MATH 227

Recitation Hour - Tuesday March 24

Recall from MATH 127

Main Theorems A&B Let $\mathbb{F}$ be a field, and let V be a vector space over $\mathbb{F}$. Then
 (I) V has a basis.
 (II) Every basis of V has the same size.

We saw in MATH 127 how one proves (II) in the finite-dimensional setting.
But how do we justify it in the infinite-dimensional setting?
And first of all, how do we justify (I) in either setting?

Proof of (I) We will look at linearly independent subsets of V.
More specifically, consider the following collection of subsets of V:

$$\mathcal{F}_L := \{ T \subseteq V : T \text{ is finite and linearly independent} \}$$

In other words, $\mathcal{F}_L$ is a subset of the powerset $\mathcal{P}(V)$ of V.
Look also at the collection

$$\mathcal{S}_L := \{ \operatorname{span}(T) : T \in \mathcal{F}_L \}.$$

Again this is a subset of the powerset $\mathcal{P}(V)$ of V, that is, it contains subsets of V, and more specifically in this case subspaces of V.

Now there are two mutually exclusive possibilities: either $V \in S_L$ or $V \notin S_L$.

If $V \in S_L$, then by how we constructed S_L we can see that there is a finite, linearly independent subset B of V such that $V = \mathrm{span}(B)$.
But then this subset B is a basis of V, as we wanted.

On the other hand:
If $V \notin S_L$, then V doesn't have a finite basis. But will it have an infinite basis?

To be able to conclude this, we need a very powerful Axiom from Set Theory (that is, a mathematical statement that we <u>cannot</u> prove from other mathematical statements we've been relying on so far, but which we have to assume true, have to take for granted, if we want to do Linear Algebra in Infinite Dimensions).

<u>Question</u> Suppose we have a (possibly infinite) collection of <u>non-empty</u> sets indexed by a set I:

$$\{X_i : i \in I\}$$

(we can suppose that all these sets are subsets of a a very big set A).

Is it possible to choose one element from each set X_i?

That is, can we define a function

$$f: I \longrightarrow \bigcup\{X_i : i \in I\}$$

such that for each $i \in I$ $f(i) \in X_i$?

<u>Answer</u> Surprisingly enough, and even though the question sounds very natural, there is no guarantee that we can find such a function f because there isn't always a "constructive" way to pick one element from each X_i.

<u>Popular example here</u> If we assumed that each X_i is a "box" containing a new pair of shoes, then we would be able to define a function with the desired properties by specifying that for each $i \in I$ we pick the left shoe in X_i.

However, if each X_i contained a new pair of socks, then we wouldn't be able to define a function as desired based only on mathematical statements that we have been using so far.

This is why the existence of such a function f is introduced as a new Axiom:

Axiom of Choice Whenever we have a collection

$$\{X_i : i \in I\}$$

of non-empty sets X_i indexed by a set I, we can find a function

$$f : I \longrightarrow \bigcup\{X_i : i \in I\}$$

such that for each $i \in I$ $f(i) \in X_i$.

There are several other statements that are equivalent to the Axiom of Choice, and some are more famous and more commonly used than others.

One of the most useful statements that are equivalent to the Axiom of Choice is what we call Zorn's lemma or the Kuratowski - Zorn lemma.

To state it, we need the following

Definition Let C be a set. An ordering or order on C is a relation "$\leq$" on C which has the following properties:

(Reflexive) For every $a \in C$, $a \leq a$

(Transitive) For every $a, b, c \in C$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

(Antisymmetric) For every $a, b \in C$, if $a \leq b$ and $b \leq a$, then $a = b$.

Moreover, if we can compare any two elements in C, that is, if for every $a, b \in C$ we have $a \leq b$ or $b \leq a$, then we call the order a **total order**.

Otherwise, we say that "$\leq$" is a **partial order** on C, and we call C a partially ordered set.

Definition Let C be a partially ordered set and let T be a subset of C. Then clearly T is also an ordered set because we can simply consider the restriction of the order "$\leq$" on C to the elements of T (and with this restricted ordering T may be again a partially ordered set or it could even be a totally ordered set).

An upper bound of T is an element γ in C that is $\geq$ every element in T.

That is, γ is an upper bound of T if

for every $b \in T$, $b \leq \gamma$.

We can now state

Zorn's Lemma Let C be a partially ordered set with the following property:

every subset T of C that is totally ordered has an upper bound in C.

Then C has a maximal element, that is, an element

γ that is <u>not</u> $<$ than any other element in C.
In other words, we cannot find another element δ in C such that $\gamma \leq \delta$ and $\gamma \neq \delta$.

Some examples of partially/totally ordered sets and of upper bounds of subsets:

1) $\mathbb{R}$, the set of real numbers, with the standard order is a totally ordered set. Thus every subset of it is also totally ordered.

If $T_1 = (-\infty, 0)$, then an upper bound of T_1 is the number 0, or the number e (or any other non-negative number). Note however that no upper bound of T_1 belongs to T_1 (because if that were the case, then that upper bound would be the maximum element of T_1, whereas T_1 does <u>not</u> have a maximum element).

On the other hand, $T_2 = \mathbb{N}$ does not have an upper bound. Thus, $(\mathbb{R}, \leq)$ fails the required property of Zorn's Lemma, which makes sense since it also fails its conclusion: no element of $(\mathbb{R}, \leq)$ is maximal.

2) Let A be a set. Then the powerset $\mathcal{P}(A)$ of A admits a very natural order: for any $S_1, S_2 \in \mathcal{P}(A)$ (that is, for any subsets S_1, S_2 of A) we define $S_1 \leq S_2$ if and only if $S_1 \subseteq S_2$.
Then this is a partial order on $\mathcal{P}(A)$ (and in general it's not a total order: e.g. if $A = \{0, 1\}$, then $\{0\}, \{1\} \in \mathcal{P}(A)$ and we have neither $\{0\} \leq \{1\}$ nor $\{1\} \leq \{0\}$).

Proof of Main Thm A & B (I) continued on Thursday