Math 227
Suggested solutions to 2nd Midterm

Problem 1. We have that

$$\begin{aligned}
p_A(t) &= \det\left(\begin{pmatrix} 6-t & 0 & 3 \\ 2 & 2-t & 5 \\ 1 & 0 & 1-t \end{pmatrix}\right) \\
&= (6-t)\cdot\det\left(\begin{pmatrix} 2-t & 5 \\ 0 & 1-t \end{pmatrix}\right) \;+\; 3\cdot\det\left(\begin{pmatrix} 2 & 2-t \\ 1 & 0 \end{pmatrix}\right) \\
&= (6-t)(2-t)(1-t) + 3(-2+t) \\
&= (2-t)\big((6-t)(1-t) - 3\big) \\
&= (2-t)(6+t^2-3) \\
&= (2-t)(t^2+3).
\end{aligned}$$

We can see right away that 2 is a root of $p_A(t)$, and thus an eigenvalue of A. Moreover, 2 is also a root of the polynomial t^2+3 (which divides $p_A(t)$), as we can check by inspection: $2^2+3 = 4+3 = 7 = 0$. But then, we also have that $(-2)^2+3 = 4+3 = 0$, and hence $-2 = 5$ is also a root of $p_A(t)$ and an eigenvalue of A.

We conclude that $t^2+3 = (t-2)(t-5)$, and hence $p_A(t) = -(t-2)^2(t-5)$, showing that the eigenvalues of A are 2 and 5.

We now find the eigenspace of A corresponding to eigenvalue 2. This is the nullspace of $A - 2I_3$, where

$$A - 2I_3 = \begin{pmatrix} 4 & 0 & 3 \\ 2 & 0 & 5 \\ 1 & 0 & -1 \end{pmatrix} \xrightarrow[4R_2 \to R_2']{2R_1 \to R_1'} \begin{pmatrix} 1 & 0 & 6 \\ 1 & 0 & 6 \\ 1 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is a Row Echelon Form of $A - 2I_3$, so it has the same nullspace. Given that the corresponding homogeneous linear system has two free variables, the variables x_2 and x_3, we obtain that the nullspace has dimension 2, and a basis for it is formed by the solutions to the homogeneous system that we get:

- by setting $x_2 = 1$ and $x_3 = 0$, which gives the solution $\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$,
- or by setting $x_2 = 0$ and $x_3 = 1$, which gives the solution $\left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$.

In other words, the eigenspace of A corresponding to eigenvalue 2 is

$$\operatorname{span}\left(\left\{\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)\right\}\right).$$

Next, we find the eigenspace of A corresponding to eigenvalue 5. This is the nullspace of $A - 5I_3$, where

$$A - 5I_3 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -3 & 5 \\ 1 & 0 & -4 \end{pmatrix} \xrightarrow{4R_2 \to R_2'} \begin{pmatrix} 1 & 0 & 3 \\ 1 & 2 & 6 \\ 1 & 0 & 3 \end{pmatrix} \xrightarrow[R_3 - R_1 \to R_3']{R_2 - R_1 \to R_2'} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is a Row Echelon Form of $A - 5I_3$, so it has the same nullspace. Given that the corresponding homogeneous linear system has only one free variable, the variable x_3, we obtain that the nullspace has dimension 1, and is spanned by any eigenvector of A corresponding to eigenvalue 5: e.g. the eigenvector $\left(\begin{smallmatrix}4\\2\\1\end{smallmatrix}\right)$ (equivalently solution to the system) that we get by setting $x_3 = 1$.

In other words, the eigenspace of A corresponding to eigenvalue 5 is

$$\operatorname{span}\left(\left\{\left(\begin{smallmatrix}4\\2\\1\end{smallmatrix}\right)\right\}\right).$$

Problem 2. (i) This statement is false.

For a counterexample, set $U = \mathbb{Z}_5^2$, and consider the following two subspaces of U: $S_1 = \operatorname{span}(\{\bar{e}_1\}) = \operatorname{span}\left(\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}\right)$ and $S_2 = \operatorname{span}(\{\bar{e}_2\}) = \operatorname{span}\left(\{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}\right)$.

Then $S_1 \cup S_2 = \{\bar{x} = \left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right) \in \mathbb{Z}_5^2 : \text{at most one of } x_1 \text{ or } x_2 \text{ is non-zero}\}$, which is not a subspace of U since it is not closed under addition (e.g. $\bar{e}_1$ and $\bar{e}_2$ are both contained in $S_1 \cup S_2$, but $\bar{e}_1 + \bar{e}_2$ is not).

(ii) This statement is true. Let S_1, S_2 be two subspaces of V. Then $S_1 \cap S_2$ is non-empty, since $\bar{0}_V \in S_1 \cap S_2$.

Moreover, $S_1 \cap S_2$ is closed under addition. Indeed, let $\bar{u}, \bar{w} \in S_1 \cap S_2$. Then $\bar{u}, \bar{w} \in S_1$, and since S_1 is a subspace of V, we must have $\bar{u} + \bar{w} \in S_1$ as well. Similarly, we can see that $\bar{u} + \bar{w} \in S_2$. But then $\bar{u} + \bar{w} \in S_1 \cap S_2$, as claimed.

Finally, $S_1 \cap S_2$ is closed under scalar multiplication too. Indeed, let $\bar{u} \in S_1 \cap S_2$ and $r \in \mathbb{R}$. Then we have that $\bar{u} \in S_1$, and since S_1 is a subspace of V, we get that $r\bar{u} \in S_1$ too. Similarly, we see that $r\bar{u} \in S_2$. Therefore, $r\bar{u} \in S_1 \cap S_2$, as claimed.

Combining the above, we can conclude that $S_1 \cap S_2$ is a subspace of V.

(iii) This statement is true. First of all, considering any finite subset Γ_2 of A which has size, say, 2 (write for example $\Gamma_2 = \{\bar{u}_1, \bar{u}_2\}$), we have that $\bar{u}_1, \bar{u}_2 \in \operatorname{span}(\Gamma_2) \subset \bigcup_{\substack{\Gamma \subset A,\ \Gamma \text{ finite} \\ |\Gamma| \text{ even}}} \operatorname{span}(\Gamma)$, so $\bigcup_{\substack{\Gamma \subset A,\ \Gamma \text{ finite} \\ |\Gamma| \text{ even}}} \operatorname{span}(\Gamma)$ is non-empty.

Next, consider $\bar{v}, \bar{w} \in \bigcup_{\substack{\Gamma \subset A,\ \Gamma \text{ finite} \\ |\Gamma| \text{ even}}} \operatorname{span}(\Gamma)$. Then there exist subsets $\Gamma_{\bar{v}}$ and $\Gamma_{\bar{w}}$ of A which are finite and have even size such that

$$\bar{v} \in \operatorname{span}(\Gamma_{\bar{v}}) \qquad \text{and} \qquad \bar{w} \in \operatorname{span}(\Gamma_{\bar{w}}).$$

Consider the subset $\Gamma_{\bar{v}} \cup \Gamma_{\bar{w}}$ of A. This is definitely finite (in fact, its size is $\leqslant |\Gamma_{\bar{v}}| + |\Gamma_{\bar{w}}|$), and there are two possibilities for its size:

- the size of $\Gamma_{\bar{v}} \cup \Gamma_{\bar{w}}$ is even, in which case we leave the set as is;

- the size of $\Gamma_{\bar{v}} \cup \Gamma_{\bar{w}}$ is odd; in this latter case we make use of the fact that A is an infinite set, so it definitely contains a vector $\bar{x} \in A \setminus \left(\Gamma_{\bar{v}} \cup \Gamma_{\bar{w}}\right)$; we then consider the set $\Gamma_{\bar{v}} \cup \Gamma_{\bar{w}} \cup \{\bar{x}\}$, which is a finite subset of A with even size.

We conclude that, in any case, we can find a finite subset $\widetilde{\Gamma}$ of A which has even size and which satisfies the inclusion $\Gamma_{\bar{v}} \cup \Gamma_{\bar{w}} \subseteq \widetilde{\Gamma}$. But then

$$\operatorname{span}(\Gamma_{\bar{v}}) \subseteq \operatorname{span}(\widetilde{\Gamma}) \qquad \text{and} \qquad \operatorname{span}(\Gamma_{\bar{v}}) \subseteq \operatorname{span}(\widetilde{\Gamma}),$$

therefore both $\bar{v}$ and $\bar{w}$ belong to $\operatorname{span}(\widetilde{\Gamma})$. Since $\operatorname{span}(\widetilde{\Gamma})$ is a subspace of W, we get that

$$\bar{v} + \bar{w} \in \operatorname{span}(\widetilde{\Gamma}) \subset \bigcup_{\substack{\Gamma \subset A,\ \Gamma \text{ finite} \\ |\Gamma| \text{ even}}} \operatorname{span}(\Gamma).$$

Since $\bar{v}, \bar{w} \in \bigcup_{\substack{\Gamma \subset A,\ \Gamma \text{ finite} \\ |\Gamma| \text{ even}}} \operatorname{span}(\Gamma)$ were arbitrary, we can conclude that $\bigcup_{\substack{\Gamma \subset A,\ \Gamma \text{ finite} \\ |\Gamma| \text{ even}}} \operatorname{span}(\Gamma)$ is closed under addition.

Similarly we check that $\bigcup_{\substack{\Gamma \subset A,\ \Gamma \text{ finite} \\ |\Gamma| \text{ even}}} \operatorname{span}(\Gamma)$ is closed under scalar multiplication. Consider $\bar{v} \in \bigcup_{\substack{\Gamma \subset A,\ \Gamma \text{ finite} \\ |\Gamma| \text{ even}}} \operatorname{span}(\Gamma)$ and $r \in \mathbb{C}$. Then there exists a subset $\Gamma_{\bar{v}}$ of A which is finite and has even size such that $\bar{v} \in \operatorname{span}(\Gamma_{\bar{v}})$. But then, since $\operatorname{span}(\Gamma_{\bar{v}})$ is a subspace of W, we get that

$$r\bar{v} \in \operatorname{span}(\Gamma_{\bar{v}}) \subset \bigcup_{\substack{\Gamma \subset A,\ \Gamma \text{ finite} \\ |\Gamma| \text{ even}}} \operatorname{span}(\Gamma).$$

Combining all the above, we get that $\bigcup_{\substack{\Gamma \subset A,\ \Gamma \text{ finite} \\ |\Gamma| \text{ even}}} \operatorname{span}(\Gamma)$ is a subspace of W.

Problem 3. (a) We have that

$$V = \{p \in \mathcal{P}_6 : p(5) = 0\} = \{p \in \mathcal{P}_6 : x - 5 \text{ divides } p(x)\}$$
$$= \{p \in \mathcal{P}_6 : \exists\, q \in \mathcal{P}_5 \text{ such that } p(x) = (x-5)q(x)\}.$$

We have then seen that a basis for V is the set

$$\mathcal{B} = \{x - 5,\ (x-5)x,\ (x-5)x^2,\ (x-5)x^3,\ (x-5)x^4,\ (x-5)x^5\}$$

(indeed, any polynomial of the form $(x-5)q(x)$ with $q \in \mathcal{P}_5$ can be written as a linear combination of the vectors in this set; moreover, if we have scalars $a_0, a_1, \ldots, a_5 \in \mathbb{R}$ such that

$$a_0(x-5)+a_1(x-5)x+a_2(x-5)x^2+a_3(x-5)x^3+a_4(x-5)x^4+a_5(x-5)x^5 = \mathbf{0},$$

then we can rewrite this as

$$a_5x^6+(a_4-5a_5)x^5+(a_3-5a_4)x^4+(a_2-5a_3)x^3+(a_1-5a_2)x^2+(a_0-5a_1)x-5a_0 = \mathbf{0},$$

and then conclude that $a_5 = a_0 = 0$, which also implies that $a_4 = 0$, $a_3 = 0$, $a_2 = 0$ and finally $a_1 = 0$; this shows that the set is linearly independent too).

It follows that any subset T of V that contains $\mathcal{B}$ will be a spanning set of V, and also, as long as T contains elements outside $\mathcal{B}$ as well, T won't be linearly independent, and hence it won't be a basis. Therefore, an example with the desired properties here is the set

$$T = \{x-5,\ (x-5)x,\ (x-5)x^2,\ (x-5)x^3,\ (x-5)x^4,\ (x-5)x^5,\ (x-5)(3+2x+x^2)\}.$$

(b) The answer here is affirmative: we necessarily have $f = g$.

To justify this, consider $\bar{x} \in V_1$. We need to show that we must have $f(\bar{x}) = g(\bar{x})$.

Since T is a spanning set of V_1, we can find vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m \in T$ (for some $m \geqslant 1$) and scalars $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{F}$ such that

$$\bar{x} = \lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \cdots + \lambda_m\bar{u}_m.$$

By the assumption that f and g are extensions of the function $\phi : T \to V_2$, we get that, for every $1 \leqslant i \leqslant m$,

$$f(\bar{u}_i) = \phi(\bar{u}_i) = g(\bar{u}_i).$$

Also based on the assumption that f and g are linear, we can write

$$\begin{aligned} f(\bar{x}) &= f\big(\lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \cdots + \lambda_m\bar{u}_m\big) \\ &= f(\lambda_1\bar{u}_1) + f(\lambda_2\bar{u}_2) + \cdots + f(\lambda_m\bar{u}_m) \\ &= \lambda_1 f(\bar{u}_1) + \lambda_2 f(\bar{u}_2) + \cdots + \lambda_m f(\bar{u}_m) \\ &= \lambda_1 g(\bar{u}_1) + \lambda_2 g(\bar{u}_2) + \cdots + \lambda_m g(\bar{u}_m) \\ &= g(\lambda_1\bar{u}_1) + g(\lambda_2\bar{u}_2) + \cdots + g(\lambda_m\bar{u}_m) \\ &= g\big(\lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \cdots + \lambda_m\bar{u}_m\big) = g(\bar{x}). \end{aligned}$$

Since $\bar{x} \in V_1$ was arbitrary, we can conclude that $f(\bar{y}) = g(\bar{y})$ for every $\bar{y} \in V_1$, or in other words that the two functions coincide.

Problem 4. (a) We need to show that P is non-empty and closed under vector addition and scalar multiplication.

Since S is a subspace of V, it contains $\bar{0}_V$, therefore $\bar{0}_W = f(\bar{0}_V) \in P$.

Consider $\bar{x}_1, \bar{x}_2 \in P$. Then there are $\bar{u}_1, \bar{u}_2 \in S$ such that $\bar{x}_i = f(\bar{u}_i)$ for $i = 1, 2$. Since S is a subspace, we have that $\bar{u}_1 + \bar{u}_2 \in S$, and thus, given also that f is linear,

$$\bar{x}_1 + \bar{x}_2 = f(\bar{u}_1) + f(\bar{u}_2) = f(\bar{u}_1 + \bar{u}_2) \in P = \{f(\bar{u}) : \bar{u} \in S\}.$$

Similarly, if $\bar{x}_1$ is as above, $\bar{x}_1 = f(\bar{u}_1)$ with $\bar{u}_1 \in S$, and $r \in \mathbb{F}$, then we have that $r\bar{u}_1 \in S$, and hence

$$r\bar{x}_1 = rf(\bar{u}_1) = f(r\bar{u}_1) \in P.$$

Combining the above, we get that P is a subspace of W.

(b) Since S is a subspace of V and P is a subspace of W (by part (a)), we have that both of them are vector spaces over $\mathbb{F}$. Also, for any $\bar{u} \in S$, $f(\bar{u}) \in P$ by the definition of P, and therefore

$$f_{|S} : S \to P$$

is a well-defined function.

Moreover, $\mathrm{Range}\big(f_{|S}\big) = \{f_{|S}(\bar{u}) : \bar{u} \in S\} = \{f(\bar{u}) : \bar{u} \in S\} = P$, therefore $f_{|S}$ is surjective.

Finally, for any $\bar{u}_1, \bar{u}_2 \in S$ and $r \in \mathbb{F}$, we have that

$$f_{|S}(r\bar{u}_1 + \bar{u}_2) = f(r\bar{u}_1 + \bar{u}_2) = rf(\bar{u}_1) + f(\bar{u}_2) = rf_{|S}(\bar{u}_1) + f_{|S}(\bar{u}_2),$$

therefore $f_{|S} : S \to P$ is a linear map.

We can now apply Main Theorem E to conclude that

$$\dim_{\mathbb{F}} S = \dim_{\mathbb{F}} \mathrm{Ker}\big(f_{|S}\big) + \dim_{\mathbb{F}} \mathrm{Range}\big(f_{|S}\big) = \dim_{\mathbb{F}} \mathrm{Ker}\big(f_{|S}\big) + \dim_{\mathbb{F}} P,$$

which shows that $\dim_{\mathbb{F}} P \leqslant \dim_{\mathbb{F}} S$.

(c) By the conclusion of Main Theorem E, which we wrote just above, we note that, if we have $\dim_{\mathbb{F}} \mathrm{Ker}\big(f_{|S}\big) = 0$, we do get $\dim_{\mathbb{F}} P = \dim_{\mathbb{F}} S$.

But $\dim_{\mathbb{F}} \mathrm{Ker}\big(f_{|S}\big) = 0$ is equivalent to having $\mathrm{Ker}\big(f_{|S}\big) = \{\bar{0}_S\} = \{\bar{0}_V\}$. We note that

$$\begin{aligned}\mathrm{Ker}\big(f_{|S}\big) = \big\{\bar{u} \in S : f_{|S}(\bar{u}) = \bar{0}_W\big\} &= \{\bar{u} \in S : f(\bar{u}) = \bar{0}_W\} \\ &= \{\bar{u} \in S : \bar{u} \in \mathrm{Ker}(f)\} = \mathrm{Ker}(f) \cap S.\end{aligned}$$

Therefore, $\mathrm{Ker}\big(f_{|S}\big) = \{\bar{0}_V\}$ is equivalent to $\mathrm{Ker}(f) \cap S = \{\bar{0}_V\}$, showing that condition (iii) is strong enough to imply $\dim_{\mathbb{F}} P = \dim_{\mathbb{F}} S$, as we want.

Moreover, the other two conditions would not be strong enough. In fact, condition (ii) is always wrong, since both $\mathrm{Ker}(f)$ and S are subspaces of V, therefore their intersection would definitely contain the zero vector $\bar{0}_V$.

Finally, to see that condition (i) is not sufficient for us to conclude that $\dim_{\mathbb{F}} P = \dim_{\mathbb{F}} S$, consider the following example: let $V = \mathbb{R}^4$ and $W = \mathbb{R}^3$ (viewed as vector spaces over $\mathbb{R}$), and set

$$f : \mathbb{R}^4 \to \mathbb{R}^3, \qquad f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ -x_1 \\ x_4 \end{pmatrix}.$$

We can then check that

$$\mathrm{Ker}(f) = \mathrm{span}\big(\{\bar{e}_2, \bar{e}_3\}\big).$$

Consider also the subspace $S = \mathrm{span}\big(\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}\big)$ of $\mathbb{R}^4$. Then clearly $\mathrm{Ker}(f) \leqslant S$.

Moreover, in this setting,

$$\begin{aligned}P = \{f(\bar{u}) : \bar{u} \in S\} &= \left\{\begin{pmatrix} u_1 \\ -u_1 \\ u_4 \end{pmatrix} : \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \in S\right\} \\ &= \left\{\begin{pmatrix} u_1 \\ -u_1 \\ u_4 \end{pmatrix} : u_4 = 0\right\} = \mathrm{span}\left(\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}\right).\end{aligned}$$

Therefore, $\dim_{\mathbb{F}} P = 1 \neq 3 = \dim_{\mathbb{F}} S$.