MATH 118 - Midterm 2 - Cheat Sheet

March 12, 2020

Important: The fact that a result or theorem is listed here does not necessarily mean it is needed for completing the exam. Conversely, there may be other results discussed in class that *are* needed, but which are not listed here.

Specific series

- $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges iff $a > 1$.
- $\sum_{n=0}^{\infty} a^n$ converges iff $|a| < 1$, and then the limit is $\frac{1}{1-a}$.
- $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for every real number x.

Theorem (Cauchy's double series theorem)**.** Let $\sum_{m,n=1}^{\infty} a_{mn}$ be a double series. If $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}|$ converges of if $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a_{mn}|$ converges, then the double series converges absolutely and

$$\sum_{m,n=1}^{\infty} a_{mn} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}$$

Radius of convergence The radius of convergence of a formal power series $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is $\frac{1}{L}$ where $L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. $f(x)$ is (absolutely) convergent if $|x - x_0| < R$, and divergent if $|x - x_0| > R$.

Theorem. A power series with positive radius of convergence is both continuous and differentiable at its centre.

Theorem (Transformation Theorem)**.** Let $f(x)$ be a formal power series centred at x_0 with positive radius of convergence R. Then at any x_1 with $|x_1 - x_0| < R$ f can be developped as a power series $f(x) = \sum_{n=0}^{\infty} b_n (x - x_1)^n$ convergent for all x with $|x - x_1| + |x_1 - |x_0| < R$ and $b_n = \frac{D^n(f)(x_1)}{n!}$.

Theorem (General Rolle Theorem)**.** Let f be continuous on $[a, b]$ and n-times continuously differentiable on at least $(a, b]$, such that $f^{(n+1)}$ exists on at least (a, b). If $f(a) = f(b)$ and $f^{(k)}(b) = 0$ for all $1 \leq k \leq n$, then there is $d \in (a, b)$ such that $f^{(n+1)}(d) = 0$.

Taylor polynomial Let f be defined on an interval containing x_0, and suppose f is n-times differentiable at x_0. Then the degree n Taylor polynomial P_{f,n,x_0} of f at x_0 is defined as

$$P_{f,n,x_0} = f(x_0) + f'(x_0)(x - x_0) + 1/2 f^{(2)}(x_0)(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

It is a polynomial function of degree at most n.

Taylor series The Taylor series of a function f at x_0 is the power series $T_{f,x_0} = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_))^n$, if it exists.

Lagrange remainder Keeping the notation above, the (nth) Lagrange remainder $R_n(x)$ (for f at x_0) is the difference $R_n(x) = f(x) - P_{f,n,x_0}$.

Theorem (Taylor's Theorem)**.** Let f be continuous on $[a, b]$ and suppose f is n-times continuously differentiable on $[a, b]$ and suppose $f^{(n+1)}$ exists on at least (a, b). Let $x \neq x_0 \in [a, b]$. Then there is d strictly between x and x_0 such that

$$R_n(x) = \frac{f^{(n+1)}(d)}{(n+1)!}(x - x_0)^{n+1}$$

Indefinite Integrals An antiderivative of f is a function F such that $F' = f$ (with the same domain). We write $\int f dx$ for an antiderivative of f.

Rules Let F, G be antiderivatives of f and g.

- $\int(\alpha f + \beta g)dx = \alpha F + \beta G$ (but note that there may be constants)
- $\int fG dx = FG - \int Fg dx$
- $\int(f \circ G)g = F \circ G$
- If $G'(x) \neq 0$ on J and $G(J) = I$, and $H = \int(f \circ G)g dx$, then $\int f dx = H \circ G^{-1}$ on I.

Riemann sequenes Let $f : [a, b] \to \mathbb{R}$ be a function $P_n = a = x_{n0} < x_{n1} < x_{n2} < \cdots < x_{n,|P_n|} < b = x_{n,|P_n|+1}$ a sequence of partitions of $[a, b]$ such that $m(P_n) \to 0$. Then for any tag vector $\mathbf{y}$

$$S(P_n, \mathbf{y}, f) = \sum_{i=0}^{|P_n|} f(y_{ni})(x_{n,i+1} - x_{ni})$$

is the associated Riemann sequence.

Theorem (First Fundamental Theorem of Calculus)**.** Let $F : [a, b] \to \mathbb{R}$ be continuous and differentiable on at least (a, b), and let $f : [a, b] \to \mathbb{R}$ be continuous or integrable. If $f = F'$ on (a, b), then

$$F(b) - F(a) = \int_a^b f(x)dx.$$

Uniform continuity A function is uniform continuous on I if for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in I$ with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

MATH 118

Midterm Exam

13 February 2020

Location: CAB 235
Time: 1:00pm - 1:50pm
Chief Exam Administrator: Jochen Kuttler

Email/CCID: ______________________________@ualberta.ca

PLEASE NOTE THAT THIS EXAM WILL BE MARKED ELECTRONICALLY

- Scrap paper is provided. Scrap paper will not be collected or marked.
- This is a closed book exam. No notes, books, or formula sheets are permitted.
- All electronic equipment, including calculators, is prohibited. Make certain that mobile phones are turned off.
- Be precise, concise, and use correct terminology in your answers.
- **Show your work!** Answers without justification may receive reduced or no credit.
- Tip: Do those problems first that you know how to do.
- This exam consists of **5 questions**, for a total of **30 points**. There is a bonus question worth 2 points.
- There are a total of 5 sheets (front and back, 10 pages). Make sure that you have a complete exam.
- The numbers in the margin list the points for each question.

If anything is unclear, please ask!

Good Luck!

Extra space

You may use this page if you require additional space for one of the exam questions. Clearly indicate which question and which part(s) of the question you are completing on this page. You may use this page for ONE question only.

[10] **Question 1.** Discuss the function $f(x) = \frac{\log x}{x}$ defined on $(0, \infty)$.

1. Compute f' and f'' where they exist.
2. Determine all local extrema of f.
3. Find all inflection points of f.
4. Determine the intervals where f is (strictly) convex or concave.
5. Compute the (one-sided) limit of f as x approaches 0. Compute $\lim_{x\to\infty} f(x)$ if it exists.
6. Repeat 5. with $f'(x)$ instead of f.

Justify all answers. Justify the limits you compute, in particular in the last two parts.

1. $$f'(x) = f\frac{(1/x)x - \log x}{x^2} = \frac{1 - \log x}{x^2}$$
$$f''(x) = \frac{(-1/x)x^2 - 2x(1 - \log x)}{x^4} = \frac{2\log x - 3}{x^3}$$

2. All potential extrema positions are interior points, so if f has an extremum at x_0 then $f'(x_0) = 0$.
$$1 - \log x = 0 \text{ iff } \log x = 1 \text{ iff } x = e$$
Since $f''(e) = \frac{2-3}{e^3} < 0$ we have a local maximum at $x_0 = e$.

3. Necessary for an inflection point: $f''(x_0) = 0$ (if f'' exists).
$$2\log x - 3 = 0$$
iff
$$x = e^{\frac{3}{2}}$$
To check that this is actually an inflection point we must verify that f changes convexity/concavity behaviour. The next part answers this in the affirmative.

4. f'' has only one zero and is continuous. Thus $f'' < 0$ or $f'' > 0$ on $(0, e^{\frac{3}{2}})$ and likewise on $(e^{\frac{3}{2}}, \infty)$. Now $f''(1) = -3 < 0$ and $1 < e^{\frac{3}{2}} > 1$. So f strictly concave on $(0, e^{\frac{3}{2}})$. Since $f''(x) > 0$ for large x, f is strictly convex on $(e^{\frac{3}{2}}, \infty)$. And it follows that $e^{\frac{3}{2}}$ is indeed an inflection point.

5. $$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log x}{x} = -\infty \cdot \infty = -\infty.$$
$$\lim_{x\to\infty} f(x) = \text{"}\infty/\infty\text{"} = (L'H) \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

6. $$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{1 - \log x}{x^2} = \text{"}\infty \cdot \infty\text{"} = \infty.$$
$$\lim_{x\to\infty} f'(x) = \text{"} - \infty/\infty\text{"} = (L'H) \lim_{x\to\infty} \frac{\frac{-1}{x}}{2x} = 0.$$

[5] **Question 2.** Show that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

METHOD 1: For $n > 1$, we have

$$\frac{n!}{n^n} \leq \frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{2}{n}\frac{1}{n} \leq \frac{2}{n^2}$$

This happens to be true if $n = 1$ as well. Thus, $\sum_{n=1}^{\infty} |\frac{n!}{n^n}|$ is bounded by the convergent series $\sum_{n=1}^{\infty} \frac{2}{n^2}$ and hence convergent itself.
METHOD 2:
The ratio test with $a_n = \frac{n!}{n^n}$ gives

$$\frac{a_{n+1}}{a_n} = (n+1)\frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = (\frac{n}{n+1})^n$$

Recall

$$(\frac{n+1}{n})^n = (1+\frac{1}{n})^n \to e$$

so the above converges to $\frac{1}{e} < 1$ because $e > 1$.
It follows the series is convergent.

[5] **Question 3.** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two absolutely convergent series. Show that $\sum_{n=1}^{\infty} a_n b_n$ is an absolutely convergent series.

For any N we have

$$S_N := \sum_{n=1}^{N} |a_n b_n| \leq (\sum_{n=1}^{N} |a_n|)(\sum_{n=1}^{N} |b_n|) \leq AB$$

where $A = \sum_{n=1}^{\infty} |a_n|$ and $B = \sum_{n=1}^{\infty} |b_n|$.
Since S_N is monotone and bounded, it is convergent. But then $\sum_{n=1}^{n} a_n b_n$ is absolutely convergent.

[6] **Question 4.** Let

$$f(x) = \begin{cases} x^4(\sin(\frac{1}{x}))^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

1. Show that f is differentiable everywhere and $f'(0) = 0$.
2. Show that f has a local minimum at 0.
3. Show that f is not monotone increasing on any interval $(0, a)$ for $a > 0$.

1. If $x \neq 0$, then around x, f is a product of compositions of differentiable functions, and hence differentiable. (You could also compute

   $$f'(x) = 3x^3(\sin(\frac{1}{x}))^2 + x^4 2\sin(\frac{1}{x})\cos(\frac{1}{x})\frac{-1}{x^2}$$

   for $x \neq 0$.

   Now for $x_0 = 0$:

   $$\frac{f(x) - f(0)}{x - x_0} = x^3 \sin(\frac{1}{x})^2 \to 0$$

   for $x \to 0$ because $\sin(\frac{1}{x})^2$ is bounded and $x^3 \to 0$. It follows $f'(0) = 0$.

2. $f \geq 0$ everywhere because $f(x)$ is a square of a real number. So at any point x_0 with $f(x_0) = 0$ f has a global and hence local minimum.

3. $f(x) = 0$ for exactly $x = 0, x = \frac{1}{k\pi}$ and $k \in \mathbb{Z} \setminus \{0\}$. Thus on $[0, a)$, f has an infinite sequence of zeros $x_n \to 0$ (monotone decreasing). And $f(x) > 0$ for some x between each x_{n+1}, x_n. Thus, f cannot be monotone increasing on $[0, x_n)$ for any n, and hence not on $(0, a)$.

[4] **Question 5.** Let f be a function defined and differentiable on $(0, 1)$. Suppose $\lim_{x\to 1} f(x) = \infty$.

1. if $\lim_{x\to 1} f'(x)$ exists, then $\lim_{x\to 1} f'(x) = \infty$.

[2] 2. (OPTIONAL BONUS QUESTION:) Does $\lim_{x\to 1} f'(x)$ always exist? You must justify your answer to receive any extra credit.

1) Let $L = \lim_{x\to 1} f'(1)$. If $L < \infty$, then there exists $M > 0$ and $\delta > 0$ such that $f'(x) < M$ for all $x \in [1 - \delta, 1)$. Then for $x \in (1 - \delta, 1)$ we have

$$\frac{f(x) - f(1 - \delta))}{x - (1 - \delta)} = f'(c) < M$$

for some $c \in (1-\delta, x)$. In particular $f(x) < M(x-(1-\delta))+f(1-\delta)$. So f is bounded on $(1-\delta, 1)$, a contradiction.
2) $\lim_{x\to 1} f'(x)$ need not exist.
Example: first construct function f such that $\lim_{x\to\infty} f(x) = \infty$, but $\lim_{x\to\infty} f'(x)$ does not exist. For example $f(x) = xe^{\sin x}$. Then $f(x) \geq x1/e \to \infty$. But $f'(x) = e^{\sin x} + x(\cos x)e^{\sin x}$. f' changes sign infinitely many often for $x \to \infty$.
Now take $h\colon (0, 1) \to \mathbb{R}$ defined as $h(x) = f(1/(1 - x))$. Then $\lim_{x\to 1} h(x) = \infty$. And $h'(x) = f'(1/(1 - x))\frac{1}{(1-x)^2}$ changes sign infinitely many often close to 1.

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