1) $f(x) = \sqrt{1-x^2} \quad -1 \le x \le 1$

then $\sqrt{1-x^2} = \sum_{n=0}^{\infty} c_n P_n(x)$

where $c_n = \frac{2n+1}{2} \int_{-1}^{1} \sqrt{1-x^2}\, P_n(x)\, dx$

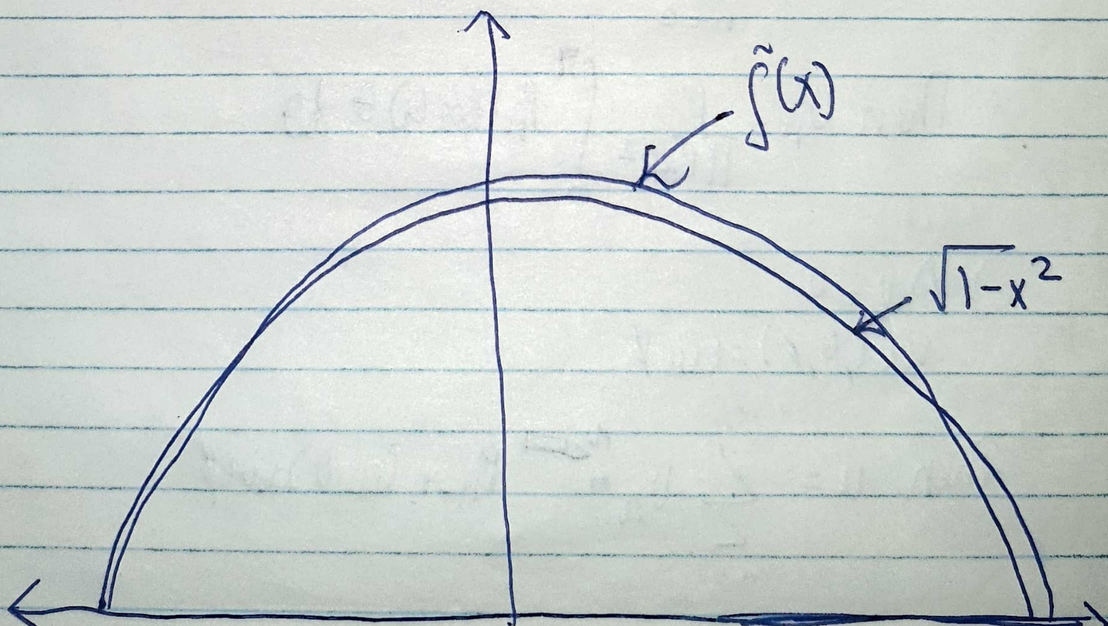
$c_0 = \frac{1}{2} \int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{4}$ | $c_2 = \frac{5}{2} \int_{-1}^{1} \sqrt{1-x^2}\, \frac{1}{2}(3x^2-1)\, dx = -\frac{5\pi}{8 \times 4}$

$c_1 = \frac{3}{2} \int_{-1}^{1} \sqrt{1-x^2} \cdot x = 0$ | $c_3 = \frac{7}{2} \int_{-1}^{1} \sqrt{1-x^2}\, P_4 x = 0$

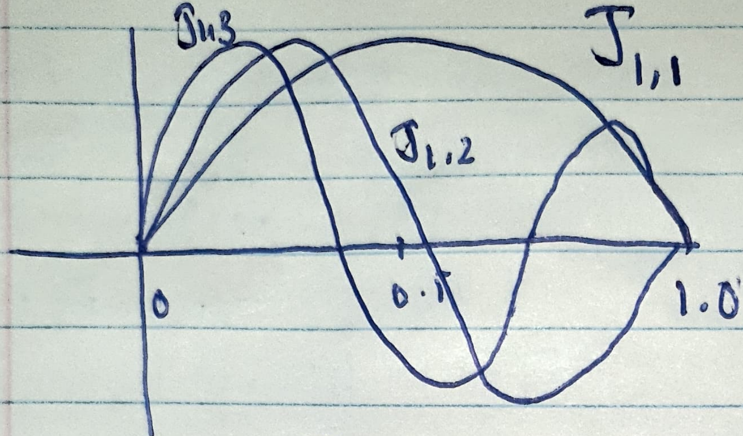
$\therefore c_{2k+1} = 0 \ \forall k \in \mathbb{N}$

$\therefore \tilde{f}(x) \approx \frac{\pi}{4} + -\frac{5\pi}{32} \cdot \left(\frac{1}{2}\right)(3x^2-1) + \ldots$

2)



2) $J_{1,1}, J_{1,2}, J_{1,3}$



$$f(x) \sim \sum_{p=1}^{\infty} f_p J_{np}(x) = \sum_{p=1}^{3} f_p J_{np}(x)$$

now $f_1 = \dfrac{1}{\|J_{11}\|_x^2} \langle f, J_{np} \rangle_x = \dfrac{1}{\|J_{11}\|_x^2} \langle 1, J_{11} \rangle_x$

$f_2 = \dfrac{1}{\|J_{1,2}\|_x^2} \langle 1, J_{12} \rangle_x$, $f_3 = \dfrac{1}{\|J_{1,3}\|_x^2} \langle 1, J_{1,3} \rangle_x$

$\therefore 1 \sim f_1 J_{11}(x) + f_2 J_{12}(x) + f_3 (J_{13})(x)$

3) a) $\Delta u = 0$

$u(1,\theta,\phi) = 1$

Then we know $u(r,\theta,\phi) = \sum_{n=0}^{\infty} U_n r^n P_n(\cos\theta)$

and $u(1,\theta,\phi) = 1 = \sum_{n=0}^{\infty} U_n P_n(\cos\theta)$

now clearly since $P_0 = 1$ we derive $U_n = 0 \ \forall n \geq 1$

& $\Rightarrow U_0 = 1$

$\therefore u = 1$ is the soln

b) $\Delta u = 0$

$u(1,\theta,\phi) = \theta$

since the boundary condition is independent of ϕ

Now again $u(r,\theta) = \sum_{n=0} U_n r^n P_n(\cos\theta)$

Applying boundary

$\Rightarrow \theta = \sum_{n=0}^{\infty} U_n P_n(\cos\theta))$

Then $U_n = \frac{1}{\|P_n\|^2} \int_0^{\pi} P_n(\cos\theta)\, \theta \, d\theta$

c) $\Delta u = 0$

$u(1,\theta,\phi) = \sin\phi$

Then $u = \sum_{n=1}^{\infty} U_n r^n P_{n,1}(\cos\theta)\sin\phi$

Applying boundary

$\sin\phi = \sum_{n} U_n P_{n,1}(\cos\theta)\sin\phi$

$\Rightarrow U_n = \frac{1}{\|P_{n,1}\|^2} \int_0^{\pi} P_{n,1}(\cos\theta)\sin\theta \, d\theta$.

4) a) $\begin{cases} \Delta u = 0 \\ u(1,\theta,\phi) = -1 \\ u(2,\theta,\phi) = 1 \end{cases}$

now since this is independent of ϕ

we get $u(r,\theta,\phi) = \sum_{n=0}^{\infty} \left[A_n r^n + B_n r^{-n-1}\right] P_n(\cos\theta)$

then $u(1,\theta,\phi) = -1 = \sum_{n=0}^{\infty} [A_n + B_n] P_n(\cos\theta)$

now clearly $n=0 \Rightarrow A_0 + B_0 = -1$

$u(2,\theta,\phi) = 1 = \sum_{n=0}^{\infty} \left[\cancel{A_n + B_n} \left[A_n 2^n + \frac{B_n}{2^{n+1}}\right] P_n(\cos\theta)\right]$

now again $n=0$

$\Rightarrow A_0 + \frac{B_0}{2} = 1$

$- \quad A_0 + B_0 = -1$

$-\frac{B_0}{2} = 2 \Rightarrow B_0 = -4$

$\Rightarrow A_0 = 3$

$\therefore u(r,\theta,\phi) = \left(3 - \frac{4}{r}\right)$

4) b) $\begin{cases} \Delta u = 0 \\ u(1,\theta,\phi) = \sin\phi \\ u(2,\theta,\phi) = \sin\theta \end{cases}$

Then soln is of the form

$$u(r,\theta,\phi) = \sum_{n,m} \left[A_{n,m} r^n + \frac{B_{n,m}}{r^{n+1}}\right] P_{n,m}(\cos\theta)\, F_{n,m}$$

where $F_{n,m} = (\tilde{A}_{n,m}\cos(m\phi) + \tilde{B}_{n,m}\sin(m\phi))$

~~Then now $u(1,\theta,\phi)$~~ = Clearly $m=1$ & no $\cos(m\phi)$ term $\Rightarrow \tilde{A}_{n,m} = 0$

$$\therefore u(r,\theta,\phi) = \sum_{n,m} \left[A_{n,m} r^n + \frac{B_{n,m}}{r^{n+1}}\right] P_{n,m}(\cos\theta)\sin\phi$$

$$u(1,\theta,\phi) = \sum_{n,m} [A_{n,m} + B_{n,m}] P_{n,m}(\cos\theta)\cancel{\sin\phi} = \cancel{\sin\phi}$$

$$= \sum_{n,m} [A_{n,m} + B_{n,m}] P_{n,m}(\cos\theta) = 1$$

$$= \sum_{n} [A_{n,1} + B_{n,1}] P_{n,1}(\cos\theta) = 1$$

and $$u(2,\theta,\phi) = \sum_{n} \left[A_{n,1} 2^n + \frac{B_{n,1}}{2^{n+1}}\right] P_{n,1}(\cos\theta) = 1$$

$$\therefore A_{n,1} + B_{n,1} = \frac{\langle P_{n,1}(\cos\theta), 1\rangle_{\sin\theta}}{\|P_{n,1}\|^2_{\sin\theta}} = k$$

$$A_{n,1} 2^n + \frac{B_{n,1}}{2^{n+1}} = \frac{\langle P_{n,1}(\cos\theta), 1\rangle_{\sin\theta}}{\|P_{n,1}\|^2_{\sin\theta}} = k$$

$$\Rightarrow A_{n,1} + B_{n,1} = k \qquad \times 2^n \Rightarrow \quad 2^n A_{n,1} + 2^n B_{n,1} = 2^n k \quad - ①$$

$$2^n A_{n,1} + \frac{B_{n,1}}{2^{n+1}} = k \qquad - ②$$

$$① - ② \Rightarrow \quad 2^n B_{n-1} - \frac{B_{n-1}}{2^{n+1}} = k(2^n - 1)$$

$$\Rightarrow B_{n,1} = \frac{k(2^n-1)}{\left(2^n - \frac{1}{2^{n+1}}\right)}$$

$\Rightarrow$

$$2^n A_{n,1} = 2^n k - \frac{2^n k(2^n-1)}{\left(2^n - \frac{1}{2^{n+1}}\right)}$$

$$\Rightarrow A_{n,1} = k - \frac{k(2^n-1)}{\left(2^n - \frac{1}{2^{n+1}}\right)}$$

$$\therefore u(r,\theta,\phi) = \sum_{n,1} \left[A_{n,1} r^n + \frac{B_{n,1}}{r^{n+1}} \right] P_{n,m}(\cos\theta) \sin\phi$$

5) $\begin{cases} \partial_t u = \Delta_{(\theta,\phi)} u - P_j(\cos\theta) & \text{for fixed } j \\ u(0,\theta,\phi) = 0 & \text{Let } u = v + w \end{cases}$

Let us first solve $\Delta v = \phi = P_j(\cos\theta)$
since this is independent of ϕ hence

then $v(\theta,\phi) = \sum_{n=1}^{\infty} V_n P_j(\cos\theta)$

Then $\Delta_{(\theta,\phi)} v = \sum_{n=1}^{\infty} -n(n+1) P_n(\cos\theta) V_n$

$\Rightarrow \sum_{n=1}^{\infty} -n(n+1) P_n(\cos\theta) V_n = P_j(\cos\theta) =$

then $V_n = \dfrac{-1}{n(n+1)\,\|P_n\|^2} \langle P_n(\cos\theta), P_j(\cos\theta)\rangle_{\sin\theta}$

and for all $n \neq j$ $V_n = 0 \Rightarrow V = \dfrac{-1}{n(n+1)} P_n(\cos\theta)$

now $\begin{cases} w \text{ we have} \\ \partial_t w = \Delta w \\ w = \dfrac{1}{n(n+1)} P_n(\cos\theta) \end{cases}$

Let $w = W(t) P_n(\cos\theta)$

$\Rightarrow P_n(\cos\theta) W' = W(-n(n+1)) P_n(\cos\theta)$

$\Rightarrow W = W(0)\, e^{-n(n+1)t}$

Then at $t=0 \Rightarrow \dfrac{1}{n(n+1)} \cancel{P_n(\cos\theta)} = W(0) \cancel{P_n(\cos\theta)}$

$\Rightarrow W = \dfrac{1}{n(n+1)} e^{-n(n+1)t} P_n(\cos\theta)$

- Hence $u = v + w$

$$= \frac{-1}{n(n+1)} P_n(\cos\theta) + \frac{e^{-n(n+1)t}}{n(n+1)} P_n(\cos\theta)$$

now at $u(1, \pi/4, \phi)$ $n=2 \Rightarrow \left(\frac{-1}{6} P_2(\cos\theta) + \frac{e^{-6t}}{6} P_2(\cos\theta)\right)$

$$= \frac{3\cos^2\pi/4 - 1}{2}\left(\frac{e^{-6t} - 1}{6}\right)$$

$$= \frac{1}{4}\left(\frac{e^{-6t} - 1}{6}\right) = \frac{1}{24}\left(e^{-6t} - 1\right)$$

B) $\begin{cases} \partial_{tt} u = \Delta(\theta,\phi) u \\ u(0,\theta,\phi) = 0 \ , \ \partial_t u(0,\theta,\phi) = f(\theta,\phi) \end{cases}$

According to the initial condition we write

$$u(t,\theta,\phi) = \sum_{n,m}^{\infty} U_n(t) P_{n,m}(\cos\theta)\sin(m\phi)$$

Then substituting into the eqn gives

$$\sum_{n,m} U_n''(t) P_{n,m}(\cos\theta)\sin(m\phi) = \sum_{n,m} -n(n+1) U_n(t) P_{n,m}(\cos\theta)\sin(m\phi)$$

$$\Rightarrow U_n''(t) = -n(n+1) U_n(t) \Rightarrow U_n(t) = A_{n,m}\cos(\sqrt{\lambda_{n,m}}t) + B_{n,m}\sin(\sqrt{\lambda_{n,m}}t)$$

$$\therefore u(t,\theta,\phi) = \sum_{n,m} \left[A_{n,m}\cos(\sqrt{\lambda_{n,m}}t) + B_{n,m}\sin(\sqrt{\lambda_{n,m}}t)\right] P_{n,m}(\cos\theta)\sin(m\phi)$$

Now $u(0,\theta,\phi) = 0 = \sum_{n,m} A_{n,m} P_{n,m}(\cos(\theta))\sin(m\phi) \Rightarrow A_{n,m} = 0$

$\therefore$ Then $\partial_t u = f(\theta,\phi) = \sum_{n,m} \sqrt{\lambda_n}\, B_{n,m} P_{n,m}(\cos\theta)\sin(m\phi)$

$$\Rightarrow B_{n,m} = \frac{1}{\|P_{n,m}\|^2} \langle P_{n,m}, f(\theta,\phi)\rangle$$

$$\therefore u(t,\theta,\phi) = \sum_{n,m} B_{n,m}\sin(\sqrt{\lambda_{n,m}}t)\, P_{n,m}(\cos\theta)\sin(m\phi)$$

Let $f(\theta, \phi) = 0.1 \cos\theta$

Then $0.1 \cos\theta$ implies $m = 0$ & $n = 1$

$$\Rightarrow \frac{1}{10}\cos\theta = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, B_n P_{n,0}(\cos\theta)$$

$$\Rightarrow B_n = \frac{1}{\|P_{n,0}\|^2} \int_{-1}^{1} \frac{1}{10}\, x P_{n,0}(x)\, dx$$

7) $\begin{cases} \partial_t u = \Delta u - \cos\theta \\ u(t,1,\theta,\phi) = 5/4 \cos\theta \\ u(0,r,\theta,\phi) = \frac{r^2}{4}\cos\theta \end{cases}$

(a) (b)

let $u = v(r,\theta,\phi) + w(t,r,\theta,\phi)$

now lets solve for $v \Rightarrow \partial_t w = (\Delta v + \Delta w) - \cos\theta$

(a)

Then $\begin{cases} \Delta v = \cos\theta \\ v(1,\theta,\phi) = 0 \end{cases}$ (c) and $\begin{cases} \Delta v = 0 \\ v(1,\theta,\phi) = \frac{5}{4}\cos\theta \end{cases}$ (d)

now (c) is solved in class by the cauchy euler eqn

let $V = V(r)\cos\theta$

$\Rightarrow r^2 V'' + 2r V' - 2V = r^2$

and $\Rightarrow V = \left(Ar + \frac{1}{4} r^2\right)$

then $v(1) = 0 \Rightarrow A + \frac{1}{4} = 0 \Rightarrow A = -\frac{1}{4}$

$\therefore v = \left(-\frac{1}{4} r + \frac{1}{4} r^2\right)\cos\theta$

now (d) is solved by the general soln

$v(r,\theta,\phi) = \sum_n V_n r^n P_n(\cos\theta)$

now clearly $n = 1 \Rightarrow V_1 = \frac{5}{4}$

$\therefore v(r,\theta,\phi) = \frac{5}{4} r\cos\theta$

$\therefore$ now $u(0,r,\theta,\phi) = v(r,\theta,\phi) + w(0,r,\theta,\phi)$

$$\Rightarrow \cancel{\frac{r^2}{4}\cos\theta} = \frac{5r\cos\theta}{4} + \left(-\frac{1}{4}r\cos\theta\right) + \cancel{\frac{r^2}{4}\cos\theta} + w(0,r,\theta,\phi)$$

$$\Rightarrow w(0,r,\theta,\phi) = -r\cos\theta$$

$\therefore$ To solve
$$\begin{cases} \partial_t w = \Delta w \\ w(0,r,\theta,\phi) = -r\cos\theta \\ w(t,1,\theta,\phi) = 0 \end{cases}$$

Let the solution be $w = \sum_{n=0}^{\infty} W(t) r^n P_n(\cos\theta)$

Then $\partial_t w = \sum_n W_n' P_n(\cos\theta) r^n = \sum_n -\lambda_n W_n P_n(\cos\theta) r^n$

$$\therefore W' = -\lambda W \Rightarrow W_n(t) = C_n e^{-\lambda_n t}$$

$$\therefore w(0,r,\theta,\phi) = -r\cos\theta = \sum_{n=0}^{\infty} C_n P_n(\cos\theta) r^n$$

$$\Rightarrow n = 1 \text{ and } C_1 = (-1) \text{ and } \lambda_n = n(n+1)$$

$$\therefore w(t,r,\theta,\phi) = \sum_{n=0}^{\infty} -e^{-\lambda_n t} r^n P_n(\cos\theta)$$

$\therefore$ transient soln $\Rightarrow w = -1 + -e^{-2t} r\cos\theta + -e^{-6t} r^2 \frac{[3\cos^2\theta - 1]}{2}$

$$\therefore u = v + w = \sum_{n=0}^{\infty} e^{-\lambda_n t} r^n P_n(\cos\theta) + \frac{5}{4} r\cos\theta - \frac{1}{4} r\cos\theta$$

$$+ \frac{r^2}{4}\cos\theta$$

8) $$\begin{cases} \partial_{tt} u + 2\varepsilon \partial_t u = c^2 \Delta u \\ u(t, l, \theta, \phi) = 0 \\ u(0, r, \theta, \phi) = 0 \\ \partial_t u(0, r, \theta, \phi) = \phi \end{cases} \qquad \varepsilon > 0$$

Since the boundary is $\emptyset$ then the soln is

$$U(t, r, \theta, \phi) = \sum_{n,p,m=1}^{\infty}{}' \, U_{n,p,m}(t) \, j_{np}(r) \underbrace{P_{n,m}(\cos\theta)\sin(n\phi)}_{Y_{nm}}$$

Plugging this into the equation. Let $t = cT$ (scaling) and hence independent $\therefore$ can solve without c^2.

$$\Rightarrow \sum \ddot{U}_{npm}(t) \, j_{np}(r) \, Y_{n,m} + \sum 2\varepsilon \dot{U}_{npm}(t) \, j_{np}(r) \, Y_{nm}$$
$$= \sum -\cancel{c^2} \lambda_{npm} U_{npm}(t) \, j_{np}(r) \, Y_{n,m}$$

$$\Rightarrow U''_{n,p,m}(t) + 2\varepsilon U'_{n,p,m}(t) + \cancel{c^2} \lambda_{npm} U_{npm} = 0$$
$\Rightarrow s^2 + 2\varepsilon s + \lambda = 0$ (characteristic eqn)
$\to s = -\varepsilon \pm \sqrt{\varepsilon^2 - \lambda}$ and since $\lambda > 0$ & $\varepsilon > 0$ sufficiently small & hence $\lambda > \varepsilon^2$. Let $\omega = \sqrt{\varepsilon^2 - \lambda}$
$$\Rightarrow U_{npm} = A_{npm} e^{-\varepsilon t} \cos(\omega t) + e^{-\varepsilon t} \sin(\omega t) B_{npm}$$

and since initial conditions are 0 $\Rightarrow$ only $\sin(\omega t)$ term

$$\Rightarrow u(t, r, \theta, \phi) = \sum_{npm} B_{npm} e^{-\varepsilon t c^2} \sin(\omega t) \, j_{np}(r) \, Y_{nm}$$

now applying boundary conditions

$$\partial_t u(0, r, \theta, \phi) = \phi = \sum_{n,p,m=1}^{\infty} B_{npm}\, w\, j_{np}(r)\, Y_{nm}$$

$$\Rightarrow B_{npm} = \frac{1}{w\, \|j_{np}\|^2\, \|P_{n,m}\|^2\, \pi} \int_{-\pi}^{\pi} \int_{0}^{\pi} \int_{0}^{1} (\phi\, j_{np}(r)\, Y_{nm}\, r^2 \sin\theta)\, dr\, d\theta\, d\phi$$

$$\text{now } u(0, r, \theta, \phi) = 0 = \sum_{npm=1}^{\infty} B_{npm} \sin(w0) = 0$$

where $B_{n,pm}$.

$\therefore$ satisfied

$$\therefore \text{soln is } u(t, r, \theta, \phi) = \sum_{n,p,m=1}^{\infty} B_{npm}\, e^{-\varepsilon c^2 t}\, j_{np}(r)\, Y_{n,m}$$