansion over the 2nd row:

$$\begin{aligned}
\det(B) &= (-1)^{2+1} b_{21} \det(M_{21}) + (-1)^{2+2} b_{22} \det(M_{22}) \\
&\qquad + (-1)^{2+3} b_{23} \det(M_{23}) \\
&= -2\cdot(1\cdot(-1) - 5\cdot 4) + (-1)\cdot(3\cdot(-1) - 5\sqrt{3}) - e\cdot(3\cdot 4 - 1\cdot\sqrt{3}) \\
&= +42 + 3 + 5\sqrt{3} - 12e + \sqrt{3}e \\
&= 45 + (5+e)\sqrt{3} - 12e.
\end{aligned}$$

Using expansion over the 3rd column:

$$\begin{aligned}
\det(B) &= (-1)^{1+3} b_{13} \det(M_{13}) + (-1)^{2+3} b_{23} \det(M_{23}) + (-1)^{3+3} b_{33} \det(M_{33}) \\
&= 5\cdot(2\cdot 4 - (-1)\cdot\sqrt{3}) - e\cdot(3\cdot 4 - 1\cdot\sqrt{3}) + (-1)\cdot(3\cdot(-1) - 1\cdot 2)
\end{aligned}$$

$$= 5\cdot(8+\sqrt{3}) - e(12-\sqrt{3}) + 5 = 40 + (5+e)\sqrt{3} - 12e + 5$$
$$= 45 + (5+e)\sqrt{3} - 12e.$$

The two answers agree indeed, as expected.

<u>Example 2</u> Let $C = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \in \mathbb{Q}^{4\times 4}$

(note that this is a diagonal matrix). What is $\det(C)$?

<u>Solution</u> Expanding over the 1st row:

$\det(C) = 4\cdot\det(M_{11})$

Note now that $M_{11} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is again a diagonal matrix (and its $(1,1)$ entry is -3).

Thus $\det(M_{11}) = (-3)\cdot\det((M_{11})_{11}) = (-3)\cdot\det(M_{\{1,2\},\{1,2\}})$

where we have adopted the following notation:
if S_1, S_2 are subsets of $\{1,2,\dots n\}$, then we write
M_{S_1,S_2} for the minor of B
that we get by removing the rows of B with index in S_1
and the columns of B with index in S_2.

Finally, $\det(M_{\{1,2\},\{1,2\}}) = \det\left(\begin{pmatrix} 0.5 & 0 \\ 0 & 5 \end{pmatrix}\right) = 0.5\cdot 5$

We conclude that
$$\det(C) = 4\cdot(-3)\cdot 0.5\cdot 5$$
that is, it is the product of its diagonal entries.

This is no accident, and we'll justify it soon for an arbitrary diagonal matrix.

Sometimes we may have to consider a matrix some of whose entries are given in terms of a parameter or variable allowed to take values in $\mathbb{F}$.
Still we can compute its determinant by treating the parameter/variable as an unknown number.

<u>Example 3</u> If $A = \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 4}$, and if we set

$A_\lambda = A - \lambda I_4 = \begin{pmatrix} 1-\lambda & 0 & 3 & 8 \\ 0 & 1-\lambda & 2 & 4 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{pmatrix}$ with λ denoting a parameter

recall that we considered A_λ when studying the eigenvalues of A.

allowed to take values in $\mathbb{R}$, what is $\det(A_\lambda)$?

<u>Solution</u> Note that here it's easier to expand over the 1st column or over the 2nd column.
Expanding over the 1st column, and then over the 2nd column, we get similarly to Example 2:

$$\det(A_\lambda) = (1-\lambda)\cdot\det(M_{11}) = (1-\lambda)^2\cdot\det(M_{\{1,2\},\{1,2\}})$$

$$= (1-\lambda)^2\cdot\det\left(\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\right) = (1-\lambda)^2\cdot(\lambda^2-1)$$

We thus see that $\det(A_\lambda)$ is a polynomial in λ (of degree? 4, equal to the number of rows or the number of columns of A).
Also $(1-\lambda)^2\cdot(\lambda^2-1) = (1-\lambda)^2(\lambda-1)(\lambda+1) = (\lambda-1)^3\cdot(\lambda+1)$,
so the only roots of this polynomial are the numbers $+1$ and -1.
Recall that we saw that these are also the only eigenvalues of A.
This is no accident, and we'll soon justify the general rule

applying here.

Now let's turn to why expansion over any row or over any column gives the same result: we will try to justify this by giving an alternative definition for the determinant, which we can show is an equivalent definition.

Going back to Example 1, we can see:

$$\det(B) = (-1)^{1+3} b_{13} \det(M_{13}) + (-1)^{2+3} b_{23} \det(M_{23}) + (-1)^{3+3} b_{33} \det(M_{33})$$

$$= b_{13}(b_{21}b_{32} - b_{22}b_{31}) - b_{23}(b_{11}b_{32} - b_{12}b_{31}) + b_{33}\cdot(b_{11}\cdot b_{22} - b_{12}\cdot b_{21})$$

$$= b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{23}b_{11}b_{32} + b_{23}b_{12}b_{31} + b_{33}b_{11}b_{22} - b_{33}b_{12}b_{21}.$$

Thus we can write $\det(B)$ as a sum of $+$ or $-$ products of entries of B, where each such product contains exactly one entry from each row and exactly one entry from each column.

In other words, each such product is of the form $b_{1j_1} b_{2j_2} b_{3j_3}$ with $\{j_1, j_2, j_3\} = \{1,2,3\}$.

Note that this corresponds to a bijective function from $\{1,2,3\}$ to itself:

$$1 \mapsto j_1, \quad 2 \mapsto j_2, \quad 3 \mapsto j_3$$

and for each such function we have the corresponding product in the sum above. (check)

Terminology We call such functions **permutations** of $\{1,2,3\}$ or 3-permutations

(the name is given because if we write the outputs of such a function in the order prescribed by the inputs we get a rearrangement or permutation of the numbers 1, 2 and 3).

Analogously we have the permutations of $\{1,2,\ldots n\}$ or n-permutations.

Possible(?) Definition of Determinant Let $A \in \mathbb{F}^{n\times n}$.

We set

$$\det(A) = \sum_{\substack{\sigma \text{ is} \\ \text{an } n\text{-permutation}}} \operatorname{sgn}(\sigma) \cdot a_{1,\sigma(1)}\, a_{2,\sigma(2)} \cdots a_{n-1,\sigma(n-1)}\, a_{n,\sigma(n)}$$

here we have a summand for each n-permutation

where, for each n-permutation σ, $\operatorname{sgn}(\sigma)$ is $+1$ or -1.

Question What is the correct choice of $\operatorname{sgn}(\sigma)$ for each n-permutation σ, so that this becomes an equivalent way of defining the determinant? Is there a correct choice here?

We will see that a "good" choice exists, and also that it does not depend on what the matrix A is.

Recall: Motivated by examples we've seen so far, but also because we can justify it using

mathematical induction

→ for a given matrix $A \in F^{n \times n}$ we can write

$$\det(A) = \sum_{\substack{\sigma \text{ is an} \\ n\text{-permutation}}} \operatorname{sgn}(\sigma) \cdot a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n-1,\sigma(n-1)} \cdot a_{n,\sigma(n)}.$$

We get one + or − product for each n-permutation

where $\operatorname{sgn}(\sigma)$ is supposed to be either $+1$ or -1.

Reminder: Terminology We call n-permutations or permutations of $\{1,2,\dots n\}$ the bijective functions from $\{1,2,\dots n\}$ to itself.

e.g. the function
$\sigma(1)=4$, $\sigma(2)=5$, $\sigma(3)=2$, $\sigma(4)=1$, $\sigma(5)=3$ and $\sigma(6)=6$
is a permutation of $\{1,2,3,4,5,6\}$

while the function
$\tau(1)=8$, $\tau(2)=6$, $\tau(3)=2$, $\tau(4)=7$, $\tau(5)=4$, $\tau(6)=1$,
$\tau(7)=5$, $\tau(8)=3$, $\tau(9)=11$, $\tau(10)=9$, $\tau(11)=12$ and $\tau(12)=10$
is a 12-permutation.

But what should $\mathrm{sgn}(\sigma)$ be in the above expression for $\det(A)$? Does the choice of $+$ or $-$ for each σ depend on A too? This would be "bad" because it would make this equivalent expression for $\det(A)$ harder to define.

Fortunately, there is a method of "choosing" correctly $\mathrm{sgn}(\sigma)$ by looking <u>only</u> at the permutations themselves.

By this method we also get "nice" representations/ ways of "compactly" writing down a permutation.

<u>Definition</u> (A special family of permutations: cyclic permutations or cycles)

Suppose we have a k-subset $\{a_1, a_2, \ldots a_k\}$ of $\{1, 2, \ldots n\}$.

We write $\boxed{(a_1\ a_2\ a_3 \ldots\ a_k)}$ for the n-permutation

$$a_1 \longmapsto a_2,\ a_2 \longmapsto a_3, \ \ldots \ a_{k-1} \longmapsto a_k,\ a_k \longmapsto a_1$$

and if $b \in \{1, 2, \ldots n\}$, $b \notin \{a_1, a_2, \ldots a_k\}$ then $b \longmapsto b$

When the subset we start with has k elements, we call this permutation

a k-cyclic permutation or k-cycle

Simplest cases of cyclic permutations:

- A 1-cycle is always the identity function: (a_1) is just another way of writing the identity function on $\{1,2,\ldots n\}$

- A 2-cycle is a permutation that <u>swaps</u> two elements of $\{1,2,\ldots n\}$ and leaves all other elements unchanged:
  e.g. on $\{1,2,\ldots 5\}$ the 2-cycle $(2\ 4)$ is the permutation
  $1 \mapsto 1,\ 2 \mapsto 4,\ 3 \mapsto 3,\ 4 \mapsto 2,\ 5 \mapsto 5$

2-cycles are also called <u>transpositions</u>.

<u>Important Remark</u> 2-cycles (or transpositions) naturally correspond to the elementary permutation matrices we have seen.

e.g. the permutation $(2\,4)$ on $\{1,2,\ldots 5\}$ corresponds to the elementary matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{F}^{5\times 5}$$

<u>Natural Generalisation</u> We can think of any permutation as corresponding to a <u>general permutation matrix</u> (which is simply a matrix with <u>exactly</u> <u>one</u> <u>non-zero</u>

entry equal to 1 in each row, and exactly one non-zero entry equal to 1 in each column).

e.g. the permutation $(1\,2\,4)$ on $\{1, 2, \ldots 5\}$ corresponds to the permutation matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{F}^{5\times 5}$$

<u>Also</u> our usual matrix multiplication now corresponds to composition of permutations (which is why we also use the terminology "product of permutations").

e.g. check that, as functions,

$$(1\,2\,4) = (1\,2) \circ (2\,4).$$

Indeed if $\sigma = (1\,2)$, $\tau = (2\,4)$,

then $(\sigma\circ\tau)(1) = \sigma(\tau(1)) = \sigma(1) = 2$

$(\sigma\circ\tau)(2) = \sigma(\tau(2)) = \sigma(4) = 4$

$(\sigma\circ\tau)(3) = \sigma(\tau(3)) = \sigma(3) = 3$

$(\sigma\circ\tau)(4) = \sigma(\tau(4)) = \sigma(2) = 1$

$(\sigma\circ\tau)(5) = \sigma(\tau(5)) = \sigma(5) = 5$

$\longrightarrow$ we thus see that $\sigma\circ\tau$ and $(1\,2\,4)$ take the same value on every element of $\{1, 2, \ldots 5\}$, and hence

$$\sigma\circ\tau = (1\,2\,4).$$

Completely analogously

$$\begin{pmatrix} 1&0&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} = \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 1&0&0&0&0\\ 0&0&0&0&1 \end{pmatrix}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$(2\,4) \qquad\qquad (1\,2) \qquad\qquad (1\,2\,4)$$

notice that we reverse the order in which we multiply the corresponding matrices

Example 1 On $\{1,2,3,4\}$ consider the 3-cycles $(1\,3\,2)$ and $(1\,2\,4)$. What permutation is the composition/product $(1\,3\,2)(1\,2\,4)$?

Solution $(1\,3\,2)(1\,2\,4) = (2\,4\,3)$.

Example 2 Check that $(1\,2\,4)(1\,3\,2) \neq (1\,3\,2)(1\,2\,4)$.

Solution We can similarly check that

$$(1\,2\,4)(1\,3\,2) = (1\,3\,4)$$

Question: Why is $(1\,3\,4) \neq (2\,4\,3)$?

Important Remark As we just saw, composition/product of permutations is not commutative.

However if we have two disjoint cycles (that is, the subsets they "start from" do not have common elements), then these cycles commute,

e.g. $(1\,2\,4)$ and $(3\,5)$ are disjoint cycles on $\{1,2,..5\}$

and $(1\,2\,4)(3\,5) = (3\,5)(1\,2\,4)$ (check!)

Note: The analogous conclusions hold true for multiplication of (general) permutation matrices. (with respect to when they commute).

Very Important and Useful Fact

- Every k-cycle can be written as a product of k-1 transpositions
  e.g. $(1\,8\,3\,2\,6) = (1\,8)(8\,3)(3\,2)(2\,6)$ (check!)

- Every permutation can be written as a product of disjoint cycles, and hence it can also be written as a product of transpositions.

e.g. recall the 6-permutation we saw at the beginning
$\sigma(1)=4,\ \sigma(2)=5,\ \sigma(3)=2,\ \sigma(4)=1,\ \sigma(5)=3,\ \sigma(6)=6.$

How do we find the disjoint cycle representation of σ?

Note that

$\underbrace{1 \mapsto 4 \mapsto 1}_{\text{one 2-cycle}}$ Also $\underbrace{2 \mapsto 5 \mapsto 3 \mapsto 2}_{\text{one 3-cycle}}$

Finally $\underbrace{6 \mapsto 6}_{\text{one 1-cycle}}$ which we can choose to omit writing

Thus $\sigma = (1\,4)(2\,5\,3) = (1\,4)(2\,5)(5\,3)$

now write the cycles as products of transpositions

Exercise 3 Find the disjoint cycle representation of the 10-permutation we saw at the beginning.

We can finally define what sign of a permutation is:

Definition Let σ be an n-permutation, that is, a bijective function from $\{1, 2, \ldots, n\}$ to itself.
We can write σ as a product of disjoint cycles in an essentially unique way (that is, up to the order in which we write the disjoint cycles), and then, based on this disjoint cycle representation of σ, we can write σ as a product of transpositions.

If the number of transpositions appearing in this representation of σ is even,
then we set $\mathrm{sgn}(\sigma) = +1$.

Otherwise, if the number of transpositions whose product gives σ is odd, then we set
$$\mathrm{sgn}(\sigma) = -1.$$

E.g. $\mathrm{sgn}\big((1\,4)(2\,5\,3)\big) = \mathrm{sgn}\big(\underbrace{(1\,4)(2\,5)(5\,3)}_{3 \text{ transpositions}}\big) = -1$

$\mathrm{sgn}\big(\underbrace{(3)}_{0 \text{ transpositions}}\big) = \mathrm{sgn}(\mathrm{id}) = +1$

$\mathrm{sgn}\big((1\,8\,3\,2\,6)\big) = \mathrm{sgn}\big(\underbrace{(1\,8)(8\,3)(3\,2)(2\,6)}_{4 \text{ transpositions}}\big) = +1$