Reminder Proposition 2 Let $F_1 = (\{\text{elements in } F_1\}, +_1, \cdot_1)$ and $F_2 = (\{\text{elements in } F_2\}, +_2, \cdot_2)$ be two fields, and let $\varphi: F_1 \to F_2$ be a field homomorphism.

Suppose φ is bijective, and hence that it has an inverse $\varphi^{-1}: F_2 \to F_1$.

Then φ^{-1} is a field homomorphism too.

Proof Let $u, v \in F_2$. We need to show

$$\varphi^{-1}(u +_2 v) = \varphi^{-1}(u) +_1 \varphi^{-1}(v)$$

and $$\varphi^{-1}(u \cdot_2 v) = \varphi^{-1}(u) \cdot_1 \varphi^{-1}(v).$$

We note that $\varphi^{-1}(u) = x \in F_1$ if $\varphi(x) = u$
and similarly $\varphi^{-1}(v) = y$ if $\varphi(y) = v$.

Since φ is a field homomorphism, we get that
$$\varphi(x +_1 y) = \varphi(x) +_2 \varphi(y) = u +_2 v.$$
Therefore $\varphi^{-1}(u +_2 v) = x +_1 y = \varphi^{-1}(u) +_1 \varphi^{-1}(v)$, as we wanted.

Again since φ is a field homomorphism, we get that
$$\varphi(x \cdot_1 y) = \varphi(x) \cdot_2 \varphi(y) = u \cdot_2 v,$$
hence $\varphi^{-1}(u \cdot_2 v) = x \cdot_1 y = \varphi^{-1}(u) \cdot_1 \varphi^{-1}(v)$, as desired.

Finally we know that $\varphi(1_{F_1}) = 1_{F_2}$, hence $\varphi^{-1}(1_{F_2}) = 1_{F_1}$.
We conclude that φ^{-1} is a field homomorphism.

Important Remark The conclusion of Proposition 2 is very useful and powerful, and does not necessarily have analogous versions in other settings. Eg.

1) We can find metric spaces X, Y and a function

$f: \mathcal{X} \to \mathcal{Y}$ that is continuous and bijective, but whose inverse $f^{-1}: \mathcal{Y} \to \mathcal{X}$ is not continuous.

Consider e.g. $\mathcal{X} = [0, 2\pi)$, $\mathcal{Y} = \{z \in \mathbb{C} : |z| = 1\}$
and $f: \mathcal{X} \to \mathcal{Y}$ given by
$\theta \in \mathcal{X} \longmapsto f(\theta) = \exp(i\theta)$
[or, equivalently because of Euler's formula,
$$\theta \in \mathcal{X} \longmapsto \cos(\theta) + i\sin(\theta)]$$
Then $f: \mathcal{X} \to \mathcal{Y}$ is continuous, but $f^{-1}: \mathcal{Y} \to \mathcal{X}$ cannot be (in particular, it fails to be continuous at $z=1$).

2) Let $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^3$. Then g is differentiable and bijective, but $g^{-1}: \mathbb{R} \to \mathbb{R}$, $g^{-1}(y) = y^{1/3}$ is not differentiable everywhere.

We now turn our attention again to vector spaces, and will explore the analogous questions in this setting.

Reminder

Vector space Let F be a field. A vector space V over F is a triple $V = (\{\text{vectors in } V\}, +, \cdot)$ of a set and two operations:
$$+ : V \times V \longrightarrow V \quad \text{(vector addition)}$$
$$\cdot : F \times V \longrightarrow V \quad \text{(scalar multiplication)}$$
such that:

i) for all $\bar{x}, \bar{y} \in V$ $\quad \bar{x} + \bar{y} = \bar{y} + \bar{x}$ (vector addition is commutative)

ii) for all $\bar{x}, \bar{y}, \bar{z} \in V$ $\quad (\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z})$ (vector addition is associative)

iii) there is a zero vector $\bar{0} \in V$ satisfying for all $\bar{x} \in V$, $\bar{0} + \bar{x} = \bar{x} = \bar{x} + \bar{0}$.

iv) For every $\bar{x} \in V$ there exists a vector $-\bar{x} \in V$ satisfying
$$\bar{x} + (-\bar{x}) = \bar{0} = (-\bar{x}) + \bar{x}.$$

v) for all $\lambda, \mu \in F$ and all $\bar{x} \in V$
$$(\lambda \cdot \mu)\bar{x} = \lambda(\mu \bar{x})$$

vi) for all $\bar{x} \in V$ $1_F \bar{x} = \bar{x}$.

vii) for all $\lambda \in F$ and $\bar{x}, \bar{y} \in V$ (scalar multiplication distributes over vector addition)
$$\lambda(\bar{x} + \bar{y}) = \lambda \bar{x} + \lambda \bar{y}$$

viii) for all $\lambda, \mu \in F$ and $\bar{x} \in V$ (scalar multiplication distributes over field addition)
$$(\lambda + \mu)\bar{x} = \lambda \bar{x} + \mu \bar{x}$$

What is a substructure here?

Definition Let V be a vector space over a field F. A subset W of V is a subspace of V if W is a vector space over F with the restricted operations to W of vector addition and scalar multiplication in V.

Remark Equivalently, $W \subseteq V$ is a subspace, if W is a nonempty set which is

i) closed under addition

and ii) closed under scalar multiplication.

Reminder: what is a structure preserving function here?

<u>Definition</u> Let F be a field, and V_1, V_2 be vector spaces over F, $V_1 = (\{\text{vectors in } V_1\}, +_1, \cdot_1)$, $V_2 = (\{\text{vectors in } V_2\}, +_2, \cdot_2)$.
A function $f: V_1 \longrightarrow V_2$ is a <u>vector space homomorphism</u> or <u>linear transformation</u> or <u>linear map</u> if

i) $\forall\, \bar{x}, \bar{y} \in V_1 \quad f(\bar{x} +_1 \bar{y}) = f(\bar{x}) +_2 f(\bar{y})$.

ii) $\forall\, r \in F, \forall\, \bar{x} \in V_1 \quad f(r \cdot_1 \bar{x}) = r \cdot_2 f(\bar{x})$.

<u>Proposition 1</u> Let V_1, V_2 be vector spaces over a field F, and let $f: V_1 \to V_2$ be a linear map.
Then $\text{Range}(f)$ is a subspace of V_2.

<u>Proposition 2</u> Let V_1, V_2, V_3 be vector spaces over a field F, and let $f: V_1 \to V_2$ and $g: V_2 \to V_3$ be linear maps. Then $g \circ f: V_1 \to V_3$ is a linear map.

<u>Proposition 3</u> Let V_1, V_2 be vector spaces over F, and let $f: V_1 \to V_2$ be a linear map.
Suppose f is bijective, and hence $f^{-1}: V_2 \to V_1$ exists. We have that $f^{-1}: V_2 \to V_1$ is a linear map too.

Linear Maps (cont.)

Reminders

Proposition 1 Let V_1, V_2 be vector spaces over a field F, and let $f: V_1 \to V_2$ be a linear map.
Then Range(f) is a subspace of V_2.

Proposition 2 Let V_1, V_2, V_3 be vector spaces over a field F, and let $f: V_1 \to V_2$ and $g: V_2 \to V_3$ be linear maps.
Then $g \circ f: V_1 \to V_3$ is a linear map.

Proposition 3 Let V_1, V_2 be vector spaces over F, and let $f: V_1 \to V_2$ be a linear map.
Suppose f is also bijective (and hence $f^{-1}: V_2 \to V_1$ is defined). Then f^{-1} is a linear map too.

Propositions 2 and 3 left as HW1, Problem 4.

Proposition 3 allows us to give the following definition:

Definition Let F be a field and let V_1, V_2 be vector spaces over F. Suppose $f: V_1 \to V_2$ is a linear map. If f is bijective, we call it a linear isomorphism.

If there exists a linear isomorphism from V_1 to V_2, we say that V_1 and V_2 are isomorphic vector spaces.

Proof of Proposition 1: We need to check that Range(f)
- is nonempty
- is closed under vector addition
- is closed under scalar multiplication

Range(ℓ) is nonempty Indeed $\ell(\bar{0}_{V_1}) \in \text{Range}(\ell)$.

Remark We can even say something more detailed here:

$$\ell(\bar{0}_{V_1}) = \bar{0}_{V_2} \text{ and hence } \bar{0}_{V_2} \in \text{Range}(\ell).$$

One way to show that $\ell(\bar{0}_{V_1}) = \bar{0}_{V_2}$ is to note that

$$\bar{0}_{V_1} = 0_F \cdot \bar{0}_{V_1} \Rightarrow \ell(\bar{0}_{V_1}) = \ell(0_F \cdot \bar{0}_{V_1}) \underset{\ell \text{ linear}}{=} 0_F \cdot \ell(\bar{0}_{V_1}) = \bar{0}_{V_2}.$$

Alternatively, we could note that

$$\bar{0}_{V_1} = \bar{0}_{V_1} + \bar{0}_{V_1} \Rightarrow \ell(\bar{0}_{V_1}) = \ell(\bar{0}_{V_1} + \bar{0}_{V_1}) \underset{\ell \text{ linear}}{=} \ell(\bar{0}_{V_1}) + \ell(\bar{0}_{V_1})$$

Also $\ell(\bar{0}_{V_1}) = \ell(\bar{0}_{V_1}) + \bar{0}_{V_2}$, therefore by the cancellation law for vector addition we must have $\ell(\bar{0}_{V_1}) = \bar{0}_{V_2}$.

Side Note Knowing that every linear map $\ell: V_1 \to V_2$ satisfies $\ell(\bar{0}_{V_1}) = \bar{0}_{V_2}$ gives us an easy criterion to check whenever we want to determine whether a function $g: V_1 \to V_2$ is linear: if it doesn't satisfy $g(\bar{0}_{V_1}) = \bar{0}_{V_2}$, then it is definitely not linear.

Returning to the Proof of Prop 1:

Range(ℓ) is closed under vector addition Let $\bar{u}, \bar{v} \in \text{Range}(\ell)$. Then we can find $\bar{x}, \bar{y} \in V_1$ such that

$$\ell(\bar{x}) = \bar{u} \text{ and } \ell(\bar{y}) = \bar{v}.$$

But then $\bar{u} + \bar{v} = \ell(\bar{x}) + \ell(\bar{y}) \underset{\ell \text{ linear}}{=} \ell(\bar{x} + \bar{y}) \in \text{Range}(\ell)$.

Thus, for the arbitrary $\bar{u}, \bar{v} \in \text{Range}(\ell)$, we have $\bar{u} + \bar{v} \in \text{Range}(\ell)$.

Range(ℓ) is closed under scalar multiplication Let $\bar{u} \in \text{Range}(\ell)$ and $r \in F$. We can find $\bar{x} \in V_1$ such that $\ell(\bar{x}) = \bar{u}$.

But then $r \cdot \bar{u} = r \cdot \ell(\bar{x}) \underset{\ell \text{ linear}}{=} \ell(r \cdot \bar{x}) \in \text{Range}(\ell)$. Thus for the arbitrary $\bar{u} \in \text{Range}(\ell)$ and $r \in F$ we have $r \cdot \bar{u} \in \text{Range}(\ell)$.

<u>Definition</u> Let ℓ be a linear map from V_1 to V_2. The <u>kernel</u> or <u>Nullspace</u> of ℓ, which we denote by $\ker(\ell)$, is the set

$$\ell^{-1}(\{\bar{0}_{V_2}\}) = \{\bar{x} \in V_1 : \ell(\bar{x}) = \bar{0}_{V_2}\}.$$

<u>Important Note</u> From the Remark included in the proof of Proposition 1, we see that $\bar{0}_{V_1} \in \ker(\ell)$.

<u>Proposition 4</u> Let $\ell : V_1 \longrightarrow V_2$ be a linear map. Then $\ker(\ell)$ is a subspace of V_1.

<u>Proof</u> $\ker(\ell)$ is nonempty since $\bar{0}_{V_1} \in \ker(\ell)$.

We check whether $\ker(\ell)$ is closed under vector addition: let $\bar{x}, \bar{y} \in \ker(\ell)$. Then by definition $\ell(\bar{x}) = \bar{0}_{V_2} = \ell(\bar{y})$.

Since ℓ is linear, we then get

$$\ell(\bar{x}+\bar{y}) = \ell(\bar{x}) + \ell(\bar{y}) = \bar{0}_{V_2} + \bar{0}_{V_2} = \bar{0}_{V_2}.$$

Thus $\bar{x}+\bar{y} \in \ker(\ell)$. Since $\bar{x}, \bar{y}$ were arbitrary elements of $\ker(\ell)$, we conclude that $\ker(\ell)$ is closed under addition.

We similarly check whether $\ker(\ell)$ is closed under scalar multiplication: Let $\bar{x} \in \ker(\ell)$ and $r \in \mathbb{F}$. Then we have $\ell(\bar{x}) = \bar{0}_{V_2}$, so $\ell(r \cdot \bar{x}) \underset{\ell \text{ linear}}{=} r \cdot \ell(\bar{x}) = r \cdot \bar{0}_{V_2} = \bar{0}_{V_2}$, thus $r \cdot \bar{x} \in \ker(\ell)$.

Since $\bar{x} \in \ker(\ell)$ and $r \in \mathbb{F}$ were arbitrary, we conclude that $\ker(\ell)$ is closed under scalar multiplication.

The above combined give the conclusion.

Propositions 1 and 4 allow us to come up with many examples of subspaces: we've just seen that if V_1, V_2 are two subspaces over the same field $\mathbb{F}$, and if $\ell : V_1 \to V_2$ is a linear map, then Range(ℓ) is a subspace of V_2 and Ker(ℓ) is a subspace of V_1.

We will see some examples like this in a bit.

Another example of a subspace is the following:

Remark Let V be a vector space over a field $\mathbb{F}$. The set $\{\bar{0}_V\}$ is a subspace of V (check that it satisfies the necessary conditions)

Proposition 5 Let $\ell : V_1 \to V_2$ be a linear map. Then ℓ is injective if and only if $\mathrm{Ker}(\ell) = \{\bar{0}_{V_1}\}$.

Proof Recall that by definition ℓ is injective if, for any $\bar{x}, \bar{y} \in V_1$,

$$\ell(\bar{x}) = \ell(\bar{y}) \quad \text{implies} \quad \bar{x} = \bar{y}.$$

We first show that

$$\ell \text{ is injective} \implies \mathrm{Ker}(\ell) = \{\bar{0}_{V_1}\}.$$

Indeed we know that $\bar{0}_{V_1} \in \mathrm{Ker}(\ell)$. Now, if $\bar{x} \in V_1$ and $\bar{x} \neq \bar{0}_{V_1}$, then, since ℓ is assumed to be injective, we must have $\ell(\bar{x}) \neq \ell(\bar{0}_{V_1}) = \bar{0}_{V_2}$. Thus $\bar{x} \notin \mathrm{Ker}(\ell)$. This shows that the only element of V_1 that is contained in $\mathrm{Ker}(\ell)$ is $\bar{0}_{V_1}$.

Now we show that

$$\mathrm{Ker}(\ell) = \{\bar{0}_{V_1}\} \implies \ell \text{ is injective.}$$

Assume $\mathrm{Ker}(\ell) = \{\bar{0}_{V_1}\}$ and consider $\bar{x}, \bar{y} \in V_1$ with $\ell(\bar{x}) = \ell(\bar{y})$. We need to show that $\bar{x} = \bar{y}$.

We have $\ell(\bar{x}) = \ell(\bar{y}) \implies$

$$\ell(\bar{x}) - \ell(\bar{y}) = \ell(\bar{x}) + (-\ell(\bar{y})) = \ell(\bar{y}) + (-\ell(\bar{y})) = \bar{0}_{V_2} \implies$$

$$\ell(\bar{x}) + (-1\cdot \ell(\bar{y})) = \bar{0}_{V_2} \Rightarrow$$ ℓ is linear
$$\ell(\bar{x}) + \ell(-1\cdot\bar{y}) = \bar{0}_{V_2} \Rightarrow$$
$$\ell(\bar{x} + (-1)\cdot\bar{y}) = \bar{0}_{V_2} \Rightarrow$$
$$\bar{x} - \bar{y} = \bar{x} + (-1)\bar{y} \in \ker(\ell) = \{\bar{0}_{V_1}\} \Rightarrow$$
$$\bar{x} - \bar{y} = \bar{0}_{V_1} \Rightarrow \bar{x} = \bar{y}.$$

Since $\bar{x}, \bar{y} \in V_1$ were arbitrary elements (assumed only to satisfy $\ell(\bar{x}) = \ell(\bar{y})$), we can conclude that ℓ is injective.

Examples of Subspaces / Remarks

1) Let V be a vector space over a field $\mathbb{F}$. We already saw that $\{\bar{0}\}$ is a subspace of V.
Also V is a subspace of V.
$\{\bar{0}\}$ and V are called the trivial subspaces of V.

2) No subspace of V is empty. Therefore $\{\bar{0}\}$ is the subspace of V with the smallest size.

3) Moreover, given any subspace S_1 of V, we have that $\bar{0}_V \in S_1$ (why?). Thus

$$\{\bar{0}_V\} \leq S_1 \leq V.$$

we use this symbol to denote "$\{\bar{0}_V\}$ is a subspace of S_1"

In addition, if S_2 is another subspace of V, then $S_1 \cap S_2$ is nonempty since $\bar{0}_V \in S_1, \bar{0}_V \in S_2 \Rightarrow \bar{0}_V \in S_1 \cap S_2$.
(we will later see that $S_1 \cap S_2$ is a subspace too).

4) Recall that if V_1, V_2 are vector spaces over the same field $\mathbb{F}$, and if $\ell: V_1 \to V_2$ is a linear map, then $\text{Range}(\ell)$ is a subspace of V_2 and $\ker(\ell)$ is a subspace of V_1.

One example: let $V_1 = V_2 = \mathcal{P}_4 = \{$real polynomials of degree at most $4\}$ (recall that this is a vector space over $\mathbb{R}$) and let $D: \mathcal{P}_4 \to \mathcal{P}_4$ be a function defined as follows:

if $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

then $D(p(x)) := a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$.

We can call D the <u>derivative operator</u>

Check that D is a linear map.

What is Range(D) and what is ker(D)?

(left as an exercise until next time)

Continuing with examples of subspaces:

Last example from previous time:
Consider the derivative operator $D: \mathcal{P}_4 \to \mathcal{P}_4$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \in \mathcal{P}_4$$
$$\longmapsto D(p(x)) := a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \in \mathcal{P}_4$$

We show that D is a linear map:
let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$
and $q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4$
be two arbitrary polynomials in $\mathcal{P}_4$.
Then
$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 + (a_4 + b_4)x^4$$
and
$$D(p+q) = (a_1 + b_1) + 2(a_2 + b_2)x + 3(a_3 + b_3)x^2 + 4(a_4 + b_4)x^3$$
$$= (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3) + (b_1 + 2b_2 x + 3b_3 x^2 + 4b_4 x^3) = D(p) + D(q).$$

Similarly we can show that, if $r \in \mathbb{R}$, then
$$D(r \cdot p(x)) = D(ra_0 + (ra_1)x + (ra_2)x^2 + (ra_3)x^3 + (ra_4)x^4)$$
$$= ra_1 + 2ra_2 x + 3ra_3 x^2 + 4ra_4 x^3$$
$$= r(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3) = r \cdot D(p(x)).$$

What is Range(D)? We see that, for every $p(x) \in \mathcal{P}_4$
$D(p(x)) \in \mathcal{P}_3 = \{$real polynomials of degree at most $3\}$.
Thus Range$(D) \subseteq \mathcal{P}_3$ (and in fact Range$(D) \leq \mathcal{P}_3$)
is a subspace

Let's now check that Range$(D) = \mathcal{P}_3$: it suffices to check that $\mathcal{P}_3 \subseteq$ Range(D).
Let $q(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \in \mathcal{P}_3$,
then $p(x) = 0 + c_0 x + \frac{c_1}{2} x^2 + \frac{c_2}{3} x^3 + \frac{c_3}{4} x^4 \in \mathcal{P}_4$

and $D(p(x)) = q(x)$. Thus $q(x) \in \text{Range}(D)$ and since $q(x)$ was an arbitrary element of $\mathcal{P}_3$, we get $\mathcal{P}_3 \subseteq \text{Range}(D)$.

Thus $\text{Range}(D) = \mathcal{P}_3$.

What is $\text{Ker}(D)$? By definition

$$\text{Ker}(D) = \{p(x) \in \mathcal{P}_4 : D(p(x)) = 0\}$$

But $0 = D(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$ if and only if $a_1 = a_2 = a_3 = a_4 = 0$.

Thus $\text{Ker}(D) = \mathcal{P}_0 = \{\text{all constant polynomials}\}$.

__Another example__

Consider the following function

$$\mathcal{E} : \mathcal{P}_4 \longrightarrow \mathbb{R}, \quad p(x) \in \mathcal{P}_4 \longmapsto \mathcal{E}(p(x)) := p(1).$$

In other words, $\mathcal{E}(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)$

$$= a_0 + a_1 + a_2 + a_3 + a_4$$

Check that $\mathcal{E}$ is a linear map.

What are $\text{Range}(\mathcal{E})$ and $\text{Ker}(\mathcal{E})$?

$\text{Range}(\mathcal{E}) = \mathbb{R}$ since for any $a_0 \in \mathbb{R}$, the polynomial $p(x) = a_0$ is in $\mathcal{P}_4$ and $\mathcal{E}(p(x)) = a_0$.

$$\text{Ker}(\mathcal{E}) = \{p(x) \in \mathcal{P}_4 : p(1) = 0\}$$
$$= \{p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \in \mathcal{P}_4 : a_0 + a_1 + a_2 + a_3 + a_4 = 0\}$$

We can show that this coincides with the set of polynomials $p(x) \in \mathcal{P}_4$ that are divided by $x-1$:

$$\text{Ker}(\mathcal{E}) = \{p(x) \in \mathcal{P}_4 : \exists\, q(x) \in \mathcal{P}_3 \text{ such that } p(x) = (x-1)q(x)\}$$

Indeed consider some $p(x) \in Ker(\mathcal{E})$,
$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \text{ with } a_0 + a_1 + a_2 + a_3 + a_4 = 0$$

Then
$$\begin{aligned} p(x) &= p(x) - p(1) \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 - (a_0 + a_1 + a_2 + a_3 + a_4) \\ &= \cancel{a_0 - a_0} + a_1(x-1) + a_2(x^2-1) + a_3(x^3-1) + a_4(x^4-1) \\ &= a_1(x-1) + a_2(x-1)(x+1) + a_3(x-1)(x^2+x+1) \\ &\quad + a_4(x-1)(x^3+x^2+x+1) \\ &= (x-1)\cdot q(x) \end{aligned}$$
where $q(x) = a_1 + a_2(x+1) + a_3(x^2+x+1) + a_4(x^3+x^2+x+1) \in \mathcal{P}_3$.

Note Recall that $Ker(\mathcal{E})$ is a subspace of $\mathcal{P}_4$.
Having this description of $Ker(\mathcal{E})$:
$$Ker(\mathcal{E}) = \{\text{polynomials in } \mathcal{P}_4 \text{ that are divided by } x-1\}$$
will be helpful when we seek to find a basis for this subspace of $\mathcal{P}_4$.

More examples of subspaces?
We have already seen some more in MATH 127, but using another way of viewing subspaces:

Subspaces coincide with Linear Spans

Reminder / More general form of the definition

Definition Let V be a vector space over a field $\mathbb{F}$, and let T be a subset of V (not necessarily finite). Then the linear span of T is the set
$$\{w \in V : \exists\, n_0 \in \mathbb{N} \text{ and } u_1, \dots, u_n \in T, \lambda_1, \dots, \lambda_n \in \mathbb{F} \text{ such that } w = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_n u_n\}.$$

We denote this set by span(T).

Important Remark By definition we are allowed to take the empty sum
$$\sum_{i \in \emptyset} u_i$$
which by convention is the zero vector $\overline{0}_V$. Therefore, span(T) is nonempty in all cases since it contains $\overline{0}_V$ (this is true even if $T = \emptyset$; of course in that case we have $\text{span}(T) = \text{span}(\emptyset) = \{\overline{0}_V\}$).

Lemma 1 Let V be a vector space over a field $\mathbb{F}$, and let S_1 be a subspace of V. Then S_1 is the linear span of some subset of V.

Proof 1st way We trivially have
$$S_1 = \text{span}(S_1).$$
Indeed, we always have $T \subseteq \text{span}(T)$, no matter what the subset T is, so we have $S_1 \subseteq \text{span}(S_1)$ in this case too.

At the same time, given that S_1 is a subspace of V, we have that it is closed under addition and scalar multiplication, so every sum of the form
$$\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_n u_n, \quad u_1, \dots u_n \in S_1, \lambda_1, \dots \lambda_n \in \mathbb{F}$$
must be in S_1.

In other words, $\text{span}(S_1) \subseteq S_1$.

Combining the above, we get that $S_1 = \text{span}(S_1)$, as we wanted.

2nd way Recall one of the main theorems we stated in MATH 127 (to be proven this term)

Main Thm B Every vector space has a basis.

Since S_1 is a vector space itself over $\mathbb{F}$, it must have

a basis $\mathcal{B}_1$, that is, a linearly independent subset $\mathcal{B}_1$ of S_1 which is also a spanning set.
Thus $S_1 = \mathrm{span}(\mathcal{B}_1)$ and we also have $\mathcal{B}_1 \subseteq S_1$ and $S_1 \subseteq V \Rightarrow \mathcal{B}_1 \subseteq V$, as we wanted.

We also want the converse result:

<u>Proposition 1</u> Let V be a vector space over a field $\mathbb{F}$, and let $T \subseteq V$. Then $\mathrm{span}(T)$ is a subspace of V.
<u>Proof</u> left for next time

Let us finally note down an important fact that can be justified by the 2nd proof we gave to Lemma 1:

<u>Remark</u> Let V be a vector space over a field $\mathbb{F}$, and let S_1 be a subspace of V. Then

$$\dim_{\mathbb{F}} S_1 \leq \dim_{\mathbb{F}} V.$$

Indeed we saw that S_1 has a basis $\mathcal{B}_1$. This is a linearly independent subset of S_1, so (by looking at the definition of a linearly independent set) we get that it is a linearly independent subset of V as well.
Thus $\dim_{\mathbb{F}} V =$ largest possible size of a linearly independent subset of V

recall this equivalent definition

$$\geq |\mathcal{B}_1| = \dim_{\mathbb{F}} S_1.$$