Math 227
Suggested solutions to Homework Set 2

Problem 1. (a) We check all the axioms of a commutative ring for $\mathcal{R}_1$.

$+_1$ **is commutative:** Consider two real polynomials $p, q \in \mathcal{P}$. There are $n \in \mathbb{N}$ and coefficients $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n \in \mathbb{R}$ such that

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad q(x) = b_0 + b_1 x + \cdots + b_n x^n$$

(here we choose n to be larger than or equal to both the degree of the polynomial p and the degree of the polynomial q, so we may need to take some of the coefficients of higher-order terms equal to 0).

We then have

$$\begin{aligned} p(x) +_1 q(x) &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \\ &= (b_0 + a_0) + (b_1 + a_1)x + \cdots + (b_n + a_n)x^n = q(x) +_1 p(x), \end{aligned}$$

where the second equality holds because addition in $\mathbb{R}$ is commutative, and where in the first and third equalities we also use generalised associativity of the addition in $\mathbb{R}$ along with generalised commutativity.

$+_1$ **is associative:** Consider three real polynomials $p, q, r \in \mathcal{P}$. There are $n \in \mathbb{N}$ and coefficients $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n, c_0, c_1, \ldots, c_n \in \mathbb{R}$ such that

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad q(x) = b_0 + b_1 x + \cdots + b_n x^n,$$
$$r(x) = c_0 + c_1 x + \cdots + c_n x^n$$

(here we choose n to be larger than or equal to the maximum of the degrees of the polynomials p, q and r).

We then have

$$\begin{aligned} &(p(x) +_1 q(x)) +_1 r(x) \\ =&\big((a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n\big) + (c_0 x + c_1 x + \cdots + c_n x^n) \\ =&((a_0 + b_0) + c_0) + ((a_1 + b_1) + c_1)x + \cdots + ((a_n + b_n) + c_n)x^n \\ =&(a_0 + (b_0 + c_0)) + (a_1 + (b_1 + c_1))x + \cdots + (a_n + (b_n + c_n))x^n \\ =&p(x) +_1 (q(x) + r(x)), \end{aligned}$$

where we have used associativity of addition in $\mathbb{R}$ for the third equality, and generalised associativity and commutativity of the addition in $\mathbb{R}$ for the remaining equalities.

Neutral element of $+_1$: We check that this is the constant function $\mathbf{0}$. Consider a polynomial $p \in \mathcal{P}$. Then there are $n \in \mathbb{N}$ and coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that $p(x) = a_0 + a_1x + \cdots + a_nx^n$. We can then write

$$\begin{aligned} p(x) + \mathbf{0} &= (a_0 + a_1x + \cdots + a_nx^n) + (0 + 0x + \cdots + 0x^n) \\ &= (a_0 + 0) + (a_1 + 0)x + \cdots + (a_n + 0)x^n = p(x), \end{aligned}$$

where we used that 0 is the neutral element of addition in $\mathbb{R}$.

Since p is an arbitrary polynomial in $\mathcal{P}$, it follows that $\mathbf{0}$ is the neutral element of $+_1$.

Additive inverses: For every polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathcal{P}$, we have that

$$p(x) + \big((-a_0) + (-a_1)x + \cdots + (-a_n)x^n\big) = (a_0 - a_0) + (a_1 - a_1)x + \cdots + (a_n - a_n)x^n = \mathbf{0},$$

therefore $p(x)$ has an additive inverse.

$\cdot_1$ is commutative: Consider two real polynomials $p, q \in \mathcal{P}$. There are $n \in \mathbb{N}$ and coefficients $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n \in \mathbb{R}$ such that

$$p(x) = a_0 + a_1x + \cdots + a_nx^n, \qquad q(x) = b_0 + b_1x + \cdots + b_nx^n.$$

We then have

$$p(x) \cdot_1 q(x) = \sum_{k=0}^{2n} \left(\sum_{\substack{0 \leqslant i,j \leqslant n \\ i+j=k}} a_ib_j \right) x^k = \sum_{k=0}^{2n} \left(\sum_{\substack{0 \leqslant i,j \leqslant n \\ i+j=k}} b_ja_i \right) x^k = q(x) \cdot_1 p(x),$$

where we use that multiplication in $\mathbb{R}$ is commutative for the second equality, and we also use generalised commutativity and associativity of addition in $\mathbb{R}$ to express the product of two polynomials by writing a double sum as above.

$\cdot_1$ is associative: Consider three real polynomials $p, q, r \in \mathcal{P}$. There are $n \in \mathbb{N}$ and coefficients $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n, c_0, c_1, \ldots, c_n \in \mathbb{R}$ such that

$$p(x) = a_0 + a_1x + \cdots + a_nx^n, \qquad q(x) = b_0 + b_1x + \cdots + b_nx^n,$$
$$r(x) = c_0 + c_1x + \cdots + c_nx^n.$$

We then have

$$
\begin{aligned}
(p(x)\cdot_1 q(x))\cdot_1 r(x) &= \left(\sum_{k=0}^{2n}\left(\sum_{\substack{0\leqslant i,j\leqslant n\\ i+j=k}} a_ib_j\right)x^k\right)\cdot_1 (c_0+c_1x+\cdots+c_nx^n)\\
&= \sum_{l=0}^{3n}\left(\sum_{\substack{0\leqslant k\leqslant 2n,\,0\leqslant t\leqslant n\\ k+t=l}}\left(\sum_{\substack{0\leqslant i,j\leqslant n\\ i+j=k}} a_ib_j\right)\cdot c_t\right)x^l\\
&= \sum_{l=0}^{3n}\left(\sum_{\substack{0\leqslant i\leqslant n,\,0\leqslant k'\leqslant 2n\\ i+k'=l}} a_i\cdot\left(\sum_{\substack{0\leqslant j,t\leqslant n\\ j+t=k'}} b_jc_t\right)\right)x^l\\
&= p(x)\cdot_1 (q(x)\cdot_1 r(x)),
\end{aligned}
$$

where we use the associativity of multiplication in $\mathbb{R}$, as well as the distributive law, in order to get the third equality.

Neutral element of $\cdot_1$: We check that this is the constant function $\mathbf{1}$. Consider a polynomial $p\in\mathcal{P}$. Then there are $n\in\mathbb{N}$ and coefficients $a_0,a_1,\ldots,a_n\in\mathbb{R}$ such that $p(x)=a_0+a_1x+\cdots+a_nx^n$. We can then write

$$
\begin{aligned}
p(x)\cdot_1\mathbf{1} &= (a_0+a_1x+\cdots+a_nx^n)\cdot_1(1+0x+\cdots+0x^n)\\
&= (a_0\cdot 1)+(a_1\cdot 1)x+\cdots+(a_n\cdot 1)x^n = p(x),
\end{aligned}
$$

where we used that 1 is the neutral element of multiplication in $\mathbb{R}$.

Since p is an arbitrary polynomial in $\mathcal{P}$, it follows that $\mathbf{1}$ is the neutral element of $\cdot_1$.

Distributive law: We first note that it suffices to check either the left distributive property or the right distributive property, given that we have already verified that $\cdot_1$ is commutative; we check the right distributive property here.

Consider three real polynomials $p,q,r\in\mathcal{P}$. There are $n\in\mathbb{N}$ and coefficients $a_0,a_1,\ldots,a_n,b_0,b_1,\ldots,b_n,c_0,c_1,\ldots,c_n\in\mathbb{R}$ such that

$$
p(x)=a_0+a_1x+\cdots+a_nx^n,\qquad q(x)=b_0+b_1x+\cdots+b_nx^n,
$$
$$
r(x)=c_0+c_1x+\cdots+c_nx^n.
$$

We then have

$$
\begin{aligned}
&(p(x) +_1 q(x)) \cdot_1 r(x) \\
&= \big((a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n\big) \cdot_1 (c_0 + c_1 x + \cdots + c_n x^n) \\
&= \sum_{k=0}^{2n} \left(\sum_{\substack{0 \leqslant i,j \leqslant n \\ i+j=k}} (a_i + b_i) c_j \right) x^k \\
&= \sum_{k=0}^{2n} \left(\left(\sum_{\substack{0 \leqslant i,j \leqslant n \\ i+j=k}} a_i c_j \right) + \left(\sum_{\substack{0 \leqslant i,j \leqslant n \\ i+j=k}} b_i c_j \right) \right) x^k = p(x) \cdot_1 r(x) +_1 q(x) \cdot_1 r(x),
\end{aligned}
$$

where we use the distributive law in $\mathbb{R}$, as well as generalised commutativity and associativity of addition in $\mathbb{R}$, to get the third equality.

We conclude that $\mathcal{R}_1$ is a commutative ring.

(b) We check the ring axioms for addition and for multiplication in the case of $\mathcal{R}_2$ now.

We first note that addition of polynomials in $\mathcal{R}_2$ is the same operation as addition in $\mathcal{R}_1$, therefore the properties we verified before still hold:

- $+_2$ is commutative.
- $+_2$ is associative.
- There exists a neutral element of $+_2$, and it is the constant function $\mathbf{0}$.
- For every polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{P}$, we can define the polynomial $\widetilde{p}(x) = (-a_0) + (-a_1)x + \cdots + (-a_n)x^n$, and we can check that it is the additive inverse of $p(x)$: $p(x) +_2 \widetilde{p}(x) = \mathbf{0}$.

We now check the ring axioms concerning multiplication:

$\cdot_2$ **is associative:** Consider three real polynomials $p, q, r \in \mathcal{P}$. Then, for every $a \in \mathbb{R}$ we have

$$
\begin{aligned}
&\big((p \cdot_2 q) \cdot_2 r\big)(a) = \big((p \circ q) \circ r\big)(a) = (p \circ q)(r(a)) \\
&\quad = p\big(q(r(a))\big) = p\big((q \circ r)(a)\big) = \big(p \circ (q \circ r)\big)(a) = \big(p \cdot_2 (q \cdot_2 r)\big)(a),
\end{aligned}
$$

therefore the functions $(p \cdot_2 q) \cdot_2 r$ and $p \cdot_2 (q \cdot_2 r)$ coincide.

Neutral element of $\cdot_2$: We check that this is the polynomial $u(x) = x$. Consider a polynomial $p \in \mathcal{P}$. Then there are $n \in \mathbb{N}$ and coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that $p(x) = a_0 + a_1 x + \cdots + a_n x^n$. We can then write

$$(p \cdot_2 u)(x) = a_0 + a_1 u(x) + \cdots + a_n (u(x))^n = a_0 + a_1 x + \cdots + a_n x^n = p(x).$$

Similarly,

$$(u \cdot_2 p)(x) = u(p(x)) = p(x).$$

Since p is an arbitrary polynomial in $\mathcal{P}$, it follows that $u(x) = x$ is the neutral element of $\cdot_2$.

Finally, we check that the right distributive property holds true in $\mathcal{R}_2$, while the left distributive property fails.

Consider three real polynomials $p, q, r \in \mathcal{P}$. There are $n \in \mathbb{N}$ and coefficients $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n, c_0, c_1, \ldots, c_n \in \mathbb{R}$ such that

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad q(x) = b_0 + b_1 x + \cdots + b_n x^n,$$
$$r(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

We then have

$$\begin{aligned}
(p(x) +_2 q(x)) \cdot_2 r(x) &= ((a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n) \circ r(x) \\
&= (a_0 + b_0) + (a_1 + b_1) r(x) + \cdots + (a_n + b_n)(r(x))^n \\
&= \big(a_0 + a_1 r(x) + \cdots + a_n (r(x))^n\big) + \big(b_0 + b_1 r(x) + \cdots + b_n (r(x))^n\big) \\
&= (p \cdot_2 r)(x) + (q \cdot_2 r)(x),
\end{aligned}$$

which we can justify by viewing the coefficients as constant polynomials as well, and then viewing addition of them and multiplication with powers of $r(x)$ as the operations in $\mathcal{R}_1$ (whose properties we studied in part (a)); then, by the distributive law in $\mathcal{R}_1$ and by generalised commutativity and associativity of addition in $\mathcal{R}_1$, we get the third equality here.

Since the polynomials p, q and r that we considered are arbitrary, we conclude that the right distributive property holds in $\mathcal{R}$.

On the other hand, if we choose $p(x) = x^2$, $q(x) = x$, $r(x) = x + 1$, then

$$\begin{aligned}
p(x) \cdot_2 (q(x) +_2 r(x)) = p(x) \circ (q(x) +_2 r(x)) &= p(q(x) + r(x)) = p(2x + 1) \\
&= (2x + 1)^2 = 4x^2 + 4x + 1 \quad \neq \quad 2x^2 + 2x + 1 \\
&= (x)^2 + (x + 1)^2 = p(q(x)) + p(r(x)) \\
&= (p(x) \cdot_2 q(x)) +_2 (p(x) \cdot_2 r(x)).
\end{aligned}$$

This shows that the left distributive property does not hold for any three real polynomials in $\mathcal{P}$.

(c) We first deal with $\mathcal{R}_1$. We are looking for polynomials $p \in \mathcal{P}$ such that there exists $q \in \mathcal{P}$ with

$$p(x) \cdot_1 q(x) = q(x) \cdot_1 p(x) = \mathbf{1}; \tag{1}$$

in fact, since we have already confirmed that $\cdot_1$ is commutative, it suffices to ask for what polynomials p, q we have $p(x) \cdot_1 q(x) = \mathbf{1}$.

Clearly, $p(x)$ cannot be the constant function $\mathbf{0}$ (because then the product of $p(x)$ with any other polynomial in $\mathcal{P}$ would be equal to the constant function $\mathbf{0}$), so we can find $n \geqslant 0$ and coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$ so that

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Similarly, if a polynomial q exists such that (1) holds true, then q cannot be equal to the constant function $\mathbf{0}$. Thus we would be able to find $m \geqslant 0$ and coefficients $b_0, b_1, \ldots, b_m \in \mathbb{R}$ with $b_m \neq 0$ so that

$$q(x) = b_0 + b_1 x + \cdots + b_m x^m$$

and so that

$$\begin{aligned} p(x) \cdot_1 q(x) &= (a_0 + a_1 x + \cdots + a_n x^n) \cdot_1 (b_0 + b_1 x + \cdots + b_m x^m) \\ &= \sum_{k=0}^{n+m} \Bigg(\sum_{\substack{0 \leqslant i \leqslant n,\, 0 \leqslant j \leqslant m \\ i+j=k}} a_i b_j \Bigg) x^k. \end{aligned}$$

For the latter product to be equal to $\mathbf{1}$, we need $n = \deg(p) = 0$, and similarly $m = \deg(q) = 0$.

Therefore, the only polynomials that could have a multiplicative inverse in $\mathcal{R}_1$ are the non-zero constant polynomials.

We now check that every non-zero constant polynomial has a multiplicative inverse in $\mathcal{R}_1$, and thus that the invertible elements of $\mathcal{R}_1$ are precisely the elements of $\mathcal{P}_0 \setminus \{\mathbf{0}\}$.

Consider a non-zero constant polynomial $p \in \mathcal{P}$; then $p(x) = a_0$ with $a_0 \in \mathbb{R}$, $a_0 \neq 0$. But then, if we set $q(x) = a_0^{-1}$, this is another constant polynomial, for which we have

$$p(x) \cdot_1 q(x) = a_0 \cdot a_0^{-1} = 1.$$

Thus the non-zero constant polynomial we considered has a multiplicative inverse in $\mathcal{R}_1$, and since this was arbitrary the conclusion we wanted follows.

We now deal with $\mathcal{R}_2$. We are looking for polynomials $p \in \mathcal{P}$ such that there exists $q \in \mathcal{P}$ with

$$p(x) \cdot_2 q(x) = q(x) \cdot_2 p(x) = x. \tag{2}$$

We recall that this is equivalent to having

$$p(q(x)) = (p \circ q)(x) = (q \circ p)(x) = q(p(x)) = x.$$

We now note that, if either p or q is a constant polynomial, then $p \circ q$ or $q \circ p$ respectively will be a constant polynomial too. Therefore, we must have $\deg(p) \geqslant 1$ for p to potentially have a multiplicative inverse, and we should also have $\deg(q) \geqslant 1$ if q were to be the multiplicative inverse of p.

We can thus find $n, m \in \mathbb{N}$ and coefficients $a_0, a_1, \dots, a_n, b_0, b_1, \dots, b_m \in \mathbb{R}$ with $a_n b_m \neq 0$ and such that

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad q(x) = b_0 + b_1 x + \cdots + b_m x^m.$$

If either n or m were greater than 1, then we would have $\deg(p \circ q) = nm > 1$, and so it wouldn't be possible for $p \cdot_2 q = p \circ q$ to be equal to a degree 1 polynomial.

Therefore, the only polynomials that could have a multiplicative inverse in $\mathcal{R}_2$ are the polynomials of degree 1.

We now check that every polynomial of degree 1 has a multiplicative inverse in $\mathcal{R}_2$, and thus that the invertible elements of $\mathcal{R}_2$ are precisely the elements of $\mathcal{P}_1 \setminus \mathcal{P}_0$.

Consider a polynomial $p \in \mathcal{P}$ of degree 1; then $p(x) = a_0 + a_1 x$ with $a_0, a_1 \in \mathbb{R}$, $a_1 \neq 0$. But then, if we set $q(x) = \frac{-a_0}{a_1} + \frac{1}{a_1}x$, this is another polynomial of degree 1, for which we have

$$p(x) \cdot_2 q(x) = p(q(x)) = a_0 + a_1 q(x) = a_0 + a_1 \left(\tfrac{-a_0}{a_1} + \tfrac{1}{a_1}x\right) = x,$$

$$\text{and also} \quad q(x) \cdot_2 p(x) = q(p(x)) = \frac{-a_0}{a_1} + \frac{1}{a_1}(a_0 + a_1 x) = x.$$

Thus the polynomial p of degree 1 that we considered has a multiplicative inverse in $\mathcal{R}_2$, and since this was arbitrary the conclusion we wanted follows.

Summarising, we have verified that the elements of $\mathcal{P}$ with a multiplicative inverse in $\mathcal{R}_1$ are the polynomials in $\mathcal{P}_0 \setminus \{\mathbf{0}\}$, while those with a multiplicative inverse in $\mathcal{R}_2$ are the polynomials in $\mathcal{P}_1 \setminus \mathcal{P}_0$. Clearly, $\mathcal{P}_0 \setminus \{\mathbf{0}\} \neq \mathcal{P}_1 \setminus \mathcal{P}_0$ (in fact, they even have no common elements).

Problem 2. Consider two arbitrary vectors $\bar{x}, \bar{y} \in V_1$, and $r \in \mathbb{F}$.

We then have

$$\begin{aligned}
(\mu_1 f_1 + \mu_2 f_2)(\bar{x} + \bar{y}) &= (\mu_1 f_1)(\bar{x} + \bar{y}) + (\mu_2 f_2)(\bar{x} + \bar{y}) && \text{(by definition of sum of functions)} \\
&= \mu_1 \cdot f_1(\bar{x} + \bar{y}) + \mu_2 \cdot f_2(\bar{x} + \bar{y}) && \text{(by definition of scalar multiplication for functions)} \\
&= \mu_1 \cdot (f_1(\bar{x}) + f_1(\bar{y})) + \mu_2 \cdot (f_2(\bar{x}) + f_2(\bar{y})) && \text{(because } f_1, f_2 \text{ are linear)} \\
&= \big(\mu_1 \cdot f_1(\bar{x}) + \mu_1 \cdot f_1(\bar{y})\big) + \big(\mu_2 \cdot f_2(\bar{x}) + \mu_2 \cdot f_2(\bar{y})\big) \\
&= (\mu_1 f_1)(\bar{x}) + (\mu_1 f_1)(\bar{y}) + (\mu_2 f_2)(\bar{x}) + (\mu_2 f_2)(\bar{y}) \\
&= \big((\mu_1 f_1)(\bar{x}) + (\mu_2 f_2)(\bar{x})\big) + \big((\mu_1 f_1)(\bar{y}) + (\mu_2 f_2)(\bar{y})\big) \\
&= (\mu_1 f_1 + \mu_2 f_2)(\bar{x}) + (\mu_1 f_1 + \mu_2 f_2)(\bar{y}),
\end{aligned}$$

which shows the additivity of $\mu_1 f_1 + \mu_2 f_2$.

Moreover,

$$\begin{aligned}
(\mu_1 f_1 + \mu_2 f_2)(r\bar{x}) &= (\mu_1 f_1)(r\bar{x}) + (\mu_2 f_2)(r\bar{x}) && \text{(by definition of sum of functions)} \\
&= \mu_1 \cdot f_1(r\bar{x}) + \mu_2 \cdot f_2(r\bar{x}) && \text{(by definition of scalar multiplication for functions)} \\
&= \mu_1 \cdot (r \cdot f_1(\bar{x})) + \mu_2 \cdot (r \cdot f_2(\bar{x})) && \text{(because } f_1, f_2 \text{ are linear)} \\
&= (\mu_1 r) \cdot f_1(\bar{x}) + (\mu_2 r) \cdot f_2(\bar{x}) \\
&= (r\mu_1) \cdot f_1(\bar{x}) + (r\mu_2) \cdot f_2(\bar{x}) \\
&= r \cdot (\mu_1 \cdot f_1(\bar{x})) + r \cdot (\mu_2 \cdot f_2(\bar{x})) \\
&= r \cdot \big((\mu_1 f_1)(\bar{x}) + (\mu_2 f_2)(\bar{x})\big) \\
&= r \cdot (\mu_1 f_1 + \mu_2 f_2)(\bar{x}).
\end{aligned}$$

Combining the above, we conclude that $\mu_1 f_1 + \mu_2 f_2$ is a linear map from V_1 to V_2.

Problem 3. We first use Gaussian elimination to find a Row Echelon Form of A:

$$
A = \begin{pmatrix} 7 & 2 & 0 & 1 & 4 \\ 0 & 0 & 3 & 2 & 6 \\ 10 & 6 & 9 & 9 & 8 \\ 0 & 0 & 5 & 7 & 0 \\ 3 & 4 & 4 & 1 & 5 \end{pmatrix}
\xrightarrow{\substack{8R_1,\,10R_3 \\ 4R_5}}
\begin{pmatrix} 1 & 5 & 0 & 8 & 10 \\ 0 & 0 & 3 & 2 & 6 \\ 1 & 5 & 2 & 2 & 3 \\ 0 & 0 & 5 & 7 & 0 \\ 1 & 5 & 5 & 4 & 9 \end{pmatrix}
$$

$$
\xrightarrow{\substack{R_3 - R_1 \to R_3' \\ R_5 - R_1 \to R_5'}}
\begin{pmatrix} 1 & 5 & 0 & 8 & 10 \\ 0 & 0 & 3 & 2 & 6 \\ 0 & 0 & 2 & 5 & 4 \\ 0 & 0 & 5 & 7 & 0 \\ 0 & 0 & 5 & 7 & 10 \end{pmatrix}
\xrightarrow{\substack{4R_2,\,6R_3 \\ R_5 - R_4 \to R_5'}}
\begin{pmatrix} 1 & 5 & 0 & 8 & 10 \\ 0 & 0 & 1 & 8 & 2 \\ 0 & 0 & 1 & 8 & 2 \\ 0 & 0 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{pmatrix}
$$

$$
\xrightarrow{\substack{R_3 - R_2 \to R_3' \\ 9R_4,\,10R_5}}
\begin{pmatrix} 1 & 5 & 0 & 8 & 10 \\ 0 & 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\xrightarrow{R_4 - R_2 \to R_4'}
\begin{pmatrix} 1 & 5 & 0 & 8 & 10 \\ 0 & 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
\xrightarrow{R_3 \leftrightarrow R_5}
\begin{pmatrix} 1 & 5 & 0 & 8 & 10 \\ 0 & 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\xrightarrow{R_5 - 9R_4 \to R_5'}
A' = \begin{pmatrix} 1 & 5 & 0 & 8 & 10 \\ 0 & 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
$$

The last matrix, matrix A', is in Row Echelon Form, and has 3 pivots. Therefore, we can conclude that $\mathrm{rank}(A) = 3$.

Moreover, we see that the pivot columns of A' are $C_1(A')$, $C_3(A')$ and $C_5(A')$, and we recall (or alternatively can check directly by doing back substitution) that these form a maximal linearly independent subset of the columns of A'. Therefore, as we discussed in class (in fact, as we saw in the proof of Main Theorem C), a maximal linearly independent subset of the columns of A, and hence a basis of $\mathrm{CS}(A)$, is the set

$$
\{C_1(A), C_3(A), C_5(A)\} = \left\{ \begin{pmatrix} 7 \\ 0 \\ 10 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 9 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 8 \\ 0 \\ 5 \end{pmatrix} \right\}.
$$

On the other hand, we recall that a basis for $\mathrm{RS}(A)$ is the set of non-zero rows of A'. Given though that here we want to find a basis formed from rows of A, we proceed in a similar way to above by considering A^T instead and by finding a Row Echelon Form of it:

$$A^T = \begin{pmatrix} 7 & 0 & 10 & 0 & 3 \\ 2 & 0 & 6 & 0 & 4 \\ 0 & 3 & 9 & 5 & 4 \\ 1 & 2 & 9 & 7 & 1 \\ 4 & 6 & 8 & 0 & 5 \end{pmatrix} \underbrace{\overset{8R_1,\,6R_2}{\underset{3R_5}{}}}\quad \begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & 3 & 9 & 5 & 4 \\ 1 & 2 & 9 & 7 & 1 \\ 1 & 7 & 2 & 0 & 4 \end{pmatrix}$$

$$\underbrace{\overset{R_2-R_1,\,R_4-R_1}{\underset{R_5-R_1}{}}}\quad \begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 9 & 5 & 4 \\ 0 & 2 & 6 & 7 & 10 \\ 0 & 7 & 10 & 0 & 2 \end{pmatrix} \underbrace{\overset{R_2\leftrightarrow R_5}{\underset{4R_3,\,6R_4}{}}}\quad \begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 7 & 10 & 0 & 2 \\ 0 & 1 & 3 & 9 & 5 \\ 0 & 1 & 3 & 9 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{\overset{R_4-R_3\to R_4'}{\underset{8R_2}{}}}\quad \begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 1 & 3 & 9 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \underbrace{R_3 - R_2 \to R_3'}\quad B = \begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The last matrix, matrix B, is in Row Echelon Form. Its pivot columns are $C_1(B)$, $C_2(B)$ and $C_4(B)$. Therefore, analogously to what we noted previously, a maximal linearly independent subset of the columns of A^T, and hence a basis of $\mathrm{CS}(A^T) = \mathrm{RS}(A)$, is the set

$$\{R_1(A), R_2(A), R_4(A)\} = \{C_1(A^T), C_2(A^T), C_4(A^T)\} = \left\{ \begin{pmatrix} 7\\2\\0\\1\\4 \end{pmatrix}, \begin{pmatrix} 0\\0\\3\\2\\6 \end{pmatrix}, \begin{pmatrix} 0\\0\\5\\7\\0 \end{pmatrix} \right\}.$$

Finally, by Main Theorem D we know that

$$\mathrm{nullity}(A) = \#\{\text{columns of } A\} \; - \; \mathrm{rank}(A) = 5 - 3 = 2.$$

We also recall that the linear system $A\bar{x} = \bar{0}$ is equivalent to the linear system $A'\bar{x} = \bar{0}$, and that Nullspace(A) coincides with the common solution set of these systems.

To find a basis for Nullspace(A), or equivalently for the solution set of the linear system $A'\bar{x} = \bar{0}$, we note that this system has 2 free variables, the variables x_2 and x_4. We also recall that a basis for the solution set of $A'\bar{x} = \bar{0}$ can be chosen to be the subset of solutions we get when we set one of the free variables equal to 1 and the remaining free variables equal to 0.

In this case, we can set

- $x_2 = 1, x_4 = 0$, which corresponds to the solution $\bar{x} = \begin{pmatrix} 6\\1\\0\\0\\0 \end{pmatrix}$;

- $x_2 = 0, x_4 = 1$, which corresponds to the solution $\bar{x} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}$.

We conclude that a basis for Nullspace(A) is the set

$$\left\{ \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Problem 4. (a) For every matrix $\left(\begin{smallmatrix} r_1 & r_2 \\ r_3 & r_4 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$,

$$f\left(\begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}\right) = \begin{pmatrix} 3 & 2 \\ -2 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} = \begin{pmatrix} 3r_1 + 2r_3 & 3r_2 + 2r_4 \\ -2r_1 + r_3 & -2r_2 + r_4 \\ 4r_3 & 4r_4 \end{pmatrix}.$$

We first note that f is **not** surjective, that is, $\mathrm{Range}(f) \neq \mathbb{R}^{3\times 2}$. Indeed, consider the matrix

$$\begin{pmatrix} 3 & 3 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 2},$$

and suppose that there were a matrix $\left(\begin{smallmatrix} r_1 & r_2 \\ r_3 & r_4 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ such that

$$\begin{pmatrix} 3r_1 + 2r_3 & 3r_2 + 2r_4 \\ -2r_1 + r_3 & -2r_2 + r_4 \\ 4r_3 & 4r_4 \end{pmatrix} = f\left(\begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}\right) = \begin{pmatrix} 3 & 3 \\ 2 & 2 \\ 0 & 0 \end{pmatrix}.$$

We would then need $r_3 = r_4 = 0$, which would then imply

$$3r_1 = 3,\ 3r_2 = 3,\ -2r_1 = 2,\ -2r_2 = 2 \quad \Rightarrow \quad r_1 = 1 = r_2 \text{ but also } r_1 = -1 = r_2.$$

Since the conclusions contradict each other, we obtain that there is no matrix $\left(\begin{smallmatrix} r_1 & r_2 \\ r_3 & r_4 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ whose image under f is the matrix $\left(\begin{smallmatrix} 3 & 3 \\ 2 & 2 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{R}^{3\times 2}$.

Alternatively, we could use Main Theorem E to conclude that f is not surjective: the theorem tells us that

$$\dim_{\mathbb{R}} \mathrm{Range}(f) \leqslant \dim_{\mathbb{R}} \mathrm{Range}(f) + \dim_{\mathbb{R}} \mathrm{Ker}(f) = \dim_{\mathbb{R}} \mathbb{R}^{2\times 2} = 4,$$

while $\dim_{\mathbb{R}} \mathbb{R}^{3\times 2} = 6$. Thus we **cannot** have $\dim_{\mathbb{R}} \mathrm{Range}(f) = \dim_{\mathbb{R}} \mathbb{R}^{3\times 2}$, and hence we **cannot** have $\mathrm{Range}(f) = \mathbb{R}^{3\times 2}$.

We now check that f is injective:

$$\begin{pmatrix} 3r_1 + 2r_3 & 3r_2 + 2r_4 \\ -2r_1 + r_3 & -2r_2 + r_4 \\ 4r_3 & 4r_4 \end{pmatrix} = f\left(\begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad r_3 = r_4 = 0, \quad \text{which in turn imply } 3r_1 = 3r_1 + 2r_3 = 0$$
$$\text{and } 3r_2 = 3r_2 + 2r_4 = 0 \quad \Rightarrow \quad r_1 = r_2 = 0 \text{ as well.}$$

Therefore, $\mathrm{Ker}(f) = \{\boldsymbol{O}\}$, where $\boldsymbol{O}$ is the zero matrix in $\mathbb{R}^{2\times 2}$.

Clearly, a basis for $\mathrm{Ker}(f)$ is the empty set $\emptyset$. As for $\mathrm{Range}(f)$, we know that the arbitrary image of f is of the form

$$\begin{aligned}
&\begin{pmatrix} 3r_1+2r_3 & 3r_2+2r_4 \\ -2r_1+r_3 & -2r_2+r_4 \\ 4r_3 & 4r_4 \end{pmatrix} \\
&= \begin{pmatrix} 3r_1 & 0 \\ -2r_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 3r_2 \\ 0 & -2r_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2r_3 & 0 \\ r_3 & 0 \\ 4r_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2r_4 \\ 0 & r_4 \\ 0 & 4r_4 \end{pmatrix} \\
&= r_1 \begin{pmatrix} 3 & 0 \\ -2 & 0 \\ 0 & 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 & 3 \\ 0 & -2 \\ 0 & 0 \end{pmatrix} + r_3 \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 0 \end{pmatrix} + r_4 \begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 0 & 4 \end{pmatrix}.
\end{aligned}$$

Therefore,

$$\mathrm{Range}(f) = \mathrm{span}\left(\left\{ \left(\begin{smallmatrix} 3 & 0 \\ -2 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 3 \\ 0 & -2 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 2 \\ 0 & 1 \\ 0 & 4 \end{smallmatrix}\right) \right\}\right).$$

We finally check that the spanning set for $\mathrm{Range}(f)$ that we just found is linearly independent too: if we have $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4 \in \mathbb{R}$ such that

$$\lambda_1 \begin{pmatrix} 3 & 0 \\ -2 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 3 \\ 0 & -2 \\ 0 & 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
\text{then} \quad & \begin{pmatrix} 3\lambda_1+2\lambda_3 & 3\lambda_2+2\lambda_4 \\ -2\lambda_1+\lambda_3 & -2\lambda_2+\lambda_4 \\ 4\lambda_3 & 4\lambda_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\Rightarrow \quad & f\left(\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\Rightarrow \quad & \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{since we showed that } f \text{ is injective} \\
\Rightarrow \quad & \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.
\end{aligned}$$

We conclude that $\left\{ \left(\begin{smallmatrix} 3 & 0 \\ -2 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 3 \\ 0 & -2 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 2 \\ 0 & 1 \\ 0 & 4 \end{smallmatrix}\right) \right\}$ is a basis for $\mathrm{Range}(f)$.

(b) We first look for a matrix representation for g: we want a matrix $B \in \mathbb{Z}_7^{3\times 3}$ satisfying

$$g\left(\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)\right) = B\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)$$

for every $\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) \in \mathbb{Z}_7^3$. We can verify that

$$B = \left(\, g(\bar{e}_1) \, \vdots \, g(\bar{e}_2) \, \vdots \, g(\bar{e}_3) \, \right) = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix}$$

is the matrix we want.

We now remark that, since $\dim_{\mathbb{Z}_7} \mathrm{Dom}(g) = \dim_{\mathbb{Z}_7} \mathrm{Codomain}(g) = 3$, Main Theorem E implies that g is injective if and only if g is surjective (see e.g. discussion from the January 23 Recitation file), so it suffices to check one of the two properties.

We check whether g is injective: this is equivalent to checking that the linear system

$$B\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$$

has only one solution, the trivial solution, which in this case is equivalent to B being invertible.

To check whether B is invertible, we use Gauss-Jordan elimination:

$$B = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 5 \\ 0 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

We thus see that a REF of B has 3 pivots (as many as its rows or columns), therefore B is invertible. Going back, this implies that g is injective, and therefore surjective too.

We conclude that $\mathrm{Ker}(g) = \{\bar{0}\}$ and $\mathrm{Range}(g) = \mathbb{Z}_7^3$, and also that g is bijective.

Clearly, a basis for $\mathrm{Ker}(g)$ is the empty set $\emptyset$, while a basis for $\mathrm{Range}(g) = \mathbb{Z}_7^3$ is the standard basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ of $\mathbb{Z}_7^3$.

Finally, we find the inverse of g by noting that the matrix representation of g^{-1} would be the matrix B^{-1} (see e.g. the discussion from the January 23 Recitation file). We start again from the Gauss-Jordan elimination steps we did before:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 5 & 5 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 5 & 5 & 1 & 0 \\ 0 & 0 & 6 & 6 & 4 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 5 & 5 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 3 & 6 & 5 \\ 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 1 & 3 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 6 & 4 \\ 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 1 & 3 & 6 \end{array}\right).$$

Therefore,

$$B^{-1} = \begin{pmatrix} 3 & 6 & 4 \\ 0 & 0 & 5 \\ 1 & 3 & 6 \end{pmatrix}$$
$$\text{and} \quad g^{-1}\left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = B^{-1}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3y_1 + 6y_2 + 4y_3 \\ 5y_3 \\ y_1 + 3y_2 + 6y_3 \end{pmatrix}$$

for every $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{Z}_7^3$.

Problem 5. We recall that A is invertible if and only if the linear system $A\bar{x} = \bar{0}$, where $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\bar{0} \in \mathbb{F}^n$, has only the trivial solution.

If we now assume that A is not invertible, then the linear system $A\bar{x} = \bar{0}$ has a non-zero solution. Therefore, there is a non-zero vector $\bar{u} \in \mathbb{F}^n$ such that

$$A\bar{u} = \bar{0} = 0 \cdot \bar{u}.$$

This shows that $\bar{u}$ is an eigenvector of A corresponding to eigenvalue 0.

Conversely, if we assume that 0 is an eigenvalue of A, then we know that there exists a non-zero vector $\bar{v} \in \mathbb{F}^n$ such that $A\bar{v} = 0 \cdot \bar{v} = \bar{0}$. But then the linear system $A\bar{x} = \bar{0}$ has more than one solutions, which shows that A is not invertible.

Combining the above, we get that

$$A \text{ is not invertible} \quad \text{ if and only if } \quad 0 \text{ is an eigenvalue of } A.$$

By taking contrapositives of the two implications forming this equivalence, we get the equivalence we wanted.

Problem 6. By our assumption, there exists a non-zero vector $\bar{u} \in \mathbb{R}^n$ such that

$$A\bar{u} = \lambda \cdot \bar{u}.$$

We now have

1. $(2A)\bar{u} = 2(A\bar{u}) = 2(\lambda \cdot \bar{u}) = (2\lambda) \cdot \bar{u}$, therefore $\bar{u}$ is an eigenvector of $2A$ corresponding to eigenvalue 2λ.

2. $(A + I_n)\bar{u} = A\bar{u} + I_n\bar{u} = \lambda \cdot \bar{u} + \bar{u} = \lambda \cdot \bar{u} + 1 \cdot \bar{u} = (\lambda + 1) \cdot \bar{u}$, which shows that $\bar{u}$ is an eigenvector of $A + I_n$ corresponding to eigenvalue $\lambda + 1$.

3. $(A^2)\bar{u} = (A \cdot A)\bar{u} = A(A\bar{u}) = A(\lambda \cdot \bar{u}) = \lambda \cdot (A\bar{u}) = \lambda \cdot (\lambda \cdot \bar{u}) = \lambda^2 \cdot \bar{u}$, which shows that $\bar{u}$ is an eigenvector of A^2 corresponding to eigenvalue λ^2.

4. Similarly to part (3), and starting from it and using mathematical induction, we can show that, for every $k > 2$, $\bar{u}$ is an eigenvector of A^k corresponding to eigenvalue λ^k. But then, for every $m \geqslant 0$ and for every set of coefficients $b_0, b_1, \ldots, b_m \in \mathbb{R}$, we have for the polynomial $p(x) = b_0 + b_1x + \cdots + b_mx^m$ that

$$\begin{aligned}
&(p(A))\bar{u} = \big(b_mA^m + b_{m-1}A^{m-1} + \cdots + b_1A + b_0I_n\big)\bar{u} \\
&= (b_mA^m)\bar{u} + (b_{m-1}A^{m-1})\bar{u} + \cdots + (b_1A)\bar{u} + (b_0I_n)\bar{u} \\
&= b_m\big(A^m\bar{u}\big) + b_{m-1}\big(A^{m-1}\bar{u}\big) + \cdots + b_1(A\bar{u}) + b_0 \cdot \bar{u} \\
&= b_m \cdot (\lambda^m \cdot \bar{u}) + b_{m-1} \cdot (\lambda^{m-1} \cdot \bar{u}) + \cdots + b_1 \cdot (\lambda \cdot \bar{u}) + b_0 \cdot \bar{u} \\
&= (b_m\lambda^m) \cdot \bar{u} + (b_{m-1}\lambda^{m-1}) \cdot \bar{u} + \cdots + (b_1\lambda) \cdot \bar{u} + b_0 \cdot \bar{u} \\
&= \big(b_m\lambda^m + b_{m-1}\lambda^{m-1} + \cdots + b_1\lambda + b_0\big) \cdot \bar{u} = p(\lambda) \cdot \bar{u}.
\end{aligned}$$

   Thus, $\bar{u}$ is an eigenvector of the matrix $p(A)$ corresponding to eigenvalue $p(\lambda)$.

5. By the assumption,

$$\begin{aligned}
&A\bar{u} = \lambda \cdot \bar{u} \quad \Rightarrow \quad \tfrac{1}{\lambda}(A\bar{u}) = \bar{u} \\
\Rightarrow \quad & A\left(\tfrac{1}{\lambda} \cdot \bar{u}\right) = \bar{u} \quad \Rightarrow \quad \tfrac{1}{\lambda} \cdot \bar{u} = A^{-1}\left(A\left(\tfrac{1}{\lambda} \cdot \bar{u}\right)\right) = A^{-1}\bar{u}.
\end{aligned}$$

   Thus, $\bar{u}$ is an eigenvector of the matrix A^{-1} corresponding to eigenvalue $1/\lambda$.