Math 322 – Fall Term 2020
Suggested solutions to the Midterm exam

Problem 1. (a) The complement $\overline{G}$ of G has the same vertex set as G. In other words, $V\big(\overline{G}\big) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

We also recall that, for each vertex $v_i \in \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, its neighbours in $\overline{G}$ are precisely those vertices from $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \setminus \{v_i\}$ which are not joined with v_i in G. Thus $\deg_{\overline{G}}(v_i) = 6 - \deg_G(v_i)$.

This shows that, to find the desired degree sequence, we could first determine the degree sequence of G. But to find the degree of each vertex v_i in G, we can simply count how many edges v_i is incident with by looking at the incidence matrix of G: the number we want is equal to the number of entries in the i-th row which are equal to 1.

We obtain that

$$\big(\deg_G(v_1), \deg_G(v_2), \deg_G(v_3), \deg_G(v_4), \deg_G(v_5), \deg_G(v_6), \deg_G(v_7)\big) = (2, 4, 4, 3, 3, 1, 3).$$

By this, we also get that

$$\big(\deg_{\overline{G}}(v_1), \deg_{\overline{G}}(v_2), \deg_{\overline{G}}(v_3), \deg_{\overline{G}}(v_4), \deg_{\overline{G}}(v_5), \deg_{\overline{G}}(v_6), \deg_{\overline{G}}(v_7)\big) = (4, 2, 2, 3, 3, 5, 3).$$

Problem 1 (cont.) (b) By looking at the adjacency matrix of H, we can draw the graph H:

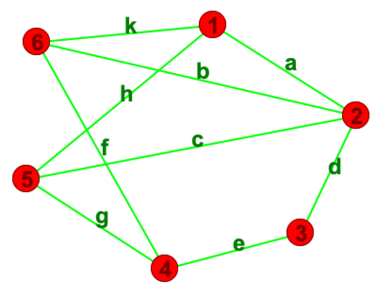


Figure 1: Graph H

Therefore, the line graph $L(H)$ of H is the graph

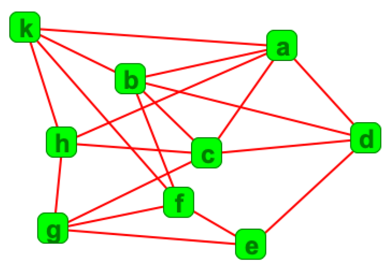


Figure 2: Graph $L(H)$

We conclude that, if we order the vertices of $L(H)$ alphabetically, its degree sequence is

$$(5, 5, 5, 4, 3, 4, 4, 4, 4).$$

Problem 2. (a) *1st way.* We show directly that Graphs G_2 and G_3 are line graphs of some other graphs, by drawing a graph H_1 whose line graph would be G_2, as well as a graph H_2 whose line graph would be G_3.

In the case of G_2 consider the following graph:

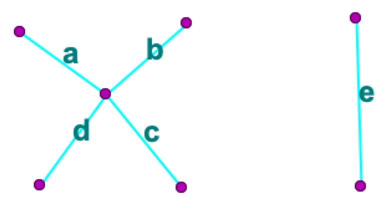


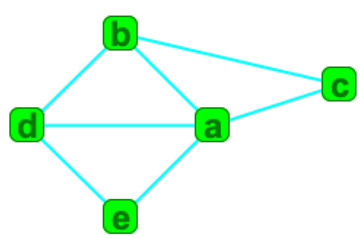
Figure 3: Graph H_1

Note that, in H_1, any two of its first 4 edges are adjacent, while the last edge is not adjacent to any other edge. This implies that $L(H_1)$ will contain 5 vertices, satisfying the following adjacencies:

- the induced subgraph on the first 4 vertices will be a complete graph, the graph K_4,
- while the last vertex will be an isolated vertex.

This shows that $L(H_1) \cong G_2$, and confirms that G_2 is a line graph.

In the case of G_3, if we consider the following labelling of it:



then we can check that it is the line graph of the following graph:

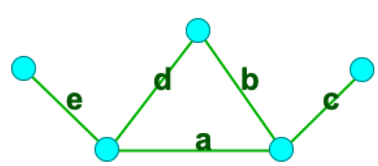


Figure 4: Graph H_2

2nd way. We can check that Graph G_1 cannot be the line graph of any graph. We recall Beineke's theorem from Lecture 8 (last slide): a graph G is a line graph if and only if none of the graphs on that slide are induced subgraphs of G.

But $K_{1,3}$ is a graph on that slide, and it is an induced subgraph of G_1 (we can get $K_{1,3}$ by deleting two of the vertices of G_1 which have degree 1). Hence, no matter what graph H we start with, G_1 will not be isomorphic to $L(H)$.

3rd way. We could alternatively use Beineke's theorem to check that G_2 and G_3 can be viewed as line graphs.

First of all, it is not hard to check that $K_{1,3}$ is not an **induced** subgraph of either G_2 or G_3.

Secondly, each of the graphs G_2 and G_3 has 5 vertices. Therefore, any of the forbidden subgraphs from Beineke's theorem which has more than 5 vertices clearly cannot be an induced subgraph of G_2 or of G_3 (more generally, it cannot be a subgraph of G_2 or of G_3).

This leaves only 2 forbidden subgraphs from Beineke's theorem which we would still have to check for (the second and the third graph in the top row of the image on the last slide of Lecture 8). We now observe that each of these two forbidden subgraphs has exactly 5 vertices. Thus, for such a graph K to be an induced subgraph of G_2 or of G_3, K would have to coincide with G_2 or G_3 respectively.

But neither G_2 nor G_3 coincide with one of these two graphs of order 5 from Beineke's theorem. Combining this with what we noted at the beginning, that neither G_2 nor G_3 contain $K_{1,3}$ as an induced subgraph, we can conclude that both G_2 and G_3 are line graphs.

Problem 2 (cont.) (b) Graphs G_4 and G_5 are isomorphic. Consider the following bijection τ from $V(G_4)$ to $V(G_5)$:

$$\tau(7) = b, \quad \tau(1) = f, \quad \tau(2) = g, \quad \tau(3) = e, \quad \tau(4) = a, \quad \tau(5) = d, \quad \text{and} \quad \tau(6) = c.$$

Observe also that

$$E(G_4) = \{12,\ 16,\ 17,\ 23,\ 34,\ 37,\ 45,\ 47,\ 56,\ 67\},$$

while

$$\begin{aligned} E(G_5) &= \{fg,\ fc,\ fb,\ ge,\ ea,\ eb,\ ad,\ ab,\ dc,\ cb\} \\ &= \big\{\tau(1)\tau(2),\ \tau(1)\tau(6),\ \tau(1)\tau(7),\ \tau(2)\tau(3),\ \tau(3)\tau(4), \\ &\qquad\qquad \tau(3)\tau(7),\ \tau(4)\tau(5),\ \tau(4)\tau(7),\ \tau(5)\tau(6),\ \tau(6)\tau(7)\big\}. \end{aligned}$$

In other words, τ preserves the adjacencies, and thus it is a graph isomorphism from G_4 to G_5.

Problem 3. (a) We first show that the vertex subset $\{A, L\}$ is a vertex cut of G_0.

Indeed, in the subgraph $G_0 - \{A, L\}$, we have, for instance, that there is no F–B path.

If there were such a path P_0, then the second vertex on the path should be a neighbour of F. But the neighbours of F in G_0 are the vertices A, L and E, so in the subgraph $G_0 - \{A, L\}$ the only neighbour of F is the vertex E.

In other words, if a path P_0 from F to B existed in $G_0 - \{A, L\}$, its first few vertices would be among the vertices of the 'outer' cycle of G_0, that is, the vertices in the set $\{C, D, E, F, K\}$. But B belongs to an 'inner' cycle of G_0, therefore as we traverse the path P_0 we have assumed exists, eventually we must arrive at a vertex among the vertices B, G and H.

If we consider the first such vertex on the path P_0, which implies that the previous vertex on P_0 would be among C, D, E, F and K, we see that one of these latter vertices should be adjacent to one of B, G, H in $G_0 - \{A, L\}$ (and hence also in G_0). But this contradicts the fact that none of the vertices in $\{C, D, E, F, K\}$ is adjacent to a vertex from $\{B, G, H\}$ in G_0. Thus the assumption that there exists a path P_0 from F to B in $G_0 - \{A, L\}$ was incorrect.

We have already shown that $\kappa(G_0) \leqslant 2$. We now show that there is no 1-vertex cut of G_0, or in other words that G_0 has no cutvertices. By the Corollary of the vertex form of Menger's theorem, it suffices to show that we can find at least two internally disjoint paths from any vertex to any other vertex (which is not adjacent to the first vertex).

To verify this, observe that the vertices C, D, E, F, K and L form a cycle in G_0, the cycle $C\,D\,E\,F\,L\,K\,C$. Thus, for any two vertices from $\{C, D, E, F, K, L\}$, we can find two internally disjoint paths (which will be subpaths of this cycle) taking us from the first vertex to the second one.

Similarly, the vertices A, B, G and H form (at least) one cycle in G_0, say the cycle $A\,B\,H\,G\,A$. So, again for any two vertices from $\{A, B, G, H\}$, we can find (at least) two internally disjoint paths taking us from one vertex to the other one.

Finally, if we consider instead a vertex from $\{C, D, E, F, K, L\}$ and a vertex from $\{A, B, G, H\}$, we note that we can find paths that contain either the edge $\{D, A\}$ in our path (along with its endvertices), or the edge $\{L, H\}$ (along with its endvertices). Thus, we can again find (at least) two internally disjoint paths taking us from any vertex in $\{C, D, E, F, K, L\}$ to any vertex in $\{A, B, G, H\}$.

Additional Remark. Here we could more simply note that there is a cycle in G_0 which contains all the vertices in G_0: say the cycle $L\,K\,C\,D\,E\,F\,A\,G\,B\,H\,L$. Then this implies that, no matter which two vertices we start with, we can find two internally disjoint paths in G_0 (which will be subpaths of this cycle) taking us from one vertex to the other vertex.

Problem 3 (cont.) (b) By part (a) we have that $\kappa(G_0) = 2$. Moreover, $\delta(G_0) = 3$ (given that e.g. $\deg(F) = 3$, while every other vertex of G_0 has degree $\geqslant 3$). Thus by Whitney's theorem $\lambda(G_0) = 2$ or $\lambda(G_0) = 3$. We will show that the latter is true by checking that $\lambda(G_0) > 2$.

Let us consider the following cycles of G_0:

$$\begin{gathered} C\,D\,A\,K\,C, \qquad K\,A\,F\,L\,K, \qquad A\,D\,E\,F\,A, \\ C\,L\,F\,E\,C, \qquad C\,D\,E\,C, \qquad \text{and} \quad K\,C\,L\,K\,. \end{gathered}$$

They have the property that the contain all the edges of the 'outer' cycle $C\,D\,E\,F\,L\,K\,C$ of G_0, as well as the edges $\{A, D\}$, $\{A, F\}$, $\{A, K\}$, $\{C, E\}$ and $\{C, L\}$ (this also implies that none of these edges could be a bridge of G_0).

Moreover, for any two of these edges, say edges e_{i_1} and e_{i_2}, we can find a cycle from the abovementioned which contains e_{i_2} but does not contain e_{i_1}. In other words, we can find a cycle in the subgraph $G_0 - e_{i_1}$ which contains e_{i_2}. But then e_{i_2} will not be a bridge of the subgraph $G_0 - e_{i_1}$, which is equivalent to saying that $\{e_{i_1}, e_{i_2}\}$ will not be an edge cut of G_0.

If in addition we consider the following cycles of G_0:

$$\begin{gathered} A\,B\,G\,A, \qquad A\,B\,H\,A, \qquad A\,G\,H\,A, \qquad B\,G\,H\,B, \\ A\,D\,C\,L\,H\,A, \qquad \text{and} \quad L\,H\,B\,A\,F\,L\,, \end{gathered}$$

then these contain all the remaining edges of G_0 as well, and we can again check that, for any two edges of G_0, say edges d_{i_1} and d_{i_2}, we can find a cycle among either the first group of cycles that we gave, or the second group, which will contain the edge d_{i_2} but will not contain the edge d_{i_1}.

In other words, we can find a cycle in $G_0 - d_{i_1}$ which contains the edge d_{i_2} (and this cycle will be one of the cycles we have already considered). As before, this implies that the edge subset $\{d_{i_1}, d_{i_2}\}$ is not an edge cut of G_0.

We conclude that G_0 doesn't have 2-edge cuts, and thus $\lambda(G_0) > 2$. By what we said above, this implies that $\lambda(G_0) = \delta(G_0) = 3$.

Problem 3 (cont.) (c) Recall that by Menger's theorem (the vertex form), we have that $\kappa(C,F) = \kappa'(C,F)$, where $\kappa'(C,F)$ is the maximum cardinality of an internally disjoint collection of paths from C to F.

Consider the following $C-F$ paths in G_0:

$$C\,E\,F, \qquad C\,D\,A\,F, \qquad \text{and} \quad C\,L\,F\,.$$

Then any two of these are internally disjoint, which shows that $\kappa'(C,F) \geqslant 3$.

On the other hand, for any $C-F$ path in G_0, we must have that its penultimate vertex is a neighbour of F. Therefore, we have up to 3 choices for what this penultimate vertex will be, which shows that any internally disjoint collection of $C-F$ paths will contain at most 3 paths.

Combining the above, we see that $\kappa(C,F) = \kappa'(C,F) = 3$.

Problem 4. (a) Let's assume that the vertex set of T is the set $\{u_1, u_2, \ldots, u_n\}$. By the Handshaking lemma, we have that

$$\sum_{i=1}^{n} \deg(u_i) = 2e(T),$$

where $e(T)$ is the size of T, that is, the total number of edges of T.

We now recall that one of the basic properties of trees gives us that $e(T) = |T| - 1 = n - 1$. Therefore,

$$\text{avgdeg}(T) = \frac{1}{n} \sum_{i=1}^{n} \deg(u_i) = \frac{2(n-1)}{n} = 2 - \frac{2}{n}\ .$$

We can now solve for $n = |T|$ in the above equality:

$$\begin{aligned} \text{avgdeg}(T) = 2 - \frac{2}{|T|} \quad &\Rightarrow \quad \frac{2}{|T|} = 2 - \text{avgdeg}(T) \\ &\Rightarrow \quad |T| = \frac{2}{2 - \text{avgdeg}(T)}\ . \end{aligned}$$

(b) We will see that <u>only Seq_3</u> can be viewed as the degree sequence of a <u>disconnected</u> graph.

- If $\text{Seq}_1 = (4, 4, 4, 4, 4)$ were the degree sequence of a graph, then the graph would have 5 vertices, and hence at least one of its vertices would be joined with each of the other vertices of the graph. This is because the sequence contains terms equal to $4 = 5 - 1$ (in fact, in this case all the terms are equal to 4, hence every vertex of the graph would be joined with each of the other vertices). Therefore, such a graph would not be disconnected.

- If we assume that there is a disconnected graph G realising $\text{Seq}_2 = (2, 2, 2, 2, 2)$, then this graph would need to have at least two connected components, say components G_1 and G_2. Also it would contain 5 vertices.

  We note that G_1 would need to contain at least 3 of these vertices, since the degree of each vertex in G_1 would be 2 (so each vertex would need to be joined with 2 more vertices in G_1). But then there are at most 2 vertices remaining that could be contained in G_2, which will contradict the assumption that the degree of each vertex in G_2 is 2.

We conclude that $\mathrm{Seq}_2 = (2, 2, 2, 2, 2)$ cannot be the degree sequence of a graph with at least two connected components.

- We use a similar reasoning to that used for Seq_2: if we assume that there is a disconnected graph H realising $\mathrm{Seq}_3 = (5, 4, 4, 3, 2, 2, 2, 2, 2)$, then this graph would need to have at least two connected components, say components H_1 and H_2.

  We can start by assuming that these are the only components of H, and hence that $H = H_1 \oplus H_2$. But then we could get the degree sequence of H if we wrote the degree sequence of the graph H_2 next to the degree sequence of the graph H_1.

  In other words, to view $\mathrm{Seq}_3 = (5, 4, 4, 3, 2, 2, 2, 2, 2)$ as the degree sequence of a disconnected graph, we could try to see whether we can break it into two subsequences which are both degree sequences of some graphs.

  Note that the subsequence that will contain the term 5 needs to contain **at least six terms** (since it would correspond to a vertex which is joined with 5 other vertices in that connected component of H). Then we will be left with **at most three terms** from Seq_3 for the remaining connected component, and hence these terms cannot be larger than 2.

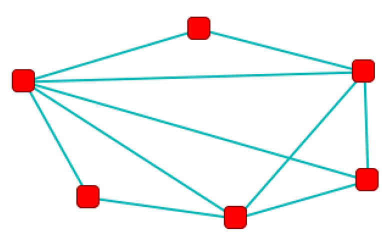
  This shows that breaking $\mathrm{Seq}_3 = (5, 4, 4, 3, 2, 2, 2, 2, 2)$ into two potentially graphical subsequences can only be done in the following way:

  the first subsequence will be $(5, 4, 4, 3, 2, 2)$,

  and the second subsequence will be $(2, 2, 2)$.

  It remains to check whether these subsequences are indeed graphical. Note that the second subsequence is the degree sequence of a 3-cycle (and thus the second connected component of H will be a 3-cycle).

  To check if the first subsequence is graphical, we could try guessing a graph on 6 vertices that would realise it: for example the graph



could be the first connected component of H, and hence H could be the following graph:

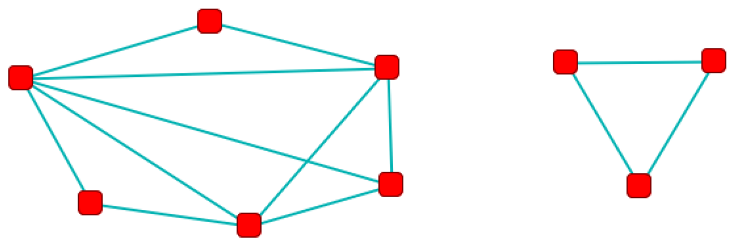


Figure 5: Graph H realising Seq_3

<u>*Alternatively,*</u> we could use the Havel-Hakimi theorem to determine whether the subsequence $(5, 4, 4, 3, 2, 2)$ is graphical or not. We have

$$\begin{gathered}(5,4,4,3,2,2) \text{ is graphical}\\ \text{if and only if}\\ (4-1,\ 4-1,\ 3-1,\ 2-1,\ 2-1) = (3,3,2,1,1) \text{ is graphical}\\ \text{if and only if}\\ (3-1,\ 2-1,\ 1-1,\ 1) = (2,1,0,1) \text{ is graphical.}\end{gathered}$$

But the last sequence, sequence $(2, 1, 0, 1)$, which can be rewritten in decreasing order as $(2, 1, 1, 0)$, is graphical, as it is the degree sequence of $P_3 \oplus N_1$ (the disjoint union of a path on 3 vertices and of the unique graph on 1 vertex).

Thus by the Havel-Hakimi theorem, the original sequence $(5, 4, 4, 3, 2, 2)$ is also graphical. This shows that both subsequences of Seq_3 that we considered can be viewed as degree sequences of different connected components of a graph H.