Applications of Determinants (cont.)

Question from Last Time: Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

which has real entries, and view it as a matrix in $\mathbb{C}^{2\times 2}$. Find its eigenvalues and eigenvectors corresponding to them.

Solution We saw that the characteristic polynomial of A is $p_A(t) = \det(A - tI_2) = (1-t)^2 + 1 = t^2 - 2t + 2$.

It has two complex roots, the numbers

$$t_1 = \frac{2+\sqrt{4-8}}{2} = 1+i \quad \text{and} \quad t_2 = 1-i.$$

Thus, these are the eigenvalues of A.

Eigenvectors?

Let's look for a non-zero vector $\bar{u} \in \mathbb{C}^2$ satisfying

$$A\bar{u} = t_1\bar{u} = (1+i)\bar{u} \iff (A - t_1 I_2)\bar{u} = \bar{0}.$$

In other words, it suffices to find a non-zero solution to the homogeneous system

$$\bar{0} = (A - t_1 I_2)\bar{u} = \begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix}\bar{u} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\bar{u}.$$

We have $\left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$

therefore this system has one pivot variable, the variable u_1, and one free variable, the variable u_2.

Setting $u_2 = 1$, we get the solution
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$
which is an eigenvector corresponding to the eigenvalue $t_1 = 1+i$. Indeed, it's a non-zero vector satisfying
$$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -1+i \\ 1+i \end{pmatrix} = (1+i)\begin{pmatrix} i \\ 1 \end{pmatrix}.$$
Practice Similarly find an eigenvector of A corresponding to the eigenvalue $t_2 = 1-i$.

Application 2: Finding the inverse of an invertible matrix A.

Terminology Let B be a matrix in $\mathbb{F}^{n\times n}$. Recall that, to find $\det(B)$, we can use Laplace expansion over the k-th row, say. In this case

$$\det(B) = (-1)^{k+1} a_{k1} \det(M_{k1}) + (-1)^{k+2} a_{k2} \det(M_{k2}) + \cdots + (-1)^{k+n} a_{kn} \det(M_{kn})$$
and so we need to find the determinants of the submatrices M_{kj} of B (which are the submatrices we get by omitting the k-th row and the j-th column of B).

We call the term $(-1)^{k+j}\det(M_{kj})$ the (k,j)-th cofactor of B.

For this reason, we also often call the above expression for $\det(B)$ the cofactor expansion of $\det(B)$ over the k-th row.

<u>Theorem</u> Let A be an invertible matrix in $\mathbb{F}^{n\times n}$, that is, we have $\det(A)\neq 0$.
Then, if

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2n} \\ \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{pmatrix}$$

cofactors of A

is the cofactor matrix of A, we have that

$$A^{-1} = \frac{1}{\det(A)} C^T = \begin{pmatrix} \frac{C_{11}}{\det(A)} & \frac{C_{21}}{\det(A)} & \cdots & \frac{C_{n1}}{\det(A)} \\ \frac{C_{12}}{\det(A)} & & & \\ \vdots & & & \\ \frac{C_{1n}}{\det(A)} & \frac{C_{2n}}{\det(A)} & \cdots & \frac{C_{nn}}{\det(A)} \end{pmatrix}$$

<u>Proof</u> We have to prove that

$$A\cdot\left(\frac{1}{\det(A)} C^T\right) = \left(\frac{1}{\det(A)} C^T\right) A = I_n.$$

In fact, it suffices to show that

$$A\cdot C^T = \det(A)\cdot I_n = \begin{pmatrix} \det(A) & 0 & 0 & \cdots & 0 \\ 0 & \det(A) & 0 & \cdots & 0 \\ & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \det(A) \end{pmatrix}$$

(why?).
We will check what happens with diagonal entries and with non-diagonal entries separately.

<u>Diagonal entries</u>: Consider $i\in\{1,2,\dots,n\}$. Then the (i,i)-th entry of the product $A\cdot C^T$ is equal to

$$\langle \mathrm{Row}_i(A), \mathrm{Col}_i(C^T)\rangle = \langle \mathrm{Row}_i(A), \mathrm{Row}_i(C)\rangle =$$

$$= \sum_{j=1}^{n} a_{ij}\, C_{ij} = \sum_{j=1}^{n} a_{ij}\,(-1)^{i+j}\det(M_{ij}(A)) = \det(A)$$

Laplace (or cofactor) expansion of $\det(A)$ over the i-th row

<u>Non-diagonal entries</u>: Let $i,k \in \{1,2,\dots n\}$, $i \neq k$. (For convenience let us assume $i<k$; the proof is completely analogous in the case $k<i$). The (i,k)-th entry of the product $A\cdot C^T$ is equal to

$$\langle \mathrm{Row}_i(A), \mathrm{Col}_k(C^T)\rangle = \langle \mathrm{Row}_i(A), \mathrm{Row}_k(C)\rangle =$$

$$= \sum_{j=1}^{n} a_{ij}\, C_{kj} = \sum_{j=1}^{n} a_{ij}\,(-1)^{k+j}\det(M_{kj}(A)) =$$

$$= \sum_{j=1}^{n} a_{ij}\,(-1)^{k+j}\det\begin{pmatrix} \widetilde{R}_1(A) \\ \widetilde{R}_2(A) \\ \vdots \\ \widetilde{R}_i(A) \\ \vdots \\ \widetilde{R}_{k-1}(A) \\ \widetilde{R}_{k+1}(A) \\ \vdots \\ \widetilde{R}_n(A) \end{pmatrix}$$

1st row of A but without its j-th entry

$$= \det\begin{pmatrix} R_1(A) \\ R_2(A) \\ \vdots \\ R_i(A) \\ \vdots \\ R_{k-1}(A) \\ R_i(A) \\ R_{k+1}(A) \\ \vdots \\ R_n(A) \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow i\text{-th row} \\ \\ \\ \leftarrow k\text{-th row} \\ \\ \\ \\ \end{matrix} = \det(B_{k,i}) = 0$$

Laplace expansion over the k-th row of the matrix $B_{k,i}$ which has the same rows as A except for the k-th row which is equal to $\mathrm{Row}_i(A)$

by the alternating property

By combining both arguments, we conclude that every entry of the product AC^T is equal to the corresponding entry of the matrix $\det(A)\cdot I_n = \begin{pmatrix} \det(A) & & \mathbf{0} \\ & \det(A) & \\ \mathbf{0} & & \ddots \ \det(A) \end{pmatrix}$.

Example 1 Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 5 & 6 \\ 0 & 6 & 3 \end{pmatrix} \in \mathbb{Z}_7^{3\times 3},$$

which is invertible. Find its inverse A^{-1} by using the cofactor matrix of A.

Solution The cofactor matrix of A is

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} 0 & 4 & 6 \\ 5 & 6 & 2 \\ 4 & 2 & 0 \end{pmatrix}.$$

Also, now that we have all the cofactors of A, we can readily compute det(A): e.g.

$$\det(A) = a_{11}\cdot C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= 2\cdot 0 + 3\cdot 4 + 0\cdot 6 = 5.$$

Thus in the end

$$A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{5}\begin{pmatrix} 0 & 5 & 4 \\ 4 & 6 & 2 \\ 6 & 2 & 0 \end{pmatrix} = 3\begin{pmatrix} 0 & 5 & 4 \\ 4 & 6 & 2 \\ 6 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 5 \\ 5 & 4 & 6 \\ 4 & 6 & 0 \end{pmatrix}$$

Practice Double check that this is A^{-1} by using the method of Gauss-Jordan elimination too.

Recall

Main Theorem D Let $\mathbb{F}$ be a field, m, n positive integers, and $A \in \mathbb{F}^{m\times n}$. Then

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = \dim_{\mathbb{F}} CS(A) + \dim_{\mathbb{F}} N(A) = n.$$

Main Theorem E Let V_1, V_2 be vector spaces over the field $\mathbb{F}$, and let $\ell : V_1 \to V_2$ be a linear map. Then

$$\dim_{\mathbb{F}} \mathrm{Range}(\ell) + \dim_{\mathbb{F}} \mathrm{Ker}(\ell) = \dim_{\mathbb{F}} V_1.$$

We will give two proofs of Main Theorem E: one in the case that the vector spaces V_1, V_2 are both finite-dimensional, and another one for arbitrary vector spaces (both finite-dimensional, or both infinite-dimensional, or one of them finite-dimensional and the other one infinite-dimensional).

Proof of Main Theorem E in the very special case that $V_1 = \mathbb{F}^n$ and $V_2 = \mathbb{F}^m$ for some positive integers n, m:

In this case we can find a matrix representation for $\ell : V_1 \to V_2$ as follows: there is $A_\ell \in \mathbb{F}^{m\times n}$ such that for every $\bar{x} \in \mathbb{F}^n$

$$A_\ell \bar{x} = \ell(\bar{x}).$$

Then we have seen that, for every $i \in \{1, 2, \dots n\}$,

$$\ell(\bar{e}_i) = Col_i(A_\ell),$$

and thus

$$\mathrm{Range}(\ell) \underset{\substack{\uparrow \\ \text{recall} \\ \text{Problem 4(ii)} \\ \text{of 1st Midterm}}}{=} \mathrm{span}(\{\ell(\bar{e}_i) : 1 \le i \le n\}) = CS(A_\ell).$$

At the same time
$$\mathrm{Ker}(\ell) = \{\bar{x} \in \mathbb{F}^n : A_\ell \bar{x} = \bar{0}_{\mathbb{F}^m}\} = N(A_\ell).$$

We conclude that
$$\begin{aligned}\dim_{\mathbb{F}} \mathrm{Range}(\ell) + \dim_{\mathbb{F}} \mathrm{Ker}(\ell) &= \dim_{\mathbb{F}} CS(A_\ell) + \dim_{\mathbb{F}} N(A_\ell) \\ &= \mathrm{rank}(A_\ell) + \mathrm{nullity}(A_\ell) \\ &= n = \dim_{\mathbb{F}} \mathbb{F}^n = \dim_{\mathbb{F}} \mathrm{Dom}(\ell).\end{aligned}$$

We will next try to see how to use a similar proof method for other finite-dimensional vector spaces. For this reason we need to understand how linear maps act on bases.

Lemma 1 Let $\mathbb{F}$ be a field and V_1 a vector space over $\mathbb{F}$. Suppose B is a subset of V_1.

The following statements are equivalent:

I) B is a basis of V_1.

II) Every vector $\bar{x}$ in V_1 can be written as a linear combination of vectors in B in a unique way.

Proof II) $\Rightarrow$ I) If every vector $\bar{x}$ in V_1 can be written as a linear combination of vectors in B, then B is a spanning set of V_1.

In addition, since every vector in V_1 can be uniquely written as a linear combination of vectors in B, the zero vector $\bar{0} \in V_1$ in particular will be uniquely written as a linear combination of vectors in B. But this implies that, whenever we have a linear combination of vectors in B equal to $\bar{0}$, the coefficients must be zero. Thus the set B must be linearly independent.

So we conclude that B is both a spanning set of V_1

and a linearly independent set, and thus it is a basis of V_1

I) $\Rightarrow$ II) Since B is a spanning set of V_1, we must have that every vector $\bar{x} \in V_1$ can be written as a linear combination of vectors in B.

Assume towards a contradiction that there is $\bar{x}_0 \in V_1$ that can be written as a linear combination of vectors in B in at least two different ways. In other words, assume that there are $k \geq 1$, $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k \in B$ distinct and scalars $\lambda_1, \lambda_2, \dots, \lambda_k, \mu_1, \mu_2, \dots, \mu_k \in F$ with

$$(\lambda_1, \lambda_2, \dots, \lambda_k) \neq (\mu_1, \mu_2, \dots, \mu_k)$$

and such that

$$\lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \dots + \lambda_k \bar{u}_k = \bar{x}_0 = \mu_1 \bar{u}_1 + \mu_2 \bar{u}_2 + \dots + \mu_k \bar{u}_k.$$

But then

$$(\lambda_1 - \mu_1)\bar{u}_1 + (\lambda_2 - \mu_2)\bar{u}_2 + \dots + (\lambda_k - \mu_k)\bar{u}_k = \bar{0}$$

while not all coefficients $\lambda_i - \mu_i$ in this last linear combination are equal to zero.

This would imply that the subset $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k\}$ of B is not linearly independent, and hence B is not linearly independent either, contradicting our assumption that B is a basis of V_1.

We conclude that every vector $\bar{x} \in V_1$ should be uniquely written as a linear combination of vectors in B.

Clarification If S is a (potentially) infinite subset of a vector space V (over a field F), what do we mean by span(S)?

Also, when would S be linearly independent?

Definition 1 Let V be a vector space over a field $\mathbb{F}$ and let S be a subset of V (potentially infinite). Then

$$\operatorname{span}(S) = \{\bar{x} \in V : \exists\, k \in \mathbb{Z}_{\geq 0},\ \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k \in S \text{ and } \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{F} \text{ such that } \bar{x} = \lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \cdots + \lambda_k\bar{u}_k\}.$$

Alternatively, we can say that

$$\operatorname{span}(S) = \bigcup_{\substack{T \subseteq S \\ T \text{ finite}}} \operatorname{span}(T).$$

Definition 2 Let V be a vector space over a field $\mathbb{F}$ and let S be a subset of V (potentially infinite). Then S is linearly independent if, whenever we have $k \geq 1$, $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k \in S$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{F}$ such that $\lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \cdots + \lambda_k\bar{u}_k = \bar{0}_V$, we obtain that $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0_{\mathbb{F}}$.

Returning to Lemma 1, Condition (II) of the lemma allows us to state and prove the following

Proposition 1 Let $\mathbb{F}$ be a field and let V_1, V_2 be vector spaces over $\mathbb{F}$. Suppose $B \subseteq V_1$ is a basis of V_1, and consider a function

$$\varphi : B \longrightarrow V_2.$$

Then there is a unique linear map $\ell : V_1 \longrightarrow V_2$ that extends φ, or, in other words, such that

$$\text{for every } \bar{u} \in B \quad \ell(\bar{u}) = \varphi(\bar{u}).$$

Proof Next time.

From last Time:

Proposition 1 Let $\mathbb{F}$ be a field and let V_1, V_2 be vector spaces over $\mathbb{F}$. Suppose $B \subseteq V_1$ is a basis of V_1, and consider a function

$$\varphi : B \longrightarrow V_2.$$

Then there is a unique linear map $\ell : V_1 \to V_2$ that extends φ (in other words, satisfying

$$\text{for every } \bar{u} \in B \quad \ell(\bar{u}) = \varphi(\bar{u})).$$

Remark The proposition allows us to conclude that

- for every linear function $\ell : V_1 \longrightarrow V_2$
  we get a function $\varphi : B \longrightarrow V_2$
  satisfying $\ell(\bar{u}) = \varphi(\bar{u})$ for every $\bar{u} \in B$
  (this will be the restriction of ℓ to the set B)
- for every function $\psi : B \longrightarrow V_2$
  we get a unique linear function $g : V_1 \longrightarrow V_2$
  satisfying $g(\bar{u}) = \psi(\bar{u})$ for every $\bar{u} \in B$.

In other words, we get a 1-1 and onto correspondence between linear functions $\ell : V_1 \longrightarrow V_2$ and functions $\varphi : B \longrightarrow V_2$.

Proof of Proposition 1 Given the function $\varphi : B \to V_2$ we first need to show how we define its linear extension $\ell : V_1 \to V_2$.

Let $\bar{x} \in V_1$. If $\bar{x} = \bar{0}_{V_1}$, we just set $\ell(\bar{x}) = \bar{0}_{V_2}$. Otherwise, if $\bar{x} \neq \bar{0}_{V_1}$, we have by Lemma 1 from last time that $\bar{x}$ can be uniquely written as a linear combination of vectors in B. In other words, there are unique $k \geq 1$

and $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k \in B$, $\lambda_1, \lambda_2, \ldots \lambda_k \in \mathbb{F}$ with all the λ_i non-zero so that

$$\bar{x} = \lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \cdots + \lambda_k \bar{u}_k.$$

Since this is the only way of writing $\bar{x}$ as the sum of non-zero scalar multiples of vectors in B, there won't be any confusion if we set

$$\ell(\bar{x}) := \lambda_1 \varphi(\bar{u}_1) + \lambda_2 \varphi(\bar{u}_2) + \cdots + \lambda_k \varphi(\bar{u}_k)$$

which is a linear combination of the vectors $\varphi(\bar{u}_i)$, $1 \le i \le k$, in V_2, thus it is an element of V_2.

In particular, for every $\bar{u} \in B$, we have

$$\bar{u} = 1 \cdot \bar{u}$$

(and this is the unique way of writing $\bar{u}$ as a linear combination of vectors in B),

thus we set $\ell(\bar{u}) = 1 \cdot \varphi(\bar{u}) = \varphi(\bar{u})$, which is what we wanted.

Now that we have defined $\ell : V_1 \to V_2$ that extends φ, we have to check that this ℓ is linear.

Let $\bar{x}, \bar{y} \in V_1$ and $r \in \mathbb{F}$.

If one of $\bar{x}, \bar{y}$ is equal to $\bar{0}_{V_1}$, say $\bar{y} = \bar{0}_{V_1}$, then we immediately see that

$$\ell(\bar{x} + \bar{y}) = \ell(\bar{x} + \bar{0}_{V_1}) = \ell(\bar{x}) = \ell(\bar{x}) + \ell(\bar{0}_{V_1}).$$

Similarly, if $r = 0_{\mathbb{F}}$, then

$$\ell(r \cdot \bar{x}) = \ell(0_{\mathbb{F}} \cdot \bar{x}) = \ell(\bar{0}_{V_1}) = \bar{0}_{V_2} = 0_{\mathbb{F}} \cdot \ell(\bar{x}).$$

Thus, from now on we can assume that both $\bar{x}, \bar{y}$ are non-zero vectors of V_1 and r is a non-zero scalar.

Then we know that there are unique $k \geq 1$, $k' \geq 1$, $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{k'} \in B$ and $\lambda_1, \lambda_2, \ldots, \lambda_k, \mu_1, \mu_2, \ldots, \mu_{k'} \in \mathbb{F}$ with all λ_i, μ_j non-zero so that

$$\bar{x} = \lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \dots + \lambda_k \bar{u}_k$$
and $\bar{y} = \mu_1 \bar{v}_1 + \mu_2 \bar{v}_2 + \dots + \mu_{k'} \bar{v}_{k'}$.

We can also suppose that we have ordered the vectors u_i and v_j in such a way that there is some $s \in \mathbb{Z}_{\geq 0}$, $0 \leq s \leq \min\{k, k'\}$ so that
$\bar{u}_i = \bar{v}_i$ if $i \leq s$, while all the other $\bar{u}_i, \bar{v}_j$ are all different.

But then
$$\bar{x} + \bar{y} = (\lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \dots + \lambda_s \bar{u}_s + \lambda_{s+1} \bar{u}_{s+1} + \dots + \lambda_k \bar{u}_k)$$
$$+ (\mu_1 \bar{v}_1 + \mu_2 \bar{v}_2 + \dots + \mu_s \bar{v}_s + \mu_{s+1} \bar{v}_{s+1} + \dots + \mu_{k'} \bar{v}_{k'})$$

$$= (\lambda_1 + \mu_1) \bar{u}_1 + (\lambda_2 + \mu_2) \bar{u}_2 + \dots + (\lambda_s + \mu_s) \bar{u}_s + \lambda_{s+1} \bar{u}_{s+1} + \dots + \lambda_k \bar{u}_k$$
$$+ \mu_{s+1} \bar{v}_{s+1} + \dots + \mu_{k'} \bar{v}_{k'}$$

is the unique way of writing $\bar{x} + \bar{y}$ as a linear combination of vectors in $\mathcal{B}$ (where it's ok to also have some zero scalar multiples since these don't affect the sum).

Thus, by the definition of ℓ,
$$\ell(\bar{x} + \bar{y}) = (\lambda_1 + \mu_1) \varphi(\bar{u}_1) + (\lambda_2 + \mu_2) \varphi(\bar{u}_2) + \dots + (\lambda_s + \mu_s) \varphi(\bar{u}_s)$$
$$+ \lambda_{s+1} \varphi(\bar{u}_{s+1}) + \dots + \lambda_k \varphi(\bar{u}_k) + \mu_{s+1} \varphi(\bar{v}_{s+1}) + \dots + \mu_{k'} \varphi(\bar{v}_{k'})$$
$$= \Big(\lambda_1 \varphi(\bar{u}_1) + \lambda_2 \varphi(\bar{u}_2) + \dots + \lambda_s \varphi(\bar{u}_s) + \lambda_{s+1} \varphi(\bar{u}_{s+1}) + \dots + \lambda_k \varphi(\bar{u}_k)\Big)$$
$$+ \Big(\mu_1 \varphi(\bar{v}_1) + \dots + \mu_s \varphi(\bar{v}_s) + \mu_{s+1} \varphi(\bar{v}_{s+1}) + \dots + \mu_{k'} \varphi(\bar{v}_{k'})\Big)$$

$$= \ell(\bar{x}) + \ell(\bar{y}).$$

Similarly $r \cdot \bar{x} = r(\lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \dots + \lambda_k \bar{u}_k) =$
$$= (r\lambda_1) \bar{u}_1 + (r\lambda_2) \bar{u}_2 + \dots + (r\lambda_k) \bar{u}_k$$
is the unique way of writing $r \cdot \bar{x}$ as a linear combination

of vectors in B, and thus

$$\begin{aligned} \ell(r\bar{x}) &= (r\lambda_1)\varphi(\bar{u}_1) + (r\lambda_2)\varphi(\bar{u}_2) + \dots + (r\lambda_k)\varphi(\bar{u}_k) \\ &= r\,(\lambda_1\varphi(\bar{u}_1) + \lambda_2\varphi(\bar{u}_2) + \dots + \lambda_k\varphi(\bar{u}_k)) \\ &= r\cdot \ell(\bar{x}). \end{aligned}$$

We conclude that the way we defined ℓ (based on the values that φ takes and based on Condition (II) from Lemma 1 of last time) leads to a linear function from V_1 to V_2.

It remains to show that this is the only linear function from V_1 to V_2 that extends φ.

Indeed, suppose $g: V_1 \to V_2$ is also linear and satisfies

$$g(\bar{u}) = \varphi(\bar{u}) \text{ for every } \bar{u} \in B.$$

Then we have $g(\bar{u}) = \ell(\bar{u})$ for every $\bar{u} \in B$.

But if $\bar{x} \in V_1$, then $\bar{x}$ can be written as a linear combination of vectors in B, that is, there are $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k \in B$ and $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{F}$ such that

$$\bar{x} = \lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \dots + \lambda_k\bar{u}_k.$$

Then, by the linearity of ℓ and g we get

$$\begin{aligned} \ell(\bar{x}) &= \ell(\lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \dots + \lambda_k\bar{u}_k) \\ &= \lambda_1\ell(\bar{u}_1) + \lambda_2\ell(\bar{u}_2) + \dots + \lambda_k\ell(\bar{u}_k) \\ &= \lambda_1 g(\bar{u}_1) + \lambda_2 g(\bar{u}_2) + \dots + \lambda_k g(\bar{u}_k) \\ &= g(\lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \dots + \lambda_k\bar{u}_k) = g(\bar{x}). \end{aligned}$$

In other words, we get $\ell(\bar{x}) = g(\bar{x})$ for every $\bar{x} \in V_1$, and thus the two functions ℓ, g coincide.