# Problems 4

*10/09/2020*

1) Let $(V, b)$ be an euclidean vector space, *i.e.* $V$ is a finite dimensional real vector space with positive definite symmetric bilinear form $b : V \times V \longrightarrow \mathbb{R}$. Show that every element of the orthogonal group $\mathrm{O}(V, b)$ can be written as a product of $\leq \dim_{\mathbb{R}} V$ reflections.

The aim of the following problem is a generalization of what has been proven in Topic 4 of the lectures. All assertions can be proven by slight modifications of the arguments used in these lectures for an euclidean vector space.

2) Let $F$ be a field of characteristic $\neq 2$ and $V$ a $F$-vector space of dimension $n < \infty$. A *quadratic form* on $V$ is a map $q : V \longrightarrow F$, such that
   (a) $q(\lambda \cdot v) = \lambda^2 \cdot q(v)$ for all $v \in V$ and $\lambda \in F$, and
   (b)
   $$b_q(v, w) := \frac{1}{2} \cdot \big(q(v + w) - q(v) - q(w)\big)$$
   is a symmetric bilinear form, *i.e.*
   $$b_q(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 b(v_1, w) + \lambda_2 b(v_2, w)$$
   for all $v_1, v_2, w \in V$ and $\lambda_1, \lambda_2 \in F$.

   The quadratic form $q$ is called *non degenerate* if $b_q(v, w) = 0$ for all $w \in V$ implies $v = 0$. If $q$ is non degenerated then $(V, q)$ is called a *quadratic space*.
   (i) Let $(V, q)$ be a quadratic space and $\mathrm{O}(V, q)$ be the set of all $F$-linear maps $f : V \longrightarrow V$ satisfying $q(f(v)) = q(v)$ for all $\in V$. Show that $\mathrm{O}(V, q)$ is a subgroup of $\mathrm{GL}(V)$.
   (ii) A vector $v \in V$ is called *anisotropic* (with respect to $q$) if $q(v) \neq 0$. Show that the map
   $$s_v : V \longrightarrow V, \; x \longmapsto x - \frac{2b_q(v, x)}{q(v)} \cdot v$$
   is in $\mathrm{O}(V, b)$ for all anisotropic vectors $v$ in $V$. The map $s_v$ is called a *reflection*.
   (iii) Show that $\mathrm{O}(V, q)$ is generated by the set of elements
   $$\Big\{ s_v \,\Big|\, v \in V \text{ a anisotropic vector} \Big\}.$$
   (iv) Determine the centre of $\mathrm{O}(V, q)$.

3) Let $< x, y > := x^t \cdot y$ be the usual scalar product on $\mathbb{R}^n$, $\mathrm{O}_n(\mathbb{R})$ the set of real $n \times n$-matrices $A$ with $< Ax, Ay > = < x, y >$ for all $x, y \in \mathbb{R}^n$, and $\mathrm{SO}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R}) \cap \mathrm{O}_n(\mathbb{R})$. Show that $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{SO}_n(\mathbb{R})$ are subgroups of $\mathrm{GL}_n(\mathbb{R})$, and that $\mathrm{O}_n(\mathbb{R})$ is isomorphic to a semidirect product of $\mathrm{SO}_n(\mathbb{R})$ and $\mathbb{Z}/2$.