On the global solvability of porous media equations with general (spatially dependent) advection terms

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Abstract

We show that advection-diffusion equations with porous media type diffusion and integrable initial data are globally solvable under very mild assumptions. Some generalizations and related results are also given.

2010 AMS Subject Classification: 35K65 (primary), 35A01, 35K15

Keywords: porous medium type equations, global solvability, weak solutions, Cauchy problem, spatially dependent advection flux, pointwise estimates

1. Introduction

In this note, we describe general results recently obtained by the authors concerning the solvability in the large of initial value problems for degenerate advection-diffusion equations of the type

$$u_t + \operatorname{div} \boldsymbol{f}(x,t,u) + \operatorname{div} \boldsymbol{g}(t,u) = \mu(t) \operatorname{div}(|u|^{\alpha} \nabla u), \tag{1.1a}$$

$$u(\cdot,0) = u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \tag{1.1b}$$

and some generalizations (see Section 2). Here, $\alpha > 0$ is constant, $\mu \in C^0([0,\infty))$ is positive, and $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)$, $\boldsymbol{g} = (g_1, g_2, \ldots, g_n)$, are given continuous advection flux fields that are locally Lipschitz in u uniformly in $x \in \mathbb{R}^n$ and bounded $t \geq 0$, with $\boldsymbol{f}$ satisfying: $\boldsymbol{f}(x,t,0) = \mathbf{0}$ for all x, t and

$$|\,\boldsymbol{f}(x,t,\mathrm{u})\,| \;\leq\; F(t)\,|\,\mathrm{u}\,|^{\kappa+1} \qquad \forall\, x \in \mathbb{R}^n,\; t \geq 0,\; \mathrm{u} \in \mathbb{R} \tag{1.2}$$

for some $F \in C^0([0,\infty))$ and some constant $\kappa \geq 0$, where $|\cdot|$ denotes the absolute value (in case of scalars) or the Euclidean norm (in case of vectors), as in (1.1*a*). By a (bounded) *solution* of the problem (1.1) in some time interval $[0, T_*)$ is meant any function $u(\cdot,t) \in C^0([0,T_*), L^1_{\text{loc}}(\mathbb{R}^n)) \cap L^\infty_{\text{loc}}([0,T_*), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ having $|\,u(\cdot,t)\,|^\alpha u(\cdot,t) \in L^2_{\text{loc}}((0,T_*), W^{1,2}_{\text{loc}}(\mathbb{R}^n))$ which satisfies (1.1*a*) in distributional sense (i.e., in $\mathcal{D}'(\mathbb{R}^n \times (0,T_*))$) and takes the initial value $u(\cdot,0) = u_0$, see e.g. [5, 16, 17]. This says, in particular, that $u(\cdot,t) \to u_0$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ as $t \to 0$, and that, for every $0 < T < T_*$ given,

$$\|\,u(\cdot,t)\,\|_{L^1(\mathbb{R}^n)} \;\leq\; M_1(T), \qquad \forall\; 0 \leq t \leq T, \tag{1.3a}$$

$$\|\,u(\cdot,t)\,\|_{L^\infty(\mathbb{R}^n)} \leq\; M_\infty(T), \qquad \forall\; 0 \leq t \leq T, \tag{1.3b}$$

for some bounds $M_1(T)$, $M_\infty(T)$ depending on T (and the solution u considered). For the *local* (in time) existence of such solutions, which are typically obtained by parabolic regularization or Galerkin approximations, see e.g. [5, 9, 13, 16, 17]. From the basic theory, many interesting solution properties are known; for example, one has $u(\cdot,t) \in C^0([0,T_*), L^1(\mathbb{R}^n))$ and

$$\int_0^T\!\!\int_{\mathbb{R}^n} |\,u(x,t)\,|^{2\alpha}\,|\,\nabla u(x,t)\,|^2\,dx\,dt \;<\; \infty \tag{1.4}$$

for every $0 < T < T_*$, see e.g. [4, 9, 12, 16, 17]. More importantly to us here, solutions $u(\cdot,t)$ decrease monotonically in $L^1(\mathbb{R}^n)$, so that, in particular,

$$\|\,u(\cdot,t)\,\|_{L^1(\mathbb{R}^n)} \leq\; \|\,u_0\,\|_{L^1(\mathbb{R}^n)}, \qquad \forall\;\; 0 < t < T_*. \tag{1.5}$$

For all that is presently known, however, little has been obtained regarding the solvability for large t in the general framework (1.1), (1.2) above, except in very special situations. Thus, for example, when the flux $\boldsymbol{f}(x,t,\mathrm{u})$ does *not* depend explicitly on x, or, when it does, if it behaves so as to satisfy special conditions like

$$\sum_{i=1}^n \mathrm{u}\,\frac{\partial f_i}{\partial x_i}(x,t,\mathrm{u}) \;\geq\; 0, \qquad \forall\, x \in \mathbb{R}^n,\, t \geq 0,\, \mathrm{u} \in \mathbb{R}, \tag{1.6}$$

then solutions are known to be globally defined, with $\|\,u(\cdot,t)\,\|_{L^q(\mathbb{R}^n)}$ monotonically decreasing for every $1 \leq q \leq \infty$, see e.g. [6, 9, 10, 12, 14, 16]. In the absence of (1.6), however, things get much more complicated to analyze. To see why, let us consider by way of illustration the simple example below. Taking $\boldsymbol{f}(x, t, \mathrm{u}) = \boldsymbol{b}(x, t)\,|\,\mathrm{u}\,|^{\kappa}\,\mathrm{u}$ for some $\kappa > 0$, and (say) $\boldsymbol{g} = \mathbf{0}$, $\mu(t) = 1$, the equation (1.1a) becomes

$$u_t + \boldsymbol{b}(x,t) \cdot \nabla(|\,u\,|^{\kappa} u) = \operatorname{div}(\,|\,u\,|^{\alpha}\,\nabla u\,) + \beta(x,t)\,|\,u\,|^{\kappa} u \tag{1.7}$$

with $\beta(x,t) = -\sum_{i=1}^{n} \partial b_i / \partial x_i$. In regions where $\beta(x,t) > 0$, it is clear that $|u(x,t)|$ tends to grow, particularly if $\beta(x,t) \gg 1$, potentially leading to finite-time blow-up. In fact, solutions are known to increase quite substantially in size in many cases, as shown in the examples below (Figs. 1, 2). In view of the constraint (1.5), however, any substantial growth of $|\,u\,|$ leads to the development of high frequency structures, as illustrated below, which in turn tend to be efficiently dissipated by the ever larger viscosity present in these regions (the local bulk viscosity is proportional to $|\,u\,|^{\alpha}$). Thus, although the basic ingredients for solution blow-up are clearly there, especially for large κ, the final outcome seems difficult to predict, be it on physical or mathematical grounds. This interesting interaction between convection and diffusion due to $\beta(x,t) > 0$ has not been studied in the literature (see e.g. [1, 7, 15]).

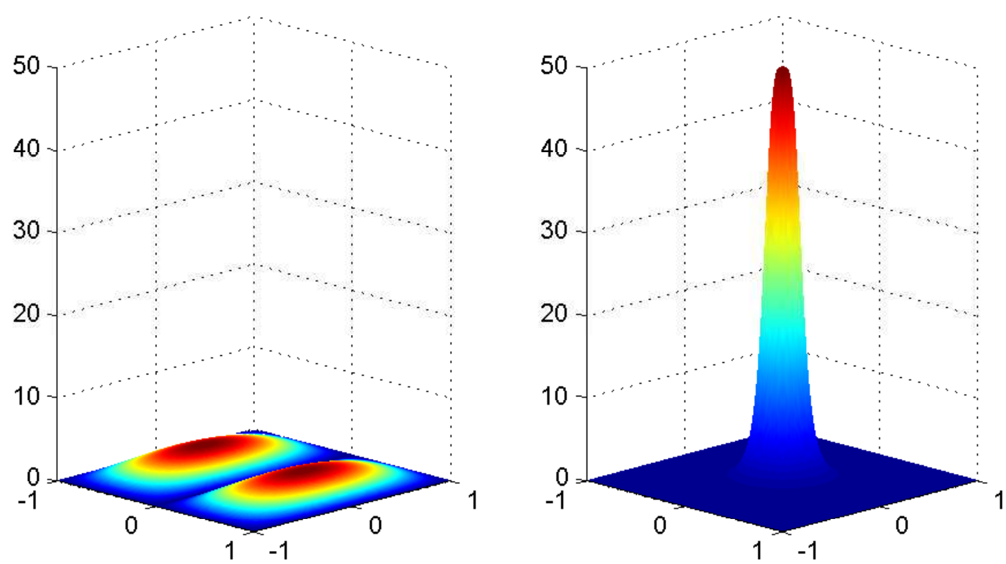

Fig. 1: The solution $u(\cdot, t)$ at time $t = 1000$ (right) for some given initial state compactly supported in the square $|\,x\,| \leq 1$, $|\,y\,| \leq 1$ (left), in the case $n = 2$, $\alpha = \kappa = 1$ and $\boldsymbol{b}(x,t) = -\,|\,x\,|^2\,x\,/\,(10^{-4} + |\,x\,|^4/4)$, showing an 18-fold increase in solution size. An impressive 18,000-fold increase in size would have been observed in this example by taking $\boldsymbol{b}(x,t) = -\,|\,x\,|^2\,x\,/\,(10^{-10} + |\,x\,|^4/4)$ instead.

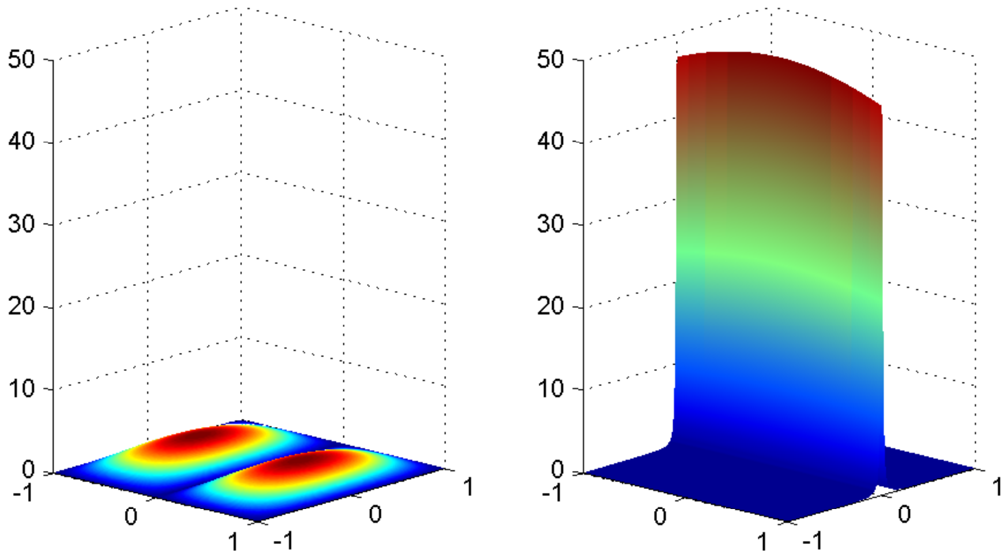

Fig. 2: The solution of (1.7) at time $t = 1000$ (right) for the same initial state considered in Fig. 1, assuming $n = 2$, $\alpha = 2$, $\kappa = 1$ and $\boldsymbol{b}(x,t) = (b_1(x), b_2(x))$ with $b_1(x) = -4/25\, x_1/[(16 + x_1^2)^2 (10^{-4} + x_2^2)]$, $b_2(x) = -4/25\, x_2/[(16 + x_1^2)\cdot (10^{-4} + x_2^2)^2]$, showing once again an 18-fold increase in solution size. A similar 18,000-fold increase in size would happen with $b_1(x) = -10^{-5}\, x_1/[(4 + x_1^2)^2 \cdot (10^{-10} + x_2^2)]$ and $b_2(x) = -10^{-5}\, x_2/[(4 + x_1^2)(10^{-10} + x_2^2)^2]$ in this example.

Going back to the general setting (1.1), (1.2), we now state our main results. Even when solutions are subject to considerable growth due to strong convection instabilities, the constraint (1.5) makes diffusion ultimately have the upper hand in *every* case, thus preventing any finite time blow-up from happening (i.e., $T_* = \infty$):

Theorem I. *Let* $\kappa \geq 0$. *Then, all solutions to the problem* (1.1), (1.2) *satisfying* (1.3) *are **globally defined*** (*i.e, defined for all* $t > 0$).

Theorem I is established after a series of technical lemmas in [11]. It significantly improves a previous result obtained in [9, 12], which was restricted to vanishing viscosity solutions and in addition required the extra assumption that $\kappa < \alpha + 1/n$. From the lemmata in [11] we also obtain an important pointwise estimate for the solutions of (1.1), (1.2) involving the quantities

$$\mathbb{U}_q(t) := \sup_{0 < \tau < t} \| u(\cdot, \tau) \|_{L^q(\mathbb{R}^n)} \qquad (1 \leq q \leq \infty) \qquad (1.8a)$$

and

$$\mathbb{F}_\mu(t) \;:=\; \sup_{0\,<\,\tau\,<\,t} \frac{F(\tau)}{\mu(\tau)} \tag{1.8b}$$

where F is given in (1.2) above. Let $a := n(\kappa - \alpha)$. Taking $p \geq 1$, $\sigma > 1$ satisfying

$$p \;>\; n(\kappa - \alpha), \qquad \sigma \;\geq\; \max\left\{\, \frac{2}{p},\; 1 + \frac{(2\kappa - \alpha)_-}{p} \,\right\} \tag{1.9}$$

it is shown in [11] the following fundamental estimate (see also [2, 3, 9, 12, 18]):

Theorem II. *Let* $p \geq 1$, $\sigma > 1$ *satisfy* (1.9). *Then*

$$\mathbb{U}_\infty(t) \;\leq\; \mathbb{K}\cdot\max\left\{ \|\, u_0 \,\|_{L^\infty(\mathbb{R}^n)};\; \mathbb{F}_\mu(t)^{\frac{n}{p-a}}\,\mathbb{U}_p(t)^{\frac{p}{p-a}} \right\} \tag{1.10}$$

for every $t > 0$, *and some constant* $\mathbb{K} = \mathbb{K}(n, \kappa, \alpha, p, \sigma) > 1$, *where* $a = n(\kappa - \alpha)$.

2. Some generalizations

We note that the results discussed above apply to more general degenerate parabolic equations of the form

$$u_t \;+\; \operatorname{div} \boldsymbol{f}(x,t,u) \;+\; \operatorname{div} \boldsymbol{g}(t,u) \;=\; \operatorname{div} \boldsymbol{A}(x,t,u,\nabla u). \tag{1.1}$$

where $\boldsymbol{f}$, $\boldsymbol{g}$ are as before and $\boldsymbol{A} \in C^0(\mathbb{R}^n \times [0,\infty) \times \mathbb{R} \times \mathbb{R}^n)$ satisfies the condition

$$\langle\, \boldsymbol{A}(x,t,\mathrm{u},\mathbf{v}), \mathbf{v} \,\rangle \;\geq\; \mu(t)\,|\,\mathrm{u}\,|^{\alpha}\,|\,\mathbf{v}\,|^{2} \quad \text{and} \quad |\,\boldsymbol{A}(x,t,\mathrm{u},\mathbf{v})\,| \;\leq\; M(t)\,|\,\mathrm{u}\,|^{\alpha}\,|\,\mathbf{v}\,| \tag{2.2}$$

for all $x \in \mathbb{R}^n$, $t \geq 0$, $\mathrm{u} \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n$, and some $\mu, M \in C^0([0,\infty))$, with $\mu(t) > 0$, as long as it can be shown that $\|\, u(\cdot,t) \,\|_{L^1(\mathbb{R}^n)}$ cannot blow up in finite time. The results also extend to the case of arbitrary initial states $u_0 \in L^1(\mathbb{R}^n)$ [not necessarily bounded], with condition (1.3b) then replaced by the assumption that $u(\cdot,t) \in L^\infty_{\mathrm{loc}}((0,T_*), L^\infty(\mathbb{R}^n))$, or even more generally to initial data $u_0 \in L^{p_0}(\mathbb{R}^n)$ for some given $1 \leq p_0 < \infty$, with only minor changes in the statements, provided once more that it can be shown that $\|\, u(\cdot,t) \,\|_{L^q(\mathbb{R}^n)}$ will not blow up in finite time for some suitable $p_0 \leq q < \infty$.

Acknowledgements. This work was partially supported by CAPES (Ministry of Education, Brazil), Grant # 88887.125079/2015. The computations in this research were performed by the SGI cluster ALTIX 1350/450 of the CENTRO NACIONAL DE PROCESSAMENTO DE ALTO DESEMPENHO EM SÃO PAULO (CENAPAD-SP), Brazil.

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