Different Matrix Representations of Linear Maps

Recall a very important fact about bases:

Let V be a vector space over a field $\mathbb{F}$, and assume that $B = \{u_1, u_2, \ldots, u_k\}$ is a basis of V (so here we also assume that $\dim_{\mathbb{F}} V = k$).

Then we have seen that, for every $x \in V$, there is $\boxed{\text{a unique}}$ choice of scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ so that

$$x = \lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_k u_k.$$

But then, having fixed the basis B that we are using here, we can identify

x with the column vector $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix}$

without any ambiguity.

We write $[x]_B = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix}$.

This motivates the following notion:

Change of basis matrix

Assume that we are also given another basis C of V, $C = \{w_1, w_2, \ldots, w_k\}$. Then, for the same vector $x \in V$ that we fixed above, we have

$$[x]_C = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{pmatrix}$$

for another column vector $\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} \in \mathbb{F}^k$.

Clearly then we can define a function $G : \mathbb{F}^k \to \mathbb{F}^k$ by setting

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = [\lambda_1 u_1 + \dots + \lambda_k u_k]_B$$

$$\longmapsto [\lambda_1 u_1 + \dots + \lambda_k u_k]_C = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{pmatrix}$$

where the $\mu_i \in \mathbb{F}$ will depend on the $\lambda_i \in \mathbb{F}$ in the following way: the μ_i are the unique choice of scalars such that

$$\mu_1 w_1 + \mu_2 w_2 + \dots + \mu_k w_k = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_k u_k.$$

<u>Remark</u> This function G is linear.

Indeed, if we have

$$G\left(\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix}\right) = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{pmatrix} \quad \text{and} \quad G\left(\begin{pmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \\ \vdots \\ \tilde{\lambda}_k \end{pmatrix}\right) = \begin{pmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \\ \vdots \\ \tilde{\mu}_k \end{pmatrix}$$

then this means that

$$\mu_1 w_1 + \mu_2 w_2 + \dots + \mu_k w_k = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_k u_k$$

and $\tilde{\mu}_1 w_1 + \tilde{\mu}_2 w_2 + \dots + \tilde{\mu}_k w_k = \tilde{\lambda}_1 u_1 + \tilde{\lambda}_2 u_2 + \dots + \tilde{\lambda}_k u_k$.

But then

$$\begin{aligned}
&(\lambda_1 + \tilde{\lambda}_1) u_1 + (\lambda_2 + \tilde{\lambda}_2) u_2 + \dots + (\lambda_k + \tilde{\lambda}_k) u_k \\
&= (\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_k u_k) + (\tilde{\lambda}_1 u_1 + \tilde{\lambda}_2 u_2 + \dots + \tilde{\lambda}_k u_k) \\
&= (\mu_1 w_1 + \mu_2 w_2 + \dots + \mu_k w_k) + (\tilde{\mu}_1 w_1 + \tilde{\mu}_2 w_2 + \dots + \tilde{\mu}_k w_k) \\
&= (\mu_1 + \tilde{\mu}_1) w_1 + (\mu_2 + \tilde{\mu}_2) w_2 + \dots + (\mu_k + \tilde{\mu}_k) w_k
\end{aligned}$$

and these are the only coefficients we could use here to write the vector $\sum_{i=1}^{k} (\lambda_i + \tilde{\lambda}_i) u_i$ as a linear combination of the w_j

Thus we should have

$$G\left(\begin{pmatrix}\lambda_1+\tilde{\lambda}_1\\ \lambda_2+\tilde{\lambda}_2\\ \vdots\\ \lambda_k+\tilde{\lambda}_k\end{pmatrix}\right) = \begin{pmatrix}\mu_1+\tilde{\mu}_1\\ \mu_2+\tilde{\mu}_2\\ \vdots\\ \mu_k+\tilde{\mu}_k\end{pmatrix} = \begin{pmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_k\end{pmatrix} + \begin{pmatrix}\tilde{\mu}_1\\ \tilde{\mu}_2\\ \vdots\\ \tilde{\mu}_k\end{pmatrix}$$

$$= G\left(\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix}\right) + G\left(\begin{pmatrix}\tilde{\lambda}_1\\ \vdots\\ \tilde{\lambda}_k\end{pmatrix}\right).$$

Analogously we check that G respects scalar multiplication.

But then, since $G: \mathbb{F}^k \to \mathbb{F}^k$ is a linear map, it has a matrix representation A_G:

$$A_G = \begin{pmatrix} | & | & & | \\ G(\bar{e}_1) & G(\bar{e}_2) & \cdots & G(\bar{e}_k) \\ | & | & & | \end{pmatrix}$$ which here equals

$$= \begin{pmatrix} | & | & & | \\ [u_1]_{\mathcal{C}} & [u_2]_{\mathcal{C}} & \cdots & [u_k]_{\mathcal{C}} \\ | & | & & | \end{pmatrix}$$

We call this matrix A_G the change of basis matrix from the basis $\mathcal{B}$ of V to the basis $\mathcal{C}$ of V. Also we can write $A_{\mathcal{B}\to\mathcal{C}}$ for this matrix.

Question What is the change of basis matrix from the basis $\mathcal{C}$ to the basis $\mathcal{B}$?

Note that with $A_{\mathcal{B}\to\mathcal{C}}$ we have

$$\left(A_{\mathcal{B}\to\mathcal{C}}\right)\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix} = \begin{pmatrix}\mu_1\\ \vdots\\ \mu_k\end{pmatrix}$$

if $\lambda_1 v_1 + \dots + \lambda_k v_k = \mu_1 w_1 + \dots + \mu_k w_k$.

Now we want $A_{\mathcal{C}\to\mathcal{B}}$ to give us

$$\left(A_{\mathcal{C}\to\mathcal{B}}\right)\begin{pmatrix}\mu_1\\ \vdots\\ \mu_k\end{pmatrix} = \begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix}.$$

Thus $A_{\mathcal{C}\to\mathcal{B}} = \left(A_{\mathcal{B}\to\mathcal{C}}\right)^{-1}$ (and this is one way of seeing that these change of basis matrices are invertible; we can also check directly that their columns are $\mathbb{F}$-linearly independent).

<u>Example</u> Consider the following two bases of P_2:

$$\mathcal{B} = \{p_1(x) = x+1,\ p_2(x) = x-2,\ p_3(x) = x^2\},$$
$$\mathcal{C} = \{q_1(x) = x^2+1,\ q_2(x) = x^2+x,\ q_3(x) = x+1\}.$$

Find $A_{\mathcal{B}\to\mathcal{C}}$.

<u>Solution</u> It suffices to figure out how to write the basis vectors of $\mathcal{B}$ as linear combinations of the basis vectors of $\mathcal{C}$.

We have $p_1(x) = x+1 = 0\cdot q_1(x) + 0\cdot q_2(x) + 1\cdot q_3(x)$

$$\rightsquigarrow [p_1]_{\mathcal{C}} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$p_3(x) = x^2 = \frac{1}{2}\left(1\cdot q_1(x) + 1\cdot q_2(x) + (-1)\cdot q_3(x)\right)$$

$$\rightsquigarrow [p_3]_e = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

Finally

$$p_2(x) = x-2 = \frac{3}{2}(x-1) - \frac{1}{2}(x+1)$$

$$= \frac{3}{2}\left(q_2(x) - q_1(x)\right) + \left(\frac{-1}{2}\right) q_3(x)$$

$$= \left(\frac{-3}{2}\right) q_1(x) + \left(\frac{3}{2}\right) q_2(x) + \left(\frac{-1}{2}\right) q_3(x)$$

$$\rightsquigarrow [p_2]_e = \begin{pmatrix} -3/2 \\ 3/2 \\ -1/2 \end{pmatrix}$$

Thus $A_{B\to e} = \begin{pmatrix} 0 & -3/2 & 1/2 \\ 0 & 3/2 & 1/2 \\ 1 & -1/2 & -1/2 \end{pmatrix}$.

For practice, let us also check how to find $[p]_e$ if we already know $[p]_B$ and if e.g. $p(x) = 3$.

Note that $p(x) = 3 = (x+1) - (x-2)$

$$= 1 \cdot p_1(x) + (-1) p_2(x) + 0 \cdot p_3(x)$$

$$\rightsquigarrow [p]_B = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

But then we should have

$$[p]_e = (A_{B\to e})[p]_B = \begin{pmatrix} 0 & -3/2 & 1/2 \\ 0 & 3/2 & 1/2 \\ 1 & -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} =$$

$$= C_1(A_{B\to e}) - C_2(A_{B\to e}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -3/2 \\ 3/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -3/2 \\ 3/2 \end{pmatrix}$$

Let us verify this

$$3/2\, q_1(x) + \left(\frac{-3}{2}\right) q_2(x) + 3/2 \cdot q_3(x) =$$

$$\frac{3}{2}(x^2+1) + \left(\frac{-3}{2}\right)(x^2+x) + \frac{3}{2}(x+1) =$$

$$\frac{3}{2} \cdot 1 + \frac{3}{2} \cdot 1 = 3.$$

But Why do we consider these change of basis matrices?

Because they allow us to consider:

- first of all, matrix representations of linear maps $f: V \to W$ where V and W are finite-dimensional vector spaces over the field $\mathbb{F}$, but are not necessarily of the form $\mathbb{F}^{k_i}$ for some $k_1, k_2 \in \mathbb{N}$;
- and secondly, [redacted] more, and perhaps "nicer" sometimes, matrix representations of a linear map $g: \mathbb{F}^n \to \mathbb{F}^m$ compared to the standard matrix representation A_g we already know how to find.