Math 227 – Recitation of January 23

During this recitation hour we discussed some examples of linear maps, and how to study them (that is, determine whether they are injective/surjective, find their kernel, their range, and their inverse linear map if this exists).

These examples also serve as motivation for formulating, discussing and then also applying some consequences of Main Theorem E (that we have stated in class, but haven't seen the proof of yet).

Problem 1. For each of the following linear maps, determine whether it is injective/surjective/bijective. If it is bijective, find its inverse.

Describe also its Kernel and its Range.

- $f_1 : \mathbb{R}^2 \to \mathbb{R}^2, \quad f_1\left(\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 2x_1 \\ 3x_1+x_2 \end{smallmatrix}\right).$
- $f_2 : \mathbb{R}^2 \to \mathbb{R}, \quad f_2\left(\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)\right) = x_1 + x_2.$
- $f_3 : \mathbb{R}^3 \to \mathbb{R}^3, \quad f_3\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 3x_1+2x_2+x_3 \\ x_1-2x_3 \\ 2x_1-2x_3 \end{pmatrix}.$
- $f_4 : \mathbb{Z}_{11}^3 \to \mathbb{Z}_{11}^3, \quad f_4\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1+4x_2-3x_3 \\ -4x_1+2x_2-x_3 \\ x_1+6x_2-2x_3 \end{pmatrix}.$
- $f_5 : \mathbb{Z}_5^4 \to \mathbb{Z}_5^5, \quad f_5\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1+x_2 \\ x_4-x_2 \\ x_3 \\ x_3-x_4 \\ x_2-x_3-x_1 \end{pmatrix}.$
- $f_6 : \mathbb{R}^3 \to \mathbb{R}^5, \quad f_6\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1+4x_2-3x_3 \\ 3x_1+2x_2+x_3 \\ 2x_1-2x_3 \\ 0 \\ 2x_1+3x_2-2x_3 \end{pmatrix}.$

It is useful to first try to see how Main Theorem E can help us approach such a problem and draw some initial conclusions even without considering all the details we have been given about each linear map.

Main Theorem E. Let $\mathbb{F}$ be a field, and suppose V_1, V_2 are vector spaces over $\mathbb{F}$. Consider a linear map $f : V_1 \to V_2$. We have that

$$\dim_{\mathbb{F}}\text{Range}(f) + \dim_{\mathbb{F}}\text{Ker}(f) = \dim_{\mathbb{F}} V_1.$$

An immediate consequence of Main Theorem E is the following proposition.

Proposition 1. Let $\mathbb{F}$ be a field, and suppose V_1, V_2 are vector spaces over $\mathbb{F}$. Consider a linear map $f : V_1 \to V_2$.
(i) If $\dim_{\mathbb{F}} V_1 < \dim_{\mathbb{F}} V_2$, then f **cannot** be surjective.
(ii) If $\dim_{\mathbb{F}} V_1 > \dim_{\mathbb{F}} V_2$, then f **cannot** be injective.

Proof. (i) By Main Theorem E, we can write

$$\begin{aligned}\dim_{\mathbb{F}} \mathrm{Range}(f) &\leqslant \dim_{\mathbb{F}} \mathrm{Range}(f) + \dim_{\mathbb{F}} \mathrm{Ker}(f)\\ &= \dim_{\mathbb{F}} V_1 < \dim_{\mathbb{F}} V_2,\end{aligned}$$

therefore $\mathrm{Range}(f)$ is a subspace of V_2 with strictly smaller dimension. Thus $\mathrm{Range}(f)$ cannot be equal to V_2, or in other words f cannot be surjective.

(ii) Assume towards a contradiction that we had $\dim_{\mathbb{F}} V_1 > \dim_{\mathbb{F}} V_2$ and an injective linear map $f : V_1 \to V_2$. Then, as we have shown in class, we would have $\mathrm{Ker}(f) = \{\bar{0}_{V_1}\}$, and thus $\dim_{\mathbb{F}} \mathrm{Ker}(f) = 0$. But then, by Main Theorem E, we would get that

$$\dim_{\mathbb{F}} \mathrm{Range}(f) = \dim_{\mathbb{F}} \mathrm{Range}(f) + \dim_{\mathbb{F}} \mathrm{Ker}(f) = \dim_{\mathbb{F}} V_1.$$

This would give us

$$\dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} \mathrm{Range}(f) \leqslant \dim_{\mathbb{F}} V_2,$$

which would contradict our assumption that $\dim_{\mathbb{F}} V_1 > \dim_{\mathbb{F}} V_2$.

We conclude that if V_1 has larger dimension than V_2, then we cannot have an injective linear map $f : V_1 \to V_2$. $\square$

Remark 1. An immediate consequence of Proposition 1 is that a linear map $f : V_1 \to V_2$ is automatically **not** a bijection if $\dim_{\mathbb{F}} V_1 \neq \dim_{\mathbb{F}} V_2$.

In cases that $\dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} V_2$, we have another very useful consequence of Main Theorem E that helps us in analysing a linear map $f : V_1 \to V_2$, and whether it is a bijection or not: note that the following proposition concerns linear maps between **finite-dimensional** vector spaces (and the assumption that the vector spaces we will be dealing with are finite-dimensional will be crucially used, and as we will see later in the term it cannot be removed).

For the proof of Proposition 2, we also need to recall the following fact which we discussed in class on the topic of "Subspaces and Bases of Subspaces".

Remark 2. Let V be a finite-dimensional vector space over a field $\mathbb{F}$, say $\dim_{\mathbb{F}} V = n$ for some integer $n \geqslant 0$.

If S is a subspace of V, and $\dim_{\mathbb{F}} S = \dim_{\mathbb{F}} V$, then $S = V$ (that is, V itself is the only subspace of V which has dimension n).

Proposition 2. Let V_1, V_2 be **finite-dimensional** vector spaces over a field $\mathbb{F}$, and assume that

$$\dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} V_2.$$

Then, for any linear map $f : V_1 \to V_2$, we have that

$$f \text{ is injective} \qquad \textit{if and only if} \qquad f \text{ is surjective.}$$

Proof. We have to show two implications:

$$f \text{ is injective} \qquad \Rightarrow \qquad f \text{ is surjective} \tag{1}$$

and

$$f \text{ is surjective} \qquad \Rightarrow \qquad f \text{ is injective.} \tag{2}$$

We first show implication (1): assume that f is injective. Then, as we have shown in class, we have that $\mathrm{Ker}(f) = \{\bar{0}_{V_1}\}$, therefore $\dim_{\mathbb{F}} \mathrm{Ker}(f) = 0$. We can now use Main Theorem E to obtain that

$$\dim_{\mathbb{F}} \mathrm{Range}(f) = \dim_{\mathbb{F}} \mathrm{Range}(f) + \dim_{\mathbb{F}} \mathrm{Ker}(f) = \dim_{\mathbb{F}} V_1.$$

But we also know that $\dim_{\mathbb{F}} V_2 = \dim_{\mathbb{F}} V_1$, therefore we get that

$$\dim_{\mathbb{F}} \mathrm{Range}(f) = \dim_{\mathbb{F}} V_2.$$

Since V_2 is finite-dimensional by our assumptions, and since $\mathrm{Range}(f)$ is a subspace of V_2, by the above remark we conclude that $\mathrm{Range}(f) = V_2$. In other words, we get that f is surjective.

We now check implication (2): assume this time that f is surjective. Then we have that $\mathrm{Range}(f) = V_2$, and thus $\dim_{\mathbb{F}} \mathrm{Range}(f) = \dim_{\mathbb{F}} V_2$. We again use Main Theorem E, and our assumption that V_1 and V_2 have the same dimension, to obtain that

$$\begin{aligned} \dim_{\mathbb{F}} \mathrm{Range}(f) &= \dim_{\mathbb{F}} V_2 \\ &= \dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} \mathrm{Range}(f) + \dim_{\mathbb{F}} \mathrm{Ker}(f). \end{aligned}$$

But since $\dim_{\mathbb{F}} \mathrm{Range}(f) = \dim_{\mathbb{F}} V_2 < \infty$, the equality

$$\dim_{\mathbb{F}} \mathrm{Range}(f) = \dim_{\mathbb{F}} \mathrm{Range}(f) + \dim_{\mathbb{F}} \mathrm{Ker}(f)$$

implies that $\dim_{\mathbb{F}} \mathrm{Ker}(f) = 0$.

This can only happen if $\mathrm{Ker}(f) = \{\bar{0}_{V_1}\}$, which is equivalent to f being injective, as we wanted. $\square$

Partial solution to Problem 1. By Proposition 1 we know that f_2 cannot be injective, while f_5 and f_6 cannot be surjective. Therefore, the only bijections here can be among the functions f_1, f_3 and f_4.

Let us now analyse in more detail the functions f_2, f_3 and f_5.

- As we've already remarked, f_2 cannot be injective. To find $\mathrm{Ker}(f_2)$, we note that

$$\begin{aligned} \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathrm{Ker}(f_2) \quad &\Leftrightarrow \quad x_1 + x_2 = 0 \quad \Leftrightarrow \\ & \quad\ \ x_2 = -x_1 \quad \Leftrightarrow \quad \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 \\ -x_1 \end{smallmatrix}\right) = x_1 \cdot \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right). \end{aligned}$$

  Thus $\mathrm{Ker}(f_2) = \mathrm{span}\big(\{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\}\big)$.

  At the same time, $\mathrm{Range}(f_2) = \mathbb{R}$, given that, for any $r \in \mathbb{R}$, we have that $r = f_2(\left(\begin{smallmatrix} r \\ 0 \end{smallmatrix}\right))$. We conclude that f_2 is surjective.

- To analyse f_3, we can first look for its matrix representation: we want a matrix $A_3 \in \mathbb{R}^{3\times 3}$ satisfying

$$f_3\left(\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)\right) = A_3 \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)$$

  for every $\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) \in \mathbb{R}^3$. By consecutively setting $\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)$ equal to each of the standard basis vectors in $\mathbb{R}^3$, we can verify that

$$A_3 = \left(\begin{array}{c:c:c} f_3(\bar{e}_1) & f_3(\bar{e}_2) & f_3(\bar{e}_3) \end{array} \right) = \left(\begin{array}{c:c:r} 3 & 2 & 1 \\ 1 & 0 & -2 \\ 2 & 0 & -2 \end{array} \right).$$

  Next we recall that, as we showed in Proposition 2, f_3 will be injective if and only if f_3 is surjective. Therefore, it suffices to check one of the two; let's check whether f_3 is injective. This is equivalent to checking that the linear system

$$A_3 \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$$

  has only one solution, the trivial solution. We have also seen, that in the case of a square matrix, as A_3 is, the latter is equivalent to A_3 being invertible.

  To check whether A_3 is invertible, we use Gauss-Jordan elimination:

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & -2 \\ 2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2/3 & 1/3 \\ 1 & 0 & -2 \\ 2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2/3 & 1/3 \\ 0 & -2/3 & -7/3 \\ 0 & -4/3 & -8/3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2/3 & 1/3 \\ 0 & -2/3 & -7/3 \\ 0 & 0 & 2 \end{pmatrix}.$$

We thus see that a REF of A_3 has 3 pivots (as many as the rows or columns of A_3), therefore A_3 is invertible. By our previous discussion, this also implies that f_3 is injective, and therefore surjective too.

We conclude that $\mathrm{Ker}(f_3) = \{\bar{0}\}$, $\mathrm{Range}(f_3) = \mathbb{R}^3$, and that f_3 is bijective.

To also find the inverse f_3^{-1} of f_3, we note that the matrix representation of f_3^{-1} would be the matrix A_3^{-1}. Indeed, we recall that f_3^{-1} should be a function from $\mathbb{R}^3$ to $\mathbb{R}^3$ that satisfies

$$f_3^{-1}\left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \Leftrightarrow \quad f_3\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

for every $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$. But the latter equality is equivalent to

$$A_3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \text{which in turn is equivalent to } \; A_3^{-1}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus, we conclude that, for every $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$, we must have

$$f_3^{-1}\left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = A_3^{-1}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

It remains to find A_3^{-1}. We repeat the Gauss-Jordan elimination steps we did before:

$$\left(\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 2 & 0 & -2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 2 & 0 & -2 & 0 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & -2/3 & -7/3 & -1/3 & 1 & 0 \\ 0 & -4/3 & -8/3 & -2/3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & -2/3 & -7/3 & -1/3 & 1 & 0 \\ 0 & 0 & 2 & 0 & -2 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 7/2 & 1/2 & -3/2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2/3 & 0 & 1/3 & 1/3 & -1/6 \\ 0 & 1 & 0 & 1/2 & 2 & -7/4 \\ 0 & 0 & 1 & 0 & -1 & 1/2 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1/2 & 2 & -7/4 \\ 0 & 0 & 1 & 0 & -1 & 1/2 \end{array}\right).$$

We obtain that

$$A_3^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1/2 & 2 & -7/4 \\ 0 & -1 & 1/2 \end{pmatrix}$$

$$\text{and } \; f_3^{-1}\left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = A_3^{-1}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -y_2 + y_3 \\ \frac{y_1}{2} + 2y_2 - \frac{7y_3}{4} \\ -y_2 + \frac{y_3}{2} \end{pmatrix}.$$

- By Proposition 1 we know that f_5 cannot be surjective. To analyse it further, we can first look for its matrix representation: since f_5 is a function from $\mathbb{Z}_5^4$ to $\mathbb{Z}_5^5$, the matrix representation A_5 of f_5 has to be a matrix in $\mathbb{Z}_5^{5\times 4}$ satisfying

$$f_5\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = A_5\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \Rightarrow \qquad A_5 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}.$$

Similarly to above, we note that f_5 is injective if and only if the linear system $A_5\bar{x} = \bar{0}$ has only the trivial solution. To check whether this is true, we use Gaussian elimination to row reduce A_5:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & -1 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This last matrix is a REF of A_5. We note that all its columns are pivot columns, therefore the linear system $A_5\bar{x} = \bar{0}$ has only the trivial solution.

We conclude that f_5 is injective, and thus that $\mathrm{Ker}(f_5) = \{\bar{0}\}$.

Finally, for any $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$,

$$f_5\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1+x_2 \\ x_4-x_2 \\ x_3 \\ x_3-x_4 \\ x_2-x_3-x_1 \end{pmatrix} = x_1\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + x_2\cdot\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_3\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} + x_4\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

thus $f_5\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) \in \mathrm{CS}(A_5)$, or in other words $\mathrm{Range}(f_5) \leqslant \mathrm{CS}(A_5)$.

At the same time, for each $i = 1, 2, 3, 4$, the i-th column of A_5 is equal to $f_5(\bar{e}_i) \in \mathrm{Range}(f_5)$, therefore $\mathrm{CS}(A_5) = \mathrm{span}(\{C_i(A_5), i = 1, 2, 3, 4\}) \leqslant \mathrm{Range}(f_5)$.

We conclude that $\mathrm{Range}(f_5) = \mathrm{CS}(A_5)$.