Math 227
Suggested solutions to the Final Exam

Problem 1. (a) We recall that the field $\mathbb{K}$ is also a commutative ring. Therefore, by HW1, Problem 3, parts (i) and (ii), we get that $\mathrm{Range}(\phi)$ is a commutative subring of $\mathcal{R}$. We also recall that the neutral element of addition in $\mathrm{Range}(\phi)$ is $0_{\mathcal{R}}$, while the neutral element of multiplication in $\mathrm{Range}(\phi)$ is $1_{\mathcal{R}}$.

Now, to show that $\mathrm{Range}(\phi)$ is a field, it remains to prove that every non-zero element in $\mathrm{Range}(\phi)$ has a multiplicative inverse.

Let $a \in \mathrm{Range}(\phi)$, $a \neq 0_{\mathcal{R}}$. Then there is $u \in \mathbb{K}$ such that $a = \phi(u)$, and since we have seen that $\phi(0_{\mathbb{K}}) = 0_{\mathcal{R}}$, we must have that $u \neq 0_{\mathbb{K}}$. But then, since $\mathbb{K}$ is a field, u has a multiplicative inverse u^{-1} in $\mathbb{K}$. We can then write

$$\begin{aligned} 1_{\mathcal{R}} = \phi\big(1_{\mathbb{K}}\big) &= \phi\big(u \cdot_{\mathbb{K}} u^{-1}\big) \\ &= \phi(u) \cdot_{\mathcal{R}} \phi(u^{-1}) = a \cdot_{\mathcal{R}} \phi(u^{-1})\,. \end{aligned}$$

This shows that the element $\phi(u^{-1})$ in $\mathrm{Range}(\phi)$ is a right inverse of a, and given that we have already recalled that multiplication within $\mathrm{Range}(\phi)$ will be commutative, $\phi(u^{-1})$ is also a left inverse of a. Thus a has a multiplicative inverse in $\mathrm{Range}(\phi)$.

Since a was an arbitrary non-zero element of $\mathrm{Range}(\phi)$, we conclude that $\mathrm{Range}(\phi)$ satisfies this field axiom too, and thus that it is a field.

(b) Let us set $\mathbb{K} = \mathbb{R}$ and $\mathcal{R} = \mathbb{R}^{2\times 2}$. We recall that $\mathbb{R}^{2\times 2}$ is a non-commutative ring, therefore it cannot be a field.

We also define

$$\phi : \mathbb{R} \to \mathbb{R}^{2\times 2}, \qquad r \in \mathbb{R} \;\mapsto\; \phi(r) := \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}.$$

Then

$$\phi\big(1_{\mathbb{R}}\big) = \phi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Id}_{\mathbb{R}^{2\times 2}}.$$

Also, for every $r_1, r_2 \in \mathbb{R}$, we have

$$\phi(r_1 + r_2) = \begin{pmatrix} r_1 + r_2 & 0 \\ 0 & r_1 + r_2 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_1 \end{pmatrix} + \begin{pmatrix} r_2 & 0 \\ 0 & r_2 \end{pmatrix} = \phi(r_1) + \phi(r_2)$$

$$\text{and} \quad \phi(r_1 \cdot r_2) = \begin{pmatrix} r_1 \cdot r_2 & 0 \\ 0 & r_1 \cdot r_2 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_1 \end{pmatrix} \cdot \begin{pmatrix} r_2 & 0 \\ 0 & r_2 \end{pmatrix} = \phi(r_1) \cdot \phi(r_2).$$

Thus ϕ is a ring homomorphism, as wanted.

Problem 2. (a) The conclusion about geometric multiplicities is true.

Justification: We recall that, given an eigenvalue λ of A, and hence also of A^T, its geometric multiplicity with respect to A is the dimension of the Nullspace of the matrix $A-\lambda I_n$, while its geometric multiplicity with respect to A^T is the dimension of the Nullspace of the matrix $A^T-\lambda I_n$.

Observe first of all that $A^T-\lambda I_n = A^T - \lambda I_n^T = \big(A-\lambda I_n\big)^T$.

Also, by Main Theorem D that we stated and proved in class, we have that

$$\begin{aligned} \dim_{\mathbb{F}} N(A-\lambda I_n) = \mathrm{nullity}(A-\lambda I_n) &= n - \mathrm{rank}(A-\lambda I_n), \\ \text{and analogously} \quad \mathrm{nullity}(A^T-\lambda I_n) &= n - \mathrm{rank}(A^T-\lambda I_n). \end{aligned}$$

At the same time, by Main Theorem C,

$$\begin{aligned} \mathrm{rank}(A-\lambda I_n) = \dim_{\mathbb{F}}\mathrm{RS}(A-\lambda I_n) &= \dim_{\mathbb{F}}\mathrm{CS}(A-\lambda I_n) \\ &= \dim_{\mathbb{F}}\mathrm{RS}\big((A-\lambda I_n)^T\big) \\ &= \dim_{\mathbb{F}}\mathrm{RS}(A^T-\lambda I_n) = \mathrm{rank}(A^T-\lambda I_n). \end{aligned}$$

Combining the above, we obtain that $\mathrm{nullity}(A-\lambda I_n) = \mathrm{nullity}(A^T-\lambda I_n)$, or in other words that the geometric multiplicity of λ with respect to A the same as with respect to A^T.

(b) Let $Q = \big(q_{ij}\big)_{1\leqslant i,j\leqslant n}$ be a stochastic matrix in $\mathbb{R}^{n\times n}$. Then

$$Q\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n q_{1,j} \\ \sum_{j=1}^n q_{2,j} \\ \vdots \\ \sum_{j=1}^n q_{n,j} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

In other words, $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector of Q corresponding to eigenvalue 1.

(c) This is false.

It suffices to give a counterexample: let us suppose $n > 1$ and write E_{ij} for the matrix in $\mathbb{R}^{n\times n}$ whose (i,j)-th entry is equal to 1, while any other entry is equal to 0; moreover, let us set

$$Q = \frac{1}{2}(E_{11}+E_{12}) + \sum_{i=2}^n E_{ii}.$$

Then Q is a row stochastic matrix, and we have that

$$Q^T = \frac{1}{2}(E_{11} + E_{21}) + \sum_{i=2}^{n} E_{ii}.$$

But then

$$\begin{aligned} Q^T \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} &= Q^T \big(\bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \cdots + \bar{e}_n\big) \\ &= \mathrm{Col}_1(Q^T) + \mathrm{Col}_2(Q^T) + \mathrm{Col}_3(Q^T) + \cdots + \mathrm{Col}_n(Q^T) \\ &= \frac{1}{2}\big(\bar{e}_1 + \bar{e}_2\big) \;+\; \bar{e}_2 \;+\; \bar{e}_3 \;+\; \cdots \;+\; \bar{e}_n = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \end{aligned}$$

Given that

$$\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \\ \vdots \\ 1 \end{pmatrix} \neq r \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

for any $r \in \mathbb{R}$, we conclude that $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ cannot be an eigenvector of Q^T.

Problem 3. By standard properties of determinants, we have that

$$\det(A) = \det\left(\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ 3 & -4 & 5 & -6 \end{pmatrix}\right) + \det\left(\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 3 & -4 & 5 & -6 \end{pmatrix}\right)$$

(by linearity in the 3rd row)

$$= 0 \quad + \quad \det\left(\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 3 & -4 & 5 & -6 \end{pmatrix}\right)$$

(by the alternating property)

$$= \det\left(\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 1 & -2 & 3 & -4 \end{pmatrix}\right) + \det\left(\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 \end{pmatrix}\right)$$

(by linearity in the 4th row)

$$= 0 \quad + \quad \det\left(\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 \end{pmatrix}\right).$$

(by the alternating property)

We now also note that

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 2 \\ 2 & 2 & 2 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix} = E_{21;2}E_{43;1}\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix} = E_{34;3} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 2 \\ 2 & 2 & 2 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & 1 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix} = E_{14;3} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix} = E_{12;2} \begin{pmatrix} 5 & 1 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix} = E_{31;5} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix}.$$

In other words,

$$\det(A) = \det \left(\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 \end{pmatrix} \right),$$

while

$$E_{31;5}E_{12;2}E_{14;3}E_{34;3}E_{21;2}E_{43;1} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix},$$

therefore, by the Multiplication Theorem, and the fact that the determinant of an elementary matrix of the type $E_{ij;\lambda}$ *(that is, an elementary matrix corresponding to the elementary row operation of adding to the i-th row the j-th row multiplied by λ)* is always equal to 1, we obtain

$$\det \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix} \right)$$

$$= \det(E_{31;5}) \cdot \det(E_{12;2}) \cdot \det(E_{14;3}) \cdot \det(E_{34;3}) \cdot \det(E_{21;2}) \cdot \det(E_{43;1}) \cdot \det \left(\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 \end{pmatrix} \right)$$

$$= \det \left(\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 \end{pmatrix} \right) = \det(A).$$

It remains to compute the determinant of the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix}$. By using Laplace expansion along the 1st row, and then along the 2nd row of the submatrix we get, we see that

$$\begin{aligned}
\det\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix}\right) &= (-1)^{1+2} \cdot 1 \cdot \det\left(\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 2 \\ 4 & 4 & 0 \end{pmatrix}\right) \\
&= -(-1)^{2+3} \cdot 2 \cdot \det\left(\begin{pmatrix} 1 & 0 \\ 4 & 4 \end{pmatrix}\right) \\
&= 2 \cdot 4 = 1.
\end{aligned}$$

We conclude that

$$\det(A) = \det\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 4 & 0 & 4 & 0 \end{pmatrix}\right) = 1.$$

Problem 4. The answer is yes.

Justification: We note that the set $\{x+1,\, x^2+x\}$ is a linearly independent subset of $\mathcal{P}_3$. In fact, we can extend this to a basis of $\mathcal{P}_3$ by including the polynomials 1 and x^3: the set

$$\mathcal{B} = \{1,\, x+1,\, x^2+x,\, x^3\}$$

is a basis of $\mathcal{P}_3$.

We now recall that, for every function $\phi : \mathcal{B} \to \mathbb{R}^{3\times 2}$, there is a unique linear extension $f : \mathcal{P}_3 \to \mathbb{R}^{3\times 2}$.

For example, here we can define

$$\phi(x+1) = \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 0.5 & 0 \end{pmatrix} \quad \text{and} \quad \phi(x^2+x) = \begin{pmatrix} 0 & -2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{as we want,}$$

$$\text{as well as} \quad \phi(1) = \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 0.5 & 0 \end{pmatrix} \quad \text{and} \quad \phi(x^3) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We then extend this linearly: if $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ is a polynomial in $\mathcal{P}_3$, then we can also write $p(x) = b_0 + b_1(x+1) + b_2(x^2+x) + b_3 x^3$ for some $b_0, b_1, b_2, b_3 \in \mathbb{R}$ (in fact, a unique choice of such coefficients, since $\mathcal{B}$ is a basis), and then we should set

$$\begin{aligned}
f(p) &= b_0 \cdot \phi(1) + b_1 \cdot \phi(x+1) + b_2 \cdot \phi(x^2+x) + b_3 \cdot \phi(x^3) \\
&= b_0 \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 0.5 & 0 \end{pmatrix} + b_1 \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 0.5 & 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 & -2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= b_0 \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 0.5 & 0 \end{pmatrix} + b_1 \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 0.5 & 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 & -2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

To figure out a formula for f, we can note that in particular we should have

$$\begin{aligned}
f(x) = f\big((x+1)-1\big) &= f(x+1) - f(1) && \text{(since } f \text{ will be linear)} \\
&= \phi(x+1) - \phi(1) && \text{(since } f \text{ extends } \phi) \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
f(x^2) = f\big(\big((x^2+x) - x\big) &= f(x^2+x) - f(x) \\
&= \phi(x^2+x) - f(x) \\
&= \begin{pmatrix} 0 & -2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

We conclude that, since f must be linear, we will have

$$\begin{aligned}
f(p) &= f\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3\big) \\
&= a_0 \cdot f(1) + a_1 \cdot f(x) + a_2 \cdot f(x^2) + a_3 \cdot f(x^3) \\
&= a_0 \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 0.5 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2a_0 & 3a_0 - 2a_2 \\ 0 & a_0 - a_2 \\ \frac{1}{2}a_0 + a_2 & 0 \end{pmatrix}.
\end{aligned}$$

It is not hard to double check that this function f from $\mathcal{P}_3$ to $\mathbb{R}^{3\times 2}$ is linear, and that it maps the polynomials $x+1$ and x^2+x to the matrices we wanted respectively.

Problem 5. (a) Let $A = (a_{ij})_{1\leqslant i,j\leqslant 3}$ and $B = (b_{ij})_{1\leqslant i,j\leqslant 3}$ be two elements of U. Then $A + B = (a_{ij} + b_{ij})_{1\leqslant i,j\leqslant 3}$ and

$$\begin{aligned} f(A+B) &= \begin{pmatrix} (a_{11}+b_{11}) + 2(a_{12}+b_{12}) + (a_{33}+b_{33}) \\ (a_{22}+b_{22}) + (a_{23}+b_{23}) \end{pmatrix} \\ &= \begin{pmatrix} a_{11}+2a_{12}+a_{33} \\ a_{22}+a_{23} \end{pmatrix} + \begin{pmatrix} b_{11}+2b_{12}+b_{33} \\ b_{22}+b_{23} \end{pmatrix} = f(A) + f(B). \end{aligned}$$

Consider also $r \in \mathbb{R}$. Then $rA = (ra_{ij})_{1\leqslant i,j\leqslant 3}$ and

$$f(rA) = \begin{pmatrix} ra_{11}+2ra_{12}+ra_{33} \\ ra_{22}+ra_{23} \end{pmatrix} = r\begin{pmatrix} a_{11}+2a_{12}+a_{33} \\ a_{22}+a_{23} \end{pmatrix} = rf(A).$$

Since $A, B \in U$ and $r \in \mathbb{R}$ were arbitrary elements, we conclude that f is linear.

(b) We first describe $\mathrm{Ker}(f)$: we have that $A = (a_{ij})_{1\leqslant i,j\leqslant 3}$ is in $\mathrm{Ker}(f)$ if and only if

$$\begin{pmatrix} a_{11}+2a_{12}+a_{33} \\ a_{22}+a_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad a_{33} = -a_{11} - 2a_{12} \quad \text{and} \quad a_{23} = -a_{22}.$$

Therefore, $A = (a_{ij})_{1\leqslant i,j\leqslant 3}$ is in $\mathrm{Ker}(f)$ if and only if

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & -a_{22} \\ a_{31} & a_{32} & -a_{11} - 2a_{12} \end{pmatrix}.$$

If we now write E_{ij} for the matrix whose (i,j)-th entry is equal to 1, while any other entry is equal to 0, we get that $A = (a_{ij})_{1\leqslant i,j\leqslant 3}$ is in $\mathrm{Ker}(f)$ if and only if

$$A = a_{11}(E_{11} - E_{33}) + a_{12}(E_{12} - 2E_{33}) + a_{22}(E_{22} - E_{23}) + \sum_{\substack{(i,j) \notin \{(1,1),(1,2), \\ (2,2),(2,3),(3,3)\}}} a_{ij}E_{ij}\,.$$

We note at the same time that the set

$$\mathcal{B}_0 = \big\{E_{11} - E_{33},\ E_{12} - 2E_{33},\ E_{22} - E_{23}\big\} \cup \{E_{ij} : (i,j) \neq (1,1),(1,2),(2,2),(2,3),(3,3)\}$$

is linearly independent (given that $\{E_{ij} : 1 \leqslant i,j \leqslant 3\}$ is a basis of U, and that the elements of the above set are linear combinations of these basis vectors, with each such linear combination containing a non-zero scalar multiple of some basis vector which doesn't appear in any of the other linear combinations).

We can then conclude that $\mathcal{B}_0$ is a basis of $\mathrm{Ker}(f)$.

We now recall (from HW5, Problem 1) that an equivalent formula for the inner product on U is the following: if $A = (a_{ij})_{1\leqslant i,j\leqslant 3}$ and $B = (b_{ij})_{1\leqslant i,j\leqslant 3}$ are in U, then

$$\langle A, B\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}b_{ij}\,.$$

Based on this, we can see that the subset

$$\mathcal{B}_{0,1} = \big\{E_{22} - E_{23}\big\} \cup \{E_{ij} : (i,j) \neq (1,1),(1,2),(2,2),(2,3),(3,3)\}$$

of $\mathcal{B}_0$ is already orthogonal, and also that each of the matrices $E_{11} - E_{33}$ and $E_{12} - 2E_{33}$ is orthogonal to each element in $\mathcal{B}_{0,1}$, which also implies that so is every matrix in $\mathrm{span}\big(\big\{E_{11} - E_{33},\ E_{12} - 2E_{33}\big\}\big)$. Therefore, to find an orthogonal basis of $\mathrm{Ker}(f)$, it suffices to find an orthogonal basis of $\mathrm{span}\big(\big\{E_{11} - E_{33},\ E_{12} - 2E_{33}\big)$. We can use the Gram-Schmidt orthogonalisation process to do this: we note that

$$\begin{aligned} &E_{12} - 2E_{33} - \frac{\langle E_{12} - 2E_{33}, E_{11} - E_{33}\rangle}{\langle E_{11} - E_{33}, E_{11} - E_{33}\rangle}\cdot(E_{11} - E_{33}) \\ &\quad = E_{12} - 2E_{33} - (E_{11} - E_{33}) = E_{12} - E_{11} - E_{33} \end{aligned}$$

is orthogonal to $E_{11} - E_{33}$, and also $\mathrm{span}\big(\big\{E_{11} - E_{33},\ E_{12} - E_{11} - E_{33}\big\}\big) = \mathrm{span}\big(\big\{E_{11} - E_{33},\ E_{12} - 2E_{33}\big\}\big)$.

We conclude that

$$\widetilde{\mathcal{B}_0} = \big\{E_{11}-E_{33},\ E_{12}-E_{11}-E_{33},\ E_{22}-E_{23}\big\}\cup\{E_{ij} : (i,j) \neq (1,1),(1,2),(2,2),(2,3),(3,3)\}$$

is an orthogonal basis of $\mathrm{Ker}(f)$.

Finally, we can see that

$$\mathrm{span}\big(\widetilde{\mathcal{B}_0} \cup \{E_{33},\ E_{23}\}\big) = \mathrm{span}\big(\{E_{ij} : 1 \leqslant i,j \leqslant 3\}\big),$$

therefore $\widetilde{\mathcal{B}_0} \cup \{E_{33},\ E_{23}\}$ is a basis of U (since this set also has size 9). Similarly to above, we can see that E_{33} is orthogonal to the set

$$\big\{E_{22} - E_{23}\big\} \cup \{E_{ij} : (i,j) \neq (1,1),(1,2),(2,2),(2,3),(3,3)\}$$

and that E_{23} is orthogonal to the set

$$\big\{E_{11} - E_{33},\ E_{12} - E_{11} - E_{33}\big\} \cup \{E_{ij} : (i,j) \neq (1,1),(1,2),(2,2),(2,3),(3,3)\}.$$

Thus, in order to find an orthogonal basis of U that extends $\widetilde{B}_0$, it suffices to find an orthogonal basis of $\mathrm{span}\big(\{E_{11}-E_{33},\ E_{12}-E_{11}-E_{33},\ E_{33}\}\big)$ that contains the first two vectors of the spanning set we already have, and an orthogonal basis of $\mathrm{span}\big(\{E_{22}-E_{23},\ E_{23}\}\big)$ that contains $E_{22}-E_{23}$. For the latter, we note that

$$E_{23}-\frac{\langle E_{23},E_{22}-E_{23}\rangle}{\langle E_{22}-E_{23},E_{22}-E_{23}\rangle}\cdot(E_{22}-E_{23}) = E_{23}+\frac{1}{2}\,(E_{22}-E_{23}) = \frac{1}{2}E_{22}+\frac{1}{2}E_{23}$$

is orthogonal to $E_{22}-E_{23}$, and that

$$\mathrm{span}\big(\{E_{22}-E_{23},\ \tfrac{1}{2}E_{22}+\tfrac{1}{2}E_{23}\}\big) = \mathrm{span}\big(\{E_{22}-E_{23},\ E_{23}\}\big).$$

Similarly, we note that

$$\begin{aligned}
&E_{33}-\frac{\langle E_{33},\ E_{11}-E_{33}\rangle}{\langle E_{11}-E_{33},\ E_{11}-E_{33}\rangle}\cdot(E_{11}-E_{33})\\
&\qquad\qquad-\frac{\langle E_{33},\ E_{12}-E_{11}-E_{33}\rangle}{\langle E_{12}-E_{11}-E_{33},\ E_{12}-E_{11}-E_{33}\rangle}\cdot(E_{12}-E_{11}-E_{33})\\
&=E_{33}+\frac{1}{2}\,(E_{11}-E_{33})+\frac{1}{3}\,(E_{12}-E_{11}-E_{33})\\
&=\frac{1}{6}E_{11}+\frac{1}{3}E_{12}+\frac{1}{6}E_{33}
\end{aligned}$$

is orthogonal to both $E_{11}-E_{33}$ and $E_{12}-E_{11}-E_{33}$. Moreover,

$$\begin{aligned}
\mathrm{span}\big(\{E_{11}-E_{33},\ E_{12}-E_{11}-E_{33},\ \tfrac{1}{6}E_{11}+\tfrac{1}{3}E_{12}+\tfrac{1}{6}E_{33}\}\big)&\\
=\mathrm{span}\big(\{E_{11}-E_{33},\ E_{12}-E_{11}-E_{33},\ E_{33}\}\big)&.
\end{aligned}$$

We conclude that

$$\widetilde{\mathcal{B}}_0\cup\{\tfrac{1}{6}E_{11}+\tfrac{1}{3}E_{12}+\tfrac{1}{6}E_{33},\ \tfrac{1}{2}E_{22}+\tfrac{1}{2}E_{23}\}$$

is an orthogonal basis of U that extends the orthogonal basis $\widetilde{\mathcal{B}}_0$ of $\mathrm{Ker}(f)$.

Problem 6. (a) Let $A = (a_{ij})_{1\leqslant i,j\leqslant 3}$ be a matrix in $V = \mathbb{Z}_5^{3\times 3}$. Then

$$\begin{aligned}
[A]_S &= \big\{B = (b_{ij})_{1\leqslant i,j\leqslant 3} \in V : B - A \in S\big\} \\
&= \big\{B = (b_{ij})_{1\leqslant i,j\leqslant 3} \in V : b_{ij} - a_{ij} = 0 \text{ if } j > i \ \text{ and } \ \mathrm{tr}(B - A) = 0\big\} \\
&= \big\{B = (b_{ij})_{1\leqslant i,j\leqslant 3} \in V : b_{ij} - a_{ij} = 0 \text{ if } j > i \ \text{ and } \ \mathrm{tr}(B) - \mathrm{tr}(A) = 0\big\} \\
&= \big\{B = (b_{ij})_{1\leqslant i,j\leqslant 3} \in V : b_{ij} = a_{ij} \text{ if } j > i \ \text{ and } \ \mathrm{tr}(B) = \mathrm{tr}(A)\big\}.
\end{aligned}$$

We then see that

$$V/S = \big\{\{C = (c_{ij})_{1\leqslant i,j\leqslant 3} \in V : c_{ij} = a_{ij} \text{ if } j > i \ \text{ and } \ \mathrm{tr}(C) = d\} : d, a_{12}, a_{13}, a_{23} \in \mathbb{Z}_5\big\}.$$

Next, we observe that $A = (a_{ij})_{1\leqslant i,j\leqslant 3}$ is in S if and only if

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & -a_{11} - a_{22} \end{pmatrix},$$

therefore a basis for S is the set

$$\mathcal{B}_S = \big\{E_{11} - E_{33},\ E_{22} - E_{33},\ E_{21},\ E_{31},\ E_{32}\big\}.$$

To extend this to a basis of the entire space V, we could include the matrices E_{33}, E_{12}, E_{13} and E_{23}: indeed,

$$\mathrm{span}\big(\mathcal{B}_S \cup \big\{E_{33},\ E_{12},\ E_{13},\ E_{23}\big\}\big) = \mathrm{span}\big(\{E_{ij} : 1 \leqslant i, j \leqslant 3\}\big)$$

and the two spanning sets have the same size. But then, by a theorem stated in class (and also in HW5, Problem 5), a basis for V/S is the set

$$\mathcal{B}_{V/S} = \big\{[E_{33}]_S,\ [E_{12}]_S,\ [E_{13}]_S,\ [E_{23}]_S\big\}.$$

(b) We have that $\mathrm{tr}(A) = 0$, therefore

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in S$ and $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \in T = \mathrm{span}\big(\big\{E_{33},\ E_{12},\ E_{13},\ E_{23}\big\}\big)$,

which shows that $[A]_S = [E_{13}]_S + 2[E_{23}]_S$.

Similarly, we note that $\mathrm{tr}(B) = 4$, therefore

$$B = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & -1-2 \end{pmatrix} + \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$
$$\text{with } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & -1-2 \end{pmatrix} \in S \ \text{ and } \ \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{pmatrix} \in T$$

which shows that $[B]_S = 3[E_{12}]_S + 3[E_{23}]_S + 4[E_{33}]_S$.

Finally, we note that $\mathrm{tr}(C) = 1$, therefore

$$C = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 4 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\text{with } \begin{pmatrix} 3 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in S \ \text{ and } \ \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix} \in T$$

which shows that $[C]_S = 2[E_{12}]_S + [E_{13}]_S + 4[E_{23}]_S + [E_{33}]_S$.

We can now observe that

$$\begin{aligned} [B]_S + [C]_S &= \big(3[E_{12}]_S + 3[E_{23}]_S + 4[E_{33}]_S\big) + \big(2[E_{12}]_S + [E_{13}]_S + 4[E_{23}]_S + [E_{33}]_S\big) \\ &= [E_{13}]_S + 2[E_{23}]_S = [A]_S. \end{aligned}$$

This shows that $[A]_S$ is a linear combination of $[B]_S$ and $[C]_S$, so the given set is linearly dependent.