Math 322, A1
Midterm Exam – October 22, 2020

General instructions.

- This exam has 4 problems, which are worth a total of 85 points. To earn maximum credit, you need to accumulate 65 points or more.
- All submitted answers must be **handwritten on paper** (any answer or part of an answer that fails this will receive 0 points with no exception).
- You may refer to your course notes, and any other files on the eClass page of the course.
- **No other internet resources are allowed.**
- **No collaboration is allowed.**
- You must show your work and justify your answers to receive full credit. A correct answer without any justification will receive little or no credit.

  In your justifications, you may simply refer to, and rely on, any results/properties that we discussed in class or that appear in the notes or in the first 2 homework assignments (and the files with suggested solutions to them), **except of course if a problem specifically asks you to explain why such a result holds**.
- The exam formally starts at 9:30am and finishes at 10:40am.

  You have until 11am to make sure your answers are submitted correctly to Assign2. **The latter is a strict deadline.**

Problem 1 (*max. 20 = 10 + 10 points*) (a) Consider the following incidence matrix of a graph G:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Find (in any way you want) the degree sequence of the complement $\overline{G}$ of G *(given according to a labelling/ordering of its vertices of your choice)*. Justify your answer.

(b) Consider the following adjacency matrix of a graph H:

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

Find (in any way you want) the degree sequence of the line graph $L(H)$ of H *(given according to a labelling/ordering of its vertices of your choice)*. Justify your answer.

Problem 2 (*max. 20 = 10 + 10 points*) (a) Consider the following 3 graphs:

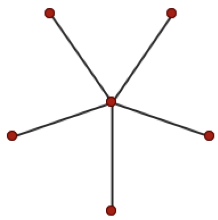

Figure 1: Graph G_1

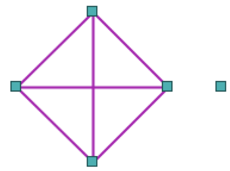

Figure 2: Graph G_2

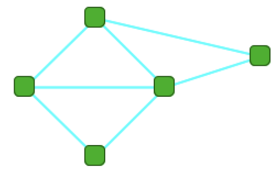

Figure 3: Graph G_3

Exactly two of them are line graphs of some other graphs, while the remaining one is not the line graph of any graph.

Determine which two are line graphs *(you can either pick the two correct ones and explain why there are graphs H_1 and H_2 such that the two graphs you picked are $L(H_1)$ and $L(H_2)$ respectively, or alternatively you can try to find which one graph cannot be written as a line graph and justify this).*

(b) Decide whether the graphs G_4 and G_5 below are isomorphic. If they are, give an explicit isomorphism. If they are not, explain why not.

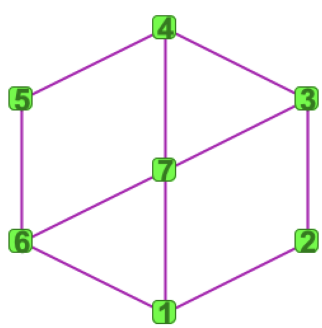

Figure 4: Graph G_4

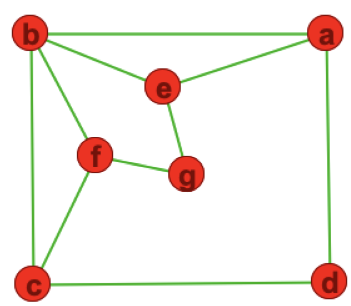

Figure 5: Graph G_5

Problem 3 (*max. 20 = 7 + 7 + 6 points*) Consider the following connected graph:

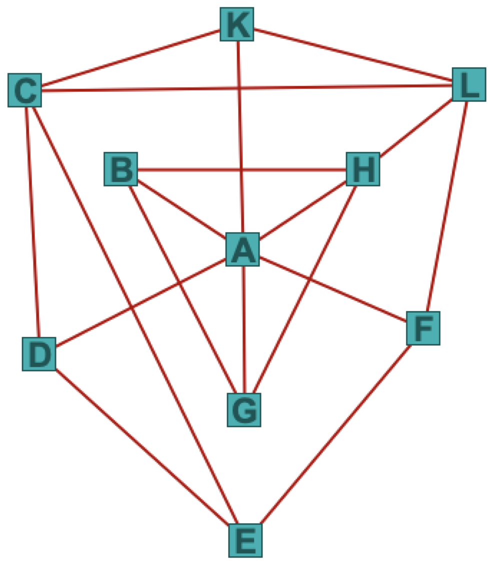

Figure 6: Graph G_0

(a) (*max. 7 points*) Show that $\kappa(G_0) = 2$. Give a full justification.

(b) (*max. 7 points*) What is $\lambda(G_0)$? Determine it precisely, and justify your answer fully.

(c) (*max. 6 points*) Determine $\kappa(C, F)$ precisely, and justify your answer fully.

Problem 4 (*max. 25 = 10 + 15 points*) (a) Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. We define the *average degree* of G as follows:

$$\operatorname{avgdeg}(G) := \frac{1}{|G|} \sum_{i=1}^{n} \deg(v_i) = \frac{1}{n} \sum_{i=1}^{n} \deg(v_i),$$

where $|G|$ is the order of the graph G, that is, the cardinality $|V|$ of the vertex set V.

Suppose now that T is a tree. Express the order of T as a function of the average degree of T.

(b) For each of the following sequences, determine whether it can be viewed as the degree sequence of a **disconnected** graph H.

$$\mathrm{Seq}_1 = (4, 4, 4, 4, 4), \qquad \mathrm{Seq}_2 = (2, 2, 2, 2, 2),$$
$$\mathrm{Seq}_3 = (5, 4, 4, 3, 2, 2, 2, 2, 2, 2)\,.$$

Justify your answers fully.