Green's Theorem

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1 Motivation

The Fundamental Theorem of Calculus is one of the most important theorems in the world that unifies calculus. The theorem basically reduces to a definite integral

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is basically the anti derivative of f. This essentially avoids the use of the limit of the Riemann sum definition to calculate the definite integral.

A natural question that follows would be *" How can we generalize this? Can this easily be extended to 2 or higher dimensions?"* And surprisingly the answer is **"YES"** which is the motivation for Green's theorem. The theorem is stated as follows

2 Green Theorem

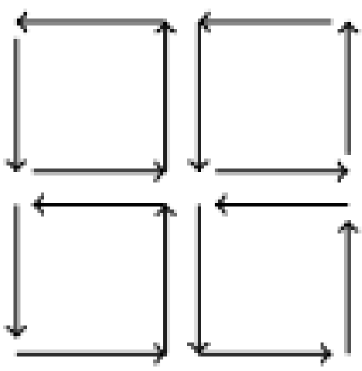
Let $S \subset \mathbb{R}^2$ be a simple, closed, positively-oriented differentiable curve(piece wise smooth boundary), and let $\mathbf{F}$ be a C^1 vector field on an open set that contains S.

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA$$

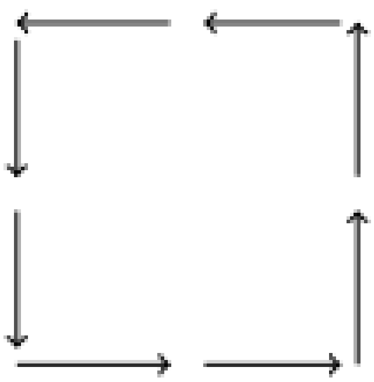
Now this may seem to be a bit hard to grasp but the physical intuition is pretty simple. Take a look at this figure 1. Let us break it up into parts to understand this more in depth. First let us take any one of the red squares, note that each red square is broken up into 4 green squares and for each arrow inside the red square, there is a second arrow right next to it pointing in the opposite direction. Therefore they *cancel* out each other leaving just the red boundary! This is demonstrated exactly in 2. Therefore in conclusion we can infer that

$$\text{macroscopic circulation} = \sum \text{microscopic circulation}$$

This is exactly the beauty of Green's Theorem.



(a) Microscopic Circulations



(b) Macroscopic Circulations 2

Figure 1: Microscopic vs Macroscopic circulations of the squares

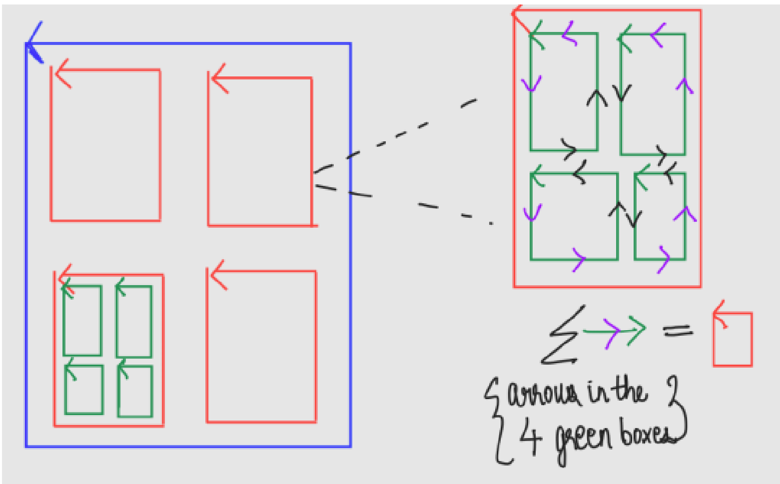


Figure 2: Macroscopic Square decomposition into subsequent microscopic sub-squares

3 Preliminaries

Now that we stated Green's theorem is, let us understand what a simple, closed, positively-oriented differentiable curve is. We start with the fundamentals and progressively build up the essential knowledge required to completely comprehend Green's theorem.

3.1 Gradient

Intuition The gradient is a simple operation that takes a scalar field f and returns a vector field, (grad f). Note that the gradient is always perpendicular to lines of equal potential(ie the lines of points with the same energy, like a circle represents all items the same distance from the centre)
Definition Let $\varnothing \neq D \subset \mathbb{R}^N$, let $x_0 \in \operatorname{int} D$, and let $f : D \to \mathbb{R}$ be partially differentiable at x_0. Then the gradient (vector) of f at x_0 is defined as [1]

$$(\operatorname{grad} f)(x_0) := (\nabla f)(x_0) := \left(\frac{\partial f}{\partial x_1}(x_0), \ldots, \frac{\partial f}{\partial x_N}(x_0)\right).$$

3.2 Divergence

Intuition The divergence operation takes a vector field F and produces a scalar field, div F. The gradient and divergence are not inverse functions tho. A velocity field is the most natural method to illustrate the divergence. Given an area (two-dimensional or three-dimensional) filled with flowing liquid, the velocity of the liquid at each location is a vector (direction and magnitude). Then it easy to notice that a velocity field's divergence at a certain place reflects the expansion, compression , or no movement of the fluid flow at that point.
Definition Let $\varnothing \neq U \subset \mathbb{R}^N$ be open, and let $F : U \to \mathbb{R}^N$ be a partially differentiable vector field. Then the divergence of F is defined as [1]

$$\operatorname{div} F := \sum_{j=1}^{N} \frac{\partial F_j}{\partial x_j}.$$

3.3 Curl

Intuition Curl is a vector field operation that takes a vector field F and returns a different vector field, curl F. Assume that F is the three-dimensional velocity field of a fluid, and that we have put a very tiny sphere with a rough surface at the point of interest in this field. Assume we can somehow prevent this sphere from moving in any direction while still allowing it to rotate in any direction without resistance. The speed of the fluid combined with the rough surface of the sphere will cause it to rotate in some velocity fields. The curl is proportional to the rotation of the sphere. *To put it simply, the curl*

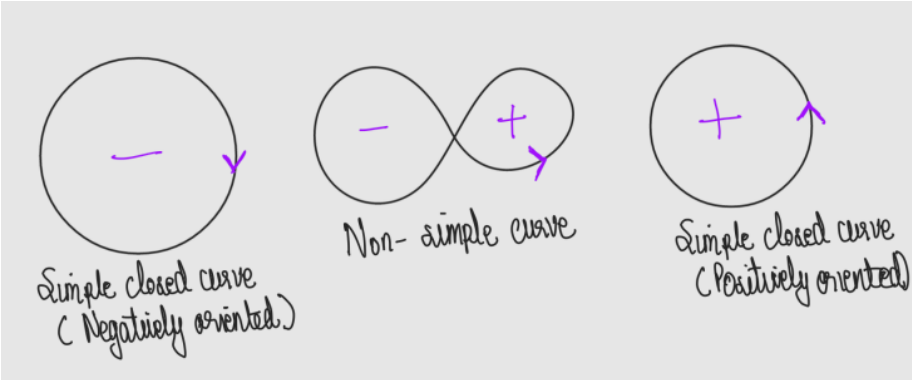


Figure 3: Simple and Non-Simple Curves Orientation

of a vector field measures "the tendency of the field to spiral about."
Definition Let $\varnothing \neq U \subset \mathbb{R}^3$ be open, and let $F : U \to \mathbb{R}^3$ be partially differentiable. Then the curl of F is defined as [1]

$$\operatorname{curl} F := \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right).$$

3.4 Simple Closed Curves

Intuition A plane simple closed curve, is a non-self-intersecting continuous loop in the plane.
Definition Say we have a curve C which can be parameterized by $x = g(t)$, $a \leq t \leq b$ and that g is continuous as well as that the parametrization has the same starting and ending values ie $g(a) = g(b)$ and that the curve does not intersect itself. This is demonstrated in 3 (Curve 1 and 3 are simple).

3.5 Piecewise Smooth

Intuition A piecewise smooth curve is one that can be divided into an infinite number of smooth parts $C_1, C_2, ..., C_N$, with the terminal point of one piece being the beginning point of the next.
Definition A simple closed curve [2] is piecewise smooth if it has a parametrization $\mathbf{g}$ as above, and there exists a finite set of points $\{t_1, \ldots, t_K\} \subset [a, b]$ such that

1. $\mathbf{g}$ is continuously differentiable with $\mathbf{g}'(t) \neq \mathbf{0}$, except at points in $\{t_1, \ldots, t_K\}$
2. $\lim_{t \to t_k^\pm} \mathbf{g}'(t)$ exists for every $k \in 1, \ldots, K$.

3.6 Piecewise Smooth Boundary

Definition For $S \subset \mathbb{R}^2$, we say that S has piecewise smooth boundary [2] if ∂S consists of finitely many (one or more) disjoint, piecewise smooth, simple closed curves.

3.7 Orientation / Stokes Orientation

Intuition The two different approaches of traversing the curve result in two different curve orientations. The left curve (the clockwise direction) is negative, whereas the right curve (the counter-clockwise direction) is positive.
Definition Assume that S is a regular area with a piecewise smooth boundary formed by one or more piecewise smooth curves. A curve is said to be positively oriented if the curve's interior is to the left when travelling on it and negatively oriented if the curve's interior is to the right when travelling on it. An example would be 3.

3.8 Line Integral

Intuition A line integral [1] is the integral of a function along a curve. A scalar-valued function can be integrated along a curve to calculate, for example, the mass of a wire from its density. A certain

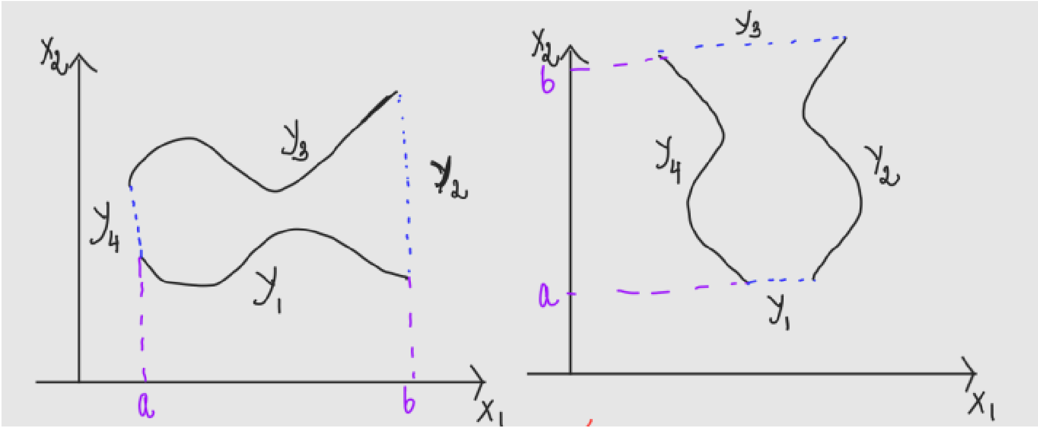


Figure 4: Natural Parametrization of ∂S for both x_1-simple and x_2-simple(left and right respectively)

type of vector-valued function can also be integrated along a curve.
Definition Let $\gamma : [a, b] \to \mathbb{R}^N$ be a $\mathcal{C}^1$-curve, and let $f : \{\gamma\} \to \mathbb{R}^N$ be continuous. Then

$$\int_\gamma f \cdot dx = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt$$

4 Proof

Although I won't get into the full detail of the proof, here is an outline of the proof for normal regions [1] [2]. This can easily be extended to general regions by considering a more general parametrization. Let $S \subset \mathbb{R}^2$ be a simple, closed, positively-oriented differentiable curve(piece wise smooth boundary), and let $\mathbf{F}$ be a C^1 vector field on an open set that contains S. We need to prove that

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA$$

Let us assume for simplicity that S is defined by the inequalities (basically S is x_1-simple or a normal region with respect to the x_1-axis)

$$\psi(x_1) \leq x_2 \leq \phi(x_1) \text{ and } a \leq x_1 \leq b$$

for some C^1 functions $\psi, \phi : [a, b] \to \mathbb{R}$ with $\psi(x_1) \leq \phi(x_1)$ for $x_1 \in [a, b]$. Now we will prove that if S is normal with respect to the x_1-axis then we have

$$\int_{\partial S} F_1(x_1, x_2)dx_1 = \iint_S -\frac{\partial F_1}{\partial x_2}(x_1, x_2)dA$$

Let us break ∂S into 4 pieces as shown in 4 by parametrization as follows

1. For $\gamma_1 = \psi(x)$ we have a parametrization $(t, \psi(t)),\ a \leq t \leq b$
2. For γ_2 we have $(b, t),\ \psi(b) \leq t \leq \phi(b)$. Notice that x is constant on this segment.
3. For $\gamma_3 = \phi(x)$ we have a parametrization $(t, \phi(t)),\ a \leq t \leq b$ and this is multiplied by -1 to preserve the orientation
4. For γ_4 we have $(a, t),\ \psi(a) \leq t \leq \phi(a)$ and this is multiplied by -1 to preserve the orientation. Similarly as γ_2, we have that x is constant on this segment.

Now γ_2 and γ_4 contribute nothing to the integral as this is normal with respect to the x-axis and so we get

$$\int_{\partial S} F_1(x_1, x_2)dx_1 = \int_a^b F_1(t, \psi(t))dt - \int_a^b F_1(t, \phi(t))dt$$

and then by the fundamental theorem of calculus we also get

$$\int_a^b F_1(t,\psi(t))dt - \int_a^b F_1(t,\phi(t))dt = -\iint_S \frac{\partial F_2}{\partial x_1} dA$$

Therefore by comparing the above equations we get

$$\int_{\partial S} F_1(x_1,x_2)dx_1 = \iint_S -\frac{\partial F_1}{\partial x_2}(x_1,x_2)dA$$

Similarly we can also obtain

$$\int_{\partial S} F_2(x_1,x_2)dx_2 = \iint_S \frac{\partial F_2}{\partial x_1}(x_1,x_2)dA$$

when S is normal with respect to the x_2-axis or is x_2-simple. Now adding both the equations we get

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dA$$

Q.E.D.

5 Conclusion

What exactly is the purpose of Green's theorem now that we know what it is? Green's theorem is applied in a number of situations. Some examples are as follows

- Finding the area and centroid of plane figures by just integrating over the boundary
- Transforming a line integral into a double integral and vice versa
- The proof of Cauchy's Integral Theorem uses Green's theorem as a subroutine
- Much of the identities in electromagnetism are derived using Green's theorem
- Calculating the area to non-simply connected regions(ie shapes with holes in them)
- To calculate the winding number which is extremely important in algebraic topology.

It is also worth noting that when the region is planar and the border is a simple curve, Green's Theorem is simply a specific version of *Stokes Theorem*. As a result, it is commonly referred to as the 2-dimensional version of Stokes Theorem. Overall, Green's theorem is incredibly beneficial to us and makes vector calculus much more simpler.

References

[1] University of Alberta MATH 217/317 Notes

[2] University of Toronto MATH 237 Notes