Math 322
Suggested solutions to Homework Set 4

Problem 1. (a) We will show that the graphs G_1 and G_2 are isomorphic.

To find an appropriate bijection from $V(G_1)$ to $V(G_2)$, we could first notice that the 'inner' vertices of the graph G_1 form a 5-cycle, the cycle $F\,H\,J\,G\,I\,F$.

Thus, we could first try to see which subgraph of G_2 could be isomorphic to the 5-cycle: we could choose the 5-cycle 3 4 8 9 10 3.

Thus we can start by defining a bijection τ from the vertex set of the first cycle to the vertex set of the second cycle:

$$\tau(F) = 3, \quad \tau(H) = 4, \quad \tau(J) = 8, \quad \tau(G) = 9, \quad \text{and } \tau(I) = 10.$$

We can then extend this to a bijection $\tau : V(G_1) \to V(G_2)$ by setting $\tau(A) = 2$ (given that A is the only neighbour of the vertex F in G_1 outside the cycle $F\,H\,J\,G\,I\,F$, and similarly 2 is the only neighbour of the vertex 3 in G_2 outside the cycle 3 4 8 9 10 3).

Analogously, we set

$$\tau(B) = 1, \quad \tau(C) = 5, \quad \tau(D) = 6, \quad \text{and } \tau(E) = 7.$$

In other words, the bijection τ we have defined is given by

$$\tau : \begin{pmatrix} A & B & C & D & E & & F & G & H & I & J \\ 2 & 1 & 5 & 6 & 7 & & 3 & 9 & 4 & 10 & 8 \end{pmatrix}.$$

Let us now check that τ is a graph isomorphism from G_1 to G_2. It is not hard to see that τ restricted to the first five vertices of G_1 is an isomorphism from the 5-cycle $A\,B\,C\,D\,E\,A$ to the 5-cycle 2 1 5 6 7 2 (indeed, the neighbours of A in the first cycle are the vertices B and E, while the vertices of $2 = \tau(A)$ in the second cycle are the vertices $1 = \tau(B)$ and $7 = \tau(E)$; we can argue analogously about the neighbours of the other vertices in the two cycles).

Similarly, we can check that τ restricted to the last five vertices of G_1 is an isomorphism from the 5-cycle $F\,H\,J\,G\,I\,F$ to the 5-cycle 3 4 8 9 10 3 (in fact, recall that this is what we based our definition of τ on, that is, this is the first thing we tried to make sure will hold): indeed, the neighbours of G in the first cycle are the vertices H and I, while the vertices of $3 = \tau(F)$ in the second cycle are the vertices $4 = \tau(H)$ and $10 = \tau(I)$; analogously we argue about the neighbours of the rest of the vertices.

Finally we can check that

A and F are neighbours in G_1, and similarly 2 and 3 are neighbours in G_2;
B and G are neighbours in G_1, and similarly 1 and 9 are neighbours in G_2;
C and H are neighbours in G_1, and similarly 5 and 4 are neighbours in G_2;
D and I are neighbours in G_1, and similarly 6 and 10 are neighbours in G_2;
E and J are neighbours in G_1, and similarly 7 and 8 are neighbours in G_2.

Based on the above, τ is a graph isomorphism from G_1 to G_2.

(b) We now justify why G_3 is not isomorphic to G_1 (or equivalently, it is not isomorphic to G_2). One idea here is to look for a subgraph of G_3 which is not isomorphic to any subgraph of G_1.

For example, we observe that G_3 contains the 4-cycle $b\,c\,d\,j\,b$. If G_1 and G_3 were isomorphic, then G_1 should also contain a 4-cycle.

We will now show that G_1 does not contain any 4-cycles. Assume towards a contradiction that $C_0 : v_1\,v_2\,v_3\,v_4\,v_1$ is a 4-cycle in G_1. Note that we can't have all the v_i to be among the 'outer' vertices of G_1 (because there is no way to form a 4-cycle using only 'outer' vertices of G_1), and similarly we cannot have all the v_i to be among the 'inner' vertices of G_1.

Thus we must have a vertex v_i on C_0 which is among the vertices A, B, C, D and E, and such that v_{i+1} is among the vertices F, G, H, I and J. In fact, given that we can consider any vertex of the cycle to be the initial vertex, we can assume without loss of generality that this is true for $i = 1$, or in other words that v_1 is an 'outer' vertex of G_1 and v_2 is an 'inner' vertex of G_1.

Observe finally that, because of symmetries of the graph G_1, we can assume that $v_1 = A$ and thus $v_2 = F$.

We can now try to determine which are the shortest cycles in G_1 which start with the vertices A and F. We have the following;

if after the vertex F we move to the vertex H, then we can get

- the 5-cycle $A\,F\,H\,C\,B\,A$,
- the 5-cycle $A\,F\,H\,J\,E\,A$,
- as well as some even longer cycles;

if after the vertex F we move to the vertex I, then we can get

- the 5-cycle $A\,F\,I\,D\,E\,A$,

- the 5-cycle $A\,F\,I\,G\,B\,A$,
- as well as some even longer cycles.

The above show that there is no 4-cycle in G_1 starting with the vertices A and F, which contradicts our initial assumption that a 4-cycle C_0 in G_1 exists. This implies that G_1 has no 4-cycles, and therefore it cannot be isomorphic to G_3.

Alternative solution (suggested by one of your classmates): In Problem 5, we will justify (in a similar way to above) that the graph G_1 is not Hamiltonian, that is, it contains no Hamilton cycle.

On the other hand, we can check directly that G_3 does contain a Hamilton cycle: e.g. the cycle

$$b\,c\,d\,e\,a\,h\,g\,f\,i\,j\,b.$$

From the above it follows that G_1 and G_3 cannot be isomorphic.

Problem 2. (a) In the first stage of the algorithm, we assign weight 0 to the vertex a, and temporary weight ∞ to every other vertex of the graph:

	a	b	c	d	e	f	g	h	i	j
Stage 1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞

.

Next we note that, in Stage 2 of the algorithm, the only vertices whose weight will change are the neighbours of a: these are vertices b, c and d.

- The (only) candidate for the new temporary weight of b is

  $\min\{\infty, \text{permanent weight of } a + \text{weight of the edge } ab\} = \min\{\infty, 3\} = 3.$

- The (only) candidate for the new temporary weight of c is

  $\min\{\infty, \text{permanent weight of } a + \text{weight of the edge } ac\} = \min\{\infty, 2\} = 2.$

- The (only) candidate for the new temporary weight of d is

  $\min\{\infty, \text{permanent weight of } a + \text{weight of the edge } ad\} = \min\{\infty, 4\} = 4.$

Based on these, we can add a second row to the above table which will contain the temporary weights of the vertices in Stage 2:

	a	b	c	d	e	f	g	h	i	j
Stage 1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
Stage 2	0	3	2	4	∞	∞	∞	∞	∞	∞

.

Also, the minimum of the new temporary weights, which is the weight of vertex c, becomes a permanent weight now (we highlight that entry of the table to demonstrate this).

Again, in Stage 3 of the algorithm, the only vertices whose weight may change are the neighbours of a and the neighbours of c (excluding c and a of course, since these already have a permanent weight): these neighbours are vertices b, d, e and f.

- The (only) candidate for the new temporary weight of b is

  $\min\{3, \text{permanent weight of } a + \text{weight of the edge } ab\} = \min\{3, 3\} = 3.$

- The (only) candidate for the new temporary weight of d is

  $\min\{4, \text{permanent weight of } a + \text{weight of the edge } ad\} = \min\{4, 4\} = 4.$

- The (only) candidate for the new temporary weight of e is

  $$\min\{\infty, \text{permanent weight of } c + \text{weight of the edge } ce\} = \min\{\infty, 3\} = 3.$$

- The (only) candidate for the new temporary weight of f is

  $$\min\{\infty, \text{permanent weight of } c + \text{weight of the edge } cf\} = \min\{\infty, 4\} = 4.$$

Out of these new temporary weights, both the weight of vertex b and the weight of vertex e have minimum value equal to 3; so in this case we can choose any of the two vertices to be the one whose weight becomes permanent; let us choose vertex b here.

Thus the third row of the table is as follows:

	a	b	c	d	e	f	g	h	i	j
Stage 1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
Stage 2	0	3	2	4	∞	∞	∞	∞	∞	∞
Stage 3	0	3	2	4	3	4	∞	∞	∞	∞

.

In Stage 4 of the algorithm, the only vertices whose weight may change are the neighbours of vertices a, b and c (excluding b, c and a of course, since these already have a permanent weight): these are the vertices d, e and f.

- The (only) candidate for the new temporary weight of d is

  $$\min\{4, \text{permanent weight of } a + \text{weight of the edge } ad\} = \min\{4, 4\} = 4.$$

- The two candidates for the new temporary weight of e are

  $$\min\{3, \text{permanent weight of } c + \text{weight of the edge } ce\} = \min\{3, 3\} = 3,$$

  and

  $$\min\{3, \text{permanent weight of } b + \text{weight of the edge } be\} = \min\{3, 5\} = 3.$$

  Thus the new temporary weight of e will be the minimum of these two candidates, which is equal to 3.

- The (only) candidate for the new temporary weight of f is

  $$\min\{4, \text{permanent weight of } c + \text{weight of the edge } cf\} = \min\{4, 4\} = 4.$$

Out of these new temporary weights, the weight of vertex e is minimum, so this becomes now permanent:

	a	b	c	d	e	f	g	h	i	j
Stage 1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
Stage 2	0	3	2	4	∞	∞	∞	∞	∞	∞
Stage 3	0	3	2	4	3	4	∞	∞	∞	∞
Stage 4	0	3	2	4	3	4	∞	∞	∞	∞

.

In Stage 5 of the algorithm, the only vertices whose weight may change are the neighbours of the vertices which already have permanent weight, that is, vertices a, b, c and e: the neighbours of these vertices (which don't already have permanent weight) are vertices d, f, g and h.

- The (only) candidate for the new temporary weight of d is

  $$\min\{4, \text{permanent weight of } a + \text{weight of the edge } ad\} = \min\{4, 4\} = 4.$$

- The (only) candidate for the new temporary weight of f is

  $$\min\{4, \text{permanent weight of } c + \text{weight of the edge } cf\} = \min\{4, 4\} = 4.$$

- The (only) candidate for the new temporary weight of g is

  $$\min\{\infty, \text{permanent weight of } e + \text{weight of the edge } eg\} = \min\{\infty, 7\} = 7.$$

- The (only) candidate for the new temporary weight of h is

  $$\min\{\infty, \text{permanent weight of } e + \text{weight of the edge } eh\} = \min\{\infty, 5\} = 5.$$

Out of these new temporary weights, both the weight of vertex d and the weight of vertex f have minimum value equal to 4; so in this case we can choose any of the two vertices to be the one whose weight becomes permanent; let us choose vertex d here.

Thus the fifth row of the table is as follows:

	a	b	c	d	e	f	g	h	i	j
Stage 1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
Stage 2	0	3	2	4	∞	∞	∞	∞	∞	∞
Stage 3	0	3	2	4	3	4	∞	∞	∞	∞
Stage 4	0	3	2	4	3	4	∞	∞	∞	∞
Stage 5	0	3	2	4	3	4	7	5	∞	∞

.

In Stage 6 of the algorithm, the vertices whose weight may change are those which don't have permanent weight yet, and are neighbours to vertices with permanent weight: these are vertices f, g, h and i.

- The two candidates for the new temporary weight of f are

  $$\min\{4, \text{permanent weight of } c + \text{weight of the edge } cf\} = \min\{4, 4\} = 4,$$

  and

  $$\min\{4, \text{permanent weight of } d + \text{weight of the edge } df\} = \min\{4, 5\} = 4.$$

  Thus the new temporary weight of f will have the same value as before, equal to 4.

- The (only) candidate for the new temporary weight of g is

  $$\min\{7, \text{permanent weight of } e + \text{weight of the edge } eg\} = \min\{7, 7\} = 7.$$

- The (only) candidate for the new temporary weight of h is

  $$\min\{5, \text{permanent weight of } e + \text{weight of the edge } eh\} = \min\{5, 5\} = 5.$$

- The (only) candidate for the new temporary weight of i is

  $$\min\{\infty, \text{permanent weight of } d + \text{weight of the edge } di\} = \min\{\infty, 7\} = 7.$$

Out of these new temporary weights, the weight of vertex f is minimum, so this becomes now permanent:

	a	b	c	d	e	f	g	h	i	j
Stage 1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
Stage 2	0	3	2	4	∞	∞	∞	∞	∞	∞
Stage 3	0	3	2	4	3	4	∞	∞	∞	∞
Stage 4	0	3	2	4	3	4	∞	∞	∞	∞
Stage 5	0	3	2	4	3	4	7	5	∞	∞
Stage 6	0	3	2	4	3	4	7	5	7	∞

.

In Stage 7 of the algorithm, the vertices whose weight may change are those which don't have permanent weight yet, and are neighbours to vertices with permanent weight: these are again vertices g, h and i.

- The (only) candidate for the new temporary weight of g is

  $$\min\{7, \text{permanent weight of } e + \text{weight of the edge } eg\} = \min\{7, 7\} = 7.$$

- There are two candidates for the new temporary weight of h: these are

  $$\min\{5, \text{permanent weight of } e + \text{weight of the edge } eh\} = \min\{5, 5\} = 5$$

  and

  $$\min\{5, \text{permanent weight of } f + \text{weight of the edge } fh\} = \min\{5, 7\} = 5.$$

  Thus the new temporary weight of h will have the same value as before, equal to 5.

- Similarly, there are two candidates for the new temporary weight of i: these are

  $$\min\{7, \text{permanent weight of } d + \text{weight of the edge } di\} = \min\{7, 7\} = 7$$

  and

  $$\min\{7, \text{permanent weight of } f + \text{weight of the edge } fi\} = \min\{7, 6\} = 6.$$

  Thus the new temporary weight of i will be the minimum of these two candidates, which is equal to 6.

Out of these new temporary weights, the weight of vertex f is minimum, so this becomes now permanent:

	a	b	c	d	e	f	g	h	i	j
Stage 1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
Stage 2	0	3	2	4	∞	∞	∞	∞	∞	∞
Stage 3	0	3	2	4	3	4	∞	∞	∞	∞
Stage 4	0	3	2	4	3	4	∞	∞	∞	∞
Stage 5	0	3	2	4	3	4	7	5	∞	∞
Stage 6	0	3	2	4	3	4	7	5	7	∞
Stage 7	0	3	2	4	3	4	7	5	6	∞

In Stage 8 of the algorithm, the weight of all vertices which still have a temporary weight may now change: these are the vertices g, i and j.

- The (only) candidate for the new temporary weight of g is

  $$\min\{7, \text{permanent weight of } e + \text{weight of the edge } eg\} = \min\{7, 7\} = 7.$$

- There are two candidates for the new temporary weight of i: these are

  $\min\{7, \text{permanent weight of } d + \text{weight of the edge } di\} = \min\{7, 7\} = 7$

  and

  $\min\{7, \text{permanent weight of } f + \text{weight of the edge } fi\} = \min\{7, 6\} = 6.$

  Thus the new temporary weight of i will stay the same as in the previous stage, equal to 6.

- The (only) candidate for the new temporary weight of j is

  $\min\{\infty, \text{permanent weight of } h + \text{weight of the edge } hj\} = \min\{\infty, 11\} = 11.$

Out of these new temporary weights, the weight of vertex i is minimum, so this becomes now permanent:

	a	b	c	d	e	f	g	h	i	j
Stage 1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
Stage 2	0	3	2	4	∞	∞	∞	∞	∞	∞
Stage 3	0	3	2	4	3	4	∞	∞	∞	∞
Stage 4	0	3	2	4	3	4	∞	∞	∞	∞
Stage 5	0	3	2	4	3	4	7	5	∞	∞
Stage 6	0	3	2	4	3	4	7	5	7	∞
Stage 7	0	3	2	4	3	4	7	5	6	∞
Stage 8	0	3	2	4	3	4	7	5	6	11

.

Again, in Stage 9 of the algorithm, the weight of all vertices which still have a temporary weight may change: these are vertices g and j.

- The (only) candidate for the new temporary weight of g is

  $\min\{7, \text{permanent weight of } e + \text{weight of the edge } eg\} = \min\{7, 7\} = 7.$

- There are two candidates for the new temporary weight of j: these are

  $\min\{11, \text{permanent weight of } h + \text{weight of the edge } hj\} = \min\{11, 11\} = 11$

  and

  $\min\{11, \text{permanent weight of } i + \text{weight of the edge } ij\} = \min\{11, 10\} = 10.$

  Thus the new temporary weight of i will be the minimum of these two candidates, which is equal to 10.

Out of these new temporary weights, the weight of vertex g is minimum, so this becomes now permanent:

	a	b	c	d	e	f	g	h	i	j
Stage 1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
Stage 2	0	3	2	4	∞	∞	∞	∞	∞	∞
Stage 3	0	3	2	4	3	4	∞	∞	∞	∞
Stage 4	0	3	2	4	3	4	∞	∞	∞	∞
Stage 5	0	3	2	4	3	4	7	5	∞	∞
Stage 6	0	3	2	4	3	4	7	5	7	∞
Stage 7	0	3	2	4	3	4	7	5	6	∞
Stage 8	0	3	2	4	3	4	7	5	6	11
Stage 9	0	3	2	4	3	4	7	5	6	10

.

Finally, in Stage 10 of the algorithm, the weight of the last vertex with temporary weight, that is, vertex j, may still change:

- there will be three candidates for the new temporary weight of j: these are

  $\min\{10, \text{permanent weight of } h + \text{weight of the edge } hj\} = \min\{10, 11\} = 10,$

  $\min\{10, \text{permanent weight of } i + \text{weight of the edge } ij\} = \min\{10, 10\} = 10$

  and

  $\min\{10, \text{permanent weight of } g + \text{weight of the edge } gj\} = \min\{10, 10\} = 10.$

  The new temporary weight of j will be the minimum of these three candidates, which is equal to 10. Also, it will finally become a permanent weight.

We can now write down the full table that captures all the stages of the algorithm, as well as each shortest distance from vertex a to any other vertex:

	a	b	c	d	e	f	g	h	i	j
Stage 1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
Stage 2	0	3	2	4	∞	∞	∞	∞	∞	∞
Stage 3	0	3	2	4	3	4	∞	∞	∞	∞
Stage 4	0	3	2	4	3	4	∞	∞	∞	∞
Stage 5	0	3	2	4	3	4	7	5	∞	∞
Stage 6	0	3	2	4	3	4	7	5	7	∞
Stage 7	0	3	2	4	3	4	7	5	6	∞
Stage 8	0	3	2	4	3	4	7	5	6	11
Stage 9	0	3	2	4	3	4	7	5	6	10
Stage 10	0	3	2	4	3	4	7	5	6	10

.

(b) As we saw in part (a), the shortest possible length, or equivalently minimum possible weight, of a path from a to j is 10.

We now try to find all these paths: if we start traversing such a path from vertex j instead, the next vertex should be among the vertices g, h and i. We have

$$\text{permanent weight of } j - \text{permanent weight of } g = 3 = \text{weight of the edge } gj,$$

so we could have a minimum weight path from j to a that first moves from j to g.

Similarly,

$$\text{permanent weight of } j - \text{permanent weight of } i = 4 = \text{weight of the edge } ij,$$

so we could have a minimum weight path from j to a that first moves from j to i.

On the other hand,

$$\text{permanent weight of } j - \text{permanent weight of } h = 5 < \text{weight of the edge } gj,$$

so there does <u>not</u> exist any minimum weight path from j to a that first moves from j to h.

Next, we examine how we could continue building a minimum weight path from j to a that starts with the vertices j and g: from g necessarily we have to move to vertex e, and then from vertex e we could move either to vertex b or to vertex c (or furthermore to vertex h; note however that we should probably already guess that this last option cannot be a good one, given that we saw above that, if we moved directly from vertex j to vertex h, we would not get a minimum weight path, so having vertex h appear in a $j - a$ path with even more edges couldn't possibly result in a path with smaller weight).

We have that

$$\text{permanent weight of } e - \text{permanent weight of } b = 0 < \text{weight of the edge } eb,$$

so there does <u>not</u> exist any minimum weight path from j to a that includes the edge eb.

On the other hand,

$$\text{permanent weight of } e - \text{permanent weight of } c = 1 = \text{weight of the edge } ec,$$

so we can find a minimum weight $j - a$ path whose initial part is $j\,g\,e\,c$.

Finally, from vertex c we could either move directly to vertex a, or move to vertex f first. However,

$$\text{permanent weight of } c - \text{permanent weight of } f = -2 < \text{weight of the edge } cf,$$

therefore moving to vertex f will not give a minimum weight path.

We conclude that the only minimum weight path from j to a that passes through vertex g is the path

$$j\,g\,e\,c\,a.$$

Next, we examine which minimum weight paths from j to a we could find if we first moved from vertex j to vertex i: from vertex i we can either move to vertex d or to vertex f.

Given that

$$\text{permanent weight of } i - \text{permanent weight of } d = 2 < \text{weight of the edge } id,$$

while

$$\text{permanent weight of } i - \text{permanent weight of } f = 2 = \text{weight of the edge } if,$$

we see that our only option here is to move to vertex f.

Next, from vertex f we can either move to vertex c or to vertex h: given that

$$\text{permanent weight of } f - \text{permanent weight of } h = -1 < \text{weight of the edge } fh,$$

while

$$\text{permanent weight of } f - \text{permanent weight of } c = 2 = \text{weight of the edge } fc,$$

we see that our only option here is to move to vertex c.

Finally, from vertex c we can either move to vertex a directly or to vertex e: given that

$$\text{permanent weight of } c - \text{permanent weight of } e = -1 < \text{weight of the edge } ce,$$

while

$$\text{permanent weight of } c - \text{permanent weight of } a = 2 = \text{weight of the edge } ca,$$

we see that our only option here is to move to vertex a directly.

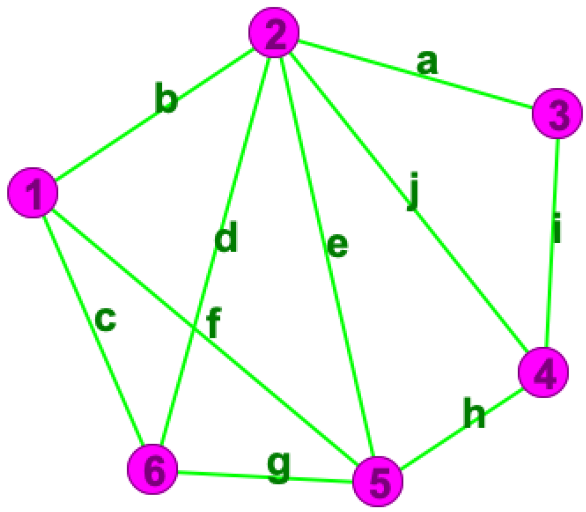
We conclude that the only minimum weight path from j to a that passes through vertex i is the path

$$j\,i\,f\,c\,a.$$

Summarising the above, we obtain that the only minimum weight paths from a to j are the paths

$$a\,c\,e\,g\,j \qquad \text{and} \qquad a\,c\,f\,i\,j.$$

Problem 3. Consider the following graph H:



which has size 10, and is clearly connected.

Its (decreasing) degree sequence is $\big(5,\ 4,\ 3,\ 3,\ 3,\ 2\big)$, and thus by Theorem 1 and Proposition 1 of Lecture 17 the graph cannot be Eulerian, and it cannot even have an Euler trail.

On the other hand, the vertex set of $L(H)$ is the set $\{a, b, c, d, e, f, g, h, i, j\}$, and a cycle passing through each of these 'vertices' is the following:

$$a\,b\,c\,d\,e\,f\,g\,h\,i\,j\,a.$$

Note that this is indeed a cycle in $L(H)$ because any pair of consecutive vertices coincides with a pair of adjacent edges in H (indeed, a and b have common endvertex 2, b and c have common endvertex 1, c and d have common endvertex 6, d and e have common endvertex 2, all four of the vertices e, f, g and h have common endvertex 5, all three of the vertices h, i and j have common endvertex 4, and finally j and a have common endvertex 3).

Problem 4. Fix $d \geqslant 2$, and consider a connected d-regular graph G. First of all, we note that G should contain **at least** $d+1$ vertices, and therefore, by the Handshaking Lemma, it should also contain **at least**

$$\frac{1}{2}(d+1) \cdot \text{common degree} = \frac{(d+1)\cdot d}{2} \geqslant \frac{3\cdot 2}{2} = 3$$

edges.

Moreover, from HW2, Problem 3, we know that $L(G)$ will be a connected graph (which, as we just saw, will have at least 3 vertices, and thus also at least 2 edges).

Consider now a vertex e of $L(G)$, or equivalently an edge e of G. Assume that the endvertices of e in G are the vertices u_1 and u_2. We recall that

$$\begin{aligned} \deg_{L(G)}(e) &= \big(\deg_G(u_1) - 1\big) \; + \; \big(\deg_G(u_2) - 1\big) \\ &= \deg_G(u_1) + \deg_G(u_2) - 2 = 2d - 2. \end{aligned}$$

Since $e \in V(L(G)) = E(G)$ was arbitrary, we can conclude that $L(G)$ will be a $(2d-2)$-regular graph, and thus the degrees of all its vertices will be even.

Therefore, we can refer to Theorem 1 of Lecture 17, which gives us that $L(G)$ will be Eulerian.

Problem 5. (a) We first show that G_0 has a Hamilton path: one example here is the path

$$b\,a\,c\,f\,d\,i\,j\,h\,e\,g.$$

Next, we justify why G_0 cannot have a Hamilton cycle: assume towards a contradiction that we could find a cycle C_H in G_0 which passes through all vertices of G_0.

Observe now that both vertex b and vertex g have degree 2 in G_0, so for each of these vertices both its neighbours must also be neighbours of the vertex in C_H. In addition, b and g have vertex e as their common neighbour, so one 'arc' of C_H must be of the form $b\,e\,g$ or $g\,e\,b$ (depending on which direction we traverse the cycle C_H in).

Given now that we can consider any vertex in the cycle as its initial vertex, and also that we can move in two possible directions along the cycle, we can assume without loss of generality that the initial vertex of C_H is vertex b, and that then we move to vertex e and then to vertex g. In other words, we can assume that we begin traversing the cycle C_H by moving along the (initial) path

$$b\,e\,g.$$

We now observe the following:

- the next vertex on the cycle should necessarily be the other neighbour of g, that is, vertex j.
- From vertex j we can either move to vertex h or to vertex i.
  - If we move to vertex h, we should then necessarily move to vertex f (given that e has already appeared in the cycle).

    From f we should then move to i, because otherwise there would be only one neighbour of i left to pass through, so we could not both arrive at i and move on from it (alternatively, we could say that, since there's only one neighbour of i left that could be adjacent to it in the cycle C_H as well, we would get that i has degree $\leqslant 1$ in C_H, which is absurd).

    But if we move from f to i (and given that we have already passed through j too), we then must move to d, and from d we must move to a.

    Finally, from a we should move to c first (because if we move to b, we will have completed a cycle, which however does not contain c). Note however that, once we move to c, there's no way to get to b without passing through a vertex we have already used.

This shows that there couldn't be a Hamilton cycle in G_0 which starts with the path

$$b\,e\,g\,j\,\mathbf{h}.$$

– We recall that the only other option that we have for the cycle C_H (given how we chose to start traversing it) is for the cycle to start with the path

$$b\,e\,g\,j\,\mathbf{i}.$$

But in this case the only neighbour of h left to pass through is vertex f, thus we cannot both arrive at h and move on from it.

This shows that we cannot have a Hamilton cycle in G_0 which start with the path

$$b\,e\,g\,j\,\mathbf{i}$$

either.

Given that we could start traversing the cycle C_H that we assumed exists by first moving along the path $b\,e\,g$, and that next this path should be extended either to the path $b\,e\,g\,j\,h$ or to the path $b\,e\,g\,j\,i$, and given that either of these two options leads to a contradiction, we conclude that no Hamilton cycle C_H can be found in G_0.

(b) We will show that graph G_1 does not have a Hamilton cycle either; we will use a similar analysis to part (a) (recall that this argument here also helps with one of the two solutions we gave for Problem 1, part (b) of this homework).

We first observe that G_1 contains two 'main' cycles: cycle $A\,B\,C\,D\,E\,A$, which we can call the "outer" cycle of G_1, and cycle $F\,H\,J\,G\,I\,F$, which we can call the "inner" cycle of G_1.

Again, we assume towards a contradiction that G_1 contains a Hamilton cycle C_H, and we note that, at least one of the "outer" vertices of G_1 must be adjacent to one of the "inner" vertices on this cycle.

– **By symmetry**, we can assume that this "outer" vertex is A, and hence the "inner vertex is F.

– We can also view A and F as the first two vertices of the cycle (given that we can start traversing C_H from any of its vertices, and that we can choose to move in either of two directions).

– Moreover, **by symmetry again**, we can assume that the last vertex that we will pass through on C_H before returning to A is the vertex B.

The above discussion shows that it suffices to restrict our attention to Hamilton cycles with first edge AF and last edge BA. But then there are only a few options regarding how we can 'order' the remaining vertices:

- If after vertices A and F we only have one more "inner" vertex before we pass through an "outer" vertex again, then the possible cycles with the above properties are the following:

$$A\,F\,H\,C\,B\,A, \qquad A\,F\,H\,C\,D\,I\,G\,B\,A, \qquad A\,F\,H\,C\,D\,E\,J\,G\,B\,A$$
$$A\,F\,I\,D\,C\,B\,A, \qquad A\,F\,I\,D\,E\,J\,G\,B\,A, \quad \text{and} \quad A\,F\,I\,D\,E\,J\,H\,C\,B\,A$$

(note that we don't also consider in the above list the cycle

$$A\,F\,H\,C\,D\,I\,G\,J\,E\,A$$

(for instance), because that cycle would not end with the edge BA; however, as we said above, it does suffice to only look at cycles which end with that edge because, if there existed a Hamilton cycle in G_1, we would also be able to find one among those cycles which start with the edge AF and end with the edge BA).

Observe now that none of the cycles is a Hamilton cycle.

- If after vertices A and F we have two more "inner" vertices before we pass through an "outer" vertex again, then the possible cycles with the above properties are the following:

$$A\,F\,H\,J\,E\,D\,I\,G\,B\,A, \qquad A\,F\,H\,J\,E\,D\,C\,B\,A, \quad \text{and} \quad A\,F\,I\,G\,B\,A.$$

Again, observe that none of these is a Hamilton cycle.

- If after vertices A and F we have three more "inner" vertices before we pass through an "outer" vertex again, then the possible cycles with the above properties are the following:

$$A\,F\,H\,J\,G\,B\,A, \quad \text{and} \quad A\,F\,I\,G\,J\,E\,D\,C\,B\,A.$$

None of these is a Hamilton cycle.

- Finally, if after vertices A and F we have all of the remaining "inner" vertices following before we pass through an "outer" vertex again, then the possible cycles with the above properties are the following:

  $$A\,F\,H\,J\,G\,I\,D\,C\,B\,A, \quad \text{and} \quad A\,F\,I\,G\,J\,H\,C\,B\,A.$$

  None of these is a Hamilton cycle.

We conclude that G_1 does not contain any Hamilton cycles with first edge AF and last edge BA. But as we explained above, this would follow from the assumption that G_1 contains at least one Hamilton cycle, and this shows that this assumption was incorrect.

Next, we show that the other two graphs, graphs H_1 and H_2, are Hamiltonian.

- A Hamilton cycle in H_1 is the cycle

  $$a\ e\ b\ f\ j\ i\ d\ k\ q\ l\ c\ g\ h\ a.$$

- A Hamilton cycle in H_2 is the cycle

  $$a\ i\ j\ k\ b\ c\ p\ l\ y\ x\ w\ g\ f\ s\ t\ z\ q\ r\ d\ e\ a.$$