Reminder from last time:

We want to prove

Main Theorem C Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{m \times n}$.
Then $\dim_{\mathbb{F}} CS(A) = \dim_{\mathbb{F}} RS(A)$.

Last time we saw examples that suggest that
$$\dim_{\mathbb{F}} RS(A) = \dim_{\mathbb{F}} RS(B) = \#\text{ of pivots of } B$$
where B is any Row Echelon Form of A.
In fact, this was based on the fact that
$$RS(A) = RS(B)$$
and a basis for $RS(B)$ is the set of its non-zero rows
(and since B is in REF, # of non-zero rows = # of pivots).

Next we look at some examples to see what happens with Column Spaces.

8) Let $A_3 = \begin{pmatrix} 1 & 0 & 3 & 8 & 6 \\ 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 10 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{4 \times 5}$

What is $\dim_{\mathbb{R}} CS(A_3)$? A basis for $CS(A_3)$?

Remark If ℓ is the number of non-zero rows of A_3, $\ell = 3$ here, then $CS(A_3)$ is contained in the subspace of $\mathbb{R}^m$ ($=\mathbb{R}^4$ here) of vectors with $(\ell+1)$-th, $(\ell+2)$-th, ... m-th components equal to 0, or in other words the subspace of $\mathbb{R}^m$ spanned by the first ℓ standard basis vectors:
$$CS(A_3) \leq \operatorname{span}(\{\bar{e}_1, \bar{e}_2, \ldots \bar{e}_\ell\}) = \left\{ \bar{x} \in \mathbb{R}^m : \cancel{\bar{x}=} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

From this we get that

$$\dim_{\mathbb{R}} CS(A_3) \leq \#\text{ of non-zero rows of } A_3$$
$$= \#\text{ of pivots of } A_3 = 3$$

On the other hand, we also have

$$\dim_{\mathbb{R}} CS(A_3) \geq \#\text{ of pivot columns of } A_3$$
$$= \#\text{ of pivots of } A_3 = 3.$$

Indeed, the pivot columns are linearly independent: if $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ are such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 8 \\ 4 \\ 10 \\ 0 \end{pmatrix} = \begin{pmatrix} \mu_1 + 8\mu_3 \\ \mu_2 + 4\mu_3 \\ 10\mu_3 \\ 0 \end{pmatrix}$$

then we can conclude that $10\mu_3 = 0 \Rightarrow \mu_3 = 0$ and also that $0 = \mu_2 + 4\mu_3 = \mu_2$, $0 = \mu_1 + 8\mu_3 = \mu_1$.

What helps us reach this conclusion? The fact that A_3 is in Row Echelon Form, and so we can do back substitution.

9) Let $A_4 = \begin{pmatrix} 6 & 5 & -1 & 2 \\ 1 & 0 & 2 & 3 \\ 0 & 6 & 4 & 2 \\ 0 & 1 & 3 & 6 \end{pmatrix} \in \mathbb{Z}_7^{4\times 4}$. A basis for $CS(A_4)$?

This matrix is not in REF, so we cannot apply the above reasoning immediately. We have

$$A_4 = \begin{pmatrix} 6 & 5 & -1 & 2 \\ 1 & 0 & 2 & 3 \\ 0 & 6 & 4 & 2 \\ 0 & 1 & 3 & 6 \end{pmatrix} \underset{6R_3}{\overset{6R_1}{\sim}} \begin{pmatrix} 1 & 2 & 1 & 5 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 6 \end{pmatrix} \underset{R_4-R_3}{\overset{R_2-R_1}{\sim}} \begin{pmatrix} 1 & 2 & 1 & 5 \\ 0 & 5 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{3R_2}{\sim} \begin{pmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{R_3-R_2}{\sim} \begin{pmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{R_4-2R_3}{\sim} \begin{pmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B_4$$

Applying the reasoning from the previous part, we have that

$$CS(B_4) = \text{span}(\{\bar{e}_1, \bar{e}_2, \bar{e}_3\})$$

and a basis for $CS(B_4)$ is the set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 4 \\ 0 \end{pmatrix} \right\}.$$

However, $CS(A_4)$ contains $C_2(A_4)$, $C_3(A_4)$ and $C_4(A_4)$ which have fourth component non-zero, so

$$CS(A_4) \neq CS(B_4).$$

But what about their dimensions? Are they equal? Note that

$B_4 \sim A_4$, and in fact we saw that we can use 7 elementary matrices to get from A_4 to B_4:

$$B_4 = \underbrace{\mathcal{E}_7 \mathcal{E}_6 \cdots \mathcal{E}_2 \mathcal{E}_1}_{\substack{Q \text{ product of} \\ \text{elementary matrices,} \\ \text{thus invertible}}} A_4 \iff A_4 = Q^{-1} B_4$$

Recall now the following result from MATH 127:

<u>Lemma</u> Let $u_1, \ldots u_k$ be linearly independent vectors in $\mathbb{F}^m$ and let $Q \in \mathbb{F}^{m \times m}$ be invertible.

Then the vectors $Qu_1, \ldots Qu_k$ are also linearly independent.

Recall now that the pivot columns of B_4, $C_1(B_4)$, $C_2(B_4)$ and $C_4(B_4)$, are linearly independent

$\rightarrow$ $C_1(A_4) = Q^{-1} C_1(B_4)$, $C_2(A_4) = Q^{-1} C_2(B_4)$ and $C_4(A_4) = Q^{-1} C_4(B_4)$ are linearly independent.

$\rightsquigarrow \dim_{\mathbb{Z}_7} CS(A_4) \geq 3.$

On the other hand, if a larger subset $C_{i_1}(A_4), C_{i_2}(A_4), \ldots, C_{i_k}(A_4)$ of the columns of A were linearly independent (where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, $k > 3$), then

$$\{Q C_{i_1}(A_4), Q C_{i_2}(A_4), \ldots, Q C_{i_k}(A_4)\}$$
$$\|$$
$$\{C_{i_1}(B_4), C_{i_2}(B_4), \ldots, C_{i_k}(B_4)\}$$

would be linearly independent, implying that

$$\dim_{\mathbb{Z}_7} CS(B_4) \geq k > 3,$$

which contradicts that $CS(B_4) = \operatorname{span}(\{\bar{e}_1, \bar{e}_2, \bar{e}_3\})$.

Thus $\dim_{\mathbb{Z}_7} CS(A_4) = 3$, and a basis for $CS(A_4)$ is the set

$$\{C_1(A_4), C_2(A_4), C_4(A_4)\}.$$

We can finally give a proof of Main Theorem C:

<u>Proof</u> We will rely on the two facts from MATH 127 that we mentioned in the previous examples

<u>Fact A</u> If $B \in \mathbb{F}^{m\times n}$ is such that $A \sim B$, then

$$RS(A) = RS(B).$$

<u>Fact B</u> If $\{u_1, \ldots, u_k\} \subseteq \mathbb{F}^m$ is a linearly independent set and $Q \in \mathbb{F}^{m\times m}$ is an invertible matrix, then

$\{Qu_1, \ldots, Qu_k\}$ is a linearly independent set too.

With these in mind, we consider a Row Echelon Form B_0 of A, and write ℓ for the number of pivots of B_0 (we also have $\ell = \#$ of non-zero rows of B_0

$= \#$ of pivot columns of B_0).

Then $RS(A) = RS(B_0)$, and hence $\dim_{\mathbb{F}} RS(A) = \dim_{\mathbb{F}} RS(B_0)$.
But $\dim_{\mathbb{F}} RS(B_0) \leq \ell = \#$ of non-zero rows of B_0.
At the same time, if $\mu_1, \mu_2, \dots \mu_\ell \in \mathbb{F}$ are such that
$$\mu_1 R_1(B_0) + \mu_2 R_2(B_0) + \dots + \mu_\ell R_\ell(B_0) = \overline{0}_{\mathbb{F}^n},$$
then:

- since the 1st pivot column of B_0 contains only the 1st pivot as a non-zero entry, we must have $\mu_1 = 0$;
- since the 2nd pivot column of B_0 contains the 2nd pivot and maybe a non-zero entry in the 1st row, and since we already have $\mu_1 = 0$, we must also have $\mu_2 = 0$;

$\vdots$

- if we know already $\mu_1 = \mu_2 = \dots = \mu_{\ell-1} = 0$, then looking at the ℓ-th pivot column (column j_ℓ) ~~would~~ give us
$$0 = \mu_1 (B_0)_{1,j_\ell} + \mu_2 (B_0)_{2,j_\ell} + \dots + \mu_{\ell-1}(B_0)_{\ell-1,j_\ell} + \mu_\ell (B_0)_{\ell,j_\ell} = \mu_\ell (B_0)_{\ell,j_\ell}$$

and since $(B_0)_{\ell,j_\ell}$ is the ℓ-th pivot, it is a non-zero entry which implies $\mu_\ell = 0$.

This shows that the non-zero rows of B_0 are linearly independent, and hence $\dim_{\mathbb{F}} RS(B_0) \geq \ell$
$\rightsquigarrow \dim_{\mathbb{F}} RS(A) = \dim_{\mathbb{F}} RS(B_0) = \ell = \#$ of pivots of B_0.

At the same time, all columns of B_0 have components equal to 0 in rows $\ell+1, \ell+2, \dots m$, therefore
$$CS(B_0) \leq \text{span}(\{\overline{e}_1, \overline{e}_2, \dots, \overline{e}_\ell\}).$$

But also, similarly to above, we can use back substitution to verify that the pivot colums of B_0 are linearly independent. Therefore

$$\ell \geqslant \dim_{\mathbb{F}} CS(B_0) \geq \# \text{ of pivot columns} = \ell.$$

$$\rightsquigarrow \quad \dim_{\mathbb{F}} CS(B_0) = \ell.$$

Now we recall that there exists a product Q_0 of elementary matrices (thus an invertible matrix Q_0) such that

$$B_0 = Q_0 A.$$

Using Fact B we can check that

$$\dim_{\mathbb{F}} CS(A) \leq \dim_{\mathbb{F}} CS(B_0) = \ell$$

(we cannot have more than ℓ columns of A which, taken together, form a linearly independent set, because then we would also have more than ℓ columns of B_0 that are linearly independent, contradicting the fact that $\dim_{\mathbb{F}} CS(B_0) = \ell$).

Similarly, given that we can also write

$$A = Q_0^{-1} B_0,$$

we get from Fact B that

$$\ell = \dim_{\mathbb{F}} CS(B_0) \leq \dim_{\mathbb{F}} CS(A).$$

(and also that the columns of A corresponding to pivot columns of B_0 are linearly independent).

We conclude that

$$\begin{aligned}\dim_{\mathbb{F}} CS(A) &= \dim_{\mathbb{F}} CS(B_0) \\ &= \# \text{ of pivots of } B_0 \\ &= \dim_{\mathbb{F}} RS(B_0) = \dim_{\mathbb{F}} RS(A).\end{aligned}$$

<u>Exercise 1</u> Consider the following vectors from $\mathbb{R}^6$:

$$u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -6 \\ -4 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 10 \\ 5 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 4 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 2 \\ -2 \end{pmatrix}$$

and $u_5 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 14 \\ 6 \end{pmatrix}$. Determine $\dim_{\mathbb{R}} \operatorname{span}(\{u_i : 1 \leq i \leq 5\})$

Also find a basis of $\operatorname{span}(\{u_i : 1 \leq i \leq 5\})$.

Idea: if we write the given vectors as rows of a matrix A, then $\operatorname{span}(\{u_i : 1 \leq i \leq 5\})$ coincides with the Row Space of A. But in the proof of Main Theorem C we did see that we can determine the dimension and also find a basis for $RS(A)$ by finding a Row Echelon Form of A.

Here $A = \begin{pmatrix} 0 & 0 & 1 & 0 & -6 & -4 \\ 1 & 1 & 0 & 1 & 10 & 5 \\ 1 & 1 & 1 & 1 & 4 & 1 \\ 1 & 0 & 3 & 1 & 2 & -2 \\ 1 & 0 & 1 & 1 & 14 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 10 & 5 \\ 0 & 0 & 1 & 0 & -6 & -4 \\ 1 & 1 & 1 & 1 & 4 & 1 \\ 1 & 0 & 3 & 1 & 2 & -2 \\ 1 & 0 & 1 & 1 & 14 & 6 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 1 & 0 & 1 & 10 & 5 \\ 0 & 0 & 1 & 0 & -6 & -4 \\ 0 & 0 & 1 & 0 & -6 & -4 \\ 0 & -1 & 3 & 0 & -8 & -7 \\ 0 & -1 & 1 & 0 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 10 & 5 \\ 0 & 0 & 1 & 0 & -6 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 8 & 7 \\ 0 & 1 & -1 & 0 & -4 & -1 \end{pmatrix} \sim$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 1 & 10 & 5 \\ 0 & 1 & -1 & 0 & -4 & -1 \\ 0 & 0 & 1 & 0 & -6 & -4 \\ 0 & 1 & -3 & 0 & 8 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 10 & 5 \\ 0 & 1 & -1 & 0 & -4 & -1 \\ 0 & 0 & 1 & 0 & -6 & -4 \\ 0 & 0 & -2 & 0 & 12 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 1 & 10 & 5 \\ 0 & 1 & -1 & 0 & -4 & -1 \\ 0 & 0 & 1 & 0 & -6 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B$$

This final matrix B is a Row Echelon Form of A, thus $RS(B) = RS(A)$.
and we've seen that a basis for $RS(B) = RS(A)$ is the set of non-zero rows of B:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 10 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -6 \\ -4 \end{pmatrix} \right\}. \longrightarrow \dim_{\mathbb{R}} RS(A) = \dim_{\mathbb{R}} \operatorname{span}(\{u_i : 1 \le i \le 5\}) = 3.$$

<u>Question</u> What if we wanted to find a basis of $RS(A) = \operatorname{span}(\{u_i : 1 \le i \le 5\})$ that contains as many of the given vectors u_i as possible?

The above basis contains two of the given u_i, but its remaining vector is not equal to any of the other u_i.
Also, even though the u_i appearing in the basis we ended up with happen to be in the first 3

rows of A, this is just a coincidence and it shouldn't suggest that we could take the first 3 rows of A to be a basis for RS(A) simply because a basis for RS(B) = RS(A) is the set of the first 3 rows of B. (in fact here it is easy to verify that the set $\{u_1, u_2, u_3\}$ is <u>not</u> linearly independent given that $u_3 = u_1 + u_2$).

The problem here is that, when we do elementary row operations, we are allowed to, and in many examples we have to, move rows up and down, so by the end of the process the first 3 rows of A may not "correspond" very well to the first 3 rows of B.

On the other hand, when we do elementary row operations, we change the columns of the matrix, but we don't move them around: if $B = E_k \cdots E_2 E_1 A$ with the E_j being elementary matrices, then

$$C_1(B) = E_k \cdots E_2 E_1 C_1(A), \quad C_2(B) = E_k \cdots E_2 E_1 C_2(A)$$

and so on.

This suggests the following idea: work instead with a matrix which arises if we write the u_i as its columns rather than its rows.

Given the notation we already have here, this would be the transpose A^T of A:

$$A^T = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ -6 & 10 & 4 & 2 & 14 \\ -4 & 5 & 1 & -2 & 6 \end{pmatrix} \sim \begin{pmatrix} -6 & 10 & 4 & 2 & 14 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ -4 & 5 & 1 & -2 & 6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -5/3 & -2/3 & -1/3 & -7/3 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -5/4 & -1/4 & 1/2 & -3/2 \end{pmatrix} \sim \begin{pmatrix} 1 & -5/3 & -2/3 & -1/3 & -7/3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 5/3 & 5/3 & 10/3 & 10/3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 5/12 & 5/12 & 5/6 & 5/6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -5/3 & -2/3 & -1/3 & -7/3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10/3 & 10/3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 5/6 & 5/6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -5/3 & -2/3 & -1/3 & -7/3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10/3 & 10/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} =: C$$

Pivot columns of C

The final matrix C is a Row Echelon Form of A^T. Clearly $cs(C) \neq cs(A^T) = \text{span}(\{u_i : 1 \leq i \leq 5\})$ since e.g. here $cs(C) = \text{span}(\{\bar{e}_1, \bar{e}_2, \bar{e}_3\})$ but neither of the columns of A^T are contained in $\text{span}(\{\bar{e}_1, \bar{e}_2, \bar{e}_3\})$.

However, we have seen in the proof of Main Thm C that if certain of the columns of C form a linearly independent set, then the corresponding columns of A^T form a linearly independent set too.

We also saw that the pivot columns of C form a maximal linearly independent subset of the columns of C, therefore

$$C_1(A^T),\ C_2(A^T) \text{ and } C_4(A^T)$$

form a maximal linearly independent subset of the columns of A^T.

We can then conclude that

$$\{u_1, u_2, u_4\} = \{C_1(A^T), C_2(A^T), C_4(A^T)\}$$

is a basis of $CS(A^T) = \text{span}(\{u_i : 1 \leq i \leq 5\})$.

Exercise 2 Why is the following subset of $\mathbb{Z}_{13}^6$ linearly independent? Do we need to use Gaussian elimination to check that it is?

$$\left\{ \begin{pmatrix} 0 \\ 7 \\ 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -6 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 8 \\ 0 \\ 1 \\ 12 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ 0 \\ 0 \\ 11 \\ 1 \end{pmatrix} \right\}$$

We could verify that this set is linearly independent by focusing on the 1st, 3rd, 4th and

6th components of the vectors: just by looking at these components, we can conclude for any linear combination equalling the zero vector, that the coefficients involved need to all be 0: in particular

- the coefficient of the first vector will be equal to the 3rd component of the linear combination (why?), so it will have to equal 0;
- similarly the coefficient of the second vector will be equal to the 1st component of the linear combination, so it must be equal to 0, and so on...

Note also that this reasoning is the same one that we would use if we had to check that the set $\{\bar{e}_3, \bar{e}_1, \bar{e}_4, \bar{e}_6\}$ is linearly independent.

We will now make use of this approach in the proof of Main Theorem D:

Recall <u>Main Theorem D</u> Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{m \times \boxed{n}}$. Then $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$

\# of columns of A.

<u>Remark</u> Recall that nullity(A) is the dimension (over $\mathbb{F}$) of the Nullspace of A. But also

Nullspace of A = solution set of the linear system $A\bar{x} = \bar{0}$.

Proof of Main Thm D We will rely on the following two facts from MATH 127:

Fact A If $A \sim B$, then $(A|\bar{0}) \sim (B|\bar{0})$ as well, and therefore the linear systems

$$A\bar{x} = \bar{0} \quad \text{and} \quad B\bar{x} = \bar{0} \qquad \left(\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \bar{0} \in \mathbb{F}^m\right)$$

are equivalent, or in other words they have the same solution set.

We can also equivalently write that

$$\mathcal{N}(A) = \mathcal{N}(B).$$

In particular, if B_0 is a Row Echelon Form of A, then the same conclusion holds.

Fact B If B_0 is a matrix in REF, then the linear system $B_0\bar{x} = \bar{0}$ is staircase, and we can write ℓ for the number of its pivot variables (which then implies that $n-\ell$ is the number of its free variables). We note that $B_0\bar{x} = \bar{0}$ is consistent (it definitely has one solution, the trivial solution, and if $n-\ell > 0$ (that is, if we have free variables), then the system has more than one solutions).

More specifically, for every choice of values for the free variables, the system has exactly one solution.

We will start by considering a REF B_0 of A. Then by Fact A we have $\mathcal{N}(A) = \mathcal{N}(B_0)$ and hence also $\quad \text{nullity}(A) = \text{nullity}(B_0)$.

We have also seen that

$$\begin{aligned} \text{rank}(A) = \text{rank}(B_0) &= \#\text{ of pivot columns of } B_0 \\ &= \#\text{ of pivots of } B_0 \\ &= \#\text{ of pivot variables of the linear system } B_0\bar{x} = \bar{0} \\ &= \ell \end{aligned}$$

Thus our goal should be to show

$$\text{nullity}(B_0) = n - \ell = \#\text{ of free variables of the linear system } B_0\bar{x} = \bar{0}$$

(to be continued next time)

Proof of Main Thm D (continued)

Recall that we have considered a REF B_0 of A, and we want to show that
$$\text{nullity}(A) = \text{nullity}(B_0) = \#\text{ of free variables of the linear system } B_0\bar{x} = \bar{0}$$

We first deal with the easier cases:

Case (i): If $B_0 = \bar{\mathbf{0}} \in \mathbb{F}^{m\times n}$, then
$$\text{rank}(A) = \text{rank}(B_0) = 0,$$
while $N(A) = N(B_0) = \mathbb{F}^n$, thus
$$\dim_{\mathbb{F}} N(A) = n = n - \ell.$$
Thus we have $\text{rank}(A) + \text{nullity}(A) = n$.

Case (ii): If B_0 has n pivots, or in other words if all variables of the linear system $B_0\bar{x} = \bar{0}$ are pivot variables, then
$$N(A) = N(B_0) = \{\bar{0}_{\mathbb{F}^n}\},$$
and thus $\dim_{\mathbb{F}} N(A) = 0 = n - \ell$.
Thus we have $\text{rank}(A) + \text{nullity}(A) = n$ again.

Case (iii): If $1 \le \ell = \#$ of pivots of $B_0 < n$, then the system $B_0\bar{x} = \bar{0}$ has $n-\ell$ free variables with $n - \ell > 0$. Let's denote these free variables as
$$x_{i_1}, x_{i_2}, \dots x_{i_{n-\ell}}$$
with $1 \le i_1 < i_2 < \dots < i_{n-\ell} \le n$.

By Fact B, we know that for every choice of values for the free variables we can find a vector $\bar{u} \in N(B_0)$ (a solution to the system $B_0\bar{x}=\bar{0}$)

We consider the following choices of values:
For every $1 \leq j \leq n-\ell$
we set $x_{i_j}=1$, while we set the remaining free variables equal to 0.

Then for every $1 \leq j \leq n-\ell$ we get a vector $\bar{u}_j \in N(B_0)$ which satisfies:
its i_j-th component is equal to 1
while for every other index i_s, $s \neq j$, its i_s-th component is equal to 0.

We conclude that the set
$$D=\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-\ell}\} \subseteq N(B_0)$$
is a subset of $N(B_0)$ like the one we discussed in Exercise 2 last time, therefore it is linearly independent.

If we show that it also spans $N(B_0)$, we will be done because we will have found a basis of $N(B_0)$ with size $n-\ell$.

Consider an arbitrary vector $\bar{w} \in N(B_0)$, or in other words a solution $\bar{w}$ to the system $B_0\bar{x}=\bar{0}$.

Then

$$w_{i_1} = \lambda_1, \; w_{i_2} = \lambda_2, \; \dots \; w_{i_{n-\ell}} = \lambda_{n-\ell}$$

for some $\lambda_1, \lambda_2, \dots, \lambda_{n-\ell} \in \mathbb{F}$.

Define now

$$\bar{z} = \lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \dots + \lambda_{n-\ell} \bar{u}_{n-\ell}.$$

But then, by the way we constructed the vectors $\bar{u}_1, \bar{u}_2, \dots \bar{u}_{n-\ell}$,

we have that the i_1-th component of $\bar{z}$ is

$$\lambda_1 \cdot (\bar{u}_1)_{i_1} + \lambda_2 \cdot (\bar{u}_2)_{i_1} + \dots + \lambda_{n-\ell}(\bar{u}_{n-\ell})_{i_1}$$
$$= \lambda_1 \cdot 1 + \lambda_2 \cdot 0 + \dots + \lambda_{n-\ell} \cdot 0 = \lambda_1,$$

its i_2-th component is $= \lambda_2$

$\vdots$

its $i_{n-\ell}$-th component is $= \lambda_{n-\ell}$.

Moreover, since $N(B_0)$ is a subspace, we have that $\bar{z} \in N(B_0)$ too, or in other words that $\bar{z}$ is a solution to the system $B_0 \bar{x} = \bar{0}$.

We thus have that both $\bar{w}$ and $\bar{z}$ are solutions of the system $B_0 \bar{x} = \bar{0}$, and that they agree in the values the free variables have been assigned:

$$w_{i_1} = z_{i_1}, \; w_{i_2} = z_{i_2}, \; \dots \;, w_{i_{n-\ell}} = z_{i_{n-\ell}}.$$

But then by Fact B we get that

$\bar{w} = \bar{z}$ and hence $\bar{w} \in \text{span}(D)$.

Since $\bar{w} \in N(B_0)$ was arbitrary, we conclude that
$D = \{\bar{u}_1, \bar{u}_2, \ldots \bar{u}_{n-\ell}\}$ spans $N(B_0)$.

Thus $\text{nullity}(A) = \dim_{\mathbb{F}} N(A) = \dim_{\mathbb{F}} N(B_0) = |D| = n-\ell$, which gives us the conclusion of the theorem.

<u>More examples of subspaces associated with linear maps and with matrices:</u>

<u>Definition</u> Let V_0 be a vector space over a field $\mathbb{F}$, and let $\ell : V_0 \longrightarrow V_0$ be a linear map.
A subspace S of V_0 is called an <u>invariant subspace</u> of ℓ if S is preserved by ℓ, that is,
if $\ell(S) := \{\ell(\bar{x}) : \bar{x} \in S\} \subseteq S$.

<u>Examples of invariant subspaces</u>

1) The trivial subspaces of V_0, that is, V_0 itself and the zero subspace $\{\bar{0}_{V_0}\}$, are always invariant subspaces of any linear map ℓ:
indeed, $\ell(V_0) = \text{Range}(\ell) \leq V_0$

and also $\ell(\{\bar{0}_{V_0}\}) = \{\ell(\bar{0}_{V_0})\} = \{\bar{0}_{V_0}\}$.

2) A very interesting and useful type of examples of invariant subspaces is based on the following notion:

MATH 227 Jan 31 (3)

Definition Let $g: V_0 \to V_0$ be a linear map.
A non-zero vector $\bar{u}$ of V_0 is called an eigenvector of g if there exists some $\lambda \in F$ such that $g(\bar{u}) = \lambda \bar{u}$.
In this case λ is called an eigenvalue of g.

Remark If $\bar{u}$ is an eigenvector of g (corresponding to eigenvalue λ), then every non-zero scalar multiple of $\bar{u}$ is also an eigenvector of g (corresponding to eigenvalue λ).

Indeed, if $r \in F$, $r \neq 0$,
then $g(\underbrace{r\bar{u}}_{\text{non-zero vector}}) \underset{\text{by linearity of } g}{=} r \cdot g(\bar{u}) = r \cdot (\lambda \bar{u}) = \lambda \cdot (r\bar{u})$

$\rightsquigarrow$ $r\bar{u}$ is eigenvector of g corresponding to eigenvalue λ.

Also $g(r\bar{u}) = (\lambda r) \cdot \bar{u} \in \operatorname{span}(\{\bar{u}\})$.
Finally, $g(0\bar{u}) = g(\bar{0}_{V_0}) = \bar{0}_{V_0} \in \operatorname{span}(\{\bar{u}\})$.

Thus, we have also shown that,
if $\bar{u}$ is an eigenvector of g
then $\operatorname{span}(\{\bar{u}\})$ is an invariant subspace of g.