Reminder V vector space over a field $\mathbb{F}$
S subspace of V

Introducing a structure on $V/S = \{[\bar{x}]_S : \bar{x} \in V\}$
$= \{\bar{x} + S : \bar{x} \in V\}$?

Given $[\bar{x}]_S, [\bar{y}]_S$ in V/S and $r \in \mathbb{F}$

$$[\bar{x}]_S + [\bar{y}]_S := [\bar{x} + \bar{y}]_S$$

and $$r \cdot [\bar{x}]_S := [r \cdot \bar{x}]_S$$

Recall that these operations are well-defined.

Recall also: V/S together with this vector addition and this scalar multiplication becomes a vector space over $\mathbb{F}$.

Examples

1. Simplest example: What if $S = \{\bar{0}_V\}$?

Then $\bar{x} \sim_S \bar{y}$ if and only if $\bar{x} - \bar{y} \in \{\bar{0}_V\}$
if and only if $\bar{x} = \bar{y}$.

Thus $[\bar{x}]_S = \{\bar{y} \in V : \bar{x} \sim_S \bar{y}\}$
$= \{\bar{x}\}$.

In other words, $V/S = \{\{\bar{x}\} : \bar{x} \in V\}$
and $\{\bar{y}\} + \{\bar{z}\} = \{\bar{y} + \bar{z}\}$, $r \cdot \{\bar{y}\} = \{r \cdot \bar{y}\}$.

Clearly V is isomorphic to this quotient space.

2. $V = \mathcal{P} = \{\text{all real polynomials}\}$,

$S = \{p \in \mathcal{P} : p(1) = 0\}$

Recall that we can rewrite S as follows:

$S = \{p \in \mathcal{P} : \exists\, q \in \mathcal{P} \text{ such that } p(x) = q(x) \cdot (x-1)\}$.

Note now that, if $p_1, p_2 \in \mathcal{P}$, then

$p_1 \sim_S p_2$ if and only if $p_1 - p_2 \in S$

if and only if $(p_1 - p_2)(1) = 0$

if and only if $p_1(1) = p_2(1)$.

Thus $[p_1]_S = \{p_2 \in \mathcal{P} : p_2(1) = p_1(1)\}$

$= \{p_2 \in \mathcal{P} : \exists\, q \in \mathcal{P} \text{ s.t. } p_2(x) = q(x) \cdot (x-1) + p_1(1)\}$

In other words,

$\mathcal{P}/S = \{\{p \in \mathcal{P} : \exists\, q \in \mathcal{P} \text{ s.t. } p(x) = q(x)(x-1) + a_0\} : a_0 \in \mathbb{R}\}$

Dimension of $\mathcal{P}/S$?

3. Let $V = C[0,1]$ = the space of all <u>continuous</u> functions defined on $[0,1]$ and taking values in $\mathbb{R}$, with pointwise addition and scalar multiplication

Note that $C[0,1]$ is a vector space over $\mathbb{R}$.

Also, interesting to know: $\dim_{\mathbb{R}}(C[0,1]) = |\mathbb{R}|$.

Consider also the subspace of $C[0,1]$

$$S = \{ f \in C[0,1] : \int_0^1 f(x)dx = 0 \}$$

Then $f \sim_S g$ if and only if $f - g \in S$

if and only if $\int_0^1 (f-g)(x)dx = 0$

if and only if $\int_0^1 f(x)dx = \int_0^1 g(x)dx$.

Thus $[f]_S = \{ g \in C[0,1] : \int_0^1 g(x)dx = \int_0^1 f(x)dx \}$

and $C[0,1]/S = \{ \{ h \in C[0,1] : \int_0^1 h(x)dx = a \} : a \in \mathbb{R} \}$.

Dimension of $C[0,1]/S$?

How do we find a basis for V/S? Recall

<u>Theorem</u> Let V be a vector space over a field F, and S a subspace of V.

Consider another subspace T of V such that

$$V = S \oplus T,$$

and moreover consider a basis B_T of T,

$B_T = \{ \bar{u}_i : i \in I \}$ where I is some index set.

Then, whenever we have two distinct indices i_1, i_2 from I, we get that

$$[\bar{u}_{i_1}]_S \neq [\bar{u}_{i_2}]_S.$$

Moreover, the set $\{ [\bar{u}_i] : i \in I \} = \{ \bar{u}_i + S : i \in I \}$ is a basis of V/S.

The above theorem shows that
$$\dim_{\mathbb{F}} V/S = |B_T| = \dim_{\mathbb{F}} T,$$

where T is such that $V = S \oplus T$.
But we have already seen that the latter implies
$$\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} S + \dim_{\mathbb{F}} T$$

Thus we can conclude that
$$\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} S + \dim_{\mathbb{F}} V/S.$$

We will now use these to give a more general proof of

Main Theorem E Let V_1, V_2 be vector spaces over the same field $\mathbb{F}$, and let $\ell : V_1 \to V_2$ be a linear map. Then
$$\dim_{\mathbb{F}} \operatorname{Ker}(\ell) + \dim_{\mathbb{F}} \operatorname{Range}(\ell) = \dim_{\mathbb{F}} V_1.$$

The theorem will be a consequence of a stronger result:

Theorem (First Isomorphism Theorem for vector spaces)
Let V_1, V_2 be vector spaces over the same field $\mathbb{F}$ and let $g : V_1 \to V_2$ be a linear map.
Then we can also define a linear map

$$G : V_1/\ker(g) \longrightarrow V_2 \text{ as follows:}$$

if $\bar{u} + Ker(g) \in V_1/Ker(g)$, then

$$G(\bar{u} + Ker(g)) := g(\bar{u}) \in V_2.$$

We can verify that G is well-defined, it is linear, and $Range(G) = Range(g)$.
Moreover, G is a linear isomorphism from $V_1/Ker(g)$ to $Range(G) = Range(g)$.

Proof of Thm (next time)

Remark How does Main Thm E follow from the 1st Isomorphism Thm?
Note that, according to the latter thm, we can find an isomorphism from $V_1/Ker(f)$ to $Range(f)$.

Thus $\dim_F(V_1/Ker(f)) = \dim_F Range(f)$. ①

But we have already seen that
$$\dim_F V_1 = \dim_F(Ker(f)) + \dim_F(V_1/Ker(f)). \quad ②$$

Combining ① and ② we get the conclusion of Main Thm E.