Math 322
Review and suggested practice problems

Topics covered before the Midterm Exam

(taken from the Review File for the Midterm Exam):

- Graphs/multigraphs/directed graphs.
- Vertices/nodes and edges of a graph; (V, E)-notation; (pictorial) representations.
- Multiple (or parallel) edges; loops.
- Order and size of a graph; finite graphs.
- Labelled/unlabelled graphs.
- Adjacent (or neighbouring) vertices; adjacent edges; endvertices of an edge (vertex incident with an edge).
- Neighbourhood of a vertex, degree of a vertex.
- Connected/disconnected graphs; connected components of a graph.
- Subgraphs and induced subgraphs.
- Forbidden subgraphs.
- Walks; paths; cycles; trails; circuits.
- Adjacency matrix of a graph; Incidence matrix of a graph.
- Graph Isomorphism; Isomorphic graphs.
- Operations on graphs *(review also the established notation, wherever applicable)*:
  - Complement $\overline{G}$ of a graph G.
  - Line graph $L(G)$ of a graph G.
  - Disjoint union $G_1 \oplus G_2$ of two graphs G_1, G_2 (with disjoint vertex sets).

  - Join $G_1 \vee G_2$ of two graphs G_1, G_2 (with disjoint vertex sets).
  - Vertex deletion.
  - Edge deletion.

- Degree sequences and graphical sequences.
- Regular graphs. *(Recall the family of 0-regular graphs, of 1-regular graphs, and of 2-regular graphs.)*
- Special families of graphs *(review also the established notation, wherever applicable)*:
  - Null graphs.
  - Complete graphs.
  - Paths; trees; forests.
  - Cycle graphs.
  - Cyclic / acyclic graphs.
  - Wheel graphs.
  - Bipartite graphs.
- Maximum and minimum degree of a graph G ($\Delta(G)$ and $\delta(G)$ respectively).
- Cutvertices (or equivalenlty, 1-vertex cuts) of a graph.
- Vertex cuts of a graph.
- Bridges (or equivalently, 1-edge cuts) of a graph.
- Edge cuts of a graph.
- k-vertex connected graphs; the vertex connectivity $\kappa(G)$ of a graph G.
- The edge connectivity $\lambda(G)$ of a graph G.
  - Review (or practise) how we determine the parameters $\kappa(G)$ and $\lambda(G)$ for important families of connected graphs (e.g. complete graphs, bipartite graphs, cycle graphs, trees).
  - Review the related results about
    * the maximum size of a disconnected graph on n vertices;

    * the minimum size of a connected graph on n vertices.

- Vertex cuts for two (non-adjacent) vertices u, v of a connected graph G.
- Edge cuts for two vertices u, v of a connected graph G.
- Local vertex connectivities.
- Local edge connectivities.
- Internally disjoint (or equivalently, vertex-disjoint) paths.
- Edge-disjoint paths.
  - Review how the local vertex connectivities relate to the vertex connectivity $\kappa(G)$ of the graph G.
  - Review how the local edge connectivities relate to the edge connectivity $\lambda(G)$ of the graph G.
- Review the equivalent characterisations of trees that we have discussed.
- Spanning trees of a connected graph G.
- Weighted graphs.
- Minimum weight spanning tree.

Topics covered after the Midterm Exam

(note that it would be useful to review some of these topics in conjuction with previous ones)

- Other important parameters of graphs *(and concepts/constructions they are related to):*
  - the independence number $\alpha(G)$ of a graph G; independent sets of vertices.
  - the maximum cardinality $\nu(G)$ of a matching in a graph G.
  - the clique number $\omega(G)$ of a graph G; cliques in G; maximal / maximum cliques.

- the chromatic number $\chi(G)$ of a graph G; (proper) vertex colourings of G.

- Given a graph H, review what it means for another graph G to be H-free.

  Similarly, given a family $\mathcal{F}$ of graphs, review what it means for a graph G to be $\mathcal{F}$-free.

- Walks, paths, cycles, trails, circuits **in a multigraph G**: review how we define and denote them.

- Review how we define the degree of a vertex in a multigraph.

- (Directed) walks, paths, cycles, trails, circuits **in a directed graph (or directed multigraph) H**: review how we define and denote them.

- Eulerian graphs or multigraphs (that is, graphs or multigraphs which contain an Euler circuit).
  - Graphs (or multigraphs) with an Euler trail.

- Hamiltonian graphs (that is, graphs which contain a Hamilton cycle).
  - Graphs with a Hamilton path.

- Oriented graphs.
  - Orientation of an undirected graph G; Tournaments (that is, orientations of complete graphs).

- Strongly connected directed graphs.
  - Strong orientations of an undirected graph G.

- Weighted graphs; Weight matrix of a graph.

- Problems about weighted graphs inspired by applications:
  - The connector problem (in other words, finding a minimum weight spanning tree (or spanning forest)) $\dashleftarrow$ Kruskal's algorithm.
  - The shortest path problem $\dashleftarrow$ Dijkstra's algorithm.

- Factors (or equivalently, spanning subgraphs) of a graph G (some special examples: spanning trees (or forests), one-factors, two-factors, etc.).

- Factorizations of a graph G (some special examples: one-factorizations, two-factorizations, etc.).
- Matchings in a graph G: perfect matchings; matchings covering given subsets of vertices.
- Stable matchings; The Stable Marriage Problem $\leftarrow$-- The Gale-Shapley algorithm.
- Vertex colourings $\leftrightsquigarrow$ Scheduling problems $\leftarrow$-- Greedy Colouring algorithm.
- Planar graphs.
- One more operation on graphs: *subdivision* of a graph G.

Some algorithms / algorithmic processes we introduced to study notions from above

- The Havel-Hakimi theorem and algorithm.
- Kruskal's algorithm.
- Dijkstra's algorithm.
- The Gale-Shapley algorithm.
- Greedy colouring algorithm.

Review also the theorems, propositions, lemmas that we stated and discussed in class regarding all these concepts/notions.

Some suggested practice problems

Problem 1. Below are some cities in North Carolina, USA, and a road network connecting them. Assume that the weights appearing in the graph give a measure of the distance and/or difficulty of each route.

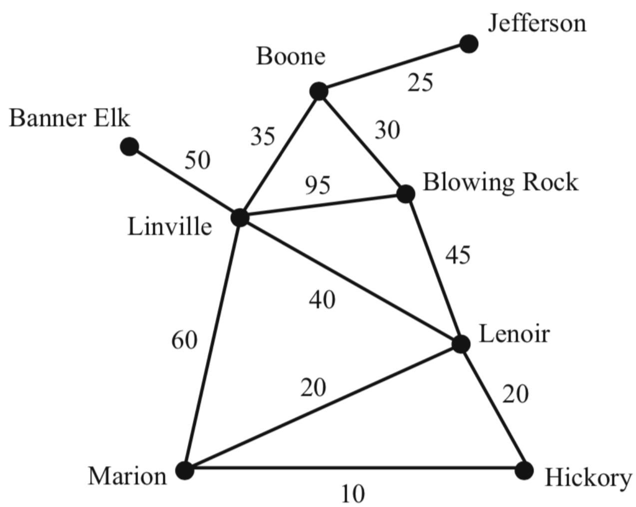


Figure 1: From the Harris-Hirst-Mossinghoff book

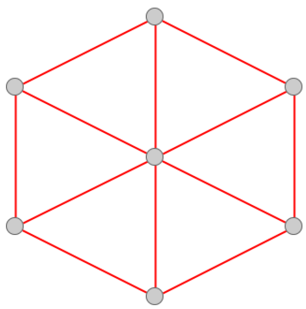
What is the shortest possible length of a path connecting the cities of Marion and Boone? Of a path connecting the cities of Marion and Jefferson? Can you find minimum weight paths too?

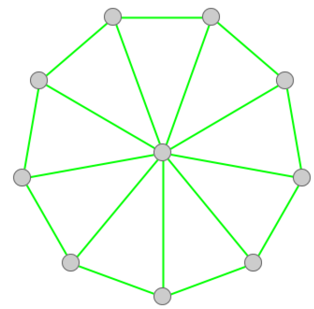
Problem 2. Solve the Stable Marriage Problem if the following tables of preferences are given:

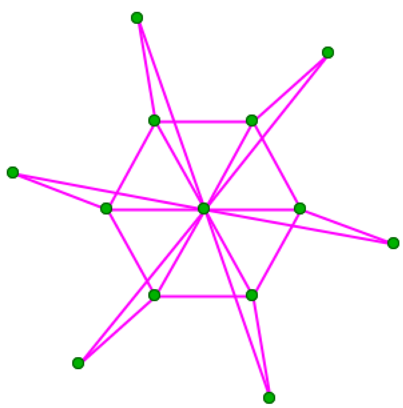
m_1	m_2	m_3	m_4	m_5
w_1	w_1	w_2	w_1	w_5
w_2	w_2	w_4	w_2	w_2
w_5	w_5	w_3	w_5	w_4
w_3	w_3	w_5	w_3	w_3
w_4	w_4	w_1	w_4	w_1

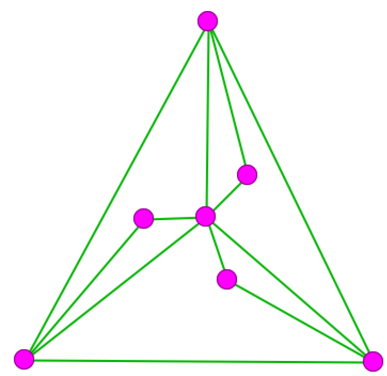
w_1	w_2	w_3	w_4	w_5
m_5	m_5	m_3	m_3	m_2
m_4	m_4	m_1	m_2	m_1
m_3	m_3	m_2	m_1	m_5
m_2	m_2	m_4	m_5	m_4
m_1	m_1	m_5	m_4	m_3

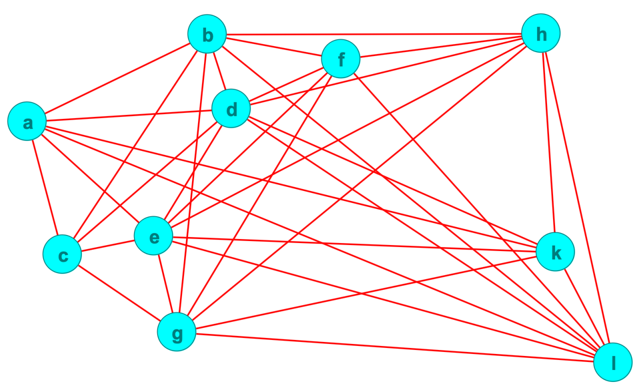
Problem 3. For each of the following 11 graphs, determine precisely the parameters $\alpha(G)$, $\nu(G)$, $\omega(G)$ and $\chi(G)$. Justify your answers (remember that, to determine such parameters, you need to provide both an upper bound and a matching lower bound).











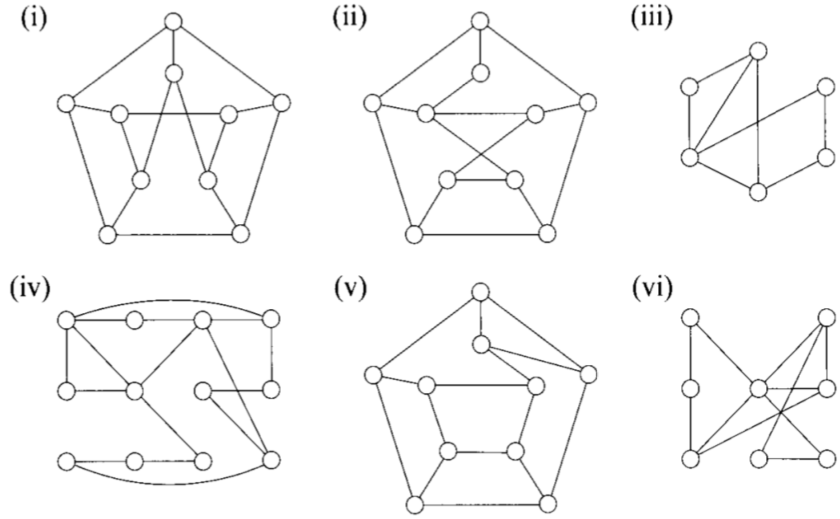
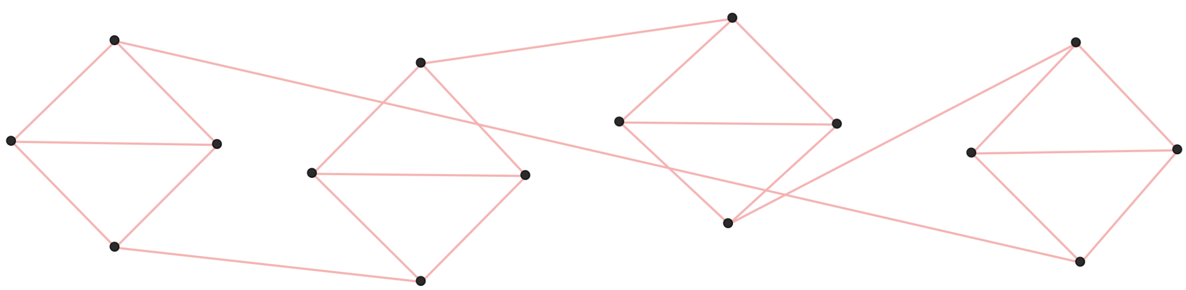
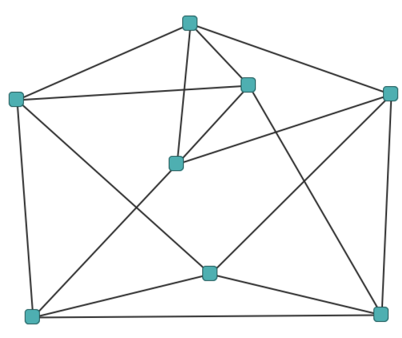


Figure 2: from Wallis' book

Problem 4. For each of the two graphs given below, answer the following questions.





(a) Is the graph Eulerian? If not, does it have an Euler trail? Justify your answer to both questions.

(b) Is the graph Hamiltonian? If not, does it have a Hamilton path? Justify your answer to both questions.

(c) Does the graph have one-factors? Does it have a one-factorization? Does it have two-factors? Does it have a two-factorization? Justify your answer to all four questions.

Problem 5. Is any of the graphs in the previous problems $K_{1,3}$-free (or in other words, claw-free)? Find all such graphs.