Math 227, Q1
Final Exam – April 21, 2020

General instructions.

- This exam has 6 problems with a total worth of 120 points. To earn maximum credit, you need to accumulate 95 points or more.
- All submitted answers must be **handwritten** (any answer or part of an answer that fails this will receive 0 points with no exception).
- You may refer to your course notes, any other files on the eclass page of the course, as well as your previous Crowdmark assignments.
- **No collaboration is allowed, nor is seeking help on other internet sites.**
- You must show your work and justify your answers to receive full credit. A correct answer without any justification will receive little or no credit.

  In your justifications, you may simply refer to, and rely on, any results/properties that we discussed in class or appeared in the homework assignments and/or the review files, **except of course if a problem specifically asks you to prove such a result**.
- The exam formally starts at 2pm and finishes at 4:30pm.

  You have until 5pm to make sure your answers are submitted correctly to Crowdmark. **The latter is a strict deadline.**

Problem 1 (*max. 20 = 12 + 8 points*) (a) Let $\mathbb{K}$ be a field, let $\mathcal{R}$ be a ring (not necessarily a field), and consider a ring homomorphism $\phi : \mathbb{K} \to \mathcal{R}$, that is, a function that satisfies

$$\text{for all } a, b \in \mathbb{K}, \quad \phi\big(a +_{\mathbb{K}} b\big) = \phi(a) \;+_{\mathcal{R}}\; \phi(b) \quad \text{and} \quad \phi\big(a \cdot_{\mathbb{K}} b\big) = \phi(a) \;\cdot_{\mathcal{R}}\; \phi(b),$$
$$\text{and} \quad \phi(1_{\mathbb{K}}) = 1_{\mathcal{R}}\,.$$

Show that Range(ϕ) is a field (even in cases that $\mathcal{R}$ is not a field).

(b) Give an example of a field $\mathbb{K}$, a ring $\mathcal{R}$ that is **<u>not</u>** a field, and of a ring homomorphism $\phi : \mathbb{K} \to \mathcal{R}$. Verify that your example has the required properties.

Problem 2 (*max. 25 = 12 + 5 + 8 points*) (a) Given a positive integer $n > 1$, a field $\mathbb{F}$, and a matrix $A \in \mathbb{F}^{n\times n}$, we have seen that A and A^T have the same eigenvalues. In fact, we have seen that the characteristic polynomial of A is the same as the characteristic polynomial of A^T, so each common eigenvalue λ of these two matrices has the same algebraic multiplicity with respect to A as with respect to A^T.

Is the analogous conclusion for geometric multiplicities TRUE OR FALSE? That is, given an eigenvalue λ of A (and hence of A^T as well), is the geometric multiplicity of λ with respect to A the same as with respect to A^T? Justify your answer fully.

(b) A matrix $Q = \big(q_{ij}\big)_{1\leqslant i,j\leqslant n} \in \mathbb{R}^{n\times n}$ is called *stochastic* (or sometimes *row stochastic*) if:

- all the entries q_{ij} are non-negative numbers, and
- for each row the total sum of the entries contained in it is 1, that is, for every $1 \leqslant i_0 \leqslant n$,
$$\sum_{j=1}^{n} q_{i_0,j} = 1.$$

Given any stochastic matrix $Q \in \mathbb{R}^{n\times n}$, show that 1 is an eigenvalue of Q and that an eigenvector of Q corresponding to eigenvalue 1 is the vector

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_n\,.$$

(c) According to what is recalled in part (a), given any stochastic matrix $Q \in \mathbb{R}^{n\times n}$, 1 will be an eigenvalue of Q^T as well.

TRUE OR FALSE: do we necessarily also have that the vector $\bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_n$ is an eigenvector of Q^T ? Justify your answer fully.

Problem 3 (*max. 15 points*) Consider the following matrix with entries from $\mathbb{Z}_7$. Compute its determinant (and show all your work).

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 3 & 4 & 5 \\ 3 & -4 & 5 & -6 \end{pmatrix} \in \mathbb{Z}_7^{4\times 4}$$

Problem 4 (*max. 15 points*) Is there a linear function f from $\mathcal{P}_3$ to $\mathbb{R}^{3\times 2}$ such that

$$f(x+1) = \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 0.5 & 0 \end{pmatrix} \quad \text{and} \quad f(x^2+x) = \begin{pmatrix} 0 & -2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} ?$$

Justify your answer fully (that is, if the answer is no, then explain why; if the answer is yes, define such a function (by giving a formula for it), and confirm it has the desired properties).

[*Note.* As usual, here $\mathcal{P}_3$ and $\mathbb{R}^{3\times 2}$ are viewed as vector spaces over $\mathbb{R}$.]

Problem 5 (*max. 20 = 5 + 15 points*) Consider the vector space U of 3×3 matrices with entries from $\mathbb{R}$, and view it as a real inner product space with inner product

$$A, B \in U \quad \mapsto \quad \langle A, B \rangle := \operatorname{tr}(AB^T)\,.$$

Consider also the function $f : U \to \mathbb{R}^2$ given by

$$A = (a_{ij})_{1 \leqslant i,j \leqslant 3} \in U \quad \mapsto \quad f(A) := \begin{pmatrix} a_{11} + 2a_{12} + a_{33} \\ a_{22} + a_{23} \end{pmatrix}.$$

(a) Show that f is a linear function.

(b) Find an orthogonal basis for $\operatorname{Ker}(f)$, and extend it to an orthogonal basis for the entire space U.

[*Note. In part (b), if you only find non-orthogonal bases for* $\operatorname{Ker}(f)$ *and then for* U*, you will get maximum 8 points.*]

Problem 6 (*max. 25 = 15 + 10 points*) Let $V = \mathbb{Z}_5^{3\times 3}$ (viewed as a vector space over $\mathbb{Z}_5$), and let S be the subspace of V consisting of all lower triangular matrices with zero trace.

(a) Describe the elements of V/S and find a basis for the quotient space.

(b) Consider the following three matrices in V:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 4 \\ 1 & 0 & 1 \end{pmatrix}.$$

Is the subset $\{[A]_S,\ [B]_S,\ [C]_S\}$ of V/S linearly independent? Justify your answer fully.