Praxis Midterm Exam

10/02/2020

1) Let G be a group and $\emptyset \neq M \subseteq G$ a finite subset, such that we have $M \cdot M = M$, where
$$M \cdot M := \left\{ m_1 \cdot m_2 \,\middle|\, m_1, m_2 \in M \right\}.$$
Show that M is a subgroup. Is this also true if M is not a finite subset?

2) Let F be a field and $n \geq 1$ an integer. Determine the centre of $\mathrm{GL}_n(F)$

3) Is $\mathbb{Q}$ a finitely generated group?

4) Let G be a finite group of order $r \geq 1$. Show that there are precisely r different homomorphisms of groups $\mathbb{Z} \longrightarrow G$, but only one homomorphism of groups $G \longrightarrow \mathbb{Z}$.

5) Let G be a non abelian group of order 8 with at least two elements of order 2. Show that there is a non trivial homomorphism $\alpha : \mathbb{Z}/2 \longrightarrow \mathrm{Aut}(\mathbb{Z}/4)$, such that G is isomorphic to $\mathbb{Z}/4 \rtimes_\alpha \mathbb{Z}/2$.

6) Let $p_1, \ldots, p_l$ be l different prime numbers and G an abelian group of order $\prod_{i=1}^{l} p_i$. Show that G is cyclic.

7) Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$, and denote by $< -, - >$ the usual scalar product on $\mathbb{R}^n$. Prove:
   (i) For $0 \neq v \in \mathbb{R}^n$ the map
   $$s_v : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ x \longmapsto x - \frac{2 < v, x >}{< v, v >} \cdot v$$
   is in $\mathrm{GL}_n(\mathbb{R})$.
   (ii) The subgroup H of $\mathrm{GL}_n(\mathbb{R})$ generated by all $s_{e_i - e_j}$, $1 \leq i \neq j \leq n$, is finite.

8) Let G be a finite group, and $S = \{ g_1, \ldots, g_l \} \subseteq G$ a subset which generates G, and which has the property that no proper subset of S generates G (*i.e.* S is a minimal set of generators for G). Show that G has at least 2^l elements.

All answers have to be justified.