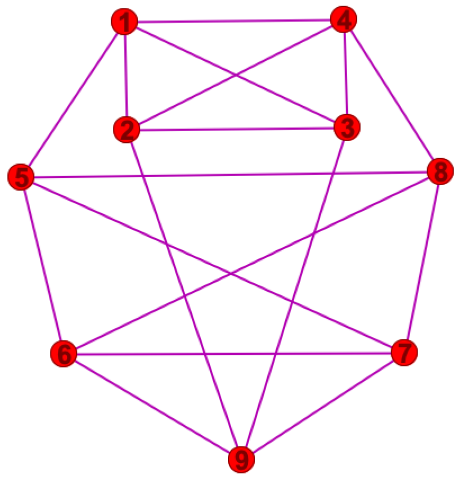
Math 322
Suggested solutions to Homework Set 3

Problem 1. (a) This is false. One counterexample is graph G_1 of Problems 4 and 5 (labelled in a way that we will also use in the solutions to these problems later):



We will check in Problem 4 that $\lambda(G_1) = 3$, and thus G_1 does not have bridges (we can also confirm the latter more simply right now, given that every edge of G_1 is found on some cycle of G_1).

However, G_1 has more than one cycle: e.g. both $1\,2\,3\,4\,1$ and $5\,6\,9\,7\,8\,5$ are cycles of G_1.

(b) This is false in the special cases where

- $G = K_2$ (that is, G is a graph on two vertices joined by an edge),
- or G has at least one connected component isomorphic to K_2 (while none of its connected components have cutvertices).

This is because K_2 is a complete graph, so it has no vertex cuts, and in particular it has no cutvertices.

At the same time, the only edge of K_2 is a bridge, since removing it gives the null graph on two vertices which is disconnected (or more generally, if G is disconnected to begin with, by removing this edge of its connected component K_2, we end up with a subgraph which has more connected components).

Remark. The statement would be correct if we additionally assume

- that G is a connected graph of order $n \geqslant 3$,

- or similarly that every connected component of G contains at least 3 vertices.

In such cases, we can argue as follows. By Whitney's theorem, we know that, for every connected graph H, $\kappa(H) \leqslant \lambda(H)$.

We also know that

(i) a connected graph H of order $n \geqslant 3$ has no cutvertices if and only if $\kappa(H) \geqslant 2$,

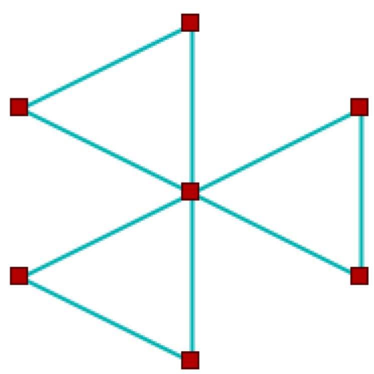
(ii) and analogously, a connected graph H has no bridges if and only if $\lambda(H) \geqslant 2$.

Consider now a graph G that satisfies the additional assumption and has no cutvertices (and again recall that, if G is disconnected, then G having no cutvertices is equivalent to saying that **none** of the connected components of G has cutvertices).

If none of the connected components of G has a cutvertex (while each has at least 3 vertices), then, from Whitney's theorem and fact (i), we will also have that, for every connected component G_i of G, $\lambda(G_i) \geqslant \kappa(G_i) \geqslant 2$.

This will imply that none of the connected components of G has a bridge, and thus G has no bridges.

(c) This is false. Consider the following graph:



Note that this graph can be seen as the union of three 'triangles' (or 3-cycles) with a single common vertex; this common vertex is a cutvertex of the graph.

On the other hand, this graph has no bridges since every edge of it is found on a cycle.

Problem 2. Let us consider two cases:

$\delta(G) = n - 1$. Then G is the complete graph K_n on n vertices, and we know that, by the definition of vertex connectivity, $\kappa(K_n) = n - 1$ too. In other words, we do have $\kappa(G) = \delta(G)$ in this case.

$\delta(G) = n - 2$. We will show that no set of $n - 3$ vertices can be a vertex cut of G. This will imply that $\kappa(G) \geqslant n - 2$, which combined with Whitney's theorem gives us that $\kappa(G) = \delta(G)$.

Let us write V for the vertex set of G, and let us assume that V' is a subset of V which contains $n - 3$ of the vertices of G. Then $V \setminus V'$ contains 3 vertices, let's say vertices u_1, u_2 and u_3.

Given that $\deg(u_1) \geqslant \delta(G) \geqslant n-2$, there must exist a neighbour of u_1 which is not in V'. In other words, at least one of the vertices u_2 and u_3 is a neighbour of u_1; let's assume without loss of generality that u_2 is adjacent to u_1.

Next, we can argue analogously about u_3: since $\deg(u_3) \geqslant \delta(G) \geqslant n-2$, there must exist a neighbour of u_3 which is not in V'; in other words one of the vertices u_1 and u_2 is a neighbour of u_3.

We can now see that, no matter if u_1 is adjacent to u_3, or u_2 is adjacent to u_3 (or both), the induced subgraph on the vertices u_1, u_2, u_3 will be connected (in fact, it will either be a path on 3 vertices, or a 3-cycle).

We conclude that $G - V'$ will be a connected graph, and hence V' won't be a vertex cut of G. Since V' was an arbitrary subset of V with $n - 3$ vertices, this completes the argument in the second case too.

Problem 3. The statement is true. To confirm it, we will consider a labelled connected graph K of order $n \geqslant 3$; note that we assume that K has at least 3 vertices since otherwise K would be a tree. Say the vertex set of K is the set $V = \{u_1, u_2, u_3, \ldots, u_{n-1}, u_n\}$.

We will prove the statement using induction in the size of K.

Base Case: The minimum size to consider here is n (indeed, we recall that, if K is a connected graph with $n-1$ edges, then it is a tree, and conversely if K contains $\geqslant n$ edges, then K cannot be a tree).

Let us then consider a connected graph K that has n vertices and n edges. Since K cannot be a tree, it contains at least one cycle, say the cycle $u_{i_1}\, u_{i_2}\, \ldots\, u_{i_{k-1}}\, u_{i_k}\, u_{i_1}$.

But then both edges $\{u_{i_1}, u_{i_2}\}$ and $\{u_{i_2}, u_{i_3}\}$ are not bridges of K (since they are found on a cycle of K), and hence both subgraphs

$$K - \{u_{i_1}, u_{i_2}\} \qquad \text{and} \qquad K - \{u_{i_2}, u_{i_3}\}$$

are connected. Also each of these subgraphs has $n-1$ edges, and hence it is a tree.

Since both these subgraphs contain all the vertices of K, they are both spanning trees of K (and they are clearly different, since u_{i_1} and u_{i_2} are not joined by an edge in the first subgraph, but they are joined by an edge in the second one).

Induction Step: Assume now that, for some s with $n \leqslant s < \binom{n}{2}$, we have already shown that

every connected graph H on the vertex set V
<u>which has size s</u> has at least **two** spanning trees.

We consider a connected graph $\widetilde{K}$ on the same set of vertices which has size $s+1$, and again we note that such a graph $\widetilde{K}$ cannot be a tree (since its size is $> n > n-1$). Thus, we can find a cycle $u_{j_1}\, u_{j_2}\, \ldots\, u_{j_{l-1}}\, u_{j_l}\, u_{j_1}$ in $\widetilde{K}$. Then the subgraph

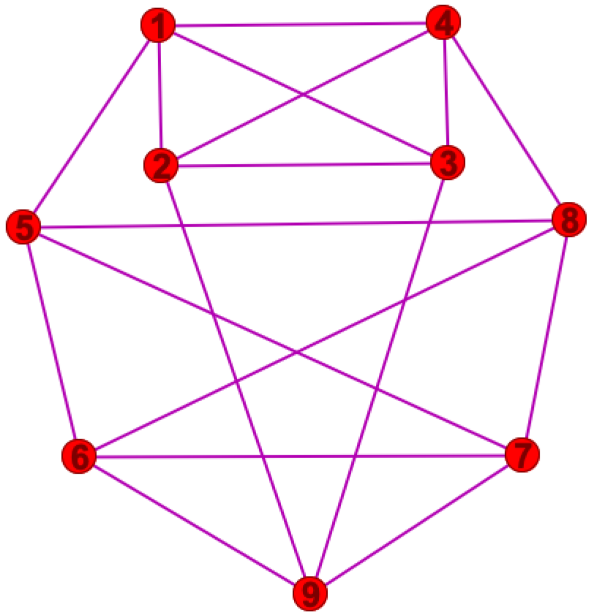
$$\widetilde{K} - \{u_{j_1}, u_{j_2}\}$$

is connected, and has size s, so we can apply the Inductive Hypothesis to it: $\widetilde{K} - \{u_{j_1}, u_{j_2}\}$ has at least two spanning trees, say subgraphs T_1 and T_2.

Both T_1 and T_2 contain all vertices of $\widetilde{K} - \{u_{j_1}, u_{j_2}\}$, which coincide with the vertices of $\widetilde{K}$, and hence T_1 and T_2 are also two different spanning trees of $\widetilde{K}$, as we wanted to show.

This completes the proof of the statement.

Problem 4. Let us consider a labelling of graph G_1:



We first note that G_1 is 4-regular, and thus $\delta(G_1) = 4$. By Whitney's theorem, we immediately obtain that $\kappa(G_1) \leqslant \lambda(G_1) \leqslant 4$. Let us now try to determine their values exactly.

Regarding $\kappa(G_1)$: We observe that the vertex set $\{1, 4, 9\}$ is a vertex cut of G_1: indeed, $G_1 - \{1, 4, 9\}$ has two connected components, the induced subgraph on the vertices 2 and 3, and the induced subgraph on the vertices $5, 6, 7$ and 8.

Thus $\kappa(G_1) \leqslant 3$. To show that it is equal to 3, we rely on the vertex form of Menger's theorem: it suffices to check that, for any two different, non-adjacent vertices i and j of G_1, there are at least **three** pairwise internally disjoint paths from i to j. We need to consider the following pairs of vertices:

- 1 and 6, and analogously 4 and 7 (by symmetry these two pairs will behave completely analogously);
- similarly, 1 and 7, and analogously 4 and 6;
- 1 and 8, and analogously 4 and 5;
- 1 and 9, and analogously 4 and 9;
- 5 and 9, and analogously 8 and 9;

– 2 and 5, and analogously 2 and 8, 3 and 5, and 3 and 8;
– 2 and 6, and analogously 2 and 7, 3 and 6, and 3 and 7.

We can now check that the following paths are three pairwise internally disjoint paths from 1 to 6:

$$1\,5\,6, \qquad 1\,2\,9\,6, \quad \text{and} \quad 1\,4\,8\,6$$

(analogously we can find three pairwise internally disjoint paths from 4 to 7).

Similarly, here are three pairwise internally disjoint paths from 1 to 7:

$$1\,5\,7, \qquad 1\,2\,9\,7, \quad \text{and} \quad 1\,4\,8\,7$$

(analogously we can find three pairwise internally disjoint paths from 4 to 6).

Here are three pairwise internally disjoint paths from 1 to 8:

$$1\,5\,8, \qquad 1\,4\,8, \quad \text{and} \quad 1\,2\,9\,7\,8$$

(analogously we can find three pairwise internally disjoint paths from 4 to 5).

Here are three pairwise internally disjoint paths from 1 to 9:

$$1\,2\,9, \qquad 1\,5\,6\,9, \quad \text{and} \quad 1\,3\,9$$

(analogously we can find three pairwise internally disjoint paths from 4 to 9).

Here are three pairwise internally disjoint paths from 5 to 9:

$$5\,6\,9, \qquad 5\,7\,9, \quad \text{and} \quad 5\,1\,2\,9$$

(analogously we can find three pairwise internally disjoint paths from 8 to 9).

Here are three pairwise internally disjoint paths from 2 to 5:

$$2\,1\,5, \qquad 2\,9\,6\,5, \quad \text{and} \quad 2\,4\,8\,5.$$

Similarly, we can find three pairwise internally disjoint paths from 2 to 8:

$$2\,4\,8, \qquad 2\,9\,7\,8, \quad \text{and} \quad 2\,1\,5\,8$$

(and analogously we can find three pairwise internally disjoint paths from 3 to 5 and from 3 to 8).

Here are three pairwise internally disjoint paths from 2 to 6:

$$2\,1\,5\,6, \qquad 2\,9\,6, \quad \text{and} \ \ 2\,4\,8\,6.$$

Similarly, we can find three pairwise internally disjoint paths from 2 to 7:

$$2\,4\,8\,7, \qquad 2\,9\,7, \quad \text{and} \ \ 2\,1\,5\,7$$

(and analogously we can find three pairwise internally disjoint paths from 3 to 6 and from 3 to 7).

We can conclude that $\kappa'(i,j) \geqslant 3$ for any two different and non-adjacent vertices of G_1, and hence, by Menger's theorem, $\kappa(G_1) \geqslant 3$.

Combining all the above, we see that $\kappa(G_1) = 3$.

Regarding $\lambda(G_1)$: We will show now that $\lambda(G_1) = 4$. We have already observed that $\lambda(G_1) \leqslant \delta(G_1) \leqslant 4$. We now use the edge version of Menger's theorem to show that, for any two different vertices i and j of G_1, there are at least **four** pairwise edge-disjoint paths from i to j.

- Let's suppose first that both i and j are in $\{1,2,3,4\}$; all cases that we can consider here are analogous to having $i=1$ and $j=3$, say. But in this case the following are four edge-disjoint paths from 1 to 3:

$$1\,3, \qquad 1\,4\,3, \qquad 1\,2\,3, \quad \text{and} \ \ 1\,5\,6\,9\,3.$$

- Next, let's suppose that both i and j are in $\{5,6,7,8\}$; all cases that we can consider here are analogous to having $i=5$ and $j=8$, say. But in this case the following are four edge-disjoint paths from 5 to 8:

$$5\,8, \qquad 5\,7\,8, \qquad 5\,6\,8, \quad \text{and} \ \ 5\,1\,4\,8.$$

- Let us now suppose that $i=9$ and $j \in \{2,3,6,7\}$; all cases here are analogous to the case where $i=9$ and $j=3$. In this case the following are four edge-disjoint paths from 9 to 3:

$$9\,3, \qquad 9\,7\,8\,4\,3, \qquad 9\,2\,3, \quad \text{and} \ \ 9\,6\,5\,1\,3.$$

- We also consider the cases where $i = 1$ and $j = 5$, or $i = 4$ and $j = 8$ (both these cases are analogous); here are four edge-disjoint paths from 1 to 5:

$$1\,5, \qquad 1\,2\,9\,6\,5, \qquad 1\,4\,8\,5, \quad \text{and} \quad 1\,3\,9\,7\,5.$$

The remaining pairs of vertices that we need to consider are non-adjacent vertices, **so pairs of vertices that we dealt with when estimating $\kappa(G_1)$ as well.** Thus, we recall that for each such pair of vertices we have already found three pairwise internally disjoint paths, which are certainly edge-disjoint paths as well. This shows that we now only need to find one more path for each such pair of vertices which won't pass by the same edges as the ones we have used so far.

- In the case of $i = 1$ and $j = 6$, one more path we can use, aside from the ones we found previously, is the path $1\,3\,9\,7\,6$ (note that none of the edges contained in this path appear in any of the other three paths we found above, and also none of the edges in those paths appears here).

  Analogously, we can find one more path when $i = 4$ and $j = 7$.
- In the case of $i = 1$ and $j = 7$, one more path we can use is the path $1\,3\,9\,6\,7$.

  Analogously, we can find one more path when $i = 4$ and $j = 6$.
- In the case of $i = 1$ and $j = 8$, one more path we can use is the path $1\,3\,9\,6\,8$.

  Analogously, we can find one more path when $i = 4$ and $j = 5$.
- In the case of $i = 1$ and $j = 9$, one more path we can use is the path $1\,4\,8\,7\,9$.

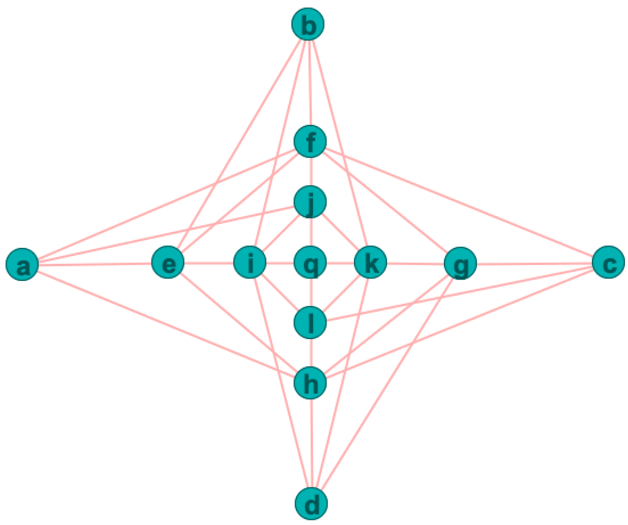
  Analogously, we can find one more path when $i = 4$ and $j = 9$, or when $i = 5$ and $j = 9$, or finally when $i = 8$ and $j = 9$.
- In the case of $i = 2$ and $j = 5$, one more path we can use is the path $2\,3\,9\,7\,5$.

  Analogously, we can find one more path when $i = 2$ and $j = 8$, or when $i = 3$ and $j = 5$, or finally when $i = 3$ and $j = 8$.
- Finally, in the case of $i = 2$ and $j = 6$, one more path we can use is the path $2\,3\,9\,7\,6$.

  Analogously, we can find one more path when $i = 2$ and $j = 7$, or when $i = 3$ and $j = 6$, or finally when $i = 3$ and $j = 7$.

We conclude that, for any two different vertices i and j of G_1, there are at least four pairwise edge-disjoint paths from i to j, and hence all the local edge connectivities are $\geqslant 4$. This also implies that $\lambda(G_1) \geqslant 4$, and finally that $\lambda(G_1) = 4$.

Let us now consider a labelling of graph G_2:



We can quickly see that the degree sequence of G_2 is

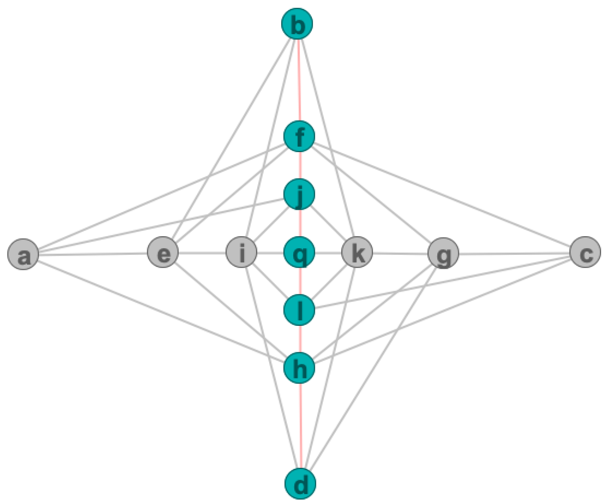
$$(4, 4, 4, 4, 5, 6, 5, 6, 5, 6, 5, 6, 5, 4)$$

(where the vertices here are ordered alphabetically). Thus $\delta(G_2) = 4$.

Next we will show that $\kappa(G_2) \geqslant 4$, which, combined with Whitney's theorem, will imply that $4 \leqslant \kappa(G_2) \leqslant \lambda(G_2) \leqslant \delta(G_2) = 4$, and will thus give us the precise value of $\kappa(G_2)$ and of $\lambda(G_2)$.

To show that $\kappa(G_2) \geqslant 4$, we can again use the vertex form of Menger's theorem: we need to show that, for every pair of different, non-adjacent vertices u, v in G_2, there are at least **four** pairwise internally disjoint paths from u to v. We can consider three cases here regarding where u and v are found on the graph:

(A) both vertices are found on the 'vertical' path in the graph, highlighted below:

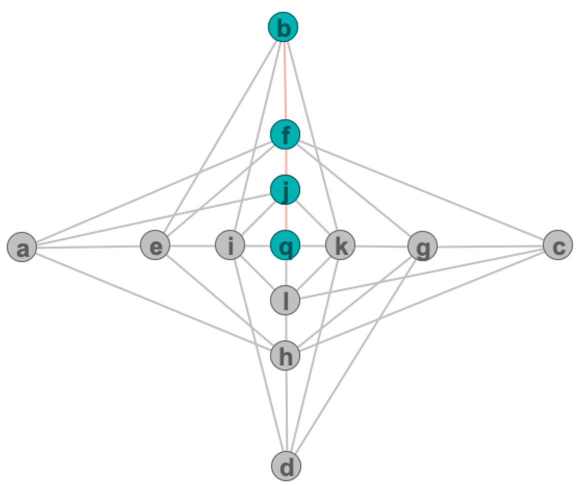


(B) neither of the two vertices is found on this 'vertical' path,

(C) or one of the vertices comes from this 'vertical' path, and the other vertex is one of the vertices a, e, i, k, g or c.

By symmetry, we can also break Case (A) into the following 2 subcases (because any other example of two different, non-adjacent vertices found on this 'vertical' path can be treated analogously to the cases already covered below):

(Ai) both vertices are found in the upper part of this 'vertical' path, highlighted below:



(Aii) one vertex is found in this upper part, while the other one is among the vertices l, h and d; in fact, again by symmetry, in this subcase it's sufficient to consider pairs of the form

- $u = b$ and $v \in \{l, h, d\}$;
- $u = f$ and $v \in \{l, h\}$;
- $u = j$ and $v = l$.

Similarly, by symmetry we can break each of Cases (B) and (C) into 2 subcases (and whatever examples are not covered in these subcases can be treated analogously to one of the examples that is covered):

(Bi) both vertices are found on the left-hand side of the 'vertical' path we have highlighted above, that is, they are among the vertices a, e and i;

(Bii) one vertex is found on the left-hand side, while the other one is on the right-hand side; in fact, again by symmetry, in this subcase it's sufficient to consider pairs of the form

- $u = a$ and $v \in \{k, g, c\}$;
- $u = e$ and $v \in \{k, g\}$;
- $u = i$ and $v = k$.

(Ci) one vertex is found in the upper part of the 'vertical' path, while the other vertex is found to the left of it, that is, the second vertex is among the vertices a, e and i;

(Cii) one vertex is found in the upper part of the 'vertical' path, while the other vertex is found to the right of it, that is, the second vertex is among the vertices k, g and c.

We conclude that the pairs of vertices we need to consider, separated into the subcases they fall in, are the following:

(Ai)
- $u = b$ and $v = j$; here are four pairwise internally disjoint paths from b to j:

$$b\,e\,a\,j, \qquad b\,i\,j, \qquad b\,k\,j, \quad \text{and} \quad b\,f\,j.$$

- $u = b$ and $v = q$; here are four pairwise internally disjoint paths from b to q:

$$b\,e\,h\,l\,q, \qquad b\,i\,q, \qquad b\,k\,q, \quad \text{and} \quad b\,f\,j\,q.$$

- $u = f$ and $v = q$; here are four pairwise internally disjoint paths from f to q:

$$f\,e\,i\,q, \qquad f\,a\,h\,l\,q, \qquad f\,j\,q, \quad \text{and} \quad f\,g\,k\,q.$$

(Aii)

- $u = b$ and $v = l$; here are four pairwise internally disjoint paths from b to l:

$$b\,e\,h\,l, \qquad b\,i\,l, \qquad b\,k\,l, \quad \text{and} \quad b\,f\,j\,q\,l.$$

- $u = b$ and $v = h$; here are four pairwise internally disjoint paths from b to h:

$$b\,e\,h, \qquad b\,i\,l\,h, \qquad b\,k\,d\,h, \quad \text{and} \quad b\,f\,g\,h.$$

- $u = b$ and $v = d$; here are four pairwise internally disjoint paths from b to d:

$$b\,e\,h\,d, \qquad b\,i\,d, \qquad b\,k\,d, \quad \text{and} \quad b\,f\,g\,d.$$

- $u = f$ and $v = l$; here are four pairwise internally disjoint paths from f to l:

$$f\,a\,h\,l, \qquad f\,e\,i\,l, \qquad b\,k\,l, \quad \text{and} \quad f\,c\,g\,k\,l.$$

- $u = f$ and $v = h$; here are four pairwise internally disjoint paths from f to h:

$$f\,a\,h, \qquad f\,e\,h, \qquad f\,g\,h, \quad \text{and} \quad f\,j\,q\,l\,h.$$

- $u = j$ and $v = l$; here are four pairwise internally disjoint paths from j to l:

$$j\,f\,c\,l, \qquad j\,i\,l, \qquad j\,k\,l, \quad \text{and} \quad j\,q\,l.$$

(Bi) the only case to consider here is $u = a$ and $v = i$; here are four pairwise internally disjoint paths from a to i:

$$a\,e\,i, \qquad a\,j\,i, \qquad a\,h\,l\,i, \quad \text{and} \quad a\,f\,g\,k\,q\,i.$$

(Bii)

- $u = a$ and $v = k$; here are four pairwise internally disjoint paths from a to k:

$$a\,e\,i\,q\,k, \qquad a\,j\,k, \qquad a\,h\,l\,k, \quad \text{and} \quad a\,f\,g\,k.$$

- $u = a$ and $v = g$; here are four pairwise internally disjoint paths from a to g:

$$a\,e\,i\,l\,c\,g, \qquad a\,j\,k\,g, \qquad a\,h\,g, \quad \text{and} \quad a\,f\,g.$$

- $u = a$ and $v = c$; here are four pairwise internally disjoint paths from a to c:

$$a\,e\,i\,l\,c, \qquad a\,j\,k\,g\,c, \qquad a\,h\,c, \quad \text{and} \quad a\,f\,c.$$

- $u = e$ and $v = k$; here are four pairwise internally disjoint paths from e to k:

$$e\,i\,q\,k, \qquad e\,b\,k, \qquad e\,h\,g\,k, \quad \text{and} \quad e\,f\,j\,k.$$

- $u = e$ and $v = g$; here are four pairwise internally disjoint paths from e to g:

$$e\,i\,l\,c\,g, \qquad e\,b\,k\,g, \qquad e\,h\,g, \quad \text{and} \quad e\,f\,g.$$

- $u = i$ and $v = k$; here are four pairwise internally disjoint paths from i to k:

$$i\,q\,k, \qquad i\,j\,k, \qquad i\,l\,k, \quad \text{and} \quad i\,b\,k.$$

(Ci)

- $u = b$ and $v = a$; here are four pairwise internally disjoint paths from b to a:

$$b\,f\,a, \qquad b\,e\,a, \qquad b\,k\,l\,h\,a, \quad \text{and} \quad b\,i\,j\,a.$$

- $u = f$ and $v = i$; here are four pairwise internally disjoint paths from f to i:

$$f\,b\,i, \qquad f\,e\,i, \qquad f\,j\,i, \quad \text{and} \quad f\,g\,k\,q\,i.$$

- $u = j$ and $v = e$; here are four pairwise internally disjoint paths from j to e:

$$j\,i\,e, \qquad j\,f\,e, \qquad j\,a\,e, \quad \text{and} \quad j\,q\,l\,h\,e.$$

- $u = q$ and $v = a$; here are four pairwise internally disjoint paths from q to a:

$$q\,i\,e\,a, \qquad q\,j\,a, \qquad q\,k\,g\,f\,a, \quad \text{and} \quad q\,l\,h\,a.$$

- $u = q$ and $v = e$; here are four pairwise internally disjoint paths from q to e:

$$q\,i\,e, \qquad q\,j\,f\,e, \qquad q\,k\,b\,e, \quad \text{and} \quad q\,l\,h\,e.$$

(Cii)

- $u = b$ and $v = g$; here are four pairwise internally disjoint paths from b to g:

$$b\,f\,g, \qquad b\,k\,g, \qquad b\,i\,d\,g, \quad \text{and} \quad b\,e\,h\,g.$$

- $u = b$ and $v = c$; here are four pairwise internally disjoint paths from b to c:

$$b\,f\,c, \qquad b\,k\,g\,c, \qquad b\,i\,l\,c, \quad \text{and} \quad b\,e\,h\,c.$$

- $u = f$ and $v = k$; here are four pairwise internally disjoint paths from f to k:

$$f\,b\,k, \qquad f\,j\,k, \qquad f\,e\,i\,q\,k, \quad \text{and} \quad f\,g\,k.$$

- $u = j$ and $v = g$; here are four pairwise internally disjoint paths from j to g:

$$j\,f\,g, \qquad j\,k\,g, \qquad j\,q\,l\,c\,g, \quad \text{and} \quad j\,a\,h\,g.$$

- $u = j$ and $v = c$; here are four pairwise internally disjoint paths from j to c:

$$j\,f\,c, \qquad j\,k\,g\,c, \qquad j\,q\,l\,c, \quad \text{and} \quad j\,a\,h\,c.$$

- $u = q$ and $v = g$; here are four pairwise internally disjoint paths from q to g:

$$q\,j\,f\,g, \qquad q\,k\,g, \qquad q\,l\,c\,g, \quad \text{and} \quad q\,i\,e\,a\,h\,g.$$

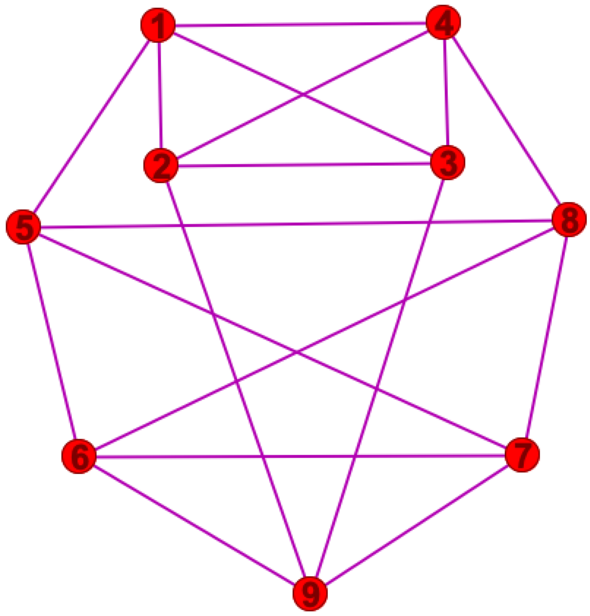
- $u = q$ and $v = c$; here are four pairwise internally disjoint paths from q to c:

$$q\,j\,f\,c, \qquad q\,k\,g\,c, \qquad q\,l\,c, \quad \text{and} \quad q\,i\,e\,a\,h\,c.$$

We conclude that, for all pairs of different, non-adjacent vertices u and v of G_2 that we considered, there are at least four pairwise internally disjoint paths from u to v. Moreover, all other pairs that should be considered here behave completely analogously to one of the above pairs (e.g. if $u = l$ and $v = a$, then we could treat this pair analogously to how we treated the case $u = j$ and $v = c$; or if $u = d$ and $v = f$, then we should treat this pair analogously to how we treated the case $u = b$ and $v = h$).

The above show that $\kappa(G_2) \geqslant 4$, and hence $\kappa(G_2) = \delta(G_2) = 4$. Moreover, $\lambda(G_2) = \kappa(G_2) = \delta(G_2) = 4$.

Problem 5. (a) Let us consider the labelling of graph G_1 that we considered in Problem 4 too:



Below are two spanning trees of graph G_1 in (V, E) notation (and also drawn):

$$\big(\{1,2,3,4,5,6,7,8,9\},\ \{14,15,29,39,48,56,69,78\}\big)$$
$$\text{and}\quad \big(\{1,2,3,4,5,6,7,8,9\},\ \{12,29,34,39,56,69,78,79\}\big).$$

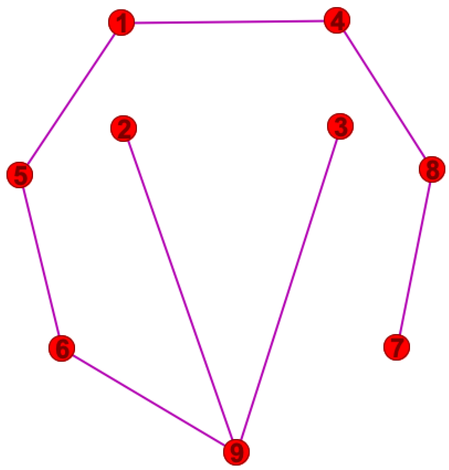


Figure 1: Spanning Tree 1

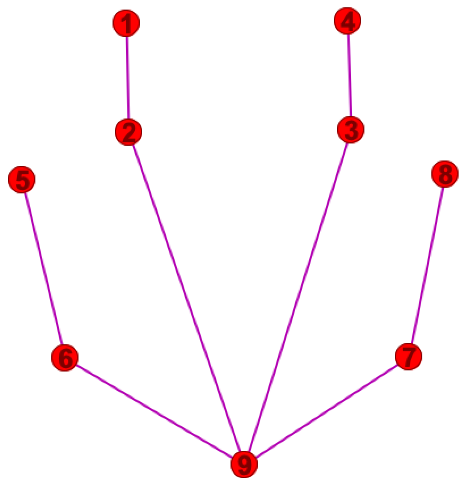
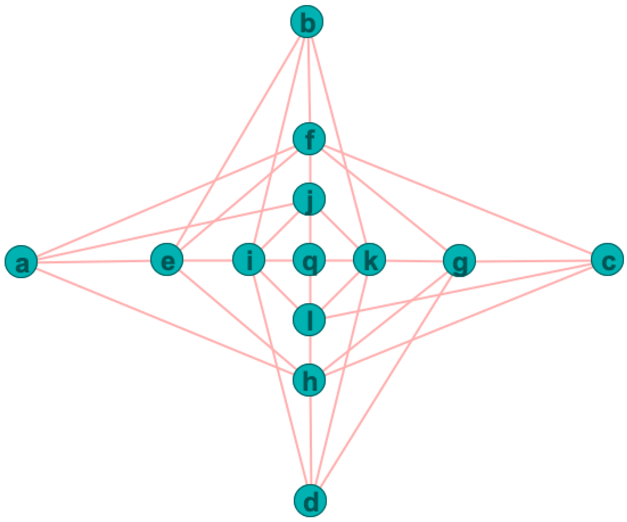


Figure 2: Spanning Tree 2

Similarly, let us consider the labelling of graph G_2 that we considered in Problem 4 too:



Below are two spanning trees of graph G_2 in (V, E) notation (and also drawn):

$\big(\{a, b, c, d, e, f, g, h, i, j, k, l, q\},\ \{bf, fj, jq, ql, lh, hd, ae, ei, iq, qk, kg, gc\}\big)$
and $\big(\{a, b, c, d, e, f, g, h, i, j, k, l, q\},\ \{bf, fj, fa, fe, ji, il, qi, lh, lk, kd, hg, hc\}\big)$.

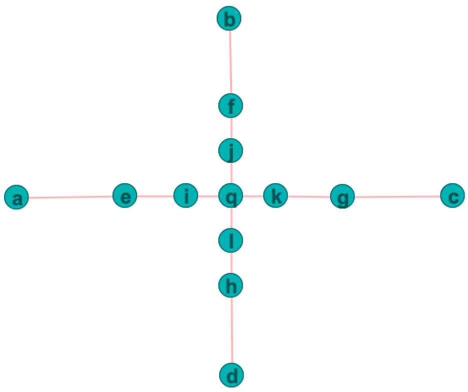


Figure 3: Spanning Tree 1

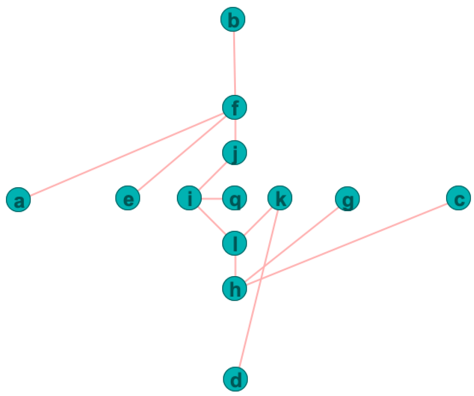
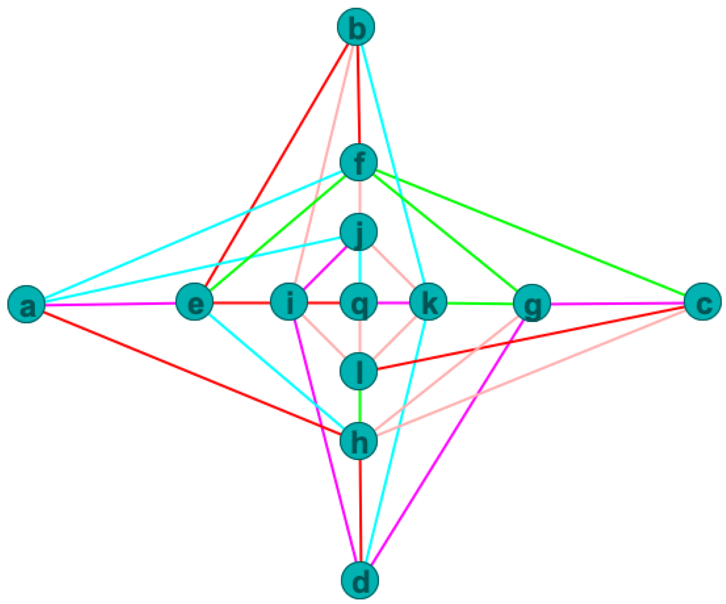
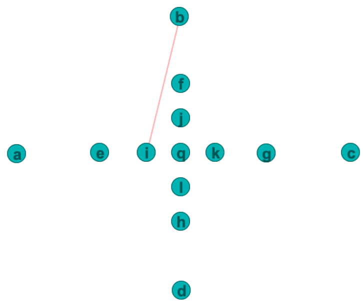


Figure 4: Spanning Tree 2

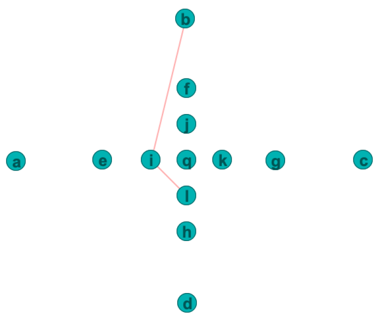
(b) We will use Kruskal's algorithm. We keep the above labelling of Graph G_2 (but with the edges coloured now):



We start with an edge of minimum possible weight, say the edge bi:

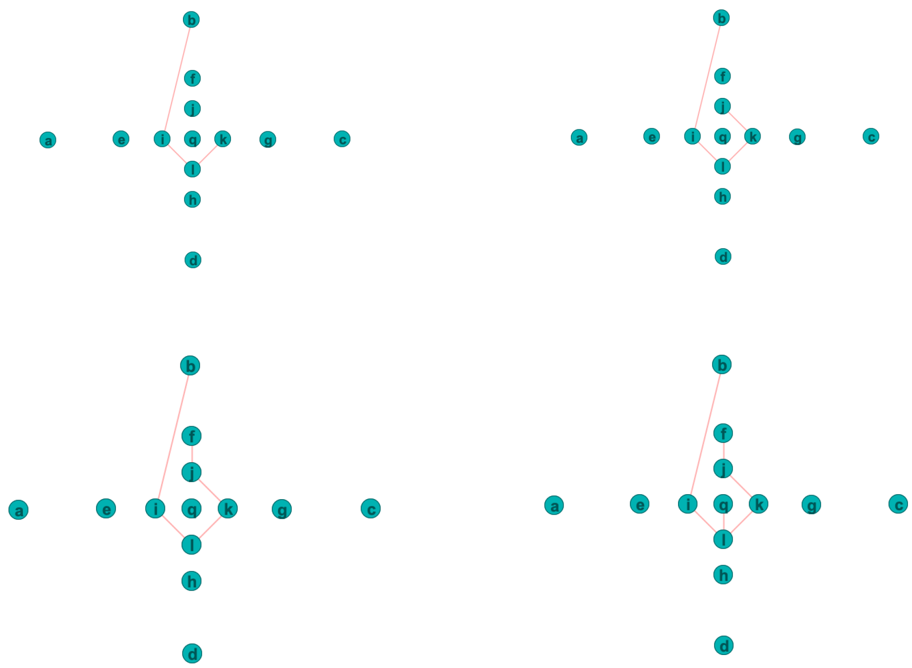


We continue by adding the edge il (which also has minimum possible weight, equal to 10):



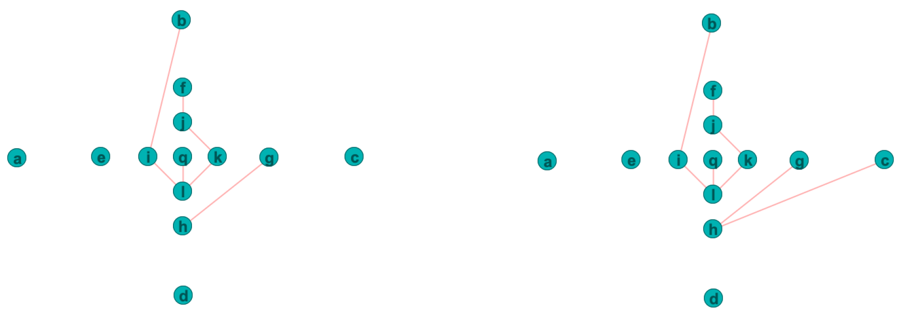
We can continue by adding the edge lk, and afterwards the edges kj, jf and

lq (note that, at each step, none of the edges we add forms a cycle together with some of the previous edges):

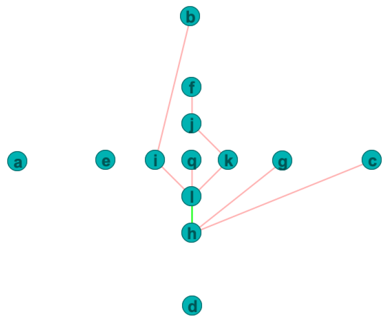


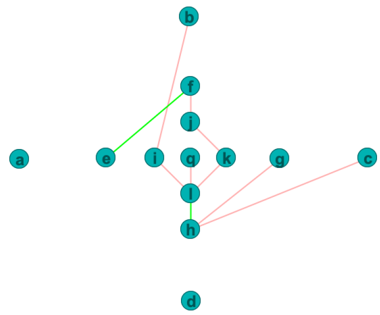
At this point, we note that we can no longer consider the edges qi, qj, ij, fb and kb (because either of these would form a cycle with some of the edges we have already selected).

On the other hand, we can pick the edge hg, and afterwards the edge hc, since the two new subgraphs we get are still acyclic:



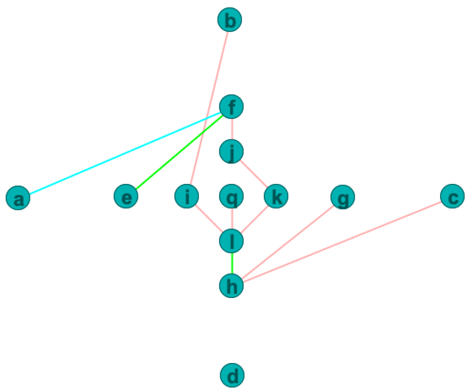
We now observe that there are no more edges with weight 10. Thus, we next choose an edge which has minimum weight out of the remaining ones; in fact, we can choose an edge with weight 25, since e.g the edge hl does not form a cycle with any of the edges we have already picked. Similarly, if we subsequently choose the edge ef, we still get an acyclic graph.

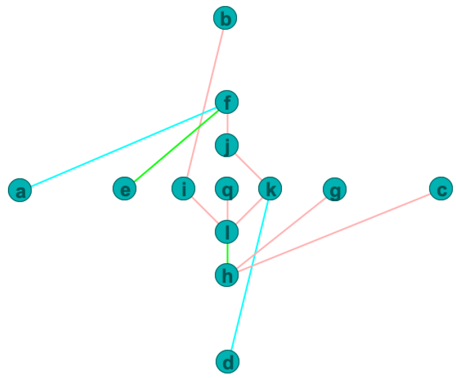




On the other hand, we can no longer use any of the edges fg, fc and kg. In other words, at this point, we can no longer use an edge with weight 25 (and of course there are no more edges with weight 10), and thus the minimum possible edge weight we could perhaps pick is 35 (or in other words, the best pick now would be an edge of turquoise colour).

Given that we can pick the edge fa, and afterwards the edge kd (without ending up with cyclic graphs), we conclude that the last graph is a minimum weight spanning graph of Graph G_2:





We also conclude that the minimum total weight is $8 \cdot 10 + 2 \cdot 25 + 2 \cdot 35 = 200$.