Math 322
Review and suggested practice problems

(I) Important Concepts/Terminology. *(review also the notation for the notions below, wherever we have introduced one)*

- Graphs/multigraphs/directed graphs.
- Vertices/nodes and edges of a graph; (V, E)-notation; (pictorial) representations.
- Multiple (or parallel) edges; loops.
- Order and size of a graph; finite graphs.
- Labelled/unlabelled graphs.
- Adjacent (or neighbouring) vertices; adjacent edges; endvertices of an edge (vertex incident with an edge).
- Neighbourhood of a vertex, degree of a vertex.
- Connected/disconnected graphs; connected components of a graph.
- Subgraphs and induced subgraphs.
- Forbidden subgraphs.
- Walks; paths; cycles; trails; circuits.
- Adjacency matrix of a graph; Incidence matrix of a graph.
- Graph Isomorphism; Isomorphic graphs.
- Operations on graphs *(review also the established notation, wherever applicable)*:
  - Complement $\overline{G}$ of a graph G.
  - Line graph $L(G)$ of a graph G.
  - Disjoint union $G_1 \oplus G_2$ of two graphs G_1, G_2 (with disjoint vertex sets).
  - Join $G_1 \vee G_2$ of two graphs G_1, G_2 (with disjoint vertex sets).

  - Vertex deletion.
  - Edge deletion.

- Degree sequences and graphical sequences.

- Regular graphs. *(Recall the family of 0-regular graphs, of 1-regular graphs, and of 2-regular graphs.)*

- Special families of graphs *(review also the established notation, wherever applicable)*:

  - Null graphs.
  - Complete graphs.
  - Paths; trees; forests.
  - Cycle graphs.
  - Cyclic / acyclic graphs.
  - Wheel graphs.
  - Bipartite graphs.

- Maximum and minimum degree of a graph G ($\Delta(G)$ and $\delta(G)$ respectively).

- Cutvertices (or equivalenlty, 1-vertex cuts) of a graph.

- Vertex cuts of a graph.

- Bridges (or equivalently, 1-edge cuts) of a graph.

- Edge cuts of a graph.

- k-vertex connected graphs; the vertex connectivity $\kappa(G)$ of a graph.

- The edge connectivity $\lambda(G)$ of a graph G.

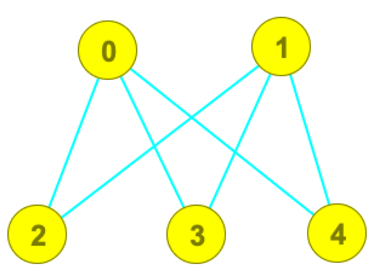
  - Review (or practise) how we determine the parameters $\kappa(G)$ and $\lambda(G)$ for important families of connected graphs (e.g. complete graphs, bipartite graphs, cycle graphs, trees).
  - Review the related results about
    * the maximum size of a disconnected graph on n vertices;
    * the minimum size of a connected graph on n vertices.

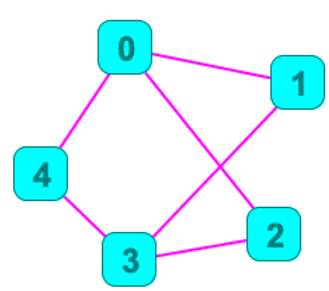
- Vertex cuts for two (non-adjacent) vertices u, v of a connected graph G.
- Edge cuts for two vertices u, v of a connected graph G.
- Local vertex connectivities.
- Local edge connectivities.
- Internally disjoint (or equivalently, vertex-disjoint) paths.
- Edge-disjoint paths.
  - Review how the local vertex connectivities relate to the vertex connectivity $\kappa(G)$ of the graph G.
  - Review how the local edge connectivities relate to the edge connectivity $\lambda(G)$ of the graph G.
- Review the equivalent characterisations of trees that we have discussed.
- Spanning trees of a connected graph G.
- Weighted graphs.
- Minimum weight spanning tree.

Review the theorems, propositions, lemmas (and possibly algorithmic methods) that we stated and discussed in class regarding these concepts/notions.

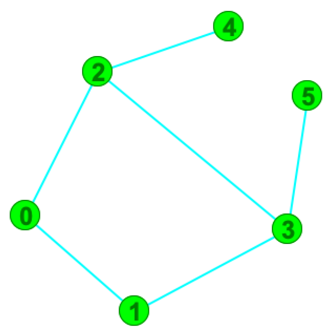
(II) Some of the practice problems suggested in the lectures.

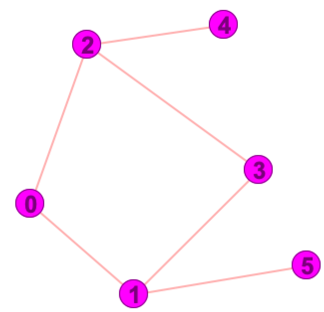
Problem A. Find an isomorphism from the first graph to the second graph in the picture below.





Problem B. Show that the following graphs are not isomorphic.





Could you use their degree sequences to conclude that they are not isomorphic?

Problem C. Determine whether the sequence

$$(8, 8, 7, 7, 6, 6, 4, 3, 2, 1, 1, 1)$$

is graphical or not.

Problem D. Give an example of a graph G which

- satisfies $\delta(G) \geqslant \frac{n-2}{2}$.
- and is disconnected.

Problem E. Let G be a connected graph which has at least 3 vertices. Show that at least one of these vertices has degree $\geqslant 2$.

Problem F. Let G be a graph, and let e_0 be an edge in G. Show that e_0 is a bridge of G **if and only if** e_0 does **not** belong to any cycle in G.

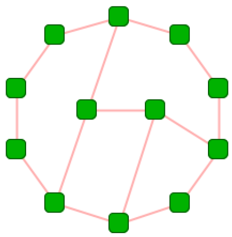
Problem G. (a) Let G be a connected graph, and let u, v be two different vertices of G.

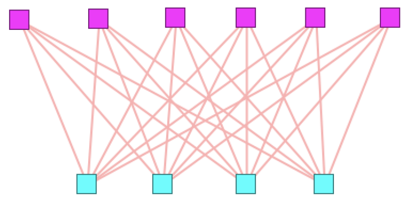
(i) Show that, if there are (at least) two internally disjoint $u-v$ paths in G, then there is a cycle in G **passing through the vertices u and v**.

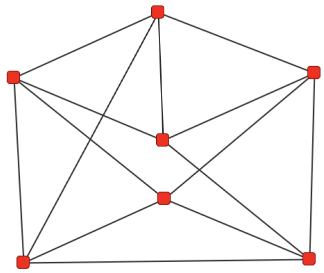
(ii) Show that, if there are (at least) two <u>different</u> (but not necessarily internally disjoint) $u-v$ paths in G, then G contains at least one cycle.

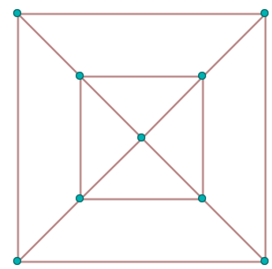
(b) Let T be a tree. Then, for every two vertices w and z in T, there is **exactly one** path connecting w and z.

Problem H. Find spanning trees for the following graphs.

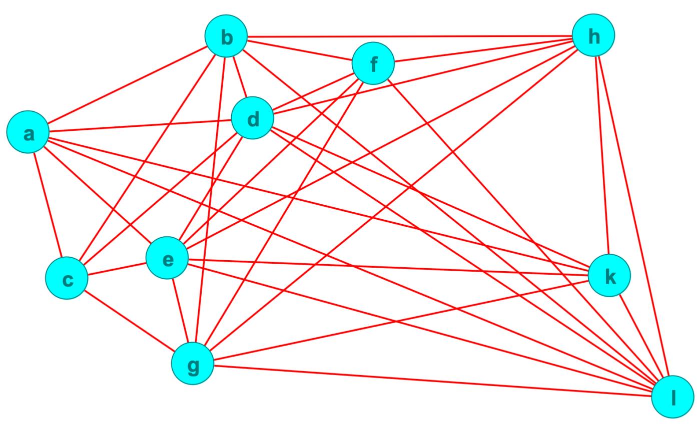








Problem J. Consider the following labelled graph:



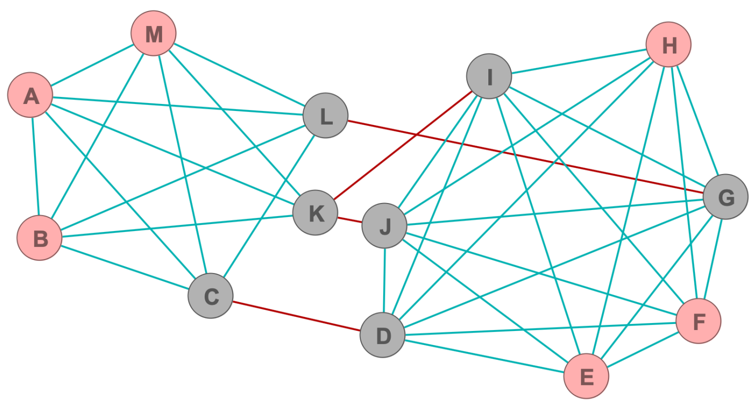
(i) Do we have $\kappa'(a, c) \geqslant 5$? Justify your answer.

(ii) Can you find 5 pairwise internally disjoint $c - h$ paths?

(iii) Can you find edge-disjoint paths that connect the vertices g and h which are not vertex-disjoint (equivalently, internally disjoint)?

(iv) Determine $\lambda'(a, k)$. Show all your work.

Problem K. (i) For the following graph G_0, determine precisely $\kappa(G_0)$ and $\lambda(G_0)$. Show all your work.



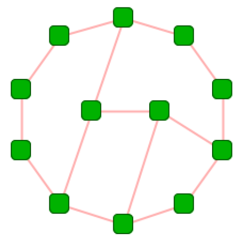
(ii) Find a pair of non-adjacent vertices u, v of G_0 such that $\kappa(u, v) = \kappa(G_0)$.

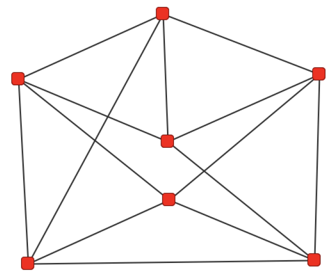
Problem L. (i) Determine the vertex connectivity and the edge connectivity of

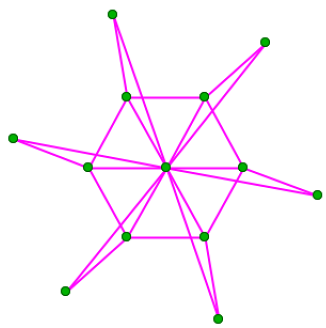
- a cycle graph C_n.
- a bipartite graph $K_{m,n}$.
- a wheel graph W_m.

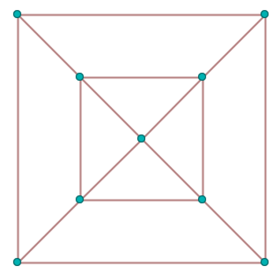
[*Hint.* The parameters in question may or may not depend on the parameters m, n which give the order and size of the graphs; if they do depend, express them as functions of m and/or n.]

(ii) For each of the following graphs G, determine precisely $\kappa(G)$ and $\lambda(G)$. Show all your work.









(III) A few more practice problems.

Problem 1. Explain why P_4 is **<u>not</u>** an induced subgraph of $K_{4,4}$.

Problem 2. For each of the following graphs, write its adjacency matrix.

(i) A path P_{2k} of odd length, where the vertices are labelled in the order that we pass by them as we traverse the path (starting at one of its ends).

(ii) A path P_{2k+1} of even length, where the vertices are labelled as before.

(iii) A cycle C_{2k} of even length, where the vertices are labelled consecutively around the cycle.

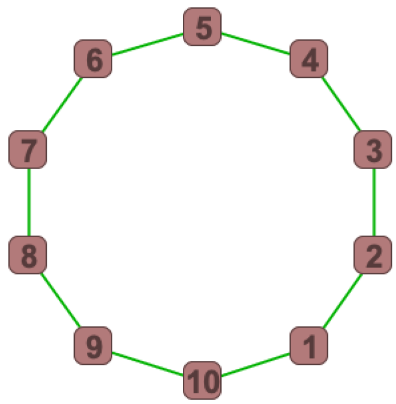
(iv) A cycle C_{2k+1} of odd length, where the vertices are labelled as before.

(v) A bipartite graph $K_{m,n}$, where the vertices in the first partite set are labelled as $v_1, v_2, \ldots, v_m$, and the remaining vertices are labelled as $v_{m+1}, v_{m+2}, \ldots, v_{m+n}$.

(vi) A complete graph K_n (labelled in any way you like).

Problem 3. (a) Consider the complete graph K_n on n vertices, and let u, v be two different vertices of K_n. For each $s \geqslant 1$, determine how many $u-v$ walks of length s we have in K_n.

(b) Consider the 10-cycle C_{10} labelled as below. Explain why either all walks from the vertex 1 to the vertex 5 will have even length, or all such walks will have odd length (determine also which of the two statements is true and which is false).



Remark. For the next problem, you may wish to look back at your answers to Problem 3.

Problem 4. (a) Without computing the matrix directly, find A_0^3 where A_0 is the adjacency matrix of K_4.

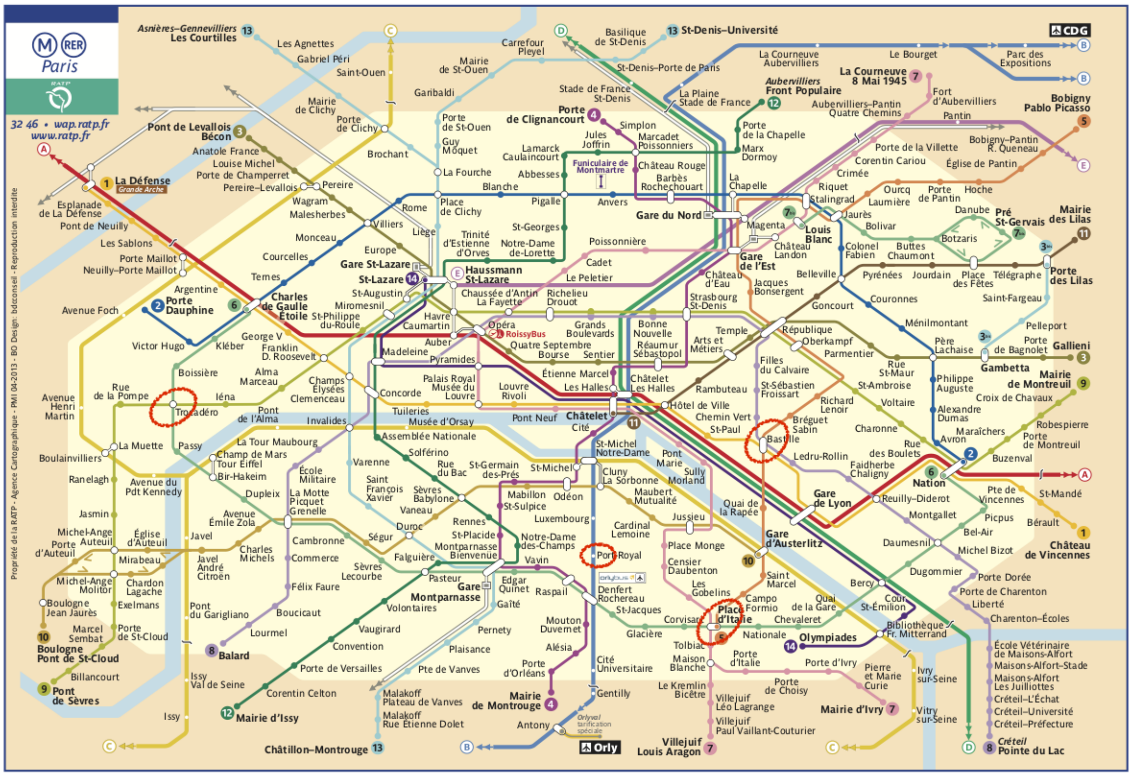
(b) Find the (1,5) entry of A_1^{2009}, where A_1 is the adjacency matrix of the cycle C_{10} above.

[*Hint.* It may help to also recall HW2, Problem 1.]

Problem 5. Recall that we can view the map of the Paris subway and regional train system as a (simple) graph,

- if we view each station as a node of this graph,
- and we assume that there is an edge connecting two such stations/nodes if we can get from one station to the other (on one of the available subway or train lines) without any intermediate stops.

(Recall also that we do not consider multiple edges for the time being, or in other words, we don't distinguish between the cases where two stations are consecutive in exactly one line/route or in more than one.)



In the above copy of this map, four stations have been highlighted: Trocadéro, Port Royal *(or in other words, royal port)*, Place d'Italie *(or in other words, the Italy square)*, and Bastille.

Assume that you are spending a semester in Paris, and have chosen not to get a car, instead relying on the subway system to get around the city.

What is the minimum number of stations that would need to be shut down for you to be forced to take the bus or a taxi

(i) if you needed to go from Port Royal to Place d'Italie? *(Assume here that these two stations are still in service, but other stations may be closed.)* In other words, what is the local vertex connectivity $\kappa(\text{Port Royal}, \text{Place d'Italie})$?

(ii) if you needed to go from Place d'Italie to Bastille? In other words, what is the local vertex connectivity $\kappa(\text{Place d'Italie},\ \text{Bastille})$?

(iii) if you needed to go from Bastille to Trocadéro? In other words, what is the local vertex connectivity $\kappa\big(\text{Trocadéro}, \text{Bastille}\big)$?

Problem 6. Consider again the following graph:

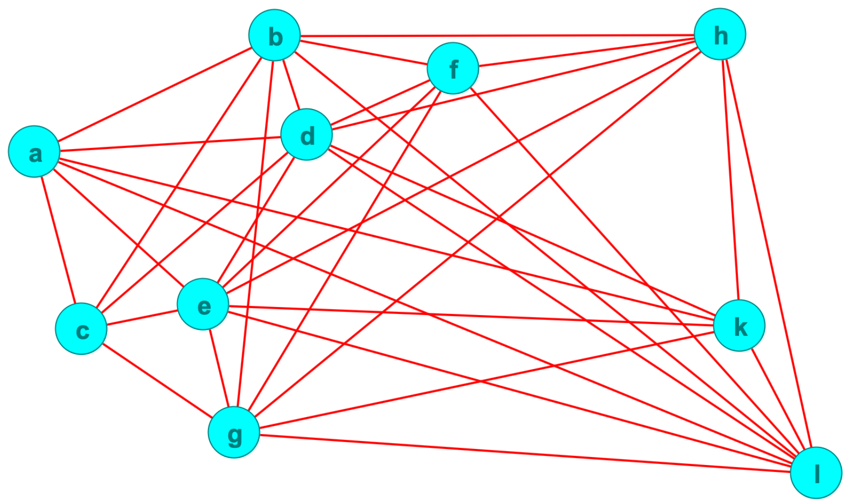


Figure 1: Graph H_0

Note that if we order the vertices of H_0 as follows:

$$\text{d, e, l, b, g, h, a, k, f, c,}$$

then the degree sequence of the graph is

$$(8,\ 8,\ 8,\ 7,\ 7,\ 7,\ 6,\ 6,\ 6,\ 5).$$

Then two consecutive applications of the Havel-Hakimi theorem give that the following sequences are also graphical

$$(8-1,\ 8-1,\ 7-1,\ 7-1,\ 7-1,\ 6-1,\ 6-1,\ 6-1,\ 5) = (7,\ 7,\ 6,\ 6,\ 6,\ 5,\ 5,\ 5,\ 5)$$

$$\text{and} \quad (7-1,\ 6-1,\ 6-1,\ 6-1,\ 5-1,\ 5-1,\ 5-1,\ 5) = (6,\ 5,\ 5,\ 5,\ 4,\ 4,\ 4,\ 5).$$

By revisiting the proof of the theorem (and determining how it would work for this particular graph H_0, that is, which cases and subcases we would need to consider), find new graphs (based on/related to the given graph H_0) that realise the other two sequences.