MATH 227 April 8 ①

Last time we discussed the notion of a change of basis matrix.

What is its main use?

It allows us to consider other matrix representations of a linear map $g: \mathbb{F}^n \to \mathbb{F}^n$ that may be "nicer" and easier to work with than the standard matrix representation.

What does "nicer" mean here?

Suppose the standard matrix representation of a linear map $g: \mathbb{F}^3 \to \mathbb{F}^3$ is the matrix

$$A_g = \begin{pmatrix} | & | & | \\ g(\bar{e}_1) & g(\bar{e}_2) & g(\bar{e}_3) \\ | & | & | \end{pmatrix}$$

which has, say, 3 eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$, and suppose u_1, u_2, u_3 are eigenvectors of A_g corresponding to $\lambda_1, \lambda_2, \lambda_3$ respectively, and that $\bar{u}_1, \bar{u}_2, \bar{u}_3$ are linearly independent.

Then the set $\mathcal{C} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is a basis of $\mathbb{F}^3$. Write also SB for the standard basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ of $\mathbb{F}^3$.

Question 1 What is the change of basis matrix $P_{\mathcal{C} \to SB}$?

We should have $[\bar{u}_1]_e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longmapsto [\bar{u}_1]_{SB}$,
and similarly $[\bar{u}_2]_e \longmapsto [\bar{u}_2]_{SB}$, $[\bar{u}_3]_e \longmapsto [\bar{u}_3]_{SB}$, thus

$$P_{e \to SB} = \begin{pmatrix} | & | & | \\ [\bar{u}_1]_{SB} & [\bar{u}_2]_{SB} & [\bar{u}_3]_{SB} \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \\ | & | & | \end{pmatrix}$$

(note that, for every $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in F^3$, $[\bar{x}]_{SB} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \bar{x}$)

<u>Question 2</u> What are the entries of the matrix

$$B_g = (P_{e \to SB})^{-1} A_g P_{e \to SB} = P_{SB \to e} A_g P_{e \to SB} ?$$

Let us try to see how the matrix B_g acts on the standard basis vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$B_g \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (P_{SB \to e} A_g P_{e \to SB}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= (P_{SB \to e} A_g) \left(P_{e \to SB} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = (P_{SB \to e} A_g) \bar{u}_1$$

$$= P_{SB \to e} (A_g \bar{u}_1) \underset{\bar{u}_1 \text{ eigenvector of } A_g}{=} P_{SB \to e} (\lambda_1 \bar{u}_1)$$

$$= \lambda_1 (P_{SB \to e} [\bar{u}_1]_{SB}) = \lambda_1 [\bar{u}_1]_e = \lambda_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\rightsquigarrow Col_1(B_g) = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix}.$$

Similarly we can see that

$$B_g\begin{pmatrix}0\\1\\0\end{pmatrix} = A_g\left(P_{SB\to e}[\bar{u}_2]_{SB}\right) = \lambda_2[\bar{u}_2]_e = \lambda_2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\\lambda_2\\0\end{pmatrix}$$

$$\rightsquigarrow \; Col_2(B_g) = \begin{pmatrix}0\\\lambda_2\\0\end{pmatrix}$$

and $B_g\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\lambda_3\end{pmatrix} \rightsquigarrow Col_3(B_g) = \begin{pmatrix}0\\0\\\lambda_3\end{pmatrix}$.

Putting everything together, we see that

$$B_g = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}$$ is a diagonal matrix

and its diagonal entries are the eigenvalues of A_g.

<u>Question 3</u> How is the matrix B_g related to the matrix A_g?

We see that there exists an invertible matrix E such that $B_g = E^{-1}A_gE$. Indeed, $E = P_{e\to SB}$ here.
Recall that we then say A_g and B_g are <u>similar</u>.
<u>Question 4</u> How is the matrix B_g related to the linear map $g: \mathbb{F}^3 \to \mathbb{F}^3$?

For every $\bar{x} \in \mathbb{F}^3$ we have

$$[\bar{x}]_e \longmapsto B_g[\bar{x}]_e = \left(P_{SB\to e}A_gP_{e\to SB}\right)[\bar{x}]_e$$
$$= \left(P_{SB\to e}A_g\right)\bar{x} = P_{SB\to e}(A_g\bar{x}) = P_{SB\to e}\left(g(\bar{x})\right) = [g(\bar{x})]_e$$

Thus B_g is another matrix representation of the linear map g as long as we now identify vectors in $\mathbb{F}^3$ with their coefficient column vectors with respect to the basis $\mathcal{E} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$.

<u>Terminology</u> Let $\mathbb{F}$ be a field, and let A be a matrix in $\mathbb{F}^{n\times n}$.

- We say that A is <u>diagonalisable</u> if there exists a diagonal matrix D in $\mathbb{F}^{n\times n}$ such that A and D are similar (that is, so that $A = \Gamma^{-1} D \Gamma$ for some invertible matrix Γ in $\mathbb{F}^{n\times n}$).

- We say that A is <u>(upper) triangularisable</u> if there exists an upper triangular matrix $\mathcal{U}$ in $\mathbb{F}^{n\times n}$ such that A and $\mathcal{U}$ are similar (that is, so that $A = E^{-1}\mathcal{U}E$ for some invertible matrix E in $\mathbb{F}^{n\times n}$).

<u>Natural Question Now</u> Under what conditions is a matrix $A \in \mathbb{F}^{n\times n}$ diagonalisable or upper triangularisable?

By adapting in the most logical way the above discussion we had for $\mathbb{F}^3$ to the general case of $\mathbb{F}^n$, we see that A would be diagonalisable if we could find a basis $\mathcal{E}$ of $\mathbb{F}^n$ consisting <u>only</u> of eigenvectors of A.

In fact the converse is also true:

<u>Theorem 1</u> A matrix $A \in \mathbb{F}^{n\times n}$ is diagonalisable if and only if there exists a basis $\mathcal{C} = \{\bar{u}_1, \dots \bar{u}_n\}$ of $\mathbb{F}^n$ consisting of eigenvectors of A.

In fact, if the latter is true, we can also check that the matrix

$$D := (P_{\mathcal{C}\to SB})^{-1} A P_{\mathcal{C}\to SB}$$

is diagonal.

<u>2nd Natural Question</u>: When can we find a basis of $\mathbb{F}^n$ consisting only of eigenvectors of a given matrix A?

Recall the following Homework Problem from HW3:

<u>Pb6</u> Let $\lambda_1, \lambda_2, \dots \lambda_k$ be distinct eigenvalues of $A \in \mathbb{F}^{n\times n}$ and let $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k \in \mathbb{F}^n$ be eigenvectors corresponding to them. Then the set

$$\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k\}$$

is linearly independent.

<u>**Remark**</u> following from this problem:

If A has <u>n</u> distinct eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n \in \mathbb{F}$, then any eigenvectors $\bar{u}_1, \bar{u}_2, \dots \bar{u}_n$ corresponding to them will form a linearly independent subset of $\mathbb{F}^n$ of size n, thus a basis of $\mathbb{F}^n$.

→ <u>Corollary 1</u> If a matrix $A \in \mathbb{F}^{n\times n}$ has n distinct eigenvalues, then A is diagonalisable.

Another very important condition in the case of inner product spaces:

Definitions I) A matrix $A \in \mathbb{R}^{n\times n}$ is called symmetric if $A^T = A$.

II) A matrix $B \in \mathbb{C}^{n\times n}$ is called Hermitian if $B^* = B$.

Alternative Terminology A symmetric or Hermitian matrix is sometimes also called self-adjoint.

III) A matrix $O \in \mathbb{R}^{n\times n}$ whose columns form an orthonormal set (wrt the standard inner product on $\mathbb{R}^n$) is called orthogonal.

In that case, we can prove that we have that O is invertible and $O^{-1} = O^T$.

IV) A matrix $U \in \mathbb{C}^{n\times n}$ whose columns form an orthonormal set (wrt the standard inner product on $\mathbb{C}^n$) is called unitary.

In that case, we can prove that we have that U is invertible and $U^{-1} = U^*$.

We can now state the following theorem.

Spectral Theorem I) (Real case) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then there exists an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^n$ consisting only of eigenvectors of A.

Thus A is diagonalisable, and moreover the change of basis matrix $P_{e\to\mathcal{B}}$ that we need so that

$$D = (P_{e\to SB})^{-1} \cdot A \cdot P_{e\to SB}$$

will be diagonal is orthogonal (since its columns are the elements of the set e).

Thus we have $D = P^{T} A P$ is diagonal.

II) (Complex case) Let $B \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Then there exists an orthonormal basis $\mathcal{E}$ of $\mathbb{C}^n$ consisting only of eigenvectors of **B**.

Thus B is diagonalisable, and moreover the change of basis matrix $Q_{\mathcal{E}\to SB}$ that we need so that

$$\tilde{D} = (Q_{\mathcal{E}\to SB})^{-1} \cdot B \cdot Q_{\mathcal{E}\to SB}$$

will be diagonal is unitary.

Thus we have $\tilde{D} = Q^{*} B Q$ is diagonal.