Math 127, A1
Final Exam – December 11, 2019

Name: ____

This exam has 6 problems with a total worth of 115 points. To earn maximum credit, you need to accumulate 85 points or more.

General instructions (important, read back of the page too).

- Notes, formula sheets, calculators, or electronic aids are **not** allowed.
- All cell phones should be turned off and left in your bags.
- You must show your work and justify your answers to receive full credit. A correct answer without any justification will receive little or no credit.

  In your justifications, you may simply refer to, and rely on, any of the results/properties that we discussed in class or we saw in the homework assignments and the review files, **except of course if a problem specifically asks you to prove such a result**.
- You may not leave the exam room until at least 30 minutes have elapsed.
- If you need extra space for a problem, use the reverse side of the page of the corresponding problem and indicate this clearly. Note that Problem 1 is split across pages 3 and 5 (and pages 4 and 6 can be used as extra space for that problem), while Problem 5 is split across pages 13 and 14.

- The last two pages (pages 17 and 18), as well as the rest of this page, can be freely used as scratch paper.

  Do not write any part of your final answers there

  because these parts will not be considered during grading.

Name and Student ID: ____________________

Problem 1 (*max. 20 points*) The following two pairs of a table of addition and a table of multiplication come from two different structures (the tables for Structure 2 can be found on page 5). You can take for granted that in each of these structures addition and multiplication are associative, and also that they have the left and right distributive properties.

Determine which, if any, of the two structures is a field *(it could be none of them, or only one of them, or both)*, which of them is a commutative ring *(again it could be none of them, or only one, or both)*, and which, if any, is neither of these. Justify your answer.

Structure 1

$+$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_1	a_6	a_8	a_4	a_3	a_7	a_1	a_5	a_2
a_2	a_8	a_6	a_7	a_5	a_4	a_2	a_3	a_1
a_3	a_4	a_7	a_6	a_1	a_8	a_3	a_2	a_5
a_4	a_3	a_5	a_1	a_6	a_2	a_4	a_8	a_7
a_5	a_7	a_4	a_8	a_2	a_6	a_5	a_1	a_3
a_6	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_7	a_5	a_3	a_2	a_8	a_1	a_7	a_6	a_4
a_8	a_2	a_1	a_5	a_7	a_3	a_8	a_4	a_6

$\cdot$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_1	a_2	a_7	a_1	a_8	a_4	a_6	a_5	a_3
a_2	a_7	a_5	a_2	a_3	a_8	a_6	a_4	a_1
a_3	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_4	a_8	a_3	a_4	a_7	a_2	a_6	a_1	a_5
a_5	a_4	a_8	a_5	a_2	a_1	a_6	a_3	a_7
a_6	a_6	a_6	a_6	a_6	a_6	a_6	a_6	a_6
a_7	a_5	a_4	a_7	a_1	a_3	a_6	a_8	a_2
a_8	a_3	a_1	a_8	a_5	a_7	a_6	a_2	a_4

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Name and Student ID: ______________________________

Problem 1 (continued)

Structure 2

$+$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
b_1	b_2	b_3	b_1	b_6	b_4	b_5	b_8	b_9	b_7
b_2	b_3	b_1	b_2	b_5	b_6	b_4	b_9	b_7	b_8
b_3	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
b_4	b_6	b_5	b_4	b_8	b_7	b_9	b_2	b_3	b_1
b_5	b_4	b_6	b_5	b_7	b_9	b_8	b_1	b_2	b_3
b_6	b_5	b_4	b_6	b_9	b_8	b_7	b_3	b_1	b_2
b_7	b_8	b_9	b_7	b_2	b_1	b_3	b_6	b_5	b_4
b_8	b_9	b_7	b_8	b_3	b_2	b_1	b_5	b_4	b_6
b_9	b_7	b_8	b_9	b_1	b_3	b_2	b_4	b_6	b_5

$\cdot$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
b_1	b_1	b_2	b_3	b_2	b_1	b_3	b_3	b_1	b_2
b_2	b_2	b_1	b_3	b_1	b_2	b_3	b_3	b_2	b_1
b_3	b_3	b_3	b_3	b_3	b_3	b_3	b_3	b_3	b_3
b_4	b_2	b_1	b_3	b_5	b_4	b_6	b_7	b_9	b_8
b_5	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
b_6	b_3	b_3	b_3	b_6	b_6	b_6	b_7	b_7	b_7
b_7	b_3	b_3	b_3	b_7	b_7	b_7	b_6	b_6	b_6
b_8	b_1	b_2	b_3	b_9	b_8	b_7	b_6	b_5	b_4
b_9	b_2	b_1	b_3	b_8	b_9	b_7	b_6	b_4	b_5

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Name and Student ID: ______________________________

Problem 2 (a) (*max. 10 points*) For each of the following matrices, determine whether it is invertible or not and justify your answer (you do not need to find its inverse).

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 5 & 6 \\ 0 & 6 & 3 \end{pmatrix} \in \mathbb{Z}_7^{3\times 3}, \qquad B = \begin{pmatrix} \boldsymbol{i}-1 & 2 & -6\boldsymbol{i} & 0 \\ 3+4\boldsymbol{i} & 4 & 1 & 3 \\ 0 & \sqrt{2}\,\boldsymbol{i} & 0 & \sqrt{18} \\ 3 & -6 & 10\boldsymbol{i} & 18\boldsymbol{i} \end{pmatrix} \in \mathbb{C}^{4\times 4}.$$

(b) (*max. 10 points*) Let $\mathbb{F}$ be a field, and $n > 1$ a positive integer. Suppose that U_1, U_2 are upper triangular matrices in $\mathbb{F}^{n\times n}$. Show that $U_1 U_2$ is also upper triangular.

Name and Student ID: ____

Problem 3 (*max. 20 points*) True of False? For each of the following statements, determine whether it is correct or not, and give a full justification for your answer.

(i) Let $\mathbb{F}$ be a field, $n > 1$ a positive integer. A matrix $A \in \mathbb{F}^{n \times n}$ is invertible if and only if A can be written as a product of elementary matrices in $\mathbb{F}^{n \times n}$.

(ii) Let $m \geqslant 3$, and let $LS1$ be a linear system with m equations in m unknowns, and coefficients from $\mathbb{Q}$. $LS1$ has a unique solution if and only if its coefficient matrix has no zero rows.

(iii) Every two upper triangular matrices in $\mathbb{R}^{5 \times 5}$ commute.

Name and Student ID: ______________________________

Problem 4 (*max. 10 points*) Let $\mathbb{F}$ be a field, and let $\{u_1, u_2, \ldots, u_m\}$ be a spanning set of $\mathbb{F}^n$. Prove that, if $A \in \mathbb{F}^{n \times n}$ is invertible, then $\{Au_1, Au_2, \ldots, Au_m\}$ is a spanning set of $\mathbb{F}^n$ too.

Name and Student ID: ____

Problem 5 (*max. 25 points*) (a) (*max. 10 points*) Solve the following linear system with coefficients from $\mathbb{Z}_5$ (if the system has more than one solutions, find all its solutions):

$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 4 \\ 2x_1 & + & 3x_2 & + & 2x_3 & = & 1 \\ x_1 & + & x_2 & + & x_3 & = & 2 \end{array}\right\}.$$

Problem 5 (continued) (b) (*max. 15 points*) Suppose the following matrices are augmented matrices of certain linear systems. For each of these systems, determine the size of its solution set. Justify your answer.

$$A_1 = \begin{pmatrix} 1 & 0 & 2 & 0 & 9 & 7 \\ 0 & -7 & 8 & -4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 5 \\ 0 & 0 & 0 & 0 & -3 & 8 \end{pmatrix} \in \mathbb{Z}_{11}^{4\times 6}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 2 & 4 & 3 \\ 0 & 3 & 0 & -4 & 0 \\ 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{Z}_5^{5\times 5},$$

$$A_3 = \begin{pmatrix} 2 & -3.5 & 17 & 0 & 9 & 1 & 2 \\ 0 & 2 & 0 & -4 & 10 & 21 & 5 \\ 0 & 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & -2 & 1 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 8 & 17 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} \in \mathbb{Q}^{6\times 7}.$$

Name and Student ID: ____________________

Problem 6 (a) (*max. 10 points*) For each of the following two sets, determine whether it is linearly independent, as well as whether it is a spanning set of the space it is a subset of (both spaces should be viewed as vector spaces over $\mathbb{R}$). Justify your answers.

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0.5 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^4,$$

$$S_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 1.5 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\} \subset \mathbb{R}^{2\times 2}.$$

(b) (*max. 10 points*) For each of the given spaces, give a basis containing as many of the vectors of the corresponding subset as possible. Justify your answer.

Name and Student ID: ____________________

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