$$\begin{array}{ccc} m \times m & m \times n & n \times n \end{array}$$

2) $A = [\hat{U}\ \ U_0] \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{V}^T \\ V_0^T \end{bmatrix}$ where rank(A) $A = k < m < n$

now I claim that if I let the diagonal elements of Σ I take the reciprocal of it and then

$$A^+ = [\hat{V}\ \ V_0] \begin{bmatrix} \hat{\Sigma}^+ & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{U}^T \\ V_0^T \end{bmatrix}$$

where $\hat{\Sigma} = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ 0 & & \\ \vdots & & \sigma_k \end{bmatrix}$

then $\hat{\Sigma}^+ = \begin{bmatrix} \sigma_1^{-1} & \cdots & 0 \\ \vdots & & \\ 0 & & \sigma_k^{-1} \end{bmatrix}$

now if $\forall i \in \{1 .. k\}$ if $\sigma_i \geq tol$ then $\frac{1}{\sigma_i}$ is taken else 0 is set

$$\therefore A^+ = [\hat{V}\ \ V_0] \begin{bmatrix} \hat{\Sigma}^+ & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{U}^T \\ U_0^T \end{bmatrix} = \hat{V} \hat{\Sigma}^+ \hat{U}^T$$

now hote note that dimension of A

$$
A = \begin{array}{cc} m\times k & m\times m-k \\ \left[\begin{array}{c|c} \hat{U} & U_0 \end{array}\right. & \left.\right] \end{array}
\begin{array}{cc} k\times k & k\times n-k \\ \left[\begin{array}{c|c} \hat{\Sigma} & 0 \\ \hline 0 & 0 \end{array}\right. & \left.\right] \\ m-k\times k & m-k\times n-k \end{array}
\begin{array}{cc} \left[\begin{array}{c} \hat{V}^T \\ \hline V_0^T \end{array}\right] & \begin{array}{c} k\times n \\ n-k\times n \end{array} \end{array}
$$

$\therefore\ A = \hat{U}\hat{\Sigma}\hat{V}^T$

$$
\therefore \text{ then } A^+ = \begin{array}{cc} n\times k & n\times n-k \\ \left[\hat{V}, \right. & \left. V\right] \end{array}
\begin{array}{cc} k\times k & k\times m-k \\ \left[\begin{array}{c|c} \hat{\Sigma}^+ & 0 \\ \hline 0 & 0 \end{array}\right. & \left.\right] \\ n-k\times k & n-k\times m-k \end{array}
\begin{array}{cc} \left[\begin{array}{c} \hat{U}^T \\ U_0^T \end{array}\right] & \begin{array}{c} k\times m \\ m-k\times m \end{array} \end{array}
$$

$\Rightarrow A^+ = \hat{V}\hat{\Sigma}^+\hat{U}^T$

Then $(AA^+)^T = (\hat{U}\hat{\Sigma}\hat{V}^T\hat{V}\hat{\Sigma}^+\hat{U}^T)^T = (\hat{U}\hat{\Sigma}\hat{V}^T\hat{V}\hat{\Sigma}^+\hat{V}^T) = AA^+$

$(A^+A)^T = (\hat{V}\hat{\Sigma}^+\hat{U}^T\hat{U}\hat{\Sigma}\hat{V}^T) = (\hat{V}\hat{\Sigma}^+\hat{U}^T\hat{U}\hat{\Sigma}\hat{V}^T) = A^+A$

$\therefore$ The properties are satisfied hence $\hat{U}\hat{\Sigma}^+\hat{U}^T$ is the pseudo inverse