Math 227 – Review of Material for 2nd Midterm

Important Topics/Concepts/Theorems to keep in mind

1. Subspaces and Construction of New Vector Spaces (i) Let V be a vector space over a field $\mathbb{F}$. If S is a subset of V, what do we need to check in order to find out whether S is a subspace of V?

(ii) Given two subspaces S, T of V, how do we define the sum $S+T$ of S and T? Is it a subspace of V too? Explain.

(iii) When is the sum $S+T$ of S and T called the *direct sum* of S and T? (Recall that we denote it by $S \oplus T$ then.)

(iv) True or False? If $V = S+T$, and $\mathcal{B}_1$ is a basis of S, $\mathcal{B}_2$ a basis of T, then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of V.

(v) True or False? If $V = S+T$, and $\mathcal{B}_1$ is a basis of S, $\mathcal{B}_2$ a basis of T, then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a spanning set of V.

(vi) True or False? If $V = S \oplus T$, and $\mathcal{B}_1$ is a basis of S, $\mathcal{B}_2$ a basis of T, then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of V.

(vii) Given a subspace S of V, recall that we define a relation on V by setting

$$\bar{x} \sim_S \bar{y} \quad \text{if and only if} \quad \bar{x} - \bar{y} \in S \qquad\qquad (\bar{x}, \bar{y} \in V).$$

How do we check that this is an equivalence relation?

(viii) Recall that we denote by $[\bar{x}]_S$ or by $\bar{x} + S$ the equivalence class of $\bar{x}$ with respect to the relation $\sim_S$. That is,

$$[\bar{x}]_S = \bar{x} + S := \{\bar{y} \in V : \bar{x} \sim_S \bar{y}\}.$$

How do we prove that

$$[\bar{x}_1]_S = [\bar{x}_2]_S \quad \text{if and only if} \quad \bar{x}_1 \sim_S \bar{x}_2 \ ?$$

(ix) Describe the elements of V/S if: (a) $S = \{\bar{0}_V\}$; (b) $S = V$.

(x) Given a subspace S of V, how do we find a basis for the quotient space V/S?

(xi) Let $\mathbb{F}$, V and $S \leqslant V$ be as above. How do we justify that

$$\dim_{\mathbb{F}} V \ = \ \dim_{\mathbb{F}} S \ + \ \dim_{\mathbb{F}} V/S \ ?$$

2. Bases and Linear Maps (i) Let V be a vector space over a field $\mathbb{F}$, and let T be a subset of V. Are all of the following statements equivalent?

1. T is a basis of V.

2. T is a linearly independent set and a spanning set of V.

3. Every vector $\bar{x}$ of V can be written as a linear combination of distinct vectors from T in an (essentially) unique way (that is, if we consider that the reordering of **non-zero** scalar multiples in the sum is still the same way of expressing $\bar{x}$ as a linear combination, and so is adding some extra **zero** scalar multiples).

Note. In the case that T is a finite set of size k for some $k \in \mathbb{N}$, $T = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k\}$, we can state the above a bit more simply:

1′ T is a basis of V.

3′ For every vector $\bar{x}$ of V, there is a unique choice of scalars $\lambda_1, \lambda_2, \dots, \lambda_k$ such that $\bar{x} = \lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \cdots + \lambda_k \bar{u}_k$.

(ii) Let V_1, V_2 be vector spaces over the same field $\mathbb{F}$, and let $\mathcal{B}$ be a basis of V_1. Can we extend any function

$$\phi : \mathcal{B} \to V_2$$

to a linear map $f : V_1 \to V_2$? If yes, how? If such an extension exists, is it unique?

(iii) Let V_1, V_2 be as above, $\mathcal{B}$ a basis of V_1, and consider a function $\phi : \mathcal{B} \to V_2$ that has a linear extension $f : V_1 \to V_2$. What properties would ϕ need to have in order for us to conclude that:

(a) f is injective?

(b) f is surjective?

(c) f is bijective?

(iv) Let $\mathbb{F}$ be a field, and let V be a finite-dimensional vector space over $\mathbb{F}$. How do we show that V is isomorphic to a vector space of the form $\mathbb{F}^k$ for some integer k? That is, how do we find a (the?) suitable k, and do we define an isomorphism from V to $\mathbb{F}^k$?

(v) Below are some finite-dimensional **real** vector spaces or subsets of such spaces:

$$\mathcal{P}_{10}, \quad \mathbb{C}^4 \textit{ (viewed as a vector space over } \mathbb{R}\textit{)},$$
$$\{p \in \mathcal{P}_5 : p \text{ has exactly degree } 2\}, \quad \{p \in \mathcal{P}_5 : p \text{ has even degree}\},$$
$$\{p \in \mathcal{P}_{20} : p \text{ has only even degree monomials}\},$$
$$\mathbb{R}^{3\times 2}, \quad \{A \in \mathbb{R}^{4\times 4} : A \text{ is upper triangular and } A_{1,1} = A_{1,2} = 1\},$$
$$\{A \in \mathbb{R}^{4\times 4} : A \text{ is upper triangular and has zero third row}\}.$$

- Determine which of these sets are vector spaces over $\mathbb{R}$ and which of them are not. Justify your answer.
- Are any two of the vector spaces you found isomorphic? Find all possible pairs. (Do you need to define an isomorphism to justify your answer?)

(vi) In what setting do Main Theorem D and Main Theorem E tell us essentially the same thing? How do we relate them?

(vii) How do we deduce Main Theorem E from the 1st Isomorphism Theorem?

(viii) Why can we say that the 1st Isomorphism Theorem gives us more information than Main Theorem E?

3. Applications of Determinants (i) Given a matrix $A \in \mathbb{F}^{n\times n}$, how do we define its characteristic polynomial $p_A(t)$?

(ii) Explain why the roots of $p_A(t)$ coincide with the eigenvalues of A.

(iii) Given a matrix $A \in \mathbb{F}^{n\times n}$, how do we show that A has at most n eigenvalues in $\mathbb{F}$?

(iv) How do we define the cofactor matrix of A?

(v) Assume that A is invertible. How can you use the cofactor matrix of A in order to find A^{-1}?

3. Eigenvalues, eigenspaces, characteristic polynomial (i) Given a matrix $A \in \mathbb{F}^{n\times n}$ and an eigenvalue λ of A, what do we define as the *algebraic multiplicity* of λ?

(ii) What do we define as the *geometric multiplicity* of λ?

(iii) How do we find the eigenspace corresponding to λ?

(iv) Is every non-zero vector in the eigenspace corresponding to λ an eigenvector of A?