Math 322, A1
Final Exam – December 11, 2020

General instructions.

- This exam has 5 problems, which are worth a total of 125 points. To earn maximum credit, you need to accumulate 95 points or more.
- All submitted answers must be **handwritten on paper** (any answer or part of an answer that fails this will receive 0 points with no exception).
- You may refer to your course notes, and any other files on the eClass page of the course.
- **No other internet resources are allowed.**
- **No collaboration is allowed.**
- You must show your work and justify your answers to receive full credit. A correct answer without any justification will receive little or no credit.

  In your justifications, you may simply refer to, and rely on, any results/properties that we discussed in class or that appear in the notes or in the homework assignments (and the files with suggested solutions to them), **except of course if a problem specifically asks you to explain why such a result holds**.
- The exam formally starts at 9am and finishes at 11:30am.

  You have until 12pm to make sure your answers are submitted correctly to Assign2. **The latter is a strict deadline.**

Problem 1 (*max. 20 = 10 + 10 points*) (a) Find the chromatic number of the following graph. Also, give a minimal colouring of the graph. Justify your answer fully.

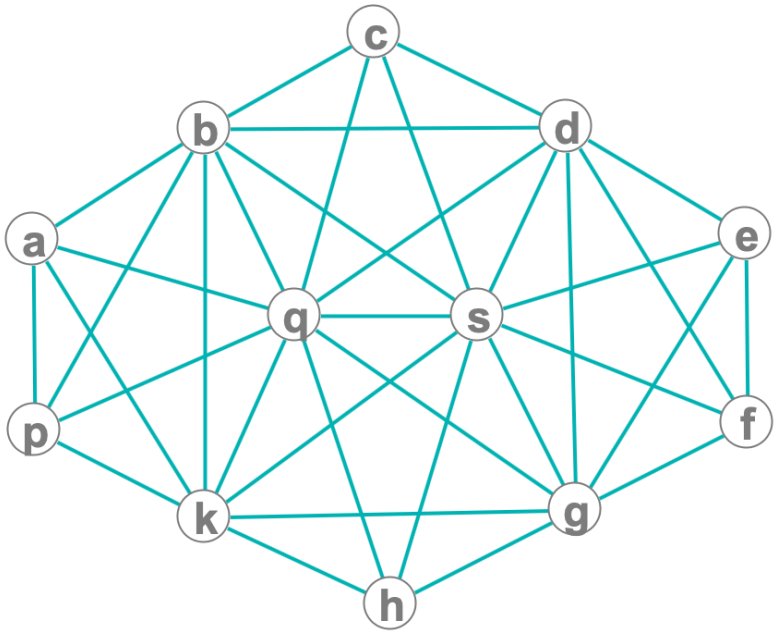

Figure 1: Graph G_0

(b) Does G_0 have a two-factor? Does it have a two-factorization? Justify your answers fully.

Problem 2 (*max. 40 = 10 + 10 + 20 points*) In each part of this problem, you will be given the exact value for one, or more, of the parameters of a certain graph G (otherwise unknown to you), and, based on this information, you will be asked to determine the value of some of the parameters of the complement $\overline{G}$ of G.

In each case, your answer should either be an explicit integer value, or it should be "Not enough information". You should justify each answer fully.

In particular, whenever you answer "Not enough information", then you should justify your answer by giving two (or more if you prefer) examples of graphs

- each of which satisfies the required properties,
- but for which the desired parameters of the complements do not have the same value.

One clarifying example. If you were given that the size of a graph G is 10, and you were asked to determine the order of the complement $\overline{G}$ of G, then your answer should be "Not enough information". In this case, you should draw (or describe) two specific graphs G_1 and G_2 such that

- each of these graphs contains 10 edges,
- but the complement of G_1 has different order from the complement of G_2.

(a) (*max. 10 points*) Assume that a graph G has order 13 and size 25. Can you find the size of the complement $\overline{G}$ of G? Justify your answer.

(b) (*max. 10 points*) Again, assume that G has order 13 and size 25. Can you find the number of connected components of the complement $\overline{G}$ of G? Justify your answer.

(c) (*max. 20 points*) Assume that a graph G satisfies $\delta(G) = 3$, $\Delta(G) = 12$, $\alpha(G) = 6$ and $\omega(G) = 4$. Can you find

(i) (*max. 7 points*) $\Delta\big(\overline{G}\big)$?

(ii) (*max. 7 points*) $\alpha\big(\overline{G}\big)$?

(iii) (*max. 6 points*) $\omega\big(\overline{G}\big)$?

Justify your answers fully.

Problem 3 (*max. 20 = 10 + 10 points*) Both parts of the following problem concern complete bipartite graphs $K_{m,n}$.

(a) Find all pairs of positive integers (m, n) for which the graph $K_{m,n}$ is Eulerian. Justify your answer fully.

(b) Find all pairs of positive integers (m, n) for which the graph $K_{m,n}$ is Hamiltonian. Justify your answer fully.

Problem 4 (*max. 20 = 10 + 10 points*) (a) Is the following graph Hamiltonian? Justify your answer fully.

(b) Find $\kappa(G_1)$ precisely. Justify your answer fully.

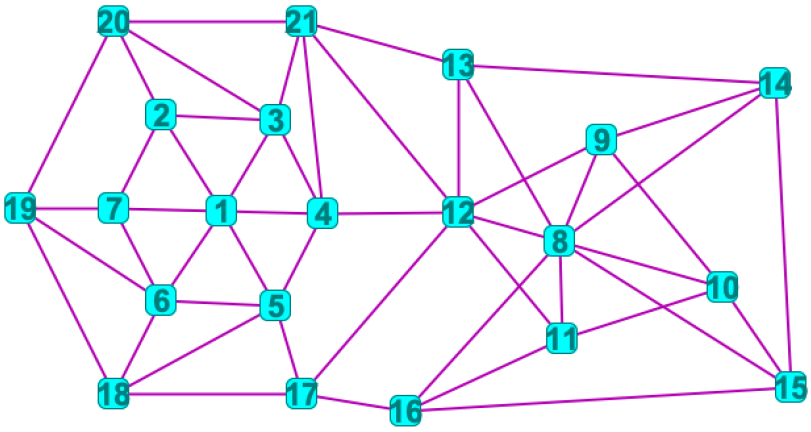

Figure 2: Graph G_1

Problem 5 (*max. 25 = 10 + 15 points*) (a) Solve the connector problem for the following graph, that is, find a minimum weight spanning tree. Also, find the total weight of such a tree. Show all your work.

Here we assume that

- edges ak, aq, cs, di, ef, hp and tu have weight 2;
- edges $ai, aj, bc, de, dj, fg, kq, lq$ and pu have weight 3;
- edges $ac, ad, ar, as, eh, gl, gu, hi, jk$ and ru have weight 4;
- edges al, bj, dh, ei, gh, ik and st have weight 5;
- and edges ab, cf and gt have weight 6.

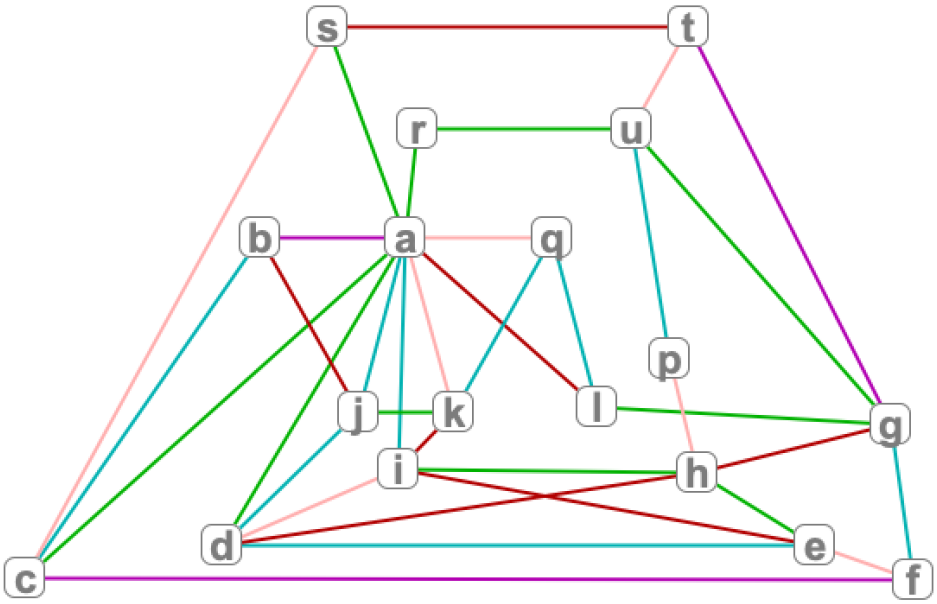

Figure 3: Graph G_2

(b) Below we have an orientation H_2 of the graph G_2.

(i) (*max. 7 points*) Show that H_2 does **not** contain a directed Hamilton path.

(ii) (*max. 8 points*) On the other hand, show that H_2 is strongly connected.

Justify your answers fully.

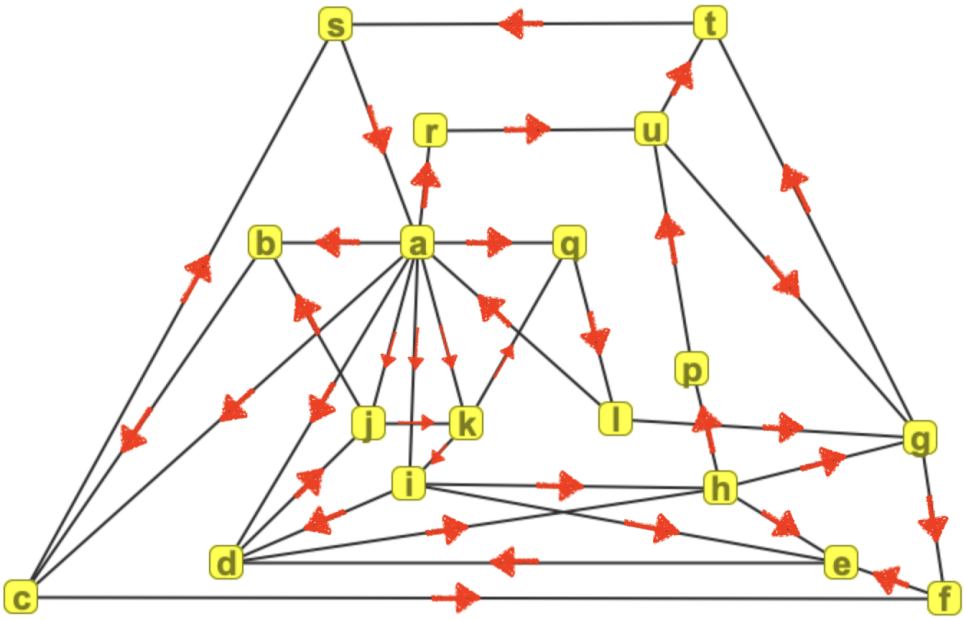

Figure 4: Digraph H_2