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On convexity of level sets of p-harmonic functions

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ABSTRACT

In this paper, we give sharp estimates of the smallest principal curvature k_1 of level sets of n-dimensional p-harmonic functions which extends the result of 2-dimensional minimal surface case due to Longinetti [Longinetti, On minimal surfaces bounded by two convex curves in parallel planes, J. Differential Equations 67 (3) (1987) 344–358]. More precisely, we prove that the function $|\nabla u|k_1^{-1}$ is a convex function with respect to the layer parameter of the level sets for all $2 \leqslant n < +\infty$ and $1 < p < +\infty$.

1. Introduction

Convexity of level sets is an important geometric feature for solutions of elliptic boundary value problems. By the work of Gabriel [7], the level sets of the Green function of a 3-dimensional bounded convex domain are strictly convex. In 1977, Lewis [13] extended Gabriel's result to p-harmonic functions in high dimensions. Caffarelli and Spruck [4] generalized Lewis's results to a class of semilinear elliptic partial differential equations. The convexity of level sets for solutions of partial differential equations has been extensively studied in the literature, we refer to [1,2,11,12,16,21] and references therein.

The results stated above are all of qualitative nature. It is natural for us to give quantitative descriptions of the convexity of level sets. For 2-dimensional harmonic function with convex level curves, Longinetti [14], Ortel and Schneider [19], Talenti [22] proved that the curvature of the level curves attains its minimum on the boundary. Longinetti [15] also studied the relation between the curvature of the convex level curves and the height of 2-dimensional minimal surface. More precisely, let Γ_0

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and Γ_1 be two convex plane curves lying in the planes $\{x_3 = 0\}$ and $\{x_3 = 1\}$, respectively. Denote by S the minimal surface between Γ_0 and Γ_1. For $0 < t < 1$, let $\Gamma_t = S \cap \{x_3 = t\}$ and let k be the curvature of Γ_t. Then the function of one variable

$$f(t) = \min_{x \in \Gamma_t} \log k(x)$$

is a concave function for $t \in (0, 1)$. From this, Longinetti proved that the curvature k of the convex curves Γ_t on the minimal surface S takes its minimum on the boundary $\Gamma_0 \cup \Gamma_1$.

Recently, Ma, Ou and Zhang [17] got the Gaussian curvature estimates of the convex level sets of high dimensional p-harmonic functions. In the paper Chang, Ma and Yang [5], they obtained principal curvature estimates for convex level sets of high dimensional harmonic functions. These results are also generalized by Guan and Xu [8] to a class of fully nonlinear elliptic equations under certain structural condition (introduced by Bianchini, Longinetti and Salani [2]) by the approach of constant rank theorem. For more results on curvature estimates, please see the papers [3,10,18,23,24].

The main focus of this paper is to give sharp estimates of the principal curvature of level sets of p-harmonic functions which extends the results of minimal surfaces case due to Longinetti [15]. Now we state our main theorem.

Theorem 1.1. *Let u satisfy*

$$\begin{cases} \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ u = 0 & \text{on } \partial\Omega_0, \\ u = 1 & \text{on } \partial\Omega_1, \end{cases}$$

where Ω_0 and Ω_1 are bounded smooth convex domains in $\mathbb{R}^n$ $(n \geqslant 2)$, $1 < p < +\infty$ and $\bar{\Omega}_1 \subset \Omega_0$. Let k_1 be the smallest principal curvature of the level sets of u. For $t \in (0, 1)$, denote $\Omega_t = \{x \in \Omega \mid u(x) = t\}$. Then the function of one variable

$$f(t) = \max_{x \in \Omega_t} \big(|\nabla u| k_1^{-1}\big)(x)$$

is a convex function for $t \in (0, 1)$.

Remark 1.2. Let u be the standard p-Green function of the ball $B_R(0) \subset \mathbb{R}^n$, i.e.

$$u(x) = \begin{cases} |x|^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}, & \text{for } 1 < p < n, \\ -\log|x| + \log R, & \text{for } p = n. \end{cases}$$

Then

$$|\nabla u|(x) = \begin{cases} \frac{n-p}{p-1}|x|^{\frac{1-n}{p-1}}, & \text{for } 1 < p < n, \\ |x|^{-1}, & \text{for } p = n, \end{cases}$$

and the smallest principal curvature of the level set through x is

$$k_1(x) = |x|^{-1}.$$

Hence, for $t = u(x)$ and $1 < p < n$,

$$\begin{aligned}\left(|\nabla u|k_1^{-1}\right)(x) &= \frac{n-p}{p-1}|x|^{\frac{p-n}{p-1}}\\ &= \frac{n-p}{p-1}\left[u(x)+R^{\frac{p-n}{p-1}}\right]\\ &= \frac{n-p}{p-1}t+\frac{n-p}{p-1}R^{\frac{p-n}{p-1}}.\end{aligned}$$

For $p = n$, we have

$$\left(|\nabla u|k_1^{-1}\right)(x) = 1.$$

From the above calculation, we know that $|\nabla u|k_1^{-1}$ is an affine function of the height of the p-Green function. In this sense, the result in Theorem 1.1 is sharp.

Remark 1.3. It is well-known that the norm of the gradient $|\nabla u|$ attains its maximum and minimum on the boundary $\partial\Omega$ (see Proposition 4.1 in Ma, Ou and Zhang [17]). So Theorem 1.1 implies a positive lower bound for the principal curvatures.

The approach to prove Theorem 1.1 is essentially a maximum principle argument. The main difficulty lies in deriving a differential inequality for a suitable test function.

The organization of this paper is as follows. In Section 2, we first give some notations relative to support function on $\mathbb{S}^{n-1}$, then obtain a fully nonlinear version of the p-Laplace equation which is stated in the language of support function. Section 3 is devoted to the proof of Theorem 1.1.

2. Support function

We start by introducing some basic notations, which appeared in [6,16].

Let Ω_0 and Ω_1 be two bounded smooth convex domains in $\mathbb{R}^n$ such that $\bar{\Omega}_1 \subset \Omega_0$ and let $\Omega = \Omega_0 \setminus \bar{\Omega}_1$. Let $u : \bar{\Omega} \to \mathbb{R}$ be a smooth function such that

$$u = 0 \quad \text{on } \partial\Omega_0, \qquad u = 1 \quad \text{on } \bar{\Omega}_1.$$

Furthermore, we assume that $|\nabla u| > 0$ in Ω and the level sets of u are strictly convex with respect to the normal direction ∇u. For $0 < t < 1$, we set

$$\bar{\Omega}_t = \{x \in \bar{\Omega}_0 \mid u \geqslant t\}.$$

Then each point $x \in \Omega$ belongs to the boundary of $\bar{\Omega}_{u(x)}$. Under these assumptions, it is possible to define a function $H : \mathbb{R}^n \times [0,1] \to \mathbb{R}$, $(X,t) \mapsto H(X,t)$ as follows. For each $t \in [0,1]$, $H(\cdot,t)$ is the support function of the convex body $\bar{\Omega}_t$. Denote by h the restriction of H to $\mathbb{S}^{n-1} \times [0,1]$.

In the rest of this section, we will derive the p-Laplace equation by means of h. Before doing that, we should reformulate the first and second derivatives of u using h and its derivatives (see [6,16,20]). For convenience of the reader, we sketch out the main steps here.

Note that h is the restriction of H to $\mathbb{S}^{n-1} \times [0,1]$. It follows that $h(\theta,t) = H(Y(\theta),t)$, where $Y \in \mathbb{S}^{n-1}$ and $\theta = (\theta_1,\dots,\theta_{n-1})$ is a local coordinate system on $\mathbb{S}^{n-1}$. Since the level sets of u are strictly convex, we can define the map

$$x(X,t) = x_{\bar{\Omega}_t}(X),$$

which for every $(X,t) \in \mathbb{R}^n \setminus \{0\} \times (0,1)$ assigns the unique point $x \in \Omega$ on the level set $\{u = t\}$ where the gradient of u is parallel to X (and orientation reversed).

If we define

$$T_i = \frac{\partial Y}{\partial \theta_i},$$

then $\{T_1, \dots, T_{n-1}\}$ is a tangent frame field on $\mathbb{S}^{n-1}$. Furthermore, we assume that $\{T_1, \dots, T_{n-1}, Y\}$ is an orthonormal frame positively oriented. It is easy to see that

$$\frac{\partial T_i}{\partial \theta_j} = -\delta_{ij} Y, \tag{2.1}$$

where δ_{ij} is the standard Kronecker delta symbol.

We denote

$$x(\theta, t) = x_{\bar{\Omega}_t}\big(Y(\theta)\big).$$

Since Y is orthogonal to $\partial\bar{\Omega}_t$ at $x(\theta, t)$, by differentiating the equality

$$h(\theta, t) = \big\langle x(\theta, t), Y(\theta)\big\rangle, \tag{2.2}$$

we obtain

$$h_i = \langle x, T_i \rangle. \tag{2.3}$$

Here, $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$. By (2.2) and (2.3), we have

$$x = hY + \sum_{i=1}^{n-1} h_i T_i. \tag{2.4}$$

Henceforth we will omit the ranges of the summation indices if they run from 1 to $n-1$. With (2.1) in hand, by differentiating (2.4), we obtain

$$\frac{\partial x}{\partial t} = h_t Y + \sum_i h_{ti} T_i;$$

$$\frac{\partial x}{\partial \theta_j} = hT_j + \sum_i h_{ij} T_i, \quad j = 1, \dots, n-1.$$

The inverse of the above Jacobian matrix is

$$\frac{\partial t}{\partial x_\alpha} = h_t^{-1} [Y]_\alpha, \quad \alpha = 1, \dots, n;$$

$$\frac{\partial \theta_i}{\partial x_\alpha} = \sum_j b^{ij} \big[T_j - h_t^{-1} h_{tj} Y\big]_\alpha, \quad \alpha = 1, \dots, n, \tag{2.5}$$

where $[\cdot]_\alpha$ denotes the α-coordinate of the vector in the bracket and (b^{ij}) denotes the inverse matrix of the inverse second fundamental form

$$b_{ij} = \left\langle \frac{\partial x}{\partial \theta_i}, \frac{\partial Y}{\partial \theta_j} \right\rangle = h\delta_{ij} + h_{ij} \tag{2.6}$$

of the level set $\partial\bar{\Omega}_t$ at $x(\theta,t)$. The eigenvalues of (b^{ij}) are the principal curvatures $k_1,\dots,k_{n-1}$ of $\partial\bar{\Omega}_t$ at $x(\theta,t)$ (see Schneider [20]).

The first equation of (2.5) can be rewritten as

$$\nabla u=\frac{Y}{h_t},$$

where the left hand side is computed at $x(\theta,t)$ and the right hand side is computed at (θ,t). It follows that

$$|\nabla u|=-\frac{1}{h_t}.$$

By chain rule and (2.5), the second derivatives of u in terms of h and its derivatives can be computed as

$$u_{\alpha\beta}=\sum_{i,j}\left[-h_t^{-2}h_{ti}Y+h_t^{-1}T_i\right]_\alpha b^{ij}\left[T_j-h_t^{-1}h_{tj}Y\right]_\beta-h_t^{-3}h_{tt}[Y]_\alpha[Y]_\beta,$$

for $\alpha,\beta=1,\dots,n$.

Thus the p-Laplace equation becomes

$$h_{tt}=\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)b^{ij}, \tag{2.7}$$

and the associated linear elliptic operator is

$$L=\sum_{i,j,p,q}\left(\frac{1}{p-1}h_t^2\delta_{pq}+h_{tp}h_{tq}\right)b^{ip}b^{jq}\frac{\partial^2}{\partial\theta_i\partial\theta_j}-2\sum_{i,j}h_{tj}b^{ij}\frac{\partial^2}{\partial\theta_i\partial t}+\frac{\partial^2}{\partial t^2}. \tag{2.8}$$

Let $u\in C^4(\mathbb{S}^{n-1})$. The following commutation formulas for covariant derivatives of u are well-known

$$\begin{aligned} u_{ijk}-u_{ikj}&=-u_k\delta_{ij}+u_j\delta_{ik},\\ u_{ijkl}-u_{ijlk}&=u_{ik}\delta_{jl}-u_{il}\delta_{jk}+u_{kj}\delta_{il}-u_{lj}\delta_{ik}. \end{aligned} \tag{2.9}$$

3. Principal curvature estimates

In this section, we will prove Theorem 1.1. Our proof involves complicated and subtle calculation. For clarity, we divide this section into three subsections. In Section 3.1, we give a refined maximum principle. In Section 3.2, we derive the formula of $L(\varphi)$. In Section 3.3, we complete the proof of Theorem 1.1.

Since the level sets of u are strictly convex with respect to the normal direction ∇u, the inverse second fundamental form (b_{ij}) is positive definite in Ω. Set

$$\varphi=\alpha\log(-h_t)+\log b_{11},$$

where b_{11} is the largest principal radius of the level sets. For $\alpha=-1$ and $\beta=1$, it follows that

$$e^{\beta\varphi}=|\nabla u|k_1^{-1},$$

where k_1 is the smallest principal curvature of the level sets. We will derive the following differential inequality (in the sense modulo the terms involving $\nabla_\theta \varphi$ with locally bounded coefficients)

$$L\big(e^{\beta\varphi}\big) \geqslant 0 \quad \mathrm{mod}\ \nabla_\theta \varphi \quad \text{in } \Omega, \tag{3.1}$$

where the elliptic operator L is given in (2.8). By a maximum principle argument, we can obtain the desired result.

3.1. A refined maximum principle

We first state a lemma which appeared in Hörmander [9]. It characterizes continuous convex functions in terms of the second differences.

Lemma 3.1. *(See Hörmander [9].)* (1) *If f is continuous but not convex in the open interval I, then one can find $y \in I$, $c \in \mathbb{R}$ and $\varepsilon > 0$ such that*

$$f(y+h) \leqslant f(y) + ch - \varepsilon h^2, \quad \text{when } |h| \text{ is small}. \tag{3.2}$$

(2) *For a continuous real-valued function f defined in the interval I, we define its generalized second derivative at any interior point $x \in I$ as*

$$D^2 f(x) = \limsup_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}. \tag{3.3}$$

Then f is convex if and only if for any interior point $x \in I$, there holds

$$D^2 f(x) \geqslant 0.$$

Proof. (1) Let $J = [a,b] \subset I$ be an interval such that for some affine g we have $f \leqslant g$ on ∂J but $\sup_J (f-g) > 0$. Then

$$f_\varepsilon(x) = f(x) - g(x) + \varepsilon(x-a)(x-b)$$

is $\leqslant 0$ on ∂J but $\sup_J f_\varepsilon > 0$ if ε is small enough. The maximum is then taken at an interior point $y \in J$, so

$$f(x) - g(x) + \varepsilon(x-a)(x-b) = f_\varepsilon(x) \leqslant f_\varepsilon(y) = f(y) - g(y) + \varepsilon(y-a)(y-b),$$

when $x \in J$. With $x = y+h$ it follows that

$$f(y+h) \leqslant f(y) + \big(g' + \varepsilon(a+b-2y)\big)h - \varepsilon h^2, \quad \text{if } y+h \in J,$$

which proves (3.2).

(2) To obtain the sufficiency, we replace h by $-h$ in (3.2), then add the two inequalities. The necessity is obvious for the second difference is $\geqslant 0$ if f is convex. $\square$

The following lemma was first proved by Longinetti for the case $n = 2$ in the appendix of [15]. Here, we will need its general form.

Lemma 3.2. *Let $\mathbb{Q} \equiv \mathbb{S}^{n-1} \times (0,1)$ and $G(\theta,t)$ be a smooth function in $\mathbb{Q}$ such that*

$$\mathscr{L}\big(G(\theta,t)\big) \geqslant 0 \quad \text{for } (\theta,t) \in \mathbb{Q}, \tag{3.4}$$

where $\mathscr{L}$ is an elliptic operator of the form

$$\mathscr{L} = \sum_{i,j} a^{ij} \frac{\partial^2}{\partial\theta_i \partial\theta_j} + \sum_i b^i \frac{\partial^2}{\partial\theta_i \partial t} + \frac{\partial^2}{\partial t^2} + \sum_i c^i \frac{\partial}{\partial\theta_i}$$

with smooth coefficients a^{ij}, b^i, c^i. Set

$$\phi(t) = \max\big\{G(\theta,t) \;\big|\; \theta \in \mathbb{S}^{n-1}\big\}.$$

Then ϕ satisfies the following differential inequality

$$D^2\phi \geqslant 0.$$

Proof. The proof is almost the same as in [15]. For completeness, we still sketch out the proof. It is clear that ϕ is continuous in the interval $(0,1)$. Let us set the function

$$\Phi(\theta,t) = \phi(t), \quad (\theta,t) \in \mathbb{Q}.$$

So by definitions, we have

$$G(\theta,t) \leqslant \Phi(\theta,t), \quad (\theta,t) \in \mathbb{Q}. \tag{3.5}$$

Now let $t \in (0,1)$ be fixed. At any point $(\bar\theta,t) \in \mathbb{Q}$ such that

$$G(\bar\theta,t) = \Phi(\bar\theta,t), \tag{3.6}$$

we have

$$\nabla_\theta G(\bar\theta,t) = 0.$$

Let us consider the generalized second order elliptic operator

$$\tilde{\mathscr{L}} = \sum_{i,j} a^{ij} \frac{\partial^2}{\partial\theta_i \partial\theta_j} + \sum_i b^i \frac{\partial^2}{\partial\theta_i \partial t} + D_{tt} + \sum_i c^i \frac{\partial}{\partial\theta_i},$$

where for any function v on $\mathbb{Q}$, $D_{tt}v$ is the generalized second derivative of $v(\theta,\cdot)$ with respect to t, $\theta \in \mathbb{S}^{n-1}$ fixed (see the definition (3.3)).

By (3.5) and (3.6), we have

$$\tilde{\mathscr{L}} G(\bar\theta,t) \leqslant \tilde{\mathscr{L}} \Phi(\bar\theta,t) \equiv D^2\phi(t). \tag{3.7}$$

Since G is smooth, we have $\mathscr{L}G = \tilde{\mathscr{L}}G$. So the conclusion follows from (3.4) and (3.7). □

3.2. Computation of the formula of $L(\varphi)$

In order to prove (3.1) at an arbitrary point $x_0 \in \Omega$, we assume that the matrix $(b_{ij}(x_0))$ is diagonal. This can be realized by choosing suitable orthonormal frame. From now on, all the calculation will be performed at the fixed point x_0.

Since

$$\varphi = \alpha \log(-h_t) + \log b_{11},$$

taking first derivatives of φ, we have

$$\frac{\partial \varphi}{\partial \theta_j} = \alpha h_t^{-1} h_{tj} + b^{11} b_{11,j}, \tag{3.8}$$

$$\frac{\partial \varphi}{\partial t} = \alpha h_t^{-1} h_{tt} + b^{11} b_{11,t}. \tag{3.9}$$

Taking second derivatives of φ, we have

$$\frac{\partial^2 \varphi}{\partial \theta_i \partial \theta_j} = -\alpha h_t^{-2} h_{ti} h_{tj} + \alpha h_t^{-1} h_{tji} - \sum_{r,s} b^{1r} b_{rs,i} b^{s1} b_{11,j} + b^{11} b_{11,ji},$$

$$\frac{\partial^2 \varphi}{\partial \theta_i \partial t} = -\alpha h_t^{-2} h_{ti} h_{tt} + \alpha h_t^{-1} h_{tti} - \sum_{r,s} b^{1r} b_{rs,i} b^{s1} b_{11,t} + b^{11} b_{11,ti},$$

$$\frac{\partial^2 \varphi}{\partial t^2} = -\alpha h_t^{-2} h_{tt}^2 + \alpha h_t^{-1} h_{ttt} - \sum_{r,s} b^{1r} b_{rs,t} b^{s1} b_{11,t} + b^{11} b_{11,tt};$$

hence

$$\begin{aligned} L(\varphi) = & -\alpha h_t^{-2} \left[\sum_{i,j} \left(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} h_{ti} h_{tj} - 2 \sum_i h_{ti}^2 b^{ii} h_{tt} + h_{tt}^2 \right] \\ & + \alpha h_t^{-1} \left[\sum_{i,j} \left(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} h_{tji} - 2 \sum_i h_{ti} b^{ii} h_{tti} + h_{ttt} \right] \\ & - \left(b^{11}\right)^2 \left[\sum_{i,j} \left(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} b_{11,i} b_{11,j} - 2 \sum_i h_{ti} b^{ii} b_{11,i} b_{11,t} + b_{11,t}^2 \right] \\ & + b^{11} L(b_{11}) \\ \triangleq & \, I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.10}$$

In the rest of this subsection, we will deal with the four terms above respectively. By (2.7), at the point x_0, we have

$$h_{tt} = \frac{1}{p-1} h_t^2 \sigma_1 + \sum_i h_{ti}^2 b^{ii}, \tag{3.11}$$

where $\sigma_1 = \sum_i b^{ii}$ is the mean curvature of the level sets. For the term I_1, we have

$$\begin{aligned}I_1&=-\alpha h_t^{-2}\left[\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)b^{ii}b^{jj}h_{ti}h_{tj}-2\sum_i h_{ti}^2b^{ii}h_{tt}+h_{tt}^2\right]\\&=-\alpha h_t^{-2}\left[\frac{1}{p-1}h_t^2\sum_i\left(h_{ti}b^{ii}\right)^2+\left(\sum_i h_{ti}^2b^{ii}-h_{tt}\right)^2\right]\\&=-\frac{\alpha}{p-1}\sum_i\left(h_{ti}b^{ii}\right)^2-\frac{\alpha}{(p-1)^2}h_t^2\sigma_1^2.\end{aligned}\tag{3.12}$$

Let us deal with the term I_2. Differentiating (2.7) with respect to t, we have

$$h_{ttt}=\frac{2}{p-1}h_th_{tt}\sigma_1+2\sum_i h_{tti}h_{ti}b^{ii}-\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)b^{ii}b^{jj}b_{ij,t}.\tag{3.13}$$

By inserting (3.13) into I_2, we get

$$\begin{aligned}I_2&=\alpha h_t^{-1}\left[\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)b^{ii}b^{jj}h_{tji}-2\sum_i h_{ti}b^{ii}h_{tti}+h_{ttt}\right]\\&=\alpha h_t^{-1}\left[\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)b^{ii}b^{jj}(h_{tji}-b_{ij,t})+\frac{2}{p-1}h_th_{tt}\sigma_1\right].\end{aligned}$$

Recalling the definition of the inverse second fundamental form, i.e. (2.6), together with Eq. (3.11), we obtain

$$\begin{aligned}I_2&=\alpha h_t^{-1}\left[\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)b^{ii}b^{jj}(-h_t\delta_{ij})+\frac{2}{p-1}h_th_{tt}\sigma_1\right]\\&=-\frac{\alpha}{p-1}h_t^2\sum_i\left(b^{ii}\right)^2-\alpha\sum_i\left(h_{ti}b^{ii}\right)^2+\frac{2\alpha}{(p-1)^2}h_t^2\sigma_1^2+\frac{2\alpha}{p-1}\sigma_1\sum_i h_{ti}^2b^{ii}.\end{aligned}\tag{3.14}$$

Combining (3.12) and (3.14),

$$\begin{aligned}I_1+I_2=&-\frac{p\alpha}{p-1}\sum_i\left(h_{ti}b^{ii}\right)^2+\frac{\alpha}{(p-1)^2}h_t^2\sigma_1^2-\frac{\alpha}{p-1}h_t^2\sum_i\left(b^{ii}\right)^2\\&+\frac{2\alpha}{p-1}\sigma_1\sum_i h_{ti}^2b^{ii}.\end{aligned}\tag{3.15}$$

In the following, we shall compute $L(b_{11})$. By differentiating (2.7) twice with respect to θ_1, we have

$$h_{tt1}=\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)_1b^{ij}+\sum_{i,j,p,q}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)\left(-b^{ip}b_{pq,1}b^{qj}\right),$$

and

$$\begin{aligned}
h_{tt11} &= \sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)_{11}b^{ij} + 2\sum_{i,j,p,q}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)_1\left(-b^{ip}b_{pq,1}b^{qj}\right) \\
&\quad + \sum_{i,j,p,q,r,s}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)\left(2b^{ir}b_{rs,1}b^{sp}b_{pq,1}b^{qj}\right) \\
&\quad + \sum_{i,j,p,q}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)\left(-b^{ip}b_{pq,11}b^{qj}\right) \\
&\triangleq J_1+J_2+J_3+J_4.
\end{aligned} \tag{3.16}$$

For the term J_1, we have

$$\begin{aligned}
J_1 &= \sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)_{11}b^{ij} \\
&= \sum_{i,j}\left(\frac{2}{p-1}h_th_{t1}\delta_{ij}+h_{ti1}h_{tj}+h_{ti}h_{tj1}\right)_1 b^{ij} \\
&= \frac{2}{p-1}h_{t1}^2\sigma_1 + \frac{2}{p-1}h_th_{t11}\sigma_1 + 2\sum_i h_{ti11}h_{ti}b^{ii} + 2\sum_i h_{ti1}^2 b^{ii}.
\end{aligned}$$

By (2.9),

$$\begin{aligned}
b_{ij,1} &= b_{i1,j}, \\
h_{ti1} &= h_{1it} = b_{1i,t} - h_t\delta_{1i}, \\
h_{ti11} &= h_{i11t} = b_{i1,1t} - h_{1t}\delta_{i1} = b_{11,it} - h_{1t}\delta_{i1}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
J_1 &= 2\sum_i h_{ti}b^{ii}b_{11,it} + 2\sum_i b^{ii}b_{1i,t}^2 + \frac{2}{p-1}h_t\sigma_1 b_{11,t} - 4h_tb^{11}b_{11,t} \\
&\quad + \frac{2p-4}{p-1}h_t^2b^{11} - \frac{2}{p-1}h_t^2\sum_{i\geqslant 2}b^{ii} + \frac{4-2p}{p-1}h_{t1}^2b^{11} + \frac{2}{p-1}h_{t1}^2\sum_{i\geqslant 2}b^{ii}.
\end{aligned} \tag{3.17}$$

For the term J_2, we have

$$\begin{aligned}
J_2 &= 2\sum_{i,j}\left(\frac{2}{p-1}h_th_{t1}\delta_{ij}+h_{ti1}h_{tj}+h_{ti}h_{tj1}\right)\left(-b^{ii}b_{ij,1}b^{jj}\right) \\
&= -\frac{4}{p-1}h_th_{t1}\sum_i\left(b^{ii}\right)^2b_{ii,1} - 4\sum_{i,j}h_{ti1}h_{tj}b^{ii}b^{jj}b_{ij,1} \\
&= -\frac{4}{p-1}h_th_{t1}\sum_i\left(b^{ii}\right)^2b_{ii,1} - 4\sum_{i,j}h_{tj}b^{ii}b^{jj}b_{ij,1}b_{1i,t} + 4h_tb^{11}\textstyle\sum_j h_{tj}b^{jj}b_{11,j}.
\end{aligned} \tag{3.18}$$

We also have

$$J_3 = 2\sum_{i,j,k}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}b^{kk}b_{ik,1}b_{jk,1}. \tag{3.19}$$

Using (2.9) again, we have the following commutation rule

$$b_{pq,11} = b_{11,pq} + b_{pq} - b_{11}\delta_{pq} + b_{1q}\delta_{1p} - b_{1p}\delta_{1q}.$$

For the term J_4, we have

$$\begin{aligned} J_4 &= -\sum_{i,j,p,q}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ip}b^{jq}(b_{11,pq} + b_{pq} - b_{11}\delta_{pq}) \\ &= -\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}b_{11,ij} - h_{tt} \\ &\quad + \frac{1}{p-1}h_t^2 b_{11}\sum_i \left(b^{ii}\right)^2 + b_{11}\sum_i \left(h_{ti}b^{ii}\right)^2. \end{aligned} \tag{3.20}$$

Note that $h_{11tt} = b_{11,tt} - h_{tt}$. By putting (3.17)–(3.20) into (3.16), and recalling the definition of the operator L, we obtain

$$\begin{aligned} L(b_{11}) = {} & 2\sum_i b^{ii}b_{1i,t}^2 + \frac{2}{p-1}h_t\sigma_1 b_{11,t} - 4h_t b^{11}b_{11,t} + \frac{2p-4}{p-1}h_t^2 b^{11} - \frac{2}{p-1}h_t^2\sum_{i\geqslant 2}b^{ii} \\ & + \frac{4-2p}{p-1}h_{t1}^2 b^{11} + \frac{2}{p-1}h_{t1}^2\sum_{i\geqslant 2}b^{ii} - \frac{4}{p-1}h_t h_{t1}\sum_i \left(b^{ii}\right)^2 b_{ii,1} \\ & - 4\sum_{i,j}h_{tj}b^{ii}b^{jj}b_{ij,1}b_{1i,t} + 4h_t b^{11}\sum_j h_{tj}b^{jj}b_{11,j} \\ & + 2\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}b^{kk}b_{ik,1}b_{jk,1} \\ & + \frac{1}{p-1}h_t^2 b_{11}\sum_i \left(b^{ii}\right)^2 + b_{11}\sum_i \left(h_{ti}b^{ii}\right)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} I_4 = {} & 2b^{11}\sum_i b^{ii}b_{1i,t}^2 + \frac{2}{p-1}h_t\sigma_1 b^{11}b_{11,t} - 4h_t\left(b^{11}\right)^2 b_{11,t} + \frac{2p-4}{p-1}h_t^2\left(b^{11}\right)^2 \\ & - \frac{2}{p-1}h_t^2 b^{11}\sum_{i\geqslant 2}b^{ii} + \frac{4-2p}{p-1}h_{t1}^2\left(b^{11}\right)^2 + \frac{2}{p-1}h_{t1}^2 b^{11}\sum_{i\geqslant 2}b^{ii} \\ & - \frac{4}{p-1}h_t h_{t1}b^{11}\sum_i \left(b^{ii}\right)^2 b_{ii,1} - 4b^{11}\sum_{i,l}h_{ti}b^{ii}b^{ll}b_{1l,i}b_{1l,t} \end{aligned}$$

$$+4h_t\left(b^{11}\right)^2\sum_j h_{tj}b^{jj}b_{11,j}+2b^{11}\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)b^{ii}b^{jj}b^{kk}b_{ik,1}b_{jk,1}$$
$$+\frac{1}{p-1}h_t^2\sum_i\left(b^{ii}\right)^2+\sum_i\left(h_{ti}b^{ii}\right)^2. \tag{3.21}$$

By putting (3.15) and (3.21) into (3.10), we obtain

$$L(\varphi)=\left(b^{11}\right)^2\left[\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)b^{ii}b^{jj}b_{11,i}b_{11,j}-2\sum_i h_{ti}b^{ii}b_{11,i}b_{11,t}+b_{11,t}^2\right]$$
$$+2b^{11}\sum_{l\geqslant 2}b^{ll}\left[\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)b^{ii}b^{jj}b_{1l,i}b_{1l,j}-2\sum_i h_{ti}b^{ii}b_{1l,i}b_{1l,t}+b_{1l,t}^2\right]$$
$$+\frac{2}{p-1}h_t\sigma_1 b^{11}b_{11,t}-4h_t\left(b^{11}\right)^2 b_{11,t}-\frac{4}{p-1}h_t h_{t1}b^{11}\sum_i\left(b^{ii}\right)^2 b_{ii,1}$$
$$+4h_t\left(b^{11}\right)^2\sum_i h_{ti}b^{ii}b_{11,i}+q_1(\alpha)h_t^2+q_2(\alpha)h_{t1}^2+\sum_{i\geqslant 2}q_{3,i}(\alpha)h_{ti}^2, \tag{3.22}$$

where

$$q_1(\alpha)=\left(\frac{\alpha}{(p-1)^2}+\frac{2p-3-\alpha}{p-1}\right)\left(b^{11}\right)^2+\left(\frac{2\alpha}{(p-1)^2}-\frac{2}{p-1}\right)b^{11}\sum_{i\geqslant 2}b^{ii}$$
$$+\frac{1-\alpha}{p-1}\sum_{i\geqslant 2}\left(b^{ii}\right)^2+\frac{\alpha}{(p-1)^2}\left(\sum_{i\geqslant 2}b^{ii}\right)^2,$$
$$q_2(\alpha)=\frac{(2-p)\alpha+3-p}{p-1}\left(b^{11}\right)^2+\frac{2(1+\alpha)}{p-1}b^{11}\sum_{i\geqslant 2}b^{ii},$$
$$q_{3,i}(\alpha)=\frac{2\alpha}{p-1}\sigma_1 b^{ii}+\frac{p-1-p\alpha}{p-1}\left(b^{ii}\right)^2.$$

3.3. Completion of the proof of Theorem 1.1

In this subsection, we shall calculate $L(e^{\beta\varphi})$ and obtain the formula (3.29), thus complete the proof of Theorem 1.1. Note that

$$L\left(e^{\beta\varphi}\right)=\beta e^{\beta\varphi}\left\{L(\varphi)+\beta\varphi_t^2\right\}+\beta^2 e^{\beta\varphi}\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj}\right)b^{ii}b^{jj}\frac{\partial\varphi}{\partial\theta_i}\frac{\partial\varphi}{\partial\theta_j}$$
$$-2\beta^2 e^{\beta\varphi}\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i}\frac{\partial\varphi}{\partial t}.$$

For $\beta=1$, $\alpha=-1$, in order to prove

$$L\left(e^{\beta\varphi}\right)\geqslant 0 \mod \nabla_\theta\varphi \quad \text{in } \Omega,$$

it suffices to prove

$$L(\varphi) + \beta\varphi_t^2 \geqslant 0 \quad \text{mod } \nabla_\theta \varphi \quad \text{in } \Omega.$$

Now we compute $\beta\varphi_t^2$. By (3.9) and Eq. (3.11), we have

$$\begin{aligned}
\beta\varphi_t^2 &= \beta\alpha^2 h_t^{-2} h_{tt}^2 + 2\beta\alpha h_t^{-1} h_{tt} b^{11} b_{11,t} + \beta\left(b^{11} b_{11,t}\right)^2 \\
&= \frac{\beta\alpha^2}{(p-1)^2} h_t^2 \sigma_1^2 + \frac{2\beta\alpha^2}{p-1}\sigma_1 \sum_i h_{ti}^2 b^{ii} + \beta\alpha^2 h_t^{-2}\left(\sum_i h_{ti}^2 b^{ii}\right)^2 \\
&\quad + \frac{2\beta\alpha}{p-1} h_t \sigma_1 b^{11} b_{11,t} + 2\beta\alpha h_t^{-1}\left(\sum_i h_{ti}^2 b^{ii}\right) b^{11} b_{11,t} + \beta\left(b^{11} b_{11,t}\right)^2. \qquad (3.23)
\end{aligned}$$

Collecting (3.22) and (3.23), we regroup the terms in $L(\varphi) + \beta\varphi_t^2$ as follows

$$L(\varphi) + \beta\varphi_t^2 \triangleq P_1 + P_2 + P_3 + P_4, \qquad (3.24)$$

where

$$\begin{aligned}
P_1 &= 2b^{11} \sum_{l\geqslant 2} b^{ll}\left(\sum_{i,j} h_{ti} h_{tj} b^{ii} b^{jj} b_{1l,i} b_{1l,j} - 2\sum_i h_{ti} b^{ii} b_{1l,i} b_{1l,t} + b_{1l,t}^2\right) \geqslant 0, \\
P_2 &= (1+\beta)\left(b^{11} b_{11,t}\right)^2 - 2\left(\sum_i h_{ti} b^{ii} b^{11} b_{11,i} - \frac{1}{p-1} h_t \sigma_1 + 2h_t b^{11} - \frac{\beta\alpha}{p-1} h_t \sigma_1\right. \\
&\quad \left. - \beta\alpha h_t^{-1} \sum_i h_{ti}^2 b^{ii}\right) b^{11} b_{11,t} \\
&\geqslant -\frac{1}{1+\beta}\left(\sum_i h_{ti} b^{ii} b^{11} b_{11,i} - \frac{1+\beta\alpha}{p-1} h_t \sigma_1 + 2h_t b^{11} - \beta\alpha h_t^{-1} \sum_i h_{ti}^2 b^{ii}\right)^2, \\
P_3 &= \left(b^{11}\right)^2 \sum_{i,j}\left(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj}\right) b^{ii} b^{jj} b_{11,i} b_{11,j} + \frac{2}{p-1} h_t^2 b^{11} \sum_{l\geqslant 2} b^{ll} \sum_i \left(b^{ii}\right)^2 b_{1l,i}^2 \\
&\quad - \frac{4}{p-1} h_t h_{t1} b^{11} \sum_i \left(b^{ii}\right)^2 b_{ii,1} + 4h_t \left(b^{11}\right)^2 \sum_i h_{ti} b^{ii} b_{11,i}, \\
P_4 &= q_1(\alpha) h_t^2 + q_2(\alpha) h_{t1}^2 + \sum_{i\geqslant 2} q_{3,i}(\alpha) h_{ti}^2 + \frac{\beta\alpha^2}{(p-1)^2} h_t^2 \sigma_1^2 + \frac{2\beta\alpha^2}{p-1} \sigma_1 \sum_i h_{ti}^2 b^{ii} \\
&\quad + \beta\alpha^2 h_t^{-2}\left(\sum_i h_{ti}^2 b^{ii}\right)^2.
\end{aligned}$$

In the following, we will make use of the first order condition, i.e. (3.8), to calculate the terms P_2 and P_3. By (3.8), we have

$$b^{11} b_{11,j} = \frac{\partial\varphi}{\partial\theta_j} - \alpha h_t^{-1} h_{tj}, \quad \text{for } j = 1, 2, \ldots, n-1,$$

hence

$$\begin{aligned}
P_2 &\geqslant -\frac{1}{1+\beta}\left(-\alpha(1+\beta)h_t^{-1}\sum_i h_{ti}^2 b^{ii} - \frac{1+\beta\alpha}{p-1}h_t\sigma_1 + 2h_t b^{11}\right)^2 + R_2(\nabla_\theta\varphi) \\
&= -\alpha^2(1+\beta)h_t^{-2}\left(\sum_i h_{ti}^2 b^{ii}\right)^2 - \frac{1}{(p-1)^2}\frac{(1+\beta\alpha)^2}{1+\beta}h_t^2\sigma_1^2 - \frac{4}{1+\beta}h_t^2\left(b^{11}\right)^2 \\
&\quad - \frac{2\alpha(1+\beta\alpha)}{p-1}\sigma_1\sum_i h_{ti}^2 b^{ii} + 4\alpha b^{11}\sum_i h_{ti}^2 b^{ii} + \frac{1}{p-1}\frac{4(1+\beta\alpha)}{1+\beta}h_t^2 b^{11}\sigma_1 + R_2(\nabla_\theta\varphi),
\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}
R_2(\nabla_\theta\varphi) = &-\frac{1}{1+\beta}\left(\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i}\right)^2 - \frac{2}{1+\beta}\left(-\alpha(1+\beta)h_t^{-1}\sum_i h_{ti}^2 b^{ii}\right. \\
&\left. - \frac{1+\beta\alpha}{p-1}h_t\sigma_1 + 2h_t b^{11}\right)\left(\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i}\right).
\end{aligned}$$

In a similar way, one can check that

$$\begin{aligned}
P_3 = &\frac{\alpha^2}{p-1}\sum_i\left(h_{ti}b^{ii}\right)^2 + \alpha^2 h_t^{-2}\left(\sum_i h_{ti}^2 b^{ii}\right)^2 + \frac{2\alpha^2}{p-1}b^{11}\sum_{i\geqslant 2}h_{ti}^2 b^{ii} \\
&+ \frac{2}{p-1}h_t^2 b^{11}\sum_{i,l\geqslant 2}b^{ll}\left(b^{ii}\right)^2 b_{1l,i}^2 + \frac{4\alpha}{p-1}h_{t1}^2\left(b^{11}\right)^2 - \frac{4}{p-1}h_t h_{t1}b^{11}\sum_{i\geqslant 2}\left(b^{ii}\right)^2 b_{ii,1} \\
&- 4\alpha b^{11}\sum_i h_{ti}^2 b^{ii} + R_3(\nabla_\theta\varphi),
\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
R_3(\nabla_\theta\varphi) = &\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}\left(\frac{\partial\varphi}{\partial\theta_i}\frac{\partial\varphi}{\partial\theta_j} - 2\alpha h_t^{-1}h_{tj}\frac{\partial\varphi}{\partial\theta_i}\right) \\
&+ \frac{2}{p-1}h_t^2 b^{11}\sum_{l\geqslant 2}b^{ll}\left(\left(\frac{\partial\varphi}{\partial\theta_l}\right)^2 - 2\alpha h_t^{-1}h_{tl}\frac{\partial\varphi}{\partial\theta_l}\right) \\
&- \frac{4}{p-1}h_t h_{t1}\left(b^{11}\right)^2\frac{\partial\varphi}{\partial\theta_1} + 4h_t b^{11}\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i}.
\end{aligned}$$

For the fourth term and sixth term in (3.26), we have

$$\begin{aligned}
&\frac{2}{p-1}h_t^2 b^{11}\sum_{i,l\geqslant 2}b^{ll}\left(b^{ii}\right)^2 b_{1l,i}^2 - \frac{4}{p-1}h_t h_{t1}b^{11}\sum_{i\geqslant 2}\left(b^{ii}\right)^2 b_{ii,1} \\
&\quad\geqslant \frac{2}{p-1}b^{11}\sum_{i\geqslant 2}b^{ii}\left[\left(h_t b^{ii}b_{ii,1}\right)^2 - 2h_t b^{ii}b_{ii,1}\cdot h_{t1}\right]
\end{aligned}$$

$$\geqslant -\frac{2}{p-1}h_{t1}^2 b^{11}\sum_{i\geqslant 2} b^{ii}. \tag{3.27}$$

By (3.26) and (3.27),

$$\begin{aligned} P_3 \geqslant{}& \frac{\alpha^2}{p-1}\sum_i \left(h_{ti}b^{ii}\right)^2 + \alpha^2 h_t^{-2}\left(\sum_i h_{ti}^2 b^{ii}\right)^2 + \frac{2\alpha^2}{p-1}b^{11}\sum_{i\geqslant 2} h_{ti}^2 b^{ii} \\ &- \frac{2}{p-1}h_{t1}^2 b^{11}\sum_{i\geqslant 2} b^{ii} + \frac{4\alpha}{p-1}h_{t1}^2\left(b^{11}\right)^2 - 4\alpha b^{11}\sum_i h_{ti}^2 b^{ii} + R_3(\nabla_\theta\varphi). \end{aligned} \tag{3.28}$$

Combining (3.24), (3.25) and (3.28), we obtain

$$\begin{aligned} L(\varphi) + \beta\varphi_t^2 \geqslant{}& r_1(\alpha,\beta)h_t^2 + r_2(\alpha,\beta)h_{t1}^2 + \sum_{i\geqslant 2} r_{3,i}(\alpha,\beta)h_{ti}^2 + R_2(\nabla_\theta\varphi) \\ &+ R_3(\nabla_\theta\varphi) + q_1(\alpha)h_t^2 + q_2(\alpha)h_{t1}^2 + \sum_{i\geqslant 2} q_{3,i}(\alpha)h_{ti}^2, \end{aligned}$$

where

$$\begin{aligned} r_1(\alpha,\beta) ={}& \left[\frac{\beta\alpha^2 - 2\beta\alpha - 1}{(p-1)^2(1+\beta)} + \frac{4\beta\alpha - 4p + 8}{(p-1)(1+\beta)}\right]\left(b^{11}\right)^2 + \left[\frac{2\beta\alpha^2 - 4\beta\alpha - 2}{(p-1)^2(1+\beta)}\right. \\ &\left. + \frac{4(1+\beta\alpha)}{(p-1)(1+\beta)}\right]b^{11}\sum_{i\geqslant 2} b^{ii} + \frac{\beta\alpha^2 - 2\beta\alpha - 1}{(p-1)^2(1+\beta)}\left(\sum_{i\geqslant 2} b^{ii}\right)^2, \\ r_2(\alpha,\beta) ={}& \frac{\alpha^2 + 2\alpha}{p-1}\left(b^{11}\right)^2 - \frac{2(1+\alpha)}{p-1}b^{11}\sum_{i\geqslant 2} b^{ii}, \\ r_{3,i}(\alpha,\beta) ={}& \frac{\alpha^2}{p-1}\left(b^{ii}\right)^2 + \frac{2\alpha^2}{p-1}b^{11}b^{ii} - \frac{2\alpha}{p-1}\sigma_1 b^{ii}. \end{aligned}$$

Thus, we finally obtain

$$\begin{aligned} L(\varphi) + \beta\varphi_t^2 \geqslant{}& h_t^2\left[\left(\frac{\beta\alpha^2 - \beta\alpha + \alpha - 1}{(p-1)^2(1+\beta)} + \frac{3\beta\alpha - \alpha + (2p-3)\beta - 2p + 5}{(p-1)(1+\beta)}\right)\left(b^{11}\right)^2\right. \\ &+ \left(\frac{2\beta\alpha^2 - 2\beta\alpha + 2\alpha - 2}{(p-1)^2(1+\beta)} + \frac{4\beta\alpha - 2\beta + 2}{(p-1)(1+\beta)}\right)b^{11}\sum_{i\geqslant 2} b^{ii} \\ &\left. + \frac{\beta\alpha^2 - \beta\alpha + \alpha - 1}{(p-1)^2(1+\beta)}\left(\sum_{i\geqslant 2} b^{ii}\right)^2 + \frac{1-\alpha}{p-1}\sum_{i\geqslant 2}\left(b^{ii}\right)^2\right] \\ &+ \frac{\alpha^2 + (4-p)\alpha + 3 - p}{p-1}\left(h_{t1}b^{11}\right)^2 \\ &+ \sum_{i\geqslant 2} h_{ti}^2\left(\frac{2\alpha^2}{p-1}b^{11}b^{ii} + \frac{\alpha^2 - p\alpha + p - 1}{p-1}\left(b^{ii}\right)^2\right) \\ &+ R_2(\nabla_\theta\varphi) + R_3(\nabla_\theta\varphi). \end{aligned} \tag{3.29}$$

For $\alpha=-1$ and $\beta=1$, we have

$$L(\varphi)+\varphi_t^2 \geqslant \frac{2}{p-1}h_t^2\sum_{i\geqslant 2} b^{ii}\left(b^{ii}-b^{11}\right)+\frac{2}{p-1}\sum_{i\geqslant 2} h_{ti}^2 b^{ii}\left(b^{11}+pb^{ii}\right)+R_2(\nabla_\theta\varphi)+R_3(\nabla_\theta\varphi)$$
$$\geqslant 0 \quad \mathrm{mod}\ \nabla_\theta\varphi.$$

Combining Lemma 3.1 and Lemma 3.2, we complete the proof of Theorem 1.1. □

Remark 3.3. For the case of 3-dimensional harmonic function, the function of one variable

$$f(t)=\min_{x\in\Omega_t}\log k_1(x)$$

is a concave function for $t\in(0,1)$. In fact, for $n=3$ and $p=2$, by setting $\alpha=\beta=0$ in (3.29), we have

$$L(\varphi)\geqslant\left(h_{t1}b^{11}\right)^2+\left(h_{t2}b^{22}\right)^2+R_2(\nabla_\theta\varphi)+R_3(\nabla_\theta\varphi)\geqslant 0 \quad \mathrm{mod}\ \nabla_\theta\varphi.$$

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