Math 322 - Graph Theory

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Definition of a Graph

A *graph* G is an ordered pair $(V(G), E(G))$,

- where $V(G)$ is a non-empty set (whose elements are called the *vertices* of G; singular number of the word is *vertex*),
- and where $E(G)$ is a subset of the set of 2-element subsets of $V(G)$ (or in other words, it is a set of unordered pairs of elements from $V(G)$). The elements of $E(G)$ are called the *edges* of G.

Terminology

Definition. The size/cardinality of the set V of vertices of G is called the *order* of G, while the cardinality of the set E of edges of G is called the *size* of G.

Let $G = (V, E)$ be a graph.

- Two different vertices $x, y \in V$ are called *adjacent* or *neighbouring* if $\{u, v\} \in E$, that is, if there is an edge e in E with endvertices x and y.
- We say that a vertex x is *incident* with an edge e of G, if x is one of the endvertices of e.
- Two different edges $e_1, e_2 \in E$ are called *adjacent* if they have a common endvertex (that is, if there exists $z \in V$ such that z is incident with e_1 and also incident with e_2.
- A vertex x of G that has no neighbours in G is called an *isolated vertex* of G (note that this happens if we have that, for every $e \in E$, $x \notin e$ (or, using words, x is not incident with e)).

Adjacency Matrix

if the set of vertices $V(G)$ of G is, say, the set $\{v_1, v_2, v_3, \ldots, v_n\}$, and thus the order of G is n, then this is an $n \times n$ square matrix with only 0 or 1 entries, such that

- the (i, j)-th entry is equal to 1 if $v_i v_j$ is an edge of G (in other words, if $\{v_i, v_j\} \in E$),
- the (i, j)-th entry is equal to 0 otherwise.

Note. From the way we define the adjacency matrix, we can conclude that

- it is a symmetric matrix (remember that the edge $v_i v_j$ can also be written as $v_j v_i$);
- its diagonal entries are all equal to 0.

More Terminology

Let $G = (V, E)$ be a vertex, and let $x \in V$.
The *(open) neighbourhood* $N(x)$ of x in G is the set of all neighbours of x in G, that is,

$$N(x) = \{y \in V : xy \in E\}.$$

Important Definition. The *degree* of a vertex x of G is the size/cardinality of the neighbourhood $N(x)$ of x.

Observe that any isolated vertex of G has degree 0 (since its neighbourhood is going to be the empty set).

Connected/Disconnected graphs

Definition. A graph $G = (V, E)$ is called *connected* if, for every two different vertices $x, y \in V$, there is a path from x to y (or equivalently from y to x) in G (namely there is a subgraph P of G which is a path from x to y).

A graph G that is not connected is called *disconnected*.

Remark. Clearly a graph G that contains isolated vertices will be disconnected. However, we can also find disconnected graphs that do not have any isolated vertices (see examples on next slide).

Subgraphs

Let $G = (V, E)$ be a graph.

Definition. A subgraph H of G is an ordered pair (V', E')

- where $\emptyset \neq V' \subseteq V$ (that is, V' is a non-empty subset of V),
- and where $E' \subseteq E$ with every edge $e \in E'$ having both endvertices in V'.

In this case, we write $H \subseteq G$.

Definition. If $H = (V', E')$ is a subgraph of G, and E' contains all the possible edges of G which have both endvertices in V', then we say that H is the subgraph of G that is *induced* or *spanned* by V'.

We denote this induced subgraph by $G[V']$.

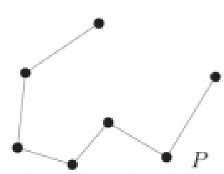
Paths

Definition. A path P is a graph of the form

$$\left(\{x_0, x_1, x_2, \ldots, x_l\},\ \{x_0x_1, x_1x_2, \ldots, x_{l-1}x_l\}\right)$$

where l is an integer $\geqslant 1$.

The number l is called the *length* of the path P (note that it is also the number of edges of P, that is, it is equal to the size of P).



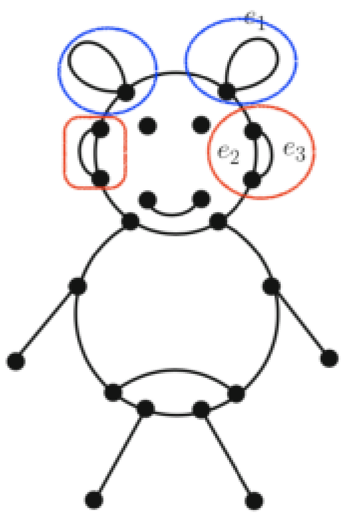
A path P on 7 vertices, thus of length 6

Number of Subgraphs

In general, if a graph $H = (V(H), E(H))$ has m edges, and we want to find all subgraphs of H with vertex set the entire $V(H)$ (that is, if we don't want to remove any vertex), **then we will have 2^m different subgraphs** (because 2^m is the number of all subsets of a set with m elements, in this case the edge set E).

Types of Graphs

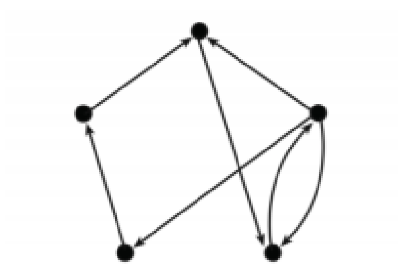
- **Multigraph**



A *multigraph* G is an ordered pair $(V(G), E(G))$,

- where $V(G)$ is a non-empty set (whose elements are called the *vertices* or *nodes* of G),
- and where $E(G)$ is a **multiset** whose elements are taken from the set of 2-element and 1-element subsets of $V(G)$. The elements of $E(G)$ are called the *edges* of G (and now we can also have 'repeated' edges connecting the same pair of vertices, which we call groups of *multiple edges* or *parallel edges*, as well as *loops*).

- **Directed Graph**



from the HHM book

A *directed graph* G is an ordered pair $(V(G), E(G))$,

- where $V(G)$ is a non-empty set (whose elements are called the *vertices* or *nodes* of G),
- and where $E(G)$ is a subset of **ordered** pairs of elements from $V(G)$). The elements of $E(G)$ are again called the *edges* of G (and each of them now has a specific *orientation* or *direction*).

Isomorphism

Let G, H be two graphs. An *isomorphism* from the graph G to the graph H is a bijective function

$$f : V(G) \to V(H)$$

that preserves adjacencies. That is, f has to be 1-1 and onto, and we must have that

$$e \in E(G) \text{ and has endvertices } v_i, v_j \in V(G)$$
$$\text{if and only if } \{f(v_i),\ f(v_j)\} \in E(H).$$

If such an isomorphism from G to H exists, we say that G and H are *isomorphic* and we denote this by $G \cong H$.

Terminology. If two graphs are not isomorphic (that is, if no such bijection from the vertex set of the first graph onto the vertex set of the second graph exists), we say that the graphs are *non-isomorphic*.

- Isomorphic graphs must have the same order (that is, they must have the same number of vertices).
- Isomorphic graphs must have the same size (that is, they must have the same number of edges).
- Isomorphic graphs must have the same degree sequences, up to reordering of the sequences *(much more about degree sequences very soon)*.

However, if two graphs G and H are isomorphic, then there exists a permutation matrix P (that is, a 0-1 matrix which has exactly one entry equal to 1 in each row, and also exactly one entry equal to 1 in each column) such that

$$A_G = P \cdot A_H \cdot P^T.$$

In other words, A_H can be turned into A_G by permuting the rows of A_H in a suitable way, and by permuting its columns too in the same way.

Complement of a Graph

Let $G = (V, E)$ be a graph. Recall that E is a subset of the set of 2-element subsets of V (sometimes denoted by $[V]^2$).

We can construct a new graph H on the same set of vertices by setting

$$H = (V,\ [V]^2 \setminus E),$$

that is, by setting the edge set of H to be the *complement* of E in $[V]^2$.

In essence, what we are doing is removing any edges/'connections' we have in G, and then we are joining any two vertices that were not joined in G.

Definition. The new graph is called the *complement* of G, and is denoted by $\overline{G}$.

Connected Components

Let $G = (V, E)$ be a graph. Define a relation on the vertex set V of G by setting

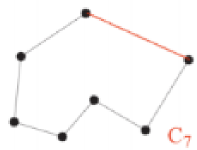
$$v_i \sim v_j \quad \textbf{if and only if}$$
$$v_i = v_j \quad \text{or} \quad \text{there is a path in } G \text{ from } v_i \text{ to } v_j.$$

Then this is an equivalence relation on V (verify this), and thus it gives us a *partition* of V: the different blocks of the partition are the different equivalence classes, where e.g. the equivalence class $[v_i]_\sim$ of a vertex v_i of G is the maximal subset of vertices that we can get to from v_i travelling on a path in G (including the vertex v_i itself).

For each such equivalence class, the induced subgraph of G that we get is one of the connected components of G (and as the name indicates, it is a (maximal) connected subgraph of G).

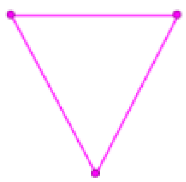
Cycle Graph

If we 'closed' the path P by joining the initial and the terminal vertex as well, then we would get what we call a *cycle graph.*

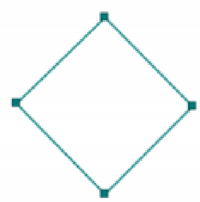


A cycle on 7 vertices, or equivalently a 7-cycle

In other words, $C_7 = \left(\{x_0, x_1, x_2, \ldots, x_6\},\ \{x_0x_1, x_1x_2, \ldots, x_5x_6,\ x_6x_0\}\right)$.

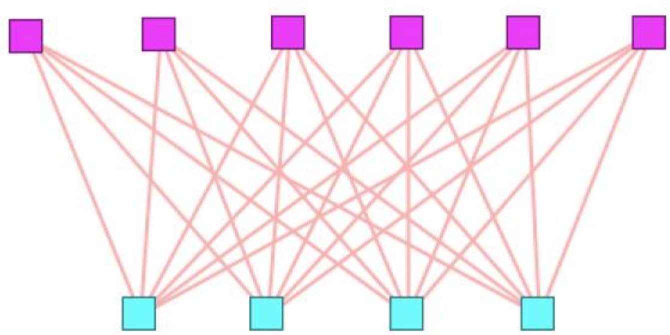


A 3-cycle, denoted by C_3



A 4-cycle, denoted by C_4

Bipartite Graphs



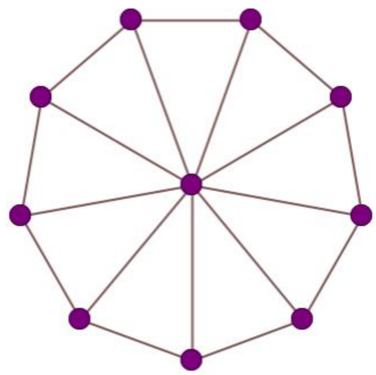
The bipartite graph $K_{6,4}$

Question. How do we construct a bipartite graph?
Answer. As the name suggests, the vertex set V of a bipartite graph is divided into **two parts**, say part V_1 and part V_2. Every vertex in V_1 is joined with every vertex in V_2; on the other hand, no two vertices in V_1 are joined, and similarly no two vertices in V_2 are joined.

If the cardinality $|V_1|$ of the part V_1 is m, and the cardinality $|V_2|$ of the part V_2 is n, then the bipartite graph we just described is denoted by $K_{m,n}$ (or equivalently, $K_{n,m}$).

Wheel Graph

Wheels

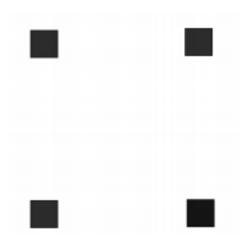


The wheel graph on 10 vertices, denoted by W_{10}

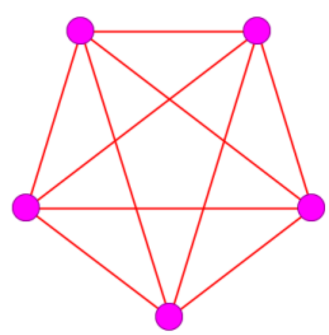
Null/Complete Graphs

Given a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$, the two extreme cases of graphs we can have with vertex set V are:

- the *null graph* on V, that is, the graph on V that has no edges at all,
- the *complete graph* on V, that is, the graph $\left(V, [V]^2\right)$, in which any two elements of V are joined.



The null graph on 4 vertices, denoted by N_4



The complete graph on 5 vertices, denoted by K_5

Disjoint Union of two Graphs

In particular, if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs with **disjoint vertex sets**, namely such that

$$V_1 \cap V_2 = \emptyset,$$

then the ordered pair

$$(V_1 \cup V_2,\ E_1 \cup E_2)$$

is a new graph whose vertices consist of all the vertices of G_1 and all the vertices of G_2.

We denote this new graph by $G_1 \oplus G_2$

(note also that, in the graph $G_1 \oplus G_2$, none of the vertices in V_1 is joined with a vertex in V_2, and vice versa, given that $E_1 \subseteq [V_1]^2$ and $E_2 \subseteq [V_2]^2$, so each edge of $G_1 \oplus G_2$ is either an unordered pair of elements of V_1, or an unordered pair of elements of V_2).

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d- Regular Graphs

Definition. A graph $G = (V, E)$ is called **regular** if all its vertices have the same degree.

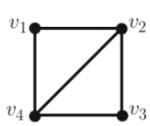
If all these degrees are equal to the non-negative integer d, then we say that G is ***d*-regular**.

Suppose now that you are given a decreasing sequence of n integers,

$$(d_1, d_2, \dots, d_n) \qquad \text{with } d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n.$$

We say that this sequence is a *graphical sequence* if there exists (at least) one graph $H = (V(H), E(H))$ of order n such that the sequence $(d_1, d_2, \dots, d_n)$ will coincide with the degree sequence of H *(after reordering if needed)*.

E.g. the sequence $(3, 3, 2, 2)$ is graphical, since it can be viewed as the degree sequence of the graph



Immediate Restrictions on Graphical Sequences. 1. If a decreasing sequence $(d_1, d_2, \dots, d_n)$ is a graphical sequence, then necessarily all the integers in the sequence must be non-negative. Equivalently we must have $d_n \geqslant 0$.

2. If a decreasing sequence $(d_1, d_2, \dots, d_n)$ of n integers is a graphical sequence, then necessarily all the integers are $\leqslant n-1$. Equivalently we must have $n - 1 \geqslant d_1$. Indeed, recall that the sequence should coincide with the degree sequence of a graph **on n vertices**. But the largest possible degree in such a graph would be $n-1$ (and this would be attained only if there is (at least) one vertex connected to all other vertices).

Handshaking Lemma

Let $G = (V, E)$ be a finite graph. Recall that the order $|G|$ of G is the cardinality of its vertex set V, and the size $e(G)$ of G is the cardinality of its edge set E.

For convenience here, let's say that $V = \{v_1, v_2, \ldots, v_n\}$ (and thus $|G| = n$).

Lemma

We always have that

$$\sum_{v_i \in V} \deg(v_i) = 2e(G)\,.$$

Corollary of the Handshaking Lemma

Let $G = (V, E)$ be a finite graph, and write

- V_{even} for the subset of all the vertices that have *even* degree,
- and V_{odd} for the subset of the remaining vertices, that is, the vertices that have *odd* degree.

The cardinality of V_{odd} will necessarily be **even**!

Let $G = (V, E)$ be a finite graph, and suppose that G has **exactly two** vertices with odd degree, say the vertices v_i and v_j.
Then we must have a path from v_i to v_j.

In any finite graph $G = (V, E)$ **which has at least two vertices**, we can find at least one pair of (different) vertices $v_i, v_j \in V$ which have the same degree.

Equivalent way of thinking about this problem

In any group of two or more people *(each of whom may be friends with some of the people in the group)*, there are at least two people who have the same number of friends.

Havel-Hakimi Theorem

Theorem

Consider a decreasing sequence $S_1 = (d_1, d_2, \ldots, d_n)$ of n non-negative integers.
Then S_1 is graphical **if and only if** the sequence

$$S_1' = \left(d_2 - 1,\, d_3 - 1, \ldots,\, d_{d_1+1} - 1,\, d_{d_1+2},\, d_{d_1+3}, \ldots,\, d_{n-1},\, d_n\right)$$

of $n - 1$ integers is graphical *(note that the purple-coloured terms here are d_1 in total).*

Join of two Graphs

Consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with **disjoint vertex sets**, namely such that

$$V_1 \cap V_2 = \emptyset.$$

The *join* of G_1 and G_2 is a new graph with vertex set

$$V_1 \cup V_2$$

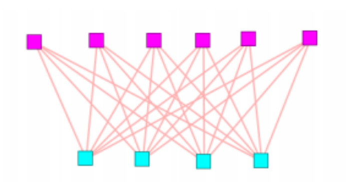
and edge set

$$E_1 \cup E_2 \cup \big\{\{v, w\} : v \in V_1, w \in V_2\big\}.$$

We denote the join of G_1 and G_2 by $G_1 \vee G_2$.

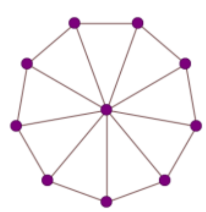
Observe that the vertices of $G_1 \vee G_2$ consist of all the vertices of G_1 and of G_2, and two such vertices v, w are connected in $G_1 \vee G_2$ **exactly when**:

- both v, w are vertices of G_1, and there is an edge in G_1 joining v and w,
- or both v, w are vertices of G_2, and there is an edge in G_2 joining v and w,
- or finally v is a vertex of G_1 and w is a vertex of G_2.



The bipartite graph $K_{6,4}$ can be viewed as the join of the null graphs N_6 and N_4:

$K_{6,4} = N_6 \vee N_4$.



The wheel graph W_{10} can be viewed as the join of the cycle C_9 and the null graph N_1:

$W_{10} = C_9 \vee N_1$.

Deletion of an edge subset

Let $G = (V, E)$ be a graph, and consider a subset E' of the edge set E of G.

We can construct a new graph, which will also be a subgraph of G, **if we simply delete the edges of G which are contained in the edge subset E'.** In other words, we can consider the subgraph

$$(V, E \setminus E')$$

of G, which we will denote by $G - E'$.

Again, if E' contains **only one element, in this case one edge of G**, say edge e_0, we will more simply write $G - e_0$ instead of $G - \{e_0\}$.

Line Graph

Let $G = (V, E)$ be a graph, which has a **non-empty** edge set E (in other words, there is at least one edge in G).

Then we can define a new graph, called the *line graph* of G, which will be denoted by $L(G)$, as follows:

- the vertex set of $L(G)$ is the edge set of G; in other words, $V(L(G)) = E(G)$;
- two 'vertices' in $L(G)$ are joined if, when we view them as edges of G, they are adjacent, or in other words, they have a common endvertex.

Complement Graph Connectedness

Proposition 1

Let G be a disconnected graph, that is, a graph that has at least two connected components.

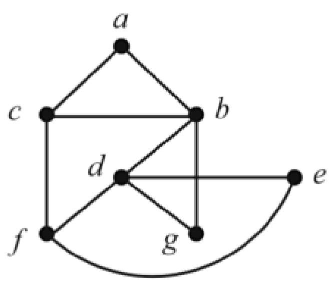
Then the complement $\overline{G}$ of G must be connected.

Maximum/Minimum Degree

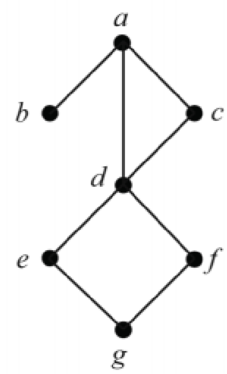
Let $G = (V, E)$ be a finite graph.

- We will denote by $\delta(G)$ the minimum degree of a vertex in G,
- and by $\Delta(G)$ the maximum degree of a vertex in G.

Two examples from HW1.



Here $\delta(G) = 2$ and $\Delta(G) = 4$



Here $\delta(G) = 1$ and $\Delta(G) = 4$

Proposition 2

Let G be a finite graph of order n, and assume that $\delta(G) \geqslant \frac{n-1}{2}$. Then G must be connected.

Proposition 3

Let G be a graph of order n, and suppose that G has size strictly less than $n - 1$.

Then G must be disconnected.

Let k be a positive integer, and let G be a graph satisfying $\delta(G) \geqslant k$.

- Show that G contains a path of length at least k.
- If $k \geqslant 2$, show that G contains a cycle of order at least $k + 1$.

Walks/Paths/Cycles/Trails/Circuits

- walks A *walk* of length k in G is a sequence of (not necessarily distinct) vertices $v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ from V, such that $v_i v_{i+1} \in E(G)$ for every $i = 0, 1, 2, \ldots, k-1$. The vertices v_{i_0} and v_{i_k} are called the *endvertices* of the walk, and we sometimes say that this is a $v_{i_0} - v_{i_k}$ walk.

  Since G is a graph (and thus, according to our convention, it does not contain multiple edges), we can completely describe the walk by writing the vertices one next to the other in the correct order: $v_{i_0} v_{i_1} v_{i_2} \cdots v_{i_k}$.
- paths Recall that a path contained in the graph G is simply a walk in which **all the vertices are distinct**.
- cycles If the endvertices v_{i_0} and v_{i_k} of the walk coincide, but all the other vertices are distinct and also different from v_{i_0}, then we get a 'closed path', which we have called a *cycle*.
- trails If the **edges** in the walk are distinct (but not necessarily the vertices), we call the walk a *trail*.
- circuits Finally, a 'closed trail' is called a *circuit*.

Theorem

Let G be a graph, and let u, v be two different vertices of G.
Any $u - v$ walk in G contains a $u - v$ path.

Trees/Forests

Definition

1. A graph G is called *cyclic* if it contains at least one cycle. It is called *acyclic* otherwise.
2. A connected acyclic graph is called a *tree*.
3. On the other hand, if an acyclic graph is disconnected, then it is called a *forest* (in other words, *forests* are disjoint unions of two or more *trees*).
4. A vertex of degree 1 in a tree or a forest is called a *leaf*.

Forbidden Subgraphs

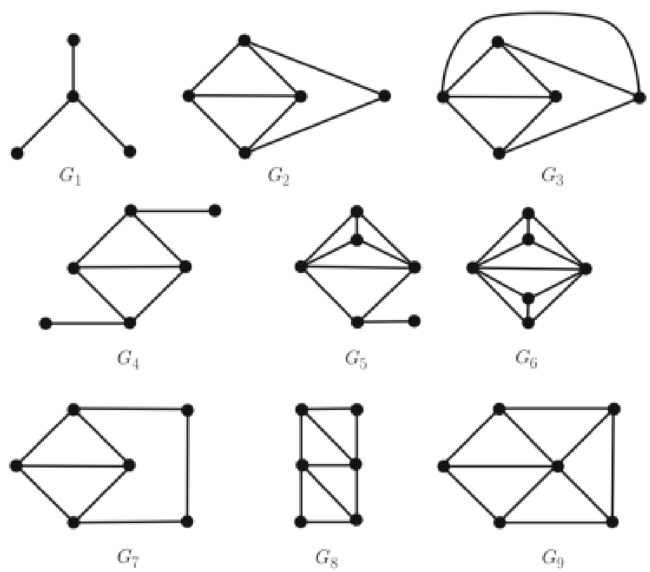
Terminology. We call a graph H a *forbidden subgraph for a property P* of graphs if the following holds true: given any graph G,

property P holds true for G **if and only if**
G does not contain H as an **induced subgraph**
(in other words, H is not isomorphic to an induced subgraph of G).

A graph G is (a subgraph of) a bipartite graph if and only if G does not contain any odd cycles (that is, if and only if none of the subgraphs of G is an odd cycle).

Theorem (Beineke, 1968)

A graph G is the *line graph* of some other graph H if and only if the following 9 graphs are forbidden subgraphs for G.



from the Balakrishnan-Ranganathan book

Cut vertex/Bridge

Definitions

Let G be a connected graph.

1. A vertex v of G is called a *cutvertex* of G if we have that
$$G - v$$
is no longer connected.
2. An edge e in G is called a *bridge* (or a *cutedge*) of G if we have that
$$G - e$$
is no longer connected.

Definition again, more generally formulated

If we start with a graph H which is not necessarily connected, then

1. a vertex $\tilde{v}$ of H is called a *cutvertex* of H if, by deleting $\tilde{v}$ (and of course all the edges $\tilde{v}$ is incident with), we increase the number of connected components of H.
2. an edge $\tilde{e}$ in H is called a *bridge* of H if, by deleting $\tilde{e}$, we increase the number of connected components of H.

Definitions

Let $G = (V, E)$ be a connected graph.

1. A subset V' of the vertex set V of G is called a *vertex cut* if we have that
$$G - V'$$
is disconnected. We call it a *k-vertex cut* if the cardinality $|V'|$ of V' is equal to k (that is, if V' contains k vertices of G).

   V' is also called a *separating set of vertices* of G.
2. A subset E' of the edge set E of G is called an *edge cut* if we have that
$$G - E'$$
is disconnected. We call it a *k-edge cut* if $|E'| = k$.

   E' is also called a *separating set of edges* of G.

K-vertex Connected

Definition 1

A connected graph G is said to be *k-vertex connected* if G has **at least** $k+1$ vertices, and **there does not exist** a vertex cut of G of cardinality $k-1$.

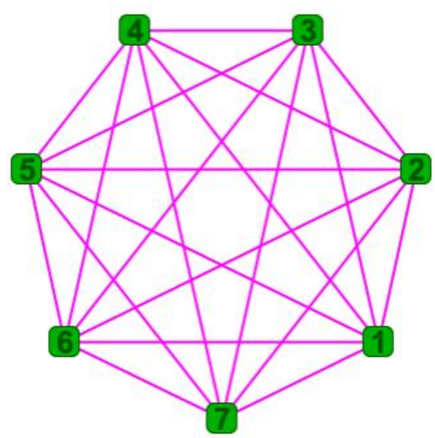
Useful Remark. Note that,

- if a connected graph H of order $n \geqslant 3$ has an s-vertex cut with $s \leqslant n-2$,
- then it also has t-vertex cuts for each cardinality t between s and $n-2$ (this is because if, by removing certain s vertices of H, we end up with a disconnected graph, then clearly by removing $t = s + (t-s)$ vertices of H in a suitable way (that is, by including the s vertices from the s-vertex cut we found before) we will end up again with a disconnected graph.

Thus a k-vertex connected graph will have no s-vertex cuts for any $s < k-1$ either. In other words, a k-vertex connected graph G is also a t-vertex connected graph for every $0 < t \leqslant k$.

According to how we gave Definition 1, we get that

the complete graph K_n is $(n-1)$-vertex connected.



Complete graph on vertices $\{1, 2, 3, 4, 5, 6, 7\}$

Important Remark. Given $n \geqslant 2$, we have that, for every (proper) subset V' of the vertex set V of K_n, the graph $K_n - V'$ is again a complete graph (on the vertices $V \setminus V'$ now), and hence it cannot be disconnected.

$\rightsquigarrow$ a complete graph does not have any vertex cuts.

Parameters- Kappa(G)/Lambda(G)

Definition 2: the parameter $\kappa(G)$

Let G be a connected graph of order $\geqslant 2$. We define the *vertex connectivity* $\kappa(G)$ of G to be the maximum integer k such that G is k-vertex connected.

Note that, since we start with a connected graph G with at least two vertices, we will have that G is 1-vertex connected, and hence $\kappa(G) \geqslant 1$.

By convention $\kappa(K_1) = 0$.

Very Useful Remark

Let G be a connected graph on n vertices which is different from K_n. The vertex connectivity $\kappa(G)$ of G coincides with **the minimum cardinality of a vertex cut** of G.

Definition 3: the parameter $\lambda(G)$

Let G be a connected graph of order $\geqslant 2$. We define the *edge connectivity* $\lambda(G)$ of G to be **the minimum cardinality of an edge cut** of G.

Whitney's Theorem

Theorem 1 (H. Whitney, 1932)

For every connected graph G (of order $\geqslant 2$), we have

$$\kappa(G) \leqslant \lambda(G) \leqslant \delta(G).$$

Let G be a connected graph of order n that is different from K_n.

Then $\lambda(G) < n - 1$ (why?).

We can reach this conclusion using Whitney's theorem: we have that

- if G is not the complete graph on n vertices, then G has at least one vertex which is not connected to every other vertex, and hence $\delta(G) < n - 1$.
- By Whitney's theorem we get that $\lambda(G) \leqslant \delta(G) < n - 1$.

Some conclusions

- The maximum size of a connected graph on n vertices is $\binom{n}{2}$.

Question 1. Which connected graphs of order n have this size? Only the complete graph K_n has this size.

- The minimum size of a connected graph on n vertices is $n - 1$.

Question 2. Which connected graphs of order n have this size? We have seen that a path on n vertices has this size; *we'll see what other connected graphs have this size soon.*

- The maximum size of a disconnected graph on n vertices is $\binom{n}{2} - (n-1) = \binom{n-1}{2}$.
- The minimum size of a disconnected graph on n vertices is 0 (and the only example here is the Null Graph N_n).

More generally, we can show...

Theorem

Let G be a graph of order n which has exactly k connected components (where $1 \leqslant k \leqslant n$).

Then the maximum possible size of G is $\binom{n-k+1}{2}$, and the minimum possible size is $n - k$.

Propositions

Proposition 1

Let G be a connected graph of order $\geqslant 3$ which contains at least one cycle, and let e be an edge in G which belongs to a cycle in G.

Then e cannot be a bridge.

Proposition 2

Let G be a connected graph of order $\geqslant 2$, and let v be a vertex of G such that $\deg(v) = 1$.

Then v cannot be a cutvertex.

Corollary of Proposition 2

Let T be a tree of order $\geqslant 2$. Then T does contain leaves (that is, vertices of degree 1), and none of these leaves is a cutvertex of T.

Proposition 3

Let T_n be a tree on n vertices.

Then we necessarily have $e(T_n) = n - 1$ (that is, T_n always has $n - 1$ edges).

Corollary 3

Let T be a tree with at least 2 vertices.

Then T contains **at least two leaves** (namely it has at least two vertices of degree 1).

Corollary 2

Let T_n be a tree on n vertices.

Every edge of T_n is a bridge of T_n *(and hence $\lambda(T_n) = 1$ when defined).*

Trees - Theorem

Theorem 2

Let K be a graph of order n. The following statements are equivalent:

(i) K is a tree (that is, K is connected and acyclic).

(ii) K is acyclic and has precisely $n-1$ edges.

(iii) K is connected and has precisely $n-1$ edges.

(iv) K is connected and each edge is a bridge.

Local Vertex/Edge Connectivity

Definition

Let G be a connected graph of order n, and let w, z be two vertices of G.

An *edge cut for w and z* is a subset E' of $E(G)$ with the property that

there is no $w - z$ path in $G - E'$.

The *local edge connectivity $\lambda(w, z)$* is the **minimum cardinality of an edge cut for w and z.**

Note. It's not hard to convince ourselves that $\lambda(w, z) = \lambda(z, w)$.

Important Observation

We have that $\lambda(G)$ equals the minimum of the quantities $\lambda(w, z)$ that we obtain if we consider all pairs (w, z) of different vertices of G:

$$\lambda(G) = \min\{\lambda(w, z) : w, z \in V(G),\ w \neq z\}.$$

Definition

Let G be a connected graph of order n (other than K_n), and let u, v be two **non-adjacent** vertices of G.

A *vertex cut for u and v* is a subset V' of $V(G) \setminus \{u, v\}$ with the property that

there is no $u - v$ path in $G - V'$.

The *local vertex connectivity $\kappa(u, v)$* is the **minimum cardinality of a vertex cut for u and v.**

Note. It's not hard to convince ourselves that $\kappa(u, v) = \kappa(v, u)$.

Important Observation

We have that $\kappa(G)$ equals the minimum of the quantities $\kappa(u, v)$ that we obtain if we consider all pairs (u, v) of **non-adjacent** vertices of G:

$$\kappa(G) = \min\{\kappa(u, v) : u, v \in V(G),\ u \neq v,\ uv \notin E(G)\}.$$

Disjoint Paths

Definition 1

Let G be a graph, and let w, z be two vertices of G. Suppose that $Q_1, Q_2, \ldots, Q_s$ are s different $w-z$ paths in G.

The collection $\{Q_1, Q_2, \ldots, Q_s\}$ is called *edge-disjoint* if, for any two different paths Q_i, Q_j in this collection, Q_i and Q_j contain **no** common edges.

We write $\lambda'(w, z)$ for the maximum possible cardinality that an edge-disjoint collection of $w-z$ paths in G can have.

Definition 1

Let G be a graph, and let u, v be two vertices of G. Suppose that $P_1, P_2, \ldots, P_l$ are l different $u-v$ paths in G.

The collection $\{P_1, P_2, \ldots, P_l\}$ is called *internally disjoint* (or alternatively *vertex-disjoint*) if, for any two different paths in this collection, **their only common vertices** are the vertices u and v (in other words, if none of the internal vertices in any one of these paths appears in another path too).

We write $\kappa'(u, v)$ for the maximum possible cardinality that an internally disjoint collection of $u-v$ paths in G can have.

Menger's Theorem

Menger's theorem *(vertex form)*

Let G be a connected graph of order n (other than K_n), and let u, v be two **non-adjacent** vertices of G.

Then the **minimum** cardinality of a vertex cut for u and v equals the **maximum** cardinality of an internally disjoint collection of $u-v$ paths in G. In other words,

$$\kappa(u, v) = \kappa'(u, v).$$

Menger's theorem *(edge form)*

Let G be a connected graph of order n, and let w, z be two vertices of G. Then the **minimum** cardinality of an edge cut for w and z equals the **maximum** cardinality of an edge-disjoint collection of $w-z$ paths in G.

In other words,

$$\lambda(w, z) = \lambda'(w, z).$$

Important Corollary of Menger's Theorem

- Let G be a connected graph of order n (other than K_n). Then $\kappa(G) \geqslant t$ if and only if, for any two **non-adjacent** vertices u, v of G, we can find **at least** t pairwise internally disjoint paths in G that connect u and v.
- Let H be a connected graph of order n. Then $\lambda(H) \geqslant s$ if and only if, for any two different vertices w, z of H, we can find **at least** s pairwise edge-disjoint paths in H that connect w and z.

Problem (based on recent material)

Let G be a connected graph, and let u, v be two different vertices of G.

- Show that, if there are (at least) two internally disjoint $u - v$ paths in G, then there is a cycle in G **passing through the vertices u and v**.
- Show that, if there are (at least) two different (but not necessarily internally disjoint) $u - v$ paths in G, then G contains at least one cycle.

Corollary to this problem

Let T be a tree. Then, for every two vertices u and v in T, there is **exactly one** path connecting u and v.

Spanning Tree

Definition

Let G be a connected graph, and let T be a subgraph of G.
T is called a *spanning tree* of G if

- T is a tree
- and T contains all vertices of G.

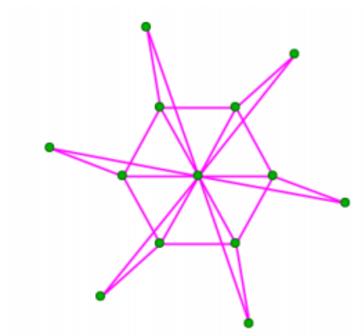
Observation 1

A spanning tree of G is a *minimal* connected subgraph of G that contains all the vertices of G (in the sense that, if we remove any more edges from T, then we will have a disconnected subgraph).

We could think of a spanning tree of a connected graph G as a very simple 'skeletal frame' for the graph.

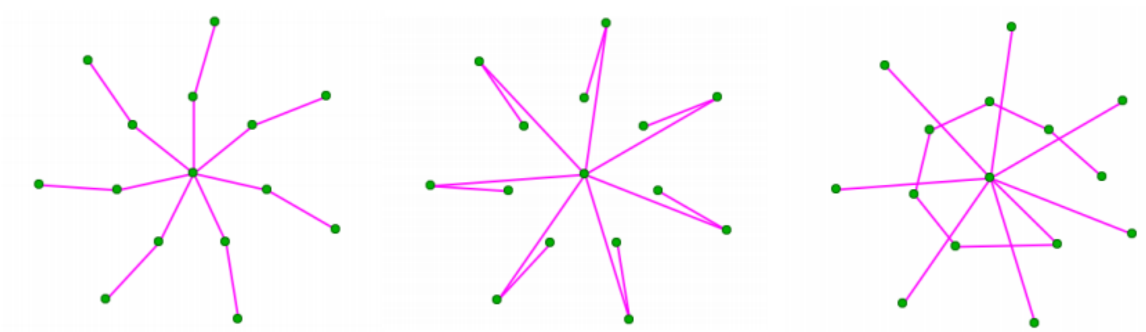
Observation 2

In general, a connected graph G may have several spanning trees.



A 'flower' graph

and three spanning trees for it:



Theorem

Every connected graph G has at least one spanning tree.

Weighted Graphs

Definition

Let $K = (V(K), E(K))$ be a graph. A *weight function* for K is a function

$$w : E(K) \to [0, +\infty)$$

mapping each edge of K to a non-negative real number.

The graph K together with a weight function for it is called a *weighted graph*.

Minimum Weight Spanning Tree

Kruskal's algorithm

Suppose you are given a weighted connected graph G, and you are looking for a minimum weight spanning tree of it. Then:

1. Begin by finding an edge of minimum weight, and mark it.
2. Out of all the edges that remain unmarked **and which do not form a cycle with any of the already marked edges**, pick an edge of minimum weight and mark it.
3. If the set of the already marked edges gives a spanning tree of G, then terminate the process. Otherwise, return to Step 2.

Theorem (analogous to above)

For every weighted connected graph G, Kruskal's algorithm gives a minimum weight spanning tree.

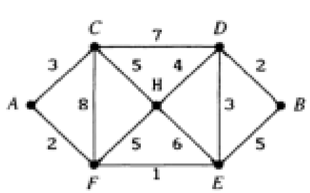
Weighted Matrix

In the process of solving the shortest path problem, we need to consider the *weight matrix* W_{G_0} of G_0.

Assume that the vertex set of G_0 is $\{v_1, v_2, \ldots, v_n\}$; then $W_{G_0} = \left(w_{i,j}\right)_{i,j}$ is an $n \times n$ matrix satisfying

$$w_{i,j} = \begin{cases} \infty & \text{if } i = j \text{, or the vertices } v_i \text{ and } v_j \text{ are non-adjacent} \\ \text{weight of the edge } \{v_i, v_j\} & \text{if } v_i \text{ and } v_j \text{ are adjacent vertices} \end{cases}$$

For the example above, we have



$$W_{G_0} = \begin{array}{cc} & \begin{matrix} A & B & C & D & E & F & H \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ H \end{matrix} & \begin{pmatrix} \infty & \infty & 3 & \infty & \infty & 2 & \infty \\ \infty & \infty & \infty & 2 & 5 & \infty & \infty \\ 3 & \infty & \infty & 7 & \infty & 8 & 5 \\ \infty & 2 & 7 & \infty & 3 & \infty & 4 \\ \infty & 5 & \infty & 3 & \infty & 1 & 6 \\ 2 & \infty & 8 & \infty & 1 & \infty & 5 \\ \infty & \infty & 5 & 4 & 6 & 5 & \infty \end{pmatrix} \end{array}.$$

Dijkstra's Algorithm

Dijkstra's algorithm

— The algorithm proceeds as follows: at each stage it assigns weights to the vertices (which are based in some sense on the weights of the edges).

— Some of these weights are called temporary *(in a sense, we're still testing out what the value of those vertices should be)*, while the rest have become permanent at some point *(and the algorithm cannot alter the latter anymore)*.

— **At each stage a new vertex is allotted a permanent weight,** with the vertex A being the first one to be allotted a permanent weight.

— The algorithm can be terminated as soon as the vertex B (the vertex that we want our path to end at) is allotted a permanent weight.

Multigraph – Definitions/Terminology

Let $H = (V, E)$ be a multigraph in H, and suppose that $V = \{v_1, v_2, \ldots, v_n\}$, while $E = \{e_1, e_2, \ldots, e_m\}$ (the latter set may include edges which have the same pair of endvertices (and thus form groups of multiple edges), as well as loops; it's even possible that we have 'parallel' loops, that is, distinct loops which are attached to the same vertex).

- walks A *walk* of length k in H is now represented by a sequence of the form

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

  where $v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are vertices from $V(H)$ (not necessarily distinct), $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ are edges from $E(H)$ (not necessarily distinct), **and for every $s = 1, 2, \ldots, k$ we have that $e_{j_s} = \{v_{i_{s-1}}, v_{i_s}\}$** (or in other words, e_{j_s} joins the vertices $v_{i_{s-1}}$ and v_{i_s}; note that now we may have cases where consecutive vertices are equal, that is, cases where $v_{i_{s-1}} = v_{i_s}$, and then e_{j_s} should be a loop attached to the vertex v_{i_s}).

  As before, the vertices v_{i_0} and v_{i_k} are called the *endvertices* of the walk, and we sometimes say that this is a $v_{i_0} - v_{i_k}$ walk.
- paths A path in H is simply a walk in which **all the vertices are distinct** *(note that, as a consequence of this definition, a path will not contain any loops)*.
- cycles A cycle is a closed path.
- trails A trail in H is a walk in which **all the edges are distinct** (but not necessarily all the vertices).
- circuits Finally, a circuit is a closed trail.

Eulerian

Definition

Let G be a connected graph (or multigraph) which is non-trivial, that is, it contains at least one edge.

- An *Euler trail* in G is a trail that passes by **all** edges in G (and hence, given that it is a trail, it passes by each edge exactly once).
- An *Euler circuit* in G is a circuit (that is, a closed trail) that passes by **all** edges in G.

G is called *Eulerian* if we can find (at least) one Euler circuit in G.

Hamiltonian

Definition

Let G be a connected graph.

- A *Hamilton path* in G is a path that passes through **all** vertices in G (and hence, given that it is a path, it passes through each vertex exactly once).
- A *Hamilton cycle* in G is a cycle (that is, a closed path) that passes through **all** vertices in G.

G is called *Hamiltonian* if we can find (at least) one Hamilton cycle in G.

The name is in honour of the mathematician William Hamilton who introduced the idea of looking for Hamilton cycles in graphs (with the first graph he considered being (the 'frame' of) the solid dodecahedron) as a new board game!

Theorems for Eulerian Graphs

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Theorem 1

Let G be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** every vertex of G has even degree.

Proposition 1

Let G be a connected graph (or multigraph).

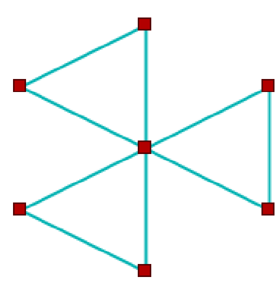
Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

Theorem 2

Let $G = (V, E)$ be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** its edge set E can be written as the disjoint union of subsets $E_1, E_2, \ldots, E_s$ each of which forms a cycle in G.

Example. In the graph below we can write the edge set as the disjoint union of three cycles:



Immediate Corollary

Every Eulerian graph G is **bridgeless** (that is, it satisfies $\lambda(G) \geqslant 2$).

Difference between sufficient and Necessary conditions

some conditions that are necessary for a graph to be Hamiltonian
(that is, if any of these conditions doesn't hold,
then the graph cannot be Hamiltonian),

and some conditions that are sufficient
(that is, it suffices to check for any one of these conditions,
and if it does hold true, then the graph will be Hamiltonian).

Theorems for Hamiltonian Graphs

Necessary Condition 1

Let G be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then G has no cutvertices.
In other words, if G is Hamiltonian, then $\kappa(G) \geqslant 2$ (or equivalently G is 2-vertex connected).

Necessary Condition 2

Let $G = (V, E)$ be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then the following holds:

for every vertex subset $S \subsetneq V$,
the subgraph $G - S$ has at most $|S|$ connected components.

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$, then G is Hamiltonian.

Remark. The lower bound $\frac{n}{2}$ for the minimum degree of G is optimal, and, perhaps surprisingly, **cannot** even be replaced by $\delta(G) \geqslant \lfloor \frac{n}{2} \rfloor$.

Theorem 2 (Ore, 1960)

Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of distinct and non-adjacent vertices u and v of G, we have that $\deg(u) + \deg(v) \geqslant n$.

Then G is Hamiltonian.

Proposition 3

Let G be a graph of order $n \geqslant 3$, and suppose that G has at least

$$\binom{n-1}{2} + 2$$

edges. Then G is Hamiltonian.

Independence Number

Definition

Let $G = (V, E)$ be a graph. A vertex subset $V' \subseteq V$ is called an *independent* set of vertices, if any two different vertices in V' are **non-adjacent**.

The *independence number* of G, denoted by $\alpha(G)$, is defined to be the maximum cardinality of an independent set of vertices of G.

Again, only a sufficient condition

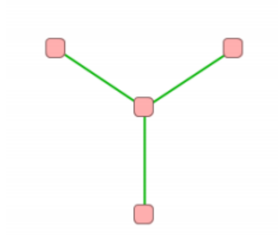
Theorem 4 (Chvátal-Erdös, 1972)

Let G be a graph of order $n \geqslant 3$ such that $\kappa(G) \geqslant \alpha(G)$. Then G is Hamiltonian.

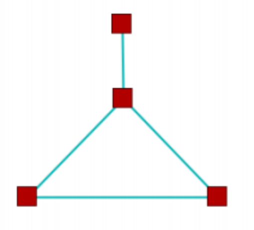
Forbidden Subgraphs

Family of possible forbidden subgraphs

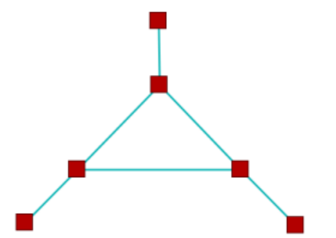
Consider the following three graphs:



Graph $K_{1,3}$



Graph Z_4



Graph Z_6

Theorem 5 (Goodman-Hedetniemi, 1974)

Let G be a graph (of order $n \geqslant 3$) which is 2-vertex connected (that is, $\kappa(G) \geqslant 2$).

If G is $\{K_{1,3}, Z_4\}$-free *(that is, none of those two graphs is an induced subgraph of G)*, then G is Hamiltonian.

Theorem 6 (Duffus-Gould-Jacobson, 1980)

Let G be a $\{K_{1,3}, Z_6\}$-free graph.

- If G is connected, then G has a Hamilton path.
- If G is 2-vertex connected, then G is Hamiltonian.

Necessary Condition $2'$

Let $H = (V, E)$ be a connected graph of order $n \geqslant 2$.

If H has a Hamilton path, then the following holds:

for every vertex subset $S \subsetneq V$,
the subgraph $H - S$ has at most $|S| + 1$ connected components.

Factor/Factorization

Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

1. A *factor*, or equivalently *spanning subgraph*, of G is a subgraph (or 'sub-multigraph') of G that contains all vertices of G (that is, a subgraph of order n).

2. A *factorization* of G is any collection of s factors (spanning subgraphs) $H_1, H_2, \ldots, H_s$ of G such that
   - any two different factors H_i and H_j are edge-disjoint;
   - every edge of G is contained in one of the factors $H_1, H_2, \ldots, H_s$, that is, $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s)$.

Some immediate observations:

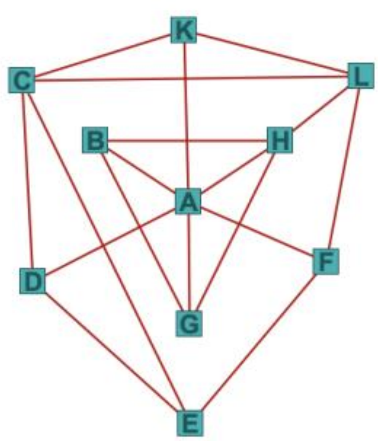
- A spanning tree (or spanning forest, in cases where G is not connected) is a factor of G.
- Every graph (or multigraph) G has a trivial factorization: since G is a factor of itself, the collection $\{G\}$ is a factorization of G.

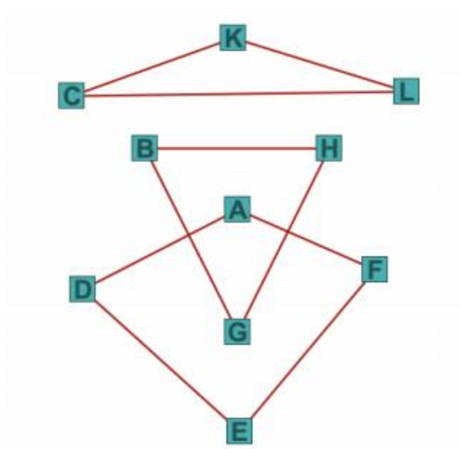
Definition

Let G be a graph (or multigraph).

- A spanning subgraph of G is called a *one-factor* of G **if it is 1-regular.** A *one-factorization* of G is a factorization of G consisting of one-factors of G.
- Similarly, a spanning subgraph of G is called a *two-factor* of G **if it is 2-regular.** A *two-factorization* of G is a factorization of G consisting of two-factors of G.

A subtle point. Note that a Hamilton cycle of G is a two-factor of G, but not every two-factor needs to be a Hamilton cycle. E.g. the graph on the right below is a two-factor of the graph on the left (but not a Hamilton cycle):





Existence of one-factors

Two (very simple) necessary conditions for the existence of one-factors

If a graph (or multigraph) G has a one-factor, then

(i) G has an even number of vertices.

(ii) G cannot have isolated vertices *(be careful that, if G is a multigraph, this condition is not immediately equivalent to $\delta(G) \geqslant 1$, given that some vertices of G may have no neighbours, but may have loops attached to them, so their degree will still be positive)*.

Necessary conditions for the existence of a one-factorization

Let G be graph of order $n = 2k$ and size m. Suppose that G has a one-factorization. Then

(I) $k = \frac{n}{2}$ must divide m.

(II) Even more restrictively, the following property must be true for G: **the graph G must be d-regular for some d which divides m.** *(Justification? Note that, if G can be decomposed into d pairwise edge-disjoint one-factors, then each vertex v of G must be incident with precisely d edges.)*

(III) **G cannot have bridges** (except if G is a 1-regular graph itself, and hence the trivial factorization $\{G\}$ of G is a one-factorization too).

Necessary Condition

Theorem (Tutte, 1947)

Let $G = (V, E)$ be a graph (or multigraph). Given a proper subset S of V, write $\mathrm{OC}(G - S)$ for the number of **odd** connected components of $G - S$ (that is, the number of those connected components of $G - S$ which have odd order).

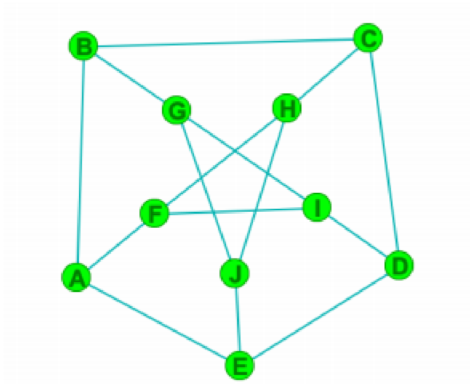
G has a one-factor **if and only if**

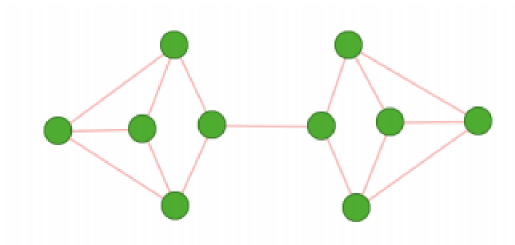
for every proper subset S of V, we have that $\mathrm{OC}(G - S) \leqslant |S|$.

Theorem (Petersen, 1891)

If G is a 3-regular graph with **no bridges**, then G has a one-factor.

Check the theorem on the following examples.





note that this graph has a one-factor, even though it has a bridge

Theorem (Petersen, 1891)

Suppose H is a d-regular graph, where d is an even positive integer (say $d = 2k$). Then H has two-factors.

Moreover, H can be decomposed into k pairwise edge-disjoint two-factors, that is, H has a two-factorization.

Proposition in terms of Complete graphs

Proposition

(i) If $s \geqslant 3$ is an odd positive integer, then K_s can be decomposed into $\frac{s-1}{2}$ pairwise edge-disjoint Hamilton cycles.

(ii) If t is an even positive integer, then K_t has a factorization consisting of $\frac{t}{2} - 1$ Hamilton cycles and a one-factor.

Proof of (i). Let $s = 2k + 1$ with $k \geqslant 1$, and suppose that the vertices of K_s are indexed by the integers $0, 1, 2, \ldots, 2k$. For each $i = 1, 2, \ldots, k$, define the walks Z_i by

$$Z_i : \; v_0 \, v_i \, v_{i+1} \, v_{i-1} \, v_{i+2} \, v_{i-2} \, \ldots \, v_{i+j} \, v_{i-j} \, \ldots \, v_{i+k} \, v_0$$

and according to the convention that, whenever $i - j \leqslant 0$ (for any of the internal vertices of the walk), then the index $i - j$ is replaced by the unique integer in $\{1, 2, \ldots, 2k-1, 2k\}$ which is congruent to $i - j$ modulo $2k$.

We can check that these walks are Hamilton cycles of K_s, and that the collection $\{Z_1, Z_2, \ldots, Z_k\}$ is a two-factorization of K_s.

In the specific case of K_{11}, we have...

$11 = 2 \cdot 5 + 1$, and hence $k = 5$. Moreover,

$$Z_1 : \; v_0 \, v_1 \, v_2 \, v_{10} \, v_3 \, v_9 \, v_4 \, v_8 \, v_5 \, v_7 \, v_6 \, v_0,$$

$$Z_2 : \; v_0 \, v_2 \, v_3 \, v_1 \, v_4 \, v_{10} \, v_5 \, v_9 \, v_6 \, v_8 \, v_7 \, v_0,$$

$$Z_3 : \; v_0 \, v_3 \, v_4 \, v_2 \, v_5 \, v_1 \, v_6 \, v_{10} \, v_7 \, v_9 \, v_8 \, v_0,$$

$$Z_4 : \; v_0 \, v_4 \, v_5 \, v_3 \, v_6 \, v_2 \, v_7 \, v_1 \, v_8 \, v_{10} \, v_9 \, v_0,$$

$$\text{and} \quad Z_5 : \; v_0 \, v_5 \, v_6 \, v_4 \, v_7 \, v_3 \, v_8 \, v_2 \, v_9 \, v_1 \, v_{10} \, v_0.$$

Note that all these are Hamilton cycles of K_{11}, and any two of them are edge-disjoint.

Proposition

(i) If $s \geqslant 3$ is an odd positive integer, then K_s can be decomposed into $\frac{s-1}{2}$ pairwise edge-disjoint Hamilton cycles.

(ii) If t is an even positive integer, then K_t has a factorization consisting of $\frac{t}{2} - 1$ Hamilton cycles and a one-factor.

Proof of (ii). Now write $t = 2r + 2$ with $r \geqslant 0$, and suppose that the vertices of K_t are indexed by the integers $0, 1, 2, \ldots, 2r$ and the symbol ∞. For each $j = 1, 2, \ldots, r$, define the walks Z_j by

$$Z_j : \; v_\infty \, v_j \, v_{j-1} \, v_{j+1} \, v_{j-2} \, v_{j+2} \, \ldots \, v_{j-l} \, v_{j+l} \, \ldots \, v_{j-r} \, v_{j+r} \, v_\infty .$$

and according to the convention that, whenever $j - l < 0$, then the index $j - l$ is replaced by the unique integer in $\{0, 1, 2, \ldots, 2r-1, 2r\}$ which is congruent to $j - l$ modulo $2r + 1$.

We can check that these walks are pairwise edge-disjoint Hamilton cycles of K_t. Moreover, consider the one-factor H_0 of K_t given by

$$H_0 = \big(V(K_t), \, \big\{\{v_\infty, v_0\}, \, \{v_1, v_{2r}\}, \, \{v_2, v_{2r-1}\}, \, \ldots, \, \{v_r, v_{r+1}\}\big\}\big).$$

Then the collection $\{Z_1, Z_2, \ldots, Z_r, H_0\}$ is a factorization of K_t.

Say, in the specific case of K_8, we have...

$8 = 2 \cdot 3 + 2$, and hence $r = 3$. Moreover,

$$Z_1 : \; v_\infty \, v_1 \, v_0 \, v_2 \, v_6 \, v_3 \, v_5 \, v_4 \, v_\infty ,$$

$$Z_2 : \; v_\infty \, v_2 \, v_1 \, v_3 \, v_0 \, v_4 \, v_6 \, v_5 \, v_\infty ,$$

$$Z_3 : \; v_\infty \, v_3 \, v_2 \, v_4 \, v_1 \, v_5 \, v_0 \, v_6 \, v_\infty ,$$

$$\text{while} \quad E(H_0) = \big\{\{v_\infty, v_0\}, \, \{v_1, v_6\}, \, \{v_2, v_5\}, \, \{v_3, v_4\}\big\}.$$

Note that all these are factors of K_8, any two of them are edge-disjoint, and combined together they give us a factorization of K_8.

Orientation/Tournament

Terminology

Let G be a graph. We can obtain a directed graph (or digraph) G' from G by assigning a direction to each edge of G *[in particular, this implies that, if $\{u,v\} \in E(G)$, only one of the ordered pairs (u,v) and (v,u) will be contained in $E(G')$].* We call the process of obtaining such a digraph G' from G, as well as G' itself, an *orientation* of G.

Moreover, an *oriented graph* H' is a digraph that can be obtained from orienting a graph H *[in other words, a digraph H' can be viewed as an oriented graph if and only if, for every two different vertices x, y of H', at most one of the pairs (x,y) and (y,x) is contained in $E(H')$].*

Terminology

Let $n \geqslant 2$. Any orientation of the complete graph K_n is called a *tournament*.

Remark. If we interpret the directed edge $x \to y$ (which formally would be written as the ordered pair (x,y)) as capturing that player/team x beat player/team y, then a one-factor of a tournament is not only a stage of a round robin, but it also encodes the results of that stage (that is, we can tell which player/team was the winner of any game that was played by looking at the direction of the corresponding edge in the one-factor).

Theorem

Every tournament on $n \geqslant 2$ vertices has a Hamilton path.

Strongly Connected/Orientation

Terminology

Let H be a digraph. Two vertices u, v of H are called *strongly connected* in H if we can find

- both a (directed) path starting at u and ending at v,
- and a (directed) path starting at v and ending at u.

H is called *strongly connected* if any two different vertices in H are strongly connected.

Let G be a graph. An orientation of G is called a *strong orientation* if the resulting oriented graph G' is strongly connected.

Strong Orientation/Connectedness

Theorem 1′

Let G be a connected graph. Then G has a strong orientation **if and only if** every edge of G belongs to at least one cycle.

Note that Theorem 1′ implies quite quickly that a complete graph K_n with at least 3 vertices does have strong orientations (that is, there exist strongly connected tournaments on n vertices for all $n \geqslant 3$). In fact, we can even state a (more or less efficient) criterion for when a tournament will be strongly connected:

Theorem 1″

Let $n \geqslant 3$. A tournament H' on n vertices is strongly connected **if and only if** H' has a Hamilton cycle.

Matching

Definition

Let $G = (V, E)$ be a graph. A subset E' of E is called a *matching* in G if, for any two different edges $e_1, e_2 \in E'$, we have that e_1, e_2 are not adjacent.

In other words, E' is a matching in G if it is the edge set of a 1-regular subgraph of G (where we consider the vertex set of the subgraph to be all the endvertices of the edges in E').

A matching E' in G is called a *perfect matching* if every vertex of G is *covered* by E', that is, if every vertex of G is the endvertex of some edge in E'. In other words, E' is a perfect matching if it is the edge set of a one-factor of G.

Definition

The maximum cardinality of a matching in G is denoted by $\nu(G)$.

Useful observation

Let G be a graph. Then $\nu(G)$ coincides with the independence number $\alpha(L(G))$ of the line graph $L(G)$ of G (why?).

Matching in Bipartite Graphs

Theorem (Hall, 1935)

Let $G = (V, E)$ be a subgraph of a bipartite graph (alternatively, a not necessarily complete bipartite graph) with partite sets A and B (that is, $V(G) = A \cup B$, and there are no edges in $E(G)$ joining two vertices in A, or two vertices in B).

Then there is a matching in G covering the set A **if and only if**

$$\text{for every } S \subseteq A, \text{ we have that } |S| \leqslant |N(S)|,$$

where $N(S)$ is the union of all neighbourhoods of vertices in S.

Proposition (Corollary to Hall's theorem and a defect version of it)

Let $G = (V, E)$ be a subgraph of a bipartite graph (alternatively, a not necessarily complete bipartite graph) with partite sets A and B.

Assume that, for some integer $d \geqslant 1$, G satisfies the following:

$$\text{for every } S \subseteq A, \text{ we have that } |N(S)| \geqslant |S| - d.$$

Then there is a matching in G of cardinality at least $|A| - d$ (in other words, a matching that covers at least $|A| - d$ of the vertices in A).

Important Remark. The only (complete) bipartite graphs for which there exists a perfect matching are the graphs $K_{n,n}$ (where $n \geqslant 1$ is some integer).

In fact, in the graph $K_{n,n}$ we will have $n!$ different perfect matchings (or one-factors of $K_{n,n}$) (can you explain why this is so?).

A slight detour. Can we also find a one-factorisation of $K_{n,n}$? And if yes, how many (and which) of the abovementioned perfect matchings should we single out to get a one-factorisation?

Answer. Let us write $\{1, 2, \ldots, n\} \cup \{1', 2', \ldots, n'\}$ for the vertex set of $K_{n,n}$ (with the two subsets here being the two partite sets of $K_{n,n}$).

Then a one-factorisation of $K_{n,n}$ can be formed by considering the following n matchings (which are pairwise disjoint edge subsets and form a partition of $E(K_{n,n})$):

$$\begin{gathered}
\{\{1,1'\}, \{2,2'\}, \ldots, \{(n-1),(n-1)'\}, \{n,n'\}\}, \\
\{\{1,2'\}, \{2,3'\}, \ldots, \{(n-1),n'\}, \{n,1'\}\}, \\
\{\{1,3'\}, \{2,4'\}, \ldots, \{(n-1),1'\}, \{n,2'\}\}, \\
\ddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots \\
\{\{1,n'\}, \{2,1'\}, \ldots, \{(n-1),(n-2)'\}, \{n,(n-1)'\}\}.
\end{gathered}$$

Stable Marriage Problem

Suppose that we denote the bipartition of $K_{n,n}$ by (M, W), that is, we write M for the first partite set of $K_{n,n}$ and W for the second one.

Suppose also that M represents men and W represents women, and we want to have each man from M to get 'engaged to be married' to a woman from W. **Here we have the additional assumption that each man from M has ranked all women from W in (descending) order of preference, and similarly each woman from W has ranked all men from M in (descending) order of preference.**

Clearly what we want is a perfect matching of $K_{n,n}$. Moreover...

we will consider such a matching ***not stable*** if it contains (at least) two pairs (m_{i_1}, w_{j_1}) and (m_{i_2}, w_{j_2}) such that

1. m_{i_1} prefers w_{j_2} over w_{j_1},
2. and w_{j_2} also prefers m_{i_1} over m_{i_2}.

That is, if both m_{i_1} and w_{j_2} would prefer each other over the partners they are currently engaged to.

If we cannot find any two such pairs in our matching, then the matching is called ***stable***.

Gale-Shapley Algorithm

The algorithm will (usually) have several stages / rounds. First, we decide which partite set will be considered the 'first' set, or 'proposing' set; the other partite set will be called the 'accepting' set.

- In the first round
  - we initially have each 'man' (that is, each member of the 'proposing' partite set) propose to the 'woman' who is ranked first in his list.
  - Then, each 'woman' who has received at least one proposal replies "maybe" to the 'man' she prefers the most **out of those who proposed to her**, and "no" to all the other 'men' who proposed to her.

By the end of this round, a 'man' and a 'woman' are **provisionally** engaged if the 'man' proposed to the 'woman' and the 'woman' did reply "maybe". Also, we might still have unengaged 'men' and 'women' (and if we do, then we need to go on with the process).

- In the subsequent round,
  - we initially have each **unengaged** 'man' propose to his most preferred 'woman' out of those he has not yet proposed to.
  - Then, each unengaged 'woman' who has received a proposal in this round replies "maybe" to the 'man' she prefers the most out of those who proposed to her, and "no" to all the other 'men' who proposed to her,
  - but also **each already engaged 'woman'** replies "maybe" to a 'man' who proposed to her in this round **if she prefers him over the 'man' she is currently engaged to** (in such a case her previous partner is 'rejected' and becomes unengaged again).

Again, by the end of the round we have some 'men' and 'women' **provisionally** engaged, and possibly some unengaged 'men' and 'women'.

- We repeat this process until everyone is engaged, **at which stage the engagements become final, and we are guaranteed to have found a stable matching.**

Colouring

Definition

Let $G = (V, E)$ be a graph. A *colouring*, or even more specifically a *vertex colouring*, of G is any function

$$\xi : V(G) \to \mathbb{N}_+.$$

The vertex subsets $V_i = \{v \in V(G) : \xi(v) = i\}$ will be called the *colour classes* of the colouring ξ. Observe that the **non-empty** colour classes form a partition of $V(G)$.

— A colouring of G is called a *proper colouring* if

no two adjacent vertices belong to the same colour class.

In other words, if each of the colour classes is an independent set of vertices.

Definition 1

Let $G = (V, E)$ be a graph. A proper colouring ξ of G is called an *n-colouring* if there are exactly n non-empty colour classes of ξ. In other words, if the range of the function ξ contains exactly n positive integers.

G will be called n-colourable if we can find an n-colouring of G *(pictorially we can think of this as follows: G is n-colourable if n colours are enough for us to find a way to colour the vertices of G so that no two adjacent vertices will end up having the same colour)*.

Definition 2

The *chromatic number* of a graph G is equal to the **smallest** integer n for which we can find an n-colouring of G.

If n_0 is this smallest integer, then we say that G is *n_0-chromatic*. We denote this smallest integer by $\chi(G)$ *(in other words, G is $\chi(G)$-chromatic)*.

Finally, a $\chi(G)$-colouring of G, that is, a proper colouring of G in $\chi(G)$ colours, is called *minimal*.

- The null graphs are the only graphs which have chromatic number equal to 1 (why?).
- For any graph G of order n, we have that $\chi(G) \leqslant n$.

  Moreover, the complete graph on n vertices is the **only** graph **of order n** which has chromatic number equal to n (why?).
- Every path has chromatic number equal to 2.

  (Practice Question: Could we make a similarly general statement about other trees?)
- Every **even** cycle has chromatic number equal to 2 (why?).
- Every **odd** cycle has chromatic number equal to 3 (why?).
- If G is a graph, and H is a subgraph of G, that is, $H \subseteq G$, then we have that

$$\chi(H) \leqslant \chi(G).$$

Important Results/Bounds of Chromatic Numbers

Theorem 1

For every graph G, we have that $\chi(G) \leqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

Remark. Note that the upper bound given by the theorem is best possible: **if G_0 is a complete graph, or G_0 is an odd cycle**, then

$$\chi(G_0) = \Delta(G_0) + 1.$$

However, it turns out these are **essentially** the only examples where the inequality is not strict. This is because of

Theorem 2 (Brooks' theorem, 1941)

Let G be a connected graph which is neither a complete graph nor an odd cycle. Then $\chi(G) \leqslant \Delta(G)$.

Clique

Definition

Let $G = (V, E)$ be a graph. A *clique* of G is a subgraph H of G which is a complete graph on the vertices it contains.

In other words, a subgraph H of G with vertex set $V(H) \subseteq V = V(G)$ is a clique of G if

- any two vertices in $V(H)$ are joined in G,
- and H is the induced subgraph on those vertices.

- A clique H of G will be called *maximal* if we cannot view H as the subgraph of a larger clique in G. That is, if we cannot find any vertex u of G outside the vertex set $V(H)$ of H such that the induced subgraph on $V(H) \cup \{u\}$ would again be a clique.
- A *maximum* clique of G is a clique of G with maximum possible order. The order of a maximum clique in G is called the *clique number* of G, and is denoted by $\omega(G)$.

Remark. Given a graph G, we can think of the cliques in G and their vertex sets as being the "complementary" concept to the different independent sets of vertices in G.

Theorem 3

For every graph G, we have that $\chi(G) \geqslant \omega(G)$.

Theorem 4

For every graph G of order n, we have that

$$\frac{n}{\alpha(G)} \leqslant \chi(G) \leqslant n + 1 - \alpha(G)$$

where $\alpha(G)$ is the independence number of G, that is, the maximum cardinality of an independent set of vertices in G.

Greedy Coloring Algorithm

Suppose that G has order n. We begin by ordering / labelling the vertices of G as $v_1, v_2, \ldots, v_n$.

— Then we colour vertex v_1 using colour 1.

— Next we examine whether v_2 is a neighbour of v_1 or not: if it is a neighbour, then we colour it using colour 2; if it is not a neighbour, then we colour v_2 using colour 1 again.

— We continue like this. If we assume that we have already coloured vertices $v_1, v_2, \ldots, v_i$ such that no two neighbours have the same colour, we then examine how many of the neighbours of v_{i+1} have already been coloured: say, j neighbours of v_{i+1} have been coloured and we have used s colours to do so, say colours $c_1, c_2, \ldots, c_s$; then we colour v_{i+1} using the **first available colour** from the set

$$\{1, 2, \ldots, n\} \setminus \{c_1, c_2, \ldots, c_s\}.$$

In fact, since we always return to the smallest integer / colour we can use, **and given that, no matter what the vertex v_{i+1} is, its already coloured neighbours cannot be more than all the neighbours of v_{i+1}, which shows that the s colours used for the already coloured neighbours cannot be more than $\deg(v_{i+1}) \leqslant \Delta(G)$,** we can conclude that we will always be able to choose a colour from the set

$$\{1, 2, \ldots, \Delta(G), \Delta(G) + 1\} \setminus \{c_1, c_2, \ldots, c_s\}.$$

(and to systematise the choice here, we might as well choose the **first available colour** from this set).

Bipartite Graphs Theorem/Coloring

Theorem (Criterion for bipartite graphs)

A graph G is (a subgraph of) a bipartite graph if and only if G does not contain any odd cycles.

Proposition 1

Let G be a graph that does not contain any odd cycles.

Then $\chi(G) \leqslant 2$. More specifically, $\chi(G) = 1$ if G is a null graph (that is, if it contains no edges), otherwise $\chi(G) = 2$.

Planar Graphs

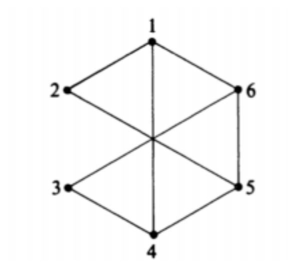
Recall that a pictorial representation of a graph $G = (V, E)$ in the plane is essentially a function $\mathcal{F}$ that maps each vertex v of G to a point $\mathcal{F}(v)$ in the plane, with different vertices being mapped to different points, and also maps each edge $\{v_1, v_2\} \in E(G)$ to a simple curve in the plane with endpoints $\mathcal{F}(v_1)$ and $\mathcal{F}(v_2)$ *(with the curve being otherwise disjoint from points that other vertices are mapped to)*.

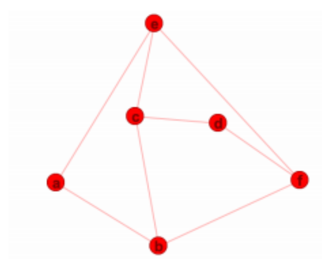
If the curves $\mathcal{F}(e_1)$ and $\mathcal{F}(e_2)$ that two different edges of G are mapped to contain common points different from a common endpoint, then we say that $\mathcal{F}(e_1)$ and $\mathcal{F}(e_2)$ have a *crossing*.

Definition

A graph $G = (V, E)$ is called *planar* if we can draw it in the plane without any crossings.
Any such pictorial representation of G will be called a *planar embedding* of G.

Example. We have seen that the following two graphs are isomorphic, and thus representations of the same unlabelled graph. Therefore this graph is planar.





Subdivision

Definition

Let $G = (V, E)$ be a graph. A *subdivision* of G is a graph H which satisfies the following:

- H contains all the vertices of G, and perhaps a few more vertices;
- every edge $e = \{v_1, v_2\}$ in G
  - either appears as an edge of H as well,
  - or is replaced in H by the path $v_1\, w_1\, w_2\, \ldots\, w_{s-1}\, w_s\, v_2$, where $w_1, w_2, \ldots, w_s$ are new vertices that we add to the vertex set of H.

    In other words, in the second case, $e = \{v_1, v_2\}$ is replaced by the edges $\{v_1, w_1\}$, $\{w_1, w_2\}$, $\ldots$, $\{w_{s-1}, w_s\}$ and $\{w_s, v_2\}$.

Kuratowski's Algorithm

Theorem (Kuratowski, 1930)

A finite graph G is planar **if and only if** none of its subgraphs is a subdivision of $K_{3,3}$ or of K_5.

Terminology

If G contains a subgraph H which is a subdivision of $K_{3,3}$ or of K_5, then H is called a *Kuratowski subgraph* of G.

With this terminology at hand, Kuratowski's theorem can be more simply stated as:

a finite graph G is planar if and only if G does not have any Kuratowski subgraphs.