Math 127
Suggested solutions to the Final Exam

Problem 1. We show that Structure 1 is a field (and hence a commutative ring too), while Structure 2 is a commutative ring.

By the given assumptions, we already know that addition and multiplication in Structure 1 are associative and that they satisfy the distributive law. We also note that:

addition is commutative: this is because, for every $1 \leqslant i \leqslant 8$, the i-th row of the table of addition is the same as the i-th column.

a_6 **is the neutral element of addition:** from the table we see that, for every $1 \leqslant i \leqslant 8$, $a_6 + a_i = a_i$.

additive inverses: for every $1 \leqslant i \leqslant 8$, we have that $a_i + a_i = a_6$, thus a_i has an additive inverse (in fact, in this structure each element is its own additive inverse).

multiplication is commutative: this is because, for every $1 \leqslant i \leqslant 8$, the i-th row of the table of multiplication is the same as the i-th column.

a_3 **is the neutral element of multiplication:** from the table we see that, for every $1 \leqslant i \leqslant 8$, $a_3 \cdot a_i = a_i$.

multiplicative inverses: from the table we see that, every row that corresponds to a non-zero element, that is, every row except for the sixth one, contains a_3 in some cell. This shows that, for every $i \in \{1, 2, 3, 4, 5, 7, 8\}$, a_i has a multiplicative inverse. In particular, we have the following table listing these inverses:

a	a_1	a_2	a_3	a_4	a_5	a_7	a_8
a^{-1}	a_8	a_4	a_3	a_2	a_7	a_5	a_1

.

Thus Structure 1 satisfies all the axioms of a field.

By the given assumptions, we already know that addition and multiplication in Structure 2 are associative and that they satisfy the distributive law. We also note that:

addition is commutative: this is because, for every $1 \leqslant i \leqslant 9$, the i-th row of the table of addition is the same as the i-th column.

b_3 is the neutral element of addition: from the table we see that, for every $1 \leqslant i \leqslant 9$, $b_3 + b_i = b_i$.

additive inverses: for every $1 \leqslant i \leqslant 9$, the i-th row of the table contains b_3 in some cell, which shows that b_i has an additive inverse: there exists some element b_j such that $b_i + b_j = b_3$. In particular, we have the following table listing the additive inverses:

b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
$-b$	b_2	b_1	b_3	b_8	b_9	b_7	b_6	b_4	b_5

.

multiplication is commutative: this is because, for every $1 \leqslant i \leqslant 9$, the i-th row of the table of multiplication is the same as the i-th column.

b_5 is the neutral element of multiplication: from the table we see that, for every $1 \leqslant i \leqslant 9$, $b_5 \cdot b_i = b_i$.

Thus Structure 2 satisfies all the axioms of a commutative ring.

However, Structure 2 is not a field, because for instance b_1 is a non-zero element, but has no multiplicative inverse: for every $1 \leqslant i \leqslant 9$, $b_1 \cdot b_i \neq b_5$.

Problem 2. Part (b) was given as a homework problem: see HW4, Problem 4(i) and suggested solution to it.

For part (a), we will use Gaussian elimination:

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 5 & 6 \\ 0 & 6 & 3 \end{pmatrix} \xrightarrow{R_2 + 3R_1 \to R_2'} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 0 & 6 \\ 0 & 6 & 3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 6 & 3 \\ 0 & 0 & 6 \end{pmatrix}.$$

The last matrix is in REF and row equivalent to A. We thus see that a REF of A has 3 pivots, which, by a theorem we proved in class, is equivalent to A being invertible.

Similarly,

$$B = \begin{pmatrix} \boldsymbol{i}-1 & 2 & -6\boldsymbol{i} & 0 \\ 3+4\boldsymbol{i} & 4 & 1 & 3 \\ 0 & \sqrt{2}\,\boldsymbol{i} & 0 & \sqrt{18} \\ 3 & -6 & 10\boldsymbol{i} & 18\boldsymbol{i} \end{pmatrix} \xrightarrow[(3-4\boldsymbol{i})R_2 \to R_2']{(\boldsymbol{i}+1)R_1 \to R_1'} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 25 & 12-16\boldsymbol{i} & 3-4\boldsymbol{i} & 9-12\boldsymbol{i} \\ 0 & \sqrt{2}\,\boldsymbol{i} & 0 & \sqrt{18} \\ 3 & -6 & 10\boldsymbol{i} & 18\boldsymbol{i} \end{pmatrix}$$

$$\xrightarrow[R_4+(3/2)R_1 \to R_4']{R_2+(25/2)R_1 \to R_2'} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & 37-9\boldsymbol{i} & 78-79\boldsymbol{i} & 9-12\boldsymbol{i} \\ 0 & \sqrt{2}\,\boldsymbol{i} & 0 & \sqrt{18} \\ 0 & -3+3\boldsymbol{i} & 3+7\boldsymbol{i} & 18\boldsymbol{i} \end{pmatrix} \xrightarrow[3R_4 \to R_4']{\frac{\boldsymbol{i}}{\sqrt{2}}R_3 \to R_3'} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & 37-9\boldsymbol{i} & 78-79\boldsymbol{i} & 9-12\boldsymbol{i} \\ 0 & -1 & 0 & 3\boldsymbol{i} \\ 0 & -9+9\boldsymbol{i} & 9+21\boldsymbol{i} & 54\boldsymbol{i} \end{pmatrix}$$

$$\xrightarrow[R_4+(-9)R_3 \to R_4']{R_2+37R_3 \to R_2'} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & -9\boldsymbol{i} & 78-79\boldsymbol{i} & 9+99\boldsymbol{i} \\ 0 & -1 & 0 & 3\boldsymbol{i} \\ 0 & 9\boldsymbol{i} & 9+21\boldsymbol{i} & 27\boldsymbol{i} \end{pmatrix}$$

$$\xrightarrow{-9\boldsymbol{i}R_3 \to R_3'} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & -9\boldsymbol{i} & 78-79\boldsymbol{i} & 9+99\boldsymbol{i} \\ 0 & 9\boldsymbol{i} & 0 & 27 \\ 0 & 9\boldsymbol{i} & 9+21\boldsymbol{i} & 27\boldsymbol{i} \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & 9\boldsymbol{i} & 9+21\boldsymbol{i} & 27\boldsymbol{i} \\ 0 & 9\boldsymbol{i} & 0 & 27 \\ 0 & -9\boldsymbol{i} & 78-79\boldsymbol{i} & 9+99\boldsymbol{i} \end{pmatrix}$$

$$\xrightarrow[R_4+R_2 \to R_4']{R_3+(-1)R_2 \to R_3'} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & 9\boldsymbol{i} & 9+21\boldsymbol{i} & 27\boldsymbol{i} \\ 0 & 0 & -9-21\boldsymbol{i} & 27-27\boldsymbol{i} \\ 0 & 0 & 87-58\boldsymbol{i} & 9+126\boldsymbol{i} \end{pmatrix} \xrightarrow{\frac{1}{3}R_3 \to R_3'} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & 9\boldsymbol{i} & 9+21\boldsymbol{i} & 27\boldsymbol{i} \\ 0 & 0 & -3-7\boldsymbol{i} & 9-9\boldsymbol{i} \\ 0 & 0 & 87-58\boldsymbol{i} & 9+126\boldsymbol{i} \end{pmatrix}$$

$$\xrightarrow{R_4 + 29R_3 \to R_4'} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & 9\boldsymbol{i} & 9+21\boldsymbol{i} & 27\boldsymbol{i} \\ 0 & 0 & -3-7\boldsymbol{i} & 9-9\boldsymbol{i} \\ 0 & 0 & -261\boldsymbol{i} & 270-135\boldsymbol{i} \end{pmatrix} \xrightarrow{(\boldsymbol{i}/261)R_4 \to R_4'} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & 9\boldsymbol{i} & 9+21\boldsymbol{i} & 27\boldsymbol{i} \\ 0 & 0 & -3-7\boldsymbol{i} & 9-9\boldsymbol{i} \\ 0 & 0 & 1 & \frac{15}{29}+\frac{30}{29}\boldsymbol{i} \end{pmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & 9\boldsymbol{i} & 9+21\boldsymbol{i} & 27\boldsymbol{i} \\ 0 & 0 & 1 & \frac{15}{29}+\frac{30}{29}\boldsymbol{i} \\ 0 & 0 & -3-7\boldsymbol{i} & 9-9\boldsymbol{i} \end{pmatrix} \xrightarrow{R_4 + (3+7\boldsymbol{i})R_3 \to R_4'} \begin{pmatrix} -2 & 2+2\boldsymbol{i} & 6-6\boldsymbol{i} & 0 \\ 0 & 9\boldsymbol{i} & 9+21\boldsymbol{i} & 27\boldsymbol{i} \\ 0 & 0 & 1 & \frac{15}{29}+\frac{30}{29}\boldsymbol{i} \\ 0 & 0 & 0 & z_4 \end{pmatrix},$$

where $z_4 = 9 - 9\boldsymbol{i} + (3+7\boldsymbol{i})\left(\frac{15}{29} + \frac{30}{29}\boldsymbol{i}\right)$. The last matrix is in REF and is row equivalent to B. Moreover, it has 4 pivots, given that $z_4 \neq 0$. Indeed

$$\Re(z_4) = 9 + \frac{45}{29} - \frac{210}{29} = \frac{261 + 45 - 210}{29} > 0.$$

Therefore, by the same theorem mentioned above, we get that B is invertible.

Problem 3. Statement (i) is true. By a theorem we proved in class, we know that A is invertible if and only if the Reduced Row Echelon Form of A is the identity matrix I_n, that is, if and only if $A \sim I_n$.

But by definition of row equivalence, $A \sim I_n$ means that there exist $k \geqslant 1$ and elementary matrices $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k \in \mathbb{F}^{n\times n}$ such that

$$I_n = \mathcal{E}_k \cdots \mathcal{E}_2 \mathcal{E}_1 A.$$

Since elementary matrices are invertible, the product $\mathcal{E}_k \cdots \mathcal{E}_2 \mathcal{E}_1$ is invertible with inverse equal to $\mathcal{E}_1^{-1}\mathcal{E}_2^{-1} \cdots \mathcal{E}_k^{-1}$. Then the above equality implies that

$$A = \mathcal{E}_1^{-1}\mathcal{E}_2^{-1} \cdots \mathcal{E}_k^{-1}(\mathcal{E}_k \cdots \mathcal{E}_2 \mathcal{E}_1 A) = \mathcal{E}_1^{-1}\mathcal{E}_2^{-1} \cdots \mathcal{E}_k^{-1} I_n = \mathcal{E}_1^{-1}\mathcal{E}_2^{-1} \cdots \mathcal{E}_k^{-1}.$$

Thus, if A is invertible, we get that A can be written as a product of elementary matrices.

Conversely, if A is a product of elementary matrices, then it is a product of invertible matrices, and hence it is invertible (and its inverse can be found as above).

Statement (ii) is false. Let $A \in \mathbb{Q}^{m\times m}$ be the coefficient matrix of $LS1$ and $(A \mid \bar{b}) \in \mathbb{Q}^{m\times(m+1)}$ be its augmented matrix.

We recall that $A\bar{x} = \bar{b}$ has a unique solution if and only if a REF of A has m pivots, which in turn is equivalent to A being invertible. But we have seen that if A has a zero row, then it cannot be invertible. Therefore, if $LS1$ has a unique solution, then A cannot have zero rows.

On the other hand, the converse is not always true, and to disprove it, we can give a counterexample. Consider

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 5 & 6 \end{pmatrix} \in \mathbb{Q}^{3\times 3} \quad \text{and} \quad \bar{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{Q}^3.$$

Then $A\bar{x} = \bar{b}$ is inconsistent, given that we would need to choose values for x_1, x_2, x_3 so that we have

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \qquad \text{and at the same time} \\ 2(x_1 + 2x_2 + 3x_3) = 2x_1 + 4x_2 + 6x_3 &= 1. \end{aligned}$$

In other words we would need to have $2 \cdot 1 = 1$, which is not true in $\mathbb{Q}$.

Statement (iii) is false. We can give a counterexample. Consider the following upper triangular matrices in $\mathbb{R}^{5\times 5}$:

$$U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(we note that these matrices are also elementary matrices: U_1 corresponds to multiplying the 2nd row of a matrix on the right by 2, while U_2 corresponds to adding twice the 2nd row of a matrix on the right to the 1st row of that matrix).

We then have that

$$U_1U_2 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = U_2U_1.$$

Thus U_1 and U_2 do not commute.

Problem 4. Since A is invertible, we can consider its inverse $A^{-1} \in \mathbb{F}^{n\times n}$.

Let $\bar{b}$ be a vector in $\mathbb{F}^n$. Then $A^{-1}\bar{b}$ is also a vector in $\mathbb{F}^n$. Since $\{u_1, u_2, \dots, u_m\}$ is a spanning set of $\mathbb{F}^n$, we can find $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{F}$ so that

$$\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_m u_m = A^{-1}\bar{b}.$$

But then

$$\begin{aligned} & A\big(\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_m u_m\big) = A\big(A^{-1}\bar{b}\big) \\ \Rightarrow\quad & A(\lambda_1 u_1) + A(\lambda_2 u_2) + \cdots + A(\lambda_m u_m) = (AA^{-1})\bar{b} \end{aligned}$$

(because of the distributive law and associativity of matrix multiplication)

$$\Rightarrow\quad \lambda_1(Au_1) + \lambda_2(Au_2) + \cdots + \lambda_m(Au_m) = \bar{b}$$

(by properties of scalar multiplication of vectors/matrices that we have seen).

Thus $\bar{b} \in \operatorname{span}(Au_1, Au_2, \dots, Au_m)$. Since the vector $\bar{b} \in \mathbb{F}^n$ was arbitrary, we conclude that $\operatorname{span}(Au_1, Au_2, \dots, Au_m) = \mathbb{F}^n$, as we wanted.

Problem 5. (a) We look at the augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right) \underset{\substack{R_2+3R_1\to R_2' \\ R_3+4R_1\to R_3'}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 4 & 0 & 3 \\ 0 & 4 & 0 & 3 \end{array}\right) \underset{R_3-R_2\to R_3'}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The last matrix is in REF and has no pivot in the last column, therefore, by a result we proved in class, it follows that the system is consistent. Moreover, there are two pivot columns, the 1st and the 2nd one, which shows that the system has two pivot variables, the variables x_1 and x_2, and one free variable, the variable x_3.

Thus we get 5 solutions, one for each choice of value for the free variable x_3. We can find each one of these by back-substitution: we first note that the 2nd equation in the final, staircase system we got is

$$4x_2 = 0x_1 + 4x_2 + 0x_3 = 3,$$

thus, regardless of the value that we assign to x_3, we must have $x_2 = 4^{-1} \cdot 3 = 2$. Plugging this into the 1st equation, we obtain that

$$x_1 + 2 \cdot 2 + x_3 = x_1 + 2x_2 + x_3 = 4 \quad \Rightarrow \quad x_1 + x_3 = 0 \quad \Rightarrow \quad x_1 = -x_3.$$

Therefore,

- if $x_3 = 0$, then we get that $x_1 = 0$, $x_2 = 2$ and $x_3 = 0$ is the corresponding solution to the system;
- if $x_3 = 1$, then we get that $x_1 = 4$, $x_2 = 2$ and $x_3 = 1$ is the corresponding solution to the system;
- if $x_3 = 2$, then we get that $x_1 = 3$, $x_2 = 2$ and $x_3 = 2$ is the corresponding solution to the system;
- if $x_3 = 3$, then we get that $x_1 = 2$, $x_2 = 2$ and $x_3 = 3$ is the corresponding solution to the system;
- and if $x_3 = 4$, then we get that $x_1 = 1$, $x_2 = 2$ and $x_3 = 4$ is the corresponding solution to the system.

(b) Matrix A_1 is in Row Echelon Form and has 4 pivot columns: columns 1, 2, 4 and 5. Thus, since the last column is not a pivot column, the corresponding system is consistent. Moreover, it has one free variable, the variable corresponding to the 3rd column, therefore the system has $|\mathbb{Z}_{11}| = 11$ solutions.

Matrix A_2 is in Row Echelon Form and has 4 pivot columns: columns 1, 2, 3 and 4. Thus, since the last column is not a pivot column, the corresponding system is consistent. Moreover, all other columns are pivot columns, therefore the corresponding system has no free variables, and thus the system has a unique solution.

Matrix A_3 is not in Row Echelon Form, so we first have to find a REF for A_3 before we can determine the size of its solution set. We have

$$A_3 = \left(\begin{array}{cccccc:c} 2 & -3.5 & 17 & 0 & 9 & 1 & 2 \\ 0 & 2 & 0 & -4 & 10 & 21 & 5 \\ 0 & 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & -2 & 1 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 8 & 17 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cccccc:c} 2 & -3.5 & 17 & 0 & 9 & 1 & 2 \\ 0 & 2 & 0 & -4 & 10 & 21 & 5 \\ 0 & 0 & 0 & -2 & 1 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 8 & 17 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc:c} 2 & -3.5 & 17 & 0 & 9 & 1 & 2 \\ 0 & 2 & 0 & -4 & 10 & 21 & 5 \\ 0 & 0 & 0 & -2 & 1 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 17 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cccccc:c} 2 & -3.5 & 17 & 0 & 9 & 1 & 2 \\ 0 & 2 & 0 & -4 & 10 & 21 & 5 \\ 0 & 0 & 0 & -2 & 1 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 17 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{145}{51} \end{array}\right).$$

Thus, a Row Echelon Form of A_3 has a pivot in its last column, which implies that the system is inconsistent, or in other words that it has no solutions.

Problem 6. (a) We recall that $\mathbb{R}^4$ has dimension 4 over $\mathbb{R}$. In other words, it has a basis $\mathcal{B}_1$ of size 4.

We also recall that, as we showed in class, we cannot have a linearly independent subset T of $\mathbb{R}^4$ with size larger than the size of this basis. Thus, S_1 is not linearly independent.

We now check whether it is a spanning set of $\mathbb{R}^4$. Let's consider an arbitrary vector $\bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \in \mathbb{R}$. We need to show that $\bar{b} \in \text{span}(S_1)$, or equivalently that there exist $\lambda_i \in \mathbb{R}$, $1 \leqslant i \leqslant 5$, such that

$$\begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0.5 & 2 \\ 1 & -1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

We look at the coefficient matrix of this system:

$$\begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0.5 & 2 \\ 1 & -1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0.5 & 2 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0.5 & 2 \\ 0 & 0 & 0.5 & 1.25 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The last matrix is in REF and has 4 pivots, as many as its rows, therefore the augmented matrix of the above system will also have a Row Echelon Form with 4 pivots none of which will be in the last column, regardless of what b_1, b_2, b_3 and b_4 are. Thus the above system will be consistent, which shows that $\bar{b} \in \text{span}(S_1)$.

Since $\bar{b} \in \mathbb{R}^4$ was arbitrary, we conclude that S_1 is a spanning set of $\mathbb{R}^4$.

Next, we need to check whether S_2 is a linearly independent subset of $\mathbb{R}^{2\times 2}$. Suppose $\lambda_i \in \mathbb{R}$, $1 \leqslant i \leqslant 4$, are such that

$$\lambda_1 \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 3 \\ 1.5 & 4 \end{pmatrix} + \lambda_4 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we must have

$$\begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_4 & -2\lambda_2 + 3\lambda_3 \\ \lambda_1 + 1.5\lambda_3 + 2\lambda_4 & -\lambda_1 + 3\lambda_2 + 4\lambda_3 + \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

or equivalently

$$\left\{ \begin{array}{rcrcrcrcl} \lambda_1 & + & \lambda_2 & & & + & \lambda_4 & = & 0 \\ & & -2\lambda_2 & + & 3\lambda_3 & & & = & 0 \\ \lambda_1 & & & + & 1.5\lambda_3 & + & 2\lambda_4 & = & 0 \\ -\lambda_1 & + & 3\lambda_2 & + & 4\lambda_3 & + & \lambda_4 & = & 0 \end{array} \right\}.$$

We have

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 3 & 0 \\ 1 & 0 & 1.5 & 2 \\ -1 & 3 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & -1 & 1.5 & 1 \\ 0 & 4 & 4 & 2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 10 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 10 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The last matrix is in REF and has 4 pivots, therefore the above linear system has a unique solution, which is the trivial solution. This shows that S_2 is linearly independent.

We also recall that $\mathbb{R}^{2\times 2}$ has dimension 4 over $\mathbb{R}$. Therefore, as we showed in HW6, a linearly independent subset T of $\mathbb{R}^{2\times 2}$ with size 4 must be a basis too. In particular, S_2 is a basis of $\mathbb{R}^{2\times 2}$, and hence a spanning set of $\mathbb{R}^{2\times 2}$.

(b) We've already seen in part (a) that S_2 is a basis of $\mathbb{R}^{2\times 2}$ over $\mathbb{R}$, given that it is a linearly independent subset with size $4 = \dim_{\mathbb{R}} \mathbb{R}^{2\times 2}$.

We have also seen in part (a) that S_1 is a spanning set of $\mathbb{R}^4$ but it is not linearly independent. Going back to the coefficient matrix of the system

$$\begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0.5 & 2 \\ 1 & -1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix},$$

we recall that its coefficient matrix is row equivalent to the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0.5 & 2 \\ 0 & 0 & 0.5 & 1.25 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which has 4 pivot columns: columns 1, 2, 3 and 5.

Therefore, if we consider only these columns of the original matrix, we get that

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0.5 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This implies that the linear system

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

has only one solution, the trivial solution.

We can thus conclude that the subset

$$\widetilde{S}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

of S_1 is linearly independent. We also note that it has size $4 = \dim_{\mathbb{R}} \mathbb{R}^4$, thus, as we have seen in homework, it is a basis of $\mathbb{R}^4$ of the form we wanted.