MATHEMATICS 117

LECTURE NOTES

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I. INTRODUCTION

§1.1. Real Numbers. We begin by recalling some elementary algebraic and order properties of the real number system. We will use the following notation.

$\mathbb{R}$: the set of real numbers

$\mathbb{Z}$: the set of integers $\{0, \pm 1, \pm 2, \dots\}$

$\mathbb{Q}$: the set of rational numbers, i.e. numbers of the form p/q where $p, q \in \mathbb{Z}$, $q \neq 0$

$\mathbb{N}$: the set of natural numbers or positive integers $\{1, 2, 3, \dots\}$

ϕ : the empty set, or set with no elements.

Evidently $\phi \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

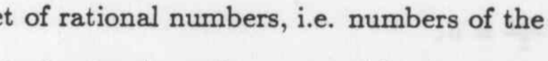
It is important to remember that

(i) Division by zero is not defined

(ii) $ab = 0 \iff a = 0$ or $b = 0$

(iii) $a < b$ ("a is less than b") is equivalent to $b > a$ ("b is greater than a"). Both are equivalent to $b - a > 0$. This is what allows us to model the real numbers on a line; the real numbers $\mathbb{R}$ are *ordered*. For example,

$\mathbb{R}$:



$$0 < 1, \quad x < y, \quad 0 > x, \quad 0 < y < 1.$$

Some order properties are:

(0_1) $a, b \in \mathbb{R} \Longrightarrow a < b$ or $a = b$ or $a > b$

and only *one* of these holds.

(0_2) $a < b,\ 0 < c \Longrightarrow ac < bc$

$a < b,\ c < 0 \Longrightarrow ac > bc$

(0_3) $a < b,\ b < c \Longrightarrow a < c$ (transitivity)

(0_4) $a < b,\ c \in \mathbb{R} \Longrightarrow a + c < b + c$

(0_5) $ab > 0 \Longrightarrow a > 0$ and $b > 0$ *or*

$a < 0$ and $b < 0$.

If $a < x < b$ we say x is between a and b.

THEOREM 1.1.1. $a < b \Longrightarrow a < \frac{1}{2}(a+b) < b$.

PROOF: $a < b \Longrightarrow a + a < a + b < b + b$

$$\Longrightarrow a = \tfrac{a+a}{2} < \tfrac{a+b}{2} < \tfrac{b+b}{2} = b.$$

REMARKS: (i) Determine where the properties (0_1) – (0_5) are used in the proof.

(ii) This theorem shows that, for example, there is no "least positive number" or number "$x \neq 1$ which is closest to 1", since between any two numbers there is another one.

§1.2. Induction. A subset S of $\mathbb{R}$ is called *inductive* if

(I_1) $1 \in S$.

(I_2) $k \in S \Longrightarrow k + 1 \in S$.

For example, the sets $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ are inductive. Think of some subsets S of $\mathbb{R}$ which are not inductive.

AXIOM 1.2.1. (THE INDUCTION AXIOM): If $S \subset \mathbb{N}$ satisfies (I_1) and (I_2), then $S = \mathbb{N}$.

Thus we see that $\mathbb{N}$ is the only inductive set with no inductive proper subset; $\mathbb{N}$ is the smallest inductive set.

In practice we will use induction as follows. We will wish to show that a certain statement $[n]$ about the natural number n holds true for all $n \in \mathbb{N}$. We prove this by showing that

(i) $[1]$ holds, and

(ii) if $[k]$ holds, then $[k+1]$ also holds.

We conclude from this that $[n]$ holds for all $n \in \mathbb{N}$. Here $S = \{n \in \mathbb{N} : [n] \text{ holds}\} \subset \mathbb{N}$ and S satisfies (I_1) by (i) and (I_2) by (ii) so that $S = \mathbb{N}$.

EXAMPLE 1.2.2: Suppose that $0 < a < b$. We will prove that

$$[n] : \quad 0 < a^n < b^n$$

holds for $n = 1, 2, 3, \ldots$.

PROOF:

(i) $[1]$ holds. This is easy, we are given $0 < a < b$.

(ii) $[k]$ holds $\Longrightarrow [k+1]$ holds. To see this, suppose $[k]$ holds, i.e. $0 < a^k < b^k$

$$\text{Then} \qquad 0a < a^k a < b^k a < b^k b \qquad \text{(why?)}$$

$$\text{so that} \qquad 0 < a^{k+1} < b^{k+1},$$

i.e. $[k+1]$ holds if $[k]$ holds.

(i), (ii) $\Longrightarrow [n]$ holds for all $n \in N$.

EXAMPLE 1.2.3. (BERNOULLI'S INEQUALITY): If $a \geq -1$, then

$$(1+a)^n \geq 1 + na, \quad n = 1, 2, 3, \ldots .$$

PROOF: We wish to show that the statement

$$[n] : \qquad (1+a)^n \geq 1 + na$$

is true for each $n \in N$, if $a \geq -1$. First [1] is true, since

$$(1+a)^1 = 1 + a. \qquad (*)$$

Secondly

$$\begin{aligned} &(1+a)^k \geq 1 + ka \\ \Longrightarrow\ &(1+a)^{k+1} \geq (1+ka)(1+a) \quad \text{(Multiply by } 1+a \geq 0 \text{, since } a \geq -1) \\ &\quad = 1 + (k+1)a + ka^2 \\ &\quad \geq 1 + (k+1)a. \qquad (**) \end{aligned}$$

Thus if $[k]$ is true, then $[k+1]$ is also true. From $(*)$, $(**)$ we find by induction that $[n]$ is true for each $n \in N$.

REMARK: Examine the proof of Bernoulli's Inequality to determine those values of n, a for which we can state that the strict inequality $(1+a)^n > 1 + na$ holds.

EXAMPLE 1.2.4: $1 + 2 + \cdots + n = \frac{1}{2}n(n+1), \quad n = 1, 2, \ldots$.

PROOF: $1 \geq \frac{1}{2}1(1+1)$. The statement is true when $n = 1$.

$$\begin{aligned} \text{Also} \qquad 1 + 2 + \cdots + k &= \frac{1}{2}k(k+1) \\ \Longrightarrow 1 + 2 + \cdots + k + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{1}{2}(k+1)(k+2), \end{aligned}$$

so that, if the statement is true for $n = k$, it is also true for $n = k+1$. We conclude, by induction, that $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$, $n = 1, 2, 3, \ldots$.

Of course some problems may be solved by many different means, some with more advantages than others. If we were faced with the problem of finding a formula for $1+2+\cdots+n$, the sum of the first n natural numbers, then to use the method of Example 1.2.4, we would need to first guess the correct formula by looking at several cases $n = 1,2,3$ etc. and then prove the formula by induction as we have done.

An alternative method of deriving the result of Example 1.2.4 is as follows. Since

$$(n+1)^2 - n^2 = 2n + 1,$$

we have

$$\begin{array}{ll}
2^2 - 1^2 = 2.1 + 1 & \qquad n = 1 \\
3^2 - 2^2 = 2.2 + 1 & \qquad n = 2 \\
4^2 - 3^2 = 2.3 + 1 & \\
\dots & \\
n^2 - (n-1)^2 = 2(n-1) + 1 & \\
(n+1)^2 - n^2 = 2n + 1. &
\end{array}$$

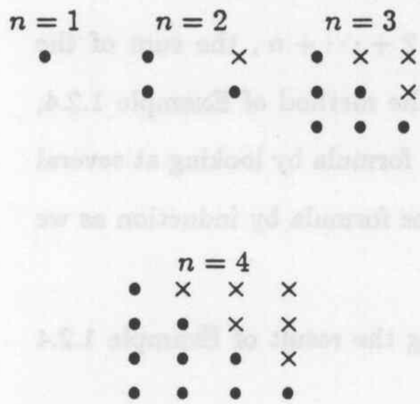
Adding and noting the cancellations on the left, we get

$$\begin{aligned}
(n+1)^2 - 1 &= 2(1+2+\cdots+n) + n \\
n^2 + n &= 2(1+2+\cdots+n) \\
\frac{1}{2}n(n+1) &= 1+2+\cdots+n.
\end{aligned}$$

A more geometric way of approaching this question is to let

$$S_n = 1 + 2 + \cdots + n.$$

Then S_n is the number of dots in the triangular arrays.



Note that the number of crosses in the nth figure is S_{n-1}. Therefore

$$S_n + S_{n-1} = n^2$$
$$S_n - S_{n-1} = n,$$

since $S_n + S_{n-1}$ is the number of points (dots and crosses) in a square array of side n and $S_n - S_{n-1}$ is the number of dots on the diagonal. Thus

$$2S_n = n^2 + n = n(n+1)$$
$$S_n = \frac{1}{2}n(n+1)$$

as before.

EXAMPLE 1.2.5: Determine the number of subsets of a set of n elements.

First we make some experiments, then we guess the general result and finally prove it by induction. Denote the set by

$$S_n = \{a_1, a_2, \dots, a_n\}$$

$n = 1: \quad S_1 = \{a_1\}$ has 2 subsets $\varphi, \{a_1\}$

$n = 2: \quad S_2 = \{a_1, a_2\}$ has 4 subsets $\varphi, \{a_1\}, \{a_2\}, \{a_1, a_2\}$

$n = 3: \quad S_3 = \{a_1, a_2, a_3\}$ has 8 subsets $\varphi, \{a_1\}, \{a_2\}, \{a_3\}$

$$\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_2, a_3\}.$$

Now we may *guess* that S_n has 2^n subsets and *prove* this by induction.

First S_1 has $2^1 = 2$ subsets (we just counted them). Second, if it is true that $S_k = \{a_1, a_2, \dots, a_k\}$ has 2^k subsets, then $S_{k+1} = \{a_1, a_2, \dots, a_k, a_{k+1}\}$ also has these 2^k subsets since $S_k \subset S_{k+1}$. The only other subsets of S_{k+1} may be obtained by throwing a_{k+1} into any of these subsets of S_k to obtain a further 2^k subset of S_{k+1} and $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ subsets altogether of S_{k+1}. Thus if S_k has 2^k subsets then S_{k+1} has 2^{k+1} subset. Therefore S_n has 2^n subsets, $n = 1, 2, \dots$.

EXAMPLE 1.2.6: "All dogs are shaggy."

PROOF: Suppose that "all groups of k dogs are shaggy". Now consider any group of $k+1$ dogs. Disregard one of the dogs; the group remaining consists of k dogs and are therefore all shaggy. The same argument shows that the disregarded dog is shaggy if we simply disregard a different dog. Thus "all groups of k dogs are shaggy" implies "all groups of $k+1$ dogs are shaggy". By induction "all groups of n dogs are shaggy" for all $n = 1, 2, 3, \dots$. What, besides the dogs, is shaggy here?

§1.3. The Binomial Theorem. We define the symbol $n!$ ("n factorial") by $0! = 1$ and $(k+1)! = k!k$, so that $n!$ is defined for $n = 0, 1, 2, \ldots$. This is called an inductive definition. Equivalently

$$0! = 1, \quad n! = 1.2 \cdot \cdots \cdot (n-1)n, \quad \text{if } n \in \mathbb{N}\,.$$

Thus $0! = 1,\ 1! = 1,\ 2! = 2,\ 3! = 6,\ 4! = 24\,.$

If $0 \le k \le n$, the binomial coefficient $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{1.2 \cdot \cdots \cdot k}.$$

For example,

$$\binom{7}{3} = \frac{7.6.5}{1.2.3} = 35, \quad \binom{7}{0} = 1, \quad \binom{7}{7} = 1$$

$$\binom{7}{1} = \frac{7}{1} = 7, \quad \binom{0}{0} = 1.$$

PROPOSITION 1.3.1 (PASCAL'S TRIANGLE LAW):

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

PROOF:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n}{(k-1)!(n-k)!}\left[\frac{1}{n-k+1} + \frac{1}{k}\right]$$

$$= \frac{n!}{(k-1)!(n-k)!}\,\frac{n+1}{k(n-k+1)}$$

$$= \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}.$$

$\square$

Pascal's Triangle gives a convenient method of computing successive binomial coefficients $\binom{n}{k}$.

n/k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

$\binom{0}{0} = 1, \quad \binom{1}{0} = 1, \quad \binom{1}{1} = 1$

$\binom{2}{0} = 1, \quad \binom{2}{1} = 2, \quad \binom{2}{2} = 1.$

All entries not equal to 1 are obtained by adding the entry immediately above to the one to its left.

THEOREM 1.3.2. (BINOMIAL THEOREM).

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

PROOF (BY INDUCTION): The theorem is true when $n = 1$, since

$$(a+b)^1 = a + b = \binom{1}{0}a + \binom{1}{1}b.$$

Next, if the theorem holds for some n, then

$$
\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\left[\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n\right] \\
&= \left[\binom{n}{0}a^{n+1} + \binom{n}{1}a^n b + \binom{n}{2}a^{n-1}b^2 + \cdots + \binom{n}{n-1}a^2b^{n-1} + \binom{n}{n}ab^n\right] \\
&\quad + \left[\binom{n}{0}a^n b + \binom{n}{1}a^{n-1}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-1} + \binom{n}{n-1}ab^n + \binom{n}{n}b^{n+1}\right] \\
&= \binom{n}{0}a^{n+1} + \left[\binom{n}{0} + \binom{n}{1}\right]a^n b + \left[\binom{n}{1} + \binom{n}{2}\right]a^{n-1}b^2 + \cdots \\
&\qquad \cdots + \left[\binom{n}{n-1} + \binom{n}{n}\right]ab^n + \binom{n}{n}b^{n+1} \\
&= \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + \binom{n+1}{2}a^{n-1}b^2 + \cdots + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^{n+1},
\end{aligned}
$$

using Pascal's Triangle laws and the fact that $\binom{n}{0} = \binom{n+1}{0} = 1$, $\binom{n}{n} = \binom{n+1}{n+1} = 1$. Thus, if the Binomial Theorem holds for some n, it also holds for $n+1$. Since we have already checked that it holds when $n = 1$, it follows by induction that it holds for all natural numbers n.

$\square$

REMARKS: (i) It is also helpful to consider the Binomial Theorem in the form

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

(ii) Note that, with $a = 1$, $b = x$, we find

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \cdots + nx^{n-1} + x^n$$

$$\geq 1 + nx, \quad \text{if } x \geq 0 \text{ and } n \geq 1 \quad \text{(Why?).}$$

In fact we already know this inequality for $x \geq -1$ (Bernoulli) but this suggests further generalizations of Bernoulli's Inequality.

§1.4. Absolute Value. We define the absolute value $|x|$ of a real number x by

$$|x| = \begin{cases} x, & \text{if } \quad x \geq 0 \\ -x, & \text{if } \quad x < 0. \end{cases}$$

Geometrically $|x|$ is the distance of x from 0 on the number line. More generally $|b-a|$ is the distance between a and b.

The absolute value has the following properties

(A1) $|x| \geq 0$

(A2) $|x| = 0 \iff x = 0$

(A3) $|-x| = |x|$

(A4) $|xy| = |x|\ |y|$

(A5) If $c \geq 0$, then $|x| \leq c \iff -c \leq x \leq c$

(A6) $-|x| \leq x \leq |x|$

(A7) $\Big|\, |x| - |y| \,\Big| \leq |x \pm y| \leq |x| + |y|$ (Triangle Inequality)

PROOF OF (A1), (A2), (A3):

Exercise

PROOF OF (A4):

Case $x = 0$ or $y = 0 \Longrightarrow xy = 0 \Longrightarrow |xy| = 0 = |x|\ |y|$

Case $x > 0$ and $y > 0 \Longrightarrow xy > 0 \Longrightarrow |xy| = xy = |x|\ |y|$

Case $x > 0$ and $y < 0 \Longrightarrow xy < 0 \Longrightarrow |xy| = -xy = x(-y) = |x|\ |y|$

Case $x < 0$ and $y > 0$ same as preceding case

Case $x < 0$ and $y < 0 \Longrightarrow xy > 0 \Longrightarrow |xy| = xy = (-x)(-y) =$ $|x|\ |y|$.

PROOF OF (A5):

$$|x| \le c \Longleftrightarrow x \le c \quad \text{and} \quad -x \le c \quad (\text{consider } x \ge 0 \text{ and } x < 0)$$
$$\Longleftrightarrow -c \le x \le c.$$

PROOF OF (A6):

$$(\text{A5}) \Longrightarrow (\text{A6}) \quad \text{if we take } c = |x|.$$

PROOF OF (A7):

$$(\text{A6}) \Longrightarrow \begin{cases} -|x| \le x \le |x| \\ -|y| \le y \le |y| \end{cases}$$
$$\Longrightarrow -(|x| + |y|) \le x + y \le (|x| + |y|) \qquad (\text{add})$$
$$\Longrightarrow |x + y| \le |x| + |y|$$

from (A5) with $c = |x| + |y|$ and x replaced by $x + y$.

We now have proved the right-hand part of (A7) with the '+' sign.

To obtain this with the '−' sign, replace y by $-y$.

$$|x - y| \le |x| + |-y| = |x| + |y|.$$

Next, consider

$$|x| = |x + y - y| \leq |x + y| + |y| \qquad \text{(right-hand half of (A7), just proved)}$$

$$\left.\begin{aligned} &\Longrightarrow |x| - |y| \leq |x + y| \\ \text{and} \quad &|y| - |x| \leq |x + y| \end{aligned}\right\} \quad \text{(Interchange } x, y\text{)}$$

$$\Longrightarrow |\,|x| - |y|\,| \leq |x + y|$$

$$\text{and} \quad |\,|x| - |y|\,| \leq |x - y| \quad \text{(Replace } y \text{ by } -y\text{).}$$

This proves the left-hand half of (A7).

§1.5. Intervals. Let $a, b \in \mathbb{R}$, $a < b$. There are four types of intervals I determined by a, b.

$$I = [a, b] = \{x : a \leq x \leq b\} \quad \text{closed interval}$$

$$I = (a, b) = \{x : a < x < b\} \quad \text{open interval}$$

$$I = [a, b) = \{x : a \leq x < b\}$$

$$I = (a, b] = \{x : a < x \leq b\}.$$

The last two intervals are neither open nor closed. The points a, b are the left endpoint and right endpoint respectively of each inteval. A closed interval contains both endpoints; an open interval contains neither endpoint. Any point of I which is not an endpoint is called an <u>interior</u> <u>point</u> of I. The point $\frac{1}{2}(a + b)$ is the midpoint of each of the intervals above since its distance from a and from b is $\frac{1}{2}(b - a)$ and $b - a$, the distance from a to b is the length of the interval.

It is convenient to extend this notation as follows.

$$(-\infty,\infty) = \mathbb{R}$$

$$[a,\infty) = \{x : x \geq a\}$$

$$(a,\infty) = \{x : x > a\}$$

$$(-\infty,a] = \{x : x \leq a\}$$

$$(-\infty,a) = \{x : x < a\}.$$

We will not, however, consider these to be intervals.

§1.6. Completeness of $\mathbb{R}$. Let $S \subset \mathbb{R}$, $b \in \mathbb{R}$. We say b is an *upper bound* of S if

$$x \leq b, \quad \text{for each} \quad x \in S.$$

S is *bounded above* if it has an upper bound and S is *unbounded above* if it has no upper bound. For example, $[0,1]$, $(-\infty,1)$ are both bounded above since 1 is an upper bound (2 or any number greater than or equal to 1 is also an upper bound). Notice that an upper bound of S may or may not be an element of the set S. The sets $(0,\infty)$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$ are all unbounded above.

If $a \in \mathbb{R}$ satisfies

$$a \leq x, \quad \text{for each} \quad x \in S,$$

then a is called a *lower bound* of S. A set which has a lower bound is *bounded below* and a set with no lower bound is *unbounded below*.

A set which is bounded above and below is called *bounded* and otherwise it is *unbounded*. Thus $S \subset \mathbb{R}$ is bounded if and only if

$$S \subset [a, b], \quad \text{for some closed interval } [a, b] .$$

DEFINITION 1.5.1: Let $S \subset \mathbb{R}$, $b \in \mathbb{R}$. Suppose

(i) $x \leq b$, for each $x \in S$,

(ii) $b \leq y$, for each upper bound y of S.

Then b is called the *least upper bound* of S or the *supremum* of S.

We write

$$b = \ell.u.b.S \quad \text{or} \quad b = \sup S.$$

Thus $b = \sup S$ means

(i) b is an upper bound of S and

(ii) b is a lower bound of the set of upper bounds of S.

For example,

$$\sup(0, 1) = 1, \quad \sup(-1, 1) = 1, \quad \sup(-\infty, 0) = 0.$$

If $b = \sup S$ *and* $b \in S$, we say b is the *maximum* of S. Thus $1 = \max[-1, 1]$, but $[-1, 1)$ has no maximum.

DEFINITION 1.5.2: Let $S \subset \mathbb{R}$, $a \in \mathbb{R}$. Suppose

(i) $a \leq x$, for each $x \in S$

(i) $y \leq a$, for each lower bound y of S.

Then a is called the *greatest lower bound* or *infimum* of S. We write

$$a = g.\ell.bS. \quad \text{or} \quad a = \inf S.$$

If $a = \inf S$ and $a \in S$ we say a is the *minimum* of S.

Axiom 1.5.3. (Completeness of $\mathbb{R}$): If $S \subset \mathbb{R}$, $S \neq \varphi$ and S is bounded above then S has a supremum.

Equivalently, the Completeness Axiom for the Real Numbers is that any nonempty set of real numbers which has an upper bound has a least upper bound.

We asserted earlier that the set $\mathbb{N}$ of natural numbers is unbounded above. This is called the *Archimedean Property* of $\mathbb{R}$. Even though it may seem intuitively clear it should be proved. We will deduce it from the Completeness Axiom. The proof is an example of a proof by contradiction. Suppose $\mathbb{N}$ has an upper bound. Then, by the completeness of $\mathbb{R}$, $\mathbb{N}$ has a least upper bound or supremum. Let $b = \sup \mathbb{N}$. Then

$$b \geq n, \quad \text{for each} \quad n \in N.$$

But $b - 1 < b$, so $b - 1$ is *not* an upper bound of $\mathbb{N}$ and there is an element $m \in \mathbb{N}$ such that

$$b - 1 < m \quad \text{and therefore} \quad b < m + 1.$$

This contradicts $b = \sup N$, since $m \in N \implies m + 1 \in \mathbb{N}$. The assumption that $\mathbb{N}$ has an upper bound leads to a contradiction and therefore the assumption is false.

Problems

1.1 Show by induction that

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1), \quad n \in \mathbb{N}.$$

1.2 Give another proof of 1.1 by considering the expression

$(n+1)^3 - n^3$.

1.3 Show by induction that $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2$, $n \in \mathbb{N}$

i.e. $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.

1.4 Check that

$$\frac{1}{1.2} = \frac{1}{2}, \quad \frac{1}{1.2} + \frac{1}{2.3} = \frac{2}{3}, \quad \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} = \frac{3}{4}.$$

Guess a general formula and prove it by induction.

1.5 Given n straight lines in the plane, no two of which are parallel and no three concurrent, determine the number of intersection points.

1.6 Let $[n]$ denote the statement $1+2+\cdots+n = \frac{1}{8}(2n+1)^2$. Show that $[n] \implies [n+1]$. Thus, by induction, $[n]$ holds for each $n \in \mathbb{N}$. But we have seen (Example 1.2.4) that $1+2+\cdots+n = \frac{1}{2}n(n+1) \neq \frac{1}{8}(2n+1)^2$. Comment.

1.7 Let S be a set of n elements and $0 \le k \le n$. Show that $\binom{n}{k}$ is the number of subsets of S each having exactly k elements.

1.8 Prove

(a) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

(b) $\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n \binom{n}{n} = 0$

Hint: Binomial Theorem.

1.9 Show $\binom{n}{k} = \binom{n}{n-k}$.

1.10 Check that

$$\left(1-\frac{1}{2}\right) = \frac{1}{2}, \ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = \frac{1}{3}, \ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right) = \frac{1}{4}.$$

Guess a general formula and prove it by induction.

1.11 Prove that, if $x, y \in \mathbb{R}$, $n \in \mathbb{N}$, then

$$x^n - y^n = (x-y)\big(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}\big).$$

Hint: Don't use induction.

1.12 Let $a_1, a_2, \ldots, a_n$ be real numbers all having the same sign and all greater than -1. Show that

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n\,.$$

The case $a_1 = a_2 = \cdots = a_n = a > -1$ is Bernoulli's Inequality $(1+a)^n \geq 1 + na$.

1.13 Solve

(a) $|x-1| = 0$ (b) $|x-1| = 2$

(b) $|x-1| < 2$ (d) $|x-1| \leq 2$.

1.14 If $a, b \in \mathbb{R}$, what x satisfy

$$|x-a| + |x-b| = |a-b|\,?$$

1.15 Determine whether each of the following is true or false. Give a reason for your decision in each case.

(a) $x < 5 \Longrightarrow |x| < 5$

(b) $|x-5| < 2 \Longrightarrow 3 < x < 7$

(c) $|1+3x| \leq 1 \Longrightarrow x \geq -\frac{2}{3}$

(d) There is no x such that $|x-1| = |x-2|$.

1.16 If A, B, C are real numbers, show that

$$|A-B| \leq |A-C| + |B-C|\,.$$

1.17 If A, B are real numbers such that $|A-B| < \frac{1}{2}$, show that

$$|A^2 - B^2| < \tfrac{1}{2}(2|B| + \tfrac{1}{2})\,.$$

1.18 Express the solution sets of the following in interval notation.

(a) $-3 < x + 3 < 5$ (b) $1 < 3x - 2 \leq 2$

(c) $|x - 7| \leq 5$ (d) $|1 + 3x| < 4$.

1.19 If $a, b \in \mathbb{R}$, prove $2ab \leq a^2 + b^2$. Under what circumstances does equality hold?

1.20 If $0 < x < y$, prove $x < \sqrt{xy} < \frac{x+y}{2} < y$.

1.21 Show $n < 2^n$, if $n \in \mathbb{N}$.

1.22 Write down $\sup S$, $\inf S$ for the following when they exist.

(a) $S = (0, 1]$ (b) $S = (-\infty, 0)$

(c) $S = \{\frac{1}{n} : n = 1, 2, \dots\}$ (d) $S = \left\{\frac{(-1)^n}{n};\ n = 1, 2, \dots\right\}$.

1.23 If $0 < a$ show that there is a natural number n such that $0 < \frac{1}{n} < a$.

1.24 If $0 < a$ show that there is a natural number n such that $0 < 2^{-n} < a$.

1.25 Prove the Well-Ordering Principle: If S is a nonempty subset of $\mathbb{N}$, then S contains a least element, i.e. S has a minimum. *Hint*: Suppose S has no least element. Consider $\mathbb{N} \backslash S = \{n : n \in \mathbb{N},\ n \notin S\}$. Use induction to show $\mathbb{N} \backslash S = \mathbb{N}$ and hence $S = \varphi$ contradicting $S \neq \varphi$.

1.26 Is the empty set φ bounded? Explain.

1.27 Let $x_1, x_2, \dots, x_n$ be real numbers. Prove

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

1.28 Let $s_n = 1 + r + r^2 + \cdots + r^n$.

(a) Show by induction that $s_n = \frac{1 - r^{n+1}}{1 - r}$, if $r \neq 1$.

(b) Give another proof of (a).

Hint: Use 1.11 or consider $s_n - rs_n$.

1.29 If $a > 0$ and $n \in \mathbb{N}$, show that the equation $x^n = a$ has at most one solution $x > 0$. We will see later (page 63) that it has one solution x . We denote this unique solution $x = a^{\frac{1}{n}}$. If $0 < a < b$, prove $a^{\frac{1}{n}} < b^{\frac{1}{n}}$.

Hint: Use # 1.11.

II. SEQUENCES

§2.1. Functions. Let $D \subset \mathbb{R}$. A *real-valued function* f of *domain* D is a rule which, with each $x \in D$ associates a real number $f(x)$. The *range* $f(D)$ of f is the set $\{f(x) : x \in D\}$. For example, the function f defined by the rule $f(x) = x^2$, $0 \le x < 2$ has domain $D = [0, 2)$ and range $f(D) = [0, 4)$.

A *real sequence* is a real-valued function of domain $\mathbb{N}$, the set of natural numbers. For example the rules

$$a(n) = n - 1, \quad b(n) = \frac{1}{2}\,n, \quad c(n) = \frac{1}{n}\,, \quad d(n) = \frac{(-1)^n}{n}$$

define sequences a, b, c, d. The expression $a(n)$ is called the n^{th} *term* of the sequence a and will usually be denoted a_n. We will normally specify a sequence by simply giving the rule and speaking of the sequence $\{n-1\}$, or $\{\frac{1}{2}\,n\}$, or $\{\frac{1}{n}\}$, or $\{\frac{(-1)^n}{n}\}$.

It is also sometimes convenient to consider sequences somewhat less precisely and to simply give a few terms which suggest a general rule:

$$\{0, 1, 2, 3, \dots\}, \qquad \{\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \dots\}$$

$$\{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4} \dots\}, \quad \{-1, \tfrac{1}{2}, -\tfrac{1}{3}, \tfrac{1}{4}, \dots\}$$

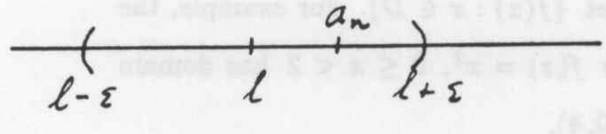
§2.2. Limit of a Sequence. Before considering limits in general, we will first discuss the special case of sequences.

DEFINITION 2.2.1: The sequence $\{a_n\}$ is *convergent with limit* ℓ if, for each $\varepsilon > 0$, there exists a natural number N such that

$$n \ge N \Longrightarrow |a_n - \ell| < \varepsilon.$$

The sequence is said to be *divergent* if no such number ℓ exists.

Note: ε is the Greek letter 'epsilon'.



REMARKS: .

(i) The statement '$\{a_n\}$ is convergent with limit ℓ' is abbreviated to

$$\lim_{n\to\infty} a_n = \ell.$$

(ii) The definition says '$\lim_{n\to\infty} a_n = \ell$' means the distance $|a_n - \ell|$ between a_n and ℓ is as small as we please provided n is large enough.

(iii) Equivalently '$\lim_{n\to\infty} a_n = \ell$' means, for each interval I with interior point ℓ, all but a finite number of terms of $\{a_n\}$ are contained in I.

(iv) The choice of N in Definition 2.2.1 depends on the number ε given and it is important to remember that we must be able to find such a number N for each positive number ε.

EXAMPLE 2.2.2: If $a_n = 1$, $n = 1, 2, 3, \ldots$, then $\lim_{n\to\infty} a_n = 1$. Let $\varepsilon > 0$. Now $n \geq 1 \Longrightarrow |a_n - 1| = |1 - 1| = 0 < \varepsilon$.

Thus, for each $\varepsilon > 0$, the choice $N = 1$ works in Definition

2.2.1 in this case. In fact *any* choice of N is satisfactory for this sequence.

EXAMPLE 2.2.3: If $a_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$, then $\lim_{n\to\infty} a_n = 0$. To see this, consider

$$|a_n - 0| = |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N}, \quad \text{if} \quad n \geq N.$$

Therefore, if $\varepsilon > 0$, we may choose N to be any number greater than $\frac{1}{\varepsilon}$. Then $N > \frac{1}{\varepsilon}$ means $\frac{1}{N} < \varepsilon$ so that

$$n \geq N \Longrightarrow |a_n - 0| < \varepsilon$$

and $\lim_{n\to\infty} a_n = 0$, as asserted.

PROPOSITION 2.2.4 (UNIQUENESS OF LIMITS). *For any sequence* $\{a_n\}$, *at most one number* ℓ *satisfies Definition 2.2.1.*

PROOF: Suppose ℓ_1, ℓ_2 are both limits of $\{a_n\}$. Thus, if $\varepsilon > 0$, then $\frac{\varepsilon}{2} > 0$ and there exist N_1, N_2 such that

$$n \geq N_1 \Longrightarrow |a_n - \ell_1| < \frac{\varepsilon}{2}$$

and

$$n \geq N_2 \Longrightarrow |a_n - \ell_1| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then $n \geq N \Longrightarrow n \geq N_1$ and $n \geq N_2 \Longrightarrow$ $|\ell_1 - \ell_2| = |\ell_1 - a_n + a_n - \ell_2| \leq |a_n - \ell_1| + |a_n - \ell_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ (Triangle Inequality).

Thus $0 \leq |\ell_1 - \ell_2| < \varepsilon$, for each $\varepsilon > 0$, so that $0 = |\ell_1 - \ell_2|$, since we have seen in §1.1 that there is no smallest positive number, and so $\ell_1 = \ell_2$.

A sequence is said to be *bounded* if there is a number K such that

$$|a_n| \leq K, \quad n = 1, 2, 3, \ldots .$$

PROPOSITION 2.2.5. *A convergent sequence is bounded.*

PROOF: Suppose $\lim_{n\to\infty} a_n = \ell$.

Let $\varepsilon = 1$. There exists N such that

$$\begin{aligned} n \geq N &\Longrightarrow |a_n - \ell| < 1 \\ &\Longrightarrow |a_n| - |\ell| \leq |a_n - \ell| < 1 \quad \text{(triangle inequality)}. \end{aligned}$$

Therefore $|a_n| \leq 1 + |\ell|$, if $n \geq N$, and
$|a_n| \leq K = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}||, 1 + |\ell|\}, \quad n = 1, 2, 3, \ldots$.

□

REMARK: (i) A bounded sequence need not be convergent. To see this, consider $\{(-1)^n\}$. This sequence is bounded, since $|(-1)^n| = 1$. However the sequence is not convergent: suppose $\lim_{n\to\infty}(-1)^n = \ell$. Then there exists N such that $n \geq N$

$$\begin{aligned} &\Longrightarrow |(-1)^n - \ell| < 1 \qquad (\varepsilon = 1) \\ &\Longrightarrow |1 - \ell| < 1 \quad \text{and} \quad |1 + \ell| = |-1 - \ell| < 1 \\ &\Longrightarrow 2 = |1 + 1| = |1 - \ell + 1 + \ell| \\ &\leq |1 - \ell| + |1 + \ell| < 1 + 1 = 2, \quad \text{that is} \quad 2 < 2. \end{aligned}$$

This contradiction shows $\{(-1)^n\}$ is not convergent.

(ii) Boundedness is a necessary but not a sufficient condition for convergence of a sequence. As we saw in (i), a bounded sequence might not be convergent. However, an unbounded sequence is necessarily divergent. For example, from the Archimedian Property, page 14 $\{n\}$ is unbounded and therefore divergent, by Proposition 2.2.5,

THEOREM 2.2.6. *Suppose* $a_n = c$, $n = 1, 2, \ldots$. *Then* $\lim_{n\to\infty} a_n = c$.

PROOF: Exercise.

THEOREM 2.2.7. *Suppose* $\lim_{n\to\infty} a_n = \ell$, $\lim_{n\to\infty} b_n = m$. *Then*

(a) $\lim_{n\to\infty}(a_n + b_n) = \ell + m$,

(b) $\lim_{n\to\infty} a_n b_n = \ell m$,

(c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\ell}{m}$, *if* $m \neq 0$.

PROOF OF (a): For all $n = 1, 2, \ldots$

$$|a_n + b_n - \ell - m| = |(a_n - \ell) + (b_n - m)| \leq |a_n - \ell| + |b_n - m|. \quad \text{(A)}$$

If $\varepsilon > 0$, there exist N_1, N_2 such that

$$n \geq N_1 \Longrightarrow |a_n - \ell| < \frac{\varepsilon}{2}$$
$$n \geq N_2 \Longrightarrow |b_n - m| < \frac{\varepsilon}{2}\,.$$

Thus $n \geq N = \max\{N_1, N_2\} \Longrightarrow |a_n + b_n - \ell - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, from (A). Hence

$$\lim_{n\to\infty}(a_n + b_n) = \ell + m.$$

PROOF OF (b): For all $n = 1, 2, \ldots$

$$\begin{aligned}
|a_n b_n - \ell m| &= |a_n b_n - \ell b_n + \ell b_n - \ell m| \\
&= |(a_n - \ell) b_n + (b_n - m)\ell| \\
&\leq |(a_n - \ell) b_n| + |(b_n - m)\ell| \\
&= |a_n - \ell|\, |b_n| + |b_n - m|\, |\ell|.
\end{aligned}$$

Now $\{b_n\}$ is convergent, so there exists a number K such that $|b_n| < K$ $n = 1, 2, \ldots$ (Proposition 2.2.5). Hence

$$|a_n b_n - \ell m| \leq |a_n - \ell| K + |b_n - m|\, |\ell|. \qquad \text{(B)}$$

If $\varepsilon > 0$, there exist N_1, N_2 such that

$$\begin{aligned}
n \geq N_1 &\Longrightarrow |a_n - \ell| < \varepsilon/(K + |\ell|) \quad \text{and} \\
n \geq N_2 &\Longrightarrow |b_n - m| < \varepsilon/(K + |\ell|).
\end{aligned}$$

Hence $n \geq N = \max\{N_1, N_2\} \Longrightarrow |a_n b_n - \ell m| < (K + |\ell|)\varepsilon/(K + |\ell|) = \varepsilon$ from (B) and therefore

$$\lim_{n \to \infty} a_n b_n = \ell_m.$$

PROOF OF (c): It is enough to prove that

$$\lim_{n \to \infty} b_n = m \Longrightarrow \lim_{n \to \infty} \frac{1}{b_n} = \frac{1}{m}, \quad \text{if} \quad m \neq 0. \qquad \text{(C)}$$

Then use Part (b) to deduce

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} (a_n \frac{1}{b_n}) = \ell \frac{1}{m}.$$

To prove (C) consider

$$\left|\frac{1}{b_n}-\frac{1}{m}\right| = \left|\frac{m-b_n}{b_n m}\right| = \frac{|b_n-m|}{|b_n|\,|m|}\,. \qquad \text{(D)}$$

There exists N_1 such that

$$\begin{aligned} n \ge N_1 &\Longrightarrow |b_n - m| < \frac{|m|}{2} \quad (\text{let} \quad \varepsilon = \frac{|m|}{2} > 0) \\ &\Longrightarrow |m| - |b_n| \le |b_n - m| < \frac{|m|}{2} \\ &\Longrightarrow 0 < \frac{|m|}{2} < |b_n| \quad (\text{case } m > 0). \end{aligned}$$

Thus, from (D),

$$\left|\frac{1}{b_n}-\frac{1}{m}\right| < \frac{2}{|m|^2}\,|b_n - m|, \quad \text{if} \quad n \ge N_1. \qquad \text{(E)}$$

There exists N_2 such that

$$|b_n - m| < \frac{|m|^2}{2}\,\varepsilon, \quad \text{if} \quad n \ge N_2. \qquad \text{(F)}$$

If $n \ge N = \max\{N_1, N_2\}$, then both (E) and (F) hold so that

$$n \ge N \Longrightarrow \left|\frac{1}{b_n}-\frac{1}{m}\right| < \frac{2}{|m|^2}\,\frac{|m|^2}{2}\,\varepsilon = \varepsilon.$$

Therefore $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{m}$ as asserted in (C). $\square$

EXAMPLE 2.2.8: $\lim_{n\to\infty}(3+\frac{1}{n}) = 3$, by (a), since

$$\lim_{n\to\infty} 3 = 3 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0.$$

EXAMPLE 2.2.9: $\lim_{n\to\infty} \frac{6n+5}{7n+8} = \frac{6}{7}$, by (a) and (c) since

$$\frac{6n+5}{7n+8} = \frac{6+\frac{5}{n}}{7+\frac{8}{n}}$$

and $\lim_{n\to\infty}(6+\frac{5}{n})=6$, $\lim_{n\to\infty}(7+\frac{8}{n})=7\neq 0$.

EXAMPLE 2.2.10: $\lim_{n\to\infty}\frac{n+1}{n^2+1}=0$, by (a), (b) and (c), since

$$\frac{n+1}{n^2+1}=\frac{n(1+\frac{1}{n})}{n^2(1+\frac{1}{n^2})}=\frac{1}{n}\cdot\frac{1+\frac{1}{n}}{1+\frac{1}{n^2}}$$

and $\lim_{n\to\infty}\frac{1}{n}=0$, $\lim_{n\to\infty}(1+\frac{1}{n})=1$, and $\lim_{n\to\infty}(1+\frac{1}{n^2})=1$, since $\frac{1}{n^2}=\frac{1}{n}\cdot\frac{1}{n}$.

If $n_1<n_2<n_3\dots$, where $n_k\in\mathbb{N}$,, then $\{a_{n_k}\}$ is a *subsequence* of $\{a_n\}$. For example, $\{\frac{1}{2n}\},\{\frac{1}{2n+1}\},\{\frac{1}{100},\frac{1}{101},\frac{1}{102},\dots\}$, $\{\frac{1}{n}\}$ are all subsequences of $\{\frac{1}{n}\}$.

PROPOSITION 2.2.11. *$\{a_n\}$ is convergent with limit $\ell \iff$ each subsequence $\{a_{n_k}\}$ of $\{a_n\}$ is convergent with limit ℓ.*

PROOF:

" $\Longrightarrow$ " Suppose $\{a_n\}$ is convergent with limit ℓ. If $\varepsilon>0$, there exists N such that

$$n\ge N\Longrightarrow |a_n-\ell|<\varepsilon. \qquad \text{(A)}$$

Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$.

$$\begin{aligned} k\ge N &\Longrightarrow n_k\ge k\ge N \quad \text{(why?)}\\ &\Longrightarrow |a_{n_k}-\ell|<\varepsilon, \text{from (A)}. \end{aligned}$$

Therefore $\lim_{k\to\infty}a_{n_k}=\ell$.

" $\Longleftarrow$ " Suppose each subsequence of $\{a_n\}$ is convergent with limit ℓ. But $\{a_n\}$ is a subsequence of itself. Therefore $\{a_n\}$ is convergent with limit ℓ.

$\square$

EXAMPLE 2.2.12: The sequence $\{(-1)^n\}$ is not convergent since $\{(-1)^{2n}\} = \{1, 1, 1, \dots\}$ and $\{(-1)^{2n+1}\} = \{-1, -1, -1, \dots\}$ and these ~~both~~ subsequences of $\{(-1)^n\}$ and have limits +1, -1 respectively.

Recall that we also showed that $\{(-1)^n\}$ is divergent directly from the definition of convergence (cf. Remark (i), p. 22).

Problems

2.1 Prove Theorem 2.2.6: $a_n = c,\ n = 1, 2, \cdots \Longrightarrow \lim_{n\to\infty} a_n = c$.

2.2 (Squeeze Principle). Suppose $x_n \le z_n \le y_n$, $n = 1, 2, \ldots$ and $\{x_n\}$, $\{y_n\}$ are both convergent with limit c. Show that $\{z_n\}$ is also convergent with limit c.

2.3 Using only results already established, show that each of the following sequences is convergent and find its limit

(a) $\{\frac{n}{n+1}\}$, (b) $\{\frac{5n+3}{8n+9}\}$,

(c) $\{\frac{2n^2+5}{3n^2-1}\}$, (d) $\{\frac{1+2+3+\cdots+n}{n^2}\}$,

(e) $\{\frac{(-1)^n}{n}\}$, (f) $\{\frac{\sin n}{n}\}$

[You may assume $|\sin n| \le 1$]

2.4 Show that $\lim_{n\to\infty} a_n = \ell \Longrightarrow \lim_{n\to\infty} |a_n| = |\ell|$. Does the reversed implication hold?

2.5 Suppose $a_n \ge 0$ and $\lim_{n\to\infty} a_n = a$. Show that $a \ge 0$. Does $a_n > 0 \Longrightarrow a > 0$?

2.6 Suppose $a_n \ge 0$ and $\{a_n\}$ is convergent with limit a. Show that $\{\sqrt{a_n}\}$ is convergent with limit $\sqrt{a}$. *Hint:* Do the case $a = 0$ separately. When $a > 0$, show $\sqrt{a_n} - \sqrt{a} = \frac{a_n - a}{\sqrt{a_n}+\sqrt{a}}$.

2.7 Let $x_n = \sqrt{n+1} - \sqrt{n}$. Show that $\{x_n\}, \{\sqrt{n}\, x_n\}$ are both convergent and find their limits. What can you say about $\{nx_n\}$?

§2.3. Monotone Sequences. The sequence $\{a_n\}$ is *increasing* if

$$a_1 \le a_2 \le a_3 \le \ldots \quad (a_n \le a_{n+1},\ n = 1, 2, \ldots)$$

and it is *decreasing* if

$$a_1 \geq a_2 \geq a_3 \geq \dots \quad (a_n \geq a_{n+1},\ n = 1, 2, \dots)$$

The sequence is *monotone* if it is increasing or decreasing.

PROPOSITION 2.3.1. *Let $\{a_n\}$ be a monotone sequence. Then $\{a_n\}$ is convergent $\iff$ $\{a_n\}$ is bounded.*

PROOF:

" $\Longrightarrow$ " Let $\{a_n\}$ be convergent. Then $\{a_n\}$ is bounded, by Proposition 2.2.5, page 22.

" $\Longleftarrow$ " Suppose $\{a_n\}$ is increasing and bounded. Let $\ell = \sup\{a_n : n = 1, 2, \dots\}$. We will show that $\ell = \lim_{n\to\infty} a_n$. If $\varepsilon > 0$, then $\ell - \varepsilon$ is not an upper bound of $\{a_n : n = 1, 2, \dots\}$, since $\ell - \varepsilon < \ell$. Therefore, there exists an element of the set which is greater then $\ell - \varepsilon$: there exists N such that

$$\ell - \varepsilon < a_N.$$

Therefore $n \geq N$ and $\{a_n\}$ increasing $\Longrightarrow$

$$\ell - \varepsilon < a_N \leq a_n \leq \ell \Longrightarrow |a_n - \ell| < \varepsilon$$

and hence $\lim_{n\to\infty} a_n = \ell$.

The case where $\{a_n\}$ is decreasing is similar.

$\square$

EXAMPLE 2.3.2: The sequence $\{c^n\}$ is convergent with limit 0, if $0 \leq c < 1$, convergent with limit 1, if $c = 1$ divergent, if $c > 1$. In fact it is unbounded if $c > 1$.

$\underline{0 \le c < 1:}$ In this case $0 \le c^{n+1} = cc^n < c^n$, so $\{c^n\}$ is a decreasing bounded sequence. Therefore

$$\ell = \lim_{n\to\infty} c^n \quad \text{exists and} \quad 0 \le \ell < 1 \quad \text{(why?)}$$

$\{c^{n+1}\}$ is a subsequence of $\{c^n\}$, so that

$$\ell = \lim_{n\to\infty} c^{n+1}, \quad \text{by Proposition 2.2.11.} \qquad \text{(A)}$$

But $c^{n+1} = cc^n$ implies, by Theorem 2.2.6(b)

$$c\ell = c \lim_{n\to\infty} c^n = \lim_{n\to\infty} c^{n+1}. \qquad \text{(B)}$$

$$\begin{aligned}\text{(A),(B)} \implies \ell = c\ell &\implies (1-c)\ell = 0 \\ &\implies \ell = 0, \quad \text{since } c \ne 1.\end{aligned}$$

$\underline{c = 1:}$ Here $\lim_{n\to\infty} c^n = \lim_{n\to\infty} 1 = 1$ (Example 2.2.2).

$\underline{c > 1:}$ In this case $c^{n+1} > c^n$, so that $\{c^n\}$ is increasing. If $\{c^n\}$ is convergent, then $1 < \ell = \lim_{n\to\infty} c^n$ and the same argument as before shows $(1-c)\ell = 0$. But $c \ne 1, \ell \ne 0$ gives a contradiciton so $\{c^n\}$ is divergent. Therefore, $\{c^n\}$ is unbounded, by Proposition 2.2.13, since it is monotone and divergent.

We give an alternative approach to this example based on the Squeeze Principle (Problem 2.2, page 30) and Bernoulli's Inequality (Example 1.2.3, page 3). The cases $c = 0$, $c = 1$ are easy.

$\underline{0 < c < 1:}$ Then $\frac{1}{c} > 1$, so $\frac{1}{c} = 1 + a$ where $a > 0$ and $c = \frac{1}{1+a}$.

Therefore $0 < c^n = \frac{1}{(1+a)^n} < \frac{1}{1+na}$ (Bernoulli's Inequality) and, since $\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{1}{1+na} = 0$, we have $\lim_{n\to\infty} c^n = 0$.

$\underline{c > 1:}$ Here $c = 1 + a$, where $a > 0$.

Thus $c^n = (1+a)^n > 1 + na$, by Bernoulli's Inequality, and this implies $\{c^n\}$ is unbounded and therefore divergent by Proposition 2.2.5.

EXAMPLE 2.3.3: $\lim_{n\to\infty} c^{1/n} = 1$, if $c > 0$.

$\underline{c > 1:}$ Let $x_n = c^{1/n}$. We will show $1 \le x_{n+1} \le x_n$ i.e. x_n is decreasing and bounded below.

$$(x_{n+1})^{n+1} = (c^{\frac{1}{n+1}})^{n+1} = c \ge 1$$
$$(x_n)^{n+1} = (c^{\frac{1}{n}})^{n+1} = c^{1+\frac{1}{n}} = cc^{\frac{1}{n}} \ge c$$

since $c \ge 1 \Longrightarrow c^{1/n} \ge 1$ (Problem 1.29, page 20).

Thus $1 \le (x_{n+1})^{n+1} \le x_n^{n+1}$ and therefore $1 \le x_{n+1} \le x_n$ (Problem 1.29, page 20). Hence $1 \le \ell = \lim_{n\to\infty} c^{1/n}$ exists. Also $\{c^{\frac{1}{2n}}\}$ is a subsequence of $\{c^{1/n}\}$ so that

$$\ell = \lim_{n\to\infty} c^{\frac{1}{2n}}.$$

But $c^{\frac{1}{2n}} = \sqrt{c^{1/n}}$ so Problem 2.6, page 30, gives

$$\sqrt{\ell} = \lim_{n\to\infty} c^{\frac{1}{2n}}.$$

Thus $1 \le \ell = \sqrt{\ell}$ which implies $\ell = 1$.

$\underline{0 < c < 1:}$ Here $c = k^{-1}$, where $k > 1$, so that

$$c^{1/n} = k^{-1/n} = 1/k^{1/n}.$$

The first part of this example and Theorem 2.2.7(c), page 25, give $\lim_{n\to\infty} c^{1/n} = 1$ in this case also.

EXAMPLE 2.3.4: (The number e)

$$e \stackrel{\text{def}}{=} \lim_{n\to\infty} (1 + \frac{1}{n})^n \quad \text{exists and} \quad 2 < e \leq 3.$$

PROOF: Let $x_n = (1 + \frac{1}{n})^n$.

STEP 1: $\{x_n\}$ is *bounded*, $2 < x_n < 3$.

The Binomial Theorem gives

$$\begin{aligned} x_n &= (1 + \frac{1}{n})^n \\ &= 1 + n(\frac{1}{n}) + \frac{n(n-1)}{1\cdot 2}(\frac{1}{n})^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(\frac{1}{n})^3 + \ldots \\ &\quad + \frac{n(n-1)\ldots 3\cdot 2\cdot 1}{1\cdot 2\cdot\ldots n}(\frac{1}{n})^n. \end{aligned}$$

Therefore, since $0 < n(n-1)\ldots(n-k+1)(\frac{1}{n})^k < 1$, if $1 < k \leq n$, we have

$$\begin{aligned} 2 < x_n &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 1 + 2(1 - \frac{1}{2^n}) < 3. \end{aligned}$$

STEP 2: $\{x_n\}$ is *increasing*.

From Step 1,

$$x_n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \dots$$
$$\dots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n})\dots(1 - \frac{n-1}{n})$$

$$x_{n+1} = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n+1}) + \frac{1}{3!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1}) + \dots$$
$$\dots + \frac{1}{(n+1)!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1})\dots(1 - \frac{n}{n+1}).$$

Starting from the left, each term in the expression for x_n is less than or equal to the corresponding term for x_{n+1} and the expression for x_{n+1} contains one more positive term. Thus $x_n < x_{n+1}$.

Steps 1 and 2 give the result asserted.

§2.4. The Bolzano-Weierstrass Theorem. We saw in Proposition 2.2.5 that a convergent sequence is necessarily bounded. While a bounded sequence need not itself be convergent the proposition has the following important partial converse.

THEOREM 2.4.1 (BOLZANO-WEIERSTRASS THEOREM). *A bounded sequence has a convergent subsequence.*

PROOF I: This proof shows that every real sequence $\{a_n\}$ has a monotone subsequence. This must be bounded and therefore convergent if $\{a_n\}$ is bounded. Consider the subsequences

$$\boxed{1} \quad \{a_1, a_2, a_3, \dots\}$$

$$\boxed{2} \quad \{a_2, a_3, a_4, \dots\}$$

$$\boxed{3} \quad \{a_3, a_4, a_5, \dots\}$$

$$\dots\dots$$

$$\boxed{N} \quad \{a_N, a_{N+1}, a_{N+2}, \dots\}$$

$$\dots\dots.$$

Notice that, if $M > N$, then $\boxed{M}$ is a subsequence of $\boxed{N}$.

There are two possibilities. Either

(i) Each sequence $\boxed{N}$ has a largest element or,

(ii) There is a sequence $\boxed{N}$ which has no largest element.

In case (i), let a_{n_1} be a largest element of $\boxed{1}$

Let a_{n_2} be a largest element of $\boxed{n_1 + 1}$

$\dots\dots$

Let a_{n_k} be a largest element of $\boxed{n_{k-1} + 1}$

Then $n_1 < n_2 < n_3 < \dots$ and

$$a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq \dots$$

so that $\{a_{n_k}\}$ is a decreasing subsequence of $\{a_n\}$.

In case (ii) some subsequence $\boxed{N}$ contains no largest element. Let $n_1 = N$. There exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$ and, inductively, $n_k > n_{k-1}$ such that $a_{n_k} > a_{n_{k-1}}$. Thus $\{a_{n_k}\}$ is an increasing subsequence of $\{a_n\}$.

□

PROOF II: Let $\{a_n\}$ be a bounded sequence. There exist $A, B \in \mathbb{R}$ such that

$$a_n \in [A, B], \quad n = 1, 2, \ldots .$$

At least one of the intervals $[A, \frac{A+B}{2}], [\frac{A+B}{2}, B]$ contains infinitely many terms a_n of the sequence. Denote such an interval $[A_1, B_1]$. At least one of $[A_1, \frac{A_1+B_1}{2}], [\frac{A_1+B_1}{2}, B_1]$ contains infinitely many a_n; call it $[A_2, B_2]$. Proceeding in this way, we obtain intervals $[A_n, B_n]$, each of which contains infinitely many terms from the sequence $\{a_n\}$. Moreover

$$[A_{n+1}, B_{n+1}] \subset [A_n, B_n], \quad B_n = A_n + \frac{B-A}{2^n}, \quad n = 1, 2, \ldots .$$

Thus $\{A_n\}$ is increasing and bounded and $\{B_n\}$ is decreasing and bounded. Since $\lim_{n\to\infty} \frac{1}{2^n} = 0$ (Example 2.3.2, page 31), both sequences have the same limit

$$\ell = \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n.$$

The construction of $[A_n, B_n]$ shows that there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$A_k \le a_{n_k} \le B_k, \quad k = 1, 2, \ldots .$$

Thus $\ell = \lim_{k\to\infty} a_{n_k}$, by the Squeeze Principle.

$\square$

§2.5. Cauchy's Criterion for Convergence. An interesting aspect of monotone sequences is that they allow us the opportunity of detemining whether or not they are convergent without the necessity of initially guessing likely candidates as limits and using the definition of convergence directly. This, for example, allowed the definition of e as the limit of a sequence.

Fortunately this type of discussion may be extended to sequences in general. Recall that a sequence $\{a_n\}$ is convergent if we can find a number ℓ such that a_n is as close as we please to ℓ provided we take n large enough. The Cauchy Criterion avoids the necessity of finding ℓ explicitly and says that $\{a_n\}$ is convergent if and only if the terms in the sequence are as close together as we please provided we omit a finite number of terms.

DEFINITION 2.5.1: The sequence $\{a_n\}$ is a *Cauchy Sequence* if, for each $\varepsilon > 0$, there exists N such that

$$m, n \geq N \Longrightarrow |a_n - a_m| < \varepsilon.$$

THEOREM 2.5.2 (CAUCHY CRITERION). *$\{a_n\}$ is convergent $\Longleftrightarrow$ $\{a_n\}$ is a Cauchy sequence.*

EXAMPLE 2.5.3: If $x_n = (-1)^n$, then $\{x_n\}$ is not a Cauchy sequence since $|x_{n+1} - x_n| = 2$, for all n.

EXAMPLE 2.5.4: If $\{a_n\} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then $\{a_n\}$ is not a Cauchy sequence and so it is divergent. To soo this, consider

$$\begin{aligned}|a_{2n} - a_n| &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \\ &> n(\frac{1}{2n}) = \frac{1}{2}\,, \quad \text{for all } n.\end{aligned}$$

Thus we conclude that the sequence $\{1+\frac{1}{2}+\cdots+\frac{1}{n}\}$ is divergent. In fact we can deduce that the sequence is unbounded since it is increasing and therefore divergent if and only if it is unbounded.

PROOF OF THE CAUCHY CRITERION:

" $\Longrightarrow$ " Suppose $\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = \ell$. If $\varepsilon > 0$, there exists N such that

$$n \geq N \Longrightarrow |a_n - \ell| < \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned} n, m \geq N \Longrightarrow |a_n - a_m| &\leq |a_n - \ell| + |a_m - \ell| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore $\{a_n\}$ is a Cauchy Sequence.

" $\Longleftarrow$ " STEP 1: A Cauchy Sequence is bounded.

PROOF: Let $\{a_n\}$ be a Cauchy Sequence. There exists N such that $m, n \geq N \Longrightarrow |a_n - a_m| < 1$. Therefore

$$\begin{aligned} n \geq N &\Longrightarrow |a_n - a_n| \leq 1 \\ &\Longrightarrow |a_n| - |a_N| \leq |a_n - a_n| \leq 1 \\ &\Longrightarrow |a_n| \leq |a_N| + 1 \end{aligned}$$

and $|a_n| \leq K = \max\{|a_1|, \ldots, |a_{N-1}|, |a_N| + 1\}$ $k = 1, 2, 3, \ldots$.

STEP 2: A Cauchy Sequence has a convergent subsequence.

This follows from Step 1 and the Bolzano-Weierstrass Theorem (page 35).

STEP 3: If $\{a_n\}$ is a Cauchy Sequence and $\lim_{k\to\infty} a_{n_k} = \ell$, then $\lim_{n\to\infty} a_n = \ell$.

Let $\varepsilon > 0$. There exists K such that

$$k \geq K \Longrightarrow |a_{n_k} - \ell| < \frac{\varepsilon}{2} \quad (\lim_{k\to\infty} a_{n_k} = \ell).$$

There exists N such that

$$m, n \geq N \Longrightarrow |a_m - a_n| < \frac{\varepsilon}{2} \quad (\{a_n\} \text{ is Cauchy}).$$

Choose a number $k \geq K$ such that

$$n_k \geq N.$$

Then

$$\begin{aligned} n \geq N \Longrightarrow |a_n - \ell| &\leq |a_n - a_{n_k}| + |a_{n_k} - \ell| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore $\lim_{n\to\infty} a_n = \ell$. Combining Steps 2 and 3, we find $\{a_n\}$ is a Cauchy Sequence $\Longrightarrow \{a_n\}$ is convergent.

□

Problems

2.8 Suppose the sequence $\{a_n\}$ is defined inductively by

$$a_1 = 0, \quad a_{n+1} = \sqrt{3 + 2a_n}, \quad n = 1, 2, 3, \ldots.$$

Show that $\lim_{n\to\infty} a_n = 3$ by showing that the sequence is increasing and bounded and hence convergent.

2.9 Let $s_n = 1 + r + r^2 + \ldots r^n$. Show that the sequence $\{s_n\}$ is convergent if $|r| < 1$ and divergent if $|r| \geq 1$. What is its limit when $|r| < 1$?

2.10 Let $a_1 = 0, a_2 = 1, a_{n+2} = \frac{1}{2}(a_n + a_{n+1}), n = 1, 2, \ldots$. Prove that $\lim_{n\to\infty} a_n = \frac{2}{3}$.

2.11 If $0 \leq a \leq b$, prove $\lim_{n\to\infty}(a^n + b^n)^{1/n} = b$.

2.12 Show $\lim_{n\to\infty} n^{1/n} = 1$ by considering the sequence $\{x_n\}$, where $x_n = n^{1/n} - 1 > 0$. Show that $n = (1 + x_n)^n > \frac{n(n-1)}{2} x_n^2$ and deduce from this that $\lim_{n\to\infty} x_n = 0$.

2.13 Show $\lim_{n\to\infty} n^{1/n} = 1$ by proving that the sequence is eventually monotone and that it is bounded. *Hint:* Consider also the subsequence $\{(2n)^{\frac{1}{2n}}\}$.

2.14 (a) Give a definition of

$$\lim_{n\to\infty} a_n = \infty \qquad \text{(infinite limit)}.$$

(b) Suppose $\lim_{n\to\infty} a_n = \infty, \lim_{n\to\infty} b_n = \infty$. Use your definition to prove

$$\lim_{n\to\infty} (a_n + b_n) = \infty, \quad \lim_{n\to\infty} a_n b_n = \infty.$$

(c) Show by examples that, with a_n, b_n as in (b),

$$\lim_{n\to\infty} (a_n - b_n), \quad \lim_{n\to\infty} \frac{a_n}{b_n}$$

may exist as a real number, as an infinite limit or may fail to exist.

(d) Show $\lim_{n\to\infty} a_n = \infty \Longrightarrow \lim_{n\to\infty} \frac{1}{a_n} = 0 \quad (a_n \neq 0)$.

(e) Show $\lim_{n\to\infty} \frac{1}{a_n} = 0 \Longrightarrow \lim_{n\to\infty} a_n = \infty$, if $a_n > 0$.

2.15 Suppose $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is a bounded sequence. Show $\lim_{n\to\infty} a_n b_n = 0$.

2.16 Let $a_n \in [A, B]$, $n = 1, 2, \dots$ and $\lim_{n\to\infty} a_n = \ell$. show $\ell \in [A, B]$.

Hint: Consider the sequences $\{a_n - A\}, \{B - a_n\}$. Use # 2.5.

2.17 Suppose $\lim_{n\to\infty} x_n = a$. Show

$$\lim_{n\to\infty} x_n^{1/3} = a^{1/3}.$$

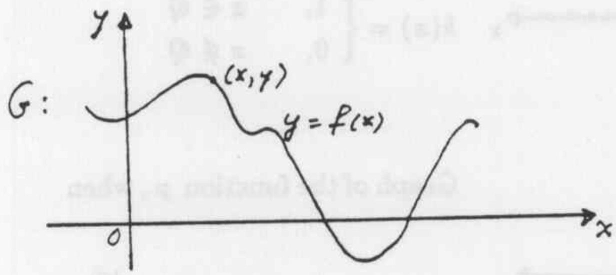
III. LIMITS AND CONTINUITY

§3.1. Graphs We denote the Cartesian coordinate plane by $\mathbb{R}^2$. Thus

$$\mathbb{R}^2 = \{(x,y) : \quad x \in \mathbb{R}, \quad y \in \mathbb{R}\}.$$

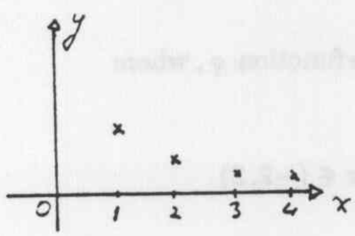
If f is a real-valued function of domain $D \subset \mathbb{R}$, then its *graph* G is the subset of $\mathbb{R}^2$ given by

$$\begin{aligned} G &= \{(x, f(x)) : \quad x \in D\} \\ &= \{(x,y) : x \in D,\ y = f(x)\}. \end{aligned}$$



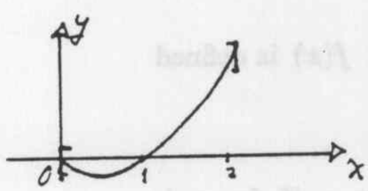
EXAMPLES 3.1.1:

(a)



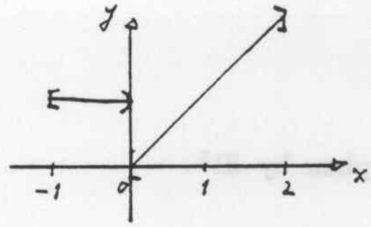
Part of the graph of the sequence $\left\{\frac{1}{n}\right\}$, i.e. $f(x) = \frac{1}{x}, \quad x = 1, 2, 3, \ldots$.

(b)



Graph of the function f is given by $f(x) = x^2 - x,\ x \in [0,2]$.

(c)

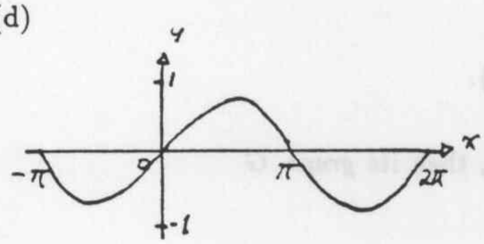


Graph of the function

g , where

$$g(x) = \begin{cases} 1, & x \in [-1, 0) \\ x, & x \in [0, 2] \end{cases}$$

(d)

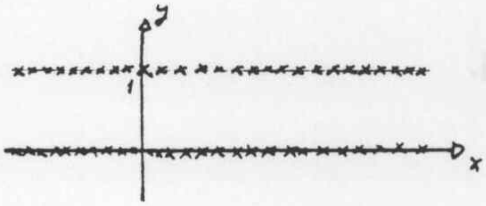


Part of the graph of

the function h , where

$h(x) = \sin x, \quad x \in \mathbb{R}$

(e)

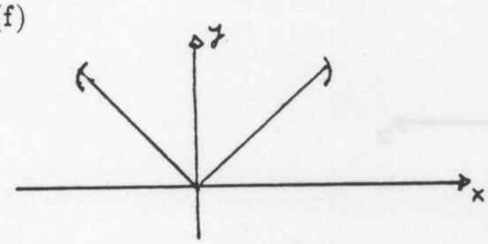


Part of the graph of

the function k , where

$$k(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

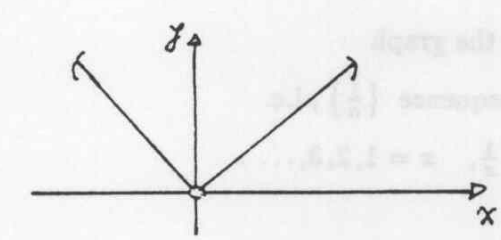
(f)



Graph of the function p , when

$p(x) = |x|, \quad x \in (-2, 2)$

(g)



Graph of the function q , where

$q(x) = |x|, \ x \in (-2, 2)$

$x \neq 0$.

We now consider a real-valued function f such that $f(x)$ is defined for all x near a but not necessarily at $x = a$.

DEFINITION 3.1.2: The function f has limit ℓ at a if, for each

$\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow |f(x) - \ell| < \varepsilon.$$

Note: δ is the Greek letter 'delta'.

REMARKS: (i) The statement 'f has limit ℓ at a' is abbreviated to

$$\lim_{x \to a} f(x) = \ell \quad \text{or} \quad \lim_{a} f = \ell.$$

(ii) f need not be defined at a and, even if it is defined, the value of $f(a)$ does not effect the existence or value of $\lim\limits_{x \to a} f(x)$.

(iii) f must however, be defined for all x close to a, $x \neq a$. Thus, for example, if f is the function in Example 3.1.1(b), then none of

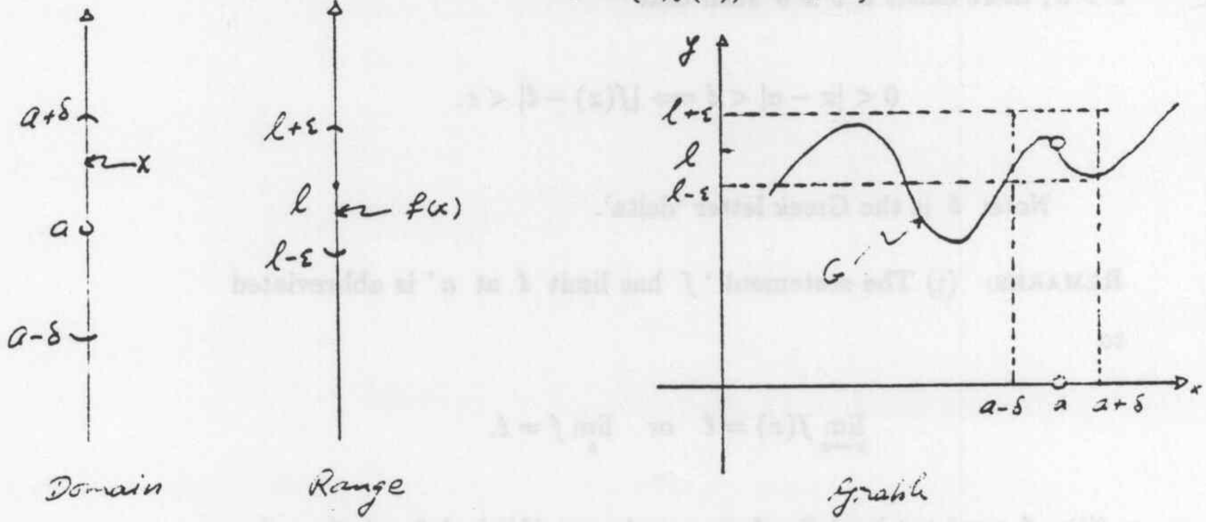
$$\lim_{0} f, \quad \lim_{2} f, \quad \lim_{3} f$$

exists, since f is not defined for all x near $0, 1$ and 3. We will see, in fact that $\lim\limits_{a} f = a^2 - a$, if $a \in (0,2)$, and $\lim\limits_{a} f$ does not exist, if $a \notin (0,2)$.

(iv) $\lim\limits_{a} f = \ell$ means that the distance $|f(x) - \ell|$ between $f(x)$ and ℓ is as small as we please provided x is close enough to but different from a.

(v) It is worthwhile taking some time now to consider the Examples 3.1.1 and discuss the limits of the functions there at various points and to identify points where the limit fails to exist.

$$G = \{x, f(x)) : x \in D\}$$



EXAMPLE 3.1.3: Let $f(x) = 3x + 5$, $x \in \mathbb{R}$. We will show

$$\lim_{1} f = 8 \quad \text{(equivalent: } \lim_{x \to 1} \ 3x + 5 = 8\text{)}.$$

Let $\varepsilon > 0$. We must show that we can find $\delta > 0$ such that

$$0 < |x - 1| < \delta \Longrightarrow |f(x) - 8| < \varepsilon.$$

Now $\quad |f(x) - 8| = |3x + 5 - 8| = |3x - 3| = 3|x - 1|$. $\qquad$ (A)

Choose $\quad \delta = \varepsilon/3$. Then, from (A)

$$0 < |x - 1| < \delta = \frac{\varepsilon}{3} \Longrightarrow |f(x) - 8| < 3 \ \frac{\varepsilon}{3} = 5.$$

Notice that, if $h(x) = \begin{cases} 3x + 5, & x \neq 1, \\ 7, & x = 1 \end{cases}$, then $\quad \lim_{1} h = 8$,

without any change in the argument of Example 3.1.3. The value of $h(1)$ is irrelevant as is the fact that 1 is in the domain of h. The expression $\lim_{1} h$ describes the behaviour of h *near* 1 but not *at* 1.

EXAMPLE 3.1.4: Let $p(x) = x^2$, $x \in \mathbb{R}$. Then $\lim_{-2} p = 4$.

Let $\varepsilon > 0$. Now, since

$$|p(x) - 4| = |x^2 - 4| = |x - 2|\,|x + 2| = |x + 2 - 4|\,|x + 2|$$
$$\leq (|x + 2| + 4)|x + 2| \qquad \text{(triangle inequality)}$$

we find

$$|x + 2| < 1 \Longrightarrow |p(x) - 4| < 5|x + 2|.$$

Therefore

$$|x + 2| < \delta = \min\left\{1,\ \frac{\varepsilon}{5}\right\} \Longrightarrow |p(x) - 4| < 5\ \frac{\varepsilon}{5} = \varepsilon$$

so that

$$\lim_{x \to -2} p(x) = 4.$$

EXAMPLE 3.1.5: Let $g(x) = c$, $x \in \mathbb{R}$. Then

$$\lim_{x \to a} g(x) = c, \quad \text{for each } a \in \mathbb{R}.$$

In this case any $\delta > 0$ works for every $\varepsilon > 0$, since

$$0 < |x - a| < \delta \Longrightarrow |g(x) - c| = |c - c| = 0 < \varepsilon.$$

EXAMPLE 3.1.6: Let $F(x) = \frac{1}{x}$, $x \neq 0$. Then

$$\lim_{3} F = \frac{1}{3}.$$

Let $\varepsilon > 0$. Our task is to produce a $\delta > 0$ such that $0 < |x - 3| < \delta \Longrightarrow |F(x) - \frac{1}{3}| < \varepsilon$.

Now

$$|F(x) - \frac{1}{3}| = |\frac{1}{x} - \frac{1}{3}| = \left|\frac{3-x}{3x}\right|, \quad \text{if } x \neq 0$$

$$= \frac{|x-3|}{3|x|}. \qquad \text{(B)}$$

The term $|x|$ in the denominator can make this expression very large if x is near zero. But

$$\begin{aligned}
|x-3| < 2 &\Longrightarrow 3 - |x| \leq |x-3| < 2 \quad \text{(triangle inequality)} \\
&\Longrightarrow 1 < |x| \\
&\Longrightarrow \left|F(x) - \frac{1}{3}\right| < \frac{|x-3|}{3}, \quad \text{from (B)}.
\end{aligned}$$

Hence

$$0 < |x-3| < \delta = \min\{2,\ 3\varepsilon\} \Longrightarrow \left|F(x) - \frac{1}{3}\right| < \varepsilon$$

$$\therefore \quad \lim_{x \to 3} F(x) = \frac{1}{3}.$$

EXAMPLE 3.1.7: Let $f(x) = \begin{cases} 1, & x < 0 \\ -1, & x > 0. \end{cases}$ Then $\lim\limits_{x \to 0} f(x)$ does not exist. To see this suppose $\lim\limits_{x \to 0} f(x) = \ell$. Let $\varepsilon = 1$. There exists $\delta > 0$ such that $0 < |x - 0| = |x| < \delta \Longrightarrow |f(x) - \ell| < 1$. Hence

$$-\delta < x < 0 \Longrightarrow |1 - \ell| < 1 \Longrightarrow 1 - \ell \leq |1 - \ell| < 1 \Longrightarrow \ell > 0$$

$$0 < x < \delta \Longrightarrow |-1 - \ell| < 1 \Longrightarrow 1 + \ell \leq |-1 - \ell| < 1 \Longrightarrow \ell < 0.$$

The contradition $\ell > 0$ and $\ell < 0$ shows that ℓ cannot exist.

The following theorem relating limits of general functions and limits of sequences is very useful.

THEOREM 3.1.8. $\lim\limits_a f = \ell$.

$\Leftrightarrow$ *f is defined near a and every sequence* $\{x_n\}$ *in the domain of f such that* $x_n \neq a$, $\lim\limits_{n\to\infty} x_n = a$ *satisfies* $\lim\limits_{n\to\infty} f(x_n) = \ell$.

In Example 3.1.7, we found that $\lim\limits_0 f$ does not exist, when $f(x) = \begin{cases} 1, & x < 0 \\ -1, & x > 0 \end{cases}$. This also follows if we observe that $\left\{-\frac{1}{n}\right\}$, $\left\{\frac{1}{n}\right\}$ both have limit 0 but $\lim\limits_{n\to\infty} f\left(-\frac{1}{n}\right) = 1$, $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = -1$ and, since these limits differ, $\lim\limits_0 f$ does not exist.

EXAMPLE 3.1.9: If $f(x) = \frac{1}{x}$, $x \neq 0$, then $\lim\limits_0 f$ does not exist, since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ but $\left\{f\left(\frac{1}{n}\right)\right\} = \{n\}$ is not convergent (it is unbounded).

If $a \neq 0$, then $\lim\limits_a f = \frac{1}{a}$ i.e. $\lim\limits_{x\to a} \frac{1}{x} = \frac{1}{a}$. This follows from Theorem 3.1.8 and Theorem 2.2.7, page 25 since, if $\{x_n\}$ is any sequence such that $x_n \neq 0$, $x_n \neq a$ and $\lim\limits_{n\to\infty} x_n = a \neq 0$, then $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} \frac{1}{x_n} = \frac{1}{a}$.

We could prove $\lim\limits_{x\to a} \frac{1}{x} = \frac{1}{a}$ directly from the definition using (ε, δ) as we did in the case $a = 3$ in Example 3.1.6. In fact, you should check that each of the Examples 3.1.3 – 3.1.6 may be easily deduced from our results on sequences if we use Theorem 3.1.8.

Using, Theorem 3.1.8 we find that the following are implied by the corresponding results for sequences.

COROLLARY 3.1.10. *Suppose* $\lim\limits_a f = \ell$, $\lim\limits_a g = m$. *Then*

(a) $\lim\limits_a (f+g) = \ell + m$,

(b) $\lim\limits_a fg = \ell m$,

(c) $\lim\limits_a \frac{f}{g} = \frac{\ell}{m}$, *if* $m \neq 0$.

COROLLARY 3.1.11. (CAUCHY CRITERION). $\lim\limits_a f$ *exists* $\Longleftrightarrow$

for each $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that, if* $0 < |x-a| < \delta$ *and* $0 < |y-a| < \delta$, *then* $|f(x) - f(y)| < \varepsilon$.

COROLLARY 3.1.12. *Suppose* $f(x) \le h(x) \le g(x)$ *when* $0 < |x-a| < r$.

Then $\lim_a f = \ell, \quad \lim_a g = \ell \Longrightarrow \lim_a h = \ell$.

Closely related to $\lim_a f$ are the one-sided limits $\lim_{a+} f$ and $\lim_{a-} f$, the limits from the right and left at a.

DEFINITION 3.1.13: (a) $\lim_{a+} f = \ell$ means, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < x - a < \delta \Longrightarrow |f(x) - \ell| < \varepsilon$

(b) $\lim_{a-} f = \ell$ means, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < a - x < \delta \Longrightarrow |f(x) - \ell| < \varepsilon$.

EXAMPLE 3.1.14: If $f(x) = \begin{cases} 1, & x < 0 \\ -1, & x > 0 \end{cases}$, then

$$\lim_{0+} f = -1, \quad \text{and} \quad \lim_{0-} f = 1.$$

The details of the proofs of the corollaries are left as exercises.

PROOF OF THEOREM 3.1.8:

"$\Longrightarrow$" Suppose $\lim_a f = \ell$ and let $\{x_n\}$ be a sequence in the domain of f such that $\lim_{n\to\infty} x_n = a \ (x_n \neq a)$. We wish to prove $\lim_{n\to\infty} f(x_n) = \ell$. If $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \Longrightarrow |f(x) - \ell| < \varepsilon \qquad \text{(A)}$$

$$(\text{since } \lim_a f = \ell).$$

Also there exists N such that

$$n \geq N \Longrightarrow 0 < |x_n - a| < \delta \qquad \text{(B)}$$

$$(\text{since } \lim_{n\to\infty} x_n = a;\ x_n \neq a).$$

From (A), (B), $n \geq N \Longrightarrow |f(x_n) - \ell| < \varepsilon$. Therefore

$$\lim_{n\to\infty} f(x_n) = \ell.$$

" $\Longleftarrow$ " Suppose $\lim_{n\to\infty} f(x_n) = \ell$ for each sequence $\{x_n\}$ in the domain of f with $x_n \neq a$ and $\lim_{n\to\infty} x_n = a$. We wish to show $\lim_a f = \ell$. Suppose that f does not have limit ℓ at a. Then there is some $\varepsilon_0 > 0$ for which no $\delta > 0$ works in Definition 3.1.2. Thus, for each $n = 1, 2, \dots$, there exists x_n, $0 < |x_n - a| < \frac{1}{n}$ and such that $|f(x_n) - \ell| \geq \varepsilon_0 > 0$. Hence $\lim_{n\to\infty} x_n = a$, $x_n \neq a$, but $\{f(x_n)\}$ is not convergent to the limit ℓ contradicting our hypothesis. Therefore $\lim_a f = \ell$.

$\square$

PROBLEMS

3.1 Sketch the graph of each of the functions. The domain is $\mathbb{R}$ unless specified otherwise.

(a) $f(x) = 3x + 2$,

(b) $g(x) = |3x + 2|$,

(c) $h(x) = x|x|$,

(d) $k(x) = \sqrt{4 - x^2}, \quad -2 \leq x \leq 2$,

(e) $\ell(x) = \begin{cases} x, & x > 0 \\ x^2, & x \leq 0 \end{cases}$

(f) $m(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x.

(g) $p(x) = |x - 1| - |x + 1|$

(h) $q(x) = \begin{cases} x, & \text{if } x = \frac{1}{n},\ n = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$

3.2 Let $F(x) = x$. If $\varepsilon > 0$, determine δ so that $0 < |x - a| < \delta \Longrightarrow |F(x) - a| < \varepsilon$.

This proves $\lim\limits_{x \to a} x = a$.

3.3 Give an (ε, δ)-proof that $\lim\limits_{x \to -1} 3x + 2 = -1$.

3.4 Let $f(x) = \begin{cases} 5x, & x < 1 \\ 2x + 3, & x > 1. \end{cases}$. Sketch the graph of this function. If $\varepsilon > 0$, determine δ so that

$$0 < |x - 1| < \delta \Longrightarrow |f(x) - 5| < \varepsilon \quad \text{i.e. prove } \lim_{x \to 1} f(x) = 5.$$

3.5 Let $f(x) = x^2$. Show that $\lim_{x \to -1} f(x) = 1$ in two ways, one using the (ε, δ) definition and the other based on sequences, using Theorem 3.1.8.

3.6 Let $p(x) = \begin{cases} 1, & x = \frac{1}{n}, \quad n = 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases}$. Explain carefully why $\lim_{x \to a} p(x)$ exists for each $a \in \mathbb{R}$ with one exception.

3.7 At what points $a \in \mathbb{R}$ does $\lim_{x \to a} q(x)$ exist, where q is the function in # 3.1(h)?

3.8 Write down the value of each of the limits

$$\lim_{x \to 0+} [x], \quad \lim_{x \to 0-} [x], \quad \lim_{x \to \frac{1}{2}} [x].$$

(See #3.1 (f).)

3.9 Show that $\lim_a f = \ell$ if and only if both $\lim_{a+} f$ and $\lim_{a-} f$ exist and equal ℓ.

3.10 Determine which of the following limits exist:

(a) $\lim_{x \to 3} \dfrac{x^2 - 4}{x - 2}$ (b) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

(c) $\lim_{x \to -2} \dfrac{x^2 - 4}{x - 2}$ (d) $\lim_{x \to 2} \dfrac{x^2 - 3}{x - 2}$

(e) $\lim_{x \to 0+} \dfrac{x}{|x|}$ (f) $\lim_{x \to 0} \dfrac{x}{|x|}$.

3.11 If $P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$, $x \in \mathbb{R}$, where $c_0, c_1, \ldots, c_n$ are constants, $c_n \neq 0$, then P is a *polynomial* of degree n. Given $\lim_{x \to a} c = c$ and $\lim_{x \to a} x = a$ (of Example 3.1.5, Problem 3.2) and Corollary 3.1.10(a), (b) show that

$$\lim_a P = P(a), \qquad \text{if} \quad a \in \mathbb{R}.$$

3.12 If $R(x) = P(x)/Q(x)$, $x \in \mathbb{R}$, $Q(x) \neq 0$, where P, Q are polynomials, then R is a *rational* function. Prove

$$\lim_a R = R(a), \quad \text{if} \quad a \in \mathbb{R}, \quad Q(a) \neq 0.$$

What can you say about $\lim_a R$, if $Q(a) = 0$?

3.13 (a) Give a definition for '$\lim_{x \to \infty} f(x) = \ell$'.

(b) Show, using your definition, that if $\lim_{x \to \infty} f(x) = \ell$ and $\ell > 0$, then there is a number B such that $x > B \Longrightarrow f(x) > 0$.

3.14 Show $\lim_{x \to a} f(x) = \ell \Longleftrightarrow \lim_{h \to 0} f(a+h) = \ell$.

§3.2. Continuity.

Let $D \subset \mathbb{R}$. A point a is an *interior point* of D if there is an open interval I such that $a \in I \subset D$. Thus, for example, $\frac{1}{2}$, $\frac{1}{3}$ are both interior points of $[0,1]$ while $0, 1$ are elements of the set but not interior points. The set $\mathbb{N}$ has no interior points.

We have seen that if a is an interior point of the domain of f then $\lim_a f$ might not exist, or it might exist and differ from $f(a)$. However, for 'nice' functions, $\lim_a f = f(a)$.

DEFINITION 3.2.1: A function f is *continuous* at an interior point a of its domain if

$$\lim_a f = f(a).$$

REMARK: There are two ways in which a function could fail to be continuous at an interior point of its domain:

(i) $\lim_a f$ might fail to exist as in the case $a = 0$ for the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

This function is not continuous at 0 since $\lim_0 f$ does not exist. It is continuous at all other points.

(ii) $\lim_a f$ might exist but be different from $f(a)$ as in the case

$$f(x) = \begin{cases} 1, & x \neq 0; \\ 0, & x = 0 \end{cases};$$

here $\lim_0 f = 1 \neq f(0)$ so f is not continuous at 0.

PROPOSITION 3.2.2. *f is continuous at a $\iff$ for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies*

$$|f(x) - f(a)| < \varepsilon.$$

[Notice that we have omitted '$0 < |x-a|$'. Why?]

PROPOSITION 3.2.3. *f is continuous at $a \iff$*

$$\lim_{h \to 0} f(a+h) = f(a).$$

PROOF: See Problem 3.14.

PROPOSITION 3.2.4. *f is continuous at an interior point a in the domain D of $f \iff$ each sequence $\{x_n\}$ such that $x_n \in D$ and $\lim_{n\to\infty} x_n = a$ satisfies*

$$\lim_{n\to\infty} f(x_n) = f(a).$$

PROOF: See Theorem 3.1.8.

THEOREM 3.2.5. *Suppose f, g are continuous at a. Then $f+g$, fg are continuous at a and f/g is continuous at a if $g(a) \neq 0$.*

COROLLARY 3.2.6. *A rational function is continuous at every point in its domain [cf. Problem 3.12, page 54].*

PROOF: Theorem 3.2.5 follows form Corollary 3.1.10, page 49 with $\ell = f(a)$, $m = g(a)$.

$\square$

Let f, g be functions. The *composition* $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x))$$

for each x in the domain of g such that $g(x)$ is in the domain of f.

THEOREM 3.2.7. *Suppose g is continuous at a and f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.*

PROOF: Let $\{x_n\}$ be a sequence in the domain of $f \circ g$ such that $\lim_{n\to\infty} x_n = a$. Then $\{g(x_n)\}$ is a sequence in the domain of f such that $\lim_{n\to\infty} g(x_n) = g(a)$ (since g is continuous at a) and $\lim_{n\to\infty} f(g(x_n)) = f(g(a))$ (since f is continuous at $g(a)$). Here we have used Proposition 3.2.4 twice. □

EXAMPLE 3.2.8: The functions f given in (a), (b), (c), (d) are continuous on their domains

(a) $f(x) = x, \quad x \in \mathbb{R}$ (b) $f(x) = x^2 - 1, \quad x \in \mathbb{R}$

(c) $f(x) = \frac{1}{x}, \quad x \neq 0$ (d) $f(x) = \frac{1}{x^2-1}$.

The function f given by $f(x) = \sqrt{x}, \quad x \geq 0$ is continuous at each $a \in (0, \infty)$. Our discussion does not yet deal with the point 0 for this function since 0 is not an interior point.

DEFINITION 3.2.9: (a) f is *continuous from the right at a* if

$$\lim_{a+} f = f(a).$$

(b) f is *continuous from the left at a* if

$$\lim_{a-} f = f(a).$$

EXAMPLE 3.2.10: (a) If $f(x) = \sqrt{x}, \quad x \geq 0$, f is continuous from the right at every point in its domain including 0.

(b) The function $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ (cf. page 55) is continuous from the right at all points in its domain including 0. It is continuous from the left at all points except 0:

$$\lim_{0+} f = 1 = f(0), \qquad \lim_{0-} f = 0.$$

(c) The function $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$ (cf. page 55) is not continuous from the left or from the right at 0 since

$$\lim_{0+} f = \lim_{0-} f = 1 \neq f(0).$$

PROPOSITION 3.2.11. *A function is continuous at an interior point a of its domain if and only if it is continuous from the left and right at a.*

PROOF: Exercise

§3.3. Continuity on a Closed Interval. The function f is said to be continuous on $[a, b]$ if f is continuous at each point in (a, b), continuous from the right at a and continuous from the left at b.

PROPOSITION 3.3.1. *f is continuous on $[a, b]$ if and only if $\lim_{n\to\infty} f(x_n) = f(c)$ for each sequence $\{x_n\}$ such that $x_n \in [a, b]$ and $\lim_{n\to\infty} x_n = c$.*

The proof of Theorem 3.1.8, extended to one-sided limits establishes Proposition 3.3.1.

LEMMA 3.3.2. *Suppose f is continuous at* a *and $f(a) > 0$ (< 0) Then there is a $\delta > 0$ such that*

$$x \in (a - \delta,\, a + \delta) \Longrightarrow f(x) > 0 \ (< 0).$$

Note: If we replace 'continuous' by 'continuous from the right' or 'continuous from the left' we must replace '$(a-\delta,\ a+\delta)$' by '$[a,\ a+\delta)$' or '$(a-\delta,\ a]$', respectively.

PROOF: There exists $\delta > 0$ such that

$$
\begin{aligned}
&|x-a| < \delta \Longrightarrow |f(x)-f(a)| < f(a) \qquad (\varepsilon = f(a)) \\
&\Longrightarrow \quad f(a) - f(x) \le |f(x)-f(a)| < f(a) \\
&\Longrightarrow \quad 0 < f(x).
\end{aligned}
$$

$\square$

THEOREM 3.3.3. (BOLZANO'S INTERMEDIATE VALUE THEOREM).

Suppose

(i) f *is continuous on* $[a,b]$.

(ii) $f(a) < 0 < f(b)$. *(or* $f(a) > 0 > f(b)$*)*

Then $f(c) = 0$ *for some* $c \in (a,b)$.

PROOF I:

Consider $f\left(\frac{a+b}{2}\right)$.

If $f\left(\frac{a+b}{2}\right) = 0$, then we may take $c = \frac{a+b}{2}$.

If $f\left(\frac{a+b}{2}\right) < 0$, let $a_1 = \frac{a+b}{2}$, $b_1 = b$.

If $f\left(\frac{a+b}{2}\right) > 0$, let $a_1 = a$, $b_1 = \frac{a+b}{2}$.

Thus we have either found c or $[a_1, b_1] \subset [a,b]$ such that $f(a_1) < 0 < f(b_1)$, $b_1 = a_1 + \frac{b-a}{2}$. Next consider $f\left(\frac{a_1+b_1}{2}\right)$ and continue. We either find c in a finite number of steps or an increasing sequence $\{a_n\}$ and a decreasing sequence $\{b_n\}$ in $[a,b]$ such that $f(a_n) < 0 < f(b_n)$, $b_n = a_n + \frac{b-a}{2^n}$. Thus $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = c \in (a,b)$.

Since f is continuous at c

$$\lim_{n\to\infty} f(a_n) = f(c) \Longrightarrow f(c) \le 0$$

$$\lim_{n\to\infty} f(b_n) = f(c) \Longrightarrow f(c) \ge 0 \quad \text{(Problem 2.5, page 30)}$$

Therefore $f(c) = 0$.

PROOF II:

Let $c = \sup\{x : a \le x \le b, \quad f(x) < 0\}$

[The set is not empty (Why?) and is bounded (Why?)]. There are three possibilities: $f(c) < 0,\ f(c) > 0,\ f(c) = 0$. Suppose $f(c) < 0$. Then $c \in [a, b)$ (Why?) and, since f is continuous from the right at c, there exists $\delta > 0$ such that

$$x \in [c,\ c+\delta) \Longrightarrow f(x) < 0 \quad \text{(Lemma 3.3.2)}$$

which contradicts the definition of c. Thus $f(c) \ge 0$. Suppose $f(c) > 0$. Then $c \in (a, b]$ and, since f is continuous from the left at c, there exists $\delta > 0$ such that

$$x \in (c-\delta,\ c] \Longrightarrow f(x) > 0 \qquad \text{(Lemma 3.3.2)}$$

again contradicting the definition of c. Therefore $f(c) = 0$.

□

THEOREM 3.3.4. (WEIERSTRASS' MAXIMUM-MINIMUM THEOREM).

Suppose f is continuous on $[a, b]$. Then f has both a maximum and a minimum value on $[a, b]$. This means that there exists

$c \in [a, b]$ *such that*

$$f(x) \leq f(c), \quad \textit{for all } x \in [a, b]$$

and there exists $d \in [a, b]$ *such that*

$$f(x) \geq f(d), \quad \textit{for all} \quad x \in [a, b].$$

PROOF:

STEP 1: f is bounded on $[a, b]$. This means there exists M such that

$$|f(x)| \leq M, \quad \text{for all} \quad x \in [a, b].$$

We will prove this by contradiction. Suppose f is not bounded on $[a, b]$ i.e. no such number M exists. Then, for $n = 1, 2, \ldots$, there exists $x_n \in [a, b]$ such that

$$|f(x_n)| \geq n. \tag{A}$$

The sequence $\{x_n\}$ is bounded $(x_n \in [a, b])$ and therefore has a convergent subsequence $\{x_{n_k}\}$ (Bolzano-Weierstrass Theorem, page 35) and, if $c = \lim_{k \to \infty} x_{n_k}$, then $c \in [a, b]$ (Problem 2.16, page 42). Hence, since f is continuous at c,

$$\lim_{k \to \infty} f(x_{n_k}) = f(c),$$

which contradicts (A). Thus f is bounded on $[a, b]$.

STEP 2: f has a maximum value in $[a, b]$.

To see this, let $M = \sup\{f(x) : x \in [a, b]\}$, which exists by Step 1.

Now, for $n = 1, 2, \dots$, there exists $x_n \in [a, b]$ such that

$$M - \frac{1}{n} < f(x_n) \le M$$

and hence

$$\lim_{x \to \infty} f(x_n) = M. \qquad \text{(B)}$$

But, as in Step 1, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ such that $c \in [a, b]$ if $c = \lim_{k \to \infty} x_{n_k}$. Again, the continuity of f at c implies

$$\lim_{k \to \infty} f(x_{n_k}) = f(c). \qquad \text{(C)}$$

Now (B), (C) $\implies f(c) = M \ge f(x),\ a \le x \le b$.

It may be proved similarly that f has a minimum value in $[a, b]$.

$\square$

These two theroems have many important corollaries. We list a few.

COROLLARY 3.3.5. *If f is continuous on $[a, b]$ and $f(a) < y < f(b)$ (or $f(a) > y > f(b)$), then there exists $c \in (a, b)$ such that $f(c) = y$.*

PROOF: This follows by replacing $f(x)$ by $f(x) - y,\ a \le x \le b$ in Theorem 3.3.3.

$\square$

COROLLARY 3.3.6. *If f is continuous on $[a, b]$, then $\{f(x) : x \in [a, b]\}$ is a closed interval or a point.*

PROOF: By Theorem 3.3.4, f achieves its maximum and minimum values in $[a, b]$ at some points $c,\ d \in [a, b]$ and every value in between these.

$\square$

COROLLARY 3.3.7. *If $a > 0$ and $n \in \mathbb{N}$, there is a unique number $x > 0$ such that $x^n = a$. This number is denoted $a^{\frac{1}{n}}$.*

PROOF: Consider the function f

$$f(x) = x^n - a, \quad x \in [0, \infty).$$

This function is continuous on its domain (Problem 3.11, page 54). Now $f(0) = -a < 0$, and

$$f(x) = x^n\Big(1 - \frac{a}{x^n}\Big) \geq x^n\Big(1 - \frac{a}{x}\Big), \quad \text{if} \quad x \geq 1$$

$$\text{since} \quad x \geq 1 \Longrightarrow x^n \geq x$$

$$\Longrightarrow \quad \frac{1}{x^n} \leq \frac{1}{x}.$$

$$\begin{aligned} \therefore f(x) &\geq x^n\Big(1 - \frac{a}{x}\Big) \\ &\geq x^n \frac{1}{2}, \quad \text{if} \quad x \geq \max\Big\{1,\ 2a\Big\} \\ &> 0. \end{aligned}$$

(You may prefer the argument given in Remark (i), page 64.) Thus, since f takes a positive and a negative value and is continuous, the Intermediate Value Theorem implies that $f(x) = 0$ for some x. Thus $x^n - a = 0$ has a solution $x > 0$. The equation $x^n - a = 0$ cannot have more than one solution since

$$0 < x_1 < x_2 \Longrightarrow x_1^n < x_2^n \quad \text{(Example 1.2.2, page 3).}$$

$\square$

REMARK: (i) If the properties of '$\lim\limits_{x\to\infty}$' are used (Problem 3.13, page 54), the existence of a point x such that $f(x) > 0$ in the preceding proof is easy to see:

Since

$$\frac{f(x)}{x^n} = 1 - \frac{a}{x^n} \quad \text{implies} \quad \lim_{x\to\infty} \frac{f(x)}{x^n} = 1,$$

it follows that $\frac{f(x)}{x^n}$, and therefore $f(x)$, is positive if x is large enough.

(ii) If n is odd, a similar argument may be used to show that there is a unique $x \in \mathbb{R}$ such that $x^n = a$ for any $a \in \mathbb{R}$ (i.e. we don't need $a > 0$ in this case).

Problems

3.15 For each of the functions in # 3.1, page 52, find all points where the function is not continuous. Explain briefly in each case.

3.16 For the functions in Example 3.1.1, (c) – (g), page 43, 44, identify all points in the domain where each function is not continuous. [In discussing (e), you may use the fact that every real interval contains rational and irrational numbers].

3.17 Let $f(x) = \frac{x^2-4}{x-2}$, $x \neq 2$. Can $f(2)$ be defined in such a way that f is continuous at 2? Explain.

3.18 Let $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$. Can $f(0)$ be defined in such a way that f is continuous at 0? Explain.

3.19 Sketch the graph of the function $h(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Show h is not continuous at 0. You may use $\sin \frac{(2n+1)\pi}{2} = (-1)^n$.

3.20 Show that $|f|$ is continuous at a if f is continuous at a.

3.21 Suppose f, g are continuous on $[0,1]$ and $f(0) < g(0)$, $f(1) > g(1)$. Show that $f(c) = g(c)$ for some $c \in (0,1)$.

3.22 Suppose $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$. Show that f is continuous at 0.

3.23 Let f be continuous on $[0,1]$ and such that $f(x) \in [0,1]$ for each $x \in [0,1]$. Show that $f(x) = x$ for some $x \in [0,1]$.

3.24 You are given:

$$|\sin x| \leq |x|, \quad |\cos x| \leq 1, \quad \text{if} \quad x \in \mathbb{R},$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right), \quad \text{if} \quad x, y \in \mathbb{R}.$$

Show that the sine function is continuous on $\mathbb{R}$.

3.25 Let p be a polynomial of odd degree. Show that the equation $p(x) = 0$ has a root.

3.26 Let p be a polynomial of even degree. Show that the equation $p(x) = c$ does not have a root for every $c \in \mathbb{R}$. [*Hint*: Show p must have either a maximum or a minimum].

3.27 Let $f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$ Show that f is continuous at 0 and is not continuous at any other point in its domain.

3.28 Show that there are antipodal points on the equator which are at the same temperature [Let $T(\theta)$ be the temperature on the Equator at Longitude θ, $0 \leq \theta \leq 2\pi$. Assume T is continuous on $[0, 2\pi]$, $T(0) = T(2\pi)$. It must be shown that $T(\theta_0) = T(\theta_0 + \pi)$ for some θ_0]. *Hint:* Consider $f(\theta) = T(\theta) - T(\theta + \pi)$, $0 \leq \theta \leq \pi$.

3.29 Let f be continuous on $[0,1]$ with $f(0) = f(1)$.

(a) If n is a natural number, show that $f(x) = f\left(x + \frac{1}{n}\right)$ for some $x \in [0, 1 - \frac{1}{n}]$. This means that the graph of f has a horizontal chord of length $\frac{1}{n}$.

(b) Suppose $0 < \alpha < 1$, $\alpha \neq \frac{1}{n}$, $n = 1, 2, \ldots$. Construct a function f, continuous on $[0,1]$ with $f(0) = f(1)$ such that $f(x) \neq f(x + \alpha)$ for all $x \in [0, 1 - \alpha]$.

3.30 A function f has the Darboux (intermediate value) property on an interval I if, when $p, q \in I$, $p < q$, f takes every value between $f(p)$ and $f(q)$ in the subinterval (p, q) of I. We have seen from Bolzano's Theorem that

f continuous on $I \Longrightarrow f$ has the Darboux property on I.

What about the converse implication '$\Longleftarrow$'?

Hint: Consider the function in Problem 3.19.

3.31 Let $f(x) = x^{\frac{1}{n}}$. Show f is continuous on its domain.

3.32 Suppose f is continuous on $(0, \infty)$ and

$$\lim_{0+} f = \lim_{\infty} = \infty.$$

Show that $f(c) \leq f(x)$ for some c and all $x \in (0, \infty)$; f has a minimum at c.

IV. DIFFERENTIATION

§4.1. The Derivative. Let a be an interior point of the domain of f. If

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exists, then f is said to be *differentiable at* a; the limit is denoted $f'(a)$ and is called the *derivative* of f *at* a.

The following two expressions for $f'(a)$ are equivalent (see Problem 3.14, p. 48):

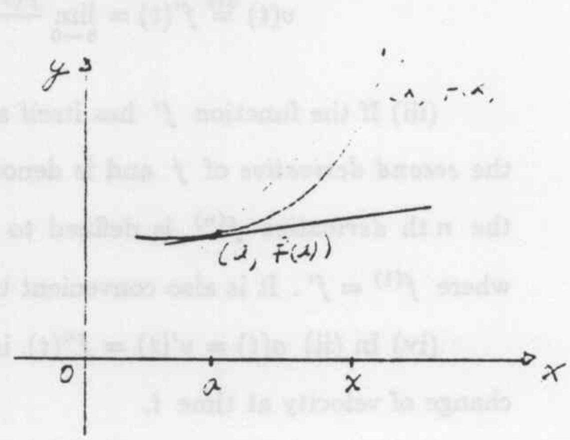
$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$
$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Thus, associated with f we have a new function f', the *derivative* of f whose domain is the set of all x in the domain of f such that $f'(x) \stackrel{\text{def}}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists.

Consideration of the derivative arises naturally in many different ways.

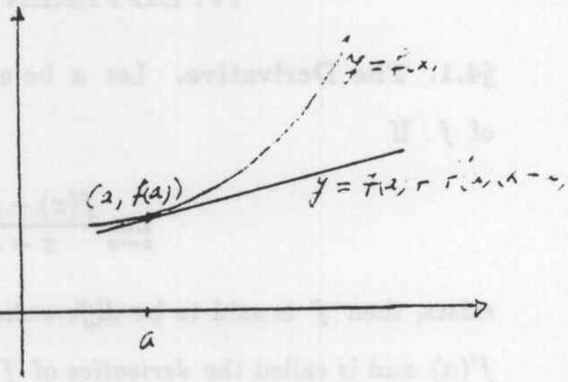
(i) The *secant* line joining the points $(x, f(x))$ and $(a, f(a))$ on the curve $y = f(x)$ has slope

$$\frac{f(x)-f(a)}{x-a}, \quad x \neq a.$$



Our intuition tells us that this slope is as close as we please to the slope of the tangent line of the curve at $(a, f(a))$ provided we take

x sufficiently close to a. In fact we take this as the *definition* of the tangent: If f is differentiable at a, then the tangent to the graph of f at $(a, f(a))$ is the straight line through $(a, f(a))$ with slope $f'(a)$. Thus the *tangent line* has equation



$$y = f(a) + f'(a)(x - a).$$

(ii) The number $f'(a)$ may also be interpreted as the rate of change of f at a. For example, suppose a particle travelling in a straight line has coordinate $f(t)$ meters at time t seconds. If h is positive or negative (but not zero) the average velocity between times t and $t + h$ is given by

$$\frac{\text{change in position coordinate}}{\text{change in time}} = \frac{f(t+h) - f(t)}{h}.$$

This leads us to define the *velocity* of the particle at time t (or instantaneous velocity) by

$$v(t) \stackrel{\text{def}}{=} f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

(iii) If the function f' has itself a derivative then this is called the *second derivative* of f and is denoted f'' or $f^{(2)}$. Inductively, the nth *derivative* $f^{(n)}$ is defined to be the derivative of $f^{(n-1)}$ where $f^{(1)} = f'$. It is also convenient to use the notation $f^{(0)} = f$.

(iv) In (ii) $a(t) = v'(t) = f''(t)$ is the *acceleration*, the rate of change of velocity at time t.

PROPOSITION 4.1.1. (DERIVATIVE OF A CONSTANT FUNCTION). *If $f(x) = c$, for all $x \in \mathbb{R}$, then $f'(x) = 0$ for all $x \in \mathbb{R}$*

PROOF: If $h \neq 0$, $\frac{f(x+h)-f(x)}{h} = \frac{c-c}{h} = \frac{0}{h} = 0$. Therefore $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0$, for all $x \in \mathbb{R}$.

□

PROPOSITION 4.1.2. (DERIVATIVE OF A LINEAR FUNCTION). *If $f(x) = mx + b$, and $x \in \mathbb{R}$, then $f'(x) = m$, for all $x \in \mathbb{R}$*

PROOF: If $h \neq 0$,

$$\frac{f(x+h)-f(x)}{h} = \frac{m(x+h)+b-(mx+b)}{h}$$

$$= \frac{mh}{h} = m.$$

Therefore $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = m$, for all $x \in \mathbb{R}$

□

PROPOSITION 4.1.3. (DERIVATIVE OF A POWER). *If $f(x) = x^n$, then $f'(x) = nx^{n-1}$, $n = 0, 1, 2$,*

PROOF I: This proof uses the Binomial Theorem. If $h \neq 0$, $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^n - x^n}{h}$

$$= \frac{1}{h}[x^n + nx^{n-1}h + \frac{n(n-1)}{1.2}x^{n-2}h^2 + \cdots + h^n - x^n] \qquad \text{(Binomial Theorem)}$$

$$= nx^{n-1} + \frac{n(n-1)}{1.2}x^{n-2}h + \cdots + h^{n-1}$$

Therefore $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = nx^{n-1}$, for each $x \in \mathbb{R}$

□

PROOF II: Here we show $f'(a) = na^{n-1}$ by using Problem # 1.11, p 18. If $x \neq a$,

$$\frac{f(x) - f(a)}{x - a} = \frac{x^n - a^n}{x - a}$$
$$= \frac{1}{(x-a)}(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})$$
$$= x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots xa^{n-2} + a^{n-1}$$

Each of the terms in this expression has limit a^{n-1} $(x \to a)$ and there are n terms in all. Therefore $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = na^{n-1}$.

□

PROPOSITION 4.1.4. (DERIVATIVE OF A ROOT FUNCTION). *If* $f(x) = x^{\frac{1}{n}}$, $x > 0$, *then* $f'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$

PROOF: Exercise.

□

THEOREM 4.1.5. *Suppose f is differentiable at a. Then f is continuous at a.*

PROOF: If $x \neq a$, then $f(x) - f(a) = \frac{f(x)-f(a)}{x-a}(x-a)$. Thus $f(x) = f(a) + \frac{f(x)-f(a)}{x-a}.(x-a)$, so that $\lim_{x \to a} f(x) = f(a) + f'(a).0 = f(a)$ and f is continuous at a.

□

The converse of Theorem 4.1.5 is not true. If f is continuous at a, it does not necessarily follow that $f'(a)$ exists as shown by the following example.

EXAMPLE 4.1.6: If $f(x) = |x|, x \in \mathbb{R}$, then $f'(0)$ does not exist.

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x| - 0}{x - 0}$$
$$= \frac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0. \end{cases}$$

Therefore $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$ does not exist; f is not differentiable at 0 even though it is continuous there.

§4.2. Alternative Notation. It is sometimes convenient to denote the differentiation operation by D:

$$f' = Df, \quad f^{(2)} = D^2 f, \ldots, f^{(n)} = D^n f.$$

If we wish to emphasize the letter being used for our independent variable we may even use D_x or D_t:

$$D_x x^3 = 3x^2, \qquad D_t t^{\frac{1}{2}} = \frac{1}{2} t^{-\frac{1}{2}}.$$

A notation introduced by Leibniz is also extremely useful. When considering

$$\frac{f(x + h) - f(x)}{h},$$

let $y = f(x)$, $h = \Delta x$, $\Delta y = f(x + h) - f(x)$. Then $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ and we write

$$f'(x) = \frac{dy}{dx}.$$

It is usual to call $\frac{\Delta y}{\Delta x}$ a 'difference quotient'. Variations of these notations may also be found. For example, the differentiation operation may be denoted by $\frac{d}{dx}$ or $\frac{d}{dt}$. Thus $D_x = \frac{d}{dx}$, $D_t = \frac{d}{dt}$.

§4.3. Some Rules for Differentiation.

THEOREM 4.3.1. *Suppose f, g are such that $f'(a)$, $g'(a)$ both exist. Let*

$$F = f + g, \quad G = fg, \quad H = \frac{f}{g} \quad (g(a) \neq 0).$$

Then $F'(a)$, $G'(a)$, $H'(a)$, all exist and are given by

(a) $F'(a) = f'(a) + g'(a)$

(b) $G'(a) = f'(a)g(a) + f(a)g'(a)$

(c) $H'(a) = \dfrac{g(a)f'(a) - f(a)g'(a)}{g(a)^2}$

More informally, this may be stated

(a) $D(f+g) = Df + Dg$ *Sum Rule*

(b) $D(fg) = gDf + fDg$ *Product Rule*

(c) $D(\frac{f}{g}) = \frac{gDf - fDg}{g^2}$ $(g \neq 0)$ *Quotient Rule*

PROOF: We are given that

$$f'(a) = \lim_{x \to 0} \frac{f(x) - f(a)}{x - a}, \quad g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \qquad \text{(A)}$$

both exist.

(a)

$$\begin{aligned} \frac{F(x) - F(a)}{x - a} &= \frac{f(x) + g(x) - f(a) - g(a)}{x - a} \\ &= \frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a}. \end{aligned}$$

Therefore $F'(a) = \lim_{x \to a} \frac{F(x) - F(a)}{x - a} = f'(a) + g'(a)$, from (A).

(b)

$$\begin{aligned}\frac{G(x)-G(a)}{x-a} &= \frac{f(x)g(x)-f(a)g(a)}{x-a}\\ &= \frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a}\\ &= \frac{f(x)-f(a)}{x-a}g(x)+f(a)\frac{g(x)-g(a)}{x-a}.\end{aligned}$$

Theorem 4.1.5 implies $\lim_{x\to a} g(x) = g(a)$ so that, from (A)

$$G'(a) = \lim_{x\to a}\frac{G(x)-G(a)}{x-a} = f'(a)g(a)+f(a)g'(a).$$

(c) It is sufficient to consider $H = \frac{1}{g}$ to prove the quotient rule. If we show $H' = \frac{-g'}{g^2}$ in this case, then the general formula (c) follows by applying the product rule (b) to $f\frac{1}{g}$.

$$\frac{H(x)-H(a)}{x-a} = \frac{\frac{1}{g(x)}-\frac{1}{g(a)}}{x-a} = \frac{-1}{g(x)g(a)}\,\frac{g(x)-g(a)}{x-a}.$$

Therefore, as before,

$$H'(a) = -\frac{g'(a)}{g(a)^2}.$$

□

COROLLARY 4.3.2.

(a) $Dx^n = nx^{n-1}$, $\quad n = 0,1,2,\ldots \quad$ *all* $x \in \mathbb{R}$.

(b) $Dx^n = nx^{n-1}$, $\quad n = -1,-2,-3, \quad x \neq 0$

(c) $Dx^\alpha = \alpha x^{\alpha-1}$, $\quad \alpha \in Q, \quad x > 0$.

OUTLINE OF PROOF:

(a) We have already seen two proofs of this in Proposition 4.1.3. A simple proof by induction is as follows. First show $Dx^0 = D1 =$

0 and $Dx^1 = Dx = 1$. Then, if $Dx^n = nx^{n-1}$, it follows from the product rule that

$$Dx^{n+1} = D(xx^n) = xnx^{n-1} + 1x^n = (n+1)x^n$$

so that (a) holds for all $n = 0, 1, 2, \ldots,$ by induction.

(b) To prove this observe that $n = -m$, where $m \in \mathbb{N}$. Use the quotient rule and (a) to find

$$\begin{aligned} Dx^n = Dx^{-m} = D\frac{1}{x^m} &= \frac{-mx^{m-1}}{x^{2m}} \\ &= -m\times^{-m-1} = nx^{n-1}. \end{aligned}$$

(c) Proposition 4.1.4 asserts $Dx^{\frac{1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}$, if $n \in \mathbb{N}$. Now use induction on m to show that

$$Dx^{m/n} = \frac{m}{n}x^{\frac{m}{n}-1}, \quad \text{if } m, n \in \mathbb{N}.$$

Finally use the quotient rule to extend the formula to negative integers m.

We have only asserted (c) for $x > 0$. However, the formula is true for all $x \in \mathbb{R}$ when $\alpha = \frac{m}{n}$, n odd, with the possible exception of $x = 0$.

The preceding theorem and corollary show that any polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = \sum_{k=0}^{n} a_kx^k$$

is differentiable at each point in $\mathbb{R}$ and

$$P'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} = \sum_{k=0}^{n} ka_kx^{k-1}.$$

More generally a rational function $P(x)/Q(x)$ is differentiable at each point in its domain.

THEOREM 4.3.3. *Consider a composition of two functions*

$$F = f \circ g \qquad [F(x) = f(g(x))].$$

Suppose that $f'(b)$, $g'(a)$ *both exist, where* $b = g(a)$. *Then* $F'(a)$ *exists and*

$$F'(a) = f'(b)g'(a) \qquad \textit{The Chain Rule.}$$

In the Leibniz notation this may be expressed as follows: If

$$y = f(u) \quad \textit{and} \quad u = g(x),$$

then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad (= f'(u)g'(x))$$

at any point x *such that* $\frac{dy}{du}, \frac{du}{dx}$ *both exist.*

$\square$

EXAMPLE 4.3.4: $D(x^2+1)^{\frac{1}{2}} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}2x$.

$y = f(u) = u^{\frac{1}{2}}$ $\qquad u = g(x) = x^2 + 1$ so that

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}, \qquad \frac{du}{dx} = 2x.$$

Thus $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{2}u^{-\frac{1}{2}}2x = \dfrac{1}{2}(x^2+1)^{-\frac{1}{2}}2x$.

EXAMPLE 4.3.5: The formula $D\frac{1}{g} = \frac{-Dg}{g^2}$ used to prove the quotient rule is a special case of the chain rule:

$$y = \frac{1}{u}, \quad u = g(x)$$
$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = -\frac{1}{u^2}g'(x) = -\frac{g'(x)}{g(x)^2}.$$

EXAMPLE 4.3.6: The volume of a spherical balloon is growing at the rate of 15 cm^2/sec. At what rate is the radius growing when the radius is 10 cm?

The Volume V and radius r are related by the formula

$$V = \frac{4}{3}\pi r^3.$$

Thus

$$15 = \frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt},$$

from the chain rule. When $r = 10$

$$15 = 4\pi 10^2\frac{dr}{dt}$$

so that, at this instant the radius is growing at the rate $\frac{15}{4\pi 10^2} = \frac{3}{80\pi}$ cm/sec.

PROOF OF THEOREM 4.3.3: The following is 'almost' a proof of the chain rule. Since $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u}\cdot\dfrac{\Delta u}{\Delta x}$, where $\Delta x = x - a$, $\Delta u = g(x) - g(a)$ and $\Delta y = f(g(x)) - f(g(a))$, and $\lim_{\Delta x\to 0}\Delta u = 0$, because g is continuous at a, it follows that

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \Big(\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\Big)\Big(\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\Big)$$

and so $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. The flaw in this 'proof' is that $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}$ is valid only as long as $0 \neq \Delta u = g(x) - g(a)$, which may fail to hold for x arbitrarily close to a. The following argument avoids this difficulty.

Since $f'(b) = \lim_{u \to b} \frac{f(u)-f(b)}{u-b}$ exists,

$$f(u) = f(b) + f'(b)(u-b) + E(u)(u-b) \tag{A}$$

where

$$E(u) = \begin{cases} \frac{f(u)-f(b)}{u-b} - f'(b), & u \neq b \\ 0, & u = b \end{cases}$$

and E is continuous at b, since $\lim_{u \to b} E(u) = 0$. With $u = g(x)$, $b = g(a)$, $\quad x \neq a$,

$$\begin{aligned} \frac{F(x)-F(a)}{x-a} &= \frac{f(g(x)) - f(g(a))}{x-a} \\ &= f'(b)\frac{g(x)-g(a)}{x-a} + E(g(x))\frac{g(x)-g(a)}{x-a}. \end{aligned}$$

Thus $F'(a) = \lim_{x \to a} \frac{F(x)-F(a)}{x-a}$ exists and

$$F'(a) = f'(b)g'(a) + 0g'(a) = f'(b)g'(a).$$

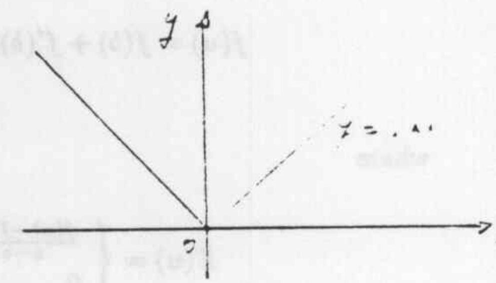
The fact that $\lim_{x \to a} E(g(x)) = 0$ follows from Theorem 3.2.7, page 57.

$\square$

§4.4. One-sided derivatives. The left-hand and right-hand derivatives of f are denoted D_-f and D_+f and are defined by

$$D_-f(a) = \lim_{x \to a-} \frac{f(x) - f(a)}{x - a}$$
$$D_+f(a) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a}.$$

EXAMPLE 4.4.1: If $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$, then



$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$$D_-f(0) = \lim_{x \to 0-} \frac{|x|}{x} = -1$$
$$D_+f(0) = \lim_{x \to 0+} \frac{|x|}{x} = 1.$$

REMARKS: (i) $f'(a)$ does not exist if a is an endpoint of the domain of f. However, $D_-f(a)$ may exist if a is a right endpoint and $D_+f(a)$ may exist if a is a left endpoint of the domain of f.

(ii) Note that, as in Theorem 4.1.5, page 72, the existence of $D_-f(a)$ implies the continuity from the left of f at a and the existence of $D_+f(a)$ implies continuity from the right of f at a.

(iii) The function f is differentiable at a if and only if $D_-f(a)$ and $D_+f(a)$ both exist and are equal and then $f'(a) = D_-f(a) = D_+f(a)$.

Problems

4.1 If $f(x) = 3x^2 - 5x + 2$, use the definition of the derivative to find $f'(2), f'(-1), f'(a)$.

4.2 If $f(x) = \sqrt{x}$, use the definition of the derivative to find $f'(2), f'(a), a > 0$.

4.3 Use the rules of differentiation to find the derivative of each of the functions

(a) $(1-2x)^5$, (b) $(1-2x)^{-1}$, (c) $\sqrt{1-2x}$

(d) $\frac{x^4-1}{x^4+1}$, (e) $\sqrt{\left(\frac{x^4-1}{x^4+1}\right)}$, (f) $\left[\left\{x^4+(2x+1)^3\right\}^6+1\right]^{\frac{1}{2}}$.

4.4 Let $f(x) = \begin{cases} x^2, & x \le 1 \\ ax+b, & x > 1 \end{cases}$.

(a) For what values of a, b is the function f continuous at 1? Explain.

(b) For what values of a, b is the function f differentiable at 1? Explain.

4.5 Let $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$, $\quad g(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

Each of f, g is continuous at exactly one point in its domain. Determine the existence or otherwise of the derivative of each of the functions at that point.

4.6 If $f^{(n)}, g^{(n)}$ exist, show

$$(fg)^{(n)} = \binom{n}{0} f^{(n)} g + \binom{n}{1} f^{(n-1)} g' + \binom{n}{2} f^{(n-2)} g^{(2)} + \cdots + \binom{n}{n} f g^{(n)}.$$

This is *Leibniz's Rule.* In the case $n = 1$ it is the Product Rule.

4.7 Find all points on the curve $y = x^2$ where the tangent line passes through $(1,0)$.

4.8 Prove Proposition 4.1.4, page 72, $Dx^{\frac{1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}, x > 0$.

Hint: Let $y = x^{1/n}$, $b = a^{1/n}$. Then $\frac{x^{1/n}-a^{1/n}}{x-a} = \frac{y-b}{y^n-b^n}$.

4.9 You are given $\lim_{x \to a} \cos x = \cos a$, $\cos 0 = 1$ and

$$\cos x < \frac{\sin x}{x} < 1, \quad \text{if } 0 < |x| < \frac{\pi}{2}.$$

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

(b) Show $D \sin x = \cos x$ [Hint: #3.24, p. 66]

(c) Use $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, (b) and the chain rule to show $D\cos x = -\sin x$.

4.10 At what points on the curve $y = x^3 - x$ is the tangent

(a) horizontal? (b) parallel to the line $y = 3x + 4$?

4.11 Show that the area of the triangle formed by the tangent line to the curve $xy = 1$ at the point $\left(a, \frac{1}{a}\right)$ and the coordinate axes is constant (i.e. independent of a).

4.12 The volume of a sperical balloon is, at a certain instant, growing at a rate of 32 cm ^{2} /sec and, at the same instant, its surface area is growing at 16 cm ^{2} /sec. Find the radius of the balloon and the rate at which it is growing at that instant.

4.13 The volume of an expanding cube is increasing at the rate of 4 cm ^{3} /sec. How fast is the surface area growing when the surface area is 24 cm ^{2} ? [Ans. 8 cm ^{2} /sec.]

4.14 A meteorite falling directly to earth has velocity inversely proportional to $\sqrt{s}$ when at a distance s from the centre of the earch. Show that its acceleration is inversely proportional to s^2.

4.15 An inverted conical tank has height 4 m and radius 1 m at the top. Oil flows in at the rate of 2 m ^{3} /min. How fast is the level rising when the depth is 2m? [Volume of cone = $\frac{1}{3}$ (base area) $\times$ (height). Ans. $\frac{8}{\pi}$ m/min].

4.16 A boat is attached by a rope from its bow to a winch on a dock 3 m above the level of the bow. If the rope is being hauled in at the rate of 2m/sec, find the rate at which the boat is travelling through the water when it is 4 m from the dock.

4.17 A boat sails parallel to a straight beach at 20 km/hr and maintains a course 4 km offshore. How fast is it approaching a lighthouse on the shore when it is 5 km from the lighthouse?

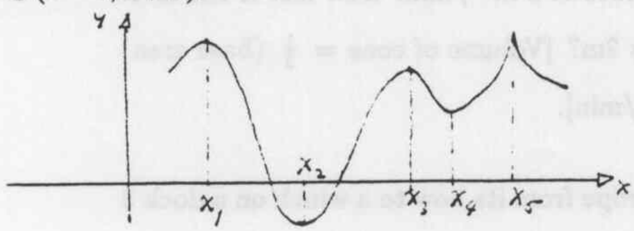
4.18 If the position of a particle in the plane at time t is $(x(t), y(t))$, then its speed is given by $[x'(t)^2 + y'(t)^2]^{\frac{1}{2}}$. Suppose a particle moves along the curve $y = x^3$ with a constant speed of 2 cm/sec and its x coordinate is increasing. Find the rate at which the x coordinate is increasing when it is at the point $(2, 8)$ on the curve.

V. PROPERTIES OF THE DERIVATIVE

§5.1. The Lagrange Mean Value Theorem. A function f is said to have an *interior local maximum (minimum)* at c if, for some $\delta > 0$,

$$f(c) \geq f(x) \qquad (f(c) \leq f(x))$$

for each $x \in (c-\delta, c+\delta)$.



The function graphed here has an interior local maximum at each of x_1, x_3, x_5 and an interior local minimum at x_2, x_4.

PROPOSITION 5.1.1. *Suppose*

(i) f *has an interior local extremum (maximum/minimum) at* c.

(ii) $f'(c)$ *exists.*

Then $f'(c) = 0$.

PROOF: Consider the case that f has an interior local maximum at c so that $f(x) \leq f(c)$, $c - \delta < x < c + \delta$. Thus

$$\frac{f(x)-f(c)}{x-c} \begin{cases} \geq 0, \ x \in (c-\delta, c) \Longrightarrow f'(c) \geq 0 \\ \leq 0, \ x \in (c, c+\delta) \Longrightarrow f'(c) \leq 0 \end{cases}$$

$$\Longrightarrow f'(c) = 0.$$

$\square$

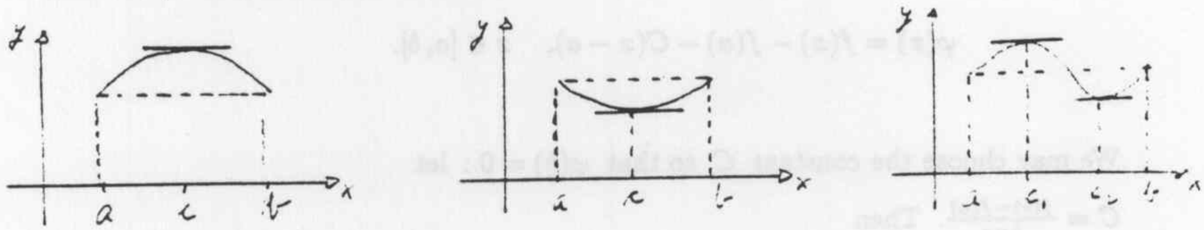
Theorem 5.1.2 (Rolle's Theorem). *Suppose*

(i) f *is continuous on* $[a, b]$,

(ii) $f'(x)$ *exists if* $x \in (a, b)$,

(iii) $f(a) = f(b)$.

Then $f'(c) = 0$ *for some* $c \in (a, b)$.



Proof: First consider the case that $f(a) = f(x) = f(b)$ if $x \in (a, b)$. Then f is constant on $[a, b]$ so that $f'(c) = 0$ for each $c \in (a, b)$.

Next suppose $f(x_0) > f(a) = f(b)$, for some $x_0 \in (a, b)$. Then Weierstrass' Theorem (page 60) shows that, since f is continuous on $[a, b]$, f has a maximum value $f(c)$. Moreover $c \in (a, b)$, since $f(c) \geq f(x_0) > f(a) = f(b)$. Thus f has an interior local maximum at c and, since $f'(c)$ exists, $f'(c) = 0$ by Proposition 5.1.1.

Finally, if $f(x_0) < f(a) = f(b)$, f has an interior minimum at some c and again $f'(c) = 0$.

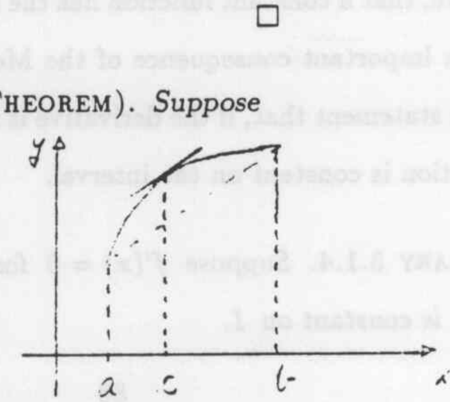
□

Theorem 5.1.3 (Lagrange Mean Value Theorem). *Suppose*

(i) f *is continuous on* $[a, b]$

(ii) $f'(x)$ *exists if* $x \in (a, b)$.

Then $f'(c) = \frac{f(b) - f(a)}{b - a}$ *for some* $c \in (a, b)$.



Geometrically, this says that, for some $c \in (a, b)$, the tangent to the graph of f at $(c, f(c))$ is parallel to the chord joining $(a, f(a))$ to $(b, f(b))$.

PROOF: Consider the function φ

$$\varphi(x) = f(x) - f(a) - C(x - a), \quad x \in [a, b].$$

We may choose the constant C so that $\varphi(b) = 0$: let $C = \frac{f(b)-f(a)}{b-a}$. Then

(i) φ is continuous on $[a, b]$ (why?)

(ii) $\varphi'(x)$ exists if $x \in (a, b)$ (why?)

(iii) $\varphi(a) = \varphi(b) = 0$.

Thus, by Rolle's Theorem, there exists $c \in (a, b)$ such that

$$\begin{aligned} 0 = \varphi'(c) &= f'(c) - C \\ &= f'(c) - \frac{f(b) - f(a)}{b - a} . \end{aligned}$$

□

We saw in Proposition 4.1.1, directly from the definition of the derivative, that a constant function has the zero function as its derivative. An important consequence of the Mean Value Theorem is the converse statement that, if the derivative is zero on some interval then the function is constant on the interval.

COROLLARY 5.1.4. *Suppose $f'(x) = 0$ for each $x \in I$, an interval. Then f is constant on I.*

PROOF: Let $u, v \in I,\ u < v$. Then, for some $c \in (u, v)$,

$$\frac{f(v) - f(u)}{v - u} = f'(c) = 0 \quad \text{(Mean Value Theorem)}$$

so that $f(v) = f(u)$ and f is constant on I.

$\square$

COROLLARY 5.1.5. *Suppose $f'(x) = g'(x)$ for each $x \in I$, an interval. Then $f(x) = g(x) + k$, for each $x \in I$, where k is a constant.*

PROOF: Consider $F(x) = f(x) - g(x)$.

$\square$

A function f is said to be *increasing (decreasing)* on an interval I if

$$u, v \in I,\ u < v \Longrightarrow f(u) \leq f(v)\ (f(u) \geq f(v)).$$

The function f is *strictly increasing (strictly decreasing)* on I if

$$u, v \in I,\ u < v \Longrightarrow f(u) < f(v) \quad (f(u) \geq f(v)).$$

Clearly a function which is strictly increasing is increasing.

COROLLARY 5.1.6. *Suppose $f'(x)$ exists for each x in the interval I. Then f is increasing (decreasing) on $I \Longleftrightarrow f'(x) \geq 0$ (≤ 0) for each $x \in I$.*

PROOF:

" $\Longrightarrow$ " Let f be increasing on I. It follows from the definition of the derivative that, since $f'(x)$ exists for each $x \in I,\ f'(x) \geq 0$.

" $\Longleftarrow$ " Conversely, suppose $f'(x) \geq 0$ for each $x \in I$.

Then, if $u, v \in I$, $u < v$, there exists $c \in (u, v)$ such that $\frac{f(v)-f(u)}{v-u} = f'(c) \geq 0 \Longrightarrow f(u) \leq f(v)$ so that f is increasing on I. The proof of the statement about f decreasing is the same.

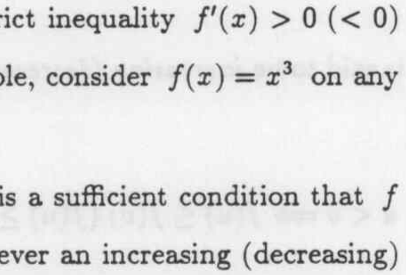
$\square$

REMARKS:

(i) If $f'(x) > 0$ (< 0) for each $x \in I$, then the preceding proof also shows that f is strictly increasing (strictly decreasing) on I. However, if f is strictly increasing (strictly decreasing) on I it does not follow that the strict inequality $f'(x) > 0$ (< 0) holds for each $x \in I$. For example, consider $f(x) = x^3$ on any interval which contains 0.

(ii) The condition $f'(x) \geq 0$ (≤ 0) is a sufficient condition that f be increasing (decreasing). However an increasing (decreasing) function need not be differentiable or even continuous. For example if $f(x) = [x]$, the greatest integer not exceeding x, then f is increasing on every interval even though it is discontinuous at every integer.

A function f is said to be *concave up (concave down)* on an interval I if f' is increasing(decreasing) on I. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$. This function is thus concave down on $(-\infty, 0]$ since f' decreases there and concave up on $[0, \infty)$,



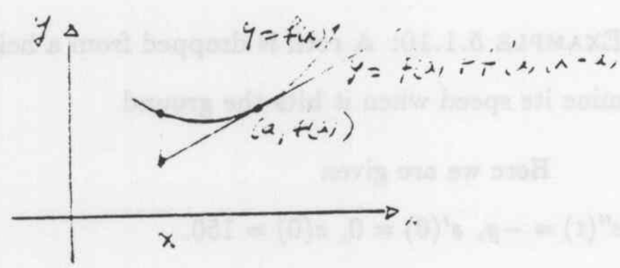
since f' increases.

Corollary 5.1.7. *Suppose $f''(x)$ exists for each $x \in I$. Then f is concave up (down) on $I \iff f''(x) \geq 0$ (≤ 0) for each $x \in I$.*

This follows by replacing f by f' in Corollary 5.1.6. However f may be concave up or down without f'' existing at all points.

Proposition 5.1.8. *Suppose f is concave up on its domain, an interval I. Then, for each $a \in I$, the graph of f is above the tangent line to the graph at $(a, f(a))$.*

Proof: The equation of the tangent at $(a, f(a))$ is
$y = f(a) + f'(a)(x - a)$.
The proposition asserts that



$$f(x) \geq f(a) + f'(a)(x - a), \quad \text{for each} \quad a,\ x \in I. \qquad \text{(A)}$$

The Mean Value Theorem shows that

$$f(x) - f(a) = f'(c)(x - a)$$

for some c between a and x. Now

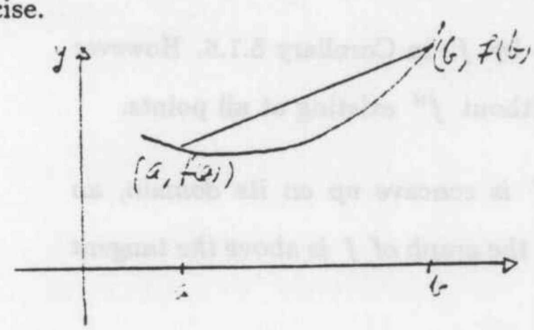
$$\begin{array}{llll} f'(c) \leq f'(a) & \text{if} \;\; x < a, & \text{since} & c \in (x, a) \\ f'(c) \geq f'(a) & \text{if} \;\; x > a, & \text{since} & c \in (a, x). \end{array}$$

In either case $f(x) - f(a) = f'(c)(x - a) \geq f'(a)(x - a)$ and (A) holds as asserted.

Proposition 5.1.9. *Suppose f is concave up on an interval I.*

Then, for each $[a,b] \subset I$, the chord joining $(a, f(a))$ to $(b, f(b))$ lies above the graph $y = f(x), a \leq x \leq b$.

PROOF: Exercise.



EXAMPLE 5.1.10: A rock is dropped from a height of 150 m. Determine its speed when it hits the ground.

Here we are given

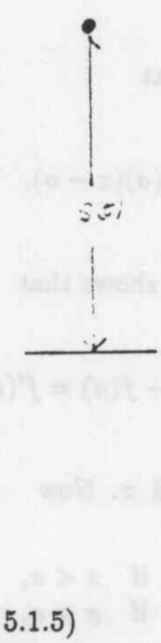
$s''(t) = -g,\ s'(0) = 0,\ s(0) = 150.$

$s(t)$ is the height of the rock above the ground t sec. after it has been droppped.

g is the acceleration due to gravity: $g \simeq 9.8\ \mathrm{m/sec}^2$.

$s(0) = 150$, the height when dropped

$s'(0) = 0$, the rock is dropped from rest.



$$\begin{aligned} s''(t) = -g &\Longrightarrow s'(t) = -gt + c \quad \text{(Corollary 5.1.5)} \\ &\Longrightarrow s'(t) = -gt, \text{ since } 0 = s'(0) = c. \\ &\Longrightarrow s(t) = -\frac{1}{2}gt^2 + k \quad \text{(Corollary 5.1.5)} \\ &\Longrightarrow s(t) = -\frac{1}{2}gt^2 + 150 \text{ since } 150 = s(0) = k. \end{aligned}$$

The time when the rock hits the ground is given by

$$0 = s(t) = -\frac{1}{2}gt^2 + 150.$$

$$\therefore \quad t = \sqrt{\frac{300}{g}}\,.$$

The speed with which the rock hits the ground is given by

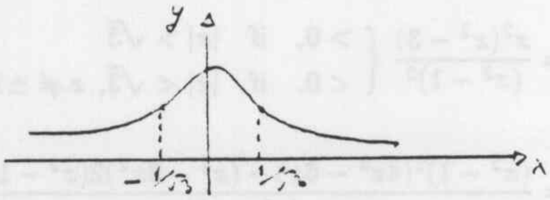
$$|s'(\sqrt{\frac{300}{g}})| = |-g\,\sqrt{\frac{300}{g}}\,| = \sqrt{300g}$$

$$\simeq 54.2 \text{ m/sec.}$$

EXAMPLE 5.1.11: Consider the function $f(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$.

$f'(x) = \frac{-2x}{(1+x^2)^2}$

$$\begin{cases} > 0, & \text{if} \quad x < 0 \\ < 0, & \text{if} \quad x > 0 \end{cases}$$



$$f'' = \frac{(1+x^2)^2(-2) + 2x \cdot 2(1+x^2)2x}{(1+x^2)^4}$$

$$= \frac{2(3x^2-1)}{(1+x^2)^3} \begin{cases} > 0, & \text{if} \quad |x| > \frac{1}{\sqrt{3}} \\ < 0, & \text{if} \quad |x| < \frac{1}{\sqrt{3}} \end{cases}$$

The function is therefore increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$.

Also f is concave up on $(-\infty, -\frac{1}{\sqrt{3}}\,]$ and on $[\frac{1}{\sqrt{3}}\,, \infty)$ and is concave down on $[-\frac{1}{\sqrt{3}}\,, \frac{1}{\sqrt{3}}\,]$.

Other observations useful in sketching the graph are $f(x) > 0$ for all x : the graph is in the upper half plane $f(-x) = f(x)$: the function is *even* so the graph is symmetric about the y-axis.

$\lim_{x\to\infty} f(x) = 0$: the graph is asymptotic to the x-axis.

EXAMPLE 5.1.12: Next consider $g(x) = \frac{x^3}{x^2-1}$, $x \neq \pm 1$.

Some simple observations are: $g(-x) = -g(x)$: this is an *odd* function so the graph is antisymmetric with respect to the y-axis.

The function changes sign at $x = 0, \pm 1$

$$\lim_{x\to 1+} g(x) = \infty, \quad \lim_{x\to 1-} g(x) = -\infty, \quad g(0) = 0 \quad \lim_{x\to\infty} g(x) = \infty;$$

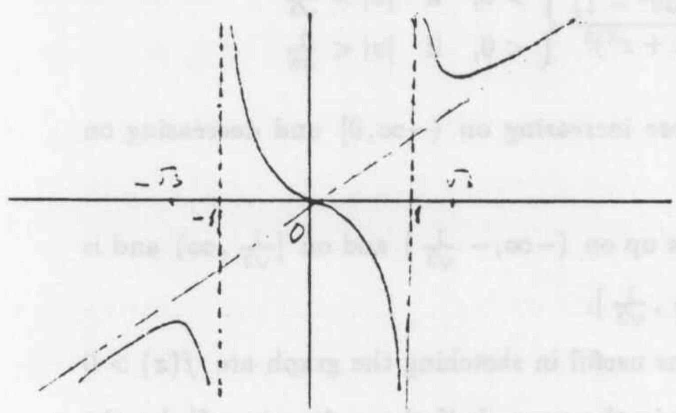
in fact $g(x) = x + \frac{x}{x^2-1}$, so if $x > 1$, $g(x) > x$ and $g(x) - x \to 0\,(x \to \pm\infty)$.

$$g'(x) = \frac{(x^2-1)3x^2 - x^3\cdot 2x}{(x^2-1)^2} = \frac{x^4-3x^2}{(x^2-1)^2}$$

$$= \frac{x^2(x^2-3)}{(x^2-1)^2} \begin{cases} > 0, & \text{if} \quad |x| > \sqrt{3} \\ < 0, & \text{if} \quad |x| < \sqrt{3},\ x \neq \pm 1 \end{cases}$$

$$g''(x) = \frac{(x^2-1)^2(4x^3-6x) - (x^4-3x^2)2(x^2-1)2x}{(x^2-1)^4}$$

$$= \frac{2x(x^2+3)}{(x^2-1)^3} \begin{cases} > 0, & \text{if} \quad -1 < x < 0 \text{ or } x > 1 \\ < 0, & \text{if} \quad x < -1 \text{ or } 0 < x < 1 \end{cases}$$



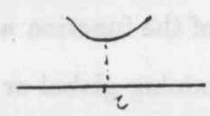
§5.2. Local Extrema. Let f be a continuous function on its domain. To determine the interior local maxima and minima of f, first observe that these occur at *critical points* of f :

(i) Points c such that $f'(c) = 0$, or

(ii) Points where f is not differentiable.

The behaviour of f' near a critical point may often be used to determine the nature of the point. The *first derivative test* is as follows

(a) $f'(x)\begin{cases} \leq 0, & x \in (c-\delta, c) \\ \\ \geq 0, & x \in (c, c+\delta) \end{cases}$



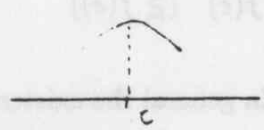
$\Longrightarrow f$ has an interior *local minimum* at c. This follows from the Mean Value Theorem, since

$$f(x) - f(c) = f'(y)(x-c),$$

for some y between x and c, and thus

$$f'(y)(x-c) \geq 0 \quad \text{implies} \quad f(x) \geq f(c) \quad \text{if} \quad x \in (c-\delta, c+\delta)$$

(b) $f'(x)\begin{cases} \leq 0, & x \in (c-\delta, c) \\ \leq 0, & x \in (c, c+\delta) \end{cases}$



$\Longrightarrow f$ has an interior *local maximum* at c.

Note that in (a), (b) it is not required that $f'(c)$ exists but only that f is continuous at c. For example, (a) shows that if $f(x) = |x|$, f has a local minimum at 0 even though $f'(0$ does not exist.

The *second derivative test* for local extrema is a consequence of these observations:

$$f'(c) = 0, \quad f''(c) > 0\ (< 0)$$

$\Longrightarrow f$ has a local minimum (maximum) at c. Since

$$f''(c) = \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = \lim_{x \to c} \frac{f'(x)}{x - c},$$

it follows that if $f''(c) > 0$ (< 0), then $\frac{f'(x)}{x-c} > 0$ (< 0) for x close to c so that f' has the behaviour described in (a)((b)).

A word of warning: the second derivative test is a local result only and describes the behaviour of the function *near* a critical point. This is mostly irrelevant in determining global or absoluted maxima and minima. In Example 5.1.11, $f'(0) = 0$, $f''(0) < 0$ shows f has a *local* maximum at 0. However, we get more information by considering f' alone: $f(x) < f(0)$, for *all* $x \neq 0$, since $f' > 0$ on $(-\infty, 0)$ and $f' < 0$ on $(0, \infty)$.

§5.3. Global Extremum Problems. Here we are interested in determining the existence and location of *absolute or global maxima and minima*: points c such that

$$f(x) \leq f(c) \quad (\geq f(c))$$

for all x in the domain of f. In general the existence or otherwise of such points is an essential part of our task. When the domain of f is a closed interval, the Weierstrass Theorem (Theorem 3.3.4, page 60) gives the following:

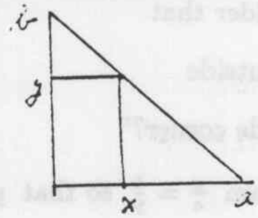
FUNDAMENTAL PRINCIPLE. If f is continuous on the closed interval $[a, b]$, then f has both a maximum value and a minimum value in $[a, b]$. These can occur only at the following points:

(i) an end-point a or b,

(ii) a point where f' does not exist,

(iii) a point where $f' = 0$.

EXAMPLE 5.3.1: Determine the rectangle of largest area that can be inscribed in a right-triangle. $A = xy$



$\frac{y}{a-x} = \frac{b}{a}$ (similar triangles)

Thus the function to be maximized is

$$A(x) = \frac{b}{a}\, x(a-x), \quad 0 \le x \le a.$$

Since A is continuous on the closed interval $[0, a]$ it has both a maximum and a minimum value

$$A'(x) = \frac{b}{a}\,(a - 2x) = 0 \Longrightarrow x = \frac{a}{2}.$$

Thus the global extrema can only occur at the end-points $0, a$ or the critical point $\frac{a}{2}$

x	0	a	$\frac{a}{2}$
$A(x)$	0	0	$\frac{ab}{4}$

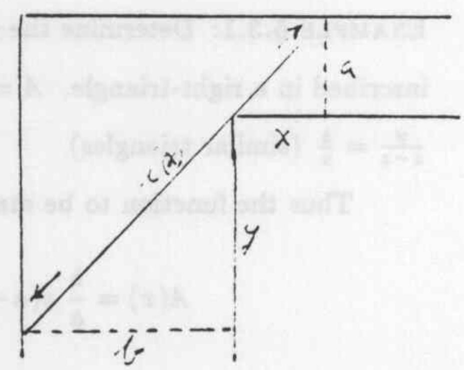
The maximum area occurs, when $x = a/2$, in which case $y = b/a$ and $A = \frac{ab}{4}$. The minimum occurs at $x = 0$ and $x = a$, when $A = 0$.

$\square$

If the domain of the function in the extremum problem is not a closed interval, then the approach must be modified.

EXAMPLE 5.3.2: Find the length of the longest ladder that can be carried horizontally round a corner from a corridor of width a to a corrider of width b.

Here it is convenient to reformulate the problem as: "What is the length of the shortest ladder that touches the two outside walls and the inside corner?"



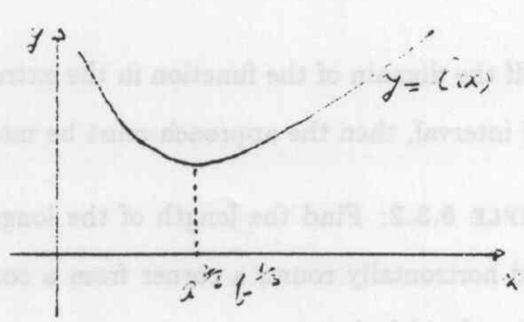
In the diagram $\frac{x}{a} = \frac{b}{y}$ so that $y = \frac{ab}{x}$

$$
\begin{aligned}
\ell(x) &= \sqrt{a^2 + x^2} + \sqrt{b^2 + y^2} \\
&= \sqrt{a^2 + x^2} + \sqrt{b^2 + \frac{a^2 b^2}{x^2}} \\
&= (1 + \frac{b}{x})\sqrt{a^2 + x^2}, \quad x > 0.
\end{aligned}
$$

The domain of the function which we wish to minimize is $(0, \infty)$. It it not clear at this stage whether a minimum exists.

$$
\begin{aligned}
\ell'(x) &= -\frac{b}{x^2}\,\sqrt{a^2 + x^2} + (1 + \frac{b}{x})\,\frac{x}{\sqrt{a^2 + x^2}} \\
&= \frac{-b(a^2 + x^2) + x^3 + bx^2}{x^2\sqrt{a^2 + x^2}} \\
&= \frac{-ba^2 + x^3}{x^2\sqrt{a^2 + x^2}} \quad \begin{cases} < 0, & \text{if} \quad x < a^{2/3}b^{1/3} \\ > 0, & \text{if} \quad x > a^{2/3}b^{1/3} \end{cases}
\end{aligned}
$$

Therefore ℓ is decreasing on $(0, a^{2/3}b^{1/3}]$ and increasing on $[a^{2/3}b^{1/3}, \infty)$, so that ℓ has a minimum (global) at $a^{2/3}b^{1/3}$ given by



$$
\begin{aligned}
\ell(a^{2/3}b^{1/3}) &= (1 + \frac{b^{2/3}}{a^{2/3}})\sqrt{a^2 + a^{4/3}b^{2/3}} \\
&= (a^{2/3} + b^{2/3})\sqrt{a^{2/3} + b^{2/3}} \\
&= (a^{2/3} + b^{2/3})^{3/2}.
\end{aligned}
$$

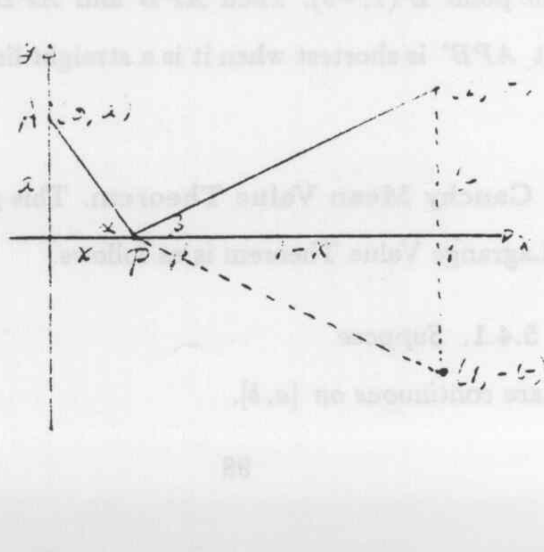
This is the length of the longest ladder that can be carried round the corner.

$\square$

In this example, it was fortunate that ℓ has only one critical point and that ℓ decreases to the left and increases to the right of this point. Just checking the critical point(s) here would not have been enough to show we had a global minimum.

An alternative approach could be to observe that $\lim_{x\to 0+} \ell(x) = \lim_{x\to\infty} \ell(x) = \infty$ which, together with the continuity of ℓ on $(0,\infty)$, implies that ℓ has a global minimum at some $c \in (0,\infty)$ (This is not trivial; Problem 3.32, page 68.) Thus c is a critical point of ℓ. Since there is only one critical point, this gives the minimum.

EXAMPLE 5.3.3:
In the diagram, show that the polygonal curve APB joining $(0,a)$ to $(1,b)$ through $(x,0)$, $0 \le x \le 1$, is shortest when the angles α, β are equal.



We must minimize

$$\ell(x) = \sqrt{a^2 + x^2} + \sqrt{(1-x)^2 + b^2}, \quad 0 \le x \le 1.$$

The function ℓ is continuous on its domain which is the closed interval $[0,1]$. Therefore ℓ has a maximum and a minimum value in $[0,1]$ which must occur at an end-point or a critical point

$$\ell'(x) = \frac{x}{\sqrt{a^2+x^2}} - \frac{(1-x)}{\sqrt{(1-x)^2+b^2}}, \; 0 < x < 1.$$

Observe that $\ell'(x) < 0$ if x is near 0 and $\ell'(x) > 0$ if x is near 1. Thus ℓ is decreasing near 0 and increasing near 1 so that the minimum does not occur at an end-point. Therefore at the minimum

$$\begin{aligned} 0 = \ell'(x) &= \frac{x}{\sqrt{a^2+x^2}} - \frac{(1-x)}{\sqrt{(1-x)^2+b^2}} \\ &= \cos\,\alpha - \cos\,\beta. \end{aligned}$$

Hence, $\cos\alpha = \cos\beta$, $0 \le \alpha$, $\beta \le \frac{\pi}{2}$, so that $\alpha = \beta$ as asserted.

□

An even nicer way to see the last result does not involve calculus. Consider the point $B'(1,-b)$. Then APB and APB' have the same length. But APB' is shortest when it is a straight line, in which case $\alpha = \beta$.

§5.4. The Cauchy Mean Value Theorem. This generalization of the Mean Lagrange Value Theorem is as follows.

THEOREM 5.4.1. *Suppose*

(i) f, g *are continuous on* $[a,b]$,

(ii) $f'(x), g'(x)$ *exist if* $x \in (a, b)$.

Then

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$$

for some $c \in (a, b)$.

PROOF: Consider $\varphi(x) = [f(x) - f(a)][g(b) - g(a)]$ $-[g(x) - g(a)][f(b) - f(a)]$.

Then φ is continuous on $[a, b]$, $\varphi'(x)$ exists if $x \in (a, b)$ and $\varphi(a) = \varphi(b) = 0$. Thus, by Rolle's Theorem, $\varphi'(c) = 0$ for some $c \in (a, b)$ which gives the result.

$\square$

COROLLARY 5.4.2 (L'HOSPITAL'S RULE). *Let* f', g' *exist near* a *(but not necessarily at* a*). Suppose*

(i) $\lim\limits_{a+} f = 0\ (\infty)$, $\lim\limits_{a+} g = 0\ (\infty)$,

(ii) $\lim\limits_{a+} \frac{f'}{g'} = k$.

Then $\lim\limits_{a+} \frac{f}{g} = k$.

Note: "$\lim\limits_{a+}$" may be replaced by "$\lim\limits_{a-}$", "$\lim\limits_{a}$", "$\lim\limits_{\infty}$", "$\lim\limits_{-\infty}$".

PROOF: The proof in the case $\lim\limits_{a+} f = \infty$, $\lim\limits_{a+} g = \infty$ is quite tricky. We prove only the case $\lim\limits_{a+} f = 0$, $\lim\limits_{a+} g = 0$ here.

Now $f(x), g(x)$ are defined and $f'(x), g'(x)$ exist if $x \in (a, a + \delta_0)$, for some $\delta_0 > 0$. Further (ii) implies $\frac{f'(x)}{g'(x)}$ is defined for x near a so we may also suppose $g'(x) \neq 0$, $x \in (a, a + \delta_0)$.

Define $f(a) = 0$, $g(a) = 0$. Then, from (i), f, g are continuous on $[a, a + \delta_0)$, $g(x) \neq 0$ if $x \in (a, a + \delta_0)$ (why?) and the Cauchy

Mean Value Theorem gives

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c_x)}{g'(c_x)}, \quad \text{for some} \quad c_x \in (a,x).$$

This, together with (ii), implies

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{x\to a+} \frac{f'(c_x)}{g'(c_x)} = k.$$

$\square$

The case when condition (i) is replaced by (i)′

$$\lim_{\infty} f = 0\ (\infty), \quad \lim_{\infty} g = 0\ (\infty)$$

may be deduced from the case of a finite point a by considering the functions

$$F(t) = f(\frac{1}{t}), \quad G(t) = g(\frac{1}{t}),$$

so that

$$F'(t) = -\frac{1}{t^2}\, f'(\frac{1}{t}), \quad G'(t) = -\frac{1}{t^2}\, g'(t)$$

and

$$\frac{F'(t)}{G'(t)} = \frac{-\frac{1}{t^2}f'(\frac{1}{t})}{-\frac{1}{t^2}g'(\frac{1}{t})} = \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})}.$$

Thus

(i) $\lim_{0+} F = \lim_{\infty} f = 0(\infty), \quad \lim_{0+} G = \lim_{\infty} g = 0(\infty)$

(ii) $\lim_{0+} \frac{F'}{G'} = \lim_{\infty} \frac{f'}{g'} = k.$

Hence $\lim_{0+} \frac{F}{G} = k$, which implies $\lim_{\infty} \frac{f}{g} = k$.

$\square$

EXAMPLE 5.4.3:

$$\lim_{x\to 1} \frac{x^{1/2}-x^{1/4}}{x-1} = \frac{1}{4};.$$

Here, let $f(x) = x^{1/2} - x^{1/4}$, $g(x) = x - 1$

(i) $\lim_{x\to 1} f(x) = 0$, $\lim_{x\to 1} g(x) = 0$

(ii) $\lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{\frac{1}{2}x^{-1/2}-\frac{1}{4}x^{-3/4}}{1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

Therefore $\lim_{x\to 1} \frac{f(x)}{g(x)} = \frac{1}{4}$.

□

EXAMPLE 5.4.4:

$$\lim_{x\to 0} \frac{\sin\ x}{x} = 1.$$

Let $f(x) = \sin\ x$, $g(x) = x$

(i) $\lim_{x\to 0} f(x) = 0$ $\lim_{x\to 0} g(x) = 0$

(ii) $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cos\ x}{1} = 1$

Hence $\lim_{x\to 0} \frac{\sin\ x}{x} = 1$.

We now have to admit that we cheated here! Notice that we used the formula $D\ \sin\ x = \cos\ x$. But to derive this formula (Problem 4.9, page 82) we needed to know that $\lim_{x\to 0} \frac{\sin\ x}{x} = 1$; this limit must be shown, strictly speaking, before we can claim to have proved the formula $D\ \sin\ x = \cos\ x$.

EXAMPLE 5.4.5: Given $D_x x^\alpha = \alpha x^{\alpha-1}$, if $x \in \mathbb{R}$, $x > 0$.

$$\begin{aligned}\lim_{x\to 1} \frac{x^\alpha - \alpha x + \alpha - 1}{(x-1)^2} &= \lim_{x\to 1} \frac{\alpha x^{\alpha-1} - \alpha}{2(x-1)} \\ &= \lim_{x\to 1} \frac{\alpha(\alpha-1)x^{\alpha-2}}{2} = \frac{1}{2}\alpha(\alpha-1).\end{aligned}$$

This used l'Hispital's Rule twice and the existence of each limit implies the existence of and its equality with the preceding one. The special cases $\alpha = 0$, $\alpha = 1$ do not require l'Hospital's Rule at all since the numerator is identically zero in these two situations.

We must take care that all of the conditions of l'Hospital's Rule are satisfied each time we use it. There is an error in the following repeated use of the rule. *Find the error.*

EXAMPLE 5.4.6:

$$\lim_{x\to 1}\frac{3x^2-4x+1}{x^2-x}=\lim_{x\to 1}\frac{6x-4}{2x-1}=\lim_{x\to 1}\frac{6}{2}=3.$$

The correct answer is $\lim_{x\to 1}\frac{3x^2-4x+1}{x^2-x}=2$.

□

§5.5. Inverse Functions and Functions Defined Implicitly.

The direct proof of the formula $Dx^{1/n}=\frac{1}{n}x^{\frac{1}{n}-1}$. (Problem 4.8, page 82) and, more generally, $Dx^{m/n}=\frac{m}{n}x^{\frac{m}{n}-1}$, when $x>0$, $m\in\mathbb{Z}, n\in\mathbb{N}$, is a somewhat tedius exercise. The following argument is flawed but more attractive:

$$y=x^{1/n}\Longleftrightarrow y^n=x,\quad\text{when}\quad x>0.$$

Therefore, by the Chain Rule, $ny^{n-1}Dy=1$ and $Dy=\frac{1}{n}y^{1-n}=\frac{1}{n}(x^{1/n})^{1-n}=\frac{1}{n}x^{\frac{1}{n}-1}$. Similarly:

$$y=x^{m/n}\Longleftrightarrow y^n=x^m,\quad\text{when}\quad x>0.$$

Again the chain rule shows $ny^{n-1}Dy=mx^{n-1}$ so that $Dy=\frac{m}{n}y^{1-n}x^{m-1}=\frac{m}{n}x^{\frac{m}{n}(1-n)}x^{m-1}=\frac{m}{n}x^{\frac{m}{n}-1}$.

The difficulty with this argument for proving the formula for $Dx^{1/n}$ is that, even though we established that the n-th root function exists, we need to know that its derivative also exists before we can apply the chain rule to find the derivative.

In this sectin we consider the general problem of when a function f has an *inverse function* g, that is when does g exist such that

$$y = f(x) \Longleftrightarrow g(y) = x?$$

This is equivalent to $g(f(x)) = x$, for all x in the domain of f and to $f(g(y)) = y$ for all y in the range of f. We discuss the existence, continuity and differentiability of inverse functions.

A function f is *one-to-one* $(1-1)$ if

$$u \neq v \Longrightarrow f(u) \neq f(v).$$

This is equivalent to

$$f(u) = f(v) \Longrightarrow u = v.$$

EXAMPLES 5.5.1:

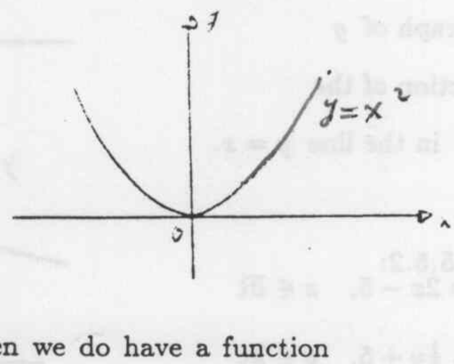
(a) If $f(x) = x^2, x \in \mathbb{R}$

then f is not $(1-1)$

on its domain. However,

if we restrict the domain to

$(-\infty, 0]$, or $[0, \infty)$,



or to any subset of one of these, then we do have a function which is $(1-1)$.

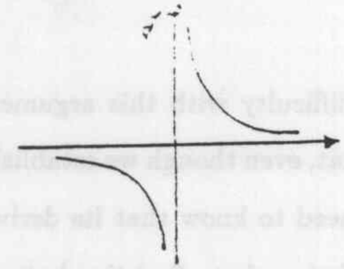
(b) If $g(x) = \frac{1}{x}$, $x \in \mathbb{R}$, $x \neq 0$,

then g is $(1-1)$ on its domain.

Can we define $g(0)$

in such a way that

this new function is $(1-1)$ on its domain?

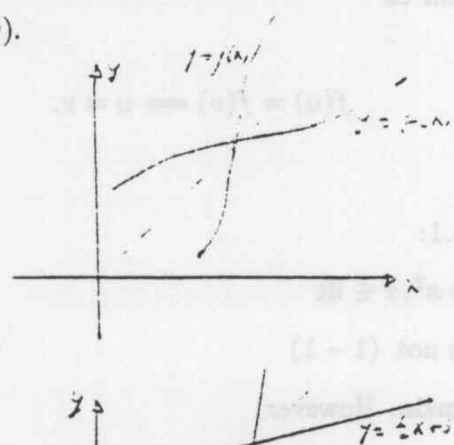


A function f has an inverse $g \iff f$ is $(1-1)$ on its domain. The domain of f is the range of g and conversely the domain of g is the range of f. Notice that g is the inverse of $f \iff f$ is the inverse of g

$$y = f(x) \iff g(y) = x.$$

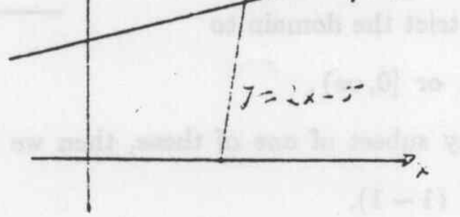
The inverse of f is often denoted f^{-1}. This should not be confused with $\frac{1}{f}$, the reciprocal of f.

If f has an inverse function g, then the graph of f, the curve $y = f(x)$, is the same curve as $x = g(y)$.
To draw the graph of g,
the curve $y = g(x)$, we
swap the x and y axes.
Thus the graph of g
is the reflection of the
graph of f in the line $y = x$.

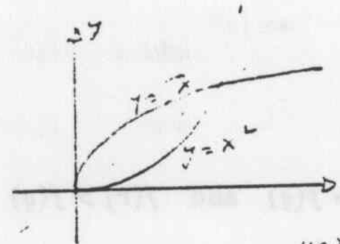


EXAMPLE 5.5.2:

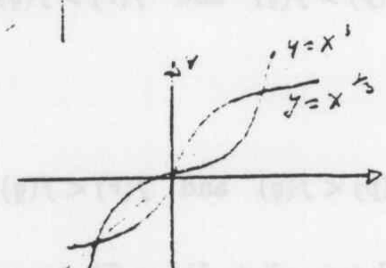
(a)
$$f(x) = 2x - 5, \quad x \in \mathbb{R}$$
$$g(y) = \tfrac{1}{2}y + 5, \quad y \in \mathbb{R}$$



(b)
$$f(x) = x^2, \quad x \geq 0$$
$$g(y) = \sqrt{y}, \quad y \geq 0$$

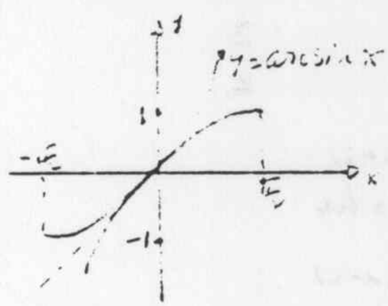


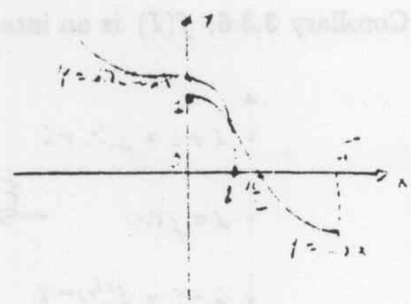
(c)
$$f(x) = x^3, \quad x \in \mathbb{R}$$
$$g(y) = y^{1/3}, \quad y \in \mathbb{R}$$



Notice that the function $g(x) = \frac{1}{x}$ (Example 5.5.1(b)) is its own inverse function. Can you think of other functions with this property?

The sine and cosine functions do not have inverses. However, if we restrict their domains to intervals in which they are increasing on decreasing, the inverse functions are denoted $\sin^{-1}$ (or arc sin) and $\cos^{-1}$ (or arc cos) respectively. Unless specified otherwise arc sin will denote the inverse of the function $\sin\, x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and arc cos will denote the inverse of the function $\cos\, x$, $x \in [0, \pi]$.





PROPOSITION 5.5.3. *Suppose f is continuous and $(1-1)$ on an interval I. Then f is either strictly increasing or strictly decreasing on I.*

PROOF: If f is not monotone on I, then we can find $p, q, r \in I$, $p <$

$q < r$ such that

$$f(p) > f(q) \quad \text{and} \quad f(r) > f(q)$$

or

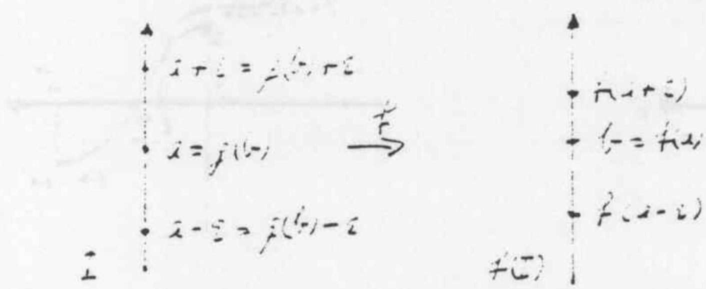
$$f(p) < f(q) \quad \text{and} \quad f(r) < f(q).$$

In either case, the Intermediate Value Theorem implies that there exist $s \in (p,q)$, $t \in (q,r)$ such that $f(s) = f(t)$, which would contradict that is f $(1-1)$.

$\square$

PROPOSITION 5.5.4. *Suppose f is continuous and $(1-1)$ on an interval I. Then its inverse function g is also continuous on $f(I) = \{f(x) : x \in I\}$.*

PROOF: By Proposition 5.5.3, f is strictly increasing or decreasing and, by Corollary 3.3.6, $f(I)$ is an interval.



We will consider only the case that f, and therefore g, is increasing. Let b be an interior point of $f(I)$, $b = f(a)$. If $\varepsilon > 0$ is sufficiently small that $[a-\varepsilon, a+\varepsilon] \subset I$, let

$$\delta = \min\{f(a+\varepsilon) - b,\ b - f(a-\varepsilon)\}.$$

Then

$$\begin{aligned}
|y-b|<\delta &\Longrightarrow f(a-\varepsilon)<y<f(a+\varepsilon) \\
&\Longrightarrow a-\varepsilon<g(y)<a+\varepsilon \ (g \text{ is increasing}) \\
&\Longrightarrow g(b)-\varepsilon<g(y)<g(b)+\varepsilon \ (a=g(b)) \\
&\Longrightarrow |g(y)-g(b)|<\varepsilon.
\end{aligned}$$

Therefore g is continuous at b.

The one-sided continuity of g at an endpoint a is proved similarly.

□

PROPOSITION 5.5.5. *Suppose f is continuous and $(1-1)$ on an interval I with inverse function g. If $b=f(a)$ and $f'(a)=0$, then $g'(b)$ does not exist.*

PROOF: Suppose $f'(x)$, $g'(f(x))$ both exist. Then, since $g(f(x))=x$, we find from the Chain Rule,

$$D_x g(f(x))=1$$
$$g'(f(x))f'(x)=1.$$

But, $f'(a)=0$, so $g'(f(a))f'(a)=1$ implies $g'(f(a))0=1$. Therefore $g'(f(a))$ does not exist.

□

PROPOSITION 5.5.6. *Suppose f is continuous and $(1-1)$ on an interval I with inverse function g. If $b=f(a)$ and f is differentiable* at a with $f'(a) \neq 0$, then g is differentiable at b and

$$g'(b) = \frac{1}{f'(a)}.$$

PROOF: If $y \neq b$

$$\frac{g(y) - g(b)}{y - b} = \frac{x - a}{f(x) - f(a)} \qquad \begin{array}{l} y = f(x) \\ x = g(y) \end{array}$$

Then, by Proposition 5.5.4, if $x_n = g(y_n)$, $n = 1, 2, \ldots,$

$$\lim_{n \to \infty} y_n = b \Longrightarrow \lim_{n \to \infty} x_n = a.$$

Therefore

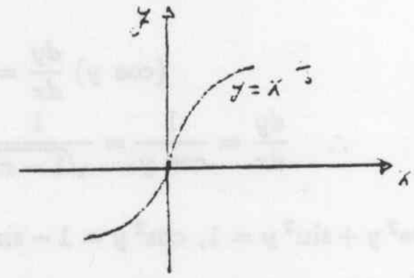
$$\lim_{y \to b} \frac{g(y) - g(b)}{y - b} = \lim_{x \to a} \frac{x - a}{f(x) - f(a)}$$
$$g'(b) = \frac{1}{f'(a)}.$$

□

The important part of this result is that $g'(b)$ exists if $f'(a) \neq 0$. It is then easy to find g'.

EXAMPLE 5.5.7: Let $f(x) = x^3$, $x \in \mathbb{R}$. Then $f'(x) = 3x^2 \neq 0$ if $x \neq 0$. The inverse of f is g where $g(x) = x^{1/3}$. The preceding propositions tell us that $g'(x)$ exist if $x \neq 0$ and $g'(0)$ does not exist

$(0 = f(0) = f'(0))$. To calculate $g'(x)$, suppose $y = g(x)$.



$$\begin{aligned} y &= x^{1/3} \\ \therefore \quad y^3 &= x \\ \therefore \quad 3y^2 \frac{dy}{dx} &= 1 \\ \therefore \quad \frac{dy}{dx} &= \frac{1}{3y^2} \\ &= \frac{1}{3}x^{-\frac{2}{3}}, \quad x \neq 0. \end{aligned}$$

It is easy to check also that, as shown above, $g'(0)$ does not exist

$$\frac{g(x) - g(0)}{x - 0} = \frac{x^{1/3} - 0^{1/3}}{x - 0} = \frac{1}{x^{2/3}}$$

which is unbounded and so does not have a limit at $x = 0$.

We now have another way of deriving the formula $Dx^{1/n} = \frac{1}{n}x^{\frac{1}{n}-1}$ (Problem 4.8, page 82. Proposition 5.5.6 ensures that if $y = x^{1/n}$, then $\frac{dy}{dx}$ exists for all $x \neq 0$ in the domain.

$$\begin{aligned} y^n &= x \\ \therefore \quad ny^{n-1}\frac{dy}{dx} &= 1 \quad \text{(Chain Rule)} \\ \therefore \quad \frac{dy}{dx} &= \frac{1}{ny^{n-1}} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1}, \quad x \neq 0 \end{aligned}$$

EXAMPLE 5.5.8: D arc sin $x = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$. To see this, consider

$$y = \text{ arc sin } x, \quad \sin y = x$$
$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad -1 \leq x \leq 1.$$

We know $\frac{dy}{dx}$ exists $-1 < x < 1$ and does not exist if $x = \pm 1$ (Why?). From the Chain Rule

$$(\cos\ y)\ \frac{dy}{dx} = 1$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

since $\cos^2 y + \sin^2 y = 1$, $\cos^2 y = 1 - \sin^2 y = 1 - x^2$ implies $\cos y = \sqrt{1-x^2}$. We must choose the *positive* square root, since $0 < \cos\ y$ if $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

the procedure we have been using here is called *implicit differentiation* and may be used in more general situations.

EXAMPLE 5.5.4: Suppose the equation $x^2 + y^2 = 1$ may be solved for y in the form $y = f(x)$, where f is differentiable, we may find $\frac{dy}{dx}$ without *explicitly* solving the equation.

$$x^2 + y^2 = 1$$
$$\therefore \quad 2x + 2y\ \frac{dy}{dx} = 0$$
$$\therefore \quad \frac{dy}{dx} = -\ \frac{x}{y}, \quad \text{if} \quad y \neq 0$$

equivalently $f'(x) = -\frac{x}{f(x)}$.

We may verify this independently: $x^2 + y^2 = 1$ has solutions $y = \pm\sqrt{1-x^2}$ and

$$\frac{dy}{dx} = \frac{-x}{\pm\sqrt{1-x^2}} = \frac{-x}{y}.$$

EXAMPLE 5.5.10: Find $\frac{dy}{dx}$, if it exists, given that

$$4x^2 + 2xy - xy^3 = 14.$$

If $\frac{dy}{dx}$ exists, then

$$\begin{aligned} 8x + 2y + 2x\,\frac{dy}{dx} - y^3 - 3xy^2\,\frac{dy}{dx} &= 0 \\ \therefore \quad \frac{dy}{dx}(2x - 3xy^2) &= y^3 - 2y - 8x \\ \therefore \quad \frac{dy}{dx} &= \frac{y^3 - 2y - 8x}{x(2 - 3y^2)} \ . \end{aligned}$$

This may be used to find the slope of the tangent to the curve determined by the equation. For example $(2, -1)$ is a point on the curve and the tangent at that point has slope

$$\frac{dy}{dx} = \frac{-1 + 2 - 16}{2(2 - 3)} = \frac{15}{2} \ .$$

Remember that in the special case of inverse functions (equations of the form $f(y) = x$), the existence of $\frac{dy}{dx}$ was proved when $f'(y) \neq 0$. For the general implicit differentiation we have not proved that $\frac{dy}{dx}$ exists and this must be assumed. The Implicit Function Theorem, a general result dealing with this problem will be discussed in Advanced Calculus.

Problems

5.1 Let $f(x) = |x|, \ -1 \leq x \leq 1$. Then $f(-1) = f(1) = 1$, but there is no $c \in (-1,1)$ for which $f'(c) = 0$. Does this contradict Rolle's Theorem?

5.2 Find $c \in (1,4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}, \quad \text{if } f(x) = x^3 - x.$$

5.3 If $f(x) = (x-a)^m(x-b)^n, \ a \leq x \leq b$, show that the "$c$" of Rolle's Theorem divides the interval $[a,b]$ in the ratio $\frac{m}{n}$ $(0 < m, n \in \mathbb{Q})$.

5.5 If $f(x) = px^2 + qx + r, \quad a \leq x \leq b$, show that the "$c$" of Lagrange's Mean Value Theorem is the midpoint of $[a,b]$.

5.5 Show that the equation

$$x^8 + 5x^2 - 7 = 0$$

has exactly two real roots.

5.6 Let $f(x) = \frac{x^2+1}{x^2-1}, \ x \neq \pm 1$. Identify these intervals where f is increasing, decreasing, concave up and down. Sketch the graph of f with particular attention to the asymptotic behaviour near $x = \pm 1$ and when $|x|$ is large.

5.7 Suppose $f'(x) = \frac{4}{x^5}$, for $x > 0$ and $f(1) = 1$. Find f. Justify your conclusion.

5.8 The strength of a beam of fixed length and rectangular cross-section is proportional to the product of the width by the square of the depth of the cross-section. Find the proportions of the beam of greatest strength that can be cut from a log of circular cross-section.

5.9 Use derivatives to prove Bernoulli's Inequality

$$(1+x)^\alpha > 1+\alpha x, \quad \text{if} \quad x \in (-1,0)\cup(0,\infty), \quad 1 \le \alpha \in \mathbb{Q}.$$

Observe that is a stronger form of the inequality which is previously proved for $1 \le \alpha \in \mathbb{N}$.

5.10 A farmer has 200m of fencing wire; find the largest rectangular area he can enclose.

5.11 If, in #5.10, the area to be fenced is bounded on one side by a straight river, it needs to be enclosed on three sides only. Find the largest area that can be fenced with 200m of wire in this case.

5.12 A body is dropped from a balloon which is rising at 1m/sec and it reaches the ground 12 seconds later. How high was the balloon when the body was dropped?

5.13 An athletic field with a 400m perimeter consists of a rectangle with a semicircle at each end. Find the dimensions of the field if the rectangular portion is to have the largest possible area.

5.14 Given a sphere of radius a, show that the right circular cylinder of largest lateral surface area that can be inscribed in the sphere has radius r and height h given by $r = h/2 = a/\sqrt{2}$.

5.15 Let $f(x) = x/(x^2+1)$.

(a) Find $f'(x)$.

(b) In what intervals is f increasing? decreasing?

(c) Where is f concave up? down?

(d) What is the behaviour of f when $|x|$ is large?

(e) Sketch the graph.

(f) What are the greatest and least values of f in $[-10, 10]$?

5.16 If $\alpha > 0$ and $0 \le a < b$, show that

$$a^\alpha(b-a) < \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha+1} < b^a(b-a).$$

Hint: What was Lagrange's first name?

5.17 Suppose f is differentiable on an interval I and satisfies $|f'(x)| \le M$ for each $x \in I$. Show that

$$|f(x) - f(y)| \le M|x-y|, \quad \text{for each} \quad x, y \in I.$$

5.18 Show that $\lim_{n\to\infty}[(n+1)^\alpha - n^\alpha] = 0$, if $\alpha < 1$.

5.19 What point on the curve $y = \sqrt{x}$ is closest to the point $(1, 0)$? Justify your conclusion.

5.20 Show $D \tan x = \sec^2 x$ [$\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$. Use #4.9].

5.21 If the equation $y^3 - y^2 + 24 = x^3 - x$ can be solved in the form $y = f(x)$ where f is differentiable, show

$$f'(x) = \frac{3x^2 - 1}{3f(x)^2 - 2f(x)}.$$

5.22 Consider the functions f_n, $n = 0, 1, 2$, defined by

$$f_n(x) = \begin{cases} 0, & \text{if } x = 0 \\ x^n \sin(\frac{1}{x}), & \text{if } x \neq 0. \end{cases}$$

(a) Sketch the graph of each of the functions.

(b) Which of the functions are continuous at 0?

(c) Determine whether or not $f_n'(0)$ exists $n = 0, 1, 2$.

5.23 Suppose $f'(x)$ exists, $a < x < b$. Must f' be continuous on (a, b) [*Hint:* Consider f_2' in #5.22].

5.24 Suppose the function $L(x)$ satisfies $L(1) = 0$ and

$$L'(x) = \frac{1}{x}$$

for each $x > 0$.

(a) Let $f(x) = L(10x)$; show $f'(x) = \frac{1}{x}$ [Chain Rule].

(b) Let $g(x) = L(x^n)$; show $g'(x) = \frac{n}{x}$.

(c) Let $h(x) = L(x^2 + 1)$; find $h'(x)$.

(d) If $a, b > 0$, show $L(ab) = L(a) + L(b)$ [Consider $L(ax)$].

5.25 Since $L'(x) > 0$, if $x > 0$, L is increasing on $(0, \infty)$ and therefore has an inverse function E (L as in #5.24).

(a) Show $E'(x) = E(x)$, for all x in the domain of E.

(b) Show $E(a)E(b) = E(a + b)$, for all a, b in the domain of E.

5.26 Let f be a polynomial of degree n. Show that the equation $f(x) = 0$ has at most n distinct solutions in $\mathbb{R}$. [*Hint:* Try induction. What was Rolle's first name?]

5.27 If $x > 0$, let $f(x) = 5x^2 + Ax^{-5}$, where A is a positive constant. Find the smallest A such that $f(x) \geq 24$ for all $x > 0$.

5.28 Find the rectangle of greatest area which has one side on the x-axis and lies under the cureve $y = \frac{1}{x^2+1}$.

5.29 A man in a row boat 3 km off a long straight shore wishes to reach a point 5 km up the shore. If he can row at 2 km/hr and walk at 4 km/hr, describe his fastest route. [*Ans.* Rows to a point $\sqrt{3}$ km upshore, then walks.]

5.30 A 5 ft. fence stands 4 ft. from a high wall. Show that the shortest ladder than can reach the wall from the ground outside the fence has length $(1 + \frac{4}{100^{1/3}})(100^{2/3} + 25)^{1/2}$.

5.31 Let f be continuous on an interval I and have a local maximum at each of two points $u, v \in I$. Show that f has a local minimum at some point between u and v. It cannot be assumed that f has a derivative.

5.32 Suppose $f'(a) > 0$. Show that, for some $\delta > 0$

$$f(x) < f(a), \quad \text{if} \quad x \in (a - \delta, a)$$
$$f(x) > f(a), \quad \text{if} \quad x \in (a, a + \delta).$$

Do not assume that $f'(x)$ exists for any $x \neq a$. [*Hint:* consider $\frac{f(x)-f(a)}{x-a}$, $x \neq a$].

5.33 Suppose f is differentiable on $(0, \infty)$ and

$$\lim_{x\to\infty} f(x), \quad \lim_{x\to\infty} f'(x)$$

both exist. Show that $\lim_{x\to\infty} f'(x) = 0$. [*Hint:* Consider $f(x+1) - f(x)$].

5.34 Suppose

(i) f is continuous at 0,

(ii) $\lim_{x\to 0} f'(x) = \alpha$ exists.

Show that $f'(0)$ exists and equals α. [*Hint:* Consider $\frac{f(x)-f(0)}{x-0}$.]

5.35 [THE DARBOUX PROPERTY OF DERIVATIVES]. Suppose $f'(x)$ exists $a \le x \le b$ and $f'(a) = \alpha$, $f'(b) = \beta$ and γ is a number between α and β. Then there exists a point $c \in (a, b)$ such that $f'(a) = \gamma$.

This problem shows that derivatives, like continuous functions have the 'Intermediate Value Property.' However $f'(x)$ may exist at each point in an interval I but f' need not be continuous on I, as shown by the function f_2' in #5.22.

[*Hint:* If $\alpha < \gamma < \beta$, by considering $g(x) = f(x) - \gamma x$, use $g'(a) < 0 < g'(b)$ with #5.32 to show that g has an interior minimum at some $c \in (a, b)$.]

5.36 For each real t, let $f(x) = -\frac{1}{3}x^3 + t^2 x$ and let $m(t)$ denote the minimum of $f(x)$ over the interval $0 \le x \le 1$. Determine the value of $m(t)$ for each t in the interval $-1 \le t \le 1$. [*Be Careful*].

5.37 Use l'Hospital's Rule to find $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n}$, where $m, n > 0$. Then find the result by some other method.

5.38 Find

(a) $\lim_{x\to 0} \frac{\tan 5x}{\tan x}$,

(b) $\lim_{x\to\frac{\pi}{2}} \frac{\tan 5x}{\tan x}$.

5.39 Show

(a) $\lim_{x\to a+} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}} = \frac{1}{\sqrt{2a}} \quad (a > 0)$,

(b) $\lim_{x\to 0}(\frac{\cot x}{x} - \frac{1}{x^2}) = -\frac{1}{3}$.

5.40 Find

(a) $\lim_{x\to 0} \frac{\sin ax}{\sin bx} \quad (b \neq 0)$,

(b) $\lim_{x\to\pi} \frac{\sin x}{x-\pi}$

(c) $\lim_{x\to 0} \frac{x-\sin x}{x^2}$,

(d) $\lim_{x\to 0} x^{-4}(1 - \cos 2x - 2x^2)$.

5.41 Suppose $100 < x < 121$. Prove

(a) $\frac{1}{22}(x - 100) < \sqrt{x} - 10 < \frac{1}{20}(x - 100)$.

(b) $\frac{1}{32}(121 - x) < 11 - \sqrt{x} < \frac{1}{20}(121 - x)$.

[*Hint:* Apply the Mean Value Theorem to $\sqrt{x}$].

5.42 Show that

(a) $\sin x < x$ for all $x > 0$,

(b) $\frac{2}{\pi}x < \sin x, \quad 0 < x < \frac{\pi}{2}$.

5.43 Sketch the curve $y = (x - 1)(x + 1)^2$, locating points on the curve which correspond to local extrema and inflection points of the function (an inflection point is a point where the function changes concavity). [*Ans.* $(-1, 0)$ local max., $(\frac{1}{3}, \frac{-32}{27})$ local min., $(-\frac{1}{3}, -\frac{16}{27})$ inflection point.]

5.44 Show that the equation

$$2x^3 - 3x^2 + 6x + 6 = 0$$

has exactly one real root and locate the interval $[n, n+1)$ which contains the root.

5.45 Show that for all real numbers a, b

$$|\sin^5 b - \sin^5 a| \leq \frac{16}{5\sqrt{5}} \, |b - a|.$$

5.46 Find

(a) $\lim_{x \to 0} [\frac{1}{x} - \frac{2}{x^2+2x}]$.

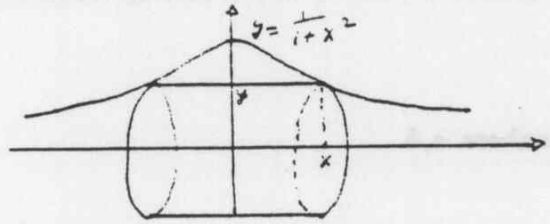
(b) $\lim_{x \to \infty} x^3 (\sin \frac{2}{x})^3$.

5.47 Given n real numbers $a_1, a_2, \ldots, a_n$, prove that the sum $\sum_{k=1}^{n} (x - a_k)^2$ is smallest when $x = \bar{x} = \frac{1}{n} \sum_{k=1}^{n} a_k$, the arithmetic mean of $a_1, \ldots, a_n$.

5.48 Suppose $f''(x) \neq 0$, $0 < x < 1$ and $f(0) < 0$, $f(1) > 1$. Show that the equation $f(x) = x$ has exactly one root in $[0, 1]$.

5.49 A wire of length L is to be cut into two pieces. One of these will then be bent to form a circle; the other will be bent into the shape of a square. How should the wire be cut so that the sum of the areas will be a minimum? a maximum?

5.50 A cylinder is generated by revolving a rectangle about the x-axis. If the base of the rectangle lies on the x-axis and the rectangle lies in the region between the x-axis and the curve $y = \frac{1}{1+x^2}$, find the maximum possible volume of the cylinder (if the maximum exists).



5.51 A conical tent (without floor) is to enclose a given volume V. Find the dimensions of the tent for which the area of canvas used is a minimum. [Volume of cone $= \frac{1}{3}\pi r^2 h$, lateral area of cone $= \pi r\sqrt{r^2 + h^2}$].

5.52 Let $f(x) = x^5 + x$; show that f has an inverse function g and find $g'(2)$.

5.53 Suppose that $f(0) = 0$, f' exists on $[0, \infty)$ and f' is increasing. Show that $g(x) = \frac{f(x)}{x}$ is increasing on $[0, \infty)$. [*Hint:* Consider $g'(x)$. Show this is positive on $(0, \infty)$ by applying the Mean Value Theorem to f on the interval $[0, x]$.

5.54 Show that the equation

$$4x^5 - 5x^4 + 2 = 0$$

has exactly one real root.

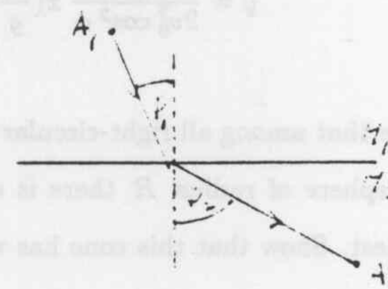
5.55 Let f be a twice differentiable function. If the chord joining two points on the graph of f intersects the graph at a third point, show that $f'' = 0$ at some point.

5.56 Show that $x^2 = x \sin x + \cos x$ for exactly two real values of x.

5.57 Show that, irrespective of the value of b, there is at most one point $x \in [-1,1]$ for which $x^3 - 3x + b = 0$.

5.58 *Fermat's Principle* states that a ray of light travels from a point A_1 to a point A_2 in such a way as to minimize the time taken. Suppose that A_1 is in a medium M_1 (air, say) and A_2 is in a medium M_2 (water) and that the interface is a plane. If c_1 is the speed of light in M_1 and c_2 is the speed of light in M_2. Prove *Snell's Law of Refraction*

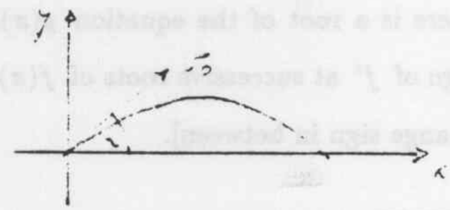
$$\frac{\sin \varphi_1}{\sin \varphi_2} = \frac{c_1}{c_2}$$



where φ_1, φ_2 are the angles between the normal line to the surface and the incident and refracted rays respectively.

5.59 Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a right-circular cone of base radius R and altitude H, [*Ans.* $r = 2R/3$, $h = \frac{1}{3}H$].

5.60 If air resistance is neglected, a cannon ball projected from 0 at time $t = 0$ with muzzle velocity v_0 at an angle α with the horizonal satisfies the equations



$$\begin{aligned} x''(t) &= 0, \quad x'(0) = v_0 \cos\,\alpha, \quad x(0) = 0 \\ y''(t) &= -g, \quad y'(0) = v_0 \sin\,\alpha, \quad y(0) = 0 \end{aligned}$$

where $y(t)$ is the height of the projectile and $x(t)$ is the distance travelled along the line of fire at time t.

(a) Show that the projectile reaches a maximum height

$y = v_0^2 \sin^2 \alpha/2g$ at time $t = v_0 \sin \alpha/g$.

(b) Show that it returns to earth when

$$x = v_0^2 \sin 2\alpha/g \quad \text{at time} \quad t = 2v_0 \sin \alpha/g.$$

(c) Show that the path of the projectile is the parabola

$$y = \frac{g}{2v_0^2 \cos^2 \alpha} x(\frac{v_0^2}{g} \sin 2\alpha - x).$$

5.61 Prove that among all right-circular cones that may be inscribed in a sphere of radius R there is one for which the volume is greatest. Show that this cone has volume $32\pi R^3/81$.

5.62 Let f, g be differentiable functions such that

$$f'g - g'f > 0.$$

Show that between any pair of roots of the equations $f(x) = 0$ there is a root of the equation $g(x) = 0$ [*Hint:* Consider the sign of f' at successive roots of $f(x) = 0$ to show that g must change sign in between].

5.63 Suppose

(i) $f''(x) - g(x)f(x) = 0, \quad a \leq x \leq b,$

(ii) $g(x) > 0, \ a \leq x \leq b,$

(iii) $f(a) = f(b) = 0,$

Show that $f(x) = 0, a \leq x \leq b.$

[*Hint:* If $f(x_0) > 0$ for some $x_0 \in (a, b)$, then f has an interior maximum. Show that this cannot occur.]

5.64 Prove

(a) $D \operatorname{arc} \cos x = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$

(b) $D \operatorname{arc} \tan x = \frac{1}{1+x^2}, \quad x \in \mathbb{R}.$

Here 'arc tan' is the inverse of 'tan' with its domain restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

5.65 If $p_n(x) = 0$ is a polynomial of degree n, show that the equation $p_n(x) = 0$ has at most n real roots. [What was Rolle's first name?]

5.66 (a) If $f(x) = (x^2 - 1)^n$, show that $f^{(k)}(-1) = f^{(k)}(1) = 0$ and $f^{(k)}(x) = 0$ has at least k roots $x \in (-1, 1)$, $k = 1, 2, \ldots, n-1$. [What was Leibniz's first name?]

(b) Show that $f^{(n)}(x)$ is a polynomial of degree n and that $f^{(n)}(x) = 0$ has exactly n real roots all of which lie in $(-1, 1)$.

5.67 Sketch the following curves. Label asymptotes, local maxima and minima, and points of inflection

(i) $y = \frac{x}{(x-1)(x-2)}$, (ii);$y = \frac{x^2}{1+x^2}$
(iii) $y = \frac{x^3}{1+x^2}$, (iv);$y = \frac{x}{(x+1)(x-2)^2}$
(v) $y = x^3 + 3x^2 - 1$, (vi);$y = x^{1/4}(x - y)^2$
(vii) $y = (x^2 - 4)^2$ (viii);$y = x^4 - x^3$

5.68 Suppose that $\lim_{x \to \infty} f'(x) = A$

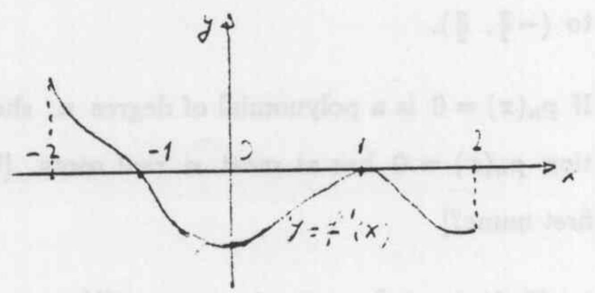
(a) Show that $\lim_{x \to \infty}[f(x + 1) - f(x)] = A$

(b) Find $\lim_{x \to \infty}[f(x + B) - f(x)]$.

5.69 Prove Proposition 5.1.9, that if f is concave up on $[a, b]$ then the chord joining $(a, f(a))$ to $(b, f(b))$ lies above the graph $y = f(x)$, $a < x < b$.

5.70 Let $y = f(x)$, $\quad -\infty < x < \infty$, be a smooth curve which does not pass through $0 = (0,0)$. If $P = (x_0, y_0)$ is a point on the curve which is closest to 0, show that OP is perpendicular to the tangent to the curve at P.

5.71 Suppose f is a differentiable function on $[-2, 2]$ and the graph of its derivative f' is given in the diagram



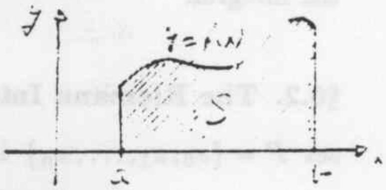
(a) Where is the maximum of f located? Why?

(b) Identify those points where the minimum might be located and justify your choice.

VI. INTEGRATION

§6.1. The Area Problem

Let $f(x) \geq 0,\ a \leq x \leq b$. If we wish to determine the area of the set



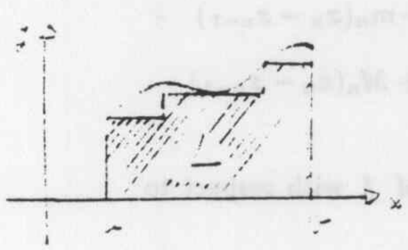
$$S = \{(x,y):\ a \leq x \leq b,\ 0 \leq y \leq f(x)\},$$

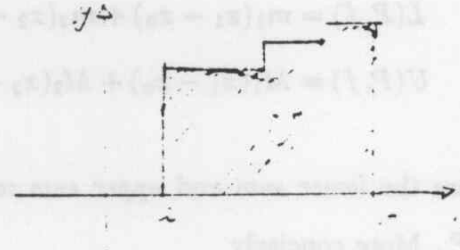
we realize that formulating a precise definition of what we mean by this area is a major step towards solving the problem.

From the definition of the area of a rectangle (length $\times$ breadth) we can easily calculate the area of a region which is the union of a finite set of nonoverlapping rectangles. It is reasonable to require our definition of area to satisfy

$$\text{area } L \leq \text{area } S \leq \text{area } U\,,$$

if L is a union of rectangles contained in S and U is a union of rectangles which contains S.





If there is a *unique* number α satisfying

$$\text{area } L \leq \alpha \leq \text{ area } U$$

for *all* such rectangular union, we *define* $\alpha =$ area S.

We also write $\alpha = \int_a^b f$, *the integral over* $[a,b]$ *of* f . Since there are many applications of the integral besides the calculation of areas, we omit the condition ' $f(x) \geq 0$ ' in the formal definition of the integral.

§6.2. The Riemann Integral. Let $[a,b]$ be a closed interval. The set $P = \{x_0, x_1, \ldots, x_n\}$ is a *partition* of $[a,b]$ if

$$a = x_0 < x_1 < \cdots < x_n = b.$$

A partition Q is a *refinement* of the partition P if $P \subset Q$.

Let f be a *bounded* function on $[a,b]$, $P = \{x_0, x_1, \ldots, x_n\}$ a partition of $[a,b]$ and

$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \quad M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

Then

$$L(P,f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$$
$$U(P,f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$$

are the *lower sum* and *upper sum* respectively of f with respect to P . More concisely

$$L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$
$$U(P,f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Clearly $L(P,f) \leq U(P,f)$, since $m_i \leq M_i$, $i = 1, \ldots, n$.

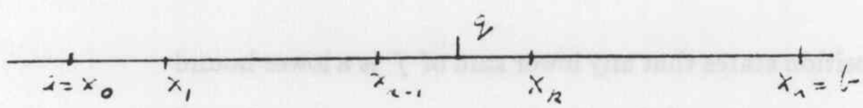
LEMMA 6.2.1. *Let P, Q be partitions of $[a, b]$ with $P \subset Q$. Then*

$$L(P, f) \leq L(Q, f),$$
$$U(P, f) \geq U(Q, f).$$

This lemma states that refinement increases lower sums and decreases upper sums.

PROOF: It is sufficient to prove the lemma when Q contains just one more point than P. Let

$$P = \{x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_n\},$$
$$Q = \{x_0, x_1, \ldots, x_{k-1}, q, x_k, \ldots, x_n\}.$$



Then

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$
$$= \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + m_k(x_k - x_{k-1}) + \sum_{i=k+1}^{m} m_i(x_i - x_{i-1})$$

and

$$L(Q, f)$$
$$= \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + m'_k(q - x_{k-1}) + m''_k(x_k - q) + \sum_{i=k+1}^{m} m_i(x_i - x_{i-1})$$

where $m'_k = \inf\{f(x) : x_{k-1} \leq x \leq q\}$, $m''_k = \inf\{f(x) : q \leq$

$x \leq x_k\}$. Note that $m_k \leq m'_k$ and $m_k \leq m''_k$. Now

$$\begin{aligned} L(Q,f) - L(P,f) &= m'_k(q - x_{k-1}) + m''_k(x_k - q) - m_k(x_k - x_{k-1}) \\ &\geq m_k(q - x_{k-1}) + m_k(x_k - q) - m_k(x_k - x_{k-1}) \\ &= 0, \end{aligned}$$

so that $L(P,f) \leq L(Q,f)$ as asserted. The statement about upper sums is proved in a similar way.

□

PROPOSITION 6.2.2. *Let P, Q be any partitions of $[a,b]$. Then*

$$L(P,f) \leq U(Q,f).$$

This proposition states that any lower sum of f is a lower bound for the set of *all* upper sums and that any upper sum is an upper bound for all the lower sums.

PROOF: Consider the partition $P \cup Q$. Then

$$P \subset P \cup Q, \qquad Q \subset P \cup Q$$

i.e., $P \cup Q$ refines both P and Q. Therefore Lemma 6.2.1 implies

$$L(P,f) \leq L(P \cup Q, f) \leq U(P \cup Q, f) \leq U(Q,f).$$

□

We define the *lower integral* and *upper integral* of f over $[a,b]$, respectively, by

$$\underline{\int_a^b} f = \sup\{L(P,f) : \ P \quad \text{partitions } [a,b]\}$$
$$\overline{\int_a^b} f = \ \inf\{U(P,f) : \ P \quad \text{partitions} \quad [a,b]\}.$$

COROLLARY 6.2.3.

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

This follows immediately from Proposition 6.2.2 and the definition of the upper and lower integrals.

DEFINITION 6.2.4. *If* $\underline{\int_a^b} f = \overline{\int_a^b} f$ *, then we say* f *is Riemann integrable on* $[ab]$ *and*

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

Equivalently, if there is a unique number α *such that*

$$L(P,f) \leq \alpha \leq U(P,f)$$

for all partitions P *of* $[a,b]$ *, then* f *is Riemann integrable on* $[a,b]$ *and*

$$\int_a^b f = \alpha.$$

The number $\int_a^b f$ *is called the Riemann integral of* f *on* $[a,b]$ *.*

THEOREM 6.2.5. $\int_a^b f = \alpha \iff$ *there exists partitions* P_n *of* $[a,b]$, $n = 1,2,\ldots$ *, such that*

$$\lim_{n\to\infty} L(P_n,f) = \alpha \quad \text{and} \quad \lim_{n\to\infty} U(P_n,f) = \alpha.$$

PROOF:

"$\Rightarrow$" : Suppose $\int_a^b f = \alpha$. Then

$$\underline{\int_a^b} f = \overline{\int_a^b} f = \alpha$$

so there exist partition Q_n, R_n of $[a,b]$ such that

$$\alpha - \frac{1}{n} < L(Q_n, f) \le U(R_n, f) < \alpha + \frac{1}{n}, \quad n = 1, 2, \dots .$$

Therefore, if $P_n = Q_n \cup R_n$, $Q_n \subset P_n$ and $R_n \subset P_n$ so that Lemma 6.2.1 implies

$$\alpha - \frac{1}{n} < L(P_n, f) \le U(P_n, f) < \alpha + \frac{1}{n}.$$

The Squeeze Principle implies that the sequences $\{L(P_n, f)\}$, $\{U(P_n, f)\}$ both converge to α.

"$\Leftarrow$" Suppose

$$\lim_{n \to \infty} L(P_n, f) = \alpha \quad \text{and} \quad \lim_{n \to \infty} U(P_n, f) = \alpha.$$

The first of these implies $\underline{\int_a^b} f \ge \alpha$ and the second implies $\overline{\int_a^b} f \le \alpha$. But $\underline{\int_a^b} f \le \overline{\int_a^b}$, and therefore

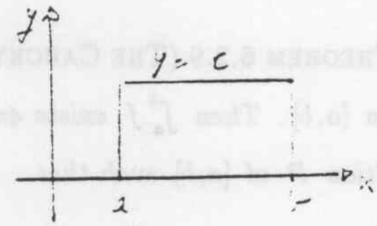
$$\underline{\int_a^b} f = \overline{\int_a^b} f = \alpha$$

so that

$$\int_a^b f = \alpha.$$

□

EXAMPLE 6.2.6: If $f(x) = c,\ a \leq x \leq b$ then $\int_a^b f = c(b-a)$. To see this let $P = \{a, b\}$. Then



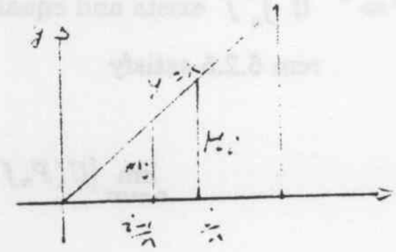
$$L(P, f) = c(b-a) \qquad U(P, f) = c(b, a)$$

so that

$$\underline{\int_a^b} f = \overline{\int_a^b} f = c(b-a).$$

□

EXAMPLE 6.2.7: If $f(x) = x,\ 0 \leq x \leq 1$ then $\int_0^1 f = \frac{1}{2}$. Let $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. Then



$$\begin{aligned} L(P_n, f) &= \sum_{i=1}^{n} \frac{(i-1)}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} (i-1) \\ &= \frac{1}{n^2}(0 + 1 + \cdots + n - 1) = \frac{(n-1)n}{2n^2} \\ U(P_n, f) &= \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2}(1 + 2 + \cdots + n) = \frac{n(n+1)}{2n^2} \end{aligned}$$

so that $\lim_{n\to\infty} L(P_n, f) = \frac{1}{2}$ and $\lim_{n\to\infty} U(P_n, f) = \frac{1}{2}$ and Theorem 6.2.5 implies $\int_0^1 f = \frac{1}{2}$.

□

EXAMPLE 6.2.8: If $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$, then $\int_a^b f$ does not exist for any interval $[a, b]$. Here $L(P, f) = 0$ and $U(P, f) = b - a$ for every partition P of $[a, b]$ so that

$$\underline{\int_a^b} f = 0 < b - a = \overline{\int_a^b} f.$$

□

As in the case of limits, it is important to determine the existence of the integral even when we cannot find its actual value. We have the Cauchy Criterion for integrability.

THEOREM 6.2.9 (THE CAUCHY CRITERION). *Suppose* f *is bounded on* $[a,b]$. *Then* $\int_a^b f$ *exists* $\iff$ *for each* $\varepsilon > 0$, *there exists a partition* P *of* $[a,b]$ *such that*

$$U(P,f) - L(P,f) < \varepsilon.$$

PROOF:

"$\Rightarrow$" If $\int_a^b f$ exists and equals α, then the partitions P_n in Theorem 6.2.5 satisfy

$$\lim_{n\to\infty} [U(P_n f) - L(P_n, f)] = \alpha - \alpha = 0.$$

Thus, if $\varepsilon > 0$, $U(P_n,f) - L(P_n,f) < \varepsilon$, if n is large enough.

"$\Leftarrow$" Suppose that for each $\varepsilon > 0$, there is a partition P such that

$$U(P,f) - L(P,f) < \varepsilon.$$

Then, from the definition of the upper and lower integrals

$$\overline{\int_a^b} f - \underline{\int_a^b} f \le \varepsilon$$

for each $\varepsilon > 0$. Thus

$$\overline{\int_a^b} f - \underline{\int_a^b} f \le 0.$$

But $\overline{\int_a^b} f - \underline{\int_a^b} f \geq 0$, so that $\overline{\int_a^b} f = \underline{\int_a^b} f$ and $\int_a^b f$ exists.

□

PROPOSITION 6.2.10. *Suppose* $a < c < b$. *Then* $\int_a^b f$ *exists* $\iff$ $\int_a^c f, \int_c^b f$ *both exists. Moreover*

$$\int_a^b f = \int_a^c f + \int_c^b f,$$

when either side of this equation exists.

PROOF: Let P be a partition of $[a, b]$ and let $c \in P$. Then if

$$P' = P \cap [a, c], \quad P'' = P \cap [c, b].$$

P' is a partition of $[a, c]$ and P'' is a partition of $[c, b]$. Furthermore

$$\left.\begin{aligned} L(P, f) &= L(P', f) + L(P'', f) \\ U(P, f) &= U(P', f) + U(P'', f) \end{aligned}\right\} \qquad \text{(A)}$$

and, therefore,

$$U(P, f) - L(P, f) = \{U(P', f) - L(P', f)\} + \{U(P'', f) - L(P'', f)\}. \qquad \text{(B)}$$

" $\Longrightarrow$ ": If $\int_a^b f$ exists and $\varepsilon > 0$ then there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

Therefore, from (B),

$$U(P', f) - L(P', f) < \varepsilon, \quad U(P'', f) - L(P'', f) < \varepsilon$$

since both of these expressions are positive. Thus $\int_a^c f$ and $\int_c^b f$ both exist.

" $\Longleftarrow$ " Suppose $\int_a^c f$ and $\int_c^b f$ both exist. Then there exist partitions P_n', P_n'' of $[a,c],[c,b]$ respectively $n = 1,2,\ldots$ such that

$$\lim_{n\to\infty} L(P_n', f) = \lim_{n\to\infty} U(P_n', f) = \int_a^c f$$

and

$$\lim_{n\to\infty} L(P_n'', f) = \lim_{n\to\infty} U(P'', f) = \int_c^b f.$$

Then, with $P_n = P_n' \cup P_n''$, we find from A

$$\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f) = \int_a^c f + \int_c^b f.$$

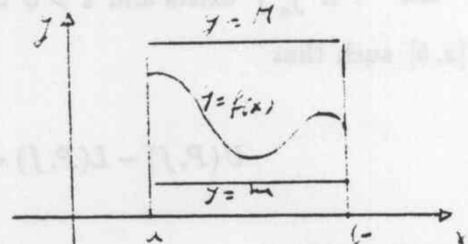
Thus, from Theorem 6.2.5 (p. 128), $\int_a^b f$ exists and equals $\int_a^c f + \int_c^b f$.

PROPOSITION 6.2.11 (LINEARITY OF THE INTEGRAL OPERATION). *Suppose $\int_a^b f, \int_a^b g$ both exist and c is a constant. Then*

(i) $\int_a^b (f+g)$ *exists and equals* $\int_a^b f + \int_a^b g$

(ii) $\int_a^b (cf)$ *exists and equals* $c\int_a^b f$.

PROOF: Exercise.



PROPOSITION 6.2.12. *Suppose*

(i) $\int_a^b f$ *exists,*

(ii) $m \le f(x) \le M, \quad a \le x \le b.$

Then

$$m(b-a) \le \int_a^b f \le M(b-a).$$

PROOF: Consider the partition $P = \{a, b\}$ of $[a, b]$. Then

$$L(P, f) \leq \int_a^b f \leq U(P, f)$$

implies the result asserted since $m(b-a) \leq (\inf\ f)(b-a) = L(P, f)$ and $M(b-a) \geq (\sup\ f)(b-a) = U(P, f)$.

□

COROLLARY 6.2.13. *If* $0 \leq f(x)$, $a \leq x \leq b$ *and* $\int_a^b f$ *exists then* $0 \leq \int_a^b f$.

PROPOSITION 6.2.14. *Suppose*

(i) $\int_a^b f$ *exists,*

(ii) $F(x) = \int_a^x f, \quad a \leq x \leq b$.

Then F *is continuous on* $[a, b]$.

PROOF: Let $[u, v] \subset [a, b]$. Then

$$\begin{aligned}
|F(v) - F(u)| &= \Big| \int_a^v f - \int_a^u f \Big| \\
&= \Big| \int_u^v f \Big| \quad \text{(Proposition 6.2.10)} \\
&\leq K|v - u| \quad \text{(Proposition 6.2.12,} \quad m = -K, M = K)
\end{aligned}$$

if $K = \sup\{|f(x)| : a \leq x \leq b\}$. This implies continuity of f on $[a, b]$, by the Squeeze Principle.

□

§6.3. Uniform Continuity. Recall that "f is continuous at a" means $\lim_{x \to a} f(x) = f(a)$. Equivalently, if $\varepsilon > 0$, there exists $\delta > 0$

such that

$$|x-a|<\delta \Longrightarrow |f(x)-f(a)|<\varepsilon.$$

Note that $\delta=\delta(\varepsilon,a)$; δ depends in general not only on the choice of ε but also on the point a in the domain of f which is being considered. When the choice of δ depends only on ε and not on the point a in consideration we say f is unformly continuous on its domain.

DEFINITION 6.3.1: Let f be a function with domain I. Then f is *uniformly continuous* on I if, for each $\varepsilon>0$, there exists $\delta>0$ with

$$x,y\in I,\ |x-y|<\delta \Longrightarrow |f(x)-f(y)|<\varepsilon.$$

EXAMPLE 6.3.2: The function $f(x)=x$, $x\in\mathbb{R}$, is uniformly continuous on $\mathbb{R}$, since $|x-y|<\delta=\varepsilon \Longrightarrow |f(x)-f(y)|=|x-y|<\varepsilon$.

□

EXAMPLE 6.3.3: The function $f(x)=\frac{1}{x}$, $x>0$, is *not* uniformly continuous on $(0,\infty)$. This function is uniformly continuous on $[1,\infty)$.

Consider $x_n=\frac{1}{n}$, $y_n=\frac{1}{2n}$. Then

$$|x_n-y_n|=\frac{1}{2n},\quad |f(x_n)-f(y_n)|=n.$$

Thus, for each $\delta>0$, there are points $x,y\in(0,\infty)$ such that

$$|x-y|<\delta,\quad |f(x)-f(y)|>1.$$

With $\varepsilon=1$, $I=(0,\infty)$ in Definition 6.3.1, so no $\delta>0$ works. To see that f is uniformly continuous on $[1, \infty)$, consider

$$|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| = \frac{|y-x|}{|xy|} \leq |y - x|$$

if $x, y \in [1, \infty)$. Therefore $|x - y| < \delta = \varepsilon \Longrightarrow |f(x) - f(y)| < \varepsilon$.

□

EXAMPLE 6.3.4: $f(x) = x^2$, $0 \leq x \leq 1$, is uniformly continuous in $[0, 1]$.

$$\begin{aligned} |f(x) - f(y)| = |x^2 - y^2| &= |x + y|\,|x - y| \\ &\leq (|x| + |y|)|x - y| \\ &\leq 2|x - y| \quad \text{if} \quad x, y \in [0, 1]. \end{aligned}$$

Therefore $x, y \in [0, 1], \quad |x - y| < \delta = \frac{\varepsilon}{2} \Longrightarrow |f(x) - f(y)| < \varepsilon$.

□

PROPOSITION 6.3.5. *Suppose f is continuous on the closed interval $[a, b]$. Then f is uniformly continuous on $[a, b]$.*

PROOF: Let f be continuous on $[a, b]$. Suppose that f is *not* uniformly continuous on $[a, b]$. Then for some $\varepsilon_0 > 0$, there exist $x_n, y_n \in [a, b]$ such that

$$|x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon_0, \quad n = 1, 2, \ldots.$$

This is simply the negation of Definition 6.3.1.

Now $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k \to \infty} x_{n_k} = c \in [a, b].$$

But then $\lim_{k\to\infty} y_{n_k} = c$ also $(|x_{n_k} - y_{n_k}| < \frac{1}{n_k})$. Since f is continuous at c

$$0 = |f(c) - f(c)| = \lim_{k\to\infty} |f(x_{n_k} - f(y_{n_k})|,$$

contradicting $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0 > 0$. This contradiction shows f is uniformly continuous on $[a, b]$.

□

The uniform continuity on $[a, b]$ of any function continuous on $[a, b]$ has the following important consequence.

THEOREM 6.3.6. *Suppose f is continuous on $[a, b]$. Then $\int_a^b f$ exists.*

PROOF: Since f is continuous on $[a, b]$, Proposition 6.3.5 states f is uniformly continuous on $[a, b]$. Therefore, if $\varepsilon > 0$, there is a $\delta > 0$ such that

$$x, y \in [a, b], \quad |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{b - a}\,.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ such that $x_i - x_{i-1} < \delta$, $i = 1, 2, \ldots, n$, and

$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \quad M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

Since f is continuous on $[x_{i-1}, x_i]$, Theorem 3.3.4 (p. 60) shows that there exist $p_i, q_i \in [x_{i-1}, x_i]$ such that

$$f(p_i) = m_i, \quad f(q_i) = M_i.$$

Thus

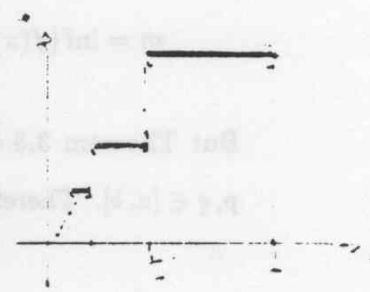
$$
\begin{aligned}
U(P,f)-L(P,f) &= \sum_{i=1}^{m} M_i(x_i - x_{i-1}) - \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \left(f(q_i) - f(p_i)\right)(x_i - x_{i-1}) \\
&< \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1}) \quad (|p_i - q_i| < \delta) \\
&= \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= \frac{\varepsilon}{b-a}(b-a) = \varepsilon.
\end{aligned}
$$

By the Cauchy Criterion, $\int_a^b f$ exists.

□

§6.4. Further Theorems on Integration. Recall that a function f is increasing (decreasing) on an interval I if

$$u, v \in I, u < v \Longrightarrow f(u) \leq f(v) \quad (f(u) \geq f(v)).$$



Note that an increasing function need not be continuous. For example, if $f(x) = \frac{1}{n}$, $\frac{1}{n+1} < x < \frac{1}{n}$ $n = 1, 2, \ldots$ and $f(0) = 0$, then f is increasing on $[0,1]$ but has infinitely many discontinuities there.

In fact, it is possible to construct an increasing function which is discontinuous at every rational number. Nevertheless, functions which are increasing (decreasing) share the following property with continuous functions.

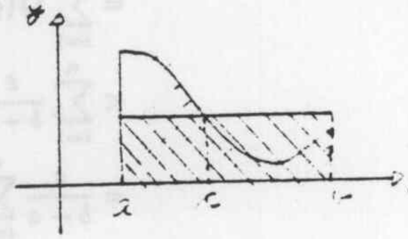
THEOREM 6.4.1. *Suppose f is increasing (decreasing) on $[a,b]$. Then*

$\int_a^b f$ *exists.*

PROOF: Exercise. □

THEOREM 6.4.2 (THE MEAN VALUE THEOREM FOR INTEGRALS). *Suppose f is continuous on $[a, b]$. Then*

$$\int_a^b f = f(c)(b-a)$$



for some $c \in [a, b]$.

PROOF: From Theorem 6.3.6, $\int_a^b f$ exists. By Proposition 6.2.12 (p. 133)

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

so that $m \leq \frac{1}{b-a} \int_a^b f \leq M$, where

$$m = \inf\{f(x) : a \leq x \leq b\}, \quad M = \sup\{f(x) : a \leq x \leq b\}.$$

But Theorem 3.3.4 (p. 60) implies $m = f(p)$, $M = f(q)$ for some $p, q \in [a, b]$. Therefore

$$f(p) \leq \frac{1}{b-a} \int_a^b f \leq f(q)$$

and hence, from the Intermediate Value Theorem (p. 59)

$$f(c) = \frac{1}{b-a} \int_a^b f$$

for some $c \in [a, b]$.

$\square$

NOTATION:

(i) To make the formula

$$\int_a^b f = \int_a^c f + \int_c^b f \qquad \text{(page 131)}$$

independent of whether or not $c \in (a, b)$ it is convenient to *define*

$$\int_v^u f = -\int_u^v f, \quad \text{if} \quad u < v$$

and

$$\int_u^u f = 0.$$

(ii) It is also convenient to use the notation $\int_a^b f(x)dx$ for $\int_a^b f$. In addition to being suggestive of the sums which define the integral, it is efficient. For example, the results of Examples 6.2.5-6 may be written, respectively,

$$\int_a^b cdx = c(b-a),$$
$$\int_0^1 xdx = \frac{1}{2}.$$

The specific letter used is irrelevant:

$$\int_a^b f = \int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du;$$

x or t or u is a 'dummy' variable.

THEOREM 6.4.3 (DIFFERENTIATION THEOREM). *Suppose* *(i)* $\int_a^b f$ *exists.*

(ii) f *is continuous at* $c \in [a, b]$. *

(iii) $F(x) = \int_a^x f,\ a \le x \le b$.

Then $F'(c)$ *exists and equals* $f(c)$

PROOF:

$$F(c+h) - F(c) = \int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f(x)dx.$$

Therefore

$$F(c+h) - F(c) - f(c)h = \int_c^{c+h} [f(x) - f(c)]dx,$$

since $f(c)$ is constant.

If $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon, \quad \text{from (ii).}$$

Therefore $|h| < \delta \Longrightarrow |\int_c^{c+h}[f(a) - f(c)]dx| < \varepsilon|h|$, from Proposition 6.2.12 (p. 133) with $m = -\varepsilon,\ M = \varepsilon,\ b - a = |h|$. From this, $|h| < \delta \Longrightarrow |\frac{F(c+h)-F(c)}{h} - f(c)| < \varepsilon$, which implies $\lim_{h\to 0} \frac{F(c+h)-F(c)}{h} = f(c)$, or

$$F'(c) = f(c).$$

□

* *Note:* When $c = a(b)$, (ii) should be taken as (ii)′ f is right-continuous at a (left continuous at b). Then the conclusion should also be modified to

$$D_+F(a) = f(a) \quad (D_-F(b) = f(b)).$$

A function F is an *antiderivative* of f on an interval I if $F'(x) = f(x)$, for each $x \in I$. If F_0 is an antiderivative of f, then F is also an antiderivative of f if and only if

$$F(x) = F_0(x) + C$$

where C is constant. Clearly $F' = F_0' = f$ if F is of this form and conversely $F' = f = F_0 \Longrightarrow F = F_0 + C$ by Corollary 5.1.5, p. 87.

THEOREM 6.4.4 (THE FUNDAMENTAL THEOREM OF CALCULUS). *Let f be continuous on an interval I. Then a function F on I satisfies*

$$F(b) - F(a) = \int_a^b f, \quad \textit{for all} \quad a, b \in I$$

$\Longleftrightarrow F' = f$ *on I (i.e. F is an antiderivative of f on I.)*

PROOF:

" $\Longrightarrow$ ": Since f is continuous at each $x \in I$, it follows from the preceding theorem that

$$F(x) - F(a) = \int_a^x f \Longrightarrow F'(x) = f(x), \quad \text{for all} \quad x \in I.$$

" $\Longleftarrow$ ": Conversely suppose $F' = f$ on I. Consider $F_0(x) = \int_a^x f$, $x \in I$. Again Theorem 6.4.3 implies $F_0' = f$ on I so that

$$F(x) = F_0(x) + C = \int_a^x f + C.$$

Setting $x = a$, we find $F(a) = C$, and therefore

$$F(x) - F(a) = \int_a^x f.$$

$\square$

A frequently used notation is

$$F(b) - F(a) = F(x)|_a^b.$$

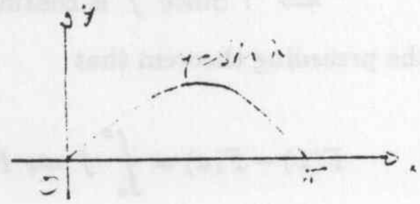
Thus, if F is an antiderivative of f on $[a,b]$,

$$\int_a^b f(x)dx = F(x)|_a^b.$$

$\square$

EXAMPLE 6.4.5: $\int_a^b x dx = \frac{x^2}{2}|_a^b = \frac{1}{2}(b^2 - a^2)$. More generally, from Corollary 4.3.2 (p. 75), if $\alpha \in \mathbb{Q}$, $\alpha \neq -1$, $\int_a^b x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1}|_a^b = \frac{1}{\alpha+1}(b^{\alpha+1} - a^{\alpha+1})$ if $[a,b]$ is an interval in the domain of x^α. Note that $0 \notin [a,b]$ if $\alpha < 0$.

EXAMPLE 6.4.6: The area enclosed by an arch of the sine curve $y = \sin x$ and the x-axis is



$$\int_0^\pi \sin x \, dx = -\cos x|_0^\pi = 2.$$

EXAMPLE 6.4.7: $L(x) = \int_1^x \frac{1}{t}dt$ is a differentiable function, $0 < x < \infty$, with $L'(x) = \frac{1}{x}$, $L(1) = 0$. Thus the function investigated in Problems 5.24, 5.25 (p. 115) exists.

EXAMPLE 6.4.8: $F(x) = \int_0^{x^2}(1+\sin^3 t)dt$ is a differentiable function on $\mathbb{R}$ and

$$F'(x) = [1 + \sin^3(x^2)]2x,$$

since $y = F(x) = \int_0^u (1 + \sin^3 t)dt$, $u = x^2$, implies

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = [1 + \sin^3 u]2x \quad \text{(Chain Rule).}$$

§6.5. Riemann Sums. Let f be a function on $[a, b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ a partition of $[a, b]$. Expressions of the form

$$S(P, f) = \sum_{i=1}^{n} f(x_i')(x_1 - x_{i-1}), \quad \text{where} \quad x_i' \in [x_{i-1}, x_i],$$

are Riemann sums of f with respect to P.

There are many such sums corresponding to a particular partition P since there are infinitely many choices of the points x_i'. Nevertheless, all the sums $S(P, f)$ satisfy

$$L(P, f) \leq S(P, f) \leq U(P, f),$$

where L and U are upper and lower sums. In proving Theorem 6.3.6 on the integrability of continuous functions f, we saw that it was sufficient to require that the points $x_i \in P$ be close together in order that $L(P, f), U(P, f)$ be close to each other and hence to $\int_a^b f$. This in turn ensures that $S(P, f)$ is close to $\int_a^b f$.

For the partition P, let

$$\|P\| = \sup\{x_i - x_{i-1} : i = 1, \ldots, n\},$$

the *norm* of P.

PROPOSITION 6.5.1. *Suppose*

(i) f *is continuous on* $[a, b]$.

(ii) P_n, $n = 1, 2, \dots$ *are partitions of* $[a, b]$ *with* $\lim_{n\to\infty} \|P_n\| = 0$.

(iii) $S(P_n, f)$, $\quad n = 1, 2, \dots$ *are Riemann sums of* f *with respect to* P_n.

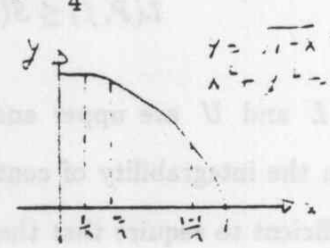
Then

$$\lim_{n\to\infty} S(P_n, f) = \int_a^b f.$$

In fact this proposition is true if (i) is replaced by '(i)' *f is Riemann integrable on* $[a, b]$'. We will not prove this more general statement, however.

EXAMPLE 6.5.2: Given that π is the area of a disc of radius 1, it follows that

$$\lim_{n\to\infty} \frac{1}{n^2}[\sqrt{n^2 - 1^2} + \sqrt{n^2 - 2^2} + \cdots + \sqrt{n^2 - (n-1)^2}] = \frac{\pi}{4}.$$



To see this, observe that

$$\int_0^1 \sqrt{1 - x^2}\, dx = \frac{\pi}{2},$$

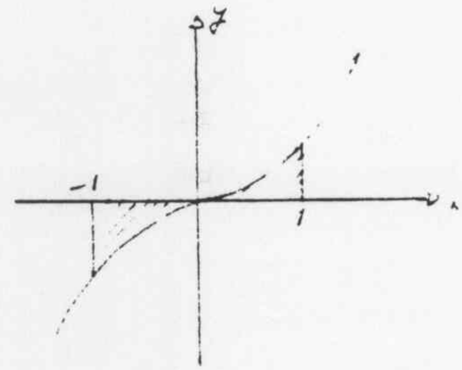
one quarter of the area of a circular disc of radius 1. If $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, n\}$, then $\|P_n\| = \frac{1}{n}$ and

$$\begin{aligned} S(P_n, f) &= \frac{1}{n}[\sqrt{1 - (\frac{1}{n})^2} + \sqrt{1 - (\frac{2}{n})^2} + \cdots + \sqrt{1 - (\frac{n-1}{n})^2} + \sqrt{1 - (\frac{n}{n})^2}] \\ &= \frac{1}{n^2}[\sqrt{n^2 - 1} + \sqrt{n^2 - 2^2} + \cdots + \sqrt{n^2 - (n-1)^2} + 0] \end{aligned}$$

is a Riemann sum (in fact the lower sum) of f with respect to P_n. Since $\lim_{n\to\infty} \|P_n\| = 0$, the assertion $\lim_{n\to\infty} S(P_n, f) = \frac{\pi}{4}$ follows from Proposition 6.5.1.

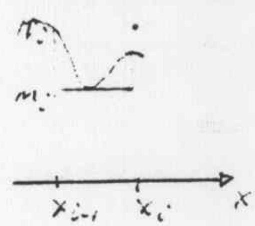
§6.5. The Integral as an Area. Recall that in §6.1 we introduced the integral $\int_a^b f$ as the area of the region bounded by the curves $y = f(x)$, $y = 0$, $x = a$, $x = b$, when $f(x) \geq 0$, $a \leq x \leq b$. This interpretation must be modified if the condition of positivity of f is omitted. For example,

$$\int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4}(1-1)$$
$$= 0$$

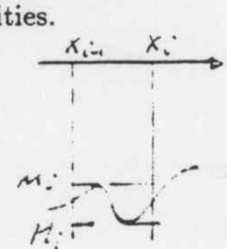


$$\int_{-1}^{0} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{0} = -\frac{1}{4}$$
$$\int_{0}^{1} x^3 dx = \frac{x^4}{4}\Big|_{0}^{1} = \frac{1}{4}\,.$$

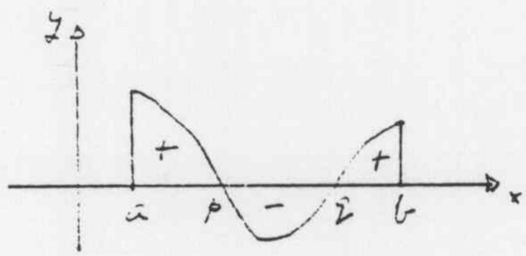
In intervals where f is positive the upper and lower sums are sums of areas of approximating rectangles. But in intervals where f is negative the sums are the negatives of these quantities.



$f > 0$



$f < 0$



$$\int_a^b f = \int_a^p f + \int_p^q f + \int_q^b f.$$

We therefore may interpret $\int_a^b f(x)dx$ to be the sum of signed areas determined by $y = f(x)$, $y = 0$, $x = a$, $x = b$, with the areas being signed positively when they are over the x-axis and negatively when they are under the x-axis.

Problems

6.1 Let $f(x) = \begin{cases} 0, & 0 \le x < 1/2 \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$. Use the definition of the integral to show $\int_0^1 f = \frac{1}{2}$.

6.2 Let $f(x) = \begin{cases} 0, & x = 0 \\ 1, & 0 < x \le 1 \end{cases}$. Use the definition of the integral to show $\int_0^1 f = 1$.

6.3 From the definition of the integral, show $\int_0^1 x^2 dx = \frac{1}{3}$.

6.4 Let $f(x) = \begin{cases} 1, & x = \frac{1}{n}, \quad n = 1, 2, 3, \\ 0, & \text{otherwise} \end{cases}$. Show $\int_0^1 f = 0$.

6.5 Suppose $\int_a^b f$ exists. Show $\int_a^b |f|$ exists.

[*Hint:* Use the triangle inequality to show

$$U(P, |f|) - L(P, |f|) \le U(P, f) - L(P, f).$$

The result then follows from the Cauchy Criterion.]

6.6 Suppose f is bounded on $[a, b]$ and continuous on $(a, b]$. Then $\int_a^b f$ exists and is independent of the value of f at a. Thus, for example, $\int_0^1 \sin(\frac{1}{x})dx$ exists.

6.7 Suppose f is increasing (decreasing) on $[a, b]$; then $\int_a^b f$ exists. Prove this statement (Theorem 6.4.1).

6.8 Use the Fundamental Theorem of Calculus to show:

(a) $\int_0^1 (2x - 6x^4 + 5)dx = \frac{2\cancel{6}4}{5}$,

(b) $\int_{-1}^1 (t-1)(t+2)dt = -\frac{10}{3}$,

(c) $\int_1^2 \frac{1}{u^2} du = \frac{1}{2}$,

(d) $\int_0^1 \frac{3}{2}\sqrt{x}\, dx = 1$,

(e) $\int_{-1}^1 2x(x^2-1)^4 dx = 0$,

(f) $\int_0^1 x^3(x^2-1)^5dx = -\frac{1}{84}$,

Hint: $Dg(x)^n = ng(x)^{n-1}g'(x)$, $x^3 = x(x^2-1)+x$

(g) $\int_{-2}^0 \frac{x}{(5-x^2)^3}\,dx = -\frac{6}{25}$,

(h) $\int_2^4 \frac{dt}{\sqrt{t}} = 2(2-\sqrt{2}\,)$,

(i) $\int_1^x \frac{3t+5}{t^3}dt = \frac{11}{2} - \frac{6x+5}{2x^2}$,

(j) $\int_1^3 (x+\frac{1}{x})(x-\frac{1}{x})dx = 8$

(k) $\int_{-3}^3 5|x+2|dx = 65$,

(l) $\int_1^2 (6-x)^{-3}dx = \frac{9}{800}$.

6.9 Suppose $\int_a^b f$, exists. Show $\int_a^b f^2$ exists.

6.10 Suppose $\int_a^b f$, $\int_a^b g$ both exist. Show

(a) $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

(b) $\int_a^b (cf) = c\int_a^b f$ (c constant)

6.11 Suppose $\int_a^b f$, $\int_a^b g$ both exist. Show $\int_a^b fg$ exists.

[*Hint:* use #6.9, 6.10 to consider $\int_a^b (f+g)^2$, $\int_a^b (f-g)^2$].

6.12 Find $f'(x)$ for each of the following

(a) $f(x) = \int_1^x \frac{dt}{t}$,

(b) $f(x) = \int_1^{x^2} \frac{dt}{t}$,

(c) $f(x) = \int_x^{x^2} \frac{dt}{t}$.

6.13 Let $f(x) = \frac{1}{1+x^3}$, $0 \le x \le 1$ and $P = \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$.

(a) Sketch the graph of f

(b) Show that $\int_0^1 f$ differs from $L(P,f)$ by at most $\frac{1}{10}$.

6.14 (a) State a result which allows you to conclude that $\int_0^2 \frac{1}{x+1}\,dx$ exists.

(b) Prove that $\frac{5}{6} < \int_0^2 \frac{1}{x+1}\,dx < \frac{3}{2}$.

6.15 Show $\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}\, dx = \frac{\pi}{4}$. [*Hint:* Example 5.5.8].

6.16 Show

(a) $\int_0^1 x\sqrt{x^2+1}\, dx = \frac{1}{3}(2^{3/2}-1)$

(b) $\int_0^1 x^3\sqrt{x^2+1}\, dx = \frac{2}{15}(2^{1/2}+1)$.

6.17 Suppose

(i) $\int_a^b f$, $\int_a^b g$ both exist.

(ii) $f(x) \le g(x)$, if $a \le x \le b$.

Show that $\int_a^b f \le \int_a^b g$ [*Hint:* Corollary 6.2.12]

6.18 If $\int_0^x f(t)dt = x^2(1+x)$, for all $x \in \mathbb{R}$, find $f(5)$, if f is continuous

6.19 Find $F'(x)$ if $\int_{x^2}^{x^3} \frac{t^5}{1+t^{10}}\, dt = F(x)$.

6.20 Using antiderivatives, compute the following

(a) $\int_1^4 (3x^2 - \frac{1}{\sqrt{x}} + x^{3/2})dx$

(b) $y(t)$ if $\frac{dy}{dt} = \sqrt{1+t} + \frac{1}{t^2}$ and $y(3) = 5$.

(c) $\int_0^1 x^3\sqrt{1+3x^2}\, dx$

6.21 (a) Prove that $\int_0^1 (1+x)^n dx = \frac{2^{n+1}-1}{n+1}$, $n = 1, 2, \ldots$.

(b) Deduce from (a) that

$$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}, \quad n = 1, 2, \ldots$$

6.22 Let $f(x) = \frac{1}{1+x^4}$, $0 \le x \le 1$.

(a) Show f is decreasing on $[0,1]$

(b) Let $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Write down (but *do not simplify*) the upper and lower sums of f corresponding to this partition.

(c) Show that

$$0 \leq \int_0^1 \frac{1}{1+x^4}\, dx - L \leq \frac{1}{6}$$

where L is the lower sum found in (b).

6.23 Let f be continuous on $[a,b]$ and g, fg be integrable on $[a,b]$ with $g(x) \geq 0$, $a \leq x \leq b$. Show that

$$\int_a^b fg = f(c) \int_a^b g$$

for some $c \in [a,b]$. Note that this is the Mean Value Theorem for integrals in the case that $g(x) = 1$, $a \leq x \leq b$.

6.24 Given that $D \sin x = \cos x$, prove

$$\lim_{n\to\infty} \frac{\pi}{2n}(\cos \frac{\pi}{2n} + \cos \frac{3\pi}{2n} + \cdots + \cos \frac{n\pi}{2n}) = 1.$$

6.25 Let $f(x) = \begin{cases} \frac{1}{q}, & \text{if } \ x = \frac{p}{q}, p, q \ \text{ integers with no} \\ & \text{common divisors, } \quad q \neq 0 \\ 0, & \text{otherwise.} \end{cases}$

Show $\int_a^b f$ exists and equals 0 for every interval $[a,b]$. Compare with Example 6.2.8, page 139.

6.26 Show

$$\lim_{n\to\infty} \frac{1}{n^5} \sum_{j=1}^{n} j^4 = \frac{1}{5}$$

$$\lim_{n\to\infty} \frac{1}{n^{3/2}} \sum_{j=1}^{n} \sqrt{j} = \frac{2}{3}\,.$$