MATH 317 PRACTICE FINAL 2

6.5.1)

$$f: \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (1, xz, xy)$$

$$\Phi: \mathbb{R}^2 \to \mathbb{R}^3, \quad (s, t) \mapsto (\cos s \cos t, \sin s \cos t, \sin t)$$

The parameter domain of Φ is $K = [0, 2\pi] \times [0, \pi/2]$. Note that $S = \{\Phi\}$.

Let γ be the natural parametrization of ∂K, then we traverse K as follows:

+ Part 1. $[0 \twoheadrightarrow 2\pi] \times \{0\}$, i.e., x goes from 0 to 2π, fix $y = 0$.

+ Part 2. $2\pi \times \left[0 \twoheadrightarrow \frac{\pi}{2}\right]$, i.e., fixes $x = 2\pi$, and y goes from 0 to $\frac{\pi}{2}$.

+ Part 3. $[2\pi \twoheadrightarrow 0] \times \left\{\frac{\pi}{2}\right\}$, i.e., x goes from 2π to 0, and fixes $y = \frac{\pi}{2}$.

+ Part 4. $\{0\} \times \left[\frac{\pi}{2} \twoheadrightarrow 0\right]$, i.e., fixes $x = 0$, and y goes from $\frac{\pi}{2}$ to 0.

Plugging part 1 to Φ, we have: $\gamma_1(t) \coloneqq (\cos t, \sin t, 0)$, t goes from 0 to 2π.

Plugging part 2 to Φ, we have $\gamma_2(t) \coloneqq (\cos t, 0, \sin t)$, t goes from 0 to $\frac{\pi}{2}$.

Plugging part 3 to Φ, we have $\gamma_3(t) \coloneqq (0,0,1)$ (just a single point in $\mathbb{R}^3$).

Plugging part 4 to Φ, we have $\gamma_4(t) \coloneqq (\cos t, 0, \sin t)$, t goes from $\frac{\pi}{2}$ to 0.

Note that part 2 and part 4 are the same curve but with reverse direction, and $\{\gamma_3\} = (0,0,1) \in \{\gamma_2\} \cap \{\gamma_4\}$. When doing the line integrals, if we included $\{\gamma_2\}$ and $\{\gamma_4\}$ together, they would cancel each other (Proposition 6.2.5). Therefore, we can exclude $\{\gamma_2\}, \{\gamma_3\}$, and $\{\gamma_4\}$ in $\Phi \circ \gamma$ and only includes $\{\gamma_1\}$ for the sake of simplicity.

Define:

$$\gamma_1: [0, 2\pi] \to \mathbb{R}^3, \quad t \mapsto (\cos t, \sin t, 0)$$

Now, Φ is clearly a $\mathcal{C}^2$-surface, and γ is the parametrization of ∂K (which is the positively oriented boundary ∂K of K) is a piecewise $\mathcal{C}^1$-curve, and $1, xz, xy \in \mathcal{C}^1$-functions on $\mathbb{R}^3$. Therefore, by the Stokes Theorem, we have:

$$\int_S (\operatorname{curl} f) \cdot n \, d\sigma = \int_{\Phi \circ \gamma} f \cdot d\vec{x} = \int_{\gamma_1} f \cdot d\vec{x} = \int_0^{2\pi} f\begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt$$
$$= \int_0^{2\pi} \begin{pmatrix} 1 \\ 0 \\ \cos t \sin t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt = \int_0^{2\pi} -\sin t \, dt = 0$$

8.1.1)

a)

$$S = \sum_{n=1}^{\infty} \frac{1}{\cos(n) + \pi}$$

Note that:

$$\frac{1}{\cos(n) + \pi} \geq \frac{1}{1 + \pi}$$

Also,

$$\sum_{n=1}^{\infty} \frac{1}{1 + \pi} = \frac{1}{1 + \pi} \sum_{n=1}^{\infty} 1 = \infty$$

Therefore, by the Comparison Test, the series S diverges.

b)

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n+4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$$

Now, $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n+4}} = 0$ and $a_n = \frac{1}{\sqrt{n+4}}$ is clearly decreasing and non-negative, and so by the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n+4}}$ converges.

On the other hand,

$$\left|\frac{\cos(\pi n)}{\sqrt{n+4}}\right| = \left|\frac{(-1)^n}{\sqrt{n+4}}\right| = \frac{1}{\sqrt{n+4}} > \frac{1}{\sqrt{2n}} = \frac{1}{\sqrt{2}\sqrt{n}}$$

$(n + 4 > 2n$ for all $n \geq 5)$.

Now, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2}\sqrt{n}} = \infty$, and so by the Comparison Test, $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n+4}}$ does not converge absolutely.

c)

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n^3}$$

Now, note that:

$$\left|\frac{(-1)^n(n+1)}{n^3}\right| = \frac{n+1}{n^3} = \frac{1}{n^2} + \frac{1}{n^3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left|\frac{(-1)^n(n+1)}{n^3}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n^3}$ converges absolutely, and hence it is also convergent.

8.2.1)

Suppose $I \coloneqq \int_a^b f(x)\,dx$, and since f is Riemann integrable on $[a,b]$, $|f(x)| \leq M$ for all $x \in [a,b]$. Let $\epsilon > 0$ and consider $c \in \mathbb{R}$ such that $b - c < \frac{\epsilon}{M}$

Then,

$$I - \int_a^c f(x)\,dx \leq \left| I - \int_a^c f(x)\,dx \right| = \left| \int_c^b f(x)\,dx \right| \leq \int_c^b |f(x)|\,dx \leq (b-c)M < \frac{\epsilon}{M}M = \epsilon$$

Therefore,

$$\lim_{c \to b^-} \int_a^c f(x)\,dx = I = \int_a^b f(x)\,dx \quad \blacksquare$$

8.2.5)

Since g is bounded, $\exists M > 0$ such that $|g(x)| < M$ for all $x \in [a,c]$ for each $c \in [a,b)$.

Note that:

$$|f(x)g(x)| \leq M|f(x)|$$

On the other hand, note that $\int_a^b M|f(x)|\,dx = M\int_a^b |f(x)|\,dx$ exists ($\int_a^b f(x)\,dx$ converges absolutely). Therefore, by the Comparison Test, $\int_a^b f(x)g(x)\,dx$ converges absolutely.

For the counterexample, let:

$$f(x) = g(x) = \frac{1}{x\sqrt{x-1}}$$

Note that $f(x) = g(x)$ are decreasing, and so on $[1,\infty)$, $f(x) = g(x)$ must be bounded.

Then,

$$\int_1^\infty f(x)\,dx = \int_1^\infty g(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x\sqrt{x-1}}\,dx = \lim_{t\to\infty} \left[2\arctan\left(\sqrt{x-1}\right)\right]_1^t = \pi$$

Now, for any $k \geq 1$ and finite, we have:

$$\int_1^k f(x)\,dx = \int_1^k g(x)\,dx = \int_1^k \frac{1}{x\sqrt{x-1}}\,dx = \left[2\arctan\left(\sqrt{x-1}\right)\right]_1^t < \infty$$

($\arctan(x)$ is defined on all $x \in \mathbb{R}$).

However,

$$\int_1^\infty f(x)g(x)\,dx = \int_1^\infty \frac{1}{x^3 - x^2}\,dx = \lim_{t\to\infty} \int_1^\infty \frac{1}{x^3 - x^2}\,dx = \lim_{t\to\infty} \left[\log\left(\frac{|x-1|}{|x|}\right) + \frac{1}{x}\right]_1^t = \infty$$

Therefore, we have a counterexample.

8.2.7)

$$\Gamma(x) \coloneqq \int_0^\infty t^{x-1}e^{-t}\,dt$$

a)

Let $t = (x+2)! + 1$, then $t \geq (x+2)!$

$$\Rightarrow \frac{t^{x+2}}{t^{x+1}} \geq (x+2)! \Rightarrow e^t \geq \frac{t^{x+2}}{(x+2)!} \geq t^{x+1}$$

(Note that for $t \geq 0$, $e^t = \sum_{n=0}^{\infty} \frac{t^n}{t!} \geq \frac{t^{x+2}}{(x+2)!}$).

$$\Rightarrow \frac{1}{e^t} \leq \frac{1}{t^{x+1}} \Rightarrow \frac{1}{e^t} \leq \frac{1}{t^2 \cdot t^{x-1}} \Rightarrow |t^{x-1}e^{-t}| = t^{x-1}e^{-t} \leq \frac{t^{x-1}}{e^t} \leq \frac{1}{t^2}$$

Since $\int_{(x+2)!+1}^{\infty} \frac{1}{t^2}\,dt$ exists, it follows from the Comparison Test that $\int_{(x+2)!+1}^{\infty} t^{x-1}e^{-t}\,dt$ converges absolutely. Since $t^{x-1}e^{-t}$ is bounded continuous on the compact interval $[0, (x+2)!+1]$, and so $\int_0^{(x+2)!} t^{x-1}e^{-t}\,dt$ exists.

Therefore, $\Gamma(x) \coloneqq \int_0^\infty t^{x-1}e^{-t}\,dt = \int_0^{(x+2)!} t^{x-1}e^{-t}\,dt + \int_{(x+2)!+1}^{\infty} t^{x-1}e^{-t}\,dt$ exists.

b)

Let $u(t) = t^x$ and $v(t) = -e^{-t}$, then $\frac{dv}{dt} = e^{-t}$

$$\Gamma(x+1) \coloneqq \int_0^\infty t^x e^{-t}\,dt = [-t^x e^{-t}]_0^\infty + x\int_0^\infty e^{-t}t^{x-1}\,dt = [-t^x e^{-t}]_0^\infty + x\Gamma(x)$$

Now, $\lim_{t\to\infty}(-t^x e^{-t}) = 0$, and $-0^t e^{-0} = 0$, and so $[-t^x e^{-t}]_0^\infty = 0$.

Therefore, $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

c)

Note that:

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\,dt = \int_0^\infty e^{-t}\,dt = \lim_{k\to\infty}\int_0^k e^{-t}\,dt = \lim_{k\to\infty}[-e^{-t}]_0^k = \lim_{k\to\infty}[-e^{-k}+1] = 0 + 1 = 1$$

Given $\Gamma(0+1) = \Gamma(1) = 1 = 0!$ and $\Gamma(n+1) = n\Gamma(n)$ for $n \in \mathbb{N} \cup \{0\}$, it follows that:

$$\Gamma(n+1) = n\Gamma(n) = n^2\Gamma(n-1) = \cdots = n!$$

9.3.1)

Suppose $F \in \mathcal{PC}_{2\pi}(\mathbb{R})$, then it follows from the Fundamental Theorem of Calculus that:

$$\int_{-\pi}^{\pi} f(t)\,dt = F(\pi) - F(-\pi) = F(\pi) - F(\pi) = 0$$

Now, suppose $\int_{-\pi}^{\pi} f(t)\,dt = 0$, then by the Fundamental Theorem of Calculus, we have that:

$$F(x+2\pi)-F(x)=\int_x^{x+2\pi} f(x)dx$$

Note that:

$$\frac{d}{dx}\left(\int_x^{x+2\pi} f(x)dx\right)=f(x+2\pi)-f(x)=0$$

Therefore, $\int_x^{x+2\pi} f(x)dx$ is a constant. Since $\int_{-\pi}^{-\pi+2\pi} f(x)dx=\int_{-\pi}^{\pi} f(t)\,dt=0$, it follows that $\int_x^{x+2\pi} f(x)dx=0$, and so $F(x+2\pi)-F(x)=0 \Leftrightarrow F(x+2\pi)=F(x)$.

Since F is differentiable, it must be continuous, and so it must be piecewise continuous that has one-sided limit. As a result, $F\in\mathcal{PC}_{2\pi}(\mathbb{R})$.

9.3.2)

Claim 1: If $f(x)$ is an even function and $g(x)$ is an odd function, then $f(x)g(x)$ is an odd function.

Proof:

$$f(-x)g(-x)=f(x)\cdot\big(-g(x)\big)=-f(x)g(x) \quad \blacksquare$$

Claim 2: If $f(x)$ is odd and Riemann integrable on $[-a,a]$ for some $a\in\mathbb{R}\setminus\{0\}$, then $\int_{-a}^{a} f(x)dx=0$.

Proof:

$$\int_{-a}^{a} f(x)dx=\int_{-a}^{0} f(x)dx+\int_0^a f(x)dx=\int_0^a f(-x)dx+\int_0^a f(x)dx=-\int_0^a f(x)+\int_0^a f(x)=0 \quad \blacksquare$$

Return to the main question:

Suppose f is odd, then $f(t)\cos(nt)$ is an odd function since $\cos(nt)$ is an even function, and so:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt=0$$

Now, suppose f is even, then since $\sin(nt)$ is an odd function, $f(t)\sin(nt)$ is an odd function, and so:

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt=0$$

9.3.4)

$$f(t):=\begin{cases}-1, & t\in(-\pi,0)\\ \ \ 1, & t\in[0,\pi]\end{cases}$$

Let $a_0,a_1,a_2,\dots,b_1,b_2,\dots$ be the Fourier Coefficients of f.

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt=-\frac{1}{\pi}\int_{-\pi}^{0}\cos(nt)\,dt+\frac{1}{\pi}\int_0^{\pi}\cos(nt)\,dt=-\frac{\sin(n\pi)}{n\pi}+\frac{\sin(n\pi)}{n\pi}=0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin(nt)\,dt = \frac{-2(\cos(n\pi)-1)}{n\pi}$$

$$\therefore f(x) \sim \sum_{n=1}^{\infty}\frac{-2(\cos(n\pi)-1)}{n\pi}\sin(nx) = \sum_{n=1}^{\infty}\frac{-2((-1)^n-1)}{n\pi}\sin(nx) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}+2}{n\pi}\sin(nx)$$
$$= \sum_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin\big((2n+1)x\big)$$

By Theorem 9.3.13, then $f(x) = \sum_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin\big((2n+1)x\big)$ pointwise for all x that are not integer multiples of π. If x is an integer multiple of π, then $\sum_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin\big((2n+1)x\big) = 0$, and so $\sum_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin\big((2n+1)x\big)$ converges pointwise to 0.

Therefore, the Fourier series of f converges pointwise on $\mathbb{R}$ to the function ϕ defined as:

$$\phi(t) \coloneqq \begin{cases} f(t), & t \neq k\pi \text{ for all } k \in \mathbb{Z} \\ 0, & t = k\pi \text{ for all } k \in \mathbb{Z} \end{cases}$$

Note that $\phi(t)$ is clearly discontinuous.

Now, note that $\sum_{k=0}^{n}\frac{4}{(2k+1)\pi}\sin\big((2k+1)x\big)$ is continuous, and so if $\sum_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin\big((2n+1)x\big)$ converges uniformly to $\phi(x)$, it follows from Theorem 9.1.3 that $\phi(x)$ is continuous, but this is clearly not the case.

Therefore, the Fourier Series of f does not converge uniformly on $\mathbb{R}$, but converges pointwise on $\mathbb{R}$. ■