MATH 118: Honors Calculus II

Winter 2020 – Assignment 1

Problem 1

Show that $\sin x$ and $\cos \mathrm{x}$ as functions on $\mathbb{R}$ are *linearly independent*, that is $\mathrm{a} \sin + \mathrm{b} \cos = 0$ for $a, b \in \mathbb{R}$ if and only if $a = b = 0$.

Solution

In a sense this is a trick question: it does not ask you to show that $a = b = 0$ whenever for a specific x, $a \sin x + b \cos x = 0$. Rather, you need to prove that $a = b = 0$ if $a \sin x + b \cos x = 0$ **for all** x.

So suppose that is true. Then evaluating at $x = 0$ we get $a \sin 0 + b \cos 0 = 0$ and thus $a \cdot 0 + b \cdot 1 = 0$, that is, $b = 0$. But then $a \sin x = 0$ for all x forces $a = 0$ (e.g. pick $x = \frac{\pi}{2}$).

Problem 2

Let f be a polynomial function on $\mathbb{R}$.

A. Suppose there is $x_0 \in \mathbb{R}$ such that
$$f(x_0) = f'(x_0) = \cdots = f^{(k)}(x_0) = 0$$
Show that $(x - x_0)^{k+1}$ divides f, that is, there is a polynomial g with $f = (x - x_0)^{k+1} g$.

B. Suppose there is x_0 such that $f^{(n)}(x_0) = 0$ for all $n > 0$. Show that $f = 0$.

C. Show that for any x_0 the numbers $f^{(n)}(x_0)$ ($n \in \mathbb{N} \cup \{0\}$) determine f uniquely: If $f^{(n)}(x_0) = g^{(n)}(x_0)$ for all n where g is also a polynomial, then $f = g$.

Warning: A., B., C. heavily rely on the fact that f (and g) are polynomials.

Recall that $f^{(n)}$ denotes the nth derivative of f, and $f^{(0)} = f$.

Solution

First a remark on polynomial functions: Recall that if f, q are polynomials with $q \neq 0$, there are unique polynomials g, r such that $f = qg + r$ with $\deg r < \deg f$ or $r = 0$. This is called **division with remainder** and a consequence of the Euclidean Division Algorithm (a.k.a Long Division). It is proved essentially the same way as the fact that for any two interges a, b with $b > 0$ there are unique integers k, ℓ such that $a = bk + \ell$ and $0 \leq \ell < b$. Of course, for this problem you **may assume all of this without proof**.

If you haven't seen this in High School this is admittedly a difficult problem. Here is a quick workaround for the part actually needed (but again, this is not required): if f is a polynomial function with $f(x_0) = 0$, then $f = (x - x_0)g$ for some polynomial function g: Put $y = x - x_0$. For any polynomial f and any $x \in \mathbb{R}$, we have $f(x) = f(y + x_0)$. Evaluating the polynomial formula for f we arrive at formula

$$f(x) = f(y + x_0) = h(y)$$

where h is some polynomial function. Then $f(x_0) = h(0) = 0$ if and only if $h(y) = h_0 + h_1 y + \cdots + h_m y^m$ with $h_0 = 0$. But that means $h(y) = y k(y)$ with $k(y) = h_1 + h_2 y + \cdots + h_m y^{m-1}$.

Going back that means $f(x) = (x - x_0)k(x - x_0)$ so if we put $g(x) = k(x - x_0)$ (which is again a polynomial in x), we are done.

We will need the following below: For every $k > 0$, and every polynomial p we have $\left(\frac{d^k}{dx^k}(x - x_0)^k p\right)(x_0) = cp(x_0)$ for some $c \neq 0$. This is shown by induction on k. If $k = 1$, this says $\left(\frac{d}{dx}(x - x_0)p\right) = p + (x - x_0)p'$. Evaluating at x_0 gives the result. Now suppose the assertion is true for some k, and we want to prove it for $k + 1$. Then

$$\frac{d^{k+1}}{dx^{k+1}}(x - x_0)^{k+1}p = \frac{d^k}{dx^k}\left((k+1)(x - x_0)^k p + (x - x_0)^{k+1}p'\right)$$

The right hand side is $\frac{d^k}{dx^k}((x - x_0)^k\left((k+1)p + (x - x_0)p'\right)$. By the induction hypothesis (applied for the case where $(k+1)p + (x - x_0)p'$ plays the role of p, evaluating the right hand side at x_0 we get $d\left((k+1)p(x_0) + (x_0 - x_0)p'(x_0)\right) = d(k+1)p(x_0)$ for some nonzero d. The claim then follows with $c = d(k+1) \neq 0$.

A. We will proceed by induction on k. If $k = 0$ (the base case), the assertion is that if$f(x_0) = 0$, then $f = (x - x_0)g$ for some polynomial g. This is precisely the content of the remarks above and therefore holds.
   Now suppose that we know for some specific $k \geq 0$ that $f'(x_0) = 0, f''(x_0) = 0, \ldots, f^{(k)}(x_0) = 0$ implies that $f = (x - x_0)^{k+1}h$ for some polynomial h. If also $f^{(k+1)}(x_0) = 0$ we must show that $f = (x - x_0)^{k+2}g$ for some polynomial g.
   One could show this using the Leibniz Formula (Problem 6), and that is perfectly fine. Or, we could argue as follows: We know that $f = (x - x_0)^{k+1}h$ for some h. By the above $\left(\frac{d^{k+1}}{dx^{k+1}}(x - x_0)^{k+1}h\right)(x_0) = ch(x_0)$ for some $c \neq 0$. Since we assume that $f^{(k+1)}(x_0) = 0$ we must have $h(x_0) = 0$ and therefore $h = (x - x_0)g$ for some g. Together this means $f = (x - x_0)^{k+2}g$ as needed.
B. This is a consequence of A. If $f^{(n)}(x_0) = 0$ for all $n \geq 0$, then by A. f is divisible by $(x - x_0)^n$ for every $n \geq 0$. In particular, f does not have a degree. But the only polynomial without a degree is 0.
C. This is B. applied to $f - g$.

Problem 3

Let $r < 0$ be a rational number, and let $f: (0, \infty) \to \mathbb{R}$ be defined as $f(x) = x^r$. Show that f is differentiable everywhere and $f'(x) = rx^{r-1}$.

Solution

There are many ways to approach it. The quickest is the following: we know the result if $r > 0$ is a rational number from class. But then $f(x) = \frac{1}{x^{-r}}$. The Chain Rule gives $f'(x) = -\frac{1}{x^{-2r}}(-r)x^{-r-1}$. We get

$$f'(x) = rx^{2r}x^{-r-1} = rx^{r-1}$$

Problem 4

Let $a_n, b_n \geq 0$ be sequences.

A. Show that $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, with equality if b_n is convergent.

B. Now suppose neither a_n nor b_n is a zero sequence. Show that

$$\limsup_{n\to\infty} a_n b_n \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right)$$

C. Show that we have equality in B. if b_n converges.

Solution

A. Let $S_n = \sup_{k\geq n}\{a_k\}, T_n = \sup_{k\geq n}\{b_k\}, U_n = \sup_{k\geq n}\{a_k + b_k\}$. Let $U = \lim_{n\to\infty} U_n, S = \lim_{n\to\infty} S_n, T = \lim_{n\to\infty} T_n$. We must show that $U \leq S + T$

Let $n \in \mathbb{N}$. Then for any $k \geq n$, $a_k + b_k \leq S_n + T_n$. Thus $U \leq U_n \leq S_n + T_n$ for all $n \in \mathbb{N}$. But then, taking the limit for $n \to \infty$, $U \leq S + T$ as needed.

Now suppose b_n converges. Then $\lim_{n\to\infty} b_n = T \in \mathbb{R}$. Let $\varepsilon > 0$ and n_0 such that for all $n > n_0$ $|T - b_n| < \varepsilon$. For $n > n_0$ then also $T - \varepsilon \leq T_n \leq T + \varepsilon$. It follows for $k \geq n$ that $a_k + b_k \leq S_n + T_n \leq S_n + T + \varepsilon$. Therefore $U_n \leq S_n + T + \varepsilon$. On other other hand, $a_k + b_k \geq a_k + T - \varepsilon$ (for all $k \geq n$) and this means $U_n \geq S + T - \varepsilon$.

For every ε we therefore find n_0 such that $|U_n - S - T| \leq \varepsilon$ for all $n > n_0$. But this means $\lim_{n\to\infty} U_n = S + T$ as needed.

B. Let S_n, T_n, S, T as in A. Let $V_n = \sup_{k\geq n}\{a_k b_k\}$, and $V = \lim_{n\to\infty} V_n$.

Note that $S, T > 0$ as a_n, b_n are not zero sequences (indeed, there is an $\varepsilon > 0$ such that for all $n_0 \in \mathbb{N}$, there is $n > n_0$ such that $a_n, b_n > \varepsilon$; why?).

Similarly to A., for $k \geq n$, $a_k b_k \leq S_n T_n$. Thus $U \leq U_n \leq S_n T_n$ for all n. And as above this means $ST \geq U$.

C. We keep the notation of B. Then $S = \lim_{n\to\infty} b_n$. Given $\varepsilon > 0$, small enough such that $S - \varepsilon > 0$, there is n_0 such that $|b_n - S| < \varepsilon$ for all $n > n_0$. Then for all $k \geq n > n_0$ we have $a_k b_k \leq T_n(S + \varepsilon)$ (where we define $\infty \cdot a = \infty$ for every real number $a > 0$). Thus, $U_n \leq T_n(S + \varepsilon)$ for all n, so $U \leq T(S + \varepsilon)$. Similarly, $a_k b_k \geq a_k(S - \varepsilon)$, and therefore $U_n \geq T_n(S - \varepsilon)$ and thus $U \geq T(S - \varepsilon)$. If $T = \infty$, this means $U = \infty = TS$. If $T < \infty$, then for every $\varepsilon > 0$ as above $T(S - \varepsilon) \leq U \leq T(S + \varepsilon)$, which means $U = TS$.

Problem 5

The Mean Value Theorem is your friend:

A. Compute $\lim_{n\to\infty} n\left(1 - \cos\left(\frac{1}{n}\right)\right)$

B. Show: Suppose f is continuous on $[a, b]$ and differentiable on at least $(c - \delta, c + \delta) \setminus \{c\} \subseteq [a, b]$. If $\lim_{x\to c} f'(x) = \alpha \in \mathbb{R}$ then $f'(c)$ exists and $f'(c) = \alpha$.

Solution

A. There are several methods:

a. Without MVT: $\cos x$ is differentiable at 0 and $\cos' 0 = -\sin 0 = 0$. Therefore $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$. Therefore, for every sequence $x_n \neq 0$ with $x_n \to 0$ we have

$$\lim_{n\to\infty} \frac{\cos x_n - 1}{x_n} = 0$$

This applies to $x_n = \frac{1}{n}$ and we find $0 = \lim_{n\to\infty} n\left(\cos\frac{1}{n} - 1\right) = -\lim_{n\to\infty} n\left(1-\cos\left(\frac{1}{n}\right)\right)$. So the limit is zero.

b. Consider the function $f(x) = x\left(1-\cos\left(\frac{1}{x}\right)\right)$ defined on $(0,\infty)$. If we show that $\lim_{x\to\infty} f(x) = 0$ then the limit of the problem is also 0.
Consider $\cos(y)$ on $[0,\infty]$. Then $\cos(y) - \cos(0) = -\sin(c)\, y$ for some $c \in (0,y)$ by the MVT.
Thus, $1-\cos\left(\frac{1}{n}\right) = \frac{\sin(c_n)}{n}$ for some $c_n \in \left(0, \frac{1}{n}\right)$. In particular, $\lim_{n\to\infty} c_n = 0$.
Then $n\left(1-\cos\left(\frac{1}{n}\right)\right) = \frac{n\sin(c_n)}{n} = \sin(c_n)$. As $c_n \to 0$ by the continuity of sin we find $n\left(1-\cos\left(\frac{1}{n}\right)\right) \to 0$ for $n\to\infty$.

B. We must show that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \alpha$. By the MVT, for every $x \in (c-\delta, c+\delta) \setminus \{c\}$ we have $\frac{f(x)-f(c)}{x-c} = f'(z)$ for some z strictly between x and c. For $\varepsilon > 0$ there is $\mu > 0$ such that for all x with $0 < |x-c| < \mu$ we have $|f'(x) - \alpha| < \varepsilon$. For any such x such find

$$\left|\frac{f(x)-f(c)}{x-c} - \alpha\right| < \varepsilon$$

Because $|z-c| < \mu$. This shows $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \alpha$.

Problem 6

Prove the Leibniz formula: If f, g are differentiable at x_0, then

$$(fg)^{(n)}(x_0) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x_0) g^{(n-k)}(x_0)$$

(For example if $n = 1$, this is the product rule.)

Recall that for a function f, $f^{(n)}$ denotes the nth derivative of$f$, and $f^{(0)} = f$.

Solution

Remark: Of course the problem should have stated that f, g must be **$\boldsymbol{n}$-times differentiable** at x_0, otherwise this makes no sense.

This is a (tedious) induction proof.

The base case here is $n = 1$ where it is the product rule.

So suppose the Leibniz formula holds for a particular n.

Then $(fg)^{(n+1)}(x_0) = \frac{d}{dx}\Big|_{x_0} (fg)^{(n)} = \frac{d}{dx}\Big|_{x_0} \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x_0) g^{(n-k)}(x_0)$

By the product rule, this is equal to

$$\sum_{k=0}^{n}\binom{n}{k}\left(f^{(k+1)}(x_0)g^{n-k}(x_0)+f^{(k)}(x_0)g^{(n-k+1)}(x_0)\right)$$

Collecting coefficients of $f^{(k)}(x_0)$, this amounts to

$$f^{(0)}(x_0)g^{(n+1)}(x_0)+$$

$$\sum_{k=1}^{n}\left(\binom{n}{k-1}f^{(k)}(x_0)g^{(n+1-k)}(x_0)+\binom{n}{k}f^{(k)}(x_0)g^{(n+1-k)}(x_0)\right)$$

$$+f^{(n+1)}(x_0)g^{(0)}(x_0)$$

Note this uses $\binom{n}{0}=\binom{n}{n}=1$.

Now for $1\le k\le n$, $\binom{n}{k-1}+\binom{n}{k}=\frac{n!}{(n-k+1)!(k-1)!}+\frac{n!}{(n-k)!k!}=\frac{n!(k+n-k+1)!}{(n+1-k)!k!}=\binom{n+1}{k}$.

If we also observe that $1=\binom{n+1}{0}=\binom{n+1}{n+1}$, the Leibniz formula for $n+1$ follows.

Problem 7 (OPTIONAL)

You may use that f is continuous below, but please do not use that f is differentiable.

Let f be an exponential function. We call $a=f(1)$ the *base* of f.

A. Show: For all $r\in\mathbb{Q}$ and all $x\in\mathbb{R}$ we have $f(rx)=f(x)^r$. In particular $f(r)=a^r$.

B. Use A. to show that the range of f is $\mathbb{R}_{>0}$.

C. Use A. to show that the set of numbers of the form a^r where r ranges over the rationals is *dense* in $\mathbb{R}_{>0}$.

D. Show that if $a>1$ and $x>0$, then $f(x)>1$. Conclude that f is strictly monotone increasing if $a>1$.

Solution

A. Clearly, we must only show that $f(rx)=f(x)^r$, as $f(r)=a^r$ is simply the case $x=1$.
Now let $r=\frac{p}{q}$ with $p\in\mathbb{Z}$ and $q\in\mathbb{N}$.

We have seen in class that $f(px)=f(x)^p$ and that $f\left(\frac{x}{q}\right)=f(x)^{\frac{1}{q}}$. But here is a proof nevertheless:
Let $p\in\mathbb{Z}$ and $x\in\mathbb{R}$. If $p>0$, then $px=x+x+\cdots+x$ (p summands). Applying $f(x+y)=f(x)f(y)$ repeatedly, we get $f(px)=f(x)f\big((p-1)x\big)=\cdots=f(x)^p$.
If $p=0$, then $f(px)=f(0)=1=f(x)^0$. (We know that $f(x)>0$ for all x.)
If $p<0$, then $f(px)=f(-p'x)$ where $p'=-p$. For any y, $1=f(0)=f(y-y)=f(y)f(-y)$ shows that $f(-y)=\frac{1}{f(y)}$. Therefore $f(px)=\frac{1}{f(x)^{p'}}=f(x)^{-p'}=f(x)^p$.

By these observations we have for any x that $f(x)=f\left(q\left(\frac{1}{q}x\right)\right)=f\left(\frac{1}{q}x\right)^q$. This shows that $f\left(\frac{1}{q}x\right)$ is a positive qth root of $f(x)$, and therefore $f\left(\frac{x}{q}\right)=f(x)^{\frac{1}{q}}$.

Taken together, $f(rx) = f\left(\frac{p}{q}x\right) = f\left(\frac{1}{q}x\right)^p = \left(f(x)^{\frac{1}{q}}\right)^p = f(x)^{\frac{p}{q}} = f(x)^r$.

B. Technically, the problem should have stated that $f(1) \neq 1$. (Otherwise the range is just $\{1\}$.)
Let $a = f(1) > 1$. Then $\lim\limits_{n\to\infty} f(n) = a^n = \infty$ by A. Similarly $\lim\limits_{n\to\infty} f(-n) = a^{-n} = 0$.
By the Intermediate Value Theorem, f must attain all values in $(0, \infty)$.
If $a = f(1) < 1$, the same applies because then $\lim\limits_{n\to\infty} f(n) = 0$ and $\lim\limits_{n\to\infty} f(-n) = \infty$.

C. Again, we assume $a \neq 1$. Let $y_0 = f(x_0) > 0$. (x_0 exists by B.) Let $r_n \in \mathbb{Q}$ be any sequence with limit x_0. This exists because $\mathbb{Q}$ is dense in $\mathbb{R}$. Then $\lim\limits_{n\to\infty} f(r_n) = f(x_0) = y_0$ because f is continuous. As $f(r_n) = a^{r_n}$, $\lim\limits_{n\to\infty} a^{r_n} = y_0$.

D. Let $x > 0$. Let $r_n \to x$ be a sequence of rational numbers. Choose $m > 0$ such that $\frac{1}{m} < x$. We may assume that $r_n > \frac{1}{m}$ for all n. Then $f(r_n) = a^{r_n} > a^{\frac{1}{m}} > 1$ for all n because $a^r > a^s$ if $r > s$ are rational numbers. f is continuous so taking the limit shows that $f(x) \geq a^{\frac{1}{m}} > 1$

There are many ways to show that f is strictly monotone increasing. Here is one: Let $x < y \in \mathbb{R}$. Then $y - x > 0$ and hence $f(y-x) > 1$ by the above. But $f(y-x) = f(y)f(-x) = \frac{f(y)}{f(x)} > 1$ means $f(y) > f(x)$ because we already know that $f > 0$.

MATH 118 Honors Calculus II

Winter 2020 – Assignment 2

Problem 1

A. $\lim_{x\to\infty} \frac{\log x}{x^\alpha}$ where $\alpha > 0$.

B. $\lim_{x\to 0^+} x^\alpha \log x$ where $\alpha > 0$.

C. $\lim_{x\to 0^+} x^\alpha (\log x)^\beta$ where $\alpha > 0$ and $\beta \in \mathbb{N}$.

D. $\lim_{x\to 0^+} x^x$.

E. $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

F. $\lim_{x\to 1^-} \log x \cdot (1 - \log x)$.

Solution

A. We apply L'Hôpital's Rule: $\frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \frac{1}{\alpha x^\alpha} \to 0$ for $x \to \infty$.
Not required: $\lim_{x\to\infty} x^\alpha = \infty$ because $x^\alpha = \exp\,(\alpha \log x) \to \infty$.

B. Again L'Hôpital's Rule applied to $\frac{\log x}{\frac{1}{x^\alpha}}$ gives $\frac{\frac{1}{x}}{-\alpha x^{-\alpha-1}} = \frac{1}{-\alpha x^{-\alpha}} = -\alpha x^\alpha \to 0$.

C. L'Hôpital's Rule on $\frac{(\log x)^\beta}{x^{-\alpha}}$ gives $\frac{\beta (\log x)^{\beta-1} x^{-1}}{-\alpha x^{-\alpha-1}} = \frac{\beta (\log \beta)^{\beta-1}}{-\alpha x^{-\alpha}}$ Now we can do induction on β to conclude this converges to 0: The case $\beta = 1$ is part B. By induction we may assume that $x^\alpha (\log x)^{\beta-1} \to 0$ and hence the same is true for $x^\alpha (\log x)^\beta$.

D. $x^x = \exp\,(x \log x)$. Now $x \log x \to 0$ by B. As $\exp x$ is continuous this means $x^x \to 1$.

E. $\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \frac{x - \sin x}{x \sin x}$
L'Hôpital's Rule gives $\frac{1-\cos x}{\sin x + x\cos x}$ for the first time and or the second $\frac{\sin x}{\cos x + \cos x - x \sin x}$ Note we must verify that the derivative of the denominator is never zero close to 0. This is true for $x \sin x$, as its derivative $\sin x + x \cos x$ is the sum of two positive or negative numbers. Similarly, its derivative $\cos x + \cos x - x \sin x$ is close to 2 for x close to 0. L'Hôpital's Rule then gives $\frac{0}{2} = 0$ as a limit.

F. $\log x \cdot \log(1-x)$. Rewriting this as $\frac{\log(1-x)}{\frac{1}{\log x}}$, L'Hôpital's Rule leads us to investigate

$$\frac{\frac{-1}{1-x}}{-\frac{1}{(\log x)^2}\cdot\frac{1}{x}} = \frac{(\log x)^2}{1-x}\frac{1}{x} = \frac{\log x}{x}\frac{\log x}{1-x} \to \frac{0}{1}\cdot(-\log' 1) = 0$$

Problem 2

Prove the *Intermediate Value Theorem for Derivatives*:

Let f be a function defined and differentiable on an interval $[a,b]$.

If $f'(a) \neq f'(b)$, then f attains every value between $f'(a)$ and $f'(b)$ on $[a,b]$.

A. If $f'(x) \neq 0$ for all $x \in [a,b]$, then f is strictly monotone increasing or decreasing.
(*Hint*: Show that $f$is injective and use results from Math 117.)

B. Conclude that if $f'(x) \neq 0$ for all $x \in [a,b]$, then $f' \geq 0$ or $f' \leq 0$.

C. Show that if $f'(a) < 0$ and $f'(a) > 0$ there is $c \in (a,b)$ $f'(c) = 0$.

D. Finish the proof by considering the function $g(x) = f(x) - \alpha x$ with α strictly between $f'(a)$ and $f'(b)$.

Solution

A. Let $x < y \in [a,b]$. Then $f(y) - f(x) = f'(c)(y - x)$ for some $c \in (x,y)$ by the MVT. By assumption $f'(c) \neq 0$, and $y \neq x$, so $f(y) - f(x) \neq 0$, and $f(x) \neq f(y)$. It follows that f is injective. By results of Math 117 we know that if a continuous function is injective on an interval it is (strictly) monotone.

B. By A. f is strictly monotone. We know that means $f' \geq 0$ or $f' \leq 0$ on (a,b) by a theorem in class. Suppose $f' \geq 0$ on (a,b). It remains to show that also $f'(a), f'(b) \geq 0$.
We have seen in class that if $f'(a) < 0$ then then there is $\delta > 0$ such that $f(x) < f(a)$ for all $x \in (a, a+\delta)$, contradicting that f is increasing. Similarly, if $f'(b) < 0$, there is $\delta > 0$ such hat $f(x) > f(b)$ for all $x \in (b - \delta, b)$. Applying the same argument to $-f$ in case $f' \leq 0$ for all $x \in (a,b)$ shows that we then also have $f'(a), f'(b) \leq 0$.

C. Suppose there is no $c \in (a,b)$ such that$f'(c) = 0$. Then there is no $c \in [a,b]$ for which $f'(c) = 0$. By A. and B. this means $f' \geq 0$ or $f' \leq 0$ on $[a,b]$. We therefore cannot have $f'(a)f'(b) < 0$. It follows there must be c such that $f'(c) = 0$.

D. Let $g(x) = f(x) - \alpha x$. Suppose $f'(a) < f'(b)$. Then g is differentiable on $[a,b]$ and $g'(a) = f'(a) - \alpha < 0$ and $g'(b) = f'(b) - \alpha > 0$. By C. there must be $c \in (a,b)$ such yhat $g'(c) = 0$. But that means $f'(c) = \alpha$. If $f'(b) > f'(a)$ apply the same reasoning to $-g$.

Problem 3

A function $f$defined on an interval i is called **convexif** for all $x_1, x_2 \in I$ and all $t \in [0,1]$ we always have

$$f\big((1-t)x_1 + tx_2\big) \leq (1-t)f(x_1) + tf(x_2)$$

(Note this inequality trivially holds if $t = 0{,}1$.)

Show: If f is convex then for all $n \in \mathbb{N}$ and all $x_1, x_2, \ldots, x_n \in I$ and all $t_1, t_2, \ldots, t_n > 0$ with $1 = t_1 + t_2 + \cdots + t_n$ we have

$$f(t_1x_1 + t_2x_2 + \cdots + t_nx_n) \leq t_1f(t_1) + t_2f(x_2) + \cdots + t_nf(x_n)$$

Solution

Suppose f satisfies the property. Then f satisfies the property in particular in case $n = 2$. But then $f\big((1-t)x_1 + tx_2\big) \leq (1-t)f(x_1) + tf(x_2)$ for all $t \in (0,1)$ and all $x_1, x_2 \in I$. This inequality is trivially true also in case $t = 0{,}1$. Thus f is convex.

Now suppose f is convex, and we will show the statement by induction.

For $n = 1$ there is nothing to do, and the statement holds for $n = 2$ by the convexity of f.

Suppose the statement holds for a given $n \geq 2$. Let $x_1, x_2, \dots, x_{n+1} \in I$ and $t_1, t_2, \dots, t_{n+1} > 0$ with $1 = t_1 + t_2 + \cdots + t_{n+1}$.

We may assume that $x_1 \leq x_2 \leq \cdots \leq x_{n+1}$. Then $x_1 \leq t_1 x_1 + \cdots + t_{n+1} x_{n+1} \leq x_{n+1}$, so $t_1 x_1 + t_2 x_2 + \cdots + t_{n+1} x_{n+1} \in I$. Let $s_1 = \frac{t_1}{1-t_{n+1}}, s_2 = \frac{t_2}{1-t_{n+1}}, \dots, s_n = \frac{t_n}{1-t_{n+1}}$. This is well defined since $t_{n+1} < 1$ because all $t_i > 0$. Then $x_1 \leq s_1 x_1 + s_2 x_2 + \cdots + s_n x_n \leq x_n$ so $x_0 := s_1 x_1 + \cdots + s_n x_n \in I$.

Now $f(t_1 x_1 + \cdots + t_{n+1} x_{n+1}) = f\big((1-t_{n+1})x_0 + t_{n+1} x_{n+1}\big) \leq (1-t_{n+1}) f(x_0) + t_{n+1} f(x_{n+1})$ by the convexity of f. Now $1 - t_{n+1} > 0$ and by the induction hypothesis

$$(1-t_{n+1}) f(s_1 x_1 + s_2 x_2 + \cdots + s_n x_n) \leq (1-t_{n+1})(s_1 f(x_1) + s_2 f(x_2) + \cdots + s_n f(x_n))$$

The right hand side is equal to

$$t_1 f(x_1) + t_2 f(x_2) + \cdots + t_n f(x_n)$$

Combining with the above we get

$$f(t_1 x_1 + \cdots + t_{n+1} x_{n+1}) \leq t_1 f(x_1) + t_2 f(x_2) + \cdots + t_{n+1} f(x_{n+1})$$

Problem 4 – Optional

Otto Stolz in Math. Ann. 15 (1879) 556-559 gave the following example:

Let $f(x) = x + \sin x \cos x$ and $g(x) = f(x) e^{\sin x}$.

A. Show that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ does not exist.
B. Suppose now that someone argues
$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{2\cos x}{x + \sin x \cos x + 2\cos x} e^{-\sin x} = 0$$
and therefore L'Hôpital's Theorem is false.
Where is the mistake in this reasoning?

Solution

A. For x large $f(x) > 0$, so $\frac{f(x)}{g(x)} = \frac{1}{e^{\sin x}}$.for large x. This has no limit, as $e^{\sin x}$ takes all values in $\left[\frac{1}{e}, e\right]$ in the intervals $[2\pi n, 2\pi(n+1)]$ for n arbitrarily large.
B. There are two problems with this reasoning.
$$f'(x) = 1 + \cos(x)^2 - \sin(x)^2 = 2\cos(x)^2$$
$$g'(x) = f(x)\cos(x)\, e^{\sin x} + f'(x) e^{\sin x}$$
Thus $\frac{f'(x)}{g'(x)} = \frac{2\cos(x)^2}{(x+\sin x \cos x)\cos x + 2\cos(x)^2} e^{-\sin(x)}$ **wherever** $g'(x) \neq 0$.
If $\cos(x) \neq 0$ we can cancel it in the numerator and denumerator and arrive at the formula
$\frac{f'(x)}{g'(x)} = \frac{2\cos(x)}{x+\sin x \cos x + 2\cos x} e^{-\sin x}$
The right hand side of this equation is defined for large x, and has limit 0 for $x \to \infty$. However, there is no interval (a, ∞) where this equation is true for all x, because $\cos(x)$ has infinitely many zeros on any such interval, so $g'(x) = 0$ on at least one point on any such interval.

It follows that $\frac{f'(x)}{g'(x)}$ does not have a limit as $x \to \infty$. What is true is the fact that $\frac{f'(x)}{g'(x)}$ in its domain (which excludes all points where $\cos x \neq 0$) agrees with a (continuous) function that has a limit (0) for $x \to \infty$. One *could* get around this by redefining limit.
This is the first problem.

The second, less subtle problem (**and the only you needed to realize to receive full credit)** is that the hypothesis $g'(x) \neq 0$ for $x \in (a, \infty)$ is not satisfies for any $a > 0$, and thus L'Hôpital's Theorem does not apply.

MATH 118 Honors Calculus II

Winter 2020 – Assignment 3

Problem 1

Let $f: (0, \infty) \to \mathbb{R}$ be defined as $f(x) = x^x$.

A. Compute f' and f''.
B. Find all critical points for f. Determine for each one whether it is a local maximum or minimum or neither.
C. Find all inflection points for f.
D. Compute $\lim_{x\to 0} f(x)$, $\lim_{x\to\infty} f(x)$$
E. Compute $\lim_{x\to 0} f'(x)$ and $\lim_{x\to\infty} f'(x)$.

Solution

A. $f(x) = \exp(x\log x)$. So $f'(x) = (\log x + 1)\exp(x \log x) = (\log x + 1)x^x$.
$f''(x) = \frac{1}{x}x^x + (\log x + 1)^2 x^x = x^{x-1} + (\log x + 1)^2 x^x$.

B. $f'(x) = 0$ iff $\log x + 1 = 0$ because $\exp(x\log x)$is always positive.
$\log x = -1$ iff $x = \frac{1}{e}$. So $\frac{1}{e}$ is the only critical point. $f''\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)-1} + 0 > 0$. So f has a local minimum at $\frac{1}{e}$. $f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}}$.

C. As f'' is defined everywhere, c is an inflection point only if $f''(c) = 0$.
But x^{x-1} and $(\log x + 1)^2 x^x$ are both nonnegative and x^{x-1} is positive, so $f'' > 0$ everywhere. Therefore, there re no inflection points and more over f is strictly convex.

D. $\lim_{x\to 0} x^x = \lim_{x\to 0} \exp(x\log x) = 1$ because $\lim_{x\to 0} x\log x = 0$. Indeed, for $x > 0$ $x\log x = \frac{(\log x)}{\frac{1}{x}}$.
It follows by L'Hôpital's rule that the limit is the limit of

$$\frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x \to 0$$

Thus $\lim_{x\to 0} f(x) = 1$.
As $x\log x \to \infty$ for $x \to \infty$, it follows that $\lim_{x\to\infty} f(x) = \infty$.

E. As $x^x \to 1$ for $x \to 0$, we are reduced to determining $\lim_{x\to 0}(\log x + 1) = -\infty$. Therefore $\lim_{x\to 0} f'(x) = -\infty$. $\lim_{x\to\infty} f'(x) = \infty$ should be clear: both x^x (by D.) and $\log x + 1$ go to ∞.

Problem 2

A. Let $t_1, t_2, \dots, t_n > 0$ such that $1 = t_1 + t_2 + \dots + t_n$. For $x_1, x_2, \dots, x_n \neq 0$ show that

$$x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n} \leq t_1x_1 + t_2x_2 + \dots + t_nx_n$$

B. Let $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that for any $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ we always have

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}}$$

This is known as the Hölder inequality.

(*Hint*: Reduce to the case that $a = \sum_{i=1}^{n} |a_i|^p$ and $b = \sum_{i=1}^{n} |b_n|^q$ are both nonzero. Now for each i consider $\left(\frac{|a_i|^p}{a}\right)^{\frac{1}{p}} \left(\frac{|b_i|^q}{b}\right)^{\frac{1}{q}}$.)

Solution

A. The problem should have stated $x_i \geq 0$ not $x_i \neq 0$.
If one of the $x_i = 0$ there is nothing to show. So we may assume all $x_i > 0$.
$\log x$ is strictly concave. From a previous homework problem we know that that means
$\log\left(x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}\right) = t_1 \log(x_1) + t_2 \log(x_2) + \cdots + t_n \log(x_n) \leq \log(t_1 x_1 + t_2 x_2 + \cdots + t_n x_n)$
As $\log x$ is monotone increasing, this means $x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n} \leq t_1 x_1 + t_2 x_2 + \cdots + t_n x_n$.

B. If all a_i or all b_i are equal to 0, there is nothing to show. We may therefore assume hat at least one a_i and at least one b_i are nonzero. Let a, b be defined as in the hint. Then $a, b > 0$.
By A. $\left(\frac{|a_i|^p}{a}\right)^{\frac{1}{p}} \left(\frac{|b_i|^q}{b}\right)^{\frac{1}{q}} \leq \frac{1}{p}\left(\frac{|a_i|^p}{a}\right) + \frac{1}{q}\left(\frac{|b_i|^q}{b}\right)$ for all i.
Hence $\sum_{i=1}^{n} \left(\frac{|a_i|^p}{a}\right)^{\frac{1}{p}} \left(\frac{|b_i|^q}{b}\right)^{\frac{1}{q}} \leq \frac{1}{p}\sum_{i=1}^{n}\left(\frac{|a_i|^p}{a}\right) + \frac{1}{q}\sum_{i=1}^{n}\left(\frac{|b_i|^q}{b}\right) = \frac{1}{p} + \frac{1}{q} = 1$.
Multiplying both sides with $a^{\frac{1}{p}} b^{\frac{1}{q}}$ it follows that
$\sum_{i=1}^{n} |a_i b_i| \leq a^{\frac{1}{p}} b^{\frac{1}{q}}$ which is what was to show.

Problem 3

Show that for every $C > 0$ the equation

$$C \exp(x) = \frac{x^2}{2} + x + 1$$

has a unique solution.

Solution

Method 1: "The Hard Way"

Let $f(x) = C \exp(x) - \frac{x^2}{2} - x - 1$. Then

$$f'(x) = C \exp(x) - x - 1$$

and

$$f''(x) = C \exp(x) - 1$$

We know that $\lim_{x \to \infty} f(x) = \infty$ because for large x, $C \exp(x) > \frac{x^2}{2} + x + 1$. Indeed,

$\frac{C \exp(x)}{p(x)} > 1$ for x large enough for any polynomial $p(x)$ with $\lim_{x \to \infty} p(x) = \infty$ by L'H.

We also know that $\lim_{x\to-\infty} f(x) = -\infty$ because $C\exp(x) \to 0$ and $-\frac{x^2}{2} - x - 1 \to -\infty$ for $x \to -\infty$.

By the IVT there must be x_0 such that $f(x_0) = 0$. So there exists at least one solution.

Now $f''(x) = 0$ has at most one solution because $C\exp(x)$ is strictly monotone increasing. Therefore f' can have at most two zeros (by Rolle's Theorem there is a root of f'' strictly between any two roots of f'), which (by the same argument) limits the number of roots of f to 3.

Let x_0 be the smallest root of f. This exists because f has at most 3 roots.

Then $0 = f(x_0) = f'(x_0) - \frac{x_0^2}{2}$. It follows that $f'(x_0) = 0 + \frac{x_0^2}{2} \geq 0$.

Suppose f has a second root $x_1 > x_0$. Then there is $y_1 \in (x_0, x_1)$ such that $f'(y_1) = 0$ (Rolle). We know there are at most two such zeros. Let y_1 be the smallest in (x_0, x_1). Then $f(y_1) = -\frac{y_1^2}{2} \leq 0$. If we assume that $x_1 > x_0$ is minimal, then $f(y_1) < 0$. But then f cannot have a local maximum on (x_0, y_1) as that would imply another zero of f' in (x_0, y_1), so f is strictly decreasing from x_0 to y_1. It follows that f has a local maximum at x_0 (x_0 is the smallest root, so $f(x) < 0$ for $x < x_0$). Therefore $f'(x_0) = 0$.

Now $f(x_0) = f'(x_0) = 0$ forces $x_0 = 0$. But then $C = 1$. And $x_0 = 0$ is the only possible position of a local extremum of f' (since $f''(x) = 0$ only for $x = 0$) and f' is (strictly) monotone increasing on $(0, \infty)$ (and thus f is as well, as then $f' > 0$ on $(0, \infty)$). In particular $f'(y_1) > 0$ a contradiction. So x_1 cannot exist.

Method 2: "The Clever Way" (as done by one of you)

As $e^x > 0$, we have

$Ce^x = \frac{x^2}{2} + x + 1$ iff

$$C = \frac{x^2}{2}e^{-x} + xe^{-x} + e^{-x}$$

Let $f(x)$ be the function defined by the formula on the right hand side. Then $f'(x) = xe^{-x} - \frac{x^2}{2}e^{-x} + e^{-x} - xe^{-x} - e^{-x} = -\frac{x^2}{2}e^{-x}$. Thus $f' \leq 0$ and f is monotone decreasing. Since $f'(x) = 0$ only if $x = 0$, it follows hat f is strictly monotone on $(-\infty, 0]$ and $[0, \infty)$ and therefore strictly monotone everywhere. But then the equation $f(x) = C$ has at most one solution.

That it has a solution is clear from $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = 0$ and the fact that $C > 0$. The first limit is clear as $f = e^{-x}\left(\frac{x^2}{2} + x + 1\right)$ and both factors go to ∞ for $x \to -\infty$. The second limit is a consequence of the fact that $\lim_{x\to\infty} \frac{e^x}{g(x)} = 0$ whenever g is a polynomial and the function approaches 0 from one side (depending on the sign of $g(x)$ for large x).

Note that there are likely other methods to solve this.

Problem 4

Let I be an interval. We say f defined on I is *Lipschitz continuous* if there is a constant L such that for all $x, y \in I$ we have $|f(x) - f(y)| \le L|x - y|$. A Lipschitz continuous function is always continuous.

A. Show that if f is continuously differentiable **and** $I = [a, b]$ is bounded and closed, then f is Lipschitz continuous.

B. Show that $f(x) = \exp(x)$ is *not* Lipschitz continuous on $I = \mathbb{R}$ (But by A. f is Lipschitz continuous on every closed bounded interval).

Solution

A. Under the assumptions f' is *bounded* on $[a, b]$. So there is L such that $|f'(x)| \le L$ for all $x \in [a, b]$. Let $x < y \in I$. Then by the MVT we have $f(y) - f(x) = f'(c)(y - x)$ for some $c \in (x, y)$.
Thus, $|f(x) - f(y)| \le |f'(c)||x - y| \le L|x - y|$. The case $x > y$ is similar.

B. Suppose for a contradiction that there is L such that $|e^x - e^y| \le L|x - y|$ for all $x, y \in \mathbb{R}$. Fix y. Then $\frac{|e^x - e^y|}{|x-y|} \le L$ for all $x \ne y$. But $\lim_{x\to\infty} \frac{|e^x - e^y|}{|x-y|} = \infty$ by e.g. L'H. This is a contradiction.

   Similar, but direct method: let $L > 0$. Then there is x_0 such that $e^{x_0} > L$. Let $x > y > x_0$, then $e^x - e^y = e^c(x - y)$ for some $c \in (y, x)$. As $c > x_0$ this means $e^c > L$. Thus, for any L there is x, y such that $|e^x - e^y| > L|x - y|$.

Problem 5 (OPTIONAL)

A. Let $f''(x) > 0$ for all x in an interval x. Then f is strictly convex.
Show that for all $x \ne y \in I$, $f(x) > f(y) + f'(y)(x - y)$. In other words, the graph of f is above each its tangents on I.

B. For all $x \ne 0$, $e^x > 1 + x$.

C. For all positive $x \ne 1$, $\log x < x - 1$.

Solution

A. Let $x < y \in I$. Then there is $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$. Thus, $f(x) = f(y) + f'(c)(x - y)$. As f' is strictly increasing, $f'(c) < f'(y)$ and $x - y < 0$ by construction. Thus $f(x) > f(y) + f'(y)(x - y)$.
Now let $x > y \in I$. Again there is $c \in (y, x)$ such that $f(x) - f(y) = f'(c)(x - y)$.
Then $f(x) = f(y) + f'(c)(x - y)$ but this time $x - y > 0$ and $f'(c) > f'(y)$. Again we conclude that $f(x) > f(y) + f'(y)(x - y)$.

B. We apply A. in case $y = 0$. We know that e^x is strictly convex on $\mathbb{R}$. Thus, for $x \ne 1$ we have $e^x > e^0 + (e^x)'(0)(x - 0) = 1 + x$.

C. We apply A. in case $y = 1$. Since $-\log x$ is strictly convex, we know $-\log x > 0 + (-\log x)'(1)(x - 1) = 1 - x$. Therefore $\log x < x - 1$.

Math 118 – Honors Calculus II

Winter 2020 – Assignment 4

Problem 1

A. Show that $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges.

B. Show that the double series $\sum_{n,m=2}^{\infty} \frac{1}{n^m}$ converges.

Solution

A. There are several methods. One is to observe that $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$, so the partial sum is a telescope sum and
$S_N = \sum_{n=2}^{N} \frac{1}{n(n-1)} = 1 - \frac{1}{N} \to 1$ for $N \to \infty$.
Another is to observe that $0 \leq \frac{1}{n(n-1)} \leq \frac{1}{(n-1)^2}$ so $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} \leq \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

B. We use A. By the Cauchy Double Series Theorem (ands since all summands are nonnegative), it suffices to show that $\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{n^m} < \infty$.
Now $\sum_{m=2}^{\infty} \frac{1}{n^m} = -1 - \frac{1}{n} + \sum_{m=0}^{\infty} \left(\frac{1}{n}\right)^m = -1 - \frac{1}{n} + \frac{1}{1-\frac{1}{n}} = \frac{n}{n-1} - \frac{n+1}{n} = \frac{n^2-n^2+1}{n(n-1)} = \frac{1}{n(n-1)}$.
It follows by A. that the iterated series converges (absolutely) and hence the double series converges.

Again, there are other methods. One could for example use a version of the ratio test for double series (where one has to first establish absolute convergence of all row and column series, which is not hard here).

Problem 2

Determine the radius of convergence of the following power series

A. $\sum_{n=1}^{\infty} n^{\frac{\log n}{n}} x^n$.

B. Let $a > 0$, $\sum_{n=1}^{\infty} n(\sqrt[n]{a} - 1)\, x^n$
(*Hint*: Show that $\lim_{n\to\infty} n\left(\sqrt[n]{a} - 1\right) = \log a$.)

Solution

A. $a_n := n^{\frac{\log n}{n}} = \exp\left(\frac{1}{n}(\log n)^2\right)$ So $\sqrt[n]{a_n} = \exp\left(\frac{(\log n)^2}{n^2}\right)$. As $n \to \infty$, this converges to 1, since $\lim_{n\to\infty} \frac{\log n}{n} = 0$ (e.g. by L.H.). The radius of convergence is 1.

B. Let $f(x) = a^x$. Then $f'(x) = (\log a) f(x)$ and $f(0) = 1$. Thus, for every sequence $0 \neq x_n \to 0$, we have $\lim_{n\to\infty} \frac{f(x_n)-1}{x_n} = f'(0) = \log a$. Then $x_n = \frac{1}{n}$ gives $\lim_{n\to\infty} n\left(\sqrt[n]{a} - 1\right) = \log a$.
Now if $a = 1$, then the series equals 0 and has radius of convergence ∞.

If $a \neq 1$, then $|\log a| - \varepsilon < a_n := \left|n\left(\sqrt[n]{a} - 1\right)\right| < |\log a| + \varepsilon$ for n large and any chosen $\varepsilon > 0$ such $|\log a| > \varepsilon$. Then $\sqrt[n]{|\log a| - \varepsilon} < \sqrt[n]{a_n} < \sqrt[n]{|\log a| + \varepsilon}$. So the limit of $n \to \infty$ $\sqrt[n]{a_n} = 1$ and the radius of convergence is 1.

Problem 3

A. (OPTIONAL for 3 points) Let $\alpha_n > 0$ be a *bounded* sequence.
Show that $\liminf_{n\to\infty} \alpha_n \leq \liminf_{n\to\infty} \sqrt[n]{\alpha_1\alpha_2\cdots\alpha_n} \leq \limsup_{n\to\infty} \sqrt[n]{\alpha_1\alpha_2\cdots\alpha_n} \leq \limsup_{n\to\infty} \alpha_n$

B. Show that if a_n is a sequence such that $a_n \neq 0$ for all n, and such that $\frac{|a_{n+1}|}{|a_n|}$ is a bounded sequence, then

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf_{n\to\infty} \sqrt[n]{|a_n|} \leq \limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

(*Hint*: Without loss of generality one may assume that $a_1 = 1$.)

C. Conclude that if the ratio test for a **series** to converge or diverge is conclusive, then so is the root test.

D. Show that there is a series for which the root test asserts convergence, but the ratio test is inconclusive.
(*Hint*: Consider $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^{n-1}}$.)

E. Suppose $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists. Show that the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is equal to $\frac{1}{L}$ (with the usual convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Solution

A. This is harder than expected.
Method 1:
We first treat the limit inferior. Let $L = \liminf_{n\to\infty} \alpha_n$. We may assume that $L > 0$. Otherwise nothing is to show. Then the assertion holds iff $\log L \leq \log \liminf_{n\to\infty} \sqrt[n]{\alpha_1\alpha_2\cdots\alpha_n}$. As $\log x$ is monotone increasing, this is true iff

$$\liminf \log \alpha_n \leq \liminf \log \sqrt[n]{\alpha_1\alpha_2\cdots\alpha_n} = \liminf \frac{1}{n}(\log \alpha_1 + \log \alpha_2 + \cdots + \log\ \alpha_n)$$

Putting $b_n = \log \alpha_n$ it therefor suffices to show that for any bounded sequence b_n we have

$$\liminf b_n \leq \liminf \frac{1}{n}(b_1 + b_2 + \cdots + b_n)$$

(Note $\log \alpha_n$is bounded below because $L > 0$ and bounded above because α_n is.) Let $M = \liminf b_n$. Then $M \neq \pm\infty$ because b_n is bounded.
Let $\varepsilon > 0$. Then there is n_0 such that for all $n > n_0$ we have $b_n > M - \frac{\varepsilon}{2}$.
Then $\frac{1}{n-n_0}\left(b_{n_0+1} + b_{n_0+2} + \cdots + b_n\right) > M - \frac{\varepsilon}{3}$.
But then

$$\begin{aligned}\frac{1}{n}(b_1 + \cdots + b_n) &= \frac{1}{n}\left(b_1 + b_2 + \cdots + b_{n_0}\right) + \frac{n-n_0}{n}\frac{1}{n-n_0}\left(b_{n_0+1} + \cdots + b_n\right)\\ &> \frac{1}{n}\left(b_1 + b_2 + \cdots + b_{n_0}\right) + \frac{(n-n_0)}{n}\left(M - \frac{\varepsilon}{3}\right)\end{aligned}$$

This is true for all $n > n_0$. In particular, for large n, $\frac{1}{n}\left(b_1 + \cdots + b_{n_0}\right) > -\frac{\varepsilon}{2}$. Likewise, for large n $\frac{n-n_0}{n}\left(M - \frac{\varepsilon}{3}\right) > M - \frac{\varepsilon}{2}$. Thus, there is m_0 such that for $n > m_0$ we have that the above is

$$> -\frac{\varepsilon}{2} + M - \frac{\varepsilon}{2} = M - \varepsilon$$

(Of course one could avoid the $\frac{\varepsilon}{p}$ business and just accept $M - k\varepsilon$ for some constant k instead.)

It follows that $\liminf \frac{1}{n}(b_1 + b_2 + \cdots + b_n) > M - \varepsilon$ for any $\varepsilon \geq 0$. Thus the limit inferior is $\geq M$ as claimed.

Method 2 (as proposed by one of you): Avoid the detour through $\log x$.
Let $L = \liminf \alpha_n$. Then $L \geq 0$. We may assume $L > 0$.
Again, as above, for a given $\varepsilon > 0$ there exists n_0 such that $\alpha_n > L - \varepsilon$. We may choose ε small enough such that $L - \varepsilon > 0$.
Then $\sqrt[(n-n_0)]{\alpha_{n_0+1}\alpha_{n_0+2}\cdots\alpha_n} > L - \varepsilon$ (because $\sqrt[p]{x}$ is monotone increasing for all $p > 0$).
It follows that

$$\sqrt[n]{\alpha_1\alpha_2\cdots\alpha_n} = \sqrt[n]{\alpha_1\alpha_2\cdots\alpha_{n_0}}\sqrt[n]{\alpha_{n_0+1}\alpha_{n_0+2}\cdots\alpha_n}$$
$$= \sqrt[n]{\alpha_1\alpha_2\cdots\alpha_{n_0}}\left(\alpha_{n_0+1}\alpha_{n_0+2}\cdots\alpha_n\right)^{\frac{1}{n-n_0}\cdot\frac{n-n_0}{n}} > \sqrt[n]{\alpha_1\alpha_2\cdots\alpha_{n_0}}\,(L-\varepsilon)^{\frac{n-n_0}{n}}$$

The right hand side has limit (for $n \to \infty$) $1 \cdot (L - \varepsilon)$. As this holds for evert $\varepsilon > 0$ small enough, the claim follows.

The claim regarding limit superior follows in a similar fashion (and only one is needed for full credit). The center inequality (that limit inferior $\leq$ limit superior is clear from the definition).

B. The limit inferior or superior of any sequence do not change if we change the first element of the sequence. So we may modify a_n if necessary to have $a_1 = 1$.
Then we note that for $n \geq 2$, $|a_n| = \frac{|a_n|}{|a_1|} = \frac{|a_n|}{|a_{n-1}|} \cdot \frac{|a_{n-1}|}{|a_{n-2}|} \cdots \frac{|a_2|}{|a_1|} \cdot \frac{|a_1|}{|a_0|}$ where we also put $a_0 = 1$.
Then $\liminf \frac{|a_n|}{|a_{n-1}|} = \liminf \frac{|a_{n+1}|}{|a_n|}$ and similar for the limit superior. Apply A. to the sequence $\alpha_n = \frac{|a_n|}{|a_{n-1}|}$ for $n > 1$ and $\alpha_1 = |a_1|$. This gives the result.

Remark: you may have put $\alpha_n = \frac{|a_{n+1}|}{|a_n|}$ (with $a_1 = 1$). Then $\alpha_1\alpha_2\cdots\alpha_n = |a_{n+1}|$. But $\liminf \sqrt[n]{|a_{n+1}|} = \liminf \sqrt[n]{|a_n|}$ (we have shown this in class for the limit superior, but the proof is essentially the same. You did not need to show that.

C. If the ratio test predicts convergence, then the root test does as well as in this case $\limsup \sqrt[n]{|a_n|} \leq \limsup \frac{|a_{n+1}|}{|a_n|} < 1$. This uses B. and the fact that both sequences here are bounded in this case. For divergence the argument is essentially the same (with limit inferior and all inequalities reversed). However, in this case the sequences may not be bounded above. But for the proof of the left-most inequality in A. (and B.) we did not need bounded above, we needed that the limit inferior was not $+\infty$. But if $\liminf \frac{|a_{n+1}|}{|a_n|} = \infty$, then for large enough n, we have $|a_{n+1}| > 2|a_n|$ and hence $\left|a_{n_0+k}\right| > 2^k\left|a_{n_0}\right|$ for some n_0 and all $k \geq 1$. Then $\sqrt[n_0+k]{\left|a_{n_0+k}\right|} \geq \frac{3}{2} > 1$ for large k.

D. Let $a_n = \frac{2+(-1)^n}{2^{n-1}}$. Then $a_n > 0$. Also $\frac{a_{n+1}}{a_n} = \frac{(2+(-1)^{n+1})2^{n-1}}{(2+(-1)^n)2^n} = \frac{\frac{1}{2}(2+(-1)^{n+1})}{2+(-1)^n}$. This alternates between $\frac{3}{2}$ and $\frac{1}{6}$. So the lim sup is $\frac{3}{2} > 1$ and lim inf is $\frac{1}{6} < 1$, so there is no statement about convergence. However, $\sqrt[n]{a_n} \leq \sqrt[n]{\frac{3}{2^{n-1}}} = \sqrt[n]{3}\left(\frac{1}{2}\right)^{\frac{n}{n-1}} \to \frac{1}{2}$ for $n \to \infty$. So $\limsup \sqrt[n]{a_n} \leq \frac{1}{2} < 1$. So the root test asserts convergenc.

E. If $L < \infty$, then by B. we have that $\limsup(\sqrt[n]{|a_n|}) = L$ (as both lim inf and lim sup are sandwiched between the lim inf and lim sup of the convergent sequence $\frac{|a_{n+1}|}{|a_n|}$. In this case the radius of convergence is therefore $\frac{1}{L}$.
If $L = \infty$, then if $|x| > 0$, the sequence $\frac{|a_{n+1}||x|^{n+1}}{|a_n||x|^n}$ has limit ∞, and the series does not converge at $\pm x$. Thus, the radius of convergence is 0.
(One could also argue using $\sqrt[n]{|a_n|}$ but we haven't shown directly above that the lim sup of that is ∞ (but this can be done).

Problem 4 (OPTIONAL)

Let $k \in \mathbb{N}_0$.

A. Show that $f(x) = \sum_{n=0}^{\infty}\binom{n+k}{n}x^n$ has radius of convergence 1.
B. With $f(x)$ as in A. show that for $|x| < 1$, $f(x) = \frac{1}{(1-x)^{k+1}}$.

Solution

There are many ways to approach this.

Method 1:

Let $g(x) = \sum_{n=0}^{\infty} x^n$. This is a series with radius of convergence 1 (from class). For $|x| < 1$ it is equal to $\frac{1}{1-x}$. Note that $f(x) = D^k(g)$. We have seen in class that the radius of convergence of $D(g)$ (and hence $D^k(g)$ is equal to the radius of convergence of g. This is A.

Next, we know that $D^k(g)$ is equal to $g^{(k)}$. Now $g^{(k)} = \frac{1}{(1-x)^{k+1}}$. This proves B.

Method 2:

We use Problem 3 E.

$$\frac{\binom{n+k+1}{n+1}}{\binom{n+k}{n}} = \frac{(n+k+1)!\,n!}{(n+1)!\,(n+k)!} = \frac{n+k+1}{n+1} \to 1$$

for $n \to \infty$. This shows A.

For B. we use induction. The case $k = 0$ was done in class.

Now suppose for a given k we have $f_k(x) = \sum\binom{n+k}{n}x^n$ equals $\frac{1}{(1-x)^{k+2}}$.

Then $(1+x)f_{k+1}(x) = \sum_{n=0}^{\infty}\binom{n+k+1}{n}(1+x)x^n = 1 + \sum_{n=1}^{\infty}\left(\binom{n+k+1}{n} + \binom{n+k}{n-1}\right)x^n = f_k(x)$.

Thus $f_{k+1}(x) = \frac{1}{1-x} f_k(x) = \frac{1}{(1-x)^{k+1}}$.

There are many more methods.

Math 118 – Honors Calculus II

Winter 2020 – Assignment 4

Problem 1

Suppose f and g are n times differentiable at x_0 ($n > 0$), and let p be a polynomial of degree $\leq n$.

Suppose $g(x_0) = 0$ and suppose $f(x) = p(x) + (x - x_0)^n g(x)$.

Show that $P = P\_(f, n, x_0)$. That is, show hat p is the Taylor polynomial of$f$ of degree n at x_0.

Solution

Note that if h is a polynomial of degree d, then $h = P_{h,n,x_0}$ for all $n \geq d$. (Because we showed in class that $h = \sum_{n=0}^{\infty} \frac{h^{(n)}(x_0)}{n!}(x - x_0)^n$, and we know that $h^{(n)}(x_0) = 0$ for all $n > d$.

It follows that we must show that for $k = 0,1, \dots, n$ we have $f^{(k)}(x_0) = p^{(k)}(x_0)$.

This is clear for $k = 0$: then $f(x_0) = p(x_0) + 0 \cdot g(x_0) = p(x_0)$.

Moreover $f^{(k)}(x_0) = p^{(k)}(x_0) + \frac{d^k}{dx^k}\Big|_{x_0} (x - x_0)^n g(x)$

By the Leibniz Rule for differentiation we have

$$\frac{d^k}{dx^k}\bigg|_{x_0} (x - x_0)^n g(x) = \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\frac{d^\ell}{dx^\ell}(x - x_0)^n\right)(x_0) \left(\frac{d^{k-\ell}}{dx^{k-\ell}} g(x_0)\right)$$

Now $\left(\frac{d^\ell}{dx^\ell}(x - x_0)^n\right)(x_0) = 0$ for $\ell < n$. So if $k < n$, this is always zero.

We are left with showing that $f^{(n)}(x_0) = p^{(n)}(x_0)$.

$\frac{d^n}{dx^n}\Big|_{x_0} (x - x_0)^n g(x) = \left(\frac{d^n}{dx^n}(x - x_0)^n\right)(x_0) g(x_0) = 0$ because $g(x_0) = 0$.

Problem 2

For $-1 \leq x \leq 0$ show that

$$\log\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Solution

Recall that for $x \in [-1,0]$, we have $\log\left(\frac{1}{1-x}\right) = \log(1) - \log(1 - x) = -\log(1 - x)$.

Let $f(x) = \log(1 + x)$. We have seen in class that on $[0,1]$, $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$

Therefore for $x \in [-1,0]$ we get

$$\log\left(\frac{1}{1-x}\right) = -f(-x) = -\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-x)^n = -\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{n}x^n = \sum_{n=1}^{\infty}\frac{x^n}{n}$$

There are other methods that work equally well.

Problem 3

A. Determine the radius of convergence of

$$f(x) = \sum_{n=0}^{\infty}(2n+1)(2x)^{2n}$$

B. Compute $f\left(\frac{1}{4}\right)$. (*Hint*: Write f as a rational function.)

Solution

A. Note that $f(x) = \sum(2n+1)2^{2n}x^{2n}$. So We may consider $g(x) = \sum(2n+1)2^{2n}x^n$ first. For g the inverse of the radius of convergence is $\limsup_{n\to\infty}\sqrt[n]{(2n+1)2^{2n}} = 4\limsup\sqrt[n]{(2n+1)} = 4$. (Recall $e^{\frac{1}{n}\log(2n+1)} \to 0$ for $n \to \infty$ e.g. by L.H.)
Thus, the radius of convergence for g is $\frac{1}{4}$. Then $f(x)$ converges iff $g(x^2)$ converges, and diverges iff $g(x^2)$ diverges. So $f(x)$ converges for converges for $|x| < \sqrt{\frac{1}{4}} = \frac{1}{2}$ and diverges for $|x| > \frac{1}{2}$. The radius of convergence is therefore $\frac{1}{2}$.
(There are other methods to do this.)

B. Note $(2n+1)(2x)^{2n} = \frac{1}{2}\frac{d}{dx}(2x)^{2n+1}$. Let $G(x) = \sum_{n=0}^{\infty}(2x)^{2n+1} = 2x\sum_{n=0}^{\infty}(2x)^{2n}$. Then G has the same radius of convergence as f and $G'(x) = 2f(x)$.
Note $\sum_{n=0}^{\infty}(2x)^{2n} = \sum(4x^2)^n = \frac{1}{1-4x^2}$, and $G(x) = \frac{2x}{1-4x^2}$. Then $G'(x) = \frac{2(1-4x^2)+16x^2}{(1-4x^2)^2} = 2\frac{1+4x^2}{(1-4x^2)^2}$, so $f(x) = \frac{(1+4x^2)}{(1-4x^2)^2}$.
Now $G'\left(\frac{1}{4}\right) = \frac{1+\frac{1}{4}}{\left(1-\frac{1}{4}\right)^2} = \frac{\frac{5}{4}}{\frac{9}{16}}$, and $f\left(\frac{1}{4}\right) = \frac{20}{9}$.

Note you could also do B. first to get A.

A. Determine the radius of convergence of

$f(x)=\sum_{n=0}^\infty (2n+1)(2x)^{2n}$.

B. Compute $f(1/4)$. (*Hint*: Write f as a rational function.)

Problem 4

Compute the following indefinite integrals

A. $\int \sqrt{2x+3}dx$

B. $\int (x^3+x^2-1)e^{2x-4}dx$

C. $\int \frac{e^x-1}{e^x+1}dx$ (*Hint*: $e^x - 1 = 2e^x - (e^x + 1)$.)

D. $\int \sin^2(x)\, dx$ and $\int \cos^2(x)\, dx$

E. $\int \sqrt{1-x^2}dx$ on $I = (-1,1)$. (*Hint*: On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ put $x = \sin t$.)

Solution

A. Recall that $\int \sqrt{y}dy = \int y^{\frac{1}{2}}dy = \frac{2}{3}y^{\frac{3}{2}}$

Now $\sqrt{2x+3} = \frac{1}{2}f\big(g(x)\big)g'(x)$ where $f(y) = \sqrt{y}$ and $g(x) = 2x + 3$ (so $g'(x) = 2$).

Therefore, $\int \sqrt{2x+3}dx = \frac{1}{2}F \circ g = \frac{1}{3}(2x+3)^{\frac{3}{2}}$

B. Repeated integration by parts:

Recall $\int Uv = UV - \int uV$.

$$\int e^{2x-4}dx = \frac{1}{2}e^{2x-4}$$

$$\int x^2e^{2x-4}dx = \frac{1}{2}x^2e^{2x-4} - \int xe^{2x-4}dx = \frac{1}{2}x^2e^{2x-4} - \left(\frac{1}{2}xe^{2x-4} - \frac{1}{2}\int e^{2x-4}dx\right)$$

$$= \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)e^{2x-4}$$

$$\int x^3e^{2x-4}dx = \frac{1}{2}x^3e^{2x-4} - \int \frac{3}{2}x^2e^{2x-4}dx = \frac{1}{2}x^3e^{2x-4} - \frac{3}{2}\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)e^{2x-4}$$

The total is therefore

$$\int (x^3 + x^2 - 1)e^{2x-4}dx = \left(\frac{1}{2}x^3 - \frac{1}{2}\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right) - \frac{1}{2}\right)e^{2x-4}$$

$$= \left(\frac{1}{2}x^3 - \frac{1}{4}x^2 + \frac{1}{4}x - \frac{5}{8}\right)e^{2x-4}$$

C. Using the hint, $\frac{e^x-1}{e^x+1} = \frac{2e^x}{e^x+1} - 1$. Therefore,

$$\int \frac{e^x-1}{e^x+1}dx = \int \frac{2e^x}{e^x+1} - 1dx = 2\log|e^x+1| - x = 2\log(e^x+1) - x$$

D. Recall that $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$. Therefore $\sin(2x) = 2\sin(\mathrm{x})\cos(\mathrm{y})$ and that $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$. Therefore $\cos(2x) = \cos^2(x) - \sin^2(y)$.

It follows that $\int \cos^2 x - \sin^2 x\, dx = \frac{1}{2}\sin(2x)$.

On the other hand $\int \cos^2 x + \sin^2 x\, dx = x$. Let $A = \int \sin^2 x\, dx$ and $B = \int \cos^2 x\, dx$.

Then $B - A = \frac{1}{2}\sin(2x) + C$ and $A + B = x + D$ where C, D are some constants.

Then $B = \frac{1}{2}(B-A) + \frac{1}{2}(A+B) = \frac{1}{4}\sin(2x) + \frac{1}{2}x$.

Similarly, $A = \frac{1}{2}(A-B) + \frac{1}{2}(A+B) = \frac{1}{2}x - \frac{1}{4}\sin(2x)$.

E. We use the substitution rule: $\sin t$ is invertible on $(-\frac{\pi}{2}, \frac{\pi}{2})$ with range $(-1,1)$. Set $x = \sin t =: g(t)$, then $\sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \cos t$ on that interval, and $g'(t) = \cos t$.

Therefore $F = \int \sqrt{1-x^2}dx = [\int(\cos t)g'(t)dt]_{t=\arcsin x} = \int \cos^2 t\, dt$ evaluated in $g^{-1}(x) = \arcsin x$.

By D. $\int \cos^2 t \, dt = \frac{1}{2}t + \frac{1}{4}\sin(2t)$. So as a first result we obtain $F = \frac{1}{2}\arcsin x + \frac{1}{4}\sin(2\arcsin x)$.

But $\sin 2t = 2\sin t \cos t$, and on the interval in question $\cos t = \sqrt{1 - \sin^2 t}$. So

$$F(x) = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1 - x^2}$$

Problem 5 (OPTIONAL)

Suppose $f, F: [a, b] \to \mathbb{R}$ are continuous.

Suppose $F' = f$ on (a, b)\$. Show: F is differentiable on $[a, b]$ and $F' = f$ on $[a, b]$.

Solution

We have seen a similar problem in a previous assignment (using the MVT) and in class (using L'H).

As f is continuous $\lim_{x \to a} f(x) = f(a)$. Thus, $\lim_{x \to a} F'(x)$ exists and is finite. We have seen that this means $F'(a)$ exists and is equal to $\lim_{x \to a} F'(x) = f(a)$. The case of $F'(b)$ is similar.

Math 118 – Honors Calculus II

Winter 2020 – Assignment 6

Problem 1

Compute the following indefinite integrals

A. $\int \frac{x}{\cos^2 x} dx$
B. $\int x^2 \sin x \, dx$
C. $\int x \sin \sqrt{x} \, dx$

Solution

Strictly speaking one should always specify the interval where the antiderivative is meant to be computed. But that was not required here.

A. Let $F(x) = \tan x$. Then we are asked to compute $\int xF'(x)dx$
$$\int xF'(x)dx = xF - \int F dx$$
Now $F(x) = \frac{\sin x}{\cos x} = -\frac{g'(x)}{g(x)}$ where $g(x) = \cos x$ So $\int F dx = -\log|\cos x|$
So $\int xF' dx = x \tan x + \log|\cos x|$

B. $\int x^2 \sin x \, dx = x^2(-\cos x) - \int 2x(-\cos x)dx = -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx$
$$= -x^2 \cos x + 2x \sin x + 2 \cos x = (2 - x^2) \cos x + 2x \sin x$$

C. Consider $u = \sqrt{x}$ and $x = u^2$. So $\frac{dx}{du} = 2u$ and $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$. So formally "$dx = 2u \, du$”.
More precisely put $g(u) = u^2$. Then $g'(u) = 2u$, and
$$g(u) \sin \sqrt{g(u)} = u^2 \sin u$$
The substitution rule then says that if $F = \int u^2 \sin u \, g'(u) du$ then $F(\sqrt{x}) = \int x \sin \sqrt{x} \, dx \quad (+C)$.

$$\int 2u^3 \sin u \, du = 2u^3(-\cos u) - 2\int 3u^2(-\cos u) \, du$$
$$= -2u^3 \cos u + 6u^2 \sin u - 6\int 2u \sin u \, du$$
$$= -2u^3 \cos u + 6u^2 \sin u + 12u \cos u - 12\int \cos u \, du$$
$$= (12u - 2u^3) \cos u + (6u^2 - 12) \sin u$$

(Test: $\frac{d}{du}$ gives $(12 - 6u^2) \cos u + (12u - 2u^3)(-\sin u) + 12u \sin u + (6u^2 - 12) \cos u = 2u^3 \sin u$, as desired.)

We conclude that
$$\int x \sin \sqrt{x} \, dx = \left(12\sqrt{x} - 2\sqrt{x^3}\right) \cos(\sqrt{x}) + (6x - 12) \sin \sqrt{x}$$

(Again $\frac{d}{dx}$ gives $\left(\frac{6}{\sqrt{x}} - 3\sqrt{x}\right) \cos(\sqrt{x}) + \frac{(12\sqrt{x} - 2\sqrt{x^3})(-\sin \sqrt{x})}{2\sqrt{x}} + 6 \sin \sqrt{x} + \frac{(6x-12) \cos \sqrt{x}}{2\sqrt{x}} = x \sin \sqrt{x}$.)

Problem 2

Compute $\int_0^\pi x^2 \sin 2x\, dx$.

Let $g(x) = 2x$. Then $g'(x) = 2$. Then $g(x)^2 \sin g(x)\, g'(x) = 8x^2 \sin 2x$.

Then $\int_0^\pi x^2 \sin(2x) dx = \int_0^\pi \frac{1}{8} g(x)^2 \sin\big(g(x)\big)\, g'(x) dx = \frac{1}{8}\int_{g(0)}^{g(\pi)} u^2 \sin u\, du$.

By problem 1B, the result is

$\frac{1}{8}[(2-u^2)\cos u + 2u \sin u]_0^{2\pi} = \frac{1}{8}\big((2-4\pi^2) - 2\cos 2\pi\big) = -\frac{\pi^2}{2}$.

Of course, one can also do integration by parts directly.

Problem 3

The *oscillation* of a bounded function on an interval I is defined as

$$\Omega_f(I) \coloneqq \sup f - \inf f$$

A. Show that if $J \subseteq I$ is an interval then $\Omega_f(J) \leq \Omega_f(I)$.
B. Let $\delta > 0$ and $x \in I$. We put $\Omega_{f,x}(\delta) \coloneqq \Omega_f(I \cap (x-\delta, x+\delta))$.
Show that $\Omega_{f,x}$ is a monotone increasing function on $(0,\infty)$ and show that $\omega_f(x) \coloneqq \lim_{\delta \to 0} \Omega_{f,x}(\delta)$ exists (and is nonnegative).
C. Show that $f \in \mathcal{B}[a,b]$ is continuous at x iff $\omega_f(x) = 0$.
D. Show that $f \in \mathcal{B}[a,b]$ is integrable if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that for all i, $\Omega_f([x_i, x_{i+1}]) < \varepsilon$.

Solution

Not that in the computation of $\Omega_f(I)$ f is viewed as a function on I (so technically it is the restriction of f to I. That is, a more precise notation would be

$$\Omega_f(I) = \sup f(I) - \inf f(I)$$

A. If $J \subseteq I$ then $\sup f(J) \leq \sup f(I)$ and $\inf f(J) \geq \inf f(I)$ (because $f(J) \subseteq f(I)$). Then $\Omega_f(J) \leq \Omega_f(I)$ is immediate.
B. $I \cap (x-\delta, x+\delta)$ is always a subinterval of I. It follows that $\Omega_{f,x}$ is a well defined function on $(0,\infty)$. If $\delta < \delta'$, then $I \cap (x-\delta, x+\delta) \subseteq I \cap (x-\delta', x+\delta')$. By A. we have $\Omega_{f,x}(\delta) \leq \Omega_{f,x}(\delta')$.
Next $\Omega_{f,x} \geq 0$ because $\sup f(J) - \inf f(J) \geq 0$ for any interval containing contained in I.
Any monotone increasing function bounded below has an infimum. But then as $\Omega_{f,x}$ is monotone increasing it follows that $\inf \Omega_{f,x} = \lim_{\delta \to 0} \Omega_{f,x}(\delta) = \omega_f(x)$. Since 0 is a lower bound, it follows that $\omega_f(x) \geq 0$.
C. For this problem it is useful to observe that for a bounded function f on some interval J
$$\sup f(J) - \inf f(J) = \sup\{f(x) - f(y) \mid x, y \in J\} = \sup\{|f(x) - f(y)| \mid x, y \in J\}$$
Indeed: first it is clear that $f(x) - f(y) \leq \sup f(J) - \inf f(J)$ because $f(x) \leq \sup f(J)$ and $f(y) \geq \inf f(J)$. Since $|f(x) - f(y)| = \pm\big(f(x) - f(y)\big)$ is of this form (maybe we must exchange x and y), als $|f(x) - f(y)| \leq \sup f(J) - \inf f(J)$.

On the other hand for any $\varepsilon > 0$, pick $x \in J$ and $y \in J$ such that $f(x) > \sup f(J) - \frac{\varepsilon}{2}$ and $f(y) < \inf f(J) + \frac{\varepsilon}{2}$. Then $f(x) - f(y) > \sup f(J) - \inf f(J) + \varepsilon$. As ε is arbitrary, therefore also $\sup\{ f(x) - f(y) \mid x, y \in J \} \geq \sup f(J) - \inf f(J)$. Of course, $\sup\{ |f(x) - f(y)| \mid x, y \in J \} \geq \sup\{ f(x) - f(y) \mid x, y \in J \}$ is clear.

Suppose f is continuous at $x \in I$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that for all $y \in I$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \frac{\varepsilon}{3}$. But then for any $y, y' \in J := (x - \delta, x + \delta) \cap I$ we have

$$|f(y) - f(y')| \leq |f(y) - f(x)| + |f(x) - f(y')| \leq \frac{2}{3}\varepsilon$$

This means also $\sup f(J) - \inf f(J) \leq \frac{2}{3}\varepsilon < \varepsilon$.
Hence $\Omega_{f,x}(\delta) < \varepsilon$. As $\Omega_{f,x}$ is monotone increasing, this means for all $\delta' \leq \delta$ we also have $\Omega_{f,x}(\delta') < \varepsilon$. But that means $\omega_f(x) = 0$.

Conversely, suppose $\omega_f(x) = 0$. Let $\varepsilon > 0$. Then there exist $\delta > 0$ such that $\Omega_{f,x}(\delta) < \varepsilon$, and hence $\Omega_f\big((x - \delta, x + \delta) \cap I\big) < \varepsilon$. Then

$$\sup\{ |f(x_1) - f(x_2)| \mid x_1, x_2 \in (x - \delta, x + \delta) \cap I \} < \varepsilon$$

But then $|f(x') - f(x)| < \varepsilon$ for every $x' \in (x - \delta, x + \delta) \cap I$. But that means f is continuous at x.

D. Let $\varepsilon > 0$. By assumption, there is a partition P such that

$$\Omega_f([x_i, x_{i+1}]) < \frac{\varepsilon}{b - a}$$

But this means

$$\mathcal{U}(P, f) - \mathcal{L}(P, f) = \sum_{i=0}^{|P|} \Omega_f([x_i, x_{i+1}])\,(x_{i+1} - x_i) < \frac{\varepsilon}{b - a} \sum_{i=0}^{|P|} (x_{i+1} - x_i) = \varepsilon$$

Equivalently, f is integrable.

Problem 4

Let $[a, b]$ be an interval and let φ be a *step function* on $[a, b]$: that is, there is a partition P of $[a, b]$ such that $\varphi(x) = \alpha_i$ is constant on (x_i, x_{i+1}). (We do not care what the values of φ in the end points of the intervals are.)

Show that φ is integrable and that $\int_a^b \varphi(x)dx = \sum_{i=0}^{|P|} \alpha_i (x_{i+1} - x_i)$

(Note: there was a second typo, there should be no c_i anywhere.)

Solution

Note now that we have the Lebesgue criterion, since φ is bounded, we know it is integrable because it is continuous at all but finitely many points.

By results from class, we know that φ is integrable iff it is integrable on all $[x_i, x_{i+1}]$, and then

$$\int_a^b \varphi dx = \sum_{i=0}^{|P|} \int_{x_i}^{x_{i+1}} \varphi(x)dx$$

It thus suffices to show that φ is integrable on $[x_i, x_{i+1}]$ and that $\int_{x_i}^{x_{i+1}} \varphi dx = \alpha_i(x_{i+1} - x_i)$.

We may therefore assume that $P = \emptyset$ (that is, there is only one interval, namely $[a.b]$ and $\varphi(x) = \alpha$ on (a, b).

Let Q be any partition of $[a, b]$ with at least one element. Then

$$\mathcal{U}(Q, \varphi) - \mathcal{L}(Q, \varphi) = |\varphi(a) - \alpha|(x_1 - a) + |\varphi(b) - \alpha|\left(b - x_{|Q|}\right) \leq m(Q)(|\varphi(a) - \alpha| + |\varphi(b) - \alpha|)$$

If Q_n is a sequence of partitions of $[a, b]$ with $m(Q_n) \to 0$, this converges to 0, and so φ is integrable.

But notice that

$$\mathcal{U}(Q_n, \varphi) = \max\{\alpha, \varphi(a)\}(x_1 - a) + \alpha(b - a) + \alpha\left(x_1 - x_{|Q_n|}\right) + \max\{\alpha, \varphi(b)\}\left(b - x_{|Q_n|}\right)$$

which converges to $\alpha(b - a)$ as $m(Q_n) \to 0$.

Math 118 – Honors Calculus II

Winter 2020 – Assignment 7

Problem 1

Let f, g be two functions defined on some interval I and let $x_0 \in I$.

Suppose f, g are both continuous at x_0. Show that $\min\{f, g\}$ is also continuous at x_0.

Solution

Method 1: Suppose $f(x_0) > g(x_0)$. Then there is $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$ we still have $f(x) > g(x)$. (This uses that both f, g are continuous at x_0. Indeed, $h = f - g$ is continuous at x_0 and $h(x_0) > 0$. Then there is such a δ such that $|h(x) - h(x_0)| < \varepsilon := \frac{h(x_0)}{2}$. Then $h(x) > h(x_0) - \varepsilon = \frac{h(x_0)}{2} > 0$.)

Thus for $x \in (x - \delta, x + \delta)$, $\min\{f, g\}(x) = g(x)$, and this is continuous at x_0. The same argument applies if $f(x_0) < g(x_0)$ (and then the roles of f, g are interchanged).

If $f(x_0) = g(x_0)$, then for any $\varepsilon > 0$ there is $\delta_1 > 0$ such that for all $x \in (x_0 - \delta_1, x_0 + \delta_1)$, we have $|f(x) - f(x_0)| < \varepsilon$. There is also $\delta_2 > 0$ such that for any $x \in (x_0 - \delta_2, x_0 + \delta_2)$, $|g(x) - g(x_0)| < \varepsilon$. Then for $|x - x_0| < \min\{\delta_1, \delta_2\}$ (and $x \in I$), we have $|\min\{f(x), g(x)\} - h(x_0)| < \varepsilon$ where $h(x_0) = f(x_0) = g(x_0) = \min\{f(x_0), g(x_0)\}$.

Method 2 (as proposed by one of you):

$\min\{f, g\} = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$. Since both f, g are continuous at x_0 and since $|y|$ (as a function of y) is continuous, $|f - g|$ is continuous at x_0. It follows that $\min\{f, g\}$ is continuous at x_0.

To see why $\min\{f, g\} = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$, consider $x \in I$. If $f(x) \geq g(x)$, then $|f(x) - g(x)| = f(x) - g(x)$, so $\frac{1}{2}(f + g)(x) - \frac{1}{2}|f - g|(x) = \frac{1}{2}\big(f(x) + g(x)\big) - \frac{1}{2}\big(f(x) - g(x)\big) = g(x)$. If $f(x) < g(x)$, then $|f(x) - g(x)| = g(x) - f(x)$ and $\frac{1}{2}(f(x) + g(x)) - \frac{1}{2}|f(x) - g(x)| = \frac{1}{2}\big(f(x) + g(x)\big) - \frac{1}{2}\big(g(x) - f(x)\big) = f(x)$.

Problem 2

Decide whether or not the following improper integrals converge:

A. $\int_1^\infty \frac{\cos x}{x} dx$
B. $\int_1^\infty \sin x^2 \, dx$

Solution

A. This is very similar to $\int_0^\infty \frac{\sin x}{x} dx$ done in class.
For $s > r > t_0 > 1$ we have

$$\int_r^s \frac{\cos x}{x} dx = \left[\frac{\sin x}{x}\right]_r^s + \int_r^s \frac{\sin x}{x^2} dx$$

Taking absolute values

$$\left|\int_r^s \frac{\cos x}{x} dx\right| \le \frac{1}{s} + \frac{1}{r} + \int_r^s \frac{1}{x^2} dx = \frac{1}{s} + \frac{1}{r} + \frac{1}{r} - \frac{1}{s} = \frac{2}{r}$$

Thus, if $t_0 > \frac{2}{\varepsilon}$, then for $r > t_0$, $\frac{2}{r} < \varepsilon$. The Cauchy Criterion does the rest.

B. Consider $\int_1^t \sin x^2 \, dx$. Let $u = x^2$. Then $\frac{du}{dx} = 2x$. And $x = \sqrt{u}$. The substitution rule then says,

$$\int_1^t \sin x^2 \, dx = \int_1^{t^2} \frac{\sin u}{2\sqrt{u}} du$$

More formally $f(x) = \sin x^2$, and $g(s) = \sqrt{s}$ on $[1, t^2]$. Then

$$\int_1^{t^2} f\big(g(s)\big) g'(s) ds = \int_1^t f(x) dx$$

We now apply the Cauchy Criterion: (u^(-3/2) -2 u^(-1/2)

For $s > r$, we have

$$\int_r^s \frac{\sin u}{2\sqrt{u}} du = \left[-\frac{\cos u}{2\sqrt{u}}\right]_r^s + \int_r^s \frac{\cos u}{2u^{\frac{3}{2}}} du$$

Again, taking absolute values:

$$\left|\int_r^s \frac{\sin u}{2\sqrt{u}} du\right| \le \frac{1}{2\sqrt{s}} + \frac{1}{2\sqrt{r}} + \int_r^s \frac{1}{2u^{\frac{3}{2}}} du = \frac{1}{2\sqrt{s}} + \frac{1}{2\sqrt{r}} + \left[-\frac{1}{\sqrt{u}}\right]_r^s = \frac{1}{2\sqrt{s}} + \frac{1}{2\sqrt{r}} + \frac{1}{\sqrt{r}} - \frac{1}{\sqrt{s}}$$

$$\le \frac{3}{2\sqrt{r}}$$

For t_0 large enough, this is small if $s > r > t_0$.

Problem 3

For which a is $\int_2^\infty \frac{1}{x(\log x)^a} dx$ convergent?

Solution

Let $f(y) = y^{-a}$ and $g(x) = \log x$.

Then $\frac{1}{x\,(\log x)^a} = f(\log x)\frac{1}{x} = f\big(g(x)\big)g'(x)$.

If $a \neq 1$, then

$$\int f\big(g(x)\big) g'(x) dx = \frac{1}{-a+1} g(x)^{1-a}$$

In particular, if $a < 1$, then $\left[\frac{g(x)^{1-a}}{1-a}\right]_2^t$ diverges for $t \to \infty$.

If $a > 1$, then $\left[\frac{g(x)^{1-a}}{1-a}\right]_2^t \to \frac{\log(2)^{1-a}}{a-1}$ for $t \to \infty$ since $g(t) \to 0$.

If $a = 1$, then $\int f\big(g(x)\big)g'(x)dx = \log\log x$

The integral clearly diverges in this case as $\log\log t \to \infty$.

Problem 4 (OPTIONAL)

The improper integral $\int_0^1 \frac{1}{\sqrt{x}} dx$ exists.

A. What about $\int_0^1 \frac{1}{\sqrt{\sin(x)}} dx$?

B. Show that if f, g both have an improper integral on some interval $(a. b]$ say, it is not true in general that f, g has a (converging) improper integral.

Solution

A. Consider $f(x) = \frac{1}{\sqrt{\sin(x)}}$ and $g(x) = \frac{1}{\sqrt{x}}$ on $(0,1]$. Tnen $\frac{f(x)}{g(x)} = \sqrt{\frac{x}{\sin x}} \to 1$ for $x \to 0^+$.

But then the limit criterion says that $\int_0^1 f(x)dx$ converges because $\int_0^1 g(x)dx$ converges.

B. Take $f = g = \frac{1}{\sqrt{x}}$ on $(0,1]$. Then $\int_0^1 f(x)dx = \left[2\sqrt{x}\right]_0^1 = 2$. But $\int_0^1 f(x)^2 dx = \int_0^1 \frac{1}{x} dx$ which does not converge.

Math 118 – Honors Calculus II

Winter 2020 – Assignment 7

Problem 1

Verify whether the following improper integrals converge. Justify your answers.

A. $\int_0^1 \frac{1}{\sqrt{\sin(x)}} dx$
B. $\int_0^1 \frac{\log x}{(1-x)\sqrt{x}} dx$

Solution

A. This was Q4 A on Homework 7.
B. Let $f(x) = \frac{\log x}{(1-x)\sqrt{x}}$ defined on $(0,1)$.

Consider $\int_0^{\frac{1}{2}} f(x)dx$ and $\int_{\frac{1}{2}}^1 f(x)dx$ separately. Then $\int_0^1 f(x)dx$ converges iff both of these converge, and is equal to the sum in this case.

Let $g(x) = x^{-\frac{3}{4}}$ on $\left(0,\frac{1}{2}\right]$. Then $\int_0^{\frac{1}{2}} g(x)$ exists.

Also $\lim\limits_{x\to 0^+} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0^+} \frac{x^{\frac{1}{4}}\log(x)}{1-x} = 0$, because $1 - x \to 1$ and $x^{\frac{1}{4}} \log x \to 0$ as in class.

But then $\int_0^{\frac{1}{2}} f(x)dx$ exists if $\int_0^{\frac{1}{2}} g(x)dx$ exists, which it does, by the limit criterion.

Next, let $\lim\limits_{x\to 1^{-1}} f(x) = \lim\limits_{x\to 1^-} \frac{\log x}{1-x}$ because $\frac{1}{\sqrt{x}} \to 1$. But $\lim\limits_{x\to 1^-} \frac{\log x}{x-1} = \frac{d}{dx} \log x$ at $x = 1$, so the limit is 1. But then $\lim\limits_{x\to 1^-} \frac{\log x}{1-x} = -1$.

It follows that f is bounded on $\left[\frac{1}{2}, 1\right)$ and therefore the improper integral exists (why?).

Altogether, the integral converges.

Problem 2

Compute the improper integral

$$\int_0^\infty \frac{1}{\sqrt{1+e^x}} dx$$

Justify your answer.

(*Hint*: The answer is $\log\left(3 + 2\sqrt{2}\right)$. Also, $\int \frac{1}{1-x^2} = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|$.)

Solution

Let $g(x) = \sqrt{1+e^x}$. Then $g'(x) = \frac{1}{2\sqrt{1+e^x}} e^x$. Also $g(x)^2 = 1 + e^x$.

Therefore $\frac{1}{\sqrt{1+e^x}} = \frac{2g'(x)}{g(x)^2-1}$. The substitution rule tells us that

$$\int_0^a \frac{2g'(x)}{g(x)^2-1}dx = 2\int_{g(0)}^{g(a)} \frac{1}{t^2-1}dt = \left[-\log\left|\frac{1+t}{1-t}\right|\right]_{g(0)}^{g(a)}$$

Now $g(0)=\sqrt{2}$ and $g(a)=\sqrt{1+e^a}$.

We find

$$\int_0^a \frac{2g'(x)}{g(x)^2-1}dx = \log\left|\frac{1+\sqrt{2}}{1-\sqrt{2}}\right| - \log\left|\frac{1+g(a)}{1-g(a)}\right|$$

Finally, we observe that $\frac{1+g(a)}{1-g(a)} \to -1$ for $a \to \infty$, so the end result is

$$\log\left|\frac{1+\sqrt{2}}{1-\sqrt{2}}\right| = \log\left(3+2\sqrt{2}\right)$$

because $\frac{1+\sqrt{2}}{1-\sqrt{2}} = 3+2\sqrt{2}$.

(Of course there may be other ways to arrive at this solution.)

Problem 3

Use the integral criterion to determine whether the following series converge:

A. $\sum_{n=1}^{\infty} n^n e^{-n^2}$
B. $\sum_{n=0}^{\infty} \frac{1}{\sqrt{1+e^n}}$

Solution

A. Let $f(x) = x^x e^{-x^2} = e^{x(\log x - x)}$. We have seen in class that $\int_0^\infty f(x)dx$ converges. Also we saw that there exists t_0 such that for all $x \geq t_0$, $\log x - x < 1$. In fact, $\log x - x$ is strictly decreasing on $[t_0, \infty)$. Then also $f(x)$ is strictly decreasing on $[t_0,\infty)$. Pick any $m \in \mathbb{N}$ with $m > t_0$. Then f satisfies the hypothesis for the integral criterion on $[m,\infty)$.
   Now $\int_m^\infty f(x)dx$ converges because $\int_0^\infty f(x)dx$ does. Therefore also $\sum_{n=m}^{\infty} f(n)$ and as a consequence $\sum_{n=1}^{\infty} f(n)$ converge.
B. By Problem 2, $\int_0^\infty \frac{1}{\sqrt{1+e^x}}dx$ converges. Also, the integrand is positive and monotone decreasing, therefore the integral criterion applies and $\sum_{n=0}^{\infty} \frac{1}{\sqrt{1+e^n}}$ converges.

Problem 4

Let f satisfy the hypotheses of the integral criterion (f positive and monotone decreasing on $[m,\infty)$)

A. Show that $A_N := \sum_{n=m}^{N} f(n) - \int_m^N f(x)dx$ is a monotone decreasing sequence and for $N \to \infty$ converges to a number in $[0, f(m)]$.
B. Show that

$$\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \log n$$

   converges to a finite limit.

Solution

A. We have seen in class that
$$\sum_{n=m}^{N} f(n) \geq \int_m^{N+1} f(x)dx \geq \sum_{n=m+1}^{N+1} f(n)$$
This is a consequence of $f(n) \geq \int_n^{n+1} f(x)dx \geq f(n+1)$.
Since $0 \leq \int_m^N f(x)dx \leq \int_m^{N+1} f(x)dx$, we find that $A_N \geq 0$ for all $N \geq m$.
Also $A_{N+1} - A_N = f(N+1) - \int_N^{N+1} f(x)\,dx \leq 0$, so A_N is monotone decreasing.
Therefore $0 \leq A_N \leq A_m = f(m)$. As A_N is monotone and bounded it converges.

B. This is an immediate application of Part A: $\int_1^n \frac{1}{x}dx = \log n$.

Problem 5 (OPTIONAL)

Let S and C be two differentiable functions defined on $\mathbb{R}$ with the following properties:

1. $S' = C$.
2. $C' = -S$.
3. $S(x)^2 + C(x)^2 = 1$ for all $x \in \mathbb{R}$.
4. $S(x+y) = C(x)S(y) + C(y)S(x)$ for all $x, y \in \mathbb{R}$.
5. $C(x+y) = C(x)C(y) - S(x)S(y)$ for all $x, y \in \mathbb{R}$.
6. $S(0) = 0$, $C(0) = 1$.
7. S is odd, C is even.

(This is not a minimal set of properties, some imply others. For instance if S is odd, then S' is even.)

Without any reference to sin or cos show the following:

A. The range of S and C is $[-1,1]$.
B. There is a smallest positive number x_0 such that $C(x_0) = 0$.
C. Let $P = 2x_0$ (x_0 as in B.). Then $S(x+2P) = S(x)$ and $C(x+2P) = C(x)$.
D. For each pair (a, b) with $a^2 + b^2 = 1$ there is $x \in [0,2P)$ such that $C(x) = a$ and $S(x) = b$.

Solution

First let us collect some general facts: since $S'(0) = C(0) = 1$, S is not constant. Both S, C are continuous. But then, since $S^2 + C^2 = 1$, C cannot be constant either. Since S' is continuous, S is strictly increasing on some interval $(-\varepsilon, \varepsilon)$. That means C is strictly decreasing on $(0, \varepsilon)$ because there $C' < 0$.

If S has a local extremum at x, then $S(x) = \pm 1$. Indeed, then $C(x) = S'(x) = 0$, and therefore by 3. $S(x) = \pm 1$. The same reasoning applies to C.

A. Suppose S is strictly increasing on $[0, a]$ for all $a > 0$. Then $0 < S(a)$ is the maximum of S on any such interval. Then C is strictly decreasing on $[0, a]$, and $C(x) > 0$ on $(0, a)$ (otherwise $S(x) = 1$ on $(0, a)$ where $C(x) = 0$, and then $S(a) \leq S(x)$). Therefore $C(x) > 0$ for all $x \geq 0$.
Let $s = \lim\limits_{x\to\infty} S(x)$ and $c = \lim\limits_{x\to\infty} C(x)$. These must exist as S, C are monotone and bounded on $[0, \infty)$ by assumption.
Note 4. means $S(2x) = 2S(x)C(x)$ and 5. means $C(2x) = C(x)^2 - S(x)^2$. Taking limits for $x \to \infty$ we obtain the two equations

$$s = 2sc$$

and

$$c = c^2 - s^2$$

Since also $s > 0$ (as $S > 0$ is increasing), we find $c = \frac{1}{2}$ and

$$\frac{1}{2} = \frac{1}{4} - s^2$$

which is absurd.

Therefore S cannot be strictly increasing on $[0, \infty)$, and in particular, S must have a local maximum at some $x_1$somewhere on $[0, \infty)$. But then $C(x_1) = 0$, and necessarily $S(x_1) = 1$. As S is odd, then $S(-x_1) = -1$ and we get that the range of S is $[-1,1]$. Since $-x_1$ is the place of a local minimum of S we also have $C(-x_1) = 0$ (C is also even)

Next, by 4. we get that $S(-x_1 + x) = C(x_1)S(x) + S(x_1)C(x) = -C(x)$.

But the range of $S(-x_1 + x)$ and the range of $-C(x)$ are clearly all equal to the range of C and S, and therefore the range of C is also $[-1,1]$.

B. We have seen in A. that C has positive roots. Since $C(0) = 1$ and C is continuous, there cannot be a sequence $x_n > 0$ such that $x_n \to 0$ and $C(x_n) = 0$. Thus, $x_0 \coloneqq \inf\{\, x > 0 \mid C(x) = 0 \,\} > 0$. By continuity also $C(x_0) = 0$.

C. First observe that $S(x) \geq 0$ on $[0, x_0]$, and that means $S(x_0) = 1$. Consider $S(x + 2P) = S(x)C(2P) + S(2P)C(x)$.

Now $C(2P) = C(P)C(P) - S(P)S(P)$, and $S(2P) = 2S(P)C(P)$.

Also $C(P) = C(x_0)^2 - S(x_0)^2 = -1$, and $S(P) = 2S(x_0)C(x_0) = 0$.

We get $S(x + 2P) = S(x)$.

Similarly, $C(x + 2P) = \frac{d}{dx}S(x + 2P) = \frac{d}{dx}S(x) = C(x)$.

D. By A.,B.,C. we know that S attains it minimum and maximum at least once on $[0,2P)$. ($S(2P = 0)$ is neither a maximum nor a minimum.) We also know that $S(x_0) = 1$, and therefore $S(-x_0) = -1$, which in turn means $S(-x_0 + 2P) = S(3x_0) = -1$. The same applies to C. We know $C(0) = 1$, and $C(2x_0) = C(P) = -1$ (then $S(2x_0) = 0$).

We know S is increasing on $[0, x_0]$. Further, by 4. $S(x_0 + x) = C(x_0)S(x) + S(x_0)C(x) = C(x)$. As C is decreasing on $[0, x_0]$, this means S is decreasing on $[x_0, 2x_0]$ (with its only zero in that interval at $P = 2x_0$). Again $S(2x_0 + x) = C(2x_0)S(x) + C(x)S(2x_0) = C(P)S(x) = -S(x)$. So S is decreasing on $[2x_0, 3x_0]$, with $S(3x_0) = -S(x_0) = -1$. The behavior of S on $[3x_0, 4x_0]$ is then clear: As S is strictly increasing from 0 to 1 on $[0, x_0]$, it is also strictly increasing from -1 to 0 on $[-x_0, 0]$ (it is odd). Therefore, it is strictly increasing on $[3x_0, 4x_0)$.

We have shown:

S is (strictly) increasing on $[0, x_0]$ (and $S(x_0) = 1$), decreasing on $[x_0, 3x_0]$ and $S(3x_0) = -1$, and increasing again on $[3x_0, 2P)$ with $S(2P) = 0$.

$$S(P - x) = C(P)S(-x) + C(-x)S(P) = C(P)S(-x) = -\big(-S(x)\big) = S(x)$$

Therefore S is strictly decreasing on $[x_0, P]$, with its only zero at P.

This means C has extrema at 0 and P on $[0, P]$ (1 and -1 respectively), and is strictly decreasing in between. C is strictly increasing on $[P, 2P)$ (as $S(x) > 0$ on $(P, 2P)$).

We have

a. $C(x), S(x) \geq 0$ for $x \in [0, x_0]$
b. $C(x) \leq 0, S(x) \geq 0$ for $x \in [x_0, P]$
c. $C(x) \leq 0, S(x) \leq 0$ for $x \in [P, 3x_0]$

d. $C(x), S(x) \leq 0$ for $x \in [3x_0, 2P)$

Now let $a^2 + b^2 = 1$.
Suppose $a \geq 0$, $b \geq 0$, then $b = \sqrt{1 - a^2}$, and for the unique $x \in [0, x_0]$ where $C(x) = a$, we must have $S(x) = b$.
For all the other possible cases ($a \leq 0, b \geq 0, a \leq 0, b \leq 0, a \geq 0, b \leq 0$) there is an interval where C, S have the same relation, since $S(x)^2 + C(x^2) = 1$, there must be an x on that interval where $(C(x), S(x)) = (a, b)$.