Math 227 – Recitation of February 13

During this recitation hour we will review the material we have already covered, and which is relevant to the 1st Midterm Exam.

Important Topics/Concepts/Theorems to keep in mind

1. Substructures (i) When is a subset of a field $\mathbb{F}$ a subfield of $\mathbb{F}$? What are sufficient properties we need to verify?

(ii) When is a subset of a ring $\mathcal{R}$ a subring of $\mathcal{R}$? What are sufficient properties we need to verify?

(iii) When is a subset of a vector space V a subspace of V? What are sufficient properties we need to verify?

(iv) Let V be a vector space over some field $\mathbb{F}$, and let T be a subset of V. How do we show that span(T) is a <u>subspace</u> of V?

(v) Given a subspace S of V, how do we show that S can be viewed as the linear span of some subset of V?

(vi) Given a subspace S of V, justify the equality $S = \text{span}(S)$.

2. Structure Preserving Functions (i) What is the definition of a field homomorphism?

(ii) When do we call two fields isomorphic?

(iii) Let $\mathbb{F}_1, \mathbb{F}_2$ and $\mathbb{F}_3$ be fields, and $f : \mathbb{F}_1 \to \mathbb{F}_2$ and $g : \mathbb{F}_2 \to \mathbb{F}_3$ be field homomorphisms. How do we show that $g \circ f : \mathbb{F}_1 \to \mathbb{F}_3$ is a field homomorphism too?

(iv) Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be fields, and let $f : \mathbb{F}_1 \to \mathbb{F}_2$ be a field homomorphism that is also bijective. How do we show that $f^{-1} : \mathbb{F}_2 \to \mathbb{F}_1$ is a field homomorphism too?

(v) Given a field homomorphism $f : \mathbb{F}_1 \to \mathbb{F}_2$, how do we show that Range(f) is a subfield of $\mathbb{F}_2$?

(vi) What is the definition of a ring homomorphism?

(vii) Formulate the analogous questions to parts 2(iii), 2(iv) and 2(v) for ring homomorphisms, and then answer them.

(viii) What is the definition of a linear map?

(ix) When do we call two vector spaces isomorphic? Do they have to be vector spaces over the same field $\mathbb{F}$?

(x) Let $\mathbb{F}$ be a field, and let V_1, V_2 and V_3 be vector spaces over $\mathbb{F}$. Consider linear maps $f : V_1 \to V_2$ and $g : V_2 \to V_3$. How do we show that $g \circ f : V_1 \to V_3$ is a linear map too?

(xi) Let $\mathbb{F}$ be a field, and let V_1, V_2 be vector spaces over $\mathbb{F}$. Suppose that $f : V_1 \to V_2$ is a linear map that is also bijective. How do we show that $f^{-1} : V_2 \to V_1$ is a linear map too?

(xii) Given a linear map $f : V_1 \to V_2$, how do we show that $\mathrm{Range}(f)$ is a subspace of V_2? How do we show that $\mathrm{Ker}(f)$ is a subspace of V_1?

3. Dimensions and Bases of Subspaces (i) What is the <u>**rank**</u> of a matrix?

(ii) What are ways to find a basis for the Row Space of a matrix?

(iii) How do we find a basis for the Column Space of a matrix?

(iv) Is the rank of a matrix always equal to the rank of its transpose?

(v) How do you find a basis for the Nullspace of a matrix?

(vi) Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{m \times n}$. Suppose B is a Row Echelon Form of A. **For each of the statements below, determine whether it is <u>always</u> true, and if it is <u>not</u>, give a counterexample to it.**

- $\mathrm{RS}(B) = \mathrm{RS}(A)$.
- $N(B) = N(A)$.
- $\mathrm{CS}(B) = \mathrm{CS}(A)$.

(vii) Let $\mathbb{F}$ be a field, and consider a linear map f from $\mathbb{F}^n$ to $\mathbb{F}^m$. Recall that we can find a matrix $A_f \in \mathbb{F}^{m \times n}$, that we call a *matrix representation* of f, which satisfies

$$f(\bar{x}) = A_f \bar{x}$$

for every $\bar{x} \in \mathbb{F}^n$.

As we have seen, important subspaces associated with f are the subspaces $\mathrm{Ker}(f)$ and $\mathrm{Range}(f)$, while important subspaces associated with the matrix A_f are the subspaces $\mathrm{RS}(A_f), \mathrm{CS}(A_f)$ and $N(A_f)$.

- Which of these subspaces are subspaces of $\mathbb{F}^n$, and which are subspaces of $\mathbb{F}^m$?
- Do any of these subspaces coincide? Find all possible pairs.

Some Practice Problems

Some of the following problems are based on problems from previous terms of MATH 227, or on problems that can be found in Strang's book, Sections 2.4 and 4.2, 4.3.

Problem 1. Recall the *derivative operator* $D : \mathcal{P}_4 \to \mathcal{P}_4$, defined by

$$D\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\big) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3.$$

(i) Find all the invariant subspaces of D.

(ii) Find the eigenvalues of D and eigenvectors corresponding to them.

Problem 2. Find the eigenvalues and the eigenvectors of the following matrices:

$$\textbf{(a)} \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix} \qquad \textbf{(b)} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} \qquad \textbf{(c)} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 7 \end{pmatrix}.$$

Problem 3. (a) Suppose that κ is a parameter allowed to take values in $\mathbb{Q}$. For which choices of values for it, are the following vectors linearly independent? Justify your answer.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} \kappa \\ \kappa^2 \\ \kappa^3 \\ \kappa^4 \end{pmatrix}, \begin{pmatrix} 2\kappa - 1 \\ 4\kappa^2 - 2 \\ 8\kappa^8 - 3\kappa \\ 16\kappa^{16} + 4\kappa^3 - 1 \end{pmatrix}, \begin{pmatrix} 2+\kappa \\ 2\kappa + 3 \\ \kappa - 3 \\ 4\kappa^2 \end{pmatrix}, \begin{pmatrix} 1/(1+\kappa^4) \\ 2/(3+\kappa^2) \\ 3/(5+2\kappa^2) \\ 4/(1+\kappa^2) \end{pmatrix}.$$

(b) Suppose now that μ is a parameter allowed to take values in $\mathbb{R}$. For which choices of values for it, do the following vectors span $\mathbb{R}^4$? Justify your answer.

$$\begin{pmatrix} 2\mu \\ 6\mu^2 - 2\mu \\ 4\mu^8 - 3\mu \\ \mu^{16} + 4\mu^3 - 1 \end{pmatrix}, \begin{pmatrix} 2+\mu \\ 2\mu + 3 \\ \mu - 3 \\ 4\mu^2 \end{pmatrix}, \begin{pmatrix} 1/(1+\mu^2) \\ 2/(2+\mu^2) \\ 3/(3+\mu^2) \\ 4/(4+\mu^2) \end{pmatrix}.$$

Problem 4. Find a basis and the dimension of the subspace of $\mathbb{R}^4$ which contains exactly those vectors whose components add to zero.

Problem 5. (a) Find bases for the Row Space, the Column Space and the Nullspace of the following matrix from $\mathbb{Z}_7^{5\times 5}$:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

(b) What is the rank of A? What are the dimensions of those three spaces?

Problem 6. Let κ and μ be parameters allowed to take values in $\mathbb{R}$, and suppose the remaining entries of the following matrices are real numbers. In each case, determine which choices of values for κ and μ, if any, give a rank-2 matrix.

$$\textbf{(a)}\ A = \begin{pmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & \kappa & 2 & 2 \\ 0 & 0 & 0 & \mu & 2 \end{pmatrix}, \qquad \textbf{(b)}\ B = \begin{pmatrix} \kappa & \mu - 1 \\ \mu - 1 & \kappa \end{pmatrix}.$$

Problem 7. Let $C \in \mathbb{Z}_{11}^{5\times 5}$ be the following matrix:

$$C = \begin{pmatrix} 2 & 0 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

Find $\det(C)$ using the Leibniz formula for the determinant.

Problem 8. Consider the following three permutations of $\{1, 2, 3, 4\}$:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

(this is alternative notation for e.g. stating that $f(1) = 1$, $f(2) = 3$, $f(3) = 4$ and $f(4) = 2$).
(a) Find the compositions $f \circ g$ and $g \circ f$ (use the above notation to describe them).
(b) Find h^{-1} (again, use the above notation to describe it).
(c) Write h as a product of transpositions (you may wish to find its disjoint cycle representation first).

(d) Find the *signs* of f, g and h.
(e) If $A \in \mathbb{R}^{4\times4}$ is the following matrix:

$$A = \begin{pmatrix} 2 & 0 & 1 & 2 \\ -1 & 2 & -4 & -1 \\ -4 & 3 & -1 & 2 \\ -5 & 8 & -4 & 3 \end{pmatrix},$$

which summands (that is, products of matrix entries together with a sign) in the Leibniz formula for $\det(A)$ do f, g and h give respectively?

Problem 9. Find the determinants of the following matrices in any way you want:

$$A = \begin{pmatrix} 2 & 0 & 1 & 2 \\ -1 & 2 & -4 & -1 \\ -4 & 3 & -1 & 2 \\ -5 & 8 & -4 & 3 \end{pmatrix} \in \mathbb{R}^{4\times4}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -3 & 9 \end{pmatrix} \in \mathbb{Z}_{11}^{3\times3}.$$

Problem 10. Find the signs of the following permutations:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 3 & 4 & 7 & 9 & 10 & 2 & 8 & 1 \end{pmatrix},$$
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 15 & 14 & 13 & 11 & 9 & 4 & 6 & 3 & 5 & 7 & 1 & 10 & 2 & 8 & 12 \end{pmatrix}.$$

Problem 11. Suppose the following two matrices have entries from $\mathbb{Q}$. For each one of them, find two ways to choose non-zero entries from four different rows and columns (note that B has zero entries in the same positions as A, so your answer should be the same in both cases).

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

Is $\det(A)$ equal to $1+1$, $1-1$ or $-1-1$? What is $\det(B)$?

Problem 12. Let $\mathbb{F}$ be a field, and consider a square matrix $A = (a_{ij})$ with entries from $\mathbb{F}$.
(i) If A is a 3×3 matrix, and we know that $a_{11} = a_{22} = a_{33} = 0$, how many

of the 6 summands in the Leibniz formula for $\det(A)$ are guaranteed to be zero?
(ii) If A is a 4×4 matrix, and we know that $a_{11} = a_{22} = a_{33} = a_{44} = 0$, how many of the 24 summands in the Leibniz formula for $\det(A)$ are guaranteed to be zero?

Problem 13. Prove that 4 is the largest real number the determinant of a 3×3 real matrix can equal if we know that all entries of the matrix are $+1$ or -1.