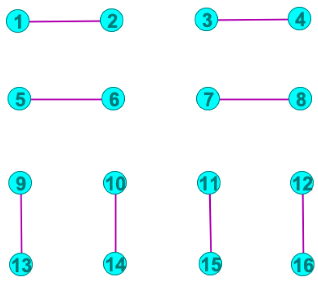
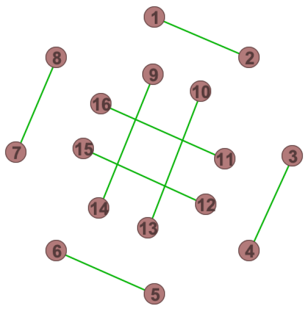
# Math 322
# Suggested solutions to Homework Set 6

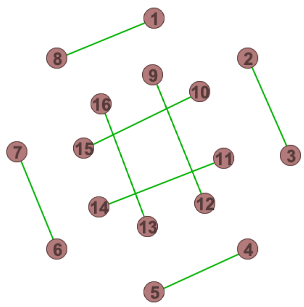
**Problem 1.** We observe that graph $H_1$ is not regular, so it cannot have a one-factorisation. On the other hand, the following is a one-factor of $H_1$:



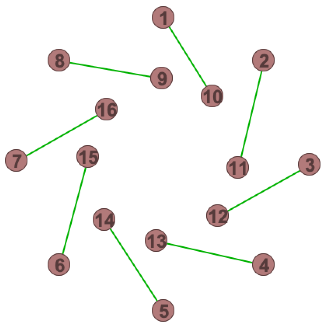

On the other hand, graph $H_2$ has both one factors and a one-factorisation. Indeed, observe that the following 3 subgraphs of $H_2$ form a one-factorisation of $H_2$ (and each one of them is a one-factor of $H_2$):




1st one-factor of the factorisation




2nd one-factor of the factorisation




3rd one-factor of the factorisation

**Problem 2.** (a) Let us denote by $A$ and $B$ the two partite sets of $K_{m,n}$.

Observe that in every subdivision of $K_{m,n}$ which is different from $K_{m,n}$ itself, each of the preexisting vertices, that is, each of the vertices in $A$ or in $B$, still has the same degree as it had in $K_{m,n}$, while each new vertex has degree 2. Moreover, the set $A$ is an independent set of vertices of the subdivision as well (given that, when constructing the subdivision, we don't add any edges joining two vertices of the original graph), and similarly the set $B$ is also an independent set of vertices of the subdivision.

If we now fix such a subdivision $G_1$ of $K_{m,n}$, we can use the above observations, as well as the greedy colouring algorithm we discussed in class, in order to come up with a proper colouring of $G_1$ which uses 3 colours. This will show that $\chi(G_1) \leqslant 3$.

We begin by ordering the vertices of the subdivision in such a way that the vertices in $A$ and $B$ will be considered first. Since both $A$ and $B$ are independent sets of vertices, we can now colour all vertices in $A$ using colour 1, and then colour all vertices in $B$ using colour 2.

Next, we consider each of the remaining vertices: each such vertex $w$ has degree 2 in $G_1$, and therefore, even if both of the neighbours of $w$ have been coloured by the time we reach vertex $w$, there will be at least one available colour in the set $\{1, 2, 3\}$ to use for $w$ (e.g. if both neighbours of $w$ have been coloured already, with, say, colours 1 and 3, then we can use colour 2 for $w$; if both of them have been coloured with the same colour, say colour 2, then we can use colour 1 for $w$; and of course we can argue even more simply if only one (or none) of the neighbours of $w$ has been coloured by the point vertex $w$ comes up).

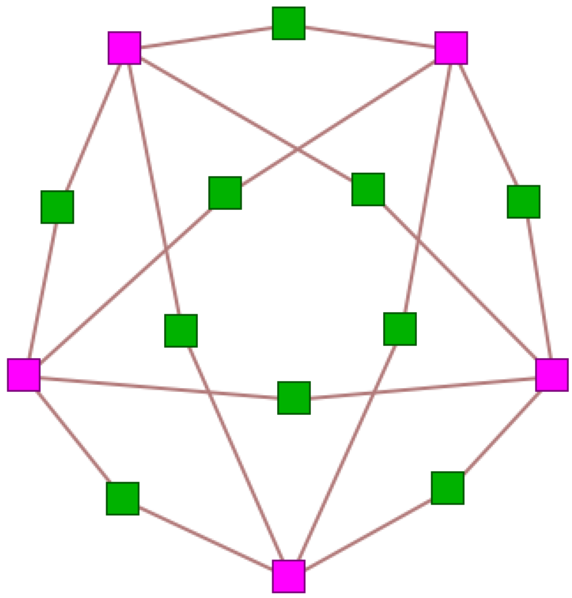
Continuing like this, we will end up with a 3-colouring of $G_1$. As already mentioned, this shows that $\chi(G_1) \leqslant 3$ (and given that $\chi(G_1)$ cannot be equal to 1, since $G_1$ is not a null graph, we conclude that $\chi(G_1) = 2$ or $\chi(G_1) = 3$).

(b) The answer here is no. We already know that $\chi(K_5) = 5$, so it suffices to check what happens with other subdivisions of $K_5$.

Similarly to above, we observe that in every subdivision $G_2$ of $K_5$ which is different from $K_5$ itself, each of the vertices which already existed in $K_5$ still has degree 4 in the subdivision, while each new vertex has degree 2.

This shows that $\Delta(G_2) = \Delta(K_5) = 4$, and hence $\chi(G_2) \leqslant \Delta(G_2) + 1 = 5$ (in fact, since we consider a subdivision of $K_5$ which is different from $K_5$, we will certainly have that $G_2$ is not a regular graph, and thus, in particular, $G_2$ will not be a complete graph nor an odd cycle; this gives us an even better conclusion compared to above: $\chi(G_2) \leqslant \Delta(G_2) = 4$).

(c) The answer here is yes. The following graph is a subdivision of $K_5$, and it's not hard to see that the colouring of its vertices (which only uses 2 colours) is a proper colouring; thus the chromatic number of this graph is 2.



**Problem 3.** (a) We observe that the induced subgraph on the vertices $a, e, f, g, h$ is a clique of $G_0$ of order 5. This shows that $\omega(G_0) \geqslant 5$.

On the other hand, if a clique with larger order existed, then necessarily we would be able to find a clique of $G_0$ with order 6. But if we consider any 6 vertices of $G_0$, then necessarily one of them must be one of the vertices $b, c$ or $d$. We now note:

- if the vertex $b$ were included in a clique of $G_0$ of order 6, then given that $\deg_{G_0}(b) = 5$, all the neighbours of $b$ should be in this clique. However, $a$ and $c$ are not joined by an edge in $G_0$, so the induced subgraph on $b$ and its neighbours would not be a clique of $G_0$.

- Similarly, if the vertex $c$ were included in a clique of $G_0$ of order 6, then given that $\deg_{G_0}(c) = 5$, all the neighbours of $c$ should be in this clique. However, $b$ and $d$ are not joined by an edge in $G_0$, so the induced subgraph on $c$ and its neighbours would not be a clique of $G_0$.

- Finally, in exactly the same way, we note that, if the vertex $d$ were included in a clique of $G_0$ of order 6, then given that $\deg_{G_0}(d) = 5$, all the neighbours of $d$ should be in this clique. However, $c$ and $e$ are not joined by an edge in $G_0$, so the induced subgraph on $d$ and its neighbours would not be a clique of $G_0$.

We conclude that there are no cliques in $G_0$ of order 6 or larger, and hence $\omega(G_0) = 5$.

(b) Note that the vertices $a, e, f, g, h$, which are contained in a clique of $G_0$, must each get its own colour if we are considering a proper colouring of $G_0$.

Next we note that

- $b$ is adjacent to the vertices $a, f, g$ and $h$, and hence it must get a different colour from the colours we have used for the latter vertices. If we use a different colour from the colour of vertex $e$ as well, then we will definitely get a colouring with 6 colours or more. Thus, to try to avoid this, we could colour $b$ with the same colour used for $e$.

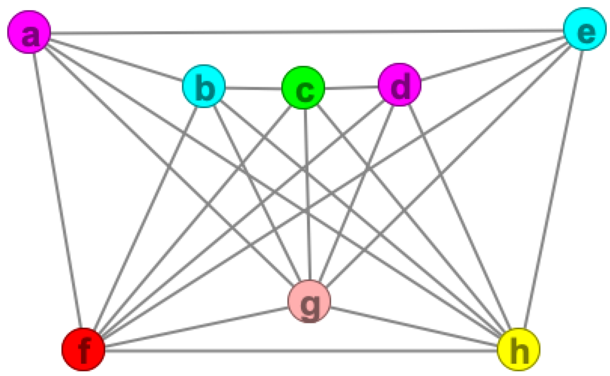
- Similarly, $d$ is adjacent to the vertices $e, f, g$ and $h$, and hence it must get a different colour from the colours we have used for the latter vertices. If we use a different colour from the colour of vertex $a$ as well, then we will definitely get a colouring with 6 colours or more. Thus, to try to avoid this, we could colour $d$ with the same colour used for $a$.

- We now observe that, by this point, we have either ended up with a proper colouring of $G_0$ which uses 6 colours or more, or, in the case that we have used only 5 colours so far, we have then been forced to use the same colour for $b$ as for $e$, and the same colour for $d$ as for $a$.

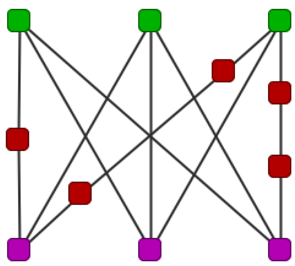
  In the latter case now, we see that the 5 neighbours of the remaining vertex $c$ are all coloured with a different colour, and hence we need to use a 6th colour for vertex $c$. In other words, in this case as well we need to use at least 6 colours.

From the above we can conclude that $\chi(G_0) \geqslant 6$.

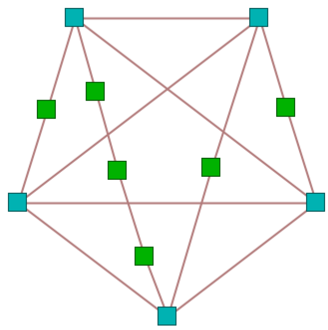
We now give a 6-colouring of $G_0$ (which will also show that $\chi(G_0) = 6$):



**Problem 4.** To make it as clear as possible which subgraphs we are considering, and also which subdivision of $K_{3,3}$ or of $K_5$ each such subgraph is, we will colour the vertices of a subdivision of $K_{3,3}$ in a similar way to the colouring in the next image, and analogously for the vertices of a subdivision of $K_5$.

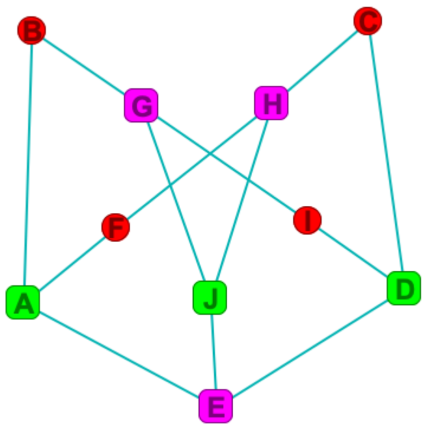


Example of a subdivision of $K_{3,3}$ with coloured vertices



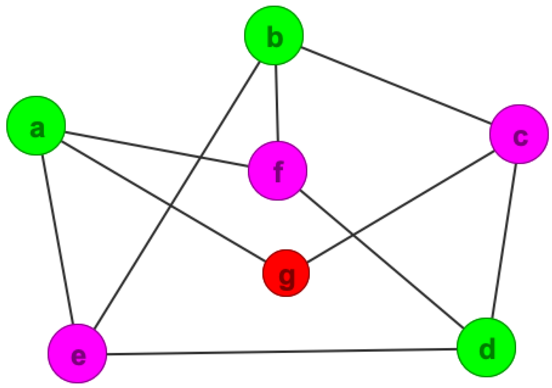
Example of a subdivision of $K_5$ with coloured vertices

We now have: graph $G_1$ contains the following subgraph



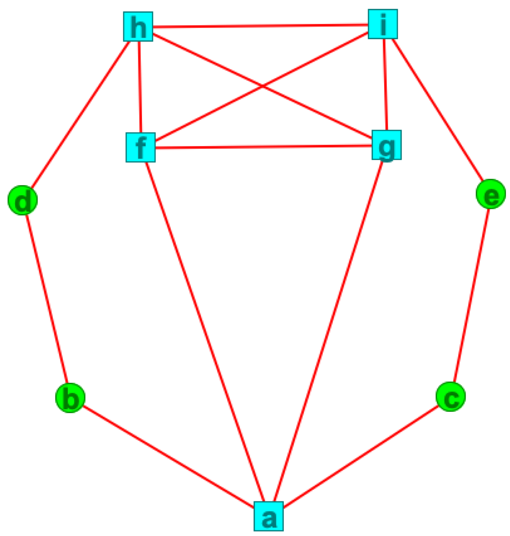

This is a subdivision of $K_{3,3}$ (which we can view as arising from subdividing preexisting edges $AG$, $AH$, $DG$ and $DH$, and replacing each of them by a path of length 2).

Graph $G_2$ contains the following subgraph



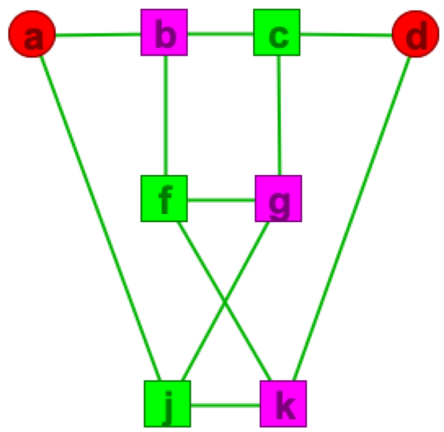

This is a subdivision of $K_{3,3}$ (which we can view as arising from subdividing a preexisting edge $ac$, and replacing it by a path of length 2).

Graph $G_3$ contains the following subgraph



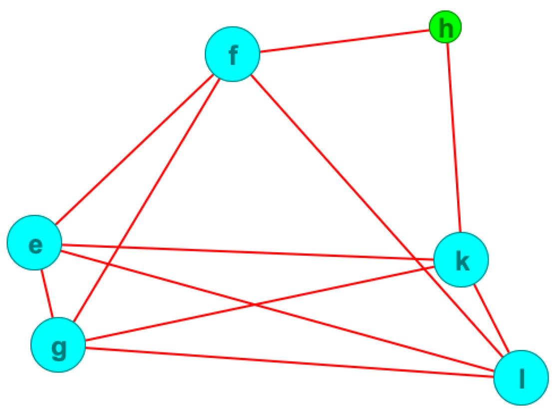

This is a subdivision of $K_5$ (which we can view as arising from subdividing preexisting edges $ha$ and $ia$, and replacing each of them by a path of length 3).

Graph $G_4$ contains the following subgraph




This is a subdivision of $K_{3,3}$ (which we can view as arising from subdividing preexisting edges $bj$ and $ck$, and replacing each of them by a path of length 2).

Finally, graph $G_5$ contains the following subgraph




This is a subdivision of $K_5$ (which we can view as arising from subdividing a preexisting edge $fk$, and replacing it by a path of length 2).