Math 322
Homework Problem Set 5

Problem 1. (a) Let H be a **strongly connected** tournament on n vertices (where $n \geqslant 3$). In the proof of Theorem $1''$ of Lecture 22, when we were showing that H being strongly connected implies that H has a (directed) Hamilton cycle, we needed to start our justification by assuming that H contains (directed) cycles, and hence we can consider such a cycle C_0 in H which has largest possible length.

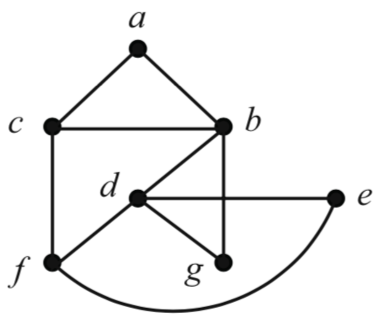
Explain why we are allowed to make this assumption, that is, explain that H must contain (directed) cycles *(even if we don't know right away if it contains Hamilton cycles too)*. **In your explanation here, you are <u>not</u> allowed to use Theorem $1''$** (since you are asked to justify something that is needed in the proof of that theorem), but you can use the previous theorem about tournaments if you want, that states that every tournament has a Hamilton path.

(b) *Now, in this part, you can use Theorem $1''$ as well if you want.* Show that every **strongly connected** tournament H' on n' vertices (where $n' \geqslant 4$) contains a (directed) 3-cycle.

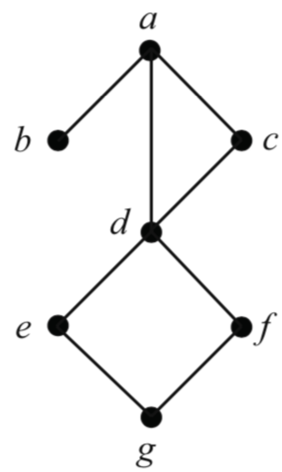
(c) Give an example of a tournament on 4 vertices which does not contain any (directed) 3-cycles *(this will show that the assumption that H' is strongly connected is needed in part (b))*.

Problem 2. For each of the undirected graphs below, determine whether it has a strong orientation or not. Justify your answers.

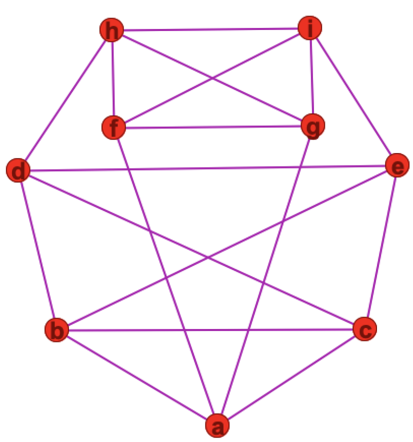
Moreover, for any of these graphs that does have a strong orientation, provide an example of such an orientation (and briefly verify that the example you gave is a strongly connected oriented graph).



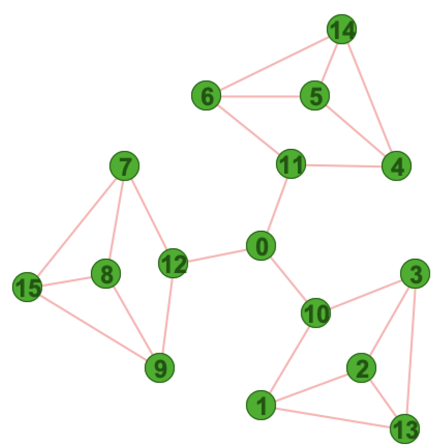
Graph H_1



Graph H_2



Graph H_3



Graph H_4

Problem 3. Nine students have enrolled to a music school for advanced singing lessons (below we denote these students by $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8$ and S_9).

There are three primary singing instructors in this school, which we will denote by T_1, T_2, T_3. We know that T_1 is planning to accept 3 students, T_2 is planning to accept 4 students, and T_3 will accept 2 students.

All instructors have listened to a brief performance by each prospective student, and based on these performances they have submitted a list with their preferences.

Similarly, each student has submitted a list indicating which instructor they would prefer to work with.

The tables below contain this information (note that the first table gives the preferences of the instructors, while the second table gives the preferences of the students).

Assume that you are the director of this school and need to decide which student is paired with which instructor. Use the Gale-Shapley algorithm to find a **stable** matching. Show all your work as you go through the stages / rounds of the algorithm.

[*Clarification.* Treat the group of instructors as the proposing set.]

T_1	T_2	T_3
S_3	S_3	S_1
S_4	S_2	S_4
S_1	S_7	S_3
S_2	S_8	S_5
S_7	S_9	S_9
S_9	S_4	S_8
S_8	S_1	S_6
S_5	S_6	S_7
S_6	S_5	S_2

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
T_1	T_2	T_3	T_2	T_1	T_1	T_2	T_2	T_3
T_2	T_3	T_1	T_3	T_3	T_2	T_1	T_1	T_2
T_3	T_1	T_2	T_1	T_2	T_3	T_3	T_3	T_1

Problem 4. Let G be a graph of order n. Recall how we define the average degree of G:

$$\operatorname{avgdeg}(G) := \frac{1}{|G|}\sum_{i=1}^{n}\deg(v_i) = \frac{1}{n}\sum_{i=1}^{n}\deg(v_i).$$

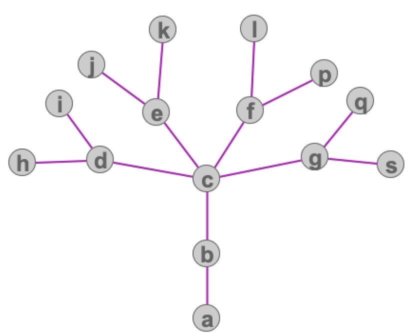
Show with an example that the quantity $1 + \operatorname{avgdeg}(G)$ is not a good upper bound for the chromatic number $\chi(G)$ of G.

In other words, find a graph G_0 which satisfies

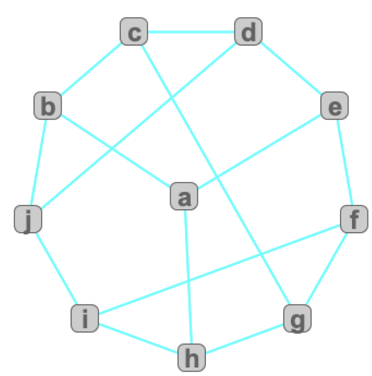
$$\chi(G_0) > 1 + \operatorname{avgdeg}(G_0)$$

(and determine precisely the quantities $\chi(G_0)$ and $\operatorname{avgdeg}(G_0)$ for your example to verify that the above inequality will hold).

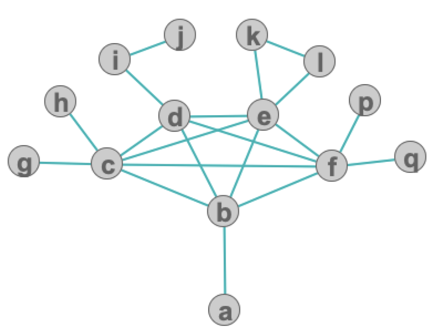
Problem 5. For each of the **five** graphs below, find its chromatic number and also give a (proper) vertex colouring that uses the minimum number of colours. Explain your answers fully.



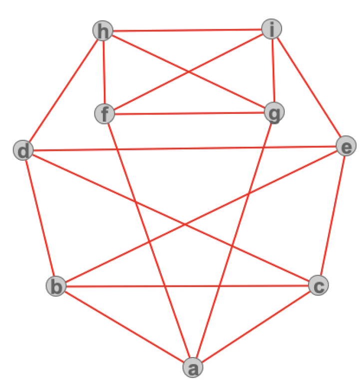
Graph G_1



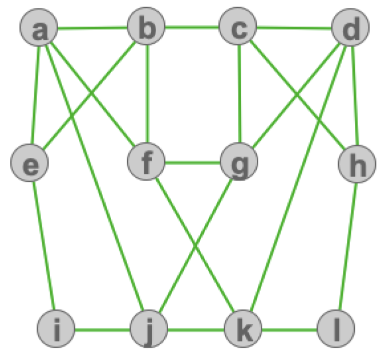
Graph G_2



Graph G_3



Graph G_4



Graph G_5