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On p-Harmonic Functions in the Plane and Their Stream Functions

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1. INTRODUCTION

This paper treats the p-harmonic equation,

$$\Delta_p \varphi \equiv \operatorname{div}(|\nabla\varphi|^{p-2} \nabla\varphi) = 0, \qquad 1 < p < \infty,$$

in the plane. A number of analogies with basic complex function theory in $\mathbb{C}$ are developed.

First, known facts about quasi-regular and quasi-conformal mappings are used to represent $\varphi_x - i\varphi_y$ and study its Hölder continuity. A study of the zeros of the Jacobian $\partial(\varphi_x, \varphi_y)/\partial(x, y)$ is included.

If $\Delta_p \varphi = 0$ in a simply connected domain D, then a q-harmonic "stream function" ψ is shown to exist in D:

$$\Delta_q \psi = 0, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

The functions φ, ψ form a *conjugate pair*, analogous to conjugate harmonic functions, and $F = \varphi + i\psi$ has some properties in common with holomorphic functions. The singular set ($\nabla\varphi = \nabla\psi = 0$) consists of isolated points. A representation formula à la Stoïlow is given for F.

The hodograph method is used to transform $\Delta_p \varphi = 0 = \Delta_q \psi$ into linear elliptic equations in the hodograph plane. The pull-back operation is analyzed. Further, we consider some particular p-harmonic functions of the form $\varphi = r^k f(\phi)$ (quasi-radial solutions). These functions are non-linear counterparts ($p \neq 2$) of the basic harmonic functions $\operatorname{Re}(z^n)$, $\operatorname{Im}(z^n)$ ($p = 2$).

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They are here used to illustrate the representation theorems. It is finally shown that these quasi-radial solutions assume a particularly simple form in the hodograph plane, obtained by separation of variables in the linear equation for φ (or ψ).

2. Basic Results for p-Harmonic Functions

Let D be a domain in $\mathbb{R}^n$, and let $p>1$. The Euler–Lagrange equation for the problem of minimizing

$$\int_D |\nabla\varphi|^p \, dx$$

over some convenient function class is written in weak form as

$$\int_D |\nabla\varphi|^{p-2} \nabla\varphi \cdot \nabla\eta \, dx = 0, \tag{2.1}$$

valid at least for all $\eta \in C_0^\infty(D)$. (Here, $|\nabla\varphi|^{p-2}\nabla\varphi$ is by definition $=0$, whenever $\nabla\varphi=0$.) If $\varphi \in C^2(D)$, this implies that

$$(p-2) \sum_{i,j=1}^{n} \varphi_{x_i}\varphi_{x_j}\varphi_{x_i x_j} + |\nabla\varphi|^2 \Delta\varphi = 0 \tag{2.2}$$

in D. The equation is often written as

$$\operatorname{div}(|\nabla\varphi|^{p-2}\nabla\varphi) = 0. \tag{2.3}$$

Either of the three equations is called the *p-harmonic equation*, or the p-Laplace equation.

A weak solution to the p-harmonic equation is a function $\varphi \in W^{1,p}_{\mathrm{loc}}(D)$ such that (2.1) holds for all $\eta \in C_0^\infty(D)$. (Observe that it is not required that φ be globally in the Sobolev space.) This is the solution concept used here.

It is well known that any such φ belongs to the Hölder class $C^{1,\alpha}_{\mathrm{loc}}(D)$, where $\alpha = \alpha(p,n) > 0$. It is also known that φ is smooth, in fact real analytic, away from the zeros of $\nabla\varphi$. Concerning these questions we refer to [LE 2] and [D] and references given there.

From now on the discussion is confined to the case $n=2$.

2.1. *The Beltrami Equation for the Complex Gradient*

Many relevant qualitative properties for the hodograph transformation are consequences of the fundamental observation that the complex gradient

$$f = \varphi_x - i\varphi_y$$

of a p-harmonic function φ is quasi-regular. This property was proved by Bojarski and Iwaniec in [B–I] on the assumption that $|f|^{(p-2)/2} f$ or $|\nabla\varphi|^{(p-2)/2} \nabla\varphi$ is in the Sobolev space $W^{1,2}_{\mathrm{loc}}$, a regularity result that they proved for $p \geqslant 2$ [B–I, Proposition 2].

We shall use the so-called analytic definition for quasi-regularity [AH, pp. 24 and 29]. We say that $f: D \to \mathbb{C}$, D being a domain in C, is *quasi-regular* in D, if

(i) $f \in C(D)$

(ii) $f \in W^{1,2}_{\mathrm{loc}}(D)$

(iii) $|f_{\bar z}| \leqslant k |f_z|$ a.e. in D for some constant $k < 1$.

We shall here call the smallest $k \geqslant 0$ the *dilatation* for f (usually one regards $K = (1+k)/(1-k)$ as dilatation). The complex notation

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \qquad 2\frac{\partial}{\partial \bar z} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

for the (weak) derivatives in (iii) is self-explanatory. If (ii) and (iii) hold, then f has a continuous representative. A quasi-regular homeomorphism is called *quasi-conformal*. For these concepts we refer to [AH, V, R].

THEOREM 1 (Bojarski-Iwaniec). *Suppose that $\varphi \in C(D) \cap W^{1,p}_{\mathrm{loc}}(D)$ satisfies the equation*

$$\int |\nabla\varphi|^{p-2} \nabla\varphi \cdot \nabla\eta \, dx = 0 \tag{2.1}$$

whenever $\eta \in C_0^\infty(D)$. Then the complex gradient $f = \varphi_x - i\varphi_y$ has a continuous representative that is quasi-regular in D with dilatation $\leqq |1 - 2/p|$.

Proof. The auxiliary mapping $G = |f|^{(p-2)/2} f$ is in $W^{1,2}_{\mathrm{loc}}(D)$ (cf. [B–I, Section 2; D, p. 828]). By the calculations in [B–I, p. 7] $|G_{\bar z}| \leqq (|p-2|/(p+2)) |G_z|$ a.e. in that part of D where $G \neq 0$. Since G belongs to the Sobolev space, G_z and $G_{\bar z}$ are zero a.e. in the set where $G = 0$ [G–T, Lemma 7.7, p. 152]. This means that $|G_{\bar z}| \leqq (|p-2|/(p+2)) |G_z|$ a.e. in D. Thus the (continuous representative for) G is quasi-regular in D.

Since f itself is the composition of G and the radial quási-conformal map $z \to |z|^{2/p-1} z$, it is quasi-regular. By the formal calculations in [B–I]

$$\frac{\partial f}{\partial \bar z} = \frac{2-p}{2p} \left\{ \frac{\bar f}{f} \frac{\partial f}{\partial z} + \frac{f}{\bar f} \frac{\overline{\partial f}}{\partial z} \right\} \tag{2.4}$$

a.e. in that part of D where $f \neq 0$, and

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leqslant \left|1 - \frac{2}{p}\right| \left|\frac{\partial f}{\partial z}\right| \tag{2.5}$$

a.e. in D. This yields the desired dilatation.

From now on *we regard $f = \varphi_x - i\varphi_y$ as defined pointwise so that f is continuous.*

COROLLARY. *f is locally Hölder continuous in D with exponent*

$$\alpha = \min\left\{p - 1, \frac{1}{p-1}\right\}$$

and $\varphi \in C^{1,\alpha}_{\mathrm{loc}}(D)$.

Proof. If $|f_{\bar{z}}| \leqslant k|f_z|$ a.e. in D and $k < 1$ is a constant, then f is locally Hölder continuous with exponent

$$\alpha = \frac{1-k}{1+k}.$$

See [L–V, Satz 3.2, p. 69] and (2.6) below. For $k = |1 - 2/p|$ we get our α.

Remark. If φ is p-harmonic and ψ is q-harmonic where $p + q = pq$, then the above corollary gives the same Hölder exponent for $\nabla\varphi$ and $\nabla\psi$. For $p \geqslant 2$ we have

$$\alpha = q - 1 = \frac{1}{p-1}.$$

The general representation formula for quasi-regular mappings in the plane yields information essential for our purpose.

THEOREM 2. *Suppose that φ is p-harmonic in D. Then the complex gradient $f = \varphi_x - i\varphi_y$ has the representation*

$$f = h \circ \chi \tag{2.6}$$

where $\chi: D \to D'$ is a quasi-conformal mapping and $h: D' \to \mathbb{C}$ is analytic. Moreover, χ has the same dilatation ($\leqslant |1 - 2/p|$) as f (except for $f \equiv$ Const.).

Proof. See, e.g., [V, Theorem 3.30, p. 211 or R, Satz 2.17, p. 48] bearing (2.4) and (2.5) in mind.

This means especially that *either f reduces to a constant or the zeros of f are isolated*, i.e., *the singular set*

$$S = \{z \in D \mid f(z) = 0\} \tag{2.7}$$

is discrete. See also [AL] for this fact.

COROLLARY. *f is real-analytic outside a discrete set, namely S.*

Proof. [LE1, p. 208].

We say that $\nabla\varphi$ has *a zero of order n* at a point $z_0 \in D$, if the analytic function h in some representation (2.6) has a zero of order n at the corresponding point $\chi(z_0)$. Observe that the order of the zero does not depend on the particular representation chosen (cf. [R, III.1, pp. 67–68]).

One can also classify *isolated singularities* by means of the representation $f = h \circ \chi$, and even the *Principle of Argument* holds, if properly formulated. See [R, III.1 and III.2, pp. 67–71] for a presentation of this kind of Complex Analysis.

2.2. *On the Zeros of the Jacobian*

If φ is p-harmonic in a domain D in the plane, then $f = \varphi_x - i\varphi_y$ is quasiregular in D and f is smooth in $D \backslash S$; the singular set S consisting of the isolated zeros of f (the case $f \equiv \text{Const.}$ is excluded). The Jacobian

$$J_f = |f_{\bar z}|^2 - |f_z|^2 = \frac{\partial(\varphi_x, \varphi_y)}{\partial(x, y)} = \varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 \tag{2.8}$$

for $\bar f = \varphi_x + i\varphi_y$ is smooth (in fact, real-analytic) in $D \backslash S$ and hence its zeros can be interpreted pointwise.

THEOREM 3. *Let φ be p-harmonic in a plane domain D. Assume that φ is not a linear function. Then there exists a set E of isolated points in D, such that φ is real-analytic and*

$$\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 \neq 0$$

in $D \backslash E$.

Proof. We know that the complex gradient f has a representation $f = h \circ \chi$ in D where h is analytic (holomorphic), $h \neq \text{Const.}$, and χ is quasiconformal in D. Put

$$S = \{z \in D \mid f(z) = 0\}, \qquad T = \{z \in D \mid h'(\chi(z)) = 0\}.$$

We claim that $J_f \neq 0$ in $D \backslash (S \cup T)$.

So take an arbitrary point $z_0 \in D\backslash(S \cup T)$ and try to prove that $\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 \neq 0$ at z_0. Put $u = \varphi_x$, $v = -\varphi_y$ so that $f = u + iv$. Since $h'(\chi(z_0)) \neq 0$, a suitable neighborhood ω_1 of $\chi(z_0)$ will be in topological correspondence with neighborhoods ω_2 of $f(z_0)$ and ω of z_0. We may assume that ω is so small that $f \in C^\infty(\omega)$.

The chosen branch $\omega_2 \ni (u, v) \rightarrow^{f^{-1}} z \in \omega$ of the inverse is quasi-conformal [AH, pp. 9 and 22]. Thus f^{-1} is differentiable a.e. in ω_2 and has first-order derivatives in L^2_{loc} [AH, pp. 27–28], i.e., $f^{-1} \in W^{1,2}_{\text{loc}}(\omega_2)$. It is known that

$$J = x_u y_v - x_v y_u \neq 0 \tag{2.9}$$

a.e., since f^{-1} is quasi-conformal [AH, pp. 27 and 34]. We can represent the identity on ω_2 as $f \circ f^{-1}$, and here the chain rule is applicable. Thus

$$u_x x_u + u_y y_u = 1, \qquad u_x x_v + u_y y_v = 0,$$

etc. This leads to the relations

$$u_x = \frac{y_v}{J}, \quad u_y = -\frac{x_v}{J}, \quad v_x = -\frac{y_u}{J}, v_y = \frac{x_u}{J}$$

holding a.e. in ω_2. Insertion of these formulas into the p-harmonic equation (2.2) gives that

$$(u^2 + v^2)(y_v + x_u) + (p-2)(u^2 y_v - 2uv y_u + v^2 x_u) = 0 \tag{2.10}$$

a.e. in ω_2. Observe that $x_v = y_u$ a.e. in ω_2, since $\varphi_{xy} = \varphi_{yx}$. This gives a system

$$\begin{aligned} (u^2 + v^2)(x_u + y_v) + (p-2)(u^2 y_v - 2uv y_u + v^2 x_u) &= 0 \\ x_v &= y_u \end{aligned} \tag{2.11}$$

satisfied a.e. in ω_2. We can assume ω_2 so chosen that $u^2 + v^2 \geqslant \varepsilon > 0$ in ω_2 for some ε. Then the above system is linear and uniformly elliptic in ω_2. It is well known that any continuous solution $x + iy$ of (2.11) is classical, in fact even real-analytic, in any domain where uniform ellipticity holds. See [B–J–S, pp. 138, 207–210, 136, and 255–256]. Being quasi-conformal in ω_2, $z = x + iy = f^{-1}(u, v)$ is continuous in ω_2. Thus $x(u, v)$ and $y(u, v) \in C^\infty(\omega_2)$.

It follows immediately from

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1,$$

an identity to be interpreted *pointwise* now, that $\partial(u, v)/\partial(x, y) \neq 0$ in ω. This proves the theorem.

3. p-Harmonic Functions and Their Stream Functions

Following [AR2, pp. 1–3] we write the p-harmonic equation for φ in divergence from

$$\frac{\partial}{\partial x}(|\nabla\varphi|^{p-2}\varphi_x)+\frac{\partial}{\partial y}(|\nabla\varphi|^{p-2}\varphi_y)=0 \tag{3.1}$$

and introduce a stream function ψ by the aid of the relations

$$\psi_x=-|\nabla\varphi|^{p-2}\varphi_y, \qquad \psi_y=|\nabla\varphi|^{p-2}\varphi_x \tag{3.2}$$

in simply connected domains, and it turns out that this stream function is q-harmonic, i.e.,

$$\frac{\partial}{\partial x}(|\nabla\psi|^{q-2}\psi_x)+\frac{\partial}{\partial y}(|\nabla\psi|^{q-2}\psi_y)=0, \tag{3.3}$$

where q is the exponent conjugate to p:

$$\frac{1}{p}+\frac{1}{q}=1.$$

Especially,

$$\nabla\varphi\cdot\nabla\psi=0. \tag{3.4}$$

In the terminology of fluid mechanics we call ψ the *stream function* corresponding to the *potential* φ. In the language of Potential Theory we may say that φ and ψ are *conjugate functions.* Clearly, φ is obtained from ψ in an analogous manner, and the system conjugate to (3.2) is

$$\varphi_x=|\nabla\psi|^{q-2}\psi_y, \qquad \varphi_y=-|\nabla\psi|^{q-2}\psi_x. \tag{3.2$'$}$$

In view of the results of Section 2, we have immediately the following result.

Theorem 4. *Let $\varphi \not\equiv$ Const. be p-harmonic in a simply connected domain D. Then there exists a q-harmonic function ψ, where $1/p+1/q=1$, such that*

$$\psi_x=-|\nabla\varphi|^{p-2}\varphi_y, \qquad \psi_y=|\nabla\varphi|^{p-2}\varphi_x.$$

Both φ and ψ belong to the Hölder class $C^{1,\alpha}_{\mathrm{loc}}(D)$ with $\alpha=\min\{p-1, 1/(p-1)\}$. The gradients $\nabla\varphi$ and $\nabla\psi$ are zero precisely in the discrete singular set S and $\nabla\varphi\cdot\nabla\psi=0$.

The theorem was first derived in [AR2, p. 3] in slightly different form. The Hölder continuity is illumiated in the remark after Theorem 1.

Remark. If (3.2) holds, i.e., if

$$\psi_x - i\psi_y = -i|f|^{p-2} f \qquad (f = \varphi_x - i\varphi_y) \tag{3.2$''$}$$

then also (3.1), (3.3), and (3.4) are valid. A convenient regularity assumption in (3.2)$''$ is that $\varphi_x - i\varphi_y$ and $\psi_x - i\psi_y$ belong to $C(D) \cap W^{1,2}_{\mathrm{loc}}(D)$.

We say that φ and ψ are conjugate functions, when they are coupled so that the reciprocity expressed by (3.1), (3.2), and (3.3) holds. The function $F = \varphi + i\psi$ reduces to a holomorphic function for $p = q = 2$!

The described conjugation can also be explained by means of Fenchel's duality for convex variational integrals. See [E–T, pp. 81–82] for this kind of analysis.

3.1. *The Stoïlow Representation*

Suppose that φ and ψ are conjugate functions in the sense described above. Then the mapping $F\colon D \to \mathbb{C}$,

$$F = \varphi + i\psi,$$

has the Jacobian

$$J_F = \frac{\partial(\varphi, \psi)}{\partial(x, y)} = |\nabla\varphi|^p = |\nabla\psi|^q, \tag{3.5}$$

a well-known formula in the classical case $p = 2$. This means that $J_F = 0$ only in the singular set S. See (2.7).

LEMMA. *If φ and ψ are conjugate functions in D and $F = \varphi + i\psi$, then either* (i) *F reduces to a constant or* (ii) *J_F is real-analytic and positive in $D \backslash S$.*

Proof. Combine (3.4), (2.7), and the Corollary of Theorem 2.

THEOREM 5. *If $F = \varphi + i\psi$, φ and ψ being conjugate functions in D, then F admits the Stoïlow representation*

$$F = H \circ \sigma. \tag{3.6}$$

Here σ is a topological mapping of D and H is analytic in $\sigma(D)$.

Proof. Case (i) in the above lemma is clear. In case (ii) the representation follows from [BE2, Lemma 8.1, p. 33], the necessary assumptions being satisfied according to the Implicit Function Theorem. This completes the proof.

Representations of this kind were called interior transformations in [S]. The above-mentioned lemma of Bers is a consequence of the General Uniformization Theorem. Actually, much more can be said about the inner function σ in the Stoïlow representation. Since

$$J_F = |H'(\sigma)|^2 J_\sigma,$$

J_σ is positive and real-analytic outside the singular set S. This implies that σ is quasi-conformal in each domain with compact closure in a fixed set

$$J_\sigma > \varepsilon \qquad (\varepsilon > 0).$$

Thus σ is *locally quasi-conformal in the domain $D \backslash S$.*

Remark. A simple calculation shows that

$$\frac{\partial F}{\partial \bar{z}} = \mu \frac{\partial F}{\partial z},$$

where

$$\mu = \frac{1 - |f|^{p-2}}{1 + |f|^{p-2}} \cdot \left(\frac{\bar{f}}{f}\right)$$

when $f = \varphi_x - i\varphi_y \neq 0$. Especially,

$$\left|\frac{\partial F}{\partial \bar{z}}\right| \leqslant \left|\frac{1 - |f|^{p-2}}{1 + |f|^{p-2}}\right| \left|\frac{\partial F}{\partial z}\right|$$

everywhere in D. (For $p = 2$ we get the Cauchy–Riemann equations: $F_{\bar{z}} = 0$.) We have

$$\left|\frac{1 - |f|^{p-2}}{1 + |f|^{p-2}}\right| = \left|\frac{1 - |\nabla\varphi|^{p-2}}{1 + |\nabla\varphi|^{p-2}}\right| = \left|\frac{1 - |\nabla\psi|^{2-q}}{1 + |\nabla\psi|^{2-q}}\right| \leqslant 1, \tag{3.7}$$

but this quantity takes the value 1 in the singular set S, if $p \neq 2$. By definition (see Section 2.1) this hinders σ in (3.5) from being quasi-conformal in the whole of D, if $p \neq 2$ and $S \neq \phi$. The case $F \equiv \text{Const.}$ is an exception. The proof is complete.

A lot can be done on the basis of the Stoïlow representation. The *Principle of Argument* can be used [R, p. 70], and one can define *the order of a zero* for F at a point z_0 as the order of the corresponding zero at $\sigma(z_0)$ for H. (This concept depends only on F and z_0 but not on the particular Stoïlow representation chosen.)

3.2. *The Order of Zeros*

There is a connection between the representations $\varphi + i\psi = H \circ \sigma$ and $\varphi_x - i\varphi_y = h \circ \chi$. The essential feature is that the orders of zeros for H and h are related:

THEOREM 6. *Let φ be p-harmonic in D $(1 < p < \infty)$ and ψ a stream function, having representations*

$$\varphi + i\psi = H \circ \sigma, \qquad \varphi_x - i\varphi_y = h \circ \chi,$$

valid in D. Let $\varphi \not\equiv$ constant. For any fixed $z_0 \in D$, denote by M (≥ 1) the order of the first non-vanishing derivative of H at $\sigma(z_0)$. (The value of $(\varphi + i\psi)(z_0)$ has no importance.) Denote by N (≥ 0) the order of the first non-vanishing derivative of h at $\chi(z_0)$. ($N = 0$ if $h(\chi(z_0)) \neq 0$). Then $N = M - 1$.

Proof. We will compute $\int_\gamma d\arg(\varphi_x + i\varphi_y)$ for a certain γ in the z-plane, constructed from the representation $\varphi + i\psi = H \circ \sigma$. Put $\zeta = \sigma(z)$, $\zeta_0 = \sigma(z_0)$. It is no restriction to assume $\zeta_0 = 0$ and $H(0) = 0$. Choose a neighborhood U_0 of z_0 such that $\nabla\varphi \neq 0$ in $U_0 \backslash z_0$, and $H' \neq 0$ in $\sigma(U_0 \backslash z_0)$. Now

$$H(\zeta) = \sum_M^\infty a_k \zeta^k = \zeta^M \left(a_M + \sum_{M+1}^\infty a_k \zeta^{k-M} \right)$$

holds in a neighborhood Ω_0 of $\zeta = 0$. We may assume that $\sigma^{-1}(\Omega_0) \subset U_0$. A branch of

$$\sqrt[M]{a_M + \sum_{M+1}^\infty a_k \zeta^{k-M}}$$

can be defined in a neighborhood $\Omega_1 \subset \Omega_0$ of $\zeta = 0$. Put

$$\zeta_1 = \zeta \sqrt[M]{a_M + \sum_{M+1}^\infty a_k \zeta^{k-M}} = \phi(\zeta).$$

Choose a neighborhood $\Omega_2 \subset \Omega_1$ of $\zeta = 0$, mapped by ϕ topologically onto some ω in the ζ_1-plane. For any $z \in \sigma^{-1}(\Omega_2)$ we have $(H \circ \sigma)(z) = (\phi(\sigma(z)))^M = (\varphi + i\psi)(z)$.

Put $\omega_1 = \{\zeta_1 \mid |\mathrm{Re}(\zeta_1^M)| < \varepsilon, |\mathrm{Im}(\zeta_1^M)| < \varepsilon\}$, and choose $\varepsilon > 0$ so small that $\bar{\omega}_1 \subset \omega$. Put $\Gamma = \partial\omega_1$. Then Γ consists of $4M$ sections of curves $\mathrm{Re}(\zeta_1^M) = \varepsilon$, $\mathrm{Im}(\zeta_1^M) = \varepsilon$, $\mathrm{Re}(\zeta_1^M) = -\varepsilon$, $\mathrm{Im}(\zeta_1^M) = -\varepsilon$, repeated in that order. The curves are easily specified in polar coordinates. They meet at $4M$ corners. The angle at each corner within ω_1 is $\pi/2$, by conformality. The mapping $\sigma^{-1} \circ \phi^{-1}$ is topological, and defined in ω. Put $\gamma = (\sigma^{-1} \circ \phi^{-1})(\Gamma)$. Clearly, γ consists of $4M$ sections of level curves of φ and ψ, alternating as above, and meeting at $4M$ corners. Since $\nabla\varphi \cdot \nabla\psi = 0$, level curves of φ and ψ meet at right angles. Put $U_1 = (\sigma^{-1} \circ \phi^{-1})(\omega_1)$. Clearly, the angle within U_1 at each corner of γ is $\pi/2$. Note that $(\sigma^{-1} \circ \phi^{-1})$ is smooth along Γ. Denote the smooth sections of γ by γ_v, $v = 1, 2, ..., 4M$. Observe that, along each γ_v, $\nabla\varphi$ is either tangent to or orthogonal to γ_v. Let γ have a positive orientation. Clearly,

$$\int_\gamma d\arg(\varphi_x + i\varphi_y) = \sum_{v=1}^{4M} \int_{\gamma_v} d\arg(\varphi_x + i\varphi_y).$$

Let e_t be the tangent vector along γ, positively oriented. Then, as is well known,

$$\sum_1^{4M} \int_{\gamma_v} d\arg e_t + \sum \Delta \arg e_t = 2\pi,$$

the last sum being taken over all corners of γ. Thus we have

$$\begin{aligned} &\int_\gamma d\arg(\varphi_x + i\varphi_y) - 2\pi \\ &\quad = \sum_1^{4M} \int_{\gamma_v} d(\arg(\varphi_x + i\varphi_y) - \arg e_t) - \sum \Delta \arg e_t \\ &\quad = -\sum \Delta \arg e_t = -4M \cdot \frac{\pi}{2} = -M \cdot 2\pi, \end{aligned}$$

since $\arg(\varphi_x + i\varphi_y) - \arg e_t$ is constant along each γ_v. This gives

$$\int_\gamma d\arg(\varphi_x + i\varphi_y) = -(M-1)\,2\pi.$$

But $f = \varphi_x - i\varphi_y = h \circ \chi$, and h has a zero at $\chi(z_0)$ of the order $N \geqslant 0$. Hence $\int_\gamma d(\arg f) = N \cdot 2\pi$. Taking into account the minus sign in f, we find $N = M - 1$ (see Fig. 1). This proves the theorem.

Remark. If $p = q = 2$ we have $F(z) = \varphi + i\psi = H(z) =$ holomorphic function, and $F'(z) = \varphi_x + i\psi_x = \varphi_x - i\varphi_y = h(z)$, also holomorphic, and $\sigma(z) = \chi(z) = z$. Thus $H'(z) = h(z)$ in this case.

4. On the Hodograph Method and Quasi-radial Solutions

Here, the hodograph method will be applied to the p-harmonic equation, thereby giving a linear elliptic differential equation in the hodograph plane. Further, the operation of going from the hodograph plane to the physical

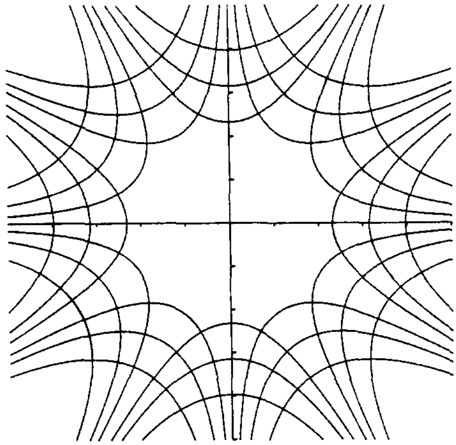

FIG. 1. Proof of Theorem 6. The curves shown are of the form $\mathrm{Re}(\zeta_1^3) = \text{const.}$, and $\mathrm{Im}(\zeta_1^3) = \text{const.}$ Thus $M = 3$, and ω_1 is the star-shaped region in the middle. Further, $\partial\omega_1 = \Gamma$ consists of $4M = 12$ arcs, always meeting at right angles (plot produced by L. Persson, Luleå).

plane will be discussed. Then p-harmonic functions of the form $\varphi = r^k f(\phi)$, "quasi-radial" solutions, will be treated and these functions will illustrate some of our previous theorems. We shall also see what some of these solutions look like in the hodograph plane.

4.1. *Hodograph Transformation for the p-Harmonic Equation*

We will here transform the p-harmonic equation in a classical manner by introducing $u = \varphi_x$, $v = \varphi_y$ as new independent variables. Let us first fix some terminology and notation. In the *physical plane* the variables are x, y and $z = x - iy$. In the *hodograph plane*: u, v and polar coordinates q, θ. In the *potential plane*: φ and ψ.

The presentation will be similar to, but not identical with, [BE1, pp. 13–14; B–S, pp. 130–132].

Assume now that $\varphi(x, y)$ is p-harmonic in a simply connected domain D in the xy-plane. Let $|\mathrm{grad}\,\varphi| \neq 0$ in D. Then φ is smooth, in fact real analytic, in D (see the second corollary in Section 2.1). Assume also that $J = \varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 \neq 0$ in D. Let ψ be a stream function for φ. Then

$$\frac{\partial(\varphi, \psi)}{\partial(x, y)} = |\nabla\varphi|^p = q^p > 0.$$

After suitably restricting the domain D, it will be in one-to-one correspondence continuously differentiable, with some domains in the hodograph and potential planes, respectively. We shall derive the "Chaplygin equations" for φ and ψ. These are differential equations for φ and ψ, considered as functions of (q, θ) in the hodograph plane. First, the chain rule gives

$$\begin{pmatrix} x_\varphi & x_\psi \\ y_\varphi & y_\psi \end{pmatrix}\begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $x_\varphi = (\partial x/\partial \varphi)_{\psi=\text{const.}}$, etc. Put

$$A=\begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix}=\begin{pmatrix} q\cos\theta & q\sin\theta \\ -q^{p-1}\sin\theta & q^{p-1}\cos\theta \end{pmatrix}.$$

The inverse of A is

$$A^{-1}=\left(\frac{\partial(x,y)}{\partial(\varphi,\psi)}\right)=\frac{1}{q^p}\begin{pmatrix} q^{p-1}\cos\theta & -q\sin\theta \\ q^{p-1}\sin\theta & q\cos\theta \end{pmatrix}$$

which gives $x_\varphi = \cos\theta/q$, etc.

Writing $z = x+iy$, we see that $\partial z/\partial\varphi = e^{i\theta}/q$ and $\partial z/\partial\psi = ie^{i\theta}/q^{p-1}$. Therefore, a line integral

$$\int \frac{e^{i\theta}}{q}\,d\varphi + \frac{ie^{i\theta}}{q^{p-1}}\,d\psi$$

is independent of the path in the potential plane. Consequently,

$$\frac{\partial}{\partial\psi}\left(\frac{e^{i\theta}}{q}\right)=\frac{\partial}{\partial\varphi}\left(\frac{ie^{i\theta}}{q^{p-1}}\right) \tag{a}$$

(= two real equations). Transforming this condition to the (q, θ)-plane will give the Chaplygin equations. The mapping $(\varphi, \psi) \leftrightarrow (q, \theta)$ is one-to-one and C^1 in both directions. Put $J_1 = \partial(q, \theta)/\partial(\varphi, \psi)$. Since $(\partial(\varphi, \psi)/\partial(q, \theta)) = (\partial(q, \theta)/\partial(\varphi, \psi))^{-1}$ we see that

$$\varphi_q=\frac{\theta_\psi}{J_1},\quad \varphi_\theta=-\frac{q_\psi}{J_1},\quad \psi_q=-\frac{\theta_\varphi}{J_1},\qquad\text{and}\qquad \psi_\theta=\frac{q_\varphi}{J_1}.$$

We can therefore also write

$$\partial_\psi = J_1(-\varphi_\theta\,\partial_q + \varphi_q\,\partial_\theta)$$
$$\partial_\varphi = J_1(\psi_\theta\,\partial_q - \psi_q\,\partial_\theta)$$

so the relation (a) is transformed into

$$(-\varphi_\theta\,\partial_q+\varphi_q\,\partial_\theta)\left(\frac{e^{i\theta}}{q}\right)=(\psi_\theta\,\partial_q-\psi_q\,\partial_\theta)\left(\frac{ie^{i\theta}}{q^{p-1}}\right),$$

or

$$\varphi_\theta \frac{e^{i\theta}}{q^2} + \varphi_q \frac{ie^{i\theta}}{q} = (1-p)\,\psi_\theta \frac{ie^{i\theta}}{q^p} + \psi_q \frac{e^{i\theta}}{q^{p-1}}.$$

Multiplying by $e^{-i\theta}$ and taking real and imaginary parts gives the system

$$\begin{aligned} \varphi_\theta &= \frac{\psi_q}{q^{p-3}} \\ \varphi_q &= \frac{(1-p)\,\psi_\theta}{q^{p-1}} \end{aligned} \tag{b}$$

of linear equations. Elimination of φ or ψ by differentiation gives

$$\begin{aligned} q^{p-3}\varphi_{\theta\theta} &= \partial_q\left(-\frac{q^{p-1}}{p-1}\varphi_q\right) \\ -\frac{p-1}{q^{p-1}}\psi_{\theta\theta} &= \partial_q\left(\frac{\psi_q}{q^{p-3}}\right). \end{aligned} \tag{c}$$

These equations are simplified to

$$\begin{aligned} \varphi_{\theta\theta} + \frac{q^2}{p-1}\varphi_{qq} + q\varphi_q &= 0 \\ \psi_{\theta\theta} + \frac{q^2}{p-1}\psi_{qq} + \frac{3-p}{p-1}q\psi_q &= 0. \end{aligned} \tag{d}$$

These "Chaplygin equations" are *a consequence of our assumptions concerning* $\varphi(x, y)$. Observe that the uncoupled equations (d) are not equivalent to (b).

Before transforming (d) further, we will prove a theorem concerning the inverse procedure, i.e., going from the hodograph plane to the physical. Strictly speaking, we consider the "polar hodograph" plane and do *not* identify points (q, θ) and $(q, \theta + 2\pi)$.

THEOREM 7. *Let there be given a function* $\varphi = \varphi(q, \theta) \in C^2(\Omega)$, *where* Ω *is a simply connected domain in the right half* $q > 0$ *of the* (q, θ)*-plane. Assume that* $\varphi_q^2 + \varphi_\theta^2 > 0$ *in* Ω *and*

$$\varphi_{\theta\theta} + \frac{q^2}{p-1}\varphi_{qq} + q\varphi_q = 0$$

there. One can then define, uniquely up to a constant, a function $\psi \in C^2(\Omega)$ *by*

$$\psi_q = q^{p-3}\varphi_\theta, \qquad \psi_\theta = -\frac{q^{p-1}}{p-1}\varphi_q.$$

Then all equations (b), (c), *and* (d) *will be satisfied in* Ω. *Further, the line integral*

$$I(\Gamma)=\int_\Gamma \frac{e^{i\theta}}{q}(\varphi_q\,dq+\varphi_\theta\,d\theta)+\frac{ie^{i\theta}}{q^{p-1}}(\psi_q\,dq+\psi_\theta\,d\theta)$$

is path independent in Ω. *Let* Γ *connect a fixed* (q_0,θ_0) *to a variable* $(q,\theta)\in\Omega$. *Put* $z=x+iy=I(\Gamma)=z(q,\theta)$. *Then* $z(q,\theta)\in C^2(\Omega)$ *and* $\partial(x,y)/\partial(q,\theta)\neq 0$ *in* Ω. *After a possible reduction of* Ω *to* Ω', Ω' *will be in one-to-one correspondence with a domain* D' *in the physical plane. Now the relations* $z=I(\Gamma)$, $\varphi=\varphi(q,\theta)$, $\psi=\psi(q,\theta)$ *implicitly define* $\varphi=\varphi(x,y)$ *and* $\psi=\psi(x,y)$ *as functions in* D'. *These functions are in* $C^2(D')$, φ *is p-harmonic, and* ψ *is a stream function for* φ. *Thus,* ψ *is p'-harmonic* $(1/p+1/p'=1)$. *Finally,* $\varphi_x+i\varphi_y=qe^{i\theta}$.

Proof. The p.d.e. for φ says that the expressions for ψ_q and ψ_θ are *compatible* in Ω. Since Ω is simply connected, ψ can be defined as stated in $C^2(\Omega)$. Clearly, Eq. (b), (c), and (d) hold in Ω. Now

$$I(\Gamma)=\int_\Gamma\left(\frac{e^{i\theta}}{q}\varphi_q+\frac{ie^{i\theta}}{q^{p-1}}\psi_q\right)dq+\left(\frac{e^{i\theta}}{q}\varphi_\theta+\frac{ie^{i\theta}}{q^{p-1}}\psi_\theta\right)d\theta.$$

It is a routine matter to verify that $I(\Gamma)$ is path independent, as a consequence of (b). Clearly

$$I(\Gamma)=\int_{\Gamma'}\frac{e^{i\theta}}{q}\,d\varphi+\frac{ie^{i\theta}}{q^{p-1}}\,d\psi,$$

where Γ' is the image of Γ in the potential plane. We further observe that $\partial(\varphi,\psi)/\partial(q,\theta)=(-q^{p-1}/(p-1))\,\varphi_q^2-q^{p-3}\varphi_\theta^2<0$ in Ω, so the map $(q,\theta)\to(\varphi,\psi)$ is invertible, *at least locally*. From the form of $I(\Gamma)$ we see that

$$\left(\frac{\partial(x,y)}{\partial(\varphi,\psi)}\right)=\begin{pmatrix}\dfrac{\cos\theta}{q} & -\dfrac{\sin\theta}{q^{p-1}}\\[2ex] \dfrac{\sin\theta}{q} & \dfrac{\cos\theta}{q^{p-1}}\end{pmatrix}$$

and the Jacobian $\partial(x,y)/\partial(\varphi,\psi)=1/q^p>0$. Since $\partial(\varphi,\psi)/\partial(q,\theta)<0$ we have $\partial(x,y)/\partial(q,\theta)<0$ in Ω. It is also clear that $z(q,\theta)\in C^2(\Omega)$. It remains to consider $\varphi=\varphi(x,y)$ and $\psi=\psi(x,y)$, defined in some convenient domain $D'\subset D$. It follows that $\varphi,\psi\in C^2(D')$, since all variable transformations are in C^2. From the form of $\partial(x,y)/\partial(\varphi,\psi)$ one has at once

$$\frac{\partial(\varphi,\psi)}{\partial(x,y)}=\begin{pmatrix}q\cos\theta & q\sin\theta\\ -q^{p-1}\sin\theta & q^{p-1}\cos\theta\end{pmatrix}. \tag{4.1}$$

Thus, $\varphi_x = q\cos\theta, \ldots, |\nabla\varphi| = q$ and $|\nabla\psi| = q^{p-1}$. The relation $\partial_x(\psi_y) = \partial_y(\psi_x)$ gives $\partial_x(|\nabla\varphi|^{p-2}\varphi_x) = \partial_y(-|\nabla\varphi|^{p-2}\varphi_y)$, i.e., φ is p-harmonic and analogously for ψ. This proves the theorem.

COROLLARY. *Suppose that, instead of φ, ψ is given in $C^2(\Omega)$, satisfying*

$$\psi_{\theta\theta} + \frac{q^2}{p-1}\psi_{qq} + \frac{3-p}{p-1}q\psi_q = 0, \tag{4.2}$$

and $\psi_q^2 + \psi_\theta^2 > 0$ in Ω. Then a conjugate function $\varphi \in C^2(\Omega)$ can be defined via the same relations as in the theorem, and all conclusions remain valid.

Proof. Analogous to the above proof.

Remark. A multiple covering of the (u, v)-plane is not excluded, as long as φ, ψ are well-defined functions of (q, θ).

4.2. *Quasi-radial Solutions*

p-harmonic functions of the form $\varphi = r^k f(\phi)$ (polar coordinates) will be called quasi-radial solutions. In [AR1] all such functions were determined for any $p > 2$, and a representation formula was derived. As is briefly explained in [AR2], this implies similar results for the dual case $q < 2$. The quasi-radial solutions $r^k f(\phi)$ fall into different categories with quite different properties, depending on k and p, and there are also "subcases" within some categories. We will now consider $p > 2$ *and "category* 3," *for which* $k > (p-2)/(p-1)$.

Then $f(\phi)$ has a parametric representation

$$\begin{aligned} \phi &= \tau - a(k-1)\int_0^\tau \frac{d\tau'}{ak - \cos^2\tau'} + C_1 \\ f &= C_2 \cdot \left(1 - \frac{\cos^2\tau}{ak}\right)^{(k-1)/2} \cdot \cos\tau, \end{aligned} \tag{4.3}$$

where $a = (p-1)/(p-2)$, C_1 and C_2 are arbitrary constants, and τ is a parameter, $-\infty < \tau < \infty$. It is easy to verify that $f(\phi)$ is a *periodic function* with period $2\bar{\phi} = 2\pi[1 - (1 - 1/k)\sqrt{ak}/\sqrt{ak-1}]$ [AR1, p. 146].

Solutions $r^k(f(\phi))$ in a full neighborhood of the origin are found by solving the equation $2\pi = m \cdot 2\bar{\phi}$, $m = \text{integer} \geqslant 1$. Then m is the number of max and min of $f(\phi)$, for $0 \leqslant \phi \leqslant 2\pi$. One finds that [AR1, p. 150]

$$k = \frac{2a - (1-1/m)^2 + (1-1/m)\sqrt{4a(a-1) + (1-1/m)^2}}{2a[1-(1-1/m)^2]} = k(m, p).$$

EXAMPLE. Choose $p=3$, $m=2$. The above formula gives $k(2,3)=(15+\sqrt{33})/12$. If φ is 3-harmonic, it follows from Section 2 that $\nabla\varphi$ satisfies a local Hölder condition with the exponent $1/(p-1)=\frac{1}{2}$. We also now find an upper estimate for the Hölder exponent, namely $k(2,3)-1=(3+\sqrt{33})/12=0{,}7287...$.

Return to the p-harmonic $\varphi=r^k f(\phi)$ $(p>2$ and $k>(p-2)/(p-1))$. Clearly (for $z\neq 0$)

$$\varphi_x+i\varphi_y=r^{k-1}(kf(\phi)+if'(\phi))\,e^{i\phi}.$$

The parametric representation gives, if we choose $C_2=1/k>0$:

$$f'(\phi)=\left(1-\frac{\cos^2\tau}{ak}\right)^{(k-1)/2}\cdot\sin\tau$$

[AR1, p. 143]. Thus

$$kf(\phi)+if'(\phi)=\left(1-\frac{\cos^2\tau}{ak}\right)^{(k-1)/2}\cdot e^{-i\tau},$$

and

$$\varphi_x+i\varphi_y=r^{k-1}\left(1-\frac{\cos^2\tau}{ak}\right)^{(k-1)/2}\cdot e^{i(\phi-\tau)}.$$

Clearly, $k=1$ implies $\varphi=Ax+By$; *a case avoided from now on.* Observe that $d\phi/d\tau=1-(a(k-1))/(ak-\cos^2\tau)$, so that $|(d\phi/d\tau)-1|\geqslant (a|k-1|)/ak>0$, for all τ.

Further, $d\phi/d\tau=(a-\cos^2\tau)/(ak-\cos^2\tau)\geqslant(a-1)/ak>0$, for all τ. Put $\rho(\tau)=(1-\cos^2\tau/ak)^{(k-1)/2}$, so that

$$\varphi_x+i\varphi_y=r^{k-1}\rho(\tau)\,e^{i(\phi-\tau)}.$$

Thus $\partial/\partial r|\varphi_x+i\varphi_y|=(k-1)\,r^{k-2}\rho(\tau)\neq 0$, and $\partial/\partial r(\arg(\varphi_x+i\varphi_y))=0$. Further, $\partial/\partial\phi(\arg(\varphi_x+i\varphi_y))=1-d\tau/d\phi\neq 0$. Therefore, $J_0=\partial(|\varphi_x+i\varphi_y|,\arg(...))/\partial(r,\phi)\neq 0$. Indeed, $C_3r^{k-2}\leqslant|J_0|\leqslant C_4r^{k-2}$, for some positive constants C_3, C_4 which only depend on p and k. This holds also if φ is not defined in a full neighborhood of the origin.

Let $\varphi=r^kf(\phi)$ be p-harmonic in a full neighborhood of the origin, and not a linear function. Then $k>1$ and $m\geqslant 2$ (see [AR1, p. 149]). In the parametric representation, choose $C_1=0$ and $C_2=1/k$. As in Section 2 we want to write $\varphi_x-i\varphi_y=(h\circ\chi)(z)$ for some holomorphic h and quasiconformal χ. We simply try

$$\varphi_x-i\varphi_y=r^{k-1}\rho(\tau)\,e^{i(\tau-\phi)}=(\chi(z))^n \tag{4.4}$$

for some integer $n\geqslant 1$.

Let ϕ increase from 0 to 2π. Then τ increases from 0 to $m \cdot 2\pi$, so that $(\tau - \phi)$ increases to $(m-1) \cdot 2\pi$. Therefore, choose $n = m-1$. Thus $\arg \chi(z) = (\tau - \phi)/(m-1)$, which is uniquely defined, modulo 2π, and continuous. Further,

$$\chi(z) = r^{(k-1)/(m-1)} \left(1 - \frac{\cos^2 \tau}{ak}\right)^{(k-1)/(2(m-1))} \cdot e^{i(\tau - \phi)/(m-1)}. \qquad (4.5)$$

It remains to verify that the mapping $z \to \chi(z)$ is quasi-conformal. It can be decomposed into three mappings: first a "radial power," then multiplication by $\rho(\tau)^{1/(m-1)}$, and finally an angular shift, $\phi \to (\tau(\phi) - \phi)/(m-1)$. It is well known that the radial power is quasi-conformal (see [L–V, p. 66]). One can easily verify that the two other mappings are so, too. We leave the details. Thus, χ is quasi-conformal.

Next, consider the problem of representing $\varphi + i\psi$ near a critical point. Let $p > 2$ and choose φ quasi-radial, defined in $\mathbb{C}^2$, with $m \geqslant 2$, so that $k > 1$. As shown in [AR2], there is a stream function $\psi = r^l g(\phi)$, where $l - 1 = (p-1)(k-1)$. It follows that $l > k > 1$, so that $\nabla\varphi = \nabla\psi = 0$ at the origin. Further,

$$g(\phi) = -\frac{1}{l} f'(\phi)[k^2 f(\phi)^2 + (f'(\phi))^2]^{(p-2)/2}.$$

Choose $C_1 = 0$, and recall that $f'(\phi) = -\text{const.} \cdot \rho(\tau) \sin \tau$. Clearly, $|(\varphi + i\psi)(z)| = A|z|^k$ (A = positive constant), if $\arg z = n \cdot \pi/m$, n = integer, since then $\tau = n \cdot \pi$. For some other angles, we find $|(\varphi + i\psi)(z)| = B \cdot |z|^l$. Clearly, this behavior cannot be produced by a mapping $H \circ \chi$, with H holomorphic and χ quasi-conformal. It is, however, possible to construct a representation $(\varphi + i\psi)(z) = (\chi(z))^m$, where χ is a topological mapping of the whole plane onto itself. Put

$$\chi(z) = (\varphi(z)^2 + \psi(z)^2)^{1/2m} \cdot e^{i(\arg(\varphi(z) + i\psi(z))/m)},$$

i.e.,

$$\chi(z) = (r^{2k} f(\phi)^2 + r^{2l} g(\phi)^2)^{1/2m} \cdot e^{i(\arg(r^k f(\phi) + i r^l g(\phi))/m)}.$$

One can easily verify that $\arg(r^k f(\phi) + i r^l g(\phi))$ is strictly increasing from 0 to $m \cdot 2\pi$, when ϕ increases from 0 to 2π, keeping $r > 0$ fixed. This explains the exponent m. Clearly, χ is well defined and continuous. We leave further details.

4.3. *The Hodograph Again*

In agreement with Theorem 7, consider the p.d.e. in the hodograph plane:

$$\varphi_{\theta\theta} + \frac{q^2}{p-1} \varphi_{qq} + q\varphi_q = 0 \qquad (q > 0).$$

The change of variables $\mu = -\log q$ gives

$$\varphi_{\theta\theta} + \frac{1}{p-1}\,\varphi_{\mu\mu} - \frac{p-2}{p-1}\,\varphi_\mu = 0. \tag{4.6}$$

We are looking for some simple solutions of the form $\varphi = F(\theta)\,G(\mu)$: $F''G + (1/(p-1))\,FG'' - ((p-2)/(p-1))\,FG' = 0$. Separation of variables gives

$$F'' + \lambda F = 0,$$

$$G'' - (p-2)\,G' - (p-1)\,\lambda G = 0,$$

where the parameter λ is at our disposal. The characteristic equation for G is

$$\alpha^2 - (p-2)\,\alpha - \lambda(p-1) = 0, \tag{4.7}$$

i.e., $\alpha = (p-2)/2 \pm \frac{1}{2}\sqrt{(p-2)^2 + 4\lambda(p-1)}$. We still have $p > 2$ and look for a solution φ approaching 0, when $q \to 0$, i.e., when $\mu \to +\infty$. Thus (4.7) must have a negative root and $\lambda > 0$ must hold. This gives $F(\theta) = A \sin\sqrt{\lambda}\,\theta + B\cos\sqrt{\lambda}\,\theta$, $G(\mu) = Ce^{\alpha_1\mu} + De^{\alpha_2\mu}$.

Put $\beta = \frac{1}{2}[\sqrt{(p-2)^2 + 4\lambda(p-1)} - (p-2)]$. It is, for the present purpose, sufficient to consider $\varphi = \sin\sqrt{\lambda}\,\theta \cdot e^{-\beta\mu}$, i.e.,

$$\varphi(q, \theta) = q^\beta \sin\sqrt{\lambda}\,\theta.$$

In order to transform φ to the z plane, we must find the conjugate function ψ according to Theorem 7. Therefore, ψ must satisfy

$$\begin{aligned} \psi_q &= q^{p-3}\cdot\sqrt{\lambda}\cdot q^\beta\cos\sqrt{\lambda}\,\theta \\ \psi_\theta &= -\frac{q^{p-1}}{p-1}\cdot\beta q^{\beta-1}\sin\sqrt{\lambda}\,\theta. \end{aligned} \tag{4.8}$$

It is an easy consequence of (4.7) (satisfied by $-\beta$) that these equations are compatible and satisfied by

$$\psi = \frac{\beta}{(p-1)\sqrt{\lambda}}\cdot q^{p+\beta-2}\cos\sqrt{\lambda}\,\theta.$$

Now φ, ψ are well defined for $q > 0$ and all θ. It remains to consider $\int_\Gamma (e^{i\theta}/q)\,d\varphi + (ie^{i\theta}/q^{p-1})\,d\psi$ along various paths Γ in the (q, θ)-plane. Let Γ_0 be a segment $\theta' \leqslant \theta \leqslant \theta''$, $q = \bar{q}$, parallel to the θ-axis. Observe that $|\varphi_\theta| \leqslant \text{const.}\cdot q^\beta$, $|\psi_\theta| \leqslant \text{const.}\cdot q^{p+\beta-2}$, and, therefore, along this segment,

$|(e^{i\theta}/q)\,\varphi_\theta + (ie^{i\theta}/q^{p-1})\,\psi_\theta| \leqslant K\cdot \bar{q}^{\beta-1}$, where K is independent of $\bar{q}$. *Suppose that* $\beta > 1$. It follows that $\int_{\Gamma_0} (e^{i\theta}/q)\,d\varphi + (ie^{i\theta}/q^{p-1})\,d\psi \to 0$, when $\bar{q} \to 0$. Now write

$$z = z(q_1, \theta_1) = \int_\Gamma \frac{e^{i\theta}}{q}\,d\varphi + \frac{ie^{i\theta}}{q^{p-1}}\,d\psi,$$

where Γ connects some fixed point (q_0, θ_0) to the variable (q_1, θ_1). The integral is path independent in the half-plane $q > 0$. Choose Γ as indicated in Fig. 2 and let $\bar{q} \to +0$.

It is clearly no serious restriction to choose $z(q_1, \theta_1) = \int_{\Gamma_1} \cdots$, where Γ_1 is the segment $0 \leqslant q \leqslant q_1$, $\theta = \theta_1$. This integral is

$$\int_0^{q_1} \left(\frac{e^{i\theta_1}}{q}\cdot \beta q^{\beta-1} \sin\sqrt{\lambda}\,\theta_1 + \frac{ie^{i\theta_1}}{q^{p-1}}\cdot K_0 q^{p+\beta-3} \cos\sqrt{\lambda}\,\theta_1\right) dq,$$

where

$$K_0 = \frac{\beta(p+\beta-2)}{(p-1)\sqrt{\lambda}} = \frac{\beta^2 + (p-2)\,\beta}{(p-1)\sqrt{\lambda}} = \sqrt{\lambda}$$

(note that (4.7) holds for $\alpha = -\beta$). The integral is simplified to

$$\begin{aligned} z = (q_1, \theta_1) &= e^{i\theta_1} \int_0^{q_1} q^{\beta-2}\,dq \cdot (\beta \sin\sqrt{\lambda}\,\theta_1 + i\sqrt{\lambda}\cos\sqrt{\lambda}\,\theta_1) \\ &= \frac{1}{\beta-1}\cdot q_1^{\beta-1}\cdot e^{i\theta_1}(\beta\sin\sqrt{\lambda}\,\theta_1 + i\sqrt{\lambda}\cos\sqrt{\lambda}\,\theta_1) \\ &= \frac{i\sqrt{\lambda}}{\beta-1}\cdot q_1^{\beta-1}\cdot e^{i\theta_1}\left(\cos\sqrt{\lambda}\,\theta_1 - i\frac{\beta}{\sqrt{\lambda}}\sin\sqrt{\lambda}\,\theta_1\right). \end{aligned}$$

The derivation was done assuming that

$$\beta = \tfrac{1}{2}\,[\sqrt{(p-2)^2 + 4\lambda(p-1)} - (p-2)] > 1,$$

and $\lambda > 0$. Clearly, β is an increasing function of λ, and $\lambda = 1$ gives $\beta = 1$. It is therefore necessary and sufficient for the validity that $1 < \lambda < \infty$, and

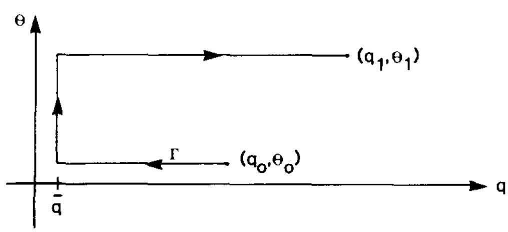

FIGURE 2

then $1 < \beta < \infty$. Further, $\beta < \lambda$ holds here, as is easily seen by looking at $d\beta/d\lambda$.

Writing $z = re^{i\phi}$, the expression for the line integral gives $\phi = \theta + H(\theta)$, with $H(\theta) = \arg(\cos\sqrt{\lambda}\,\theta - i(\beta/\sqrt{\lambda})\sin\sqrt{\lambda}\,\theta) + \pi/2$ (θ_1 replaced by θ). A trivial calculation gives $H'(\theta) = -1/((1/\beta)\cos^2\sqrt{\lambda}\,\theta + (\beta/\lambda)\sin^2\sqrt{\lambda}\,\theta)$. Put $K_1 = \min(1/\beta, \beta/\lambda)$ and $K_2 = \max(1/\beta, \beta/\lambda)$. Then $0 < K_1 \leqslant K_2 < 1$, and

$$-\frac{1}{K_1} \leqslant H'(\theta) \leqslant -\frac{1}{K_2} < -1.$$

Thus, *ϕ is a strictly decreasing function of θ.*

LEMMA. *The p-harmonic function $\varphi = \varphi(z)$ represented by $z = z(q, \theta)$ (as above) and $\varphi = q^\beta \sin\sqrt{\lambda}\,\theta$ is a quasi-radial function and so is the stream function ψ.*

Proof. Clearly, $\phi = \theta(\phi)$ has a smooth inverse $\theta = \theta(\phi)$. The formula for z gives $|z| = r = q^{\beta-1}\cdot F(\phi) > 0$, i.e., $q = r^{1/(\beta-1)}\cdot F(\phi)^{-1/(\beta-1)}$. Thus, $\varphi(re^{i\phi}) = r^{\beta/(\beta-1)}\cdot G(\phi)$, and similarly for ψ. This proves the lemma.

Remark. The solution $\varphi(z)$ falls into case 3 in [AR1], since $\beta/(\beta-1) > 1 > (p-2)/(p-1)$.

It is convenient to summarize.

THEOREM 8. *Let $1 < p < \infty$ and $\lambda > 1$. Put*

$$\beta = \tfrac{1}{2}[\sqrt{(p-2)^2 + 4\lambda(p-1)} - (p-2)].$$

Then $\beta > 1$. Consider the formulas

$$\begin{aligned} \varphi &= q^\beta \sin\sqrt{\lambda}\,\theta \\ \psi &= \frac{\beta}{(p-1)\sqrt{\lambda}}\cdot q^{p+\beta-2}\cos\sqrt{\lambda}\,\theta \qquad (4.9) \\ z &= \frac{i\sqrt{\lambda}}{\beta-1}\cdot q^{\beta-1}e^{i\theta}\left(\cos\sqrt{\lambda}\,\theta - i\frac{\beta}{\sqrt{\lambda}}\sin\sqrt{\lambda}\,\theta\right). \end{aligned}$$

Let D be a domain in the z-plane and Ω a domain in the (q, θ)-plane such that D and Ω are in 1–1 correspondence through the last of the three formulas. Then the three formulas define a p-harmonic function φ in D such that $\varphi_x + i\varphi_y = qe^{i\theta}$, and a p'-harmonic function ψ in D, where $1/p + 1/p' = 1$. Further, ψ is the stream function of φ. Finally, φ and ψ are quasi-radial functions.

It is not necessary for the conclusion that D and Ω are in 1–1 correspondence, as long as the formulas define φ and ψ uniquely in D.

Remarks. Although the preceding discussion was, at a few places, focused on the case $p > 2$, it is easy to see that the theorem holds for $1 < p < \infty$. Applications of these solutions will be given elsewhere.

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