Math 227
Suggested solutions to Homework Set 5

Problem 1. (i) Let $A = \big(a_{ij}\big)_{1\leqslant i,j\leqslant n}$ and $B = \big(b_{ij}\big)_{1\leqslant i,j\leqslant n}$ in $\mathbb{F}^{n\times n}$. Then $A+B = \big(a_{ij}+b_{ij}\big)_{1\leqslant i,j\leqslant n}$, and hence

$$\mathrm{tr}(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n} a_{ii} \ + \ \sum_{i=1}^{n} b_{ii} = \mathrm{tr}(A) \ + \ \mathrm{tr}(B).$$

Similarly, if $\lambda \in \mathbb{F}$, then $\lambda A = \big(\lambda a_{ij}\big)_{1\leqslant i,j\leqslant n}$, and hence

$$\mathrm{tr}(\lambda A) = \sum_{i=1}^{n} \lambda a_{ii} = \lambda \cdot \sum_{i=1}^{n} a_{ii} = \lambda \cdot \mathrm{tr}(A).$$

Since $A, B \in \mathbb{F}^{n\times n}$ and $\lambda \in \mathbb{F}$ were arbitrary, we can conclude that $\mathrm{tr} : \mathbb{F}^{n\times n} \to \mathbb{F}$ is a linear function.

(ii) Consider the following two matrices in $\mathbb{F}^{2\times 2}$: $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (note that these will be contained in $\mathbb{F}^{2\times 2}$ no matter what the field $\mathbb{F}$ is). Then

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and hence $\mathrm{tr}(AB) = 0 \neq 1 = \mathrm{tr}(A)\cdot\mathrm{tr}(B)$.

More generally, we can set

$$A_n = \begin{pmatrix} \color{red}{1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad B_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \color{red}{1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with $A_n, B_n \in \mathbb{F}^{n\times n}$, and then we will have $A_nB_n = \boldsymbol{O}$, the zero matrix in $\mathbb{F}^{n\times n}$, and consequently $\mathrm{tr}(A_nB_n) = 0 \neq 1 = \mathrm{tr}(A_n)\cdot\mathrm{tr}(B_n)$.

(iii) Given $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{F}^n$, let us write now $\bar{x} \bullet \bar{y} := \sum_{s=1}^{n} x_s y_s$. We recall that

$$AB^T = \Big(R_i(A) \bullet C_j(B^T)\Big)_{1\leqslant i,j\leqslant n} = \Big(R_i(A) \bullet R_j(B)\Big)_{1\leqslant i,j\leqslant n}.$$

Therefore, the i-th diagonal entry of AB^T is $R_i(A) \bullet R_i(B) = \sum_{s=1}^n a_{is}b_{is}$, and we can conclude that

$$\operatorname{tr}(AB^T) = \sum_{i=1}^n R_i(A) \bullet R_i(B) = \sum_{i=1}^n \left(\sum_{s=1}^n a_{is}b_{is} \right) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ij}.$$

(iv) Let us write $A^T = \left(\widetilde{a}_{ij}\right)_{1\leqslant i,j\leqslant n}$ and $B^T = \left(\widetilde{b}_{ij}\right)_{1\leqslant i,j\leqslant n}$, where $\widetilde{a}_{ij} = a_{ji}$ and $\widetilde{b}_{ij} = b_{ji}$ for any $1 \leqslant i, j \leqslant n$. Then, from part (iii) we have that

$$\begin{aligned}
\operatorname{tr}(AB) = \operatorname{tr}\left(A\left(B^T\right)^T\right) &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}\widetilde{b}_{ij} \\
&= \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ji} = \sum_{i'=1}^n \sum_{j'=1}^n a_{j'i'}b_{i'j'} \\
&= \sum_{i'=1}^n \sum_{j'=1}^n b_{i'j'}\widetilde{a}_{i'j'} = \operatorname{tr}\left(B\left(A^T\right)^T\right) = \operatorname{tr}(BA).
\end{aligned}$$

Problem 2. (i) Clearly ℓ_∞ is nonempty since it contains all the constant infinite sequences of real numbers (for example).

Also, if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two bounded infinite sequences of real numbers, then there exist positive numbers M_1 and M_2 such that

$$\text{for every } n \in \mathbb{N}, \quad |a_n| \leqslant M_1 \text{ and } |b_n| \leqslant M_2.$$

But then, for every $n \in \mathbb{N}$, $|a_n + b_n| \leqslant |a_n| + |b_n| \leqslant M_1 + M_2$, and thus the sequence $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}}$ is a bounded sequence. This shows that ℓ_∞ is closed under addition.

Similarly, if $r \in \mathbb{R}$, then for every $n \in \mathbb{N}$, $|ra_n| = |r| \cdot |a_n| \leqslant |r| \cdot M_1 < (|r| + 1)M_1$ (where $(a_n)_{n\in\mathbb{N}}$ is as before), which shows that $(|r| + 1)M_1$ (say) is a positive upper bound for the absolute values of the terms of the sequence $r \cdot (a_n)_{n\in\mathbb{N}} = (ra_n)_{n\in\mathbb{N}}$, or in other words that $r \cdot (a_n)_{n\in\mathbb{N}} = (ra_n)_{n\in\mathbb{N}} \in \ell_\infty$. We conclude that ℓ_∞ is closed under scalar multiplication too.

Combining the above, we obtain that ℓ_∞ is a subspace of $\mathbb{R}^{\mathbb{N}}$.

(ii) Again, we can check that c is nonempty since it contains all the constant infinite sequences.

Now, if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two convergent infinite sequences of real numbers, then we know from Calculus/Analysis that the sequence $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}}$ also converges (in fact, we know that $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$). Therefore, c is closed under addition.

Moreover, if $r \in \mathbb{R}$ and $(a_n)_{n\in\mathbb{N}} \in c$, then again from Calculus/Analysis we know that the sequence $r \cdot (a_n)_{n\in\mathbb{N}} = (ra_n)_{n\in\mathbb{N}}$ is convergent (in fact, we know that $\lim\limits_{n\to\infty} ra_n = r \cdot \lim\limits_{n\to\infty} a_n$). Therefore, c is closed under scalar multiplication too.

Combining the above, we obtain that c is a subspace of $\mathbb{R}^{\mathbb{N}}$.

(iii) We have that c_0 is nonempty since it contains the constant zero sequence, that is, the sequence all of whose terms are equal to 0.

Also, if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences of real numbers that converge to 0, then from what we recalled in part (ii) we have that $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = 0 + 0 = 0$. Therefore, the sequence $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}}$ also converges to 0, and thus c_0 is closed under addition.

Similarly, if $r \in \mathbb{R}$ and $(a_n)_{n\in\mathbb{N}} \in c_0$, then we have that $\lim\limits_{n\to\infty} ra_n = r \cdot \lim\limits_{n\to\infty} a_n = r \cdot 0 = 0$, and thus the sequence $r \cdot (a_n)_{n\in\mathbb{N}} = (ra_n)_{n\in\mathbb{N}}$ also converges to 0. Thus, c_0 is closed under scalar multiplication.

Combining the above, we obtain that c_0 is a subspace of $\mathbb{R}^{\mathbb{N}}$.

(iv) We have that c_{00} is nonempty since it contains the constant zero sequence.

Also, if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences in c_{00}, then there exist indices $n_1, n_2 \in \mathbb{N}$ such that

$$a_n = 0 \text{ for every } n > n_1, \quad b_n = 0 \text{ for every } n > n_2.$$

But then, we will have that $a_n + b_n = 0$ for every $n > \max\{n_1, n_2\}$ (since for those indices we will have both that $a_n = 0$ and that $b_n = 0$). Thus the sequence $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}}$ will only have at most $\max\{n_1, n_2\}$ non-zero terms. This shows that c_{00} is closed under addition.

Similarly, if $r \in \mathbb{R}$, then we have that $ra_n = 0$ for every $n > n_1$ (where $(a_n)_{n\in\mathbb{N}}$ is as before), and thus the sequence $r \cdot (a_n)_{n\in\mathbb{N}} = (ra_n)_{n\in\mathbb{N}}$ has at most n_1 non-zero terms. This shows that c_{00} is closed under scalar multiplication too.

Combining the above, we obtain that c_{00} is a subspace of $\mathbb{R}^{\mathbb{N}}$.

(v) We have that ℓ_1 is nonempty since it contains the constant zero sequence.

Also, if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences in ℓ_1, then we know that $\lim\limits_{m\to\infty} \sum_{n=1}^m |a_n|$ and $\lim\limits_{m\to\infty} \sum_{n=1}^m |b_n|$ exist in $\mathbb{R}$ *(note also that these limits coincide with* $\sup\left\{ \sum_{n=1}^m |a_n| : m \in \mathbb{N}\right\}$ *and* $\sup\left\{ \sum_{n=1}^m |b_n| : m \in \mathbb{N}\right\}$ *respectively, since the sequences of partial sums* $\left(\sum_{n=1}^m |a_n|\right)_{m\in\mathbb{N}}$ *and* $\left(\sum_{n=1}^m |b_n|\right)_{m\in\mathbb{N}}$ *are increasing).*

But then, for every $m_0 \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{m_0} |a_n + b_n| &\leqslant \sum_{n=1}^{m_0} (|a_n| + |b_n|) \\ &= \sum_{n=1}^{m_0} |a_n| \; + \; \sum_{n=1}^{m_0} |b_n| \leqslant \sup\left\{\sum_{n=1}^{m} |a_n| : m \in \mathbb{N}\right\} + \sup\left\{\sum_{n=1}^{m} |b_n| : m \in \mathbb{N}\right\},\end{aligned}$$

and thus the sequence of partial sums $\left(\sum_{n=1}^m |a_n + b_n|\right)_{m\in\mathbb{N}}$ is bounded. Since it is also increasing, we know from Calculus/Analysis that $\lim\limits_{m\to\infty} \sum_{n=1}^m |a_n + b_n|$ exists in $\mathbb{R}$, and therefore the sequence $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}}$ will belong to ℓ_1 as well. In other words, ℓ_1 is closed under addition.

Similarly, if $r \in \mathbb{R}$ and $(a_n)_{n\in\mathbb{N}} \in \ell_1$, then we can check that

$$\lim_{m\to\infty} \sum_{n=1}^{m} |ra_n| = \lim_{m\to\infty} \sum_{n=1}^{m} |r| \cdot |a_n| = \lim_{m\to\infty} \left(|r| \cdot \sum_{n=1}^{m} |a_n| \right) = |r| \cdot \lim_{m\to\infty} \sum_{n=1}^{m} |a_n|,$$

and thus that the sequence $r \cdot (a_n)_{n\in\mathbb{N}} = (ra_n)_{n\in\mathbb{N}}$ is also in ℓ_1. We conclude that ℓ_1 is closed under scalar multiplication too.

Combining the above, we obtain that ℓ_1 is a subspace of $\mathbb{R}^{\mathbb{N}}$.

We finally verify that these subspaces form a sequence of subspaces of $\mathbb{R}^{\mathbb{N}}$ (with respect to set inclusion).

If $(a_n)_{n\in\mathbb{N}} \in c_{00}$, then the sequence of partial sums $\left(\sum_{n=1}^{m} |a_n|\right)_{m\in\mathbb{N}}$ is eventually constant (given that, if $a_n = 0$ for every $n > n_1$, then

$$\sum_{n=1}^{m} |a_n| = \sum_{n=1}^{n_1} |a_n| \; + \sum_{n=n_1+1}^{m} 0 = \sum_{n=1}^{n_1} |a_n|$$

for every $m > n_1$). Thus, this sequence of partial sums will be a convergent sequence, which means that $(a_n)_{n\in\mathbb{N}}$ will be in ℓ_1 too.

Since $(a_n)_{n\in\mathbb{N}} \in c_{00}$ was arbitrary, we conclude that $c_{00} \subset \ell_1$. Given also that they are both subspaces of the same vector space (which implies that the vector space operations are defined in the same way, for any inputs we are allowed to consider in each case), we can also conclude that c_{00} is a subspace of ℓ_1, and thus we can write $c_{00} \leqslant \ell_1$.

Next, consider a sequence $(b_n)_{n\in\mathbb{N}}$ in ℓ_1. Then, by definition of ℓ_1, we have that the series $\sum_{n=1}^{\infty} |b_n|$ converges. From Calculus/Analysis, we know then that the sequence of terms of the series $(|b_n|)_{n\in\mathbb{N}}$ must converge to 0. In other words, we get that

$$|b_n| \xrightarrow[n\to\infty]{} 0 \qquad \Leftrightarrow \qquad b_n \xrightarrow[n\to\infty]{} 0,$$

and thus the sequence $(b_n)_{n\in\mathbb{N}}$ belongs to c_0.

Since $(b_n)_{n\in\mathbb{N}} \in \ell_1$ was arbitrary, we conclude that $\ell_1 \subset c_0$ and also that $\ell_1 \leqslant c_0$.

Next, we note that any sequence of real numbers that converges to 0 is a convergent sequence of real numbers, thus we immediately get that $c_0 \leqslant c$.

Finally, we know from Calculus/Analysis that every convergent sequence of real numbers is a bounded sequence, therefore we have that $c \leqslant \ell_\infty$.

Problem 3. First of all, we note that the zero vector of the vector space $C[0,1]$ is the constant zero function $\mathbf{0}$, that is, the function that takes the value 0 everywhere on the interval $[0,1]$.

We recall that, in order to show that T is linearly independent, we need to check that, for every $k \geqslant 1$, for every subset $\{f_{\lambda_1}, f_{\lambda_2}, \dots, f_{\lambda_k}\}$ of T of size k, and for every k-tuple of real numbers $(r_1, r_2, \dots, r_k)$,

$$\text{if we have } r_1 f_{\lambda_1} + r_2 f_{\lambda_2} + \cdots + r_k f_{\lambda_k} = \mathbf{0}, \quad \text{then } r_1 = r_2 = \cdots = r_k = 0.$$

We will show this using induction in k.

<u>**Base case: $k = 1$**</u> In this case, we only consider one function from T. But such a function will have the form $f_\lambda(x) = e^{\lambda x}$ for some $\lambda \in \mathbb{R}$, so it will certainly be a non-zero function. Thus, whenever we have $r f_\lambda = \mathbf{0}$ for some $r \in \mathbb{R}$, it will immediately follow that $r = 0$.

<u>**Induction Step**</u> Assume that, for some $k \geqslant 1$, we have already shown that,

if $\{f_{\mu_1}, f_{\mu_2}, \dots, f_{\mu_k}\}$ is a subset of T of size k, then this subset is linearly independent.

Consider now a subset $\{f_{\lambda_1}, f_{\lambda_2}, \dots, f_{\lambda_k}, f_{\lambda_{k+1}}\}$ of T of size $k+1$ (where necessarily we have $\lambda_i \neq \lambda_j$ if $1 \leqslant i, j \leqslant k+1$, $i \neq j$). We will show that this subset is also linearly independent.

To this end, consider also real numbers $r_1, r_2, \dots, r_k, r_{k+1}$ such that

$$r_1 f_{\lambda_1} + r_2 f_{\lambda_2} + \cdots + r_k f_{\lambda_k} + r_{k+1} f_{\lambda_{k+1}} = \mathbf{0}. \tag{1}$$

This is an equality of functions, so it implies an analogous equality at every point in the common domain of these functions: for every $x \in [0,1]$,

$$r_1 f_{\lambda_1}(x) + r_2 f_{\lambda_2}(x) + \cdots + r_k f_{\lambda_k}(x) + r_{k+1} f_{\lambda_{k+1}}(x) = 0,$$
$$\text{or in other words} \quad r_1 e^{\lambda_1 x} + r_2 e^{\lambda_2 x} + \cdots + r_k e^{\lambda_k x} + r_{k+1} e^{\lambda_{k+1} x} = 0.$$

We now multiply both sides of the latter equality by $e^{-\lambda_{k+1} x} = \frac{1}{e^{\lambda_{k+1} x}}$ (note that the images of the function $x \mapsto e^{\lambda_{k+1} x}$ are always non-zero, so we can take their reciprocals): we obtain that, for every $x \in [0,1]$

$$r_1 e^{(\lambda_1 - \lambda_{k+1})x} + r_2 e^{(\lambda_2 - \lambda_{k+1})x} + \cdots + r_k e^{(\lambda_k - \lambda_{k+1})x} + r_{k+1} = 0.$$

We now observe that all functions appearing in the sum on the LHS are differentiable on the open interval $(0,1)$, and so is of course the constant function $\mathbf{0}$. Therefore, by taking the derivatives of both sides, and noting that they should be equal (since the functions whose derivatives we are taking are equal), we see that, for every $x \in (0,1)$,

$$r_1(\lambda_1 - \lambda_{k+1})e^{(\lambda_1-\lambda_{k+1})x} + r_2(\lambda_2 - \lambda_{k+1})e^{(\lambda_2-\lambda_{k+1})x} + \cdots \\ \cdots + r_k(\lambda_k - \lambda_{k+1})e^{(\lambda_k-\lambda_{k+1})x} = 0 \quad (2)$$

(note also that on the LHS now we have k summands, because the final summand just previously was the constant function r_{k+1}, so its derivative was zero).

Moreover, by continuity we get that (2) remains true also for $x = 0$ or $x = 1$.

Now, (2) holding true for every $x \in [0,1]$ gives equivalently that

$$r_1(\lambda_1-\lambda_{k+1})\cdot f_{\lambda_1-\lambda_{k+1}} + r_2(\lambda_2-\lambda_{k+1})\cdot f_{\lambda_2-\lambda_{k+1}} + \cdots + r_k(\lambda_k-\lambda_{k+1})\cdot f_{\lambda_k-\lambda_{k+1}} = \mathbf{0}.$$

We can finally apply the Inductive Hypothesis: it gives that the set

$$\{f_{\lambda_1-\lambda_{k+1}}, f_{\lambda_2-\lambda_{k+1}}, \ldots, f_{\lambda_k-\lambda_{k+1}}\},$$

which is a subset of T of size k, is linearly independent, and therefore we must have

$$r_1(\lambda_1 - \lambda_{k+1}) = r_2(\lambda_2 - \lambda_{k+1}) = \cdots = r_k(\lambda_k - \lambda_{k+1}) = 0.$$

But given that, for every $1 \leqslant i \leqslant k$, we have by our initial assumption that $\lambda_i \neq \lambda_{k+1}$, all the factors $\lambda_i - \lambda_{k+1}$ appearing here are non-zero, and thus we must have

$$r_1 = r_2 = \cdots = r_k = 0.$$

Finally, going back to (1), we see that it now becomes

$$r_{k+1} \cdot f_{\lambda_{k+1}} = 0 \cdot f_{\lambda_1} + 0 \cdot f_{\lambda_2} + \cdots + 0 \cdot f_{\lambda_k} + r_{k+1} \cdot f_{\lambda_{k+1}} = \mathbf{0},$$

which implies that $r_{k+1} = 0$ as well.

We conclude that the set $\{f_{\lambda_1}, f_{\lambda_2}, \ldots, f_{\lambda_k}, f_{\lambda_{k+1}}\}$ is linearly independent too, and since it was an arbitrary subset of T of size $k+1$, the proof of the inductive step is complete.

By mathematical induction, the conclusion that T is a linearly independent subset of $C[0,1]$ follows.

Problem 4. (i) Let $p(x) = \sum_{i=0}^{10} a_i x^i$ be a polynomial in $\mathcal{P}_{10}$ (where $a_0, a_1, \ldots, a_9, a_{10}$ are some real numbers). Then

$$\begin{aligned}
[p]_S &= \big\{q \in \mathcal{P}_{10} : q - p \in \operatorname{span}\big(\{x, x^3, x^5, x^7, x^9\}\big)\big\} \\
&= \left\{q \in \mathcal{P}_{10} : \exists\, b_1, b_3, b_5, b_7, b_9 \in \mathbb{R} \text{ such that } q(x) = \sum_{i=0}^{5} a_{2i}x^{2i} + \sum_{j=0}^{4} b_{2j+1}x^{2j+1}\right\} \\
&= \left\{q \in \mathcal{P}_{10} : \begin{array}{c}\text{the coefficient of any even-degree monomial in } q \\ \text{equals the corresponding coefficient of } p\end{array}\right\}.
\end{aligned}$$

We can conclude that

$$V/S = \left\{\left\{q \in \mathcal{P}_{10} : \exists\, \bar{b} = \begin{pmatrix}\widetilde{b}_1\\ \widetilde{b}_2\\ \widetilde{b}_3\\ \widetilde{b}_4\\ \widetilde{b}_5\end{pmatrix} \in \mathbb{R}^5 \text{ such that } q(x) = \sum_{j=0}^{4}\widetilde{b}_{j+1}x^{2j+1} + \sum_{i=0}^{5}\widetilde{a}_{i+1}x^{2i}\right\} : \bar{a} = \begin{pmatrix}\widetilde{a}_1\\ \widetilde{a}_2\\ \widetilde{a}_3\\ \widetilde{a}_4\\ \widetilde{a}_5\\ \widetilde{a}_6\end{pmatrix} \in \mathbb{R}^6\right\},$$

with the equivalence classes corresponding to different vectors $\bar{a} = \begin{pmatrix}\widetilde{a}_1\\ \widetilde{a}_2\\ \widetilde{a}_3\\ \widetilde{a}_4\\ \widetilde{a}_5\\ \widetilde{a}_6\end{pmatrix} \in \mathbb{R}^6$ being different.

Furthermore, based on the theorem which Problem 5 of this homework set asks us to give a proof for, we can find a basis for V/S as follows. We note that, if $T = \operatorname{span}\big(\{1, x^2, x^4, x^6, x^8, x^{10}\}\big)$, then $S \oplus T = \mathcal{P}_{10}$. Given also that the set $\{1, x^2, x^4, x^6, x^8, x^{10}\}$ is a basis of T, we obtain that the set

$$\{[1]_S,\ [x^2]_S,\ [x^4]_S,\ [x^6]_S,\ [x^8]_S,\ [x^{10}]_S\} =$$

$$\left\{\left\{q \in \mathcal{P}_{10} : \exists\, \bar{b} = \begin{pmatrix}\widetilde{b}_1\\ \widetilde{b}_2\\ \widetilde{b}_3\\ \widetilde{b}_4\\ \widetilde{b}_5\end{pmatrix} \in \mathbb{R}^5 \text{ such that } q(x) = x^{2i} + \sum_{j=0}^{4}\widetilde{b}_{j+1}x^{2j+1}\right\} : 0 \leqslant i \leqslant 5\right\}$$

is a basis of V/S.

(ii) First of all, we observe that

$$\begin{aligned} S &= \{p \in \mathcal{P}_{10} : (x-1)(x-2)(x-3) \text{ divides } p\} \\ &= \big\{p \in \mathcal{P}_{10} : \exists\, q \in \mathcal{P}_7 \text{ such that } p(x) = (x-1)(x-2)(x-3)\cdot q(x)\big\}. \end{aligned}$$

From this it follows that any non-zero polynomial in S has degree 3 or above (why?), therefore $S \cap \mathcal{P}_2 = S \cap \operatorname{span}\big(\{1, x, x^2\}\big) = \{\mathbf{0}\}$.

We can also see that a basis for S would be the set

$$\big\{(x-1)(x-2)(x-3)\cdot x^i : i = 0, 1, \ldots, 7\big\}.$$

(Indeed, we could write any polynomial of the form $(x-1)(x-2)(x-3)\cdot q(x)$ with $q \in \mathcal{P}_7$ as a linear combination of the elements of this set, while, if we have $\sum_{i=0}^{7} a_i(x-1)(x-2)(x-3)x^i = \mathbf{0}$ for some coefficients $a_0, a_1, \ldots, a_7 \in \mathbb{R}$, then

- *the highest-degree monomial would be a_7x^{10} (coming only from the polynomial $a_7(x-1)(x-2)(x-3)x^7$, given that all other polynomials in the sum have degree less than 10), and it would have to equal $\mathbf{0}$, thus we would get $a_7 = 0$;*
- *this would then imply that the next highest-degree monomial is equal to a_6x^9 (coming only from the polynomial $a_6(x-1)(x-2)(x-3)x^6$, since we have already established that $a_7(x-1)(x-2)(x-3)x^7 = \mathbf{0}$), and it would have to equal $\mathbf{0}$, thus we would get $a_6 = 0$ too;*
- *continuing like this, we would obtain in the end that $a_i = 0$ for all the remaining coefficients a_i;*

this of course shows that the set $\big\{(x-1)(x-2)(x-3)\cdot x^i : i = 0, 1, \ldots, 7\big\}$ is linearly independent as well.)

We now have that, for every polynomial $p \in \mathcal{P}_{10}$,

$$\begin{aligned} [p]_S &= \{q \in \mathcal{P}_{10} : q - p \in S\} \\ &= \{q \in \mathcal{P}_{10} : (q-p)(1) = (q-p)(2) = (q-p)(3) = 0\} \\ &= \{q \in \mathcal{P}_{10} : q(1) - p(1) = q(2) - p(2) = q(3) - p(3) = 0\} \\ &= \{q \in \mathcal{P}_{10} : q(1) = p(1) \text{ and } q(2) = p(2) \text{ and } q(3) = p(3)\}. \end{aligned}$$

Thus, we can write

$$V/S = \big\{\{q \in \mathcal{P}_{10} : q(1) = c_1 \text{ and } q(2) = c_2 \text{ and } q(3) = c_3\} : c_1, c_2, c_3 \in \mathbb{R}\big\}$$

(and the fact that we get a different equivalence class for each combination of values $c_1, c_2, c_3 \in \mathbb{R}$ follows because for each such combination there exist polynomials $q \in \mathcal{P}_{10}$ such that $q(1) = c_1$, $q(2) = c_2$ and $q(3) = c_3$, for instance the polynomial $q_0(x) = \frac{c_1}{2}(x-2)(x-3) - c_2(x-1)(x-3) + \frac{c_3}{2}(x-1)(x-2)$).

Finally, based on the observations that $S \cap \mathcal{P}_2 = \{\mathbf{0}\}$ and that $\dim(S) = 8$, we are allowed to consider the direct sum $S \oplus \mathcal{P}_2$ and we can conclude that this is a subspace of $\mathcal{P}_{10}$ with dimension $\dim(S) + \dim(\mathcal{P}_2) = 11$. This gives that $S \oplus \mathcal{P}_2 = \mathcal{P}_{10}$.

But then, a basis for $\mathcal{P}_{10}/S$ is the set

$$\big\{[1]_S,\ [x]_S,\ [x^2]_S\big\} =$$

$$\Big\{\{q \in \mathcal{P}_{10} : q(1) = q(2) = q(3) = 1\},\ \{q \in \mathcal{P}_{10} : q(1) = 1,\ q(2) = 2,\ q(3) = 3\},\ \{q \in \mathcal{P}_{10} : q(1) = 1,\ q(2) = 4,\ q(3) = 9\}\Big\}.$$

(iii) Here it is much more difficult to find a basis for $V = c$ or for $S = c_0$. We can however describe the elements of V/S: given a sequence $(a_n)_{n\in\mathbb{N}} \in c$, we have that

$$\begin{aligned}
\big[(a_n)_{n\in\mathbb{N}}\big]_{c_0} &= \big\{(b_n)_{n\in\mathbb{N}} \in c : (b_n)_{n\in\mathbb{N}} - (a_n)_{n\in\mathbb{N}} \in c_0\big\} \\
&= \Big\{(b_n)_{n\in\mathbb{N}} \in c : \lim_{n\to\infty}(b_n - a_n) = 0\Big\} \\
&= \Big\{(b_n)_{n\in\mathbb{N}} \in c : \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0\Big\} \\
&= \Big\{(b_n)_{n\in\mathbb{N}} \in c : \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n\Big\}.
\end{aligned}$$

We thus see that

$$c/c_0 = \Big\{\Big\{(b_n)_{n\in\mathbb{N}} \in c : \lim_{n\to\infty} b_n = a\Big\} : a \in \mathbb{R}\Big\} = \Big\{\big[(a)_{n\in\mathbb{N}}\big]_{c_0} : a \in \mathbb{R}\Big\}$$

where $(a)_{n\in\mathbb{N}}$ stands for the constant infinite sequence all of whose terms are equal to a.

But then we can define a map $f : \mathbb{R} \to c/c_0$ as follows:

$$a \in \mathbb{R} \quad \mapsto \quad f(a) := \big[(a)_{n\in\mathbb{N}}\big]_{c_0}.$$

We can check that this map is bijective: clearly, all equivalence classes in $\Big\{\big[(a)_{n\in\mathbb{N}}\big]_{c_0} : a \in \mathbb{R}\Big\}$ end up being images of this map f, while the equivalence classes corresponding to different constant sequences will be different (given that the limits of the sequences are then different), thus f will be injective as well.

We finally check that f is a linear map: if $a, b, t \in \mathbb{R}$, then

$$\begin{aligned}
f(a+b) = \big[(a+b)_{n\in\mathbb{N}}\big]_{c_0} &= \big[(a)_{n\in\mathbb{N}} + (b)_{n\in\mathbb{N}}\big]_{c_0} \\
&= \big[(a)_{n\in\mathbb{N}}\big]_{c_0} + \big[(b)_{n\in\mathbb{N}}\big]_{c_0} = f(a) + f(b),
\end{aligned}$$

$$\text{while} \quad f(t\cdot a) = \big[(ta)_{n\in\mathbb{N}}\big]_{c_0} = \big[t\cdot(a)_{n\in\mathbb{N}}\big]_{c_0} = t\cdot\big[(a)_{n\in\mathbb{N}}\big]_{c_0} = t\cdot f(a)$$

(here we also made use of the way vector space operations are defined in the quotient space).

Thus f is a linear isomorphism, and so $\mathbb{R}$ and c/c_0 have the same dimension, which is equal to 1. It follows that any singleton containing a non-zero equivalence class in c/c_0 will be a basis for c/c_0 (e.g. the set $\Big\{\big[(1)_{n\in\mathbb{N}}\big]_{c_0}\Big\}$ is a basis for c/c_0).

Problem 5. We first prove that the set $\{[\bar{u}_1]_S, [\bar{u}_2]_S, \ldots, [\bar{u}_k]_S\}$ has size k. Consider $\bar{u}_i, \bar{u}_j \in \mathcal{B}_T$ with $\bar{u}_i \neq \bar{u}_j$. Then since $\mathcal{B}_T$ is a basis of the subspace T, and hence a linearly independent subset of V, we get that

$$\bar{u}_i - \bar{u}_j \neq \bar{0}_V$$

(since the coefficients used for this linear combination are not all zero).

At the same time, $\bar{u}_i - \bar{u}_j \in T$, since T is closed under addition and scalar multiplication as a subspace of V. Given now that $S \cap T = \{\bar{0}_V\}$ according to our assumptions, we see that we cannot have $\bar{u}_i - \bar{u}_j \in S$ (because if we did, it would be a non-zero element in the intersection $S \cap T$, and this would contradict the fact that $S \cap T$ does not contain non-zero elements).

This by definition means that $\bar{u}_i \not\sim_S \bar{u}_j \Leftrightarrow [\bar{u}_i]_S \neq [\bar{u}_j]_S$, as we wanted. In other words, all the elements in the set $\{[\bar{u}_1]_S, [\bar{u}_2]_S, \ldots, [\bar{u}_k]_S\}$ are pairwise different, and thus we have k different elements in it.

We now verify that the set $\{[\bar{u}_1]_S, [\bar{u}_2]_S, \ldots, [\bar{u}_k]_S\}$ is a basis for V/S. We first check that it spans the quotient space: let $\bar{X} \in V/S$; then there exists $\bar{x} \in V$ such that $\bar{X} = [\bar{x}]_S$. Since $V = S \oplus T$, we can find vectors $\bar{z} \in S$ and $\bar{y} \in T$ such that $\bar{x} = \bar{z} + \bar{y}$. Then we can write

$$\bar{x} - \bar{y} = \bar{z} \in S \quad \Rightarrow \quad \bar{x} \sim_S \bar{y} \quad \Rightarrow \quad [\bar{x}]_S = [\bar{y}]_S.$$

Moreover, since $\mathcal{B}_T$ is a basis for T, we can find scalars $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{F}$ such that

$$\bar{y} = \lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \cdots + \lambda_k\bar{u}_k.$$

But then

$$\begin{aligned}\bar{X} = [\bar{x}]_S = [\bar{y}]_S &= [\lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \cdots + \lambda_k\bar{u}_k]_S \\ &= [\lambda_1\bar{u}_1]_S + [\lambda_2\bar{u}_2]_S + \cdots + [\lambda_k\bar{u}_k]_S \\ &= \lambda_1[\bar{u}_1]_S + \lambda_2[\bar{u}_2]_S + \cdots + \lambda_k[\bar{u}_k]_S \in \operatorname{span}\big(\{[\bar{u}_1]_S, [\bar{u}_2]_S, \ldots, [\bar{u}_k]_S\}\big)\end{aligned}$$

(here we make use of the way vector space operations are defined in the quotient space).

Since $\bar{X} \in V/S$ was arbitrary, we conclude that $\{[\bar{u}_1]_S, [\bar{u}_2]_S, \ldots, [\bar{u}_k]_S\}$ spans the entire V/S.

Consider now scalars $\mu_1, \mu_2, \ldots, \mu_k \in \mathbb{F}$ such that

$$\mu_1[\bar{u}_1]_S + \mu_2[\bar{u}_2]_S + \cdots + \mu_k[\bar{u}_k]_S = \bar{0}_{V/S},$$

and recall that $\bar{0}_{V/S} = [\bar{0}_V]_S = S$. By the way vector operations are defined in the quotient space, it follows that

$$[\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k]_S = [\bar{0}_V]_S,$$

and thus $\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k \sim_S \bar{0}_V$, or in other words

$$\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k = \big(\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k\big) - \bar{0}_V \in S.$$

Since the vectors $\bar{u}_i$ belong to the subspace T, we also have that $\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k \in T$, and thus

$$\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k \in S \cap T.$$

Given that $S \cap T = \{\bar{0}_V\}$, we obtain that

$$\mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \cdots + \mu_k \bar{u}_k = \bar{0}_V.$$

Finally, given that the set $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\}$ is linearly independent, we can conclude that

$$\mu_1 = \mu_2 = \cdots = \mu_k = 0_{\mathbb{F}}.$$

This shows that the set $\big\{[\bar{u}_1]_S, [\bar{u}_2]_S, \ldots, [\bar{u}_k]_S\big\}$ is linearly independent too.

Combining the above, we see that $\big\{[\bar{u}_1]_S, [\bar{u}_2]_S, \ldots, [\bar{u}_k]_S\big\}$ is a basis of the quotient space V/S.