MATH 227 March 9 (1)

Recall: Linear functions and Bases

Proposition 1 Let $\mathbb{F}$ be a field and let V_1, V_2 be vector spaces over $\mathbb{F}$. Suppose $\mathcal{B} \subseteq V_1$ is a basis of V_1, and consider a function

$$\varphi : \mathcal{B} \longrightarrow V_2.$$

Then there is a unique linear map $f : V_1 \to V_2$ that extends φ (in other words, satisfying

$$\text{for every } \bar{u} \in \mathcal{B} \quad f(\bar{u}) = \varphi(\bar{u})).$$

Question What can the function φ tell us about whether the linear map f has particularly nice properties or not?

Proposition 2 (given as a HW4 Problem) Let $\mathbb{F}, V_1, V_2$ and $\mathcal{B} \subseteq V_1$ be as before, and consider functions

$$\varphi, \psi : \mathcal{B} \longrightarrow V_2.$$

Write f and g for the unique linear extensions from V_1 to V_2 of φ and ψ respectively.

(I) f is injective if and only if

φ is injective and the set $\{\varphi(\bar{u}) : \bar{u} \in \mathcal{B}\}$ is a linearly independent subset of V_2

(or in other words if and only if any two images of φ (corresponding to different inputs) are different and the set of images of φ is a linearly independent subset of V_2).

(II) g is surjective if and only if

the set $\{\psi(\bar{u}) : \bar{u} \in \mathcal{B}\}$ is a spanning set of V_2.

Corollary 1 Let $\mathbb{F}, V_1, V_2$ and $\mathcal{B} \subseteq V_1$ be as before and consider $\varphi : \mathcal{B} \to V_2$ and its unique linear extension $f : V_1 \to V_2$.

We have that

f is a linear isomorphism (that is, it is a bijection)

if and only if

the set $\mathcal{C} = \{\varphi(\bar{u}) : \bar{u} \in \mathcal{B}\}$ is a basis of V_2 and $\varphi : \mathcal{B} \to \mathcal{C}$ is a bijection.

Corollary 2 Let $\mathbb{F}$ be a field, and let V_1, V_2 be vector spaces over $\mathbb{F}$. Then V_1, V_2 are isomorphic if and only if their dimensions over $\mathbb{F}$ are equal.

Proof Assume first that

$$\dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} V_2.$$

We need to define an isomorphism $f : V_1 \to V_2$, or in other words a linear map $f : V_1 \to V_2$ that is also bijective.

Consider a basis $\mathcal{B}$ of V_1 and a basis $\mathcal{C}$ of V_2.

Since $\dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} V_2$, the sets $\mathcal{B}$ and $\mathcal{C}$ have the same size (the same cardinality), and thus there exists a bijection $\varphi : \mathcal{B} \to \mathcal{C}$.

Clearly, we can also view φ as a function from $\mathcal{B}$ to V_2.

But then, by Proposition 1, there is a unique linear map $f : V_1 \to V_2$ extending φ.

Also, by Corollary 1 to Proposition 2, f is bijective, and thus a linear isomorphism.

We conclude that V_1 and V_2 are isomorphic in this case.

Assume now that we know V_1 and V_2 are isomorphic. Then by definition there exists a bijective linear map $f : V_1 \to V_2$.

Also we can find a basis $\mathcal{B}$ of V_1.
From ℓ we can obtain a function $\varphi : \mathcal{B} \to V_2$ by setting $\varphi(\bar{v}) := \ell(\bar{v})$ for every $\bar{v} \in \mathcal{B}$.
But then ℓ is a linear map from V_1 to V_2 that extends φ, thus ℓ is the unique linear extension of φ.
We then obtain by Corollary 1 that,

since ℓ is a linear isomorphism,
the set $\mathcal{C} = \{\varphi(\bar{v}) : \bar{v} \in \mathcal{B}\}$ must be a basis of V_2
and $\varphi : \mathcal{B} \to \mathcal{C}$ must be a bijection.

This shows that $\mathcal{B}$ and $\mathcal{C}$ have the same size, or in other words that V_1 and V_2 have bases of the same size.
We conclude that $\dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} V_2$.

Corollary 3 (Special case of Corollary 2) Let $\mathbb{F}$ be a field. Every finite-dimensional vector space V over $\mathbb{F}$ is isomorphic to some space $\mathbb{F}^k$ for some $k \geq 1$, or V has dimension 0.

Proof Either $\dim_{\mathbb{F}} V = 0$ or it is equal to some positive integer k (since V is assumed to be finite-dimensional).
In the latter case, V and $\mathbb{F}^k$ have the same dimension over $\mathbb{F}$, thus by Corollary 2 we can find a linear isomorphism $\ell : V \to \mathbb{F}^k$.

Recall from last times

Proposition 1 Let $\mathbb{F}$ be a field and let V_1, V_2 be vector spaces over $\mathbb{F}$. Suppose $B \subseteq V_1$ is a basis of V_1, and consider a function
$$\varphi : B \to V_2.$$
Then there is a unique linear map $\ell : V_1 \to V_2$ that extends φ (in other words, satisfying $\ell(\bar{u}) = \varphi(\bar{u})$ for every $\bar{u} \in B$).

Based on this Proposition and Proposition 2 we obtained:

Corollary Let V_1, V_2 be two vector spaces over the same field $\mathbb{F}$. Then V_1 and V_2 are isomorphic if and only if
$$\dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} V_2.$$
In particular, if V_1 is finite-dimensional, then either $\dim_{\mathbb{F}} V_1 = 0$ or V_1 is isomorphic to some $\mathbb{F}^k$ for some $k \geq 1$.

The corollary allows us to give a full proof of Main Thm E in the case of finite-dimensional spaces.

Main Theorem E Let V_1, V_2 be vector spaces over a field $\mathbb{F}$. Consider a linear map $g : V_1 \to V_2$. Then
$$\dim_{\mathbb{F}} \text{Range}(g) + \dim_{\mathbb{F}} \text{Ker}(g) = \dim_{\mathbb{F}} V_1.$$

Proof of Main Thm E in the case that V_1 and V_2 are both finite-dimensional:

Assume $\dim_{\mathbb{F}} V_1 < \infty$ and $\dim_{\mathbb{F}} V_2 < \infty$.

We first deal with the very simple cases where one of V_1, V_2 has dimension 0.

If $\dim_{\mathbb{F}} V_1 = 0$, then $V_1 = \{\bar{0}_{V_1}\}$, and hence $\text{Range}(g) = \{\bar{0}_{V_2}\}$ and $\text{Ker}(g) = V_1$. Thus
$$\dim_{\mathbb{F}} \text{Range}(g) + \dim_{\mathbb{F}} \text{Ker}(g) = 0 + 0 = 0 = \dim_{\mathbb{F}} V_1$$

in this case.

If $\dim_{\mathbb{F}} V_2 = 0$, then $V_2 = \{\bar{0}_{V_2}\}$ and hence $\mathrm{Range}(g) = V_2 = \{\bar{0}_{V_2}\}$ and $\mathrm{Ker}(g) = V_1$. Thus

$$\dim_{\mathbb{F}} \mathrm{Range}(g) + \dim_{\mathbb{F}} \mathrm{Ker}(g) = 0 + \dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} V_1.$$

Finally, assume that $\dim_{\mathbb{F}} V_1 = n$ for some $n \geq 1$ and $\dim_{\mathbb{F}} V_2 = m$ for some $m \geq 1$. Then $V_1 \cong \mathbb{F}^n$ and $V_2 \cong \mathbb{F}^m$, and thus we can find bijective linear maps $h_1 : V_1 \to \mathbb{F}^n$ and $h_2 : V_2 \to \mathbb{F}^m$.

Consider now the function $\ell = h_2 \circ g \circ h_1^{-1} : \mathbb{F}^n \to \mathbb{F}^m$. Then we have seen that this ℓ is linear.

Recall also that we have already proved Main Thm E for linear maps from $\mathbb{F}^n$ to $\mathbb{F}^m$, and thus we have

$$\dim_{\mathbb{F}} \mathrm{Range}(\ell) + \dim_{\mathbb{F}} \mathrm{Ker}(\ell) = \dim_{\mathbb{F}} \mathbb{F}^n = \dim_{\mathbb{F}} V_1.$$

Therefore, we can get the desired conclusion if we now show that

$$\dim_{\mathbb{F}} \mathrm{Range}(\ell) = \dim_{\mathbb{F}} \mathrm{Range}(g),$$
$$\dim_{\mathbb{F}} \mathrm{Ker}(\ell) = \dim_{\mathbb{F}} \mathrm{Ker}(g).$$

For what $\bar{z} \in \mathbb{F}^m$ do we have $\bar{z} \in \mathrm{Range}(\ell)$?

There must exist $\bar{x} \in \mathbb{F}^n$ such that

$$\bar{z} = \ell(\bar{x}) = (h_2 \circ g \circ h_1^{-1})(\bar{x}) = (h_2 \circ g)(h_1^{-1}(\bar{x})) = h_2(g(h_1^{-1}(\bar{x})))$$

$\Rightarrow$ there must exist some $\bar{y} \in V_1$, $\bar{y} = h_1^{-1}(\bar{x})$, such that

$$\bar{z} = h_2(g(\bar{y}))$$

$\Rightarrow$ there must exist some $\bar{w} \in \mathrm{Range}(g)$, $\bar{w} = g(\bar{y})$, such that

$$\bar{z} = h_2(\bar{w}).$$

Thus $\bar{z} \in \{h_2(\bar{w}) : \bar{w} \in \mathrm{Range}(g)\}$ and $\mathrm{Range}(\ell) \subseteq \{h_2(\bar{w}) : \bar{w} \in \mathrm{Range}(g)\}$

Conversely, let $\bar{u} \in Range(g)$. Then there exists $\bar{v} \in V_1$ such that $\bar{u} = g(\bar{v})$, and thus we can write

$$h_2(\bar{u}) = h_2(g(\bar{v})) = (h_2 \circ g)(\bar{v}) = h_2 \circ g\left(h_1^{-1}\left(\underbrace{h_1(\bar{v})}_{\text{element of } \mathbb{F}^n}\right)\right)$$

$$= (h_2 \circ g \circ h_1^{-1})(h_1(\bar{v})) \in \underset{\substack{\shortparallel \\ Range(\ell)}}{Range(h_2 \circ g \circ h_1^{-1})}.$$

Thus $\{h_2(\bar{u}) : \bar{u} \in Range(g)\} \subseteq Range(\ell)$, and we can conclude that $\{h_2(\bar{u}) : \bar{u} \in Range(g)\} = Range(\ell)$.

But then $h_{2|Range(g)} : Range(g) \longrightarrow Range(\ell)$

subspace of V_2

is a well-defined linear map, and it is surjective. It is also injective as the restriction to a subspace of V_2 of the injective function $h_2 : V_2 \longrightarrow \mathbb{F}^m$.

Thus $h_{2|Range(g)} : Range(g) \longrightarrow Range(\ell)$ is a linear isomorphism, and thus by the Corollary we know that $\dim_{\mathbb{F}} Range(g) = \dim_{\mathbb{F}} Range(\ell)$.

Similarly, we try to find a linear isomorphism from $Ker(\ell)$ to $Ker(g)$.

For what $\bar{y} \in \mathbb{F}^n$ do we have $\bar{y} \in Ker(\ell)$?
We must have $(h_2 \circ g \circ h_1^{-1})(\bar{y}) = \ell(\bar{y}) = \bar{0}_{\mathbb{F}^m}$.

But then $(h_2 \circ g)(h_1^{-1}(\bar{y})) = \bar{0}_{\mathbb{F}^m} \Rightarrow h_2(g(h_1^{-1}(\bar{y}))) = \bar{0}_{\mathbb{F}^m} \Rightarrow$
$g(h_1^{-1}(\bar{y})) = h_2^{-1}(\bar{0}_{\mathbb{F}^m}) = \bar{0}_{V_2} \Rightarrow h_1^{-1}(\bar{y}) \in Ker(g)$.

Thus $\{h_1^{-1}(\bar{q}) : \bar{q} \in Ker(\ell)\} \subseteq Ker(g)$.

Conversely, if $\bar{u} \in Ker(g)$, then $g(\bar{u}) = \bar{0}_{V_2} \Rightarrow (h_2 \circ g)(\bar{u}) = h_2(g(\bar{u})) = h_2(\bar{0}_{V_2}) = \bar{0}_{\mathbb{F}^m} \Rightarrow$

$$(h_2 \circ g)(h_1^{-1}(h_1(\bar{u}))) = \bar{0}_{\mathbb{F}^m} \Rightarrow (h_2 \circ g \circ h_1^{-1})(h_1(\bar{u})) = \bar{0}_{\mathbb{F}^m}$$

$\Rightarrow h_1(\bar{u}) \in Ker(h_2 \circ g \circ h_1^{-1}) = Ker(\ell)$ and hence
$\bar{u} = h_1^{-1}(h_1(\bar{u})) \in \{h_1^{-1}(\bar{q}) : \bar{q} \in Ker(\ell)\}$.

Thus, $Ker(g) \subseteq \{h_1^{-1}(\bar{q}) : \bar{q} \in Ker(\ell)\}$ as well, and we can conclude that

$$Ker(g) = \{h_1^{-1}(\bar{q}) : \bar{q} \in Ker(\ell)\}.$$

But then $h_1^{-1}|_{Ker(\ell)} : Ker(\ell) \longrightarrow Ker(g)$

Subspace of $\mathbb{F}^n$

is a well-defined linear map, and it is also surjective. Moreover, it is injective as restriction of the injective function $h_1^{-1} : \mathbb{F}^n \to V_1$. Thus $h_1^{-1}|_{Ker(\ell)} : Ker(\ell) \to Ker(g)$ is a linear isomorphism, which allows us to conclude that $\dim_{\mathbb{F}} Ker(\ell) = \dim_{\mathbb{F}} Ker(g)$.

Combining all the above now, we have that

$$\dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} \mathbb{F}^n = \dim_{\mathbb{F}} Range(\ell) + \dim_{\mathbb{F}} Ker(\ell)$$

$$= \dim_{\mathbb{F}} Range(g) + \dim_{\mathbb{F}} Ker(g).$$