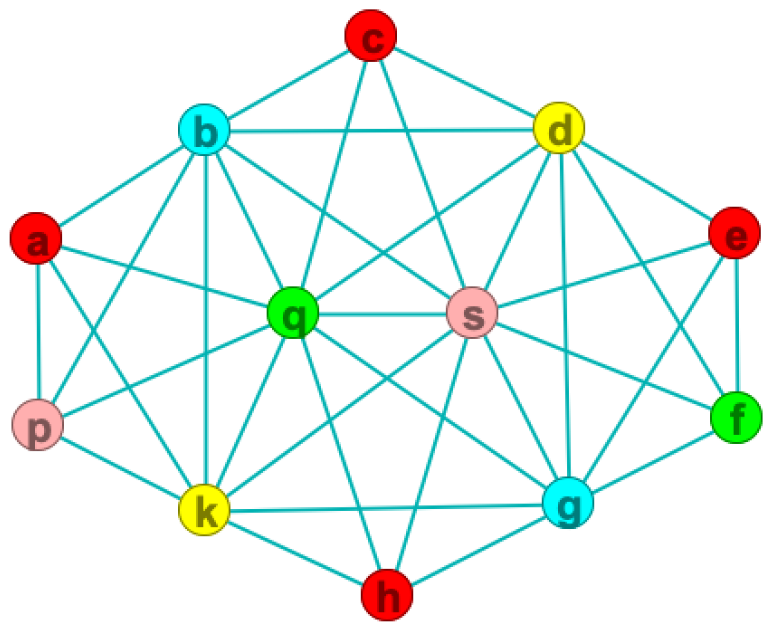
Math 322 – Fall Term 2020
Suggested solutions to the Final exam

Problem 1. (a) We observe that the induced subgraph on the vertices b, c, d, q and s is a clique of order 5. Thus, $\chi(G_0) \geqslant \omega(G_0) \geqslant 5$.

We now give a proper colouring of G_0 which uses 5 colours:



This also shows that $\chi(G_0) \leqslant 5$, which combined with our initial observation allows us to conclude that $\chi(G_0) = 5$. Thus, the above colouring is minimal.

(b) G_0 has two-factors, that is, 2-regular spanning subgraphs.

- One example is the Hamilton cycle

$$c\,b\,a\,p\,k\,q\,h\,s\,g\,f\,e\,d\,c.$$

- Another example is the disjoint union of the following cycles:

$$a\,b\,s\,k\,p\,a \qquad \text{and} \qquad c\,q\,h\,g\,f\,e\,d\,c.$$

On the other hand, we recall that, for G_0 to have a two-factorisation, G_0 needs to be a regular graph. However, $\deg_{G_0}(a) = 4 \neq 7 = \deg_{G_0}(b)$, thus G_0 is not regular, and it cannot have a two-factorisation.

Problem 2. (a) The answer here is yes, we know for sure that the size of $\overline{G}$ will be 53.

Indeed, the complement $\overline{G}$ of G will have the same 13 vertices as G, while any two of these vertices will be joined in $\overline{G}$ if and only if they are not joined in G. In other words, the edge sets of G and of $\overline{G}$ are disjoint sets, and their union will give us the edge set of the complete graph on these 13 vertices.

Since the complete graph on 13 vertices contains $\binom{13}{2} = \frac{13 \cdot 12}{2} = 78$ edges, we conclude that

$$e\big(\overline{G}\big) = 78 - e(G) = 78 - 25 = 53.$$

(b) Here we don't have enough information. Each of the graphs below contains 13 vertices and 25 edges:

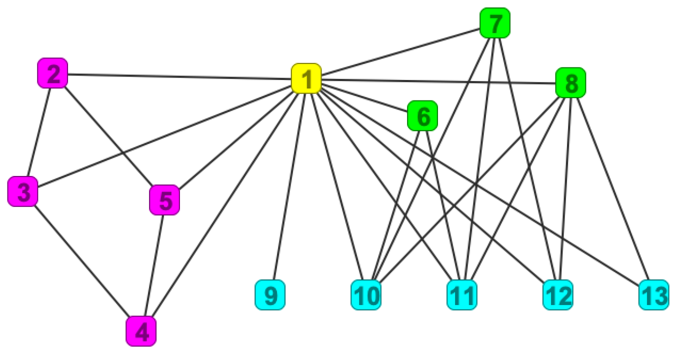


Figure 1: Graph G_1'

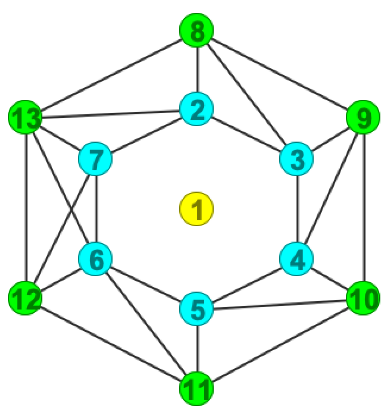


Figure 2: Graph G_2'

Indeed, the degree sequence of G_1' is

$$\big(12,\ 3,\ 3,\ 3,\ 3,\ 3,\ 4,\ 5,\ 1,\ 4,\ 4,\ 3,\ 2\big)$$

(where the vertices are ordered as labelled), and hence, by the Handshaking Lemma, we have

$$e\big(G_1'\big) = \frac{1}{2}\big(12 + 6 \cdot 3 + 3 \cdot 4 + 1 + 2 + 5\big) = \frac{1}{2}\,50 = 25.$$

Similarly, we can check that the degree sequence of G_2' is

$$\big(0,\ 4,\ 4,\ 4,\ 4,\ 5,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4,\ 5\big),$$

and hence, by the Handshaking Lemma, we obtain that

$$e\big(G_2'\big) = \frac{1}{2}\big(0 + 10 \cdot 4 + 2 \cdot 5\big) = \frac{1}{2}\,50 = 25.$$

We now note that G_1' has a vertex, in particular vertex 1, which is joined with each of the other vertices. This implies that 1 will become an isolated vertex in $\overline{G_1'}$, and hence $\overline{G_1'}$ will have at least 2 connected components (in fact, it will have exactly 2 connected components, given that vertex 9 is only joined with vertex 1 in G_1', and hence vertex 9 will be joined with all other vertices except vertex 1 in $\overline{G_1'}$).

On the other hand, G_2' has an isolated vertex, in particular vertex 1, and hence this vertex will be joined with each of the other vertices in the complement $\overline{G_2'}$. This shows that $\overline{G_2'}$ will be connected, or in other words that it will have only 1 connected component.

(c) (i) Here we don't have enough information. Each of the graphs below satisfies $\delta(G) = 3$, $\Delta(G) = 12$, $\alpha(G) = 6$ and $\omega(G) = 4$, as desired:

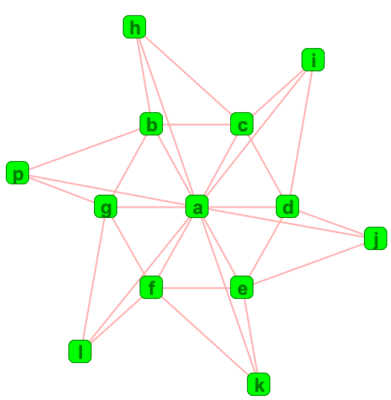


Figure 3: Graph G_3'

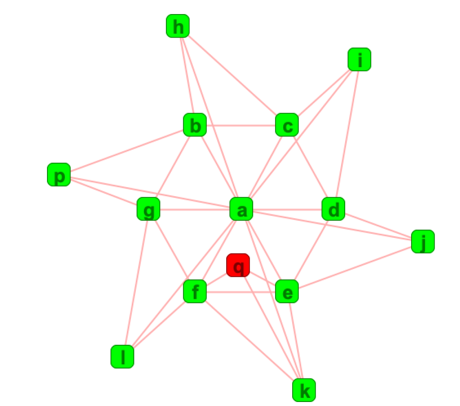


Figure 4: Graph G_4'

Indeed, by simple inspection we can find that both graphs have

- minimum degree 3 (in fact, this is the degree of each of the 'outer' vertices in the first graph, and it's the degree of vertex q, as well as each of the 'outer' vertices except vertex k, in the second graph);
- and maximum degree 12 (which is the degree of vertex a in both graphs).

At the same time, both graphs have clique number equal to 4:

- for the first graph, the induced subgraph on, say, the vertices a, b, c and h is a clique of order 4. On the other hand, if we considered 5 vertices from G_3' while looking for a clique, then we shouldn't consider any of the 'outer' vertices (since each of them only has 3 neighbours); this forces us to consider at least 4 vertices from the set $\{b, c, d, e, f, g\}$; however note that any of these vertices is joined with only two other vertices from this set, and hence any such subgraph of G_3' would not be a clique.

  Combining all observations here, we conclude that $\omega\big(G_3'\big) = 4$, as claimed.

- We can argue similarly about the second graph: the induced subgraph of G_4' on, say, the vertices a, b, c and h is a clique of order 4.

  On the other hand, if we considered 5 vertices from G_4' while looking for a clique, then, out of the 'outer' vertices, we could only consider vertex k which has 4 neighbours; however, k and its 4 neighbours don't form a clique in G_4'.

  Moreover, if we instead considered at least 3 vertices from the set $\{b, c, d, e, f, g\}$, then at least 2 of them would not be joined, and again we would not end up with a clique.

  These show that $\omega\big(G_4'\big) = 4$ as well.

Finally, we can check that both graphs have independence number equal to 6:

- for the first graph, we note that the subset of the 'outer' vertices forms an independent set of 6 vertices, therefore $\alpha\big(G_3'\big) \geqslant 6$.

  On the other hand, if we considered a subset V' of 7 vertices, then at least one of these vertices would be from the subset $\{a, b, c, d, e, f, g\}$. If $a \in V'$, then V' would certainly not be independent since a is joined with every other vertex. If instead we had one of the vertices b, c, d, e, f, g in V', then, for V' to be independent, two of the 'outer' vertices would definitely need to be outside V'; but this would imply that at least one more of the vertices b, c, d, e, f, g is contained in V', and this would then give that two more of the 'outer' vertices need to be omitted if we want V' to be independent; continuing like this, we would obtain that at least 3 of the vertices from $\{b, c, d, e, f, g\}$ need to be contained in V', which would then imply that all 'outer' vertices need to be outside V' if we want it to be independent; this would finally contradict that V' contains 7 vertices. In other words, any subset of the vertices which contains at least 7 of them cannot be independent.

We can conclude that $\alpha\big(G_3'\big) = 6$.

- Similarly for G_4' we have that the subset of the 'outer' vertices is an independent set of vertices, which gives that $\alpha\big(G_4'\big) \geqslant 6$.

  We can also check similarly to above that if V'' is a vertex subset of G_4' with 7 or more vertices, then V'' cannot be independent: indeed, it could not contain vertex a if it were independent, and also if it contained any of the vertices from $\{b, c, d, e, f, g\}$, then we wouldn't be able to find enough pairwise non-adjacent vertices to include in V''. Finally, if V'' included vertex q, then we could only include 5 of the 'outer' vertices in V'' if we wanted it independent, and again this would not give us 7 vertices or more.

  We conclude that $\alpha\big(G_4'\big) = 6$.

We now observe that $|G_3'| = 13$, and hence $\Delta\big(\overline{G_3'}\big) = 13 - 1 - \delta\big(G_3'\big) = 9$.

On the other hand, $|G_4'| = 14$, and hence $\Delta\big(\overline{G_4'}\big) = 14 - 1 - \delta\big(G_4'\big) = 10$.

(ii) We have that $\alpha\big(\overline{G}\big) = \omega(G) = 4$.

Indeed, if v_1, v_2, v_3, v_4 are 4 vertices in G that form a clique, then we know that any two of them are joined in G. This implies that no two of these vertices will be joined in $\overline{G}$, and hence the set $\{v_1, v_2, v_3, v_4\}$ will be an independent set of vertices of $\overline{G}$.

Moreover, if $\overline{G}$ contained an independent set of vertices with 5 vertices, then no two of those vertices would be joined in $\overline{G}$. This is equivalent to saying that any two of these 5 vertices would be joined in G, which would imply that G has a clique of order 5. This contradicts the given assumption, that $\omega(G) < 5$, and it thus shows that $\overline{G}$ cannot have independent sets with 5 (or more) vertices.

(iii) Analogously to part (ii), we can see that $\omega\big(\overline{G}\big) = \alpha(G) = 6$.

Indeed, if $\{w_1, w_2, w_3, w_4, w_5, w_6\}$ is an independent set of 6 vertices in G, then any two of these vertices will be joined in $\overline{G}$, and hence they will form a clique of order 6 in $\overline{G}$. This will show that $\omega\big(\overline{G}\big) \geqslant 6$.

At the same time, if $\overline{G}$ contained a clique of order 7, then this would mean that $\overline{G}$ contains a subset of 7 vertices any two of which are joined in $\overline{G}$. This would then imply that these same vertices are pairwise non-adjacent in G, and it would contradict the assumption that G does not contain independent sets of vertices with 7 vertices (or more). Thus $\omega\big(\overline{G}\big) \leqslant 6$, which finally leads to the desired conclusion.

Problem 3. (a) We saw in class that a graph G, which contains at least one edge, is Eulerian if and only if each of the vertices of G has even degree.

At the same time, for each vertex of the complete bipartite graph $K_{m,n}$ we know that its degree is either m or n (depending on which of the partite sets the vertex belongs to). Thus the degree sequence of $K_{m,n}$ contains n terms equal to m, and m terms equal to n.

Since $K_{m,n}$ will be Eulerian if and only if all these terms are even integers, we conclude that both m and n must be even. Moreover, the converse is true as well: if the positive integers m and n are both even, then the degree sequence of $K_{m,n}$ has only even terms, and hence $K_{m,n}$ is Eulerian.

Summarising the above, we see that the only Eulerian complete bipartite graphs are the graphs $K_{2r,2s}$ with r, s positive integers.

(b) Let us write $u_1, u_2, \ldots, u_m$ for the vertices of $K_{m,n}$ which form one of the two partite sets, and $u_{m+1}, u_{m+2}, \ldots, u_{m+n}$ for the remaining vertices.

If $u_{j_1}\, u_{j_2}\, u_{j_3}\, \cdots\, u_{j_{k-1}}\, u_{j_k}\, u_{j_1}$ is a cycle in $K_{m,n}$, and say $j_1 \leqslant m$, then necessarily the indices $j_1, j_3, j_5, \ldots, j_{k-1}$ are all $\leqslant m$, while the remaining indices must be $> m$. This is because consecutive vertices on the cycle must belong to different partite sets, otherwise they would not be joined by an edge. Analogously we can argue if $j_1 > m$ (then we will have that $j_1, j_3, j_5, \ldots, j_{k-1}$ are all $> m$, while the remaining indices must be $\leqslant m$). From this it also follows that k, the number of vertices on the cycle, must be even.

We conclude that $K_{m,n}$ should contain only even cycles, and more specifically that in each such cycle we should have as many vertices from one of the partite sets as we have from the other partite set too.

Now, if $K_{m,n}$ contains a Hamilton cycle, then all vertices of each partite set will be contained in this cycle, and thus we should have that the number of vertices in the first partite set should be equal to the number of vertices in the second partite set. In other words, we should have $m = n$.

Moreover, this cycle should contain at least 4 vertices (given that any cycle contains at least 3 vertices, and at the same time any cycle contained in $K_{m,n}$ must be even). Therefore, we need $m = n \geqslant 2$.

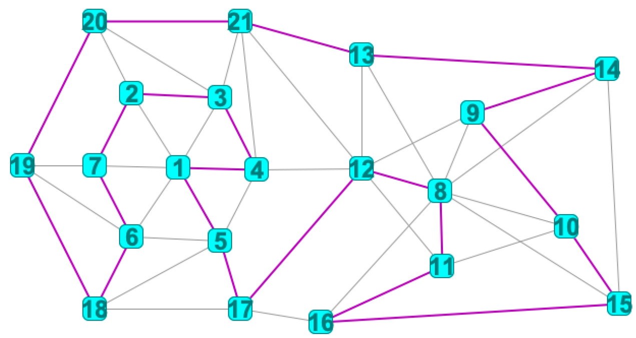
Conversely, we can directly check that the graph $K_{n,n}$ is Hamiltonian when $n \geqslant 2$: again, if we write $u_1, u_2, \ldots, u_n$ for the vertices of $K_{n,n}$ which form one of the two partite sets, and $u_{n+1}, u_{n+2}, \ldots, u_{2n}$ for the remaining vertices, then the following cycle is a Hamilton cycle in $K_{n,n}$:

$$u_1\, u_{n+1}\, u_2\, u_{n+2}\, \ldots\, u_{n-1}\, u_{2n-1}\, u_n\, u_{2n}\, u_1.$$

We conclude that the only Hamiltonian complete bipartite graphs are the graphs $K_{n,n}$ with $n \geqslant 2$.

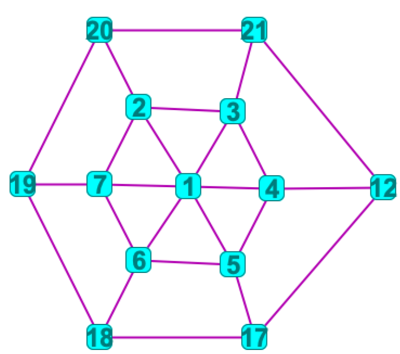
Problem 4. (a) G_1 is Hamiltonian because it contains the following Hamilton cycle (see also image below):

$$21\ 20\ 19\ 18\ 6\ 7\ 2\ 3\ 4\ 1\ 5\ 17\ 12\ 8\ 11\ 16\ 15\ 10\ 9\ 14\ 13\ 21\,.$$



(b) We first show that, for each pair of non-adjacent vertices u, v, there are at least 3 internally disjoint $u - v$ paths.

- Consider first the vertices on the cycle 2 3 4 5 6 7 2. Given any two of them that are non-adjacent, say, $i, j \in \{2, 3, 4, 5, 6, 7\}$ with $i < j$, by traversing the cycle from i to j either in a clockwise manner or in a counterclockwise manner we get two internally disjoint $i - j$ paths. Moreover, the path i 1 j is a third path which is intenally disjoint from either of the previous two.
- Next, we consider the vertices on the cycle 12 17 18 19 20 21 12. Again, given any two non-adjacent vertices s, t here, we can find two internally disjoint paths from s to t by traversing the cycle. A third path internally disjoint from the other two can be found by looking at the following subgraph of G_1:



In this subgraph, each vertex s on the 'outer' cycle is joined with a unique vertex i_s on the 'inner' cycle (and this vertex i_s is not joined with any other vertex on the 'outer' cycle). Thus, another $s - t$ path could be the path

$$s\ i_s\ 1\ i_t\ t,$$

which is clearly internally disjoint from the other two $s - t$ paths that we found.

- By working with this subgraph of G_1 again, we show that we can find at least 3 internally disjoint $u - v$ paths, when u is a vertex on the 'outer' cycle and $v = 1$.

  Let r be the neighbour of u on the 'outer' cycle which we get to when moving clockwise, and let t be the neighbour of u on the 'outer' cycle which we get to when moving counterclockwise. Also, again let i_u, i_r and i_t be the vertices on the 'inner' cycle that u, r and t are joined with respectively in the subgraph.

  Then the following paths are pairwise internally disjoint $u - 1$ paths:

  $$u\ i_u\ 1, \qquad u\ r\ i_r\ 1, \quad \text{and} \quad u\ t\ i_t\ 1.$$

- Similarly we show that we can find at least 3 internally disjoint $u - w$ paths, when u is a vertex on the 'outer' cycle, and w is a vertex on the 'inner' cycle which is non-adjacent to u.

  Again, let r be the neighbour of u on the 'outer' cycle which we get to when moving clockwise, and let t be the neighbour of u on the 'outer' cycle which we get to when moving counterclockwise.

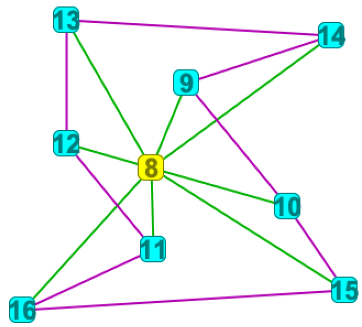
  Then one path from u to w is the path $u\ i_u\ 1\ w$.

  We get another path by moving from u to r, then to i_r and then moving along the 'inner' cycle towards w in a clockwise manner; clearly this path does **not** have any common **internal vertices** with the previous path.

  Finally, we get a third path which is internally disjoint from the previous two by first moving from u to t, then moving to i_t, and then moving along the 'inner' cycle towards w in a counterclockwise manner.

- Now we focus on the vertices on the cycle 13 12 11 16 15 10 9 14 13. Given any two of them that are non-adjacent, say, $i, j \in \{9, 10, 11, 12, 13, 14, 15, 16\}$ with $i < j$, by traversing this cycle from i to j either in a clockwise manner or in a counterclockwise manner we get two internally disjoint $i - j$ paths.

  Moreover, the path $i\ 8\ j$ is a third path which is intenally disjoint from either of the previous two; see also the subgraph of G_1 below:

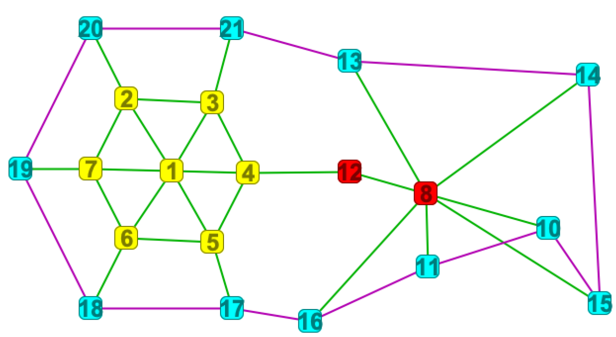


- Next we focus on the vertices on the cycle 21 20 19 18 17 16 11 10 15 14 13 21. Note that we have already found 3 internally disjoint paths connecting two vertices from the subset $\{17, 18, 19, 20, 21\}$, and analogously we have found 3 internally disjoint paths connecting two vertices from the subset $\{10, 11, 13, 14, 15, 16\}$.

  Therefore, here we only consider the case where $u \in \{17, 18, 19, 20, 21\}$ and $v \in \{10, 11, 13, 14, 15, 16\}$. By traversing the above cycle from u to v both clockwise and counterclockwise, we obtain two $u - v$ paths which are internally disjoint. Finally, a third path which is internally disjoint from the previous two is the path

  $$u\ i_u\ 1\ 4\ 12\ 8\ v,$$

  where i_u is the unique vertex from the subset $\{2, 3, 4, 5, 6, 7\}$ which is joined with u in the following subgraph of G_1:



- By looking at this subgraph of G_1 again, we can now also find 3 internally disjoint paths from any vertex $w \in \{1, 2, 3, 4, 5, 6, 7\}$ to any vertex $v \in \{10, 11, 13, 14, 15, 16\}$.

  Indeed, if $w = 1$, then we can consider the following paths:

  - the path 1 4 12 8 v;

- the path which starts from 1, then moves to 3 and 21, then moves to v along the cycle 21 20 19 18 17 16 11 10 15 14 13 21 in a 'clockwise' manner;
- the path which starts from 1, then moves to 5 and 17, then moves to v along the cycle 21 20 19 18 17 16 11 10 15 14 13 21 in a 'counterclockwise' manner.

Moreover, if $w = 4$, then we can consider the following paths:

- the path 4 12 8 v;
- the path which starts from 4, then moves to 3 and 21, then moves to v along the cycle 21 20 19 18 17 16 11 10 15 14 13 21 in a 'clockwise' manner;
- the path which starts from 4, then moves to 5 and 17, then moves to v along the cycle 21 20 19 18 17 16 11 10 15 14 13 21 in a 'counterclockwise' manner.

Similarly, if $w = 3$, then we can consider the following paths:

- the path 3 4 12 8 v;
- the path which starts from 3, then moves to 21, then moves to v along the cycle 21 20 19 18 17 16 11 10 15 14 13 21 in a 'clockwise' manner;
- the path which starts from 3, then moves to vertices $1, 5$ and 17, and finally reaches v by continuing along the cycle 21 20 19 18 17 16 11 10 15 14 13 21 in a 'counterclockwise' manner.

Analogously to the last case, we treat the case where $w = 5$.

Finally, if $w \in \{2, 6, 7\}$, then we can consider the following paths:

- the path w 1 4 12 8 v;
- the path which starts from w, then moves to 3 along the cycle 2 3 4 5 6 7 2 in a clockwise manner, then moves to 21, and finally reaches v by continuing along the cycle 21 20 19 18 17 16 11 10 15 14 13 21 in a 'clockwise' manner again;
- the path which starts from w, then moves to 5 along the cycle 2 3 4 5 6 7 2 in a counterclockwise manner, then moves to 17, and finally reaches v by continuing along the cycle 21 20 19 18 17 16 11 10 15 14 13 21 in a 'counterclockwise' manner.

- The only remaining cases to consider are when we have vertices u, v with $u \in \{8, 9\}$ and $v \in \{1, 2, 3, 4, 5, 6, 7,\ 17, 18, 19, 20, 21\}$. If we first consider one of these cases with $u = 8$, we can consider paths which begin in the following ways:

  - 8 12 4 $\ldots$;
  - 8 16 17 $\ldots$;
  - 8 13 21 $\ldots$.

It is not hard to check that we can extend each of these paths so that it ends at the other vertex v that we are considering, and so that the 3 paths we will get in the end will be internally disjoint.

Similarly we deal with the remaining cases, in which $u = 9$. Then the paths that we must consider could begin in the following ways:

- 9 12 4 ...;
- 9 8 16 17 ...;
- 9 14 13 21 ....

It is not hard to check that we can extend each of these paths so that it ends at the other vertex v that we are considering, and so that the 3 paths we will get in the end will be internally disjoint.

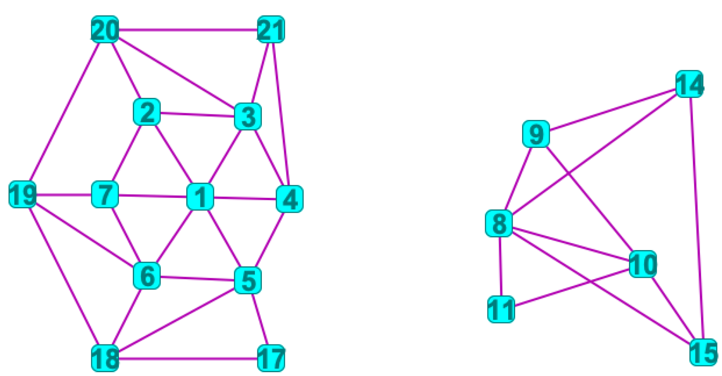
By the above, we see that, for any two non-adjacent and different vertices u, v in G_1, there are at least 3 pairwise internally disjoint $u - v$ paths. By the vertex form of Menger's theorem, we now get that

$$\kappa(u, v) = \kappa'(u, v) \geqslant 3$$

for any such pair of vertices, and hence

$$\kappa(G_1) = \inf\{\kappa(u, v) : u, v \text{ non-adjacent and different}\} \geqslant 3.$$

We now also check that $\kappa(G_1) \leqslant 3$: observe that the subgraph $G_1 - \{12, 13, 16\}$ is disconnected:

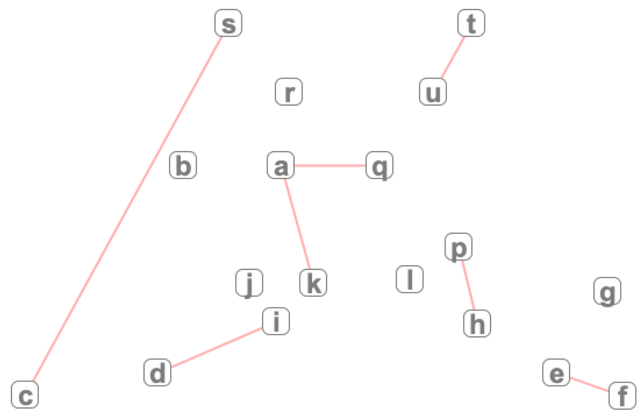


and hence G_1 has a vertex cut of cardinality 3.

Based on all the above, we conclude that $\kappa(G_1) = 3$.

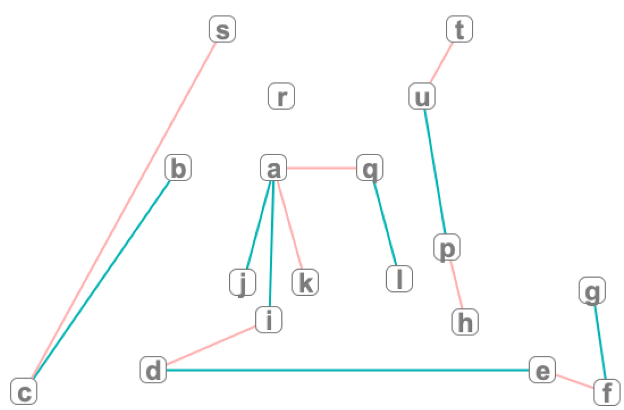
Problem 5. (a) We will find a minimum weight spanning tree by applying Kruskal's algorithm.

- We first observe that the subgraph containing all the edges with minimum weight, and only these edges, has no cycles. Therefore, we can start by including all these edges in the spanning tree we are trying to find.



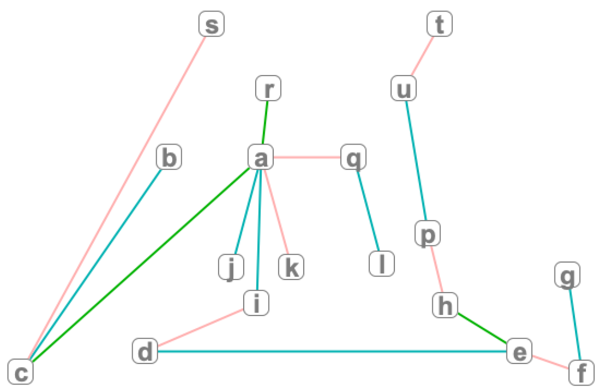
- Next, we can try to include as many of the edges which have weight 3 (which is the minimum weight among the remaining edges) as long as we don't introduce any cycles to our subgraph. We note that, if we include edges ai and aj, then we cannot also include edge dj. Similarly, because we have already added edges ak and aq to our subgraph, we cannot include edge kq.

  However, we could include all other edges that have weight 3, and this is the subgraph we get after this stage of the algorithm too:



- Next, we again try to include as many of the edges which have weight 4 (which is the minimum weight of any edges that we are still allowed to consider). We note that
  - since we have already included edges ak and aj, we cannot include edge kj;
  - since we have already included edges ai and di, we cannot include edge ad;
  - since we have already included edges ai, aq, ql, di, de, ef and fg, we can no longer consider edge gl;
  - we can still add edge ac, but once we do, then we will have to omit edge as;
  - we can also include edge ar;
  - finally, if we also include edge eh, we can no longer consider edges gu, hi and ru.

  According to the above, we get the following subgraph of G_2:



We now observe that this last subgraph is a spanning tree of G_2, and hence we can terminate the process.

We now calculate the total weight of this spanning tree (which will be the minimum possible weight a spanning tree of G_2 could have):

we have seven edges with weight 2, seven edges with weight 3,
and three edges with weight 4,

therefore the total weight is 47.

(b) (i) We assume towards a contradiction that H_2 contains a directed Hamilton path P_0. We begin by listing some pairs of vertices which have to be consecutive in such a path:

- Given that both vertices r and p can only be followed by vertex u in a path, we note that not both of them can have a 'successor' in P_0. Thus necessarily either r or p must be the terminal vertex of P_0.

- Since t cannot be the terminal vertex of P_0 based on what we just observed, it must be followed by vertex s (given that $t \to s$ is the only edge of H_2 directed away from t). Similarly, we see that s must be followed by a in P_0. In other words, P_0 must necessarily contain the subpath $t \to s \to a$.

- Given the previous observation, we now see that c cannot be followed by s, and therefore P_0 must instead contain the directed edge $c \to f$. Moreover, from f we must necessarily move to e, and then from e we must move to d. In other words, P_0 must also contain the subpath $c \to f \to e \to d$.

- Similarly we see that, because P_0 must contain the directed edge $c \to f$, it cannot also contain the directed edge $g \to f$. In other words, g must have a 'successor' in P_0 different from the vertex f. But then this forces g to be followed by t in P_0.

- By the previous observation we also obtain that u cannot be followed by t in P_0, and thus it has to be followed by g instead. In other words, P_0 will contain the subpath $u \to g \to t \to s \to a$.

- We can finally reach a contradiction: since P_0 must contain the directed edge $u \to g$, it cannot also contain the edge $l \to g$. At the same time, since P_0 must contain the directed edge $s \to a$, it cannot also contain the edge $l \to a$. But then l must be the terminal vertex of P_0, which contradicts the initial observation we made, that either r or p must be the terminal vertex of P_0.

We conclude that the assumption that H_2 contains a directed Hamilton path P_0 was incorrect.

(ii) We show that H_2 is strongly connected, or in other words that, given any two different vertices z_1, z_2 in H_2, there is a directed path from z_1 to z_2, and also a directed path from z_2 to z_1.

We begin by observing that H_2 contains the following directed cycle:

$$C_1: \ a \to b \to c \to f \to e \to d \to j \to k \to i \to h \to p \to u \to g \to t \to s \to a.$$

Therefore, if z_1 and z_2 are two of the vertices on the cycle, by traversing the cycle from z_1 to z_2, and then from z_2 to z_1, we obtain the two directed paths we wanted to find. In other words, any two vertices on the cycle are strongly connected in H_2.

Furthermore, we observe that H_2 contains the directed cycles

$$C_2: \ a \to r \to u \to t \to s \to a \qquad \text{and} \qquad C_3: \ a \to q \to l \to a.$$

Thus, any two vertices on the cycle C_2 are strongly connected, and similarly any two vertices on the cycle C_3 are strongly connected (in particular, both q and l are strongly connected with a).

Finally, if $z_1 \in \{q, l\}$ and z_2 is a vertex on C_1 different from a, then we can find a directed path from z_1 to z_2 by first traversing a directed path from z_1 to a (which we already saw exists), and then by continuing on a directed path from a to z_2 (which will be a subpath of the cycle C_1). Similarly, we can find a directed path from z_2 to z_1 by first traversing C_1 from z_2 to a, and then by continuing on a directed path from a to z_1 (which will be a subpath of the cycle C_3).

Analogously, if $z_1 = r$ and z_2 is a vertex on C_1 outside the set $\{a, u, t, s\}$, then we can find a directed path from r to z_2 by first moving from r to u, and then by continuing on a directed path from u to z_2 (which will be a subpath of the cycle C_1). Similarly, we can find a directed path from z_2 to r by first traversing a directed path from z_2 to a (which will be a subpath of the cycle C_1 again), and then by adding the directed edge $a \to r$.

It remains to check that r is strongly connected with any of the vertices q or l, which we can verify by combining parts of the cycles C_2 and C_3: the paths

$$r \to u \to g \to t \to s \to a \to q \to l \qquad \text{and} \qquad q \to l \to a \to r$$

are either directed paths from r to one of the vertices q and l, or conversely from q or l to r, or they contain such a path.