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MATH117

Final EXAM

STUDY GUIDE

Lecture Notes

Math 117 - Lecture 36 - Introduction to Functions and the Terminology

Topics of Functions:

1. Terminology
2. Limits
3. Continuity
4. Differentiation
5. Integration

Terminology

a. Definition of Function
   i. Giving two sets A and B, we say that f is a function from A to B and write $f:\ A \longrightarrow\ B$ if f associates a point of B to every point of A.
   ii. For every $a\ \in\ A$, $f\,(a)$ is an element of B, $a\ \longrightarrow\ f\,(a)$. A is called the domain of f, write $A\ =\ dom f$, B is called co-domain of f, write $B\ =\ codom f$. For $a\ \in\ A$, $f\,(a)$ is called the image of a under f.
   iii. Map = function
   iv. $graph\ f\ =\ \{a,\ f\,(a)\,,\ a\ \in\ A\}$ is a subset of $A\ \times B$
   v. One can identify f with $graph\ f$

b. Range
   i. $range f = \{f\,(a) :\ a\ \in A\}$

ii. $rangef$ is a subset of $codomf$, the set of values that the functions actually takes.

c. Injective

i. f is one to one or injective if $x \neq x_2 \Rightarrow f(x) \neq f(x)_2$

ii. f never sends 2 arguments into the same value.

iii. Onto, every point of b is an arrow coming into it.

iv. One to one, no 2 arrows end up in the same point, think of it as parallel.

v. Contrapositive; of $f(x) = f(x)_2 \quad then \quad x_1 = x_2$

Examples

1. $f(x) = x^2$

In math 117/118 we mostly deal with 'real' functions, which means that $domf$ and $codomf$, are either $\mathbb{R}$ or an internal in $\mathbb{R}$, or a union of a collection of intervals.

$domf = \mathbb{R} \quad codomf = \mathbb{R}$

$graph\ f\ is\ a\ subset\ of\ \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$\{(x\ ,\ x^2) :\ x \in R\}\ is\ a\ graph\ of\ f$

<u>Recall</u>**:** a sequence in $\mathbb{R}$ is a function $\mathbb{N}$ to $\mathbb{R}$

$x_n - x_{(n)} \qquad x : \mathbb{N} \longrightarrow \mathbb{R} \quad domx = \mathbb{N} \quad codomx = \mathbb{R}$

Which subset of $\mathbb{R}^2$ are graph functions?

A subset A of R$\mathbb{R}^2$ is a $graph\ f$ if it satisfies the “vertical line test”. Each vertical line meets A at most one point.

$$rangef \ = \ (o, +\infty)$$

If $rangef \ = codomf$, we say that f is $onto$ or f is a surjection.

f is not onto. Not one to one, $f(1) = f(-1)$

2. $f(x) = x^2 \quad f : \mathbb{R} \longrightarrow (0, +\infty)$

$domf = \mathbb{R} \quad codomf = (0, +\infty)$

Same formula, same graph, set different function.

$rangef \ = \ (o, +\infty)$, f is onto.

f is not one to one $f(-1) = f(1)$

3. $f(x) = x^2 \quad f : \mathbb{R} \longrightarrow (0, +\infty) \quad domf = (0, +\infty)$

$codomf = (0, +\infty)$

$rangef$, the same $(0, +\infty)$ so f is onto

+$\mathbb{R}$ if one to one because every +$\mathbb{R}$ has exactly one square root, if

$$x_1^2 - x_2^2 \ \ the\ x_1 = x_2\ provided\ \ x_1, x_2 \geq 0$$

4) $f(x) = \sqrt{4}\ f : (0,\infty) \longrightarrow (0,\infty)$ could be $\mathbb{R}$

$dom f = (0,\infty)\ codom f = (0,\infty)$

$range f = (0,\infty)$ so onto, depends on your choice of domain.

5) $f(x) = \frac{1}{x}\quad f : \mathbb{R}_\backslash\{0\} \longrightarrow \mathbb{R}$

$dom f = \mathbb{R}_\backslash\{0\}\quad codom f = \mathbb{R}$

$range f = \mathbb{R} / \{0\}$, not onto

One on one because it is $\frac{1}{x_1} = \frac{1}{x_2}$ then $x_1 = x_2$

6) $f(x) = \lfloor x \rfloor\ f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{R} \to \mathbb{Z}$

$range f = \mathbb{Z}$, not onto, $f : \mathbb{R} \to \mathbb{R}$ is unto

Not one to one, $\lfloor 1 \rfloor = \lfloor 1.5 \rfloor$

7) $f(x) = x - \lfloor x \rfloor\ f : \mathbb{R} \to \mathbb{R}$

$range f\ (0,\ 1)$ not onto.

The function is not one to one, $0.5 = f(1.5) = f(2.5)$

<u>Math 117 - Lecture 37 - Terminology: Inverse and Restrictions</u>

Recall

$f:\ A \longrightarrow\ B$ means " f is a function from A to B"

$A = dom f$

$B\ = codom f$

A real function is a function for which $codom f\ \leq \mathbb{R}$

$dom f$ is a union of intervals in $\mathbb{R}$

Let $f:\ A \longrightarrow\ B$ $\quad range f = \{f(a):\ a \in A\}$

$range f \leq codom f$

If the range is all domain (contains all possible values), it is onto.

f is onto or surjective of $range f = codom f$

f is one to one or injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$

f is <u>one to one correspondence</u> or a <u>bijection</u> if it is both onto and one-to-one, both injective and subjective.

Example: $f(x) = x^2$ $f: \mathbb{R} \to \mathbb{R}$

$codom f = \mathbb{R}$

$range f = (0, \infty)$

f is not onto

f is not one to one because $f(-2) = f(2)$

Horizontal Line Test

A real function is one-to-one if the graph satisfies the horizontal line test. Whenever a horizontal line meets the graph at most one point.

Example:

$$f(x) = x^2 \quad f(0,\infty) \rightarrow (0,\infty)$$

f is onto... and one to one, its a bijection.

Let A be a set, by i_A we denote the identity function of A: $i_A : A \rightarrow A \quad i_A(a) = a$

Composition:

Given functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ we define

$$g \circ f : A \longrightarrow C$$

by $(g \circ f)(a) = g(f(a))$

Warning: $g \circ f$ does not equal to $f \circ g$

Example

Consider $f, g : \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x) = x + 2 \ and\ g(x) = x^2$$

$$(f \circ g)(x) = f(g(x)) = f\left(x^2\right) = x^2 + 2$$

$$(g \circ f)(x) = g(x + 2) = (x + 2)^2$$

So $f \circ g \neq g \circ f$

f is a function

$f(x)$ is a number

Inverse:

Given two functions, $f \ and \ g$, we say that g is the inverse of f is

$$dom g \ = codom f$$

$$codom g \ = dom f$$

$g(f(x)) = x$ for all x in $dom f$ ⇔ $g \ \circ \ f$ = $subdom f$

$f(g(y)) = y$ for all y in $codom f$ ⇔ $f \ \circ \ g$ = $subcodom f$

Write $g = \ f^{-1}$

Example

$$f, \ g \ : \ (0, \infty) \ \longrightarrow (0 \inf), \ f(x) = x^2 - \sqrt{x}$$

$g = f^{-1}$ ⇔ $f = g^{-1}$ that is, the inverse of the inverse is the original function. f is said to be invertible if f^{-1} exists (it is unique)

Proposition

A function is invertible if f is a bijection.

<u>Proof</u>

Suppose f is a function from $A \ \longrightarrow \ B$

Suppose f is invertible, f^{-1} is the inverse of f

First i must see if f is onto, take $b \in B$, then

$b = f\left(f^{-1}(b)\right) = f(a)$

Where $a = f^{-1}(b)$ then $b \in range f$ so $B \subseteq f$

Hence, f is onto.

Show that f is one to one, suppose $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$

Apply f^{-1} get $f^{-1}(f(a_2)) = f^{-1}(f(a_2)),\ so\ a_1 = a_2$.

Hence, one to one, therefore it is a bijection.

Suppose f is a bijection, take $b \in B$, since f is onto, $b \in range f$, so $b = f(a)$ for some $a \in A$

Since f is one to one, such that a is unique, $a = g(b)$ This g is well defined. $G : b \longrightarrow A \quad b = f(g(g))\ for\ all\ \ b \in B$

And $g(f(a)) = a\ for\ all \quad a \in A,\ so\ \ g\ =\ f^{-1}$

Hence, f is invertible.

Restriction

Let $f:\ A \longrightarrow B,$, let $c \subseteq A$

Define $g:\ c \longrightarrow B\ \ by\ g(x) - f(x)\ \ for\ all\ x \in C$

We say that g is the restriction of f to C, write $g = f \mid C$

Example

$f(x) = x^2 \qquad f : \mathbb{R} \to \mathbb{R}$

The restriction of f to $(0, \infty)$ is $g : (0, \infty) \to \mathbb{R}$

Same formula, different domain.

f is not one to one, g is one to one.

$f(c) = \{f(x) : \ x \in C\}$

From now on, let f be a real function

f is increasing if for any $x_1, \ x_2 \ \in dom f$

If $x_1 \ < \ x_2 \ then \ (x_1) \ \leq (\ x_2)$

f is strictly increasing .... $f(x_1) < \ f(x_2)$

f is decreasing .... $f(x_1) \geq \ f(x_2)$

f is strictly decreasing......... $f(x_1) > \ f(x_2)$

Let a contain dom f, we say that f is increasing on A if the restriction $f \ | \ A$ is increasing.

f is monotone if it is either increasing or decreasing. Similarly it is strictly monotone if it is strictly decreasing or strictly increasing.

Examples (Reference last class)

1) $f(x) = x^2 \qquad f : \mathbb{R} \to \mathbb{R}$

   a) Neither increasing or decreasing

b) Show that $f(x) = x^2$ $f:$ is strictly increasing on $(0,\ \infty)$

And decreasing a ($(-\inf, 0)$)

2) $f(x) = x^2$ $f:(0,\infty) \to \mathbb{R}$

Strictly increasing.

3) $f(x) = \lfloor x \rfloor$

f is increasing, but not strictly.

Proposition

If f is strictly monotone then f is one to one

Proof

Suppose f is strictly increasing, let $x_1,\ x_2\ \in dom f$, $x_1\ =\ x_2$.

Say $x_1\ <\ x_2$ then $f(x_1) < f(x_2)$, hence $f(x_1) \neq f(x_2)$

Is the converse true?

Suppose f is one to one, does f have to be strictly monotone?? No.

Example: $$f(x)\ = \begin{cases} x + 2 when 0 \le x < 1 \\ x - 1 wnen 1 \le x \le 2 \end{cases}$$

It is one to one, but clearly not monotone.

Important Class Note: Practice these terminology, as it is the foundation of what we will learn next class.

Math 117 - Lecture 38 - Midterm Review and Limits of Functions

Review of Midterm 2

1.
    a. Every non empty set bounded above subset of $\mathbb{R}$ has a supremum.
    b. Every non empty set bounded below subset of $\mathbb{R}$ has infinum
2. The Monotone Convergence Theorem
    a. Review MCT proof from your notes
    b. Don't try to memorize it → rather understand it in your own words and get a feel for the underlying backbone of the proof.
3. Using the definition of limits, $\dfrac{3n+5}{2n+8} \longrightarrow \dfrac{3}{2}$
    a. Using scratch paper, giving £ > 0, provide a clean solution.
        i. Let $\varepsilon > 0$ take n_0 = $\left\lceil \dfrac{7}{2\varepsilon} \right\rceil$ if $n \geq n_0$
        ii. Then $\left|\left(\dfrac{3n+5}{2n+8}\right) - \dfrac{3}{2}\right| = \dfrac{7}{2n+8} < \dfrac{7}{2n} < \varepsilon$
        iii. Because $n > \dfrac{7}{2\varepsilon}$
4. Find the limits of $\sqrt{n^4 + 6n^2 + 1 - \sqrt{n^4 + n^2 + 1}}$
    a. Common mistake → can only conclude $x_n \leq \dfrac{5}{3}$ but does not mean it converges to $\dfrac{5}{3}$ unless you prove it.

Functions

- Recall

- $f:\ A\ \longrightarrow\ B,\ A = domain\quad B\ = codomain$
- $rangef = \{f(a) : a \in A\}$ the set of values of f
- f is bounded if $range\ f$ is a bounded set, that is, there exists number a and b, such that $range\ f\ \subseteq [a, b]$

* $a\ \leq f(x) \leq b$ for all $x \in domf$
  - f is bounded ⇔ there exists M element $\mathbb{R}$ such that u $\forall \in domf\ |f(x)| \leq M$
  - supf:=sup range f , inff:=inf range f
  - By (c) if f is bounded above, then $\sup f$ and if f is bounded below, then $\inf f$ exists.
  - $\min f\ = \min rangef\quad \max f = \max rangef$
  - $L = \max f$, means $L = \max rangef$ ⇔ $L = f(c)$ for some c and $L\ \geq f(x)$ for all $x \in domf$
  - In this case we say that f attains its max at c, L is the maximal value of f.
* **Examples**
  - Example 1
    - $f(x) = \sin x$ for all x, $-1\ \leq \sin x \leq 1$, hence $\sin$ is bounded
    - Max $\sin\ (x)$ is 1, attained at $\frac{\pi}{2} + 2k\pi\ \ k \in \mathbb{Z}$

- Similarly, the minimal value of $\sin\ (x)$ is -1, attained at

  $\frac{-\pi}{2} + 2k\pi \quad k \in \mathbb{Z}$

- Example 2
  - $f(x) = 1 + x^2$
  - f is bounded below, $f(x) = 1 + x^2 \ \geq 1$ for all x
  - f is not bounded above, hence $\sup \infty$
  - $\inf f = \min f = 1$, attained at $x = 0$
    - Side note: look at 1 interval and restrict it within that interval so we can look at properties of just that-so we can understand the local properties

      f has a local maximum at c if there is an open interval containing c such that the restriction of f is this interval attains a max at c.

- **Limits of Functions**
  - Recall
    - $\lim\limits_{x \to \infty} x_n = c$ means $\forall \in > 0 \ \exists\, n_0 \forall n \geq n_0$
    - $|x_n - c| < \varepsilon$
    - Given a function of f, $\lim f = f(x) = L$ means

      $\forall \varepsilon > 0 \ \exists\, M \forall x \ \geq m$
    - $|f(x) - L| < \varepsilon$

- Geometrically this means we have the Limit L, it gives us a pipe of ε and we want it to stay in this pipe, so M geometric is that after M, so that it stays in that pipe.
  - If $x \geq M$ then $f(x)$ is within ε, if $x \simeq \infty \quad f(f\infty) \simeq L$

- **Examples**
  - $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$
    - Same language as sequences but instead of naturals we talk about reals.
    - Proof
      - Fix $\varepsilon > 0$. Let $M = \dfrac{1}{\sqrt{\varepsilon}}$
      - If $x > M$ then $x^2 > m^2 \; to \; \dfrac{1}{x^2} < \dfrac{1}{m^2} \; so \left|\dfrac{1}{x^2 - 0}\right| < \dfrac{1}{m^2} = \varepsilon$
      - So $|f(x) - 0| < \varepsilon$
      - $\lim\limits_{n\to\infty} f(x) = 1$ means $\forall\varepsilon > 0 \quad \exists M \forall x < M$
      - $|f(x) - L| < \varepsilon$
      - Very formal: $x \in dom f$
      - The difference from sequences and functions if that with functions, we can find limit
      - $\lim\limits_{x\to c} f(x) = L \quad c, L \in \mathbb{R}$ means for every $\forall > 0 \exists\delta > 0 \quad \forall x \in dom f$ if $x \neq 0$ and $|x - c| < 0$ then $|f(x) - L| < \varepsilon$

- $\lim_{x \to 5} 3x = 15$
  - as x approaches 5, $3x$ approaches 15
  - Let $\varepsilon > 0$, write $\delta = \frac{\varepsilon}{3}$
  - Suppose $x \neq 5$ and $|x-5| < \varepsilon$
  - Then $|f(x) - L| = |3x - 15| \quad f(x) = 3x,\ L = 15$
  - So $3 \cdot |x-5| < 3 \cdot \varepsilon = 3 \cdot \frac{\varepsilon}{3} = \varepsilon$
  - So $|f(x) - L| < \varepsilon$ so definition is satisfied.

Math 117 - Lecture 39 - Continuous Functions

Continuity

- A function f is continuous at an interior point a of its domain if $f(x) = f(c)$.
- $\lim f(x) = L$ means $\forall \varepsilon > 0 \; \exists \partial > 0 \; \forall x \in (dom f) \setminus \{c\}$ if $|\mathrm{x} - c| < \partial \; then \; |f(x) - 1| < \varepsilon$
  - f is continuous at c if $\lim\limits_{x \to c} \; f(x) = f(c)$
- Examples
  - Show that the $\lim\limits_{x \to 9} \sqrt{x} = 3$
    - Since $3 = \sqrt{9}$ this really means $\lim\limits_{x \to 9} \sqrt{x} = \sqrt{9}$
    - Hence root is a continuous function
- The function is Discontinuous when $L = d.n.e$ or $L \neq f(c)$
- f is continuous at a ⇐⇒ for every $\varepsilon > 0$ there exists a $\partial > 0$ such that $|x - c| < \partial \Rightarrow |f(x) - f(a)| < \varepsilon$.
- $f(x) = x$ continuous at every point a of its $dom f \mathbb{R}$ since $\lim\limits_{x \to c} f(x) = f(c)$ for all $c \in \mathbb{R}$
- f $f(x) = x^2$ is continuous since $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} f(x)^2 = \left(\lim\limits_{x \to} x\right) \cdot \left(\lim\limits_{x \to} x\right) = c \cdot c \; = \; c^2 = f(c)$

Math 117 - Lecture 40 - Thursday Homework and Present Class

- $$x_n = 1 + q + \ldots + q^n = \sum_{k=0}^{n} q^k$$
    - We need to understand geometric series, , but we will go in depth later.

- Examples:
    - Note: we will talk about series at the end of 118, however this is a series example.
        - $f\ [0, 2]\ - \mathbb{R}$
        - $$f(x) = \begin{cases} 2 + x^2 & 0 \leq x \leq 1 \\ x - 1 & 1 < x \leq 2 \end{cases}$$
        - $range f = [0, 1] \cup [2, 3]$
        - f is not a subjective but an injection
        - f is not invertible
        - f is bounded above and below
        - f is neither increasing or decreasing
        - $\max f = 3$
        - $\sup f = 3$
        - $\min f = d.n.e$
        - $\inf f = 0$
        - Local max of $x - 1$ and $x \leq 2$, min of $x > 0$
    - $f : (0, 1) \longrightarrow \mathbb{R}$, $f(x) = \sin \frac{1}{x}$

- $range f = [-1, 1]$
- f is not onto, one to one or a bijection
- f is not a bijection
- f is not increasing or decreasing, does both in intervals of sin 1/x
- $\max f = \sup f = 1$
- $\min f = \inf f = -1$
- You don't have to draw a graph, however it may help you construct your proof

- When $x \in [1, 1]$
  - $x^2 \leq 1$
  - $$-2 < 0 \leq \frac{x^2}{1+x^4} < \frac{1+x^2}{1+x^4} < \frac{1+1}{1+x^4} < 2$$
  - When $x \in (-\infty, -1) \cup (1, +\infty)$
  - $x^4 > x^2 \Rightarrow x^4 + 1 > x^2 + 1$
  - $$-2 < 0 \leq \frac{x^2}{1+x^4} < \frac{1+x^2}{1+x^4} < \frac{1+x^4}{1+x^2} = 1 < 2$$
  - $\Rightarrow$ when $x \in \mathbb{R} \left| \frac{1+x^2}{1+x^2} \right| < 2$
- The equation $f(x) = y$ has solutions exactly when $y \in (0, 1) \cup (2, 3)$ if $y \in (0, 1)$
  - Take $x = y + 1,\ x \in (1, 2)$
  - $f(x) = x - 1 = y + 1 - 1 = y$

- $\lim_{x \to -\infty} \sqrt{-x} = \infty \quad \forall k > 0m\, \forall x \in doMf\, is\, x > M\, then\, f(x) > k$
  - Let $k > 0$
  - 1. Find M
    - $(\sqrt{-x})^2 > (k)^2$
    - $-x < k^2$
    - $x < -k^2$
    - Let $M = k^2$
  - 2. Let $M = -k^2$
    - $x < \; -k^2$
    - $x- < \; k^2$
    - $\sqrt{-x} > \sqrt{k^2}$
    - $\sqrt{-x} > k$
    - $f(x) > k$
- $\lim_{x \to 2} \frac{1}{x} - \frac{1}{2}$
  - $f(x) = \frac{1}{x}$ $c = 2$ $L = \frac{1}{2}$
  - Give $\varepsilon > 0$, take ∂ =?
  - Suppose $|x - 2| < \partial$ and

$$|f(x) - L| = \left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| =$$

$$\left|\frac{2-x}{2x}\right| = \left|\frac{x-2}{2x}\right| < \frac{\partial}{2|x|} \leq \frac{\partial}{2 \cdot 1}$$

- We need to separate |x| from 0, need a lower bound for |x|
- X is within ∂ from 2
  - $1 \leq 2 - \partial < x < 2 + \partial$
- $|x| \geq 1$
- give $\varepsilon > 0,\ \partial \lim \{1, 2\varepsilon\}$

Math 117 - Lecture 41 - Sequence Function Duality

- $\lim f(x) = L$ means $\forall \varepsilon > 0 \, \exists \partial > 0 \, \forall x \in (dom f) \setminus \{c\}$ if $|\mathrm{x} - c| < \partial \, then \, |f(x) - 1| < \varepsilon$
  - f is continuous at c if $\lim\limits_{x \to c} \; f(x) = f(c)$
- Examples
  - Show that the $\lim\limits_{x \to 9} \sqrt{x} = 3$
    - Since $3 = \sqrt{9}$ this really means $\lim\limits_{x \to 9} \sqrt{x} = \sqrt{9}$
    - Hence root is a continuous function
  - $\lim\limits_{x \to -2} x^3 = -8$
    - Take $\varepsilon > 0$,take $\partial = \min\left\{-, \frac{\varepsilon}{19}\right\} if \; 3\varepsilon$
    - $|x - (-2)| < \partial$ then
    - $|f(x) - 8| = |x^3 + 8| = |x + 2| \cdot |x^2 - 2x + 4|$

      $< \partial (3^2 + 2 + 2 \cdot 3 + 4) = 19\partial \leq \varepsilon$

      $< \varepsilon \cdot |x^2 - 2x + 4| < \partial \left(|x|^2 + 2|x| + 4\right)$

      Since delta is

      $\leq 1, \; |x - (-2)| < 1 \qquad -3 < 3 < 1 \; hence \; |x| < 3$

  - $Formula \; for \; cubes:$

    $\boxed{a^2 + b^2 = (a + b)\left(a^2 - ab + b^2\right)}$

- Let $f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$
    - $\lim_{x \to 0} f(x)\ d.n.e,\ let\ \varepsilon = \frac{1}{2}\quad no\ \partial\ will\ satisfy$

- **Sequence Function Duality**
    - There is a unique relationship between the limits of functions and units of sequences, which we will name, as a class, *Sequence Function Duality*.

I. $\lim_{x \to c} f(x) = L$

II. For every sequence (x_n) in $dom f\ /\ \{c\}$, $x_n \longrightarrow c$ then $f(x_n) \longrightarrow L$ that is, f maps sequences which converges to c to sequence which converges to L

Examples from above

- $\lim_{x \to -2} x^3 = -8$ if $x_n \longrightarrow -2$ then x_n^3
    - $f(x) = x^3 \quad c = -2 \quad L = -8 \quad x_n^3 \to (-2)^3 = -8$

      by properties of limits of sequence.
- $\lim_{x \to 9} \sqrt{x} = 3$
    - If $x_n \longrightarrow 9$ then $\sqrt{x_n} \longrightarrow \sqrt{9}$
    - Suppose $\lim_{x \to 0} f(x) = L$ take $x_n = \frac{1}{n}$ then $x_n \longrightarrow 0$
    - By SFD, $f(x_n) \longrightarrow L$ but

      $f(x_n) = L\quad for\ all\ n,\ so\ L = 1$

- Put $y_n = \frac{1}{2}, \; then \; y_n \longrightarrow 0$
  - By SFD, $f(y_n) \longrightarrow L$
  - $f(y_n) = -1 \; for \; all \; n \; so \; L = -1$
  - No ∂ will satisfy, no such L
  - So $\lim_{x \to c} f(x) \quad d.n.e.$

- SFD Proof
  - Now we must prove the theorem of $C, L \in \mathbb{R}$.
    - I ⇒ II
      - Suppose $\lim_{x \to c} f(x) = L$
      - Let (x_n) be a sequence in $(dom f) / \{c\}$ assume that $x_n \longrightarrow c$
      - We need to show that $f(x_n) \longrightarrow L$
      - Let $\varepsilon > 0$ since $\lim_{c \to L} f(x)$ we can find ∂ such that for all x in $(dom f) / \{c\}$ if $|x - c| < \partial \; then \; |f(x) - L| < \varepsilon$
      - We look at the interval between $L + \varepsilon$ and $L - \varepsilon$, and $c + \varepsilon$ and $c - \varepsilon$
        - This sequence converges to C, so we know a tail of (x_n) is contained in the interval of $c + \varepsilon$ and $c - \varepsilon$
      - There exists $n_o \; such \; that \; \forall n \geq n_o$

- We have $|x_n - c| < \partial$ then
  $f(x_n) \in (L - \varepsilon,\ L + \varepsilon)$ that is
  $|f(x_n) - L| < \varepsilon$
- There is, for ever $\varepsilon > 0$ there exists n_o such that for all
  $n > n_o$ *we have* $|f(x_n) - L| < 3 \quad f(x_n \longrightarrow L)$
  - II ⇒ I
    - Suppose that II is satified, assume for sake of contradiction, that I fails
    - $\exists \varepsilon > 0$
      $\boxed{\forall \partial > 0\ \exists x \in (dom f) / \{c\} \quad |x - c| < \partial}$
    - Let $|f(x) - L| > \varepsilon$ fix this "bad" ε
    - let $n \in \mathbb{N}$, apply content of above red box with $\partial = \dfrac{1}{n}$
      then there exists for some $x =: xn$ in
      $(dom f) \setminus \{c\}$ such that $|x_n - c| < \dfrac{1}{n}$ for all n
    - Hence $x_n \longrightarrow c$
    - By II $f(x_n) \longrightarrow L$
    - This contradicts $|f(x_n) - L| > \varepsilon$
  - HOMEWORK: Case $C, L = \pm\infty$
- Corollary
  - f is continuous at c if f for every sequence (x_n) in $dom f$ if $x_n \longrightarrow c$ then $f(x_n) \longrightarrow f(c)$. Replace L with $f(c)$

- **Properties of Limits of Functions**

- Let $C \in \mathbb{R}$
  - If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exists then
    $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
  - $\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
  - $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ provided that
    $\lim_{x \to c} g(x) \neq 0$
- Domination Principle (same as sequence)
  - If $\lim_{x \to c} f(x) = 0$ and $|g(x)| \leq |f(x)|$ for all x in
    $x \longrightarrow c$ a neighborhood of c (but $x \neq c$) then
    $\lim_{x \to c} (x) = c$
- Squeeze Law (same as sequence)
  - If $f(x) \leq q(x) \leq h(x)$ for all x in a neighborhood of c
    $(x \neq c)$ and $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$ then $\lim_{x \to c} g(x)$ exists.
- If $\lim_{x \to c} f(x) = 0$ and g is bounded on a neighborhood of c then
  $\lim_{x \to c} f(x) g(x) = 0$
- If $\lim_{x \to c} f(x) = \infty$ and g is bounded on a neighborhood of c then
  $\lim_{x \to c} (f(x) + g(x)) = \infty$

- If $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x)$ exists then $+\infty \, if \, \lim_{x \to c} g(x) > 0, \; -\infty \, if \, \lim_{x \to c} g(x) < 0$
- If $\lim_{x \to c} f(x) = \infty$ and $g(x) \le f(x)$ for all x in a neighborhood of c, then $\lim_{x \to c} g(x) = \infty$

Math 117 - Lecture 42 - Domination Principles and Proof and Review Properties of Limits

Recall

- $\lim f(x) = L$ means $\forall \varepsilon > 0\ \exists \partial > 0\ \forall x \in (dom f) \setminus \{c\}$ if $|\mathrm{x} - c| < \partial\ then\ |f(x) - 1| < \varepsilon$
- This definition extends naturally to $c, L\ = l \pm \infty$
- More generally, for $c, L\ \in \overline{\mathbb{R}}$, $\lim\limits_{x \to c} f(x) = L$ if for every neighborhood u of L there exists a neighborhood of V of C such that $c\ \neq x \in V$ implies $f(x) \in u$
- $f(v \setminus \{c\}) \subseteq u$
- f is continuous at C if $\lim\limits_{x \to c} f(x) = f(c)$ limit=value.
- f is continuous if it is continuous at every point of its domain

Sequence function Duality

- $\lim\limits_{x \to c} f(x)$ if f for every sequence (x_n) in $dom f\ /\ \{c\}$, $x_n \longrightarrow c$ then $f(x_n)\ \longrightarrow L$

Review: Properties of Limits

- Let $C \in \mathbb{R}$
    - If $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ exists then $\lim\limits_{x \to c} (f(x) + g(x)) = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$
        - Proof of the Sum Rule using SFDT
            - Let (x_n) be a sequence that (x_n) is in $dom f$ and $dom g$ excluding C if $x_n \longrightarrow c$

- Then $f\left(x_n\right) \longrightarrow \lim_{x \to c} f\left(x\right)$
- $g\left(x_n\right) \longrightarrow \lim_{x \to c} g\left(x\right)$
- So
  $$f\left(x_n\right) + g\left(x_n\right) \longrightarrow \lim_{x \to c} f\left(x\right) + \lim_{x \to c} g\left(x\right)$$
- Let $L = \lim_{x \to c} f\left(x\right) \ and \ k = \lim_{x \to c} g\left(x\right)$
- Suppose $x_n \longrightarrow c$
- Since $\lim_{x \to c} f\left(x\right)$ by SFDT, $f\left(x_n\right) \longrightarrow L$
- Since $\lim_{x \to c} g\left(x\right) = k$ by SFDT $g\left(x\right) \longrightarrow k$
- Then $f\left(x_n\right) + g\left(x_n\right) \longrightarrow L + M$ by properties of limits of sequences.
- Then, by SFDT again,
  $$\lim_{x \to c}\left(f\left(x\right) + g\left(x\right)\right) = L + K$$
- Hence $\left(\lim_{x \to c} f\left(x\right)\right) + \left(\lim_{x \to c} g\left(x\right)\right)$

  - $\lim_{x \to c}\left(f\left(x\right) - g\left(x\right)\right) = \lim_{x \to c} f\left(x\right) - \lim_{x \to c} g\left(x\right)$
  - $\lim_{x \to c} \dfrac{f\left(x\right)}{g\left(x\right)} = \dfrac{\lim_{x \to c} f\left(x\right)}{\lim_{x \to c} g\left(x\right)}$ provided that $\lim_{x \to c} g\left(x\right) \neq 0$

- Domination Principles
  - Suppose that $\left|g\left(x\right)\right| \leq \left|f\left(x\right)\right|$ for all $x \neq c$
  - $\lim_{x \to c} f\left(x\right) = 0$ then $\lim_{x \to c} g\left(x\right) = 0$

- $g(x) \leq f(x)$ for all $x \neq c$ and $\lim_{x \to c} g(x) = \infty$ then $\lim_{x \to c} f(x) = \infty$
- Proof
  - Case $C \in \mathbb{R}$, given $\varepsilon > 0$, since $\lim_{x \to c} f(x) = 0$ we can find $\partial > 0$ such that $|f(x)| < \varepsilon$ whenever $|x - c| < \partial,\ x \neq c$
    - But then $|g(x)| < |f(x)|$ yields that $|g(x)| < \varepsilon$ so that $\lim_{x \to c} g(x) = 0$
  - Case $C = \infty$, given $\varepsilon > 0$, since $\lim_{x \to c} f(x) = 0$ we can find M such that $|f(x)| < \varepsilon$ whenever $x > M$
    - But then $|q(x)| \leq |f(x)|$ yields that $|q(x)| < \varepsilon$ whenever $x > M$ there $\lim_{m \to \infty} g(x) = L$

- Squeeze Law
  - If $f(x) \leq q(x) \leq h(x)$ for all $x \in c\,(x \neq c)$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x)$ then $\lim_{x \to c} g(x)$ exists.
- If $f(x) \leq g(x)$ for all $x \neq c$ and $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$
- If $\lim_{x \to c} f(x) = 0$ and g is bounded on a neighborhood of c then $\lim_{x \to c} f(x)\,g(x) = 0$

- If $\lim_{x \to c} f(x) = \infty$ and g is bounded on a neighborhood of c then $\lim_{x \to c} (f(x) + g(x)) = \infty$
- If $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x)$ exists then $+\infty \; if \; \lim_{x \to c} g(x) > 0, \; -\infty \; if \; \lim_{x \to c} g(x) < 0$
- If $\lim_{x \to c} f(x) = \infty$ and $g(x) \leq f(x)$ for all x in a neighborhood of c, then $\lim_{x \to c} g(x) = \infty$

Examples

1) $\lim_{x \to 0} x \cdot \sin \frac{1}{x} = 0$
   a) Use domination principle, $f(x) = x \; and \; g(x) = x \cdot \sin \frac{1}{x}$
      $\lim_{x \to c} (x) = o$
   b) $|g(x)| = \left| x \cdot \sin \frac{1}{x} \right| = |x| \cdot \left| \sin \frac{1}{x} \right| \leq |x| = |f(x)|$
   c) Hence, $\lim_{x \to 0} x \cdot \sin \frac{1}{x} = 0$
2) $\lim_{x \to 2} (3x^2 + 5) = \lim_{x \to 2} (3x^2) + \lim_{x \to 2} (5)$
   a) $3 \cdot \left(\lim_{x \to 2} \right)^2 + 5 \; = \; 3 \cdot 2^2 + 5 = 17$

b) If $f(x) = 3x^2 + 5,$ then $\lim_{x \to 2} f(x) = 17$ so f is continuous at 2.

c) Some argument shows that if $p(x)$ is and polynomial at x then p is continuous at every point $\lim_{x \to c} p(x) = p(c)$

3) Rational Function, ration of Polynomials

i) $f(x) = \dfrac{p(x)}{q(x)}$ where q and p are polynomials

ii) $$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{p(x)}{q(x)} = \frac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)}$$

$\dfrac{p(c)}{q(c)} = f(c)$ provided $q(c) \neq 0 \qquad c \in dom f$

iii) So f is continuous at every $c \in dom f$

iv) hence f is continuous

v) Hence, every rational function is continuous, on their domain.

Math 117 - Lecture 43 - One Sided Limits

Recall:

$\lim f\left(x\right)=L$ means $\forall\varepsilon>0\;\exists\partial>0\;\forall x\in\left(domf\right)\setminus\{c\}$ if $|\mathrm{x}-c|<\partial\;then\;\left|f\left(x\right)-1\right|<\varepsilon$

One Sided Limits:

- Motivation
  - $f(x)=\begin{cases}x+2 & x<0\\ x^2+1 & x>0\end{cases}$
  - $f\left(x\right)$ approaches 2 when x approaches 0 from the left, $f\left(x\right)$ approaches 1 when x approaches 0 from the right.
  - $\forall x\in\left(domf\right)\setminus\{c\}\;if\;x\in\left(c-\partial,c\right)\;or\;\left(c,c+\partial\right)$
  - $\lim\limits_{x\to c+}f\left(x\right)$ means approach from the right / above.

    $\forall\partial>0\;\exists\partial>0\;\forall x\in\;domf\;is\;x\in\left(c,c+\partial\right)\;then\;\left|f\left(x\right)-L\right|<\varepsilon$
  - $\lim\limits_{x\to c+}f\left(x\right)$ means approach from the left / below.

    $\forall\varepsilon>0\;\exists\partial>0\;\forall x\in\;domf\;if\;x\in\left(c-\partial,c\right)\;then\;\left|f\left(x\right)-L\right|<\varepsilon$
- Proposition:
  - $\lim\limits_{x\to c}f\left(x\right)=L\iff\lim\limits_{x\to c-}f\left(x\right)=L=\lim\limits_{x\to c^+}f\left(x\right)=L$
  - That is, if $\lim\limits_{x\to c}f\left(x\right)\iff$ the two one sided limits exist and are equal.
  - In this case all the limits are the same.
- Proof

- ⇒ Suppose $\lim_{x \to c} f(x) = L$ then $\lim_{x \to c+} f(x) = L$ and $\lim_{x \to c-} f(x) = L$
- ⇐ suppose $\lim_{x \to c+} f(x) = L$ and $\lim_{x \to c-} f(x) = L$ let $\varepsilon > 0$
  since $\lim_{x \to c+} f(x) = L$ we can find $\partial_1 > 0$ such that for all $x \in dom f$ if $x \in (c, c+\partial)$ then $|f(x) - L| < \varepsilon$
  - Since $\lim_{x \to c-} f(x) = L$, we can find $\partial_2 > 0$ such that for all $x \in dom f$ if $x \in (c - \partial_2, c)$ then $|f(x) - L| < \varepsilon$
- Put $\partial = \min\{\partial_1, \partial_2\}$ then if $x \in (c - \partial, c)$ or $(c, \partial + c)$ then $x \in (c - \partial_2, c)$ or $x \in (c, c+\partial)$ in either case we have $|f(x) - L| < \varepsilon$

- Note: in $\mathbb{R}$ you can only approach from left and right, otherwise there are infinite directions on a plane.
- Example: $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (x^2 + 1)$
  - Because $f(x) = x^2 + 1 \ on \ (0, +\infty)$
  - $x^2 + 1$ is a polynomial, hence continuous, has a $\lim_{x \to 0} (x^2 + 1) = 0^2 + 1 = 1$
  - Note: continuous means value is the limit
  - By proposition, $\lim_{x \to 0+} (x^2 + 1) = \lim_{x \to 0} (x^2 + 1) = 1$

- Combined, we get $\lim_{x \to 0+} f(x) = 1$ simularily $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x+2) = \lim_{x \to 0} (x+2) = 0 + 2 = 2$
- Since $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$ by proposition, $\lim_{x \to 0} f(x) \ d.n.e$

Variants at SFDT for one sided limits

- $\lim_{x \to c-} f(x) = L \iff$ for every sequence (x_n) in $(dom f) \cap (-\infty, c)$ if $x_n \longrightarrow c$ then $f(x_n) \longrightarrow L$
- Similarly $\lim_{x \to c+} f(x) = L \iff$ for every sequence (x_n) in $(dom f) \cap (+\infty, c)$ if $x_n \longrightarrow c$ then $f(x_n) \longrightarrow L$

Homework

Fact: $\lim_{x \to c} f(x) = \lim_{h \to 0} f(c+h) = \lim_{h \to 0} g(h)$

Where $g(h) \ = f(c+h)$

Think of this as a chance of variables, such that,

$h = x - c$

$x = c + h$

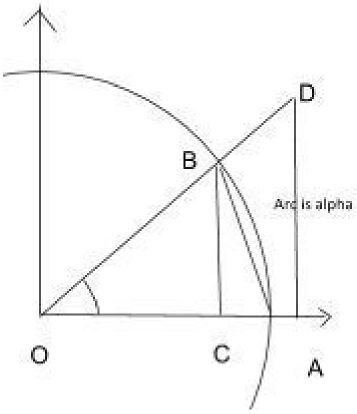
Exercize:

Prove this by expanding definition.

Trig Limits

Lemma,

If $0 < \alpha \ < \dfrac{\pi}{2}$ then $0 < \sin\alpha < \alpha < \tan\alpha$

- Proof CASE A , $0 < \sin\alpha$
  - 

  - $\alpha = arc\ length\ \ \widehat{AB}$
  - ABC is a right triangle
  - AB is the hypotenuse
  - $CB\ <\ AB$ as hypotenuse is the longest side
  - $CB\ <\ AB\ < \widehat{AB} = \alpha$ as we know straight leg is longer than arc.
  - $0 < \sin\ \alpha\ trivial$
- Proof Case B, $\alpha < \tan\alpha$
  - $\dfrac{DA}{OA} = \tan\alpha$ if $\triangle ODA,\ OA = 1$
  - $\tan\alpha = DA$
  - Formal proof that the legs are longer than the arc
    - Compare the areas of the $\triangle OAD$ and the sector OAB
    - Area of the sector $OAB\ \leq\ area\ (\triangle OAD)$ because the sector is properly contained in the triangle.
    - $area\ (\triangle OAD) = \frac{1}{2}OA \cdot OD = \frac{1}{2} \cdot 1 \cdot \tan\alpha = \frac{1}{2}\tan\alpha$
  - Calculate Sector

- Length of circle is 2π, area of the circle is π
- Take the proportion $\frac{\alpha}{x\pi}$ of the area of the dish: π
- The area of the sector is $\frac{\alpha}{x\pi} \cdot \pi = \frac{\alpha}{2}$
- Therefor $\frac{\alpha}{2} < \frac{1}{2}\tan\alpha,\ so\ \alpha < \tan\alpha$

  - Corollary
    - If $-\frac{\pi}{2} < \ -\alpha < \frac{\pi}{2}$ then $|\sin\ \alpha| \leq |\alpha| \leq |\tan\alpha|$

- Proof Case C,
  - If $0 < \alpha\ < \frac{\pi}{2}$ then by lemma, $0 < \sin\alpha < \alpha < \tan\alpha$
  - Hence, $|\sin\ \alpha| < |\alpha| < |\tan\alpha|$
  - If $\alpha = 0$then $\sin\alpha = 0$ and $\tan\alpha = 0$ so

    $|\sin\ \alpha| = |\alpha| = |\tan\alpha| = 0$
  - If $-\frac{\pi}{2} < \alpha < 0$ then $\sin\alpha = -\sin(-\alpha)$

    $\tan\alpha = -\tan(-\alpha)$ $0 < -\alpha < \frac{\pi}{2}$
  - By Lemma, $0\ < \sin(-\alpha) < -\alpha < \tan(-\alpha)$
  - Hence $|\sin\alpha| = |-\sin(-\alpha)\ |=|\ \sin(-\alpha)|$

    $|\tan\alpha| = |-\tan(-\alpha)\ |=|\ \tan(-\alpha)|$
  - Hence $|\sin\alpha| < |\alpha\ |<|\ \tan\alpha|$
- Corollary

- $\lim_{\alpha \to 0} \sin \alpha = 0 = \sin 0$ So $\sin 0 = \lim_{\alpha \to 0} \sin \alpha$
- Proof:
  - For all α in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $|\sin \alpha| \leq |\alpha|$
  - Also $\lim_{\alpha \to 0} \alpha = 0$
  - By Domination Principle, $\lim_{\alpha \to 0} \sin \alpha = 0$

- Corollary $\lim_{\alpha \to 0} \cos \alpha = 1$
  - Proof:
    - When $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\cos \alpha = \sqrt{1 - \sin^2 \alpha}$ so using properties of limits,
    - $\lim_{\alpha \to 0} \cos \alpha = \sqrt{1 - \sin^2 \alpha} = \sqrt{\lim_{\alpha \to} 0}$

      $= \sqrt{1 - \left(\lim_{\alpha \to 0} \sin \alpha\right)^2} = \sqrt{1 - 0^2} = 1$
    - Same logic cosine is continuous at 0
    - Hence by SFDT, if $f(x) > 0$ for all x the

      $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)}$

- Theorem: $\sin x$ and $\cos x$ are continuous (everywhere)
  - Proof
    - $\lim_{x \to c} \sin x = \lim_{h \to 0} \sin(c + h) = \lim_{h \to 0} \sin c \cdot \cos\ h + \cos c \cdot \sin\ h$

- h is a variable so rearranged as

$$= \sin c \left(\lim_{h \to 0} \cos\ h \right) + \cos c\ \cdot \left(\lim_{h \to 0} \sin\ h \right)$$

- $= \sin c \cdot 1 + \cos c \cdot 0 = \sin c$
- $\lim_{x \to c} \sin x = \sin c$ hence c is continuous at c
- For cosine, similar- do as homework

Math 117 - Lecture 44 - Thursday Practice and Present Class

a)

- Let $\varepsilon > 0$, let $\partial = \min\left\{1, \frac{\varepsilon}{4}\right\}$ if $|x-1| = \partial||$
- $|f(x) - L| = \left|\left(x^2 - 3x + 7\right) - 5\right| = \left|x^2 - 3x + 2\right|$

  $= |(x-1)(x-2)| < \partial|x-2| \le \partial|x| + 2\partial$
- Because $|x-1| \le \partial \le 1, \ |x-1| \le 1, |x| < 2||$

b)

- $f(x) = \sqrt{x^2 + 5} \ domf =$ $\mathbb{R}$
- Then we can find a sequence (x_n) in $\mathbb{R}$ such that $x_n \longrightarrow 2$.
- Then if $x_n \longrightarrow 2$ then $f(x_n) = \sqrt{x_n^2} + 5 \longrightarrow 3$
- As $n \longrightarrow \infty$, then by SFDT, $\lim_{x\to 2} f(x) = 3$.
- Formally
  - $x_n \longrightarrow 2$ then properties of sequences,
  - then $x_n^2 \longrightarrow 4$,
  - $x_n^2 + 5 \longrightarrow 9$
  - $\sqrt{x_n^2 + 5} \longrightarrow 3$

c)

- $f(x) = \begin{cases} x+1 & x \le 2 \\ x^2 & x > 2 \end{cases}$
- Suppose $\lim_{x\to 2} f(x)$ exists, $\lim_{x\to 2} f(x) = L$

- Let $x_n \longrightarrow 2 \; x \in (-8, 2)$
- By SFDT $f(x_n) \longrightarrow L \forall n \in \mathbb{N}$
- $x_n + 1 = 2 + 1 = 3$
- $f(x_n) \longrightarrow 3$
- $\lim_{x \to 2} f(x) = 3$
- $L = 3$

* Let $y_n \longrightarrow 2, \; y \in (2, +\infty)$
  - bySFDT, $f(x_n) \longrightarrow \forall n \in \mathbb{N}$
  - $y_n^2 = 2^2 = 4$
  - $f(y_n) = 4$
  - $\lim_{x \to 2} f(x) = 4$
  - $L = 4$
* By contradiction, l $\lim_{x \to 2} f(x) \; d.n.e.$

d)

* $f(x) = x^2 \left(\sin \frac{1}{x}\right)$
  - say $g(x) = x^2$
  - $h(x) = \sin \frac{1}{x}$
  - $\max h(x) = 1$
  - $\min h(x) = -1$
* $\lim_{x \to 0} g(x) = 0$

- $\lim_{x \to 0} x^2 \longrightarrow 0$
- $\lim_{x \to 0} -x^2 \longrightarrow 0$
- Squeeze Law
  - $\lim_{x \to 2} x^2 \left(\sin \frac{1}{x}\right) = 0$

e)I

- Suppose the limit exists.
- $\lim_{x \to 0} \left(\sin \frac{1}{x}\right)^2$
- $x_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \quad x_n \longrightarrow 0$
  - $\lim f(x_n) = \lim \left(\sin\left(\frac{\pi}{2} + 2\pi n\right)\right)^2 = 1$
- $y_n = \frac{1}{\pi n} \quad y_n \longrightarrow 0$
  - $\lim f(y_n) = \lim (\sin(\pi n))^2 = 0$
- By SFDT $\lim_{x \to 0} f(x) = 1 \; but \; \lim_{x \to 0} f(x) = 0$
- Contradiction, since limits are unique, $\lim_{x \to 0} \left(\sin \frac{1}{x}\right) = d.n.e$

f)

- take $\varepsilon = \frac{1}{\varepsilon} \forall \in domf \; x \in (c - \partial, c) \cup (c, c + \partial)$

- $\exists x_1$, irrational
- $\exists x_2$,rational
- $|f(x_1) - L| = |-L| = |L| < \frac{1}{3}$
- $|f(x_2) - L| = |1 - L| = |L| < \frac{1}{3}$
- $1 < |1 - c| + |c| < \frac{2}{3}$
- False.

Math 117 - Lecture 45 - Properties of Continuous Function

Recall

1) For every $\alpha \in \left(0, \frac{\pi}{2}\right)$ $0 < \sin\alpha < \alpha < \tan\alpha$
2) $\sin x$ and $\cos x$ are continuous functions.

Theorem:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Proof

- $x \in \left(0, \frac{\pi}{2}\right)$ then $0 < \sin x < x < \frac{\sin x}{\cos x}$
- Device by $\sin x$
- $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$
- Take reciprocals
- $\cos x < \frac{\sin x}{x} < 1$
- True for all $x \in \left(0, \frac{\pi}{2}\right)$
- If $x \in \left(-\frac{\pi}{2}, 0\right)$ same inequality holds.
- $\cos(-x) - \cos\ x$ cos is an even function
- $\frac{\sin(-x)}{(-x)} = \frac{-\sin x}{-x} - \frac{\sin x}{x}$

- Hence, $\cos x < \frac{\sin x}{x} < 1$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \ / \{0\}$
- Squeeze Law, $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\sin x \approx x$ when x is small.

Examples

1. $$\lim_{t \to 0} \frac{t^s \cos t}{\sin^2 t} = \lim_{t \to 0} \frac{\cos t}{\left(\frac{\sin t}{t}\right)^2} = \frac{\lim_{t \to 0} \cos t}{\lim_{t \to 0} \left(\frac{\sin t}{t}\right)^2} = \frac{1}{1^2} = 1$$
2. $$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2 \cdot ((1 + \cos x))}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2 (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2 (1 + \cos x)} = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{1 + \cos x}$$

$1 \cdot \frac{1}{1+1} = \frac{1}{2}$ $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ when x is small

$\frac{1 - \cos x}{x^2} \approx \frac{1}{2}$ so that $\cos x \approx 1 - \frac{x^2}{2}$

3. $$\lim_{x \to 0} \frac{\sin(3x)}{x} = 3 \lim_{x \to 0} \frac{\sin 3(x)}{3x}$$
   a. Change of Variables

i. $t = 3x \; as \; x \longrightarrow 0 \; then \; t \longrightarrow 0$ so

$$3 \cdot \lim_{x \to 0} \frac{\sin t}{t} = 3 \cdot 1 = 3$$

ii. SFDT: suppose $x_n \longrightarrow 0$ put $t_n = 3x_n$ then $f_n \longrightarrow 0$ since $\lim_{x \to 0} \frac{\sin t}{t} = 1$. By SFDT we have $\frac{\sin t}{t_n} \longrightarrow 1$

iii. Hence, $\frac{\sin(3x_n)}{3x_n} \longrightarrow 1$, thus if $x_n \longrightarrow 0$ then $\frac{\sin(3x_n)}{3x_n} \longrightarrow 1$

iv. Apply SFDT to the function $\frac{\sin 3x}{3x}$ hence $\lim_{x \to 0} \frac{\sin 3x}{3x} = 1$

4. $\lim_{x \to 1} (1 - x) \cdot \tan \frac{\pi x}{2}$

   a. Another Example involving change of variable

      i. $t = 1 - x$, $as \; x \longrightarrow 1 \; then \; t \longrightarrow 0$

ii. $$\lim_{t\to 0} \cdot \tan\left(\frac{\pi}{2}(1-t)\right) = \lim_{t\to 0} t \cdot \frac{\sin\left(\frac{pi}{2} - \frac{\pi}{2}t\right)}{\cos\left(\frac{\pi}{2} - \frac{\pi}{2}t\right)}$$

$$= \lim_{t\to 0} t \cdot \frac{\cos\ \frac{\pi}{2}t}{\sin\ \frac{\pi}{2}t} = \lim_{t\to 0} \frac{\cos\ \frac{pi}{2}t}{\frac{\sin\ \frac{\pi}{2}t}{2}}$$

$$= \frac{2}{\pi} \lim_{t\to 0} \frac{\cos\left(\frac{\pi}{2}t\right)}{\frac{\sin\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}t}} = \frac{2}{\pi} \frac{\lim_{t\to 0} \cos\left(\frac{pi}{2}t\right)}{\lim_{t\to 0} \frac{\sin\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}t}} =$$

iii. $$s = \frac{\pi}{2}t,\ as\ t \longrightarrow 0,\ then\ s\ \longrightarrow 0$$

iv. $$\frac{\lim_{s\to 0} \cos s}{\lim_{s\to 0} \frac{\sin s}{s}} = \frac{2}{\pi} \cdot \frac{1}{1} = \frac{2}{\pi}$$

Continuous Functions and its Properties

Recall,

- f is continuous at c if $\lim_{x\to c} f(x) = f(c)$ By SFDT, f is continuous at c.
- f is continuous if it is $x_n \longrightarrow c$ then $f(x_n) \longrightarrow f(c)$, continuous at every point of its $dom f$
- f is continuous at every interval, if the restriction is f, this interval is continuous.
- Polynomials are continuous on $\mathbb{R}$, everywhere.

- A Rational function such as $\frac{p(x)}{q(x)}$ is continuous on its domain.
- | $\mathbb{R}$ | {the roots at q)
- $\sqrt{x}$ is continuous (on its domain $[o, +\infty)$)
- $\sin x$ and $\cos x$ are continuous

Properties

- The sum, difference and the products of two continuous functions are continuous
  - $\lim_{x \to c} f(x) + g(x) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

    $= f(c) + g(c)$ hence $f + g$ is continuous at c
- Example 1
  - $3x^2 + 5x + \sqrt{x}$ is continuous on its domain $[o, +\infty)$
- If f is continuous at c and g is continuous at c and $g(c) \neq 0$ then

  $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)}$ hence $\frac{f}{g}$ is continuous at c.
- In particular, if f is continuous and g is continuous then $\frac{f}{g}$ is continuous.
- Example 2
  - $\frac{3x^2 + 5x + 7}{\sqrt{x}}$
  - Continuous (on its domain). Domain is $[o, \infty)$
- Example 3
  - $\tan x = \frac{\sin x}{\cos x}$ is continuous. Domain $\tan x$

- Example 4
  - $f\left(x\right)=\frac{1}{x}$ continuous on its domain, $domf=\mathbb{R}_\backslash\{0\}$
  - f has a 'singularity' at zero (one sided limits are infinite)
  - f is continuous on $(0,+\infty)$ and on $(-\infty,0)$
- Example 5
  - $\lceil x\rceil and\lfloor x\rfloor$
  - Continuous except at integer points, $\mathbb{R}_\backslash\mathbb{Z}$
  - Discontinuity at every integer point
- Example 6
  - $f(x)=\begin{cases}1 & x \quad rational\\ 0 & x \quad irrational\end{cases}$
  - $\lim\limits_{x\to c}f\left(x\right)$ d.n.e. For every $c\in\mathbb{R}$
  - Hence, f is discontinuous at every point.
  - $f(x)=\begin{cases}1 & x \quad rational\\ 0 & x \quad irrational\end{cases}$ and
    $x=\frac{m}{n}\ \ m\in Z,\ n\in N$ have no common divisors.
  - Claim
    - f is continuous at every irrational point and discontinuous at every rational point.
- Theorem: Composition of two continuous functions is continuous.
  - More precisely, if f is continuous at c and g is continuous at $f\left(c\right)$, then $g\circ f$ is continuous at c

- Proof
  - If $x_n \longrightarrow c$ then $f(x_n) \longrightarrow f(c)$ because f is continuous at c.
  - Then $g(f(x_n)) \longrightarrow g(f(c))$ because g is continuous at $f(c)$.
  - Hence if $y_n \longrightarrow f(c)\ then\ g(y_n) \longrightarrow g(f(c))$
  - Apply with $y_n = f(x_n)$
  - ($g \circ f$)$(x_n) \longrightarrow$ ($g \circ f$)(c)
  - Thus, if $x_n \longrightarrow c$ then ($g \circ f$)$(x_n) \longrightarrow$ ($g \circ f$)(c)
  - This implies that $g \circ f$ is continuous at c
  - Essentially if $x_n \longrightarrow c$ then $f(x_n) \longrightarrow f(c)$ then $g(f(x_n)) \longrightarrow g(f(c))$
  - Example
    - $f(x) = \sqrt{3x^2 + 5}$ is continuous.
    - It is a composition at $3x^2 + 5$ and $\sqrt{x}$

Exam Notes

University of Alberta

MATH117

Honors Calculus I

Fall 2017

Term Test 1

Exam Guide

Elementary Concepts from Set Theory

Numbers:

- R – denotes the set of all real numbers
- Q – denotes the set of all rational numbers i.e. fractions, p/q
- Z – denotes the set of all integers
- N – denotes the set of all natural numbers, these are the nonnegative integers. (Thus 0 is the smallest natural number)

Intervals:

- The **Open** interval is denoted by (a,b)
- The **Closed** interval is denoted by [a,b]
- The **Clopen** interval is denoted by [a,b) or (a,b]
- Infinity is not a real number
- The Infinite interval is (-infinity, infinity)

Sets:

- A set is a collection of elements
- A U B is the union of A and B and is the set of elements belonging to at least one of the sets
- A ∩ B intersection of A and B and is the set of elements belonging to both of the sets

Function:

- A function f from set A to set B (or simply function) is a rule that assigns to each element x of the domain set A to exactly one element called f(x) of the codomain set B
- All inputs of x must have a unique output
- Must pass the Vertical Line Test

Absolute Function:

- The absolute function is a piecewise function
- a if a >= 0
- -a if a < 0

One-to-One Function:

- A one-to-one function is a function that passes both the vertical line test and the horizontal line test
- A function is 1-1 if f(a) = f(b) => a = b

Example: $f(x) = 2x^2 + 5$

$2a^2 + 5 = 2b^2 + 5$
$2a^2 = 2b^2$
$a^2 = b^2$
$a = +b$ or $-b$

Therefore, this function is not 1-1 as one value of a can produce two values of b.

Composition of Functions:

- Notation is f[g(x)]
- Can be thought of as nested functions
- The domain only exists for the intersection of the domains of the original functions

Operations with Functions:

- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x) * g(x)$
- $(f / g)(x) = f(x) / g(x)$

Piecewise functions:

- Functions that have different rules according to the domain that is given
- The absolute function is an example of a piecewise function

Floor and Ceiling functions:

- This function's range can only be integers
- The domain is still all real numbers

Transcendental Functions

Exponential Functions:

- In high school, the common form was y = a^x
- The more formal form is y = A(e)^(kx), where k is an element of real numbers
- The domain and range both consist of real numbers

Transcendental vs. Algebraic:

- Transcendental functions cannot be further broken down using algebraic operations
- Algebraic functions can be broken down using algebraic operations, such as addition, subtraction, multiplication, and division

Examples

- y = 2e^(2x) is a transcendental function because it meets the formal form
- y = e^(-x) + 12 is not a transcendental function because it includes the algebraic operation of addition
- y = -2(3^x) is a transcendental function
- ln(x – 1) > 0 => x – 1 > 1 => x > 3

Logarithmic Functions:

- Logarithmic and exponential functions are both one-to-one functions
- Logarithmic and exponential functions are inverses of each other
- "log" means a power we have to raise the base to in order to match the arguement

Examples

- $\log_2(6) - \log_2(15) + \log_2(20) = \log_2(6/15 * 20) = \log_2 8 = 3$
- log(sqrt(10)) = ½(log10) = ½

Trigonometric Functions:

- Standard positon is when the initial edge rests along the x-axis
- The terminal edge represents the arm that points to any point on the unit circle
- The unit circle has a radius of 1
- If the angle theta is in the standard position and its terminal edge intersects unit circle at some point (x,y) in the XY-plane, then the sin of theta is defined as y-coordinate of that point
- If the angle theta is in the standard position and its terminal edge intersects unit circle at some point (x,y) in the XY-plane, then the cos of theta is defined as x-coordinate of that point

Principal Values

- y = sinx has principal values between -pi/2 and pi/2
- y = cosx has principal values between 0 and pi

- y = tanx has principal values between -pi/2 and pi/2
- y = cscx has principal values between -pi/2 and pi/2
- y = secx has principal values between 0 and pi
- y = cotx has principal values between 0 and pi

Example => $4\sin^2 x - 1 = 0$

$$4\sin^2 x - 1 = 0$$
$$(2\sin x - 1)(2\sin x + 1) = 0$$
x_1 = pi/6 + 2pi * k, where k is an integer
x_2 = 5pi/6 + 2pi * k, where k is an integer
x_3 = 7pi/6 + 2pi * k, where k is an integer
x_4 = 11pi/6 + 2pi * k, where k is an integer

Angles:

- Angles are always displayed with respect to standard position
- The angle between x-axis and terminal edge is the reference angle

Formulas:

- tanx = sinx / cosx
- cscx = 1 / sinx
- secx = 1 / cosx
- cotx = 1 / tanx
- sin(-x) = -sinx
- cos(-x) = cosx
- tan(-x) = -tanx
- cos (pi/2 – x) = sinx = cos(pi/2 + x)
- sin (pi/2 – x) = cosx = sin(pi/2 + x)
- tan (pi/2 – x) = cotx = tan(pi/2 + x)
- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\cot^2 x + 1 = \csc^2 x$

Logic and Mathematic Thinking

More Trigonometric Formulas

- $sin(\alpha + \beta) = sin(\alpha)cos(\beta) + cos(\alpha)sin(\beta)$
- $sin(\alpha - \beta) = sin(\alpha)cos(\beta) - cos(\alpha)sin(\beta)$
- $cos(\alpha + \beta) = cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta)$
- $cos(\alpha - \beta) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$
- $sin(2x) = 2sin(x)cos(x)$
- $cos(2x) = cos^2(x) - sin^2(x) = 1 - 2sin^2(x) = 2cos^2(x) - 1$

Example #1

ln((x – 1) / 2) > 0
e^(ln((x – 1) / 2)) > e^0
(x – 1) / 2 > 1
x > 3

Example #2

sin(-3pi/4)
= sin (pi – pi/4)
= sin(-pi/4)
= -sin(pi/4)
= -sqrt(2)/2

Example #3

sin(105) = sin(60 + 45)
= sin60cos45 + sin45cos60
= sqrt(3)/2 * sqrt(2) / 2 + sqrt(2)/2 * ½
=sqrt(2)[sqrt(3) + 1] / 4

Example #4

$sin(cos^{-1}x)$ = sin(arccosx)
= sqrt(1 – x^2)

Example #5

$sin^{-1}(cos1/2)$
= arcsin(cos1/2)
= pi / 3

Example #6

$sin(sec^{-1}(3 / x))$
= sin(arcsec(3 / x))
= sqrt(9 – x^2) / 3

Language of Proofs

- $\forall$ - "for all"
- $\exists$ = "there exists
- R – set of real numbers
- I – set of irrational numbers
- Q – set of rational numbers
- Z – set of integers
- N – natural numbers (nonnegative integers and 0)
- $\Rightarrow$ - "implies"
- $\Leftrightarrow$ - "equivalent", "iff"
- $\in$ - "belong to" "is in …"
- $\notin$ - "is not in…"
- | - "such that"

Calculational Proof

Prove that x^2 – y^2 = (x – y)(x + y)

$$(x - y)(x + y)$$
$$= x^2 - xy + xy - y^2$$
$$= x^2 - y^2$$

QED – "which was to be demonstrated"

Direct Proof

Prove that the sum of odd and odd number is even.

Given: x = 2m + 1, m $\in$ z; y = 2n + 1, n $\in$ z

Show: x + y = 2k, k $\in$ z

Proof:

$$x + y$$
$$= 2m + 1 + 2n + 1$$
$$= 2 + 2(m + n)$$
$$= 2(m + n + 1)$$

Since m, n, and 1 are all integers and the sum of all integers must be an integer also. Therfore the proof is complete.

QED

Proof by Contradiction

Let f(x) = 2x + 5. Prove that f(x) is a one-to-one function.
Given: f(x) = 2x + 5

Show: $\exists$ x_1 and x_2 | $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

Proof: Suppose that $f(x) = 2x + 5$ is not one-to-one. Then there exists such x_1 and x_2 that $f(x_1) = f(x_2)$ for some x_1 not equal to x_2

$$2x_1 + 5 = 2x_2 + 5$$
$$2x_1 = 2x_2$$
$$x_1 = x_2$$

The conclusion shows that for $f(x) = 2x + 5$, $f(x_1) = f(x_2)$ iff $x_1 = x_2$, which contradicts the supposition. Therefore, the supposition is incorrect and $f(x) = 2x + 5$ is a one-to-one function.

QED

Real Numbers, Absolute Values & Completing the Square)

Real Numbers

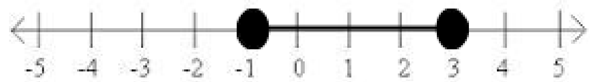
Real Number: a value that represents a quantity along line.

There are three ways to depict *real numbers* but essentially, they all mean the same thing.

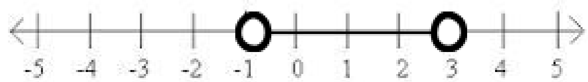
- Interval Notation: using brackets and parenthesis.
    - [] - brackets indicate that the interval includes the maximum limit as a solution to the expression.
        - Example: [-1,3] shows that the solutions to the expression are all real numbers in between -1 and 3 including -1 and 3.
    - () - parentheses indicate that the intervals do not include the maximum limit as a solution to the expression.
        - Example: (-1,3) shows that the solutions to the expression are all real numbers in between -1 and 3 but not including -1 and 3.
    - Both a bracket and parenthesis can be used at one time.
        - Example: (-1,3] shows that the solutions range from -1 to 3 including 3 but not -1.
        - Example: Vice versa, [-1,3) shows that the solutions range from -1 to 3 including -1 but not 3.
    - Interval notation should be written from smallest to largest number. [3,-1] is an incorrect format.

- Graphing
    - In the following examples, you will see two different types of circles; solid, and unfilled.
        - A solid dot indicates that the number it is above is included in the solutions.
        - An unfilled dot indicates that the number it is above is not included in the solutions.

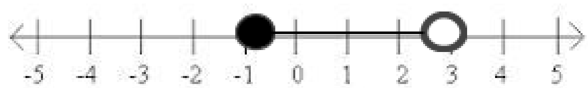
- The following number lines show the same interval notation used above in a different way.
  - [-1,3]



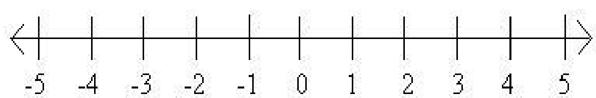
  - (-1,3)



  - [-1,3)

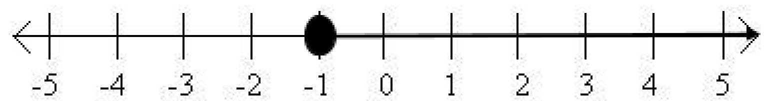


  - (-1,3]



- <u>Inequality Notation</u>
  - < - less than
  - > - greater than
  - ≤ - less than or equal to
  - ≥ - greater than or equal to
    - $-1 \leq x \leq 3$ means that the solutions range from -1 to 3 including -1 and 3.
    - $-1 < x < 3$ means that the solutions range from -1 to 3 not including -1 and 3.
    - $-1 < x \leq 3$ means that the solutions range from -1 to 3, including 3 but not -1.
    - $-1 \leq x < 3$ means that the solutions range from -1 to 3, including -1 but not 3.

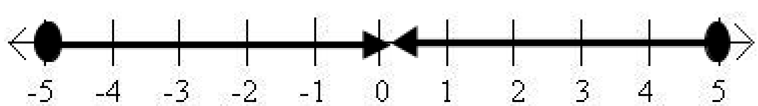
- Infinite Solutions
  - If there are an infinite number of solutions to an expression
    - An open parenthesis should be used instead of square brackets because brackets are used to enclose numbers and an infinite amount of solutions will never end.
      - Interval Notation: [-1, ∞)
      - Graph



      - Inequality Notation: $-1 \leq x$

Absolute Values

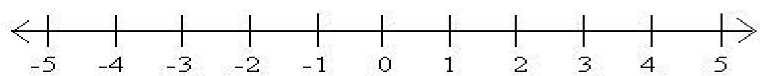
- Absolute value: the distance between x, a point on the number line, and zero.



- The absolute value of 5 is 5 because it is five spaces away from point zero.
- The absolute value of -5 is also 5 because it is also five spaces away from point zero. Direction does not matter, absolute value is strictly quantitative.
- If |x| = 7, then the two solutions are x= -7 and x=7 because they are both equidistant from zero.
- If |x| = 0, then the only solution is x= 0.
- If |x| = -7 , then there are no solutions because there is no such thing as negative distance.

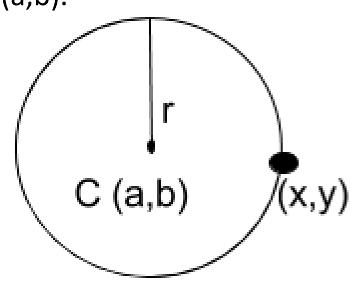
- To solve equations involving absolute values, you need to make sure you create two separate equations and solve.
  - Example: |x-3| ≤ 2
    - The two separate equations would be x - 3 = 2 and x - 3 = -2. This is because there are two values where x is equidistant and from zero. Then, you can solve algebraically.
    - x - 3 = 2 x - 3 = -2

      x= 5 **x = 1**
    - Interval Notation: [1,5]
    - Graph:



Completing the Square

- All sets of (x,y) all have the same distance, r, from the center (a,b).
- Distance Formula: $(x-a)^2+(y-b)^2=r^2$, with center (a,b).
- Important foil patterns to know and recognize:
  - $(c+d)^2 = c^2 + 2\,cd + d^2$
  - $(c-d)^2 = c^2 - 2cd + d^2$
  - $(c^2-d^2) = (c-d)(c+d)$



- Completing the Square is a method used to find a circle's center and radius when the information is given in another format, and not the distance formula format.
  - Example: Find the center and radius of a circle with the equation $4x^2+4y^2-4x+8y-5=0$.
    - To complete the square:
      - Separate the variables. $4x^2-4x+4y^2+8y-5=0$
      - Factor.

        $4(x^2-x)\ +\ 4(y^2+2y)\ -\ 5=0$

- Identify the coefficient of x alone. *The coefficient is 1 because there isn't a number in front of x.*
- Half the coefficient. *(½)*
- Square it. $(1/2)^2 = ¼$
- Add it and subtract it into the factored eq. $4(x^2 - x + 1/4 \ - \ 1/4)$
- Factor. $4((x - 1/2)^2 - 1/4)$
- Foil. $4(x - 1/2)^2 - 1$
- Repeat the squaring method for the y variable.
    - Factor. $4(y^2 + 2y)$
    - Identify the coefficient of only the y variable. *The coefficient is 2.*
    - Half the coefficient. (1)
    - Square the coefficient. $(1)^2 = 1$
    - Add and subtract 1 into the factored eq. $4(y^2 + 2y + 1 - 1)$
    - Factor. $4(y + 1)^2 - 1$
    - Foil. $4(y + 1)^2 - 4$

- Replace the old expressions for x and y with the ones you created.
    - Original Equation: $4(x^2 - x) \ + \ 4(y^2 + 2y) \ - \ 5 = 0$
    - Combined equation: $4(x - 1/2)^2 - 1 + 4(y + 1)^2 - 4 - 5 = 0$
    - Combine like terms. $4(x - 1/2)^2 + 4(y + 1)^2 - 10 = 0$
    - Add 10 to both sides. $4(x - 1/2)^2 + 4(y + 1)^2 = 10$
    - Divide both sides by 4. $(x - 1/2)^2 + (y + 1)^2 = 10/4$

- Congratulations! You've completed the square. Compare this with the distance formula given to you at the beginning of this section. They are the same!

- Now, you can find the center and the radius.
  - To find the center, solve for x and y.
    - $x - 1/2 = 0$

      $x = 1/2$
    - $y + 1 = 0$

      $y = -1$
    - The center of this circle has the coordinates (½,-1).

- Find the radius by finding the square root of 10/4. (√5/2 or ≈1.58)

Mathematical Induction

Mathematical Induction (MI) is an extremely important tool in Mathematics.

First of all you should never confuse MI with Inductive Attitude in Science. The latter is just a process of establishing general principles from particular cases.

MI is a way of *proving* math statements for all integers (perhaps excluding a finite number) says:

Statements proved by math induction all depend on an integer, say, n. For example,

(1) $1 + 3 + 5 + ... + (2n - 1) = n^2$
(2) If $x_1, x_2, ..., x_n > 0$ then $(x_1 + x_2 + ... + x_n)/n \geq (x_1 \cdot x_2 \cdot ... \cdot x_n)^{1/n}$

Note: In this case, *n* is an arbitrary integer.

It is convenient to talk about a statement P(n). For (1), P(1) says that $1 = 1^2$ which is incidentally true. P(2) says that $1 + 3 = 2^2$, P(3) means that $1 + 3 + 5 = 3^2$. And so on. These particular cases are obtained by substituting specific values 1, 2, 3 for n into P(n).

Assume you want to prove that for some statement P, P(n) is true for all n starting with n=1.

The *Principle* of *Math Induction* states that, to this end, one should accomplish just two steps:

1. Prove that P(1) is true.
2. Assume that P(k) is true for some k. Derive from here that P(k+1) is also true.

The idea of MI is that a finite number of steps may be needed to prove an infinite number of statements P(1), P(2), P(3), ...

Let's prove (1). We already saw that P(1) is true. Assume that, for an arbitrary k, P(k) is also true, i.e. $1 + 3 + ... + (2k-1) = k^2$. Let's derive P(k+1) from this assumption. We have:

$$\begin{aligned} 1 + 3 + ... + (2k-1) + (2k+1) &= [1 + 3 + ... + (2k-1)] + (2k+1) \\ &= k^2 + (2k+1) \\ &= (k+1)^2 \end{aligned}$$

Which exactly means that P(k+1) holds. (For 2k+1 = 2(k+1)-1.) Therefore, P(n) is true for all n starting with 1.

Intuitively, the inductive (second) step allows one to say, look P(1) is true and implies P(2). Therefore P(2) is true. But P(2) implies P(3). Therefore P(3) is true which implies P(4) and so on. Math induction is just a shortcut that collapses an infinite number of such steps into the two above.

In Science, *inductive attitude* would be to check a few first statements, say, P(1), P(2), P(3), P(4), and then assert that P(n) holds for all n. The inductive step "P(k) implies P(k + 1)" is missing. Needless to say nothing can be proved this way.

Remark

1. Often it's impractical to start with n = 1. MI applies with any starting integer n_0. The result is then proved for all n from n_0 on.
2. Sometimes, instead of 2., one assumes 2':

Assume that P(m) is true for all m < (k + 1).

Derive from here that P(k+1) is also true. The two approaches are equivalent, because one may consider statement Q: Q(n) = P(1) and P(2) and ... and P(n), so that Q(n) is true iff P(1), P(2), ..., P(n) are all true.

This variant goes by the name of *Complete Induction* or *Strong Induction*.

Additional Explanation:

Below, $|S|$ will denote the number of elements in a finite (or empty) set S. So, for example, $|\{\}| = 0$ and $|\{0\}| = 1$. The empty set {} is denoted Ø.

Sum Rule

If A and B are disjoint, i.e., if $A \cap B = \emptyset$, then

(1) $|A \cup B| = |A| + |B|$.

Comment: behind the set-theoretic symbolism stands a simple fact without which ***counting*** would be impossible: *it does not matter how you count*, i.e., as long as you do not make a mistake of, say, missing an object or counting an object twice. It says this: if before counting objects one splits them into two groups and then counts the elements of one of the groups before proceeding to count the elements of the other, the result will be the same - the total number of objects to be counted. (Naturally, it does not depend on how the objects have been split into two groups.)

Example 1

In a class of 30 students, there are 16 boys and 14 girls (16 + 14 = 30). Of these, 23 persons wear pants and only 7 wear skirts (23 + 7 = 30). On the last exam 20 students received a passing grade, while 10 failed (20 + 10 = 30).

By ***induction***, the sum rule is easily extended to any finite number of mutually disjoint sets:

(1') $|A \cup B \cup C \cup D \ldots| = |A| + |B| + |C| + |D| + \ldots$

Example 2

An electronic book of 472 pages has been stored in separate files - 1 file per page - in two folders. One folder contained 305 files, the other 167 files (305 + 167 = 472.)

Product Rule

For a ***direct product*** A×B of two finite sets A and B,

(2) $|A\times B| = |A|\times|B|$.

Comment: By induction the rule extends to any finite number of sets:

$$|A \times B \times C \times D ...| = |A| \times |B| \times |C| \times |D| ...$$
(2')

An essential point here is how the tuples of objects are formed: an object is picked out from one of the given sets regardless of which objects have been drawn from the other sets. Why the rule is called *sequential*? Because in a tuple, the objects (components) are ordered: there is the first one, the second, and so on.

Example 3

The are two drawers. One contains 12 shirts, the other 7 neckties. There are $84 = 12 \times 7$ ways to combine a shirt and a necktie.

It is possible to examine the drawers sequentially: first-second, first-second... It is also possible to form combinations using two hands: left for a shirt, right for a necktie. As long as all possible combinations shirt/necktie have been counted, the exact procedure is of no consequence.

Example 4

A test consists of 6 mutiple-choice questions. Each question has 4 possible answers. There are

$$4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6$$

ways to answer all 6 questions.

Counting ***poker hands* provides multiple additional examples.

Proof By Induction

Please note that:
WTS = "want to show" that
[] = substitute for the square that indicates that the proof is finished

Q1. Write a good proof for the following theorem:
The sum of two odd numbers is even.

A1. Proof:
Let n, m ∈ Z
I assume n, m are odd
WTS n + m is even. This can also be written as WTS ∃ x ∈ Z s.t. n + m = 2x

- n is odd. By definition, ∃ a ∈ Z s.t. n = 2a + 1
- m is odd. By definition, ∃ b ∈ Z s.t. m = 2b + 1

Then n + m = 2a + 1 + 2b + 1 = 2a + 2b + 2
n + m = 2(a+b+1)
n + m = 2x
Since a + b + 1 ∈ Z, this proves n + m is even. []

<u>Q2. Variations on Induction #1</u>
Let S_n be a statement that depends on a positive integer *n*.
In each of the following cases, which statements are guaranteed to be true?

1. We have proven:
   - S_3 is true.
   - $\forall n \geq 1$, S_n is true $\Rightarrow$ S_{n+1} is true.
2. We have proven:
   - S_1 is true.
   - $\forall n \geq 3$, S_n is true $\Rightarrow$ S_{n+1} is true .
3. We have proven:
   - S_1 is true.
   - $\forall n \geq 1$, S_n is true $\Rightarrow$ S_{n+3} is true .
4. We have proven:
   - S_1 is true.
   - $\forall n \geq 1$, S_{n+1} is true $\Rightarrow$ S_n is true .

A2. For each case, we know that the following are true:

1. S_3, S_4, S_5, S_6, …
2. S_1.
3. S_1, S_4, S_7 …
4. S_1.

Q3. Variations on Induction #2

We want to prove that $\forall n \geq 1$, S_n is true

So far we have proven

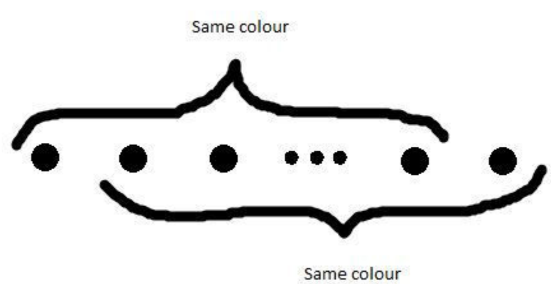
- S_1 is true.
- $\forall n \geq 1$, S_n is true $\Rightarrow$ S_{n+3} is true.

What else do we need to do?

A3. We also need to prove S_2 and S_3.

Q4. What is wrong with this proof by induction?
Theorem: $\forall N \in N$, in every set of N cars, all the cars are of the same colour.
Proof:

- Base case: It is clearly true for N = 1.
- Induction step.
    - Assume it is true for N. I'll show it is true for N + 1.
    - Take a set of N + 1 cars. By induction hypothesis:
        - The first N cars are of the same colour.
        - The last N cars are of the same colour.



        - Hence the N + 1 cars are all of the same colour.

A4. We've proven that S_1 is true, and that

- $\forall N \geq 2$, $S_N \Rightarrow S_{N+1}$ is true
    - note that this is *different* from proving that simply S_{N+1} is true
- The only error: The induction step in the proof above must start at n = 2, because of what we've proven and know (i.e., "$\forall N \geq 2$, $S_N \Rightarrow S_{N+1}$ is true").

Q5. What is wrong with the following description of proof by induction?

A proof by induction has 3 parts:

1. Base case: Prove the statement is true for 1 (for example)
2. Induction hypothesis: Assume the statement is true for n.
3. Induction step: Prove the statement is true for n + 1.

A5. A proof by induction actually has 2 parts:

1. Initial proof (e.g., true for n = 1)
2. Induction (e.g., true for n ⇒ true for n + 1)
    - Again, note that there is a difference between…
        i. $S_n \Rightarrow S_{n+1}$ is true
        ii. S_{n+1} is true

Watch:

- For Friday: 2.4 (absolute
- For Monday: 2.1, 2.2, 2.3

Binomial functions & Taylor series

The Taylor theorem

- The taylor polynomial and taylor series are obtained from a generalization fo the Mean Value Theorem.
- If f: [a,b] approaching R is differentiable, then there exists a c on [a,b] such that (f(b) - f(a))/(b-a) = f'(c), which is equivalent to:
  - f(b) = f(a) + f'(c)(b-a)
- Theorem (Taylor's theorem):
  - If f: [a,b] approaching R is (n+1) times continuously differentiable, then there exists c on (a,b) such that
    - f(b) = f(a) + f'(a)(b-a) + f''(a)/2 * $(b-a)^2$ + ….. + $\frac{f^n a}{n!}(b-a)^n + \frac{f^{n+1} c}{(n+1)!}(b-a)^{n+1}$
- Remark:
  - The taylor theorem is usually applied for a fixed point a, while the point b = x is used as an independent variable.
  - f(x) = f(a) + f'(a)(x-a) + ... + $\frac{f^n a}{n!}(x-a)^n + R_n x$
  - The remainder function is given by:
  - $R_n(x) = \frac{f^{n+1} c(x)}{(n+1)!}(x-a)^{n+1}$ $with\ c\ on\ (a,x)$
- The point c on (a,x) is also dependent on x.
- We can use the taylor polynomial to write that f(x) = $T_n(x) + R_n(x)$

The Binomial Function

- Definition:
  - The binomial function is a function of the form:
    - $f_m(x) = (1+x)^m,\ with\ m\ on\ R.$
- Remark:
  - If m is a positive integer, then the binomial function f_m is a polynomial, therefore the Taylor series is the same polynomial, hence the Taylor series has only the first m + 1 terms non zero.
  - If m is not a positive integer, then the taylor series of the binomial function has infinitely many non-zero terms.

Binomial Theorem:

- The Taylor series for the binomial function f_m (x) = $(1+x)^m$ with m not a positive integer, converges for |x| < 1, and is given by:
  - $T(x) = 1 + \sum_{n=1}^{infinity} (m\ n)x^n$
- With the binomial coefficients (m 1) = m, (m 2) =
  - $\frac{m(m-1)}{2!}$
  - (m n) = $\frac{m(m-1)...(m-(n-1))}{n!}$

Evaluating non elementary integrals

- Remark: non elementary integrals can be evaluated by integrating term by term the integrand taylor series.

Euler Identity

- The taylor definition suggests the definition:
  - $e^{i\theta} = cos\theta + isin\theta$

Continuity, Intervals & Bounds

The Extreme Value Theorem

- If f(x**) is continuous on closed interval [a,b],** then there exist some values **M** and **m** in the interval [**a,b]** such that f(M) **is the maximum value of** f(x) **on [a,b]** and f(m) **is the minimum value of** f(x) **on [a,b]**

The Intermediate Value Theorem

- If f(x) **is continuous on closed interval [a,b], then for any K strictly between** f(a) **and** f(b) **there exists at least one** c in the interval (a,b**) such that** f(c) = K

Corollary to ITV

- For any k in the interval (f(b), f(a)) there exists a c in the interval (a, b) such that f(c) = K. If sign f(b) < 0, sign f(a) > 0, then we can always find c in the interval (a,b) such that k = 0.
- A function can change its sign only at roots or points of discontinuities

Example

Show that f(x) = x^3 – x – 1 has a root on interval [1, 2]

Conditions:

- f(x) – continuous as every polynomial is on the interval (-infinity, infinity)
- f(x) is given on closed interval [1,2]
- f(x) has opposite signs at the endpoints of the interval and according to IVT there exists c in the interval (1,2) such that f(c) = 0

Example of Finding Roots by Bisection

(1,2)

c1 = (2 + 1) / 2 = 1.5
f(1.5) > 0
f(1) < 0
So the root is in (1, 1.5)

(1,1.5)

c2 = (1.5 + 1) / 2 = 1.25
f(1.25) > 0
f(1) < 0
So the root is in (1, 1.25)

(1,1.25)

c3 = (1.25 + 1) / 2 = 1.125
f(1.125) < 0

f(1) < 0
So the root is in (1.125, 1.25)

(1.125,1.25)

c3 = (1.25 + 1.125) / 2 = 1.1875
f(1.18755) < 0
f(1) < 0
So the root is in (1.1875, 1.25)

Therefore, we can conclude that the root is approximately at 1.2.

Supremum in Sets

S in the subset of R is bounded from above if there exists M that for all x is the set of M, x <= M

- M – upper bound for S and [M, infinity) are also upper bound for S
- Supremum can also be known as least upper bound (lub)

Examples

- {1, 2, 3, 4} M = 4
    - lub = 4
- {0,1,2}
    - lub = 2
- (0, 2)
    - lub = 2
    - Max does not exist since 2 is not part of the domain
- (-infinity, 5)
    - bounded from above by any number [5, infinity)
    - Max does not exist since 5 is not part of the domain

Infimum in Sets

S in the subset of R is bounded from below if there exists m that for all x is the set of m, x >= m

- m – lower bound for S and (-infinty, m] are also lower bound for S
- Infimum can also be known as greatest lower bound (glb)

Examples

- {1, 2, 3, 4} m = 1
    - glb = 1
- {0,1,2}
    - glb = 0
- (0, 2)
    - lub = 0
    - Min does not exist since 0 is not part of the domain
- (-infinity, 5)
    - No glb or min since it is inbounded from below

GLB & LUB Axiom

Review of Supremum and Infimum

Supremum in Sets

S in the subset of R is bounded from above if there exists M that for all x is the set of M, x <= M

- M – upper bound for S and [M, infinity) are also upper bound for S
- Supremum can also be known as least upper bound (lub)

Infimum in Sets

S in the subset of R is bounded from below if there exists m that for all x is the set of m, x >= m

- m – lower bound for S and (-infinty, m] are also lower bound for S
- Infimum can also be known as greatest lower bound (glb)

Least Upper Bound Axiom

- Every nonempty set of real numbers that is bounded from above has a supremum
- Axioms do not need to be proven
- They are meant to be clear and understood by all mathematicians

Theorem

If Sup S = M and epsilon > 0, then there exists at least one number s in S such that m - ε < s <= M

Sup S = M => s <= M

We need to prove that M - ε < s.

Two choices: either we have M - ε or we don't

Assume that there is no M - ε < s => x subset of S, x <= M - ε, but if x <= M - ε, then M - ε is Sup S, which contradicts the hypothesis

Greatest Lower Bound Axiom

- Every nonempty set of real numbers that is bounded from below has an infimum
- Axioms do not need to be proven
- They are meant to be clear and understood by all mathematicians

Theorem

If Inf S = m and epsilon > 0, then there exists at least one number s in S such that $m < s <= m + \varepsilon$

$$\text{Inf } S = m \Rightarrow s >= m$$

We need to prove that $m + \varepsilon > s$.

Two choices: either we have $M + \varepsilon$ or we don't

Assume that there is no $M + \varepsilon > s$ => x subset of S, $x >= m + \varepsilon$, but if $x >= m + \varepsilon$, then $m + \varepsilon$ is Inf S, which contradicts the hypothesis

Monotone Sequences

Test for Monotonicity

- Let $\{a_n\}$ be an infinite sequence. We say $\{a_n\}$ is strictly increasing if it satisifes any one of the following:
  1. Difference test: if $a_{n+1} - a_n > 0 \forall n \in \mathbb{N}$, then $\{a_n\}$ is strictly increasing
  2. Derivative test: let $a_n = f(n)$. If $f(n)$ is differentiable on $[1, \infty)$, and $f(n)' > 0$ on $[1, \infty)$, then $\{a_n\}$ is strictly increasing
  3. Ratio test: if $a_n > 0 \forall n \in \mathbb{N}$, and $\dfrac{a_n + 1}{a_n} > 1 \forall n \in \mathbb{N}$, then $\{a_n\}$ is strictly increasing
- We have similar tests for increasing, strictly decreasing, and decreasing sequences
  - Increasing: replace $>$ with $\geq$
  - Strictly decreasing: replace $>$ with $<$
  - Decreasing: replace $>$ with $\leq$

Bounded Monotone Convergence Theorem

- Let $\{a_n\}$ be a sequence. If $\{a_n\}$ is bounded and monotone, then $\{a_n\}$ converges
- In particular,
  1. $\{a_n\}$ bounded above, and (strictly) increasing, OR
  2. $\{a_n\}$ bounded below, and (strictly) decreasing
- Proof of #1:
  - Suppose $\{a_n\}$ is strictly increasing (#1), and bounded above (#2)
  - Want to show $\lim_{n\to\infty} a_n$ exists, i.e. $\{a_n\}$ converges
  - Want to show $\exists l \in \mathbb{R}, \forall \epsilon > 0, \exists N > 0$ such that $\forall n \in \mathbb{N}$, if $n > N$, then $|a_n - l| < \epsilon$
  - Let $A = \{a_n | n \in \mathbb{N}\} \subset \mathbb{R}$
  - Notice $A \neq \varnothing$, because $a_1 \in A$
  - Moreover, A is bounded above, by #2
  - $\therefore$ By completeness axiom, we know that $sup(A)$ exists
  - Let $\alpha = sup(A)$
  - $l = \alpha$; choose $l = \alpha = sup(A) \in \mathbb{R}$
  - Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}, N > 0$ such that $\alpha - \epsilon < a_N$ by definition of $sup(A)$

- Suppose $n > N$
- $\implies \alpha - \epsilon < a_N$, by choice of N with $sup(A)$
- $a_N < a_n$, as $N < n$ by #1
- $a_n \leq \alpha$, because α is an upper bound of a_n
- $\implies \alpha - \epsilon < a_n < \alpha + \epsilon$
- $\iff -\epsilon < a_n - \alpha < \epsilon$
- $\iff |a_n - \alpha| < \epsilon$
- $\therefore \{a_n\}$ converges to α as wanted

Good Luck!!

Summary

1 Real Numbers

Induction: Show first case and that case n implies case $n+1$.

Binomial Theorem: For $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

$\mathbb{R}$ is complete: Every *nonempty* subset of $\mathbb{R}$ with an upper bound has a *least* upper bound in $\mathbb{R}$.

2 Limits

Limit: $\lim_{x \to a} f(x) = L$ means for every $\epsilon > 0$ we can find a $\delta > 0$ such that

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

One-Sided Limit: $\lim_{x \to a+} f(x) = L$ means for every $\epsilon > 0$ we can find a $\delta > 0$ such that

$$x \in (a, a+\delta) \Rightarrow |f(x) - L| < \epsilon.$$

Vertical Asymptote: $\lim_{x \to a+} f(x) = \infty$ means for every $M > 0$ we can find a $\delta > 0$ such that

$$x \in (a, a+\delta) \Rightarrow f(x) > M.$$

Horizontal Asymptote: $\lim_{x \to \infty} f(x) = L$ means for every $\epsilon > 0$ we can find a number N such that

$$x > N \Rightarrow |f(x) - L| < \epsilon.$$

Infinite Limit: $\lim_{x \to \infty} f(x) = \infty$ means for every $M > 0$ we can find a number N such that

$$x > N \Rightarrow f(x) > M.$$

Cauchy Criterion: $\lim_{x \to a} f(x)$ exists $\iff$ for every $\epsilon > 0$ we can find a $\delta > 0$ such that $x, y \in (a-\delta, a) \cup (a, a+\delta) \Rightarrow |f(x) - f(y)| < \epsilon$.

Sequences: $a_n = f(n)$ is a function on the domain $\mathbb{N}$.

Cauchy Criterion for Sequences: $\lim_{n\to\infty} a_n$ exists $\iff$ for every $\epsilon > 0$ we can find a number N such that $m, n > N \Rightarrow |a_m - a_n| < \epsilon$.

Convergent $\Rightarrow$ Bounded.

Monotone Sequences: Convergent $\iff$ Bounded.

Convergent $\iff$ All Subsequences Convergent.

Bounded $\Rightarrow \exists$ Convergent Subsequence.

Limit Properties: $\lim_{x\to a}(f(x)+g(x))\exists = \lim_{x\to a} f(x)+\lim_{x\to a} g(x)$ **if** these individual limits exist.

Continuity: $\lim_{x\to a} f(x) = f(\lim_{x\to a} x) = f(a)$.

Intermediate Value Theorem: If

(i) f is continuous on [a,b],

(ii) $f(a) < y < f(b)$,

then there exists a number $c \in (a, b)$ such that $f(c) = y$.

Closed intervals: Continuous $\Rightarrow$ bounded; maximum and minimum values achieved.

Theorems

Theorem 1.1 (Binomial Theorem): *For all $n \in \mathbb{N}$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Theorem 2.1 (Convergent $\Rightarrow$ Bounded): *A convergent sequence is bounded.*

Theorem 2.2 (Properties of Limits): *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Let $L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$. Then*

(a) $\lim_{n\to\infty} (a_n + b_n) = L + M$;

(b) $\lim_{n\to\infty} a_n b_n = LM$;

(c) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{M}$ *if* $M \neq 0$.

Corollary 2.2.1 (Case $L \neq 0$, $M = 0$): Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. If $\lim_{n\to\infty} a_n \neq 0$ and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ does not exist.

Theorem 2.3 (Monotone Sequences: Convergent $\iff$ Bounded): *Let $\{a_n\}$ be a monotone sequence. Then $\{a_n\}$ is convergent $\iff$ $\{a_n\}$ is bounded.*

Theorem 2.4 (Convergent $\iff$ All Subsequences Convergent): *A sequence $\{a_n\}_{n=1}^{\infty}$ is convergent with limit L $\iff$* **each** *subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ is convergent with limit L.*

Theorem 2.5 (Bolzano–Weierstrass Theorem): *A bounded sequence has a convergent subsequence.*

Theorem 2.6 (Cauchy Criterion): *$\{a_n\}$ is convergent $\iff$ $\{a_n\}$ is a Cauchy sequence.*

Theorem 3.1 (Equivalence of Function and Sequence Limits): $\lim_{x\to a} f(x) = L \iff f$ *is defined near a and* **every** *sequence of points $\{x_n\}$ in the domain of f, with $x_n \neq a$ but $\lim_{n\to\infty} x_n = a$, satisfies $\lim_{n\to\infty} f(x_n) = L$.*

Corollary 3.1.1 (Properties of Function Limits): Suppose $L = \lim_{x\to a} f(x)$ and $M = \lim_{x\to a} g(x)$. Then

(a) $\lim_{x\to a}(f(x) + g(x)) = L + M$;

(b) $\lim_{x\to a} f(x)g(x) = LM$;

(c) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$.

University of Alberta

MATH 117

Honors Calculus I

Fall 2017

Final Exam

Prof: Vladimir Troitsky

Exam Guide

Part 1 of 2

Table of Contents:

ii. $range f$ is a subset of $codom f$, the set of values that the functions actually takes.

c. Injective

i. f is one to one or injective if

$x \neq x_2 \Rightarrow f(x) \neq f(x)_2$

ii. f never sends 2 arguments into the same value.

iii. Onto, every point of b is an arrow coming into it.

iv. One to one, no 2 arrows end up in the same point, think of it as parallel.

v. Contrapositive; of $f(x) = f(x)_2 \quad then \quad x_1 = x_2$

Examples

1. $f(x) = x^2$

In math 117/118 we mostly deal with 'real' functions, which means that $dom f$ and $codom f$, are either $\mathbb{R}$ or an internal in $\mathbb{R}$, or a union of a collection of intervals.

$dom f = \mathbb{R} \quad codom f = \mathbb{R}$

$graph\ f\ is\ a\ subset\ of\ \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$\{(x\ ,\ x^2) :\ x \in R\}\ is\ a\ graph\ of\ f$

<u>Recall</u>**:** a sequence in $\mathbb{R}$ is a function $\mathbb{N}$ to $\mathbb{R}$

$x_n - x_{(n)} \qquad x : \mathbb{N} \longrightarrow \mathbb{R} \quad dom x = \mathbb{N} \quad codom x = \mathbb{R}$

Which subset of $\mathbb{R}^2$ are graph functions?

A subset A of R$\mathbb{R}^2$ is a $graph\ f$ if it satisfies the "vertical line test". Each vertical line meets A at most one point.

$range f = (o, +\infty)$

If $range f = codom f$, we say that f is $onto$ or f is a surjection.

f is not onto. Not one to one, $f(1) = f(-1)$

2. $f(x) = x^2 \quad f : \mathbb{R} \longrightarrow (0, +\infty)$

$dom f = \mathbb{R} \quad codom f = (0, +\infty)$

Same formula, same graph, set different function.

$range f = (o, +\infty)$, f is onto.

f is not one to one $f(-1) = f(1)$

3. $f(x) = x^2 \quad f : \mathbb{R} \longrightarrow (0, +\infty) \quad dom f = (0, +\infty)$

$codom f = (0, +\infty)$

$range f$, the same $(0, +\infty)$ so f is onto

+$\mathbb{R}$ if one to one because every +$\mathbb{R}$ has exactly one square root, if

$x_1^2 - x_2^2 \ the\ x_1 = x_2\ provided\ x_1, x_2 \geq 0$

4) $f(x) = \sqrt{4}\ f: (0, \infty) \longrightarrow (0, \infty)$ could be $\mathbb{R}$

$dom f = (0, \infty)\ codom f = (0, \infty)$

$range f = (0, \infty)$ so onto, depends on your choice of domain.

5) $f(x) = \dfrac{1}{x}\quad f: \mathbb{R}_\backslash\{0\} \longrightarrow \mathbb{R}$

$dom f = \mathbb{R}_\backslash\{0\}\quad codom f = \mathbb{R}$

$range f$ = $\mathbb{R}$ / $\{0\}$, not onto

One on one because it is $\dfrac{1}{x_1} = \dfrac{1}{x_2}$ then $x_1 = x_2$

6) $f(x) = \lfloor x \rfloor\ f: \mathbb{R} \to \mathbb{R}$ or $f: \mathbb{R} \to \mathbb{Z}$

$range f$ = $\mathbb{Z}$, not onto, $f: \mathbb{R} \to \mathbb{R}$ is unto

Not one to one, $\lfloor 1 \rfloor = \lfloor 1.5 \rfloor$

7) $f(x) = x - \lfloor x \rfloor\ f: \mathbb{R} \to \mathbb{R}$

$range f\ (0,\ 1)$ not onto.

The function is not one to one, $0.5 = f(1.5) = f(2.5)$

Terminology: Inverse and Restrictions

Recall

$f: A \longrightarrow B$ means " f is a function from A to B"

$A = dom f$

$B = codom f$

A real function is a function for which $codom f \leq \mathbb{R}$

$dom f$ is a union of intervals in $\mathbb{R}$

Let $f: A \longrightarrow B$ $range f = \{f(a) : a \in A\}$

$range f \leq codom f$

If the range is all domain (contains all possible values), it is onto.

f is onto or surjective of $range f = codom f$

f is one to one or injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$

f is <u>one to one correspondence</u> or a <u>bijection</u> if it is both onto and one-to-one, both injective and subjective.

Example: $f(x) = x^2$ $f: \mathbb{R} \rightarrow \mathbb{R}$

$codom f = \mathbb{R}$

$range f = (0, \infty)$

f is not onto

f is not one to one because $f(-2) = f(2)$

Horizontal Line Test

A real function is one-to-one if the graph satisfies the horizontal line test. Whenever a horizontal line meets the graph at most one point.

Example:

$f(x) = x^2 \quad f(0,\infty) \rightarrow (0,\infty)$

f is onto... and one to one, its a bijection.

Let A be a set, by i_A we denote the identity function of A: $i_A : A \rightarrow A \quad i_A(a) = a$

Composition:

Given functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ we define

$g \circ f : A \longrightarrow C$

by $(g \circ f)(a) = g(f(a))$

Warning: $g \circ f$ does not equal to $f \circ g$

Example

Consider $f, g : \mathbb{R} \rightarrow \mathbb{R}$,

$f(x) = x + 2 \ and \ g(x) = x^2$

$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 + 2$

$(g \circ f)(x) = g(x+2) = (x+2)^2$

So $f \circ g \neq g \circ f$

f is a function

$f(x)$ is a number

Inverse:

Given two functions, $f\ and\ g$, we say that g is the inverse of f is

$domg\ = codomf$

$codomg\ = domf$

$g(f(x)) = x$ for all x in $domf$ ⇔ $g\ \circ\ f$ = $subdomf$

$f(g(y)) = y$ for all y in $codomf$ ⇔ $f\ \circ\ g$ = $subcodomf$

Write $g = \ f^{-1}$

Example

$$f,\ g\ :\ (0, \infty)\ \longrightarrow (0\,\mathrm{inf})\,,\ f(x) = x^2 - \sqrt{x}$$

$g = f^{-1}$ ⇔ $f = g^{-1}$ that is, the inverse of the inverse is the original function. f is said to be invertible if f^{-1} exists (it is unique)

Proposition

A function is invertible if f is a bijection.

<u>Proof</u>

Suppose f is a function from $A\ \longrightarrow\ B$

Suppose f is invertible, f^{-1} is the inverse of f

First i must see if f is onto, take $b \in B$, then

$b = f\left(f^{-1}(b)\right) = f(a)$

Where $a = f^{-1}(b)$ then $b \in range f$ so $B \subseteq f$

Hence, f is onto.

Show that f is one to one, suppose $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$

Apply f^{-1} get $f^{-1}(f(a_2)) = f^{-1}(f(a_2)),\ so\ a_1 = a_2$.

Hence, one to one, therefore it is a bijection.

Suppose f is a bijection, take $b \in B$, since f is onto, $b \in range f$, so $b = f(a)$ for some $a \in A$

Since f is one to one, such that a is unique, $a = g(b)$ This g is well defined. $G : b \longrightarrow A \quad b = f(g(g))\ for\ all\ \ b \in B$

And $g(f(a)) = a\ for\ all \quad a \in A,\ so\ \ g = f^{-1}$

Hence, f is invertible.

Restriction

Let $f : A \longrightarrow B,$, let $c \subseteq A$

Define $g : c \longrightarrow B\ \ by\ g(x) - f(x)\ \ for\ all\ x \in C$

We say that g is the restriction of f to C, write $g = f \mid C$

Example

$f(x) = x^2 \qquad f : \mathbb{R} \to \mathbb{R}$

The restriction of f to $(0, \infty)$ is $g : (0, \infty) \to \mathbb{R}$

Same formula, different domain.

f is not one to one, g is one to one.

$f(c) = \{f(x) : \; x \in C\}$

From now on, let f be a real function

f is increasing if for any $x_1, \; x_2 \; \in dom f$

If $x_1 \; < \; x_2 \; then \; (x_1) \; \leq (\; x_2)$

f is strictly increasing .... $f(x_1) < \; f(x_2)$

f is decreasing .... $f(x_1) \geq \; f(x_2)$

f is strictly decreasing.......... $f(x_1) > \; f(x_2)$

Let a contain dom f, we say that f is increasing on A if the restriction $f \;\; | \;\; A$ is increasing.

f is monotone if it is either increasing or decreasing. Similarly it is strictly monotone if it is strictly decreasing or strictly increasing.

Examples (Reference last class)

1) $f(x) = x^2 \qquad f : \mathbb{R} \to \mathbb{R}$

a) Neither increasing or decreasing

Introduction to Functions

Topics of Functions:

1. Terminology
2. Limits
3. Continuity
4. Differentiation
5. Integration

Terminology

a. <u>Definition of Function</u>

i. Giving two sets A and B, we say that f is a function from A to B and write $f:\ A \longrightarrow B$ if f associates a point of B to every point of A.

ii. For every $a \in A$, $f(a)$ is an element of B, $a \longrightarrow f(a)$. A is called the domain of f, write $A = domf$, B is called co-domain of f, write $B = codomf$. For $a \in A$, $f(a)$ is called the image of a under f.

iii. Map = function

iv. $graph\ f = \{a,\ f(a),\ a \in A\}$ is a subset of $A \times B$

v. One can identify f with $graph\ f$

b. Range

i. $rangef = \{f(a) :\ a \in A\}$

b) Show that $f(x) = x^2$ $f:$ is strictly increasing on $(0,\ \infty)$

And decreasing a ($(-\inf, 0)$)

2) $f(x) = x^2$ $f:(0,\infty) \rightarrow \mathbb{R}$

Strictly increasing.

3) $f(x) = \lfloor x \rfloor$

f is increasing, but not strictly.

Proposition

If f is strictly monotone then f is one to one

Proof

Suppose f is strictly increasing, let $x_1,\ x_2\ \in dom f$, $x_1\ =\ x_2$.

Say $x_1\ <\ x_2$ then $f(x_1) < f(x_2)$, hence $f(x_1) \neq f(x_2)$

Is the converse true?

Suppose f is one to one, does f have to be strictly monotone?? No.

Example: $f(x)\ = \begin{cases} x + 2 when 0 \leq x < 1 \\ x - 1 wnen 1 \leq x \leq 2 \end{cases}$

It is one to one, but clearly not monotone.

Functions

- $f: A \longrightarrow B,\ A = domain \quad B = codomain$
- $rangef = \{f(a) : a \in A\}$ the set of values of f
- f is bounded if $range\ f$ is a bounded set, that is, there exists number a and b, such that $range\ f \subseteq [a, b]$

* $a \le f(x) \le b$ for all $x \in domf$
  - f is bounded ⇔ there exists M element $\mathbb{R}$ such that u $\forall \in domf\ |f(x)| \le M$
  - supf:=sup range f , inff:=inf range f
  - By (c) if f is bounded above, then $\sup f$ and if f is bounded below, then $\inf f$ exists.
  - $\min f = \min rangef \quad \max f = \max rangef$
  - $L = \max f$, means $L = \max rangef$ ⇔ $L = f(c)$ for some c and $L \ge f(x)$ for all $x \in domf$
  - In this case we say that f attains its max at c, L is the maximal value of f.
* **Examples**
  - Example 1
    - $f(x) = \sin x$ for all x, $-1 \le \sin x \le 1$, hence $\sin$ is bounded
    - Max $\sin\ (x)$ is 1, attained at $\frac{\pi}{2} + 2k\pi\ \ k \in \mathbb{Z}$

- Similarly, the minimal value of $\sin\ (x)$ is -1, attained at $\frac{-\pi}{2} + 2k\pi \quad k \in \mathbb{Z}$
- Example 2
  - $f(x) = 1 + x^2$
  - f is bounded below, $f(x) = 1 + x^2 \ \geq 1$ for all x
  - f is not bounded above, hence $\sup \infty$
  - $\inf f = \min f = 1$, attained at $x = 0$
    - Side note: look at 1 interval and restrict it within that interval so we can look at properties of just that-so we can understand the local properties

      f has a local maximum at c if there is an open interval containing c such that the restriction of f is this interval attains a max at c.

- **Limits of Functions**
  - Recall
    - $\lim\limits_{x \to \infty} x_n = c$ means $\forall \in > 0 \ \exists\, n_0 \forall n \geq n_0$
    - $|x_n - c| < \varepsilon$
    - Given a function of f, $\lim f = f(x) = L$ means $\forall \varepsilon > 0 \ \exists\, M \forall x \ \geq m$
    - $|f(x) - L| < \varepsilon$

- Geometrically this means we have the Limit L, it gives us a pipe of ε and we want it to stay in this pipe, so M geometric is that after M, so that it stays in that pipe.
  - If $x \geq M$ then $f(x)$ is within ε, if $x \simeq \infty \quad f(f\infty) \simeq L$
- **Examples**
  - $\lim\limits_{x \to \infty} \frac{1}{x^2} = 0$
    - Same language as sequences but instead of naturals we talk about reals.
    - Proof
      - Fix $\varepsilon > 0$. Let $M = \frac{1}{\sqrt{\varepsilon}}$
      - If $x > M$ then $x^2 > m^2 \; to \; \frac{1}{x^2} < \frac{1}{m^2} so \left| \frac{1}{x^2 - 0} \right| < \frac{1}{m^2} = \varepsilon$
      - So $|f(x) - 0| < \varepsilon$
      - $\lim\limits_{n \to \infty} f(x) = 1$ means $\forall \varepsilon > 0 \quad \exists M \forall x < M$
      - $|f(x) - L| < \varepsilon$
      - Very formal: $x \in dom f$
      - The difference from sequences and functions if that with functions, we can find limit
      - $\lim\limits_{x \to c} f(x) = L \quad c, L \in \mathbb{R}$ means for every $\forall > 0 \exists \delta > 0 \quad \forall x \in dom f$ if $x \neq 0$ and $|x - c| < 0$ then $|f(x) - L| < \varepsilon$

- $\lim_{x \to 5} 3x = 15$
  - as x approaches 5, $3x$ approaches 15
  - Let $\varepsilon > 0$, write $\delta = \frac{\varepsilon}{3}$
  - Suppose $x \neq 5$ and $|x-5| < \varepsilon$
  - Then $|f(x) - L| = |3x - 15| \quad f(x) = 3x, \ L = 15$
  - So $3 \cdot |x-5| < 3 \cdot \varepsilon = 3 \cdot \frac{\varepsilon}{3} = \varepsilon$
  - So $|f(x) - L| < \varepsilon$ so definition is satisfied.

Continuous Functions

Continuity

- A function f is continuous at an interior point a of its domain if $f(x) = f(c)$
  .
- $\lim f(x) = L$ means $\forall \varepsilon > 0\ \exists \partial > 0\ \forall x \in (dom f) \setminus \{c\}$ if $|\mathrm{x} - c| < \partial\ then\ |f(x) - 1| < \varepsilon$
  - f is continuous at c if $\lim\limits_{x \to c} \quad f(x) = f(c)$
- Examples
  - Show that the $\lim\limits_{x \to 9} \sqrt{x} = 3$
    - Since $3 = \sqrt{9}$ this really means $\lim\limits_{x \to 9} \sqrt{x} = \sqrt{9}$
    - Hence root is a continuous function
- The function is Discontinuous when $L = d.n.e$ or $L \neq f(c)$
- f is continuous at a $\Leftarrow\Rightarrow$ for every $\varepsilon > 0$ there exists a $\partial > 0$ such that $|x - c| < \partial \Rightarrow |f(x) - f(a)| < \varepsilon$.
- $f(x) = x$ continuous at every point a of its $dom f \mathbb{R}$ since $\lim\limits_{x \to c} f(x) = f(c)$ for all $c \in \mathbb{R}$
- f $f(x) = x^2$ is continuous since $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} f(x)^2 = \left(\lim\limits_{x \to} x\right) \cdot \left(\lim\limits_{x \to} x\right) = c \cdot c \ = \ c^2 = f(c)$

Practice

- $$x_n = 1 + q + \ldots + q^n = \sum_{k=0}^{n} q^k$$
    - We need to understand geometric series, , but we will go in depth later.

- Examples:
    - Note: we will talk about series at the end of 118, however this is a series example.
        - $f\,[0,2] \;-\mathbb{R}$
        - $f(x) = \begin{cases} 2+x^2 & 0 \le x \le 1 \\ x-1 & 1 < x \le 2 \end{cases}$
        - $range f = [0,1] \cup [2,3]$
        - f is not a subjective but an injection
        - f is not invertible
        - f is bounded above and below
        - f is neither increasing or decreasing
        - $\max f = 3$
        - $\sup f = 3$
        - $\min f = d.n.e$
        - $\inf f = 0$
        - Local max of $x-1$ and $x \le 2$, min of $x > 0$
    - $f:(0,1) \longrightarrow \mathbb{R}$, $f(x) = \sin\dfrac{1}{x}$

- $range f = [-1, 1]$
- f is not onto, one to one or a bijection
- f is not a bijection
- f is not increasing or decreasing, does both in intervals of sin 1/x
- $\max f = \sup f = 1$
- $\min f = \inf f = -1$
- You don't have to draw a graph, however it may help you construct your proof

◦ When $x \in [1, 1]$

- $x^2 \leq 1$
- $-2 < 0 \leq \frac{x^2}{1+x^4} < \frac{1+x^2}{1+x^4} < \frac{1+1}{1+x^4} < 2$
- When $x \in (-\infty, -1) \cup (1, +\infty)$
- $x^4 > x^2 \Rightarrow x^4 + 1 > x^2 + 1$
- $-2 < 0 \leq \frac{x^2}{1+x^4} < \frac{1+x^2}{1+x^4} < \frac{1+x^4}{1+x^2} = 1 < 2$
- ⇒ when $x \in \mathbb{R} \left| \frac{1+x^2}{1+x^2} \right| < 2$

◦ The equation $f(x) = y$ has solutions exactly when $y \in (0, 1) \cup (2, 3)$ if $y \in (0, 1)$

- Take $x = y + 1,\ x \in (1, 2)$
- $f(x) = x - 1 = y + 1 - 1 = y$

- $\lim_{x\to-\infty} \sqrt{-x} = \infty \;\; \forall k > 0m\, \forall x \in doMf\, is\, x > M\, then\, f(x) > k$
  - Let $k > 0$
  - 1. Find M
    - $(\sqrt{-x})^2 > (k)^2$
    - $-x < k^2$
    - $x < -k^2$
    - Let $M = k^2$
  - 2. Let $M = -k^2$
    - $x < \; -k^2$
    - $x- < \; k^2$
    - $\sqrt{-x} > \sqrt{k^2}$
    - $\sqrt{-x} > k$
    - $f(x) > k$
- $\lim_{x\to 2} \frac{1}{x} - \frac{1}{2}$
  - $f(x) = \frac{1}{x}$ $c = 2$ $L = \frac{1}{2}$
  - Give $\varepsilon > 0$, take ∂ =?
  - Suppose $|x-2| < \partial$ and

$$|f(x) - L| = \left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| =$$

$$\left|\frac{2-x}{2x}\right| = \left|\frac{x-2}{2x}\right| < \frac{\partial}{2|x|} \leq \frac{\partial}{2\cdot 1}$$

- We need to separate |x| from 0, need a lower bound for |x|
- X is within ∂ from 2
  - $1 \leq 2 - \partial < x < 2 + \partial$
    - $|x| \geq 1$
    - give $\varepsilon > 0,\ \partial \lim \{1, 2\varepsilon\}$

Sequence Function Duality

- $\lim f(x) = L$ means $\forall \varepsilon > 0\ \exists \partial > 0\ \forall x \in (dom f) \setminus \{c\}$ if $|\mathrm{x} - c| < \partial\ then\ |f(x) - 1| < \varepsilon$
  - f is continuous at c if $\lim_{x \to c} f(x) = f(c)$
- Examples
  - Show that the $\lim_{x \to 9} \sqrt{x} = 3$
    - Since $3 = \sqrt{9}$ this really means $\lim_{x \to 9} \sqrt{x} = \sqrt{9}$
    - Hence root is a continuous function
  - $\lim_{x \to -2} x^3 = -8$
    - Take $\varepsilon > 0$,take $\partial = \min\left\{-, \frac{\varepsilon}{19}\right\} if\ 3\varepsilon$
    - $|x - (-2)| < \partial$ then
    - $|f(x) - 8| = |x^3 + 8| = |x + 2| \cdot |x^2 - 2x + 4|$

      $< \partial (3^2 + 2 + 2 \cdot 3 + 4) = 19\partial \leq \varepsilon$

      $< \varepsilon \cdot |x^2 - 2x + 4| < \partial \left(|x|^2 + 2|x| + 4\right)$

      Since delta is

      $\leq 1,\ |x - (-2)| < 1 \qquad -3 < 3 < 1\ hence\ |x| < 3$

- $Formula\ for\ cubes:$

  $\boxed{a^2 + b^2 = (a + b)(a^2 - ab + b^2)}$

- Let $f(x)=\begin{cases}1 & x>0\\ -1 & x<0\end{cases}$
  - $\lim_{x\to0} f(x)\ d.n.e,\ let\ \varepsilon=\frac{1}{2}\quad no\ \partial\ will\ satisfy$

- **Sequence Function Duality**
  - There is a unique relationship between the limits of functions and units of sequences, which we will name, as a class, *Sequence Function Duality*.

I. $\lim_{x\to c} f(x)=L$

II. For every sequence (x_n) in $dom f\ /\ \{c\},\ x_n\longrightarrow c$ then $f(x_n)\longrightarrow L$ that is, f maps sequences which converges to c to sequence which converges to L

Examples from above

- $\lim_{x\to-2} x^3=-8$ if $x_n\longrightarrow -2$ then x_n^3
  - $f(x)=x^3\quad c=-2\quad L=-8\quad x_n^3\to(-2)^3=-8$

    by properties of limits of sequence.
- $\lim_{x\to9}\sqrt{x}=3$
  - If $x_n\longrightarrow 9$ then $\sqrt{x_n}\longrightarrow\sqrt{9}$
  - Suppose $\lim_{x\to0} f(x)=L$ take $x_n=\frac{1}{n}$ then $x_n\longrightarrow 0$
  - By SFD, $f(x_n)\longrightarrow L$ but

    $f(x_n)=L\quad for\ all\ n,\ so\ L=1$

- Put $y_n = \frac{1}{2},\ then\ y_n \longrightarrow 0$
  - By SFD, $f(y_n) \longrightarrow L$
  - $f(y_n) = -1\ for\ all\ n\ so\ L = -1$
  - No ∂ will satisfy, no such L
  - So $\lim_{x \to c} f(x) \quad d.n.e.$

- SFD Proof
  - Now we must prove the theorem of $C, L \in \mathbb{R}$.
    - I ⇒ II
      - Suppose $\lim_{x \to c} f(x) = L$
      - Let (x_n) be a sequence in $(dom f) /\ \{c\}$ assume that $x_n \longrightarrow c$
      - We need to show that $f(x_n) \longrightarrow L$
      - Let $\varepsilon > 0$ since $\lim_{c \to L} f(x)$ we can find ∂ such that for all x in $(dom f) /\ \{c\}$ if $|x - c| < \partial\ then\ |f(x) - L| < \varepsilon$
      - We look at the interval between $L + \varepsilon$ and $L - \varepsilon$, and $c + \varepsilon$ and $c - \varepsilon$
        - This sequence converges to c, so we know a tail of (x_n) is contained in the interval of $c + \varepsilon$ and $c - \varepsilon$
      - There exists $n_o\ such\ that\ \forall n \geq n_o$

- We have $|x_n - c| < \partial$ then
  $f(x_n) \in (L - \varepsilon,\ L + \varepsilon)$ that is
  $|f(x_n) - L| < \varepsilon$
- There is, for ever $\varepsilon > 0$ there exists n_o such that for all
  $n > n_o\ we\ have\ |f(x_n) - L| < 3 \qquad f(x_n \longrightarrow L)$

▪ II ⇒ I

- Suppose that II is satified, assume for sake of contradiction, that I fails
- $\exists \varepsilon > 0$
  $\boxed{\forall \partial > 0\ \exists x \in (dom f) / \{c\} \quad |x - c| < \partial}$
- Let $|f(x) - L| > \varepsilon$ fix this "bad" ε
- let $n \in \mathbb{N}$, apply content of above red box with $\partial = \frac{1}{n}$
  then there exists for some $x =: xn$ in
  $(dom f) \setminus \{c\}$ such that $|x_n - c| < \frac{1}{n}$ for all n
- Hence $x_n \longrightarrow c$
- By II $f(x_n) \longrightarrow L$
- This contradicts $|f(x_n) - L| > \varepsilon$

  - HOMEWORK: Case $C, L = \pm\infty$

- Corollary
  - f is continuous at c if f for every sequence (x_n) in $dom f$ if $x_n \longrightarrow c$ then $f(x_n) \longrightarrow f(c)$. Replace L with $f(c)$

- **Properties of Limits of Functions**

- Let $C \in \mathbb{R}$
  - If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exists then

    $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
  - $\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
  - $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ provided that

    $\lim_{x \to c} g(x) \neq 0$
- Domination Principle (same as sequence)
  - If $\lim_{x \to c} f(x) = 0$ and $|g(x)| \leq |f(x)|$ for all x in

    $x \longrightarrow c$ a neighborhood of c (but $x \neq c$) then

    $\lim_{x \to c} (x) = c$
- Squeeze Law (same as sequence)
  - If $f(x) \leq q(x) \leq h(x)$ for all x in a neighborhood of c

    $(x \neq c)$ and $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$ then $\lim_{x \to c} g(x)$ exists.
- If $\lim_{x \to c} f(x) = 0$ and g is bounded on a neighborhood of c then

  $\lim_{x \to c} f(x) g(x) = 0$
- If $\lim_{x \to c} f(x) = \infty$ and g is bounded on a neighborhood of c then

  $\lim_{x \to c} (f(x) + g(x)) = \infty$
- If $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x)$ exists then

  $+\infty \; if \; \lim_{x \to c} g(x) > 0, \; -\infty \; if \; \lim_{x \to c} g(x) < 0$
- If $\lim_{x \to c} f(x) = \infty$ and $g(x) \leq f(x)$ for all x in a neighborhood of

  c, then $\lim_{x \to c} g(x) = \infty$

Domination Principles and Proof and Review of Properties of Limits

Recall

- $\lim f(x) = L$ means $\forall \varepsilon > 0\ \exists \partial > 0\ \forall x \in (domf) \setminus \{c\}$ if $|\mathrm{x} - c| < \partial\ then\ |f(x) - 1| < \varepsilon$
- This definition extends naturally to $c, L = l \pm \infty$
- More generally, for $c, L \in \overline{\mathbb{R}}$, $\lim\limits_{x \to c} f(x) = L$ if for every neighborhood u of L there exists a neighborhood of V of c such that $c \neq x \in V$ implies $f(x) \in u$
- $f(v \setminus \{c\}) \subseteq u$
- f is continuous at c if $\lim\limits_{x \to c} f(x) = f(c)$ limit=value.
- f is continuous if it is continuous at every point of its domain

Sequence function Duality

- $\lim\limits_{x \to c} f(x)$ if f for every sequence (x_n) in $domf\ /\ \{c\}$, $x_n \longrightarrow c$ then $f(x_n) \longrightarrow L$

Review: Properties of Limits

- Let $C \in \mathbb{R}$
    - If $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ exists then $\lim\limits_{x \to c}(f(x) + g(x)) = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$
        - Proof of the Sum Rule using SFDT
            - Let (x_n) be a sequence that (x_n) is in $domf$ and $domg$ excluding c if $x_n \longrightarrow c$

- Then $f(x_n) \longrightarrow \lim_{x \to c} f(x)$
- $g(x_n) \longrightarrow \lim_{x \to c} g(x)$
- So
  $f(x_n) + g(x_n) \longrightarrow \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
- Let $L = \lim_{x \to c} f(x) \ and\ k = \lim_{x \to c} g(x)$
- Suppose $x_n \longrightarrow c$
- Since $\lim_{x \to c} f(x)$ by SFDT, $f(x_n) \longrightarrow L$
- Since $\lim_{x \to c} g(x) = k$ by SFDT $g(x) \longrightarrow k$
- Then $f(x_n) + g(x_n) \longrightarrow L + M$ by properties of limits of sequences.
- Then, by SFDT again,
  $\lim_{x \to c} (f(x) + g(x)) = L + K$
- Hence $\left(\lim_{x \to c} f(x)\right) + \left(\lim_{x \to c} g(x)\right)$

  - $\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
  - $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ provided that $\lim_{x \to c} g(x) \neq 0$

- Domination Principles
  - Suppose that $|g(x)| \leq |f(x)|$ for all $x \neq c$
  - $\lim_{x \to c} f(x) = 0$ then $\lim_{x \to c} g(x) = 0$

- $g(x) \leq f(x)$ for all $x \neq c$ and $\lim_{x\to c} g(x) = \infty$ then $\lim_{x\to c} f(x) = \infty$
- Proof
  - Case $C \in \mathbb{R}$,given $\varepsilon > 0$, since $\lim_{x\to c} f(x) = 0$ we can find $\partial > 0$ such that $|f(x)| < \varepsilon$ whenever $|x - c| < \partial,\ x \neq c$
    - But then $|g(x)| < |f(x)|$ yields that $|g(x)| < \varepsilon$ so that $\lim_{x\to c} g(x) = 0$
  - Case $C = \infty$, given $\varepsilon > 0$, since $\lim_{x\to c} f(x) = 0$ we can find M such that $|f(x)| < \varepsilon$ whenever $x > M$
    - But then $|q(x)| \leq |f(x)|$ yields that $|q(x)| < \varepsilon$ whenever $x > M$ there $\lim_{m\to\infty} g(x) = L$

- Squeeze Law
  - If $f(x) \leq q(x) \leq h(x)$ for all $x \in C\ (x \neq c)$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$ then $\lim_{x\to c} g(x)$ exists.
- If $f(x) \leq g(x)$ for all $x \neq c$ and $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$
- If $\lim_{x\to c} f(x) = 0$ and g is bounded on a neighborhood of c then $\lim_{x\to c} f(x)\,g(x) = 0$

- If $\lim_{x \to c} f(x) = \infty$ and g is bounded on a neighborhood of c then $\lim_{x \to c} (f(x) + g(x)) = \infty$
- If $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x)$ exists then $+\infty \; if \; \lim_{x \to c} g(x) > 0, \; -\infty \; if \; \lim_{x \to c} g(x) < 0$
- If $\lim_{x \to c} f(x) = \infty$ and $g(x) \le f(x)$ for all x in a neighborhood of c, then $\lim_{x \to c} g(x) = \infty$

Examples

1) $\lim_{x \to 0} x \cdot \sin \dfrac{1}{x} = 0$
   a) Use domination principle, $f(x) = x \; and \; g(x) = x \cdot \sin \dfrac{1}{x}$
      $\lim_{x \to c} (x) = o$
   b) $|g(x)| = \left| x \cdot \sin \dfrac{1}{x} \right| = |x| \cdot \left| \sin \dfrac{1}{x} \right| \le |x| = |f(x)|$
   c) Hence, $\lim_{x \to 0} x \cdot \sin \dfrac{1}{x} = 0$
2) $\lim_{x \to 2} (3x^2 + 5) = \lim_{x \to 2} (3x^2) + \lim_{x \to 2} (5)$
   a) $3 \cdot \left(\lim_{x \to 2} \right)^2 + 5 \; = \; 3 \cdot 2^2 + 5 = 17$

b) If $f(x) = 3x^2 + 5,$ then $\lim_{x \to 2} f(x) = 17$ so f is continuous at 2.

c) Some argument shows that if $p(x)$ is and polynomial at x then p is continuous at every point $\lim_{x \to c} p(x) = p(c)$

3) Rational Function, ration of Polynomials

i) $f(x) = \dfrac{p(x)}{q(x)}$ where q and p are polynomials

ii) $\lim_{x \to c} f(x) = \lim_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)}$

$\dfrac{p(c)}{q(c)} = f(c)$ provided $q(c) \neq 0 \quad c \in dom f$

iii) So f is continuous at every $c \in dom f$

iv) hence f is continuous

v) Hence, every rational function is continuous, on their domain.

One -Sided Limits

Recall:

$\lim f(x) = L$ means $\forall \varepsilon > 0\ \exists \partial > 0\ \forall x \in (dom f) \setminus \{c\}$ if

$|\mathrm{x} - c| < \partial\ then\ |f(x) - 1| < \varepsilon$

One Sided Limits:

- Motivation
  - $$f(x) = \begin{cases} x+2 & x < 0 \\ x^2+1 & x > 0 \end{cases}$$
  - $f(x)$ approaches 2 when x approaches 0 from the left, $f(x)$ approaches 1 when x approaches 0 from the right.
  - $\forall x \in (dom f) \setminus \{c\}\ if\ x \in (c - \partial, c)\ or\ (c, c + \partial)$
  - $\lim_{x \to c+} f(x)$ means approach from the right / above.

    $\forall \partial > 0\ \exists \partial > 0\ \forall x \in\ dom f\ is\ x \in (c, c + \partial)\ then\ |f(x) - L| < \varepsilon$
  - $\lim_{x \to c+} f(x)$ means approach from the left / below.

    $\forall \varepsilon > 0\ \exists \partial > 0\ \forall x \in\ dom f\ if\ x \in (c - \partial, c)\ then\ |f(x) - L| < \varepsilon$
- Proposition:
  - $\lim_{x \to c} f(x) = L \iff \lim_{x \to c-} f(x) = L = \lim_{x \to c^+} f(x) = L$
  - That is, if $\lim_{x \to c} f(x) \iff$ the two one sided limits exist and are equal.
  - In this case all the limits are the same.

- Proof
  - ⇒ Suppose $\lim_{x \to c} f(x) = L$ then $\lim_{x \to c+} f(x) = L$ and $\lim_{x \to c-} f(x) = L$
  - ⇐ suppose $\lim_{x \to c+} f(x) = L$ and $\lim_{x \to c-} f(x) = L$ let $\varepsilon > 0$ since $\lim_{x \to c+} f(x) = L$ we can find $\partial_1 > 0$ such that for all $x \in dom f$ if $x \in (c, c+\partial)$ then $|f(x) - L| < \varepsilon$
    - Since $\lim_{x \to c-} f(x) = L$, we can find $\partial_2 > 0$ such that for all $x \in dom f$ if $x \in (c - \partial_2, c)$ then $|f(x) - L| < \varepsilon$
  - Put $\partial = \min\{\partial_1, \partial_2\}$ then if $x \in (c - \partial, c)$ or $(c, \partial + c)$ then $x \in (c - \partial_2, c)$ or $x \in (c, c + \partial)$ in either case we have $|f(x) - L| < \varepsilon$
- Note: in $\mathbb{R}$ you can only approach from left and right, otherwise there are infinite directions on a plane.
- Example: $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (x^2 + 1)$
  - Because $f(x) = x^2 + 1 \; on \; (0, +\infty)$
  - $x^2 + 1$ is a polynomial, hence continuous, has a $\lim_{x \to 0} (x^2 + 1) = 0^2 + 1 = 1$
  - Note: continuous means value is the limit
  - By proposition, $\lim_{x \to 0+} (x^2 + 1) = \lim_{x \to 0} (x^2 + 1) = 1$

- Combined, we get $\lim_{x \to 0+} f(x) = 1$ simularily
  $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x+2) = \lim_{x \to 0} (x+2) = 0 + 2 = 2$
- Since $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$ by proposition,
  $\lim_{x \to 0} f(x) \;\; d.n.e$

Variants at SFDT for one sided limits

- $\lim_{x \to c-} f(x) = L \iff$ for every sequence (x_n) in $(domf) \cap (-\infty, c)$ if $x_n \longrightarrow c$ then $f(x_n) \longrightarrow L$
- Similarly $\lim_{x \to c+} f(x) = L \iff$ for every sequence (x_n) in $(domf) \cap (+\infty, c)$ if $x_n \longrightarrow c$ then $f(x_n) \longrightarrow L$

Homework

Fact: $\lim_{x \to c} f(x) = \lim_{h \to 0} f(c+h) = \lim_{h \to 0} g(h)$

Where $g(h) \;\; = f(c+h)$

Think of this as a chance of variables, such that,

$h = x - c$

$x = c + h$

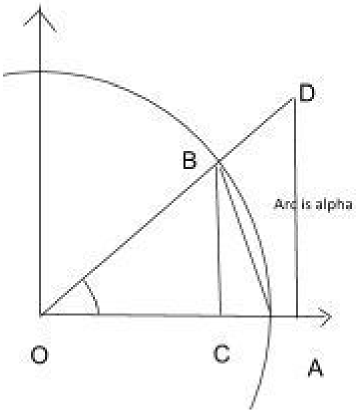
Exercize:

Prove this by expanding definition.

Trig Limits

Lemma,

If $0 < \alpha \; < \dfrac{\pi}{2}$ then $0 < \sin\alpha < \alpha < \tan\alpha$

- Proof CASE A , $0 < \sin\alpha$
  - 

  - $\alpha = arc\ length\ \ \widehat{AB}$
  - ABC is a right triangle
  - AB is the hypotenuse
  - $CB\ <\ AB$ as hypotenuse is the longest side
  - $CB\ <\ AB\ < \widehat{AB} = \alpha$ as we know straight leg is longer than arc.
  - $0 < \sin\ \alpha\ trivial$

$$\alpha < \tan\alpha$$

- Proof Case B,
  - $\dfrac{DA}{OA} = \tan\alpha$ if $\triangle ODA,\ OA = 1$
  - $\tan\alpha = DA$
  - Formal proof that the legs are longer than the arc
    - Compare the areas of the $\triangle OAD$ and the sector OAB
    - Area of the sector $OAB\ \leq\ area\ (\triangle OAD)$ because the sector is properly contained in the triangle.
    - $area\ (\triangle OAD) = \dfrac{1}{2}OA \cdot OD = \dfrac{1}{2} \cdot 1 \cdot \tan\alpha = \dfrac{1}{2}\tan\alpha$
  - Calculate Sector

- Length of circle is 2π, area of the circle is π
- Take the proportion $\frac{\alpha}{x\pi}$ of the area of the dish: π
- The area of the sector is $\frac{\alpha}{x\pi} \cdot \pi = \frac{\alpha}{2}$
- Therefor $\frac{\alpha}{2} < \frac{1}{2}\tan\alpha,\ so\ \alpha < \tan\alpha$
  - Corollary
    - If $-\frac{\pi}{2} < \ -\alpha < \frac{\pi}{2}$ then $|\sin\ \alpha| \leq |\alpha| \leq |\tan\alpha|$
- Proof Case C,
  - If $0 < \alpha\ < \frac{\pi}{2}$ then by lemma, $0 < \sin\alpha < \alpha < \tan\alpha$
  - Hence, $|\sin\ \alpha| < |\alpha| < |\tan\alpha|$
  - If $\alpha = 0$then $\sin\alpha = 0$ and $\tan\alpha = 0$ so
    $|\sin\ \alpha| = |\alpha| = |\tan\alpha| = 0$
  - If $-\frac{\pi}{2} < \alpha < 0$ then $\sin\alpha = -\sin(-\alpha)$
    $\tan\alpha = -\tan(-\alpha)$ $0 < -\alpha < \frac{\pi}{2}$
  - By Lemma, $0\ < \sin(-\alpha) < -\alpha < \tan(-\alpha)$
  - Hence $|\sin\alpha| = |-\sin(-\alpha)\ | = |\sin(-\alpha)|$
    $|\tan\alpha| = |-\tan(-\alpha)\ | = |\tan(-\alpha)|$
  - Hence $|\sin\alpha| < |\alpha\ | < |\tan\alpha|$

- Corollary
  - $\lim_{\alpha \to 0} \sin \alpha = 0 = \sin 0$ So $\sin 0 = \lim_{\alpha \to 0} \sin \alpha$
  - Proof:
    - For all α in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $|\sin \alpha| \leq |\alpha|$
    - Also $\lim_{\alpha \to 0} \alpha = 0$
    - By Domination Principle, $\lim_{\alpha \to 0} \sin \alpha = 0$
- Corollary $\lim_{\alpha \to 0} \cos \alpha = 1$
  - Proof:
    - When $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\cos \alpha = \sqrt{1 - \sin^2 \alpha}$ so using properties of limits,
    - $\lim_{\alpha \to 0} \cos \alpha = \sqrt{1 - \sin^2 \alpha} = \sqrt{\lim_{\alpha \to} 0}$
      $= \sqrt{1 - \left(\lim_{\alpha \to 0} \sin \alpha\right)^2} = \sqrt{1 - 0^2} = 1$
    - Same logic cosine is continuous at 0
    - Hence by SFDT, if $f(x) > 0$ for all x the
      $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)}$
- Theorem: $\sin x$ and $\cos x$ are continuous (everywhere)
  - Proof
    - $\lim_{x \to c} \sin x = \lim_{h \to 0} \sin(c + h) = \lim_{h \to 0} \sin c \cdot \cos\ h + \cos c \cdot \sin\ h$

- h is a variable so rearranged as

$$= \sin c \left(\lim_{h \to 0} \cos\ h \right) + \cos c\ \cdot \left(\lim_{h \to 0} \sin\ h \right)$$

- $= \sin c \cdot 1 + \cos c \cdot 0 = \sin c$
- $\lim_{x \to c} \sin x = \sin c$ hence c is continuous at c
- For cosine, similar- do as homework

Practice

a)

- Let $\varepsilon > 0$, let $\partial = \min\left\{1, \frac{\varepsilon}{4}\right\}$ if $|x - 1| = \partial||$
- $|f(x) - L| = |(x^2 - 3x + 7) - 5| = |x^2 - 3x + 2|$

  $= |(x-1)(x-2)| < \partial|x-2| \leq \partial|x| + 2\partial$
- Because $|x - 1| \leq \partial \leq 1, \; |x-1| \leq 1, |x| < 2||$

b)

- $f(x) = \sqrt{x^2 + 5} \; domf = \mathbb{R}$
- Then we can find a sequence (x_n) in $\mathbb{R}$ such that $x_n \longrightarrow 2$.
- Then if $x_n \longrightarrow 2$ then $f(x_n) = \sqrt{x_n^2} + 5 \longrightarrow 3$
- As $n \longrightarrow \infty$, then by SFDT, $\lim_{x \to 2} f(x) = 3$.
- Formally
  - $x_n \longrightarrow 2$ then properties of sequences,
  - then $x_n^2 \longrightarrow 4$,
  - $x_n^2 + 5 \longrightarrow 9$
  - $\sqrt{x_n^2 + 5} \longrightarrow 3$

c)

- $f(x) = \begin{cases} x+1 & x \leq 2 \\ x^2 & x > 2 \end{cases}$
- Suppose $\lim_{x \to 2} f(x)$ exists, $\lim_{x \to 2} f(x) = L$

- Let $x_n \longrightarrow 2 \ x \in (-8, 2)$
- By SFDT $f(x_n) \longrightarrow L \forall n \in \mathbb{N}$
- $x_n + 1 = 2 + 1 = 3$
- $f(x_n) \longrightarrow 3$
- $\lim_{x \to 2} f(x) = 3$
- $L = 3$

- Let $y_n \longrightarrow 2, \ y \in (2, +\infty)$
  - bySFDT, $f(x_n) \longrightarrow \forall n \in \mathbb{N}$
  - $y_n^2 = 2^2 = 4$
  - $f(y_n) = 4$
  - $\lim_{x \to 2} f(x) = 4$
  - $L = 4$
- By contradiction, l $\lim_{x \to 2} f(x) \ d.n.e.$

d)

- $$f(x) = x^2 \left(\sin \frac{1}{x} \right)$$
  - say $g(x) = x^2$
  - $h(x) = \sin \frac{1}{x}$
  - $\max h(x) = 1$
  - $\min h(x) = -1$
- $\lim_{x \to 0} g(x) = 0$

- $\lim_{x \to 0} x^2 \longrightarrow 0$
- $\lim_{x \to 0} -x^2 \longrightarrow 0$
- Squeeze Law
    - $\lim_{x \to 2} x^2 \left(\sin \frac{1}{x}\right) = 0$

e)I

- Suppose the limit exists.
- $\lim_{x \to 0} \left(\sin \frac{1}{x}\right)^2$
- $x_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \quad x_n \longrightarrow 0$
    - $\lim f(x_n) = \lim \left(\sin\left(\frac{\pi}{2} + 2\pi n\right)\right)^2 = 1$
- $y_n = \frac{1}{\pi n} \quad y_n \longrightarrow 0$
    - $\lim f(y_n) = \lim (\sin(\pi n))^2 = 0$
- By SFDT $\lim_{x \to 0} f(x) = 1 \; but \; \lim_{x \to 0} f(x) = 0$
- Contradiction, since limits are unique, $\lim_{x \to 0} \left(\sin \frac{1}{x}\right) = d.n.e$

f)

- take $\varepsilon = \frac{1}{\varepsilon} \forall \in dom f \; x \in (c - \partial, c) \; \cup (c, c + \partial)$

- $\exists x_1$, irrational
- $\exists x_2$,rational
- $|f(x_1) - L| = |-L| = |L| < \frac{1}{3}$
- $|f(x_2) - L| = |1 - L| = |L| < \frac{1}{3}$
- $1 < |1 - c| + |c| < \frac{2}{3}$
- False.

Properties of Continuous Functions

Recall

1) For every $\alpha \in \left(0, \frac{\pi}{2}\right)$ $0 < \sin\alpha < \alpha < \tan\alpha$
2) $\sin x$ and $\cos x$ are continuous functions.

Theorem:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Proof

- $x \in \left(0, \frac{\pi}{2}\right)$ then $0 < \sin x < x < \frac{\sin x}{\cos x}$
- Device by $\sin x$
- $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$
- Take reciprocals
- $\cos x < \frac{\sin x}{x} < 1$
- True for all $x \in \left(0, \frac{\pi}{2}\right)$
- If $x \in \left(-\frac{\pi}{2}, 0\right)$ same inequality holds.
- $\cos(-x) - \cos\ x$ cos is an even function
- $\frac{\sin(-x)}{(-x)} = \frac{-\sin x}{-x} - \frac{\sin x}{x}$

- Hence, $\cos x < \frac{\sin x}{x} < 1$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \ / \ \{0\}$
- Squeeze Law, $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\sin x \approx x$ when x is small.

Examples

1. $$\lim_{t \to 0} \frac{t^s \cos t}{\sin^2 t} = \lim_{t \to 0} \frac{\cos t}{\left(\frac{\sin t}{t}\right)^2} = \frac{\lim_{t \to 0} \cos t}{\lim_{t \to 0} \left(\frac{\sin t}{t}\right)^2} = \frac{1}{1^2} = 1$$

2. $$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2 \cdot ((1 + \cos x))}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2 (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2 (1 + \cos x)} = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{1 + \cos x}$$

$$1 \cdot \frac{1}{1+1} = \frac{1}{2} \quad \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$ when x is small

$$\frac{1 - \cos x}{x^2} \approx \frac{1}{2}$$ so that $$\cos x \approx 1 - \frac{x^2}{2}$$

3. $$\lim_{x \to 0} \frac{\sin(3x)}{x} = 3 \lim_{x \to 0} \frac{\sin 3(x)}{3x}$$

   a. Change of Variables

i. $t = 3x \; as \; x \longrightarrow 0 \; then \; t \longrightarrow 0$ so

$$3 \cdot \lim_{x \to 0} \frac{\sin t}{t} = 3 \cdot 1 = 3$$

ii. SFDT: suppose $x_n \longrightarrow 0$ put $t_n = 3x_n$ then $f_n \longrightarrow 0$ since $\lim_{x \to 0} \frac{\sin t}{t} = 1$. By SFDT we have $\frac{\sin t}{t_n} \longrightarrow 1$

iii. Hence, $\frac{\sin(3x_n)}{3x_n} \longrightarrow 1$, thus if $x_n \longrightarrow 0$ then $\frac{\sin(3x_n)}{3x_n} \longrightarrow 1$

iv. Apply SFDT to the function $\frac{\sin 3x}{3x}$ hence $\lim_{x \to 0} \frac{\sin 3x}{3x} = 1$

4. $\lim_{x \to 1} (1 - x) \cdot \tan \frac{\pi x}{2}$

a. Another Example involving change of variable

i. $t = 1 - x$, $as \; x \longrightarrow 1 \; then \; t \longrightarrow 0$

ii. $$\lim_{t\to 0} \cdot \tan\left(\frac{\pi}{2}(1-t)\right) = \lim_{t\to 0} t \cdot \frac{\sin\left(\frac{pi}{2} - \frac{\pi}{2}t\right)}{\cos\left(\frac{\pi}{2} - \frac{\pi}{2}t\right)}$$

$$= \lim_{t\to 0} t \cdot \frac{\cos\ \frac{\pi}{2}t}{\sin\ \frac{\pi}{2}t} = \lim_{t\to 0} \frac{\cos\ \frac{pi}{2}t}{\frac{\sin\ \frac{\pi}{2}t}{2}}$$

$$= \frac{2}{\pi} \lim_{t\to 0} \frac{\cos\left(\frac{\pi}{2}t\right)}{\frac{\sin\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}t}} = \frac{2}{\pi} \frac{\lim_{t\to 0} \cos\left(\frac{pi}{2}t\right)}{\lim_{t\to 0} \frac{\sin\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}t}} =$$

iii. $$s = \frac{\pi}{2}t,\ as\ t \longrightarrow 0,\ then\ s\ \longrightarrow 0$$

iv. $$\frac{\lim_{s\to 0} \cos s}{\lim_{s\to 0} \frac{\sin s}{s}} = \frac{2}{\pi} \cdot \frac{1}{1} = \frac{2}{\pi}$$

Continuous Functions and its Properties

Recall,

- f is continuous at c if $\lim_{x\to c} f(x) = f(c)$ By SFDT, f is continuous at c.
- f is continuous if it is $x_n \longrightarrow c$ then $f(x_n) \longrightarrow f(c)$, continuous at every point of its $dom f$
- f is continuous at every interval, if the restriction is f, this interval is continuous.
- Polynomials are continuous on $\mathbb{R}$, everywhere.

- A Rational function such as $\frac{p(x)}{q(x)}$ is continuous on its domain.
- | $\mathbb{R}$ | {the roots at q)
- $\sqrt{x}$ is continuous (on its domain $[o, +\infty)$)
- $\sin x$ and $\cos x$ are continuous

Properties

- The sum, difference and the products of two continuous functions are continuous
  - $\lim_{x \to c} f(x) + g(x) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

    $= f(c) + g(c)$ hence $f + g$ is continuous at c
- Example 1
  - $3x^2 + 5x + \sqrt{x}$ is continuous on its domain $[o, +\infty)$
- If f is continuous at c and g is continuous at c and $g(c) \neq 0$ then

  $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)}$ hence $\frac{f}{g}$ is continuous at c.
- In particular, if f is continuous and g is continuous then $\frac{f}{g}$ is continuous.
- Example 2
  - $\frac{3x^2 + 5x + 7}{\sqrt{x}}$
  - Continuous (on its domain). Domain is $[o, \infty)$
- Example 3
  - $\tan x = \frac{\sin x}{\cos x}$ is continuous. Domain $\tan x$

- Example 4
  - $f(x) = \frac{1}{x}$ continuous on its domain, $dom f = \mathbb{R}_{\setminus}\{0\}$
  - f has a 'singularity' at zero (one sided limits are infinite)
  - f is continuous on $(0, +\infty)$ and on $(-\infty, 0)$
- Example 5
  - $\lceil x \rceil and \lfloor x \rfloor$
  - Continuous except at integer points, $\mathbb{R}_{\setminus}\mathbb{Z}$
  - Discontinuity at every integer point
- Example 6
  - $f(x) = \begin{cases} 1 & x \quad rational \\ 0 & x \quad irrational \end{cases}$
  - $\lim_{x \to c} f(x)$ d.n.e. For every $c \in \mathbb{R}$
  - Hence, f is discontinuous at every point.
  - $f(x) = \begin{cases} 1 & x \quad rational \\ 0 & x \quad irrational \end{cases}$ and

    $x = \frac{m}{n} \;\; m \in Z, \, n \in N$ have no common divisors.
  - Claim
    - f is continuous at every irrational point and discontinuous at every rational point.
- Theorem: Composition of two continuous functions is continuous.
  - More precisely, if f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c

- Proof
  - If $x_n \longrightarrow c$ then $f(x_n) \longrightarrow f(c)$ because f is continuous at c.
  - Then $g(f(x_n)) \longrightarrow g(f(c))$ because g is continuous at $f(c)$.
  - Hence if $y_n \longrightarrow f(c)$ $then$ $g(y_n) \longrightarrow g(f(c))$
  - Apply with $y_n = f(x_n)$
  - ($g \circ f$)$(x_n) \longrightarrow$ ($g \circ f$)(c)
  - Thus, if $x_n \longrightarrow c$ then ($g \circ f$)$(x_n) \longrightarrow$ ($g \circ f$)(c)
  - This implies that $g \circ f$ is continuous at c
  - Essentially if $x_n \longrightarrow c$ then $f(x_n) \longrightarrow f(c)$ then $g(f(x_n)) \longrightarrow g(f(c))$
  - Example
    - $f(x) = \sqrt{3x^2 + 5}$ is continuous.
    - It is a composition at $3x^2 + 5$ and $\sqrt{x}$

Max Principle and Intermediate Value Theorem

Recall

Every bounded sequence has a convergent subsequence.

f is continuous at c ⇔ $x_n \longrightarrow c$ implies $f(x_n) \longrightarrow f(c)$ for every sequence in $dom f$.

Theorem (Max Principle)

- A continuous function on a closed interval is bounded, and attain max and min.
- Proof:
  - Let $f:[a,b] \longrightarrow \mathbb{R}$ be a continuous function
  - Put $L = \sup f$. At this point, $L \in \mathbb{R}$ or $L = \infty$
  - Let z_n be a sequence in $\mathbb{R}$ such that $\forall n \, \exists n < L$ and $z_n \longrightarrow L$
  - If $L \in \mathbb{R}$, one could take $z_n = L = \dfrac{1}{n}$
  - If $L = \infty$, then one could take $z_n = n$
  - For each n we have $z_n < L$ hence z_n is not an upper bound for f, hence there exists a point, denoted by x_n such that $x_n \in [a,b]$ and $f(x_n) < z_n$ we get a sequence x_n in $[a,b]$ such that $z_n < f(x_n) \leq L$ for all n
  - By squeeze law $f(x_n) \longrightarrow L$ since $x_n \subseteq [a,b]$ it's a bounded sequence.
  - By BWT, (x_n) has a convergent subsequence. Say $x_{n_k} \longrightarrow c$
  - Since $f(x_n) \longrightarrow L$ and $f(x_{n_k})$ is a subsequence at $(f(x_n))$ then $f(x_{n_k}) \longrightarrow L$

- Since $a \leq x_{n_k} \leq b$ for all k, and $x_{n_k} \longrightarrow c$ then $a \leq c \leq b$ so $c \in [a, b]$. Here we use the closedness at the interval.
- Since f is continuous at c, and $x_{n_k} \longrightarrow c$, we conclude that $f(x_{n_k}) \longrightarrow f\ (c)$
- Therefore $L = f(c)$ hence, $L \in \mathbb{R}$ so that $\sup f$ exists in $\mathbb{R}$, so f is bounded above.
- $f(c) \sup f$, hence f has a max at c
- Apply previous argument to $-f$
- Then $-f$ is bounded above and attains $\max$
- Hence f is bounded below and attains $\min$

- Examples

1. $f(x) = \dfrac{1}{x}$ on $(0, 1)$
   a. f is continuous, yet not bounded
   b. The theorem may fail if the interval is not closed or if the function is not continuous.
2. $f(x) = \begin{cases} 1/x & x > 0 \\ 0 & x < 0 \end{cases}$ on $[0, 1]$
   a. f is not bounded, f is not continuous.
   b. This theorem does not apply as this function is not continuous.

Intermediate Value Theorem

Let $f:[a,b] \longrightarrow \mathbb{R}$ be a continuous function and L a number between $f(a)$ and $f(b)$ then $L=f(c)$ for some $c \in [a,b]$

- Proof
  - If $L=f(a)$ or $L=f(b)$ then proof is done.
  - Suppose $f(a)<L<f(b)$
  - Let m = the midpoint at $[a,b]$ that is $m=\dfrac{a+b}{2}$
  - If $f(m)=L$ then proof is done. Take ($c=m$)
  - If $L<f(m)$ put $a_1=a$ and $b_1=m$
  - Then $f(a_1)<L<f(b_1)$ put $a_1=m$ and $b_1=b$
  - Then again $f(a_1)<L<f(b_1)$
  - That is $[a_1,b_1]$ is half of $[a,b]$ such that $f(a_1)<L<f(b)$
  - Let $[a_2,b_2]$ be the half of $[a_1,b_1]$ such that $f(a_2)<L<f(b_2)$
  - Proceed inductively, we get a nested decreasing sequence of intervals $[a_n,b_n]$ such that $f(a_n)<L<f(b_n)$ for all n
  - The length of $[a_n,b_n]=\dfrac{1}{2}n=\dfrac{b-a}{2_n}$
  - By Nested Interval Theorem, there exists c such that $c=\sup a_n=\lim a_n=\lim b_n$
  - In particular $\forall n\ a_n \le c \le b_n$ hence $a \le c \le b$ so $c \in [a,b]$
  - Since f is continuous at c and $a_n \longrightarrow c$, then $f(a_n) \longrightarrow f(c)$
  - Since $f(a_n)<L$ for all n, we get $f(c) \ge L$ thus $f(c)=L$

- Example
  - Prove that the Equation
    - $x^7 + 3x^5 - 8 = 0$ has roots
  - Proof
    - $f(x) = x^7 + 3x^5 - 8$ is continuous on $\mathbb{R}$
    - $\lim\limits_{x \to \infty} f(x)$, hence $f(b) > 0$ for some b
    - $\lim\limits_{x \to -\infty} f(x) = -\infty,$ hence $f(a) < 0$ for some a
    - We can take $a = 1, b = 2$
    - Appy the IVT on $[a, b]$
    - There exists c in $[a, b]$ such that $f(c) = 0$ so c is a root.
    - Moreover, f has a room in $(1, 2)$

Additional Material

Derivatives

Definition of the Derivative:

The derivative of a function f is another function f' whose value at any number a is:

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided that this limit exists.

Other Forms of the Definition of the Derivative:

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$	$f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}$

Table of Key Derivatives:

Exponent and Log functions	$\frac{d}{dx} e^x = e^x$	$\frac{d}{dx} a^x = a^x \ln a$	$\frac{d}{dx} \ln x = \frac{1}{x}$
Trigonometric functions	$\frac{d}{dx} \sin x = \cos x$ $\frac{d}{dx} \csc x = -\csc x \cot x$	$\frac{d}{dx} \cos x = -\sin x$ $\frac{d}{dx} \sec x = \sec x \tan x$	$\frac{d}{dx} \tan x = \sec^2 x$ $\frac{d}{dx} \cot x = -\csc^2 x$
Inverse Trig functions	$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

Derivative Rules

- $\frac{d}{dx} c = 0$ — derivative of ANY constant (anything without an x)
- $\frac{d}{dx}(c \cdot f) = c \cdot f'$ — derivative of a constant times a function
- $\frac{d}{dx}(x^n) = nx^{n-1}$ — the Power Rule
- $(f \pm g)' = f' \pm g'$ — sum or difference of functions
- $(f \cdot g)' = f' \cdot g + f \cdot g'$ — the Product Rule
- $\left(\frac{f}{g}\right)' = \frac{f' g - f g'}{g^2}$ — the Quotient Rule
- $[f(g(x))]' = f'(g(x)) \cdot g'(x)$ — the Chain Rule

Implicit Differentiation

If we want to find $\frac{dy}{dx}$, we think of y as implicitly defined as a function of x.

- When we differentiate x, we get 1.
- When we differentiate y, we get $\frac{dy}{dx}$ or y' (either is fine).
- Then we solve for $\frac{dy}{dx}$.

Logarithmic Differentiation

Used when the function is complicated or for functions with an x in base and in the exponent.

Option 1: Take the log of both sides, simplify with log properties, differentiate (implicit chain rule on y will always happen on the left side), then solve for y'.

Ex. $y = x^x \Rightarrow \ln y = \ln x^x = x \ln x \Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x) \Rightarrow$

$\frac{1}{y} y' = 1 * \ln x + x * \frac{1}{x} = \ln x + 1 \Rightarrow y' = y(\ln x + 1) = x^x(\ln x + 1)$

Option 2: Take e^ln(your equation), simplify with log properties, differentiate (not implicit).

Ex. $y = x^x \Rightarrow y = e^{\ln x^x} = e^{x \ln x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(e^{x \ln x}\right) \Rightarrow$

$y' = e^{x \ln x}\left(1 * \ln x + x * \frac{1}{x}\right) = e^{x \ln x}(\ln x + 1)$

First Derivative Test

Let f be a differentiable function on a closed interval [a,b]

- if $f' > 0$ for all x-values on [a,b], then f is ***increasing*** on [a,b].
- if $f' < 0$ for all x-values on [a,b], then f is ***decreasing*** on [a,b].
- if $f' = 0$ for all x-values on [a,b], then f is ***a constant function*** on [a,b].

Example:

I	$\boldsymbol{f'}$	$\boldsymbol{f}$
$(-\infty, 0)$	+	↑
$(0, +\infty)$	+	↑

$y = x^3$

$y' = 3x^2$

$y' = 0, \quad x = ?$

$0 = 3x^2$

$x = 0$

$\rightarrow y' > 0\ on\ (-\infty, 0) \cup (0, +\infty)$

$Therefore, y\ is\ increading\ on\ (-\infty, 0) \cup (0, +\infty).$

- $Therefore\ y\ is\ increasin\ on\ (-\infty, +\infty)$

First Derivative Test

Use this to find intervals for increasing or decreasing or to find local extrema.

1. Find critical numbers (when $f'(x) \nexists\ or\ f'(x) = 0$).

2. Make sure critical numbers are within interval/domain.

3. Test values around critical numbers to see if there is a change in sign for f'.

Example:

Find the local extrema values using 1st derivative test.

$f(x) = 1 + 3x^2 - 2x^2$ Domain: $(-\infty. +\infty)$

$f'(x) = 6x - 6x^2$

$f'(x) = 6x(1 - x)$ 6x=0 1-x=6

$f'(x) = 0, \ \ x = ?$ X=0 x=1

I	f'	f
$(-\infty, 0)$	-	↓
$(0,1)$	+	↑
$(1, +\infty)$	-	↓

Local minimum is at x=0 because f' changes from (-) to (+)

Local maximum is at x=1 because f' changes from (+) to (-)

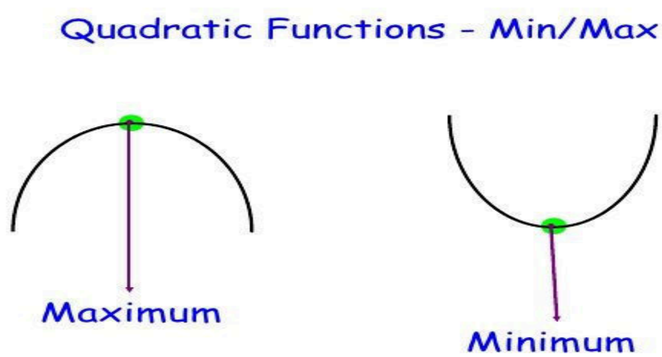
Local minimum: (0,1)

Local maximum: (1,2)

Second-Derivative Test

If $f'(a) = 0$ *and* $f''(a) \leq 0,$ *then f has a relative maximum at a*

$if\ f''(a)\ and\ f''(a) \geq 0\ then\ f\ has\ a\ relative\ at\ minimum\ at\ a$



Example:

Find relative min/min (local)

For $y = \frac{1}{3}x^{3-x^2}$+6

$y' = x^2 - 2x$ The critical values are $x = 0\ x = 2$

$= x(x-2)$

$y'' = 2x - 2$

$(0) = -2$ ≤0

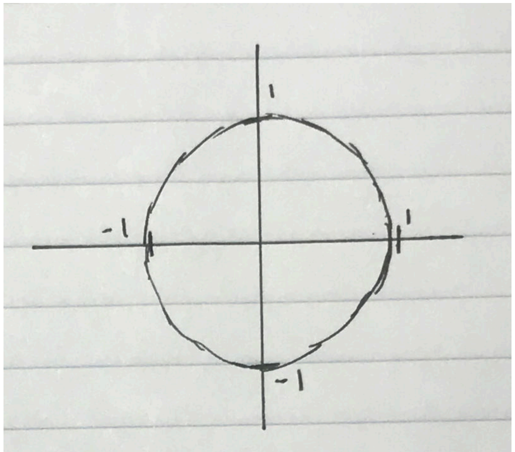
$f''(2) = 2 \geq 0$

(0,6) is the local maximum

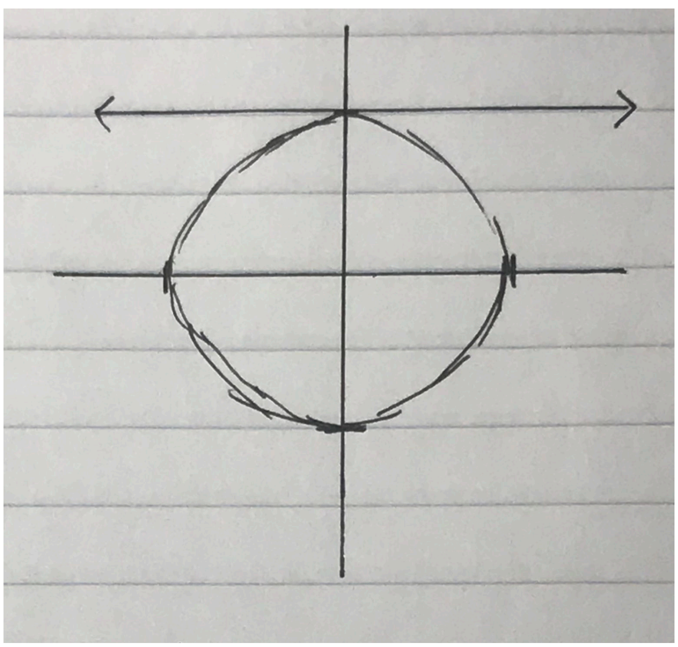
(2, $\frac{14}{3}$) =minimum

Implicit Differentiation and Maxima and Minima

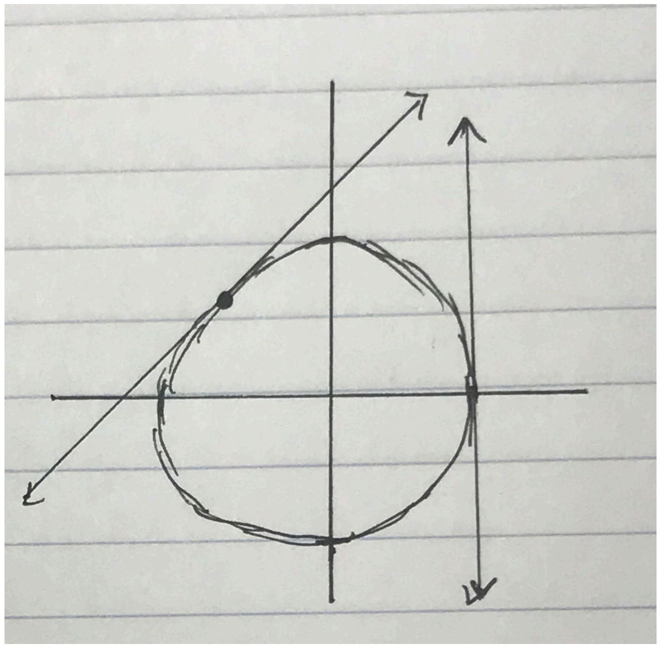
- **11.6 Implicit Differentiation**
  - Example
    - Consider a curve that cannot be represented as a function.
    - For example, the unit circle is given the equation $x^2 + y^2 = 1$
    - The slope of any curve is always $\frac{dy}{dx}$
    - How do we do this if y is not a function of x?
      - Treat y as a function and use the chain rule when differentiating.
        - Ex: $x^2 + y^2 = 1$
        - $\frac{dy}{dx}[x^2 + y(x)^2] = \frac{d}{dx}[1]$
        - $2x + 2y(x)\,\frac{dy}{dx} = 0$
          - Since we must find the derivative before we continue, we will use the chain rule.
        - $2y\frac{dy}{dx} = -2x$
        - $\frac{dy}{dx} = -\frac{x}{y} = slope\ of\ the\ tangent\ line\ on\ the\ curve$
        - $\frac{dy}{dx}[\mathrm{x}^2 + \mathrm{y}^2] = \frac{\mathrm{d}}{\mathrm{dx}}[1]$
        - $2x + 2yy' = 0$
        - $2yy' = -2\mathrm{x}$
        - $y' = -\frac{x}{y}$
        - Photo example on the next page



- Example
  - Find the slope of the tangent line of the unit circle at each point.
  - (0,1)
  - $\frac{dy}{dx} = -\frac{x}{y} = -\frac{0}{1} = 0$



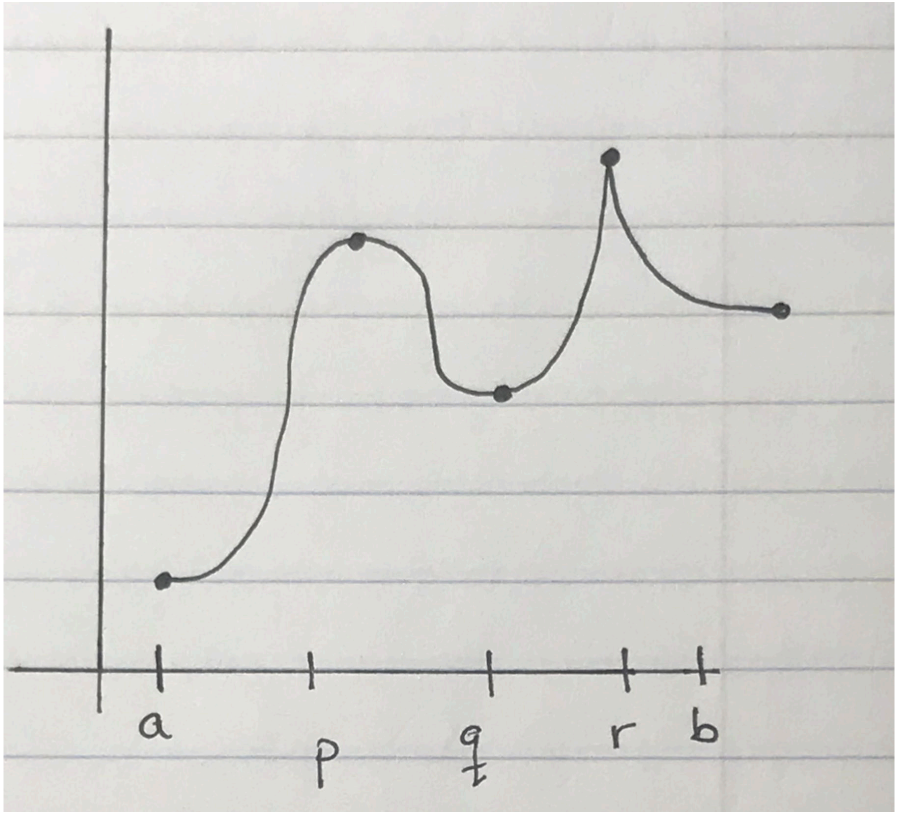
- Example
  - $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



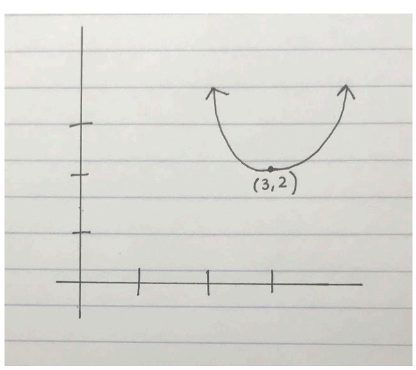
- Written Rules
  - Take derivative of each side of equation.
  - Differentiate terms with y just like x, but remember to multiply each term by y'.
  - Algebraically solve for y'.
- Example
  - Find y' for $y^5 + 3y = 9x^4 - y^2$
  - $\frac{d}{dx}[y^5 + 3y] = \frac{d}{dx}[9x^4 - y^2]$
  - $5x^4y' + 3y' = 36x^3 - 2yy'$
  - $5y^4y' + 3y' + 2yy' = 36x^3$
  - $y'(5y^4 + 3 + 2y) = 36x^3$
  - $y' = \frac{36x^3}{5y^4+2y+3}$
- Example
  - Find y' for $x^3y^4 + 7y = 6 - x$
  - $\frac{d}{dx}[x^3y^4 + 7y] = \frac{d}{dx}[6 - x]$
  - $3x^2y^4 + 7 + x^3 * 4y^3y' + 7y' = -1$

  - $7y' + 4x^3y^3y' = -1 - 3x^2y^4$
  - $y'(4x^3y^3 + 7) = -1 - 3x^2y^4$
  - $y' = \frac{-1-3x^2y^4}{4x^3y^3+7}$
- Example
  - Find y' for $x^2 \ln(y) = e^x y$
  - $\frac{d}{dx}[x^2 ln * y] = \frac{d}{dx}[e^x y]$
  - $2x * ln * y + x^2 * \frac{1}{y} * y' = e^x y + e^x y'$
  - $\frac{x^2}{y} * y' - e^x y' = e^x y - 2x \ln y$
  - $y'\left(\frac{x^2}{y} - e^x\right) = e^x y - 2x \ln y$
  - $y' = \frac{e^x y - 2x \ln y}{\frac{x^2}{y} - e^x}$

- **Maxima and Minima**
  - Example
    - Let f(x) be a function defined by [a,b]
    - (x=a): relative minima and absolute minima
      - These relationships will not always go hand in hand.
    - (x=p): relative max
    - (x=r): absolute/relative maxima
    - (x=b): relative minima

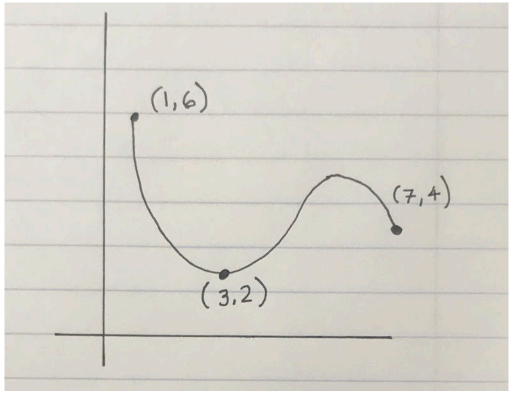


- Relative (or Local) Extrema
  - f has a relative minimum at x=r if $f(r) \leq f(x)$ for all x near r.
  - f has relative maximum at x=r if $f(r) \geq f(x)$ for all x near r.
- Absolute Extrema
  - This may not exist if domain is infinite.
  - f has an absolute minimum at x=r if $f(r) \leq f(x)$ for all x in the domain.
  - f has an absolute maximum at x=r if $(r) \geq f(x)$ for all x in the domain.
- Example
  - Find the absolute extrema for each function.



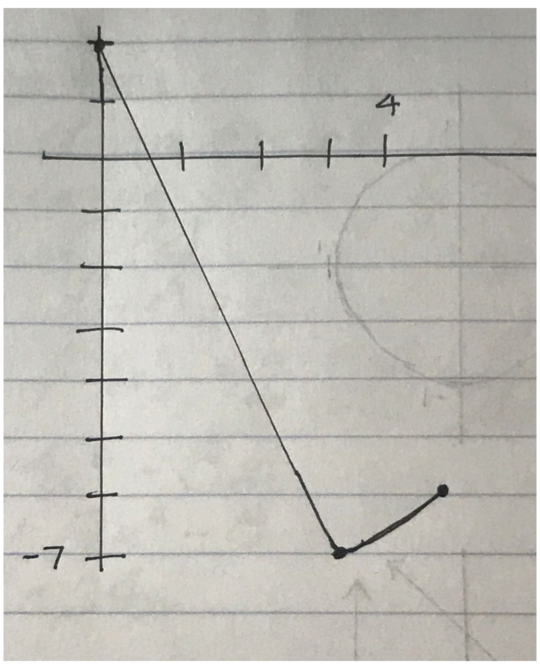
Absolute minimum = 2 @ x=3
Absolute maximum = DNE

and



Absolute minimum = 2 @ x=3
Absolute maximum = 6 @ x=1

- Candidates for extrema
  - Stationary Points: Any point where $f'(x) = 0$
  - Singular Points: Any point where $f'(x) is\ not\ defined, but\ f(x) is.$
  - End Points: The boundary points of the domain.



X=0 relative maximum
X=3 relative minimum
X=4 relative maximum

- Example
  - Find relative and absolute extrema for the function:
    - $f(x) = x^2 - 6x + 2, on\ the\ interval\ [0,4].$
  - Find stationary points
    - $f'(x) = 0$
    - $f'(x) = 2x - 6 = 0$

    - $f'(x) = 2x = 6$
    - $f'(x) = x = 3$
- Evaluate f at each candidate point.
    - $f(3) = 3^2 - 6 * 3 + 2 = -7$
    - $f(0) = 2$
    - $f(4) = 4^2 - 6 * 4 + 2 = -6$

L'Hopital's Rule

$$\text{If } \lim_{x\to c}\frac{f(x)}{g(x)} \text{ is of the form } \tfrac{0}{0} \text{ or } \tfrac{\infty}{\infty} \text{ then}$$

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f^{'}(x)}{g^{'}(x)}$$

Reasoning:

Let's assume that $\lim_{\to c} f(x) = 0$ and $\lim_{\to c} g(x) = 0$.
Liberalization of f(x) and g(x) in the neighborhood of c is,

$$\begin{aligned}
f(x) &= f(c) + f^{'}(c)(x-c)\\
and\, g(x) &= g(c) + g^{'}(c)(x-c)\\
\lim_{x\to c}\frac{f(x)}{g(x)} &= \lim_{x\to c}\frac{f(c)+f^{'}(c)(x-c)}{g(c)+g^{'}(c)(x-c)}\\
&= \frac{f^{'}(c)}{g^{'}(c)}
\end{aligned}$$

If $\lim_{x\to c}\frac{h(x)}{p(x)}$ is of the form $\frac{\infty}{\infty}$, then
We can consider the $\lim_{x\to c}\frac{\frac{1}{h(x)}}{\frac{1}{p(x)}}$

$$\begin{aligned}
(1)\lim_{x\to c}(\ln(f(x))) &= L\, then\\
\lim_{x\to c}(f(x)) &= e^{L}\\
(2)\lim_{x\to c}(\ln(f(x))) &= \infty\, then\\
\lim_{x\to c}(f(x)) &= \infty\\
(3)\lim_{x\to c}(\ln(f(x))) &= -\infty\, then\\
\lim_{x\to c}(f(x)) &= 0
\end{aligned}$$

Example:

$$\begin{aligned}
(i)\lim_{x\to 0}(\frac{1}{x} - \frac{1}{e^x-1}) &= \lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)}\\
&= \frac{0}{0}\\
&= L'Hopital's\, Rule\\
&= \lim_{x\to 0}\frac{e^x-1}{e^x-1+xe^x}\\
&= \frac{0}{0} = L'Hopital's\, Rule\\
&= \lim_{x\to 0}\frac{e^x}{e^x+xe^x+e^x} = \frac{1}{2}.
\end{aligned}$$

To find $\lim_{x\to c}(f(x))^{g(x)}$

$$(1)\lim_{x\to c} e^{\ln(f(x))^{g(x)}} = e^{\lim_{x\to c}\ln(f(x))^{g(x)}}$$

$$(2)\ \text{Aside: } \lim_{x\to c}\frac{\ln(f(x))}{\frac{1}{g(x)}}$$

Example:

$$\begin{aligned}(2)\ \lim_{x\to 1^+}(x-1)^{\ln x} &= e^{\ln(x-1)^{\ln x}}\\ &= e^{\ln x\ln(x-1)}\\ &= e^{\lim\limits_{x\to 1^+}\ln x\ln(x-1)}\end{aligned}$$

$$\begin{aligned}Aside:\ \lim_{x\to 1^+}\ln x\ln(x-1) &= \lim_{x\to 1^+}\frac{\ln(x-1)}{\frac{1}{\ln x}}\\ &= \frac{\infty}{\infty} = L'H\\ &= \lim_{x\to 1^+}\frac{\frac{1}{(x-1)}}{\frac{1}{\frac{-1}{\ln^2 x}}\frac{1}{x}}\\ &= \lim_{x\to 1^+}\frac{-x\ln^2 x}{x-1} = \frac{0}{0} = L'H\\ &= \lim_{x\to 1^+}\frac{-\ln^2 x - x.(2\ln x)\frac{1}{x}}{1} = \frac{0}{1} = 0\end{aligned}$$

Sketch the graph of $y = \ln\frac{e^x - e^{-x}}{2}$

$$\begin{aligned}&(1) Domain\\ &e^x - e^{-x} > 0\\ &e^x > e^{-x}\\ &e^{2x} > 1\\ &e^{2x} > e^0\\ &x > 0\\ &Dom(f(x)) = (0,\infty)\end{aligned}$$

$$\begin{aligned}&(2) Symmetry\\ &Neither\ even\ nor\ odd\\ &(3) Intercept\\ &0 = \ln 1\\ &\Rightarrow \frac{e^x - e^{-x}}{2} = 1\\ &\Rightarrow e^x - e^{-x} = 2\\ &\Rightarrow e^{2x} - 2e^x - 1 = 0\\ &\Rightarrow e^x = 1 \pm \sqrt{1+1}\\ &\Rightarrow e^x = 1 \pm \sqrt{2}\\ &e^x\ can\ not\ be\ negative.\\ &\Rightarrow e^x = 1 \pm \sqrt{2}\\ &x = \ln(1+\sqrt{2})\\ &X\ intercept\ is\ (\ln(1+\sqrt{2}), 0)\end{aligned}$$

$$\begin{aligned}
&No\,Y\ intercept.\\
&(4)Asymptote\\
&VA:\ x=0\\
&SA:\ y=kx+h\\
&No\,left\,SA
\end{aligned}$$

$$\begin{aligned}
k &= \lim_{x\to\infty}\frac{f(x)}{x}\\
&= \lim_{x\to\infty}\frac{\ln\frac{e^x-e^{-x}}{2}}{x}\\
&= \frac{\infty}{\infty} = L'H\\
&= \lim_{x\to\infty}\frac{\frac{2}{e^x-e^{-x}}(e^x+e^{-x})}{1}\\
&= \lim_{x\to\infty}\frac{2e^x+2e^{-x}}{(e^x-e^{-x})}\frac{e^x}{e^{-x}}\\
&= \lim_{x\to\infty}\frac{2e^{2x}+2}{(e^{2x}-1)} = 2\\
b &= \lim_{x\to\infty}(f(x)-kx) = -\ln 2
\end{aligned}$$

Taylor Theorem

Taylor Series of a Function

- The taylor series center at a on D, of an infinitely differentiable function f : D on R approaching R is given by:
  - $\sum_{n=0}^{infinity} \frac{f^n a}{n!}(x-a)^n$

Taylor Polynomials

- The Taylor polynomial of order n centered at a at D, of an n differentiable function f: D R at R is given by:
  - Tn(x) = $\sum_{k=0}^{n} \frac{f^k a}{k!}(x-a)^k$
- Remarks:
  - f(a) + f'(a)(x-a) is called the linearization of f.
  - The taylor polynomial is called order of n instead degree n, because $f^n a$ may vanish.
  - The taylor polynomial of order n centered at a = 0, is called the n +1 maclaurin polynomial.

The Taylor theorem

- The taylor polynomial and taylor series are obtained from a generalization of the Mean Value Theorem.
- If f: [a,b] approaching R is differentiable, then there exists a c on [a,b] such that (f(b) - f(a))/(b-a) = f'(c), which is equivalent to:
  - f(b) = f(a) + f'(c)(b-a)
- Theorem (Taylor's theorem):
  - If f: [a,b] approaching R is (n+1) times continuously differentiable, then there exists c on (a,b) such that
    - f(b) = f(a) + f'(a)(b-a) + f''(a)/2 * $(b-a)^2$ + ….. + $\frac{f^n a}{n!}(b-a)^n + \frac{f^{n+1} c}{(n+1)!}(b-a)^{n+1}$
- Remark:
  - The taylor theorem is usually applied for a fixed point a, while the point b = x is used as an independent variable.
  - f(x) = f(a) + f'(a)(x-a) + ... + $\frac{f^n a}{n!}(x-a)^n + R_n x$
  - The remainder function is given by:
  - $R_n(x) = \frac{f^{n+1} c(x)}{(n+1)!}(x-a)^{n+1}$ $with\ c\ on\ (a,x)$
- The point c on (a,x) is also dependent on x.

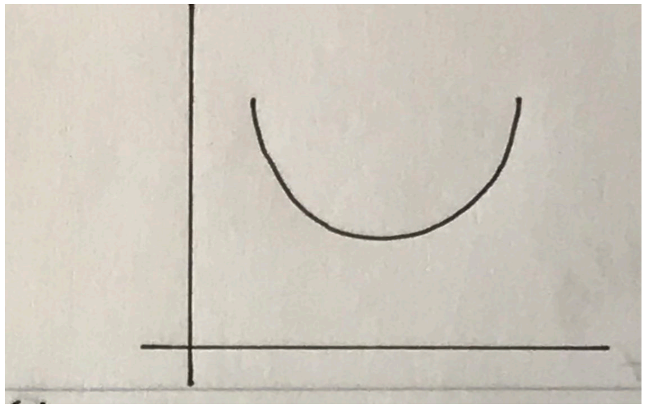
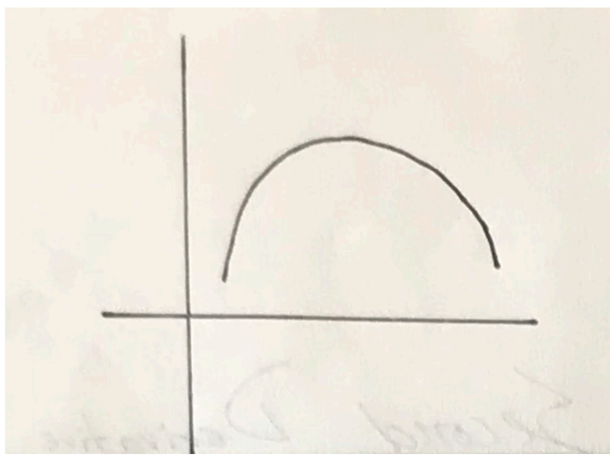
- We can use the taylor polynomial to write that f(x) = $T_n(x) + R_n(x)$

Estimating the remainder

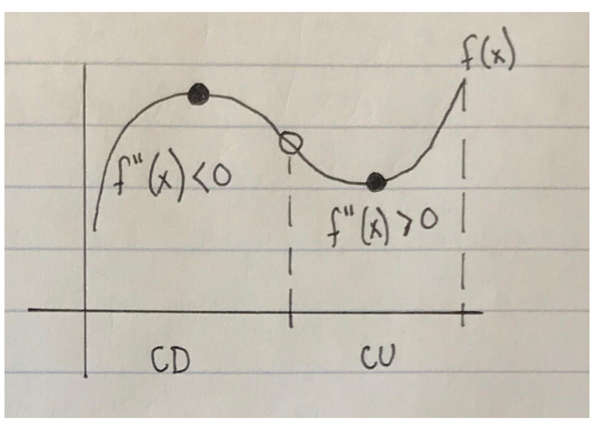
- Theorem:
- Let f: D approaching R be infinitely differentiable with taylor polynomials and Remainders Rn centered at A on D, that is for n>1 holds:
    - f(x) = $T_n(x) + R_n(x)$
- If $f^{n+1}y \leq M$ for all such y, such that |y-a| < |x-a| then:
- $R_n x \leq M \frac{|x-a|^{n+1}}{(n+1)!}$
- Furthermore, if the inequality above holds true for every n > 1, then the taylor series converges towards f(x).

Second Derivative Test and Concavity

- **Second Derivatives and Concavity**
  - Example
    - Find $\frac{d^2y}{dx^2}$ for $y = x^2e^{-3x}$
    - $\frac{dy}{dx} = 2xe^{-3x} + x^2e^{-3x}(-3)$
      - $=e^{-3x}(2x - 3x^2)$
    - $\frac{d^2y}{dx^2} = \frac{d}{dx}[e^{-3x}(2x - 3x^2)]$
    - $= -3e^{3x}(2x - 3x^2) + e^{-3x}(2x - 3x^2) + e^{-3x}(2 - 6x)$
    - $=e^{3x}[-6x + 9x^2 + 2 - 6x]$
    - $=e^{3x}(9x^2 - 12x + 2)$
  - Example
    - A $1,000 loan has a 9% interest rate and is compounded continuously. Find the acceleration of the balance after 3 years.
    - Balance: $A(t) = Aoe^{rt} = 1000e^{0.09t}$
    - $A'(t) = 1000e^{0.09t}(0.09) = 90e^{0.09t}$
    - $A'(3) = 90e^{0.09(3)} \approx \frac{\$117.90}{year}$
    - $A''(t) = \frac{d}{dt}[90e^{0.09t}] = 90e^{0.09t}\,(0.09)$
    - $= 8.1e^{0.09t}$
    - $A''(3) = 8.1e^{0.09(3)} \approx \frac{\frac{\$10.61 dollars}{year}}{year}$
- **Inflection Points**
  - An inflection point of a function f(x) is any point where f”(x)=0.
  - An inflection point of f(x) is a stationary point of f’(x).
  - Example
    - Find all inflection points of $f(x) = x^3 - 3x^2$.
- **Concavity**
  - The sign of f’’(x) determines the concavity of the graph of f(x).
  - There are two types of concavity

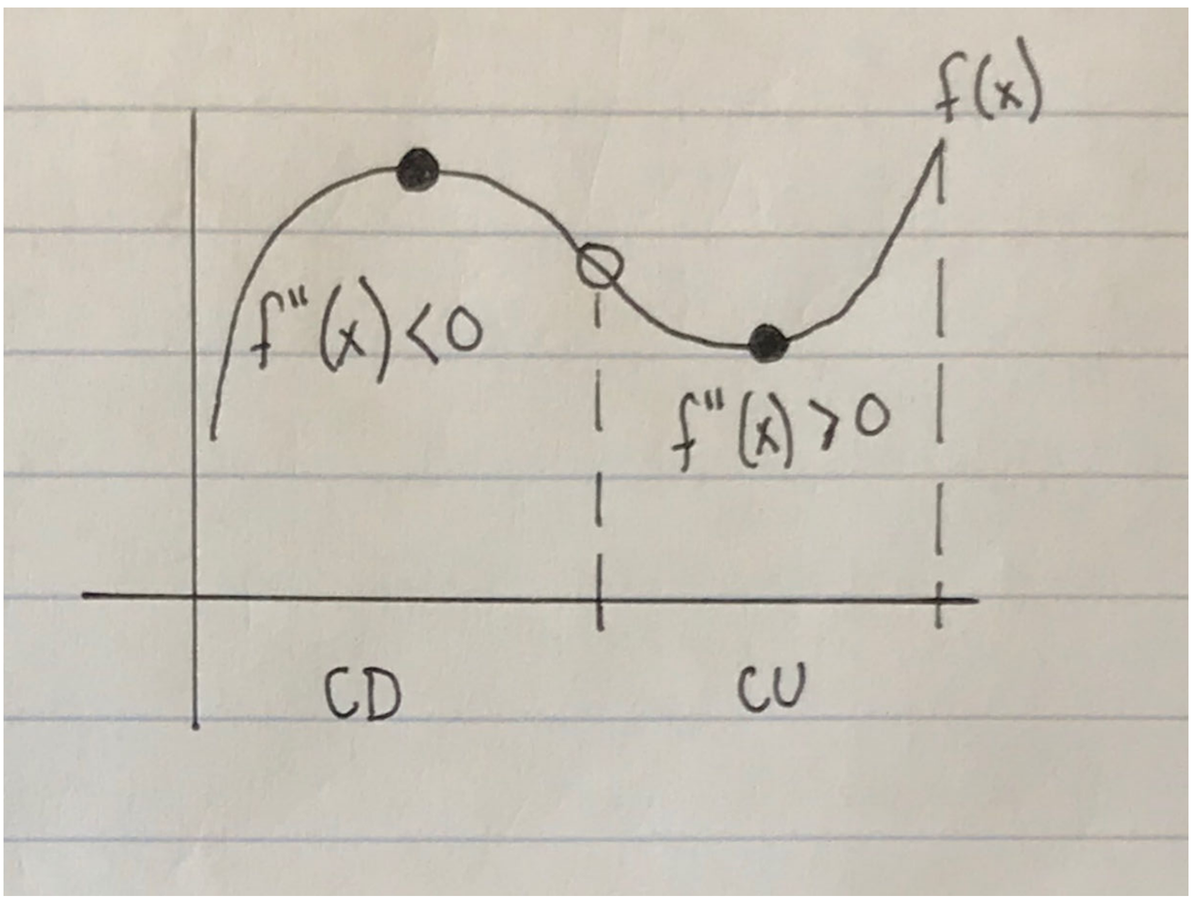
- Concave up
  - Tangent line is below curve
  - F"(x)>0
  - 
- Concave down
  - Tangent line is above curve
  - F"x<0
  - 

○ A function changes concavity at the inflection points.

○ 

Interval	Test Point	F"(x)	Concavity
$(-\infty, 0)$	X=-1	36	Concave Up
(0,2)	X=1	-12	Concave Down
$(2, \infty)$	X=3	36	Concave Up

- Determine the intervals where $f(x) = x^4 - 4x^3$ is concave up/down.
  - $f'(x) = 4x^3 - 12x^2$
  - $f''(x) = 12x^2 - 24x = 0$
  - $= 12x(x-2) = 0$
  - $x = 2 \text{ and } x = 0 \text{ are the inflection points.}$



- **Second Derivative Test**
  - Let c be a critical point of f(x).
  - If f''(c)>0, there c is a relative maximum
  - If f"(c)<0, then c is a relative minimum

- Example
  - Let $f(x) = 8x^3 - 8x = 0$
  - $= 8x(x^2 - 1) = 0$
  - $x = 0\ or\ x \pm 1$
  - $f''(x) = 24x^2 - 8$
- Inflection points:
- $f''(x) = 24x^2 - 8 = 0$
- $24x^2 = 8$
- $x^2 = \frac{8}{24} = \frac{1}{3}$
- $x = \pm\sqrt{\frac{1}{3}}$
- Concavities:
- $f''(0) = -8 < 0 \rightarrow concave\ down \rightarrow x = 0\ relative\ maximum$
- $f''(-1) = 16 > 0 \rightarrow concave\ up \rightarrow x = -1\ is\ a\ relative\ minimum$
- $f''(1) = 16 > 0 \rightarrow concave\ up \rightarrow x = 1\ is\ a\ relative\ minimum$

End of Part 1

University of Alberta

MATH 117

Honors Calculus I

Fall 2017

Final Exam

Prof: Vladimir Troitsky

Exam Guide

Part 2 of 2

Table of Contents:

Midterm 2 Review

General Review of Midterm 2

1.
    a. Every non empty set bounded above subset of $\mathbb{R}$ has a supremum.
    b. Every non empty set bounded below subset of $\mathbb{R}$ has infinum
2. The Monotone Convergence Theorem
    a. Review MCT proof from your notes
    b. Don't try to memorize it → rather understand it in your own words and get a feel for the underlying backbone of the proof.
3. Using the definition of limits, $\frac{3n+5}{2n+8} \longrightarrow \frac{3}{2}$
    a. Using scratch paper, giving £ > 0, provide a clean solution.
        i. Let $\varepsilon > 0$ take n_0 = $\lceil \frac{7}{2\varepsilon} \rceil$ if $n \geq n_0$
        ii. Then $\left| \left(\frac{3n+5}{2n+8} \right) - \frac{3}{2} \right| = \frac{7}{2n+8} < \frac{7}{2n} < \varepsilon$
        iii. Because $n > \frac{7}{2\varepsilon}$
4. Find the limits of $\sqrt{n^4 + 6n^2 + 1 - \sqrt{n^4 + n^2 + 1}}$
    a. Common mistake → can only conclude $x_n \leq \frac{5}{3}$ but does not mean it converges to $\frac{5}{3}$ unless you prove it.

Property of Infinity Limits

1. **Consider:**

(i) If $x_n \to \infty$ and $y_n \to \infty$ then $x_n + y_n \to \infty$ and $xy_n \to \infty$

(ii) if $x_n \to \infty$ and (y_n) is bounded, then $x_n + y_n \to \infty$

Proof

Since (y_n) is bounded, it has a lower bound, say ν.

$y_n = \nu$ for all n

Given any m, since $x_n \to \infty$ we can find n,

such that $X_n > m - \nu$ for all $n \geq n_0$

Then $x_n + yn > (m - v) + v = m$

So $x_n + y_n > m$ whenever $n \geq n_0$

Hence, $x_n + y_n \to \infty$

(iii) if $x_n \to \infty$ and $y_n \to b$ then

$x_n y_n \to +\infty$ if $b > 0$

$x_n y_n \to -\infty$ if $b < 0$

Anything may happen if $b = 0$.

In particular, if $x_n \to \infty$ and $x < 0$, then $\lambda x_n \to -\infty$

Proof : case $b > 0$

Recall Separation Principle since $y_n \to b$ since we can find n

Such that $y_n > \frac{b}{2}$ whenever $n \geq n$ given any $m > 0$

Since $x_n \to \infty$ we can find m_2 such that $X_n > \frac{m}{\left(\frac{b}{2}\right)}$

Whenever $n \geq n_2$.

Let $n_0 = \max\{n_1, n_2\}$ if $n \geq n_0$ then $n \geq n$ and $n = n_2$

To that $y_n \geq \frac{b}{2}$ and $x_n \geq m/\left(\frac{b}{2}\right)$

hence $x_n y_n \rangle^m / \left(\frac{b}{2}\right) \cdot \frac{b}{2} = m$

Hence $X_n Y_n > m$ whenever $n \geq n_0$ so $x_n y_n \to \infty$

Note: is y_n positive? It has to be, as when multiplied with the other, it ends larger.

Proof: Case $b < 0$

Since $y_n \to b$, we can separate by principal once again,

we can find that n, such that $y_n < \frac{b}{2}$ whenever $n \geq n$.

Given $m < 0$ since $x_n \to \infty$ we can find n_2

Such that $x_n > m/\left(\frac{b}{2}\right)$ whenever $n \geq n_2$

Note: $m/\left(\frac{b}{2}\right)$ is positive.

Let $n_0 = \max(n_1, n_2)$, let $n \geq n_0$, then $n \geq n$, and $n \geq n_2$, to

$y_n = \frac{b}{x}$ and $x_n > m/\left(\frac{b}{2}\right)$ so

$x_n y_n = x_n \frac{b}{2} < m/\left(\frac{b}{2}\right) \cdot \frac{b}{2} = M$ so $x_n y_n < m$ whenever

$n \geq 110,$ hence $x_n y_n \to -\infty$

(iv) If $x_n \to \infty$ and $y_n \geq x_n$ for all in, then $y_n \to \infty$

(v) Let $x_n > 0$ to all in, then $x_n \to \infty$ if $\frac{1}{xn} \to 0$

2. Examples:

Example A:

$$\frac{n3 + n^2 + 1}{7 + 3n^2}$$

The first step is to get the highest values out.

$$\frac{n3 + n^2 + 1}{7 + 3n^2} = \frac{n^3 \left(1 + \frac{1}{n} + \frac{1}{n^3}\right)}{n^2 \left(\frac{7}{n^2} + 3\right)} = n \cdot \frac{\left(1 + \frac{1}{n} + \frac{1}{n^3}\right)}{\frac{7}{n^2} + 3}$$

$$\longrightarrow \frac{1 + 0 + 0}{0 + 3} = \frac{1}{3} > 0$$

Note: Refer to (iii)

Example B.

$$\frac{n3 + n^2 + 1}{7 - 3n^2} = = n \cdot \frac{\left(1 + \frac{1}{n} + \frac{1}{n^3}\right)}{\frac{7}{n^2} + 3} = -\frac{1}{3} \to -\infty$$

Example C.

More generally, let $x_n = \frac{p(n)}{q(n)}$ where p and q are polynomials of degrees k and m, respectively. We will find the $\lim$ for x_n.

Strategy:

$$p(n) = a_0 + a_1\eta + a_2n^2 + \ldots + a_kn^k$$

$$q(n) = b_0 + b_1n + b_2n^2 + \ldots + b_mn^m$$

$$x_n \frac{\rho(n)}{q(n)} = \frac{\left(a_0 + a_1n + a_2n^2 + \ldots + a_kn^k\right)}{\left(b_0 + b_1\eta + b_2n^2 + \ldots + b_mn^m\right)}$$

$$\frac{a_kn^k\left(\frac{a_0}{ax} + \frac{1}{n^k} + 0\right)}{b_mn^m\left(\frac{b_0}{b_m} + \frac{1}{n^n} + \frac{b_1}{b_m} + \frac{1}{n^{m-1}}\right)}$$

$$\lim x_n = \lim_{n\to\infty} \qquad \frac{a_kn^k}{b_mx^m}$$

$\frac{a^k}{b_m}$ if $k = m$

Exponentials vs. Polynomials

Proposition: let $b > 1$, then $b^n \to \infty$

Moreover, if $m \in N$ then $\lim \frac{b_n}{n_m} = \infty$

Proof

Recall Bernoulli's inequality

If $a > -1\ n \in \mathbb{N}$ then $(1+a)^n \geq 1 + na$

Since $b > 1$ then $b - 1 > 0$ hence $b - 1 > -1$.

$1 + n(b-1) > n(b-1)$ hence $b^n > n(6-1)$

hence $\lim b_n = \infty$

Monotone Convergence Theorem

Recall:

Polynomial vs Exponential Growth

If b > 1 and m = 0 or $m \in \mathbb{N}$ then $\lim_{n\to\infty} \frac{b^n}{n^m} = \infty$

and $\lim_{n-\infty} \frac{n^m}{b^n} = 0$

Examples:

1) $\frac{(3n^2 + 7n + 1)}{2^n} = 3 \cdot \frac{n^2}{2^n} + 7\frac{n}{2^n} + \frac{1}{2^n} = 0$

Exponential Growth always wins.

2) $\frac{2^n}{3n^2 + 7n + 1} \to \infty$

3) $0.9^n \cdot n^2 = \frac{n^2}{\left(\frac{1}{0.9}\right)^n} \to 0$ $\quad b = \frac{1}{0.9} > 1$

If $-1 < a < 1$ and $m = 0$ or $m \in \mathbb{N}$ then $\lim_{n\to\infty} a^n n^m = 0$

Exponential decay wins over polynomial growth.

4) if $a = 0$,

suppose $a \neq 0$. take $6 = \dfrac{1}{|a|}$ since $|a| < 1$ then $b > 1$ and

$$|a^n n^m| = |a|^n \cdot n^m = \frac{n^m}{B^n} \to$$ and $n \to \infty$

Hence, $a^n n^m \to 0$ and $n \to \infty$.

5) Find $\lim \dfrac{5n}{n!}$

Put $x_n = \dfrac{5n}{n!}$ then

$$0 < x_n = (5 \cdot 5 \cdot 5 \cdot 5 \cdots \cdot 5) = \frac{5}{1} \cdot \frac{5}{2} \cdot \frac{5}{3} \cdot \frac{5}{4} \frac{5}{5} \cdots \frac{5}{n}$$

Take the first 5 terms, and the rest of the terms are $\dfrac{5}{n}$ as they are n factors. Each of them are less than the other one before it, this sequence is decreasing.

$$\frac{5^5}{5!} \cdot \left(\frac{5}{6}\right)^{n-6} = \frac{5^5}{5!}\left(\frac{5}{6}\right)^{-6} \cdot \left(\frac{5}{6}\right)^n$$

Dominated by a sequence to zero , by domination principle $x_n \to 0$

Challenge Question

$$\lim \frac{n^2}{(2n)!}$$

Let (x^n) be a sequence.

i) If (x^n) is bounded above then $\sup x_n$ exists in $\mathbb{R}$

ii) If (x^n) is not bounded above, we write $\sup x_n = +\infty$

iii) If (x^n) is bounded below then $\inf (x^n)$ exists in $\mathbb{R}$

iv) If (x^n) is not bounded below then we write $\inf (x^n) = -\infty$

Exactly one of the following is true:

- (x^n) converges to a number; $x_n \to c \quad \lim x_n = c$
- (x^n) tends to $+\infty$, $\lim x_n = +\infty$
- (x^n) tends to $-\infty$, $\lim x_n = +\infty$
- However, if none of the above 3 is true, it is oscillating,

  then $\lim x_n$ does not exist (D.N.E.)

Monotone sequences cannot oscillate, hence the last option is excluded for monotone sequences, so $\lim x_n$ always exists (though it maybe $\pm\infty$)

There are 3 cases:

- If (x^n) is increasing, limit of (x^n) = $\sup x_n$

- If (x^n) m is decreasing, limit of (x^n) = $\inf x_n$
- Every monotone bounded sequence is convergent

Proof

Suppose (x^n) is increasing. Always bounded below by x.

Case (x^n) is not bounded above, we write $\sup x_n = +\infty$

For every real number, M, if there is no upper bound of (x^n)

So there exists a n_0 such that $x_{n_0} > M$.

since (x^n) is increasing for all $n \geq n$, then $x_n \geq x_{n_0}$

So $x_n > M$. A tail of (x^n) is in the interval $(m, +\infty)$

Hence, $x_n \to +\infty$ so $\lim x_n = +\infty$

hence $\lim x_n = \sup x_n$

Part 1

(x^n) is bounded above then $\sup x_n$ exists in $\mathbb{R}$, put $c = \sup x$.

We need to prove that $\lim x_n = c$

Use the definition of convergence, let $\varepsilon > 0$

Since $c = \sup x$ *n,* $c - \varepsilon$ is not an upper bound of x_n

Hence, there exists n_0 so that $x_{n_0} > c - \varepsilon$ since (x^n) is increasing for every $n \geq n_0$ we have $x_n \geq x_{n_o}$

Hence, $x_n > c - \varepsilon$, also since $c = \sup x_n$. We have $x_n \leq c$ so $x_n < c + \varepsilon$.

Hence $x_n \in (c - \varepsilon, c + \varepsilon)$ whenever $n \geq n_0$

By definition of convergence this means $x_n \to c$

Part 2 - Exercise

Case:

x^n is not bounded below, then $\inf (x^n) = -\infty$ and $\lim (x^n) = -\infty$

Case:

(x^n) is bounded below, then $c = \inf (x^n)$, exists on $\mathbb{R}$ show that $x_n \to c$;

Part 3 - Exercise

Suppose (x^n) is monotone and bounded.

Case: (x^n) is increasing (and bounded above)

- The we already proved that (x^n) converges.

Case: (x^n) is decreasing (and bounded below)

- Then we should prove that (x^n) converges.

Examples

Why would the Monotone Convergence Theorem be used?

- As a theory, it is helpful to find whether a sequence converges or not.
- It is not useful in finding the limit.

1. Let x = 10 and $$x_n + 1 = \frac{1}{2}\left(x_{\ n} \ + \ \frac{2}{x_n}\right)$$

   Determine whether it converges; find the limit if it does.

   Write the first few terms

   x_1 =10, x_2 = 5.1, x_3 =2.746… , x_4 = 1.737…, x_5 = 1.464…,

   x_6 = 1.4145…, x_7 = 1.4142…

   If Conjunctive, (x_n) is decreasing and convergent.

   Suppose (x_n) is convergent, say $x_n \to c$. Then

   $$x_n + 1 = \frac{1}{2}\left(x_{\ n} \ + \ \frac{2}{x_n}\right)$$ which is

   $$c \ = \ \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$ so we can solve

   $$2c = c + \frac{3}{2}$$

   $$c = \ \frac{2}{c}$$

$c = \pm\sqrt{2}$ since $x_n > 0$ for all n

$C \geq 0$ so $c = \sqrt{2}$

If (x_n) converges, then $\lim x_n = \sqrt{2}$

We use MCT (Monotone Convergence Theorem) to show that (x_n) converges.

Claim: x_n is bounded below by $\sqrt{2}$, so $x_1 > \sqrt{2}$

$$x_{n-1} - 2 = \left[\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right]^2 - \frac{8}{4}$$

$$x_{n-1} - 2 = \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) - 2$$

$$x_{n-1} - 2 = \frac{1}{4}\left(x_n^2 - 4 + \frac{4}{x_n^2}\right)$$

$$\frac{1}{4}\left(x_n - \frac{2}{x_n}\right)^2 \geq 0$$

$x_{n+1}^2 - 2 \geq 0$ so $x_{n+1} \geq \sqrt{2}$ hence (x_n) is bounded below by $\sqrt{2}$

Claim: (x_n) is decreasing

Since $x_n \geq \sqrt{2}$ so $x_n^2 \geq 2$ so $x_n \geq \frac{2}{x_n}$ so

$$x_n + 1 = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \leq \frac{1}{2}(x_n + x_n) = x_n$$

$x_n + 1 \geq x_n$ by MCT (x_n) is convergent, hence $x_n \to \sqrt{2}$

Practice

6.2(b)

x_n = (-1, 1, -1, 1, ...) = $(-1)^n$

y_n = (1,-1,1,-1...) = $(-1)^{n+1}$

$x_n \bullet y_n$ (-1,-1,-1, ...) = $(-1)^{n+1} \to -1$

Claim: x_n is divergent

Proof: suppose not, suppose $x_n \to c$

In Particular, take $\varepsilon = 1/2$. We can find the tail of the x_n in sequence

$\left(c - \frac{1}{2}, c + \frac{1}{2}\right)$ then all the terms in this tail are within from each other.

6.2 (e)

If $\sup x_n = +\infty$ then $\lim x_n = +\infty$. This statement is false.

Take x_n = (0,1,0,2,0,3, ...)

x_n is not bounded above $\longrightarrow \sup x_n = +\infty$, however $\lim x_n$ does not exists.

6.3 (a)

$$\frac{(3n - n^5)}{1+n+n\sqrt[4]{n}} = \frac{\left(n\sqrt[4]{n}\left(\frac{3n}{n\sqrt[4]{n}} - \frac{n^5}{n\sqrt[4]{n}}\right)\right)}{n\sqrt[4]{n}\left(\frac{1}{n\sqrt[4]{n}} + \frac{n}{n\sqrt[4]{n}} + 1\right)}$$

$$= \frac{\left(\frac{3}{n\sqrt[3]{n}} - \sqrt{n}\right)}{\frac{1}{n\sqrt[4]{n}} + \frac{1}{n\sqrt[3]{n}} + 1} = \lim \frac{\left(\frac{3}{n\sqrt[3]{n}} - \sqrt{n}\right)}{\frac{1}{n\sqrt[4]{n}} + \frac{1}{n\sqrt[3]{n}} + 1}$$

$$\frac{\left(3 \cdot \lim\left(\frac{1}{n\sqrt[4]{n}}\right) - \lim\sqrt{n}\right)}{\lim\ \frac{1}{n\sqrt[4]{n}} + \frac{\lim 1}{n\sqrt[3]{n}} + 1} = \frac{(\ 0 - \infty)}{1} = -\infty$$

6.3 (g)

$$\lim\ \frac{\left(3^{n+1} \cdot 7^{n+3}\right)}{4^{n} \cdot 5^{n-2}} = \lim\ \frac{(3 \cdot 3^{n} \cdot 7 \cdot 7 \cdot 7 \cdot 7^{n})}{4^{5}\ \cdot 5^{-1} \cdot 5^{-2} \cdot 5^{n}}$$

$$= \lim\ 5^2 \cdot 3 \cdot 7^4 \cdot \left(\frac{21}{20}\right)^n = \lim\ 5 \cdot \lim 5 \cdot \lim 3 \cdot \lim 7 \cdot \lim 7 \cdot \lim 7 \cdot \lim \left(\frac{21}{20}\right)^n$$

If $b > 1,\ then\ b^{\,n} \longrightarrow +\infty$ then $= \lim\ 5^2 \cdot 3 \cdot 7^4 \cdot \infty$ hence

$\lim x_n = +\infty$ if $x_n \longrightarrow +\infty$ and $y_n \longrightarrow c\ with\ c > 0$ then

$x_n y_n \longrightarrow \infty$

6.3 (k)

$$(2.3,\ 2.33,\ 2.333,\ \ldots\ )$$

$$x_n = 2 + \frac{1}{3}(0.999\ldots9)\ \ so\ x_n = 2 + \frac{1}{3}(n)$$

$$x_n = 2 + \frac{1}{3}\left(1 - \frac{1}{10^n}\right),\ x_n = 2 + \frac{1}{3} - \frac{1}{3\ \cdot 10^n}$$

$$x_n = \frac{7}{3} - \frac{1}{3 \cdot 10^n}$$ when $n \longrightarrow +\infty$, $3 \cdot 10^n \longrightarrow \infty$

$$x_n \longrightarrow \frac{7}{3} - 0 = \frac{7}{3}$$

$$\lim x_n = \frac{7}{3}$$

6.3 (l)

$$x_n = \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}\right)$$

$$\frac{1}{2}x_n = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}\right) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2 \cdot 2^n}$$

$$x_n - \frac{1}{2}x_n = \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}\right) - \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} + \frac{1}{2^{n+1}}\right)$$

$$= \frac{1}{2} - \frac{1}{2^{n+1}}, \quad \frac{1}{2}x_n == \frac{1}{2} - \frac{1}{2^{n+1}}, \quad x_n = 1 - \frac{1}{2^n}$$

$$\lim(x_n) = \lim\left(1 - \frac{1}{2^n}\right) = \lim(1) - \lim\left(\frac{1}{2^n}\right) = 1 - 0 = 1$$

Can also be proved by induction.

Nested Intervals and the Bolzano Weierstrass Theorem

Recall : The Monotone Convergence Theorem

i) if (x_n) is increasing then $\lim x_n = \sup x_n$

ii) if (x_n) is decreasing then $\lim x_n = \inf x_n$

iii) if (x_n) is monotone and bounded then (x_n) converges.

Review : Nested Intervals

Suppose that $I_1 \geq I_2 \geq I_3 \geq I_4 \geq \dots$ is a nested decreasing sequence of closed intervals, $I_n = [a_n, b_n]$ then a_n increases and b_n decreases.

$\sup a_n$ and $\sup b_n$ exists. $n^{\infty} I_n = [\sup a_n, \inf b_n]$, non empty.

However if $\inf length\ (I_n) = 0$ where $length\ (I_n) = b_n - a_n$

then $n = 1,\ I_n = \{c\}$ where a_n increases and $\sup a_n = \inf b_n$

Since a_n is increasing, $\sup a_n$ = $= \lim a_n$ and since is decreasing,

$\inf b_n = \lim b_n$. $length\ (I_n)$ decreases, $\lim$ also decreases.

NOTE: $\sup\ and\ \inf$ are the limits.

Maybe the entire sequence does not converge. However, given a divergent sequence, can one extract a convergent subsequence?

Example

1.

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right)$$

the entire sequence converges, hence every subsequence converges.

2. (0,1,0,1,0,1,0,1,...) the entire sequence diverges, but it has many convergent subsequences. In particular, (1,1,1,...) and (0,0,0,0) are convergent.

3. $\left(1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \dots\right)$ the entire sequence diverges, however $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\right)$ is convergent. So we can take the subsequence of even terms. $\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\right)$ *or* $\left(\frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}\right)$ as it doesn't matter where we start in the sequence as long as we follow the sequence.

4. $(1, 2, 3, 4, 5, 6, \dots)$ this sequence does not converge and has no subsequence that could. More generally, if $\lim x_n = \infty$ then $\lim x_{n_k} = \infty$ for every subsequence (x_{n_k}), hence (x_n) has no convergent sequence.

The Bolzano Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Proof:

Suppose (x_n) is a bounded sequence, it's within an interval of an upper bound and a lower bound. Then exists a and b such that $x_n = [a, b] = I$ for all n.

Let I_1 be half of I which contains infinitely many terms of the sequence. Consider the Pigeonhole Principle when understanding the infinite terms without bounds.

The Pigeonhole Principle :

if n amount of pigeons were to be put into an m amount of holes, with $n > m$, then at least one container must contain more infinite amount of pigeons.

Let I_2 be half of I_1 such that I_2 contains infinatly as many terms of the sequence. Keep cutting the interval in half. Proceed inductively, we get a nested decreasing sequences of intervals.

$I_1 \geq I_2 \geq I_3 \geq I_4 \geq \ldots$ such that I_k contains infinitely many terms of (x_n).

$$Length\ (I_1) = \frac{1}{2} Length\ (1)$$

$$Length\ (I_2) = \frac{1}{4} Length\ (1)$$

$$Length\ (I_k) = \frac{1}{2^k} Length\ (1) \longrightarrow 0$$

Let I_k be the interval of $[a_k, b_k]$

By Nested Interval Theorem: $\lim a_k = \lim b_k = c$

Claim:

There is a subsequence of (x_n) which converges to c. I_1 contains infinitely as many terms of x_n, so pick only n, such that $x_{n}, \in I$.

I_2 contains infinitely as many terms of (x_n), that is, there are infitely many n's such that $x_n, \in I$, and $n > n_1$, so find any n_2 such that $n_2 > n$, and $x_{n_2} \in I_2$.

Simularily, I_3 continues infinitely as many terms of (x_n), that is, there are infinitely many n's such that $x_n, \in I$, and $n > n_2$. Pick any n_3 such that, $x_{n_3} \in I_3$ and $n_2 > n$. Proceed inductively, we produce indices $n_1 < n_2 < n_3 < n_4 < n_5 < n_6 < \dots$ such that

$x_{n_k}, \in I_k$ for each k.

Thus (x_{n_k}) is a subsequence of (x_n) thus,

$x_{n_k} \Rightarrow x_{n_k} \in [a_k, b_k]$ which implies

$a_k \leq x_{n_k} \leq b_k$ *for every k*

Since $\lim a_k = \lim b_k = c$ we can conclude by Squeeze Theorem, that $\lim x_{n_k} = c$.

Example

Let $x_n = \sin n + 4\cos$

Claim:

(x_n) has a convergent subsequence because (x_n) is bounded:

$$|x_n| \leq |\sin n| + 4|\cos| \leq 1 + 4 - 1 = 5,\ x_n \in [-5,\ 5]$$

Cauchy's Criterion

Let (x_n) be a sequence, then TFAE:

1. x_n is convergent,

   $\exists c \quad \forall \varepsilon > 0 \quad \exists_{n_0} \quad \forall_n \geq n_0 \quad |x_n - c| < \varepsilon$

2. $\forall \varepsilon > 0 \quad \exists_{n_0} \quad \forall_m, \quad k \geq n_0$

$$|x_m - x_k| \ < \varepsilon$$

3. $\forall\varepsilon > 0 \ \exists_{n_0} \quad \forall_k \ \geq n_0 \qquad |x_k - x_{k_0}| < \varepsilon$

We use this as a tool to see if they converges into something. Now the second question would be, "what is the limit"?

Example

Let $x_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{10} + \ldots \frac{1}{2_n} = \sum_{k=1}^{n} \frac{1}{2^k}$

we know $x_n \longrightarrow 1$. Define sequence (y_n)

$$y_n = \frac{\sin 1}{2} + \frac{\sin 2}{4} + \frac{\sin 3}{8} + \frac{\sin 4}{16} + \ldots + \frac{\sin k}{2^k}$$

$= \sum_{k=1}^{n} \frac{\sin k}{2^k}$ does (y_n) converge?

Let $m > n,$ find $y_m - y_n$

$$y_m = \frac{\sin 1}{2} + \frac{\sin 2}{4} + \frac{\sin 3}{8} + \ldots + \frac{\sin k}{s^k} + \frac{\sin(k+1)}{2^{k+1}} + \frac{\sin m}{2^m}$$

$$y_m - y_n = \sin \frac{(n+1)}{2^{n+1}} + \ldots + \frac{\sin m}{2^m}$$

$$|y_m - y_n| \leq \frac{|\sin(n+1)|}{2^{n+1}} + \ldots + \frac{|\sin m|}{2^n} \leq \frac{1}{2^{m+1}} + \ldots \frac{1}{2^m}$$

Simularily $x_m - x_n = \frac{1}{2^{n+1}} + \ldots + 2\,\frac{1}{2^m}$ hence,

$|y_m - y_n| \leq |x_m - x_n|$.

Term of y sequence stay closer together than the x sequence, since (x_n) converges, for every $\varepsilon > 0$, there exists n_0 such that for all η, m, $\geq n_0$ we have $(x_m - x_n) < \varepsilon$ hence by Cauchy's Criteria, $y_n \ is\ convergent$

Cauchy's Criteria and Decimal Expansions

Recall: Monotone Convergence Theorem

i) if (x_n) is increasing then $\lim x_n = \sup x_n$

ii) if (x_n) is decreasing then $\lim x_n = \inf x_n$

iii) every monotone bounded sequences converges.

Recall: The Bolzano Weierstrass Theorem

Every bounded sequence has a convergent subsequence

Cauchy's Criterion TFAE

i) x_n is convergent,

ii) $\forall \varepsilon > 0 \quad \exists_{n_0} \quad \forall_k,\ m \geq n_0 \quad |x_k - x_m| < \varepsilon$

iii) $\forall \varepsilon > 0\ \exists_{n_0} \quad \forall_k \geq n_0 \quad |x_k - x_{n_0}| < \varepsilon$

Cauchy's Criterion Proof:

- Must prove all 3 statements above.

1. (i) $\Rightarrow$ (ii) :

Suppose x_n is convergent, then $x_n \longrightarrow c$ for some number.

Take an arbitrary $\varepsilon > 0$

Since $x_n \longrightarrow c$, we can find n_o such that $(x_n - c) < \dfrac{\varepsilon}{2}$ whenever $n \geq n_0$. Hence, if $k, m \geq n_0$, then $(x_n - c) < \dfrac{\varepsilon}{2}$ and $(x_m - c) < \dfrac{\varepsilon}{2}$

Hence,

$|x_k - x_m| = |x_k - c + c - x_m| \leq |x_k - c| \ + \ |c - x_m| <$

$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ = \ \varepsilon$ then $|x_k - x_m| < \varepsilon$ whenever $k, m \ \geq n_0$

2.

(ii) $\Rightarrow$ (iii) replace m with n_0 so $m = n_o$. Trivial, as (iii) is a special case of (ii)

3.

(iii) $\Rightarrow$ (i) suppose (iii) is satisfied.

First observe that (x_n) is bounded. Why? Fix any $\varepsilon > \ 0$, by (iii) find a tail ($(x_{n_0} \ , \ x_{n_0+1} \ , \ x_{n_0+2} \ , \ \ldots \ )$ in which every term (ii) within ε from x_{n_0} , that is, $\forall k \ \geq n_o \ \ |x_k - x_{n_0}| < \varepsilon$, This means the tail is contained in $(x_{n_0} - \varepsilon, \ x_{n_0} + \varepsilon, \ )$ it follows that the tail is bounded. Having only infinitely many terms, the head is bounded. Therefore, the entire sequence is bounded.

Second, using The Bolzano Weierstrass Theorem**,** x_n has convergent subsequences. Say, $x_n \ \longrightarrow \ c$.

Third, we claim $x_n \ \longrightarrow \ c$, fix $\varepsilon > \ 0$. By (iii), find n_0 such that $\forall k \ \geq n_o$,

$|x_k - x_{n_o}| < \frac{\varepsilon}{3}$ since $x_{n_i} \ \longrightarrow \ c$ we find i_o such that $\forall i \ \geq i_o$

$|x_{n_i} \ - c| < \frac{\varepsilon}{3}$.

Take i large enough so that $i \ \geq i_o \ \ and \, n_i \ \ \geq n_o$.

Then $|x_{n_i} - c| < \frac{\varepsilon}{3} \ and \ |x_{n_i} - x_{n_o}| < \frac{\varepsilon}{3}$ therefore, if

$k \geq n_o$ then

$$|x_k - c| = |x_k - x_{n_o} + x_{n_o} - x_{n_i} + x_{n_i} 0c| < \frac{\varepsilon}{3}$$

NOTE: used Triangle Inequality to find.

$$\leq \ |x_k - x_{n_o}| + |x_{n_o} - x_{n_i}| + |x_{n_i} - c|$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Hence, $|x_k - c| < \ \varepsilon$ whenever $k \geq n_o$ Hence, $x_n \longrightarrow c$

Examples

1. The sequence (-1,1,-1,1,-1,1…) diverges because it fails cauchy's criterem with $\varepsilon \ = \ 1$
2. Put $x_n = \sqrt{n}$, $(x_n) \ is \ divergent \ yet$

   $$x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \sqrt{n+1} - \sqrt{n}$$

   $$\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n}+1+\sqrt{n}} = \frac{n+1-n}{\sqrt{n}+1+\sqrt{n}} = \frac{1}{\sqrt{n}+1+\sqrt{n}}$$

   $\longrightarrow 0$. So $x_{n+1} - x_n \longrightarrow 0,$ so generally

   $x_{n+1} - x_n \longrightarrow 0,$ does not imply that (x_n) is convergent.

Binomial Formula

Theorem; let $a, \ b \in \mathbb{R}$ and $n \in \mathbb{N}$ then

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

$(a+b)^2 = a^2 + 2ab + b^2 \; \binom{2}{0} a^2 + \binom{2}{1} ab + \binom{2}{2} b^2$

NOTE:

$\binom{n}{k}$ Reads "$n \; chooses \; k$" and is called a binomial coefficient.

$$\binom{n}{k} \frac{n!}{k!\;(n-k)!} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-k) \cdot (n-k+1) \cdot \ldots \cdot n}{k!\;(n-k)!}$$

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k}$$ so essentially

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5 \; three \; terms \; decending}{1 \cdot 2 \cdot 3 \; three \; terms \; going \; up}$$

$\binom{n}{k}$ is the number of ways to pick or to choose k objects out of n.

Unique Complication:

Recall Bernoulli's inequality. For $a > 0$, Bernoulli's inequality follows easily for the binomial formula. $(1+a)^n = 1 + na + \frac{n(n-1)}{2} a^2 + \ldots$ hence $(1+a)^n > 1 + na$ giving you the ability to find Bernoulli's inequality.

Exercise

Proof 1 **-** Use induction.

Proof 2 - Expand everything $(a+b)^n$. Get a sum of “$cross\ products$” of the form $a^{n-k}b^k$. Observe that $a^{n-k}b^k$ occurs exactly $\begin{pmatrix} n \\ k \end{pmatrix}$ times .

Decimal Expansions

Take $123.45678910111213...$ and put $a_0 = 123$

$a_0 = 123,\ a_1 = 123.4,\ a_2 = 123.45,\ a_3 = 123.456,\ \ a_4 = 123.4567\ and\ so\ on$

Put these into clean rational numbers, such as

$$a_1 = 123.4 = \frac{1234}{10},\ a_2 = 123.45 = \frac{12345}{100},\ a_3 = 123.456 = \frac{123456}{1000},$$

And so on. a_n is increasing, $a_n \ \leq 124$ for each n and by MCT, $\lim a_n$ exists. Let $x = \lim a_n,\ x$ is the real number that corresponds to this decimal expansion.

<u>Caution.</u>

A real number may have two different decimal expansions. Example $0.\bar{9} = 1.\bar{0}$

a_n =0.999..9 n times = $1 - \left(\frac{1}{10}\right)^n$ $\lim a_n = 1\ hence\ 0.\bar{9} = 1$

Clearly we have $1.\bar{0} \ = \ 1$

Exercise:

Prove that every real number admits a decimal expansion.

Hint

- consider $a_0 \ = \lfloor x \rfloor$
- Consider $\lfloor 10x \rfloor, \lfloor 100x \rfloor, \lfloor 1000x \rfloor, \lfloor 10000x \rfloor,\ and\ so\ on$

In a similar way one can define binary expansion or expansion base in where m is any natural number.

University of Alberta

MATH117

Honors Calculus I

Fall 2017

Term Test 1

Exam Guide

Elementary Concepts from Set Theory

Numbers:

- R – denotes the set of all real numbers
- Q – denotes the set of all rational numbers i.e. fractions, p/q
- Z – denotes the set of all integers
- N – denotes the set of all natural numbers, these are the nonnegative integers. (Thus 0 is the smallest natural number)

Intervals:

- The **Open** interval is denoted by (a,b)
- The **Closed** interval is denoted by [a,b]
- The **Clopen** interval is denoted by [a,b) or (a,b]
- Infinity is not a real number
- The Infinite interval is (-infinity, infinity)

Sets:

- A set is a collection of elements
- A U B is the union of A and B and is the set of elements belonging to at least one of the sets
- A ∩ B intersection of A and B and is the set of elements belonging to both of the sets

Function:

- A function f from set A to set B (or simply function) is a rule that assigns to each element x of the domain set A to exactly one element called f(x) of the codomain set B
- All inputs of x must have a unique output
- Must pass the Vertical Line Test

Absolute Function:

- The absolute function is a piecewise function
- a if a >= 0
- -a if a < 0

One-to-One Function:

- A one-to-one function is a function that passes both the vertical line test and the horizontal line test
- A function is 1-1 if f(a) = f(b) => a = b

Example: f(x) = 2x^2 + 5

2a^2 + 5 = 2b^2 + 5
2a^2 = 2b^2
a^2 = b^2
a = +b or –b

Therefore, this function is not 1-1 as one value of a can produce two values of b.

Composition of Functions:

- Notation is f[g(x)]
- Can be thought of as nested functions
- The domain only exists for the intersection of the domains of the original functions

Operations with Functions:

- (f + g)(x) = f(x) + g(x)
- (f - g)(x) = f(x) - g(x)
- (fg)(x) = f(x) * g(x)
- (f / g)(x) = f(x) / g(x)

Piecewise functions:

- Functions that have different rules according to the domain that is given
- The absolute function is an example of a piecewise function

Floor and Ceiling functions:

- This function's range can only be integers
- The domain is still all real numbers

Transcendental Functions

Exponential Functions:

- In high school, the common form was y = a^x
- The more formal form is y = A(e)^(kx), where k is an element of real numbers
- The domain and range both consist of real numbers

Transcendental vs. Algebraic:

- Transcendental functions cannot be further broken down using algebraic operations
- Algebraic functions can be broken down using algebraic operations, such as addition, subtraction, multiplication, and division

Examples

- y = 2e^(2x) is a transcendental function because it meets the formal form
- y = e^(-x) + 12 is not a transcendental function because it includes the algebraic operation of addition
- y = -2(3^x) is a transcendental function
- ln(x – 1) > 0 => x – 1 > 1 => x > 3

Logarithmic Functions:

- Logarithmic and exponential functions are both one-to-one functions
- Logarithmic and exponential functions are inverses of each other
- "log" means a power we have to raise the base to in order to match the arguement

Examples

- $\log_2(6) – \log_2(15) + \log_2(20) = \log_2(6/15 * 20) = \log_2 8 = 3$
- log(sqrt(10)) = ½(log10) = ½

Trigonometric Functions:

- Standard positon is when the initial edge rests along the x-axis
- The terminal edge represents the arm that points to any point on the unit circle
- The unit circle has a radius of 1
- If the angle theta is in the standard position and its terminal edge intersects unit circle at some point (x,y) in the XY-plane, then the sin of theta is defined as y-coordinate of that point
- If the angle theta is in the standard position and its terminal edge intersects unit circle at some point (x,y) in the XY-plane, then the cos of theta is defined as x-coordinate of that point

Principal Values

- y = sinx has principal values between -pi/2 and pi/2
- y = cosx has principal values between 0 and pi

- y = tanx has principal values between -pi/2 and pi/2
- y = cscx has principal values between -pi/2 and pi/2
- y = secx has principal values between 0 and pi
- y = cotx has principal values between 0 and pi

Example => $4\sin^2 x - 1 = 0$

$$4\sin^2 x - 1 = 0$$
$$(2\sin x - 1)(2\sin x + 1) = 0$$
$$x_1 = pi/6 + 2pi * k, \text{ where k is an integer}$$
$$x_2 = 5pi/6 + 2pi * k, \text{ where k is an integer}$$
$$x_3 = 7pi/6 + 2pi * k, \text{ where k is an integer}$$
$$x_4 = 11pi/6 + 2pi * k, \text{ where k is an integer}$$

Angles:

- Angles are always displayed with respect to standard position
- The angle between x-axis and terminal edge is the reference angle

Formulas:

- tanx = sinx / cosx
- cscx = 1 / sinx
- secx = 1 / cosx
- cotx = 1 / tanx
- sin(-x) = -sinx
- cos(-x) = cosx
- tan(-x) = -tanx
- cos (pi/2 – x) = sinx = cos(pi/2 + x)
- sin (pi/2 – x) = cosx = sin(pi/2 + x)
- tan (pi/2 – x) = cotx = tan(pi/2 + x)
- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\cot^2 x + 1 = \csc^2 x$

Logic and Mathematic Thinking

More Trigonometric Formulas

- $sin(\alpha + \beta) = sin(\alpha)cos(\beta) + cos(\alpha)sin(\beta)$
- $sin(\alpha - \beta) = sin(\alpha)cos(\beta) - cos(\alpha)sin(\beta)$
- $cos(\alpha + \beta) = cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta)$
- $cos(\alpha - \beta) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$
- $sin(2x) = 2sin(x)cos(x)$
- $cos(2x) = cos^2(x) - sin^2(x) = 1 - 2sin^2(x) = 2cos^2(x) - 1$

Example #1

ln((x – 1) / 2) > 0
e^(ln((x – 1) / 2)) > e^0
(x – 1) / 2 > 1
x > 3

Example #2

sin(-3pi/4)
= sin (pi – pi/4)
= sin(-pi/4)
= -sin(pi/4)
= -sqrt(2)/2

Example #3

sin(105) = sin(60 + 45)
= sin60cos45 + sin45cos60
= sqrt(3)/2 * sqrt(2) / 2 + sqrt(2)/2 * ½
=sqrt(2)[sqrt(3) + 1] / 4

Example #4

$\sin(\cos^{-1}x)$ = sin(arccosx)
= sqrt(1 – x^2)

Example #5

$\sin^{-1}$(cos1/2)
= arcsin(cos1/2)
= pi / 3

Example #6

$\sin(\sec^{-1}(3 / x))$
= sin(arcsec(3 / x))
= sqrt(9 – x^2) / 3

Language of Proofs

- ∀ - "for all"
- ∃ = "there exists
- R – set of real numbers
- I – set of irrational numbers
- Q – set of rational numbers
- Z – set of integers
- N – natural numbers (nonnegative integers and 0)
- ⇒ - "implies"
- ⇔ - "equivalent", "iff"
- ∈ - "belong to" "is in …"
- ∉ - "is not in…"
- | - "such that"

Calculational Proof

Prove that $x^2 - y^2 = (x - y)(x + y)$

$$(x - y)(x + y)$$
$$= x^2 - xy + xy - y^2$$
$$= x^2 - y^2$$

QED – "which was to be demonstrated"

Direct Proof

Prove that the sum of odd and odd number is even.

Given: $x = 2m + 1$, $m \in z$; $y = 2n + 1$, $n \in z$

Show: $x + y = 2k$, $k \in z$

Proof:

$$x + y$$
$$= 2m + 1 + 2n + 1$$
$$= 2 + 2(m + n)$$
$$= 2(m + n + 1)$$

Since m, n, and 1 are all integers and the sum of all integers must be an integer also. Therfore the proof is complete.

QED

Proof by Contradiction

Let $f(x) = 2x + 5$. Prove that $f(x)$ is a one-to-one function.
Given: $f(x) = 2x + 5$

Show: $\exists$ x_1 and x_2 | $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

Proof: Suppose that $f(x) = 2x + 5$ is not one-to-one. Then there exists such x_1 and x_2 that $f(x_1) = f(x_2)$ for some x_1 not equal to x_2

$$2x_1 + 5 = 2x_2 + 5$$
$$2x_1 = 2x_2$$
$$x_1 = x_2$$

The conclusion shows that for $f(x) = 2x + 5$, $f(x_1) = f(x_2)$ iff $x_1 = x_2$, which contradicts the supposition. Therefore, the supposition is incorrect and $f(x) = 2x + 5$ is a one-to-one function.

QED

Real Numbers, Absolute Values & Completing the Square)

Real Numbers

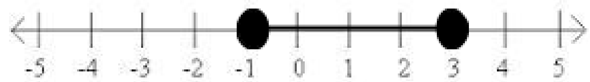
Real Number: a value that represents a quantity along line.

There are three ways to depict *real numbers* but essentially, they all mean the same thing.

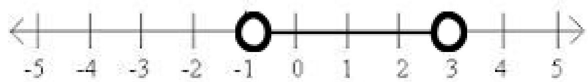
- Interval Notation: using brackets and parenthesis.
  - [] - brackets indicate that the interval includes the maximum limit as a solution to the expression.
    - Example: [-1,3] shows that the solutions to the expression are all real numbers in between -1 and 3 including -1 and 3.
  - () - parentheses indicate that the intervals do not include the maximum limit as a solution to the expression.
    - Example: (-1,3) shows that the solutions to the expression are all real numbers in between -1 and 3 but not including -1 and 3.
  - Both a bracket and parenthesis can be used at one time.
    - Example: (-1,3] shows that the solutions range from -1 to 3 including 3 but not -1.
    - Example: Vice versa, [-1,3) shows that the solutions range from -1 to 3 including -1 but not 3.
  - Interval notation should be written from smallest to largest number. [3,-1] is an incorrect format.

- Graphing
  - In the following examples, you will see two different types of circles; solid, and unfilled.
    - A solid dot indicates that the number it is above is included in the solutions.
    - An unfilled dot indicates that the number it is above is not included in the solutions.

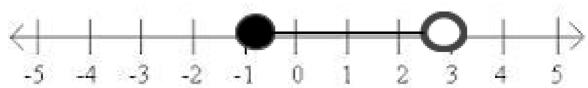
- The following number lines show the same interval notation used above in a different way.
  - [-1,3]



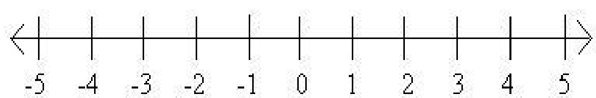
  - (-1,3)



  - [-1,3)

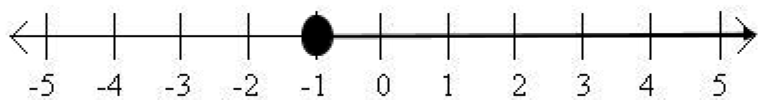


  - (-1,3]



- <u>Inequality Notation</u>
  - < - less than
  - > - greater than
  - ≤ - less than or equal to
  - ≥ - greater than or equal to
    - $-1 \leq x \leq 3$ means that the solutions range from -1 to 3 including -1 and 3.
    - $-1 < x < 3$ means that the solutions range from -1 to 3 not including -1 and 3.
    - $-1 < x \leq 3$ means that the solutions range from -1 to 3, including 3 but not -1.
    - $-1 \leq x < 3$ means that the solutions range from -1 to 3, including -1 but not 3.

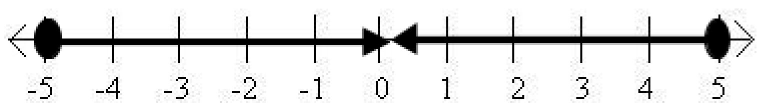
- Infinite Solutions
    - If there are an infinite number of solutions to an expression
        - An open parenthesis should be used instead of square brackets because brackets are used to enclose numbers and an infinite amount of solutions will never end.
            - Interval Notation: [-1, ∞)
            - Graph



            - Inequality Notation: $-1 \leq x$

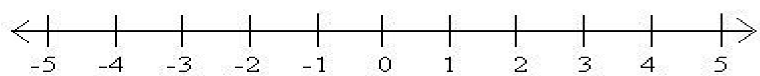
Absolute Values

- Absolute value: the distance between x, a point on the number line, and zero.



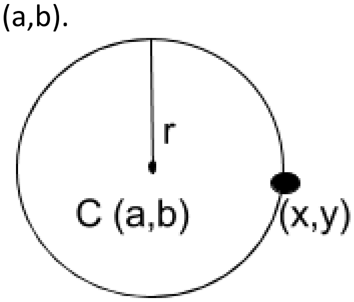
- The absolute value of 5 is 5 because it is five spaces away from point zero.
- The absolute value of -5 is also 5 because it is also five spaces away from point zero. Direction does not matter, absolute value is strictly quantitative.
- If |x| = 7, then the two solutions are x= -7 and x=7 because they are both equidistant from zero.
- If |x| = 0, then the only solution is x= 0.
- If |x| = -7 , then there are no solutions because there is no such thing as negative distance.

- To solve equations involving absolute values, you need to make sure you create two separate equations and solve.
  - Example: |x-3| ≤ 2
    - The two separate equations would be x - 3 = 2 and x - 3 = -2. This is because there are two values where x is equidistant and from zero. Then, you can solve algebraically.
    - x - 3 = 2 x - 3 = -2
      x= 5 **x = 1**
    - Interval Notation: [1,5]
    - Graph:



<u>Completing the Square</u>

- All sets of (x,y) all have the same distance, r, from the center (a,b).
- Distance Formula: $(x-a)^2+(y-b)^2=r^2$, with center (a,b).
- Important foil patterns to know and recognize:
  - $(c+d)^2 = c^2+2\,cd+d^2$
  - $(c-d)^2 = c^2-2cd+d^2$
  - $(c^2-d^2) = (c-d)(c+d)$



- Completing the Square is a method used to find a circle's center and radius when the information is given in another format, and not the distance formula format.
  - Example: Find the center and radius of a circle with the equation $4x^2+4y^2-4x+8y-5=0$.
    - To complete the square:
      - Separate the variables. $4x^2-4x+4y^2+8y-5=0$
      - Factor.
        $4(x^2-x)\ +\ 4(y^2+2y)\ -\ 5=0$

- Identify the coefficient of x alone. *The coefficient is 1 because there isn't a number in front of x.*
- Half the coefficient. *(½)*
- Square it. $(1/2)^2 = ¼$
- Add it and subtract it into the factored eq. $4(x^2 - x + 1/4 - 1/4)$
- Factor. $4((x - 1/2)^2 - 1/4)$
- Foil. $4(x - 1/2)^2 - 1$
- Repeat the squaring method for the y variable.
    - Factor. $4(y^2 + 2y)$
    - Identify the coefficient of only the y variable. *The coefficient is 2.*
    - Half the coefficient. (1)
    - Square the coefficient. $(1)^2 = 1$
    - Add and subtract 1 into the factored eq. $4(y^2 + 2y + 1 - 1)$
    - Factor. $4(y + 1)^2 - 1$
    - Foil. $4(y + 1)^2 - 4$

- Replace the old expressions for x and y with the ones you created.
    - Original Equation: $4(x^2 - x) + 4(y^2 + 2y) - 5 = 0$
    - Combined equation: $4(x - 1/2)^2 - 1 + 4(y + 1)^2 - 4 - 5 = 0$
    - Combine like terms. $4(x - 1/2)^2 + 4(y + 1)^2 - 10 = 0$
    - Add 10 to both sides. $4(x - 1/2)^2 + 4(y + 1)^2 = 10$
    - Divide both sides by 4. $(x - 1/2)^2 + (y + 1)^2 = 10/4$

- Congratulations! You've completed the square. Compare this with the distance formula given to you at the beginning of this section. They are the same!

- Now, you can find the center and the radius.
    - To find the center, solve for x and y.
        - $x - 1/2 = 0$

          $x = 1/2$
        - $y + 1 = 0$

          $y = -1$
        - The center of this circle has the coordinates (½,-1).

- Find the radius by finding the square root of 10/4. ($\sqrt{5}/2$ or ≈1.58)

Mathematical Induction

Mathematical Induction (MI) is an extremely important tool in Mathematics.

First of all you should never confuse MI with Inductive Attitude in Science. The latter is just a process of establishing general principles from particular cases.

MI is a way of *proving* math statements for all integers (perhaps excluding a finite number) says:

Statements proved by math induction all depend on an integer, say, n. For example,

(1) $1 + 3 + 5 + ... + (2n - 1) = n^2$
(2) If $x_1, x_2, ..., x_n > 0$ then $(x_1 + x_2 + ... + x_n)/n \geq (x_1 \cdot x_2 \cdot ... \cdot x_n)^{1/n}$

Note: In this case, *n* is an arbitrary integer.

It is convenient to talk about a statement P(n). For (1), P(1) says that $1 = 1^2$ which is incidentally true. P(2) says that $1 + 3 = 2^2$, P(3) means that $1 + 3 + 5 = 3^2$. And so on. These particular cases are obtained by substituting specific values 1, 2, 3 for n into P(n).

Assume you want to prove that for some statement P, P(n) is true for all n starting with n=1.

The *Principle* of *Math Induction* states that, to this end, one should accomplish just two steps:

1. Prove that P(1) is true.
2. Assume that P(k) is true for some k. Derive from here that P(k+1) is also true.

The idea of MI is that a finite number of steps may be needed to prove an infinite number of statements P(1), P(2), P(3), ...

Let's prove (1). We already saw that P(1) is true. Assume that, for an arbitrary k, P(k) is also true, i.e. $1 + 3 + ... + (2k-1) = k^2$. Let's derive P(k+1) from this assumption. We have:

$$\begin{aligned} 1 + 3 + ... + (2k-1) + (2k+1) &= [1 + 3 + ... + (2k-1)] + (2k+1) \\ &= k^2 + (2k+1) \\ &= (k+1)^2 \end{aligned}$$

Which exactly means that P(k+1) holds. (For 2k+1 = 2(k+1)-1.) Therefore, P(n) is true for all n starting with 1.

Intuitively, the inductive (second) step allows one to say, look P(1) is true and implies P(2). Therefore P(2) is true. But P(2) implies P(3). Therefore P(3) is true which implies P(4) and so on. Math induction is just a shortcut that collapses an infinite number of such steps into the two above.

In Science, *inductive attitude* would be to check a few first statements, say, P(1), P(2), P(3), P(4), and then assert that P(n) holds for all n. The inductive step "P(k) implies P(k + 1)" is missing. Needless to say nothing can be proved this way.

Remark

1. Often it's impractical to start with n = 1. MI applies with any starting integer n_0. The result is then proved for all n from n_0 on.
2. Sometimes, instead of 2., one assumes 2':

Assume that P(m) is true for all m < (k + 1).

Derive from here that P(k+1) is also true. The two approaches are equivalent, because one may consider statement Q: Q(n) = P(1) and P(2) and ... and P(n), so that Q(n) is true iff P(1), P(2), ..., P(n) are all true.

This variant goes by the name of *Complete Induction* or *Strong Induction*.

Additional Explanation:

Below, $|S|$ will denote the number of elements in a finite (or empty) set S. So, for example, $|\{\}| = 0$ and $|\{0\}| = 1$. The empty set {} is denoted Ø.

Sum Rule

If A and B are disjoint, i.e., if $A \cap B = \emptyset$, then

(1) $|A \cup B| = |A| + |B|$.

Comment: behind the set-theoretic symbolism stands a simple fact without which ***counting*** would be impossible: *it does not matter how you count*, i.e., as long as you do not make a mistake of, say, missing an object or counting an object twice. It says this: if before counting objects one splits them into two groups and then counts the elements of one of the groups before proceeding to count the elements of the other, the result will be the same - the total number of objects to be counted. (Naturally, it does not depend on how the objects have been split into two groups.)

Example 1

In a class of 30 students, there are 16 boys and 14 girls (16 + 14 = 30). Of these, 23 persons wear pants and only 7 wear skirts (23 + 7 = 30). On the last exam 20 students received a passing grade, while 10 failed (20 + 10 = 30).

By ***induction***, the sum rule is easily extended to any finite number of mutually disjoint sets:

(1') $|A \cup B \cup C \cup D \ldots| = |A| + |B| + |C| + |D| + \ldots$

Example 2

An electronic book of 472 pages has been stored in separate files - 1 file per page - in two folders. One folder contained 305 files, the other 167 files (305 + 167 = 472.)

Product Rule

For a ***direct product*** A×B of two finite sets A and B,

(2) $|A\times B| = |A|\times|B|$.

Comment: By induction the rule extends to any finite number of sets:

$$|A \times B \times C \times D \ldots| = |A| \times |B| \times |C| \times |D| \ldots \quad (2')$$

An essential point here is how the tuples of objects are formed: an object is picked out from one of the given sets regardless of which objects have been drawn from the other sets. Why the rule is called *sequential*? Because in a tuple, the objects (components) are ordered: there is the first one, the second, and so on.

Example 3

The are two drawers. One contains 12 shirts, the other 7 neckties. There are $84 = 12 \times 7$ ways to combine a shirt and a necktie.

It is possible to examine the drawers sequentially: first-second, first-second... It is also possible to form combinations using two hands: left for a shirt, right for a necktie. As long as all possible combinations shirt/necktie have been counted, the exact procedure is of no consequence.

Example 4

A test consists of 6 mutiple-choice questions. Each question has 4 possible answers. There are

$$4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6$$

ways to answer all 6 questions.

Counting ***poker hands* provides multiple additional examples.

Proof By Induction

Please note that:
WTS = "want to show" that
[] = substitute for the square that indicates that the proof is finished

Q1. Write a good proof for the following theorem:
The sum of two odd numbers is even.

A1. Proof:
Let n, m ∈ Z
I assume n, m are odd
WTS n + m is even. This can also be written as WTS ∍ x ∈ Z s.t. n + m = 2x

- n is odd. By definition, ∍ a ∈ Z s.t. n = 2a + 1
- m is odd. By definition, ∍ b ∈ Z s.t. m = 2b + 1

Then n + m = 2a + 1 + 2b + 1 = 2a + 2b + 2
n + m = 2(a+b+1)
n + m = 2x
Since a + b + 1 ∈ Z, this proves n + m is even. []

Q2. Variations on Induction #1
Let S_n be a statement that depends on a positive integer *n*.
In each of the following cases, which statements are guaranteed to be true?

1. We have proven:
   - S_3 is true.
   - $\forall n \geq 1$, S_n is true $\Rightarrow$ S_{n+1} is true.
2. We have proven:
   - S_1 is true.
   - $\forall n \geq 3$, S_n is true $\Rightarrow$ S_{n+1} is true .
3. We have proven:
   - S_1 is true.
   - $\forall n \geq 1$, S_n is true $\Rightarrow$ S_{n+3} is true .
4. We have proven:
   - S_1 is true.
   - $\forall n \geq 1$, S_{n+1} is true $\Rightarrow$ S_n is true .

A2. For each case, we know that the following are true:

1. S_3, S_4, S_5, S_6, …
2. S_1.
3. S_1, S_4, S_7 …
4. S_1.

Q3. Variations on Induction #2

We want to prove that $\forall n \geq 1$, S_n is true

So far we have proven

- S_1 is true.
- $\forall n \geq 1$, S_n is true $\Rightarrow S_{n+3}$ is true.

What else do we need to do?

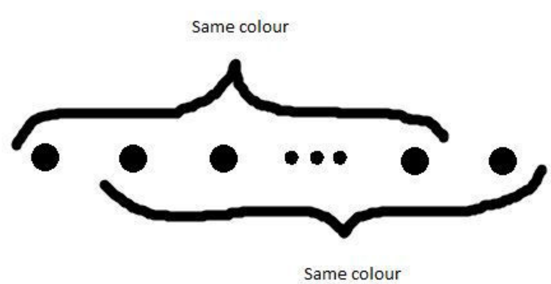
A3. We also need to prove S_2 and S_3.

Q4. What is wrong with this proof by induction?

Theorem: $\forall N \in \mathbb{N}$, in every set of N cars, all the cars are of the same colour.

Proof:

- Base case: It is clearly true for N = 1.
- Induction step.
  - Assume it is true for N. I'll show it is true for N + 1.
  - Take a set of N + 1 cars. By induction hypothesis:
    - The first N cars are of the same colour.
    - The last N cars are of the same colour.



    - Hence the N + 1 cars are all of the same colour.

A4. We've proven that S_1 is true, and that

- $\forall N \geq 2$, $S_N \Rightarrow S_{N+1}$ is true
  - note that this is *different* from proving that simply S_{N+1} is true
- The only error: The induction step in the proof above must start at n = 2, because of what we've proven and know (i.e., "$\forall N \geq 2$, $S_N \Rightarrow S_{N+1}$ is true").

Q5. What is wrong with the following description of proof by induction?

A proof by induction has 3 parts:

1. Base case: Prove the statement is true for 1 (for example)
2. Induction hypothesis: Assume the statement is true for n.
3. Induction step: Prove the statement is true for n + 1.

A5. A proof by induction actually has 2 parts:

1. Initial proof (e.g., true for n = 1)
2. Induction (e.g., true for n ⇒ true for n + 1)
    - Again, note that there is a difference between…
        1. $S_n \Rightarrow S_{n+1}$ is true
        2. S_{n+1} is true

Watch:

- For Friday: 2.4 (absolute
- For Monday: 2.1, 2.2, 2.3

Binomial functions & Taylor series

The Taylor theorem

- The taylor polynomial and taylor series are obtained from a generalization fo the Mean Value Theorem.
- If f: [a,b] approaching R is differentiable, then there exists a c on [a,b] such that (f(b) - f(a))/(b-a) = f'(c), which is equivalent to:
  - f(b) = f(a) + f'(c)(b-a)
- Theorem (Taylor's theorem):
  - If f: [a,b] approaching R is (n+1) times continuously differentiable, then there exists c on (a,b) such that
    - f(b) = f(a) + f'(a)(b-a) + f''(a)/2 * $(b-a)^2$ + ….. + $\frac{f^n a}{n!}(b-a)^n + \frac{f^{n+1} c}{(n+1)!}(b-a)^{n+1}$
- Remark:
  - The taylor theorem is usually applied for a fixed point a, while the point b = x is used as an independent variable.
  - f(x) = f(a) + f'(a)(x-a) + …. + $\frac{f^n a}{n!}(x-a)^n + R_n x$
  - The remainder function is given by:
  - $R_n(x) = \frac{f^{n+1} c(x)}{(n+1)!}(x-a)^{n+1}$ *with c on* (a, x)
- The point c on (a,x) is also dependent on x.
- We can use the taylor polynomial to write that f(x) = $T_n(x) + R_n(x)$

The Binomial Function

- Definition:
  - The binomial function is a function of the form:
    - $f_m(x) = (1+x)^m$, *with m on R.*
- Remark:
  - If m is a positive integer, then the binomial function f_m is a polynomial, therefore the Taylor series is the same polynomial, hence the Taylor series has only the first m + 1 terms non zero.
  - If m is not a positive integer, then the taylor series of the binomial function has infinitely many non-zero terms.

Binomial Theorem:

- The Taylor series for the binomial function f_m (x) = $(1+x)^m$ with m not a positive integer, converges for |x| < 1, and is given by:
  - $T(x) = 1 + \sum_{n=1}^{infinity} (m\ n) x^n$
- With the binomial coefficients (m 1) = m, (m 2) =
  - $\frac{m(m-1)}{2!}$
  - (m n) = $\frac{m(m-1)...(m-(n-1))}{n!}$

Evaluating non elementary integrals

- Remark: non elementary integrals can be evaluated by integrating term by term the integrand taylor series.

Euler Identity

- The taylor definition suggests the definition:
  - $e^{i\theta} = cos\theta + isin\theta$

Continuity, Intervals & Bounds

The Extreme Value Theorem

- If f(x**) is continuous on closed interval [a,b],** then there exist some values **M** and **m** in the interval [**a,b]** such that f(M) **is the maximum value of** f(x) **on [a,b]** and f(m) **is the minimum value of** f(x) **on [a,b]**

The Intermediate Value Theorem

- If f(x) **is continuous on closed interval [a,b], then for any K strictly between** f(a) **and** f(b) **there exists at least one** c in the interval (a,b**) such that** f(c) = K

Corollary to ITV

- For any k in the interval (f(b), f(a)) there exists a c in the interval (a, b) such that f(c) = K. If sign f(b) < 0, sign f(a) > 0, then we can always find c in the interval (a,b) such that k = 0.
- A function can change its sign only at roots or points of discontinuities

Example

Show that f(x) = x^3 – x – 1 has a root on interval [1, 2]

Conditions:

- f(x) – continuous as every polynomial is on the interval (-infinity, infinity)
- f(x) is given on closed interval [1,2]
- f(x) has opposite signs at the endpoints of the interval and according to IVT there exists c in the interval (1,2) such that f(c) = 0

Example of Finding Roots by Bisection

(1,2)

c1 = (2 + 1) / 2 = 1.5
f(1.5) > 0
f(1) < 0
So the root is in (1, 1.5)

(1,1.5)

c2 = (1.5 + 1) / 2 = 1.25
f(1.25) > 0
f(1) < 0
So the root is in (1, 1.25)

(1,1.25)

c3 = (1.25 + 1) / 2 = 1.125
f(1.125) < 0

f(1) < 0
So the root is in (1.125, 1.25)

(1.125,1.25)

c3 = (1.25 + 1.125) / 2 = 1.1875
f(1.18755) < 0
f(1) < 0
So the root is in (1.1875, 1.25)

Therefore, we can conclude that the root is approximately at 1.2.

Supremum in Sets

S in the subset of R is bounded from above if there exists M that for all x is the set of M, x <= M

- M – upper bound for S and [M, infinity) are also upper bound for S
- Supremum can also be known as least upper bound (lub)

Examples

- {1, 2, 3, 4} M = 4
    - lub = 4
- {0,1,2}
    - lub = 2
- (0, 2)
    - lub = 2
    - Max does not exist since 2 is not part of the domain
- (-infinity, 5)
    - bounded from above by any number [5, infinity)
    - Max does not exist since 5 is not part of the domain

Infimum in Sets

S in the subset of R is bounded from below if there exists m that for all x is the set of m, x >= m

- m – lower bound for S and (-infinty, m] are also lower bound for S
- Infimum can also be known as greatest lower bound (glb)

Examples

- {1, 2, 3, 4} m = 1
    - glb = 1
- {0,1,2}
    - glb = 0
- (0, 2)
    - lub = 0
    - Min does not exist since 0 is not part of the domain
- (-infinity, 5)
    - No glb or min since it is inbounded from below

GLB & LUB Axiom

Review of Supremum and Infimum

Supremum in Sets

S in the subset of R is bounded from above if there exists M that for all x is the set of M, x <= M

- M – upper bound for S and [M, infinity) are also upper bound for S
- Supremum can also be known as least upper bound (lub)

Infimum in Sets

S in the subset of R is bounded from below if there exists m that for all x is the set of m, x >= m

- m – lower bound for S and (-infinty, m] are also lower bound for S
- Infimum can also be known as greatest lower bound (glb)

Least Upper Bound Axiom

- Every nonempty set of real numbers that is bounded from above has a supremum
- Axioms do not need to be proven
- They are meant to be clear and understood by all mathematicians

Theorem

If Sup S = M and epsilon > 0, then there exists at least one number s in S such that m - ε < s <= M

Sup S = M => s <= M

We need to prove that M - ε < s.

Two choices: either we have M - ε or we don't

Assume that there is no M - ε < s => x subset of S, x <= M - ε, but if x <= M - ε, then M - ε is Sup S, which contradicts the hypothesis

Greatest Lower Bound Axiom

- Every nonempty set of real numbers that is bounded from below has an infimum
- Axioms do not need to be proven
- They are meant to be clear and understood by all mathematicians

Theorem

If Inf S = m and epsilon > 0, then there exists at least one number s in S such that $m < s <= m + \varepsilon$

Inf S = m => s >= m

We need to prove that $m + \varepsilon > s$.

Two choices: either we have $M + \varepsilon$ or we don't

Assume that there is no $M + \varepsilon > s$ => x subset of S, $x >= m + \varepsilon$, but if $x >= m + \varepsilon$, then $m + \varepsilon$ is Inf S, which contradicts the hypothesis

Monotone Sequences

Test for Monotonicity

- Let $\{a_n\}$ be an infinite sequence. We say $\{a_n\}$ is strictly increasing if it satisifes any one of the following:
  1. Difference test: if $a_{n+1} - a_n > 0 \forall n \in \mathbb{N}$, then $\{a_n\}$ is strictly increasing
  2. Derivative test: let $a_n = f(n)$. If $f(n)$ is differentiable on $[1, \infty)$, and $f(n)' > 0$ on $[1, \infty)$, then $\{a_n\}$ is strictly increasing
  3. Ratio test: if $a_n > 0 \forall n \in \mathbb{N}$, and $\dfrac{a_n + 1}{a_n} > 1 \forall n \in \mathbb{N}$, then $\{a_n\}$ is strictly increasing
- We have similar tests for increasing, strictly decreasing, and decreasing sequences
  - Increasing: replace $>$ with $\geq$
  - Strictly decreasing: replace $>$ with $<$
  - Decreasing: replace $>$ with $\leq$

Bounded Monotone Convergence Theorem

- Let $\{a_n\}$ be a sequence. If $\{a_n\}$ is bounded and monotone, then $\{a_n\}$ converges
- In particular,
  1. $\{a_n\}$ bounded above, and (strictly) increasing, OR
  2. $\{a_n\}$ bounded below, and (strictly) decreasing
- Proof of #1:
  - Suppose $\{a_n\}$ is strictly increasing (#1), and bounded above (#2)
  - Want to show $\lim\limits_{n\to\infty} a_n$ exists, i.e. $\{a_n\}$ converges
  - Want to show $\exists l \in \mathbb{R}, \forall \epsilon > 0, \exists N > 0$ such that $\forall n \in \mathbb{N}$, if $n > N$, then $|a_n - l| < \epsilon$
  - Let $A = \{a_n | n \in \mathbb{N}\} \subset \mathbb{R}$
  - Notice $A \neq \varnothing$, because $a_1 \in A$
  - Moreover, A is bounded above, by #2
  - $\therefore$ By completeness axiom, we know that $sup(A)$ exists
  - Let $\alpha = sup(A)$
  - $l = \alpha$; choose $l = \alpha = sup(A) \in \mathbb{R}$
  - Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}, N > 0$ such that $\alpha - \epsilon < a_N$ by definition of $sup(A)$

- Suppose $n > N$
- $\implies \alpha - \epsilon < a_N$, by choice of N with $sup(A)$
- $a_N < a_n$, as $N < n$ by #1
- $a_n \leq \alpha$, because α is an upper bound of a_n
- $\implies \alpha - \epsilon < a_n < \alpha + \epsilon$
- $\iff -\epsilon < a_n - \alpha < \epsilon$
- $\iff |a_n - \alpha| < \epsilon$
- $\therefore \{a_n\}$ converges to α as wanted

Good Luck!!

Summary

1 Real Numbers

Induction: Show first case and that case n implies case $n+1$.

Binomial Theorem: For $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

$\mathbb{R}$ is complete: Every *nonempty* subset of $\mathbb{R}$ with an upper bound has a *least* upper bound in $\mathbb{R}$.

2 Limits

Limit: $\lim_{x \to a} f(x) = L$ means for every $\epsilon > 0$ we can find a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

One-Sided Limit: $\lim_{x \to a+} f(x) = L$ means for every $\epsilon > 0$ we can find a $\delta > 0$ such that

$$x \in (a, a + \delta) \Rightarrow |f(x) - L| < \epsilon.$$

Vertical Asymptote: $\lim_{x \to a+} f(x) = \infty$ means for every $M > 0$ we can find a $\delta > 0$ such that

$$x \in (a, a + \delta) \Rightarrow f(x) > M.$$

Horizontal Asymptote: $\lim_{x \to \infty} f(x) = L$ means for every $\epsilon > 0$ we can find a number N such that

$$x > N \Rightarrow |f(x) - L| < \epsilon.$$

Infinite Limit: $\lim_{x \to \infty} f(x) = \infty$ means for every $M > 0$ we can find a number N such that

$$x > N \Rightarrow f(x) > M.$$

Cauchy Criterion: $\lim_{x \to a} f(x)$ exists $\iff$ for every $\epsilon > 0$ we can find a $\delta > 0$ such that $x, y \in (a - \delta, a) \cup (a, a + \delta) \Rightarrow |f(x) - f(y)| < \epsilon$.

Sequences: $a_n = f(n)$ is a function on the domain $\mathbb{N}$.

Cauchy Criterion for Sequences: $\lim_{n\to\infty} a_n$ exists $\iff$ for every $\epsilon > 0$ we can find a number N such that $m, n > N \Rightarrow |a_m - a_n| < \epsilon$.

Convergent $\Rightarrow$ Bounded.

Monotone Sequences: Convergent $\iff$ Bounded.

Convergent $\iff$ All Subsequences Convergent.

Bounded $\Rightarrow \exists$ Convergent Subsequence.

Limit Properties: $\lim_{x\to a}(f(x)+g(x))\exists = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ **if** these individual limits exist.

Continuity: $\lim_{x\to a} f(x) = f(\lim_{x\to a} x) = f(a)$.

Intermediate Value Theorem: If

(i) f is continuous on [a,b],
(ii) $f(a) < y < f(b)$,

then there exists a number $c \in (a, b)$ such that $f(c) = y$.

Closed intervals: Continuous $\Rightarrow$ bounded; maximum and minimum values achieved.

Theorems

Theorem 1.1 (Binomial Theorem): *For all $n \in \mathbb{N}$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Theorem 2.1 (Convergent $\Rightarrow$ Bounded): *A convergent sequence is bounded.*

Theorem 2.2 (Properties of Limits): *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Let $L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$. Then*

(a) $\lim_{n\to\infty}(a_n + b_n) = L + M$;

(b) $\lim_{n\to\infty} a_n b_n = LM$;

(c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ *if $M \neq 0$.*

Corollary 2.2.1 (Case $L \neq 0$, $M = 0$): Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. If $\lim_{n\to\infty} a_n \neq 0$ and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n}$ does not exist.

Theorem 2.3 (Monotone Sequences: Convergent $\iff$ Bounded): *Let $\{a_n\}$ be a monotone sequence. Then $\{a_n\}$ is convergent $\iff$ $\{a_n\}$ is bounded.*

Theorem 2.4 (Convergent $\iff$ All Subsequences Convergent): *A sequence $\{a_n\}_{n=1}^{\infty}$ is convergent with limit L $\iff$* **each** *subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ is convergent with limit L.*

Theorem 2.5 (Bolzano–Weierstrass Theorem): *A bounded sequence has a convergent subsequence.*

Theorem 2.6 (Cauchy Criterion): *$\{a_n\}$ is convergent $\iff$ $\{a_n\}$ is a Cauchy sequence.*

Theorem 3.1 (Equivalence of Function and Sequence Limits): $\lim_{x\to a} f(x) = L \iff f$ *is defined near a and* **every** *sequence of points $\{x_n\}$ in the domain of f, with $x_n \neq a$ but $\lim_{n\to\infty} x_n = a$, satisfies $\lim_{n\to\infty} f(x_n) = L$.*

Corollary 3.1.1 (Properties of Function Limits): Suppose $L = \lim_{x\to a} f(x)$ and $M = \lim_{x\to a} g(x)$. Then

(a) $\lim_{x\to a}(f(x) + g(x)) = L + M$;

(b) $\lim_{x\to a} f(x)g(x) = LM$;

(c) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$.

University of Alberta

MATH 117

Honors Calculus I

Fall 2017

Term Test 2

Prof: Vladimir Troitsky

Exam Guide

Topics Included

Sequences

Limit of a Sequence

Function: A function f is a rule maps only one unique y-value for every x-value.

Domain: All x-values for which the function exists, left and right when looking at a graph

Range: All y-values for which the function exits, up and down when looking at a graph

Sequence: A list of numbers that follow a pattern. You can also think of it as a function on the domain of $\mathbb{N}$ where each term a_n is found by function $f(n)$.

Convergent: If a sequence gets closer and closer to a particular value then we say that the sequence is convergent and $\lim_{n\to\infty} a_n = L$

Divergent: If the sequence doesn't get closer to a particular value (meaning it will continue to $-\infty$ or ∞) then we say that the sequence diverges

Uniqueness of Limits: If $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$ then $L_1 = L_2$

Bounded Sequence: A sequence is bounded if the $f(n)$ is finite. A convergent sequence is always bounded, but a bounded sequence is not always convergent

Unbounded: An unbounded sequence is divergent

Properties of Limits: Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Let $L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$. Then:

1. $\boldsymbol{\lim_{n\to\infty}(a_n + b_n) = L + M}$
2. $\boldsymbol{\lim_{n\to\infty}(a_n b_n) = LM}$
3. $\boldsymbol{\lim_{n\to\infty}\frac{a_n}{b_n)} = \frac{L}{M}}$ **if** $\boldsymbol{M \neq 0}$

Example A: Determine whether the sequence converges. If it is convergent, find the limit.

$$\{a_n\} = \left\{\frac{n^2 - 3n + 2}{1 - 2n^2}\right\}$$

Solution:

$\lim_{n\to\infty}\frac{n^2-3n+2}{1-2n^2}$ **Take the limit as n approaches infinity**

$\lim_{n\to\infty}\frac{1-\frac{3n}{n^2}+\frac{2}{n^2}}{\frac{1}{n^2}-2}$ **Divide each term by n^2**

$\lim\limits_{n\to\infty} \frac{1-\frac{3}{n}+\frac{2}{n^2}}{\frac{1}{n^2}-2} = -\frac{1}{2}$ **The sequence converges and the limit is $-\frac{1}{2}$**

Example B: Determine whether the sequence converges. If it is convergent, find the limit.

$$\{a_n\} = \left\{\frac{n^2 - \frac{3}{n}}{n}\right\}$$

Solution:

$\lim\limits_{n\to\infty} \frac{n^2-\frac{3}{n}}{n}$ **Take the limit as n approaches infinity**

$\lim\limits_{n\to\infty} 1 - \frac{3}{n^3} = 1$ **Divide the numerator by n^2 and take the limit (numerator only)**

$\lim\limits_{n\to\infty} \frac{1}{n} = 0$ **Divide the denominator by n^2 and take the limit (denominator only)**

Since the limit of the numerator is 1 and the limit of the denominator is 0 the limit does not exist and the sequence diverges.

Monotone Sequences

Increasing Sequence: A sequence is increasing if each successive term is greater than or equal to the previous term. It is **strictly increasing** if each successive term is strictly greater than the previous term.

Decreasing Sequence: A sequence is decreasing is each successive term is less than or equal to the previous term. It is **strictly decreasing** if each successive term is strictly less than the previous term.

Monotone: A sequence is monotone if it either an increasing sequence or a decreasing sequence

Bounded/Convergent Monotone Sequence: A convergent monotone sequence is always bounded and a bounded monotone sequence is always convergent

Example A: Show that the sequence $\{a_n\} = \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is strictly increasing.

Solution:

$\frac{n+1}{n+2} - \frac{n}{n+1}$	**Subtract the nth term from the $n+1$ term**
$\frac{(n+1)^2}{(n+2)(n+1)} - \frac{n(n+2)}{(n+2)(n+1)}$	**Find the common denominator**
$\frac{n^2+2n+1-(n^2+2n)}{(n+2)(n+1)}$	**Subtract the fractions**
$\frac{1}{(n+2)(n+1)} > 0$, when $n \geq 1$	**Therefore the sequence is strictly increasing**

Example B: Show that the sequence $\{a_n\} = \left\{\frac{3n^3-4}{2n^3+3}\right\}_{n=1}^{\infty}$ is bounded.

Solution:

$\frac{3(1)^3-4}{2(1)^3+3} = -\frac{1}{5}$	**This is the smallest value of the sequence and shows that it is bounded below.**
$\lim_{n\to\infty} \frac{3n^3-4}{2n^3+3}$	**Take the limit as n approaches infinity**
$\lim_{n\to\infty} \frac{3-\frac{4}{n^3}}{2+\frac{3}{n^3}}$	**Divide each term by n^3**
$\lim_{n\to\infty} \frac{3-\frac{4}{n^3}}{2+\frac{3}{n^3}} = \frac{3}{2}$	**The sequence converges and the limit is $\frac{3}{2}$**

$\frac{3(n+1)^3-4}{2(n+1)^3+3} > \frac{3n^3-4}{2n^3+3}$ **Since the next term is always greater than the previous term, the sequence is monotonic and strictly increasing**

A convergent monotone sequence is always bounded, in this case between $-\frac{1}{5}$ and $\frac{3}{2}$

Subsequences

Subsequence: A subsequence is a smaller sequence within a larger sequence. For example, positive odd numbers would be a subsequence of the sequence of natural numbers.

Convergent: If the sequence if convergent, then the subsequence will also be convergent to the same limit.

Example: Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined inductively by $a_1 = 0$, and $a_{n+1} = \sqrt{a_n + 6}$ for $n \geq 1$. Prove $\{a_n\}_{n=1}^{\infty}$ is increasing, converges, and find $\lim_{n\to\infty}\{a_n\}_{n=1}^{\infty}$

Solution:

$a^2{}_{n+1} - a^2{}_n = a_n + 6 - a^2{}_n$	**Subtract a_n from a_{n+1} square both terms to eliminate the square root (it makes the expression simpler)**
$-(a^2{}_n - a_n - 6)$	**Factor out a negative and write in standard order**
$-(a_n - 3)(a_n + 2)$	**Factor the quadratic**
For $a_n \in [-2,3]$, $-(a_n - 3)(a_n + 2) \geq 0$	**Remember that $a_1 = 0$ and is positive since it's between $[-2, 3]$**
$a_n \leq 3$	**Suppose that $a_n \leq 3$**
$a_{n+1} = \sqrt{3+6}$	**Find a_{n+1}**
$a_{n+1} = \sqrt{9} = 3$	**Since $a_{n+1} = 3$ when $a_n = 3$, 3 is an upper bound and every a_n will be between $[0, 3]$. Therefore, $a^2{}_{n+1} - a^2{}_n$ will always be positive. This means that $\{a_n\}_{n=1}^{\infty}$ is increasing.**
	Since $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence that is bounded we know that it converges.
$\lim_{n\to\infty}\{a_n\}_{n=1}^{\infty} = \mathbf{3}$	**Since the upper bound is 3, we know that the $\lim_{n\to\infty}\{a_n\}_{n=1}^{\infty}$ is also 3.**

Bolzano-Weierstrass Theorem

Bolzano-Weierstrass Theorem: A bounded sequence has a convergent subsequence.

Example: Prove that there is a convergent subsequence of $\{a_n\} = \sin n$.

Solution:

Peaks are defined as if $n < m$ then $x_n > x_m$ and according to this definition $\sin n$ has no peaks, which means it has a finite number of peaks (0). Let N be the last peak and $n_1 = N + 1$. Then n_1 is not a peak because $N < n$, which implies the existence of n_2 with $n_1 < n_2$ and $x_{n_1} \leq x_{n_2}$. This leads to a decreasing subsequence. Since the subsequence is decreasing (monotone) and bounded it is a convergent subsequence. **Note: You do not need to use this proof in the future. Now that you have learned about the Bolzano-Weierstrass Theorem you would state that: By the Bolzano-Weierstrass Theorem, since $\boldsymbol{\sin n}$ is bounded, there exists a convergent subsequence.**

Cauchy Criterion

Cauchy Sequence: A sequence if for all $\epsilon > 0$, there exists a number N such that

$$m, n > N \to |a_m - a_n| < \epsilon$$

Cauchy Criterion: A Cauchy sequence is also convergent and a convergent sequence is a Cauchy sequence.

1. $\{a_n\}$ is bounded
2. $\{a_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$
3. $\{a_n\}$ converges to the limit $L = \lim_{k\to\infty} a_{n_k}$

Infinite limits can occur when dividing by a very small number. Often this happens when x approaches 0 and x is in the denominator.

If $\lim_{n\to\infty} a_n = \infty$ **then** $\lim_{n\to\infty} \frac{1}{a_n} = 0$

$\lim_{x\to a} f(x) = \infty$ **or** $\lim_{x\to a} f(x) = -\infty$

Example A: Show that the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$ for all $n \geq 1$, satisfies the Cauchy criterion.

Solution:

$|a_{n+1} - a_n| = \left|\frac{1}{a_n} - \frac{1}{a_{n-1}}\right|$ **Find the difference between a_n and a_{n-1}**

$|a_n a_{n-1}| = \left|\left(1 + \frac{1}{a_{n-1}}\right) a_{n-1}\right|$ **Find the product of a_n and a_{n-1}**

$|a_{n-1} + 1\,| \geq 2$ Simplify the product

This implies that $|a_{n+1} - a_n| \leq \frac{1}{2}|a_{n+1} - a_n|$ and therefore satisfies the Cauchy criterion.

Example B: Evaluate $\lim_{x\to\infty} 5x^2$

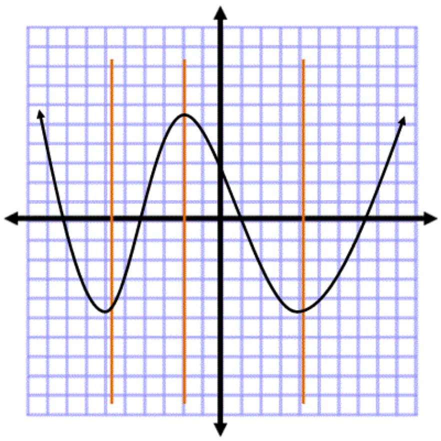
Solution:

$\lim_{x\to\infty} 5x^2 = \infty$ **As x approaches infinity the function gets larger and also approaches infinity.**

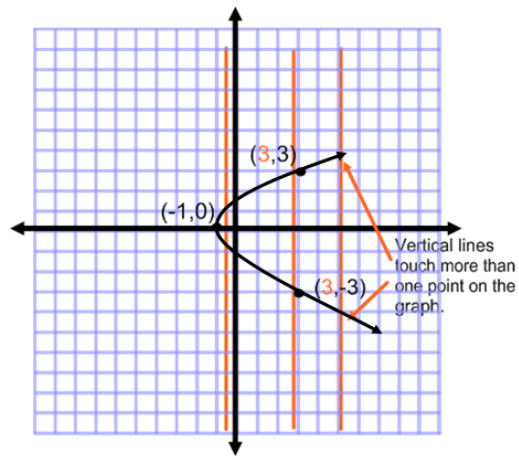
Functions

Examples of Functions

Function: A rule maps only one unique y-value for every x-value. Visually, when you look at a graph, you can see that every x-value maps to only one y-value by using the vertical line test. Any vertical line may only pass through the graph once. If it crosses the graph more than once then the graph has failed the vertical line test and it is not a function.



Function
all vertical lines cross graph only once



Not a Function
vertical lines cross the graph more than once

Domain: all x-values for which the function exists, left and right when looking at a graph

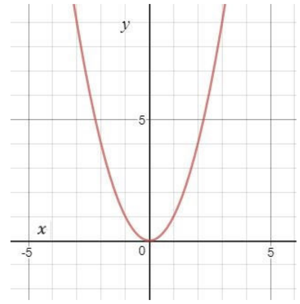
Range: all y-values for which the function exits, up and down when looking at a graph

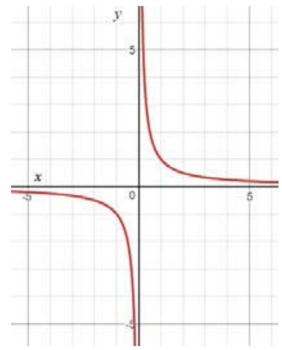
Basic Functions:

Function Name	$f(x)$	Characteristics
Constant	$f(x) = c$, where c is a constant Ex)$f(x) = 3$	Horizontal line at $y = c$
Linear	$f(x) = mx + b$ Ex) $f(x) = x$	Special case of a polynomial, x can only be raised to the power of 1
Quadratic	$f(x) = ax^2 + bx + c$ Ex) $f(x) = x^2$	Special case of a polynomial, the highest exponent x can have is $2(x^2$
Polynomial	$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$ Ex) $f(x) = 2x^3 + x^2 - 4$	Multiple terms, x can be raised to any positive exponent or zero
Rational	$f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, $Q(x) \neq 0$	There must be an x term in the denominator

	Ex) $f(x)\frac{x^2-2}{x+1}$	

Even: Symmetric about the y-axis, $f(-x) = f(x)$ (when you plug in negative x the function will simplify to be the same as the original function





Odd: Symmetric about the origin, $f(-x) = -f(x)$ (when you plug in negative x the function will simplify to be negative of the original function. Be sure to visually compare quadrants that are diagonal from each other (Quad I & III and Quad II & IV).

Sum: $(f+g)(x) = f(x) + g(x)$

Difference: $(f-g)(x) = f(x) - g(x)$

Product: $(fg)(x) = f(x) \cdot g(x)$

Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\ for\ g(x) \neq 0$ Remember you cannot divide by 0 (it is undefined)

Composition: $(f \circ g)(x) = f(g(x))$

Piecewise Function: different equations used over different intervals of the graph

$$h(x) = \begin{cases} x+3 & \text{if } x < -2 \\ x^2 & \text{if } -2 \le x < 1 \\ -x+2 & \text{if } x \ge 1 \end{cases}$$

Example A: What is the domain of the following functions?

(a) $f(x) = \frac{1}{x}$ (b) $f(x) = \sqrt{x-2}$ (c) $f(x) = 2x^2 - 6$

Solution:

(a) $\boldsymbol{x \neq 0}$ **or** $\boldsymbol{(-\infty, 0) \cup (0, \infty)}$ **The denominator is undefined at $\boldsymbol{x = 0}$**
(b) $\boldsymbol{x \geq 2}$ **or** $\boldsymbol{[2, \infty)}$ **The argument of a square root must be positive**
(c) $\boldsymbol{x \in \mathbb{R}}$ **or** $\boldsymbol{(-\infty, \infty)}$ **There are no restrictions on the domain**

Example B: If $f(x) = \sqrt{x-2}$ and $g(x) = 3x^2 - 3x$, then what is $g(f(3)$?

Solution: $f(3) = \sqrt{3-2}$ **Find $\boldsymbol{f(3)}$.**

$f(3) = \sqrt{1}$

$f(3) = 1$

$g(1) = 3(1)^2 - 3(1)$ **Find $\boldsymbol{g(1)}$**

$g(1) = 3 - 3$

$\boldsymbol{g(1) = 0}$ **Solution**

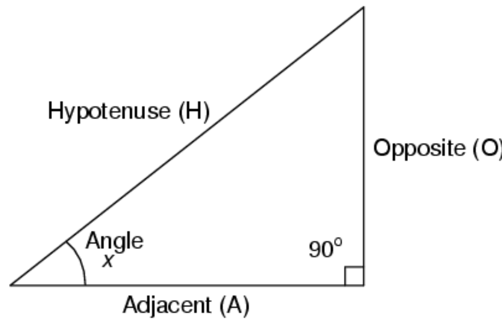
Trigonometric Functions

Basic Trig Functions:

$\sin\theta = \frac{O}{H}$

$\cos\theta = \frac{A}{H}$

$\tan\theta = \frac{O}{A}$



Reciprocal Trig Functions:

$\csc\theta = \frac{1}{\sin\theta} = \frac{H}{O}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{H}{A}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{A}{O}$

Pythagorean Identities:

$\sin^2\theta + \cos^2\theta = 1$

$1 + \cot^2\theta = \csc^2\theta$

$\tan^2\theta + 1 = \sec^2\theta$

Complementary Angle Identities:

$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

Supplementary Angle Identities:

$\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

Symmetries:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin(\theta + 2\pi) = \sin\theta$

$\cos(\theta + 2\pi) = \cos\theta$

Sum and Difference Formulas:

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan\tan A\tan B}$

Double-Angle Formulas:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Example A: Develop the identity for $\sin 2A$ by using the identity for $\sin(A + B)$

Solution:

$\sin A + \sin A = \sin A\cos A + \cos A\sin A$	**Use the sine of a sum formula and plug in A for both angles**
$\mathbf{\sin A + \sin A = 2\sin A\cos A}$	**Simplify**

Example B: Show that $\sec x + \sec x\tan^2 x = \sec^3 x$

Solution:

$\sec x\,(1+\tan^2 x) = \sec^3 x$ **Factor out $\sec x$ (when proving identities only work with one side of the equation)**

$\sec x\,(\sec^2 x) = \sec^3 x$ **$\tan^2\theta + 1 = \sec^2\theta$, Pythagorean Identity**

$\sec^3 x = \sec^3 x$ **QED, it has been shown**

Limit of a Function

Definition of Limit: The limit of $f(x)$ as x approaches a is the number L, provided that we can make the values $f(x)$ as close as we like to L, by taking x close to, but different from a. The limit describes the behavior of a function near a, not at a.

$$\lim_{x \to a} f(x) = L$$

Infinite limits can occur when dividing by a very small number. Often this happens when x approaches 0 and x is in the denominator. Even though you can write the limit as approaching infinity, technically the limit can never reach infinity so the limit does not exist.

$$\lim_{x \to a} f(x) = \infty \qquad \text{or} \qquad \lim_{x \to a} f(x) = -\infty$$

Limits that describe the end behavior of a graph occur as x approaches $\pm\infty$

$$\lim_{x \to \infty} f(x) \qquad \text{or} \qquad \lim_{x \to -\infty} f(x)$$

Example A: Evaluate the limit: $\lim_{x \to 2}(8 - 3x + 12x^2)$

Solution:

$8 - 3(2) + 12(2^2)$	**Since there are no domain restrictions you can solve the limit by plugging in 2 for x.**
$8 - 6 + 12(4)$	**Simplify**
$2 + 48$	
$\mathbf{50}$	$\lim_{x \to 2}(8 - 3x + 12x^2) = 50$

Example B: Evaluate the limit: $\lim_{x \to 1} \frac{x^2-1}{x-1}$

Solution:

We cannot just plug in 1 to evaluate since $x = 1$ is not in the domain since $1 - 1 = 0$ and we can't divide by 0.

$\frac{(x-1)(x+1)}{x-1}$	**Simplify by factoring the numerator.**
$x + 1$	**$x - 1$ will cancel**
$\lim_{x \to 1} x + 1$	**Evaluate the limit of the simplified expression**
$\mathbf{1 + 1 = 2}$	$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

Properties of Limits

Limit Properties:

4. $\lim_{x\to a} c = c$
5. $\lim_{x\to a} x^n = a^n$
6. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
7. $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
8. $\lim_{x\to a}[cf(x)] = c \cdot \lim_{x\to a} f(x)$
9. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
10. If f is a polynomial function, then $\lim_{x\to a} f(x) = f(a)$

Cauchy Criterion for Sequences:

$\lim_{n\to\infty} a_n$ exists $\leftrightarrow$ for every $\epsilon > 0$ we can find a number N such that
$m, n > N \to |a_m - a_n| < \epsilon$

Squeeze Principle for Functions:

If $f(x) \le h(x) \le g(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$, then $\lim_{x\to a} h(x) = L$

Example A: $\lim_{t\to 0} \frac{1}{t} - \frac{1}{2t^2+t}$

Solution:

$\lim_{t\to 0} \frac{2t+1}{t(2t+1)} - \frac{1}{t(2t+1)}$ **Multiply first fraction by $\frac{2t+1}{2t+1}$ (Common Denominator)**

$\lim_{t\to 0} \frac{2t}{t(2t+1)}$ **Combine fractions**

$\lim_{t\to 0} \frac{2}{(2t+1)} = \frac{2}{(2(0)+1)} = 2$ **Simplify and plug 0 in for t**

$\mathbf{2}$ $\lim_{t\to 0} \frac{1}{t} - \frac{1}{2t^2+t} = 2$

Example B: $\lim_{x\to 0} \frac{x}{\sqrt{4+5x}-2}$

Solution:

$\lim_{x\to 0} \frac{x(\sqrt{4+5x}+2)}{(\sqrt{4+5x}-2)(\sqrt{4+5x}+2)}$ **Multiply by $\frac{(\sqrt{4+5x}+2)}{(\sqrt{4+5x}+2)}$ (Conjugate of the bottom)**

$\lim_{x\to 0} \frac{x(\sqrt{4+5x}+2)}{(4+5x-4)} = \lim_{x\to 0} \frac{x(\sqrt{4+5x}+2)}{5x}$ **Multiply out the denominator (FOIL) & Simplify**

$\lim_{x\to 0} \frac{\sqrt{4+5x}+2}{5} = \frac{\sqrt{4+5(0)}+2}{5} = \frac{2}{5}$ **Simplify and plug 0 in for x**

$\mathbf{\frac{2}{5}}$ $\mathbf{\lim_{x\to 0} \frac{x}{\sqrt{4+5x}-2} = \frac{2}{5}}$

Continuity

Continuous Function: A function f is continuous at a if and only if the following three conditions are met:

1. $\boldsymbol{f(a)}$ **exists**
2. $\boldsymbol{\lim_{x \to a} f(x)}$ **exists**
3. $\boldsymbol{\lim_{x \to a} f(x) = f(a)}$

*Can you trace the graph left to right without picking up your pencil?

Discontinuous Function: If f is not continuous at a, then it is discontinuous. This can happen at holes, asymptotes, or breaks

Continuous on an Interval: A function is continuous on an interval if it is continuous at each point over the interval

Continuous on its Domain: A function is continuous on its domain if it is continuous on every interval

Properties of Continuous Functions: Suppose f and g are continuous at a. Then $f + g$ and fg are continuous at a and f/g is continuous at a if $g(a) \neq 0$.

Continuity of Rational Functions: A rational function is continuous on its domain.

Composition of Continuous Functions: Suppose g is continuous at a and f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.

Example A: Determine whether the function is continuous at the given points.

$$\frac{x+3}{x-4};\, x = 4, x = 0$$

Solution:

$f(4) = \frac{4+3}{4-4} = \frac{7}{0}$ Undefined $\quad$ $\boldsymbol{f(4)}$ **does not exist so the function is discontinuous at 4.**

$f(0) = \frac{0+3}{0-4} = \frac{-3}{4}$ $\quad$ $\boldsymbol{f(0)}$ **does exist, now we see if the** $\boldsymbol{\lim_{x \to 0} f(x)}$ **exists**

$\lim_{x \to 0} x + 3 \cdot \lim_{x \to 0} \frac{1}{x-4} = 3 \cdot \frac{-1}{4}$ $\quad$ **Find the** $\boldsymbol{\lim_{x \to 0} f(x)}$

$\boldsymbol{\frac{-3}{4}}$ $\quad$ **Since the limit and** $\boldsymbol{f(0)}$ **are equal the function is continuous at 0.**

Example B: Show that the function $f(x) = \frac{3x^2+1}{2x^2+x+4}$ is continuous for all $x \in \mathbb{R}$

Solution:

$2x^2 + x + 4 > 0$ **Since the denominator never equals 0 the domain is $x \in \mathbb{R}$**

Since the domain is $x \in \mathbb{R}$ and the $\lim_{x\to c} f(x) = f(c)$ the function $f(x) = \frac{3x^2+1}{2x^2+x+4}$ is continuous for all $x \in \mathbb{R}$

One-Sided Limits

Sometimes limits can be different depending on whether we are approaching from the right or left as x approaches a.

The limit as x approaches a from right is written as: $\lim_{x \to a^+} f(x)$ and the function f is **continuous** from the right at a if $\lim_{x \to a^+} f(x) = f(a)$

The limit as x approaches a from left is written as: $\lim_{x \to a^-} f(x)$ and the function f is **continuous** from the left at a if $\lim_{x \to a^-} f(x) = f(a)$

If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$ then $\lim_{x \to a} f(x) = L$

A function f is continuous on $[a, b]$ if f is continuous at each point in (a, b) and continuous from the right at a and from the left at b

Example A: $\lim_{x \to -2^+} \frac{x+3}{x+2}$

Solution:

$\lim_{x \to -2^+} (x+3) \frac{1}{x+2}$ **Rewrite as a product**

$\lim_{x \to -2^+} (x+3) \lim_{x \to -2^+} \frac{1}{x+2}$ **Take the limit of each factor**

$(-2+3) \frac{1}{-2+2} = 1 \cdot \infty = \infty$ **Plug -2 in for x**

$\lim_{x \to -2^+} \frac{x+3}{x+2} = \infty$ **Remember that when 1 is divided by a very small number it becomes very large, so as the denominator approaches 0 the fraction will approach ∞**

Example B: $\lim_{x \to -\infty} \frac{x^3+3x}{4x^3-2x^2+6}$

Solution:

$\lim_{x \to -\infty} \frac{x^3}{4x^3}$ **When taking the limits of a rational function, you only need to take the limit of the terms with the highest exponent in the numerator and denominator**

$\lim_{x \to -\infty} \frac{x^3}{4x^3} = \frac{1}{4}$ **If the exponents are equal then the limit is the coefficient of the numerator over the coefficient of the denominator**

Properties of Continuous Functions

Intermediate Value Theorem: If f is continuous on $[a, b]$ and $f(a) < y < f(b)$, then there exits a number c where $a < c < b$ such that $f(c) = y$

Boundedness: A function is bounded on interval $[a, b]$ if the range is finite in both directions (up and down)

Extreme Value Theorem (Weierstrass Max/Min Theorem): If a function f is continuous over $[a, b]$ then there are numbers c and d in $[a, b]$ such that $f(c)$ is an absolute minimum over $[a, b]$ and $f(d)$ is an absolute maximum over $[a, b]$.

Example A: Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0,1]$

Solution:

$f(0) = 0^3 + 2(0) - 1 = -1$ **Start by plugging the endpoints**

$f(1) = 1^3 + 2(1) - 1 = 2$

Since $\boldsymbol{f(0) < 0}$ and $\boldsymbol{f(1) > 0}$, using the Intermediate Value Theorem, you can conclude that there must be some $\boldsymbol{c}$ in $\boldsymbol{[0, 1]}$ where $\boldsymbol{f(c) = 0}$.

Example B: $\lim\limits_{x \to -3} \frac{x^2 - x - 12}{x + 3}$

Solution:

We cannot evaluate the limit without simplifying since $-3 + 3 = 0$

$\lim\limits_{x \to -3} \frac{(x+3)(x-4)}{x+3}$ **Factor the numerator**

$\lim\limits_{x \to -3} = x - 4$ **Cancel the $\boldsymbol{x + 3}$**

$-3 - 4 = -7$ $\boldsymbol{\lim\limits_{x \to -3} \frac{x^2 - x - 12}{x + 3} = -7}$

Good Luck!!