Recall from last Time

First Isomorphism Theorem Let V_1, V_2 be vector spaces over the same field F and let $g: V_1 \to V_2$ be a linear map. Then we can also define a linear map

$$G: V_1/\ker(g) \longrightarrow V_2 \text{ as follows:}$$

for $\bar{u} + \ker(g) \in V_1/\ker(g)$,
$$G(\bar{u} + \ker(g)) := g(\bar{u}) \in V_2.$$

We can verify that G is well-defined, it is linear, and $\mathrm{Range}(G) = \mathrm{Range}(g)$.
Moreover, G is a linear isomorphism from $V_1/\ker(g)$ to $\mathrm{Range}(G) = \mathrm{Range}(g)$.

Proof We first show that the definition that we gave of G is unambiguous:
let $\bar{u} + \ker(g) \in V_1/\ker(g)$ and assume $\bar{u} + \ker(g) = \bar{u}_1 + \ker(g)$.
We need to show that $g(\bar{u}) = g(\bar{u}_1)$ (in other words, that it doesn't matter which representative from the equivalence class $\bar{u} + \ker(g)$ we take in order to define $G(\bar{u} + \ker(g))$, still the result will be the same).

Since $\bar{u} + \ker(g) = \bar{u}_1 + \ker(g)$, we have $\bar{u} - \bar{u}_1 \in \ker(g)$, and thus $g(\bar{u} - \bar{u}_1) = \bar{0}_{V_2} \Rightarrow g(\bar{u}) - g(\bar{u}_1) = \bar{0}_{V_2} \Rightarrow g(\bar{u}) = g(\bar{u}_1)$.

Next, we check that G is linear.
Let $\bar{u}+\ker(g)$, $\bar{v}+\ker(g) \in V_1/\ker(g)$, and $r \in \mathbb{F}$.

Then $G((\bar{u}+\ker(g))+(\bar{v}+\ker(g))) \underset{\substack{\uparrow \\ \text{definition of } + \text{ in } V_1/\ker(g)}}{=} G((\bar{u}+\bar{v})+\ker(g))$

$\underset{\substack{\uparrow \\ \text{definition of } G}}{=} g(\bar{u}+\bar{v}) \underset{\substack{\uparrow \\ \text{linearity of } g}}{=} g(\bar{u})+g(\bar{v}) \underset{\substack{\uparrow \\ \text{definition of } G \text{ again}}}{=} G(\bar{u}+\ker(g))+G(\bar{v}+\ker(g))$

Similarly $G(r\cdot(\bar{u}+\ker(g))) \underset{\substack{\uparrow \\ \text{definition of } \cdot \text{ in } V_1/\ker(g)}}{=} G((r\bar{u})+\ker(g))$

$\underset{\substack{\uparrow \\ \text{definition of } G}}{=} g(r\bar{u}) \underset{\substack{\uparrow \\ \text{linearity of } g}}{=} r\cdot g(\bar{u}) = r\cdot G(\bar{u}+\ker(g))$.

We conclude that G is linear.

Moreover, for every $\bar{u}+\ker(g) \in V_1/\ker(g)$
$G(\bar{u}+\ker(g)) = g(\bar{u}) \in \text{Range}(g)$.
Thus $\text{Range}(G) \leq \text{Range}(g)$.
Conversely if $\bar{w} \in \text{Range}(g)$, then $\exists\ \bar{v} \in V_1$ s.t. $\bar{w} = g(\bar{v})$.
But then $\bar{w} = g(\bar{v}) = G(\bar{v}+\ker(g)) \in \text{Range}(G)$.
Thus $\text{Range}(g) \leq \text{Range}(G)$ as well.
We conclude that $\text{Range}(G) = \text{Range}(g)$.

To finish the proof of the theorem, that is, to show that $G: V_1/\ker(g) \longrightarrow \text{Range}(g)$
is a linear isomorphism, it remains to show that G is injective.

We recall that (for linear maps such as G) this is equivalent to showing that

$$\ker(G) = \{\bar{0}_{V_1/\ker(g)}\}.$$

Let $\bar{u} + \ker(g) \in \ker(G)$. Then

$$\bar{0}_{V_2} = G(\bar{u} + \ker(g)) = g(\bar{u}) \Rightarrow \bar{u} \in \ker(g)$$
$$\Rightarrow \bar{u} + \ker(g) = \bar{0}_{V_1} + \ker(g) = \bar{0}_{V_1/\ker(g)}.$$

Thus there are no other elements in $\ker(G)$ besides $\bar{0}_{V_1/\ker(g)}$.

This completes the proof of the Theorem.

Next Topic: Orthogonality and Inner Product Spaces

From now on, our scalar field is $\mathbb{R}$ or $\mathbb{C}$.

Definition A real inner product space is a vector space V over $\mathbb{R}$ together with an inner product defined on it, that is,
a positive-definite, symmetric, bilinear form $<\cdot,\cdot>: V\times V\to\mathbb{R}$
For every $\bar{x}_1,\bar{x}_2,\bar{y}\in V$ and for every $\lambda\in\mathbb{R}$, we want

Combining all first 3 properties, we also get linearity in the second argument

- $<\bar{x}_1+\bar{x}_2,\bar{y}> = <\bar{x}_1,\bar{y}> + <\bar{x}_2,\bar{y}>$ linear in the first argument
- $<\lambda\bar{x}_1,\bar{y}> = \lambda<\bar{x}_1,\bar{y}>$
- $<\bar{x}_1,\bar{y}> = <\bar{y},\bar{x}_1>$ $\to$ symmetric
- $<\bar{x}_1,\bar{x}_1>\geq 0$ and $<\bar{x}_1,\bar{x}_1>=0$ iff $\bar{x}_1=\bar{0}_V$ positive-definite

Recall Classical Example we had seen in MATH 127

$V=\mathbb{R}^n$, $<\cdot,\cdot>$ is the standard dot product

$$<\begin{pmatrix}x_1\\x_2\\x_n\end{pmatrix},\begin{pmatrix}y_1\\y_2\\y_n\end{pmatrix}> = \sum_{i=1}^{n} x_iy_i.$$ (recall that we have verified all the above properties for the dot product)

Other examples:

1) $V=\mathbb{R}^{n\times n}$. We can define
$<A,B>:=tr(AB^T)$ = the trace of the product AB^T (where $A,B\in\mathbb{R}^{n\times n}$)

2) $V = C[0,1]$. We can define
$$\langle f, g \rangle := \int_0^1 (f \cdot g)(x)\,dx \qquad (\text{where } f, g \in C[0,1])$$

Practice: check that in both of the above examples, the given vector space becomes an inner product space (that is, verify the properties of the inner product).

Recall also from MATH 127 that, when we have an inner product, we can define the length (or norm) of vectors too.

Definition Let $(V, \langle \cdot, \cdot \rangle)$ be a real inner product space. For every $\bar{x} \in V$, we define the length (or norm) of $\bar{x}$ to be
$$\|\bar{x}\| := \sqrt{\langle \bar{x}, \bar{x} \rangle}.$$
(makes sense b/c of the positive-definiteness of the inner product)

Remark We should think of $\|\bar{x}\|$ as some type of magnitude of $\bar{x}$ or of distance from the origin which is the zero vector $\bar{0}_V$.

Next we see how we can consider analogous notions in complex vector spaces (vector spaces over $\mathbb{C}$)

Definition A complex inner product space is a vector space W over $\mathbb{C}$ together with an inner product defined on it, that is, a positive-definite, conjugate symmetric form $<\cdot,\cdot>: W \times W \to \mathbb{C}$ which is linear in the first argument.

For every $\bar{x}_1, \bar{x}_2, \bar{y} \in W$ and for every $\mu \in \mathbb{C}$, we want

- $<\bar{x}_1 + \bar{x}_2, \bar{y}> = <\bar{x}_1, \bar{y}> + <\bar{x}_2, \bar{y}>$ } linear in the first argument
- $<\mu \bar{x}_1, \bar{y}> = \mu <\bar{x}_1, \bar{y}>$
- $<\bar{x}_1, \bar{y}> = \overline{<\bar{y}, \bar{x}_1>}$ $\longrightarrow$ conjugate symmetry
- $<\bar{x}_1, \bar{x}_1> \geq 0$ and $<\bar{x}_1, \bar{x}_1> = 0$ iff $\bar{x}_1 = \bar{0}_W$ } positive-definiteness

Combining all first 3 properties we also get that $<\cdot,\cdot>$ is conjugate linear in the 2nd argument:

$<\bar{y}, \bar{x}_1 + \bar{x}_2> = <\bar{y}, \bar{x}_1> + <\bar{y}, \bar{x}_2>$

and $<\bar{y}, \mu \bar{x}_1> = \bar{\mu} \cdot <\bar{y}, \bar{x}_1>$.

Examples 0') $W = \mathbb{C}^n$, $<\cdot,\cdot>$ is the standard dot product

$$<\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}> := \sum_{i=1}^{n} x_i \bar{y}_i .$$

Verify that the properties of inner product hold. (complex case)

1') $W = \mathbb{C}^{n \times n}$. For every $B = (b_{ij}) \in \mathbb{C}^{n \times n}$, the conjugate transpose B^* of B is the matrix

$$B^* := (\bar{b}_{ji})_{1 \leq i,j \leq n} .$$

We define $<A, B> := tr(A \cdot B^*)$

Important example for Fourier Analysis

2') Let $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ (the unit circle in the complex plane). Set $W = C(\mathbb{T})$ [continuous functions $f: \mathbb{T} \to \mathbb{C}$ — these can be identified with continuous fcns $\tilde{f}: [0, 2\pi] \to \mathbb{C}$ with the property $\tilde{f}(0) = \tilde{f}(2\pi)$]

We define $<f, g> := \int_{\mathbb{T}} f \cdot \bar{g} = \frac{\int_0^{2\pi} \tilde{f}\, \bar{\tilde{g}}}{2\pi}$