MATH 227 Feb 10 (1)

Recap from last Week: we saw that we can define the <u>determinant</u> of a square matrix in different but equivalent ways.

<u>Definition</u> Let $\mathbb{F}$ be a field and let $A \in \mathbb{F}^{n \times n}$. Then

(I) $$\det(A) = \sum_{\substack{\sigma \text{ is an} \\ n\text{-permutation}}} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n-1,\sigma(n-1)} \cdot a_{n\sigma(n)}$$

We call this the Leibniz formula for det(A).

(II) Alternatively, if $n=2$, then $\det(A)$ is defined as above, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$, while if $n>2$, then

a) for any $1 \le k \le n$,

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det(M_{kj})$$

kj-th minor of A, in $\mathbb{F}^{(n-1)\times(n-1)}$, so we assume we already know how to find $\det(M_{kj})$

We call this the expansion of det(A) over the k-th row.

b) for any $1 \le \ell \le n$,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+\ell} a_{i\ell} \det(M_{i\ell})$$

$i\ell$-th minor of A, in $\mathbb{F}^{(n-1)\times(n-1)}$, so we already have defined $\det(M_{i\ell})$

We call this the expansion of det(A) over the ℓ-th column.

We call the second definition the Laplace expansion or formula.

Recall that we can show that the two definitions give the exact same result by using mathematical induction in n:
more specifically, we can show for any $1 \le k \le n$ that the Laplace expansion of $\det(A)$ over the k-th row or the k-th column gives the same result as the Leibniz formula for the determinant of A.

Side Consequence of this: it doesn't matter which row or column of A we use to expand the determinant, the result we will find will be the same and equal to what the Leibniz formula gives us.

Our goal now is to use either formula for the determinant (whichever is more convenient in each case) in order to prove/establish different useful properties of determinants.

Property 1 Let $A \in \mathbb{F}^{n \times n}$ and recall that A^T stands for its transpose. We have that

$$\det(A^T) = \det(A).$$

Remark The proof of the property will be based on the following very important fact:

Fact 1 For every n-permutation σ,

$$\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma).$$

Proof of Fact 1

Consider first of all a k-cycle, say
$$(a_1\ a_2\ \ \ a_k).$$
Then
$$\big((a_1\ a_2 \dots a_k)\big)^{-1} = \underbrace{(a_k\ a_{k-1} \dots a_2\ a_1)}_{\text{reverse the order in which you write the elements}}$$
$$= (a_1\ a_k\ a_{k-1} \dots a_2).$$

Consider now an arbitrary permutation σ of $\{1,2,\dots n\}$. Then we can write σ as a product of disjoint cycles (where we omit 1-cycles for convenience): suppose we need ℓ such cycles, say
$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_\ell$$
with σ_i being a k_i-cycle ($k_i \geq 2$).

But then
$$\sigma^{-1} = \sigma_\ell^{-1} \cdots \sigma_2^{-1} \sigma_1^{-1},$$
and thus this is the disjoint cycle representation of σ^{-1}.

But as we just saw,

σ_1^{-1} is a k_1-cycle, exactly as σ_1,

σ_2^{-1} is a k_2-cycle, exactly as σ_2,

$\vdots$

σ_ℓ^{-1} is a k_ℓ-cycle, exactly as σ_ℓ,

thus when we write σ^{-1} next as a product of transpositions, we end up with the same number of transpositions as for σ $\longrightarrow$ $sgn(\sigma^{-1}) = sgn(\sigma)$.

<u>Proof of Property 1</u> For our notation to be clear, let's write $A^T = B = (b_{ij})_{1\le i,j\le n}$

Thus $b_{ij} = a_{ji}$.

Recall now that, by the Leibniz formula

$$\det(A) = \sum_{\sigma \text{ n-permutation}} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \text{ n-permutation}} \operatorname{sgn}(\sigma)\, a_{\sigma^{-1}(\sigma(1)),\sigma(1)}\, a_{\sigma^{-1}(\sigma(2)),\sigma(2)} \cdots a_{\sigma^{-1}(\sigma(n)),\sigma(n)}$$

$$= \sum_{\sigma \text{ n-permutation}} \operatorname{sgn}(\sigma)\, a_{\sigma^{-1}(1),1}\, a_{\sigma^{-1}(2),2} \cdots a_{\sigma^{-1}(n),n}$$

$$= \sum_{\sigma \text{ n-permutation}} \operatorname{sgn}(\sigma)\, b_{1,\sigma^{-1}(1)}\, b_{2,\sigma^{-1}(2)} \cdots b_{n,\sigma^{-1}(n)}$$

$$= \sum_{\sigma \text{ n-permutation}} \operatorname{sgn}(\sigma^{-1})\, b_{1,\sigma^{-1}(1)}\, b_{2,\sigma^{-1}(2)} \cdots b_{n,\sigma^{-1}(n)}$$

$$= \det(B) = \det(A^T).$$

<u>Property 2</u> Let $D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{pmatrix} \in F^{n\times n}$ a diagonal matrix, with $d_1, d_2, \ldots d_n$ its diagonal entries.

Then $\det(D) = \prod_{i=1}^{n} d_i$.

<u>Proof</u> By the Leibniz formula we have

$$\det(D) = \sum_{\sigma \text{ n-perm.}} \operatorname{sgn}(\sigma)\, d_{1,\sigma(1)}\, d_{2,\sigma(2)} \cdots d_{n,\sigma(n)}$$

Note now that, if $\sigma \neq id$, then we can find $i_0 \in \{1, 2, \ldots n\}$ such that $\sigma(i_0) \neq i_0$, therefore $d_{i_0,\sigma(i_0)} = 0 \Rightarrow d_{1,\sigma(1)} d_{2,\sigma(2)} \ldots d_{n,\sigma(n)} = 0$.

In other words, the only product in the above sum that is not automatically 0 is the product corresponding to the identity permutation: since $sgn(id) = +1$ we get $\det(D) = d_{11} d_{22} \ldots d_{nn} = \delta_1 \delta_2 \ldots \delta_n$.

<u>Property 3</u> If $A \in \mathbb{F}^{n\times n}$ has a zero row or a zero column, then $\det(A) = 0$.

<u>Proof</u> Here we use Laplace expansion, and we choose the zero row (or the zero column) of A to expand over: say the k-th row of A is zero; then

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det(M_{kj})$$

$$= \sum_{j=1}^{n} (-1)^{k+j} 0 \cdot \det(M_{kj}) = 0.$$

<u>Property 4</u> Let $A \in \mathbb{F}^{n\times n}$ and let $\lambda \in \mathbb{F}$. Write B for the matrix

$$\begin{pmatrix} - R_1(A) - \\ - R_2(A) - \\ \vdots \\ - \lambda R_k(A) - \\ \vdots \\ - R_n(A) - \end{pmatrix},$$

that is, the matrix we get by multiplying the k-th row of A by λ. Then

$$\det(B) = \lambda \det(A).$$

<u>Proof</u> We will use Laplace expansion over the k-th row: we have that

$$\det(B) = \sum_{j=1}^{n} (-1)^{k+j}\, b_{kj} \det(M_{kj}(B))$$

minor of B that we get by removing the k-th row and the j-th column

$\to$ since A and B have the same rows in the same positions except for their k-th rows (which are scalar multiples of each other) and which we are removing here, we get $M_{kj}(B) = M_{kj}(A)$

$$= \sum_{j=1}^{n} (-1)^{k+j} \cdot \lambda a_{kj} \det(M_{kj}(A))$$

$$= \lambda \cdot \left(\sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det(M_{kj}(A)) \right) = \lambda \cdot \det(A).$$

Similarly we show

Property 5 Let C be a matrix of the form

$$\begin{array}{r} \\ \\ \\ \text{k-th row} \to \\ \\ \\ \end{array} \begin{pmatrix} - R_1(C) - \\ - R_2(C) - \\ \vdots \\ - \bar{v}_1 + \bar{v}_2 - \\ \vdots \\ - R_n(C) - \end{pmatrix}$$

in $F^{n\times n}$, and write C_1 and C_2 for the matrices $\begin{pmatrix} - R_1(C) - \\ - R_2(C) - \\ \vdots \\ - \bar{v}_1 - \\ \vdots \\ - R_n(C) - \end{pmatrix}$ and $\begin{pmatrix} - R_1(C) - \\ - R_2(C) - \\ \vdots \\ - \bar{v}_2 - \\ \vdots \\ - R_n(C) - \end{pmatrix}$ respectively.

Then $\det(C) = \det(C_1) + \det(C_2)$

Proof We use Laplace expansion over the k-th row:

$$\det(C) = \sum_{j=1}^{n} (-1)^{k+j} (\bar{v}_1 + \bar{v}_2)_j \cdot \det(M_{kj}(C)) = \sum_{j=1}^{n} (-1)^{k+j} (\bar{v}_1)_j \cdot \det(M_{kj}(C)) +$$

$$+ \sum_{j=1}^{n} (-1)^{k+j} (\bar{v}_2)_j \det(M_{kj}(C)) \underset{\substack{\uparrow \\ \text{for every } 1\le j\le n \;\; M_{kj}(C) = M_{kj}(C_1) = M_{kj}(C_2)}}{=\!=\!=} \det(C_1) + \det(C_2).$$

From last Time: Important Properties of Determinants

Property 4 If $A \in \mathbb{F}^{n\times n}$ and $\lambda \in \mathbb{F}$, and we set

$$B = \begin{pmatrix} - R_1(A) - \\ - R_2(A) - \\ \vdots \\ - \lambda R_k(A) - \\ \vdots \\ - R_n(A) - \end{pmatrix} \quad \text{for some } 1 \le k \le n$$

then we have $\det(B) = \lambda \det(A)$.

Property 5 If A is as above, and $\bar{v}_1, \bar{v}_2 \in \mathbb{F}^n$ are (row) vectors satisfying $\bar{v}_1 + \bar{v}_2 = R_k(A)$, and we set

$$A_1 = \begin{pmatrix} - R_1(A) - \\ - R_2(A) - \\ \vdots \\ - \bar{v}_1 - \\ \vdots \\ - R_n(A) - \end{pmatrix}, \quad A_2 = \begin{pmatrix} - R_1(A) - \\ - R_2(A) - \\ \vdots \\ - \bar{v}_2 - \\ \vdots \\ - R_n(A) - \end{pmatrix},$$

then we have $\det(A) = \det(A_1) + \det(A_2)$.

Considering the two properties together, we can state that $\det : \mathbb{F}^{n\times n} \to \mathbb{F}$ is linear in the k-th row if we keep all other rows fixed.

Since k was arbitrary, we can say that det is linear in each row of A if we keep all other rows fixed.

Terminology Because $\det : \mathbb{F}^{n\times n} \to \mathbb{F}$ has these two properties, we call it a multilinear map or multilinear function of the rows of $A \in \mathbb{F}^{n \times n}$.

<u>Attention</u> If $n \geq 2$, that is, if we have at least two rows, then det is <u>not</u> a linear function of the matrix A. Indeed, we have

$$1 = \det(I_n) = \det\left(\begin{pmatrix} 1 & & & & \\ & 0 & & \mathbf{0} & \\ & & 1 & & \\ & \mathbf{0} & & \ddots & \\ & & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & & & & \\ & 1 & & \mathbf{0} & \\ & & 0 & & \\ & \mathbf{0} & & \ddots & \\ & & & & 0 \end{pmatrix} \right)$$

while
$$\det\left(\begin{pmatrix} 1 & & & & \\ & 0 & & \mathbf{0} & \\ & & 1 & & \\ & \mathbf{0} & & \ddots & \\ & & & & 1 \end{pmatrix} \right) + \det\left(\begin{pmatrix} 0 & & & & \\ & 1 & & \mathbf{0} & \\ & & 0 & & \\ & \mathbf{0} & & \ddots & \\ & & & & 0 \end{pmatrix} \right)$$

$$= 0 + 0 = 0,$$

therefore
$$\det\left(\begin{pmatrix} 1 & & & & \\ & 0 & & \mathbf{0} & \\ & & 1 & & \\ & \mathbf{0} & & \ddots & \\ & & & & 1 \end{pmatrix} \right) + \det\left(\begin{pmatrix} 0 & & & & \\ & 1 & & \mathbf{0} & \\ & & 0 & & \\ & \mathbf{0} & & \ddots & \\ & & & & 0 \end{pmatrix} \right)$$
$$\neq \det\left(\begin{pmatrix} 1 & & & & \\ & 0 & & \mathbf{0} & \\ & & 1 & & \\ & \mathbf{0} & & \ddots & \\ & & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & & & & \\ & 1 & & \mathbf{0} & \\ & & 0 & & \\ & \mathbf{0} & & \ddots & \\ & & & & 0 \end{pmatrix} \right).$$

This is why we say det is <u>multilinear</u> in the rows and not linear.

Multilinearity of the determinant is a property

that stands out (we'll see soon why).
Another such property is the following

<u>Property 6</u> Let $A \in F^{n\times n}$. If two rows of A are equal, then $\det(A) = 0$.

<u>Proof</u> Assume that $R_k(A) = R_\ell(A)$ with $1 \le k < \ell \le n$.

We use the Leibniz formula for $\det(A)$:

$$\det(A) = \sum_{\sigma\ n\text{-permutation}} \operatorname{sgn}(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)} \cdots a_{\ell\sigma(\ell)} \cdots a_{n\sigma(n)}$$

Let us first fix an n-permutation σ_0 and understand the product of matrix entries corresponding to it better: this is

$$a_{1\sigma_0(1)} a_{2\sigma_0(2)} \cdots a_{k\sigma_0(k)} \cdots a_{\ell\sigma_0(\ell)} \cdots a_{n\sigma_0(n)}.$$

But since $R_k(A) = R_\ell(A)$, this is equal to

$$a_{1\sigma_0(1)} a_{2\sigma_0(2)} \cdots a_{k\underline{\sigma_0(\ell)}} \cdots a_{\ell\underline{\sigma_0(k)}} \cdots a_{n\sigma_0(n)}$$

$$= a_{1\tau_0(1)} a_{2\tau_0(2)} \cdots a_{k\underline{\tau_0(k)}} \cdots a_{\ell\underline{\tau_0(\ell)}} \cdots a_{n\tau_0(n)}$$

where τ_0 is another n-permutation satisfying

$$\tau_0(k) = \sigma_0(\ell), \quad \tau_0(\ell) = \sigma_0(k)$$

and $\tau_0(i) = \sigma_0(i)$ for every $i \notin \{k, \ell\}$.

We can write down very explicitly how τ_0 is related to σ_0: we have

$$\tau_0 = \sigma_0 \cdot (k\ \ell).$$

Indeed if we write h for the transposition $(k\ell)$, we see that

$$(\sigma_0 \circ h)(k) = \sigma_0(h(k)) = \sigma_0(\ell) = \tau_0(k),$$
$$(\sigma_0 \circ h)(\ell) = \sigma_0(h(\ell)) = \sigma_0(k) = \tau_0(\ell)$$

and if $i \notin \{k, \ell\}$, then

$$(\sigma_0 \circ h)(i) = \sigma_0(h(i)) = \sigma_0(i) = \tau_0(i).$$

Therefore $\tau_0 = \sigma_0 \circ h = \sigma_0 \circ (k\ell)$ as functions.

If we now write σ_0 as a product of s transpositions, then immediately we can also write $\tau_0 = \sigma_0(k\ell)$ as a product of $\underline{s+1}$ transpositions.

This implies (we can justify this rigorously) that

$$\operatorname{sgn}(\tau_0) = -\operatorname{sgn}(\sigma_0)$$

(because if s is even, and hence $\operatorname{sgn}(\sigma_0) = +1$, then $s+1$ is odd, and hence (as we can justify, even though the corresponding product of transpositions that we wrote τ_0 equal to may not be the one coming from the disjoint cycle representation of τ_0) we get that $\operatorname{sgn}(\tau_0) = -1$.

Similarly, if s is odd, and hence $\operatorname{sgn}(\sigma_0) = -1$, then $s+1$ is even, and we get $\operatorname{sgn}(\tau_0) = +1$.)

We can now use these observations as follows when going back to the formula for $\det(A)$:

$$\det(A) = \sum_{\sigma \text{ n-permutation}} \operatorname{sgn}(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)} \cdot a_{\ell\sigma(\ell)} \cdots a_{n\sigma(n)}$$

$$= \sum_{\substack{\sigma \text{ n-permutation} \\ \operatorname{sgn}(\sigma) = +1}} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)} - a_{\ell\sigma(\ell)} \cdots a_{n\sigma(n)} \cdot \left[\operatorname{sgn}(\sigma) + \operatorname{sgn}(\sigma \circ (k\ell))\right]$$

$$= \sum_{\substack{\sigma \text{ n-permutation} \\ \operatorname{sgn}(\sigma) = +1}} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)} \cdots a_{\ell\sigma(\ell)} \cdots a_{n\sigma(n)} \cdot \underbrace{\left[+1 + (-1)\right]}_{= 0}$$

$$= 0,$$

which is what we wanted.

<u>Terminology</u> A multilinear map satisfying Property 6 is usually called <u>alternating</u>.

Summarising the above, we get

<u>Remark 1</u> The function

$$\det : F^{n \times n} \longrightarrow F$$

is a <u>multilinear</u> <u>alternating</u> function in the rows of $A \in \mathbb{F}^{n \times n}$ which also satisfies the condition

$$\det(I_n) = 1.$$

<u>Very Important Remark</u> It turns out that there is a $\boxed{\text{unique}}$ function from $F^{n \times n} \longrightarrow F$ which is <u>multilinear</u> and <u>alternating</u> in the rows and satisfies $\det(I_n) = 1$.

In other words, these properties characterise the determinant.

Additional Note Since $\det(A^T) = \det(A)$, det is also a multilinear, alternating function in the columns of A.

One more very useful property of determinants is the following:

Property 7 Let $A \in \mathbb{F}^{n \times n}$ and let $1 \leq k < l \leq n$.

$$\text{Set } B = \begin{array}{r} \\ \\ k\text{-th row} \\ \\ l\text{-th row} \\ \\ \end{array} \begin{pmatrix} - R_1(A) - \\ - R_2(A) - \\ \vdots \\ - R_l(A) - \\ \vdots \\ - R_k(A) - \\ \vdots \\ R_n(A) \end{pmatrix}$$

(that is, B is the matrix we get if we swap the k-th and l-th rows of A).

Then $\det(B) = -\det(A)$.

Proof We could give a proof using either the Leibniz formula or the Laplace expansion for the determinant, and then by analysing the sum we get very similarly to above.

However, now that we have stated that Properties 4, 5 and 6 characterise the determinant, it's worth giving a proof of Property 7 that utilises only Properties 4, 5, 6.

By Property 6 we have that

$$0 = \det \begin{array}{r} \\ \\ k\text{-th row} \\ \\ l\text{-th row} \\ \\ \end{array} \begin{pmatrix} - R_1(A) - \\ - R_2(A) - \\ \vdots \\ - R_k(A) + R_l(A) - \\ \vdots \\ - R_k(A) + R_l(A) - \\ \vdots \\ - R_n(A) - \end{pmatrix} \underset{\substack{\text{Property 5} \\ \text{applied to} \\ \text{the } k\text{-th row}}}{=\!=\!=} \det \begin{pmatrix} - R_1(A) - \\ - R_2(A) - \\ \vdots \\ - R_k(A) - \\ \vdots \\ - R_k(A) + R_l(A) - \\ \vdots \\ - R_n(A) - \end{pmatrix} +$$

$$+\det\begin{pmatrix} -R_1(A)- \\ -R_2(A)- \\ \vdots \\ -R_\ell(A)- \\ \vdots \\ -R_k(A)+R_\ell(A)- \\ \vdots \\ -R_n(A)- \end{pmatrix} \underline{\qquad\qquad\qquad}$$

Property 5 applied to the ℓ-th row now

$$= \left[\underbrace{\cancel{\det\begin{pmatrix} R_1(A) \\ R_2(A) \\ \vdots \\ R_k(A) \\ \vdots \\ R_k(A) \\ \vdots \\ R_n(A) \end{pmatrix}}}_{=0 \text{ by using Property 6 again}} + \det\begin{pmatrix} R_1(A) \\ R_2(A) \\ \vdots \\ R_k(A) \\ \vdots \\ R_\ell(A) \\ \vdots \\ R_n(A) \end{pmatrix}\right] + \left[\det\begin{pmatrix} R_1(A) \\ R_2(A) \\ \vdots \\ R_\ell(A) \\ \vdots \\ R_k(A) \\ \vdots \\ R_n(A) \end{pmatrix} + \underbrace{\cancel{\det\begin{pmatrix} R_1(A) \\ R_2(A) \\ \vdots \\ R_\ell(A) \\ \vdots \\ R_\ell(A) \\ \vdots \\ R_n(A) \end{pmatrix}}}_{=0 \text{ by Property 6}}\right]$$

$$= \det(A) + \det(B).$$

In other words, we showed

$$\det(A) + \det(B) = 0 \implies \det(B) = -\det(A).$$

We can finally use the properties we established (and in particular Properties 3, 4, 5, 6 and 7) in order to prove the most commonly mentioned and used properties of determinants.

<u>Theorem 1</u> Let $A \in \mathbb{F}^{n\times n}$. We have that

$\det(A) = 0$ if and only if A is <u>not</u> invertible.

__Theorem 2__ (Multiplication Theorem) Let $A, B \in F^{n\times n}$. We have that $\det(AB) = \det(A)\cdot\det(B)$.

The proofs for both theorems will rely on us understanding how elementary row operations affect the determinant.

Recall the notation we were using in MATH 127 for elementary matrices:

$D_{k;\lambda}$ corresponds to the elementary row operation of multiplying the k-th row by λ.

$$D_{k;\lambda} = \begin{array}{c} \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \lambda & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow k\text{-th row} \\ \\ \\ \end{array}$$

$E_{i,j;\mu}$ corresponds to adding to the i-th row the j-th row multiplied by μ

$$E_{i,j;\mu} = \begin{array}{c} \begin{pmatrix} 1 & & & & \\ 0 & \ddots\; 1 \ldots 0 \; 0 \ldots & \mu & 0 \; \cdots & \\ 0 & & 1 & 0 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \\ \uparrow \\ j\text{-th column} \end{array} \begin{array}{l} \\ \leftarrow i\text{-th row} \\ \\ \\ \\ \\ \\ \end{array}$$

P_{ij} corresponds to swapping the i-th row and the j-th row

$$P_{ij} = \begin{pmatrix} \overline{e_1} \\ \overline{e_2} \\ \vdots \\ \overline{e_j} \\ \vdots \\ \overline{e_i} \\ \vdots \\ \overline{e_n} \end{pmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i\text{-th row} \\ \\ \leftarrow j\text{-th row} \\ \\ \\ \end{array}$$

We have the following:

__Theorem 0__ Let $A \in F^{n\times n}$, $\lambda, \mu \in F$, $1 \le i < j \le n$ or $1 \le j < i \le n$.

Then (a) $\det(D_{i;\lambda}A) = \lambda\cdot\det(A) = \det(D_{i;\lambda})\cdot\det(A)$.

(b) $\det(E_{i,j;\mu}A) = \det(A) = \det(E_{i,j;\mu})\cdot\det(A)$.

(c) $\det(P_{ij}A) = -\det(A) = \det(P_{ij})\cdot\det(A)$.