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Inhalt/Contents

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Mathematische Annalen

Diffeomorphisms of a K3 Surface

Ciprian Borcea

Department of Mathematics, National Institute for Scientific and Technical Creation, Bd. Păcii 220, RO-79622 Bucharest, Romania

Let X be a compact connected complex surface with trivial canonical bundle: thus, as proved in [3], X is a complex torus, a Kodaira surface or a K3 surface.

Let L denote the lattice $\mathrm{im}(H^2(X,\mathbb{Z})\to H^2(X,\mathbb{R}))$: it is an even, unimodular, indefinite lattice.

The orthogonal group $O(p,q)$ of a real bilinear symmetric indefinite form of signature (p,q) is a Lie group which has four connected components: fix a positive p-plane α and a negative q-plane β; a transformation $T\in O(p,q)$ will be said to preserve $(+)$ or reverse $(-)$ the orientation of positive p-planes if the projection of $T\alpha$, parallel with β, determines an orientation preserving or reversing automorphism of α. (This is correct since positive p-planes and negative q-planes define connected subsets of the corresponding Grassmannians.) Thus:

$$O(p,q)=O^{++}(p,q)\cup O^{+-}(p,q)\cup O^{-+}(p,q)\cup O^{--}(p,q),$$

where the first and the second sign refer to the orientation behaviour with respect to positive p-planes and negative q-planes.

Accordingly, the automorphism group Γ of L may be written as a disjoint union: $\Gamma=\Gamma^{++}\cup\Gamma^{+-}\cup\Gamma^{-+}\cup\Gamma^{--}$, by taking intersections of Γ with the components of the real orthogonal group.

We shall prove:

Theorem A. *Let X be a K3 surface (i.e. $b_1(X)=0$, $c_1(X)=0$). Then, any element of the subgroup $\Gamma^{++}\cup\Gamma^{+-}$ is induced by a diffeomorphism of X (which necessarily preserves the orientation).*

It was known for long [8] that $\mathrm{im}\,(\mathrm{Diff}(X)\to\Gamma)$ has finite index in Γ: this index is at most 2 by the above theorem.

Theorem A will be derived from (known results and the proof of) the following version of the "Torelli theorem", which holds as well for 2-complex tori and Kodaira surfaces:

Theorem B. *Let X_0 be the underlying oriented differential manifold of a compact connected complex surface with trivial canonical bundle; let Ω denote the "period* *domain" of* X_0, *that is:*

$$\Omega=\{x\in H^2(X_0,\mathbb{C})|\langle x,x\rangle=0 \quad and \quad \langle x,\bar{x}\rangle>0\}/\mathbb{C}^*.$$

Then: (i) *any element of* Ω *may be represented by a nowhere vanishing holomorphic 2-form corresponding to a suitable complex structure on* X_0*;*

(ii) *any two such representatives of a given element come from isomorphic complex structures.*

Proof. For 2-complex tori this is "well known" [5]. For Kodaira surfaces this is proved in [2].

Let therefore X_0 come from a K3 surface.

We refer to [1] for the relevant material on K3 surfaces. See also [0].

Now, (ii) follows directly from the fact that any K3 surface is Kähler and two Kähler K3 surfaces with the same period are isomorphic.

(It will be remarked however that if in the statement of Theorem B the Kähler condition is imposed on the complex structures involved, one may circumvent the result of Siu that every K3 surface is Kähler. In the argument below, the openness of Ω_e (with Kähler condition) would follow from the stability of the Kähler property. Theorem A, therefore, does not rely on Siu's result.)

Denote by Ω_e the subset of Ω where (i) holds.

We have to prove that $\Omega_e=\Omega$.

By the "local Torelli theorem", Ω_e is open in Ω.

Let γ be an automorphism of the lattice $L=H^2(X_0,\mathbb{Z})$, different from $\pm$id. Since $\Gamma=\mathrm{Aut}(L)$ acts holomorphically on Ω, the set F_γ of fixed points of γ is a proper analytic subset of Ω. By the Baire category theorem, the complement of the union $F=\cup F_\gamma$, $\gamma\in\Gamma-\{\mathrm{id}, -\mathrm{id}\}$ is dense in Ω.

Suppose p to be a boundary point of Ω_e in Ω.

Then, one may find $\gamma\in\Gamma$ such that $\gamma\cdot p\in\Omega_e$, as a consequence of the "surjectivity of the period map" (and the fact that all K3 surfaces are diffeomorphic).

Let $q\in(\Omega_e\cap\gamma^{-1}\cdot\Omega_e)-F$ (which is non-empty).

Let J_q and $J_{\gamma\cdot q}$ be complex structures with trivial canonical bundle on X_0, representing q and $\gamma\cdot q$.

Since with marking γ, respectively id, (X_0,J_q) and $(X_0,J_{\gamma\cdot q})$ have the same period, they are isomorphic. In fact, there is a unique isomorphism $g:(X_0,J_{\gamma\cdot q})\to(X_0,J_q)$, for otherwise q would lie in F. (Recall that an automorphism of a K3 surface X which induces the identity on $H^2(X,\mathbb{Z})$ is necessarily the identity map.) In the same way we obtain $g^*=\pm\gamma$, hence the direct image by g of any representative $J_{\gamma\cdot p}$ of $\gamma\cdot p\in\Omega_e$ will be a representative for p.

Thus Ω_e cannot have boundary points in Ω and (i) follows from the connectivity of Ω.

Proof of Theorem A. Let $q\in\Omega-F$ and $\gamma\in\Gamma$. Represent q and $\gamma\cdot q$ by K3 complex structures J_q and $J_{\gamma\cdot q}$ on X_0. The proof above shows that there exists a unique isomorphism $g:(X_0,J_{\gamma\cdot q})\to(X_0,J_q)$ and g induces $\pm\gamma$. Thus, the natural morphism $\mathrm{Diff}(X_0)\to\Gamma/\pm\mathrm{id}$ is surjective. In particular, im $(\mathrm{Diff}(X_0)\to\Gamma)$ is normal in Γ.

For a (Kähler) K3 complex structure J_p on X_0, with period p, define the "Kähler cone" $C(J_p)$ to be that one of the two components in the set of real $(1,1)$ classes x, with $\langle x, x\rangle > 0$, which contains the Kähler classes. We have:

Lemma. *$-\mathrm{id} \in \Gamma$ is induced by a diffeomorphism of X_0 if and only if for some point $p \in \Omega$ (and hence for all points in Ω) there exist (Kähler) K3 complex structures J_p and J'_p representing p and with opposite Kähler cones: $C(J_p) = -C(J'_p)$.*

Proof. One implication is clear by transport of structure through any diffeomorphism realizing $-\mathrm{id} \in \Gamma$.

Conversely, since the Kähler cone "varies continuously" in a deformation, we may suppose $p \in \Omega - F$.

Then, the isomorphism between (X_0, J_p) and (X_0, J'_p) necessarily induces $-\mathrm{id} \in \Gamma$.

Now, we produce some elements in Γ^{+-} induced by diffeomorphisms: let $\delta \in L$, $\langle \delta, \delta \rangle = -2$ and consider the involution

$$\sigma_\delta(x) = x + \langle x, \delta \rangle \delta . \qquad (*)$$

For a certain δ, this involution appears as the monodromy of a Lefschetz pencil of quartics in P_3 [4, Sect. 6]. Since all such involutions are conjugate in Γ [7] and $\mathrm{im}(\mathrm{Diff}(X_0) \to \Gamma)$ is stable under conjugation, they are all induced by diffeomorphisms.

Let finally $\gamma \in \Gamma^{++}$ and $\gamma_t \in O^{++}\left(L \bigotimes_{\mathbb{Z}} R\right)$ be an arc in the real orthogonal group, from $\gamma_0 = \mathrm{id}$ to $\gamma_1 = \gamma$.

For $q \in \Omega$, let J_q be a representative and $\gamma_t \cdot q$ the corresponding arc in Ω from q to $\gamma \cdot q$.

The following alternative holds: either

(α) all cones $\gamma_t \cdot C(J_q)$ are Kähler cones for suitable representatives of $\gamma_t \cdot q$,

or

(β) there is a t, $0 < t \leqq 1$, such that all representatives of $\gamma_t \cdot q$ have the Kähler cone equal to $-\gamma_t \cdot C(J_q)$ and t is minimal with this property.

Case (β) would imply that the whole of Γ comes from diffeomorphisms since small deformations of any representative of $\gamma_t \cdot q$ would show the condition in the lemma above to be satisfied.

Thus, we are confronted with (α).

Let $J_{\gamma \cdot q}$ be a representative for $\gamma \cdot q$ such that $\gamma \cdot C(J_q) = C(J_{\gamma \cdot q})$ and denote by $\Delta(J_{\gamma \cdot q})$ the Chern classes of effective divisors on $(X_0, J_{\gamma \cdot q})$ of self-intersection -2.

The composition of γ with suitable involutions (reflections) $(*)$ with $\delta \in \Delta(J_{\gamma \cdot q})$ (which, as shown above, are induced by diffeomorphisms) becomes then an "effective Hodge isomorphism" $H^2((X_0, J_q), \mathbb{Z}) \to H^2((X_0, J_{\gamma \cdot q}), \mathbb{Z})$; [1]. By the "(strong) Torelli theorem", it comes from a (unique) isomorphism $(X_0, J_{\gamma \cdot q}) \to (X_0, J_q)$.

Consequently, γ is induced by a diffeomorphism and Theorem A follows.

Remark. For 2-complex tori $\mathrm{im}(\mathrm{Diff}(X_0) \to \Gamma) = \Gamma^{++}$; [5]. For Kodaira surfaces $\mathrm{im}(\mathrm{Diff}(X_0) \to \Gamma)$ is a subgroup of $\Gamma^{++} \cup \Gamma^{--}$ of infinite index; [2].

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Note added in proof. Professor Takao Matumoto, from Hiroshima, sent to the author his manuscript "On diffeomorphisms of a K3 surface", in which Theorem A is obtained with less analytical, but more topological machinery. His paper is part of the volume "Algebraic and topological theories", dedicated to Dr. Miyata.

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Mathematische Annalen

Asymptotics of Matrix Coefficients, Perverse Sheaves and Jacquet Modules

Luis G. Casian [1]

Institute for Advanced Study, Princeton, NJ 08540, USA

1. Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{b}=\mathfrak{h}+\mathfrak{n}$ a Borel subalgebra with Cartan subalgebra $\mathfrak{h}$ and nilradical $\mathfrak{n}$. As a consequence of the statement and proof of the Kazhdan-Lusztig conjectures, it has become clear, that the study of $\mathfrak{n}$-homology of $\mathfrak{g}$-modules can be tightly connected to the geometry of the flag variety. This is emphasized in [18]. There it is observed, that the stalks of certain intersection cohomology complex, associated to an irreducible Harish-Chandra module M, can be interpreted as "antidominant weight spaces" of the $\mathfrak{n}$-homology of M (where $\mathfrak{n}$ varies along the flag variety). A second, more algebraic way of dealing with $\mathfrak{n}$-homology, where $\mathfrak{n}$ is now associated to a maximally split Cartan of a semisimple real Lie group $G_{\mathbb{R}}$, is the use of a "$\mathfrak{g}$-module analogue of $\mathfrak{n}$-homology", the Jacquet functor $J_{\mathfrak{n}}$ (see [8]) which transfers the problem of computing $\mathfrak{n}$-homology of Harish-Chandra modules, to the more manageable problem of computing $\mathfrak{n}$-homology of objects in the category $\mathcal{O}'$ (closely connected to the $B-G-G$ category $\mathcal{O}$) (see for instance [17]). Since the $\mathfrak{n}$-homology of irreducible modules in the category $\mathcal{O}$ is known (for integral infinitesimal character [18]) from the Kazhdan-Lusztig conjectures, it is hoped that unraveling the complicated structure of the Jacquet module of an irreducible Harish-Chandra module, will lead to important new information about $\mathfrak{n}$-homology. This approximation to the problem of computing $\mathfrak{n}$-homology is becoming more promising, for instance, by the results in [7], which lead to a concrete conjecture for the socle filtration of the Jacquet module of an irreducible Harish-Chandra module. Let $\mathfrak{b}_x=\mathfrak{h}_x+\mathfrak{n}_x$ be a Borel subalgebra attached to a point x in the flag variety. In this work, we study generalized Jacquet modules $J_{\mathfrak{n}_x}M$, defined even when $\mathfrak{n}_x$ is not associated to the maximally split Cartan subalgebra. We set:

$$J_{\mathfrak{n}_x}M=\{f\in \operatorname{Hom}_{\mathbb{C}}(M,\mathbb{C}), \mathfrak{n}_x^k f=0 \text{ for some } k\}^0 .$$

[1] Supported in part by NSF grant MCS-8108814 (A03)

The notation "0" indicates that there is a twist in the $\mathfrak{g}$-action, by Adg for some $g \in G$, introduced to make the modules $J_{\mathfrak{n}_x}M$ $\mathfrak{b}$-locally finite, where $\mathfrak{b}$ is our fixed Borel subalgebra introduced at the beginning.

For almost all $\mathfrak{n}_x$, the object $J_{\mathfrak{n}_x}M$ is the usual Jacquet module. However, in general, higher derived Jacquet modules $J^q_{\mathfrak{n}_x}M$ appear. These *generalized Jacquet modules* satisfy

$$\sum_{p,q \geqq 0} (-1)^{p+q} \operatorname{tr} H^p(\mathfrak{n}, J^q_{\mathfrak{n}_x}M) = \sum_{r \geqq 0} (-1)^r \operatorname{tr} H_q(\mathfrak{n}_x, M)^0 .$$

Here "tr" denotes the trace of the $\mathfrak{h}$-action (and as before "0" is the appropriate twist interchanging $\mathfrak{h}_x$ with $\mathfrak{h}$). By the Osborne character formula the modules $J^q_{\mathfrak{n}_x}M$ can be used, as in [6], to deal with characters of Harish-Chandra on arbitary θ-stable Cartan subgroups, where the modules $J^q_{\mathfrak{n}_x}M$ take the place of the usual Jacquet module.

It then becomes natural to try to tie up these two different approaches to the study of $\mathfrak{n}$-homology, the geometric Beilinson-Bernstein theory, and the "$\mathfrak{g}$-module analogue of $\mathfrak{n}$-homology" approach of using the Jacquet module (tied up in turn with asymptotics by [13]). In this article we show how from the fact that the localization $\mathscr{m}$ of a Harish-Chandra module M (in the sense of [1]), is a holonomic $\mathscr{D}$-module with regular singularities, it is possible to obtain, not only the "antidominant weight spaces" of the $\mathfrak{n}$-homology of M, in the solution complex of a $\mathscr{D}$-module, but also the generalized Jacquet modules. This is done by localizing M in a different way to obtain a new $\mathscr{D}$-module and considering again its solution complex. The consequence of doing this is that the Beilinson-Bernstein theory is turned explicitly into a "highest weight theory" and, as a first application, the finite dimensionality of $\mathfrak{n}$-homology groups for arbitrary $\mathfrak{n}$, can be recovered from the finiteness theorems of Kashiwara [15] for holonomic $\mathscr{D}$-modules. (The finiteness of $\mathfrak{n}$-homology for arbirary $\mathfrak{n}$ is implicit in [18] without the use of $\mathscr{D}$-modules).

As a second and new application we relate the generalized Jacquet modules $J_{\mathfrak{n}_x}M$ to asymptotics of matrix coefficients, by using holomorphic continuation to the complex group G, thus extending the results of [13].

We now give a more detailed account of the results.

In order to transform the Kazhdan-Lusztig conjectures into a geometric problem, Beilinson and Bernstein in [1] and Vogan in [18], consider the localization of Harish-Chandra module as follows: Let $\mathscr{B}$ be the flag variety of $\mathfrak{g}$, consisting of all the Borel subalgebras of $\mathfrak{g}$. We fix a connected algebraic group G over $\mathbb{C}$, whose Lie algebra is $\mathfrak{g}$, and a real form $G_{\mathbb{R}} \subset G$. Let $K_{\mathbb{R}}$ be the maximal compact subgroup of $G_{\mathbb{R}}$ and $K \subset G$ its complexification. The pair $(\mathfrak{g}, K)$ defines the category of Harish-Chandra modules $\mathscr{HC}$ associated to $G_{\mathbb{R}}$. We now fix a finite dimensional representation F of G, and let $\mathscr{HC}_F$ be the category of Harish-Chandra modules with the same infinitesimal character as F. If χ_F is the infinitesimal character of F, $U(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$, the new object $R = R_\chi = U(\mathfrak{g})/U(\mathfrak{g}) \ker \chi$ acts on any $\mathfrak{g}$-module with infinitesimal character χ_F. Therefore any object in $\mathscr{HC}_F$ becomes in this way an (R, K)-module.

Let $\mathscr{L}_F$ be the line bundle on $\mathscr{B}$ with sheaf of holomorphic sections $\mathscr{O}_F$, whose global sections can be identified with F as in the Borel-Weil theorem. Let $\mathscr{D}_F$ be the sheaf of differential operators acting on $\mathscr{O}_F$. Then, as observed in [1], the module R

can be identified with the global sections of $\mathscr{D}_F$ and we can define the localization $\Delta(M)$ of $M \in Ob\,\mathscr{H}\mathscr{C}_F$ by

$$M \to \mathscr{D}_F \bigotimes_R M \stackrel{\text{def}}{=} \Delta(M).$$

This operation transforms a Harish-Chandra module into a holonomic $\mathscr{D}_F$-module with regular singularities.

The second step, in connecting $\mathscr{H}\mathscr{C}_F$ with the geometry of $\mathscr{B}$, is to transform $\Delta(M)$ into a complex of sheaves with constructible cohomology, via the Riemann-Hilbert correspondence of Kashiwara-Mebkhout [14, 16]. We can define the solution complex of $m = \Delta(M)$ by

$$\begin{aligned} \mathrm{Sol}(m) &= \mathbb{R}\, \mathscr{H}om_{\mathscr{D}_F}(m, \mathscr{O}_F) \\ &= \mathbb{R}\, \mathrm{Hom}_R(M, \mathscr{O}_F). \end{aligned}$$

This is obtained by taking a projective resolution of m and then applying the functor $\mathscr{H}om_{\mathscr{D}_F}(\ldots, \mathscr{O}_F)$ to it. The complexes $\mathrm{sol}(m)$ are perverse, and in the case when M is irreducible, $\mathrm{sol}(m)$ is an intersection cohomology complex with local coefficients in the sense of [11], which has been computed in a Grothendieck group in [18]. The reader is encouraged to think of this intersection cohomology complex, as the "highest weight" of the dual of M. For instance, if $M = F$ is finite dimensional, we obtain a constant sheaf whose stalks are isomorphic to $(F/\mathfrak{n}F)^*$, as $\mathfrak{n}$-varies.

We fix a point $x \in \mathscr{B}$, which corresponds to a Borel subalgebra $\mathfrak{b}_x = \mathfrak{h}_x + \mathfrak{n}_x$. Let $-\lambda$ be the lowest weight of F with respect to $\mathfrak{n}_x : F/\mathfrak{n}_x F \approx \mathbb{C}_{-\lambda}$. The fact that $\Delta(M) = m$ has regular singularities for M in $\mathscr{H}\mathscr{C}_F$, becomes in [18] the following

(1.1) **Theorem** (Beilinson-Bernstein-Vogan). *Let M be in $\mathscr{H}\mathscr{C}_F$. Then*

$$\mathscr{E}xt^i_{\mathscr{D}_F}(m, \mathscr{O}_F)_x \approx \mathrm{Hom}_{\mathfrak{h}}(H_i(\mathfrak{n}_x M), \mathbb{C}_{-\lambda}).$$

Here $x \in \mathscr{B}$ corresponds to $\mathfrak{b}_x = \mathfrak{h}_x + \mathfrak{n}_x$.

In particular the holonomicity of m implies that the $-\lambda$-weight space of $H_i(\mathfrak{n}_x, M)$ is finite dimensional by [15].

We now fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, and consider the category $m_{\mathfrak{b}}$ of sheaves of $\mathfrak{g}$-modules with a $\mathfrak{b}$-locally finite action. Similarly we let $m_{\mathfrak{b},F}$ be the subcategory of $m_{\mathfrak{b}}$, where $Z(\mathfrak{g})$ acts with the same infinitesimal character as F. We let $D^b(m_{\mathfrak{b}})$, $D^b(m_{\mathfrak{b},F})$ denote the corresponding derived categories (consisting of bounded complexes of objects in $m_{\mathfrak{b}}$ or $m_{\mathfrak{b},F}$).

Let $\mathscr{O}_{\mathfrak{b}}$ be the sheaf of holomorphic functions on G, which are $\mathfrak{b}$-locally finite with respect to their left $\mathfrak{g}$-action. We now consider the complex of sheaves of $\mathfrak{g}$-modules in $D^b(m_{\mathfrak{b}})$

$$\mathbb{J}(M) \stackrel{\text{def}}{=} \mathbb{R}\, \mathrm{Hom}_{U(\mathfrak{g})}(M, \mathscr{O}_{\mathfrak{b}}).$$

Here $\mathrm{Hom}_{U(\mathfrak{g})}(\ldots, \mathscr{O}_{\mathfrak{b}})$ refers to the maps which intertwine the action on M with the action on $\mathscr{O}_{\mathfrak{b}}$ by *right* translation. We leave to the reader the simple task of rewriting $\mathbb{J}(M)$ as a solution complex of a localization of M choosing an appropriate sheaf of differential operators.

Let $\mathcal{O}'$ be the category of $\mathfrak{g}$-modles of finite length which are $\mathfrak{b}$-locally finite. Similarly $\mathcal{O}'_{F^*}$, the subcategory of $\mathcal{O}'$, consisting of modules where $Z(\mathfrak{g})$ acts by the same scalar as on F^*. We now introduce a functor $\Gamma_{\mathfrak{b}_x}$ defined on $\mathfrak{g}$-modules as

$$\Gamma_{\mathfrak{b}_x} Z = \mathfrak{b}_x\text{-locally finite vectors in } Z,$$

and its right derived functors $\Gamma^q_{\mathfrak{b}_x}$. For convenience we set

$$J^q_{\mathfrak{n}_x} M \overset{\text{def}}{=} [\Gamma^q_{\mathfrak{b}_x} \operatorname{Hom}_{\mathbb{C}}(M, \mathbb{C})]^0$$

where, it should be recalled that "0" means that we have twisted the $\mathfrak{g}$-action by Adg, for some g, to make it $\mathfrak{b}$-locally finite, instead, of $\mathfrak{b}_x$-locally finite.

Denote by $\mathbb{J}^q$ the q^{th} cohomology sheaf of $\mathbb{J}$. Our first result is

(1.2) **Theorem.** *Let M be in $\mathcal{HC}_F$. Then for each $g \in G$, we have a $\mathfrak{g}$-module isomorphism:*

$$\mathbb{J}^q(M)_g \approx J^q_{\mathfrak{n}_g}(M);$$

where $\mathfrak{n}_g = \operatorname{Adg}^{-1}\mathfrak{n}$. Moreover both sides are in the category $\mathcal{O}'_{F^*}$.

(1.3) *Remark.* The fact that $J^q_{\mathfrak{n}_g} M$ is in the category $\mathcal{O}_{F^*}$ is deduced in our proof of (1.2) from the holonomicity of $\mathscr{m} = \Delta(M)$, and it immediately implies the finiteness of $H_q(\mathfrak{n}_x, M)$ for all $x \in \mathscr{B}$ by the use of the spectral sequence.

$$\text{(1.4)} \qquad H^q(\mathfrak{n}_x, \Gamma^q_{\mathfrak{b}_x} M) \Rightarrow H^{p+q}(\mathfrak{n}_x, M)$$

(see Section 3).

(1.5) *Example.* Consider the case of SU(1, 1). Then $\mathscr{B} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, and $K = \{\left(\begin{smallmatrix} z & \\ & z^{-1}\end{smallmatrix}\right) : z \in \mathbb{C}^x\}$ acts on $\mathbb{P}^1$ by $\left(\begin{smallmatrix} z & \\ & z^{-1}\end{smallmatrix}\right) \cdot x = z^2 \cdot x$. We obtain three K-orbits on $\mathbb{P}^1$, namely $\{0\}$, $\{\infty\}$, $\mathbb{C}^x$. Since for any M in $\mathcal{HC}_F$, K acts on M, we only have to compute $J^q_{\mathfrak{n}_x} M$ for one representative in each K-orbit. If $T = K$ is the compact Cartan, $\{0\}$, $\{\infty\}$ correspond to the two Borel subalgebras associated to T, denoted $\mathfrak{b}_0 = \mathfrak{h}_0 + \mathfrak{n}_0$ and $\mathfrak{b}_\infty = \mathfrak{h}_\infty + \mathfrak{n}_\infty$. We choose a Borel subalgebra in the open K-orbit, $\mathfrak{b}_I = \mathfrak{h}_I + \mathfrak{n}_I$. Let M be the discrete series associated to $\{0\}$. Since M is $\mathfrak{n}_0$-locally nilpotent, we obtain

$$J^0_{\mathfrak{n}_\infty} M \approx M^0, \qquad J^1_{\mathfrak{n}_\infty} = 0$$

(recall our notation "0" for the twist introduced to make all our modules $\mathfrak{b}$-locally finite). Using (1.4), we also have

$$J^0_{\mathfrak{n}_0} M = 0, \qquad J^1_{\mathfrak{n}_0} \approx V^0.$$

Here V is the dual of a reducible Verma module. For $\mathfrak{n}_I$ we have:

$$J^0_{\mathfrak{n}_I} M \approx M^0.$$

This last isomorphism is an illustration of (1.2) because $J^0_{\mathfrak{n}_\infty} M \approx \operatorname{Hom}_{U(\mathfrak{g})}(M, \mathcal{O}_{\mathfrak{b}})_z$ for some $z \in G$.

Therefore any map $T \in \operatorname{Hom}_{U(\mathfrak{g})}(M, \mathcal{O}_{\mathfrak{b}})_z)$ can be holomorphically continued to a map $T: M \to \mathcal{O}_{\mathfrak{b}}(U)$ for U an open set, and we get a contribution on $J^0_{\mathfrak{n}_x} M$, for $\mathfrak{n}_x$ in the open K-orbit. Hence

$$M^0 \subset J^0_{\mathfrak{n}_I} M.$$

By $\mathfrak{n}$-homology considerations;

$$M^0 \approx J^0_{\mathfrak{n}_I} M .$$

The fact that the only stalk of $\mathbb{J}M$ with a dominant weight occurs over $\{0\}$, namely $J^1_{\mathfrak{n}_0} M \approx V^0$, accounts for the fact that the perverse sheaf that the Beilinson-Bernstein theory associates to M is a skyscraper sheaf on degree one, supported at $\{0\}$.

We now recall that $\mathbb{J}(M)$ is a complex of $\mathfrak{g}$-modules with a $\mathfrak{b}$-locally finite action. Thus $\mathbb{J}(M)$ breaks up into a direct sum of complexes of $\mathfrak{h}$-modules

$$\text{(1.6)} \qquad \mathbb{J}(M) = \bigoplus_{\mu \in \mathfrak{h}^*} \mathbb{J}(M)_\mu$$

the *weight components* of $\mathbb{J}(M)$. Here notation is being abused, but (1.6) makes sense on the level of representatives of the corresponding quasi-isomorphism classes. Each $\mathbb{J}(M)_\mu$ defines an object in the derived category of complexes of sheaves of $\mathfrak{h}$-modules. By a slight change in the definition of $\mathbb{J}(M)$, which will be discussed in (4.9), we may assume that $\mathbb{J}M$ is a complex of $\mathfrak{b}$-locally finite $\mathfrak{g}$-modules, where $Z(\mathfrak{g})$ acts by a fixed scalars. By the Casselman-Osborn theorem [9], we obtain that for any weight μ such that $\mathbb{J}(M)_\mu \neq 0$, $\lambda - \mu$ is a sum of positive roots with non-negative coefficients. This gives us a notion of *heighest weight component* of $\mathbb{J}M$, namely the complex $\mathbb{J}(M)_\lambda$ of $\mathfrak{h}$-modules. Let $\pi : G \to B$ be the projection $\pi(g) = \mathrm{Ad}g^{-1}\mathfrak{b}$. We have

(1.7) **Theorem.** *Let M be in $\mathscr{HC}_F$. Then the all weight components $\mathbb{J}(M)_\mu$ are perverse sheaves on G. The highest weight component is quasi-isomorphic to* $\pi^* \operatorname{Sol}(\Delta(M))$.

We refer the reader to our example (1.5), where (1.7) is illustrated.

Finally in Sect. 5, the *asymptotic sheaf* of a Harish-Chandry module M, denoted $\mathbb{J}_{as}M$ is introduced. This is constructed in terms of the asymptotic expansion of matrix coefficients, by holomorphic continuation to the complex group G. We show the following generalization of the results in [13]

(1.8) **Theorem.** *Let M be in $\mathscr{HC}_F$. Then there is an isomorphism*

$$\mathbb{J}_{as}M \approx \mathbb{J}^0 M .$$

2. Convergence of Formal Solutions

If Z is a Lie algebra bi-module, having left and right actions, we consider the two actions as *left* actions (by putting a "minus" sign in front of the right action). These will be denoted as "ℓ" or "r", the original left action and the new left action coming from the right respectively.

Here is the basic idea for the proof of (1.2). We note first that there is an isomorphism

$$\text{(2.1)} \qquad \hat{\mathcal{O}}_{G,g} \underset{\rightarrow}{\approx} U(\mathfrak{g})^* , \quad g \in G .$$

Here $\hat{\mathcal{O}}_{G,g}$ is the completion of the stalk of $\mathcal{O}_G$ at $g \in G$. The isomorphism intertwines l-actions but r-actions are only intertwined up to a twist. Hence

$$\hat{\mathcal{O}}_{\mathfrak{b},g} \approx \Gamma_{\mathfrak{b}} U(\mathfrak{g})^* .$$

We thus have

$$\text{(2.2)} \qquad \mathbb{R}\operatorname{Hom}_{U(\mathfrak{g})}(M, \hat{\mathcal{O}}^r_{\mathfrak{b},g}) \approx \mathbb{R}\operatorname{Hom}_{U(\mathfrak{g})}(M^g, \Gamma_{\mathfrak{b}} U(\mathfrak{g})^{*r}) \approx \mathbb{R}\Gamma_{\mathfrak{b}}(\operatorname{Hom}_{\mathbb{C}}(M^g, \mathbb{C})) .$$

The last isomorphism is given by evaluation at $1 \in U(\mathfrak{g})$. The notation M^g means that action on M has been twisted by Adg (to correct the twist in (2.1) in the r-actions). The last term in (2.2) is a complex whose cohomology is what we defined as $J^q_{\mathrm{Adg}\ ^{1}\mathfrak{n}}(M)$, $q = 0, 1, \ldots, \dim \mathfrak{n}$. Therefore, to prove the first part of Theorem (1.2) we have to show that formal solutions are convergent, to be able to substitute $\mathcal{O}_{\mathfrak{b},g}$ instead of $\hat{\mathcal{O}}_{\mathfrak{b},g}$. From the fact that $\Delta(M)$ has regular singularities, we have

$$\mathbb{R}\operatorname{Hom}_R(M, \mathcal{O}_{F,x}) \approx \mathbb{R}\operatorname{Hom}_R(M, \hat{\mathcal{O}}_{F,x}), \qquad x \in \mathcal{B} .$$

Hence we have to deal a) with the change of ring $U(\mathfrak{g}) \leftrightarrow R$ and b) with the change $\mathcal{O}_F \leftrightarrow \mathcal{O}_{\mathfrak{b}}$. In turn $\mathcal{O}_{\mathfrak{b}}$ has a filtration whose subquotients are roughly, pull backs to G, of sheaf of sections of locally defined line bundles on $\mathcal{B}$. Therefore b) roughly amounts to substituting $\mathcal{O}_F$ by *any* sheaf of sections of a (locally defined) line bundle on $\mathcal{B}$.

Change of Ring Spectral Sequence

Let χ be a character of $Z(\mathfrak{g})$, and recall $R_\chi = U(\mathfrak{g})/U(\mathfrak{g}) \ker \chi$. Denote by $\mathbb{R}_\chi$ functor on $\mathfrak{g}$-modules given by

$$\mathbb{R}_\chi Z = \{v \in Z : (z - \chi(z)) \cdot v = 0 \text{ for all } z \in Z(\mathfrak{g})\}$$

and by $\mathbb{R}^q_\chi$ its right derived functors. We have the following change of ring spectral sequence:

(2.3) **Proposition.** *Let A be an R_χ-module and B a $U(\mathfrak{g})$-module. Then there is a spectral sequence*

$$E_2^{p,q} = \operatorname{Ext}^p_{R_\chi}(A, \mathbb{R}^q_\chi B) \Rightarrow \operatorname{Ext}^{p+q}_{U(\mathfrak{g})}(A, B)$$

Moreover, if χ' is a second character of $Z(\mathfrak{g})$ and B has infinitesimal character χ', then $\mathbb{R}^q_\chi B = 0$ if $\chi \neq \chi'$ and $\mathbb{R}^q_\chi B = \Lambda^q V_0^* \otimes B$ if $\chi = \chi'$. Here V_0 is a vector subspace of $Z(\mathfrak{g})$ which generates $Z(\mathfrak{g})$ as a polynomial algebra.

Proof. The first part is the spectral sequence for the composition of functors [5]. For the second, observe that

$$\text{(2.4)} \qquad \mathbb{R}^q_\chi B = \operatorname{Ext}^q_{Z(\mathfrak{g})}(\mathbb{C}_\chi, B) ,$$

and the right side of (2.4) has an action of $\mathfrak{g}$: if $B \to I_*$ is an injective resolution of B as $U(\mathfrak{g})$-module, and $T \in \operatorname{Hom}_{Z(\mathfrak{g})}(\mathbb{C}_\chi, I_q)$, then $(X \cdot T)(a) = X(T(a))$, for $X \in \mathfrak{g}$ and $a \in \mathbb{C}_\chi$. This action is well defined because X commutes with $Z(\mathfrak{g})$. On the other hand, evaluation at $1 \in \mathbb{C}_\chi$ gives an isomorphism of $\mathfrak{g}$-modules

$$\mathrm{Hom}_{Z(\mathfrak{g})}(\mathbb{C}_\chi, I_q) \approx \mathbb{R}_\chi I_q$$

giving (2.4).

Let V_0 be a vector space in $Z(\mathfrak{g})$ which generates $Z(\mathfrak{g})$ as a polynomial algebra. We obtain a Koszul resolution

$$Z(\mathfrak{g}) \otimes \Lambda^q V_0 \otimes \mathbb{C}_\chi \to Z(\mathfrak{g}) \otimes \Lambda^{q-1} V_0 \otimes \mathbb{C}_\chi \to \ldots \to \mathbb{C}_\chi$$

and $\mathrm{Ext}^q_{Z(\mathfrak{g})}(\mathbb{C}_\chi, B)$ is the cohomology of the complex

$$\mathrm{Hom}_{Z(\mathfrak{g})}(Z(\mathfrak{g}) \otimes \Lambda^q V_0 \otimes \mathbb{C}_\chi, B) \approx \mathrm{Hom}_{\mathbb{C}}(\Lambda^q V_0 \otimes \mathbb{C}_\chi, \mathbb{C}_{\chi'}) \otimes B .$$

Regarding V_0 as a commutative algebra, this complex has cohomology

$$H^q(V, \mathbb{C}_{\chi'} \otimes \mathbb{C}^*_\chi) = \begin{cases} 0 & \text{if } \chi' \neq \chi \\ (\Lambda^q V_0)^* \otimes B & \text{if } \chi' = \chi \end{cases} \quad \text{Q.E.D.}$$

(2.5) **Corollary.** *Let M_1, M_2 be $U(\mathfrak{g})$-modules of finite length, where $Z(\mathfrak{g})$ acts by different generalized infinitesimal characters. Then*

$$\mathrm{Ext}^q_{U(\mathfrak{g})}(M_1, M_2) = 0 , \qquad q = 0, 1, \ldots .$$

Let V be a $\mathfrak{b}$-module. We define a sheaf on G by

$$\mathcal{O}(V) = (\mathcal{O}^{\ell}_G \otimes V)^{\mathfrak{b}} ,$$

that is, the $\mathfrak{b}$-invariants of the tensor product action, with $\mathcal{O}_G$ considered with its action by left translation. When $V = \mathbb{C}_\mu$, we denote $\mathcal{O}(V) = \mathcal{O}(\mu)$.

Assume now that V is $\mathfrak{b}$-locally finite, and let N be a neighborhood of a point $g \in G$ of the form $B \times W$. Here B is a Borel subgroup with the Lie algebra $\mathfrak{b}$, and W a complex submanifold of G. Setting $g = (b_0, w_0) \in B \times W$, an element of $\mathcal{O}(V)_g$ is, by definition, a germ of a holomorphic function with values in V, satisfying:

$$\text{(2.6)} \qquad \frac{d}{dt} f(\exp(\mathrm{X}t)\mathrm{b}_0, \mathrm{w}_0) = \sigma(\mathrm{X}) f(\mathrm{b}_0, \mathrm{w}_0)$$

$$f(b_0, w_0) = v_0 \in V, \qquad X \in \mathfrak{b} .$$

Since locally, the action of $\mathfrak{b}$ exponentiates to B, and the solution to (2.6) is unique, locally f has the form

$$f(bb_0, w) = \sigma(b) f(b_0, w) ,$$

and is therefore determined by its restriction to $b_0 \times W$. Hence $\mathcal{O}(V)_g \approx \mathcal{O}_{W, w_0} \otimes V$, with $\mathcal{O}_W$ the sheaf of holomorphic functions on W. We obtain

(2.7). **Lemma.** *The functor $V \to \mathcal{O}(V)$ is exact on locally finite $\mathfrak{b}$-modules.*

Since $\hat{\mathcal{O}}_{G,g} \approx U(\mathfrak{g})^*$, $g \in G$, and $H^i(\mathfrak{b}, U(\mathfrak{g})^* \otimes V) = 0$ (by injectivity of $U(\mathfrak{g})^* \otimes V$), we obtain the exactness of

$$V \to \hat{\mathcal{O}}(V)_g .$$

Denote by $(\hat{\mathcal{O}}/\mathcal{O})_g(V)$ the quotient $\hat{\mathcal{O}}(V)_g / \mathcal{O}(V)_g$. We obtain

(2.8) **Lemma.** *The functor $V \to (\hat{\mathcal{O}}/\mathcal{O})_g(V)$ is exact on locally finite* $\mathfrak{b}$*-modules.*

For any χ, character of $Z(\mathfrak{g})$, and F' finite dimensional representation of G, we let $\mathbb{P}_\chi$ be the functor, defined on $\mathfrak{g}$-modules with a locally finite $Z(\mathfrak{g})$-action, which picks out the generalized χ-component (where $Z(\mathfrak{g})$ acts with all its eigenvalues equal to χ). We also denote by $\mathbb{P}_{F',\chi}$ the composition of functors $\mathbb{P}_{F',\chi}(\ldots) = \mathbb{P}_\chi(F' \otimes (\ldots))$. Let ξ be the Harish-Chandra homomorphism of $Z(\mathfrak{g})$ onto the W-invariant polynomials on $\mathfrak{h}$, were $W = W(\mathfrak{g}, \mathfrak{h})$ is the Weyl group. We also set $\varrho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, ($\Delta^+$ the positive roots in $\mathfrak{b}$), and denote by $(,)$ the killing form on $\mathfrak{g}$.

(2.9) **Lemma.** *Let μ be a weight and η the highest weight of a finite dimensional representation F' of G, such that $\mu - \eta$ is antidominant. Then for any $g \in G$*

$$\mathbb{P}_{F',\chi}(\hat{\mathcal{O}}/\mathcal{O})_g(\lambda - \eta), \qquad \chi = \xi(\mu - \varrho)$$

has a filtration whose subquotients are isomorphic as $\mathfrak{g}$-modules to $(\hat{\mathcal{O}}/\mathcal{O})_g(\sigma)$, $\sigma \in \mathfrak{h}^*$, $(\sigma, \varrho) \leqq (\mu, \varrho)$, with $\sigma = \mu$ appearing exactly once.

Proof. We consider a filtration of F' by $\mathfrak{b}$-submodules with one dimensional subquotients

$$0 = V_0 \subset V_1 \subset \ldots \subset V_k = F', \qquad V_1 \approx \mathbb{C}_\eta .$$

By (2.8), we obtain a filtration:

$$0 \to (\hat{\mathcal{O}}/\mathcal{O})_g(\mathbb{C}_\eta \otimes \mathbb{C}_{\mu-\eta}) \subset (\hat{\mathcal{O}}/\mathcal{O})_g(V_2 \otimes \mathbb{C}_{\mu-\eta}) \subset \ldots$$

of $(\hat{\mathcal{O}}/\mathcal{O})_g(F' \otimes \mathbb{C}_{\mu-\eta}) \approx F' \otimes (\hat{\mathcal{O}}/\mathcal{O})_g(\mu - \eta)$. By applying the exact functor $\mathbb{P}_\chi$, we obtain as subquotients $\mathfrak{g}$-modules of the form $(\hat{\mathcal{O}}/\mathcal{O})_g(\sigma)$, with $\sigma - \varrho$ W-conjugate to $\mu - \varrho$. Since $\sigma = \mu - \eta + \eta^i$, with η^i a weight of F', we have $\sigma = \mu - A$, A a sum of positive roots. Therefore $(\mu - A, \varrho) = (\mu, \varrho) - (A, \varrho) \leqq (\mu, \varrho)$, with equality occurring only when $\eta^i = \eta$, weight of multiplicity one in F'. Q.E.D.

(2.10) **Lemma.** *Let M be in $\mathcal{HC}_F$, with the notation as in* (2.9),

$$\operatorname{Ext}^q_{U(\mathfrak{g})}(M, \mathbb{P}_{F',\chi}(\hat{\mathcal{O}}/\mathcal{O})_g(\mu - \eta)) = 0$$

for any q.

Proof. We have:

$$\begin{aligned}\operatorname{Ext}^q_{U(\mathfrak{g})}(M, \mathbb{P}_\chi[F' \otimes (\hat{\mathcal{O}}/\mathcal{O})_g(\mu - \eta)]) &= \operatorname{Ext}^q_{U(\mathfrak{g})}(M, F' \otimes (\hat{\mathcal{O}}/\mathcal{O})_g(\mu - \eta)) \\ &\approx \operatorname{Ext}^q_{U(\mathfrak{g})}((F')^* \otimes M, (\hat{\mathcal{O}}/\mathcal{O})_g(\mu - \eta)) .\end{aligned}$$

Using long exact sequences in $\operatorname{Ext}^*_{U(\mathfrak{g})}(\ldots, (\hat{\mathcal{O}}/\mathcal{O})_g(\mu - \eta))$, we are reduced to showing that for M' irreducible $\operatorname{Ext}^q_{U(\mathfrak{g})}(M', (\hat{\mathcal{O}}/\mathcal{O})_g(\mu - \eta)) = 0$, $q = 0, 1, \ldots$. By (2.3), it is enough to prove that $\operatorname{Ext}^q_{R_{\chi'}}(M', (\hat{\mathcal{O}}/\mathcal{O})_g(\mu - \eta)) = 0$, $q = 0, 1, \ldots$. Here χ' is the character of $Z(\mathfrak{g})$ given by $\xi(\mu - \eta - \varrho)$. But now $\mu - \eta$ is antidominant and $\mathcal{O}(\mu - \eta)$ is the pull-back of some $\mathcal{O}_{F''}$, with F'' a representation of G with lowest weight $\mu - \eta$. We now use the fact that $\mathcal{D}_{F''} \bigotimes_{U(\mathfrak{g})} M'$ has regular singularities and

$$\operatorname{Ext}^q_{R_{\chi'}}(M, (\hat{\mathcal{O}}_{F''})_x / (\mathcal{O}_{F''})_x) = 0 \qquad q = 0, 1, \ldots$$

for all $x \in \mathscr{B}$. This finishes the proof since as $\mathfrak{g}$-modules $(\hat{\mathcal{O}}_{F''})_x \approx \mathcal{O}(\mu-\eta)_g$ for any g in the fiber $\pi^{-1}(x)$ of the map $\pi: G \to \mathscr{B}$. Q.E.D.

(2.11) **Proposition.** *Let M be in $\mathscr{HC}_F$. Then for any weight $\mu \in \mathfrak{h}^*$,*

$$\operatorname{Ext}^q_{U(\mathfrak{g})}(M, (\hat{\mathcal{O}}/\mathcal{O})_g(\mu)) = 0, \qquad q = 0, 1, \ldots$$

Proof. We proceed by induction on (μ, ϱ). By (2.3), we only have to consider $\mu \in A$,

$$A = \{w(-\lambda-\varrho)+\varrho, w \in W\}$$

(recall that $-\lambda$ is the lowest weight of F). Pick F', η as in (2.9), so that we have an exact sequence:

$$0 \to (\hat{\mathcal{O}}/\mathcal{O})_g(\mu) \to \mathbb{P}_\chi(F' \otimes (\hat{\mathcal{O}}/\mathcal{O})_g(\mu-\eta)) \to Z \tag{2.12}$$

where Z has a filtration with subquotients $(\hat{\mathcal{O}}/\mathcal{O})_g(\sigma)$, such that $(\sigma, \varrho) < (\mu, \varrho)$, $\sigma \in A$. By induction, we can assume that for all $\sigma \in A$ with $(\sigma, \varrho) < (\mu, \varrho)$, the conclusion of (2.11) is true. We finish the proof by using the long exact sequence in $\operatorname{Ext}^p_{U(\mathfrak{g})}(M, \ldots)$ that results from (2.12), and by the use of (2.11). Q.E.D.

(2.13) *Remark.* We can write $\mathcal{O}_\mathfrak{b}$ as $\mathcal{O}(V)$, where V is $\Gamma_\mathfrak{b}\operatorname{Hom}(U(\mathfrak{b}), \mathbb{C})$. The isomorphism is just the restriction of

$$\mathcal{O}_G \approx \operatorname{Hom}_\mathfrak{b}(U(\mathfrak{b}), \mathcal{O}_G)$$

$$\text{which gives } \mathcal{O}_\mathfrak{b} \approx (\Gamma_\mathfrak{b}\operatorname{Hom}_\mathbb{C}(U(\mathfrak{b}), \mathbb{C}) \otimes \mathcal{O}_G)^\mathfrak{b}.$$

(2.14) **Proposition** *Let M be in $\mathscr{HC}_F$. Then for any $g \in G$, there is a quasi-isomorphism*

$$\mathbb{R}\operatorname{Hom}_{U(\mathfrak{g})}(M, \mathcal{O}_{\mathfrak{b},g}) \to \mathbb{R}\operatorname{Hom}_{U(\mathfrak{g})}(M, \hat{\mathcal{O}}_{\mathfrak{b},g}).$$

Proof. By the remark above, $\mathcal{O}_{\mathfrak{b},g} = \mathcal{O}(V)_g$, where V is a direct limit of $\mathfrak{b}$-modules, which in turn, have a filtration with one dimensional subquotients of the form $\mathcal{O}(\sigma)$, $\sigma \in \mathfrak{h}^*$. Using long exact sequences in $\operatorname{Ext}^q_{U(\mathfrak{g})}(M, \ldots)$, the proof is reduced to (2.11) Q.E.D.

By the remarks at the beginning of this section, this proves:

(2.15) **Corollary.** *Let M be in $\mathscr{HC}_F$. Then*

$$\mathbb{J}^q(M)_g \approx J^q_{\mathfrak{n}_g} M$$

where $\mathfrak{n}_g = \operatorname{Ad} g^{-1} \mathfrak{n}$.

3. Finiteness

We now show that the stalks of $\mathbb{J}^q M$ are in the category $\mathcal{O}'_{F^*}$. In the course of doing so, the constructibility of the solution complex of a holonomic $\mathscr{D}$-module will be used to show the finite dimensionality of the homology groups $H_q(\mathfrak{n}_x, M)$, $x \in \mathscr{B}$. If we additionally take into account that these solution complexes are perverse, we can obtain a bound on the dimension of the support of $\mathbb{J}^q M$ (3.8), necessary for the proof of (1.7). An equivalent formulation of this support condition is the following well known vanishing result for $\mathfrak{n}$-homology (which also follows from the work in [18, 17]). Let λ be the highest weight of F^* with respect to $\mathfrak{n}$. Denote by $[\mu, H_q(\mathfrak{n}_x, M)^0]$ the multiplicity of μ in $H_q(\mathfrak{n}_x, M)^0$. Then

$$(3.1) \qquad [w(-\lambda-\varrho)+\varrho, H_q(\mathfrak{n}_x, M)^0]=0 \quad \text{if} \quad q>\ell(w)+d_x$$
$$d_x=\text{codimension of } K\cdot \mathfrak{b}_x \text{ in } \mathscr{B}.$$

We now proceed to explain the proofs. Let $\mathscr{HC}_{\text{finite}}$ be the category of Harish-Chandra modules where $Z(\mathfrak{g})$ acts with eigenvalues which are infinitesimal characters of finite dimensional representations of G. The functor $\mathbb{J}$ extends to $\mathscr{HC}_{\text{finite}}$ by

$$(3.2) \qquad \mathbb{J}M \overset{\text{def}}{=} \mathbb{R}\,\mathrm{Hom}_{U(\mathfrak{g})}(M, \mathcal{O}_{\mathfrak{b}}), \qquad M\in O_b\mathscr{HC}_{\text{finite}}.$$

By induction on the length of M, (2.19) and (2.15) still hold if M is allowed to be in $\mathscr{HC}_{\text{finite}}$. We have the following lemma whose proof we postpone until the end of this section:

(3.3) **Lemma.** *Let M be in $\mathscr{HC}_{\text{finite}}$, λ the highest weight of F^*. Then for all $x\in\mathscr{B}$*

a) $$[\lambda, H^0(\mathfrak{n}, J^q_{\mathfrak{n}_x}M)]=[-\lambda, H_q(\mathfrak{n}_x, M)^0]$$

b) $$J^q_{\mathfrak{n}_x}\circ \mathbb{P}_{F,\chi}M\approx \mathbb{P}_{F^*,\chi^*}\circ J^q_{\mathfrak{n}_x}M, \qquad q=0,1,\ldots.$$

Proof. Postponed.

The following lemma is straightforward:

(3.4) **Lemma.** *Let L be any functor from $\mathscr{HC}_{\text{finite}}$ to the category of $\mathfrak{g}$-modules with a locally finite $\mathfrak{b}$-action, such that*

$$L\circ \mathbb{P}_{F,\chi}(M)\approx \mathbb{P}_{F^*,\chi^*}\circ L(M) \quad \textit{for any} \quad M\in O_b\mathscr{HC}_{\text{finite}} \quad (\text{e.g. } L=J^q_{\mathfrak{n}_x}).$$

Then for any weight μ

$$[\mu, H^0(\mathfrak{n}, LM)]\leqq[\mu+\eta, H^0(\mathfrak{n}, \mathbb{P}_{F,\chi}LM)]$$

where η is the highest weight of F.

(3.5) **Corollary.** *Assuming* (3.3), *the $\mathfrak{g}$-modules $J^q_{\mathfrak{n}_x}M$ are in the category $\mathcal{O}'$ if M is in $\mathscr{HC}_{\text{finite}}$.*

Proof. Choose F in (3.4) such that $\mu+\eta$ is dominant. Then $[\mu+\eta, J^q_{\mathfrak{n}_x}(\mathbb{P}_{F,\chi}M)] = [-\mu-\eta, H_q(\mathfrak{n}_x, \mathbb{P}_{F,\chi}M)^0]$. Using long exact sequences for $H_q(\mathfrak{n}_x, \ldots)$, we obtain that $[-\mu-\eta, H_q(\mathfrak{n}_x, \mathbb{P}_{F,\chi}M)^0]$ is finite dimensional. Here we used that $-\mu-\eta$ is antidominant plus (1.1) and the finiteness results for holonomic $\mathscr{D}$-modules in [15]. By (3.4) we obtain that $H^0(\mathfrak{n}, J^q_{\mathfrak{n}_x}M)$ is finite dimensional. Since $J^q_{\mathfrak{n}_x}M$ is $\mathfrak{b}$-locally finite, this immediately implies that $J^q_{\mathfrak{n}_x}M$ is in $\mathcal{O}'$. Q.E.D.

In order to obtain the finite dimensionality of $H_q(\mathfrak{n}_x, M)$, we need the following lemma whose proof we also postpone to the end of this section:

(3.6) **Lemma.** *There is a spectral sequence,*

$$E_2^{q,p}=H^q(\mathfrak{n}, J^p_{\mathfrak{n}_x}M)\Rightarrow H_{p+q}(\mathfrak{n}_x, M)^0$$

We obtain from (3.5) and (3.6) the following

(3.7) **Corollary.** *Assuming* (3.3) *and* (3.6) *if* M *is in* $\mathscr{H}\mathscr{C}_{\text{finite}}$, *the homology groups* $H_q(\mathfrak{n}_x, M)$ *are finite dimensional for all* $x \in \mathscr{B}$.

Finally the vanishing conditions (3.1) can be obtained from perversity conditions: Let $A^{\cdot}$ be a complex of sheaves of vector spaces on a smooth algebraic variety Y over $\mathbb{C}$, regarded as complex analytic manifold. Assume that the cohomology sheaves $\mathscr{H}^q A^{\cdot}$ of $A^{\cdot}$ are constructible with respect to an algebraic stratification of Y. Then $A^{\cdot}$ is said to be perverse (or a perverse sheaf) if it satisfies the perversity conditions:

P I) $\mathscr{H}^q A^{\cdot} = 0$ ffor $q < 0$

P II) dim support $\mathscr{H}^q A^{\cdot} \leqq \dim Y - q$

P III) The Verdier dual $DA^{\cdot} \stackrel{\text{def}}{=} \mathbb{R}\,Hom(A^{\cdot}, \mathbb{C})$ satisfies P I and P II.

Kashiwara and Mebkhout have shown [14, 16], that holonomic $\mathscr{D}$-modules with regular singularities correspond under the functor, which assigns to a $\mathscr{D}$-module its solution complex, to certain complexes of sheaves on Y with constructible cohomology. These complexes have been described by Deligne [3] as consisting exactly of all perverse sheaves. We now show how P I) and P II) imply (3.1). Using (3.3) a) (3.4) and (1.1) these perversity conditions translate into the following

(3.8) **Corollary.** *If* M *is in* $\mathscr{H}\mathscr{C}_{\text{finite}}$, *then*

$$J^q_{\mathfrak{n}_x} M = 0 \quad \text{if} \quad q > d_x$$

d_x = *codimension of* $K \cdot \mathfrak{b}_x$ *in* $\mathscr{B}$ *(assuming* (3.3) *and* (3.6)*)*.

Proof. Use (3.3) a), (3.4), (1.1) and the perversity conditions P I, P II.

Recall the vanishing conditions for $\mathfrak{n}$-cohomology of modules in $\mathcal{O}'$ due to Schmid [17]. These imply that for λ dominant with respect to $\mathfrak{n}$

$$[w(\lambda+\varrho)-\varrho, H^p(\mathfrak{n}, J^q_{\mathfrak{n}_x} M)] = 0 \quad \text{for} \quad p > \ell(w).$$

Now using (3.3) a) and (3.6), we obtain:

(3.9) **Corollary.** *Let* M *be in* $\mathscr{H}\mathscr{C}_F$. *Then, assuming* (3.3) *and* (3.6), *we have*

$$[w(-\lambda-\varrho)+\varrho, H_q(\mathfrak{n}_x, M)^0] = 0 \quad \text{if} \quad q > l(w) + d_x$$

d_x = *codimension of* $K \cdot \mathfrak{b}_x$ *in* $\mathscr{B}$.

It should be noted that (3.8) is equivalent (by (2.15)) to the following perversity condition for $\mathbb{J}M$:

(3.10) $$\text{dim support}\,\mathbb{J}^q M \leqq \dim G - q\,.$$

Homological Algebra

We now establish (3.3) and (3.6). Define functors $\gamma_{\mathfrak{n}}$, $\Gamma_{\mathfrak{h}}$ by:

$$\gamma_{\mathfrak{n}} Z = \{w \in Z : \mathfrak{n}^r \cdot w = 0 \text{ for some } r\} \ (Z \text{ an } \mathfrak{n}\text{-module})$$

$$\Gamma_{\mathfrak{h}} Z = \{w \in Z : U(\mathfrak{h}) \cdot w \text{ is finite dimensional}\} \ (Z \text{ an } \mathfrak{h}\text{-module})$$

and their right derived functors $\gamma_{\mathfrak{n}}^q Z$, $\Gamma_{\mathfrak{b}}^q Z$, $q=0,1,\ldots$. Note that each of the cohomology groups $\gamma_{\mathfrak{n}}^q Z$, $\Gamma_{\mathfrak{b}}^q Z$ carries a $\mathfrak{g}$-action if Z is a $\mathfrak{g}$-module. From the spectral sequence for the composition of functors [5]. We have

(3.10) **Lemma.** *For any $\mathfrak{g}$-module Z, there is a spectral sequence, whose differentials are $\mathfrak{g}$-maps*

$$E_2^{p,q}=\Gamma_{\mathfrak{b}}^p \gamma_{\mathfrak{n}}^q Z \Rightarrow \Gamma_{\mathfrak{b}}^{p+q} Z .$$

(3.11) **Lemma** [6]. *For any $\mathfrak{g}$-module Z, there is a spectral sequence, whose differentials are $\mathfrak{h}$-maps*

$$E_2^{p,q}=H^p(\mathfrak{n},\gamma_{\mathfrak{n}}^q Z) \Rightarrow H^{p+q}(\mathfrak{n},Z)$$

In order to prove (3.6), we have to relate $\gamma_{\mathfrak{n}}^q$ with $\Gamma_{\mathfrak{b}}^q$. We define the functor on $\mathfrak{h}$-modules given by:

$$\gamma_{\mathfrak{h}} Z=\{w\in Z\,;\mathfrak{h}^r\cdot w=0 \text{ for some } r\} .$$

Let $H_1,\ldots,H_n$ be a basis for $\mathfrak{h}$. For any multi-index $A=(H_{j_1},\ldots,H_r)$, with $j_1<\ldots<j_k$, we let

$$D_A U(\mathfrak{h})=\{H_{j_i}^{-s_1}\ldots H_{j_k}^{-s_r}\cdot w,\ w\in U(\mathfrak{h}),\ s_i\geqq 0\}$$

with $U(\mathfrak{h})$ the enveloping algebra of $\mathfrak{h}$). Note that if $A\subset B$ is an inclusion of ordered sets, then there are inclusions $D_A U(\mathfrak{h})\to D_B U(\mathfrak{h})$. In particular if A is obtained from B by deleting the q^{th} place in B, we denote the inclusion by ∂_B^q. Let ∂_r be defined as

$$\bigoplus_{|A|=r} D_A U(\mathfrak{h}) \xrightarrow{\partial_r} \bigoplus_{|B|=r+1} D_B U(\mathfrak{h})$$

where $|(H_{j_1},\ldots,H_{j_r}|=r, j_1<\ldots<j_r$, and $\partial_k|_{D_A U(\mathfrak{h})}=\sum_{\substack{A\subset B\\|B|=r+1}}(-1)^q\partial_B^q$. We obtain a complex:

$$\tilde{U}(\mathfrak{h})\overset{\text{def}}{=}U(\mathfrak{h})\to\bigoplus_{i=1}^m D_{H_i}U(\mathfrak{h})\to\ldots$$

of $\mathfrak{h}$-bimodules. If Z is a left $\mathfrak{h}$-module $\tilde{U}(\mathfrak{h})\bigotimes_{U(\mathfrak{h})} Z$ is a complex of $\mathfrak{h}$-modules whose cohomology we denote by $\tilde{\gamma}_{\mathfrak{h}}^q Z$. Using the spectral sequence attached to the composition of two functors [5], we obtain:

(3.12) **Lemma.** *For any $\mathfrak{h}$-module Z,*

$$\tilde{\gamma}_{\mathfrak{h}}^q Z\approx\gamma_{\mathfrak{h}}^q Z .$$

Moreover, if $\mathfrak{h}=\mathfrak{h}_2+\mathfrak{h}_2$ as vector spaces, there is a spectral sequence

(3.13) $$E_2^{p,q}=\gamma_{\mathfrak{h}_1}^p\gamma_{\mathfrak{h}_2}^q Z\Rightarrow\gamma_{\mathfrak{h}}^{p+q} Z$$

where $\gamma_{\mathfrak{h}_i}^q$, $i=1,2$ is defined similarly to $\gamma_{\mathfrak{h}}^q$.

We can now prove:

(3.14) **Lemma.** *Let Z be an $\mathfrak{h}$-module which is locally finite. Then $\Gamma_{\mathfrak{h}}^q Z=0$ if $q>0$.*

Proof. Since the $\mathfrak{h}$-action is locally finite, Z breaks up into a direct sum of generalized weight spaces $Z = \bigoplus_{\mu \in \mathfrak{h}^*} Z_\mu$. Note that as vector spaces $Z_{-\mu} \approx \gamma_{\mathfrak{h}}(\mathbb{C}_\mu \otimes Z)$. Thus, we obtain an isomorphism:

$$\Gamma_{\mathfrak{h}}^q Z \approx \bigoplus_{\mu \in \mathfrak{h}^*} \gamma_{\mathfrak{h}}^q(\mathbb{C}_\mu \otimes Z)$$

and, it suffices to show that $\gamma_{\mathfrak{h}}^q(\mathbb{C}_\mu \otimes Z) = 0$ for $q > 0$. This can be reduced to showing that $\gamma_{\mathfrak{h}}^q Z = 0$ for $q > 0$, in the case when $Z = Z_\mu$ for some μ. Let $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$ with $\mathfrak{h}_1 = \ker \mu$. Then since for any $H \in \mathfrak{h}_1$, $D_H Z = 0$, from (3.12) we obtain that $\gamma_{\mathfrak{h}_1}^q Z = 0$ if $q > 0$. On the other hand, the action of $\mathfrak{h}_2$ is by invertible operators, and the complex $\tilde{U}(\mathfrak{h}_2) \bigotimes_{U(\mathfrak{h}_2)} Z$, which computes $\gamma_{\mathfrak{h}_2}^q Z$ has no cohomology. Therefore, by (3.12), we obtain that if $\mathfrak{h}_2 \neq 0$, $\gamma_{\mathfrak{h}_2}^q \gamma_{\mathfrak{h}_1}^0 Z = 0$ for all q; in the case $\mathfrak{h}_2 = 0$, $\gamma_{\mathfrak{h}_1}^q Z = \gamma_{\mathfrak{h}}^q Z$. Therefore in either case $\gamma_{\mathfrak{h}}^q Z = 0$ if $q > 0$. Q.E.D.

(3.15) **Corollary.** *Let Z be a finitely generated R-module, $R = R_\chi$ for some χ, character of $Z(\mathfrak{g})$. Then $\gamma_{\mathfrak{n}}^q \operatorname{Hom}_{\mathbb{C}}(Z, \mathbb{C}) \approx \Gamma_{\mathfrak{b}}^q \operatorname{Hom}_{\mathbb{C}}(Z, \mathbb{C})$ as $\mathfrak{g}$-modules.*

Proof. On the $\mathfrak{b}$-modules $R/\mathfrak{n}^r R$ the action of $\mathfrak{h}$ is locally finite (by [9]). Hence it is locally finite on $\gamma_{\mathfrak{n}} \operatorname{Hom}_{\mathbb{C}}(R, \mathbb{C})$. If Z is an R-module, we can take a free R-resolution of Z to compute $\gamma_{\mathfrak{n}}^q \operatorname{Hom}_{\mathbb{C}}(Z, \mathbb{C})$. Therefore each of those cohomology groups is $\mathfrak{h}$-finite, and by (3.10) and (3.14) we conclude. Q.E.D.

(3.16) **Corollary.** *The Lemmas* (3.3) *and* (3.6) *are true for any M in $\mathscr{HC}_{\text{finite}}$.*

Proof. Using (3.15) and the fact that $\gamma_{\mathfrak{n}}^q$ commutes with tensoring by finite dimensional representations [6], we obtain $J_{\mathfrak{n}_x}^q \mathbb{P}_{F,\chi} M \approx \mathbb{P}_{F^*,\chi} J_{\mathfrak{n}_x}^q M$. Again by (3.15) and (3.11) we obtain the spectral sequence of (3.6). If M has generalized infinitesimal character $-\lambda - \varrho$, with λ dominant for $\mathfrak{n}$, the weight λ only occurs in $J_{\mathfrak{n}_x}^q M$ in the $\mathfrak{n}$-invariant vectors. Using the complex $\{\Lambda^r \mathfrak{n}^* \otimes J_{\mathfrak{n}_x}^q M\}$ to compute $H^*(\mathfrak{n}, J_{\mathfrak{n}_x}^q M)$, it follows that λ cannot occur in $H^r(\mathfrak{n}, J_{\mathfrak{n}_x}^q M)$, $r > 0$ (since $\Lambda^r \mathfrak{n}^*$ will shift all the generalized weights). Thus

$$[\lambda, H^0(\mathfrak{n}, J_{\mathfrak{n}_x}^q M)] = [-\lambda, H_q(\mathfrak{n}_x, M)^0]$$

by the spectral sequence (3.6). Q.E.D.

4. Perversity of Weight Components

Recall the decomposition $\mathbb{J}M = \bigoplus_{\mu \in \mathfrak{h}^*} \mathbb{J}(M)_\mu$ as complex of $\mathfrak{h}$-modules. We now show that each weight component $\mathbb{J}(M)_\mu$ is perverse. By (3.8), it is enough to show that the Verdier dual $D\mathbb{J}(M)_\mu$ satisfies the perversity conditions P I) and P II). The way we proceed is as follows. We construct a covariant functor

$$\tilde{\mathbb{J}} : \mathscr{HC}_{\text{finite}} \to D^b(\mathscr{m}_{\mathfrak{n}})$$

whose weight component $\tilde{\mathbb{J}}(M)_\mu$ is the Verdier dual $D\mathbb{J}(M)_\mu$ of $\mathbb{J}(M)_\mu$ for all μ. Moreover if $\tilde{\mathbb{J}}^q(M)_g$ is the stalk of $\tilde{\mathbb{J}}^q(M)$ at $g \in G$, we will show that the following is true:

(4.1) **Lemma.** *Let M be in $\mathscr{HC}_{\text{finite}}$ with $Z(\mathfrak{g})$ acting by scalars, with infinitesimal character $-\lambda-\varrho$. Then*

a) $$[\lambda, H_0(\mathfrak{n}, \tilde{\mathbb{J}}^q(M)_g)] = [-\lambda, H_q(\mathfrak{n}_x, \tilde{M})^0]$$

for some $\tilde{M}$ in $\mathscr{HC}_{\text{finite}}$ with the same infinitesimal character $-\lambda-\varrho$.

b) $$\tilde{\mathbb{J}}^q(\mathbb{P}_{F,\chi}M)_g \approx \mathbb{P}_{F^*,\chi^*}\tilde{\mathbb{J}}^q(M)_g, \qquad g \in G.$$

Condition a) means that stalkwise the λ-weight component of $\tilde{\mathbb{J}}M$ can be identified with the solution complex $\mathrm{Sol}(\Delta(\tilde{M}))$. Hence by the perversity conditions that this solution complex satisfies, we will obtain

(4.2) $$[\lambda, H^0(\mathfrak{n}_x, \tilde{\mathbb{J}}^q(M)_g)] = 0 \quad \text{if} \quad q > d_x, \qquad x = \pi(g).$$

By induction on the length of a module, (4.2) extends to modules in $\mathscr{HC}_{\text{finite}}$ (using long exact sequences involving the generalized λ weight space of $H^0(\mathfrak{n}_x, \tilde{\mathbb{J}}^q(\ldots)_g)$). Now we can use (4.1) b) and argue as in Sect. 3 to obtain:

$$\dim \operatorname{support} \mathbb{J}^q M \leqq \dim G - q$$

which will imply the perversity of the weight components of $\mathbb{J}M$.

We now construct $\tilde{\mathbb{J}}$. Let $A^{\cdot}$ be a complex of sheaves of $\mathfrak{g}$-modules with a $\mathfrak{b}$-locally finite action. Then the Verdier dual, $\mathbb{R}\,\mathit{Hom}(A^{\cdot}, \mathbb{C})$ carries a $\mathfrak{g}$-action:

Let $\mathscr{I}_*$ be an injective resolution of $\mathbb{C}$. Then $\mathbb{R}\,\mathit{Hom}(A^{\cdot}, \mathbb{C}) \approx \mathit{Hom}(A^{\cdot}, \mathscr{I}_*)$. The action of $\mathfrak{g}$ on $A^{\cdot}$, gives a $\mathfrak{g}$-action on $\mathit{Hom}(A^{\cdot}, \mathscr{I}_*)$. If we consider the extension of the functor $\Gamma_{\mathfrak{b}}$ to sheaves of $\mathfrak{g}$-modules we can define a new complex of $\mathfrak{g}$-modules

$$\tilde{\mathbb{J}}M \stackrel{\text{def}}{=} \Gamma_{\mathfrak{b}}\,\mathit{Hom}(A^{\cdot}, \mathscr{I}_*)^0 = \mathbb{R}\Gamma_{\mathfrak{b}} \circ \mathit{Hom}(A^{\cdot}, \mathbb{C})^0,\ A^{\cdot} = \mathbb{J}M$$

Again "0" denotes a twist in the $\mathfrak{g}$-action by a fixed automorphism σ which interchanges $\mathfrak{b}$ with its opposite Borel subalgebra, making the $\mathfrak{g}$-action $\mathfrak{b}$-locally finite.

(4.3) **Lemma.** *If F is a finite dimensional representation of G, and $M \in O_{\mathfrak{b}}\text{-}\mathscr{HC}_{\text{finite}}$, then*

a) $$\mathbb{J}(\mathbb{P}_{F,\chi}M) \approx \mathbb{P}_{F^*,\chi^*}\mathbb{J}M, \quad \text{and}$$

b) $$\tilde{\mathbb{J}}(\mathbb{P}_{F,\chi}M) \approx \mathbb{P}_{F^*,\chi^*}\tilde{\mathbb{J}}M$$

for any character χ of $Z(\mathfrak{g})$.

Proof. We first show a). Let $P_* \to M$ be a projective resolution of M. Then $F \otimes P_* \to F \otimes M$ is a projective resolution also. On the other hand $F \otimes M \approx \mathbb{P}_{F,\chi}(M) \oplus M'$, and we can choose projective resolutions $P^1_* \to \mathbb{P}_{F,\chi}M$, $P^2_* \to M'$, and a chain homotopy equivalence $L\colon P^1_* \oplus P^2_* \to F \otimes P_* M$. Identify $F^* \approx \mathrm{Hom}_{U(\mathfrak{g})}(F, \Gamma\mathcal{O}_G)$, ($\Gamma$ the global sections). We define a map

$$S\colon F^* \otimes \mathrm{Hom}_{U(\mathfrak{g})}(P_*, \mathcal{O}_{\mathfrak{b}}) \to \mathrm{Hom}_{U(\mathfrak{g})}(F \otimes P_*, \mathcal{O}_{\mathfrak{b}})$$
$$S(f \otimes T)(w \otimes m) = f(w)T(w) \text{ (multiplication of functions).}$$

Using the homotopy equivalence L, we obtain a map

$$\mathrm{Hom}_{U(\mathfrak{g})}(F \otimes P_*, \mathcal{O}_{\mathfrak{b}}) \to \mathrm{Hom}_{U(\mathfrak{g})}(P^1_*, \mathcal{O}_{\mathfrak{b}}),$$

and applying $\mathbb{P}r(\chi^*)$ we get

$$\mathbb{P}_{F^*,\chi^*}\operatorname{Hom}_{U(\mathfrak{g})}(P_*, \mathcal{O}_{\mathfrak{b}}) \to \operatorname{Hom}_{U(\mathfrak{g})}(P^1_*, \mathcal{O}_{\mathfrak{b}}). \tag{4.4}$$

The left side of (3.5) is $\mathbb{P}_{F^*,\chi^*}\mathbb{J}(M)$, and the right is $\mathbb{J}(\mathbb{P}_{F,\chi}M)$. We leave to the reader to verify using (2.15) that on the level of stalks, this gives the isomorphism

$$\mathbb{P}_{F^*,\chi^*}J^q_{\mathfrak{n}_x}M \approx J^q_{\mathfrak{n}_x}\mathbb{P}_{F,\chi}M$$

shown in Sect. 3. For b) note that there is an isomorphism:

$$\mathcal{H}om(F\otimes\mathbb{J}M, \mathcal{I}_*) \approx \operatorname{Hom}_{\mathbb{C}}(F, \mathcal{H}om(\mathbb{J}M, \mathcal{I}_*))$$

with $\mathcal{I}_*$ an injective resolution of the constant sheaf.

Thus we obtain:

$$\mathcal{H}om(F\otimes\mathbb{J}M, \mathcal{I}_*) \approx F^*\otimes\mathcal{H}om(\mathbb{J}M, \mathcal{I}_*)$$

and this is as sheaves of $\mathfrak{g}$-modules. We now apply $\Gamma_{\mathfrak{h}}$ to both sides, and then the projection $\mathbb{P}_\chi$ to the generalized infinitesimal character corresponding to χ. After twisting the $\mathfrak{g}$-action, (which interchanges F^* with F), we obtain

$$\tilde{\mathbb{J}}(\mathbb{P}_{F,\chi}M) \approx \mathbb{P}_{F^*,\chi^*}\tilde{\mathbb{J}}m \quad \text{Q.E.D.}$$

In order to obtain (4.1) a), we need to obtain a global identification (not just on the level of stalks) of the weight component $\mathbb{J}(M)_\lambda$, where M has infinitesimal character $-\lambda-\varrho$, and $\pi^*\operatorname{Sol}(\Delta(M))$. The following lemmas are geared in that direction. We go back to our fixed representation F of G with lowest weight $-\lambda$.

(4.5) **Lemma.** *Let V be a $\mathfrak{b}$-locally finite module such that $\mathfrak{n}V=0$. Then for any $g\in G$, $\mathcal{O}(V)_g$ is a direct summand of $\hat{\mathcal{O}}(V)_g$ as $Z(\mathfrak{g})$-modules.*

Proof. As in the discussion before (2.6), there is a neighborhood $N\approx B\times W$, with W open in the flag variety $\mathcal{B}$, such that there are linear isomorphism

$$T: \mathcal{O}(V)_g \approx \mathcal{O}_{W,w_0}\otimes V, \quad g=(b_0, w_0);$$
$$\hat{T}: \hat{\mathcal{O}}(V)_g \approx \hat{\mathcal{O}}_{W,w_0}\otimes V.$$

Here, as before, $\mathcal{O}_W$ is the sheaf of holomorphic functions on W, and $\mathcal{O}_{W,w_0}$ is the stalk at $w_0\in W$. These linear maps are given by restriction. The center $Z(\mathfrak{g})$ acts, via the Harish-Chandra homomorphism on V, and therefore on $\mathcal{O}_{W,w_0}\otimes V$ and $\hat{\mathcal{O}}_{W,w_0}\otimes V$, in a way that makes the maps T, $\hat{T}$ become $Z(\mathfrak{g})$-maps. Let L be a linear complement of $\mathcal{O}_{W,w_0}$ in $\hat{\mathcal{O}}_{W,w_0}$. Then $L\otimes V$ is a $Z(\mathfrak{g})$-submodule of $\hat{\mathcal{O}}_{W,w_0}\otimes V$, and so is $\mathcal{O}_{W,w_0}\otimes V$. Since

$$\hat{\mathcal{O}}(V)_g = \hat{\mathcal{O}}_{W,w_0}\otimes V = L\otimes V\oplus\mathcal{O}_{W,w_0}\otimes V.$$

This proves the lemma. Q.E.D.

(4.6) **Lemma.** *Let $V\approx\Gamma_{\mathfrak{h}}\operatorname{Hom}_{\mathbb{C}}(U(\mathfrak{h}), \mathbb{C})$, viewed as $\mathfrak{b}$-module where $\mathfrak{n}$ acts trivially. Then for any $g\in G$,*

$$\mathbb{R}^q_\chi(\hat{\mathcal{O}}(V)_g)=0 \quad \textit{if} \quad q>0$$

(the functors $\mathbb{R}^q_\chi$ as in (2.3)).

Proof. The modules $\mathbb{R}^q_\chi(\hat{\mathcal{O}}(V)_g)$ can be rewritten as $\mathrm{Ext}^q_{U(\mathfrak{g})}(R_\chi, \hat{\mathcal{O}}(V)_g)$, as can be seen easily. On the other hand, $\hat{\mathcal{O}}(V)_g \approx [U(\mathfrak{g})^* \otimes \Gamma_\mathfrak{h} U(\mathfrak{h})^*]^\mathfrak{b}$ which embedds into $\mathrm{Hom}_\mathfrak{b}(U(\mathfrak{g}), \Gamma_\mathfrak{h} U(\mathfrak{h})^*)$; and evaluating at $1 \in U(\mathfrak{g})$, we obtain an isomorphism

$$\mathrm{Hom}_{U(\mathfrak{g})}(P_*, \hat{\mathcal{O}}(V)_g) \approx \mathrm{Hom}_\mathfrak{b}(P_*, \Gamma_\mathfrak{h} U(\mathfrak{h})^*) \tag{4.7}$$

with $P_* \to R_\chi$ a projective $U(\mathfrak{g})$-resolution of R_χ. Note that the right side of (4.7) is $\mathrm{Hom}_\mathfrak{h}(P_*/\mathfrak{n}P_*, \Gamma_\mathfrak{h} U(\mathfrak{h})^*)$. Since $H_q(\mathfrak{n}, R_\chi) = 0$ for $q > 0$, we have that $P_*/\mathfrak{n}P_*$ is a free $U(\mathfrak{h})$-resolution of $R_\chi/\mathfrak{n}R_\chi$. Therefore, we obtain (evaluating at $1 \in U(\mathfrak{h})$)

$$\mathbb{R}^q_\chi(\hat{\mathcal{O}}(V)_g) \approx \Gamma^q_\mathfrak{h}((R_\chi/\mathfrak{n}R_\chi)^*)\,.$$

By the Casselman-Osborne theorem of [9], $R_\chi/\mathfrak{n}R_\chi$ is $\mathfrak{h}$-locally finite, hence by (3.14), $\Gamma^q_\mathfrak{h}(R_\chi/\mathfrak{n}R_\chi)^* = 0$ if $q > 0$. This concludes our proof. Q.E.D.

(4.8) **Lemma.** *We have $\mathbb{R}^q_\chi \mathcal{O}_\mathfrak{b} = 0$ if $q > 0$, if χ is a character of $Z(\mathfrak{g})$.*

Proof. This follows from the fact $\mathcal{O}_\mathfrak{b} = \mathcal{O}(\Gamma_\mathfrak{b} U(\mathfrak{b})^*)$ is a direct limit of sheaves which, in turn, have a filtration with subquotients of the form $\mathcal{O}(\Gamma_\mathfrak{b} U(\mathfrak{h})^* \otimes W)$. Here W is a finite dimensional $\mathfrak{h}$-module (the $\mathfrak{n}$-action on $\Gamma_\mathfrak{b} U(\mathfrak{h})^* \otimes W$ being trivial). Using (4.6) we conclude. Q.E.D.

(4.9) **Corollary.** *Let M be in $\mathscr{HC}_F$. Then*

$$\mathbb{R}\,\mathrm{Hom}_{U(\mathfrak{g})}(M, \mathcal{O}^r_\mathfrak{b}) \approx \mathbb{R}\,\mathrm{Hom}_{R_\chi}(M, \mathbb{R}^0_\chi \mathcal{O}^r_\mathfrak{b})$$

with χ the character of $Z(\mathfrak{g})$ which corresponds to F.

Proof. This follows from (2.3) and (4.8). Q.E.D.

The upshot of (4.9) is that if λ is highest weight of F, the λ-component of $\mathbb{R}^0_\chi \mathcal{O}_\mathfrak{b}$ under the action by left translation, is always $\mathfrak{n}$-invariant (say by Casselman and Osborne [9]). But the generalized λ-weight space of $(\mathcal{O}^l_\mathfrak{b})^\mathfrak{n}$ is just $\mathcal{O}(\lambda)$. Thus we obtain:

(4.10) **Corollary.** *Let M be in $\mathscr{HC}_F$, and recall the projection $\pi: G \to \mathscr{B}$. Then $\pi^* \mathrm{Sol}(\Delta(M)) \approx \mathbb{J}(M)_\lambda$ if λ is the highest weight of F^*.*

Finally, since the λ-component of $\tilde{\mathbb{J}}(M)$ is by construction, the Verdier dual of $\mathbb{J}(M)_\lambda$, we can identify

$$\begin{aligned} \tilde{\mathbb{J}}(M)_\lambda &\approx \pi^* D\,\mathrm{sol}(\Delta(M)) \\ &\approx \pi^*\,\mathrm{sol}(\Delta(\tilde{M})) \end{aligned}$$

where $\tilde{M}$ is another object in $\mathscr{HC}_F$. This finishes the proof of (4.1), a) and b).

(4.11) **Corollary.** *Let M be in $\mathscr{HC}_{\mathrm{finite}}$. Then the weight components of $\mathbb{J}(M)$ are perverse.*

Proof. We only need to show the constructibility of the weight components. By (3.5) and (3.16) the stalks of the cohomology sheaves $\mathbb{J}^q(M)_\mu$ are finite dimensional, and we only have to show that on the pullback of the K-orbits to G, $\mathbb{J}^q(M)_\mu$ is locally constant. This follows the same kind of arguments as the proof of (2.11) and (2.14) and is left to the reader. Q.E.D.

This finishes the proof of (1.7).

5. Asymptotics

As in [13], there is a connection of $\mathbb{J}(M)$ with asymptotics of matrix coefficients. If M^c denotes the K-finite dual of M, choosing $w \in M^c$, $v \in M$, one can form the matrix coefficient on $G_{\mathbb{R}}$.

$$f_{w,v}(g) = \langle w, g \cdot v \rangle$$

which only makes sense after taking appropriate globalizations of M and M^c so that $G_{\mathbb{R}}$ acts and "$g \cdot v$" is defined.

Using Harish-Chandra's asymptotic expansion of $f_{w,v}$ along a "chamber" (defined below) $A_{\mathbb{R}}^- \subset A_{\mathbb{R}}$ ($G_{\mathbb{R}} = N_{\mathbb{R}} A_{\mathbb{R}} K_{\mathbb{R}}$ an Iwasawa decomposition) [10], there is a family of functions $\{\zeta_{w,v}^\nu\}$, $\nu \in \mathscr{E}$ (a countable family of weights for $A_{\mathbb{R}}$), such that $f_{w,v}$ can be expanded in an open dense subset of $G_{\mathbb{R}}$, namely $K_{\mathbb{R}} A_{\mathbb{R}} K_{\mathbb{R}}$, as an infinite sum [10],

$$f_{w,v}(g) = \sum_{\nu \in \mathscr{E}} \zeta_{w,v}^\nu, \quad (\mathscr{E} \text{ a countable set of weights of } A_{\mathbb{R}}).$$

which converges on $K_{\mathbb{R}} A_{\mathbb{R},\varepsilon}^- K_{\mathbb{R}}$ absolutely and uniformly, for all $\varepsilon > 0$, with

$$A_{\mathbb{R},\varepsilon} = \{a \in A_{\mathbb{R}}, |e^\alpha(a)| < 1 - \varepsilon \text{ for all positive restricted real roots } \alpha\}.$$

$$A_{\mathbb{R}}^- \overset{\text{def}}{=} A_{\mathbb{R},0}^- .$$

On $A_{\mathbb{R}}$, the functions $\zeta_{w,v}^\nu$ have the form

$$\zeta_{w,v}^\nu(a) = P_{w,v}^\nu(\log a) e^\nu(a)$$
$$a \in A_{\mathbb{R}}$$

where $P_{w,v}^\nu$ is a polynomial. For $g \in G_{\mathbb{R}}$, $g = nak$, the Iwasawa decomposition of g, we have

$$\zeta_{w,v}^\nu(g) = P_{n^{-1} \cdot w, k \cdot v}^\nu(\log a) e^\nu(a).$$

By considering holomorphic continuations $\tilde{\zeta}_{w,v}^\nu$ of $\zeta_{w,v}^\nu$ to G, as possibly multi-valued functions, we obtain that locally

$$v \to \tilde{\zeta}_{w,v}^\nu$$

is an element in $\mathrm{Hom}_{U(\mathfrak{g})}(M, \mathcal{O}_{\mathfrak{b}})$. We denote this locally defined map as $\tilde{\zeta}_w^\nu$. This gives rise to a sheaf of $\mathfrak{g}$-modules, the *asymptotic sheaf* of M, defined on each open subset U by

$$(\mathbb{J}_{as} M)_U = \text{linear span of the maps } \tilde{\zeta}_w \text{ defined on } U.$$

The proof that this is a sheaf of $\mathfrak{g}$-modules is as in [13]. Consider now $f \in J_{\mathfrak{n}_x}^0 M$, $\mathfrak{n}_x$ in the open K-orbit in $\mathscr{B}$ (the usual Jacquet module of M). For any pair (f, v), $v \in M$ one can define an analytic function on $G_{\mathbb{R}}$: first observe that if we assume $A_{\mathbb{R}}N_{\mathbb{R}} \subset B$, the group corresponding to $\mathfrak{b}_x = \mathfrak{b}$, then the action of its Lie algebra exponentiates to $A_{\mathbb{R}}N_{\mathbb{R}}$. We put $g = nak = \tilde{n}(g)\tilde{a}(g)\tilde{k}(g)$, with $\tilde{k}$, $\tilde{a}$, $\tilde{n}$ analytic functions on $G_{\mathbb{R}}$, as in the Iwasawa decomposition. The pair (f, v) gives rise to an analytic function on $G_{\mathbb{R}}$:

$$\sigma_{f,v}(g) = \langle (na)^{-1} f, k \cdot v \rangle$$

(a $\mathfrak{b}$-*finite matrix coefficient*). Note that the simply connected cover of B acts on $J^0_{\mathfrak{n}_x} M = J^0_{\mathfrak{n}} B$, and K acts on M. Hence $\sigma_{f,v}$ extends as a multi-valued function to at least $BK \subset G$. We consider (locally defined) holomorphic continuations $\tilde{\sigma}_{f,v}$ of the functions $\sigma_{f,v}$ to G. By a theorem of Hecht and Schmid [13], the space $J^0_{\mathfrak{n}} M$ is spanned by the linear maps $V \to \zeta_{w,v}(e)$, $e \in G$, the identity. Therefore by (1.2) the sheaf of $\mathfrak{b}$-finite matrix coefficient maps $\tilde{\sigma}_f$, (given by $v \to \tilde{\sigma}_{f,v}$) is spanned by the locally defined maps $\tilde{\zeta}_w$. We obtain

(5.2) **Corollary.** *Let M be in $\mathscr{HC}_{\text{finite}}$. Then the map*

$$\mathbb{J}_{as} M \to \mathbb{J}^0 M$$

is an isomorphism of sheaves of $\mathfrak{g}$-modules.

Proof. This is a consequence of (1.2) and the discussion above. The crux is that for any $y \in \mathscr{B}$, if $f \in J^0_{\mathfrak{n}_y} M \approx Hom_{U(\mathfrak{g})}(M, \mathcal{O}_{\mathfrak{b}})_y$, then f contributes to $J^0_{\mathfrak{n}_x} M$, for x in a neighborhood of y. *Therefore* f contributes to the (usual) Jacquet module of M.

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Fibrés exceptionnels et suite spectrale de Beilinson généralisée sur $\mathbb{P}_2(\mathbb{C})$

J.-M. Drezet

UER de Mathématiques, Université Paris 7, Tour 45–55, 2 Place de Jussieu, F-75221 Paris Cedex 05, France

Introduction

Soient r, c_1, c_2 des entiers, avec $r \geqq 1$, $M(r, c_1, c_2)$ la variété de modules des faisceaux algébriques semi-stables sur $\mathbb{P}_2$, de rang r et de classes de Chern c_1, c_2, M_s l'ouvert des faisceaux stables. On dit que $M(r, c_1, c_2)$ est *exceptionnelle* si $M_s(r, c_1, c_2)$ est de dimension nulle. Dans ce cas, $M(r, c_1, c_2)$ est réduite à un point, qui est une classe d'isomorphisme de fibrés vectoriels homogènes. Les fibrés vectoriels qui sont dans une telle classe sont dits *exceptionnels*. Ils jouent un rôle fondamental dans la détermination des entiers r, c_1, c_2 tels que $M(r, c_1, c_2)$ soit non vide.

Soit E un faisceau cohérent sur $\mathbb{P}_2$. On dit que E est *rigide* si $\mathrm{Ext}^1(E, E) = 0$, et *simple* si ses seuls endomorphismes sont les homothéties. Un fibré exceptionnel est, par exemple, rigide et simple. Réciproquement, on prouvera le

Théorème 1. *Soit E un faisceau cohérent rigide et simple sur $\mathbb{P}_2$. Alors E est un fibré exceptionnel.*

On donnera ensuite une construction des fibrés exceptionnels, car jusqu'à présent seuls les plus triviaux d'entre eux sont connus. Cette construction permettra en outre de généraliser la suite spectrale de Beilinson classique [1, 2, 8].

Si E est un faisceau cohérent de rang $r \geqq 1$, on pose $\mu(E) = c_1(E)/r$, qu'on appelle la *pente* de E. Soit $\mathscr{E}$ l'ensemble des pentes des fibrés exceptionnels. D'après [4], un fibré exceptionnel est entièrement déterminé, à isomorphisme près, par sa pente. Pour tout $\alpha \in \mathscr{E}$, soit E_α un fibré exceptionnel de pente α. On note $\mathscr{D}$ l'ensemble des nombres rationnels diadiques, c'est à dire pouvant se mettre sous la forme $p/2^n$, avec p entier et n entier positif ou nul. On définit dans [4] une bijection croissante

$$\varepsilon:\mathscr{D}\to\mathscr{E} \quad \text{(voir I.3)}.$$

La construction des fibrés exceptionnels est basée sur les théorèmes 2 et 4 qui suivent. Les théorèmes 3 et 5 se déduisent des théorèmes 2 et 4 respectivement par dualité.

Théorème 2. *Soient n, p des entiers, avec $n \geqq 0$, $\alpha=\varepsilon(p/2^n)$, $\beta=\varepsilon((p+1)/2^n)$. Alors le morphisme canonique*

$$\mathrm{ev}: E_\alpha \otimes \mathrm{Hom}(E_\alpha, E_\beta) \to E_\beta$$

est surjectif. Son noyau est le fibré exceptionnel de pente

$$\varepsilon\left(\frac{2p-1}{2^{n+1}}\right) \quad \textit{si } p \textit{ est impair ou} \quad n=0,$$

$$\varepsilon\left(\frac{p-2}{2^n}\right) \quad \textit{si} \quad p\equiv 2 \bmod 4,$$

$$\varepsilon\left(\frac{p+4}{2^n}-3\right) \quad \textit{si} \quad p\equiv 0 \bmod 4.$$

Théorème 3. *Sous les mêmes hypothèses, le morphisme canonique*

$$\mathrm{ev}^*: E_\alpha \to E_\beta \otimes \mathrm{Hom}(E_\alpha, E_\beta)^*$$

est injectif. Son conoyau est le fibré exceptionnel de pente

$$\varepsilon\left(\frac{2p+3}{2^{n+1}}\right) \quad \textit{si } p \textit{ est pair ou} \quad n=0,$$

$$\varepsilon\left(\frac{p+3}{2^n}\right) \quad \textit{si} \quad p\equiv 1 \bmod 4,$$

$$\varepsilon\left(\frac{p-3}{2^n}+3\right) \quad \textit{si} \quad p\equiv 3 \bmod 4.$$

Théorème 4. *Soient n, p des entiers, avec $n \geqq 0$, $\alpha=\varepsilon(p/2^n)$, $\beta=\varepsilon\left(3+\frac{p-1}{2^n}\right)$. Alors le morphisme canonique*

$$\mathrm{ev}: E_\alpha \otimes \mathrm{Hom}(E_\alpha, E_\beta) \to E_\beta$$

est surjectif. Son noyau est le fibré exceptionnel de pente $\varepsilon\left(\frac{4p-1}{2^{n+2}}\right)$.

Théorème 5. *Sous les mêmes hypothèses, le morphisme canonique*

$$\mathrm{ev}^*: E_\alpha \to E_\beta \otimes \mathrm{Hom}(E_\alpha, E_\beta)^*$$

est injectif. Son conoyau est le fibré exceptionnel de pente $\varepsilon\left(3+\frac{4p-3}{2^{n+2}}\right)$.

On déduit de ce qui précède une construction des fibrés exceptionnels, puisque pour tout entier n on a $\varepsilon(n)=n$ et $E_n \simeq \mathcal{O}(n)$.

On obtient un certain nombre de résultats intéressants au cours de la démonstration des théorèmes 2, 3, 4, 5.

Théorème 6. *Soient E, F des fibrés exceptionnels tels que $\mu(E)<\mu(F)$. Alors*

1 – On a $\operatorname{Ext}^i(E,F)=0$ pour $i>0$.

2 – Pour tout point x de $\mathbb{P}_2$, et tout sous-espace vectoriel non nul H de E_x, si

$$K = \sum_{f\in \operatorname{Hom}(E,F)} f_x(H),$$

on a $\dfrac{\dim(H)}{\operatorname{rg}(E)} \leqq \dfrac{\dim(K)}{\operatorname{rg}(F)}$, *l'inégalité étant stricte si $H \neq E_x$.*

On déduit de 2 que le morphisme canonique $\mathrm{ev}: E\otimes \operatorname{Hom}(E,F)\to F$ est surjectif. D'autre part, la condition 2 signifie que l'application

$$\mathrm{ev}_x: E_x\otimes \operatorname{Hom}(E,F)\to F_x$$

est stable sous l'action du groupe $SL(E_x)\times SL(F_x)$ (cf. [3]).

On déduit du théorème 6 un amélioration du théorème 1:

Corollaire 7. *Soit E un faisceau cohérent rigide sur $\mathbb{P}_2$. Alors E est une somme directe de fibrés exceptionnels.*

On verra que cependant la réciproque est fausse.

On définit ensuite la suite spectrale de Beilinson généralisée. Soient p, n des entiers, avec $n \geqq 0$, $\alpha=\varepsilon(p/2^n)$, $\beta=\varepsilon((p+1)/2^n)$. On définit dans [4] un nouvel élément de $\mathscr{E}$, noté $\alpha\cdot\beta$. On considère des triplets de rationnels d'un des types suivants:

$$(\alpha, \alpha\cdot\beta, \beta), (\beta-3, \alpha, \alpha\cdot\beta), (\alpha\cdot\beta, \beta, \alpha+3).$$

A de tels triplets sont associés des triplets de fibrés exceptionnels, appelés *triades*. On conviendra d'appeler aussi triades des triplets du type $(\mathcal{O}(k), \mathcal{O}(k+1), \mathcal{O}(k+2))$. Soit $\mathscr{T}=(E,G,F)$ une triade, M le conoyau du morphisme canonique

$$G\to F\otimes \operatorname{Hom}(G,F)^*,$$

qui est un fibré exceptionnel. On a alors des isomorphismes canoniques

$$\operatorname{Hom}(E,G)^* \simeq \operatorname{Hom}(M, E(3)),$$

$$\operatorname{Hom}(G,F)^* \simeq \operatorname{Hom}(F,M).$$

On note p_1, p_2 les projections $\mathbb{P}_2\times\mathbb{P}_2\to\mathbb{P}_2$, et si E, F sont des faisceaux cohérents sur $\mathbb{P}_2$, on pose

$$E\boxtimes F = p_1^* E\otimes p_2^* F.$$

Des isomorphismes précédents on déduit la suite de morphismes de fibrés vectoriels sur $\mathbb{P}_2 \times \mathbb{P}_2$:

$$\begin{array}{ccccccccc} R^{\cdot}_{\mathscr{F}}: 0 \to & E \boxtimes E^*(-3) & \to & G \boxtimes M^* & \to & F \boxtimes F^* & \to 0 \\ & \| & & \| & & \| & \\ & R^{-2}_{\mathscr{F}} & & R^{-1}_{\mathscr{F}} & & R^{0}_{\mathscr{F}}. & \end{array}$$

Soit Δ la diagonale de $\mathbb{P}_2 \times \mathbb{P}_2$, $\mathcal{O}_\Delta$ le faisceau structural de Δ, vu comme quotient de $\mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}$. On a alors le

Théorème 8. *La suite $R^{\cdot}_{\mathscr{F}}$ est un complexe, et c'est une résolution de $\mathcal{O}_\Delta$, via le morphisme canonique*

$$\mathrm{trace}_\Delta : F \boxtimes F^* \to \mathcal{O}_\Delta .$$

On peut en déduire toute une série de résultats, dont voici un exemple: si $\mathscr{F}$ est un faisceau cohérent sur $\mathbb{P}_2$, on déduit de $R^{\cdot}_{\mathscr{F}}$ un complexe

$$E \otimes \mathrm{Ext}^1(\mathscr{F}, E)^* \xrightarrow{A_{\mathscr{F}}} G \otimes \mathrm{Ext}^1(M, \mathscr{F}) \xrightarrow{B_{\mathscr{F}}} F \otimes \mathrm{Ext}^1(F, \mathscr{F}),$$

et on a le

Corollaire 9. *Si* $\mathrm{Hom}(F, \mathscr{F}) = \mathrm{Hom}(\mathscr{F}, E) = 0$, *le morphisme de faisceaux $A_{\mathscr{F}}$ est injectif, $B_{\mathscr{F}}$ est surjectif, et* $\mathrm{Ker}(B_{\mathscr{F}})/\mathrm{Im}(A_{\mathscr{F}})$ *est isomorphe à $\mathscr{F}$.*

On utilisera ce résultat dans [3]. On en déduira une description de certaines variétés de faisceaux semi-stables sur $\mathbb{P}_2$, permettant d'en étudier les propriétés et d'en calculer le groupe de Picard.

I. Préliminaires

1. Formulaire général

Si E est un faisceau cohérent sur $\mathbb{P}_2$, de rang $r > 0$, on pose

$$\Delta(E) = \frac{1}{r}\left(c_2(E) - \left(1 - \frac{1}{r}\right) c_1(E)^2/2\right),$$

qu'on nomme le *discriminant* de E. On pose

$$P(X) = \frac{X^2}{2} + \frac{3X}{2} + 1 .$$

Si E, F sont des faisceaux cohérents sur $\mathbb{P}_2$, on pose

$$\chi(E, F) = \sum_{0 \leq i \leq 2} (-1)^i \dim(\mathrm{Ext}^i(E, F)) .$$

Si $\mathrm{rg}(E) > 0$ et $\mathrm{rg}(F) > 0$, on a d'après la proposition (1.1) de [4]

$$\chi(E, F) = \mathrm{rg}(E)\,\mathrm{rg}(F)(P(\mu(F) - \mu(E)) - \Delta(E) - \Delta(F)) .$$

En fait, la démonstration de ce résultat conduit au suivant: en ne supposant plus nécessairement $\mathrm{rg}(E)$ strictement positif, on a

$$\chi(E,E)=r^2+(r-1)c_1^2-2rc_2,$$

avec $r=\mathrm{rg}(E)$, $c_i=c_i(E)$ pour $i=1,2$.

On note Q le fibré quotient canonique de $\mathcal{O}\otimes H^0(\mathcal{O}(1))^*$ sur $\mathbb{P}_2$.

2. *Faisceaux semi-stables*

Soit E un faisceau cohérent sur $\mathbb{P}_2$. On dit que E est *semi-stable* (resp. *stable*) s'il est sans torsion et si pour tout sous-faisceau propre F de E on a

$$\frac{c_1(F)}{\mathrm{rg}(F)}\leqq\frac{c_1(E)}{\mathrm{rg}(E)},$$

et en cas d'égalité

$$\frac{\chi(F)}{\mathrm{rg}(F)}\leqq\frac{\chi(E)}{\mathrm{rg}(E)}\quad(\text{resp. }<).$$

Cette notion de stabilité est celle de Gieseker [5], reprise par Maruyama [7].

Soit E un faisceau cohérent sans torsion sur $\mathbb{P}_2$. Il existe une unique filtration de E par des sous-faisceaux cohérents

$$0=F_0\subset F_1\subset\ldots\subset F_m=E$$

telle que pour $i=1,\ldots,m$, F_i/F_{i-1} soit l'unique sous-faisceau $\mathcal{G}$ de E/F_{i-1} tel que pour tout sous-faisceau $\mathcal{H}$ de E/F_{i-1} on ait $\mu(\mathcal{H})\leqq\mu(\mathcal{G})$, que si $\mu(\mathcal{H})=\mu(\mathcal{G})$ on ait $\Delta(\mathcal{H})\geqq\Delta(\mathcal{G})$, et que si $\mu(\mathcal{H})=\mu(\mathcal{G})$ et $\Delta(\mathcal{H})=\Delta(\mathcal{G})$, on ait $\mathrm{rg}(\mathcal{H})\leqq\mathrm{rg}(\mathcal{G})$. On l'appelle la *filtration de Harder-Narasimhan* de E.

3. *Fibrés exceptionnels*

Les résultats qui suivent sont démontrés dans [4].

On donne d'abord une définition de l'application $\varepsilon:\mathcal{D}\to\mathcal{E}$ évoquée dans l'introduction.

Soit α un nombre rationnel. On note r_α le plus petit entier $r>0$ tel que r_α soit un entier, et Δ_α le nombre rationnel

$$\frac{1}{2}\left(1-\frac{1}{r_\alpha^2}\right).$$

Soit (α,β) un couple de nombres rationnels tels que $3+\alpha-\beta\neq0$. On pose

$$\alpha\cdot\beta=\frac{\alpha+\beta}{2}+\frac{\Delta_\beta-\Delta_\alpha}{3+\alpha-\beta}.$$

L'application ε est définie de manière unique par les deux relations

1) $\varepsilon(n)=n$ pour tout entier n,
2) $\varepsilon\left(\frac{2p+1}{2^{q+1}}\right)=\varepsilon\left(\frac{p}{2^q}\right)\cdot\varepsilon\left(\frac{p+1}{2^q}\right)$ pour tous entiers p,q avec $q\geqq0$.

On dit qu'un couple (α, β) d'éléments de $\mathscr{E}$ est *admissible* si

$$0 \leqq \beta - \alpha \leqq 2,$$

$$P(\alpha - \beta) = \Delta_\alpha + \Delta_\beta \quad (P \text{ étant le polynôme défini en } 1\text{–}).$$

(Cette définition diffère légèrement de celle de [4]). Pour que (α, β) soit admissible, il suffit que $\beta - \alpha$ soit de la forme $1/2^n$, avec n entier, $n \geqq -1$. Si (α, β) est un couple admissible, on a

$$\mathrm{Ext}^q(E_\beta, E_\alpha) = 0 \quad \text{pour} \quad q \geqq 0,$$

$$\mathrm{Ext}^2(E_\alpha, E_\beta) = 0.$$

Pour tout élément α de $\mathscr{E}$, on a

$$\mathrm{rg}(E_\alpha) = r_\alpha, \qquad \Delta(E_\alpha) = \Delta_\alpha.$$

4. Formulaire concernant les fibrés exceptionnels

Soit (α, β) un couple admissible d'éléments de $\mathscr{E}$. Soit $\gamma = \alpha \cdot \beta$. Alors les couples (α, γ) et (γ, β) sont admissibles (proposition (5.1) de [4]). Posons $\alpha' = \alpha \cdot \gamma$, $\beta' = \gamma \cdot \beta$. On a alors les égalités suivantes:

Caractéristique d'Euler-Poincaré

(1) $\chi(E_\alpha, E_\gamma) = 3r_\beta$,
(2) $\chi(E_\gamma, E_\beta) = 3r_\alpha$,
(3) $\chi(E_\alpha, E_{\alpha'}) = 3r_\gamma$,
(4) $\chi(E_\gamma, E_{\beta'}) = 3r_\beta$,
(5) $\chi(E_\beta, E_\alpha(3)) = 3r_\gamma$.

Rang

(6) $r_{\alpha'} + r_\beta = 3r_\alpha r_\gamma$,
(7) $r_{\beta'} + r_\alpha = 3r_\beta r_\gamma$.

Pentes

(8) $3\gamma r_\alpha r_\gamma = \alpha' r_{\alpha'} + \beta r_\beta$,
(9) $3\alpha r_\gamma r_\alpha = \alpha' r_{\alpha'} + (\beta - 3) r_\beta$,
(10) $3\gamma r_\gamma r_\beta = \alpha r_\alpha + \beta' r_{\beta'}$,
(11) $3\beta r_\gamma r_\beta = \beta' r_{\beta'} + (3 + \alpha) r_\alpha$.

Il est immédiat que (2), (3), (4) découlent de (1). Démontrons (1). On a $\chi(E_\gamma, E_\alpha) = 0$, d'où on déduit d'après 1 –

$$\chi(E_\alpha, E_\gamma) = 3r_\alpha r_\gamma(\gamma - \alpha).$$

D'après la démonstration du lemme (5.4) de [4], on a

$$\gamma - \alpha = \frac{1}{r_\alpha^2(3 + \alpha - \beta)},$$

et d'après la proposition (5.1) de [4],

$$r_\gamma = r_\alpha r_\beta (3+\alpha-\beta),$$

d'où on déduit $\chi(E_\alpha, E_\gamma) = 3r_\beta$, c'est à dire (1).

Démontrons (5). On a $\chi(E_\beta, E_\alpha) = 0$, donc par dualité de Serre, $\chi(E_\alpha(3), E_\beta) = 0$, d'où on déduit avec 1 –

$$\chi(E_\alpha, E_\gamma) = 3r_\alpha r_\beta (3+\alpha-\beta) = 3r_\gamma$$

d'après la proposition (5.1) de [4].

Démontrons (6). On a, puisque $\chi(E_\beta, E_\alpha) = 0$,

$$\chi(E_\alpha, E_\beta) = 3r_\alpha r_\beta (\beta-\alpha) = 9r_\alpha r_\beta - 3r_\gamma$$

d'après ce qui précède, et en remplaçant (α, β, γ) par $(\alpha, \gamma, \alpha')$, on obtient

$$\chi(E_\alpha, E_\gamma) = 9r_\alpha r_\gamma - 3r_{\alpha'},$$

et comme $\chi(E_\alpha, E_\gamma) = 3r_\beta$ d'après (1), on obtient $3r_\alpha r_\gamma = r_{\alpha'} + r_\beta$, c'est à dire (6).

On obtient (7) de la même façon, en remplaçant (α, β, γ) par (γ, β, β').

Les formules (8) et (10) (resp. (9) et (11)) sont équivalentes par dualité.

Démontrons (8). D'après (2), avec $(\alpha, \gamma, \alpha')$ à la place de (α, β, γ), on a

$$\chi(E_{\alpha'}, E_\gamma) = 3r_\alpha .$$

Mais $\chi(E_{\alpha'}, E_\gamma) = 3r_{\alpha'} r_\gamma (\gamma - \alpha')$, d'où

$$r_{\alpha'}(\alpha' - \gamma) = -\frac{r_\alpha}{r_\gamma}.$$

De même, d'après (2) on a

$$r_\beta(\beta-\gamma) = \frac{r_\alpha}{r_\gamma},$$

et en additionnant ces égalités on a

$$\alpha' r_{\alpha'} + \beta r_\beta - 3\gamma r_\alpha r_\gamma = 0,$$

c'est à dire (8).

Démontrons (11). On a d'après ce qui précède

$$r_{\beta'}(\beta' - \beta) = -\frac{r_\gamma}{r_\beta},$$

$$r_\alpha(\alpha + 3 - \beta) = \frac{r_\gamma}{r_\beta},$$

et par addition on obtient, compte tenu de (7),

$$\beta' r_{\beta'} + (3+\alpha) r_\alpha - 3\beta r_\beta r_\gamma = 0,$$

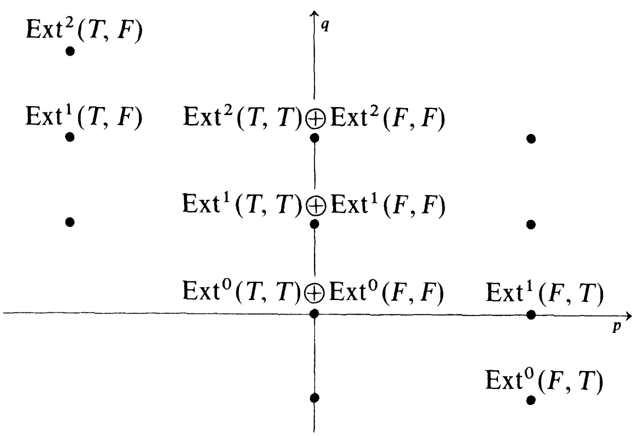
c'est à dire (11).

II. Caractérisation des fibrés exceptionnels

On démontre ici le théorème 1, c'est à dire qu'un faisceau cohérent E sur $\mathbb{P}_2$, rigide et simple est un fibré exceptionnel.

Lemme 10. *Soit E un faisceau cohérent rigide sur $\mathbb{P}_2$. Alors E est sans torsion.*

Soit T le sous-faisceau de torsion de E. A la filtration $T \subset E$ de E est associée une suite spectrale $E_r^{p,q}$ convergeant vers $\operatorname{Ext}^{p+q}(E, E)$. On va décrire les termes de niveau E_1 pouvant éventuellement être non nuls. Posons $F = E/T$. Alors on a $\operatorname{Hom}(T, F) = 0$ car F est sans torsion, et $\operatorname{Ext}^2(F, T) = 0$ car par dualité de Serre, on a $\operatorname{Ext}^2(F, T) \simeq \operatorname{Hom}(T, F(-3))^*$, qui est nul car $F(-3)$ est sans torsion (pour cette version de la dualité de Serre, cf. proposition (1.2) de [4]). Les termes $E_1^{p,q}$ éventuellement non nuls sont figurés ci-dessous



Toutes les flèches d_r^{01} et $d_r^{-r,r}$ sont nulles, donc $E_\infty^{0,1} \simeq E_1^{0,1}$. Mais $E_\infty^{0,1} = 0$, puisque $\operatorname{Ext}^1(E, E) = 0$. Donc $E_1^{0,1} = 0$, c'est à dire

$$\operatorname{Ext}^1(T, T) \oplus \operatorname{Ext}^1(F, F) = 0.$$

Il reste à montrer que si $T \neq 0$, on a $\operatorname{Ext}^1(T, T) \neq 0$. Le rang de T est nul, donc on a d'après I.1, $\chi(T, T) = -c_1(T)^2 \leqq 0$. Puisque $T \neq 0$, $\operatorname{End}(T) \neq 0$ (il contient les homothéties), donc $\dim(\operatorname{Ext}^1(T, T)) > 0$.

Ceci démontre le lemme 10.

Soit E un faisceau cohérent simple et rigide. Il est sans torsion d'après le lemme 10. On considère la filtration de Harder-Narasimhan de E

$$0 = F_0 \subset F_1 \subset \ldots \subset F_m = E.$$

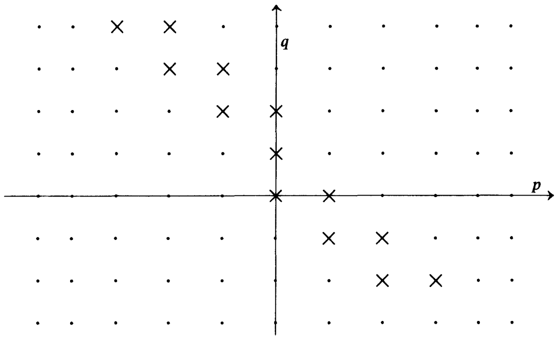
Les gradués $Gr_i = F_i/F_{i-1}$ sont des faisceaux semi-stables. On pose $\mu_i = \mu(Gr_i)$ et $\Delta_i = \Delta(Gr_i)$. Alors on a $\mu_1 \geqq \ldots \geqq \mu_m$, et $\Delta_i > \Delta_{i-1}$ si $\mu_i = \mu_{i-1}$. Considérons la suite spectrale de cette filtration de terme E_1

$$E_1^{p,q} = \bigoplus_{1 \leqq i \leqq m} \operatorname{Ext}^{p+q}(Gr_i, Gr_{i-p})$$

(où $Gr_j = 0$ si $j \neq 1, \ldots, m$), convergeant vers $\mathrm{Ext}^{p+q}(E, E)$. On a, si $p<0$, $\mathrm{Hom}(Gr_i, Gr_{i-p}) = 0$ pour tout i, donc $E_1^{p,p} = 0$. Si $p \geqq 0$, on a

$$\mathrm{Ext}^2(Gr_i, Gr_{i-p}) \simeq \mathrm{Hom}(Gr_{i-p}, Gr_i(-3))^* = 0,$$

car si Gr_i et Gr_{i-p} sont non nuls, ils sont semi-stables et $\mu(Gr_{i-p}) > \mu(Gr_i(-3))$. Les termes $E_1^{p,q}$ éventuellement non nuls sont figurés ci-dessous



Toutes les flèches $d_r^{0,1}$ et $d_r^{-r,r}$ sont nulles, donc $E_1^{0,1} = E_\infty^{0,1} = 0$, car c'est un terme du gradué d'une filtration de $\mathrm{Ext}^1(E, E)$, qui est nul. Donc

$$\bigoplus_{1 \leqq i \leqq m} \mathrm{Ext}^1(Gr_i, Gr_i) = 0,$$

ce qui prouve que Gr_i est de la forme $\mathbb{C}^{n_i} \otimes E_{\alpha_i}$, avec α_i dans $\mathscr{E}$ (proposition (4.4) de [4]).

Montrons que $\mathrm{Ext}^2(E, E) = 0$. On a $\mathrm{Ext}^2(E, E) \simeq \mathrm{Hom}(E, E(-3))^*$. Soit $f: E \to E(-3)$ un morphisme, s un morphisme non nul $\mathcal{O}(-3) \to \mathcal{O}$. On en déduit $I_E \otimes s: E(-3) \to E$, et $(I_E \otimes s) \circ f$ est un endomorphisme de E s'annulant sur le lieu des zéros de s, donc partout, puisque $\mathrm{End}(E)$ est réduit aux homothéties, donc $f = 0$. Donc $\mathrm{Ext}^2(E, E) = 0$.

On définit une sous-suite spectrale $'E_r^{p,q}$ de $E_r^{p,q}$ en posant

$$\begin{aligned} 'E_r^{p,q} &= E_r^{p,q} \quad \text{si} \quad p<0, \\ &= 0 \qquad \text{sinon}, \end{aligned}$$

les morphismes $'d_r^{p,q}$ entre les termes non nuls étant identiques aux $d_r^{p,q}$. Puisque $\mathrm{Ext}^2(E, E) = 0$, cette suite est d'aboutissement nul. On en déduit que

$$\sum_{p,q} (-1)^{p+q} \dim('E_1^{p,q}) = 0,$$

d'où

$$\sum_{i<j} \chi(Gr_i, Gr_j) = 0.$$

D'après I.1, on a $\chi(Gr_i, Gr_j) = r_i r_j (P(\mu_j - \mu_i) - \Delta_i - \Delta_j)$, r_i désignant le rang de Gr_i, et par conséquent

$$\sum_{i \neq j} \chi(Gr_i, Gr_j) = \sum_{i<j} 3 r_i r_j (\mu_i - \mu_j) \geqq 0.$$

On a
$$\begin{aligned}\chi(E,E)&=1\\&=\sum_{i,j}\chi(Gr_i,Gr_j)\\&=\sum_i\chi(Gr_i,Gr_j)+\sum_{i<j}3r_ir_j(\mu_i-\mu_j),\end{aligned}$$
et puisque $\chi(Gr_i,Gr_j)=n_i^2\geqq 1$, on a $n_1^2+\ldots+n_m^2=1$, d'où on déduit que $m=1$ et $n_1=1$, c'est à dire que $E=E_{\alpha_1}$. Donc E est exceptionnel.

Ceci démontre le théorème 1.

III. Construction des fibrés exceptionnels

On démontre ici les théorèmes 2 à 6, ainsi que le corollaire 7.

Si α,β sont des éléments de $\mathscr{E}$ et x un point de $\mathbb{P}_2$, on note $\mathrm{ev}_{\alpha,\beta,x}$ le morphisme canonique $E_{\alpha x}\otimes\mathrm{Hom}(E_\alpha,E_\beta)\to E_{\beta x}$, et on dira pour simplifier qu'il est *stable* s'il vérifie la condition 2 – du théorème 6, c'est à dire si pour tout sous-espace vectoriel non nul H de $E_{\alpha x}$, si
$$K=\sum_{f\in\mathrm{Hom}(E_\alpha,E_\beta)}f_x(H),$$
on a $\dfrac{\dim(H)}{\mathrm{rg}(E_\alpha)}\leqq\dfrac{\dim(K)}{\mathrm{rg}(E_\beta)}$, l'inégalité étant stricte si $H\neq E_{\alpha x}$.

Soient α,β des éléments de $\mathscr{E}$, avec $\alpha<\beta$. Alors on a $\mathrm{Ext}^2(E_\alpha,E_\beta)=0$, car $\mathrm{Ext}^2(E_\alpha,E_\beta)\simeq\mathrm{Hom}(E_\beta,E_\alpha(-3))^*$, et ce dernier est nul car $\beta>\alpha-3$. Par conséquent le théorème 6 équivaut à l'assertion suivante: on a

1 – $\mathrm{Ext}^1(E_\alpha,E_\beta)=0$,

2 – L'application $\mathrm{ev}_{\alpha,\beta,x}$ est stable.

Notons C_n l'assertion suivante: si α,β sont des éléments de $\mathscr{E}$ qui peuvent s'écrire $\alpha=\varepsilon(p/2^n)$, $\beta=\varepsilon(q/2^n)$, avec p,q,n entiers, $n\geqq 0$ et $p<q$, alors les conditions 1 – et 2 – sont satisfaites.

Le théorème 3 (resp. 5) se déduit du théorème 2 (resp. 4) par dualité. Il suffit donc de démontrer les théorèmes 2 et 4.

Notons A_n l'assertion suivante: soient m,p des entiers, avec $0\leqq m\leqq n$, $\alpha=\varepsilon(p/2^m)$, $\beta=\varepsilon((p+1)/2^m)$. Alors le morphisme canonique $\mathrm{ev}:E_\alpha\otimes\mathrm{Hom}(E_\alpha,E_\beta)\to E_\beta$ est surjectif. Son noyau est le fibré exceptionnel de pente $\varepsilon\left(\dfrac{2p-1}{2^{m+1}}\right)$ si p est impair ou $n=0$, $\varepsilon\left(\dfrac{p-2}{2^m}\right)$ si $p\equiv 2\bmod 4$, $\varepsilon\left(-3+\dfrac{p+4}{2^m}\right)$ si $p\equiv 0\bmod 4$.

On définit de même l'assertion B_n à l'aide de l'énoncé du théorème 4. Il faut montrer que pour tout n, A_n, B_n et C_n sont vraies. Pour cela on utilisera un procédé de récurrence. Plus précisément on prouvera

(i) $C_n \Rightarrow A_n$ et B_n,

(ii) A_{n-1}, B_{n-2} et $C_{n-1} \Rightarrow C_n$,

(iii) C_0 et C_1 sont vraies.

1. Démonstration de (i)

Supposons C_n vraie et démontrons A_n et B_n. Soit p un entier et $\alpha=\varepsilon\left(\dfrac{p}{2^n}\right)$. $\beta=\varepsilon\left(\dfrac{p+1}{2^n}\right)$ $\left(\text{resp. }\beta=\varepsilon\left(3+\dfrac{p-1}{2^n}\right)\right)$. D'après C_n l'application linéaire
$$\mathrm{ev}_x=\mathrm{ev}_{\alpha,\beta,x}:E_{\alpha x}\otimes\mathrm{Hom}(E_\alpha,E_\beta)\to E_\beta$$

est stable pour tout point x de $\mathbb{P}_2$, donc on a

$$\frac{\dim(\mathrm{Im}(\mathrm{ev}_x))}{\mathrm{rg}(E_\beta)} \geqq \frac{\dim(E_{\alpha x})}{\mathrm{rg}(E_\alpha)} = 1,$$

donc ev_x est surjective. Soit N le noyau de ev. Montrons que N est simple et rigide. Pour cela considérons le complexe

$$\begin{array}{ccccccc} D^{\cdot}: 0 \longrightarrow & E_\alpha \otimes \mathrm{Hom}(E_\alpha, E_\beta) & \xrightarrow{\mathrm{ev}} & E_\beta & \longrightarrow 0 \\ & \| & & \| & \\ & D^0 & & D^1. & \end{array}$$

Au complexe $Hom^{\cdot}(D^{\cdot}, D^{\cdot})$ est associée une suite spectrale $E_r^{p,q}$ d'aboutissement $H^{p+q}(\mathbb{P}_2, Hom(N,N))$, et de terme E_2

$$E_2^{p,q} = H^p(H^q(\mathbb{P}_2, Hom^{\cdot}(D^{\cdot}, D^{\cdot}))).$$

On a $\mathrm{Ext}^i(E_\alpha, E_\beta) = 0$ pour $i > 0$ d'après C_n. On a aussi $\mathrm{Ext}^i(E_\beta, E_\alpha) = 0$ pour $i > 0$ (cela découle du fait que (α, β) ou $(\beta - 3, \alpha)$ est admissible). Donc $E_2^{p,q} = 0$ pour $q > 0$, et la suite spectrale est dégénérée. Il en découle qu'on a

$$\mathrm{End}(N) \simeq E_2^{0,0} \quad \text{et} \quad \mathrm{Ext}^1(N,N) \simeq E_2^{1,0}.$$

Ecrivons maintenant le complexe $C^{\cdot} = H^0(\mathbb{P}_2, Hom^{\cdot}(D^{\cdot}, D^{\cdot}))$:

$$\begin{array}{ccccc} 0 \longrightarrow & \mathbb{C} \oplus \mathrm{End}(H) & \xrightarrow{d} & \mathrm{Hom}(H \otimes E_\alpha, E_\beta) & \longrightarrow 0 \\ & \| & & \| & \\ & C^0 & & C^1. & \end{array}$$

On a $\mathrm{Hom}(H \otimes E_\alpha, E_\beta) = \mathrm{End}(H)$, et d est définie par $d(t,f) = tI_H - f$, pour tout $t \in \mathbb{C}$ et $f \in \mathrm{End}(H)$. Donc $E_2^{0,0} \simeq \mathbb{C}$, et $E_2^{1,0} = 0$, ce qui prouve que N est rigide et simple.

D'après le théorème 1, N est un fibré exceptionnel. Pour achever la démonstration de A_n et B_n, il suffit d'en trouver la pente. Posons $N = E_\lambda$, avec $\lambda \in \mathscr{E}$. On va utiliser les égalités (1) à (11) de I.4.

Supposons que $\beta = \varepsilon\left(\frac{p+1}{2^n}\right)$ et que p est impair. D'après (2) on a $\chi(E_\alpha, E_\beta) = 3r_\mu$, avec $\mu = \varepsilon\left(\frac{p-1}{2^n}\right)$, et puisque $\mathrm{Ext}^i(E_\alpha, E_\beta) = 0$ pour $i > 0$ d'après C_n, on a

$$\dim(\mathrm{Hom}(E_\alpha, E_\beta)) = 3r_\mu,$$

d'où

$$\lambda = \frac{3r_\mu \alpha r_\alpha - \beta r_\beta}{3r_\mu r_\alpha - r_\beta}.$$

Posons $\alpha' = \varepsilon\left(\frac{2p-1}{2^{n+1}}\right)$. D'après (6) et (8), on a

$$\alpha' = \frac{3r_\mu \alpha r_\alpha - \beta r_\beta}{3r_\mu r_\alpha - r_\beta},$$

donc $\lambda = \alpha' = \varepsilon\left(\frac{2p-1}{2^{n+1}}\right)$, On obtient la même formule avec $n = 0$, p quelconque.

Si $p\equiv 2 \bmod 4$ et $\beta=\varepsilon\left(\frac{p+1}{2^n}\right)$, on trouve $\lambda=\varepsilon\left(\frac{p-2}{2^n}\right)$ à l'aide de (4), (7), (10), et si $p\equiv 0 \bmod 4$, on trouve $\lambda=\varepsilon\left(-3+\frac{p+4}{2^n}\right)$ en utilisant (3), (6), (9).

Si $\beta=\varepsilon\left(3+\frac{p-1}{2^n}\right)$, on trouve $\lambda=\varepsilon\left(\frac{4p-1}{2^n}\right)$ en utilisant (5), (7) et (11).

Ceci montre que A_n et B_n sont vraies et prouve (i).

2. Démonstration de (ii)

Supposons A_{n-1}, B_{n-2} et C_{n-1} vraies. Soient α,β des éléments de $\mathscr{E}$, avec $\alpha<\beta$, $\alpha=\varepsilon\left(\frac{p}{2^n}\right)$, $\beta=\varepsilon\left(\frac{q}{2^n}\right)$. Il faut montrer que $\operatorname{Ext}^1(E_\alpha, E_\beta)=0$ et que le morphisme canonique

$$\mathrm{ev}_{\alpha,\beta,x}: E_{\alpha x}\otimes \operatorname{Hom}(E_\alpha, E_\beta)\to E_{\beta x}$$

est stable. On peut supposer que l'un des entiers p,q est impair. Par dualité, les deux cas où un des entiers p,q est pair sont équivalents. Il y a donc en tout deux cas: celui où p est impair et q pair, et celui où p et q sont impairs. Posons $p=2k-1$.

Lemme 11. *Soit (λ,μ) un couple d'éléments de $\mathscr{E}$. On suppose que E_α est le noyau du morphisme canonique*

$$\mathrm{ev}: E_\lambda\otimes \operatorname{Hom}(E_\lambda, E_\mu)\to E_\mu .$$

Alors, si $\mathrm{ev}_{\lambda,\mu,x}$ et $\mathrm{ev}_{\lambda,\beta,x}$ sont stables, il en est de même de $\mathrm{ev}_{\alpha,\beta,x}$.

Soient E', F' des sous-espaces vectoriels de $E_{\alpha x}, E_{\beta x}$ respectivement, avec $F'\neq E_{\beta x}$, $E'\neq 0$, tels que

$$\mathrm{ev}_{\alpha,\beta,x}(E'\otimes \operatorname{Hom}(E_\alpha, E_\beta))\subset F' .$$

On a un diagramme commutatif

$$\begin{tikzcd}
E_{\lambda x}\otimes \operatorname{Hom}(E_\lambda, E_\beta) \arrow[r, "\mathrm{ev}_{\lambda,\beta,x}"] & E_{\beta x} \\
E_{\alpha x}\otimes \operatorname{Hom}(E_\alpha, E_\lambda)\otimes \operatorname{Hom}(E_\lambda, E_\beta) \arrow[u, "\mathrm{ev}_{\alpha,\lambda,x}\otimes I_{\operatorname{Hom}(E_\lambda, E_\beta)}"] \arrow[r, "I_{E_{\alpha x}}\otimes\tau"] & E_{\alpha x}\otimes \operatorname{Hom}(E_\alpha, E_\beta) \arrow[u, "\mathrm{ev}_{\alpha,\beta,x}"']
\end{tikzcd},$$

τ étant le morphisme canonique. Donc si $G'=\mathrm{ev}_{\alpha,\lambda,x}(E'\otimes \operatorname{Hom}(E_\alpha, E_\lambda))$, on a

$$\mathrm{ev}_{\lambda,\beta,x}(G'\otimes \operatorname{Hom}(E_\lambda, E_\beta))\subset F' .$$

Supposons que $\mathrm{ev}_{\alpha,\lambda,x}$ soit stable. Alors on a $G'\neq 0$, donc, $\mathrm{ev}_{\lambda,\beta,x}$ étant stable, on a

$$\frac{\dim(G')}{r_\lambda}<\frac{\dim(F')}{r_\beta}.$$

D'autre part, $\mathrm{ev}_{\alpha,\lambda,x}$ étant stable, on a

$$\frac{\dim(E')}{r_\alpha}\leqq\frac{\dim(G')}{r_\lambda},$$

d'où finalement

$$\frac{\dim(E')}{r_\alpha} < \frac{\dim(F')}{r_\beta},$$

ce qui prouve que $\mathrm{ev}_{\alpha,\beta,x}$ est stable.

Il reste à prouver que $\mathrm{ev}_{\alpha,\lambda,x}$ est stable. Soient E', G' des sous-espaces vectoriels de $E_{\alpha x}$, $E_{\lambda x}$ respectivement, avec $E' \neq 0$, $G' \neq E_{\lambda x}$, tels que

$$\mathrm{ev}_{\alpha,\lambda,x}(E' \otimes \mathrm{Hom}(E_\alpha, E_\lambda)) \subset G'.$$

On a une injection canonique $\mathrm{Hom}(E_\lambda, E_\mu)^* \subset \mathrm{Hom}(E_\alpha, E_\lambda)$, donc si

$$H' = \mathrm{ev}_{\lambda,\mu,x}(G' \otimes \mathrm{Hom}(E_\lambda, E_\mu)),$$

on a

$$E' \subset \mathrm{Ker}(G' \otimes \mathrm{Hom}(E_\lambda, E_\mu) \to H').$$

On en déduit qu'avec $m = \dim(\mathrm{Hom}(E_\lambda, E_\mu))$, on a

$$\dim(H') \leqq m \cdot \dim(G') - \dim(E').$$

Puisque $E_\alpha \subset E_\lambda \otimes \mathrm{Hom}(E_\alpha, E_\lambda)^*$, on ne peut pas avoir $G' = 0$, et comme $G' \neq E_{\lambda x}$, on a

$$\frac{\dim(G')}{r_\lambda} < \frac{\dim(H')}{r_\mu} \leqq \frac{m \cdot \dim(G') - \dim(E')}{m r_\lambda - r_\alpha}$$

(car $\mathrm{ev}_{\lambda,\mu,x}$ est stable). On a donc

$$\frac{\dim(E')}{r_\alpha} < \frac{\dim(G')}{r_\lambda},$$

ce qui prouve la stabilité de $\mathrm{ev}_{\alpha,\lambda,x}$.

Ceci achève la démonstration du lemme 11.

Démontrons maintenant (ii). Supposons que q est pair. On a alors une suite exacte

$$0 \to E_\alpha \to E_\lambda \otimes \mathrm{Hom}(E_\lambda, E_\mu) \to E_\mu \to 0,$$

avec

$$\lambda = \varepsilon\left(\frac{k}{2^{n-1}}\right), \quad \mu = \varepsilon\left(\frac{k+1}{2^{n-1}}\right) \qquad \text{si} \quad p \equiv 1 \bmod 4,$$

$$\lambda = \varepsilon\left(\frac{k+1}{2^{n-1}}\right), \quad \mu = \varepsilon\left(3 + \frac{k-1}{2^{n-1}}\right) \quad \text{si} \quad p \equiv 3 \bmod 4.$$

Si $p \equiv 1 \bmod 4$, c'est une application de A_{n-1}, et sinon de B_{n-2}.

D'après C_{n-1}, $\mathrm{ev}_{\lambda,\beta,x}$ et $\mathrm{ev}_{\lambda,\mu,x}$ sont stables. Il découle du lemme 11 que $\mathrm{ev}_{\alpha,\beta,x}$ l'est. Il reste à montrer que $\mathrm{Ext}^1(E_\alpha, E_\beta) = 0$. C'est une conséquence des égalités

$$\mathrm{Ext}^1(E_\lambda, E_\beta) = 0,$$

$$\mathrm{Ext}^2(E_\mu, E_\beta) = 0.$$

La première découle du fait que $\lambda \leqq \beta$ et que C_{n-1} est vraie. La seconde vient de ce que $\mu \leqq \beta$ si $p \equiv 1 \bmod 4$, et sinon $\mu - 3 < \beta$ et on applique le théorème de dualité de Serre.

Supposons maintenant que q soit impair. On a une suite exacte

$$0 \to E_\alpha \to E_\lambda \otimes \mathrm{Hom}(E_\lambda, E_\mu) \to E_\mu \to 0$$

comme précédemment, et on procède de la même façon en utilisant ce qu'on vient de démontrer: C_n est vraie si p ou q est pair.

On a donc démontré (ii).

3. Démonstration de (iii)

On a une suite exacte

$$0 \to Q^* \to \mathcal{O} \otimes \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) \to \mathcal{O}(1) \to 0,$$

donc C_1 se déduit de C_0 comme précédemment C_n se déduisait de C_{n-1}. Il suffit donc de montrer que C_0 est vraie, ce qui est immédiat.

Les théorèmes 2, 3, 4, 5, 6 sont donc démontrés.

4. Démonstration du corollaire 7

On reprend la démonstration du théorème 1. Soit E un faisceau cohérent rigide sur $\mathbb{P}_2$. Alors E est sans torsion d'après le lemme 10. Soit

$$0 = F_0 \subset F_1 \subset \ldots \subset F_m = E$$

la filtration de Harder-Narasimhan de E, $Gr_i = F_i/F_{i-1}$ pour $1 \leqq i \leqq m$. On a déjà vu qu'on pouvait écrire

$$Gr_i = \mathbb{C}^{n_i} \otimes E_{\alpha_i},$$

avec $n_i \in \mathbb{N}^*$, $\alpha_i \in \mathscr{E}$. On a $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_m$, donc d'après le théorème 6 on a

$$\mathrm{Ext}^1(Gr_i, Gr_j) = 0 \quad \text{pour} \quad 1 \leqq j < i \leqq m.$$

Il en découle que

$$E \simeq \bigoplus_{i=1}^{m} Gr_i,$$

ce qui démontre le corollaire 7.

Le réciproque du corollaire 7 est fausse: on a $H^1(Q^*(-1)) \simeq \mathbb{C}$, et $\mathcal{O} \oplus Q^*(-1) = E$ est une somme directe de fibrés exceptionnels. Cependant on a $\mathrm{Ext}^1(E, E) \simeq \mathbb{C}$.

IV. Résolutions de la diagonale de $\mathbb{P}_2 \times \mathbb{P}_2$

1. Triades

Soient p, n des entiers, avec $n \geqq 0$, $\alpha = \varepsilon\left(\frac{p}{2^n}\right)$ et $\beta = \varepsilon\left(\frac{p+1}{2^n}\right)$. On considère des triplets de rationnels du type suivant

$$1: (\alpha, \alpha \cdot \beta, \beta),$$

$$2: (\beta - 3, \alpha, \alpha \cdot \beta),$$

$$3: (\alpha \cdot \beta, \beta, \alpha + 3).$$

Pour $n=0$, on considère aussi les triplets du type $(p, p+1, p+2)$. A de tels triplets sont associés des triplets de fibrés vectoriels, appelés *triades*. L'entier n s'appelle *niveau* de la triade. Soit $\mathbb{T}$ l'ensemble des triades. On pose, pour tout entier m et toute triade $\mathscr{T}=(E, G, F)$,

$$\mathscr{T}(m)=(E(m), G(m), F(m)),$$

ce qui définit une bijection $t_m: \mathbb{T} \rightarrow \mathbb{T}$.

On définit une application

$$j: \mathbb{T} \rightarrow \mathbb{T}$$

par

$$j(E, F, G)=(F(-3), E, G).$$

Cette application associe à une triade de type 1 (resp. 2, 3) une triade de type 2 (resp. 3, 1). Pour tout entier $n>0$, j^n désigne la composée nième de j. On a

$$j^3=t_{-3},$$

donc j est une bijection.

Lemme 12. *Soit (E, G, F) une triade. Soit N le noyau du morphisme canonique*

$$\mathrm{ev}: E \otimes \operatorname{Hom}(E, G) \rightarrow G$$

et M le conoyau du morphisme canonique

$$\mathrm{ev}^*: G \rightarrow F \otimes \operatorname{Hom}(G, F)^*.$$

Alors $M \simeq N(3)$ et les triplets (N, E, F) et (E, F, M) sont des triades.

Posons $\alpha=\varepsilon\left(\frac{p}{2^n}\right)$, $\beta=\varepsilon\left(\frac{p+1}{2^n}\right)$, avec p, n entiers et $n \geqq 0$. On supposera que

$$\begin{aligned} (E, G, F) &=(E_\alpha, E_{\alpha \cdot \beta}, E_\beta) && \text{si } (E, G, F) \text{ est de type } 1, \\ &=(E_{\beta-3}, E_\alpha, E_{\alpha \cdot \beta}) && \text{si } (E, G, F) \text{ est de type } 2, \\ &=(E_{\alpha \cdot \beta}, E_\beta, E_{\alpha+3}) && \text{si } (E, G, F) \text{ est de type } 3. \end{aligned}$$

On supposera que $n>0$, le cas $n=0$ étant immédiat. Il suffit, en appliquant les théorèmes 2, 3, 4, 5, de calculer les indices de $\mathscr{D}$ correspondant à M et N. Ils sont donnés par le tableau suivant

Type de (E, G, F)	1 $p \equiv 0 \bmod 2$	1 $p \equiv 1 \bmod 2$	2	3
Indice de M	$\frac{p+2}{2^n}$	$3+\frac{p-1}{2^n}$	$\frac{4p+3}{2^{n+2}}$	$3+\frac{4p+1}{2^{n+2}}$
Indice de N	$-3+\frac{p+2}{2^n}$	$\frac{p-1}{2^n}$	$-3+\frac{4p+3}{2^{n+2}}$	$\frac{4p+1}{2^{n+2}}$
Type de (N, E, F)	2	1	3	3
Type de (E, F, M)	1	3	2	2

Ceci démontre le lemme 12.

On appelle *triade dérivée* de $\mathcal{T}=(E,G,F)$ la triade

$$\mathcal{T}'=(F,M,E(3))$$

et on définit ainsi une application $d:\mathbb{T}\to\mathbb{T}$.

Lemme 13. *On a* $d^2=t_3$.

Ou encore, pour toute triade $\mathcal{T}=(E,G,F)$,

$$\mathcal{T}''=\mathcal{T}(3).$$

Soit N le noyau de $\mathrm{ev}:E\otimes\mathrm{Hom}(E,G)\to G$, M le conoyau de $\mathrm{ev}^*:G\to F\otimes\mathrm{Hom}(G,F)^*$. L'isomorphisme $M\simeq N(3)$ induit une suite exacte

$$\begin{tikzcd}[column sep=small] & & 0 \arrow[dr] & & 0 & & \\ & & & M \arrow[ur] \arrow[dr, "\psi_0"] & & & \\ 0 \arrow[r] & G \arrow[r, "\mathrm{ev}^*"] & F\otimes\mathrm{Hom}(G,F)^* \arrow[ur, "\varphi_0"] \arrow[rr] & & E(3)\otimes\mathrm{Hom}(E,G) \arrow[r, "\mathrm{ev}"] & G(3) \arrow[r] & 0. \end{tikzcd}$$

On a des isomorphismes canoniques

$$\mathrm{Hom}(G,F)^*\simeq\mathrm{Hom}(F,M),\qquad \mathrm{Hom}(E,G)\simeq\mathrm{Hom}(M,E(3))^*,$$

de sorte que le diagramme suivant est commutatif

$$\begin{tikzcd} F\otimes\mathrm{Hom}(G,F)^* \arrow[dr, "\varphi_0"] \arrow[d, equal] & & E(3)\otimes\mathrm{Hom}(E,G) \arrow[d, equal] \\ F\otimes\mathrm{Hom}(F,M) \arrow[r, "\mathrm{ev}"'] & M \arrow[ur, "\psi_0"] \arrow[dr, "\mathrm{ev}^*"'] & \\ & & E(3)\otimes\mathrm{Hom}(M,E(3))^*. \end{tikzcd}$$

Par suite, $\mathcal{T}''=(E(3),G(3),F(3))$.

Il découle du lemme 13 que d est une bijection.

On appelle *triade duale* de $\mathcal{T}=(E,G,F)$ la triade.

$$\mathcal{T}^*=(F^*,G^*,E^*).$$

On a

$$\mathcal{T}^{*\prime}=\mathcal{T}^{\prime *}(3).$$

On note $\mathcal{T}^\vee$ la triade $(E^*(-3),M^*,F^*)=\mathcal{T}'^*$.

2. *Morphismes de fibrés sur* $\mathbb{P}_2\times\mathbb{P}_2$

A une triade $\mathcal{T}=(E,G,F)$ on associe la suite de morphismes de fibrés sur $\mathbb{P}_2\times\mathbb{P}_2$

$$\begin{array}{ccccccc} R_{\mathcal{T}}^{\cdot}:0\longrightarrow & E\boxtimes E^*(-3) & \xrightarrow{a} & G\boxtimes M^* & \xrightarrow{b} & F\boxtimes F^* & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & R^{-2} & & R^{-1} & & R^0, & \end{array}$$

où $(F,M,E(3))=\mathcal{T}'$, a et b étant les morphismes associés aux éléments de $\mathrm{Hom}(E,G)\otimes\mathrm{Hom}(M,E(3))$ et $\mathrm{Hom}(G,F)\otimes\mathrm{Hom}(F,M)$ définis par les dualités $\mathrm{Hom}(E,G)\simeq\mathrm{Hom}(M,E(3))^*$ et $\mathrm{Hom}(G,F)\simeq\mathrm{Hom}(F,M)^*$.

On va maintenant démontrer le théorème 8, c'est à dire que si Δ désigne la diagonale de $\mathbb{P}_2 \times \mathbb{P}_2$, $R^{\cdot}_{\mathscr{T}}$ est un complexe et c'est une résolution de $\mathcal{O}_\Delta$, via le morphisme canonique $\mathrm{trace}_\Delta : F \boxtimes F^* \to \mathcal{O}_\Delta$.

Lemme 14. *1– Soit $\sigma : \mathbb{P}_2 \times \mathbb{P}_2 \to \mathbb{P}_2 \times \mathbb{P}_2$ la symétrie par rapport à la diagonale. Alors on a*

$$\sigma^* R^{\cdot}_{\mathscr{T}} \simeq R^{\cdot}_{\mathscr{T}^\vee} .$$

2– On a

$$R^{\cdot *}_{\mathscr{T}} \simeq R_{\mathscr{T}^*} \otimes p_2^*(\mathcal{O}(-3)) .$$

Immédiat.

Si $R^{\cdot}$ est une suite de morphismes de fibrés vectoriels sur $\mathbb{P}_2 \times \mathbb{P}_2$:

$$R^{\cdot} : \ldots \longrightarrow R^j \xrightarrow{\delta_j} R^{j+1} \longrightarrow \ldots ,$$

on note $dec(R^{\cdot})$ la suite de morphismes de fibrés vectoriels

$$S^{\cdot} : \ldots \longrightarrow S^j \xrightarrow{\nu_j} S^{j+1} \longrightarrow \ldots$$

définie par $S^j = R^{j+1}$, $\nu_j = \delta_{j+1}$ pour tout j.

Lemme 15. *Si $\mathscr{T}$ est une triade, la suite $R^{\cdot}_{\mathscr{T}}$ est un complexe et il existe un complexe acyclique de fibrés vectoriels sur $\mathbb{P}_2 \times \mathbb{P}_2$, et une suite exacte*

$$0 \to R^{\cdot}_{\mathscr{T}} \to K^{\cdot} \to Dec(R^{\cdot}_{j\mathscr{T}}) \to 0 .$$

Supposons le lemme 15 démontré. On va en déduire le théorème 8.

L'ensemble $\mathbb{T}$ est homogène sous le groupe des automorphismes de $\mathbb{T}$ engendré par j et d: en effet, si $\mathscr{T}_0 = (\mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1))$, l'orbite de $\mathscr{T}_0$ contient la translatée $j(\mathscr{T}_0) = \mathscr{T}_0(-1)$, et donc aussi toutes les triades de niveau 0. D'autre part, il découle du tableau du lemme 12 que si l'orbite de $\mathscr{T}_0$ contient toutes les triades de niveau n, elle contient aussi toutes les triades de niveau $n+1$. Il en découle que cette orbite est égale à $\mathbb{T}$ tout entier.

Le théorème 8 est vrai pour $\mathscr{T}_0$, c'est la résolution connue de $\mathcal{O}_\Delta$ conduisant à la suite spectrale de Beilinson classique. Par conséquent il suffit de montrer que si le théorème 8 est vrai pour $\mathscr{T}$, il l'est aussi pour $\mathscr{T}'$ et $j\mathscr{T}$.

D'après le lemme 14, si le théorème 8 est vrai pour $\mathscr{T}$, il l'est pour $\mathscr{T}^\vee$, ainsi que pour $\mathscr{T}^{\vee *} = \mathscr{T}'$. D'autre part, il découle du lemme 15 que $R^{\cdot}_{j\mathscr{T}}$ est un complexe, et qu'on a des isomorphismes $H^i(R^{\cdot}_{j\mathscr{T}}) \simeq H^i(R^{\cdot}_{\mathscr{T}})$. On verra que ces isomorphismes sont compatibles avec l'augmentation. Le théorème 8 en découle pour $j\mathscr{T}$.

Il reste donc à prouver le lemme 15, et à montrer qu'on a un triangle commutatif

$$(T) \qquad \begin{tikzcd} H^0(R^{\cdot}_{j\mathscr{T}}) \arrow[dd, equal, "f"'] \arrow[dr, "\lambda_0"] & \\ & \mathcal{O}_\Delta \\ H^0(R^{\cdot}_{\mathscr{T}}) \arrow[ur, "\mu_0"'] & \end{tikzcd} ,$$

le morphisme f provenant de la suite exacte du lemme 15, λ_0 et μ_0 des applications trace_Δ.

3. *Préliminaires à la démonstration du lemme 15*

Soit $\mathscr{T}=(E,G,F)$ une triade. Soient M,K,H les conoyaux de ev* dans les suites exactes

$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & 0 & & \\
 & & & M \arrow[ur] \arrow[dr] & & & \\
D_0\colon 0 \arrow[r] & G \arrow[r, "\mathrm{ev}^*"] & F\otimes\mathrm{Hom}(G,F)^* \arrow[ur] \arrow[rr] & & E(3)\otimes\mathrm{Hom}(E,G) \arrow[r, "\mathrm{ev}"] & G(3) \arrow[r] & 0,
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & 0 & & \\
 & & & K \arrow[ur] \arrow[dr] & & & \\
D_1\colon 0 \arrow[r] & F(-3) \arrow[r, "\mathrm{ev}^*"] & E\otimes\mathrm{Hom}(F,E(3))^* \arrow[ur] \arrow[rr] & & G\otimes\mathrm{Hom}(G,F) \arrow[r, "\mathrm{ev}"] & F \arrow[r] & 0,
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & 0 & & \\
 & & & H \arrow[ur] \arrow[dr] & & & \\
D_2\colon 0 \arrow[r] & E \arrow[r, "\mathrm{ev}^*"] & G\otimes\mathrm{Hom}(E,G)^* \arrow[ur] \arrow[rr] & & F\otimes\mathrm{Hom}(F,E(3)) \arrow[r, "\mathrm{ev}"] & E(3) \arrow[r] & 0.
\end{tikzcd}$$

A la suite $D_0^{\cdot}$ correspond un élément w_0 de $\mathrm{Ext}^2(G(3),G)$. On verra dans le lemme 16 qu'il est non nul. De même on définit des éléments w_1 de $\mathrm{Ext}^2(F,F(-3))$ et w_2 de $\mathrm{Ext}^2(E(3),E)$ associés à $D_1^{\cdot}$ et $D_2^{\cdot}$ respectivement. Quitte à modifier les injections $M\to E(3)\otimes\mathrm{Hom}(E,G)$ et $H\to F\otimes\mathrm{Hom}(F,E(3))$, on peut supposer que la propriété suivante est vérifiée:

$$(P)\qquad \begin{cases}\text{Dans}\quad H^2(\mathcal{O}(-3))\simeq\mathbb{C}, \quad \text{on a}\\ \mathrm{trace}\,(w_0)=\mathrm{trace}\,(w_1)=\mathrm{trace}\,(w_2).\end{cases}$$

On note $\mathrm{Mor}^1(D_0^{\cdot},D_2^{\cdot})$ l'espace des morphismes de degré 1.

Lemme 16. *On suppose la propriété* (P) *vérifiée. Alors tout* $f\in\mathrm{Mor}^1(D_0^{\cdot},D_2^{\cdot})$ *est de la forme* $(I_G\otimes u,\gamma,I_{F(3)}\otimes u)$, *avec* $\gamma\colon F\otimes\mathrm{Hom}(G,F)^*\to F\otimes\mathrm{Hom}(F,E(3))$, $u\in\mathrm{Hom}(E,G)^*$, *et l'application* $f\mapsto u$ *est un isomorphisme de* $\mathrm{Mor}^1(D_0^{\cdot},D_2^{\cdot})$ *sur* $\mathrm{Hom}(E,G)^*$.

L'élément w_0 associé à $D_0^{\cdot}$ est défini de la manière suivante: on considère la suite spectrale d'aboutissement $\mathrm{Ext}^{\cdot}(D_0^{\cdot},D_0^{\cdot})$ dont le terme E_1 est donné par

$$E_1^{p,q}(D_0^{\cdot},D_0^{\cdot})=\bigoplus_j \mathrm{Ext}^q(D_0^j,D_0^{j+p}).$$

Compte tenu de III les termes $E_1^{p,q}$ non nuls pour $p\leqq 0$ sont $E_1^{-3,2}$ et $E_1^{0,0}$. Il en découle qu'on a un isomorphisme

$$d_3\colon E_1^{-3,2}=\mathrm{Ext}^2(G(3),G)\to\mathrm{Ker}(d_1^{0,0})=\mathrm{End}(D_0^{\cdot})$$

et w_0 est défini par $d_3(w_0)=I_{D_0^{\cdot}}$. On définit de même w_1 et w_2.

Soit $f\colon D_0^{\cdot}\to D_2^{\cdot}$ un morphisme de degré 1, de la forme $(I_F\otimes u, w, I_{E(3)}\otimes v)$, avec u,v dans $\mathrm{Hom}(E,G)^*$. On va montrer que $u=v$. Considérons pour cela la suite spectrale d'aboutissement $\mathrm{Ext}^{\cdot}(D_0^{\cdot},D_2^{\cdot})=0$, de terme E_1

$$E_1^{p,q}(D_0^{\cdot},D_2^{\cdot})=\bigoplus_j \mathrm{Ext}^q(D_0^j,D_2^{j+p}).$$

Compte tenu de III, on obtient un isomorphisme

$$d_3\colon E_2^{-2,2}(D_0^{\cdot},D_2^{\cdot})\to\mathrm{Ker}(d_1^{1,0})=\mathrm{Mor}^1(D_0^{\cdot},D_2^{\cdot}),$$

et une suite exacte

$$(S)\qquad \begin{array}{ccccccc} 0\longrightarrow & \mathrm{Ext}^2(G(3),E) & \xrightarrow{d_1} & \begin{array}{c}\mathrm{Ext}^2(E(3)\otimes\mathrm{Hom}(E,G),E)\\ \bigoplus\\ \mathrm{Ext}^2(G(3),G\otimes\mathrm{Hom}(E,G)^*)\end{array} & \xrightarrow{\phi} & E_2^{-2,2} & \longrightarrow 0\\ & \| & & \| & & & \\ & E_1^{-3,2} & & E_1^{-2,2}. & & & \end{array}$$

La multiplication à gauche par f induit un diagramme anticommutatif

$$\begin{tikzcd}
& \operatorname{Ext}^2(G(3), G) = E_2^{-3,2}(D_0^{\cdot}, D_0^{\cdot}) \arrow[r, "d_3"] \arrow[d] & \operatorname{End}(D_0^{\cdot}) \arrow[d] \\
\begin{smallmatrix} \operatorname{Ext}^2(E(3)\otimes \operatorname{Hom}(E,G), E) \\ \bigoplus \\ \operatorname{Ext}^2(G(3), G\otimes \operatorname{Hom}(E,G)^*) \end{smallmatrix} \arrow[r, "\phi"] & E_2^{-2,2}(D_0^{\cdot}, D_2^{\cdot}) \arrow[r, "d_3"] & \operatorname{Mor}^1(D_0^{\cdot}, D_2^{\cdot})
\end{tikzcd}$$

d'où on déduit que $f = d_3\phi(-w_0 \otimes u)$.

En considérant la multiplication à droite par f, on obtient aussi

$$f = d_3\phi(w_2 \otimes v).$$

On a donc

$$\phi(w_2 \otimes v - w_0 \otimes u) = 0.$$

D'autre part on a un diagramme commutatif

$$\begin{tikzcd}
\operatorname{Hom}(E,G)\otimes \operatorname{Ext}^2(G(3),E) \arrow[r, "\text{Yoneda}"] \arrow[d, "\text{Yoneda}"] & \operatorname{Ext}^2(G(3),G) \arrow[d, "\text{trace}"] \\
\operatorname{Ext}^2(E(3),E) \arrow[r, "\text{trace}"] & H^2(\mathcal{O}(-3)),
\end{tikzcd}$$

d'où on déduit le diagramme commutatif

$$\begin{tikzcd}
\operatorname{Ext}^2(G(3),E) \arrow[r, "\lambda"] \arrow[d, "\mu"'] & \operatorname{Ext}^2(G(3), G\otimes \operatorname{Hom}(E,G)^*) \arrow[d, "\text{trace}"] \\
\operatorname{Ext}^2(E(3)\otimes \operatorname{Hom}(E,G), E) \arrow[r, "\text{trace}"] & \operatorname{Hom}(E,G)^*\otimes H^2(\mathcal{O}(-3)).
\end{tikzcd}$$

Le morphisme d_1 de la suite exacte (S) ci-dessus n'est autre que (λ, μ). Dans $\operatorname{Hom}(E,G)^*\otimes H^2(\mathcal{O}(-3))$ on a donc

$$u \cdot \operatorname{trace}(w_0) - v \cdot \operatorname{trace}(w_2) = 0,$$

d'où $u = v$, puisqu'on a supposé que $\operatorname{trace}(w_0) = \operatorname{trace}(w_2)$.

Réciproquement, si u est un élément de $\operatorname{Hom}(E,G)^*$, on pose $f = -d_3\phi(w_0 \otimes u)$. Ce morphisme est de la forme $(I_G \otimes u', w', I_{E(3)} \otimes u')$, et on a

$$f = -d_3\phi(w_0 \otimes u'),$$

d'où

$$\phi(w_0 \otimes u) = \phi(w_0 \otimes u'),$$

d'où $\operatorname{trace}(w_0)\otimes u = \operatorname{trace}(w_0)\otimes u'$, et $u = u'$.

Ceci achève la démonstration du lemme 16.

4. *Dualités*

Considérons les suites exactes

(1) $$0 \longrightarrow G \xrightarrow{\mathrm{ev}^*} F\otimes \operatorname{Hom}(G,F)^* \longrightarrow M \longrightarrow 0,$$

(2) $$0 \longrightarrow H \longrightarrow F\otimes \operatorname{Hom}(F, E(3)) \xrightarrow{\mathrm{ev}} E(3) \longrightarrow 0$$

déduites de $D_0^{\cdot}$ et $D_2^{\cdot}$, et les isomorphismes

$$\begin{tikzcd}[row sep=small] & \operatorname{Hom}(G,H) \\ \operatorname{Hom}(E,G)^* \arrow[ur, "\simeq" , "\varphi"'] \arrow[dr, "\simeq", "\psi"'] & \\ & \operatorname{Hom}(M,E(3)) \end{tikzcd}$$

associés aux deux autres suites exactes courtes extraites de $D_0^{\cdot}$ et $D_2^{\cdot}$. A tout élément u de $\operatorname{Hom}(E,G)^*$ est associé d'après le lemme 16 un morphisme de degré 1 $D_0^{\cdot} \to D_2^{\cdot}$, qui conduit à un morphisme (1)→(2), de la forme $(\varphi(u), I_F \otimes \theta(u), \psi(u))$, avec $\theta(u): \operatorname{Hom}(G,F)^* \to \operatorname{Hom}(H,F)^*$.

Lemme 17. *Considérons le diagramme*

$$\begin{tikzcd} \operatorname{Hom}(G,F)^* \arrow[r, "\psi'", "\simeq"'] \arrow[d, "\delta"'] \arrow[dr, "\theta(u)"] & \operatorname{Hom}(F,M) \arrow[d, "\varepsilon_0"] \arrow[dl, phantom, "\mathrm{I}"] \\ \operatorname{Hom}(H,F)^* \arrow[r, "\varphi'", "\simeq"'] \arrow[ur, phantom, "\mathrm{II}"] & \operatorname{Hom}(F,E(3)), \end{tikzcd}$$

où les flèches horizontales sont déduites de (1) *et* (2), ε_0 *est la multiplication à gauche par* $\psi(u)$, δ *la transposée de la multiplication à droite par* $\varphi(u)$. *Alors, si la propriété* (P) *est vérifiée, ce diagramme est commutatif.*

La multiplication à gauche par $\psi(u)$ donne un diagramme commutatif

$$\begin{array}{ccc} F \otimes \operatorname{Hom}(F,M) & \to & M \\ \downarrow & & \downarrow \\ F \otimes \operatorname{Hom}(F,E(3)) & \to & E(3), \end{array}$$

et si on compose avec ψ' on obtient un morphisme (1)→(2), se relevant en un morphisme de degré 1 de $D_0^{\cdot}$ dans $D_2^{\cdot}$, de la forme $(w, w', I_{E(3)} \otimes u)$. D'après le lemme 16, ce morphisme coïncide avec celui qui est défini par u, ce qui entraine la commutativité du triangle I. Celle du triangle II s'obtient de même en considérant $\varphi(u)$.

Ceci démontre le lemme 17.

5. *Définition de la suite* $K^{\cdot}$

Soit u un élément de $\operatorname{Hom}(E,G)^*$. On en déduit un morphisme de suites exactes courtes

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ E^*(-3) \arrow[r, "{}^t\psi(u)"] \arrow[d] & M^* \arrow[d] \\ F^* \otimes \operatorname{Hom}(F,E(3))^* \arrow[r, "I_{F^*} \otimes {}^t\theta(u)"] \arrow[d] & F^* \otimes \operatorname{Hom}(G,F) \arrow[d] \\ H^* \arrow[r, "{}^t\varphi(u)"] \arrow[d] & G^* \arrow[d] \\ 0 & 0. \end{tikzcd}$$

Soit (e_i) une base de $\mathrm{Hom}(E, G)$, (e^i) la base duale dans $\mathrm{Hom}(E, G)^*$. On pose pour tout i, $u^i = \varphi(e^i)$, $v^i = \psi(e^i)$. Alors les morphismes $a_{\mathscr{T}}: E \boxtimes E^*(-3) \to G \boxtimes M^*$ et $b_{j\mathscr{T}}: E \boxtimes H^* \to G \boxtimes G^*$ sont

$$a_{\mathscr{T}} = \sum_i e_i \boxtimes {}^t v^i, \qquad b_{j\mathscr{T}} = \sum_i e_i \boxtimes {}^t u^i.$$

On a alors un morphisme de suites exactes de fibrés sur $\mathbb{P}_2 \times \mathbb{P}_2$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
E \boxtimes E^*(-3) \arrow[r, "a_{\mathscr{T}}"] \arrow[d] & G \boxtimes M^* \arrow[d] \\
E \boxtimes (F^* \otimes \mathrm{Hom}(F, E(3))^*) \arrow[r, "A"] \arrow[d] & G \boxtimes (F^* \otimes \mathrm{Hom}(G, F)) \arrow[d] \\
E \boxtimes H^* \arrow[r, "b_{j\mathscr{T}}"] \arrow[d] & G \boxtimes G^* \arrow[d] \\
0 & 0,
\end{tikzcd}$$

où $A: \alpha \boxtimes (\beta \otimes \eta) \to \sum_i e_i(\alpha) \boxtimes (\beta \otimes {}^t\theta(e^i)\eta)$. Ce diagramme se complète en le diagramme (R) (cf. Fig. 1)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & E \boxtimes E^*(-3) \arrow[r, "a_{\mathscr{T}}"] \arrow[d] & G \boxtimes M^* \arrow[r, "b_{\mathscr{G}}"] \arrow[d] \arrow[dr, phantom, "\mathrm{III}"] & F \boxtimes F^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F(-3) \boxtimes F^* \arrow[r, "B'"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{IV}"] & E \boxtimes (F^* \otimes \mathrm{Hom}(F, E(3))^*) \arrow[r, "A"] \arrow[d] & G \boxtimes (F^* \otimes \mathrm{Hom}(G, F)) \arrow[r, "B"] \arrow[d] & F \boxtimes F^* \arrow[r] & 0 \\
0 \arrow[r] & F(-3) \boxtimes F^* \arrow[r, "a_{j\mathscr{T}}"] & E \boxtimes H^* \arrow[r, "b_{j\mathscr{T}}"] \arrow[d] & G \boxtimes G^* \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

Fig. 1. Diagramme (R)

où B est défini par

$$B(\alpha' \boxtimes \beta \otimes \xi) = \xi(\alpha') \boxtimes \beta,$$

pour $\alpha' \in G$, $\beta \in F^*$ et $\xi \in \mathrm{Hom}(G, F)$, et B' par

$$\langle B'(\alpha'' \boxtimes \beta), \eta \rangle = \eta(-3)\alpha'' \boxtimes \beta$$

pour $\alpha'' \in F(-3)$, $\beta \in F^*$ et $\eta \in \mathrm{Hom}(F, E(3))$.

Le lemme 15 découle des assertions suivantes: si (P) est satisfaite, on a

a – le diagramme (R) est commutatif;

b – la seconde ligne de (R) est isomorphe à $D_1^{\cdot} \boxtimes F^*$.

Cette seconde ligne est donc le complexe acyclique du lemme 15.

6. Démonstration de a-

Il faut montrer que les diagrammes III et IV de (R) sont commutatifs.

Démontrons que III est commutatif. Soit (ε_j) une base de $\operatorname{Hom}(G,F)$, (ε^j) la base duale dans $\operatorname{Hom}(G,F)^*$, $w^j=\psi'(\varepsilon^j)$, élément de $\operatorname{Hom}(F,M)$. Alors on

$$b_{\mathscr{T}}=\sum_j \varepsilon_j \boxtimes {}^t w^j .$$

Soient $\alpha\in G$, $\beta\in M^*$. Alors l'image de $\alpha\boxtimes\beta$ dans $G\boxtimes(F^*\otimes\operatorname{Hom}(G,F))$ est

$$\sum_j \alpha\boxtimes({}^t w^j(\beta)\otimes\varepsilon_j),$$

élément dont l'image dans $F\boxtimes F^*$ est

$$\sum_j \varepsilon_j(\alpha)\boxtimes{}^t w^j(\beta)=b_{\mathscr{T}}(\alpha\boxtimes\beta),$$

ce qui prouve la commutativité de III.

Démontrons que IV est commutatif. Soit (η_l) une base de $\operatorname{Hom}(F,E(3))$, (η^l) la base duale dans $\operatorname{Hom}(F,E(3))^*$, (y_l) son image réciproque par l'isomorphisme ${}^t\varphi':\operatorname{Hom}(H,F)\simeq\operatorname{Hom}(F,E(3))^*$, qui est une base de $\operatorname{Hom}(H,F)$. Alors, pour $\alpha''\in F(-3)$, $\beta\in F^*$, on a

$$B'(\alpha''\boxtimes\beta)=\sum_l \eta_l(-3)(\alpha'')\boxtimes(\beta\otimes\eta^l),$$

donc cet élément a pour image dans $E\boxtimes H^*$

$$\sum_l \eta_l(-3)\alpha\boxtimes{}^t y_l(\beta)=a_{j\mathscr{T}}(\alpha\boxtimes\beta),$$

ce qui démontre la commutativité de IV.

7. Démonstration de b-

La seconde ligne de (R) est isomorphe à $L^{\cdot}\boxtimes F^*$, $L^{\cdot}$ étant la suite

$$0\longrightarrow F(-3)\xrightarrow{\mathrm{ev}^*}E\otimes\operatorname{Hom}(F,E(3))^*\xrightarrow{\Theta}G\otimes\operatorname{Hom}(G,F)\xrightarrow{\mathrm{ev}}F\longrightarrow 0$$

où Θ est défini par

$$\alpha\otimes\beta\mapsto\sum_i e_i(\alpha)\otimes{}^t\theta(e^i)\eta,$$

pour $\alpha\in E$, $\beta\in\operatorname{Hom}(F,E(3))^*$. D'après le lemme 17, Θ se factorise suivant le diagramme

$$\begin{tikzcd}
E\otimes\operatorname{Hom}(F,E(3))^* \arrow[d, equal, "I_E\otimes{}^t\varphi'"'] \arrow[dr, "\Theta"] & \\
E\otimes\operatorname{Hom}(H,F) \arrow[r, "\Theta'"] & G\otimes\operatorname{Hom}(G,F),
\end{tikzcd}$$

où

$$\Theta'(\alpha\otimes\beta)=\sum_i e_i(\alpha)\otimes\beta u^i,$$

pour $\alpha\in E$, $\beta\in\operatorname{Hom}(H,F)$. L'image de $\Theta'(\alpha\otimes\beta)$ dans F est donc

$$\sum_i \beta u^i e_i(\alpha).$$

Or, en vertu de la suite exacte extraite de $D_2^{\cdot}$:

$$0 \to E \to G \otimes \mathrm{Hom}(E, G)^* \to H \to 0$$

et de la définition de u^i, on a

$$\sum_i u^i e_i(\alpha) = 0$$

dans H. Par conséquent on a $\mathrm{ev} \circ \Theta = 0$.

De même, Θ se factorise suivant le diagramme

$$\begin{tikzcd} E \otimes \mathrm{Hom}(F, E(3))^* \arrow[r, "\Theta''"] \arrow[dr, "\Theta"'] & G \otimes \mathrm{Hom}(F, M)^* \arrow[d] \\ & G \otimes \mathrm{Hom}(G, F), \end{tikzcd}$$

où

$$\langle \Theta''(\alpha \otimes \beta), \xi \rangle = \sum_i e_i(\alpha) \langle \beta, v^i \xi \rangle$$

pour $\alpha \in E$, $\beta \in \mathrm{Hom}(F, E(3))^*$ et $\xi \in \mathrm{Hom}(F, M)$. On a alors, pour $\alpha'' \in F(-3)$,

$$\begin{aligned} \langle \Theta'' \,\mathrm{ev}^*(\alpha''), \xi \rangle &= \sum_{i,l} e_i \eta_l(-3) \alpha'' \langle \eta^l, v^i \xi \rangle \\ &= \sum_i e_i(v^i \xi)(-3) \alpha''. \end{aligned}$$

Mais, d'après la suite exacte déduite de $D_0^{\cdot}$:

$$0 \to M(-3) \to E \otimes \mathrm{Hom}(E, G) \to G \to 0$$

et la définition de v_i, le morphisme

$$\sum_i e_i v^i(-3) : M \to G$$

est nul. Donc $\Theta \circ \mathrm{ev} = 0$.

On a $a_{\mathscr{T}} \neq 0$, donc $\Theta \neq 0$. Le morphisme Θ induit donc un endomorphisme non nul du fibré exceptionnel K, et comme celui-ci est simple, cet endomorphisme est une homothétie de rapport non nul. Il en découle que $L^{\cdot} \simeq D_1^{\cdot}$, et que $L^{\cdot} \boxtimes F^*$ est un complexe acyclique.

Le lemme 15 est donc démontré.

8. *Fin de la démonstration du théorème 8*

Il reste à montrer que le triangle (T) de 2 – est commutatif. Cela découle du fait que (T) est induit par le diagramme commutatif

$$\begin{tikzcd} G \boxtimes (F^* \otimes \mathrm{Hom}(G, F)) \arrow[r, "B"] \arrow[d] & F \boxtimes F^* \arrow[d, "\mathrm{trace}"] \\ G \boxtimes G^* \arrow[r, "\mathrm{trace}"] & \mathcal{O}_\Delta . \end{tikzcd}$$

Le théorème 8 est donc démontré.

V. Suite spectrale de Beilinson généralisée

On démontre ici le corollaire 9. Soit $\mathcal{T}=(E,G,F)$ une triade, M le conoyau du morphisme canonique $\mathrm{ev}^*: G \to F \otimes \mathrm{Hom}(G,F)^*$. Soit $\mathcal{F}$ un faisceau cohérent sur $\mathbb{P}_2$ tel que $\mathrm{Hom}(F,\mathcal{F})=\mathrm{Hom}(\mathcal{F},E)=0$.

Puisque $\mathrm{Tor}_i^{\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}}(\mathcal{O}_\Delta, p^*(\mathcal{F}))=0$ pour $i>0$, la suite $R^{\cdot}_{\mathcal{T}}\otimes(\mathcal{O}\boxtimes\mathcal{F})$ est une résolution de $(\mathcal{O}\boxtimes\mathcal{F})\otimes\mathcal{O}_\Delta=\mathcal{F}_\Delta$. On a

$$p_{2*}\mathcal{F}_\Delta \simeq \mathcal{F}, \quad \text{et} \quad R^i p_{2*}\mathcal{F}_\Delta=0 \quad \text{si} \quad i \neq 0,$$

et la suite spectrale du foncteur dérivé Rp_{2*} appliquée à la résolution précédente de $\mathcal{F}_\Delta$ converge vers $\mathcal{F}$ en degré 0, et vers 0 en les autres degrés. Les termes E_1^{pq} de cette suite sont

$$E_1^{p,q}=E_p\otimes H^q(\mathcal{F}\otimes F_p),$$

avec $E_{-2}=E$, $F_{-2}=E^*(-3)$, $E_{-1}=G$, $F_{-1}=M^*$, $E_0=F$, $F_0=F^*$, et $E_i=F_i=0$ si $i\neq 0,-1,-2$.

D'après les hypothèses, les seuls termes éventuellement non nuls sont $E_1^{-2,1}$, $E_1^{-1,1}$ et $E_{-1}^{0,1}$. On en déduit immédiatement le corollaire 9 (cf. Verdier [8, proposition 7.1] et Le Potier [6, proposition 20]).

Remerçiement. Je tiens à remercier J. Le Potier pour les simplifications notables qu'il a apportées aux démonstrations de ces résultats, ainsi que pour l'amélioration de certains d'entre eux: je lui dois les versions finales des théorèmes 1 et 6, ainsi que le corollaire 7.

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An Approximate Riemann Mapping Theorem in $\mathbb{C}^n$

B. L. Fridman

Mathematics Department, Wichita State University, Wichita, KS 67208, USA

Introduction

Let $T_0(n)$ be the class of all domains in $\mathbb{C}^n$ diffeomorphic to the unit ball B^n [1]. If $n=1$ any two domains $G_1, G_2 \in T_0(1)$, with the exception of the entire plane $\mathbb{C}$, are biholomorphically equivalent due to the Riemann Mapping Theorem. There is no direct analogue of this result when $n>1$. In this paper we prove that the Riemann Mapping Theorem is almost true for $T_0(n)$, $n>1$; see Corollary 1 below for the precise statement.

In [5] we constructed a bounded domain $D \subset \mathbb{C}^n$ such that every domain $G \subset \mathbb{C}^n$ is a monotone union of biholomorphic images of D. The topology of D is of course very complicated. Here we will investigate the possibility of the existence of such a domain in a given (diffeomorphism) class, so it can be used to approximate domains from that class.

Let $G, D \subset \mathbb{C}^n$ be domains. We will say that G can be exhausted by D if for any compact $K \subset G$ there exists a biholomorphic imbedding $F: D \to G$ such that $F(D) \supset K$.

We prove that there exists an "almost canonical" domain in $T_0(n)$. Let $n \geqq 1$ be fixed.

Theorem 1. *There exists a domain $D \in T_0(n)$ such that every $G \in T_0(n)$ can be exhausted by D.*

Let T be a class of domains in $\mathbb{C}^n$ diffeomorphic to a given domain. We will say that $G_1, G_2 \in T$ are approximately H-equivalent if for any two compacts $K_i \subset G_i$, $i=1,2$, there exist a domain $D \in T$ and two biholomorphic imbeddings $F_i: D \to G_i$ such that $F_i(D) \supset K_i$, $i=1,2$.

The next statement follows directly from Theorem 1.

Corollary 1. *Any two domains $G_1, G_2 \in T_0(n)$ are approximately H-equivalent.*

---

[1] According to the existance of unique differential structure on $\mathbb{R}^k$ when $k \neq 4$, the class $T_0(n)$ for $n \neq 2$ coincides with the class of all domains homeomorphic to B^n. The question of whether a domain $D \subset \mathbb{C}^2$ homeomorphic to B^2 is diffeomorphic to B^2 is open. I am grateful to R. Kirby and P. Parker for clarifying this

We note here that $T_0(n)$ for $n>1$ contains an uncountable set of holomorphically nonequivalent domains (see [3, 7]; for further discussions on biholomorphic or proper holomorphic mappings see [2, 8]).

The property described in Corollary 1 does not hold for many (diffeomorphism) classes other than $T_0(n)$. We present one such example for $n=1$. Let $T_1(1)$ be the set of all domains in $\mathbb{C}$ of connectivity 2, that is all domains diffeomorphic to the annulus $\{w \mid 1<|w|<2\}$.

Theorem 2. *$T_1(1)$ contains a continuum of domains any two of which are not approximately H-equivalent.*

Though the case of the ball B^n and the polydisc U^n is covered by Theorem 1 and Corollary 1 we nevertheless consider this particular example separately. In this case the exhausting domain D can be chosen explicitly (see Theorem 1.1 below).

Notations. If $z\in\mathbb{C}^n$, $n>1$, then $z=(z_1,z')$, where $z'=(z_2,\dots,z_n)$. $B(z,r)=\{w\in\mathbb{C}^n \mid |w-z|<r\}$; $B=B^n=B(0,1)$ is the unit ball in $\mathbb{C}^n$. $U^n=\{z \mid |z_k|<1,\ k=1,2,\dots,n\}$. In case $n=1$, B^1 is also denoted as Δ. ∂D is the boundary of D, $\bar{D}$ is the closure of D. $\operatorname{Aut}(D)$ is the group of holomorphic automorphisms of D. The description and properties of $\operatorname{Aut}(B)$ one can find in [10]. $p,q\in\partial B^n$, $p=(1,0,\dots,0)$, $q=-p$.

1. The Case of U^n and B^n

Let $n\geqq 2$ be fixed and $S=\{z\in B^n \mid \operatorname{Re} z_s>0 \text{ for } s\geqq 2\}$.

1

Theorem 1.1. *U^n and B^n can be exhausted by S.*

Remark. One might compare this result with the statement that neither of the two domains B^n, U^n can be exhausted by the other [1, 4, 9].

In the proof of this Theorem 1.1 and later we will need a lemma from [5]. Let $n\geqq 1$ be fixed. We will use notations $B=B^n$, $U=U^n$.

2

Lemma 1.2. *For any $\varepsilon>0$ and R, $1>R>0$, there exist $r>0$ and $A\in\operatorname{Aut}(B)$ such that*

(a) $R>r>0$

(b) $A(B\backslash B(p,R))\subset B(q,\varepsilon)$

(c) $A(\bar{B}(p,r)\cap B)\subset B(p,\varepsilon)$.

Proof. Consider for $0<\lambda<1$

$$A_\lambda(z)=\left\{\frac{z_1-\lambda}{1-z_1\lambda},\sqrt{1-\lambda^2}\,\frac{z'}{1-z_1\lambda}\right\}.$$

$A_\lambda\in\operatorname{Aut}(B)$. Let $V=B\backslash B(p,R)$. If $\lambda\to 1$ then $A_\lambda(z)\to q$ uniformly on V. So, one can find λ_0 such that $A=A_{\lambda_0}$ will satisfy (b). $A(p)=p$ and A is continuous at p. Therefore (a) and (c) can be satisfied now by choosing a small enough $r>0$.

3

We will say that $\{D_k\}_1^\infty$ is an exhausting system for a set D if for every compact $K \subset D$ there exists an s such that $D \supset D_s \supset K$.

4. Proof of Theorem 1.1

(A) To prove that B can be exhausted by S we take the point $p_1 = (1/\sqrt{n}, \ldots, 1/\sqrt{n}) \in \partial S$ and an automorphism $Q \in \mathrm{Aut}(B)$ such that $Q(p_1) = p$. For $R_1 = 1/(2\sqrt{n})$

$$S \cap B(p_1, R_1) = B \cap B(p_1, R_1).$$

Therefore one can find an $0 < R < 1$ such that

$$Q(S) \cap B(p, R) = B \cap B(p, R). \tag{1}$$

Let $\varepsilon_k = 1/k$. Using Lemma 1.2 for $\varepsilon = \varepsilon_k$ and R one can find an automorphism $A = A_k \in \mathrm{Aut}(B)$ that satisfies the condition (b) of that lemma. Consider now $\phi_k = A_k \circ Q \in \mathrm{Aut}(B)$. Using (1), property (b) of Lemma 1.2, and the fact $S \subset B$ one can see now that

$$B \backslash B(q, \varepsilon_k) \subset A_k(B \cap B(p, R)) \subset \phi_k(S) \subset B.$$

Therefore $\{\phi_k(S)\}_1^\infty$ is an exhausting system for B.

(B) To prove that U can be exhausted by S we introduce $E = \{w \in \mathbb{C} \,|\, |w| < 1, \mathrm{Re}\, w > 0\}$. E is conformally equivalent to the unit disc Δ. Every conformal map $\varphi: E \to \Delta$ extends continuously to the boundary ∂E, so $\varphi: \bar{E} \to \bar{\Delta}$. Let $\varphi: E \to \Delta$ be such a conformal map that $\varphi(0) = -1$. We take

$$\phi: z \mapsto (z_1, \varphi(z_2), \ldots, \varphi(z_n))$$

$$A_k: z \mapsto \left(z_1, \frac{z_2 + \lambda_k}{1 + \lambda_k z_2}, \ldots, \frac{z_n + \lambda_k}{1 + \lambda_k z_n}\right),$$

where $\lambda_k = 1 - 1/2k$.

We prove now that $\{A_k \circ \phi(S)\}_1^\infty$ is an exhausting system for U. Let $K \subset U$ be a compact. There exists an $\varepsilon > 0$ such that $U_{1-\varepsilon} \supset K$, where $U_{1-\varepsilon} = \{z \in \mathbb{C}^n \,|\, |z_i| < 1 - \varepsilon, 1 \leqq i \leqq n\}$. Take a $\delta > 0$ such that $(1-\varepsilon)^2 + (n-1)\delta^2 < 1$. Let

$$W_{\varepsilon\delta} = \{z \in \mathbb{C}^n \,|\, |z_1| < 1 - \varepsilon,\ |z_s| < \delta \text{ and } \mathrm{Re}\, z_s > 0 \text{ for } s \geqq 2\}.$$

One can see that $W_{\varepsilon\delta} \subset S$. Since $\varphi: \bar{E} \to \bar{\Delta}$ is analytic at $0 \in \partial E$ we can find a $\delta' > 0$ such that $V'_{\delta'} \subset \varphi(V_\delta)$ where

$$V'_{\delta'} = \{w \in \Delta \,|\, |w + 1| < \delta'\}, \qquad V_\delta = E \cap \{w \,|\, |w| < \delta\}.$$

As in Lemma 1.2 for $n = 1$ one can see that there exists a k such that $g_k(V'_{\delta'}) \supset \{w \,|\, |w| < 1 - \varepsilon\}$ where $g_k(w) = \dfrac{w + \lambda_k}{1 + \lambda_k w}$, $\lambda_k = 1 - 1/2k$. Now

$$U \supset \phi(W_{\varepsilon\delta}) \supset \{|z_1| < 1 - \varepsilon\} \times \prod_{i=2}^{n} \{|z_i| < \delta'\}.$$

And

$$U \supset A_k \circ \phi(S) \supset A_k \circ \phi(W_{\varepsilon\delta}) \supset U_{1-\varepsilon} \supset K .$$

2. Proof of Theorem 1

If D is a bounded domain in $\mathbb{C}^n$ then $\partial_1 D$ denotes the C^∞ part of the boundary ∂D. We introduce now two sets of domains $T_0'(n)$ and $\mathfrak{B}(n)$. Let $D \subset\subset \mathbb{C}^n$.

$D \in T_0'(n)$ if there exists a homeomorphism $\varphi: \bar{D} \to \bar{B}$ such that φ is diffeomorphic on $D \cup \partial_1 D$. Evidently $T_0'(n) \subset T_0(n)$.

$M \in \mathfrak{B}(n)$ if $M \in T_0'(n)$ and M is a union of a finite number of open balls with rational centers and rational radii.

1

Lemma 2.1. *Let $G \in T_0(n)$. Then for every compact $K \subset G$ there exists a domain $M \in \mathfrak{B}(n)$ such that $K \subset M \subset\subset G$.*

Proof. First one can find a domain M_1, $K \subset M_1 \subset\subset G$ such that $M_1 \in T_0'(n)$ and the boundary ∂M_1 is a smooth nondegenerate surface. Now we take a small normal tubular neighborhood V of ∂M_1 contained in G. Let $1 > \varepsilon > 0$ be a rational number such that if $z \in \partial M_1$ then $B(z, \varepsilon) \subset V$. We find a finite number of rational points $\{z_i\}_1^N \subset M_1$ such that the distance of every point $z \in \bar{M}_1$ to at least one of the points $\{z_i\}$ is less than ε^2. Take now $M = \bigcup_{i=1}^{N} B(z_i, \varepsilon)$. One can see that M satisfies all the requirements of the lemma.

2

Lemma 2.2. *Let $M \in \mathfrak{B}(n)$, $K \subset M$ be a compact. Then there exists an $\varepsilon > 0$ and $F: M \to \mathbb{C}^n$, F is holomorphic, such that*

(a) $[F(M) \cap B] \in T_0'(n)$
(b) $W(\varepsilon) \supset F(K)$ *where* $W(\varepsilon) = B \setminus (B(p, \varepsilon) \cup B(q, \varepsilon))$
(c) $F(M) \supset [B(p, \varepsilon) \cup B(q, \varepsilon)]$
(d) F^{-1} *is one-to-one on* $F(M) \cap B$.

Proof. Let B_1 be a ball of minimal radius that contains M. $\partial B_1 \cap \partial M$ contains at least two different points ξ, η. One can find a linear transformation $L: \mathbb{C}^n \to \mathbb{C}^n$ such that $L(B_1) = B$ and then an $A \in \mathrm{Aut}(B)$ such that $A(L\xi) = p$, $A(L\eta) = q$. We stretch now the image $A \circ L(M)$ by a linear transformation $L_\delta: z \mapsto (1+\delta)z$. Let $F = L_\delta \circ A \circ L$. For a small $\delta > 0$ one can find an $\varepsilon > 0$ such that all statements of this lemma hold.

3

Below we are going to use the following notations.

$$\Omega(R, r) = (B(p, R) \setminus \bar{B}(p, r)) \cap B, \ \omega(R) = B \cap \partial B(p, R),$$

$$\hat{\Omega}(R, r) = \omega(R) \cup \Omega(R, r) \cup \omega(r), \ \Lambda = [-1, 1] \times \mathbb{R}^{2n-1} .$$

Lemma 2.3. *Let $M \in \mathfrak{B}(n)$, $K \subset M$ be a compact and R, $0<R<1$, be a given number. Then there exists an r, $0<r<R$ and a closed set V such that*

(a) *$V \subset \Omega(R', r')$ where $R > R' > r' > r$.*

(b) *There exists a diffeomorphism $\psi : (\hat{\Omega}(R,r) \setminus V) \to \Lambda$ such that*

$$\psi(\omega(R) \cup \omega(r)) = \partial \Lambda .$$

(c) *There exists a biholomorphic imbedding $\phi : (B \setminus V) \to M$ such that $\phi(\Omega(R,r) \setminus V) \supset K$.*

Proof. Using Lemma 2.2 we can find an $\varepsilon > 0$ and F (see Lemma 2.2). For this $\varepsilon > 0$ and the given R we use Lemma 1.2 to find $r > 0$ and $A \in \mathrm{Aut}(B)$. Let $V = \bar{B} \setminus A^{-1}((F)M))$ and $\phi = F^{-1} \circ A$. One can see that the statements (a), (c) hold. ψ can now be constructed by using the diffeomorphism related to the fact $[F(M) \cap B] \in T_0'(n)$.

4

We consider now $\mathfrak{B}(n)$. Evidently this set is countable. So, let this set be $\{M_1, M_2, \ldots, M_s, \ldots\}$. Every M_s we represent now as $M_s = \bigcup_{j=1}^{\infty} K_{sj}$ where $K_{sj+1} \supset K_{sj}$ for all j and K_{sj} is a compact in M_s. If we take all $\{K_{sj}\}$, then this is again a countable set $\{K^{(1)}, K^{(2)}, \ldots, K^{(s)}, \ldots\}$.

5

We are going to construct now sequences $\{R_s\}$, $\{r_s\}$ and closed sets $\{V_s\}$ such that

(a) $R_s > r_s > R_{s+1}$ for all $s \geqq 1$.

(b) $V_s \subset \Omega(R_s', r_s')$ where $R_s > R_s' > r_s' > r_s$.

(c) There exists a diffeomorphism $\psi_s : (\hat{\Omega}(R_s, r_s) \setminus V_s) \to \Lambda$ such that

$$\psi_s(\omega(R_s) \cup \omega(r_s)) = \partial \Lambda .$$

(d) Let $K^{(s)}$ from above be equal $K^{(s)} = K_{ij} \subset M_i$. Then there exists a biholomorphic imbedding $\phi_s : (B \setminus V_s) \to M_i$ such that $\phi_s(\Omega(R_s, r_s) \setminus V_s) \supset K^{(s)}$.

We construct R_s inductively and then construct r_s, V_s, ψ_s, ϕ_s. For $s=1$ we take $R_1 = 1/2$. Suppose that R_s for $s \geqq 1$ has been constructed. r_s, V_s, ψ_s, and ϕ_s can now be constructed using Lemma 2.3. Now we take $R_{s+1} = r_s/2$.

6

Let $D = B \setminus \left(\bigcup_{s=1}^{\infty} V_s \right)$. One can see from the construction that any domain M_s (from step 4) and therefore any domain $G \in T_0(n)$ (see Lemma 2.1) can be exhausted by D. We have to prove now that $D \in T_0(n)$. We are going to construct a diffeomorphism from D to $\Lambda^0 = (-1, 1) \times \mathbb{R}^{2n-1} \in T_0(n)$. Let $r_0 = 2$ and for all $k \geqq 0$ $l_{2k} = 1 - r_k$, $l_{2k+1} = 1 - R_{k+1}$, $E_{2k} = \emptyset$, $E_{2k+1} = V_{k+1}$, where $\{r_k\}$, $\{R_k\}$, and $\{V_k\}$ are from step 5. We introduce now for $s \geqq 0$

$$\Omega_s = \Omega(1-l_s, 1-l_{s+1}) \setminus E_s, \qquad \omega_s = \omega(1-l_s),$$
$$\hat{\Omega}_s = \omega_s \cup \Omega_s \cup \omega_{s+1}.$$

$\pi_0 = \emptyset$ and $\pi_s = \{(t, y) \mid t = l_s, y \in \mathbb{R}^{2n-1}\}$ for $s \geqq 1$. One can check that $D = \bigcup_{s=0}^{\infty} \hat{\Omega}_s$. We take now diffeomorphisms $\{\varphi_s\}$, $\varphi_0 \colon \hat{\Omega}_0 \to (-1, l_1] \times \mathbb{R}^{2n-1}$ and for $s \geqq 1$ φ_s: $\hat{\Omega}_s \to [l_s, l_{s+1}] \times \mathbb{R}^{2n-1}$, with the following property (for every $s \geqq 0$) $\varphi_s(\omega_i) = \pi_i$ for $i = s, s+1$. The existance of φ_s for $s = 2k+1$ one can see from step 5; for $s = 2k$ one can construct such diffeomorphisms independently. φ_{s-1} and φ_s do not necessarily agree on the common "side" of $\hat{\Omega}_{s-1}$ and $\hat{\Omega}_s$, namely on ω_s and therefore we have to change the sequence $\{\varphi_s\}$. Consider now $\phi_0 = \varphi_0$ and inductively for $s \geqq 1$

$$\phi_s = \Psi_s \circ \varphi_s : \hat{\Omega}_s \to [l_s, l_{s+1}] \times \mathbb{R}^{2n-1}$$

where

$$\Psi_s \colon [l_s, l_{s+1}] \times \mathbb{R}^{2n-1} \to [l_s, l_{s+1}] \times \mathbb{R}^{2n-1};$$
$$\Psi_s(t, y) = (t, P_Y \circ [\phi_{s-1} \circ \varphi_s^{-1}(l_s, y)]),\ t \in [l_s, l_{s+1}],\ y \in \mathbb{R}^{2n-1}$$

and P_Y is the projection on $\mathbb{R}^{2n-1}$, $P_Y(t, y) = y$. One can check now that all ϕ_s are diffeomorphisms, $\phi_s|_{\omega_s} = \phi_{s-1}|_{\omega_s}$ and $\phi_s(\omega_s) = \pi_s$. Therefore if we choose $\phi(z) = \phi_s(z)$ for $z \in \Omega_s$ then $\phi : D \to \Lambda^0$ is a homeomorphism which is diffeomorphic on all Ω_s and maps ω_s onto π_s. One can now use the smoothing theorem (see Morris W. Hirsch, Differential Topology, Springer-Verlag, 1976, p. 181–182) to change ϕ into a diffeomorphism between D and Λ^0.

7

Remark. The construction of D can be made in such a way that the boundary ∂D is C^∞ everywhere except one point (namely, $p \in \partial B^n$).

8

Remark. If $K \subset\subset G \in T_0(n)$ then the biholomorphic imbedding $F \colon D \to G$ such that $F(D) \supset K$ can be chosen to be a rational mapping.

3. Proof of Theorem 2

We are going to use the results of Chap. 5 in [6]. Every $G \in T_1(1)$ is conformally equivalent to an annulus $G' = \{w \mid r < |w| < R\}$ [6, Theorem 1, Sect. 1]. Let $T' \subset T_1(1)$ be the set of all bounded domains G that do not have isolated boundary points. If $G \in T'$ then any annulus G' conformally equivalent to G also belongs to T' and therefore $0 < r < R < \infty$. Let $M(G) = R/r$. $G_1, G_2 \in T'$ are conformally equivalent if and only if $M(G_1) = M(G_2)$. Therefore there is a continuum of non-equivalent annuli in T'. All this is well known.

We will prove the Theorem 2 if we prove the following.

1

Lemma 3.1. *If $G_1, G_2 \in T'$ are approximately H-equivalent then they are conformally equivalent.*

Proof. Let G_1, G_2 be annuli that are approximately H-equivalent and suppose $M(G_1) > M(G_2)$. Using a linear transformation if necessary we can assume that $G_i = \{w \mid r_i < |w| < R_i\}$, $i = 1, 2$ and $r_1 < r_2 < R_2 < R_1$. So, $\bar{G}_2 \subset G_1$. Let $K_1 = \bar{G}_2$, $K_2 = \{w \mid r_3 \leqq |w| \leqq R_3\}$, where $r_2 < r_3 < R_3 < R_2$. Since G_1, G_2 are approximately H-equivalent there exist a domain $D \in T_1(1)$ and biholomorphic imbeddings $F_i : D \to G_i$ such that $F_i(D) \supset K_i$. One can see that $D \in T'$. Using Theorem 3, Sect. 1 in [6] we find that $M(D) > M(G_2)$ since $F_1(D) \supset \bar{G}_2$ and G_2 divides the boundary components of $F_1(D)$. On the other hand $M(D) \leqq M(G_2)$ from a similar argument, $K_2 \subset F_2(D) \subset G_2$. The last two inequalities give us a contradiction that proves the Lemma 3.1 and therefore the Theorem 2.

2

Remark. Theorem 2 can be generalized for many (diffeomorphism) classes. It seems very likely, but is an open question, that the statement of Theorem 2 will hold for a class T in $\mathbb{C}^n$ that contains a domain with C^∞ boundary if $T \neq T_0(n)$.

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Carleman Approximation on Riemann Surfaces*

André Boivin**

Department of Mathematics, University College, Gower Street, London WCIE 6BT, UK

0. Introduction

A closed subset E of a non-compact (connected) Riemann surface R is called a set of holomorphic (respectively meromorphic) Carleman approximation if whenever f is continuous on E and holomorphic in the interior E^0 of E and ε is continuous and positive on E, there exists a holomorphic (respectively meromorphic) function g on R such that

$$|f(p)-g(p)|<\varepsilon(p), \quad \text{for all} \quad p\in E.$$

We will give a complete characterization of the sets of holomorphic Carleman approximation on an arbitrary non-compact Riemann surface, thus generalizing a result of A.H. Nersesjan in the planar case, and we will give new conditions in terms of fine topology and Gleason parts for sets to be sets of meromorphic Carleman approximation.

In 1927, Carleman [10], in his attempt to generalize the approximation theorem of Weierstrass, actually proved that the real line is a Carleman set of approximation by holomorphic functions in $\mathbb{C}$, and in 1971, Nersesjan [31] gave a complete characterization of sets of holomorphic Carleman approximation in the planar case.

The main ideas here were an argument of convexity (subharmonicity) used by Gauthier in 1969 [22] to show the necessity of the conditions and a lemma of Nersesjan that played a key role in the construction of the approximant. This lemma, which we call the "0-1 Lemma," replaces the Fusion Lemma of Roth [33] used in the theory of uniform approximation on closed sets.

By generalizing the lemma, we were able to prove Nersesjan's result on an arbitrary non-compact Riemann surface and to obtain a sufficient condition on sets of meromorphic Carleman approximation. The original idea of Gauthier [22] also led us to the establishment of a necessary condition to be satisfied by such sets. Unfortunately, unlike the holomorphic case, these arguments lead to conditions that no longer coincide.

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** Present address: Dept. of Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B7

Notation

In this article, R will always denote a non-compact connected Riemann surface, K a compact subset of R and E a closed subset of R; if S is an arbitrary subset of R, then $\mathrm{Hol}(S)$ (respectively $\mathrm{Mer}(S)$) will be the set of all functions holomorphic (respectively meromorphic) on S; if S is compact or closed, then $H(S)$ will denote the uniform closure in $C(S)$ ($=$continuous functions on S) of the functions in $\mathrm{Hol}(S)$; similarly $M(S)$ will be the uniform closure in $C(S)$ of the functions meromorphic on R without poles on S; $A(S)$ is the algebra of functions continuous on S and holomorphic in the interior S^0 of S. The fine topology is the coarsest topology on R for which all superharmonic functions on R are continuous. The terms *fine* and *finely* will mean "in the fine topology." For any subset S of R, we will denote by S' the fine interior of S, by $\tilde{S}$ the fine closure of S and by $\partial_f S$ the fine boundary of S in R.

By Runge's theorem [4], the uniform closure in $C(K)$ of $\mathrm{Hol}(K)$ is equal to $M(K)$. Also if K is compact, $\partial_f K$ is the set of stable boundary points for the Dirichlet problem [9, p. 60] and [13, p. 113].

1. Statement of the Results

Definition 1. A closed subset E of a non-compact Riemann surface R is called a set of holomorphic (respectively meromorphic) uniform approximation if for each constant $\varepsilon>0$ and for each $f\in A(E)$, there exists $g\in \mathrm{Hol}(R)$ (respectively $g\in\mathrm{Mer}(R)$) such that

$$|f(p)-g(p)|<\varepsilon, \quad \text{for all} \quad p\in E.$$

It is when ε can be chosen to be an arbitrary positive continuous function on E instead of a constant that the set is called a *Carleman approximation set.* In particular, ε is then allowed to decrease arbitrarily rapidly at "infinity." Sinclair [35] even showed that with appropriate restrictions, the function ε can be allowed zeros on E^0 and at some points of ∂E.

It follows from the definition that every set of Carleman approximation is a set of uniform approximation. In the holomorphic case, this has some interesting implications. Denote by $R^*=R\cup\{*\}$ the one-point compactification of R.

Definition 2. We say that a subset S of R satisfies the condition $\mathscr{R}-\mathscr{K}$ if $R^*\backslash S$ is connected and locally connected.

Introduced in the theory of approximation by Roth and Keldysh, this condition is necessarily satisfied by all sets of holomorphic uniform approximation [3, 24] and, a fortiori, by all holomorphic Carleman sets. In fact, if E is a relatively closed subset of a domain D in the complex plane $\mathbb{C}$, then E is a set of holomorphic uniform approximation if and only if E satisfies $\mathscr{R}-\mathscr{K}$ (Arakelyan's theorem [3]); but on an arbitrary non-compact Riemann surface the condition $\mathscr{R}-\mathscr{K}$ is, in general, not sufficient [24]. For more background on uniform approximation on closed sets (holomorphic or meromorphic) we refer the interested reader to [16, 25].

Definition 3. Let S be a subset of R. If for every compact set $K \subset R$, there is a compact Q, $R \supset Q \supset K$, such that no component of the interior S^0 of S meets both K and $R \backslash Q$, we then say that S satisfies condition $\mathscr{G}$.

Introduced in 1969 by Gauthier [22], condition $\mathscr{G}$ is also necessary for Carleman approximation (holomorphic *or* meromorphic). In fact, in [22], the necessity was established only for the holomorphic case with R a disk in $\mathbb{C}$, but one can easily verify that the proof remains valid in the general case.

We can now state our first result.

Theorem 1. *Let R be a non-compact Riemann surface and let E be a subset of R. Then the following are equivalent:*

(a) *E is a set of holomorphic Carleman approximation.*

(b) *E is a closed subset of R which satisfies conditions $\mathscr{R}-\mathscr{K}$ and $\mathscr{G}$.*

(c) *E is a set of holomorphic uniform approximation satisfying condition $\mathscr{G}$.*

Suppose now that E is a set of meromorphic Carleman approximation. As already mentioned, E must be a set of meromorphic uniform approximation and E must still satisfy condition $\mathscr{G}$. However, it is known [6] that these alone are no longer sufficient in the meromorphic case.

Definition 4. Let S be a subset of R. We say that S satisfies condition $\mathscr{G}_F$ if for every compact set $K \subset R$, there is a compact Q, $R \supset Q \supset K$, such that no component of the fine interior S' of S meets both K and $R \backslash Q$.

Theorem 2. *Let E be a set of (meromorphic) Carleman approximation. Then E satisfies condition $\mathscr{G}_F$.*

Remark. Since the fine topology is finer than the usual Euclidean topology, condition $\mathscr{G}_F$ implies condition $\mathscr{G}$; also condition $\mathscr{G}_F$ must be satisfied by every holomorphic Carleman approximation set (see Sect. 4). However in the holomorphic case, since the complement of E must also be connected, the fine interior and the interior of E coincide [28, Theorems 8.26, 10.12].

The next condition is in terms of Gleason parts. Background material on uniform algebras can be found in [18, 36]. The fact that the maximal ideal space of $M(K)$ is K is in [32]. Note that Harnack's inequality shows that if p and q belong to the interior of the same component of the interior of K, then p and q are in the same Gleason part of $M(K)$.

Definition 5. Let E be a closed subset of R and let K_n be an exhaustion of R by compacta. We define the parts of E to be the limit as n tends to ∞ of the Gleason parts of $M(E \cap K_n)$; that is, two points of E are in the same part of E if eventually they are in the same Gleason part of $M(E \cap K_n)$. A part is called trivial if it consists of only one point.

Lemma 2 (Sect. 5) shows that this definition makes sense and is independent of the choice of the exhaustion K_n.

Definition 6. Let E be a closed subset of R. We will say that E satisfies condition $\mathscr{G}_P$ if for every compact set $K \subset R$, there exists a compact set Q, $R \supset Q \supset K$, such that no (non-trivial) part of E meets both K and $R \backslash Q$.

Theorem 3. *Let E be a closed subset of a non-compact Riemann surface R. If E is a set of meromorphic uniform approximation and if E satisfies condition $\mathcal{G}_P$ then E is a set of meromorphic Carleman approximation.*

Remarks

– Condition $\mathcal{G}_P$ implies condition $\mathcal{G}_F$ (thus condition $\mathcal{G}$). Indeed the points in one component of the fine interior of a compact set K are all contained in the same Gleason part of $M(K)$, the Jensen measures at those points being mutually absolutely continuous [12].
– Based on an example of Davie and Garnett [11] of a compact set with a disconnected Gleason part, one can construct a set of meromorphic uniform approximation satisfying condition $\mathcal{G}_F$ but not $\mathcal{G}_P$. We do not know if such sets are sets of Carleman approximation.
– If E is a set of meromorphic uniform approximation and if E has no interior, then E is a set of meromorphic Carleman approximation (Roth's theorem [33, 23]). Indeed since, by hypothesis, $M(E \cap K) = C(E \cap K)$, each point of E is a peak point for $M(E \cap K)$ for some K, and thus must be a trivial part of E [36, p. 312]. Condition $\mathcal{G}_P$ is therefore trivially satisfied.

2. Preliminary Results and the 0-1 Lemma

We are interested in conditions under which one can find a sequence of uniformly bounded functions in $M(K)$ converging to zero on the interior of some subset S of K and converging to one on, say, the complement of its closure (i.e. on $K \backslash \bar{S}$). This is impossible if there exist two points, one in S and the other in $K \backslash \bar{S}$, that are in the same Gleason part of $M(K)$ [18, VI.2.1]. We will show that this is essentially the only restriction and it will be a consequence of the following two results.

Wilken's Theorem [37], [8, Theorem 6.3]. *Let P be a Gleason part of $M(K)$. Then every representing measure μ for a point $p \in P$ is supported on the closure $\bar{P}$ of P.*

Forelli's Lemma. *Let X be a compact Hausdorff space, A a uniform algebra of $C(X)$ and ϕ an element of the maximal ideal space of A. If $F \subset X$ is a countable union of compact sets E_n such that $\lambda(F) = 0$ for each representing measure λ for ϕ, then there exists a sequence $\{f_n\}$ of functions in A such that*
 i) $|f_n| \leqq 1$ *on* X.
 ii) $|f_n| < e^{-n}$ *on* E_n *(i.e.* $f_n \to 0$ *on* F*).*
 iii) $f_n \to 1$ *a.e.* λ*, for each representing measure* λ *for* ϕ.

The preceding lemma, due to Glicksberg [26], is an analog of a version of Forelli's lemma [14] obtained by Ahern [1].

Corollary 1. *Let K be a compact subset of R, let P be a non-trivial Gleason part of $M(K)$ and let $\{E_n\}$ be a sequence of compact subsets such that $\bigcup_{n=1}^{\infty} E_n = K \backslash \bar{P}$. Then there is a sequence $f_n \in M(K)$ such that*
 i) $|f_n| \leqq 1$ *on* K.
 ii) $|f_n| < e^{-n}$ *on* E_n.
 iii)′ $f_n \to 1$ *on* P*, uniformly on compacta of* P^0.

Proof. Fix $p \in P$ and let ϕ_p, in the maximal ideal space of $M(K)$, be the evaluation functional at the point p. By Wilken's theorem, $\lambda(K\backslash\bar{P})=0$ for each representing measure λ for ϕ_p. By Forelli's lemma, there is a sequence $\{f_n\}$ of functions in $M(K)$ which satisfies i), ii), iii). We claim that the sequence actually satisfies iii)$'$.

Since δ_p, the point mass at p, is a representing measure for ϕ_p, iii) implies that $f_n(p)\to 1$. Let $q \in P$, $q \neq p$. By [18, IV.1.2], there exists a representing measure λ_0 for ϕ_p such that δ_q, the point mass at q, is absolutely continuous with respect to λ_0. Thus $\lambda_0(\{q\})\neq 0$ and $f_n(q)\to 1$, i.e. f_n converges pointwise to 1 on P. Since the sequence is a normal family, iii)$'$ follows immediately.

We would like to reverse the role of the "zero" and the "one" in the preceding result.

Lemma 0-1. *Let K, P, and E_n be as above. Then there is a sequence $\{g_n\}$ of functions in $M(K)$ such that*

1) $|g_n|\leqq 1$ *on* K.
2) $|1-g_n|<1/n$ *on* E_n *(i.e.* $g_n\to 1$ *on* $K\backslash\bar{P}$*).*
3) $g_n\to 0$ *on* P^0*, uniformly on compacta.*

Proof. There is a sequence $\{f_k\}$ of functions in $M(K)$ satisfying i), ii), and iii)$'$. Let $x_n=1-1/n$ and

$$t_n(z)=\frac{1}{2}\left(\frac{x_n-z}{1-x_n z}\right)+\frac{1}{2}.$$

Then t_n maps the unit disk $\bar{U}=\{z\colon |z|\leqq 1\}$ conformally onto the disk $\{w\colon |w-1/2|<1/2\}$. Also $t_n(1)=0$ and $t_n(0)=1-1/2n$.

First notice that $t_n\circ f_k\in M(K)$ and $|t_n\circ f_k|\leqq 1$ on K. Let $\{Q_n\}$ be a sequence of compacta such that $Q_n\subset Q_{n+1}$ and $\bigcup_{n=1}^{\infty} Q_n=P^0$. Fix n. Since t_n is uniformly continuous on $\bar{U}$, there exists a small constant $\delta(n)$, depending only on n, such that whenever $z_1, z_2\in\bar{U}$ and $|z_1-z_2|<\delta(n)$, we have $|t_n(z_1)-t_n(z_2)|<1/2n$. Also there are constants $A(n)$ and $B(n)$, such that whenever $k\geqq A(n)$, $|f_k-1|<\delta(n)$ on Q_n and whenever $k\geqq B(n)$, then $|f_k|<\delta(n)$ on E_n. Take $k(n)=\max(A(n), B(n))$. Set $g_n=t_n\circ f_{k(n)}$. Then the sequence $\{g_n\}$ satisfies 1), 2), and 3). Indeed

$$|g_n|=|t_n\circ f_{k(n)}-t_n(1)|<\frac{1}{2n}\quad\text{on } Q_n$$

and

$$|g_n-1|\leqq|t_n\circ f_{k(n)}-t_n(0)|+|t_n(0)-1|<\frac{1}{n}\quad\text{on } E_n\,.$$

Definition 7. A pair $(\mathscr{P}, K)$ of sets where $\mathscr{P}\subset K\subset R$, K is compact and $\mathscr{P}$ can be written as $\bigcup_{k=1}^{N} P_k$, for some N, $1\leqq N\leqq\infty$, with each pair (P_k, K) satisfying 1), 2), and 3), is called a Nersesjan pair.

The pair (P, K) of the preceding lemma is a Nersesjan pair (with $N=1$).

Corollary 2. *Let $(\mathscr{P}, K)$ be a Nersesjan pair with $\mathscr{P}=\bigcup_{k=1}^{N} P_k$, for some N, $1\leqq N\leqq\infty$, each pair (P_k, K) satisfying* 1), 2) *and* 3). *If $\{E_n\}$ is a sequence of compact sets such that $\bigcup_{n=1}^{\infty} E_n=K\backslash(\bigcup_{k=1}^{N}\bar{P}_k)$, then there is a sequence $\{g_n\}$ of* *functions in $M(K)$ such that*

(I) $|g_n|\leqq 1$ *on* K;

(II) $|g_n-1|<1/n$ *on* E_n *(i.e.* $g_n\to 1$ *on* $K\setminus(\bigcup \bar{P}_k)$*)*;

(III) $g_n\to 0$ *on* $\bigcup_{k=1}^N P_k^0$, *uniformly on compacta.*

Proof. Let $\{Q_n\}$ be a sequence of compacta such that $Q_n\subset Q_{n+1}$ and $\bigcup_{n=1}^\infty Q_n = \bigcup_{k=1}^N P_k^0$. By hypothesis, there exists a sequence $\{f_{k,n}\}$ of functions in $M(K)$ such that

1) $|f_{k,n}|\leqq 1$ on K, 2) $|f_{k,n}-1|<\varepsilon_n$ on E_n, 3) $|f_{k,n}|<1/n$ on $Q_n\cap P_k^0$. Set

$$g_n=\prod_{k=1}^n f_{k,n}.$$

Then $|g_n|\leqq 1$ on K and $|g_n(p)|<1/n$ for $p\in(\bigcup_{k=1}^n P_k^0)\cap Q_n$. It suffices then to choose ε_n so small that $|1-g_n(p)|<1/n$ on E_n.

Lemma 1. *Let P be a union of Gleason parts of $M(K)$, then $\partial P\subset\partial K$.*

Proof. Let $x\in\partial P$. If $x\notin\partial K$, then there is an open connected neighbourhood V of x such that $V\subset K$. Since $x\in\partial P$, $V\cap P\neq\emptyset$, i.e. there is a Gleason part P_0 of $M(K)$ and a point $y\in P_0$ such that $y\in V$. But V is open and connected so V must be contained in P_0 and $x\notin\partial P$. This is a contradiction which establishes the lemma.

3. Proof of Theorem 1

Definition 8. We say that a subset S of a Riemann surface R is of finite co-connectivity if its complement, i.e. $R\setminus S$, has only a finite number of components.

Theorem [19, p. 157]. *Let K be a compact subset of a non-compact Riemann surface R. If $R^*\setminus K$ is connected then $H(K)$ is a hypodirichlet algebra on ∂K.*

Corollary 3. *If K is of finite co-connectivity, then $M(K)$ is a hypodirichlet algebra on ∂K.*

Proof. Let $U_1,\dots,U_n$ be the bounded components of $R\setminus K$ and let p_j be a point in U_j, $1\leqq j\leqq n$. Then $R_0=R\setminus(\bigcup_{j=1}^n\{p_j\})$ is a Riemann surface such that $R_0^*\setminus K$ is connected. By Runge's theorem [4], the closure in $C(K)$ of the functions holomorphic on R_0 is equal to $M(K)$ and consequently, by the previous theorem, $M(K)$ is hypodirichlet on ∂K.

Proposition 1. *Let K be a compact subset of finite co-connectivity of a non-compact Riemann surface R. The non-trivial Gleason parts of $M(K)$ are precisely the components of the interior of K.*

Proof. It is known [17] that every Gleason part of a hypodirichlet algebra is either a one-point part or a finite (connected) bordered Riemann surface. Let $K\subset R$ be of finite co-connectivity. It also follows from the abstract theory of hypodirichlet algebra (e.g. [18, p. 149]) that if $p\in\partial K$ then p is a peak point for $M(K)$. In particular $\{p\}$ is a one-point part [36, p. 312]. The non-trivial Gleason parts of $M(K)$ must then be the components of the interior of K.

A proof along the lines of [2] can be found in [7].

Remark. If K is of finite co-connectivity and if G is an open subset of K such that $\partial G \subset \partial K$, then (G, K) is a Nersesjan pair. Indeed, since $\partial G \subset \partial K$, G is a union of (at most countably many) G_k, where each G_k is a connected component of the interior of K, and thus a Gleason part of $M(K)$. This result was proved by Nersesjan [31] in the case where K is a subset of $\mathbb{C}$ by an entirely different method.

Proof of Theorem 1 (see [31]). We have already noticed in Sect. 1 that c) implies b) and that a) implies c). To show that b) implies a), we will construct an exhaustion of R so as to satisfy the hypothesis of the 0-1 Lemma.

Let A_0 be the closure of a pre-compact open subset U_0 of R bounded by a finite number of Jordan curves. B_0 will be the closure of the pre-compact components of $R \backslash (E \cup A_0)$. C_0 is the closure of the union of all components of E^0 whose closure meet $A_0 \cup B_0$. Let

$$D_0 = A_0 \cup B_0 \cup C_0 .$$

First notice the following properties of D_0.

a) D_0 is closed.

b) D_0 is compact. This follows from the fact that E satisfies conditions $\mathscr{G}$ and $\mathscr{R}-\mathscr{K}$.

c) $\partial D_0 \subset \partial(D_0 \cup E)$. Indeed, since $\partial D_0 \subset (\partial A_0 \cup \partial B_0 \cup \partial C_0)$, a point $p \in \partial D_0 \cap (\partial A_0 \cup \partial B_0)$ is a limit point of the set $R \backslash (D_0 \cup E)$ because clearly $p \in \overline{R \backslash D_0}$ and $p \in \overline{R \backslash E}$. Similarly a point $p \in \partial C_0$ cannot be an interior point of E and $p \in \partial D_0 \cap \partial C_0$ implies $p \in \partial(D_0 \cup E)$.

d) No component of $R \backslash D_0$ or $R \backslash (D_0 \cup E)$ is pre-compact in R.

Following the same procedure, we obtain the set D_1 from a pre-compact open set U_1 bounded by finitely many Jordan curves and containing D_0. Then we construct $D_2, D_3, \ldots, D_k, \ldots$ inductively making sure they form an exhaustion of R.

Now choose compact sets $\hat{D}_k$, $k \geqq 0$ so that

e) $D_k \subset (\hat{D}_k)^0$; $\hat{D}_k \subset (D_{k+1})^0$.

f) $R \backslash \hat{D}_k$ has only a finite number of components, none pre-compact.

Remarks

g) Set $K = (D_k \cup E) \cap \hat{D}_j$, where $j > k$. K is compact and of finite co-connectivity in R. Consequently $(D_k^0, D_k \cup (E \cap \hat{D}_j))$, $j > k$, is a Nersesjan pair.

h) $R \backslash (E \cap \hat{D}_k)$ has only a finite number of components since none is pre-compact. By the Mergelyan-Bishop-Kodama Theorem [30] [5] [29], each function in $A(E \cap \hat{D}_k)$ can be uniformly approximated by functions holomorphic on R.

i) Since no component of the complement of $D_k \cup (E \cap \hat{D}_j)$, $j > k$, is pre-compact, the functions in the 0-1 Lemma can be chosen to be holomorphic on R [4].

Let $f \in A(E)$ and let ε be a positive continuous function on E. Set

$$\varepsilon_k = \min_{p \in (\hat{D}_k \cap E)} \varepsilon(p) .$$

It follows from h) that there is a function u_1 holomorphic on R such that

$$|f - u_1| < \frac{\varepsilon_1}{4} \quad \text{on} \quad E \cap \hat{D}_1 .$$

There exists also a function $v_2 \in \mathrm{Hol}(R)$ such that

$$|(f-u_1)-v_2)| < \frac{\varepsilon_2}{4} \quad \text{on} \quad (\hat{D}_2 \backslash \hat{D}_0) \cap E;$$

g) and i) imply that for each $\delta > 0$, there is a holomorphic function n_2 (depending on δ) such that

$$\begin{aligned} |n_2| &< \delta && \text{on} \quad \hat{D}_0, \\ |1-n_2| &< \delta && \text{on} \quad (\hat{D}_2 \backslash \hat{D}_1) \cap E, \\ |n_2| &\leqq 1 && \text{on} \quad D_1 \cup (E \cap \hat{D}_2). \end{aligned}$$

Choose δ so small that

$$|(f-u_1)-n_2 v_2| < \frac{\varepsilon_2}{4} \quad \text{on} \quad (\hat{D}_2 \backslash \hat{D}_1) \cap E,$$

$$|(f-u_1)-n_2 v_2| < |f-u_1| + |n_2|\,|v_2| < \frac{\varepsilon_1}{4} + 1\left(\frac{\varepsilon_1}{4} + \frac{\varepsilon_2}{4}\right) \tag{1}$$

$$< \varepsilon_1 \quad \text{on} \quad (\hat{D}_1 \backslash \hat{D}_0) \cap E,$$

$$|(f-u_1)-n_2 v_2| < \frac{\varepsilon_1}{4} + \delta |v_2| < \varepsilon_0 \quad \text{on} \quad \hat{D}_0 \cap E \tag{2}$$

and

$$|n_2 v_2| < \delta |v_2| < \frac{1}{2^2} \quad \text{on} \quad D_0.$$

Set $u_2 = n_2 v_2$. There exists a function v_3 holomorphic on R such that

$$|(f-u_1-u_2)-v_3| < \frac{\varepsilon_3}{4} \quad \text{on} \quad (\hat{D}_3 \backslash \hat{D}_1) \cap E.$$

Again there is, for each $\delta > 0$, a function $n_3 \in \mathrm{Hol}(R)$ such that

$$\begin{aligned} |n_3| &< \delta && \text{on} \quad \hat{D}_1, \\ |1-n_3| &< \delta && \text{on} \quad (\hat{D}_3 \backslash \hat{D}_2) \cap E, \\ |n_3| &\leqq 1 && \text{on} \quad D_2 \cup (E \cap \hat{D}_3). \end{aligned}$$

Choose δ so small that

$$\begin{aligned} |f-u_1-u_2-n_3 v_3| &< \frac{\varepsilon_3}{4} && \text{on} \quad (\hat{D}_3 \backslash \hat{D}_2) \cap E, \\ |f-u_1-u_2-n_3 v_3| &< \varepsilon_2 && \text{on} \quad (\hat{D}_2 \backslash \hat{D}_1) \cap E, \\ |f-u_1-u_2-n_3 v_3| &< \varepsilon_1 && \text{on} \quad (\hat{D}_1 \backslash \hat{D}_0) \cap E, \\ |f-u_1-u_2-n_3 v_3| &< \varepsilon_0 && \text{on} \quad \hat{D}_0 \cap E, \end{aligned}$$

and

$$|n_3 v_3| < \frac{1}{2^3} \quad \text{on} \quad D_1.$$

This can be done since the inequalities in (1) and (2) were strict. Set $u_3 = n_3 v_3$.

Thus by induction, we can find u_k holomorphic on R such that

$$|u_k| < \frac{1}{2^k} \quad \text{on} \quad D_{k-2}, \qquad k \geqq 2,$$

$$|f - u_1 - u_2 - \cdots - u_k| < \varepsilon_j \quad \text{on} \quad E \cap (\hat{D}_j \backslash \hat{D}_{j-1}), \qquad j = 1, 2, \ldots, k-1,$$

$$|f - u_1 - u_2 - \cdots - u_k| < \frac{\varepsilon_k}{4} \quad \text{on} \quad E \cap (\hat{D}_k \backslash \hat{D}_{k-1}).$$

Then the function $g = \sum_{k=1}^{\infty} u_k$ is holomorphic on R and satisfies

$$|f(p) - g(p)| < \varepsilon(p), \qquad p \in E.$$

4. Proof of Theorem 2

Definition 9. Let S be a proper subset of a non-compact Riemann surface R. S is called a set of uniqueness if there exists a positive continuous function ε on S such that whenever f is a meromorphic function on R with

$$|f(p)| < \varepsilon(p), \qquad p \in S,$$

then f is identically zero.

Theorem [22]. *If S is a set of uniqueness, then S is not a set of Carleman approximation (holomorphic or meromorphic).*

Proof of Theorem 2. Since a set of Carleman (or for that matter uniform) approximation is by definition closed, we will assume without loss of generality that E is closed on R. Assuming E does not satisfy condition $\mathscr{G}_F$, we will then show that E is a set of uniqueness.

Let D_n be an exhaustion of R by pre-compact domains with $\bar{D}_n \subset D_{n+1}$. Since E does not satisfy $\mathscr{G}_F$, there is a compact set K and a sequence E'_n of fine components of the fine interior of E such that, for each n, E'_n meets both K and $R \backslash \bar{D}_n$. We can assume $K \subset D_n$, for all n. Also note that the E'_n are not necessarily distinct.

Let A_n be a compact subset of $\overline{E'_n}$ such that A'_n, the fine interior of A_n, is connected and satisfies $\overline{A'_n} = A_n$, $A'_n \cap K \neq \emptyset$, $A'_n \backslash \bar{D}_n \neq \emptyset$. To see that such sets exist, first notice that $A'_n \subset E'_n$ since $E'_n = (\bar{E}'_n)'$; then choose $p_1 \in E'_n \cap K$, $p_2 \in E'_n \backslash \bar{D}_n$; E'_n is connected (in the usual topology) [20]; so there is a path $\Gamma \subset E'_n$ joining p_1 and p_2; there is an n_0 for which $\Gamma \subset D_{n_0}$. Choose A'_n to be the component of $(\bar{D}_{n_0})' \cap E'_n$ containing Γ.

For each $p \in A_n$, let $v_{p,n}$ be the Keldysh measure on A_n at the point p [21]. Suppose that f is a meromorphic function on R and that

$$|f(p)| < \varepsilon(p) \leqq 1, \qquad p \in E,$$

where $\varepsilon(p)$ is a continuous positive function to be specified. Set

$$\varepsilon_n = \sup_{E \backslash D_n} \varepsilon(p).$$

Since $f \in \mathrm{Hol}(A_n)$ and $\nu_{p,n}$ is a Jensen measure supported on $\partial_f A_n$ (=stable boundary points for the Dirichlet problem on A_n), we have

$$\begin{aligned}\log|f(p)| &\leqq \int_{\partial_f A_n} \log|f| d\nu_{p,n} \\ &\leqq \int_{(\partial_f A_n)\setminus D_n} \log|f| d\nu_{p,n} + \int_{(\partial_f A_n)\cap D_n} \log|f| d\nu_{p,n} \\ &\leqq (\log \varepsilon_n)\nu_{p,n}((\partial_f A_n)\setminus D_n) + 0, \quad \text{for all} \quad p \in A_n.\end{aligned}$$

We would like to show that $\nu_{p,n}((\partial_f A_n)\setminus D_n) > 0$. Let V_n be a small open neighbourhood of $\bar{D}_n$, so small that $A_n' \setminus \bar{V}_n \neq \emptyset$. Let g be a continuous function on ∂A_n such that $0 \leqq g(p) \leqq 1$ and

$$g(p) = \begin{cases} 0 & \text{on} \quad (\partial A_n)\cap D_n, \\ 1 & \text{on} \quad (\partial A_n)\setminus V_n. \end{cases}$$

Set

$$\hat{g}(p) = \int_{\partial_f A_n} g d\nu_{p,n}$$

so that $\hat{g}(p) \leqq \nu_{p,n}((\partial_f A_n \setminus D_n)$. But $\hat{g}$ is the solution of the fine Dirichlet problem, that is $\hat{g}$ is finely continuous on $\tilde{A}_n'$, $\hat{g}$ is finely harmonic on A_n' and

$$\hat{g}|_{\partial_f A_n'} = g|_{\partial_f A_n'} \quad \text{(see [15])}.$$

Moreover $\hat{g} \geqq 0$. However $\hat{g}$ is not constant on A_n'. So by the minimum principle for finely harmonic functions [15, p. 150], $\hat{g} > 0$ on A_n', and thus

$$\nu_{p,n}((\partial_f A_n)\setminus D_n) > 0, \qquad p \in A_n'.$$

We claim there exists an arc Γ_n, joining ∂K to ∂D_1, $\Gamma_n \subset A_n'$ and such that

$$\inf_{p \in \Gamma_n} \nu_{p,n}((\partial_f A_n)\setminus D_n) = \beta_n > 0.$$

Indeed, let $p_1 \in \partial K \cap A_n'$, $p_2 \in \partial D_1 \cap A_n'$. For $\delta > 0$, set

$$B_\delta = \{p \in A_n' | \hat{g}(p) \leqq \delta\}.$$

We claim that p_1 and p_2 are in the same fine component of $A_n' \setminus B_\delta$ for sufficiently small δ's. Suppose this fails. Then let C_k be the component of $A_n' \setminus B_{1/k}$ containing p_1. Note that $C_k \supseteqq C_{k'}$ if $k > k'$ and that $C = \bigcup_{k=1}^\infty C_k$ is connected. $A_n' \setminus B_{1/k}$ is finely open since $\hat{g}$ is finely continuous, thus C is finely open. Let $p \in A_n' \setminus C$. Since $A_n' = \bigcup_{k=1}^\infty (A_n' \setminus B_{1/k})$, p is in $A_n' \setminus B_{1/k}$ for large enough k, say k greater than or equal to k_0. For $k \geqq k_0$, let D_k be the fine component of $A_n' \setminus B_{1/k}$ containing p. We are assuming that $D_k \cap C_k = \emptyset$, $k \geqq k_0$. But since $D_k \supset D_{k'}$, if $k > k' \geqq k_0$, we have that $D_{k_0} \cap C_k = \emptyset$, for all k, and thus $D_{k_0} \cap C = \emptyset$. So the complement of C is finely open and C is finely closed, and since A_n' is finely connected, C must actually be equal to A_n' (being simultaneously open and closed). But this contradicts $p_2 \notin C$. So p_1, p_2 are in the same fine component of $A_n' \setminus B_\delta$ for a certain δ, and there is a path Γ_n joining ∂K to ∂D_1 such that

$$\beta_n = \inf_{p \in \Gamma_n} \nu_{p,n}((\partial_f A_n)\setminus D_n) \geqq \inf_{p \in \Gamma_n} \hat{g}(p) \geqq \delta.$$

Returning to f, we have

$$\log|f(p)| \leqq (\log \varepsilon_n)\nu_{p,n}((\partial_f A_n)\setminus D_n).$$

If $p \in \Gamma_n$, we have (since $\log \varepsilon_n < 0$)

$$\log|f(p)| \leqq \beta_n \log \varepsilon_n,$$

$$|f(p)| \leqq \varepsilon_n^{\beta_n}, \quad \text{for all} \quad p \in \Gamma_n.$$

Now choose ε_n so small that

$$|f(p)| < \frac{1}{n}, \quad \text{for each} \quad p \in \Gamma_n.$$

Since $\{\Gamma_n\}$ is a sequence of arcs joining ∂K to ∂D_1, it follows that f must be identically zero. This completes the proof.

Remark. Results in [20] show that Theorem 2 is the best one can hope to prove in this direction using Jensen's inequality.

5. Proof of Theorem 3

To prove Theorem 3, we will build, as in the holomorphic case, an exhaustion of R that will allow us to use the 0-1 Lemma. To do so, we will need one more result on Gleason parts.

Lemma 2. *Let K be a compact subset of a non-compact Riemann surface R, let P be a Gleason part of $M(K)$, and let J be a compact subset of R such that $J \cap \bar{P} = \emptyset$. Then P is a Gleason part of $M(K \cup J)$.*

Proof. Let $p \in P$ and $q \in K \setminus P$. To show that p and q are not in the same Gleason part of $M(K \cup J)$, it suffices to find a sequence $\{g_n\}$ of functions in $M(K \cup J)$ such that $|g_n| \leqq 1$ on $K \cup J$, $g_n(p) \to 1$ and $g_n(q) \to 0$ [18, Theorem VI.2.1]. By hypothesis, such a sequence exists when $K \cup J$ is replaced by K. Combining this with Corollary 1, Sect. 2, we obtain a sequence $\{f_n\}$ of functions in $M(K)$ such that $|f_n| \leqq 1$ on K, $f_n(p) \to 1$, $f_n(q) \to 0$ and $f_n \to 0$ on compact subsets of $K \setminus \bar{P}$.

Let W_1, W_2 be open subsets of R such that $\bar{P} \subset W_1 \subset \subset W_2 \subset \subset R$ and $W_2 \cap J = \emptyset$. Let ϕ be a C^∞-function such that $\phi = 1$ on W_1, $\phi = 0$ on $R \setminus W_2$ and $0 \leqq \phi \leqq 1$. Extend f_n to be zero outside of K. We will modify f_n to be analytic on J using an analog on Riemann surfaces of the Vitushkin localization operator.

Let ϱ be the global local uniformizer of Gunning and Narasimhan [27]. Define, as in [8],

$$(T_\phi f_n)(q) = \phi(q) f_n(q) + \frac{1}{2\pi i} \int\int f_n(p) F(p,q)(\bar{\partial}\phi(p) \wedge \partial\varrho(p)),$$

where $F(p,q)$ is meromorphic on $R \times R$, $F(p,q) - (\varrho(p) - \varrho(q))^{-1}$ is analytic in a neighbourhood of the diagonal and the only singularities of F are simple poles with residues ± 1 on the diagonal. Then, [8, Corollary 4.6], $T_\phi f_n$ is in $M(K)$ and [8, Theorem 4.2] $T_\phi f_n$ is analytic off the closed support of ϕ, so, in particular, on J. Also in the local coordinates system given by ϱ, where it makes sense to talk about the derivatives of ϕ, $\operatorname{grad}\phi$ is bounded and F is locally integrable. Moreover the sequence $\{f_n\}$ converges to zero on the support of $\bar{\partial}\phi$. Thus we have

$$|T_\phi f_n - f_n| \to 0, \quad \text{as} \quad n \to \infty.$$

Dividing $T_\phi f_n$ by its sup norm on $K\cup J$ if necessary, one obtains a sequence $\{g_n\}$ of functions in $M(K\cup J)$ such that $|g_n|\leqq 1$ on $K\cup J$, $g_n(p)\to 1$, $g_n(q)\to 0$ and $g_n\to 0$ on compact subsets of $(K\cup J)\setminus\bar{P}$. This proves the lemma.

Proposition 2. *Let R be a non-compact Riemann surface and let E be a closed subset of R. Then E satisfies condition $\mathscr{G}_P$ if and only if there exists an exhaustion $\{D_n\}$ of R satisfying, for all n,*

(*) $\quad D_n$ *is pre-compact and* $\bar{D}_n\subset D^0_{n+1}$,

(**) D_n *is a union of (at most countably many) non-trivial Gleason parts of*

$$M(\bar{D}_n\cup(E\cap\bar{D}_{n+1})).$$

Proof. Suppose first that there is an exhaustion $\{D_n\}$ of R satisfying (*) and (**). Let K be a compact subset of R and fix n so that $K\subset D_n$. Since $M(\bar{D}_n\cup(E\cap\bar{D}_{n+1}))\subset M(E\cap\bar{D}_{n+1})$, we know that if P is a (non-trivial) Gleason part of $M(E\cap\bar{D}_{n+1})$ that meets K, then P is contained in a Gleason part $\hat{P}$ of $M(\bar{D}_n\cup(E\cap\bar{D}_{n+1}))$. Since $P\cap K\neq\emptyset$, we also have $\hat{P}\cap K\neq\emptyset$ and $\hat{P}\cap D_n\neq\emptyset$. Condition (**) then implies that $\hat{P}\subset D_n$, i.e. $P\subset D_n$ and with (*) this implies that condition $\mathscr{G}_P$ is satisfied for K with $Q=\bar{D}_{n+1}$.

Conversely suppose that $\mathscr{G}_P$ is satisfied. Then the following condition, called $\mathscr{G}_{\bar{P}}$, is also satisfied:

> For every compact $K\subset R$, there is a compact Q, $R\supset Q\supset K$, such that the *closure* of no part of E meets both K and $R\setminus Q$.

Indeed, fix K, and take $\hat{K}$ compact such that $K\subset(\hat{K})^0$. Let $\hat{Q}$ such that condition $\mathscr{G}_P$ is satisfied for $\hat{K}$. Let P be a part of E whose closure meets K. Then P meets $\hat{K}$ and so does not meet $R\setminus\hat{Q}$. Condition $\mathscr{G}_{\bar{P}}$ is thus satisfied for K with $\hat{Q}$.

Now fix $K_0=\hat{K}$ and $Q_0=\hat{Q}$ as in $\mathscr{G}_{\bar{P}}$. If P is a part of E that meets $R\setminus Q_0$, then $\bar{P}$ does not meet K_0. Thus, by the previous lemma, the closure of no part of $E\cup K_0$ that meets $R\setminus Q_0$ can meet K_0 and $E\cup K_0$ satisfies condition $\mathscr{G}_{\bar{P}}$ (thus $\mathscr{G}_P$).

Let D_0 be the union of all non-trivial parts of $E\cup K_0$ whose closure meet K_0. Note that $K^0_0\subset D_0\subset Q_0$. Inductively, choose K_n compact such that $Q_{n-1}\subset K^0_n$, and Q_n so that condition $\mathscr{G}_{\bar{P}}$ is satisfied and such that the sequence $\{D_n\}$, where D_n is the union of all non-trivial parts of $E\cup K_n$ whose closure meet K_n, forms an exhaustion of R. Clearly the sequence $\{D_n\}$ satisfies (*) and (**).

Proof of Theorem 3. Let $\{D_k\}$, $k\geqq 0$, be an exhaustion of R satisfying (*) and (**). Choose compact sets $\hat{D}_k$, $k\geqq 0$, with $\bar{D}_k\subset(\hat{D}_k)^0$ and $\hat{D}_k\subset(D_{k+1})^0$. Then by Lemma 2, $(D_k, \bar{D}_k\cup(E\cap\hat{D}_{k+1}))$ is a Nersesjan pair.

Let $f\in A(E)$ and ε be a positive continuous function on E. Set

$$\varepsilon_k=\min_{p\in(\hat{D}_k\cap E)}\varepsilon(p).$$

There exists a function u_1, meromorphic on R and without poles on E such that

$$|f-u_1|<\frac{\varepsilon_1}{4}\quad\text{on}\quad E\cap\hat{D}_1.$$

Similarly, there is a meromorphic function v_2 without poles on E such that

$$|(f-u_1)-v_2)|<\frac{\varepsilon_2}{4} \quad \text{on} \quad (\hat{D}_2 \backslash \hat{D}_0) \cap E.$$

If V is a component of $(\hat{D}_0)^0$ containing a pole of v_2, then $V \not\subset E$ since v_2 is without poles on E and since $\bar{D}_0 \subset (\hat{D}_0)^0$, V is not strictly contained in $\bar{D}_0$; also since $\partial D_0 \subset \partial(D_0 \cup E)$ (Lemma 1), $V \backslash \bar{D}_0 \not\subset E$. So by [34], we can and do choose v_2 without poles on $\bar{D}_0 \cup (E \cap \hat{D}_0)$.

Now the proof follows that of Theorem 1, if we are careful to choose, at each step, n_k without poles on E and v_k without poles on $\bar{D}_{k-2} \cup E$ so as to avoid any accumulation of singularities on R.

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Surfaces of Small Degree

Miles Reid

Mathematical Institute, University of Warwick, Coventry CV4 7AL, UK

1. Introduction

It is well known that the minimal model of a ruled or rational surface F is non-unique; however, in many cases, given a suitable divisor D on F, there is a morphism $f: F \to F^{\#}$ such that $F^{\#}$ is almost minimal, and f is uniquely specified by the condition that $f_* D = D^{\#}$ has singularities of small multiplicity. This is explained in Sect. 2; the phenomenon occured originally in the cases treated by Iitaka [5], see also [3].

The aim of this paper is to apply this theory of #-minimal models to give minor restatements and simplified proofs of two remarkable theorems of Xiao Gang.

(1.0) Let F be a non-singular projective surface, and D a nef and big divisor on F. From the classification of surfaces, it is obvious that $DK_F < 0$ implies that F is ruled or rational; on the other hand, by RR,

$$DK_F = D^2 - 2\chi(\mathcal{O}_F(D)) + 2\chi(\mathcal{O}_F).$$

Therefore, $D^2 < 2(\chi(\mathcal{O}_F(D)) - \chi(\mathcal{O}_F))$ implies that F is ruled or rational. Here is a quantitative version of this:

Theorem. *Suppose that*

$$(s+1)D^2 < 2s(\chi(\mathcal{O}_F(D)) - \chi(\mathcal{O}_F)) - 9s^2/4$$

for some $s \in \mathbb{Q}$, $s > 0$. Then F is ruled by rational curves of degree $< s$, that is, there exists a morphism $F \to B$ with general fibre $\Gamma \cong \mathbb{P}^1$ and $D\Gamma < s$.

[The inequality in the theorem is equivalent to

$$D^2 + 2rDK_F + 9r^2 = (D + rK_F)^2 + r^2(9 - K_F^2) < 0,$$

where $2r = s$.]

(1.1) **Corollary** [10, Sect. 1, Lemma 1]. *Let $F = F_d \subset \mathbb{P}^n$ be a surface of degree d spanning $\mathbb{P}^n$. Suppose that*

$$d < 2n - 2; \tag{1}$$

then F is ruled or rational. In addition, suppose that

$$(s+1)d<2sn-9s^2/4 \tag{2}$$

for some $s\in\mathbb{Q}$, $s>0$. Then F has a pencil of rational curves of degree $<s$.

(1.2) As will be seen in Sect. 3, the coefficient 9 appears before $s^2/4$ because $\mathbb{P}^2$ has $K^2=9$; this is just an upper bound for $(K_{F^\#})^2$, which is what makes the proof work. Thus if it is known in advance that the #-minimal model $F^\#$ of F will be a ruled surface of genus $q(F)\geqq 0$, [this is automatic if $q(F)>0$, or if n is large compared with s, see (3.4)], the condition in the lemma can be weakened to

$$(s+1)d<2sn-2(1-q)s^2.$$

In particular if $q(F)>0$,

$$d/n<2s/(s+1) \Rightarrow F \text{ has a ruling by rational curves of degree } <s.$$

(1.3) The second theorem concerns hyperelliptic surfaces of general type with small K^2, essentially $K^2<4\chi$. More precisely, let X be a smooth minimal model of a surface of general type, and ι an involution of X; write $F=X/\iota$ for the quotient surface, and

$$\nu=\nu_2=\#\{\text{isolated fixed points of } \iota\}=\#\{\text{nodes of } F\}.$$

It will often happen that we already know that F is rational or ruled, so that $\chi(\mathcal{O}_F)\leqq 1$.

(1.4) **Theorem** [9, Sect. 1, Theorem 1]. *Suppose that*

$$(1/2)K_X^2<2\chi(\mathcal{O}_X)-4\chi(\mathcal{O}_F)+\nu/2; \tag{1}$$

then F is ruled or rational. Moreover, suppose that for some $s\in\mathbb{Q}$, $s>0$,

$$((s+1)/2)K_X^2<2s\chi(\mathcal{O}_X)-4s\chi(\mathcal{O}_F)+s\nu/2-9s^2/4; \tag{2}$$

then F has a ruling by a morphism $F\to B$ such that the composite $X\to B$ is a pencil of hyperelliptic curves of genus $<s+1$.

(1.5) *Remarks.* (a) The results of both (1.1) and (1.4) are essentially sharp, except for the substitution $9\mapsto 8$ mentioned in (1.2) and proved in (3.4); see (3.5) and (4.3) below.

(b) Corollary (1.1) is similar in spirit to Nagata's papers [7] in which he characterises Del Pezzo surfaces and elliptic scrolls as the only surfaces $F_n\subset\mathbb{P}^n$; the basic idea there is to obtain F by blowing up from a minimal model of the ruled or rational surface, and arguing on the base points of the linear systems.

(c) However, there are distinct advantages in approaching rational and ruled surfaces "top-down", that is from a given (non-minimal) surface F, to be reduced by successively contracting (-1)-curves. In particular, I do not need to assume here that surfaces with $\kappa=-\infty$ are ruled or rational, although this is the case of interest. The arguments are phrased in terms of Mori's Theorem on the Cone (see [6] or [11]); the reactionary reader will have no trouble in reworking these via the traditional methods of the classification of surfaces.

(d) For hyperelliptic surfaces, the corresponding classical method of studying X is in terms of F and the ramification divisor of X/F; this is a bit messy, but has been used successfully by Persson and Horikawa in a similar context. The method used here for the proof of (1.4) consists of applying the #-minimal model method to F and D, where D is the $\mathbb{Q}$-Cartier divisor on F for which $K_X = \pi^* D$.

(e) Xiao has further applications of this type of result [9, 10].

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Conventions and Terminology

F will denote a non-singular projective surface. A divisor D on F is *nef* if $D\Gamma \geqq 0$ for every curve $\Gamma \subset F$; if in addition $D^2 > 0$ then D is *nef and big*.

Recall that a Del Pezzo surface is a surface F for which $-K_F$ is ample, and a conic bundle is a surface fibred over a base curve $F \to B$ such that $-K_F$ is relatively ample; the fibres of $F \to B$ are then $\mathbb{P}^1$ or line-pairs. A surface F is a *weak Del Pezzo surface* if $-K_F$ is nef and big; it is well-known (see for example [2]) that F is then either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$ or isomorphic to $\mathbb{P}^2$ blown up $\leqq 8$ times in weakly general position. A fibre space $\pi: F \to B$ over a curve B is a *weak conic bundle* if $-K_F$ is relatively nef and big for π (this means that $-K_F\Gamma \geqq 0$ for every curve Γ in a fibre of π, and $-K_F\Gamma > 0$ for the general fibre); it follows easily that every fibre of π is either $\mathbb{P}^1$, or a simple chain of copies of $\mathbb{P}^1$ with a (-1)-curve at each end, joined (optionally) by (-2)-curves.

2. #-Minimal Models

(2.1) **Theorem.** *Let F be a smooth projective surface, D a nef and big divisor on F, and suppose that $DK_F < 0$. Define $\varrho = \varrho(F \text{ and } D) \in \mathbb{R}$ by*

$$\varrho = \sup\{m \in \mathbb{Q} \mid D + mK_F \text{ is an effective } \mathbb{Q}\text{-divisor}\};$$

then ϱ, 2ϱ or $3\varrho \in \mathbb{Z}$, and $D + \varrho K_F$ has a Zariski decomposition

$$D + \varrho K_F = P + N$$

with $P^2 = 0$.

The negative part N can be contracted out by a sequence of contractions $F \to F_1 \to \ldots \to F_k = F^\#$, where each step contracts out a single (-1)-curve l_i, with $l_i D_i = \mu_i < \varrho$; here D_i is the direct image of D on F_i. The final surface $F^\#$ is either a weak conic bundle with $D^\# \cdot (\text{fibre}) = 2\varrho$ or a weak Del Pezzo surface with $D^\# + \varrho K_{F^\#} = 0$.

The model $F^\#$ and $D^\#$ is called the *#-minimal model* of F and D; in view of its description in terms of Zariski decomposition it is clearly uniquely determined by F and D.

(2.2) *Remarks.* (a) To pass to a minimal model of F in the usual sense, one must make a choice; for example, if $F^\#$ is a conic bundle, a minimal ruled surface is obtained by contracting down one of the two lines in each degenerate fibre. If $F^\#$ is a weak Del Pezzo surface then there will be a finite number of different ways of reducing $F^{\#}$ to $\mathbb{P}^2$. [In either case, I have to make a choice of (-1)-curves with the minimal value of $D^{\#}l=\varrho$.] Strictly speaking, it is this usual (but non-unique) minimal model which Itaka calls the #-minimal model [5]; for practical purposes, such as the proof of (1.0), there is no particular advantage in my choice of model.

(b) The condition $DK_F<0$ in the theorem is just a convenient sufficient condition for the existence of the #-minimal model. (The necessary and sufficient condition is that F is ruled or rational.)

(c) *Exercise*. Check that the theory also applies to a surface F with at worst Du Val singularities, and a $\mathbb{Q}$-Cartier divisor D. [Hint: D only appears in the statement as a $\mathbb{Q}$-divisor. As for the singularities, take the minimal resolution $\varphi: F'\to F$, and define $D'=\varphi^*D$ to reduce the problem down to the smooth case, in view of $K_F=\varphi^*K_{F'}$; figure out what happens to the exceptional (-2)-curve of φ under the reduction of F' and D' to #-minimal model.]

(2.3) The idea of the proof is simply to contract down the (-1)-curves, starting with those having the smallest value of the *multiplicity* $\mu=Dl$ (if D is effective then μ is the multiplicity of the resulting singularity of the image of D). If I'm working with a particular value of μ then the procedure of blowing down successively all (-1)-curves with $Dl=\mu$ is unique provided that $(\mu K_F+D)^2>0$, (or equivalently, provided that the curves contracted in this procedure have negative definite intersection matrix). The exceptional cases $(\mu K_F+D)^2=0$ lead directly to a weak conic bundle structure or to a Del Pezzo surface. I now give the formal proof.

(2.4) Let F be a surface for which K_F is not nef and D a nef and big divisor on F. There exists $\mu\in\mathbb{R}$, $\mu\geqq 0$ such that $D+\mu K_F$ is nef but not $D+mK_F$ for any $m>\mu$; it follows at once from Mori's Theorem on the Cone ([6], but see also [11, Theorem 3.1] for a simpler method of proof) that there is an extremal rational curve l of F with $(D+\mu K_F)l=0$. By the classification of extremal rays, $K_Fl=1, 2$ or 3, so that μ, 2μ or $3\mu\in\mathbb{Z}$. Consider the three cases:

Case $D+\mu K_F\approx 0$. Clearly in this case, for any $m>\mu$, $D(D+mK_F)<0$, so that $D+mK_F$ is not numerically equivalent to an effective divisor; then by definition, $\varrho=\mu$. In this case $-K_F$ is nef and big, that is, F is a weak Del Pezzo surface.

Case $(D+\mu K_F)^2=0$ but $D+\mu K_F\not\approx 0$. Here I clearly have $K_F(D+\mu K_F)<0$, and by RR, $h^0(a(D+\mu K_F))\to\infty$ with a; as usual in the classification of surfaces, some multiple of $D+\mu K_F$ then defines a ruling on F. Now if Γ is a fibre of the ruling then $\Gamma(D+mK_F)<0$ for any $m>\mu$, so that $D+mK_F$ is not numerically equivalent to an effective divisor, and again $\varrho=\mu$ by definition. In this case, since $-K_F$ is nef and non-zero on the fibres of the ruling, F is a weak conic bundle.

Case $(D+\mu K_F)^2>0$. By the index theorem, any extremal rational curve with $(D+\mu K_F)l=0$ must have $l^2<0$, and so there exists at least one (-1)-curve l with $(D+\mu K_F)l=0$. In particular, $\mu\in\mathbb{Z}$ in this case.

Claim. There exists a uniquely defined birational morphism $f: F\to F'$ of F to a nonsingular surface F' such that setting $D'=f_*D$, I have

(i) $D+\mu K_F=f^*(D'+\mu K_{F'})$;

and (ii) $D'+mK_{F'}$ is nef for some $m\in\mathbb{Q}$ with $m>\mu$.

In this case, if $K_{F'}$ is again not nef, then since $D'+mK_{F'}$ is nef for some $m>\mu$, I can plug F' and D' back into the argument.

Proof of Claim. If $\mu>0$ then any curve $\Gamma\subset F$ satisfying $(D+\mu K_F)\Gamma=0$ must be a (-1)-curve l with $lC=\mu$ or a (-2)-curve l with $lC=0$, and there must be at least one (-1)-curve. The set Σ of all curves Γ with $(D+\mu K_F)\Gamma=0$ has negative definite intersection matrix, and it follows that any connected component of Σ containing a (-1)-curve is a simple chain consisting of just one (-1)-curve and a tail of (-2)-curves. Then the birational morphism $f: F\to F'$ is given by just contracting all of these components, and it is easy to check the claim. To prove that $K_{F'}+mD'$ is nef for some $m\in\mathbb{Q}$ with $m>\mu$, note that $K_{F'}+\mu D'$ is nef and big, so that there are only finitely many curves Γ with $(K_{F'}+\mu D')\Gamma=0$, and by construction $K_{F'}\Gamma\geqq 0$ for each of them; the statement follows easily from this.

The case $\mu=0$, although not formally covered by the proof just given, is in fact very trivial: f can be described as the composite of successive contractions $F=F_0\to F_1\to\ldots\to F_n=F'$, where each step contracts a (-1)-curve l_i with $D_il_i=0$, where $D_i=f_{i*}D$. [In either case, the morphism f can be described as the contraction of the negative part of the Zariski decomposition of K_F relative to the finite set Σ of curves Γ such that $(D+\mu K_F)\Gamma=0$. In more detail, write

$$K_F=P+\sum a_i\Gamma_i$$

with $\Gamma_i\in\Sigma$, $a_i\in\mathbb{Q}$, $a_i\geqq 0$ such that $P\Gamma\geqq 0$ for every $\Gamma\in\Sigma$, and $P\Gamma_i=0$ for every Γ_i with $a_i>0$. Then $f: F\to F'$ contracts exactly the curves $\Gamma_i\in\Sigma$ with $a_i>0$. This argument shows why $f: F\to F'$ is unique.]

Proof of Theorem (2.1). Since D is nef and $DK_F<0$, it follows that there exists $m>0$ such that $D+mK_F$ is not numerically equivalent to an effective divisor. Hence the procedure of contracting all (-1)-curves with a given multiplicity μ, then passing on to some $m>\mu$ must eventually come to an end, that is, lead to the case $(D+\mu K_F)^2=0$. Q.E.D.

(2.5) Write k for the number of blow-downs required to get to the #-minimal model. The following simple lemma shows how to use Theorem 2.1.

Lemma. *For any* $r\in\mathbb{Q}$,

(i) $$(D+rK_F)D\leqq 0 \Rightarrow \varrho\leqq r;$$

and

(ii) $$(D+rK_F)^2+r^2k\leqq 0 \Rightarrow \varrho\leqq r.$$

(Both of the implications also work with strict inequalities, by an obvious continuity argument.)

Proof. (i) Since D is nef and $K_FD<0$, the condition $(D+rK_F)D\leqq 0$ implies that $D+r'K_F$ cannot be effective for any $r'\geqq r$; hence $\varrho\leqq r$ directly by definition of ϱ.

(ii) Consider the quadratic polynomial

$$q(\lambda)=(\lambda D+K_F)^2;$$

then obviously,

$$dq/d\lambda = 2(\lambda D + K_F)D.$$

Hence $q(\lambda)$ is an increasing function of λ whenever $(\lambda D + K_F)D \geqq 0$. On the other hand, it follows from (i) that $(D + \varrho K_F)D \geqq 0$. Now if $r < \varrho$, then $q(\lambda)$ is an increasing function for $\lambda \in [1/\varrho, 1/r]$, so

$$((1/r)D + K_F)^2 > ((1/\varrho)D + K_F)^2.$$

The point of the proof is to prove the following assertion.

(2.6) *Claim.* $(D + \varrho K_F)^2 \geqq -\varrho^2 k$.

This proves (2.5), (ii), since using it I have just proved

$$r < \varrho \Rightarrow (D + rK_F)^2 > -r^2 k.$$

By good fortune, $D + \varrho K_F$ is the divisor for which Theorem (2.1) asserts there is a nice Zariski decomposition. To prove the claim, introduce the following notation [in addition to that of Theorem (2.1)]: for $i = 0, \ldots, k-1$, let

$$l_i = \text{exceptional curve of } F_i \to F_{i+1},$$

$$L_i = \text{total transform of } l_i \text{ on } F.$$

Of course, $L_i^2 = -1$ and $L_i L_j = 0$ for $i \neq j$.

Write $f: F \to F^{\#}$ for the reduction to the #-minimal model. Then by the adjunction formula for a blow-up,

$$K_F = f^* K_{F^{\#}} + \sum L_i;$$

and by definition of μ_i,

$$D = f^* D^{\#} - \sum \mu_i L_i.$$

Hence the Zariski decomposition of $D + \varrho K_F$ is given by

$$D + \varrho K_F = f^*(D^{\#} + \varrho K_{F^{\#}}) + N,$$

where

$$N = \sum (\varrho - \mu_i) L_i.$$

Therefore,

$$(D + \varrho K_F)^2 = N^2 = -\sum (\varrho - \mu_i)^2 = -\varrho^2 k + \sum \mu_i (2\varrho - \mu_i)$$

and since $0 \leqq \mu_i < \varrho$, the sum is $\geqq 0$. This proves the claim and with it Lemma (2.5).

3. Ruled Surfaces of Small Degree

(3.1) *Proof of Theorem* (1.0). To apply Lemma (2.5), (ii), evaluate the left-hand side:

$$(D + rK_F)^2 + rk = D^2 + 2rDK_F + r^2(K_F^2 + k).$$

Now $K_F^2+k=K_{F^\#}^2$, since $F^\#$ is got from F by blowing down k (-1)-curves, and by RR, $DK_F=D^2-2\chi(\mathcal{O}_F(D))+2\chi(\mathcal{O}_F)$, so that

$$(D+rK_F)^2+rk=(2r+1)D^2-4r\chi(\mathcal{O}_F(D))+4r\chi(\mathcal{O}_F)+r^2K_{F^\#}^2.$$

Now F is a surface with $p_g=0$, so that by the Noether formula $K_{F^\#}^2\leqq 9$. Therefore, the condition given in Theorem (1.0) is exactly what's required to ensure that

$$(D+rK_F)^2+rk<0, \quad \text{with} \quad s=2r.$$

So by Lemma (2.6), $\varrho(F \text{ and } D)<r=s/2$.

Theorem (2.1) now say that $F^\#$ is either a weak conic bundle with $D^\#\cdot$(fibre) $=2\varrho<s$, or a weak Del Pezzo surface with $D^\#+\varrho K_{F^\#}=0$. The following result shows that the second case is not possible under the condition of Theorem (1.0).

(3.2) **Proposition.** *Let F be a non-singular projective surface and D a nef and big divisor on F such that*

(i) $(2r+1)D^2\leqq 4r(\chi(\mathcal{O}_F(D))-\chi(\mathcal{O}_F))-9r^2$ *for some* $r\in\mathbb{Q}$, $r>0$;

(ii) *the* $\#$*-minimal model* $F^\#$ *is a weak Del Pezzo surface with* $D^\#+\varrho K_{F^\#}=0$ *for some* $\varrho\in\mathbb{Q}$.

Then equality holds in (i); *furthermore* $F^\#\cong\mathbb{P}^2$, $3\varrho\in\mathbb{Z}$, $r=\varrho$ *and* $D=f^*D^\#$ *where* $f\colon F\to F^\#$ *is the reduction map.*

Proof. The same argument as above proves that inequality (i) implies that $\varrho\leqq r$. Unwrapping (i) gives

$$(D+rK_F)^2+r^2(9-K_F^2)\leqq 0;$$

Substituting in this the expressions for D and K_F of Sect. 2 gives

$$(D^\#+rK_{F^\#}+\sum(r-\mu_i)L_i)^2+r^2(9-K_{F^\#}^2+k)\leqq 0,$$

and multiplying out, taking the final r^2k into the middle term, I get:

$$(D^\#+rK_{F^\#})^2+\sum\mu_i(2r-\mu_i)+r^2(9-K_{F^\#}^2)\leqq 0.$$

However, each of the terms on the left is $\geqq 0$ (since $\mu_i<\varrho\leqq r$). This gives all I want.

(3.3) *Proof of Corollary* (1.1). Write F for the (minimal) resolution of $F_d\subset\mathbb{P}^n$, and D for the pull-back of $\mathcal{O}(1)$; then D is nef and big, and $|D|$ contains an irreducible curve D. The linear system $|D|_{|D}$ has dimension $n-1$ but degree $<2(n-1)$, so that by Clifford's theorem it cannot be special; therefore $H^1(\mathcal{O}_D(D))=0$. I will also need the irregularity $q=h^1(\mathcal{O}_F)$ and $h^1=h^1(\mathcal{O}_F(D))$; obviously $h^1\leqq q$, from $H^1(\mathcal{O}_D(D))=0$.

From the exact sequence

$$0\to\mathcal{O}_F\to\mathcal{O}_F(D)\to\mathcal{O}_D(D)\to 0$$

it follows that $h^0(\mathcal{O}_D(D))=n+q-h^1=n^+$, so that

$$\chi(\mathcal{O}_F(D))-\chi(\mathcal{O}_F)=\chi(\mathcal{O}_D(D))=n^+.$$

Therefore Corollary (1.1) is a particular case of the theorem.

(3.4) **Addendum.** *Let $F = F_d \subset \mathbb{P}^n$ be a surface of degree d spanning $\mathbb{P}^n$. Suppose that*

$$(s+1)d < 2sn - 2s^2$$

for some $s \in \mathbb{Q}$, $s > 0$. Then either

(a) *F has a pencil of rational curves of degree $< s$;*

or (b) *F is represented on $\mathbb{P}^2$ by a linear system of planes curves of degree a with $s < a < 2s$, and in particular, $n \leqq (2s-1)(s+1)$.*

Proof. The proof of (1.1) gives that if $(K_{F^\#})^2 \leqq 8$, the given inequality implies (a). So if (a) fails, the #-minimal model of F and D is $\mathbb{P}^2$ and $\mathcal{O}(a)$ for some a. But $\deg \mathcal{O}(a) = a^2$, $\dim |\mathcal{O}(a)| = a(a+3)/2$, and arguing as in (2.6) I get

$$(s+1)a^2 < sa(a+3) - 2s^2,$$

that is

$$(a-s)(a-2s) < 0. \quad \text{Q.E.D.}$$

(3.5) There follows a brief discussion of special initial cases of Corollary (1.1), including the limiting cases.

Special Cases. Surfaces with $\varrho \leqq 1$ are very special, since as already mentioned, I can contract out the (-1)-curves with $\mu = 0$ and assume $F = F^\#$; then if ϱ is not an integer, there cannot be any (-1)-curves orthogonal to $D + \varrho K_F$, so F must be $\mathbb{P}^2$ or a scroll. So $\varrho = 1/3$ and $2/3$ correspond to $\mathbb{P}^2$ and its Veronese embedding, $\varrho = 1/2$ to rational scrolls. So the $\varrho < 1$ cases are:

$d < n^+ \Rightarrow F$ is ruled by lines or is the Veronese surface;

$3d < 4n^+ - 9 \Rightarrow$ ditto.

The limiting case $\varrho = 1$ is represented by the 3-fold embedding of $\mathbb{P}^2$ with $d = n = 9$, $3d = 4n - 9$; and the scroll $\mathbb{F}_v$ embedded by $|aA + 2B|$ having $d = 4(a-v)$ and $n = 3(a-v) + 2$, so $3d = 4n - 8$.

Other small values of ϱ correspond to the following implications:

$11d \leqq 16n^+ - 48 \Rightarrow \varrho \leqq 4/3 \Rightarrow F$ is $\mathbb{P}^2$ embedded by $\mathcal{O}(4)$ or is ruled by lines or conics.

$2d < 3n^+ - 81/8 \Rightarrow \varrho < 3/2 \Rightarrow F$ is ruled by lines or conics. In fact $2d < 3n^+ - 9 \Rightarrow F$ is either ruled by lines or conics, or $\mathbb{P}^2$ embedded by $\mathcal{O}(4)$ or $\mathcal{O}(5)$. Note that $(\mathbb{P}^2, \mathcal{O}(4))$ gives $d = 16$, $n = 14$, and $(\mathbb{P}^2, \mathcal{O}(5))$ gives $d = 25$, $n = 20$, so that $2d < 3n - 9$ (with nothing to spare in either case).

$13d \leqq 20n^+ - 75 \Rightarrow \varrho \leqq 5/3 \Rightarrow F$ is $\mathbb{P}^2$ embedded by $\mathcal{O}(5)$ or is ruled by curves of degree $\leqq 3$.

$5d < 8n^+ - 36 \Rightarrow \varrho < 2 \Rightarrow F$ is ruled by curves of degree $\leqq 3$, and $5d < 8n^+ - 32 \Rightarrow F$ is either ruled by curves of degree $\leqq 3$, or is $\mathbb{P}^2$ birationally embedded by a linear system of curves of degree 5, 6 or 7, (possibly with base points).

$3d < 5n^+ - 225/8 \Rightarrow \varrho < 5/2 \Rightarrow F$ is ruled by curves of degree $\leqq 4$.

$7d < 12n^+ - 81 \Rightarrow \varrho < 3 \Rightarrow F$ is ruled by curves of degree $\leqq 5$.

For each ϱ with $2\varrho \in \mathbb{Z}$, a limiting case is the scroll $\mathbb{F}_v$ embedded by $|aA + 2\varrho B|$ having $d = 4\varrho a - 4\varrho^2 v$ and $n = 2\varrho a + 2\varrho + a - \varrho(2\varrho + 1)v$, so $(2\varrho + 1)d = 4\varrho n - 8\varrho^2$. For each ϱ with $3\varrho \in \mathbb{Z}$, a limiting case is $\mathbb{P}^2$ mapped by $|\mathcal{O}(3\varrho)|$, having $d = 9\varrho^2$ and $n = (9/2)(\varrho^2 + \varrho)$, so that $(2\varrho + 1)d = 4\varrho n - 9\varrho^2$.

4. Hyperelliptic Surfaces with $K^2 < 4\chi$

(4.1) I start on the proof of Theorem (1.4). Write $\pi: X \to F = X/\iota$; then the eigensheaf decomposition of $\pi_* \mathcal{O}_X$ is

$$\pi_* \mathcal{O}_X = \mathcal{O}_F \oplus \mathcal{O}_F(-E),$$

where E is a Weil divisor class on F. The double cover $X \to F$ is ramified in the v nodes of F and in the ramification divisor $R \sim 2E$. Also $K_X \approx \pi^*(K_F + E)$; note that writing π^* is an abuse of notation, since E is only a Weil divisor. However, $2E$ is a Cartier divisor, and the numerical equivalence statement is meaningful. Set $D = K_F + E$; then D is a $\mathbb{Q}$-Cartier divisor, and since $K_X = \pi^* D$, it is nef and big. The idea is simply to apply the previous method, especially Theorem (2.1) and Lemma (2.5) to D.

(4.2) *Invariants.* Obviously,

$$D^2 = (1/2) K_X^2;$$

next, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_F) + \chi(\mathcal{O}_F(-E))$, and $\chi(\mathcal{O}_F(-E))$ can be computed by RR:

$$\begin{aligned}\chi(\mathcal{O}_F(-E)) &= \chi(\mathcal{O}_F) + (1/2)(-E)(-E-K_F) - v/4 \\ &= \chi(\mathcal{O}_F) + (1/2)(D^2 - DK_F) - v/4 .\end{aligned}$$

Note that the sheaf $\mathcal{O}_F(-E)$ is singular at the v nodes of F, and the final term in this formula is the appropriate correction (compare [1, p. 212]). Hence

$$\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_F) + (1/2)(D^2 - K_F D) - v/4 .$$

This can be rewritten

$$\begin{aligned}K_F D &= D^2 - 2\chi(\mathcal{O}_X) + 4\chi(\mathcal{O}_F) - v/2 \\ &= (1/2) K_X^2 - 2\chi(\mathcal{O}_X) + 4\chi(\mathcal{O}_F) - v/2 .\end{aligned}$$

Therefore, inequality (1) of Theorem (1.4) is just the condition that $K_F D < 0$, and since D is nef and big, this implies that F is ruled or rational.

In exactly the same way, inequality (2) of (1.4) is easily recognised as the condition that

$$D^2 + sK_F D + 9s^2/4 < 0 .$$

Let $\varrho = \varrho(F \text{ and } D)$ be as in Theorem (2.1); see Remark (2.2), (c) if you're worried about the singularities. Then by Lemma (2.5), $\varrho < r = s/2$. This means that the $\#$-minimal model $F^\#$ of F is either a weak conic bundle with $D^\# \cdot (\text{fibre}) = 2\varrho < s$, or a weak Del Pezzo surface with $D^\# + \varrho K_{F^\#} = 0$; exactly as in Proposition (3.2), the case of a weak Del Pezzo surface does not satisfy the inequality.

Therefore F has a (free) pencil of rational curves of degree $< s$ with respect to D. Since $K_X = \pi^* D$, it follows from this that X has a pencil of curves E with $E^2 = 0$, $K_X E < 2s$, so that $2g(E) - 2 < 2s$, that is, $g < s+1$. Q.E.D.

(4.3) *Remarks.* (a) Standard examples of scrolls etc. show that the theorem is close to being sharp: if $X \to \mathbb{F}_n$ is a non-singular double cover with $K_X = \pi^*(aA + bB)$

then $K_X^2 = 4ab - 2b^2n$, and $\chi(\mathcal{O}_X) = 1 + (a+1)(b+1) - nb(b+1)/2$, so that

$$((b+1)/2)K_X^2 - 2b\chi(\mathcal{O}_X) + 4b + 2b^2 = 0\,,$$

hence for $2r = b$,

$$((2r+1)/2)K_X^2 - 4r\chi(\mathcal{O}_X) + 8r\chi(\mathcal{O}_F) - rv/2 + 8r^2 = 0\,;$$

and of course there is a pencil of genus $b+1 = 2r+1$, but none of smaller genus. Similarly for the non-singular $X \to \mathbb{P}^2$ with $K_X = \pi^*\mathcal{O}(a)$, we get $K^2 = 2a^2$, $\chi = 1 + (a+2)(a+1)/2$, so that

$$((2a/3+1)/2)K^2 - (4a/3)\chi + (8a/3)\chi(\mathcal{O}_F) + a^2 = 0\,,$$

hence v

$$((2r+1)/2)K_X^2 - 4r\chi(\mathcal{O}_X) + 8r\chi(\mathcal{O}_F) - rv/2 + 9r^2 = 0\,,$$

with $3r = a$.

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Isomorphisms Between Certain Function Fields Over Compact Riemann Surfaces

Pascual Cutillas Ripoll

Sección de Matemáticas, Universidad de Salamanca, Salamanca, Spain

1. Introduction

In a previous paper [1] the author has introduced the concept of a field verifying the Weierstrass property (or, in short, a W-field) in an open connected subset U of a compact Riemann surface $\mathscr{V}$, in order to study and solve a problem related with the classical theory of functions. We say that a subfield k of the field $\mathscr{M}(U)$ of meromorphic functions on U is a W-field in U, if k contains the field $\mathscr{M}(\mathscr{V})$ of meromorphic functions on $\mathscr{V}$ (which, by restriction of functions, can be considered as a function field over U) and if, δ being a given finite divisor on U (i.e. a divisor supported on a finite subset of U), there exists a function belonging to k whose divisor (on U) is δ.

Let $\mathscr{V}'$ be an open subset of $\mathscr{V}$, whose complementary set is a nonempty finite set. In the above said paper was proved the existence of an isomorphism between every two minimal W-fields in $\mathscr{V}'$ and, even, the existence of isomorphisms between minimal W-fields corresponding to different sets of the same type of $\mathscr{V}'$. The fundamental purpose of this paper is to prove an analogous theorem for the W-fields, on open sets of $\mathscr{V}'$ type, generated over $\mathscr{M}(\mathscr{V})$ by all meromorphic functions on $\mathscr{V}'$ having a finite divisor.

2. Notations

The following notations will be held through this paper.

S will denote a finite nonempty subset of the compact Riemann surface $\mathscr{V}$; $\mathscr{V}'$ will be $\mathscr{V}-S$.

For every connected open subset U of $\mathscr{V}$, $\mathscr{M}(U)$ is the field of meromorphic functions on U. $\mathscr{M}^*(U)$ is the multiplicative group of meromorphic functions not identically zero on U. $\mathcal{O}(U)$ is the ring of holomorphic functions on U, and $E(U) = \{e^h\colon h \varepsilon \mathcal{O}(U)\}$.

The genus g of $\mathscr{V}$ is supposed to be greater than 0. $A_1, \ldots, A_g$, $B_1, \ldots, B_g$ will denote closed, piecewise analytic curves in $\mathscr{V}$ defining a canonical system of generators of the fundamental group of $\mathscr{V}$ (following, for instance, the terminology of Gunning [4]). Δ will be the open simply connected subset of $\mathscr{V}$ complementary to the union of the curves $A_1, \ldots, A_g, B_1, \ldots, B_g$ (considered, of course, as subsets of $\mathscr{V}$). We shall always suppose (without loss of generality) that $S \subset \Delta$. The symbol ∞ will denote a fixed point in Δ, and we shall put $\Delta' = \Delta - \{\infty\}$.

For every pair of points $a, b \in \Delta$, ϑ_{ab} will be the normal differential of the third kind associated to the points a, b and to the system of curves $A_1, \ldots, A_g, B_1, \ldots, B_g$ (i.e. the unique holomorphic differential on $\mathscr{V} - \{a, b\}$ with simple poles at a, b, with residues $+1, -1$ respectively, and whose integrals along the curves A_i are zero).

$G(\mathscr{V}')$ will be the group of functions in $\mathscr{M}^*(\mathscr{V}')$ whose divisor is finite. $K(\mathscr{V}')$ will denote the subfield of $\mathscr{M}(\mathscr{V}')$ generated over $\mathscr{M}(\mathscr{V})$ by the functions in $G(\mathscr{V}')$, i.e. the field of quotients of the ring of finite sums of functions in $G(\mathscr{V}')$. $T(\mathscr{V}')$ will be the quotient group $G(\mathscr{V}')/\mathscr{M}^*(\mathscr{V})$, and, for every $f \in G(\mathscr{V}')$, $\bar{f}$ will denote the class of f in the quotient $T(\mathscr{V}')$.

3. Integral Types

We begin this section with a useful definition.

Definition 3.1. Let $f \in G(\mathscr{V}')$ and let δ be the divisor (on $\mathscr{V}'$) of f. The extended divisor of f is the divisor on $\mathscr{V}$ defined by the sum $\delta + \sum_{p \in S} \operatorname{Res}(d\log f, p)p$ (where $\operatorname{Res}(d\log f, p)$ is the residue at p of $d\log f$).

It is an evident consequence of the residue theorem that the degree of the extended divisor of every function in $G(\mathscr{V}')$ is 0.

Let f be a function in $G(\mathscr{V}')$; then, there is another function $h \in \mathscr{M}^*(\mathscr{V})$ such that the extended divisor of the product fh is supported on Δ and such that $d\log(fh)$ has zero integrals along $A_1, \ldots, A_g, B_1, \ldots, B_g$. If $\sum_{i=1}^{r} n_i(a_i - \infty)$, with $n_i \in \mathbb{Z}$, $a_i \in \Delta'$, is the extended divisor of fh, it is not hard to see that the numbers $\lambda_j = \int_{B_j} \sum_{i=1}^{r} n\vartheta_{a_i\infty}, j = 1, \ldots, g$, do not depend on the election of h. Therefore, they are correctly associated to the function f.

Definition 3.2. The integral type of $f \in G(\mathscr{V}')$ (with respect to the point ∞ and to the system of curves $A_1, \ldots, A_g, B_1, \ldots, B_g$) is the vector in $\mathbb{C}^g$ whose j-th component is the number λ_j associated to f as explained in the above paragraph.

From now on, throughout the rest of this paper, we shall suppose that all integral types considered are related to the same point ∞ and to the same system of curves $A_1, \ldots, A_g, B_1, \ldots, B_g$.

Let $p_1 + \ldots + p_g$, where all $p_1, \ldots, p_g \in \Delta'$ are different, a nonspecial divisor (i.e. such that no holomorphic differential on $\mathscr{V}$ is null at $p_1, \ldots, p_g$), and let $(\lambda_1, \ldots, \lambda_g)$ be an arbitrary vector in $\mathbb{C}^g$. Whenever $n \in \mathbb{N}$ is big enough, there exist points $q_1, \ldots, q_g \in \Delta'$ such that $\lambda_j = n \int_{B_j} \sum_{i=1}^{g} \vartheta_{p_i q_i}$, for every $j \in \{1, \ldots, g\}$ (see, for instance, in [2], the proof of Jacobi's inversion theorem); therefore, the sum $\vartheta + n \sum_{i=1}^{g} \vartheta_{p_i q_i}$, where ϑ is a holomorphic differential on $\mathscr{V}'$ whose residues at every point in S are null and such that $\int_{A_j} \vartheta = 0$, $\int_{B_j} \vartheta = -\lambda_j$, is the logarithmic differential of a function in $G(\mathscr{V}')$ which has $(\lambda_1, \ldots, \lambda_g)$ as its integral type. We have thus obtained, up to easy details, the proof of the following.

Proposition 3.3. *The mapping from $G(\mathscr{V}')$ into $\mathbb{C}^g$ which assigns to every $f \in G(\mathscr{V}')$ its integral type is an exhaustive group homomorphism whose kernel is the subgroup of $G(\mathscr{V}')$ generated by the functions in $\mathscr{M}^*(\mathscr{V})$ and the functions in $E(\mathscr{V}')$.*

The following definition makes sense as a consequence of Proposition 3.3.

Definition 3.4. Let $\bar{f}$ be an arbitrary element in $T(\mathscr{V}')$; then, the integral type of $\bar{f}$ is the integral type of any function in $G(\mathscr{V}')$ whose class in the quotient group $T(\mathscr{V}')$ is $\bar{f}$.

If $T_0(\mathscr{V}')$ is the subgroup of $T(\mathscr{V}')$ defined by the classes of the functions in $E(\mathscr{V}')$, another immediate consequence of Proposition 3.3 is the following.

Proposition 3.5. *The mapping from $T(\mathscr{V}')$ into $\mathbb{C}^g$ which assigns to every element of $T(\mathscr{V}')$ its integral type is an exhaustive group homomorphism whose kernel is $T_0(\mathscr{V}')$.*

Remark. It is easy to see that $T_0(\mathscr{V}')$ is isomorphic to the quotient group $\mathscr{O}(\mathscr{V}')/\mathbb{C}$; therefore, it is a divisible group. We can deduce from this that there exists a subgroup H of $T(\mathscr{V}')$ such that $T(\mathscr{V}') = H \oplus T_0(\mathscr{V}')$. The mapping in Proposition 3.5 restricted to H is a group isomorphism between the groups H and $\mathbb{C}^g$.

4. The Theorem

Throughout this section S_1 and S_2 will be any two nonempty finite subsets of $\mathscr{V}$. We shall always suppose (without loss of generality) that $S_1 \cup S_2 \subset \Delta$. $\mathscr{V}_1'$ and $\mathscr{V}_2'$ will stand for $\mathscr{V} - S_1$ and $\mathscr{V} - S_2$ respectively, and we shall use freely the analogous notations to those explained in Sect. 2.

The following auxiliary results will prove very useful in the sequel.

Lemma 4.1. *If $f_1 \in G(\mathscr{V}_1')$ and $f_2 \in G(\mathscr{V}_2')$ have the same integral type, there exist $h \in \mathscr{M}^*(\mathscr{V})$ and $\Phi \in E(\mathscr{V}_1' \cap \mathscr{V}_2')$ such that $f_1 = f_2 h \Phi$.*

Proof. We can suppose that the extended divisors of both f_1 and f_2 are supported in Δ and that $d \log f_1$ and $d \log f_2$ have zero integrals along $A_1, \ldots, A_g$, $B_1, \ldots, B_g$. Let $\delta_1 = \sum_{j=1}^{r} n_j(a_j - \infty)$ and $\delta_2 = \sum_{k=1}^{s} m_k(b_k - \infty)$, with $a_j, b_k \in \Delta'$, be the extended divisors of f_1 and f_2 respectively. By hypothesis, $\sum_{j=1}^{r} n_j \vartheta_{a_j\infty} - \sum_{k=1}^{s} m_k \vartheta_{b_k\infty}$ is the logarithmic differential of a function $h \in \mathscr{M}^*(\mathscr{V})$. Let Φ be the function $f_1 f_2^{-1} h^{-1}$; then, the integral of $d \log \Phi$ along any closed curve in $\mathscr{V}_1' \cap \mathscr{V}_2'$ is null. Therefore, $d \log \Phi$ is an exact differential in $\mathscr{V}_1' \cap \mathscr{V}_2'$, and $\Phi \in E(\mathscr{V}_1' \cap \mathscr{V}_2')$.

Lemma 4.2. *There exists an integral type preserving isomorphism between $T(\mathscr{V}_1')$ and $T(\mathscr{V}_2')$ (in the sense that the integral type of every element of $T(\mathscr{V}_1')$ coincides with that from its image in $T(\mathscr{V}_2')$).*

Proof. It is a consequence of the final remark in Sect. 3.

Lemma 4.3. *If $f_1+\dots+f_n=0$, with $f_1,\dots,f_n\in G(\mathscr{V}')$, any sum of all functions in the set $\{f_1,\dots,f_n\}$ having the same class in the quotient group $T(\mathscr{V}')$ is also null.*

A proof of Lemma 4.3 can be founded in [1]. A more general result can be seen in [3].

The theorem which we have referred to in the introduction of this paper is the following

Theorem 4.4. *There exists an isomorphism between $K(\mathscr{V}_1')$ and $K(\mathscr{V}_2')$ whose restriction to $\mathscr{M}(\mathscr{V})$ is the identity.*

Proof. Let p be a fixed point in $\mathscr{V}_1'\cap\mathscr{V}_2'$. Let $\bar{\tau}: T(\mathscr{V}_1')\to T(\mathscr{V}_2')$ be an integral type preserving isomorphism, and let f be an arbitrary function in $G(\mathscr{V}_1')$. From Lemma 4.1 we deduce that there exists a unique function $\tau(f)\in G(\mathscr{V}_2')$ such that $\bar{\tau}(\bar{f})=\overline{\tau(f)}$ (where the bars above f and $\tau(f)$ mean that we are taking the corresponding classes in the quotient groups $T(\mathscr{V}_1')$ and $T(\mathscr{V}_2')$ respectively), such that $\tau(f)f^{-1}\in E(\mathscr{V}_1'\cap\mathscr{V}_2')$, and such that $\tau(f)f^{-1}$ takes the value 1 at p. It is easy to see that the mapping $f\to\tau(f)$ is an isomorphism between $G(\mathscr{V}_1')$ and $G(\mathscr{V}_2')$ whose restriction to $\mathscr{M}^*(\mathscr{V})$ is the identity.

Let $\mathfrak{A}_i$ $(i=1,2)$ be the ring of finite sums of functions in $G(\mathscr{V}_i')$, and let $f_1+\dots+f_n$, with $f_1,\dots,f_n\in G(\mathscr{V}_1')$, be an arbitrary function in $\mathfrak{A}_1$. We can extend the isomorphism $\tau: G(\mathscr{V}_1')\to G(\mathscr{V}_2')$ to a mapping, which we shall also call τ, from $\mathfrak{A}_1$ into $\mathfrak{A}_2$, by putting $\tau(f_1+\dots+f_n)=\tau(f_1)+\dots+\tau(f_n)$. This definition is a correct one, for, let us suppose that $f_1+\dots+f_n=0$; then, by Lemma 4.3, we can think that $\bar{f}_1=\dots=\bar{f}_n$. But in this case we have

$$\begin{aligned}\tau(f_1)+\dots+\tau(f_n)&=\tau(f_1)(1+\tau(f_2f_1^{-1})+\dots+\tau(f_nf_1^{-1}))\\&=\tau(f_1)(1+f_2f_1^{-1}+\dots+f_nf_1^{-1})=0\,.\end{aligned}$$

Now, a similar reasoning shows that $\tau:\mathfrak{A}_1\to\mathfrak{A}_2$ is injective, and, this mapping being clearly an onto ring homomorphism, we deduce that it is a ring isomorphism between $\mathfrak{A}_1$ and $\mathfrak{A}_2$. By noting that, for $i=1,2$, $K(\mathscr{V}_i')$ is the field of fractions of the ring $\mathfrak{A}_i$, it is evident that we can obtain an isomorphism between $K(\mathscr{V}_1')$ and $K(\mathscr{V}_2')$ which is an extension of τ.

The kind of isomorphism between $K(\mathscr{V}_1')$ and $K(\mathscr{V}_2')$ which we have considered in Theorem 4.4 is characterized in various ways in the following

Theorem 4.5. *Let $\tau: K(\mathscr{V}_1')\to K(\mathscr{V}_2')$ be a ring isomorphism. The following conditions on τ are equivalent:*

1) *The restriction of τ to $\mathscr{M}(\mathscr{V})$ is the identity.*
2) *For every $f\in G(\mathscr{V}_1')$, $\tau(f)f^{-1}\in E(\mathscr{V}_1'\cap\mathscr{V}_2')$.*
3) *τ preserves extended finite divisors, i.e. for every $f\in G(\mathscr{V}_1')$, $\tau(f)$ has the same extended divisor as f.*

Proof. 1) $\Rightarrow$ 2). If $f\in G(\mathscr{V}_1')$, for every $n\in\mathbb{N}$ there exist $f_n\in G(\mathscr{V}_1')$ and $h_n\in\mathscr{M}^*(\mathscr{V})$ such that $f=f_n^nh_n$. From this equality we obtain $\tau(f)=\tau(f_n)^nh_n$. Therefore, for every $n\in\mathbb{N}$, the function $\tau(f)f^{-1}$ is the n-th power of a function in $\mathscr{M}^*(\mathscr{V}_1'\cap\mathscr{V}_2')$, and from this we deduce that $\tau(f)f^{-1}\in E(\mathscr{V}_1'\cap\mathscr{V}_2')$.

2) $\Rightarrow$ 3). It is obvious.

3) $\Rightarrow$ 1). If $h \in \mathcal{M}^*(\mathcal{V})$, we have $\tau(h)+\tau(1-h)=1$; and, being $\tau(h), \tau(1-h) \in G(\mathcal{V}_2')$, Lemma 4.3 implies that $\tau(h) \in \mathcal{M}^*(\mathcal{V})$. Therefore, $\tau(h)h^{-1} \in \mathbb{C}^*$; but this is only possible, for every $h \in \mathcal{M}^*(\mathcal{V})$, if $\tau(h)=h$.

Final Remarks. 1) On the assumption that the genus of the Riemann surface $\mathcal{V}$ is 0, the analogue of Theorem 4.4 is also valid and its proof is much more easy.

2) It is well known (see, for instance, [5]) that being given a discrete valuation (of rank 1) v on $\mathcal{M}(\mathcal{V}')$ (notation as explained in Sect. 2), there is a point $p \in \mathcal{V}'$ such that $v(f)=\mathrm{ord}_p(f)$ for every $f \in \mathcal{M}(\mathcal{V}')$. This property is not valid for the field $K(\mathcal{V}')$ for, we can choose a point $q \in \mathcal{V}'$ and obtain via an isomorphism $K(\mathcal{V}') \approx K(\mathcal{V} - \{q\})$, as in Theorem 4.4, a discrete valuation on $K(\mathcal{V}')$ which is associated with no point of $\mathcal{V}'$ in the above explained sense.

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New Unitary Representations of Loop Groups

Vyjayanthi Chari[1] and Andrew Pressley[2]

[1] Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Bombay 400 005, India
[2] Department of Mathematics, King's College, Strand, London WC2R 2LS, UK

Introduction

In [1] the first author classified the irreducible, integrable modules with finite dimensional weight spaces for the non-twisted affine Lie algebras. It was shown that the modules on which the centre acts non-trivially are precisely the standard modules and their duals. Several explicit realizations of these modules are known [4–6]. The purpose of this paper is to give an equally explicit construction of those modules on which the centre acts trivially. Using it, we shall obtain necessary and sufficient conditions for the modules to be unitarizable. In contrast to the standard modules, which are always unitarizable, those considered here are usually not. The construction also makes it clear how to lift to a representation of the corresponding Kac-Moody group, and to various completions of it.

The most obvious integrable modules for the loop algebra $L(\mathfrak{g})$ of a finite dimensional complex simple Lie algebra $\mathfrak{g}$ are the loop spaces $L(V)$, where V is an irreducible, integrable $\mathfrak{g}$-module, but these are obviously highly reducible. However, there is a natural derivation d of $L(\mathfrak{g})$, given by $(df)(z)=zdf/dz$ for $f\in L(\mathfrak{g})$, and then $L(V)$ is an irreducible module for the semi-direct product $\bar{L}(\mathfrak{g})=L(\mathfrak{g})\tilde{\oplus}\mathbb{C}\cdot d$.

More generally, fix a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ and a positive system for $(\mathfrak{g},\mathfrak{h})$ and let $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_k)$ be a k-tuple of dominant integral weights in $\mathfrak{h}^*$. Denote by $V(\lambda_i)$ the irreducible $\mathfrak{g}$-module with highest weight λ_i. Then one can twist the obvious action of $\bar{L}(\mathfrak{g})$ on $L\left(\bigotimes_{i=1}^{k} V(\lambda_i)\right)$ by reparametrizing loops independently on each factor in the tensor product. Namely, given a k-tuple $\mathbf{a}=(a_1,\dots,a_k)$ of distinct non-zero complex numbers, one takes

$$(f\cdot\Omega)(z)=\left(\sum_{i=1}^{k}1\otimes\dots\otimes f(a_iz)\otimes\dots\otimes 1\right)\Omega(z)$$

for $f\in L(\mathfrak{g})$, $\Omega\in L\left(\bigotimes_{i=1}^{k} V(\lambda_i)\right)$. Let $V(\boldsymbol{\lambda},\mathbf{a})$ be the resulting $\bar{L}(\mathfrak{g})$-module. Define its "character"

$$\chi_{(\boldsymbol{\lambda}, \mathbf{a})}: U(L(\mathfrak{h})) \to L(\mathbb{C}) = \mathbb{C}[t, t^{-1}]$$

by extending

$$\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(f)(z) = \sum_{i=1}^{k} \lambda_i(f(a_i z)), \qquad f \in L(\mathfrak{h}),$$

to an algebra homomorphism. The image of $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ is always a Laurent subring $\mathbb{C}[t^r, t^{-r}]$ of $\mathbb{C}[t, t^{-1}]$ for some $r \geqq 1$. We shall prove in Sects. 1 and 4 that $V(\boldsymbol{\lambda}, \mathbf{a})$ is then a direct sum of r irreducible $\bar{L}(\mathfrak{g})$-submodules which are isomorphic as $L(\mathfrak{g})$-modules, though not as $\bar{L}(\mathfrak{g})$-modules. The case $r = 1$ occurs for a dense open set of values $\mathbf{a} \in (\mathbb{C}^*)^k$.

There is one further degree of freedom to be exploited. Namely, we can change the action of d on the module to

$$d\Omega = z \frac{d\Omega}{dz} + b\Omega$$

for any $b \in \mathbb{C}$. This corresponds to twisting the action on $V(\boldsymbol{\lambda}, \mathbf{a})$ by a character of the one dimensional Lie algebra $\mathbb{C} \cdot d$.

We shall prove in Sects. 2 and 4 that the irreducible modules arising from the above construction are precisely the integrable modules described in [1] and we use the results of that paper to describe the isomorphism classes of the loop modules. In Sect. 3 we prove that the modules are unitarizable if and only if $|a_i| = |a_j|$ for all i, j and b is real.

Let G be the simply-connected complex Lie group with Lie algebra $\mathfrak{g}$. One can associate in a natural way a group $\bar{L}(G)$ with the Lie algebra $\bar{L}(\mathfrak{g})$. The loop modules lift to representations of $\bar{L}(G)$ if and only if b is an integer, and if also $|a_i| = |a_j|$ for all i, j, the representation is unitary for a "compact form" $\bar{L}(K)$ of $\bar{L}(G)$, where K is a maximal compact subgroup of G.

Finally, let us list a few differences between the theory of the modules considered here and that of the standard modules:

(a) The standard modules are uniquely determined by their highest weight just as in the finite dimensional case. Our modules depend on extra continuous parameters.

(b) The standard modules are irreducible for $L(\mathfrak{g})$ itself (or, rather, a central extension of it), whereas our modules are irreducible only for $\bar{L}(\mathfrak{g})$.

(c) The standard modules have positive energy, in the sense that the derivation d acts on the modules with positive eigenvalues. This is not so for our modules, in fact all positive and negative integer eigenvalues occur in $V(\boldsymbol{\lambda}, \mathbf{a})$.

1. Construction of Some Loop Modules

For any complex vector space V, let $L(V)$ denote the space of algebraic maps $f: \mathbb{C}^* \to V$, where $\mathbb{C}^*$ is the set of non-zero complex numbers. Obviously $L(V)$ is isomorphic to $V \bigotimes_{\mathbb{C}} L$, where $L = \mathbb{C}[t, t^{-1}]$ is the ring of Laurent polynomials in an indeterminate t. This description makes it clear that $L(V)$ is a $\mathbb{Z}$-graded vector space, $L(V)=\bigoplus_{n\in\mathbb{Z}} V\otimes t^n$.

If $\mathfrak{a}$ is any complex Lie algebra, $L(\mathfrak{a})$ is itself a $\mathbb{Z}$-graded Lie algebra under pointwise operations:

$$[f_1,f_2](z)=[f_1(z),f_2(z)]$$

for $f_1,f_2\in L(\mathfrak{a})$, $z\in\mathbb{C}^*$. $L(\mathfrak{a})$ is called the *loop algebra* of $\mathfrak{a}$. Define a derivation d of $L(\mathfrak{a})$ by

$$(df)(z)=z\frac{df}{dz}.$$

In terms of the description of $L(\mathfrak{a})$ as $\mathfrak{a}\otimes L$,

$$d(x\otimes t^n)=nx\otimes t^n$$

for $x\in\mathfrak{a}$, $n\in\mathbb{Z}$. Set

$$\bar{L}(\mathfrak{a})=L(\mathfrak{a})\oplus\mathbb{C}\cdot d.$$

Then $\bar{L}(\mathfrak{a})$ has a natural semi-direct product Lie algebra structure:

$$[f_1+a_1d,f_2+a_2d]=[f_1,f_2]+a_1df_2-a_2df_1,$$

where $f_1,f_2\in L(\mathfrak{a})$, $a_1,a_2\in\mathbb{C}$.

Now let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Δ the set of roots of $(\mathfrak{g},\mathfrak{h})$, $S=\{\alpha_1,\dots,\alpha_n\}$ a set of simple roots for Δ, and Δ_+ (resp. Δ_-) the corresponding set of positive (resp. negative) roots. For $\alpha\in\Delta$, let $\mathfrak{g}_\alpha$ denote the corresponding root space, and set

$$\mathfrak{n}_\pm=\bigoplus_{\alpha\in\Delta_\pm}\mathfrak{g}_\alpha.$$

Then

$$\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$$

and

$$\bar{L}(\mathfrak{g})=L(\mathfrak{n}_-)\oplus(L(\mathfrak{h})\oplus\mathbb{C}\cdot d)\oplus L(\mathfrak{n}_+).$$

Define an element $\delta\in(\mathfrak{h}\oplus\mathbb{C}\cdot d)^*$ by

$$\delta|_{\mathfrak{h}}=0,\qquad \delta(d)=1.$$

The space $\mathfrak{h}^*$ is identified with a subspace of $(\mathfrak{h}\oplus\mathbb{C}\cdot d)^*$ by setting $\lambda(d)=0$ for $\lambda\in\mathfrak{h}^*$.

For any $\lambda\in\mathfrak{h}^*$, let $M(\lambda)$ denote the Verma module for $\mathfrak{g}$ with highest weight λ and highest weight vector v_λ. Let

$$\Gamma_+=\left\{\sum_{i=1}^{n}k_i\alpha_i\in\mathfrak{h}^*:\ k_i\in\mathbb{Z},\ k_i\geqq 0\right\}.$$

Then

$$M(\lambda)=\bigoplus_{\eta\in\Gamma_+}M(\lambda)_{\lambda-\eta},$$

where the weight space

$$M(\lambda)_\mu=\{m\in M(\lambda):\ h\cdot m=\mu(h)m\ \forall h\in\mathfrak{h}\}$$

for $\mu \in \mathfrak{h}^*$. Further, $\dim M(\lambda)_\lambda = 1$ and $M(\lambda)$ is a free $U(\mathfrak{n}_-)$-module. All the results used concerning Verma modules may be found in [3].

Let k be any positive integer. For every pair

$$(\boldsymbol{\lambda}, \mathbf{a}) \in (\mathfrak{h}^*)^k \times (\mathbb{C}^*)^k, \quad \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k), \quad \mathbf{a} = (a_1, \dots, a_k),$$

define an $\bar{L}(\mathfrak{g})$-module structure on the loop space $L\left(\bigotimes_{i=1}^k M(\lambda_i)\right)$ as follows:

$$(f \cdot \Omega)(z) = \left(\sum_{i=1}^k 1 \otimes \dots \otimes f(a_i z) \otimes \dots \otimes 1\right) \Omega(z),$$

$$(d\Omega)(z) = z\frac{d\Omega}{dz},$$

for all $f \in L(\mathfrak{g})$, $\Omega \in L\left(\bigotimes_{i=1}^k M(\lambda_i)\right)$, $z \in \mathbb{C}^*$. Let $M(\boldsymbol{\lambda}, \mathbf{a})$ denote the corresponding $\bar{L}(\mathfrak{g})$-module.

The following lemma is trivial.

Lemma (1.1). (i) $M(\boldsymbol{\lambda}, \mathbf{a})$ *is a weight module; in fact*

$$M(\boldsymbol{\lambda}, \mathbf{a}) = \bigoplus_{\substack{\eta \in \Gamma_+ \\ m \in \mathbb{Z}}} M(\boldsymbol{\lambda}, \mathbf{a})_{\lambda - \eta + m\delta},$$

where $\lambda = \sum_{i=1}^k \lambda_i$ *and*

$$M(\boldsymbol{\lambda}, \mathbf{a})_\mu = \{\Omega \in M(\boldsymbol{\lambda}, \mathbf{a}) \colon h \cdot \Omega = \mu(h)\Omega \ \forall h \in \mathfrak{h} \oplus \mathbb{C} \cdot d\}.$$

Further, $\dim M(\boldsymbol{\lambda}, \mathbf{a})_{\lambda + m\delta} = 1$ *for all* $m \in \mathbb{Z}$.

(ii) $L(\mathfrak{n}_+) \cdot \left(\bigotimes_{i=1}^k v_{\lambda_i} \otimes t^m\right) = 0$ *for all* $m \in \mathbb{Z}$, *where* v_{λ_i} *is the highest weight vector of* $M(\lambda_i)$.

(iii) *If* $h \in \mathfrak{h}$, $m, n \in \mathbb{Z}$, *then*

$$(h \otimes t^n) \cdot \left(\bigotimes_{i=1}^k v_{\mu_i} \otimes t^m\right) = \left(\sum_{i=1}^k \mu_i(h) a_i^n\right) \left(\bigotimes_{i=1}^k v_{\mu_i} \otimes t^{m+n}\right),$$

where $v_{\mu_i} \in M(\lambda_i)_{\mu_i}$, $i = 1, \dots, k$.

For any pair $(\boldsymbol{\lambda}, \mathbf{a})$, define a "character"

$$\chi_{(\boldsymbol{\lambda}, \mathbf{a})} \colon U(L(\mathfrak{h})) \to L$$

of the universal enveloping algebra of $L(\mathfrak{h})$ by extending

$$\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(h \otimes t^n) = \left(\sum_{i=1}^k \lambda_i(h) a_i^n\right) t^n$$

to an algebra homomorphism. The algebra $U(L(\mathfrak{h}))$ inherits a natural $\mathbb{Z}$-grading from $L(\mathfrak{h})$ and then $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ is a graded algebra homomorphism.

It was shown in [1, Sect. 4] that the image of $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ is always a Laurent subring $\mathbb{C}[t^r, t^{-r}]$ of L, for some $r \geqq 0$. *Until Sect. 4 we shall deal only with pairs* $(\boldsymbol{\lambda}, \mathbf{a})$ *such that* $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ *is surjective.*

For all $n\in\mathbb{Z}$, let $\Omega_{\boldsymbol{\lambda},n}$ denote the element $\bigotimes_{i=1}^{k} v_{\lambda_i}\otimes t^n$ of $M(\boldsymbol{\lambda},\mathbf{a})$, where v_{λ_i} is the highest weight vector in $M(\lambda_i)$. Set $\Omega_{\boldsymbol{\lambda},0}=\Omega_{\boldsymbol{\lambda}}$.

Proposition (1.2). *If the numbers $a_1,\ldots,a_k$ are distinct, the $\bar{L}(\mathfrak{g})$-module $M(\boldsymbol{\lambda},\mathbf{a})$ is generated by the element $\Omega_{\boldsymbol{\lambda}}$.*

Proof. It suffices to check that all elements of the form

$$y_1v_{\lambda_1}\otimes\ldots\otimes y_kv_{\lambda_k}\otimes t^m,$$

for $m\in\mathbb{Z}$, $y_i\in U(L(\mathfrak{n}_-))$, are in the $\bar{L}(\mathfrak{g})$-submodule generated by $\Omega_{\boldsymbol{\lambda}}$.

Choose elements $Q, Q_*\in U(L(\mathfrak{h}))$ such that

$$\chi_{(\boldsymbol{\lambda},\mathbf{a})}(Q)=t,\qquad \chi_{(\boldsymbol{\lambda},\mathbf{a})}(Q_*)=t^{-1}.$$

By Lemma 1.1 (iii), it is clear that

$$\begin{aligned}Q^n\Omega_{\boldsymbol{\lambda}}&=\Omega_{\boldsymbol{\lambda},n},\\ Q_*^n\Omega_{\boldsymbol{\lambda}}&=\Omega_{\boldsymbol{\lambda},-n}\end{aligned}\tag{1.3}$$

for all $n\in\mathbb{Z}$, $n\geqq 0$. This proves that $\Omega_{\boldsymbol{\lambda},n}\in U(L(\mathfrak{h}))\cdot\Omega_{\boldsymbol{\lambda}}$ for all $n\in\mathbb{Z}$.

Now let $y\in\mathfrak{n}_-$, $m\in\mathbb{Z}$, and consider the k equations

$$(y\otimes t^j)\cdot\Omega_{\boldsymbol{\lambda},m-j}=\sum_{i=1}^{k}a_i^j(v_{\lambda_1}\otimes\ldots\otimes yv_{\lambda_i}\otimes\ldots\otimes v_{\lambda_k})\otimes t^m,\tag{1.4}$$

$j=1,\ldots,k$. Since the a_i are distinct, the matrix $(a_i^j)_{i,j=1,\ldots,k}$ is invertible, so the equations can be solved for the elements $v_{\lambda_1}\otimes\ldots\otimes yv_{\lambda_i}\otimes\ldots\otimes v_{\lambda_k}\otimes t^m$. From (1.3) we see that

$$v_{\lambda_1}\otimes\ldots\otimes yv_{\lambda_i}\otimes\ldots\otimes v_{\lambda_k}\otimes t^m\in U(\bar{L}(\mathfrak{g}))\cdot\Omega_{\boldsymbol{\lambda}}$$

for $i=1,\ldots,k$, $m\in\mathbb{Z}$. A repeated application of this argument proves the Proposition.

Remark (1.5). The same argument shows that for any $n\in\mathbb{Z}$ the element $\Omega_{\boldsymbol{\lambda},n}$ generates $M(\boldsymbol{\lambda},\mathbf{a})$ as $\bar{L}(\mathfrak{g})$-module.

Remark (1.6). It is not difficult to see that the converse of Proposition (1.2) holds. Since we shall not need this, we shall omit the proof.

From now on we shall always assume that the scalars $a_1,\ldots,a_k$ are distinct, so that $M(\boldsymbol{\lambda},\mathbf{a})$ is a cyclic module.

For every $\lambda\in\mathfrak{h}^*$ let $V(\lambda)$ be the unique irreducible quotient of $M(\lambda)$, and $p_\lambda: M(\lambda)\to V(\lambda)$ the natural map. For any pair $(\boldsymbol{\lambda},\mathbf{a})$ with $a_1,\ldots,a_k$ distinct, define an $\bar{L}(\mathfrak{g})$-module structure on $L\left(\bigotimes_{i=1}^{k}V(\lambda_i)\right)$ by

$$(f\cdot\Omega)(z)=\left(\sum_{i=1}^{k}1\otimes\ldots\otimes f(a_iz)\otimes\ldots\otimes 1\right)\cdot\Omega(z),$$

$$(d\Omega)(z)=z\frac{d\Omega}{dz},$$

where $f \in L(\mathfrak{g})$, $\Omega \in L\left(\bigotimes_{i=1}^{k} V(\lambda_i)\right)$. Denote the corresponding $\bar{L}(\mathfrak{g})$-module by $V(\boldsymbol{\lambda}, \mathbf{a})$.

Theorem (1.7). *The $\bar{L}(\mathfrak{g})$-module $V(\boldsymbol{\lambda}, \mathbf{a})$ is the unique irreducible quotient of $M(\boldsymbol{\lambda}, \mathbf{a})$.*

Proof. It is easy to check that the map $p: M(\boldsymbol{\lambda}, \mathbf{a}) \to V(\boldsymbol{\lambda}, \mathbf{a})$ defined by

$$p(\Omega)(z) = (p_{\lambda_1} \otimes \ldots \otimes p_{\lambda_k})(\Omega(z))$$

is a surjective $\bar{L}(\mathfrak{g})$-module homomorphism.

Let W be any proper submodule of $M(\boldsymbol{\lambda}, \mathbf{a})$. Then $\Omega_{\boldsymbol{\lambda}, n} \notin W$ for any $n \in \mathbb{Z}$ by Remark (1.5). Since $\dim M(\boldsymbol{\lambda}, \mathbf{a})_{\lambda + n\delta} = 1$ for all $n \in \mathbb{Z}$, it follows from standard arguments that $M(\boldsymbol{\lambda}, \mathbf{a})$ has a unique maximal proper submodule, and hence a unique irreducible quotient. The theorem will follow if we establish that $V(\boldsymbol{\lambda}, \mathbf{a})$ is irreducible.

Let $\tilde{\Omega}_{\boldsymbol{\lambda}, n}$ be the image of $\Omega_{\boldsymbol{\lambda}, n}$ in $V(\boldsymbol{\lambda}, \mathbf{a})$. Then $V(\boldsymbol{\lambda}, \mathbf{a})$ is cyclically generated by each $\tilde{\Omega}_{\boldsymbol{\lambda}, n}$, so it is enough to prove that for every element $\Omega \in V(\boldsymbol{\lambda}, \mathbf{a})$ there exists $g_\Omega \in U(\bar{L}(\mathfrak{g}))$ such that

$$g_\Omega \cdot \Omega = \tilde{\Omega}_{\boldsymbol{\lambda}, n}$$

for some $n \in \mathbb{Z}$. Further, since $V(\boldsymbol{\lambda}, \mathbf{a})$ is a weight module, it suffices to do this when Ω is a weight vector, i.e. $\Omega \in V(\boldsymbol{\lambda}, \mathbf{a})_{\lambda - \eta + m\delta}$ for some $\eta \in \Gamma_+$, $m \in \mathbb{Z}$.

For $\eta \in \Gamma_+$, $\eta = \sum_{i=1}^{n} k_i \alpha_i$, set $ht\eta = \sum_{i=1}^{n} k_i$. The proof proceeds by induction on $ht\eta$. If $ht\eta = 0$ there is nothing to prove.

Now let $ht\eta > 0$. For the remainder of the proof we restrict ourselves to the case $k = 2$. The general case is only notationally more complicated.

Let $\{v_r\}$ (resp. $\{w_s\}$) be a basis of $V(\lambda_1)$ (resp. $V(\lambda_2)$) consisting of weight vectors, and let

$$v = \left(\sum_{r,s} c_{rs} v_r \otimes w_s\right) \otimes t^m$$

be a non-zero vector in $V(\boldsymbol{\lambda}, \mathbf{a})_{\lambda - \eta + m\delta}$. It is enough to show that for some $x \in \mathfrak{n}_+$, $p \in \mathbb{Z}$, $(x \otimes t^p) \cdot v \neq 0$. Suppose then that

$$(x \otimes t^p) \cdot v = 0 \quad \forall x \in \mathfrak{n}_+, \quad p \in \mathbb{Z}.$$

This gives

$$a_1^p \left(\sum_{r,s} c_{rs} x v_r \otimes w_s\right) + a_2^p \left(\sum_{r,s} c_{rs} v_r \otimes x w_s\right) = 0$$

for all $p \in \mathbb{Z}$. As $a_1 \neq a_2$, this forces

$$\sum_{r,s} c_{rs} x v_r \otimes w_s = 0, \qquad \sum_{r,s} c_{rs} v_r \otimes x w_s = 0$$

for all $x \in \mathfrak{n}_+$. As the $\{v_r\}$ and $\{w_s\}$ are linearly independent sets of vectors,

$$x \cdot \sum_r c_{rs} v_r = 0, \qquad x \cdot \sum_s c_{rs} w_s = 0$$

for all r, s. Choose r_0, s_0 such that $c_{r_0s_0} \neq 0$. Then

$$\tilde{v} = \sum_r c_{rs_0} v_r,$$

$$\tilde{w} = \sum_s c_{r_0s} w_s$$

are non-zero vectors of highest weight in $V(\lambda_1)$ and $V(\lambda_2)$, respectively. This implies

$$\tilde{v} = a v_{\lambda_1}, \qquad \tilde{w} = b v_{\lambda_2}$$

for some $a, b \in \mathbb{C}^*$. Since the v_r and w_s are weight vectors, v_{r_0} and w_{s_0} must be highest weight vectors in $V(\lambda_1)$ and $V(\lambda_2)$. Finally, as v is a weight vector, it must have weight $\lambda_1 + \lambda_2 + m\delta$, contradicting $ht\eta > 0$.

2. Integrability

In this section we shall relate the modules $M(\boldsymbol{\lambda}, \mathbf{a})$ and $V(\boldsymbol{\lambda}, \mathbf{a})$ to the modules $M(\lambda, I)$ and $V(\lambda, I)$ defined in [1].

Let us recall the definition of $M(\lambda, I)$ and $V(\lambda, I)$. Let $\lambda \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*$ and let I be a maximal graded ideal in $U(\bar{L}(\mathfrak{h}))$ such that $I \cap U(\mathfrak{h} \oplus \mathbb{C} \cdot d)$ is the ideal in $U(\mathfrak{h} \oplus \mathbb{C} \cdot d)$ generated by $\{h - \lambda(h) : h \in \mathfrak{h} \oplus \mathbb{C} \cdot d\}$. The module $M(\lambda, I)$ is defined to be the quotient of $U(\bar{L}(\mathfrak{g}))$ by the left ideal generated by $L(\mathfrak{n}_+) \cup I$. Let $\Omega(\lambda, \mathrm{I})$ be the image of I in $M(\lambda, I)$. The module $M(\lambda, I)$ is $L(\mathfrak{n}_-)$-free, and it has a unique irreducible quotient $V(\lambda, I)$ [1, Theorem 3.5].

Let V be any weight module for $\bar{L}(\mathfrak{g})$ such that there exists $v \in V$ satisfying

$$L(\mathfrak{n}_+) \cdot v = 0, \qquad I \cdot v = 0, \qquad V = U(\bar{L}(\mathfrak{g})) \cdot v.$$

Then V is a quotient of $M(\lambda, I)$. If, in addition, V is irreducible, then V must be isomorphic to $V(\lambda, I)$.

Let $(\boldsymbol{\lambda}, \mathbf{a}) \in (\mathfrak{h}^*)^k \times (\mathbb{C}^*)^k$ and assume that the scalars $a_1, \ldots, a_k$ are distinct as usual. Extend $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ to a homomorphism

$$\chi_{(\boldsymbol{\lambda}, \mathbf{a})} : U(\bar{L}(\mathfrak{h})) \to L$$

by setting $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(d) = 0$, and let $I_{(\boldsymbol{\lambda}, \mathbf{a})}$ be its kernel. Set $\lambda = \sum_{i=1}^{k} \lambda_i$.

Proposition (2.1). *The irreducible $\bar{L}(\mathfrak{g})$-modules $V(\boldsymbol{\lambda}, \mathbf{a})$ and $V(\lambda, I_{(\boldsymbol{\lambda}, \mathbf{a})})$ are isomorphic.*

Proof. The proof is immediate. Just observe that $V(\boldsymbol{\lambda}, \mathbf{a})$ is generated by the element $\tilde{\Omega}_{\boldsymbol{\lambda}}$ and

$$L(\mathfrak{n}_+) \cdot \tilde{\Omega}_{\boldsymbol{\lambda}} = 0, \qquad I_{(\boldsymbol{\lambda}, \mathbf{a})} \cdot \tilde{\Omega}_{\boldsymbol{\lambda}} = 0.$$

Proposition (2.2). *Let $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ be dominant integral, i.e. $\langle \lambda_i, \check{\alpha}_j \rangle \in \mathbb{Z}_+$ for all $i = 1, \ldots, k$, $j = 1, \ldots, n$. Then $V(\boldsymbol{\lambda}, \mathbf{a})$ is an irreducible integrable $\bar{L}(\mathfrak{g})$-module. In fact, there exists an integer $N \geqq 0$ such that*

$$(x \otimes t^m)^N \cdot V(\boldsymbol{\lambda}, \mathbf{a}) = 0$$

for all $m \in \mathbb{Z}$ and all $x \in \mathfrak{n}_+ \cup \mathfrak{n}_-$.

Proof. Recall that $V(\boldsymbol{\lambda}, \mathbf{a}) = L\left(\bigotimes_{i=1}^{k} V(\lambda_i)\right)$, where $V(\lambda_i)$ is the unique irreducible quotient of the Verma module $M(\lambda_i)$. If $\lambda_i \in \mathfrak{h}^*$ is dominant integral, then $V(\lambda_i)$ is finite dimensional, so there exists an integer $r_i \geqq 0$ such that

$$x^{r_i} \cdot V(\lambda_i) = 0 \quad \text{for all} \quad x \in \mathfrak{n}_+ \cup \mathfrak{n}_- .$$

Set $N = \sum_{i=1}^{k} r_i$. It is easy to check that

$$(x \otimes t^m)^N \cdot L\left(\bigotimes_{i=1}^{k} V(\lambda_i)\right) = 0$$

for all $m \in \mathbb{Z}$ and $x \in \mathfrak{n}_+ \cup \mathfrak{n}_-$.

Although it is not true that every module $V(\lambda, I)$ is isomorphic to some $V(\boldsymbol{\lambda}, \mathbf{a})$, this is true whenever $V(\lambda, I)$ is integrable, $\lambda \in \mathfrak{h}^*$. To see this we recall the following result from [1]. Let $\check{\alpha}_i$ be the coroot corresponding to α_i.

Theorem (2.3) [1, Theorem 4.1]. *The $\bar{L}(\mathfrak{g})$-module $V(\lambda, I)$ is integrable if and only if*

(i) $\langle \lambda, \check{\alpha}_i \rangle$ *is a non-negative integer for* $i = 1, 2, \ldots, n$, *and*

(ii) *there exist non-zero scalars* (a_{ij}), $1 \leqq j \leqq \langle \lambda, \check{\alpha}_i \rangle$, $1 \leqq i \leqq n$, *such that I is the kernel of the homomorphism* $\Lambda_I : U(L(\mathfrak{h})) \to L$ *defined by extending*

$$\Lambda_I(\check{\alpha}_i \otimes t^m) = \left(\sum_{j=1}^{\langle \lambda, \check{\alpha}_i \rangle} a_{ij}^m \right) t^m .$$

Suppose then that $V(\lambda, I)$ is integrable. Let $\{a_1, \ldots, a_k\}$ be the set of distinct elements among the a_{ij} and let m_{ir} be the number of times a_r occurs in the set $\{a_{ij}, 1 \leqq j \leqq \langle \lambda, \check{\alpha}_i \rangle\}$. Set $\lambda_r = \sum_{i=1}^{k} m_{ir} \mu_i$, where $\mu_1, \ldots, \mu_n$ are the fundamental weights of $\mathfrak{h}^*$ (i.e. $\langle \mu_i, \check{\alpha}_j \rangle = \delta_{ij}$). Clearly, $\lambda = \sum_{r=1}^{k} \lambda_r$, $\Lambda_I = \chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ and $I = I_{(\boldsymbol{\lambda}, \mathbf{a})}$. Assuming that $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ is surjective, we deduce from Proposition (2.1) that $V(\lambda, I)$ is isomorphic to $V(\boldsymbol{\lambda}, \mathbf{a})$. We summarize the discussion in

Theorem (2.4). *Assume (λ, I) is such that $V(\lambda, I)$ is integrable, $\lambda \in \mathfrak{h}^*$, and that Λ_I is surjective. Then $V(\lambda, I)$ is isomorphic to $V(\boldsymbol{\lambda}, \mathbf{a})$, where $(\boldsymbol{\lambda}, \mathbf{a})$ is as defined above.*

The image of Λ_I is always a Laurent subring of L. The case when it is a proper subring will be discussed in Sect. 4.

Let G be the simply-connected complex Lie group with Lie algebra $\mathfrak{g}$. Denote by $L(G)$ the group of algebraic maps $\mathbb{C}^* \to G$. Equivalently, $L(G)$ consists of the points of G with values in the Laurent polynomial ring L, in the sense of algebraic geometry. The non-zero complex numbers acts as a group of automorphisms of $L(G)$ by reparametrization:

$$(a \cdot f)(z) = f(az), \qquad a \in \mathbb{C}^*, \qquad f \in L(G).$$

Let $\bar{L}(G)$ be the semi-direct product $L(G) \tilde{\times} \mathbb{C}^*$. We may think of $\bar{L}(\mathfrak{g})$ as the Lie algebra of $\bar{L}(G)$.

If $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ are dominant integral weights, the action of $\mathfrak{g}$ on the $V(\lambda_i)$ lifts to an action of G. If $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$, $\mathbf{a} = (a_1, \ldots, a_k)$, with $a_i \in \mathbb{C}^*$, the action of $\bar{L}(\mathfrak{g})$ on $V(\boldsymbol{\lambda}, \mathbf{a})$ obviously lifts to an action of $\bar{L}(G)$. Namely

$$(f \cdot \Omega)(z) = \left(\bigotimes_{i=1}^{k} f(a_i z) \right) \cdot \Omega(z),$$

$$(a \cdot \Omega)(z) = \Omega(az)$$

for $a \in \mathbb{C}^*$, $f \in L(G)$, $\Omega \in V(\boldsymbol{\lambda}, \mathbf{a})$.

Proposition (2.5). *If $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ are dominant integral and $a_1, \ldots, a_k$ are distinct then $V(\boldsymbol{\lambda}, \mathbf{a})$ is an irreducible representation of $\bar{L}(G)$.*

Proof. By Theorem (1.7) it is sufficient to prove that any subspace $W \subseteqq V(\boldsymbol{\lambda}, \mathbf{a})$ which is stable under the action of $\bar{L}(G)$ is also stable under $\bar{L}(\mathfrak{g})$.

Any element $f \in L(\mathfrak{n}_+) \cup L(\mathfrak{n}_-)$ can be exponentiated to an element of $L(G)$. Thus for any $a \in \mathbb{C}^*$, $\exp(af) W \subseteqq W$. With N as in the Proposition (2.2), we have

$$\left(\sum_{s=0}^{N} \frac{a^s f^s}{s!} \right) W \subseteqq W.$$

Taking $N+1$ distinct values for a, we deduce that $fW \subseteqq W$. Since $L(\mathfrak{n}_+) \cup L(\mathfrak{n}_-)$ generates $L(\mathfrak{g})$ as a Lie algebra it follows that W is $L(\mathfrak{g})$-stable.

Next, we claim that W is a graded subspace of $V(\boldsymbol{\lambda}, \mathbf{a})$. In particular, $dW \subseteqq W$ and the proof will be complete.

Let $w \in W$ and write

$$w = \sum_{s=-M}^{M} w_s \otimes t^s$$

with $w_s \in \bigotimes_{i=1}^{k} V(\lambda_i)$. Since W is stable under the $\mathbb{C}^*$-action, $\sum_{-M}^{M} a^s w_s \otimes t^s \in W$ for all $a \in \mathbb{C}^*$. Taking $2M+1$ different values for a, we see that $w_s \otimes t^s \in W$ for each s, as claimed.

The modules $V(\boldsymbol{\lambda}, \mathbf{a})$ can be completed to obtain representations of larger loop groups. For example, let us replace Laurent polynomials by smooth functions on S^1 in the construction of $V(\boldsymbol{\lambda}, \mathbf{a})$. This gives a representation $V^{sm}(\boldsymbol{\lambda}, \mathbf{a})$ of the group $\bar{L}^{sm}(G) = L^{sm}(G) \tilde{\times} S^1$, where $L^{sm}(G)$ is the group of smooth maps $S^1 \to G$, on which S^1 acts by reparametrization. $V^{sm}(\boldsymbol{\lambda}, \mathbf{a})$ is provided with the usual C^∞-topology.

We shall not pursue such completions here, but let us observe that $V^{sm}(\boldsymbol{\lambda}, \mathbf{a})$ is an irreducible representation of $\bar{L}^{sm}(G)$ whenever $V(\boldsymbol{\lambda}, \mathbf{a})$ is irreducible for $\bar{L}(G)$. Of course, irreducible now means there is no proper closed $\bar{L}^{sm}(G)$-invariant subspace of $V^{sm}(\boldsymbol{\lambda}, \mathbf{a})$. Indeed, if $0 \neq W \subset V^{sm}(\boldsymbol{\lambda}, \mathbf{a})$ is closed and $\bar{L}^{sm}(G)$-invariant, then $W \cap V(\boldsymbol{\lambda}, \mathbf{a})$ is an $\bar{L}(G)$-invariant subspace of $V(\boldsymbol{\lambda}, \mathbf{a})$, and is *dense* in W. [For as a representation of S^1, acting by reparametrization, $W \cap V(\boldsymbol{\lambda}, \mathbf{a})$ is the set of S^1-finite vectors in W.] Thus $W \supset V(\boldsymbol{\lambda}, \mathbf{a})$, and since $V(\boldsymbol{\lambda}, \mathbf{a})$ is dense in $V^{sm}(\boldsymbol{\lambda}, \mathbf{a})$, we must have $W = V^{sm}(\boldsymbol{\lambda}, \mathbf{a})$.

3. Unitarizability

In this section it will be necessary to consider the loop algebras of real Lie algebras. So, if V is a real vector space, we shall denote by $L(V)$ the space of maps from the unit circle $S^1 = \{z \in \mathbb{C}: |z| = 1\}$ into V given by Laurent polynomials in z.

Let $\mathfrak{k}$ be the compact real form of $\mathfrak{g}$ corresponding to a conjugate-linear anti-involution θ of $\mathfrak{g}$. Thus,

$$\mathfrak{k} = \{x \in \mathfrak{g}:\ \theta(x) = -x\}.$$

It is convenient to write $\theta(x) = x^*$. Extend θ to a conjugate-linear anti-involution $\bar{\theta}$ of $\bar{L}(\mathfrak{g})$ by

$$\bar{\theta}(x \otimes t^n) = \theta(x) \otimes t^{-n},$$

$$\bar{\theta}(d) = d.$$

Again we shall write $\bar{\theta}(f) = f^*$ for $f \in L(\mathfrak{g})$. Notice that in general $f^*(z) = f(z)^*$ only if $|z| = 1$. Then there is a natural isomorphism

$$L(\mathfrak{k}) \cong \{f \in L(\mathfrak{g}):\ f^* = -f\}.$$

Let $\bar{L}(\mathfrak{k})$ denote the semi-direct product

$$\bar{L}(\mathfrak{k}) = L(\mathfrak{k}) \oplus i\mathbb{R}d$$

of real Lie algebras. This is precisely the fixed point set of $-\bar{\theta}$ and is called the compact real form of $\bar{L}(\mathfrak{g})$ [2, Sect. 4].

Following the usual definition of unitarizability for modules over an arbitrary finite dimensional semi-simple Lie algebra we have

Definition (3.1). An $\bar{L}(\mathfrak{g})$-module V is said to be *unitarizable* if there exists a positive-definite Hermitian form $\langle\ ,\ \rangle$ on V satisfying

$$\langle f \cdot v_1, v_2 \rangle = \langle v_1, f^* \cdot v_2 \rangle$$

for all $f \in \bar{L}(\mathfrak{g})$, $v_1, v_2 \in V$.

Theorem (3.2). *Let $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ be dominant integral weights, $a_1, \ldots, a_k$ be distinct elements of $\mathbb{C}^*$. Then the $\bar{L}(\mathfrak{g})$-module $V(\boldsymbol{\lambda}, \mathbf{a})$ is unitarizable if and only if $|a_i| = |a_j|$ for all i, j.*

Proof. ($\Leftarrow$) Suppose $|a_i| = c^{-1}$ for $i = 1, \ldots, k$. Since the λ_i are dominant integral weights, $\bigotimes_{i=1}^{k} V(\lambda_i)$ is a unitarizable $\overbrace{\mathfrak{g} \oplus \ldots \oplus \mathfrak{g}}^{k}$-module. Let $(\ ,\ )$ be a positive-definite Hermitian form on $\bigotimes_{i=1}^{k} V(\lambda_i)$ preserved by the $\mathfrak{g} \oplus \ldots \oplus \mathfrak{g}$ action. The formula

$$\langle \Omega_1, \Omega_2 \rangle = \frac{1}{2\pi i} \int_{S^1} (\Omega_1(cz), \Omega_2(cz)) \frac{dz}{z}, \qquad \Omega_1, \Omega_2 \in L\left(\bigotimes_{i=1}^{k} V(\lambda_i)\right),$$

defines a positive-definite Hermitian form on $V(\boldsymbol{\lambda}, \mathbf{a})$. Now if $f \in L(\mathfrak{g})$, then

$$\begin{aligned}\langle f\Omega_1, \Omega_2 \rangle &= \frac{1}{2\pi i} \sum_{j=1}^{k} \int_{S^1} (1 \otimes \ldots \otimes f(ca_j z) \otimes \ldots \otimes 1 \cdot \Omega_1(cz), \Omega_2(cz)) \frac{dz}{z} \\ &= \frac{1}{2\pi i} \sum_{j=1}^{k} \int_{S^1} (\Omega_1(cz), 1 \otimes \ldots \otimes f^*(ca_j z) \otimes \ldots \otimes 1 \cdot \Omega_2(cz)) \frac{dz}{z}\end{aligned}$$

(using the invariance of $(\ ,\ )$ and the fact that $|ca_j z| = 1$ for all $z \in S^1$)

$$= \langle \Omega_1, f^*\Omega_2 \rangle.$$

Similarly, integration by parts shows that

$$\langle d\Omega_1, \Omega_2\rangle = \langle \Omega_1, d\Omega_2\rangle .$$

This proves that $V(\boldsymbol{\lambda}, \mathbf{a})$ is unitarizable.

($\Rightarrow$) Conversely, suppose that $\langle\,,\rangle$ is an $\bar{L}(\mathfrak{g})$-invariant positive-definite Hermitian form on $V(\boldsymbol{\lambda}, \mathbf{a})$. Choose an element $Q \in U(L(\mathfrak{h}))$ such that $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(Q) = t$. Set $Q^* = \bar{\theta}(Q)$. [We denote the extension of $\bar{\theta}$ to an anti-involution of $U(\bar{L}(\mathfrak{g}))$ also by $\bar{\theta}$.] Clearly, $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(Q^*) = c^2 t^{-1}$ for some $c \in \mathbb{C}$. Also

$$Q^n \cdot \tilde{\Omega}_{\boldsymbol{\lambda}, m} = \tilde{\Omega}_{\boldsymbol{\lambda}, m+n},$$
$$(Q^*)^n \cdot \tilde{\Omega}_{\boldsymbol{\lambda}, m} = c^{2n} \tilde{\Omega}_{\boldsymbol{\lambda}, m-n}$$

for all $m, n \in \mathbb{Z}$, $n \geqq 0$. Now compute both sides of the equation

$$\langle Q^* \cdot \tilde{\Omega}_{\boldsymbol{\lambda}, 1}, \tilde{\Omega}_{\boldsymbol{\lambda}}\rangle = \langle \tilde{\Omega}_{\boldsymbol{\lambda}, 1}, Q \cdot \tilde{\Omega}_{\boldsymbol{\lambda}}\rangle .$$

This gives

$$\|\tilde{\Omega}_{\boldsymbol{\lambda}, 1}\|^2 = c^2 \|\tilde{\Omega}_{\boldsymbol{\lambda}}\|^2$$

and hence $c \neq 0$. A repeated application of this argument proves that

$$\|\tilde{\Omega}_{\boldsymbol{\lambda}, n}\|^2 = c^{2n} \|\tilde{\Omega}_{\boldsymbol{\lambda}}\|^2 \tag{3.3}$$

for all $n \in \mathbb{Z}$.

For every $i = 1, \ldots, k$, choose a simple root $\alpha_i \in S$ such that $\langle \lambda_i, \check{\alpha}_i\rangle \neq 0$. (This is possible, since there is no loss of generality in assuming all the λ_i are non-zero.) Then if y_i is a non-zero vector in $\mathfrak{g}_{-\alpha_i}$, $y_i v_{\lambda_i}$ is a non-zero element of $V(\lambda_i)_{\lambda_i - \alpha_i}$. Set

$$v_i = v_{\lambda_1} \otimes \ldots \otimes y_i v_{\lambda_i} \otimes \ldots \otimes v_{\lambda_k} .$$

Claim. $\|v_i \otimes t\|^2 = c^2 \|v_i\|^2$.

Before proving the claim, let us deduce Theorem (3.2) from it. We compute both sides of the equation

$$\langle (\check{\alpha}_1 \otimes t) \cdot \tilde{\Omega}_{\boldsymbol{\lambda}}, \tilde{\Omega}_{\boldsymbol{\lambda}, 1}\rangle = \langle \tilde{\Omega}_{\boldsymbol{\lambda}}, (\check{\alpha}_1^* \otimes t^{-1}) \cdot \tilde{\Omega}_{\boldsymbol{\lambda}, 1}\rangle .$$

This gives [using (3.3)],

$$\left(\sum_{i=1}^{k} \langle \lambda_i, \check{\alpha}_1\rangle a_i\right) c^2 = \sum_{i=1}^{k} \langle \lambda_i, \check{\alpha}_1\rangle a_i^* ,$$

where a_i^* means $\bar{a}_i^{-1}$. Replacing $\tilde{\Omega}_{\boldsymbol{\lambda}}$ (resp. $\tilde{\Omega}_{\boldsymbol{\lambda}, 1}$) by v_1 (resp. $v_1 \otimes t$) we get the same equations with λ_1 replaced by $\lambda_1 - \alpha_1$. Subtracting the two equations and noting that $\langle \alpha_1, \check{\alpha}_1\rangle = 2 \neq 0$ proves that $c^2 a_1 = a_1^*$. Similarly, one proves that $c^2 a_i = a_i^*$ for all $i = 1, \ldots, k$, and hence $|a_i| = |a_j|$ for all i, j.

We now prove the claim in the case when $i = 1$. The proof for arbitrary i is similar.

Consider the equations

$$(y_1 \otimes t^{-i}) \cdot \tilde{\Omega}_{\boldsymbol{\lambda}, i} = \sum_j a_j^{-i} (v_{\lambda_1} \otimes \ldots \otimes y_1 v_{\lambda_j} \otimes \ldots \otimes v_{\lambda_k}), \tag{3.4}$$

where the summation is over the set

$$J=\{j\in\mathbb{Z}:\ 1\leqq j\leqq k,\ \lambda_j(\check{\alpha}_1)\neq 0\}.$$

Note that $1\in J$. Let r be the cardinality of J. Let $(b_{ji})_{i=1,\dots,r;\,j\in J}$ be the inverse of the matrix $(a_j^{-i})_{i=1,\dots,r;\,j\in J}$, and set $b_i=b_{1i}$. Solving (3.4) for $i=1,\dots,r$ gives

$$v_1=\sum_{i=1}^{r} b_i(y_1\otimes t^{-i})\cdot\tilde{\Omega}_{\lambda,i}.$$

Writing down similar equations with $\tilde{\Omega}_{\lambda,i+1}$ replacing $\tilde{\Omega}_{\lambda,i}$, we find

$$v_1\otimes t=\sum_{i=1}^{r} b_i(y_1\otimes t^{-i})\cdot\tilde{\Omega}_{\lambda,i+1}.$$

Hence,

$$\|v_1\otimes t\|^2=\sum_{p,q=1}^{r} b_p\bar{b}_q\langle(y_1\otimes t^{-p})\cdot\tilde{\Omega}_{\lambda,p+1},(y_1\otimes t^{-q})\cdot\tilde{\Omega}_{\lambda,q+1}\rangle$$

$$=\sum_{p,q=1}^{r} b_p\bar{b}_q\langle(\check{\alpha}_1\otimes t^{q-p})\cdot\tilde{\Omega}_{\lambda,p+1},\tilde{\Omega}_{\lambda,q+1}\rangle$$

(we may assume $[y_1^*,y_1]=\check{\alpha}_1$)

$$=\sum_{p,q=1}^{r} b_p\bar{b}_q\left(\sum_{i=1}^{k}\langle\lambda_i,\check{\alpha}_1\rangle a_i^{q-p}\right)\|\tilde{\Omega}_{\lambda,q+1}\|^2$$

[by Lemma 1.1(iii)]

$$=\sum_{p,q=1}^{r} b_p\bar{b}_q\left(\sum_{i=1}^{k}\langle\lambda_i,\check{\alpha}_1\rangle a_i^{q-p}\right)c^{2q+2}\|\tilde{\Omega}_{\lambda}\|^2$$

[by (3.3)]

$$=c^2\|v_1\|^2,$$

by repeating the above computation with $v_1\otimes t$ replaced by v_1. This completes the proof of the claim, and hence of Theorem (3.2).

We conclude this section by observing that the conditions stated in Theorem (3.2) are also necessary and sufficient for $V(\boldsymbol{\lambda},\mathbf{a})$ to be unitarizable as a representation of the compact form of the loop group $\tilde{L}(G)$.

Let $K\subset G$ be the compact simply-connected Lie group with Lie algebra $\mathfrak{k}$, and $L(K)$ the group of maps $S^1\to K$ which are restrictions of algebraic maps $\mathbb{C}^*\to G$. The circle group S^1 acts on $L(K)$ by reparametrization, and we denote by $\tilde{L}(K)$ the semi-direct product $L(K)\tilde{\times}S^1$. If $\lambda_1,\dots,\lambda_k\in\mathfrak{h}^*$ are dominant integral, $\tilde{L}(K)$ acts on $V(\boldsymbol{\lambda},\mathbf{a})$. To say that $V(\boldsymbol{\lambda},\mathbf{a})$ is unitarizable as a representation of $\tilde{L}(K)$ means that there exists a positive-definite Hermitian form $\langle\,,\rangle$ on $V(\boldsymbol{\lambda},\mathbf{a})$ such that

$$\langle f\cdot\Omega_1,f\cdot\Omega_2\rangle=\langle\Omega_1,\Omega_2\rangle,$$

$$\langle\Omega_1^w,\Omega_2^w\rangle=\langle\Omega_1,\Omega_2\rangle$$

for all $\Omega_1,\Omega_2\in V(\boldsymbol{\lambda},\mathbf{a})$, $f\in L(K)$, $w\in S^1$, where Ω^w is the element of $V(\boldsymbol{\lambda},\mathbf{a})$ obtained by reparametrizing $\Omega\in V(\boldsymbol{\lambda},\mathbf{a})$ by w:

$$\Omega^w(z)=\Omega(wz).$$

Theorem (3.5). *$V(\boldsymbol{\lambda}, \mathbf{a})$ is unitarizable as a representation of $\bar{L}(K)$ if and only if*
(i) *$\lambda_1, \ldots, \lambda_k$ are dominant integral, and*
(ii) *$|a_i| = |a_j|$ for all i, j.*

The "if" part of this result follows from a computation similar to, but simpler than, that used to prove the corresponding part of Theorem (3.2). For the converse, one observes that any $\bar{L}(K)$-invariant Hermitian form on $V(\boldsymbol{\lambda}, \mathbf{a})$ is also $\bar{L}(\mathfrak{k})$-invariant, and hence $\bar{L}(\mathfrak{g})$-invariant. This is true because one can choose a set of generators for $\bar{L}(\mathfrak{k})$ which can be exponentiated to $\bar{L}(K)$. We omit the details.

4. The Case $r > 1$: Twisted Loop Modules

In this section we study the case when the image of the character $\chi_{(\boldsymbol{\lambda}, \mathbf{a})} : U(L(\mathfrak{h})) \to L$ is a proper Laurent subring $L_r = \mathbb{C}[t^r, t^{-r}]$ of L, $r > 1$. We shall restrict ourselves to stating the results and giving brief indications of the proofs, which in most cases are trivial modifications of those given in the first three sections.

When $r = 1$, the $\bar{L}(\mathfrak{g})$-module $M(\boldsymbol{\lambda}, \mathbf{a})$ is cyclic. If $r > 1$ this is no longer true.

Proposition (4.1) *$M(\boldsymbol{\lambda}, \mathbf{a})$ is generated as an $\bar{L}(\mathfrak{g})$-module by the elements $\Omega_{\boldsymbol{\lambda}, i}$ for $0 \leqq i \leqq r-1$. In fact, any set of r elements $\Omega_{\boldsymbol{\lambda}, i}$ will do, provided no two of the integers i differ by a multiple of r.*

The proof is the same as that of Proposition (1.2), except that one must choose $Q, Q_* \in U(L(\mathfrak{h}))$ such that

$$\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(Q) = t^r, \qquad \chi_{(\boldsymbol{\lambda}, \mathbf{a})}(Q_*) = t^{-r}.$$

Then

$$Q^n \Omega_{\boldsymbol{\lambda}} = \Omega_{\boldsymbol{\lambda}, nr},$$
$$Q_*^n \Omega_{\boldsymbol{\lambda}} = \Omega_{\boldsymbol{\lambda}, -nr}$$

and the result follows by inverting (1.4) as before.

Proposition (4.2). (i) *$V(\boldsymbol{\lambda}, \mathbf{a})$ is a quotient of $M(\boldsymbol{\lambda}, \mathbf{a})$.*
(ii) *Any vector $v \in V(\boldsymbol{\lambda}, \mathbf{a})$ which is annihilated by $L(\mathfrak{n}_+)$ is a linear combination of the elements $\tilde{\Omega}_{\boldsymbol{\lambda}, n}$.*

This is proved as in Theorem (1.7).

When $r = 1$ we proved in Theorem (1.7) that the $\bar{L}(\mathfrak{g})$-module $V(\boldsymbol{\lambda}, \mathbf{a})$ is irreducible. This is no longer true when $r > 1$. To describe its irreducible components we shall need a simpler number theoretic result.

Since $\operatorname{im} \chi_{(\boldsymbol{\lambda}, \mathbf{a})} = L_r$, it follows that

$$\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(h \otimes t^n) = 0$$

for all $h \in \mathfrak{h}$, $n \in \mathbb{Z}$, $n \not\equiv 0(r)$, and hence that

$$\sum_{i=1}^{k} \lambda_i a_i^n = 0 \quad \text{if} \quad n \not\equiv 0(r). \tag{4.3}$$

There is no loss of generality in assuming the λ_i are all non-zero. Fix a primitive r^{th} root of unity ε.

Lemma (4.4). *Let $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ be non-zero weights and $a_1, \ldots, a_k \in \mathbb{C}^*$ distinct non-zero complex numbers. Assume that* (4.3) *holds for some $r>1$. Then $k \equiv 0(r)$. Moreover, there exists a permutation τ of $\{1, 2, \ldots, k\}$ such that*

$$\begin{gathered}
\lambda_{\tau(1)} = \lambda_{\tau(2)} = \ldots = \lambda_{\tau(r)}, \\
\lambda_{\tau(r+1)} = \lambda_{\tau(r+2)} = \ldots = \lambda_{\tau(2r)}, \\
\vdots \\
\lambda_{\tau(k-r+1)} = \ldots = \lambda_{\tau(k)},
\end{gathered}$$

and complex numbers $a_{(1)}, \ldots, a_{(p)}$ such that

$$\begin{gathered}
a_1 = \varepsilon a_{(1)},\ a_2 = \varepsilon^2 a_{(1)}, \ldots,\ a_r = \varepsilon^r a_{(1)}, \\
a_{r+1} = \varepsilon a_{(2)}, \ldots,\ a_{2r} = \varepsilon^r a_{(2)}, \\
\vdots \\
a_{k-r+1} = \varepsilon a_{(p)}, \ldots,\ a_k = \varepsilon^r a_{(p)},
\end{gathered}$$

where $p = k/r$.

The proof of this Lemma is elementary and will be omitted.

For any k-tuple $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^k$ (resp. $\mathbf{a} \in (\mathbb{C}^*)^k$) and any permutation τ of $\{1, 2, \ldots, k\}$, let $\boldsymbol{\lambda}_\tau$ (resp. $\mathbf{a}_\tau$) be the k-tuple obtained by permuting the entries in $\boldsymbol{\lambda}$ (resp. $\mathbf{a}$) according to τ.

The following result is obvious.

Lemma (4.5). *For any permutation τ of $\{1, 2, \ldots, k\}$*

$$M(\boldsymbol{\lambda}, \mathbf{a}) \cong M(\boldsymbol{\lambda}_\tau, \mathbf{a}_\tau)$$

as $\bar{L}(\mathfrak{g})$-modules.

In view of this Lemma, we may assume that the permutation in Lemma 4.4 is the identity. Thus,

$$\lambda_1 = \lambda_2 = \ldots = \lambda_r,\ \lambda_{r+1} = \ldots = \lambda_{2r}, \ldots,\ \lambda_{k-r+1} = \ldots = \lambda_k.$$

Let σ be the permutation

$$\sigma = (1, 2, \ldots, r)(r+1, r+2, \ldots, 2r) \ldots (k-r+1, \ldots, k)$$

of $\{1, 2, \ldots, k\}$. Then σ induces an automorphism of $M(\boldsymbol{\lambda}, \mathbf{a}) = L\left(\bigotimes_{i=1}^{k} M(\lambda_i)\right)$ by permuting the factors in the tensor product. For $0 \leqq i \leqq r-1$, set

$$M^i(\boldsymbol{\lambda}, \mathbf{a}) = \{\Omega \in M(\boldsymbol{\lambda}, \mathbf{a}) \colon \Omega(\varepsilon z) = \varepsilon^i \sigma(\Omega(z)) \text{ for all } z \in \mathbb{C}^*\}$$

and define $V^i(\boldsymbol{\lambda}, \mathbf{a})$ similarly. It is easy to check that the $M^i(\boldsymbol{\lambda}, \mathbf{a})$ (resp. $V^i(\boldsymbol{\lambda}, \mathbf{a})$) are $\bar{L}(\mathfrak{g})$-submodules of $M(\boldsymbol{\lambda}, \mathbf{a})$ (resp. $V(\boldsymbol{\lambda}, \mathbf{a})$). They are isomorphic as $L(\mathfrak{g})$-modules, though *not* as $\bar{L}(\mathfrak{g})$-modules. In fact, multiplication by z^i is an isomorphism

$M^0 \to M^i$ (resp. $V^0 \to V^i$) of $L(\mathfrak{g})$-modules. It is clear that

$$M(\boldsymbol{\lambda}, \mathbf{a}) = \bigoplus_{i=0}^{r-1} M^i(\boldsymbol{\lambda}, \mathbf{a}),$$

$$V(\boldsymbol{\lambda}, \mathbf{a}) = \bigoplus_{i=0}^{r-1} V^i(\boldsymbol{\lambda}, \mathbf{a}),$$

as $\bar{L}(\mathfrak{g})$-modules. Note that $\Omega_{\boldsymbol{\lambda}, n} \in M^i(\boldsymbol{\lambda}, \mathbf{a})$, $\tilde{\Omega}_{\boldsymbol{\lambda}, n} \in V^i(\boldsymbol{\lambda}, \mathbf{a})$, if $n \equiv i(r)$.

The following is now an immediate consequence of Proposition (4.1).

Corollary (4.6). *$M^i(\boldsymbol{\lambda}, \mathbf{a})$ (resp. $V^i(\boldsymbol{\lambda}, \mathbf{a})$) is generated as an $\bar{L}(\mathfrak{g})$-module by the element $\Omega_{\boldsymbol{\lambda}, i}$ (resp. $\tilde{\Omega}_{\boldsymbol{\lambda}, i}$) and $V^i(\boldsymbol{\lambda}, \mathbf{a})$ is a quotient of $M^i(\boldsymbol{\lambda}, \mathbf{a})$.*

The next result is proved by the same methods as in Theorem (1.7), Proposition (2.2), and Theorem (2.4).

Theorem (4.7). (i) *Each $V^i(\boldsymbol{\lambda}, \mathbf{a})$ is an irreducible $\bar{L}(\mathfrak{g})$-module.*

(ii) *If $\lambda_1, \ldots, \lambda_k$ are dominant integral, then each $V^i(\boldsymbol{\lambda}, \mathbf{a})$ is integrable.*

(iii) *$V^i(\boldsymbol{\lambda}, \mathbf{a}) \cong V(\lambda + i\delta, I_{(\boldsymbol{\lambda}, \mathbf{a})})$ as $\bar{L}(\mathfrak{g})$-modules, where $\lambda = \sum_{j=1}^{k} \lambda_j \in \mathfrak{h}^*$.*

(iv) *Any integrable module $V(\lambda, I)$, with $\lambda \in \mathfrak{h}^*$, such that the image of $\Lambda_I : U(L(\mathfrak{h})) \to L$ is L_r, is isomorphic to $V^0(\boldsymbol{\lambda}, \mathbf{a})$ for some pair $(\boldsymbol{\lambda}, \mathbf{a})$.*

We shall deal shortly with the realization of the integrable modules $V(\lambda, I)$ with $\lambda \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*$. But before doing so, we consider the question of unitarizability.

Theorem (4.8). *If $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ are dominant integral and $0 \leqq i \leqq r-1$, then $V^i(\boldsymbol{\lambda}, \mathbf{a})$ is unitarizable if and only if the complex numbers $a_1, \ldots, a_k$ all have the same modulus.*

Proof. ($\Leftarrow$) The proof of the corresponding part of Theorem (3.2) shows that $V(\boldsymbol{\lambda}, \mathbf{a})$ is unitarizable, and hence so is the submodule $V^i(\boldsymbol{\lambda}, \mathbf{a})$.

($\Rightarrow$) Suppose for example that $V^0(\boldsymbol{\lambda}, \mathbf{a})$ is unitarizable. The proof for any $V^i(\boldsymbol{\lambda}, \mathbf{a})$ is similar. Let c be the positive number such that $c^r = \dfrac{\|\tilde{\Omega}_{\boldsymbol{\lambda}, r}\|}{\|\tilde{\Omega}_{\boldsymbol{\lambda}}\|}$, and let $\langle , \rangle_0$ be an $\bar{L}(\mathfrak{g})$-invariant positive definite Hermitian form on $V^0(\boldsymbol{\lambda}, \mathbf{a})$. Multiplication by z^i defines an isomorphism of $L(\mathfrak{g})$-modules, $V^0(\boldsymbol{\lambda}, \mathbf{a}) \to V^i(\boldsymbol{\lambda}, \mathbf{a})$ so we can transport the form $\langle , \rangle_0$ to a form $\langle , \rangle_i$ on $V^i(\boldsymbol{\lambda}, \mathbf{a})$ by

$$\langle \Omega_1, \Omega_2 \rangle_i = c^i \langle z^{-i}\Omega_1, z^{-i}\Omega_2 \rangle_0, \qquad \Omega_1, \Omega_2 \in V^i(\boldsymbol{\lambda}, \mathbf{a}).$$

Let $\langle , \rangle$ be the direct sum form on $V(\boldsymbol{\lambda}, \mathbf{a}) = \bigoplus_{i=0}^{r-1} V^i(\boldsymbol{\lambda}, \mathbf{a})$. Then $\langle , \rangle$ is positive-definite Hermitian and $L(\mathfrak{g})$-invariant [though not $\bar{L}(\mathfrak{g})$-invariant].

The proof of Theorem (3.2) can now be carried through as before. As in the proof of Proposition (4.1), we must choose $Q \in U(L(\mathfrak{h}))$ such that $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(Q) = t^r$.

Let us now consider again the $L(\mathfrak{g})$-module isomorphism $V^0(\boldsymbol{\lambda}, \mathbf{a}) \to V^i(\boldsymbol{\lambda}, \mathbf{a})$ given by multiplication by z^i. This map does not commute with the action of the derivation d; the action of d on $V^i(\boldsymbol{\lambda}, \mathbf{a})$ corresponds to the action

$$(d\Omega)(z) = i\Omega(z) + z\frac{d\Omega}{dz}$$

on $V^0(\boldsymbol{\lambda}, \mathbf{a})$. This suggests that we should exploit the extra degree of freedom available in defining the derivation action on $M(\boldsymbol{\lambda}, \mathbf{a})$ and $V(\boldsymbol{\lambda}, \mathbf{a})$. Namely, fix a complex number $b \in \mathbb{C}$ and define

$$(d\Omega)(z) = b\Omega(z) + z\frac{d\Omega}{dz}$$

for $\Omega \in M(\boldsymbol{\lambda}, \mathbf{a})$. Denote the resulting $\bar{L}(\mathfrak{g})$-module by $M(\boldsymbol{\lambda}, \mathbf{a}, b)$ and define $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ similarly. Thus, $M(\boldsymbol{\lambda}, \mathbf{a}, b) = M(\boldsymbol{\lambda}, \mathbf{a})$ as $L(\mathfrak{g})$-modules, and $M(\boldsymbol{\lambda}, \mathbf{a}, 0) = M(\boldsymbol{\lambda}, \mathbf{a})$ as $\bar{L}(\mathfrak{g})$-modules. Further, define $M^i(\boldsymbol{\lambda}, \mathbf{a}, b)$ and $V^i(\boldsymbol{\lambda}, \mathbf{a}, b)$ in the obvious way. Then we have

$$M^i(\boldsymbol{\lambda}, \mathbf{a}, b) \cong M^0(\boldsymbol{\lambda}, \mathbf{a}, b+i),$$

$$V^i(\boldsymbol{\lambda}, \mathbf{a}, b) \cong V^0(\boldsymbol{\lambda}, \mathbf{a}, b+i)$$

as $\bar{L}(\mathfrak{g})$-modules.

The methods used in this paper together with [1, Proposition 3.9] give the following result.

Theorem (4.9). (i) *$V^0(\boldsymbol{\lambda}, \mathbf{a}, b) \cong V(\lambda, I_{(\boldsymbol{\lambda}, \mathbf{a})})$ as $\bar{L}(\mathfrak{g})$-modules, where $\lambda \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*$ is defined by $\lambda = \sum_{i=1}^{k} \lambda_i + b\delta$.*

(ii) *$V(\boldsymbol{\lambda}, \mathbf{a}, b) \cong V(\boldsymbol{\lambda}', \mathbf{a}', b')$ as $\bar{L}(\mathfrak{g})$-modules if and only if*

(a) *$b' \equiv b(r)$, where $\operatorname{im} \chi_{(\boldsymbol{\lambda}, \mathbf{a})} = L_r$, and*

(b) *there exists $c \in \mathbb{C}^*$ and a permutation σ of $\{1, 2, \ldots, k\}$ such that $\boldsymbol{\lambda}' = \boldsymbol{\lambda}_\sigma$ and $\mathbf{a}' = c\mathbf{a}_\sigma$, where $c\mathbf{a}$ means $(ca_1, \ldots, ca_k)$.*

(iii) *Any integrable module $V(\lambda, I)$, with $\lambda \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*$, is isomorphic to $V^0(\boldsymbol{\lambda}, \mathbf{a}, b)$ for suitable $(\boldsymbol{\lambda}, \mathbf{a}, b)$.*

(iv) *If $\lambda_1, \ldots, \lambda_k$ are dominant integral, then $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ is unitarizable if and only if $a_1, \ldots, a_k$ all have the same modulus and b is real.*

(v) *$V(\boldsymbol{\lambda}, \mathbf{a}, b)$ lifts to a representation of $\bar{L}(G)$ if and only if $b \in \mathbb{Z}$ and $\lambda_1, \ldots, \lambda_k$ are dominant integral.*

5. Tensor Products and Direct Integrals

It is well-known that a tensor product of standard modules is a direct sum of standard modules (see [4]). In this section we prove that a tensor product of loop modules is never completely reducible, but can always be written as a direct integral of loop modules.

Consider then a tensor product of the form $V(\boldsymbol{\lambda}, \mathbf{a}, b) \otimes V(\boldsymbol{\lambda}', \mathbf{a}', b')$. For any $s, s' \in \mathbb{C}^*$ there is a homomorphism of $\bar{L}(\mathfrak{g})$-modules

$$p_{(s,s')}: V(\boldsymbol{\lambda}, \mathbf{a}, b) \otimes V(\boldsymbol{\lambda}', \mathbf{a}', b') \to V((\boldsymbol{\lambda}, \boldsymbol{\lambda}'), (s\mathbf{a}, s'\mathbf{a}'), b + b')$$

given by

$$p_{(s,s')}(\Omega \otimes \Omega')(t) = \Omega(st) \otimes \Omega'(s't),$$

where, if

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in (\mathfrak{h}^*)^k, \qquad \boldsymbol{\lambda}' = (\lambda_1', \ldots, \lambda_l') \in (\mathfrak{h}^*)^l,$$

we set

$$(\boldsymbol{\lambda}, \boldsymbol{\lambda}') = (\lambda_1, \ldots, \lambda_k, \lambda_1', \ldots, \lambda_l') \in (\mathfrak{h}^*)^{k+l},$$

and similarly for $(s\mathbf{a}, s'\mathbf{a}')$. It is clear from Proposition (4.1) that $p_{(s,s')}$ is always surjective. Further, by Theorem (4.9), to obtain essentially different projections, we might as well assume $s' = 1$.

By analogy with the direct integral of unitary representations of locally compact groups, we define the *algebraic direct integral*

$$\int^{\oplus} V((\boldsymbol{\lambda}, \boldsymbol{\lambda}'), (s\mathbf{a}, \mathbf{a}'), b + b')ds$$

of the $\tilde{L}(\mathfrak{g})$-modules $V((\boldsymbol{\lambda}, \boldsymbol{\lambda}'), (s\mathbf{a}, \mathbf{a}'), b + b')$ to be the space of algebraic maps

$$\omega : \mathbb{C}^* \to L\left(\bigotimes_{i=1}^{k} V(\lambda_i) \otimes \bigotimes_{j=1}^{l} V(\lambda_j') \right),$$

the $\tilde{L}(\mathfrak{g})$-module structure being given by

$$(x \cdot \omega)(s) = x \cdot \omega(s), \qquad x \in \tilde{L}(\mathfrak{g}),$$

the action on the right being that in $V((\boldsymbol{\lambda}, \boldsymbol{\lambda}'), (s\mathbf{a}, \mathbf{a}'), b + b')$. [We do not assume that $V(\boldsymbol{\lambda}, a, b)$ or $V(\boldsymbol{\lambda}', a', b')$ is unitarizable.] Explicitly, we may regard ω as a map

$$\omega : \mathbb{C}^* \times \mathbb{C}^* \to \bigotimes_{i=1}^{k} V(\lambda_i) \otimes \bigotimes_{j=1}^{l} V(\lambda_j')$$

and then the action is given by

$$(f \cdot \omega)(s,t) = \left(\sum_{i=1}^{k} 1 \otimes \ldots \overset{\longleftarrow i \longrightarrow}{\otimes f(sa_i t)} \otimes \ldots \otimes 1 + \sum_{j=1}^{l} 1 \otimes \ldots \overset{\longleftarrow l-j+1 \longrightarrow}{\otimes f(a_j' t) \otimes \ldots \otimes 1} \right) \omega(s,t),$$

$$(d \cdot \omega)(s,t) = (b + b')\omega(s,t) + t \frac{\partial \omega}{\partial t}(s,t),$$

for $f \in L(\mathfrak{g})$.

Theorem (5.1). *There is an isomorphism of* $\tilde{L}(\mathfrak{g})$*-modules*

$$V(\boldsymbol{\lambda}, \mathbf{a}, b) \otimes V(\boldsymbol{\lambda}', \mathbf{a}', b') \to \int^{\oplus} V((\boldsymbol{\lambda}, \boldsymbol{\lambda}'), (s\mathbf{a}, \mathbf{a}'), b + b')ds.$$

Proof. Assign to any element $\Omega \otimes \Omega' \in V(\boldsymbol{\lambda}, \mathbf{a}, b) \otimes V(\boldsymbol{\lambda}', \mathbf{a}', b')$ the map ω given by $\omega(s,t) = \Omega(st) \otimes \Omega'(t)$. Checking that this defines an isomorphism of $\tilde{L}(\mathfrak{g})$-modules presents no difficulty.

Because the tensor product has uncountably many quotients, we have

Corollary (5.2). *A tensor product of loop modules is never completely reducible.*

Remark. In the twisted case, the restriction of $p_{(s,s')}$ to $V^0(\boldsymbol{\lambda}, \mathbf{a}, b) \otimes V^0(\boldsymbol{\lambda}', \mathbf{a}', b')$ will usually be surjective, since $V((\boldsymbol{\lambda}, \boldsymbol{\lambda}'), (s\mathbf{a}, s'\mathbf{a}'), b + b')$ will usually be irreducible even if $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ and $V(\boldsymbol{\lambda}', \mathbf{a}', b')$ are not. All one can say is that a tensor product of twisted loop modules is a submodule of a direct integral of loop modules.

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Syzygies of Canonical Curves and Special Linear Series

Frank-Olaf Schreyer

Fachbereich Mathematik der Universität, Erwin-Schrödinger-Strasse, D-6750 Kaiserslautern, Federal Republic of Germany

0. Introduction

(0.1) In this paper we study the syzygies of canonical curves C of genus $g \leqq 8$. The main result (0.5) relates the ranks of the syzygy-modules to the existence of special linear series on C. These findings confirm and were partially motivated by a conjecture of Green, see (0.7). Before we formulate the main result we recall some algebraic and geometric notation and some classical results.

(0.2) Let C be a smooth algebraic curve of genus $g \geqq 3$ defined over an algebraically closed field $\mathfrak{k}$. ω_C denoted the canonical sheaf on C and

$$j : C \to \mathbb{P}(H^0(C, \omega_C)) = \mathbb{P}^{g-1}$$

the canonical map.

j is an embedding unless C is hyperelliptic. Furthermore

Theorem (Noether [N]). *If C is not hyperelliptic then*

$$\Omega = \sum_{n \geqq 0} H^0(C, \omega_C^{\otimes n})$$

is the homogeneous coordinate ring of $C \subset \mathbb{P}^{g-1}$.

If C is hyperelliptic then $j : C \to \mathbb{P}^{g-1}$ is a $2:1$ map onto a rational normal curve

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "2:1"'] & \mathbb{P}^{g-1} \\ \mathbb{P}^1 \arrow[ur, hook] & \end{tikzcd}$$

and $\Omega = \sum_{n \geq 0} H^0(C, \omega_C^{\otimes n})$ regarded as a module over the homogenous coordinate ring $S = \operatorname{Sym} H^0(C, \omega_C)$ of $\mathbb{P}^{g-1}$ is the module of global sections of the rank 2 vectorbundle $j_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-g-1)$ on the rational normal curve $\mathbb{P}^1 \subset \mathbb{P}^{g-1}$.

(0.3) Our basic object of studies are the syzygies of

$$\Omega = \sum_{n \geq 0} H^0(C, \omega_C^{\otimes n})$$

regarded as a graded module over the homogeneous coordinate ring

$$S = \operatorname{Sym} H^0(C, \omega_C) \cong \mathfrak{k}[x_0, \dots, x_{g-1}]$$

of $\mathbb{P}^{g-1}$. Let

$$F_* : 0 \to F_{g-2} \to \dots \to F_1 \to F_0 \to \Omega \to 0$$

be the *minimal free resolution* of Ω, i.e. an exact complex of graded free S-modules

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}$$

such that $\operatorname{Im}(F_i) \subset \mathfrak{m} F_{i-1}$, where $\mathfrak{m}$ denotes the homogenous maximal ideal of S and $S(-j)$ the free S-module with one generator in degree $+j$, i.e. $S(-j)_n = S_{n-j}$.

The minimality gives

$$\beta_{ij} = \dim \operatorname{Tor}_i^S(\Omega, \mathfrak{k})_j,$$

we call these integers the *graded betti-numbers* of Ω (resp. C). $\operatorname{Im}(F_i) \subset F_{i-1}$ is the i^{th} *syzygy-module* of Ω. By the formula of Auslander-Buchsbaum-Serre the length of the resolution is $g-2$, since Ω is a Cohen-Macaulay module. Frequently we will replace F_* by the corresponding sequence of sheaves of $\mathcal{O} = \mathcal{O}_{\mathbb{P}^{g-1}}$-modules:

$$0 \to \bigoplus_j \mathcal{O}(-j)^{\beta_{g-2j}} \to \dots \to \bigoplus_j \mathcal{O}(-j)^{\beta_{0j}} \to j_* \mathcal{O}_C \to 0.$$

Since F_* is just the sequence of global sections, we loose no information.

(0.4) With this notation Noether's result says: $\beta_{0j} = 0$ for $j > 0$ iff C is not hyperelliptic. A classical result of Petri is a statement about the β_{1j}'s:

Theorem (Petri [P]). *The homogenous ideal I_C of a non-hyperelliptic canonical curve C is generated by quadrics unless*

(a) *C is trigonal (i.e. $C \xrightarrow{3:1} \mathbb{P}^1$) or*

(b) *C is isomorphic to a smooth plane quintic, $g = 6$.*

In the exceptional cases the quadrics contained in I_C generated the homogenous ideal of a surface of minimal degree, which is

(a) *a rational normal scroll or*

(b) *the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.*

For a proof see also [S-D].

Thus $\beta_{1j} = 0$ for $j > 2$ unless C is one of the exceptional curves.

(0.5) These results suggest that the values of the β_{ij} are related to the existence of special linear series of divisors on C. The symbol g_d^r will denote an *r-dimensional linear series of divisors of degree d* on C.

For $g \leqq 6$ all possible values of the β_{ij} can be deduced from the results of Noether and Petri. Our main result is

Theorem. *For C a curve of genus $g=7$ or $g=8$ the values of the graded betti-numbers β_{ij} depend on and determine the existence of a*

$$g_2^1,\ g_3^1,\ g_6^2 \text{ or } g_4^1$$

on C.

The precise values of the β_{ij} in either case are given in the following table:

Table 1. The graded betti-numbers of Ω for curves up to genus 8 for char$(\mathfrak{k}) \neq 2$

genus	β_{01}	β_{12}	β_{23}			special linear series
	β_{00}	β_{11}	β_{22}			
g = 3		1				general case
	1					
		1				g_2^1
	1					
		1				
	1					
g = 4			1			general case
		1				
		1				
	1					
			1			g_2^1
	2	3				
		3	2			
	1					
g = 5				1		general case
			3			
		3				
	1					
				1		g_3^1
		2	3			
		3	2			
	1					
				1		g_2^1
	3	8	6			
		6	8	3		
	1					
g = 6					1	general case
			5	6		
		6	5			
	1					
					1	g_3^1 or g_5^2
		3	8	6		
		6	8	3		
	1					
					1	g_2^1
	4	15	20	10		
		10	20	15	4	
	1					

genus	β_{01}	β_{12}	β_{23}					special linear series
	β_{00}	β_{11}	β_{22}					
g = 7						1		general case *
				16	10			
		10	16					
	1							
						1		g_4^1
			3	16	10			
		10	16	3				
	1							
						1		g_6^2
			9	16	10			
		10	16	9				
	1							
						1		g_3^1
		4	15	20	10			
		10	20	15	4			
	1							
						1		g_2^1
	5	24	45	40	15			
		15	40	45	24	5		
	1							
g = 8							1	general case
				21	35	15		
		15	35	21				
	1							
							1	g_4^1
			4	25	35	15		
		15	35	25	4			
	1							
							1	g_6^2
			14	35	35	15		
		15	35	35	14			
	1							
							1	g_3^1
		5	24	45	40	15		
		15	40	45	24	5		
	1							
							1	g_2^1
	6	35	84	105	70	21		
		21	70	105	84	35	6	
	1							

* For char$(\mathfrak{k})=2$: As above, but the general case for $g=7$:

						1		
g =7			1	10	16			general case
		16	10	1				
	1							

(0.6) *Remarks.* 1) The symmetry of the values

$$\beta_{ij} = \beta_{g-2-i,\, g+1-j}$$

reflects the self duality of F_*:

$$\operatorname{Hom}(F_*, S(-g-1))[g-2] \cong F_*,$$

which follows from the adjunction formula

$$j_*\omega_C = \mathscr{E}xt_{\mathscr{O}}^{g-2}(j_*\mathscr{O}_C, \omega_{\mathbb{P}^{g-1}})$$

and

$$\omega_{\mathbb{P}^{g-1}} = \mathscr{O}(-g), \qquad \omega_C = j^*\mathscr{O}_C(1).$$

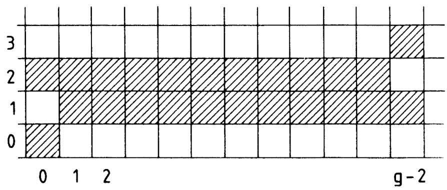
2) The graded betti-number satisfy the identities

$$\sum_i (-1)\beta_{ij} = \sum_i (-1)^i \binom{g-1}{i} h^0(\omega_C^{\otimes j-i})$$

by a result of Hilbert [Hi].

3) In general for a graded module little is known what kind of information is carried by its graded betti-numbers other than the Hilbert function and the depth.

(0.7) For curves of higher genus it is easy to see that the tuples $(i, j-i)$ such that $\beta_{ij} \neq 0$ lie in the range indicated below:



Green gives a conjecture on the precise range, which generalizes the result of Noether and Petri:

Conjecture (Green [G]). *For* $\operatorname{char}(\mathfrak{k}) = 0$ *the following is equivalent:*

(1) $\beta_{i,i+1} \neq 0$;

(2) *there exists a linear series* g_d^r *on* C *with* $r \geqq 1$, $d \leqq g-1$ *and Clifford-index* $d - 2r = g - 2 - i$.

A proof of (2) $\Rightarrow$ (1) is contained in [G]. Our result verifies this conjecture for $g \leqq 8$ over fields of $\operatorname{char}(\mathfrak{k}) \neq 2$. If $\operatorname{char}(\mathfrak{k}) = 2$ the corresponding statement is false for curves of genus 7.

(0.8) To prove the result we take the following approach to the syzygies of Ω. Start with a base point free complete pencil g_d^1 of divisors of degree d on C. The variety swept out by the linear spans of these divisors

$$X = \bigcup_{D \in g^1_d} \bar{D} \subset \mathbb{P}^{g-1}$$

is a rational normal scroll of dimension $d-1$ by a classical result of Bertini, cf. Sects. 1, 2, 4.

We resolve $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}$-module in two steps:

1) resolve $\mathcal{O}_C$ as an $\mathcal{O}_X$-module by direct sums of line bundles on X;

2) take the resolution of each of these line bundles as $\mathcal{O}_{\mathbb{P}^{g-1}}$-module and make a mapping cone construction to obtain a resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}$-module. (cf. Sects. 1, 3, 4). This approach to the syzygies of was suggested to me by E. Sernesi. For small d at least in as much as equations of C are concerned it can already be found in Petri's paper.

(0.9) For $d=3$, $d=4$ we obtain the minimal resolution of $\mathcal{O}_C$ in this way. We study these cases in Sect. 6 in some details. For example we obtain:

Theorem. *A morphism $C \to \mathbb{P}^1$ of degree 4 on a curve of genus g factors through a curve E of genus g' with $6g' < g+3$, i.e.*

$$\begin{tikzcd} C \arrow[rr, "4:1"] \arrow[dr, "2:1"'] & & \mathbb{P}^1 \\ & E \arrow[ur, "2:1"'] & \end{tikzcd}$$

iff

$$\beta_{i,i+1} = i\binom{g-3}{i+1} \quad \textit{for} \quad i > g-2-2g'$$

and

$$\beta_{i,i+1} > i\binom{g-3}{i+1} \quad \textit{for} \quad i = g-2-2g'.$$

(0.10) Unfortunately for $d \geqq 5$ the same approach gives only a non-minimal resolution of Ω. However in Sect. 7 a detailed study of these complexes for $d=5$ and $g=7$ or $g=8$ allows us to deduce the existence of a rational determinantal surface Y on X with

$$C \subset Y \subset X \subset \mathbb{P}^{g-1}$$

whose degree depends on the values of the β_{ij}. The classification of these surfaces (Sect. 5) allows us to finish the proof.

1. Line Bundles on Scrolls

(1.1) Let $\mathscr{E} = \mathcal{O}(e_1) \oplus \ldots \oplus \mathcal{O}(e_d)$ be a locally free sheaf of rank d on $\mathbb{P}^1$ and let

$$\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$$

denote the corresponding $\mathbb{P}^{d-1}$-bundle. A *rational normal scroll* X of type $S(e_1, \ldots, e_d)$ with $e_1 \geqq \ldots \geqq e_d \geqq 0$ and

$$f = e_1 + \ldots + e_d \geqq 2$$

is the image of $\mathbb{P}(\mathscr{E})$ in $\mathbb{P}^r = \mathbb{P}H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$:

$$j: \mathbb{P}(\mathscr{E}) \to X \subset \mathbb{P}^r, \qquad r = f + d - 1.$$

X is a non-degenerate irreducible variety of minimal degree

$$\deg X = f = r - d + 1 = \operatorname{codim} X + 1$$

in $\mathbb{P}^r$ [Ha].

If all $e_i > 0$ then X is smooth and $j: \mathbb{P}(\mathscr{E}) \to X$ an isomorphism. If some of the $e_i = 0$, then X is singular and $j: \mathbb{P}(\mathscr{E}) \to X$ is a resolution of singularities. The singularities of X are rational, i.e.

$$j_* \mathscr{O}_{\mathbb{P}(\mathscr{E})} = \mathscr{O}_X, \qquad R^i j_* \mathscr{O}_{\mathbb{P}(\mathscr{E})} = 0 \quad \text{for} \quad i > 0$$

[Ke]. Consequently we may replace X by $\mathbb{P}(\mathscr{E})$ for most cohomological considerations, even if X is singular.

(1.2) The *Picard group* of $\mathbb{P}(\mathscr{E})$ is generated by the *hyperplane class* $H = [j^* \mathscr{O}_{\mathbb{P}^r}(1)]$ and the *ruling* $R = [\pi^* \mathscr{O}_{\mathbb{P}^1}(1)]$:

$$\operatorname{Pic} \mathbb{P}(\mathscr{E}) = \mathbb{Z}H \oplus \mathbb{Z}R,$$

the intersection product is given by

$$H^d = f, \qquad H^{d-1} \cdot R = 1, \qquad R^2 = 0$$

[Ha].

In this section we recall from [E2] the description of the *syzygies* of the sheaves

$$\mathscr{O}_X(aH + bR) := j_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR), \qquad a, b \in \mathbb{Z}$$

regarded as $\mathscr{O}_{\mathbb{P}^r}$-modules, at least in case $b \geqq -1$.

(1.3) The *cohomology* of a line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)$ can explicitly be calculated with the Leray spectral sequence:

$$H^i(\mathbb{P}^1, R^j \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)) \;\Rightarrow\; H^{i+j}(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)).$$

For example, if $a \geqq 0$ and $S_a \mathscr{E}$ denotes the a^{th}-symmetric power of $\mathscr{E}$ we have

$$H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)) \cong H^0(\mathbb{P}^1, (S_a \mathscr{E})(b)).$$

More precisely: Let $\mathfrak{k}[s, t]$ denote the homogenous coordinate ring of $\mathbb{P}^1$ and let

$$\varphi_i \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R)), \qquad i = 1, \ldots, d$$

denote the *basic section* obtained from the inclusion of the i^{th}-summand

$$\mathscr{O}_{\mathbb{P}^1} \to \mathscr{E}(-e_i) \cong \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R).$$

Then we can identify sections

$$\psi \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR))$$

with homogenous polynomials

$$\psi = \sum_{\alpha} P_\alpha(s, t) \varphi_1^{\alpha_1} \cdot \ldots \cdot \varphi_d^{\alpha_d}$$

of degree $a=\alpha_1+\ldots+\alpha_d$ in the φ_i's and coefficients homogenous polynomials $P_\alpha \in \mathfrak{k}[s,t]$ of degree

$$\deg P_\alpha = \alpha_1 e_1 + \ldots + \alpha_d e_d + b.$$

In particular we notice that for $b \geqq -1$ the dimension

$$h^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(aH+bR)) = f\binom{a+d-1}{d} + (b+1)\binom{a+d-1}{d-1}$$

does not depend on the type $S(e_1, \ldots, e_d)$ of the scroll (resp. $\mathscr{E}$), but only on its degree $f=e_1+\ldots+e_d$.

(1.4) The defining equations of X are *determinantal*: Choose the basis

$$x_{ij} = t^j s^{e_i - j} \varphi_i$$

for $i=1, \ldots, d$, $j=0, \ldots, e_i$ of $H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H) \cong H^0\mathcal{O}_{\mathbb{P}^r}(1)$ and consider the matrix

$$\Phi = \begin{pmatrix} x_{10} \cdots x_{1e_1-1} & x_{20} \cdots & \cdots x_{de_d-1} \\ x_{11} \cdots x_{1e_1} & x_{21} \cdots & \cdots x_{de_d} \end{pmatrix}.$$

By definition the 2×2 minors of Φ vanish identically on X. We will see in (1.6) that they actually generate the homogenous ideal of X.

In more intrinsic terms we can obtain Φ from the multiplication map

$$H^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(R)) \otimes H^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-R)) \to H^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H)).$$

(1.5) Let

$$\Phi: F \to G$$

be a map of locally free sheaves of rank f and g, $f \geqq g$, respectively on a smooth variety V. We recall from [B-E 2] the family of complexes $\mathscr{C}^b$, $b \geqq -1$, of locally free sheaves on V, which resolve the b^{th}-symmetric power of coker Φ under suitable hypothesis on Φ.

Define the j^{th}-term in the complex $\mathscr{C}^b$ by

$$\mathscr{C}_j^b = \begin{cases} \bigwedge^j F \otimes S_{b-j} G, & \text{for } 0 \leqq j \leqq b \\ \bigwedge^{j+g-1} F \otimes D_{j-b-1} G^* \otimes \bigwedge^g G^*, & \text{for } j \geqq b+1 \end{cases}$$

and differential

$$\mathscr{C}_j^b \to \mathscr{C}_{j-1}^b$$

by the multiplication with $\Phi \in H^0(V, F^* \otimes G)$ for $j \neq b+1$ and $\bigwedge^g \Phi \in H^0\left(V, \bigwedge^g F^* \otimes \bigwedge^g G\right)$ for $j=b+1$ in the appropriate term of the exterior $(\wedge F)$, symmetric (S.G) or divided power (D.G) algebra.

E.g.: For an open set $U \subset V$ the differential of a term

$$f_1 \wedge \ldots \wedge f_j \otimes g \in H^0(U, \mathscr{C}_j^b); \quad j \leqq b,$$

with $f_i \in H^0(U, F)$, $g \in H^0(U, S_{b-j}G)$ is given by

$$\sum_{i=1}^{j} (-1)^i f_1 \wedge \ldots \hat{f}_i \ldots \wedge_j \otimes \Phi(f_i) \cdot g \in H^0(U, \mathscr{C}_{j-1}^b).$$

$\mathscr{C}^0$ is the well-known *Eagon-Northcott* complex [E-N] associated to Φ. The image of $\mathscr{C}_1^0 \to \mathscr{C}_0^0 \cong \mathscr{O}_V$ is the idealsheaf $I_g(\Phi)$ generated by the $g \times g$ minors of Φ. $\mathscr{C}^1$ is the *Buchsbaum-Rim* complex. The cokern M of $\mathscr{C}_1^1 \to \mathscr{C}_0^1$ is supported on $V(I_g(\Phi))$. For $b > 1$ the cokern $\mathscr{C}_1^b \to \mathscr{C}_0^b$ is (by definition) the b^{th}-symmetric power of M.

Theorem (Buchsbaum-Eisenbud [B-E2]). *The complexes $\mathscr{C}^b$ for $b \geqq -1$ is acyclic if the ideals $I_{g-k}(\Phi)$ of $(g-k)\times(g-k)$ minors of Φ have*

$$\operatorname{depth} I_{g-k}(\Phi) \geqq f - g + k + 1$$

for $k=0$ and $1 \leqq k \leqq b - f + g - 1$. □

(1.6) We regard the matrix Φ from (1.4) as a map of bundles

$$\Phi : F(-1) \to G$$

on $\mathbb{P}^r$ with $F = \mathscr{O}_{\mathbb{P}^r}^f$, $G = \mathscr{O}_{\mathbb{P}^r}^2$. Since the condition (1.5) on the depth of the ideal of the minors on Φ is satisfied, all of the complexes $\mathscr{C}^b$ and their twist $\mathscr{C}^b(a) = \mathscr{C}^b \otimes \mathscr{O}_{\mathbb{P}^r}(a)$, are acyclic. Hence

Corollary (Eisenbud [E2]). *$\mathscr{C}^b(a)$ for $b \geqq -1$ is the minimal resolution of $\mathscr{O}_X(aH + bR)$ as an $\mathscr{O}_{\mathbb{P}^r}$-module.*

Proof. It remains to identify

$$\operatorname{coker}(\mathscr{C}_1^b \to \mathscr{C}_0^b) \cong \mathscr{O}_X(bR).$$

For $b=0$, we obtain from the exactness of $\mathscr{C}^0$ that the variety X' defined by the 2×2-minors of Φ has dimension d and degree f. Furthermore, since the length of the complex $\mathscr{C}^0$ is $f-1 = \operatorname{codim} X'$, X' is arithmetically Cohen-Macaulay. Since $X' \supset X$ and both varieties have the same degree and dimension, they coincide. In particular we obtain that the 2×2-minors of Φ generate the homogenous ideal of X.

For $b=1$ we identify

$$\operatorname{coker}(\mathscr{C}_1^1 \to \mathscr{C}_0^1) = \operatorname{coker} \Phi \cong \mathscr{O}_X(R)$$

via global sections: Choose an orientation

$$\bigwedge^2 (H^0 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \cong \mathfrak{k}$$

with the induced isomorphism

$$H^0 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R) \cong (H^0 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))^*.$$

With this identification the diagram

$$\begin{array}{ccccccc}
H^0(\mathbb{P}^r, F(a-1)) & \longrightarrow & H^0(\mathbb{P}^r, G(a)) & \longrightarrow & H^0(\mathbb{P}^r, (\operatorname{coker}\Phi)(a)) & \to & 0 \\
\downarrow \wr & \square & \downarrow \wr & & \exists \downarrow \wr & & \\
H^0\mathscr{O}_{\mathbb{P}^r}(a-1) \otimes H^0 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R) & \to & H^0\mathscr{O}_{\mathbb{P}^r}(a) \otimes (H^0\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))^* & \longrightarrow & H^0\mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH+R) & \longrightarrow & 0
\end{array}$$

commutes, i.e. gives the desired isomorphism.

For $b>1$ the result follows from

$$\operatorname{coker}(\mathscr{C}_1^b \to \mathscr{C}_0^b) = S_b(\operatorname{coker}\Phi) \cong \mathscr{O}_X(bR).$$

Finally for $b=-1$ we use the duality

$$\mathscr{H}om(\mathscr{C}^{f-2-i}, \mathscr{O}_{\mathbb{P}^r}) \cong \mathscr{C}^i(f)$$

for $i=-1,\ldots,f-1$ of the complexes. From the case $i=f-2$ we obtain the dualizing sheaf of X:

$$\begin{aligned}\omega_X = H_0(\mathscr{H}om(\mathscr{C}^0, \omega_{\mathbb{P}^r})) &\cong H_0(\mathscr{C}^{f-2}(-d))\\ &\cong \mathscr{O}_X(-dH+(f-2)R).\end{aligned}$$

Consequently

$$\begin{aligned}H_0(\mathscr{C}^{-1}) &\cong H_0(\mathscr{H}om(\mathscr{C}^{f-1}, \omega_{\mathbb{P}^r}(d))\\ &\cong \mathscr{H}om(\mathscr{O}_X(f-1)R, \omega_X(dH))\end{aligned}$$

by duality,

$$\begin{aligned}&\cong \mathscr{H}om(\mathscr{O}_X(f-1)R, \mathscr{O}_X(f-2)R)\\ &\cong \mathscr{O}_X(-R). \quad \square\end{aligned}$$

(1.7) *Remarks.* a) We calculated

$$\omega_X \cong \mathscr{O}_X(-dH)+(f-2)R)$$

during the proof of (1.6).

b) Notice that the complexes $\mathscr{C}^b$, $b\geqq -1$, for our matrix are 1-*regular*, i.e. all but one differential is given in terms of a basis by a matrix with *linear* entries, and the exceptional differential $\mathscr{C}_{b+1}^b \to \mathscr{C}_b^b$ is given by a matrix with *quadratic* entries.

c) For the line bundles $\mathscr{O}_X(aH+bR)$ with $b<-1$ a similarly simple description of the syzygies is not possible. One reason is that the dimensions $h^0\mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH+bR)$ for $b<-1$ depend on the type $S(e_1,\ldots,e_d)$ of the scroll.

2. Scrolls and Pencils

(2.1) In this section we survey for a smooth variety V and linearly normal map

$$j: V \to \mathbb{P}^r = \mathbb{P}(H^0(V, \mathscr{O}_V(H))$$

the rational normal scrolls $X\subset\mathbb{P}^r$ which contain the image $j(V)$, cf. [Ha].

Let $X\subset\mathbb{P}^r$ be a scroll of degree f containing $j(V)$. The ruling R on X cuts out on V a pencil of divisors (possibly with basepoints)

$$\{D_\lambda\}_{\lambda\in\mathbb{P}^1} \subset |D|$$

with $h^0(V, \mathscr{O}_V(H-D))=f$.

(2.2) Conversely we can construct from any pencil of divisors $\{D_\lambda\}_{\lambda\in\mathbb{P}^1}$ on V which satisfies $h^0(V, \mathscr{O}_V(H-D))=f\geqq 2$ a scroll of degree f as follows: Let $G\subset H^0(V, \mathscr{O}_V(D))$ be the 2-dimensional subspace which defines the pencil. The multiplication map

$$G \otimes H^0(V, \mathcal{O}_V(H-D)) \to H^0(V, \mathcal{O}_V(H))$$

yields a $2 \times f$ matrix Φ with linear entries whose 2×2 minors vanish on $j(V)$. The variety X defined by these minors contains $j(V)$ and is a scroll of degree f [cf. (1.4)].

(2.3) Geometrically we may construct X as follows: Let

$$\overline{D_\lambda} := \bigcap_{\substack{H \text{ with} \\ j^*H \geqq D_\lambda}} H$$

be the linear span of $j(D_\lambda)$ in $\mathbb{P}^r$, i.e. the linear space defined by the linear forms of

$$H^0(V, \mathcal{O}_V(H-D_\lambda)) \to H^0(V, \mathcal{O}_V(H)) = H^0(\mathbb{P}^r, \mathcal{O}(1)).$$

Then X is the variety swept out by these linear spaces:

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \overline{D_\lambda} \subset \mathbb{P}^r.$$

[Proof: The 2×2-minors of the matrix Φ vanish in a point $p \in \mathbb{P}^r$ iff the rank $\Phi(p) \leqq 1$, i.e. iff the two rows of $\Phi(p)$ are linearly dependent. Say the first row vanishes identically. Then by definition $p \in \overline{D_{(1,0)}}$.]

(2.4) The type $S(e_1, \ldots, e_d)$ of the scroll can be determined as follows. Decompose the pencil

$$D_\lambda = F + E_\lambda, \quad \lambda \in \mathbb{P}^1$$

into its fixed and moving part and consider the following partition of $r+1$:

$$\begin{aligned} d_0 &:= h^0\mathcal{O}_V(H) - h^0\mathcal{O}_V(H-D), \\ d_1 &:= h^0\mathcal{O}_V(H-D) - h^0\mathcal{O}_V(H-F-2E), \\ &\;\vdots \\ d_i &:= h^0\mathcal{O}_V(H-F-iE) - h^0\mathcal{O}_V(H-F-(i+1)E), \\ &\;\vdots \end{aligned}$$

We use the dual partition to define the number e_i:

$$e_i + 1 = \# \{j \mid d_j \geqq i\}.$$

(2.5) **Theorem** (Harris, Bertini). *With the notation as above* (2.2–2.4) *X is a d_0-dimensional rational normal scroll of type $S(e_1, \ldots, e_{d_0})$.*

Proof. We are looking for a basis

$$x_{ij}, \quad i = 1, \ldots, d_0, \quad j = 0, \ldots, e_i,$$

of $H^0\mathcal{O}_{\mathbb{P}^r}(1) \cong H^0\mathcal{O}_V(H)$ as in (1.4). Let G now denote the 2-dimensional subspace of $H^0(V, \mathcal{O}_V(E))$ corresponding to the pencil $\{E_\lambda\}_{\lambda \in \mathbb{P}^1}$ without fixed components. We consider the exact sequences (cf. pencil trick [S-D]):

$$\begin{aligned} 0 \to \bigwedge^2 G \otimes \mathcal{O}_V(H-F+(i-2)E) &\to G \otimes \mathcal{O}_V(H-F+(i-1)E) \\ &\to \mathcal{O}_V(H-f+iE) \end{aligned}$$

and take global sections. The resulting sequence

$$0 \to \bigwedge^2 G \otimes M \to G \otimes M \to M$$

with

$$M = \sum_{i \in \mathbb{Z}} H^0 \mathcal{O}_V(H - F + iE)$$

is exact, where we regard M as a graded module over the polynomial ring $S_* G$. So

$$\mathrm{Tor}_j^{S_* G}(M, \mathfrak{k}) = 0 \quad \text{for} \quad j > 0$$

and

$$M \cong S_* G \bigotimes_{\mathfrak{k}} M/GM$$

as a graded $S_* G$-module. In particular

$$M_{-1} = \bigoplus_{i: e_i \geqq 1} S_{e_i - 1} G$$

and the composition

$$G \otimes M_{-1} = G \otimes H^0 \mathcal{O}_V(H - D) \to M_0 = H^0 \mathcal{O}_V(H - F) \subset H^0 \mathcal{O}_V(H)$$

gives a $2 \times h^0 \mathcal{O}_V(H - D)$ matrix of the desired type. □

3. Syzygies of a Subvariety of a Scroll

(3.1) Let $V \subset \mathbb{P}(\mathcal{E})$ be a subvariety of a $\mathbb{P}^{d-1}$-bundle $\mathbb{P}(\mathcal{E})$ over $\mathbb{P}^1$. In the first part of this section we want to construct a resolution F_* of $\mathcal{O}_V$ by locally free $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-modules F_i, which restricts to the *minimal* resolution of $\mathcal{O}_{V_\lambda}$ on each fibre $\mathbb{P}(\mathcal{E})_\lambda \cong \mathbb{P}^{d-1}$, $\lambda \in \mathbb{P}^1$.

Necessary and sufficient for the existence of such a complex F_* is that V has *constant betti-numbers* $\beta_{ij} = \beta_{ij}(\lambda)$ *over* $\mathbb{P}^1$, i.e. the betti-numbers $\beta_{ij}(\lambda)$ of V_λ do not depend on $\lambda \in \mathbb{P}^1$. This is a somewhat stronger condition than flatness of V over $\mathbb{P}^1$.

In the second part of this section we consider the image $V' \subset X \subset \mathbb{P}^r$ of V in a rational normal scroll X corresponding to $\mathbb{P}(\mathcal{E})$ and construct a resolution of $\mathcal{O}_{V'}$ as an $\mathcal{O}_{\mathbb{P}^r}$-module using the complex F_*, the complexes $\mathcal{C}^b(a)$ (1.6) and a mapping cone construction provide F_* satisfies a suitable hypothesis.

(3.2) **Theorem.** *Let $V \subset \mathbb{P}(\mathcal{E})$ be a subvariety with constant betti-numbers $\beta_{ij} = \beta_{ij}(\lambda)$ over $\mathbb{P}^1$. $\mathcal{O}_V$ has a resolution*

$$0 \to F_C \to F_{C-1} \to \dots \to F_1 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_V \to 0$$

by vectorbundles F_i which admit a filtration

$$\dots \subset F_{ij-1} \subset F_{ij} \subset F_{ij+1} \subset \dots \subset F_i$$

such that the quotients

$$F_{ij}/F_{ij-1}=\sum_{k=1}^{\beta_{ij}}\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-jH+b_{ij}^{(k)}R)$$

split as above. If the fibres V_λ have pure resolutions (i.e. for each i $\beta_{ij}(\lambda)\neq 0$ for at most one j) then F_* is unique up to isomorphism.

(3.3) *Remark.* The integers $b_{ij}^{(k)}$ satisfy a system of linear equations, which can be obtained from the identity

$$\chi\mathcal{O}_V(\nu H)=\sum_i(-1)^i\chi(F_i(\nu H))$$

of the Hilbert polynomials.

(3.4) *Proof.* Let $F_*^{k(t)}$ be the minimal resolution of the homogeneous coordinate ring of the generic fibre V_ξ over $\mathfrak{k}(t)[\varphi_1,\dots,\varphi_d]$. Clearing denominators we obtain a complex $F_*^{\mathfrak{k}[t]}$ defined over $R=\mathfrak{k}[t][\varphi_1,\dots,\varphi_d]$ whose homology is finite dimensional in each degree:

$$\dim_{\mathfrak{k}}H_*(F_*^{\mathfrak{k}[t]})_j<\infty\,.$$

We will modify the differential of $F_*^{\mathfrak{k}[t]}$ to obtain exactness. Suppose $H_*(F_*^{\mathfrak{k}[t]})_j\neq 0$, while the homology vanishes in all degrees $<j$, say $H_i(F_*^{\mathfrak{k}[t]})_j$ has support at a point $\lambda\in\operatorname{Spec}\mathfrak{k}[t]$. Then there exists a free summand

$$F_{i+1}^{\mathfrak{k}[t]}=\tilde{F}_{i+1}^{\mathfrak{k}[t]}\oplus R(-j)$$

whose image under the differential

$$d_{i+1}(R(-j))\subset\lambda\cdot F_i^{\mathfrak{k}[t]}.$$

(Indeed, if no summand of this type exists, than since V_ξ and V_λ have the same betti-numbers and $F_*^{\mathfrak{k}[t]}$ is exact in all degrees $<j$ the $\mathfrak{k}$-vector spaces

$$\operatorname{Im}(d_{i+1})_j\otimes\mathfrak{k}[t]/\lambda\subset\ker(d_i)_j\otimes\mathfrak{k}[t]/\lambda$$

have the same dimension, hence are equal. But this implies the exactness of $F_*^{\mathfrak{k}[t]}$ in degree ij at λ.)

Let $(t-p)\in\lambda$ be a local parameter. The differential d_{i+1} factors as follows

$$\begin{tikzcd}
F_{i+2}^{\mathfrak{k}[t]} \arrow[r, "d_{i+2}"] \arrow[dr, "d'_{i+2}"'] & \tilde{F}_{i+1}^{\mathfrak{k}[t]}\oplus R(-j) \arrow[r, "d_{i+1}"] \arrow[d, "\mathrm{id}\oplus(t-p)\mathrm{id}"] & F_i^{\mathfrak{k}[t]} \arrow[r, "d_i"] & F_{i-1}^{\mathfrak{k}[t]} \\
& \tilde{F}_{i+1}^{\mathfrak{k}[t]}\oplus R(-j) \arrow[ur, "d'_{i+1}"'] & &
\end{tikzcd}$$

By the resulting complex $(F_*^{\mathfrak{k}[t]},d')$ satisfies:

$$\dim_k H_*(F_*^{\mathfrak{k}[t]},d')_j=\dim H_*(F_*^{\mathfrak{k}[t]},d)_j-1\,.$$

Using this construction repeatedly we obtain exactness in degree j and by induction on j exactness in all degrees after finitely many steps.

Similarly we can construct an exact complex $F_*^{\mathfrak{k}[t^{-1}]}$ defined over $\mathfrak{k}[t^{-1}][\psi_1,\dots,\psi_d]$. After identifying

$$\mathfrak{k}[t,t^{-1}][\varphi_1,\dots,\varphi_d]\cong\mathfrak{k}[t,t^{-1}][\psi_1,\dots,\psi_d]$$

via the transition functions

$$\psi_i = t^{e_i} \varphi_i$$

of the bundle $\mathbb{P}(\mathscr{E})$ we obtain comparison maps

$$F^{\mathfrak{k}[t]} \otimes \mathfrak{k}[t, t^{-1}] \to F^{\mathfrak{k}[t^{-1}]} \otimes \mathfrak{k}[t, t^{-1}]$$

over $\operatorname{Spec} \mathfrak{k}[t, t^{-1}]$, that is the sequence

$$M_i = (M_i^{jl})$$

of invertible block-matrices with M_i^{jl} a $\beta_{ij} \times \beta_{il}$-matrix with entries in $\mathfrak{k}[t, t^{-1}][\varphi_1, \ldots, \varphi_d]$ of degree $l-j$ in the φ_i's. In particular

$$(M_i^{jl}) = 0 \quad \text{for} \quad l < j$$

and

$$M_{il}^{ij} \in GL(\beta_{ij}, \mathfrak{k}[t^{-1}, t]).$$

Since bundles on $\mathbb{P}^1$ split, we may assume that M_i^{jj} are diagonal matrices. Thus the sheafications $F_*^{\mathfrak{k}[t]}$ and $F_*^{\mathfrak{k}[t^{-1}]}$ glue together to give a resolution of $\mathcal{O}_V$ of the desired type.

For the uniqueness we notice that given two complexes F_* and $\tilde{F}_*$ as in (3.2), we always have comparison maps locally over $\mathbb{P}^1$, i.e. for any affine $U \subset \mathbb{P}^1$ there exist comparison maps $F_*^U \to \tilde{F}_*^U$ of graded $\mathcal{O}_{\mathbb{P}^1}(U)[\varphi_1, \ldots, \varphi_d]$-modules. These are unique up to a homotopy. By our additional assumption every homotopy is zero for degree reasons. Hence these comparison maps glue to give a uniquely determined isomorphism $F_* \to \tilde{F}_*$. $\square$

(3.5) Suppose $V \subset \mathbb{P}(\mathscr{E})$ is a subvariety as in (3.2) and assume furthermore that all invariants $b_{ij}^{(k)} \geqq -1$. We consider now the image V' of V in a rational normal scroll X:

$$j : \mathbb{P}(\mathscr{E}) \to X \subset \mathbb{P}^r.$$

Starting from the resolution F_* of V in $\mathbb{P}(\mathscr{E})$ we can construct a resolution of $\mathcal{O}_{V'}$ as an $\mathcal{O}_{\mathbb{P}^r}$-module using the complexes $\mathscr{C}^b(a)$. First notice that the induced complex

$$0 \to j_* F_C \to \ldots \to j_* F_1 \to \mathcal{O}_X \to j_* \mathcal{O}_V \to 0$$

is still exact, in particular $j_* \mathcal{O}_V = \mathcal{O}_{V'}$. This is clear, if $j : \mathbb{P}(\mathscr{E}) \to X \subset \mathbb{P}^r$ is an embedding. Otherwise it follows from the fact that $j : \mathbb{P}(\mathscr{E}) \to X$ is a rational resolution of singularities, i.e.

$$R^i j_* \mathcal{O}_{\mathbb{P}(\mathscr{E})} = 0 \quad \text{for} \quad i > 0,$$

which implies the vanishing

$$R^i j_* \mathcal{O}_{\mathbb{P}(\mathscr{E})}(aH + bR) = 0 \quad \text{for} \quad i > 0$$

for all $a \in \mathbb{Z}$ and $b \geqq -1$.

Since each of the terms $j_* F_l$ is an extension of sheaves $\mathcal{O}_X(aH + bR)$, $b \geqq -1$, a suitable mapping cone, between the Eagon-Northcott type complexes $\mathscr{C}^b(a)$'s (1.6) gives a resolution

$$\mathscr{A}_*^{(l)} : \ldots \to \mathscr{A}_1^{(l)} \to \mathscr{A}_0^{(l)} \to j_* F_l \to 0.$$

Then finally an iterated mapping cone

$$[[\dots[\mathscr{A}^c_* \to \mathscr{A}^{c-1}_*] \to \dots] \to \mathscr{A}^{(1)}_*] \to \mathscr{C}^0 \to \mathcal{O}_{V'} \to 0$$

is a resolution of $\mathcal{O}_{V'}$ as an $\mathcal{O}_{\mathbb{P}^r}$-module.

Caution. In general the resulting complex is *not* minimal.

(3.6) *Example.* Let $C \subset X \subset \mathbb{P}^r$ be a "complete intersection" of divisors

$$Y_i \sim a_i H - b_i R, \qquad i = 1, \dots, d-1$$

on a d-dimensional rational normal scroll X of degree f with $b_i \geqq 0$. The resolution of $\mathcal{O}_C$ as an $\mathcal{O}_X$-module is a Koszul-complex:

$$0 \to \mathcal{O}_X(-(a_1 + \dots + a_{d-1})H + (b_1 + \dots + b_{d-1})R) \to \dots$$
$$\dots \to \sum_{i_1 < i_2} \mathcal{O}_X(-(a_{i_1} + a_{i_2})H + (b_{i_1} + b_{i_2})R) \to \sum_i \mathcal{O}_X(-a_i H + b_i R) \to \mathcal{O}_X .$$

If all $a_i \geqq 2$, i.e. $C \subset \mathbb{P}^r$ is non-degenerate, then, because the complexes $\mathscr{C}^b$ are 1-regular, the resulting mapping cone is the *minimal* resolution of $\mathcal{O}_C$. C has

$$\deg C = a_i \cdot \ldots \cdot a_{d-1} f - \sum_i a_i \cdot \ldots \cdot a_{i-1} b_i \cdot a_{i+1} \cdot \ldots \cdot a_{d-1}$$

and arithmetic genus $p_a C$

$$2p_a C - 2 = \deg C \cdot (a_1 + \dots + a_{d-1} - d) + (a_1 \cdot \ldots \cdot a_{d-1})(b_1 + \dots + b_{d-2} - f + 2) .$$

C is arithmetically Cohen-Macaulay iff

$$b_1 + \dots + b_{d-1} \leqq f - 1 .$$

4. Canonical Curves

In this section we construct for a canonical curve $C \subset \mathbb{P}^{g-1}$ with a basepoint free complete pencil of divisors of degree d an approximation of the minimal resolution with methods of Sects. 2, 3.

(4.1) Let

$$C \subset \mathbb{P}^{g-1}$$

be a canonical curve,

$$g^1_d = \{D_\lambda\}_{\lambda \in \mathbb{P}^1}$$

a basepoint free complete pencil of divisors of degree d, $d \leqq g-1$, on C. By the *geometric version* of the theorem of *Riemann-Roch*:

$$\dim \bar{D} = \deg D - 1 - \dim |D|$$

for D an effective divisor, $\bar{D}$ its linear span in $\mathbb{P}^{g-1}$ and $|D|$ the complete linear series (cf. [G-H, p. 248]), we have

$$\dim \overline{D_\lambda} = d - 2 .$$

So by Sect. 2

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \overline{D_\lambda} \subset \mathbb{P}^{g-1}$$

is a $(d-1)$-dimensional rational normal scroll of degree $f=g-d+1$, whose type depends on and determines the dimensions $h^0(C, \mathcal{O}_C(iD))$ for $i \geqq 0$.

Let $\mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ denote the corresponding $\mathbb{P}^{d-1}$-bundle. Since the g_d^1 has no base points, C does not intersect the (possibly empty) singular set of X and we may regard C as a subvariety of $\mathbb{P}(\mathscr{E})$. We will construct an $\mathcal{O}_{\mathbb{P}(\mathscr{E})}$-module resolution of $\mathcal{O}_C$.

(4.2) Consider

$$D_\lambda \subset \overline{D_\lambda} \cong \mathbb{P}^{d-1}$$

as a zero-dimensional subscheme.

Lemma. *Let $D \subset \mathbb{P}^{d-1}$ be a zero-dimensional non-degenerate subscheme of degree $d-2$. The following assertions are equivalent:*

1) *The homogenous coordinate ring S_D is Gorenstein.*
2) *$\mathcal{O}_D$ has an $\mathcal{O}_{\mathbb{P}^{d-2}}$-module resolution of type:*

$$0 \to \mathcal{O}(-d) \to \mathcal{O}(-d+2)^{\beta_{d-3}} \to \dots \to \mathcal{O}(-3)^{\beta_2} \to \mathcal{O}(-2)^{\beta_1} \to \check{\mathcal{O}} \to \mathcal{O}_D \to 0$$

with

$$\beta_i = \frac{i(d-2-i)}{d-1}\binom{d}{i+1}.$$

3) *No subscheme $E \subset D$ of degree $d-1$ is contained in a hyperplane of $\mathbb{P}^{d-2}$.*

Proof. The equivalence of 1) and 2) is well known see e.g. [Be]. The equivalence of 1) and 3) is proved in [G-O] for a collection of d distinct points. To prove the equivalence of 1) and 3) in general, we choose coordinates $x_0, \dots, x_{d-2}$ of $\mathbb{P}^{d-2}$ such that x_0 is a non-zero divisor of the homogenous coordinate ring S_D, i.e. the hyperplane $\{x_0=0\}$ does not intersect D. $R=S_D/x_0S_D$ is a graded artinian ring with Hilbert function $(1, d-1, 1, 0, 0, \dots)$. Recall that S_D is Gorenstein iff the multiplication

$$R_1 \times R_1 \to R_2 \cong k$$

gives a non-degenerate pairing [Be].

If a subscheme $E \subset D$ of degree $d-1$ is contained in a hyperplane $\{y_0=0\}$, then $y_0 \notin (x_0)$ but $y_0 \cdot R_1 = 0$. So the pairing is degenerate.

Conversely suppose the pairing is degenerate, say $y_0 R_1 = 0$ for a linear form $y_0 \notin (x_0)$. Then there exist linear forms $y_1, \dots, y_{d-2}$ such that the quadrics

$$x_i y_0 - x_0 y_i$$

are contained in the homogenous ideal I_D of D. Since x_0 in a non-zero divisor, the relations

$$x_0(x_i y_j - x_j y_i) = x_i(x_0 y_j - x_j y_0) - x_j(x_0 y_i - x_i y_0)$$

imply that all 2×2 minors of the matrix

$$\begin{pmatrix} x_0 x_1 \dots x_{d-2} \\ y_0 y_1 \dots y_{d-2} \end{pmatrix}$$

are contained in I_D. Since $x_0, \dots, x_{d-2}$ is a basis of $H^0\mathcal{O}_{\mathbb{P}^{d-2}}(1)$ the entries of the second row become linearly dependent, if we add a suitable multiple of the first row to the second. Thus without loss of generality we may assume that $y_{k+1}, \dots, y_{d-2}$ are identically zero while $y_0, \dots, y_k$ are linearly independent for some k with $0 \leqq k < d-2$. Consequently D is contained in the union of two linear spaces

$$L_1 = \{y_0 = \dots = y_k = 0\} \quad \text{and} \quad L_2 = \{x_{k+1} = \dots = x_{d-2} = 0\}$$

of dimension $d-3-k$ and k respectively. Since

$$\deg(L_1 \cap D) + \deg(L_2 \cap D) \geqq \deg D = d$$

we conclude

$$\deg(L_i \cap D) \geqq \dim L_i + 2$$

for $i=1$ or $i=2$. Thus a suitable subscheme E with $L_i \cap D \subset E \subset D$ of degree $d-1$ is contained in a linear space of dimension $d-3$, i.e. a hyperplane. □

(4.3) **Proposition.** *$C \subset \mathbb{P}(\mathscr{E})$ has constant betti-numbers over $\mathbb{P}^1$.*

Proof. For each $\lambda \in \mathbb{P}^1$ it suffices to prove that the divisor $D_\lambda \subset \overline{D_\lambda}$ regarded as a zero-dimensional subscheme satisfies the equivalent conditions of Lemma 4.2. We prove (3): If a subscheme $E \subset D_\lambda$ is contained in a hyperplane of $\overline{D_\lambda}$ then the corresponding divisor $E \subset C$ moves in a pencil by the geometric version of Riemann-Roch (cf. [G-H, p. 248]).

Since the g_d^1 is complete this forces the remaining point $D_\lambda - E$ to be a fixed point of the g_d^1, a contradiction to our assumption. □

(4.4) **Corollary.** i) *$C \subset \mathbb{P}(\mathscr{E})$ has a resolution F_* of type*

$$0 \to \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-dH + (f-2)R) \to \sum_{k=1}^{\beta_{d-2}} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-(d+2)H + b_{d-2}^{(k)}R) \to$$

$$\to \sum_{k=1}^{\beta_1} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H + b_1^{(k)}R) \to \mathcal{O}_{\mathbb{P}(\mathscr{E})} \to \mathcal{O}_C \to 0.$$

ii) *F_* is self dual:*

$$\mathscr{H}om(F_*, \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-dH + (f-2)R) \cong F_*.$$

iii) *If all $b_i^{(k)} \geqq -1$ then an iterated mapping cone*

$$\left[\dots \left[\mathscr{C}^{(f-2)}(-d) \to \sum_{k=1}^{\beta_{d-3}} \mathscr{C}^{b_{d-3}^{(k)}}(-d+2) \right] \to \dots \right] \to \mathscr{C}^0$$

is a (not necessarily minimal) resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}$-module.

Proof. i) follows from (3.2), iii) from (3.5). The duality follows from adjunction:

$$\mathscr{E}xt^i_{\mathbb{P}(\mathscr{E})}(\mathcal{O}_C, \omega_{\mathbb{P}(\mathscr{E})}) = \begin{cases} \omega_C & i = d-2 \\ 0 & \text{otherwise} \end{cases}$$

(cf. [Gr, Corollary 2]), $\omega_C \cong \mathcal{O}_C(H)$ and

$$\omega_{\mathbb{P}(\mathscr{E})} \cong \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-(d-1)H + (f-2)R)$$

and the uniqueness of F_* (3.2). □

Remark. In Sect. 6 we will discuss this result for small d in further details.

5. On the Equation of Rational Surfaces

Summary

For H a sufficiently positive divisor on a rational surface S the image S' of $j: S \to \mathbb{P}H^0(S, \mathcal{O}_S(H))$ can be described as a subvariety of a scroll X: It will be a "*rational normal curve bundle*." We give an outline of this approach to the equation of S' in the first part of this section (5.1)–(5.5).

What will be important to us is partial converse of this approach: Given a certain determinantal subvariety of a scroll, we may conclude that it is a rational surface and how it can be obtained from minimal model via a rational map with assigned basepoints. In particular we can calculate the "number" of exceptional curves.

(5.1) We consider a rational ruled surface

$$\pi: P_k = \mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}) \to \mathbb{P}^1, \quad k \geqq 0,$$

and a surface S which is obtained from P_k via a sequence of blow ups:

$$\sigma: S \to P_k.$$

Every rational smooth surface S except $\mathbb{P}^2$ can be obtained in this way (cf. [Beau]).

With A and B we denote the hyperplane class and the ruling of P_k, and by abuse of notation also their pullbacks to S. $E = \cup E_i$ denotes the exceptional divisor of σ with its components. Let

$$H \sim dA + dB - \sum e_i E_i$$

be a divisor such that the complete linear series H has no base points. We consider the image S' of

$$j: S \to \mathbb{P}H^0(S, \mathcal{O}_S(H)) = \mathbb{P}^r.$$

Suppose that

i) $h^0(\mathcal{O}_S(H-B)) \geqq 2$,

ii) $H^1(\mathcal{O}_S(kH-B)) = 0$, $k \geqq 1$, and

iii) the map $S_k H^0 \mathcal{O}_S(H) \to H^0 \mathcal{O}_S(kH)$ is surjective.

By Sect. 2, i), ii) the variety

$$X = \bigcup_{B_\lambda \in |B|} \overline{B_\lambda} \subset \mathbb{P}^r$$

is a $(d+1)$-dimensional rational normal scroll. Let $\pi: \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ denote the corresponding $\mathbb{P}^d$-bundle and S'' the strict transform of S' in $\mathbb{P}(\mathcal{E})$. Blowing-up S further we may assume that $S \to S'$ factors through S'':

$$\begin{tikzcd}
 & S \arrow[r] \arrow[dl] \arrow[dr] \arrow[ddr, "\pi_S"'] & S' \arrow[r, hook] & X \arrow[r, hook] & \mathbb{P}^r \\
P_k \arrow[drr, "\pi_{P_k}"'] & & S'' \arrow[u] \arrow[r] \arrow[d] & \mathbb{P}(\mathcal{E}) \arrow[u] \arrow[dl, "\pi_{\mathbb{P}(\mathcal{E})}"] & \\
 & & \mathbb{P}^1 & &
\end{tikzcd}$$

We want to describe the syzygies of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathbb{P}(\mathscr{E})}$-modules with the methods of Sect. 3.

(5.2) **Lemma.** *Let B be an e-dimensional non-degenerate subvariety of degree d in $\mathbb{P}^{d+e-1}$. The following assertions are equivalent:*

1) *The homogenous coordinate ring S_B is Cohen-Macaulay.*
2) *$\mathcal{O}_B$ has an $\mathcal{O}=\mathcal{O}_{\mathbb{P}^{d+e-1}}$-module resolution of type*

$$0\to\mathcal{O}(-d)^{\beta_{d-1}}\to\ldots\to\mathcal{O}(-2)^{\beta_1}\to\mathcal{O}\to\mathcal{O}_B\to 0$$

with

$$\beta_i=i\binom{d}{i+1}.$$

For a proof see e.g. [E-R-S].

(5.3) A general fibre $B_\lambda\subset\overline{B_\lambda}=\mathbb{P}^d$ is a rational normal curve, so 1) of Lemma 5.2 is satisfied. To prove this condition for all fibres we consider the commutative diagram

$$\begin{array}{ccc} H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(kH) & \longrightarrow & H^0\mathcal{O}_{\overline{B_\lambda}}(kH) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ H^0\mathcal{O}_S(kH) & \xrightarrow[\alpha]{} & H^0\mathcal{O}_{B_\lambda}(kH) \end{array}$$

α is surjective by ii), β as a consequence of iii). Thus γ is surjective, which proves that $B_\lambda\subset\overline{B_\lambda}$ is arithmetically Cohen-Macaulay.

Consequently:

$$S''\subset\mathbb{P}(\mathscr{E}) \quad \textit{has constant betti-numbers over } \mathbb{P}^1.$$

(5.4) **Corollary.** *$\mathcal{O}_{S''}$ has an $\mathcal{O}_{\mathbb{P}(\mathscr{E})}$-module resolution of type F_**

$$0\to\sum_{j=1}^{\beta_{d-1}}\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-dH+b_{d-1}^{(j)}R)\to\ldots\to\sum_{j=1}^{\beta_1}\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H+b_1^jR)$$
$$\to\mathcal{O}_{\mathbb{P}(\mathscr{E})}\to\mathcal{O}_{S''}\to 0$$

with

$$\beta_i=i\cdot\binom{d}{i+1}.$$

(5.5) In some cases the resolution of S'' on $\mathbb{P}(\mathscr{E})$ is actually given by an Eagon-Northcott complex: Let

$$\mathscr{F}'=\pi_{S*}\mathcal{O}_S(H-A), \qquad \mathscr{G}'=\pi_{S*}\mathcal{O}_S(A)$$

$\mathscr{F}'$ is locally free of rank d, say

$$\mathscr{F}'=\mathcal{O}(a_1)\oplus\ldots\oplus\mathcal{O}(a_d)$$

and

$$\mathscr{G}'\cong\mathcal{O}(k)\oplus\mathcal{O}.$$

Consider

$$\mathscr{F}=\pi^*_{\mathbb{P}(\mathscr{E})}\mathscr{F}', \qquad \mathscr{G}=\pi^*_{\mathbb{P}(\mathscr{E})}\mathscr{G}'$$

and the map

$$\psi: \mathscr{F}(-H) \to \mathscr{G}^*$$

obtained from the composition

$$\mathscr{F} \otimes \mathscr{G} \to \pi^*_{\mathbb{P}(\mathscr{E})}(\pi_{S*})\mathcal{O}_S(H) \cong \pi^*_{\mathbb{P}(\mathscr{E})}\mathscr{E} \to \mathcal{O}_{\mathbb{P}(\mathscr{E})}(H).$$

ψ is given by a matrix with entries as indicated below

$$(\psi) \sim \begin{pmatrix} H-a_1R, & \ldots, H-a_dR \\ H-(a_1+k)R, & \ldots, H-(a_d+k)R \end{pmatrix}.$$

The Eagon-Northcott complex associated to ψ resolves $\mathcal{O}_{S''}$ iff this is true for every fibre. A necessary condition is that every fibre of $S'' \subset \mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$ is determinantal. We refer to [X] for a description of determinantal curves of minimal degree.

(5.6) We now give a partial converse of the result above: Let $X \subset \mathbb{P}^r$ denote a rational normal scroll of dimension $d+1$ and $\mathbb{P}(\mathscr{E})$ the corresponding $\mathbb{P}^d$-bundle over $\mathbb{P}^1$. Let $S'' \subset \mathbb{P}(\mathscr{E})$ be an irreducible surface defined by the 2×2 minors of a matrix ψ with entries section in line bundles as indicated below:

$$(\psi) \sim \begin{pmatrix} H-a_1R, & \ldots, H-a_dR \\ H-(a_1+k)R, & \ldots, H-(a_d+k)R \end{pmatrix}.$$

We assume that the general fibre of $S'' \subset \mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$ is a rational normal curve of degree d. We want to describe the image S' of S'' in $\mathbb{P}^r$.

Set

$$a = a_1 + \ldots + a_d, \qquad f = \deg X$$

and let P_k denote the rational ruled surface $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O})$ with hyperplane class A and ruling B as above. Then we have:

(5.7) **Theorem.** *$S' \subset \mathbb{P}^r$ is the image of P_k under a rational map defined by a subseries of*

$$H^0(P_k, \mathcal{O}_{P_k}(dA + (f - dk - a)B)$$

which has

$$\delta = d \cdot f - \frac{d(d+1)}{2} \cdot k - (d+1)a$$

assigned base points. Furthermore, if $S' \subset X \subset \mathbb{P}^r$ contains a canonical curve C of genus $r+1$, then the ruling of X cuts on C a g^1_{d+2} and the strict transform C' of C in P_k is a divisor of class

$$C' \sim (d+2)A + (f - (d+1)k - a + 2)B$$

and arithmetic genus

$$P_a C' = r + 1 + \delta.$$

Proof. Let

$$\varphi_i \in H^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R) \quad \text{for} \quad i = 1, \ldots, d+1$$

denote a collection of basic sections of $\mathbb{P}(\mathscr{E})$ [cf. (1.3)]. The entries of the matrix ψ are linear forms

$$\psi_{1j} = \sum_{i=1}^{d+1} p_{ij}(s,t)\varphi_i, \qquad \psi_{2j} = \sum_{i=1}^{d+1} q_{ij}(s,t)\varphi_i$$

whose coefficients $p_{ij}, q_{ij} \in \mathfrak{k}[s,t]$ are homogenous polynomials of degree

$$\deg p_{ij} = e_i - a_j, \qquad \deg q_{ij} = e_i - a_j - k.$$

A fibre $S''_{(s,t)}$ of S'' is given by the union

$$S''_{(s,t)} = \bigcup_{(\lambda:\mu)\in\mathbb{P}^1} \{\lambda\psi_{1i}(s,t) + \mu\psi_{2i}(s,t) = 0;\ i = 1, \dots, d+1\}.$$

Consider the $(d+1)\times d$ matrix of coefficients:

$$M = (\lambda p_{ij} + \mu q_{ij})_{\substack{i=1,\dots,d+1 \\ j=1,\dots,d}}.$$

The fibre $S''_{(s,t)}$ is singular iff rank $M(s,t,\lambda,\mu) < d$ for a suitable choice of $(\lambda:\mu)\in\mathbb{P}^1$.

The main point of the proof is simply to identify

$$\lambda \in H^0\mathcal{O}_{P_k}(A - kB), \qquad \mu \in H^0\mathcal{O}_{P_k}(A)$$

with a collection of basic section of P_k. We then may interpret M as a map of bundles on P_k and can consider the associated Eagon-Northcott complex:

$$0 \longrightarrow \sum_{j=1}^{d} \mathcal{O}_{P_k}(-A + (a_j + k)B) \xrightarrow{{}^tM} \sum_{i=1}^{d+1} \mathcal{O}_{P_k}(e_iB)$$

$$\xrightarrow{\overset{d}{\wedge} M} \mathcal{O}_{P_k}(dA + (f - dk - a)B). \tag{$*$}$$

Claim. S' is the image of P_k under the rational map defined by the subseries V given by

$$H^0\mathcal{O}_{P_k}(e_iB) \twoheadrightarrow V \subset H^0\mathcal{O}_{P_k}(dA + (f - dk - a)B). \tag{$**$}$$

We first treat the case that $(*)$ is exact, i.e. the subseries defined by V has no fixed components cf. (1.5). In this case $(*)$ resolves the structure sheaf of a zero-dimensional subscheme Δ, which is the base locus of V. Its degree is

$$\delta = h^0\mathcal{O}_\Delta = h^0(\mathcal{O}_{P_k}(dA + (f - dk - a)B) - \sum_{i=1}^{d+1}(e_i + 1)$$

$$= df - \frac{d(d+1)}{2}k - da.$$

Clearly the image of V in $\mathbb{P}^r$ is contained in a scroll of type $S(e_1, \dots, e_{d+1})$. To exhibit the defining equation of the strict transform in $\mathbb{P}(\mathscr{E})$ we proceed in the same manner as above (5.5):

Let $\sigma: S \to P_k$ be the blow-up of Δ. We have to prove that the multiplication map

$$\pi_{S*}\mathcal{O}_S(H - A) \otimes \pi_{S*}\mathcal{O}_S(A) \to \pi_{S*}\mathcal{O}_S(H)$$

induces the matrix ψ.

$$\pi_{S*}\mathcal{O}_S(A) \cong \pi_{P_k*}\mathcal{O}_{P_k}(A) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}.$$

For $\pi_{S*}\mathcal{O}_S(H)$ we notice

$$\sigma_*\mathcal{O}_S(H) \cong \mathscr{J} = \operatorname{Im}\left(\overset{d}{\bigwedge} M\right) \subset \mathcal{O}_{P_k}(dA + (f - dk - a)B)$$

thus

$$\pi_{S*}\mathcal{O}_S(H) \cong \pi_{P_k*}\mathcal{J}\,.$$

For the first row of ψ consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \sum_{j=1}^{d} \mathcal{O}_{P_k}(-A+a_jB) \arrow[r, "{}^tM"] & \sum_{i=1}^{d+1} \mathcal{O}_{P_k}(e_iB) \arrow[r] & \mathcal{J} \arrow[r] & 0 \\
0 \arrow[r] & \sum_{j=1}^{d} \mathcal{O}_{P_k}(-2A+a_jB) \arrow[r, "{}^tM"] \arrow[u, "\cdot\mu"] & \sum_{i=1}^{d+1} \mathcal{O}_{P_k}(-A+e_iB) \arrow[r] \arrow[u, "\cdot\mu"] & \mathcal{J}(-A) \arrow[r] \arrow[u, "\cdot\mu"] & 0
\end{tikzcd}$$

and apply $(\pi_{P_k})_*$. We obtain a map

$$\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^1}(a_d)\cong(\pi_{P_k})_*\mathcal{J}(-A)\to(\pi_{P_k})_*\mathcal{J}\cong\mathcal{O}_{\mathbb{P}^1}(e_1)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^1}(e_{d+1})$$

which we will identify with the map given by the matrix ${}^t(p_{ij})$.

Since both maps are well defined it suffices to prove this generically. Over the generic point of $\mathbb{P}^1$ this follows easily, chasing e.g. the corresponding diagram of Čech-cocycles. Similarly we can identify the second row.

Consequently the minors defining S'' vanish on the image of P_k in $\mathbb{P}(\mathcal{E})$. Since S'' is irreducible and has the same dimension as the image of P_k they coincide. This proves the first part of the theorem in case V has no fixed component.

If V has fixed components, then the complex $(*)$ is no longer exact, it has homology in the middle. But if we replace $\mathcal{J}$ by coker$({}^tM)$ then the calculation above goes through, and all what remains is to prove that the map

$$\sum_{i=1}^{d+1} H^0\mathcal{O}_{P_k}(e_iB)\to H^0\mathcal{O}_{P_k}(dA+(f-dk-a)B)$$

is still injective, hence has codimension δ.

Consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \sum_j \mathcal{O}_{P_k}(-A+(a_j+k)B) \arrow[r] \arrow[d, equal] & \operatorname{Ker}\bigwedge^d M \arrow[r] \arrow[d] & H_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sum_j \mathcal{O}_{P_k}(-A+(a_j+k)B) \arrow[r] & \sum_i \mathcal{O}_{P_k}(e_iB) \arrow[r] \arrow[d] & \operatorname{Coker}({}^tM) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{J} \arrow[r, equal] \arrow[d] & \mathcal{J} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

of exact sequences.

H_1 has support at the fixed components of V. These are finitely many fibres of $P_k \to \mathbb{P}^1$ since the general fibre of S'' is a rational normal curve. Thus $(\pi_{P_k})_* H_1$ is concentrated in at most finitely many points of $\mathbb{P}^1$.

Since the sheaf

$$\pi_{P_k *} \operatorname{coker}({}^t M) \cong (\pi_{P_k})_* \sum_i \mathcal{O}_{P_k}(e_i B) \cong \sum_i \mathcal{O}(e_i)$$

has no torsion, we conclude $(\pi_{P_k})_* H_1 = 0$ and hence

$$H^0\left(\ker\left(\bigwedge^d M\right)\right) \cong H^0(P_k, H_1) \cong H^0(\mathbb{P}^1, \pi_{P_k *} H_1) = 0.$$

This finishes the proof of the first part of the theorem.

For the second part of the theorem we notice that if $C \subset S' \subset X \subset \mathbb{P}^r$ is a canonical curve, then the rational map

$$P_k \dashrightarrow S' \subset \mathbb{P}^r$$

is defined by the adjoint series

$$V \subset H^0(P_k, \omega_{P_k}(C'))$$

which has assigned base points at the singularities of the strict transform C' of C. Hence

$$C' \sim H - K_{P_k} \sim (d+2)A + (f-(d+1)k - a + 2)B$$

and

$$P_a C' = \tfrac{1}{2} C'(C' + K_{P_k}) + 1 = r + 1 + \delta. \quad \square$$

6. d-Gonal Curves for $d \leqq 5$

In this section we discuss the result of Sect. 4 for $d \leqq 5$ in more details.

(6.1) *Trigonal Curves* (cf. [Ma, P]). A trigonal canonical curve C is contained in a two-dimensional rational normal scroll

$$X = \bigcup_{D \in g_3^1} \bar{D} \subset \mathbb{P}^{g-1}$$

of type $S(e_1, e_2)$ and degree $f = e_1 + e_2 = g - 2$ (cf. Sect. 2). From $H^0(C, \omega_C(-nD)) = 0$ for $n > \frac{2g-2}{3}$ we obtain the bounds

$$\frac{2g-2}{3} \geqq e_1 \geqq e_2 \geqq \frac{g-4}{3}$$

by (2.5). C is a divisor of class

$$C \sim 3H - (f-2)R$$

on X (cf. Sect. 4). The mapping cone

$$\mathscr{C}^{f-2}(-3) \to \mathscr{C}^0$$

is a *minimal* resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}$-module. The contribution of $\mathscr{C}^0$ is distinguished by its degrees from the part of $\mathscr{C}^{f-2}(-3)$. This proves the well-known fact that X and hence g_3^1 is uniquely determined by C for a trigonal curve of genus $g \geqq 5$.

(6.2) *Tetragonal Curves* (cf. [P]). A tetragonal canonical curve C of genus $g \geqq 5$ is contained in a 3-dimensional rational normal scroll

$$X = \bigcup_{D \in g_4^1} \bar{D} \subset \mathbb{P}^{g-1}$$

of type $S(e_1, e_2, e_3)$ with

$$\frac{2g-2}{4} \geqq e_1 \geqq e_2 \geqq e_3 \geqq 0$$

and degree $f = e_1 + e_2 + e_3 = g - 3$ (cf. Sect. 2).

By Sect. 4 C is a complete intersection of two divisors

$$Y \sim 2H - b_1 R, \qquad Z \sim 2H - b_2 R$$

on X with

$$b_1 + b_2 = f - 2$$

(4.4), say $b_1 \geqq b_2$. We will verify

$$f - 1 \geqq b_1 \geqq b_2 \geqq -1$$

in (6.3) below. Hence we can obtain a *minimal* resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^{g-1}}$-module via an iterated mapping cone:

$$[\mathscr{C}^{f-2}(-4) \to \mathscr{C}^{b_1}(-2) \oplus \mathscr{C}^{b_2}(-2)] \to \mathscr{C}^0 .$$

In particular we note that the invariants b_1, b_2 are determined by the graded-betti-numbers of C. X and hence g_4^1 is uniquely determined by C unless $b_1 \geqq f-2$. It is the support of the cokernel of the dual map to the term

$$\mathcal{O}^{\beta_{g-4,g-3}}(-g+3) \to \mathcal{O}^{\beta_{g-5,g-4}}(-g+4)$$

in the resolution of C. For a similar reason the surface Y is uniquely determined by C if $b_1 > b_2$ (e.g. if g is even).

(6.3) We study the geometry of Y next. In terms of a collection of basic sections (1.3)

$$\varphi_i \in H^0(\mathbb{P}(\mathscr{E}, \mathcal{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R)), \qquad i = 1, 2, 3$$

for the corresponding $\mathbb{P}^2$-bundle $\mathbb{P}(\mathscr{E})$, the surface Y is defined by a quadratic form

$$\psi = \sum_{i,j} P_{ij}(s,t) \varphi_i \varphi_j$$

with coefficients homogenous polynomials $P_{ij} \in \mathfrak{k}[s,t]$ of degree

$$\deg P_{ij} = e_i + e_j - b_1$$

with C also Y is irreducible. This gives a bound on b_1: Certainly we must have

$$2e_2 - b_1 \geqq 0.$$

In particular $f \geqq e_1 + e_2 \geqq 2e_2 \geqq b_1$. This bound can be slightly improved: Suppose $b_1 = f$ then $e_1 = e_2$ and $e_3 = 0$, hence Y is defined by a quadratic form in φ_1, φ_2 only, which has constant coefficients. Since the ground field $\mathfrak{k}$ is algebraically closed Y would be reducible. So

$$f - 1 \geqq b_1 \geqq b_2 \geqq -1.$$

Actually $f-1 = b_1$ occurs only for $g \leqq 0$ and in that case there exists a g_3^1 or a g_5^2 on C, which can easily be checked with the methods below.

(6.4) The fibres of $Y' \subset \mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$ are conics. If the general fibre is a non-singular conic, then the number of singular fibres is given by

$$\delta = 2f - 3b_1$$

i.e. the degree of the determinant of the associated symmetric matrix of coefficients of the quadratic form (char $\mathfrak{k} \neq 2$).

One can derive the same result with methods of Sect. 5. Suppose ψ is given by determinant of a matrix with entries as indicated below:

$$\begin{pmatrix} H - a_1 R & H - a_2 R \\ H - (a_1 + k)R & H - (a_2 + k)R \end{pmatrix}$$

then (5.7) gives us the same number of singular fibres. We may identify Y with image of $P_k = \mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O})$ under a rational map defined by a linear series with δ basepoints.

Moreover the composition

$$C \subset Y \to P_k \to \mathbb{P}(H^0(P_k, \mathcal{O}_{P_k}(A))) \cong \mathbb{P}^{k+1}$$

defines a linear series of degree

$$b_2 + 2 + 2(k+1) \quad (= A \cdot C', \text{ cf. } (5.7))$$

hence of *Clifford-index* $b_2 + 2$.

However, a determinantal presentation of ψ is not unique. For example taking the transposed matrix corresponds to blow-down the "other" line in each singular fibre of Y. In general the different determinantal presentations correspond to the various different ways Y can be blown down to a minimal smooth surface, such that the image of C is a curve with admost double point singularities. We leave it to reader to establish these facts with the methods of Sect. 5 and in particular to prove the existence of a determinantal presentation for a suitable choice of a_1, a_2, k with $a_1 + a_2 + k = b_1$.

(6.5) If all fibres of Y are degenerate conics the picture is quite different. In that case the g_4^1 is composed by an elliptic or hyperelliptic involution

$$C \xrightarrow{2:1} E \xrightarrow{2:1} \mathbb{P}^1$$

and Y is a birational ruled surface over E with a rational curve $\tilde{E}$ of double points. Since C is smooth $\tilde{E} \cap Z = \emptyset$, in particular

$$0 = \tilde{E} \cdot (2H - b_2 R) = 2\deg \tilde{E} - b_2$$

i.e. b_2 is *even*. A general divisor of class $H - \frac{b_2}{2} R$ intersects Y in a smooth curve isomorphic to E, so the geometric genus is given by

$$2p_a E - 2 = \left(H - \frac{b_2}{2} R\right)(2H - b_1 R)\left(f - 2 - b_1 - \frac{b_2}{2}\right) R$$

[cf. (3.6)] i.e.

$$p_a E = \frac{b_2}{2} + 1$$

(and we may identify $\tilde{E}$ with the canonical image of E). Similarly to the first case (6.4) we may use Y to discover a linear series with *Clifford-index* $b_2 + 2$ on C: The composition of $C \xrightarrow{2:1} E$ with the map induced by a general divisor of degree $2p_a E$ on E induces a g^r_d with $r = p_a E$, $d = 4p_a E$, hence $d - 2r = 2p_a E = b_2 + 2$.

(6.6) Finally we note the following numerical characterization of decomposable g^1_4's:

Theoerm. *A g^1_4 on a curve of genus g is composed by an involution of genus g' with $6g' < g - 3$ iff*

$$\beta_{i,i+1} = i\binom{g+3}{i+1} \quad \text{for} \quad i > g - 2 - 2g'$$

and

$$\beta_{i,i+1} > i\binom{g-3}{i+1} \quad \text{for} \quad i = g - 2 - 2g'.$$

Proof. The condition on the graded betti-numbers gives $b_2 = 2g' - 2$. So our assumption $6g' < g - 3$ means that

$$\delta = 2f - 3b_1 < 0.$$

So by (6.4) a general fibre of Y over $\mathbb{P}^1$ cannot be non-singular. The factorization

$$\begin{tikzcd} C \arrow[r, "4:1"] \arrow[dr, "2:1"', "\pi"] & \mathbb{P}^1 \arrow[d, "2:1"] \\ & E \end{tikzcd}$$

over a curve of genus $g' = \frac{b_2}{2} + 1$ follows then from (6.5). Conversely given a factorization we obtain a conic bundle

$$Y = \overline{\bigcup_{P \in E} \pi^{-1}(P)} \subset X \subset \mathbb{P}^{g-1}$$

with a rational curve $\tilde{E}$ of singular points of degree $g'-1$. As above $\tilde{E} \cap Z = \emptyset$ implies

$$b_2 = 2g'-2, \quad b_1 = g-5-b_2$$

and these invariants determine the graded betti-numbers of C. $\square$

(6.7) *Pentagonal Curves* (cf. [P]). From a complete base point free g_5^1 on a canonical curve of genus $g \geqq 7$, we obtain a 4-dimensional rational normal scroll

$$X = \bigcup_{D \in g_5^1} \bar{D} \subset \mathbb{P}^{g-1}$$

of type $S(e_1, e_2, e_3, e_4)$ with

$$\frac{2g-2}{5} \geqq e_1 \geqq e_2 \geqq e_3 \geqq e_4 \geqq 0$$

and degree $f = e_1 + e_2 + e_3 + e_4 = g-4$ (cf. Sect. 2).

The resolution of $\mathcal{O}_C$ on the corresponding $\mathbb{P}^3$-bundle $\mathbb{P}(\mathscr{E})$ is of type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-5H+(f-2)R) \longrightarrow \sum_{i=1}^{5} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+b_i R)$$

$$\xrightarrow{\psi} \sum_{i=1}^{5} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H+a_i R) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathscr{E})} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

with

$$a_i + b_i = f-2$$

since the complex is "selfdual," and

$$a_1 + \ldots + a_5 = 2g-12$$

by (3.3). The corresponding iterated mapping cone (in case $-1 \leqq a_i$, $b_i \leqq f-1$)

$$\left[\left[\mathscr{C}^{f-2}(-5) \to \sum_i \mathscr{C}^{b_i}(-3)\right] \to \sum_i \mathscr{C}^{a_i}(-2)\right] \to \mathscr{C}^0$$

is a *not necessarily minimal* resolution of C.

From the structure theorem for Gorenstein ideals in codimension 3 [B-E3] we obtain further information:

The matrix ψ is shew-symmetric and its 5 Pfaffians generate the ideal of C in $\mathbb{P}(\mathscr{E})$, i.e. form the entries of

$$\sum_{i=1}^{5} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H + a_i R) \to \mathcal{O}_{\mathbb{P}(\mathscr{E})}.$$

Thus C is determined by the entries of ψ. If one of the off-diagonal entries is zero, say $\psi_{12} = \psi_{21} = 0$, then C is contained in the determinantal surface Y defined by the matrix

$$\begin{pmatrix} \psi_{13} & \psi_{14} & \psi_{15} \\ \psi_{23} & \psi_{24} & \psi_{25} \end{pmatrix}$$

since in this case the 2×2 minors of that matrix are among the Pfaffians of ψ. With C also Y is irreducible, so a general fibre of $Y \subset \mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$ is a twisted cubic. Consequently Y is one of the rational surfaces classified in (5.7). We will take this approach to the syzygies of canonical curves of genus $g = 7, 8$ in Sect. 7.

(6.8) Finally we note:

Corollary (Petri). *The Hurwitz-scheme*

$$H_{d,g}=\{C\to\mathbb{P}^1\colon d\text{-sheeted covers of curves } C \text{ of genus } g \text{ over } \mathbb{P}^1\}$$

is unirational for $d=3$, $g\geqq 4$, $d=4$, $g\geqq 5$ *and* $d=5$, $g\geqq 6$.

Proof. One can describe C by sections

$$\psi\in H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(3H-(f-2)R),$$

$$\psi_i\in H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(2H-b_iR),\qquad i=1,2$$

or

$$\psi_{ij}\in H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-(b_j-a_i)R),\qquad 1\leqq i<j\leqq 5$$

respectively. The corresponding open subsets of these spaces, which describe smooth curves give for a suitable choice of $e_1,\dots,e_{d-1}$ and the b_i's a rational variety which dominates $H_{d,g}$. □

7. Proof of the Main Result

(7.1) **Theorem.** *For a curve C of genus* $g=7$ *or* $g=8$ *the distribution of the graded betti-numbers depends on and determines the existence of a* g_2^1, g_3^1, g_6^2 *or* g_4^1 *on* C.

Proof. Every curve C of genus $g=7$ or $g=8$ admits a basepoint free complete pencil g_d^1 for some d with $2\leqq d\leqq\left[\frac{g}{2}+1\right]=5$ by [Ke, K-L]. For a curve with a g_2^1, g_3^1 or g_4^1 we have a description of the minimal resolution (cf. Sect. 6). In case of a g_4^1 there are two cases: Either the g_4^1 is uniquely determined by the resolution of C or C is contained in a surface Y of degree $g-3$. Either Y is a Del-Pezzo surface, i.e. $\mathbb{P}^2$ blown-up in $10-g$ points, and the composition

$$C\to Y\to\mathbb{P}^2$$

gives a g_6^2 on C or Y is a cone over an elliptic curve E and the composition

$$C\xrightarrow{2:1}E\hookrightarrow\mathbb{P}^2$$

with any embedding of E as a plane cubic gives a g_6^2 on C [cf. (6.4–5)].

It remains to handle the case that C admits a g_5^1.

(7.2) Given a g_5^1 on C let $\quad X=\bigcup_{D\in g_5^1}\bar{D}\subset\mathbb{P}^{g-1}$

be the associated 4-dimensional rational normal scroll [cf. (6.7)].

We first prove the theorem for curves of genus $g=7$, whose invariants

$$(a_1,\dots,a_5)=(0,0,0,1,1),$$

i.e. the resolution of C on the corresponding $\mathbb{P}^3$-bundle $\mathbb{P}(\mathscr{E})$ is of type

$$0\to\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+R)\to \begin{matrix}\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 3}\\ \oplus\\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H)^{\oplus 2}\end{matrix} \xrightarrow{\psi} \begin{matrix}\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 3}\\ \oplus\\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 2}\end{matrix} \to\mathscr{O}_{\mathbb{P}(\mathscr{E})}\to\mathscr{O}_C\to 0$$

with ψ a skew-symmetric matrix with appropriated entries. We shall see that $(a_1, \dots, a_5)$ take these values for a "general" curve.

The only non-minimal map in the mapping cone

$$\left[\left[\mathscr{C}^1(-5)\to \begin{matrix}\mathscr{C}^1(-3)^{\oplus 3}\\ \oplus\\ \mathscr{C}^0(-3)^{\oplus 2}\end{matrix}\right]\to \begin{matrix}\mathscr{C}^0(-2)^{\oplus 3}\\ \oplus\\ \mathscr{C}^1(-2)^{\oplus 2}\end{matrix}\right]\to\mathscr{C}^0$$

arises from the 3×3 block

$$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 3} \xrightarrow{\begin{pmatrix}0 & \psi_{12} & \psi_{13}\\ -\psi_{12} & 0 & \psi_{23}\\ -\psi_{13} & -\psi_{23} & 0\end{pmatrix}} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 3}$$

of ψ.

$$\begin{array}{ccc} G(-3)^{\oplus 3} & \longrightarrow & \mathscr{O}(-2)^{\oplus 3}\\ \uparrow & & \uparrow\\ F(-4)^{\oplus 3} & \xrightarrow{\alpha} & \bigwedge^2 F(-4)^{\oplus 3}\\ \uparrow & & \uparrow\\ \bigwedge^3 F(-6)^{\oplus 3} & \longrightarrow & \bigwedge^3 F\otimes G^*(-5)^{\oplus 3}. \end{array}$$

The graded betti-numbers if C depend on the rank of α. Recall (1.4), (1.6)

$$\psi_{12}, \psi_{13}, \psi_{23} \in H^0\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R) \cong H^0(\mathbb{P}^{g-1}, F).$$

α is given by the wedge-product with the corresponding section of F:

$$F^{\oplus 3} \xrightarrow{\begin{pmatrix}0 & \psi_{12} & \psi_{13}\\ -\psi_{12} & 0 & \psi_{23}\\ -\psi_{13} & -\psi_{23} & 0\end{pmatrix}} \bigwedge^2 F^{\oplus 3}.$$

Thus α has maximal possible rank, i.e. is invertible for $\operatorname{char}(\mathfrak{k}) \neq 2$ and has rank 8 for $\operatorname{char}(\mathfrak{k}) = 2$ iff $\psi_{12}, \psi_{13}, \psi_{23}$ are linearly independent. In that case the betti-numbers take the minimal value as predicted in Table 1.

On the other hand, if $\psi_{12}, \psi_{13}, \psi_{23}$ are dependent, then there exist a g^1_4: After suitable row and column operations on ψ we may assume $\psi_{12} = 0$. Thus C is contained in the determinantal surface Y defined by the matrix

$$\begin{pmatrix}\psi_{13} & \psi_{14} & \psi_{15}\\ \psi_{23} & \psi_{24} & \psi_{25}\end{pmatrix} \sim \begin{pmatrix}H-R & H & H\\ H-R & H & H\end{pmatrix}$$

Table 2. Canonical curves of genus 7 with a g_5^1

$a_1, \ldots, a_5$	Special linear series	Determinantal surface Y $C \subset Y \subset X \subset \mathbb{P}^{g-1}$	deg Y
0,0,0,1,1	General case	$\mathbb{P}^2$ blown-up in the 8 double points of a g_7^2	8
0,0,0,1,1	g_4^1	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in 5 double points of $g_5^1 \times g_4^1$	7
$-1,0,1,1,1$	g_6^2	$\mathbb{P}^2$ blown-up in 3 double points of a g_6^2, the g_5^1 is a projection	6
$-1,0,0,1,2$	g_6^2	As above, but the double points lie on a line	6
$-1,-1,1,1,2$	g_3^1	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in the double point of a $g_5^1 \times g_3^1$	5

on $\mathbb{P}(\mathscr{E})$. By Theorem (5.6) Y is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in 5 points and the image of C in $\mathbb{P}^1 \times \mathbb{P}^1$ is a divisor of class

$$C' \sim 5A + 4B.$$

The projection onto the second factor of $\mathbb{P}^1 \times \mathbb{P}^1$ is a g_4^1.

(7.3) Other values $(a_1, \ldots, a_5)$ for curves of genus $g=7$ are possible. But in all other cases we can deduce already from $h^0\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R)=3$ the vanishing of an off-diagonal entry of ψ (possibly after row and column operations on ψ), hence we deduce the existence of a special linear series g_3^1, g_4^1 or g_6^2 by Theorem (5.3). The result is summarized in Table 2.
All other distributions of $(a_1, \ldots, a_5)$ are impossible: In those case it is easy to see that at least 2 off-diagonal entries of ψ contained in one row are zero for degree reasons. So the Pfaffians of ψ would contain an reducible element. But with C also the generators of the ideal $\mathscr{J}_C$ of C in $\mathbb{P}(\mathscr{E})$ have to be irreducible.

(7.4) For $g=8$ we proceed similarly. For a general curve with a g_5^1 it will turn out that

$$(a_1, \ldots, a_5) = (0, 1, 1, 1, 1).$$

The resolution of C on the associated $\mathbb{P}^3$-bundle $\mathbb{P}(\mathscr{E})$ is of type

$$0 \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+2R) \to \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(3H+2R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 4} \end{matrix} \xrightarrow{\psi} \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R) \end{matrix} \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C \to 0$$

with ψ a skew-symmetric matrix with entries as indicated below:

$$(\psi)\sim\begin{pmatrix} \emptyset & H-R & H-R & H-R & H-R \\ H-R & \emptyset & H & H & H \\ H-R & H & \emptyset & H & H \\ H-R & H & H & \emptyset & H \\ H-R & H & H & H & \emptyset \end{pmatrix}.$$

The only non-minimal maps in the corresponding mapping cone arise from the first row of ψ and its dual in the resolution, that is the first column:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 4} \arrow[r, "(\psi_{12},\psi_{13},\psi_{14},\psi_{15})"] & \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H) \\
G(-3)^{\oplus 4} \arrow[r] \arrow[u, dashed] & \mathcal{O}(-2) \arrow[u, dashed] \\
F(-4)^{\oplus 4} \arrow[r, "\alpha"] \arrow[u] & \bigwedge^2 F(-4) \arrow[u] \\
\bigwedge^3 F(-6)^{\oplus 4} \arrow[r] \arrow[u] & \bigwedge^3 F\otimes G^*(-5) \arrow[u] \\
\bigwedge^4 F\otimes G(-7)^{\oplus 4} \arrow[r] \arrow[u] & \bigwedge^4 F\otimes D_2G^*(-6).\arrow[u]
\end{tikzcd}$$

Identifying

$$\psi_{12},\dots,\psi_{15}\in H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-R)\cong H^0(\mathbb{P}^{g-1},F),$$

α is given by the wedge-product with the corresponding section of F. Thus α has maximal rank, consequently the betti-numbers of C the minimal value, if $\psi_{12},\dots,\psi_{15}$ span an at least 3-dimensional subspace of $H^0(\mathbb{P}^{g-1},F)$. But this is satisfied, since if two off-diagonal entries in one row or column of ψ vanish, we obtain a reducible generator of the ideal of C in $\mathbb{P}(\mathscr{E})$, which contradicts the irreducibility of C.

Notice that in case $\psi_{12},\psi_{13},\psi_{14},\psi_{15}$ only span a 3-dimensional subspace, C is contained in a determinantal surface Y of type

$$\begin{pmatrix} H & H & H \\ H-R & H-R & H-R \end{pmatrix}.$$

By Theorem (5.3) Y is isomorphic to $\mathbb{P}^2$ blown-up in the 7 double points of a g^2_- on C.

(7.5) Other values $(a_1,\dots,a_5)$ are possible. The result is summarized in Table 3:

Table 3. Canonical curves of genus 8 with a basepoint free g_5^1

$a_1, \ldots, a_5$	Special linear series	Determinantal surface Y $C \subset Y \subset X \subset \mathbb{P}^{g-1}$	$\deg Y$
$0,1,1,1,1$	General case	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in 8 double points of a $g_5^1 \times g_5^1$	10
$0,1,1,1,1$	g_7^2	$\mathbb{P}^2$ blown-up in 7 double points of a g_7^2, the g_5^1 is a projection	9
$0,0,1,1,2$	g_7^2	As above, but all double points but the projection point lie on a conic	9
$0,0,1,1,2$	g_4^1	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in 4 double points of a $g_5^1 \times g_4^1$	8
$0,0,0,2,2$	g_4^1	As above, but the double points lie on a divisor of type $(0,2)$	8
$-1,0,1,2,2$	g_6^2	$\mathbb{P}^2$ blown-up in 2 double points of a g_6^2	7
$-1,-1,2,2,2$	g_3^1	$\mathbb{P}^1 \times \mathbb{P}^1$, C embedded by a $g_5^1 \times g_3^1$	6

Most interesting is the distinction in case

$$(a_1, \ldots, a_5) = (0,0,1,1,2).$$

In this case the skew-symmetric matrix ψ is of type

$$(\psi) \sim \begin{pmatrix} \emptyset & H-2R & H-R & H-R & H \\ H-2R & \emptyset & H-R & H-R & H \\ H-R & H-R & \emptyset & H & H+R \\ H-R & H-R & H & \emptyset & H+R \\ H & H & H+R & H+R & \emptyset \end{pmatrix}.$$

The only non-minimal map in the corresponding mapping cone arises from the first 2×4-block of ψ and its dual.

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+2R)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+R)^{\oplus 2} & \longrightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H)^{\oplus 2} \\
\uparrow & & \uparrow \\
S_2G(-3)^{\oplus 2} \oplus G(-3)^{\oplus 2} & \longrightarrow & \mathcal{O}(-2)^{\oplus 2} \\
\uparrow & & \uparrow \\
F \otimes G(-4)^{\oplus 2} \oplus F(-4)^{\oplus 2} & \xrightarrow{\alpha_1 \oplus \alpha_2} & \bigwedge^2 F(-4)^{\oplus 2} \\
\uparrow & & \uparrow \\
\bigwedge^2 F(-5)^{\oplus 2} \oplus \bigwedge^3 F(-6)^{\oplus 2} & \xrightarrow{\beta_1 \oplus \beta_2} & \bigwedge^3 F \otimes G^*(-5)^{\oplus 2} \\
\uparrow & & \uparrow \\
\bigwedge^4 F(-7)^{\oplus 2} \oplus \bigwedge^4 F \otimes G^*(-7)^{\oplus 2} & \longrightarrow & \bigwedge^4 F \otimes D_2G^*(-6)^{\oplus 2}.
\end{array}$$

Notice that α_1 and β_1 are dual to each other.

It suffices to determine the rank of $\alpha_1 \oplus \alpha_2$. Clearly, if $\psi_{12}=0$, then $\alpha_1 \oplus \alpha_2$ is not surjective and C is contained in a determinantal surface of type

$$\begin{pmatrix} H-R & H-R & H \\ H-R & H-R & H \end{pmatrix}.$$

By Theorem (5.3) there exists a g_4^1 on C. If $\psi_{12} \neq 0$ we have to prove that $\alpha_1 \oplus \alpha_2$ is surjective. Identifying

$$H^0 \mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-R) = H^0(\mathbb{P}^{g-1}, F)$$

we obtain from $\psi_{12} \in H^0 \mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-2R)$ a 2-dimensional subspace

$$G \cdot \psi_{12} \subset F.$$

Thus one component of α_1 is defined by

$$F \otimes G \to \bigwedge^2 F$$
$$f \otimes g \to f \wedge g \cdot \psi_{12}.$$

α_2 is given by

$$F^{\oplus 2} \xrightarrow{\begin{pmatrix} \psi_{13} & \psi_{14} \\ \psi_{23} & \psi_{24} \end{pmatrix} \wedge -} \bigwedge^2 F^{\oplus 2}.$$

Since we can achieve $\psi_{14}=0$ after row and column operations on ψ and $\psi_{13}, \psi_{24} \notin G \cdot \psi_{12}$, because no two entries on one row or column of ψ can be zero $\alpha_1 \oplus \alpha_2$ is surjective, i.e. the betti-numbers of C are minimal.

Finally we note that C is contained in a determinantal surface Y of type

$$\begin{pmatrix} H-R & H & H+R \\ H-2R & H-R & H \end{pmatrix}$$

that is $\mathbb{P}^2$ blown-up in the 7 double points of a g_7^2 on C. The special position of the double points follows from Theorem (2.5) applied to Y.

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Small Deformations of Normal Singularities

Shihoko Ishii

Department of Mathematics, Sangyo-Ika University, Kita-Kyushu-city 807, Japan

Introduction

In this article, we will study the behavior, under deformations, of normal analytic singularities and their numerical invariants.

Let $\pi:(X,x)\to(C,0)$ be a germ of deformation of normal singularity of relative dimension $n\geqq 2$ with the singular locus S over a 1-dimensional parameter space C. Our main result is that, for each $m\in\mathbb{N}$ and any holomorphic m-ple $(n+1)$-form θ on an analytic space $X-S$ such that $\theta|_{X_0-S_0}$ is $L^{2/m}$-integrable, there exists an $L^{2/m}$-integrable $(n+1)$-form θ' on $X-S$ such that $\theta|_{X_0-S_0}=\theta'|_{X_0-S_0}$. As a consequence of the main theorem, if (X_0,x) is isolated, it follows that the function $C\to\mathbb{N}$ defined by $\tau\to\sum_{y\in S_\tau}\delta_m(X_\tau,y)$ is upper semi-continuous for each $m\in\mathbb{N}$. As another consequence of the theorem, we obtain that every small deformation of 2-dimensional quotient singularity is again a quotient singularity, which was already shown by Esnault and Viehweg [1].

By the way, Steenbrink, in his paper [8], posed a problem; is every small deformation of a Du Bois singularity again Du Bois? The answer is yes for a deformation of an isolated Gorenstein Du Bois singularity by the main theorem and the characterization of an isolated Gorenstein Du Bois singularity [3]. However, without the Gorenstein condition, the answer is no. In Sect. 4 we give an example of deformation $\pi:(X,x)\to(C,0)$ of a Du Bois singularity (X_0,x) with X_τ not Du Bois for each $\tau\in C$, $\tau\neq 0$.

1. Preliminaries

We will denote by a pair (X,x) a germ of an analytic space X at a point x (frequently, x a singular point). The symbol X will also denote a sufficiently small Stein neighbourhood of the germ (X,x).

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A. Basic Facts on Normal Singularities

Definition 1.1. Let X be an analytic space with the singular locus S. A proper birational morphism $f: \tilde{X} \to X$ is called a good resolution of the singularities on X, if

(1) $\tilde{X}$ is non-singular,
(2) the restriction $\tilde{X} - f^{-1}(S) \to X - S$ of f is an isomorphism, and
(3) the reduced fibre $f^{-1}(S)_{\text{red}}$ is a divisor of simple normal crossings.

Definition 1.2. For a normal isolated singularity (X, x) of dimension $n \geqq 2$, we define a pluri-genera $\{\delta_m\}_{m \in \mathbb{N}}$ by

$$\delta_m(X, x) = \dim_{\mathbb{C}} \Gamma(X - \{x\}, \mathcal{O}(mK))/L^{2/m}(X - \{x\}),$$

where, $L^{2/m}(X - \{x\})$ denotes the set of all $L^{2/m}$-integrable m-ple holomorphic n-form on $X - \{x\}$.

Proposition 1.3 (Watanabe [9]). *The pluri-genus $\delta_m(X, x)$ is represented as*

$$\begin{aligned}\delta_m(X, x) &= \dim_{\mathbb{C}} \Gamma(\tilde{X} - E, \mathcal{O}(mK_{\tilde{X}}))/\Gamma(\tilde{X}, \mathcal{O}(mK_{\tilde{X}} + (m-1)E)) \\ &= \dim_{\mathbb{C}} \mathcal{O}(mK_X)/f_*\mathcal{O}(mK_{\tilde{X}} + (m-1)E),\end{aligned}$$

where $f: \tilde{X} \to X$ is a good resolution of the singularity and $E = f^{-1}(x)_{\text{red}}$.

Definition 1.4. Let (X, x) be a normal singularity (not necessarily isolated). We call the sheaf $\mathcal{O}(mK_X)/f_*\mathcal{O}(mK_{\tilde{X}} + (m-1)E)$ the $L^{2/m}$-quotient sheaf on X and denote it by $\Delta_m(X)$, where $f: \tilde{X} \to X$ is a good resolution of the singularities and $E = f^{-1}(\text{singular locus})_{\text{red}}$.

We note that the sheaf $f_*\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}} + (m-1)E)$ is independent of the choice of a good resolution f, by the log-ramification formula [5, Lemma 1.6].

B. Deformations of Normal Singularities

Definition 1.5. Let (Z, z) be a normal singularity. A germ of a morphism $\pi: (X, x) \to (C, 0)$ is called a small deformation of the singularity (Z, z) with the singular locus S, if the followings hold:

(1) X and C are normal and $\dim C = 1$;
(2) $\pi: X \to C$ is a flat surjective morphism;
(3) an analytic subspace S on X is the minimal analytic set such that π is smooth outside of S; and
(4) $(X_0, x) \simeq (Z, z)$.

Under the notation of Definition 1.5, we will call the fiber X_τ of $\tau \in C$ near to 0 a small deformation of (Z, z) too.

Here we note that a small deformation X_τ is normal by the definition.

Definition 1.6. Let $\pi: (X, x) \to (C, 0)$ be a small deformation of a normal singularity (X_0, x). A proper birational morphism $f: \tilde{X} \to X$ is called a good resolution of a small deformation π, if

(1) f is a good resolution of the singularities on X and
(2) the restriction $f|_{[X_\tau]}: [X_\tau] \to X_\tau$ of f onto the proper transform $[X_\tau]$ of X_τ are good resolution for any $\tau \in C$.

As easily checked, there is always a good resolution for a small deformation by Hironaka's big theorem. We may assume that the proper transform $[X_\tau]$ coincides with the total fiber $f^{-1}(X_\tau)$ for $\tau \neq 0$, since we work on a germ at x.

Definition 1.7. Let $f:\tilde{X}\to X$ be a good resolution of a small deformation $\pi:(X,x)\to(C,0)$. Denote the reduced divisor $f^{-1}(S)_{\mathrm{red}}$ by E and decompose it into irreducible components $\{E_1,\ldots,E_r\}$. We call a component mapped to the point $0\in C$ by $\pi\circ f$ a vertical component. On the other hand, if a component E_i is mapped onto the curve C, we call E_i a horizontal component. Denote the divisor $\sum_{E_i:\text{vertical}} m_iE_i=\tilde{X}_0-[X_0]$, $m_i\geqq 0$, by E_v and call it the vertical divisor.

C. A Covering Procedure

Here, we introduce a cyclic covering of a resolution space which was developed in [1]. It will be an important technique in order to show our main theorem (2.1).

Let X be a non-singular analytic space and D be an effective divisor belonging to a linear system $|\mathscr{L}^m|$ for some invertible sheaf $\mathscr{L}$ on X and a positive integer m. Then the divisor determines a section $s\in\Gamma(X,\mathscr{L}^m)$. By this section s we have an m-ple cyclic cover

$$Y'=\operatorname{Spec}\Big(\bigoplus_{n\geqq 0}\mathscr{L}^{-n}T^n\Big)/(s^{-1}T^m-1)\xrightarrow{\varphi'} X$$

branched on D, where T is indeterminate.

Definition 1.8. Under the above notation, take the normalization $\sigma: Y\to Y'$ of Y'. We call the composition $\varphi=\varphi'\circ\sigma: Y\to X$ the normal cyclic m-ple cover associated to D.

We will consider the case D_{red} is of normal crossings.

Lemma 1.9 [1, 6]. *Let $\varphi: Y\to X$ be the normal m-ple cyclic cover associated to a divisor $D\in|\mathscr{L}^m|$.*

Assume D_{red} is of normal crossings.

Then,

(i) *Y has only rational singularities.*

(ii) $\varphi_*\mathcal{O}_Y=\bigoplus_{k=0}^{m-1}\mathscr{L}^{-k}([kD/m])$, *and*

(iii) $\varphi_*\mathcal{O}_Y(K_Y)=\bigoplus_{k=0}^{m-1}\mathcal{O}_X(K_X)\otimes\mathscr{L}^k(-[kD/m])$,

where $[kD/m]$ is the largest divisor (with coefficients in $\mathbb{Z}$) satisfying $[kD/m]\leqq kD/m$.

1.10. Now we practice the covering procedure on a good resolution space. Let X be a normal variety with the singular locus S, $f:\tilde{X}\to X$ be a good resolution and $E=\sum_{i=1}^r E_i$ be the reduced exceptional divisor $f^{-1}(S)_{\mathrm{red}}$. Assume, for a form $\theta\in\Gamma(\tilde{X}-E,\mathcal{O}(mK_{\tilde{X}}))$, $D_\theta\cup E$ is of normal crossings, where D_θ is the closure of zeros of $\theta|_{\tilde{X}-E}$ on $\tilde{X}$. Then, for suitable $\{a_i\}_{i=1,\ldots,r}$ and $\{n_i\}_{i=1,\ldots,r}$ ($a_i, n_i\geqq 0$ for any i), $D_\theta+\sum_{i=1}^r n_iE_i$ is a member of a linear system $|m(K_{\tilde{X}}+\sum a_iE_i)|$. Therefore we can take the normal cyclic m-ple cover $\varphi: Y_\theta \to \tilde{X}$ associated to $D_\theta + \sum n_i E_i$. As θ gives an isomorphism $\mathcal{O}_{\tilde{X}} \tilde{\to} \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}} - D_\theta - \sum v_i(\theta) E_i)$, we have $\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}) \simeq \mathcal{O}_{\tilde{X}}(D_\theta + \sum v_i(\theta) E_i)$, where v_i is the valuation at E_i. Here, we remark that $(m, n_i) = (m, v_i(\theta))$, which we call d_i, and the covering φ is independent of the choice of $\{a_i\}$ and $\{n_i\}$.

Lemma 1.11. *Let $\varphi: Y_\theta \to \tilde{X}$ be the normal m-ple cyclic cover constructed by θ as above.*

Then, the $(m-1)$-th direct summand of $\varphi_* \mathcal{O}_{Y_\theta}(K_{Y_\theta})$ (which means the direct summand of $\varphi_* \mathcal{O}_{Y_\theta}(K_{Y_\theta})$ for $k = m-1$ in Lemma 1.9) is isomorphic to

$$\mathcal{O}_{\tilde{X}}\left(mK_{\tilde{X}} - \sum_{i=1}^{r} v_i'(\theta) E_i\right),$$

where $v_i'(\theta) = v_i(\theta) - \ulcorner v_i(\theta)/m \urcorner$, and $\ulcorner\ \urcorner$ means the round up.

Proof. This is an easy consequence of (iii) of Lemma 1.9.

Lemma 1.12. *The integer $v_i'(\theta)$ varies according to $v_i(\theta)$ as follows:*
(i) *if $0 < v_i(\theta)$, then $0 \leqq v_i'(\theta) < v_i(\theta)$;*
(ii) *if $-m+1 \leqq v_i(\theta) \leqq 0$, then $v_i'(\theta) = v_i(\theta)$; and*
(iii) *if $v_i(\theta) \leqq -m$, then $v_i(\theta) < v_i'(\theta) \leqq 0$.*

2. The $L^{2/m}$-Quotient Sheaf

We introduce the $L^{2/m}$-quotient sheaf on an analytic space in Sect. 1 (Definition 1.4). In this section, we will clarify the relation between the $L^{2/m}$-quotient sheaves on a deformation space and on a fiber.

Assume we are given a small deformation $\pi: (X, x) \to (C, 0)$ of a normal singularity (X_0, x), which is not necessarily isolated, with the singular locus S. Denote the relative dimension of π by n. Restricting π onto $X - S$, we have a smooth deformation $\pi': X - S \to C$. Since the normal bundle of each fiber is trivial, we have a canonical morphism;

$$\Gamma(X - S, \mathcal{O}(mK_X)) \xrightarrow{\gamma_\tau} \Gamma(X_\tau - S_\tau, \mathcal{O}(mK_{X_\tau})),$$

for each $\tau \in C$.

Then, we have a following theorem. The technique in the first part of the proof can be found in [1]. However, for readers' conveniene, we will write down a complete proof.

Theorem 2.1. *If $\gamma_0(\tilde{\theta})$ belongs to $L^{2/m}(X_0 - S_0)$ for an element $\tilde{\theta}$ of $\Gamma(X - S, \mathcal{O}(mK_X))$, then there exists an element $\tilde{\theta}'$ of $L^{2/m}(X - S)$ such that $\gamma_0(\tilde{\theta}) = \gamma_0(\tilde{\theta}')$.*

Proof. It is sufficient to show that, for a suitable good resolution $f: \tilde{X} \to X$ of π' if $\gamma_0(\tilde{\theta})$ belongs to $\Gamma([X_0], \mathcal{O}(mK_{[X_0]} + (m-1)E|_{[X_0]})$, there exists an element $\tilde{\theta}'$ of $\Gamma(\tilde{X}, \mathcal{O}(mK_{\tilde{X}} + (m-1)E))$ with $\gamma_0(\tilde{\theta}) = \gamma_0(\tilde{\theta}')$, where E is the reduced exceptional divisor on $\tilde{X}$.

Denote $\operatorname{Im} \gamma_0 \cap L^{2/m}(X_0 - S_0)$ by H.

For a general element $\tilde{\theta}$ of $\gamma_0^{-1}(H)\subset\Gamma(X-S,\mathcal{O}(mK_X))$, the zero-loci of $\tilde{\theta}$ on $X-S$ and of $\theta=\gamma_0(\tilde{\theta})$ on X_0-S_0 are both non-singular. Now, by taking a suitable good resolution $f:\tilde{X}\to X$ of π, we may suppose $D_{\tilde{\theta}}\cup E\cup[X_0]$ is of normal crossings. Then, according to (1.10) we can construct a normal cyclic m-ple cover $\varphi: Y_{\tilde{\theta}}\to\tilde{X}$ by $\tilde{\theta}$. Here, we note that the restriction $\varphi|_{Y_0}: Y_0=\varphi^{-1}([X_0])\to[X_0]$ is the normal cyclic m-ple cover associated to θ. Let F_v be the divisor $\varphi^*(E_v)$ where E_v is the vertical divisor on $\tilde{X}$. The exact sequence;

$$0\to\mathcal{O}(K_{Y_{\tilde{\theta}}})\xrightarrow{\times t}\mathcal{O}(K_{Y_{\tilde{\theta}}}-F_v)\to\mathcal{O}(K_{Y_0})\to 0$$

on $Y_{\tilde{\theta}}$ induces an exact sequence;

$$0\to\varphi_*\mathcal{O}(K_{Y_{\tilde{\theta}}})\to\varphi_*\mathcal{O}(K_{Y_{\tilde{\theta}}}-F_v)\to\varphi_*\mathcal{O}(K_{Y_0})\to 0$$

on $\tilde{X}$, where t is a local parameter on C at 0. Since, $Y_{\tilde{\theta}}$ has only rational singularities [(i), Lemma 1.9], $\mathcal{O}(K_{Y_{\tilde{\theta}}})=h_*\mathcal{O}(K_{\tilde{Y}})$, where $h:\tilde{Y}\to Y_{\tilde{\theta}}$ is a resolution of singularities on $Y_{\tilde{\theta}}$.

By the trivial edge sequence of Leray's spectral sequence:

$$E_2^{p,q}=H^p(Y_{\tilde{\theta}},R^qh_*\mathcal{O}(K_{\tilde{Y}}))\to H^{p+q}(\tilde{Y},\mathcal{O}(K_{\tilde{Y}}))\,,$$

we have an injection $H^1(Y_{\tilde{\theta}},\mathcal{O}(K_{Y_{\tilde{\theta}}}))\subset H^1(\tilde{Y},\mathcal{O}(K_{\tilde{Y}}))$.

As $H^1(\tilde{Y},\mathcal{O}(K_{\tilde{Y}}))=0$ by Grauert-Riemenschneider vanishing theorem, we have $H^1(Y_{\tilde{\theta}},\mathcal{O}(K_{Y_{\tilde{\theta}}}))=0$, which yields $H^1(\tilde{X},\varphi_*\mathcal{O}(K_{Y_{\tilde{\theta}}}))=0$. Therefore,

$$\Gamma(\tilde{X},\varphi_*\mathcal{O}(K_{Y_{\tilde{\theta}}}-F_v))\to\Gamma([X_0],\varphi_*\mathcal{O}(K_{Y_0}))\text{ is surjective}.\tag{1}$$

By taking the $(m-1)$-th direct summands of the surjective map (1), we have the surjectivity of the map which is the restriction of γ_0;

$$\Gamma(\tilde{X},\mathcal{O}(mK_{\tilde{X}}-\sum v_i'(\tilde{\theta})E_i-E_v)\to\Gamma([X_0],\mathcal{O}(mK_{[X_0]}-\sum v_{0i}'(\theta)E_i|_{[X_0]}))\,,\tag{2}$$

where v_i and v_{0i} are the valuation corresponding to the components E_i and $E_i|_{[X_0]}$, and $v_i'(\tilde{\theta})$ and $v_{0_i}'(\theta)$ are determined by $v_i(\tilde{\theta})$ and $v_{0_i}(\theta)$ respectively as in Lemma 1.11. We note that $v_i'(\tilde{\theta})>v_i(\tilde{\theta})$ if $v_i(\tilde{\theta})\leqq -m$, by 1.12.

Here, we will show that the second module in (2) coincides with

$$H=\operatorname{Im}\gamma_0\cap\Gamma([X_0],\mathcal{O}(mK_{[X_0]}+(m-1)E|_{[X_0]})\,.$$

By the surjectivity (2), it is clear that it is contained in $\operatorname{Im}\gamma_0$. Since θ is an element of

$$\Gamma([X_0],\mathcal{O}(mK_{[X_0]}+(m-1)E|_{[X_0]}))\,,$$

$$\Gamma([X_0],\mathcal{O}(mK_{[X_0]}-\sum v_{0i}'(\theta)E_i|_{[X_0]})\subset\Gamma([X_0],\mathcal{O}(mK_{[X_0]}+(m-1)E|_{[X_0]}))$$

by 1.11 and 1.12. Therefore,

$$\Gamma([X_0],\mathcal{O}(mK_{[X_0]}-\sum v_{0i}'(\theta)E_i|_{[X_0]}))\subset H\,.$$

On the other hand, by the generality of θ in H, v_{0i} takes a minimal value at θ among the elements of H for each i. Considering that θ belongs to

$$\Gamma([X_0],\mathcal{O}(mK_{[X_0]}-\sum v_{0i}'(\theta)E_i|_{[X_i]}))\,,$$

we have the opposite inclusion $\supset$ as desired. Now we have the surjectivity of $\Gamma(\tilde{X}, \mathcal{O}(mK_{\tilde{X}}-\sum a_iE_i))\to H$, where $a_i>v_i(\tilde{\theta})$ if $v_i(\tilde{\theta})\leqq -m$. Next, by taking a general element $\tilde{\theta}_1$ of $\Gamma(\tilde{X}, \mathcal{O}(mK_{\tilde{X}}-\sum a_iE_i))$ and a suitable blowing up $g_1: \tilde{X}^{(1)}\to\tilde{X}$ with the divisor $D_{\tilde{\theta}_1}\cup E^{(1)}\cup[X_0]^{(1)}$ of normal crossings, where $E^{(1)}$ is the reduced exceptional divisor on $\tilde{X}^{(1)}$ with respect to $f\circ g_1$ and $[X_0]^{(1)}$ is the proper transform of X_0 on $\tilde{X}^{(1)}$.

By practicing the same procedure with $\tilde{\theta}_1$ as above, we have a surjection

$$\Gamma(\tilde{X}, \mathcal{O}(mK_{\tilde{X}}-\sum a_i^{(1)}E_i^{(1)}))\to H,$$

where $a_i^{(1)}>v_i(\tilde{\theta}_1)$ if $v_i(\tilde{\theta}_1)\leqq -m$. By carrying out successively the same process as above, we finally get a good resolution $f\circ g_1g_2\dots g_s: \tilde{X}^{(s)}\to X$ and a surjective morphism $\Gamma(\tilde{X}^{(s)}, \mathcal{O}(mK_{\tilde{X}^{(s)}}-\sum \alpha_iE_i^{(s)}))\to H$ where if $E_i^{(s)}$ is the proper transform of an exceptional component on $\tilde{X}$, $\alpha_i\geqq -m+1$. By the bound of coefficients α_i of $E_i^{(s)}$ noted above, we have

$$\Gamma(\tilde{X}^{(s)}, \mathcal{O}(mK_{\tilde{X}^{(s)}}-\sum \alpha_iE_i^{(s)}))\subset\Gamma(\tilde{X}, \mathcal{O}(mK_{\tilde{X}}+(m-1)E)).$$

This completes the proof of our theorem.

Corollary 2.2. *There exists a surjective morphism from a subsheaf of $\Delta_m(X_0)$ onto $\Delta_m(X)\otimes\mathbb{C}(0)$.*

Proof. Let ι be the composite map $\mathcal{O}(mK_X)\otimes\mathbb{C}(0)\to\mathcal{O}(mK_{X_0})\to\Delta_m(X_0)$ and I_0 be the kernel of ι. Then we have a subsheaf $\mathcal{O}(mK_X)\otimes\mathbb{C}(0)/I_0$ of $\Delta_m(X_0)$. Here, by Theorem 2.1, I_0 is contained by the image of $f_*\mathcal{O}(mK_{\tilde{X}}+(m-1)E)$ in $\mathcal{O}(mK_X)\otimes\mathbb{C}(0)$. Therefore, we have a canonical surjection $\mathcal{O}(mK_X)\otimes\mathbb{C}(0)/I_0\to\Delta_m(X)\otimes\mathbb{C}(0)$.

3. Applications to Isolated Singularities

Let $\pi:(X,x)\to(C,0)$ be a small deformation of a normal singularity with the singular locus S. Denote the relative dimension of π by n. Fix a good resolution $f:\tilde{X}\to X$ of π. Here, we note that $\tilde{X}_\tau=[X_\tau]$ for $\tau\in C$ $\tau\neq 0$.

Lemma 3.1. (i) *The canonical map $\mathcal{O}(mK_X)\otimes\mathbb{C}(\tau)\xrightarrow{k_\tau}\mathcal{O}(mK_{X_\tau})$ of coherent sheaves on X_τ is injective for any $\tau\in C$ and, in particular, if $\pi|_S: S\to C$ is generically of finite fiber, k_τ is bijective for $\tau\neq 0$ near to 0.*

(ii) *If $\pi|_S: S\to C$ is generically of finite fiber, k_τ induces the isomorphism $\Delta_m(X)\otimes\mathbb{C}(\tau)\simeq\Delta_m(X_\tau)$ for $\tau\neq 0$ near to 0.*

Proof. Consider the following canonical exact sequence;

$$0\to\mathcal{O}_X(mK_X)\xrightarrow{\times(t-\tau)}\mathcal{O}_X(mK_X)\xrightarrow{\gamma_\tau}\mathcal{O}_{X_\tau}(mK_{X_\tau}),$$

where $t-\tau$ is a local parameter on C at 0.

Here, $\mathcal{O}_X(mK_X)\otimes\mathbb{C}(\tau)=\mathcal{O}_X(mK_X)/(t-\tau)\mathcal{O}_X(mK_X)$ is nothing but the image of γ_τ. This asserts the first statement of (i).

For the second statement of (i), we may assume that the restriction

$$\pi|_{S-S_0}: S-S_0\to C^*=C-\{0\}$$

is finite and the restriction $E|_{C^*}\to C^*$ of f is projective. Let $E'\subset\tilde{X}$ be a divisor supported on the exceptional divisor such that $\mathcal{O}(-E')$ is relatively very ample with respect to f.

Now, claim that, if we delete C suitably, there exists an integer $r_0>0$ such that, on C^*, $R^1f_*\mathcal{O}(mK_{\tilde{X}}+rE')$ is torsion free for any $r\geqq r_0$.

To simplify the notation, we denote the divisor $mK_{\tilde{X}}+rE'$ by L_r. Since the restriction $E'_*=E'|_{C^*}\to C^*$ of f is a projectively embeddable $C-M$ morphism, by the relative duality theorem (III, 11.2, (f) in [2]), we get

$$\mathrm{Ext}^i_{\mathcal{O}_{E'_*}}(\mathcal{O}_{E'_*}(-L_r+K_{E'_*}),\mathcal{O}(K_{E'_*}))\simeq\mathrm{Hom}_{\mathcal{O}_{C^*}}(R^{n-i}f_*\mathcal{O}_{E'_*}(-L_r+K_{E'_*}),\mathcal{O}_{C^*}).$$

As easily checked, the left hand side is isomorphic to $H^i(E'_*,\mathcal{O}_{E'_*}(L_r))$. Remarking that $\mathcal{O}_{E'_*}(-E')$ is relatively ample with respect to f, we get that there exists an integer r_0 such that, for any $r\geqq r_0$, $f_*\mathcal{O}_{E'_*}(L_r)=0$ on C^* and $R^1f_*\mathcal{O}_{E'_*}(L_r)$ is zero (resp. torsion free) if $n\geqq 3$ (resp. $n=2$). Here, we note that if $n=2$, $R^2f_*\mathcal{O}_{\tilde{X}}(L_r)=0$ on C^* for any r.

Now, in any case on the relative dimension n, we have an exact sequence on C^*;

$$0\to R^1f_*\mathcal{O}_{\tilde{X}}(L_r)\to R^1f_*\mathcal{O}_{\tilde{X}}(L_{r+1})\to R^1f_*\mathcal{O}_{E'}(L_{r+1})\to 0,$$

for any $r\geqq r_0$.

Therefore, $R^1f_*\mathcal{O}_{\tilde{X}}(L_r)$ is torsion free for each $r\geqq r_0$ at a point of C^*, if and only if $R^1f_*\mathcal{O}(L_{r_0})$ is torsion free at the point. Now we get the claim.

Here, we assume that C is deleted enoughly. Then, for a point $\tau\in C^*$ and $r\geqq r_0$, we have an exact sequence;

$$\begin{aligned}0\to f_*\mathcal{O}_{\tilde{X}}(L_r)&\xrightarrow{\times(t-\tau)} f_*\mathcal{O}_{\tilde{X}}(L_r)\xrightarrow{\gamma_\tau} f_*\mathcal{O}_{\tilde{X}_\tau}(L_r)\\ &\to R^1f_*\mathcal{O}_{\tilde{X}}(L_r)\xrightarrow{\psi_\tau} R^1f_*\mathcal{O}_{\tilde{X}}(L_r),\end{aligned}$$

where ψ_τ is the map which multiplies $(t-\tau)$ which is a local parameter on C at τ. Now ψ_τ is injective since $R^1f_*\mathcal{O}_{\tilde{X}}(L_r)$ is torsion free at τ. Therefore the map;

$$f_*(\mathcal{O}_{\tilde{X}}(L_r))\otimes\mathbb{C}(\tau)\to f_*\mathcal{O}_{\tilde{X}_\tau}(L_r)$$

is bijective for any integer $r\geqq r_0$.

Here, remarking that $\mathcal{O}_X(mK_X)\otimes\mathbb{C}(\tau)=f_*\mathcal{O}_{\tilde{X}}(L_r)\otimes\mathbb{C}(\tau)$ and $\mathcal{O}_{X_\tau}(mK_{X_\tau})=f_*\mathcal{O}_{\tilde{X}_\tau}(L_r)$ for sufficiently large r, we have an isomorphism

$$\mathcal{O}_X(mK_X)\otimes\mathbb{C}(\tau)\simeq\mathcal{O}_{X_\tau}(mK_{X_\tau})$$

as desired.

For the assertion (ii), we recall the diagram, which was shown in Corollary 2.2;

$$\Delta_m(X_\tau)\overset{j_\tau}{\supset}\mathcal{O}(mK_X)\otimes\mathbb{C}(\tau)/I_\tau\overset{i_\tau}{\twoheadrightarrow}\Delta_m(X)\otimes\mathbb{C}(\tau).$$

For $\tau\neq 0$, j_τ is surjective by (*i*) of this lemma. Besides, I_τ is just the image of $f_*(mK_{\tilde{X}}+(m-1)E)$, because $\tilde{X}_\tau=[X_\tau]$. This shows the bijectivity of i_τ.

Theorem 3.2 *(Upper semi-continuity of δ_m). Let $\pi:(X,x)\to(C,0)$ be a small deformation of a normal isolated singularity (X_0,x). Then,*

$$\delta_m(X_0,x)\geqq\sum_{\substack{y:\text{singular}\\ \text{point of }X_\tau}}\delta_m(X_\tau,y)\quad \textit{for } \tau\in C \textit{ near to } 0.$$

Proof. By the property of δ_m,

$$\delta_m(X_0, x) = \dim_{\mathbb{C}} \Delta_m(X_0) \quad \text{and} \quad \sum \delta_m(X_\tau, y) = \dim_{\mathbb{C}} \Delta_m(X_\tau).$$

Here, by using Corollary 2.2 and Lemma 3.1, we have

$$\dim_{\mathbb{C}} \Delta_m(X_0) \geqq \dim_{\mathbb{C}} \Delta_m(X) \otimes \mathbb{C}(0),$$

and

$$\dim_{\mathbb{C}} \Delta_m(X_\tau) = \dim_{\mathbb{C}} \Delta_m(X) \otimes \mathbb{C}(\tau) \quad \text{for } \tau \neq 0 \text{ near to } 0.$$

Considering a coherent sheaf Δ_m on a small neighbourhood of x, it is generated by its sections which generate $\Delta_m(X)\otimes\mathbb{C}(0)$. This means the inequality as desired.

Theorem 3.3. *Let (Z, x) be a normal isolated singularity with a hyperplane section (H, x) again a normal isolated singularity.*

Then, $\delta_m(H, x) \geqq \delta_m(Z, x)$ for every $m \in \mathbb{N}$.

Proof. Remember the family $X \to A^1_{\mathbb{C}}$ constructed in [7, Lemma 2.7] such that $X_0 \simeq H \times A^1_{\mathbb{C}}$ and $X_\tau \simeq Z$ for $\tau \neq 0$. By Lemma 3.1,

$$\delta_m(Z, x) = \delta_m(X_\tau, x) = \dim_{\mathbb{C}} \Delta_m(X) \otimes \mathbb{C}(\tau) \quad \text{for} \quad \tau \neq 0. \tag{1}$$

On the other hand, by Corollary 2.2, we have

$$\Delta_m(X_0) \supset (\mathcal{O}(mK_X) \otimes \mathbb{C}(0))/I_0 \twoheadrightarrow \Delta_m(X) \otimes \mathbb{C}(0). \tag{2}$$

Here, $\Delta_m(X_0) = \Delta_m(H) \otimes_{\mathbb{C}} \mathcal{O}_{A^1_{\mathbb{C}}}$ which is a free $\mathcal{O}_{A^1_{\mathbb{C}}}$-module of rank $\delta_m(H, x)$. Therefore, the subsheaf $(\mathcal{O}(mK_X)\otimes\mathbb{C}(0))/I_0$ is also a torsion free coherent sheaf on $A^1_{\mathbb{C}}$. Since a torsion free coherent sheaf on a non-singular curve is locally free, the subsheaf is locally free of rank r $(r \leqq \delta_m(H, x))$. Now, by the diagram (2), $\Delta_m(X)\otimes\mathbb{C}(0)$ is generated by at most r-sections. By applying Nakayama's Lemma to a coherent $\mathcal{O}_X$-module $\Delta_m(X)$, we have $\delta_m(H, x) \geqq r \geqq \delta_m(Z, x)$.

Now, we mention a consequence about 2-dimensional quotient singularities, which was already shown by Esnault and Viehweg [1].

Corollary 3.4. *Let $\pi: (X, x) \to (C, 0)$ be a small deformation of a quotient singularity (X_0, x) of dimension* 2.

Then the singularities on X_τ are all quotient singularities for $\tau \neq 0$ near to 0.

Proof. In the case of surface singularities, a quotient singularity is equivalent to $\delta_m = 0$ for all $m \in \mathbb{N}$, which is also equivalent to $\delta_r = 0$ for an integer $r > 0$ such that $\mathcal{O}(rK)$ is invertible.

Since (X_0, x) is a rational singularity, X_τ has only rational singularities by Elkik, which are $\mathbb{Q}$-Gorenstein by Artin.

Since, for general $\tau \in C$, the configurations of E_τ are the same, X_τ has $\mathbb{Q}$-Gorenstein singularities with invertible $\mathcal{O}(rK_{X_\tau})$ for a common r. Applying the upper semi-continuity of δ_r, we get X_τ has only quotient singularities for τ near to 0.

4. Deformations of Du Bois Singularities

Definition 4.1 [8]. A singularity (X, x) is called a Du Bois singularity if the complex $Gr^0_F \underline{\Omega}^{\cdot}_X$ is quasi-isomorphic to $\mathcal{O}_X$, at the point x, where $(\underline{\Omega}^{\cdot}_X, F)$ is the Du Bois' filtered complex.

Proposition 4.2 [4]. *The exceptional curve of a normal Du Bois singularity of dimension two is always of normal crossings.*

Proposition 4.3 [4]. *Let $f: \tilde{X} \to X$ be a good resolution of a normal singularity (X, x) of dimension two with $E = f^{-1}(x)_{\mathrm{red}}$ normal crossings. If, for any component E_i of E,*

$$E_i^2 < -2\left(E_i \sum_{j \neq i} E_j + \max\{g(E_i) - 1, 0\}\right),$$

then the singularity (X, x) is Du Bois.

Proposition 4.4. *Let $\pi: (X, x) \to (C, 0)$ be a small deformation of an isolated Gorenstein Du Bois singularity (X_0, x).*

Then X_τ is also Gorenstein Du Bois for $\tau \in C$ near to 0.

Proof. It is well known that the singularities on X_τ are all Gorenstein $\tau \in C$ near to 0 and (X, x) also turns out to be a Gorenstein singularity. Now, by the characterization of isolated Gorenstein singularity ([3]), $\delta_m(X_0, x) \leqq 1$ for all $m \in \mathbb{N}$. Therefore $\dim_{\mathbb{C}} \Delta_m(X) \otimes \mathbb{C}(0) \leqq 1$ by Corollary 2.2. This means the singularity (X, x) is log-canonical. So, we obtain $K_{\tilde{X}} = f^* K_X + \sum n_i E_i$ $(n_i \geqq -1)$ for a good resolution $f: \tilde{X} \to X$ of π. Remarking that $f|_{\tilde{X}_\tau}: \tilde{X}_\tau \to X_\tau$ is a good resolution for $\tau \neq 0$, we have $K_{\tilde{X}_\tau} = f^* K_{X_\tau} + \sum n_i E_i|_{\tilde{X}_\tau}$ $(n_i \geqq -1)$ which means X_τ has only Gorenstein Du Bois singularities by [3].

Here, one may expect the similar assertion without the Gorenstein condition. Unfortunately, however, it is not true, as we will see below.

Theorem 4.5. *There exists a small deformation $\pi: (X, x) \to (C, 0)$ of a normal isolated Du Bois singularity (X_0, x), such that X_τ is not Du Bois for any $\tau \in C$ $\tau \neq 0$.*

Proof. We will construct an example of the case the relative dimension n of π is 2.

First, let

$$\begin{tikzcd} S' \arrow[r] \arrow[dr, "p'"'] & \mathbb{P}^2 \times \mathbb{P}^1 \arrow[d, "p_2"] \\ & \mathbb{P}^1 \end{tikzcd}$$

be a family of cubic curves defined by $z^3 t_0 + z x^2 t_1 + y^3 t_1 = 0$, where $(x; y; z)$ and $(t_0; t_1)$ are homogeneous coordinates of $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively. Then, if we put $0 = (0; 1) \in \mathbb{P}^1$, the fiber $S_0' = p'^{-1}(0)$ is a rational curve with a cusp at $(0; 0; 1) \times (0; 1)$ and $S_t' = p'^{-1}(t)$ is non-singular elliptic curve for general t.

Next, take a divisor H on S' defined by

$$t_0 Q(x; y; z, t_0; t_1) + z t_1^n = 0,$$

where Q is homogeneous of degree 1 in x, y, z and of degree n in t_0, t_1 for sufficiently large n. By taking Q sufficiently general, we may assume H is non-singular outside of $P = (1; 0; 0) \times (0; 1)$ and $S_0' \cap H = 3P$. Now, let $b: S \to S'$ be the blowing up at P, ℓ be the exceptional curve and p be the composition $p' \circ b: S \to \mathbb{P}^1$. Then the proper transform $e = [S_0']$ has the self-intersection number -1. Therefore, the curve e can be contracted to a normal isolated singularity, which is not Du Bois by Proposition 4.2. By the way, we note that the proper transform $[H]$ of H intersects ℓ at three points and does not intersect e.

On the next step, we will construct a family of surfaces by using S. Let $g: V \to S \times \mathbb{C}$ be a blowing up at $[H] \times \{0\}$ and $\pi: V \to \mathbb{C}$ be the canonical projection. Then, the fiber $V_0 = \pi^{-1}(0)$ is the sum $[S \times \{0\}] + F$, where $[S \times \{0\}]$ is the proper transform of a divisor $S \times \{0\}$ and F is the exceptional divisor isomorphic to a $\mathbb{P}^1$-bundle on $[H]$. Put $E = [e \times \mathbb{C}] + [S \times \{0\}]$. Since n was chosen enoughly large and $[H]$ intersects ℓ at three points, we can easily show that the conormal bundle $\mathcal{N}^*_{E/V}$ is relatively ample with respect to π. Therefore E is contracted to a variety V' which is a family of varieties with isolated singularities over a parameter space $\mathbb{C}$. For, $t \neq 0$ the singularity on V'_t is not Du Bois because it is a normal singularity obtained by contracting a curve with a cusp. On the other hand, the singularity on V'_0 is a normal singularity obtained by contracting the minimal section Z on $\mathbb{P}^1$-bundle F (although the normality of V'_0 is not trivial, a careful calculation of the graded ring of V'_0 makes it clear). Here, by calculating Z^2 and $g(Z)$, we have $Z^2 < -2g(Z) + 2$, which implies that the singularity V'_0 is Du Bois by Proposition 4.3.

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Congruences for Eigenvalues of Hecke Operators on Siegel Modular Forms of Degree Two

Shin-ichiro Mizumoto

Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo, 152 Japan

Introduction

Let

$$f(z)=\sum_{n=1}^{\infty} a(n)\exp(2\pi inz)$$

be a normalized cuspidal Hecke eigenform of integral weight k with respect to $SL(2,\mathbf{Z})$. For $\mathrm{Re}(s)>k$, let

$$L_2(s,f)=\prod_{p:\text{ prime}}(1-\alpha_p^2p^{-s})^{-1}(1-\alpha_p\beta_pp^{-s})^{-1}(1-\beta_p^2p^{-s})^{-1}$$

be the symmetric square zeta function associated with f. Here complex numbers α_p, β_p are taken so that $\alpha_p+\beta_p=a(p)$ and $\alpha_p\beta_p=p^{k-1}$. We put

$$L_2^*(s,f)=L_2(s,f)(2\pi)^{-(2s-k+2)}\Gamma(s)\langle f,f\rangle^{-1},$$

where $\langle\,,\rangle$ is the normalized Petersson inner product (cf. Notation 2° below). By Shimura [19], Zagier [22], and the author [17], $L_2(s,f)$ has a holomorphic continuation to the whole s-plane. By Sturm [20], Zagier [22], and [17], $L_2^*(m,f)$ for each even integer m with $k\leqq m\leqq 2k-2$ belongs to the totally real algebraic number field $\mathbf{Q}(a(n)|n\geqq 1)$.

Now, let $[f]$ be the Eisenstein series associated with f with respect to $Sp(2,\mathbf{Z})$ in the sense of Langlands [13] and Klingen [6]. This $[f]$ is an eigenform of weight k satisfying $\Phi([f])=f$ where Φ is the Siegel operator; conversely, $[f]$ is characterized by these properties [11]. A Siegel modular form is called an eigenform if it is a non-zero common eigenfunction of all Hecke operators.

In [9], Kurokawa raised the following conjecture (after some numerical data): For a certain prime $\mathfrak{l}$ in $\bar{\mathbf{Q}}$ dividing the numerator of $L_2^*(2k-2,f)$, there will exist an eigen cusp form F of weight k and degree 2 satisfying

$$\lambda(m,F)\equiv\lambda(m,[f]) \bmod \mathfrak{l}$$

for all positive integers m. Here $\lambda(m,F)$ (or $\lambda(m,[f])$) is the eigenvalue of the m-th Hecke operator $T_k(m)$ [1] on the eigenform F (or $[f]$). We note that the numerator of $L_2^*(2k-2,f)$ is equal to the common denominator of the Fourier coefficients of $[f]$ up to elementary factors [16, Theorem 4].

In this paper we show that this conjecture is true under some conditions. For the precise statement, see Sect. 1 below. From our result, for example, it follows that:

$$\lambda(m,\chi_{22}^{(4)}) \equiv \lambda(m,[\Delta_{22}]) \bmod 61 \cdot 103 \quad \text{for all} \quad m \geqq 1 .$$

Here Δ_{22} is the normalized elliptic cusp form of weight 22, and $\chi_{22}^{(4)}$ is a cuspidal Siegel eigenform of degree 2 and weight 22 which does not belong to the Maass space; such $\chi_{22}^{(4)}$ is uniquely determined up to a constant factor. For other numerical examples, see Sect. 3 below.

But our aim here is to show that this kind of phenomenon actually occurs for every weight k (under a multiplicity one condition), and to see how these congruences arise. Integrality properties of algebraic numbers associated with modular forms as were treated in [16] play important roles in several places of our proof. So this paper may be considered as a continuation of [16].

The author wishes to express his hearty thanks to Professor N. Kurokawa for encouragement, and to Takakazu Satoh who calculated some numerical values of Fourier coefficients of $\varphi_4^4\varphi_6-\varphi_4\varphi_6^3$ which were necessary to find the value of $a\left(\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, [\Delta_{22}]\right)$ in Sect. 3, Example (4).

Notation

1

$\mathbf{Z}$ is the ring of rational integers, $\mathbf{Q}$ the field of rational numbers, $\mathbf{R}$ the field of real numbers, $\mathbf{C}$ the field of complex numbers, $\bar{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, and $\bar{\mathbf{Z}}$ the ring of algebraic integers in $\bar{\mathbf{Q}}$.

2

For a positive integer n, $\Gamma_n = \mathrm{Sp}(n,\mathbf{Z})$ is the Siegel modular group of degree n and $\mathfrak{H}_n$ the Siegel upper half space of degree n. For a non-negative integer k, $M_k(\Gamma_n)$ (or $S_k(\Gamma_n)$) is the $\mathbf{C}$-vector space consisting of Siegel modular (or cusp) forms of degree n and weight k. Each $F \in M_k(\Gamma_n)$ has a Fourier expansion of the form: $F = \sum_{T \geqq 0} a(T,F) q^T$, where $q^T = \exp(2\pi i \,\mathrm{trace}(TZ))$ with a variable Z on $\mathfrak{H}_n$, and T runs over all symmetric positive semi-definite semi-integral matrices of size n. For each subring R of $\mathbf{C}$, we put

$$M_k(\Gamma_n)_R = \{F \in M_k(\Gamma_n) | a(T,F) \in R \text{ for all } T \geqq 0\},$$

and $S_k(\Gamma_n)_R = S_k(\Gamma_n) \cap M_k(\Gamma_n)_R$ (R-modules). The algebra of Hecke operators over R (acting on $M_k(\Gamma_n)$) is denoted by $\mathbf{T}_R$. For $F, G \in M_k(\Gamma_n)$ such that FG is a cusp form, we put

$$\langle F, G\rangle = \mathrm{vol}(\Gamma_n \backslash \mathfrak{H}_n)^{-1} \int_{\Gamma_n \backslash \mathfrak{H}_n} F(Z)\overline{G(Z)} |Y|^k \frac{dX\,dY}{|Y|^{n+1}}$$

(the Petersson inner product), where $Z = X + iY$ with real matrices $X = (x_{lm})$, $Y = (y_{lm})$; $dX = \prod_{l \leqq m} dx_{lm}$, $dY = \prod_{l \leqq m} dy_{lm}$, and $|Y|$ is the determinant of Y.

3

For an eigenform $f \in M_k(\Gamma_n)$, the eigenvalue of $T \in \mathbf{T_C}$ on f is denoted by $\lambda(T, f)$, i.e., $Tf = \lambda(T, f)f$. If $T = T_k(m)$, the m-th Hecke operator [1], then we write simply $\lambda(T_k(m), f) = \lambda(m, f)$. For an algebraic number field K, we put

$$K(f) = K(\lambda(T, f) | T \in \mathbf{T_Q}).$$

For eigenforms $f_1, \ldots, f_r \in M_k(\Gamma_n)$, we denote by $K(f_1, \ldots, f_r)$ the composite of the fields $K(f_1), \ldots, K(f_r)$.

1. Statement of Results

Let n and k be positive integers such that $S_k(\Gamma_n) \neq \{0\}$. We consider the following multiplicity one condition.

$$\begin{aligned}(\mathrm{MOC})_{n,k}\colon\quad &\dim_{\mathbf{C}}\{F \in S_k(\Gamma_n) | TF = \lambda(T, f)F \text{ for all } T \in \mathbf{T_C}\} = 1\\ &\text{for each eigenform} \quad f \in S_k(\Gamma_n).\end{aligned}$$

This condition is satisfied at least for $k \leqq 22$ if $n = 2$, and for all k if $n = 1$.

Concerning $\mathbf{Q}(f)$ and $\lambda(T, f)$ for an eigenform f, the following facts are known:

Proposition 1 [10]. *Let k be an even integer and $f \in S_k(\Gamma_n)$ be an eigenform.*

(1) *The field $\mathbf{Q}(f)$ is a totally real finite extension of $\mathbf{Q}$.*

(2) *If $(\mathrm{MOC})_{n,k}$ holds, then there exists a non-zero constant $\gamma \in \mathbf{C}$ such that all the Fourier coefficients of γf belong to $\mathbf{Q}(f)$.*

(3) *If $n \leqq 2$, then $\lambda(T, f)$ is an algebraic integer for each $T \in \mathbf{T_Z}$.*

Let K be an algebraic number field with integer ring $\mathcal{O} = \mathcal{O}_K$. For an eigenform $f \in S_k(\Gamma_n)$ of even weight k, let $\mathcal{O}(f)$ be the integer ring of $K(f)$ and let $\kappa(f, K)$ be the exponent of the finite abelian group $\mathcal{O}(f)/\mathcal{O}[\lambda(T, f) | T \in \mathbf{T_Z}]$.

If k is even and $(\mathrm{MOC})_{n,k}$ holds, we define the integer $\nu(f, K)$ as follows: Let V be the $\mathbf{T}_K$-irreducible K-subspace of $S_k(\Gamma_n)_K$ such that $V \otimes_K \mathbf{C} \ni f$. Such V exists and is uniquely determined by f under $(\mathrm{MOC})_{n,k}$, cf. Proposition 2 below. We put $V_{\mathcal{O}} = V \cap S_k(\Gamma_n)_{\mathcal{O}}$ and $V_{\mathcal{O}}^{\perp} = V^{\perp} \cap S_k(\Gamma_n)_{\mathcal{O}}$ where $V^{\perp}$ is the orthogonal complement of V in $S_k(\Gamma_n)$ with respect to the Petersson inner product. Since $S_k(\Gamma_n)_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C} = S_k(\Gamma_n)$ by [2, 4, 5], we see that $S_k(\Gamma_n)_{\mathcal{O}}/(V_{\mathcal{O}} \oplus V_{\mathcal{O}}^{\perp})$ is a finite abelian group, the exponent of which we denote by $\nu(f, K)$. If $K = \mathbf{Q}$, we write $\kappa(f, \mathbf{Q}) = \kappa(f)$ and $\nu(f, \mathbf{Q}) = \nu(f)$.

In the above procedure to define $\nu(f, K)$ we used the former part of the following

Proposition 2. *Let k be an even integer, $f \in S_k(\Gamma_n)$ an eigenform, and K an algebraic number field. Suppose that $(\mathrm{MOC})_{n,k}$ holds. Then:*

(1) *There exists a $\mathbf{T}_K$-irreducible K-subspace V of $S_k(\Gamma_n)_K$ such that $V \otimes_K \mathbf{C} \ni f$. Such V is uniquely determined by f.*

(2) *Let V be as above. Suppose f has a Fourier coefficient equal to 1. Then:*

$$V \otimes_K \mathbf{C} = \bigoplus_{\sigma} \mathbf{C} f^{\sigma},$$

where σ runs over all the embeddings of $K(f)$ into $\mathbf{C}$ over K.

Here f^σ is defined as follows: Under the assumption of (2), the Fourier coefficients of f belong to $\mathbf{Q}(f)$ by Proposition 1(2). So writing $f=\sum_{T>0} a(T,f)q^T$, we put $f^\sigma=\sum_{T>0} a(T,f)^\sigma q^T$. The proof of Proposition 2 will be given in Sect. 2.

Now we state our main result. Let k be an even integer such that $(\mathrm{MOC})_{2,k}$ holds. Let $f\in S_k(\Gamma_1)$ be a normalized eigenform (hence $k\geqq 12$ and $k\neq 14$), and let $\{f_i|i=1,\dots,r\}$ (resp. $\{F_j|j=1,\dots,s\}$) be an eigen basis of $S_k(\Gamma_1)$ (resp. of $S_k(\Gamma_2)$) with $f_1=f$. We put

$$c_k(f)=6\,\mathrm{Num}\left(\frac{B_kB_{2k-2}}{k(2k-2)}\right)\cdot\prod_{i=1}^{r}\kappa(f_i)\nu(f_i)d(\mathbf{Q}(f_i))$$
$$\times\prod_{j=1}^{s}\kappa(F_j,\mathbf{Q}(f))\nu(F_j,\mathbf{Q}(f))d(\mathbf{Q}(f,F_j)/\mathbf{Q}(f)).$$

Here $B_k=-k\zeta(1-k)$ is the k-th Bernoulli number, $\mathrm{Num}(*)$ denotes the reduced positive numerator of the non-zero rational number $*$, $d(\mathbf{Q}(f_i))$ is the discriminant of $\mathbf{Q}(f_i)$, and $d(\mathbf{Q}(f,F_j)/\mathbf{Q}(f))$ is the relative discriminant of $\mathbf{Q}(f,F_j)$ over $\mathbf{Q}(f)$.

As in the introduction, $[\ ]: M_k(\Gamma_1)\to M_k(\Gamma_2)$ is the Eisenstein lifting. We put $E=\mathbf{Q}(f_1,\dots,f_r)$ and $\mathcal{O}_E=E\cap\bar{\mathbf{Z}}$.

Theorem. *Let k be an even integer such that: $k\geqq 12$, $k\neq 14$, and $(\mathrm{MOC})_{2,k}$ holds. Let $f\in S_k(\Gamma_1)$ be a normalized eigenform. Suppose a prime $\mathfrak{p}$ in $\mathbf{Q}(f)$ satisfies the following conditions (i)–(iii):*

(i) *The prime $\mathfrak{p}$ does not divide $c_k(f)$.*

(ii) *There exists a positive integer α such that: The Fourier coefficients of $[f]$ belong to $\mathfrak{p}^{-\alpha}\mathbf{Z}(f)$ and $\mathrm{ord}_{\mathfrak{p}}(a(T_0,[f]))=-\alpha$ for some T_0; here $\mathrm{ord}_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-order.*

(iii) *For each $i\geqq 2$, the Fourier coefficients of $[f_i]$ are $\mathfrak{P}_j$-integral for $j=1,\dots,g$. Here $\mathfrak{p}\mathcal{O}_E=\mathfrak{P}_1\dots\mathfrak{P}_g$ is the prime decomposition of $\mathfrak{p}$ in E.*

Then, there exists an eigenform $F\in S_k(\Gamma_2)$ such that

$$N_{\mathbf{Q}(f,F)/\mathbf{Q}(f)}(\lambda(m,F)-\lambda(m,[f]))\equiv 0\ (\mathrm{mod}\,\mathfrak{p}^\alpha)\quad \textit{for all}\quad m\geqq 1\,.$$

Here $N_{\mathbf{Q}(f,F)/\mathbf{Q}(f)}$ denotes the norm map with respect to $\mathbf{Q}(f,F)/\mathbf{Q}(f)$.

The proof of Theorem will be given in Sect. 2, and some numerical examples in Sect. 3.

Remarks. (1) Concerning the condition (ii) on $\mathfrak{p}$, we note that the Fourier coefficients of $[f]$ always belong to $\mathbf{Q}(f)$ and their denominators are bounded, cf. [12, 15]. Moreover, $a(T,[f])\in\mathbf{Z}(f)$ if $|T|=0$, so the above T_0 should necessarily be positive definite.

(2) The condition (ii) on $\mathfrak{p}$ is "almost equivalent" to $\mathrm{ord}_{\mathfrak{p}}(L_2^*(2k-2,f))=\alpha$ by [16, Theorem 4], as we have noted in the introduction.

(3) In (iii), observe that $\mathfrak{p}$ is unramified in E by the condition (i).

(4) By a slight modification of the proof of Theorem, we obtain the congruences between the eigenvalues of Hecke operators on φ_k and an eigenform in $S_k(\Gamma_2)$ modulo each prime factor of $\mathrm{Num}\left(\dfrac{B_{2k-2}}{2k-2}\right)$. Here φ_k is the Siegel's

Eisenstein series in $M_k(\Gamma_2)$. But these congruences are proved more naturally by the "lifting of congruences" as in [9].

(5) On the integer $v(f, K)$ we note the following. Let n be a positive integer and k be an even integer such that $S_k(\Gamma_n) \neq \{0\}$. Suppose that $(\mathrm{MOC})_{n,k}$ holds. As above, $v(f, K)$ is defined for each eigenform $f \in S_k(\Gamma_n)$ and an algebraic number field K with $\mathcal{O} = K \cap \bar{\mathbf{Z}}$. Then there exist non-zero $h_i \in S_k(\Gamma_n)_{\mathcal{O}}$ $(i=1,2)$ satisfying the following conditions (a)–(c):

(a) $\langle h_1, h_2 \rangle = 0$.

(b) $a(T, h_1) \equiv a(T, h_2) \bmod v(f, K)$ for all $T > 0$.

(c) For each divisor d of $v(f, K)$ such that $1 < d \in \mathbf{Z}$, at least one of $d^{-1}h_i$ $(i=1,2)$ does not belong to $S_k(\Gamma_n)_{\mathcal{O}}$.

The existence of such h_i $(i=1,2)$ follows directly from the definition of $v(f, K)$. For example, a congruence of the above type modulo $2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ occurs in $S_{20}(\Gamma_2)_{\mathbf{Z}}$, cf. [16, Example 2].

2. Proofs

2.1. *Proof of Proposition 2*

(1) The representation of $\mathbf{T}_K$ on $S_k(\Gamma_n)_K$ is completely reducible, so

$$S_k(\Gamma_n)_K = \bigoplus_l W_l$$

with $\mathbf{T}_K$-irreducible K-subspaces W_l. By $\mathbf{T}_{\mathbf{C}} = \mathbf{T}_K \otimes_K \mathbf{C}$, we see that $W_l \otimes_K \mathbf{C}$ is $\mathbf{T}_{\mathbf{C}}$-invariant. So by the commutativity of $\mathbf{T}_{\mathbf{C}}$, there exists an eigenbasis $\{f_{l,1}, \ldots, f_{l,n_l}\}$ of $W_l \otimes_K \mathbf{C}$ over $\mathbf{C}$. By $(\mathrm{MOC})_{n,k}$, for some l and t $(1 \leqq t \leqq n_l)$ we have $f = c f_{l,t}$ with a non-zero constant $c \in \mathbf{C}$. The space $W_l = V$ satisfies the required conditions. The uniqueness follows from the assertion (2), since V is the fixed subspace of $V \otimes_K \mathbf{C}$ under the action of $\mathrm{Aut}(\mathbf{C}/K)$, the group of all ring automorphisms of $\mathbf{C}$ over K, $\Rightarrow$. Namely,

$$V = (V \otimes_K \mathbf{C})^{\mathrm{Aut}(\mathbf{C}/K)}.$$

(2) For each embedding $\sigma : K(f) \to \mathbf{C}$ over K, we have $V \otimes_K \mathbf{C} \ni f^\sigma$. Observing the Fourier coefficient at T_1 where $a(T_1, f) = 1$, we see that $g = \sum_\sigma f^\sigma \neq 0$, where σ runs over all such embeddings. So

$$0 \neq g \in (V \otimes_K \mathbf{C})^{\mathrm{Aut}(\mathbf{C}/K)} = V.$$

Since V is $\mathbf{T}_K$-irreducible, $V = \sum_i K(T_i g)$ for some $T_i \in \mathbf{T}_K$. Thus

$$V \otimes_K \mathbf{C} = \sum_i \sum_\sigma \mathbf{C}(T_i f^\sigma) \subset \sum_\sigma \mathbf{C} f^\sigma$$

since each f is an eigenform. This proves the assertion (2).

2.2. *Proof of Theorem*

Let

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{d \mid n \\ d > 0}} d^{k-1} \Bigg) q^n \in M_k(\Gamma_1)$$

be the Eisenstein series of degree 1, and let $\varphi_k \in M_k(\Gamma_2)$ be the Eisenstein series of degree 2 normalized so that $\Phi(\varphi_k)=E_k$.

Throughout the proof, the notation is the same as in the statement of Theorem. First we collect some preliminary facts.

Lemma 1. *Let R be a subring of $\mathbf{C}$ such that $\frac{1}{6}\in R$. We set*

$$\{(a(i),b(i))\in \mathbf{Z}^2 | i=0,\dots,r\}=\{(a,b)\in\mathbf{Z}^2 | a\geqq 0,\ b\geqq 0,\ 4a+6b=k\}.$$

Then we have:

$$M_k(\Gamma_1)_R=\bigoplus_{i=0}^{r} RE_4^{a(i)}E_6^{b(i)}.$$

Using the well-known fact for the case $R=\mathbf{C}$, one easily checks this assertion by induction with respect to k.

Lemma 2. *Let $\mathcal{O}_E$ be the integer ring of the number field $E=\mathbf{Q}(f_1,\dots f_r)$, and let $\mathfrak{p}\cap\mathbf{Z}=p\mathbf{Z}$ with $p>0$. Put $S=\{x\in\mathcal{O}_E | p\mathcal{O}_E+x\mathcal{O}_E=\mathcal{O}_E\}$. We denote by $\mathbf{Z}_{(p)}$ the localization of $\mathbf{Z}$ at $p\mathbf{Z}$. Then, under the assumption (i) on $\mathfrak{p}$, we have:*

$$M_k(\Gamma_1)_{\mathbf{Z}_{(p)}}\subset(S^{-1}\mathcal{O}_E\cdot E_k)\oplus\bigoplus_{i=1}^{r}(S^{-1}\mathcal{O}_E\cdot f_i),$$

where $S^{-1}\mathcal{O}_E$ is the localization of $\mathcal{O}_E$ with respect to S.

Proof. For $g\in M_k(\Gamma_1)_{\mathbf{Z}_{(p)}}$, put $g^*=g-a(0,g)E_k$. By the assumption (i) on $\mathfrak{p}$, we have $g^*\in S_k(\Gamma_1)_{\mathbf{Z}_{(p)}}$. From [16, Theorem 1(1)] (cf. Lemma 3 below),

$$g^*=\sum_{i=1}^{r} c_i f_i \quad \text{with} \quad c_i\in S^{-1}\mathcal{O}_E,$$

again by the assumption (i).

The following integrality result will frequently be used:

Lemma 3. *Suppose $n=1$ or 2. Let $k>0$ be an even integer such that $S_k(\Gamma_n)\neq\{0\}$ and $(\mathrm{MOC})_{n,k}$ holds. Let $F\in S_k(\Gamma_n)$ be an eigenform with a Fourier coefficient equal to 1. Let L be an algebraic number field with integer ring $\mathcal{O}_L$. Then, for each $G\in S_k(\Gamma_n)_{\mathcal{O}_L}$ we have:*

$$\frac{\langle G,F\rangle}{\langle F,F\rangle}\in(\kappa(F,L)\nu(F,L)\mathcal{D}(L(F)/L))^{-1}\mathcal{O}_L(F),$$

where $\mathcal{D}(L(F)/L)$ is the different of $L(F)/L$.

Proof is entirely similar to that of [16, Theorem 2] where the case $L=\mathbf{Q}$ is treated.

The following lemma is the key to the proof of Theorem.

Lemma 4. *Let k, f, $\mathfrak{p}$ be as in the statement of Theorem. Then there exists a modular form G^* in $M_k(\Gamma_2)$ satisfying the following conditions (a)–(c):*

(a) $G^*\in\mathbf{C}[f]\oplus S_k(\Gamma_2)$.

(b) *The Fourier coefficients of G^* belong to $\mathbf{Z}(f)_{\mathfrak{p}}$, the localization of $\mathbf{Z}(f)$ at $\mathfrak{p}$.*

(c) *In the Fourier expansion $\Phi G^*=\sum_{n=1}^{\infty}c(n)q^n$, the first coefficient $c(1)$ belongs to $\mathbf{Z}(f)_{\mathfrak{p}}^{\times}$. Here $\mathbf{Z}(f)_{\mathfrak{p}}^{\times}$ is the unit group of $\mathbf{Z}(f)_{\mathfrak{p}}$.*

Proof. In the notation of Lemma 2, we put $S^{-1}\mathcal{O}_E=R$. Since $p\geqq 5$, each of E_k, $f_1,\dots,f_r$ is an R-linear combination of $E_4^{a(i)}E_6^{b(i)}$ $(i=0,\dots,r)$ by Lemma 1. Moreover, by Lemma 2, each of $E_4^{a(i)}E_6^{b(i)}$ $(i=0,\dots,r)$ is an R-linear combination of E_k, $f_1,\dots,f_r$. So

$$(E_k,f_1,\dots,f_r)=(E_4^{a(0)}E_6^{b(0)},\dots,E_4^{a(r)}E_6^{b(r)})A \quad \text{with some} \quad A\in \mathrm{GL}(r+1,R). \tag{2.1}$$

In the expression

$$E_4^{a(i)}E_6^{b(i)}=E_k+\sum_{j=1}^{r} d_{ij}f_j$$

with $d_{ij}\in R$, observe that

$$d_{i1}\in \mathbf{Z}(f)_{\mathfrak{p}} \qquad (i=0,\dots,r)$$

by Lemma 3 and the assumption (i). From (2.1) it follows that d_{i1} is a $\mathfrak{p}$-unit for some i. Without loss of generality we suppose $d_{01}\in \mathbf{Z}(f)_{\mathfrak{p}}^{\times}$. Put

$$G^*=\varphi_4^{a(0)}\varphi_6^{b(0)}-\varphi_k-\sum_{j=2}^{r} d_{0j}[f_j].$$

We show that this G^* is the desired one. By

$$\Phi G^*=E_4^{a(0)}E_6^{b(0)}-E_k-\sum_{j=2}^{r} d_{0j}f_j=d_{01}f,$$

we see

$$G^*\in \mathbf{C}[f]\oplus S_k(\Gamma_2),$$

and the first Fourier coefficient of ΦG^* is $d_{01}\in \mathbf{Z}(f)_{\mathfrak{p}}^{\times}$. So it remains to show that the above G^* satisfies the condition (b). Let

$$S_k(\Gamma_1)_{\mathbf{Q}}=\bigoplus_l W_l$$

be the decomposition into $\mathbf{T}_{\mathbf{Q}}$-irreducible $\mathbf{Q}$-subspaces. By Proposition 2, for each l there exists a normalized eigenform $g_l\in W_l\otimes_{\mathbf{Q}}\mathbf{C}$ and the following decomposition holds:

$$W_l\otimes_{\mathbf{Q}}\mathbf{C}=\bigoplus_{\sigma}\mathbf{C}g_l^{\sigma},$$

where σ runs over all embeddings of $\mathbf{Q}(g_l)$ into $\mathbf{R}$. We may suppose $f=g_1$. According to this decomposition, we write

$$E_4^{a(0)}E_6^{b(0)}-E_k=\sum_l\sum_{\sigma}\varrho_l(\sigma)g_l^{\sigma} \quad \text{with} \quad \varrho_l(\sigma)\in\mathbf{C}. \tag{2.2}$$

As in the proof of Theorem 1(1) of [16], $\varrho_l(\sigma)=\varrho_l^{\sigma}$ with some $\varrho_l\in\mathbf{Q}(g_l)$. By (2.2),

$$H=\varphi_4^{a(0)}\varphi_6^{b(0)}-\varphi_k-\sum_l\sum_{\sigma}\varrho_l^{\sigma}[g_l]^{\sigma}\in S_k(\Gamma_2)_{\mathbf{Q}}$$

since the Eisenstein lifting maps $S_k(\Gamma_1)_{\mathbf{Q}}$ into $M_k(\Gamma_2)_{\mathbf{Q}}$ [14, 15]. Noting that $\{\varrho_l^{\sigma}\}_{l,\sigma}=\{d_{0j}|j=1,\dots,r\}$, we have

$$G^*=H+\varrho_1[g_1],$$

so the Fourier coefficients of G^* belong to $\mathbf{Q}(g_1)=\mathbf{Q}(f)$. Since $d_{0j}\in R$, the Fourier coefficients of G^* belong to $\mathbf{Z}(f)_{\mathfrak{p}}$ by the assumptions (i) and (iii) on $\mathfrak{p}$. This completes the proof of Lemma 4.

Let G^* be as in Lemma 4 and put $G=d_{01}^{-1}G^*$ where $d_{01}\in\mathbf{Z}(f)_{\mathfrak{p}}^{\times}$ is the same as in the above proof of Lemma 4. Then the Fourier coefficients of G belong to $\mathbf{Z}(f)_{\mathfrak{p}}$ and $\Phi G=f$. So

$$G=[f]+\sum_{j=1}^{s}c_jF_j \tag{2.3}$$

with $c_j\in\mathbf{C}$ where $\{F_1,\dots,F_s\}$ is an eigenbasis of $S_k(\Gamma_2)$. Let T_0 be as in the condition (ii) on $\mathfrak{p}$. Without loss of generality we may assume

$$a(T_0,F_j)=\begin{cases}1 & \text{for}\quad 1\leqq j\leqq s_0,\\ 0 & \text{for}\quad s_0<j\leqq s.\end{cases}$$

Observe that $s_0\geqq 1$, since $a(T_0,G)\in\mathbf{Z}(f)_{\mathfrak{p}}$ and $a(T_0,[f])\notin\mathbf{Z}(f)_{\mathfrak{p}}$ in (2.3). Moreover we may assume each of F_j $(s_0<j\leqq s)$ has a Fourier coefficient equal to 1. By Proposition 1, the Fourier coefficients of F_j lie in $\mathbf{Q}(F_j)$ for all $j=1,\dots,s$.

Lemma 5. *Under the above normalization of* $\{F_j|j=1,\dots,s\}$, *each* c_j *belongs to* $\mathbf{Q}(f,F_j)$. *Moreover, for some* j_0, *there exists a prime* $\mathscr{P}$ *of* $\mathbf{Q}(f,F_{j_0})$ *such that* $\mathrm{ord}_{\mathscr{P}}(c_{j_0})\leqq-\alpha$.

Proof. Let

$$S_k(\Gamma_2)_{\mathbf{Q}(f)}=\bigoplus_i V_i$$

be the decomposition into $\mathbf{T}_{\mathbf{Q}(f)}$-irreducible $\mathbf{Q}(f)$-subspaces. Let

$$G-[f]=\sum_i h_i\quad\text{with}\quad h_i\in V_i.$$

By Proposition 2, there is an eigenform $H_i\in V_i\otimes_{\mathbf{Q}(f)}\mathbf{C}$ such that the Fourier coefficients of H_i belong to $\mathbf{Q}(H_i)$, and for such H_i, we have

$$V_i\otimes_{\mathbf{Q}(f)}\mathbf{C}=\bigoplus_\tau \mathbf{C}H_i^\tau,$$

where τ runs over all embeddings of $\mathbf{Q}(f,H_i)$ into $\mathbf{R}$ over $\mathbf{Q}(f)$. Then as in the proof of Lemma 4,

$$h_i=\sum_\tau\gamma_i^\tau H_i^\tau\quad\text{with}\quad\gamma_i\in\mathbf{Q}(f,H_i).$$

We may assume $\{H_i^\tau\}_{i,\tau}=\{F_j|j=1,\dots,s\}$. By Lemma 3,

$$\mathfrak{p}^\alpha\cdot\frac{\left\langle F_j,\sum_i h_i\right\rangle}{\langle F_j,F_j\rangle}\subset[\kappa(F_j,\mathbf{Q}(f))\nu(F_j,\mathbf{Q}(f))\mathscr{D}(\mathbf{Q}(f,F_j)/\mathbf{Q}(f))]^{-1}S_j^{-1}\mathbf{Z}(f,F_j),$$

where $S_j=\{x\in\mathbf{Z}(f,F_j)|(x,\mathfrak{p})=1\}$ and $\mathbf{Z}(f,F_j)$ is the integer ring of $\mathbf{Q}(f,F_j)$. Hence

$$\mathfrak{p}^\alpha c_j\subset S_j^{-1}\mathbf{Z}(f,F_j)$$

by the condition (i) on $\mathfrak{p}$. Comparing the Fourier coefficients at T_0 of the both sides of (2.3), we obtain

$$a(T_0,G)-a(T_0,[f])=\sum_{i\in I}\mathrm{tr}_{\mathbf{Q}(f,H_i)/\mathbf{Q}(f)}(\gamma_i), \tag{2.4}$$

where I is defined by $\{H_i^\tau | i \in I,\ \tau$: embeddings of $\mathbf{Q}(f, H_i)$ into $\mathbf{R}$ over $\mathbf{Q}(f)\} = \{F_1, \ldots, F_{s_0}\}$ and tr denotes the trace. Since $\{c_i | i=1, \ldots, s\} = \{\gamma_i^\tau\}_{i,\tau}$, we have

$$\mathfrak{p}^\alpha \operatorname{tr}_{\mathbf{Q}(f, H_i)/\mathbf{Q}(f)}(\gamma_i) \subset \mathbf{Z}(f)_\mathfrak{p}.$$

By

$$\mathfrak{p}^{\alpha-1}(a(T_0, G) - a(T_0, [f])) \not\subset \mathbf{Z}(f)_\mathfrak{p},$$

for some i_0 we have

$$\operatorname{ord}_\mathfrak{p}(\operatorname{tr}_{\mathbf{Q}(f, H_{i_0})/\mathbf{Q}(f)}(\gamma_{i_0})) = -\alpha. \tag{2.5}$$

We decompose the principal ideal (γ_{i_0}) in $\mathbf{Q}(f, H_{i_0})$ as

$$(\gamma_{i_0}) = (\mathscr{P}_1^{e_1} \ldots \mathscr{P}_a^{e_a}\mathfrak{a})^{-1} \mathscr{P}_{a+1}^{e_{a+1}} \ldots \mathscr{P}_b^{e_b}\mathfrak{b},$$

where $\mathscr{P}_i$ $(i=1, \ldots, b)$ are distinct prime ideals lying above $\mathfrak{p}$, and $\mathfrak{a}, \mathfrak{b}$ are integral ideals of $\mathbf{Q}(f, H_i)$ coprime to $\mathfrak{p}$. Then $e_u \geqq \alpha$ for some $u \leqq a$. For if $e_1, \ldots, e_a \leqq \alpha - 1$, then

$$\mathfrak{p}^{\alpha-1}(\gamma_{i_0}) \subset S_0^{-1}\mathbf{Z}(f, H_{i_0}),$$

where $S_0 = \{x \in \mathbf{Z}(f, H_{i_0}) | (x, \mathfrak{p}) = 1\}$. Thus

$$\mathfrak{p}^{\alpha-1} \operatorname{tr}_{\mathbf{Q}(f, H_{i_0})/\mathbf{Q}(f)}(\gamma_{i_0}) \subset S^{-1}\mathcal{O}_E \cap \mathbf{Q}(f) \subset \mathbf{Z}(f)_\mathfrak{p}$$

in the notation of Lemma 2, which implies

$$\operatorname{tr}_{\mathbf{Q}(f, H_{i_0})/\mathbf{Q}(f)}(\gamma_{i_0}) \in \mathfrak{p}^{1-\alpha},$$

but this contradicts to (2.5). Lemma 5 is proved.

Now, by (2.3) we obtain

$$T_k(m)G - \lambda(m, [f])G = \sum_{j=1}^{s} c_j(\lambda(m, F_j) - \lambda(m, [f]))F_j \tag{2.6}$$

for each $m = 1, 2, \ldots$.

Since $S_k(\Gamma_2)_{\mathbf{Z}(f)_\mathfrak{p}}$ is $T_k(m)$-stable, the left-hand side of (2.6) belongs to $S_k(\Gamma_2)_{\mathbf{Z}(f)_\mathfrak{p}}$. Applying Lemma 3 to (2.6) we have

$$c_j(\lambda(m, F_j) - \lambda(m, [f])) \in S_j^{-1}\mathbf{Z}(f, F_j)$$

for all j in the notation of the proof of Lemma 5. By Lemma 5, for $j = j_0$ we have

$$\lambda(m, F_{j_0}) \equiv \lambda(m, [f]) \pmod{\mathscr{P}^\alpha} \quad \text{in} \quad \mathbf{Z}(f, F_{j_0}).$$

Hence, putting $F_{j_0} = F$, we obtain

$$N_{\mathbf{Q}(f, F)/\mathbf{Q}(f)}(\lambda(m, F) - \lambda(m, [f])) \equiv 0 \pmod{\mathfrak{p}^\alpha},$$

since $\mathfrak{p}$ is unramified in $\mathbf{Q}(f, F)$ by the condition (i) on $\mathfrak{p}$. This completes the proof of Theorem.

3. Numerical Examples

Let Δ_k be the normalized elliptic cusp form of weight $k = 12, 16, 20$, and 22. For the notation for eigenforms of degree 2, we refer to [7]. As we have stated, $(\mathrm{MOC})_{2,k}$ is satisfied at least for $k \leqq 22$. A table of the prime decomposition of $\frac{B_k B_{2k-2}}{k(2k-2)}$ is contained in [21].

(1) $k=12$.

We have

$$c_{12}(\Delta_{12})=2\cdot 3\cdot 131\cdot 593\cdot 691\,.$$

By [14],

$$a\left(\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, [\Delta_{12}]\right)=7^{-1}\cdot 2^2\cdot 23\,.$$

So by Theorem,

$$\lambda(m,[\Delta_{12}])\equiv\lambda(m,\chi_{12}) \pmod 7 \quad \text{for all} \quad m\geqq 1\,.$$

This coincides with a result of Kurokawa [8, Theorem 3].

(2) $k=16$.

We have the following prime decomposition:

$$c_{16}(\Delta_{16})=2^{13}\cdot 3^3\cdot 1721\cdot 3617\cdot(51349)^2\cdot 1001259881\,.$$

Note that $\mathbf{Q}(\chi_{16}^{(\pm)})=\mathbf{Q}(\sqrt{51349})$. Moreover

$$a\left(\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, [\Delta_{16}]\right)=(7^2\cdot 11)^{-1}\cdot 2^2\cdot 31\,.$$

So by Theorem,

$$N_{\mathbf{Q}(\sqrt{51349})/\mathbf{Q}}(\lambda(m,\chi_{16}^{(\pm)})-\lambda(m,[\Delta_{16}]))\equiv 0 \pmod{7^2\cdot 11} \quad \text{for all} \quad m\geqq 1\,.$$

(3) $k=20$.

By [16, Example 2], we have the following prime decomposition:

$$c_{20}(\Delta_{20})=2^{20}\cdot 3^5\cdot 5\cdot 7\cdot 11\cdot 19\cdot(181)^2\cdot 283\cdot(349)^2\cdot 617\cdot(1009)^2 \times 154210205991661\,.$$

Note that

$$\kappa(\chi_{20}^{(1)})=\kappa(\chi_{20}^{(2)})=2^5\cdot 3\,,\qquad \kappa(\chi_{20}^{(3)})=1\,,$$

$$\mathbf{Q}(\chi_{20}^{(1)})=\mathbf{Q}(\chi_{20}^{(2)})=\mathbf{Q}(\sqrt{181\cdot 349\cdot 1009})\,,\qquad \mathbf{Q}(\chi_{20}^{(3)})=\mathbf{Q}\,,$$

$$\nu(\chi_{20}^{(1)})=\nu(\chi_{20}^{(2)})=1\,,\quad \text{and} \quad \nu(\chi_{20}^{(3)})=2^9\cdot 3^2\cdot 5\cdot 7\cdot 11\,.$$

On the other hand, we have

$$a\left(\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, [\Delta_{20}]\right)=(11\cdot 71^2)^{-1}\cdot 2^2\cdot 19\,.$$

Applying Theorem, for some i we have

$$N_{\mathbf{Q}(\chi_{20}^{(i)})/\mathbf{Q}}(\lambda(m,[\Delta_{20}])-\lambda(m,\chi_{20}^{(i)}))\equiv 0 \pmod{71^2} \quad \text{for all} \quad m\geqq 1\,.$$

One easily checks that the above congruence (even in the case $m=2$) does not hold for $\chi_{20}^{(i)}$ in the Maass space, so $i \neq 1, 2$. Thus

$$\lambda(m, [\Delta_{20}]) \equiv \lambda(m, \chi_{20}^{(3)}) \pmod{71^2} \quad \text{for all} \quad m \geqq 1 .$$

This is Theorem 1 in Kurokawa [9].

(4) $k=22$.

We have $S_{22}(\Gamma_2)=S_{22}^{I}(\Gamma_2) \oplus S_{22}^{II}(\Gamma_2)$ with $\dim S_{22}^{I}(\Gamma_2)=3$ and $\dim S_{22}^{II}(\Gamma_2)=1$, where $S_{22}^{I}(\Gamma_2)$ denotes the Maass space and $S_{22}^{II}(\Gamma_2)$ its orthogonal complement with respect to the Petersson inner product. (Hence $(\mathrm{MOC})_{2,22}$ holds.) Let $\{\chi_{22}^{(i)} | i=1,2,3\}$ be an eigenbasis of $S_{22}^{I}(\Gamma_2)$. We put

$$\chi_{22}^{(4)}=30\varphi_4\varphi_6\chi_{12}-5\varphi_6^2\chi_{10}-61\varphi_4^3\chi_{10}+80870400\chi_{10}\chi_{12}$$

[18]. Then $S_{22}^{II}(\Gamma_2)=\mathbf{C}\chi_{22}^{(4)}$, so $\chi_{22}^{(4)}$ is an eigenform. On the other hand, we have

$$a\left(\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, [\Delta_{22}]\right) = -(7 \cdot 13 \cdot 17 \cdot 61 \cdot 103)^{-1} \cdot 2^2 \cdot 5 \cdot 37 .$$

First we note that

$$\text{g.c.d.}\,(13 \cdot 17 \cdot 61 \cdot 103, c_{22}(\Delta_{22}))=1 . \tag{3.1}$$

To see this, we proceed as follows:

(i) By Wagstaff [21, Table 2],

$$\text{g.c.d.}\left(13 \cdot 17 \cdot 61 \cdot 103, \mathrm{Num}\left(\frac{B_{22}B_{42}}{22 \cdot 42}\right)\right)=1 .$$

(ii) Let $\{\Delta_{42}^{(i)} | i=1,2,3\}$ be the normalized eigenbasis of $S_{42}(\Gamma_1)$ such that $\Delta_{42}^{(i)}$ corresponds $\chi_{22}^{(i)} (i=1,2,3)$ under the Saito-Kurokawa lifting. Then $\kappa(\chi_{22}^{(i)}) =\kappa(\Delta_{42}^{(i)})$, $\nu(\chi_{22}^{(i)})=\nu(\Delta_{42}^{(i)})=1$, and $d(\mathbf{Q}(\chi_{22}^{(i)}))=d(\mathbf{Q}(\Delta_{42}^{(i)}))$ for each $i=1,2,3$. Since $\mathbf{Q}(\Delta_{42}^{(i)})=\mathbf{Q}(a(2, \Delta_{42}^{(i)}))$, the discriminant of the algebraic integer $a(2, \Delta_{42}^{(i)})$ is equal to

$$d(\mathbf{Q}(\Delta_{42}^{(i)})) \cdot (\mathbf{Z}(\Delta_{42}^{(i)}) : \mathbf{Z}[a(2, \Delta_{42}^{(i)})])^2 .$$

Hence, to see

$$p \nmid \prod_{i=1}^{3} \kappa(\chi_{22}^{(i)})\nu(\chi_{22}^{(i)})d(\mathbf{Q}(\chi_{22}^{(i)})) \quad \text{for} \quad p=13, 17, 61, 103 , \tag{3.2}$$

we have only to show that the discriminant of the characteristic polynomial $P(X)$ of $T(2)$ on $S_{42}(\Gamma_1)$ is not divisible by p. We have

$$P(X)=X^3+344688X^2-6374982426624X-520435526440845312 .$$

Considering this modulo p, one checks (3.2) immediately.

(iii) The same method as in [16, Example 2] shows that $\nu(\chi_{22}^{(4)}) =2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 1423$. Thus we obtain (3.1).

Next we note that the matrix of $(T(2)|S_{22}^{I}(\Gamma_2))-\lambda(2, [\Delta_{22}]) \cdot \mathrm{id}$ (id = the identity map on $S_{22}^{I}(\Gamma_2)$) with respect to $\{\chi_{22}^{(i)} | i=1,2,3\}$ is

$$\begin{pmatrix} 305133360 & 2197843974144 & 3428554079010816 \\ 1 & 305924256 & -5706215424 \\ 0 & -576 & 304005408 \end{pmatrix} .$$

A computation modulo p shows that the determinant of the above matrix is divisible by 13, 17, and not divisible by 61, 103. On the other hand, by [18], we obtain

$$\lambda(2,\chi_{22}^{(4)})-\lambda(2,[\Delta_{22}])=299774496,$$

which is divisible by 61, 103, and not divisible by 13, 17. Hence by Theorem we have

$$\lambda(m,\chi_{22}^{(4)})\equiv\lambda(m,[\Delta_{22}])\ (\bmod 61\cdot 103)\quad\text{for all}\quad m\geqq 1,$$

and for $i=1,2,3$,

$$N_{K_i/\mathbf{Q}}(\lambda(m,\chi_{22}^{(i)})-\lambda(m,[\Delta_{22}]))\equiv 0\ (\bmod 13\cdot 17)\quad\text{for all}\quad m\geqq 1.$$

Here $K_i=\mathbf{Q}(\chi_{22}^{(i)})$ is the cubic field generated over $\mathbf{Q}$ by a root of the above $P(X)$, and K_1, K_2, K_3 are the conjugates over $\mathbf{Q}$.

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On the Homology of Metacyclic Coverings

Antonio F. Costa[1] and Jesús M. Ruiz[2]

Departamentos [1]Geometría y Topología y [2]Algebra, Facultad de Ciencias Matemáticas, Universidad Complutense, E-28040 Madrid, Spain

Introduction

The purpose of this paper is to describe a nice property of irregular metacyclic coverings. These coverings, and the particular type of dihedral coverings, have often appeared in the literature in several connections: [B–Z], [F2], [H], [M], [P], [Ch].

Let G be a metacyclic group, i.e. an extension of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}/m\mathbb{Z}$, some n, m. Then G contains a copy C_m of $\mathbb{Z}/m\mathbb{Z}$. Assume n is an odd prime and G acts on a PL connected manifold N in such a way that $\Sigma = N/G$ is simply connected and $q: N \to \Sigma$ a branched covering with branch index m in the singular locus. Then we have another n-fold branched covering $p: M = N/C_m \to \Sigma$ and this latter is called metacyclic.

Our main result is Theorem (2.1) that gives a congruence relation for $\dim H_1(M, F)$, F a field of characteristic $\neq n$, in terms of n, m and a third number α that determines G.

In particular, if F is $\mathbb{Q}$ or a finite prime field whose characteristic is a generator of the group $(\mathbb{Z}/n\mathbb{Z})^*$, we find:

Theorem. $\dim H_1(M, F) \equiv 0 \bmod \dfrac{n-1}{m}$.

Applying this to the dihedral group $G = D_n$ we get $\dim H_1(M, F) \equiv 0 \bmod \dfrac{n-1}{2}$. We remark that $\dfrac{n-1}{2} = [F(\xi + \xi^{-1}) : F]$, ξ being an n-th root of unity. The relation between this field extension and the homology of dihedral coverings was first noticed and used in [Ch], with different techniques and results.

1. Embedding of Metacyclic Groups in Symmetric Groups

A group G is called *metacyclic* if its commutator G' and its abelianization G/G' are both cyclic. There is the following classical description of this kind of groups:

Theorem (1.1) [B]. *A finite group G is metacyclic if and only if it has two generators a, b with relations*

$$a^n = b^m = 1, \qquad bab^{-1} = a^\alpha$$

where

$$gcd(n, \alpha - 1) = 1, \qquad \alpha^m \equiv 1 \bmod n.$$

Let S_n stand for the symmetric group of the elements $0, 1, \ldots, n-1$. It is not difficult to prove:

Proposition (1.2). Let G be a metacyclic group as in 1.1. Assume m is the order of α in the cyclic group $(\mathbb{Z}/n\mathbb{Z})^*$ of units of $\mathbb{Z}/n\mathbb{Z}$. Then there is a (unique) monomorphism

$$e: G \to S_n,$$

given by:

$$e(a) = (0, 1, \ldots, n-1);$$

$$e(b)(j) = j\alpha \bmod n \qquad (j = 0, \ldots, n-1).$$

Notice that $e(b)$ is a product of cycles $(j, j\alpha, \ldots, j\alpha^{m-1})$, $j = 1, \ldots, n-1$.

Remark (1.3). $\langle g \in G : e(g)(0) = j \rangle = \langle a^j b^l : l = 1, \ldots, m \rangle$.

2. Homology of Metacyclic Coverings

Let us be given the following data:

(1) Σ, a simply connected PL manifold; L, a PL submanifold of Σ of cod 2.

(2) n a positive integer and α an element of order m of $(\mathbb{Z}/n\mathbb{Z})^*$; $e: G \to S_n$ the metacyclic group and embedding described in Sect. 1;

(3) $\omega: \pi_1(\Sigma \backslash L) \to e(G) \subset S_n$, a representation which maps each meridian of L to a cycle of order m.

This kind of representation is called metacyclic. The associated branched covering $p: M \to \Sigma$ is also called metacyclic.

An equivalent description was given in the introduction, following the picture:

$$\begin{tikzcd} & N \arrow[ld, "m:1"'] \arrow[dd, "nm:1"] \\ N/C_m = M \arrow[rd, "n:1"'] & \\ & \Sigma = N/G \end{tikzcd}$$

We can now state our main result.

Theorem (2.1). *Assume n is an odd prime. Let F be a field of characteristic $\neq n$. Let $M \to \Sigma$ be a metacyclic covering as above. If $[F(\xi): F] = n-1$, then:*

$$\dim_F H_1(M, F) \equiv 0 \bmod [F(\xi^\alpha + \ldots + \xi^{\alpha^m}) : F],$$

where ξ is a primitive n-th root of unity.

We separate the proof of (2.1) in several paragraphs.

(2.2) Let $0 \in \Sigma \backslash L$ be a base point and

$$\langle m_1, \dots, m_s; r_1, \dots, r_s \rangle$$

a presentation of $\pi_1(\Sigma \backslash L)$, where the m_i's are the meridians of L. After Fox, [F1], we obtain a presentation of

$$\pi_1(M) * H_{n-1}, H_{n-1} \equiv \text{free group with } n-1 \text{ generators},$$

as follows:

(a) Generators: all liftings m_{ij} of the meridians of L

(b) Relations: all liftings r_{ij} of the relations of $\pi_1(\Sigma \backslash L)$, plus the *branch relations*

$$m_{ij_1} \dots m_{ij_n} \quad \text{for} \quad \omega(m_i) = \dots (j_1, \dots, j_n) \dots, i = 1, \dots, s.$$

Finally, if μ_{ij} is the element in $H_1(M, F) \oplus F^{\oplus(n-1)}$ given by m_{ij}, the collection $\{\mu_{ij}\}$ is a system of generators of that F-vector space.

(2.3) Now we turn to the diagram of coverings

$$\begin{tikzcd} & N \arrow[dl, "\tilde{p}"'] \arrow[dd, "q"] \\ M \arrow[dr, "p"'] & \\ & \Sigma \end{tikzcd}$$

already described, where q is regular with automorphisms group G.

Let $\{\tilde{0}_g\}_{g \in G}$ be the q-fiber over the base point $0 \in \Sigma$. The indices in this fiber can be chosen so that the action of G on N reads

(2.3.1) $$g(\tilde{0}_{g'}) = \tilde{0}_{g'g} \quad \text{(cf. [ST])}.$$

Therefore

(2.3.2) $$e(g)(0) = j \quad \text{if and only if} \quad \tilde{p}(\tilde{0}_g) = 0_j$$

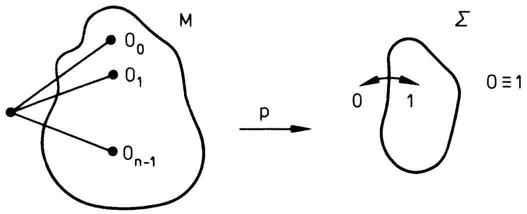
where $\{0_j\}_{j=0,\dots,n-1}$ is the p-fiber over 0.

Finally, let $(m_i)_g^{\sim}$ stand for the lifting of m_i with origin $\tilde{0}_g$, and $(\mu_i)_g^{\sim}$ for the corresponding element in

$$H_1(N, F) \oplus F^{\oplus(nm-1)}.$$

As in (2.2), $\{(\mu_i)_g^{\sim}\}$ is a system of generators of that F-vector space.

(2.4) The space $H_1(M, F) \oplus F^{\oplus(n-1)}$ is to be viewed as the first homology group of a branched covering of $[0,1] \cup \Sigma/\{1 \equiv 0\}$, which coincides with p on Σ and has a branch point over 0 of index n:



A similar description works for $H_1(N, F) \oplus F^{\oplus(nm-1)}$.

Let $a \in G$ be of order n such that $e(a) = (0, 1, \dots, n-1)$. We consider the associated automorphism

$$a_{\#}: H_1(N, F) \oplus F^{\oplus(nm-1)} \to H_1(N, F) \oplus F^{\oplus(nm-1)},$$

and also the transfer homomorphism

$$\mathrm{tr}: H_1(M, F) \oplus F^{\oplus(n-1)} \to H_1(N, F) \oplus F^{\oplus(nm-1)}$$

and its left inverse $\tilde{p}_{\#}$ corresponding to $\tilde{p}: N \to M$.

(2.4.1) *Definition of an endomorphism h of $H_1(M, F) \oplus F^{\oplus(n-1)}$.*

We define h by means of the following commutative square:

$$\begin{array}{ccc}
H_1(N, F) \oplus F^{\oplus(nm-1)} & \xrightarrow{a_{\#}} & H_1(N, F) \oplus F^{\oplus(nm-1)} \\
{\scriptstyle \mathrm{tr}}\uparrow & & \downarrow{\scriptstyle \tilde{p}_{\#}} \\
H_1(M, F) \oplus F^{\oplus(n-1)} & \xrightarrow[h]{} & H_1(M, F) \oplus F^{\oplus(n-1)},
\end{array}$$

i.e. $h = \tilde{p}_{\#} a_{\#} \,\mathrm{tr}$.

The key fact is:

(2.4.2) $$h(\mu_{ij}) = \sum_{l=1}^{m} \mu_{i, j+\alpha^l} \quad (j + \alpha^l \text{ reduced mod } n).$$

Endeed, by (2.3.2):

$$\mathrm{tr}(\mu_{ij}) = \sum_{l=1}^{m} (\mu_i)\tilde{}_{g_l}, \quad \text{with} \quad e(g_l)(0) = j,$$

so that by 1.3:

$$\mathrm{tr}(\mu_{ij}) = \sum_{l=1}^{m} (\mu_i)\tilde{}_{a^j b^l}.$$

Now by (2.3.1)

$$a_{\#}\,\mathrm{tr}(\mu_{ij}) = \sum_{l=1}^{m} (\mu_i)\tilde{}_{a^j b^l a},$$

and finally

$$\tilde{p}_{\#} a_{\#}\,\mathrm{tr}(\mu_{ij}) = \sum_{l=1}^{m} \mu_{i, e(a^j b^l a)(0)} = \sum_{l=1}^{m} \mu_{i, j+\alpha^l},$$

since:

$$e(a^j b^l a)(0) = e(a^j b^l)(1) = e(a^j)(\alpha^l) = j + \alpha^l.$$

(2.5) *Minimal polynomial of h.* Let ξ be a primitive n-th root of unity and put

$$\zeta = \sum_{l=1}^{m} \xi^{\alpha^l}.$$

Let $\Phi \in F[t]$ be the irreducible polynomial of ζ over F. We consider the multiplication by ζ as an endomorphism.

$$k: F(\xi) \to F(\xi)$$

so that

$$k(\xi^j) = \sum_{l=1}^{m} \xi^{j+\alpha^l}.$$

Clearly $\Phi(k)=0$.

Now we look at $F(\xi)$ as an F-vector space with generators $\xi, \ldots, \xi^{n-1}, \xi^n$. Since

$$\xi + \ldots + \xi^n = 0 \quad \text{and} \quad \sum_{l=1}^{m} \mu_{ij} = 0 \quad \text{(by adding the relations 2.2.b)}$$

the following homomorphism is well-defined:

$$v: F(\xi)^{\oplus s} \to H_1(M,F) \oplus F^{\oplus(n-1)} : (0, \ldots, \underset{i}{\xi^l}, \ldots, 0) \to \mu_{il}.$$

We get a commutative diagram:

$$\begin{array}{ccc} F(\xi)^{\oplus s} & \xrightarrow{k^{\oplus s}} & F(\xi)^{\oplus s} \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} \\ H_1(M,F) \oplus F^{\oplus(n-1)} & \xrightarrow[h]{} & H_1(M,F) \oplus F^{\oplus(n-1)}. \end{array}$$

As $\Phi(k)=0$ it follows easily $\Phi(h)=0$. But Φ is irreducible of degree $[F(\zeta):F]$, so that

$$\dim_F H_1(M,F) \oplus F^{\oplus(n-1)} \equiv 0 \bmod [F(\zeta):F].$$

Thus the proof of Theorem (2.1) is finished.

(2.6) The theorem stated in the introduction is now immediate. For, under those hypothesis the Galois group of $F(\xi)$ over F is $(\mathbb{Z}/n\mathbb{Z})^*$, and $[F(\xi):F]=n-1$, [A]. Then one checks that the fixed field of the automorphism $\xi \mapsto \xi^\alpha$ is exactly $F(\zeta)$ and so $[F(\xi):F(\zeta)]=m$, since m is the order of α in $(\mathbb{Z}/n\mathbb{Z})^*$. Consequently

$$[F(\zeta):F] = \frac{n-1}{m}.$$

(2.7) In the dihedral case: $m=2$, $\alpha=n-1$, and $[F(\zeta):F]=\frac{n-1}{2}$. Let us remark in this case that from the very definition of h in 2.4, it comes naturally the following sequence of polynomials:

$$\text{(2.7.1)} \qquad \begin{gathered} \Phi_0(t)=1, \quad \Phi_1(t)=t+1, \\ \Phi_l(t) = t\Phi_{l-1}(t) - \Phi_{l-2}(t), \quad l \geqq 2, \end{gathered}$$

and it is not difficult to check that

$$\Phi_l(h)=0 \quad \text{for} \quad l=\frac{n-1}{2}.$$

Consequently, this Φ_l is the irreducible polynomial of ζ over F when F is $\mathbb{Q}$ or a finite prime field whose characteristic generates the units of $\mathbb{Z}/n\mathbb{Z}$, i.e. when the cyclotomic polynomial of degree $n-1$ is irreducible. Notice that this comes also from the identity $x^l\Phi_l(x+x^{-1})=1+x+\ldots+x^{n-1}$.

(2.7) *Application.* An analogue of Hilden-Montesinos' theorem, [H] [M], cannot be true for dimensions >3: $M=S^1\times\ldots^n\times S^1$, n prime >3, is not an n-sheeted dihedral branched covering of S^n.

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Transformationsgruppen mit affinen Bahnen

Rainer Felix

Mathematisch-Geographische Fakultät, Katholische Universität Eichstätt, Ostenstraße 26–28, D-8078 Eichstätt, Bundesrepublik Deutschland

Einleitung

Sei M eine lokalkompakte Teilmenge von $\mathbb{R}^n$ und sei G eine zusammenhängende Liesche Gruppe. G operiere (von rechts) als topologische Transformationsgruppe auf M, so daß alle Bahnen von G affin (d. h. lineare Mannigfaltigkeiten in $\mathbb{R}^n$) sind. Die Euklidischen Maße auf den G-Bahnen seien invariant unter der Wirkung von G.

In der vorliegenden Arbeit wollen wir das topologische Erzeugnis $\mathscr{M}^{\perp\perp}$ der Menge $\mathscr{M}$ der Euklidischen Maße auf den G-Bahnen im Raum der (Radonschen) Maße bzw. im Raum der Distributionen untersuchen. Und zwar werden wir die Elemente von $\mathscr{M}^{\perp\perp}$ explizit (d. h. via Desintegration) ausdrücken in Termen der Elemente von $\mathscr{M}$.

Ideal ist die Situation im Raum der Maße (Abschn. 1). Hier besteht $\mathscr{M}^{\perp\perp}$ gerade aus den G-invarianten Maßen, und jedes G-invariante Maß läßt sich nach einem Maß auf $\mathscr{M}$ desintegrieren. [Auf der Menge $\mathscr{M}$ haben wir eine natürliche Topologie, die $\mathscr{M}$ zu einem lokalkompakten Raum macht (Satz 1.10), so daß wir auf $\mathscr{M}$ (Radonsche) Maße betrachten können.] Wir können das Desintegrationsmaß auf $\mathscr{M}$ konstruktiv angeben, indem wir M in Schichten mit konstanter Bahnendimension zerlegen.

Im Raum der Distributionen (Abschn. 2) läßt sich ein Vorgehen wie im Raum der Maße nicht ohne weiteres nachspielen. Zwar bleibt die Situation im Fall konstanter Bahnendimension auch hier ideal (2.3, 2.4 und 2.5); im allgemeinen Fall jedoch treten an den Nahtstellen, wo sich Bahnen verschiedener Dimension berühren, etliche Probleme auf (s. 2.1). Um eine Desintegration zu erhalten, müssen wir daher ein gewisses „Wohlverhalten" der Bahnen an den Nathstellen fordern.

Anders als im Raum der Maße ist $\mathscr{M}^{\perp\perp}$ im Raum der Distributionen im allgemeinen echt kleiner als die Menge der G-invarianten Distributionen, wie die Beispiele 4.5 in [2] und 1.7 in [4] zeigen. In Übereinstimmung mit diesen Beispielen werden wir jedoch sehen, daß jede G-invariante Distribution modulo $\mathscr{M}^{\perp\perp}$ von den Punkten mit niederdimensionaler Bahn getragen wird (Korollar 2.8).

In Abschn. 3 behandeln wir den Spezialfall, daß G eine in natürlicher Weise auf $M = \mathbb{R}^n$ wirkende Matrizengruppe ist. Dann muß G notwendigerweise unipotent sein (Satz 3.2). Das Interesse an dieser Situation stammt aus der Darstellungstheorie nilpotenter Liegruppen. Unsere Resultate lassen sich nämlich im Falle einer nilpotenten Liegruppe N, deren sämtliche Kirillov-Bahnen affin sind, auf die Charaktere irreduzibler Darstellungen anwenden. Die Charaktere irreduzibler Darstellungen bilden dann einen lokalkompakten Raum (Satz 3.9), und eine zentrale positiv definite Distribution auf N läßt sich über den Charakteren desintegrieren (Theorem 3.10).

Hat unsere Matrizengruppe eine spezielle Form (s. 3.12), dann haben die Bahnen das oben erwähnte „Wohlverhalten" an den Nahtstellen so daß die Desintegrationstheorie aus Abschn. 2 angewendet werden kann (Theorem 3.13). Da die koadjungierte Gruppe einer Heisenberggruppe N hinsichtlich einer geeigneten Basis eine Matrizengruppe von jener speziellen Form ist, erhalten wir die Desintegrationsformel für zentrale Distributionen auf N von Rothschild-Wolf [8] als Spezialfall. Schließlich gewinnen wir noch für G-invariante Differentialoperatoren $P(D)$ mit konstanten Koeffizienten eine Aussage über die Existenz G-invarianter Lösungen der Gleichung $P(D)u = f$.

1. Topologische Transformationsgruppen mit affinen Bahnen

1.1. Wir bezeichnen mit $\mathscr{B}$ die Menge der G-Bahnen in M und mit B die kanonische Abbildung von M nach $\mathscr{B}$. Für $x \in M$ sei $E(x)$ der lineare Unterraum von $\mathbb{R}^n$ mit $B(x) = x + E(x)$ und λ_x das Euklidische Maß auf $B(x)$. [Ist $\dim E(x) = 0$, so sei λ_x das Einpunktmaß im Punkte x.] Wir können λ_x auch als Maß auf M oder auf $\mathbb{R}^n$ auffassen. Die Menge $\mathscr{M}$ aller Maße λ_x, $x \in M$, kann dann als Teilmenge von $\mathscr{K}'(M)$, $\mathscr{K}'(\mathbb{R}^n)$, $\mathscr{D}'(\mathbb{R}^n)$ oder $\mathscr{S}'(\mathbb{R}^n)$ verstanden werden. Hierbei bezeichne $\mathscr{K}$ den Raum aller stetigen Funktionen mit kompaktem Träger, versehen mit der üblichen Topologie, und $\mathscr{K}'$ dessen Dualraum, also den Raum der (komplexen) Radonmaße, versehen mit der vagen Topologie; die Symbole $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{E}$, $\mathscr{E}'$ seien im Sinne von L. Schwartz verstanden. Die Räume $\mathscr{K}'(M)$, $\mathscr{K}'(\mathbb{R}^n)$, $\mathscr{D}'(\mathbb{R}^n)$ und $\mathscr{S}'(\mathbb{R}^n)$ induzieren alle dieselbe Topologie auf $\mathscr{M}$; denn die Lemmata 1.4 und 1.9 gelten für jede dieser Topologien.

Wir bezeichnen mit $\mathscr{M}^{\perp}$ das Orthogonal von $\mathscr{M}$ in $\mathscr{K}(M)$ und mit $\mathscr{M}^{\perp\perp}$ das Orthogonal von $\mathscr{M}^{\perp}$ in $\mathscr{K}'(M)$, also den von $\mathscr{M}$ erzeugten abgeschlossenen Unterraum von $\mathscr{K}'(M)$. Die Menge $\mathscr{M}$ erzeugt genau die Menge $\mathscr{K}'_G(M)$ aller G-invarianten Maße auf M, also $\mathscr{M}^{\perp\perp} = \mathscr{K}'_G(M)$; denn jedes Maß $\mu \in \mathscr{K}'_G(M)$ zerfällt in positive G-invariante Maße, und daß diese zu $\mathscr{M}^{\perp\perp}$ gehören, ergibt sich wie in [4, 1.8].

Wir zerlegen die Menge M in die G-invarianten Teilmengen M_d, $d = 0, \dots, n$, bestehend aus den Punkten mit d-dimensionaler Bahn. Für alle $d = 0, \dots, n$ ist die Menge $\tilde{M}_d := \bigcup_{k \leq d} M_k$ abgeschlossen in M. Denn für $p \in M \setminus \tilde{M}_d$ existieren Gruppenelemente $a_1, \dots, a_{d+1}$, so daß die Vektoren $p \cdot a_1 - p, \dots, p \cdot a_{d+1} - p$ linear unabhängig sind; dann sind aber auch die Vektoren $x \cdot a_1 - x, \dots, x \cdot a_{d+1} - x$ linear unabhängig für alle Punkte x einer Umgebung von p. Insbesondere ist demnach M_d, $d=0,\ldots,n$, Durchschnitt einer offenen und einer abgeschlossenen Menge, also lokalkompakt.

Natürlich operiert G auch auf M_d als topologische Transformationsgruppe. Wir bezeichnen die assoziierten Objekte mit $\mathscr{B}_d$, B_d und $\mathscr{M}_d$. Die Menge $\mathscr{B}_d$ kann dann als Teilmenge der Mannigfaltigkeit $G(d,n)$ aller d-dimensionalen Ebenen in $\mathbb{R}^n$ betrachtet werden. Auf $\mathscr{B}_d$ stimmt die von M_d deduzierte Quotiententopologie mit der von $G(d,n)$ induzierten Topologie überein (s. 1.2). Sei λ_ξ das Euklidische Maß auf $\xi\in G(d,n)$. Die Menge $\mathscr{M}_d=\{\lambda_\xi\,|\,\xi\in\mathscr{B}_d\}$ interpretieren wir als Teilmenge von $\mathscr{M}$. Nach dem oben Gesagten induzieren die Räume $\mathscr{K}'(M_d)$ und $\mathscr{K}'(M)$ dieselbe Topologie auf $\mathscr{M}_d$. [Andererseits können wir das Erzeugnis $\mathscr{M}_d^{\perp\perp}$ von $\mathscr{M}_d$ in $\mathscr{K}'(M_d)$ nicht identifizieren mit dem Erzeugnis von $\mathscr{M}_d$ in $\mathscr{K}'(M)$, da M_d in M nicht abgeschlossen zu sein braucht.]

1.2. In den folgenden Abschnitten (bis einschließlich 1.7) nehmen wir an, daß die Dimension aller Bahnen konstant gleich d ist, $1\leqq d\leqq n$. (Der Fall $d=0$ ist trivial.)

Bezeichnet nun $\sigma(x)$ den zu $E(x)$ orthogonalen Punkt aus $B(x)$ („orthogonal" im Sinne des üblichen Skalarproduktes $\langle\,|\,\rangle$ auf $\mathbb{R}^n$), so ist die Abbildung $x\to\sigma(x)$ stetig auf M. Um das einzusehen, bilden wir für $p\in M$ eine Basis von $E(p)$ der Form $p\cdot a_1-p,\ldots,p\cdot a_d-p$ mit geeigneten Elementen $a_1,\ldots,a_d\in G$. Dann sind aber für alle Punkte x einer Umgebung von p die Vektoren $x\cdot a_1-x,\ldots,x\cdot a_d-x$ linear unabhängig, bilden also eine Basis von $E(x)$. Orthonormierung liefert nun die Existenz einer stetig von x abhängenden Orthonormalbasis $e_1(x),\ldots,e_d(x)$ von $E(x)$, woraus wegen $\sigma(x)=x-\sum_{j=1}^{d}\langle x\,|\,e_j(x)\rangle\, e_j(x)$ die Stetigkeit von σ folgt.

Die Abbildung $\bar\sigma:\mathscr{B}\to M$ mit $\sigma=\bar\sigma\circ B$ ist ein Schnitt auf $\mathscr{B}$ (d. h. $B\circ\bar\sigma=\mathrm{Id}_{\mathscr{B}}$). Die Menge $S:=\{x\in M\,|\,\sigma(x)=x\}$ ist die Schnittmenge (d. h. das Bild von $\bar\sigma$), und die Abbildung $\bar\sigma:\mathscr{B}\to S$ ist bijektiv; die Einschränkung $B|_S$ von B auf S ist die Inverse. Wegen der Stetigkeit von σ ist S abgeschlossen in M, und $\mathscr{B}$ und S sind vermöge $\bar\sigma$ homöomorph. Insbesondere ist S und folglich auch $\mathscr{B}$ lokalkompakt.

[Es sei bemerkt, daß keineswegs ein stetiger Schnitt zu existieren braucht, wenn man auf die Voraussetzung, alle G-Bahnen seien affin, verzichtet. Im allgemeinen existiert schon kein stetiger Schnitt mehr, wenn die Bahnen durch quadratische Gleichungen gegeben sind. Wirkt zum Beispiel $G=\mathbb{R}$ auf $M=\{x\in\mathbb{R}^3\,|\,x_1^2+x_2^2\neq 0\}$ vermöge $x\cdot t:=\left(x_1,x_2+tx_1,x_3+tx_2+\frac{t^2}{2}x_1\right)$, $x\in M$, $t\in G$, so ist die Quotiententopologie auf $\mathscr{B}$ nicht Hausdorffsch, da $B\left(\frac{1}{k},1,0\right)$ für $k\to\infty$ sowohl gegen $B(0,1,0)$ als auch gegen $B(0,-1,0)$ konvergiert.]

1.3. Gemäß den Überlegungen von 1.2 können wir für jeden Punkt $y_0\in S$ eine relativ kompakte Umgebung U in S finden und eine stetig von $y\in U$ abhängende Orthonormalbasis $e_1(y),\ldots,e_d(y)$ von $E(y)$ konstruieren. Für $x\in\sigma^{-1}(U)$ setzen wir $z(x):=(\langle x-\sigma(x)\,|\,e_j(\sigma(x))\rangle)_{j=1,\ldots,d}\in\mathbb{R}^d$. Dann ist die Abbildung $\Phi_U(x):=(\sigma(x),z(x))$ ein Homöomorphismus von $\sigma^{-1}(U)$ auf $U\times\mathbb{R}^d$; die Umkehrabbildung lautet: $\Phi_U^{-1}(y,z)=y+\sum_{j=1}^{d} z_j e_j(y)$. Nun wählen wir eine lokalendliche Überdeckung $(U_i)_{i\in\mathbb{N}}$ von S, bestehend aus Umgebungen obiger Art, und eine dieser Überdeckung untergeordnete Partition der Eins $(\alpha_i)_{i\in\mathbb{N}}$; wir dürfen annehmen, daß jede Funktion α_i die Einschränkung auf S einer Funktion aus $\mathscr{D}(\mathbb{R}^n)$ ist. Wir bilden die G-invarianten Funktionen $\beta_i := \alpha_i \circ \sigma$ auf M. Der Träger von β_i ist in $\sigma^{-1}(U_i)$ enthalten, und es gilt $\sum_{i=1}^{\infty} \beta_i = 1$.

1.4. **Lemma.** *Die Abbildung $\Lambda: y \to \lambda_y$ von S nach $\mathscr{M}$ ist ein Homöomorphismus.*

Beweis. Sei $U \subseteq S$ offen, $y_0 \in U$. Es existiert eine nichtnegative reellwertige Funktion $\varphi \in \mathscr{D}(\mathbb{R}^n)$ mit $\varphi(y_0) > 0$ derart, daß der Träger von $\varphi|_M$ in $\sigma^{-1}(U)$ enthalten ist. Die Menge $\{\lambda \in \mathscr{M} \mid \lambda(\varphi) > 0\}$ ist dann eine in $\Lambda(U)$ enthaltene offene Umgebung von $\lambda_{y_0} \in \mathscr{M}$. Die Abbildung Λ ist also offen.

Sei umgekehrt $(y_k)_{k \in \mathbb{N}}$ eine Folge in S, die gegen $y_0 \in S$ konvergiert. Wir wählen eine Umgebung U von y_0 gemäß 1.3 und leiten aus $\lambda_y(\varphi) = \int_{\mathbb{R}^d} \varphi\left(y + \sum_{j=1}^{d} z_j e_j(y)\right) dz$ ohne Schwierigkeiten ab, daß λ_{y_k} gegen λ_{y_0} konvergiert in $\mathscr{K}'(\mathbb{R}^n)$, $\mathscr{D}'(\mathbb{R}^n)$ und $\mathscr{S}'(\mathbb{R}^n)$.

1.5. Wir definieren für $\varphi \in \mathscr{K}(M)$ eine Funktion $\tilde{\varphi} \in \mathscr{K}(S)$ durch $\tilde{\varphi}(y) := \lambda_y(\varphi)$. Der Träger von $\tilde{\varphi}$ ist enthalten im Bild des Trägers von φ unter σ. Man sieht leicht ein, daß die Abbildung $\varphi \to \tilde{\varphi}$ von $\mathscr{K}(M)$ nach $\mathscr{K}(S)$ stetig ist.

Wir wählen eine Funktion $\gamma \in \mathscr{D}(]0, \infty[)$ mit $\gamma \geqq 0$ und $\int_{\mathbb{R}^d} \gamma(|z|) dz = 1$; nun definieren wir für $f \in \mathscr{K}(S)$ eine stetige Funktion Rf auf M durch $Rf(x) := f(\sigma(x)) \cdot \gamma(|x - \sigma(x)|)$. Man sieht leicht ein, daß Rf kompakten Träger hat und daß die Abbildung $R: \mathscr{K}(S) \to \mathscr{K}(M)$ stetig ist; denn zu jeder kompakten Menge $K \subseteq S$ existiert ein $r \in \mathbb{N}$ mit $f = \sum_{i=1}^{r} \alpha_i f$ für alle $f \in \mathscr{K}(S)$, deren Träger in K enthalten ist (1.3), und es gilt $R(\alpha_i f) \circ \Phi_{U_i}^{-1}(y, z) = (\alpha_i f)(y)\, \gamma(|z|)$.

Offenbar gilt $\widetilde{Rf} = f$ für $f \in \mathscr{K}(S)$. Folglich ist die Abbildung $\varphi \to \tilde{\varphi}$ von $\mathscr{K}(M)$ nach $\mathscr{K}(S)$ surjektiv. Ferner ist $P: \mathscr{K}(M) \to \mathscr{K}(M)$, definiert durch $P\varphi := R\tilde{\varphi}$, eine Projektion in $\mathscr{K}(M)$. Der Kern von P ist $\mathscr{M}^{\perp}$, und R definiert einen Isomorphismus von $\mathscr{K}(S)$ auf das Bild von P. Die zu P transponierte Abbildung ist eine Projektion in $\mathscr{K}'(M)$, deren Bild genau aus $\mathscr{M}^{\perp\perp} = \mathscr{K}'_G(M)$ besteht.

1.6. **Satz.** *Die zu $\varphi \to \tilde{\varphi}$ transponierte Abbildung liefert einen Isomorphismus von $\mathscr{K}'(S)$ auf $\mathscr{K}'_G(M)$.*

1.7. Unter Beachtung von Lemma 1.4 erhalten wir das folgende

Korollar. *Die Abbildung $\nu \to \int_{\mathscr{M}} \lambda\, d\nu(\lambda)$ von $\mathscr{K}'(\mathscr{M})$ nach $\mathscr{K}'(M)$ liefert eine Parametrisierung des Raumes der G-invarianten Maße auf M.*

1.8. Im folgenden verzichten wir auf die Voraussetzung, daß alle Bahnen in M dieselbe Dimension haben. Dann wird die Aussage von Korollar 1.7 im allgemeinen nicht mehr gelten; denn erstens hat für $\varphi \in C_c(M)$ die Funktion $\tilde{\varphi}$ auf $\mathscr{M}$, definiert durch $\tilde{\varphi}(\lambda) := \lambda(\varphi)$, nicht notwendig kompakten Träger, und zweitens braucht es nicht zu jedem $f \in \mathscr{K}(\mathscr{M})$ ein $\varphi \in \mathscr{K}(M)$ zu geben mit $f = \tilde{\varphi}$.

Wir können nichtsdestoweniger die Resultate der Abschnitte 1.2 bis 1.7 auf M_d, $0 \leqq d \leqq n$, anwenden. Die assoziierten Objekte seien σ_d, S_d, R_d und P_d.

1.9. **Lemma.** *$\mathscr{M}_d$ ist offen und abgeschlossen in $\mathscr{M}$ für $d = 0, \ldots, n$.*

Beweis. Offenbar genügt es, die Abgeschlossenheit zu beweisen. Sei $(\lambda_i)_{i\in I}$ ein gegen $\lambda\in\mathscr{M}$ konvergierendes Netz in $\mathscr{M}_d$. Dann ist das Netz $(y_i)_{i\in I}$ in S_d mit $\lambda_i=\lambda_{y_i}$, $i\in I$, beschränkt, besitzt also ein in $\mathbb{R}^n$ konvergierendes Teilnetz; wir können also $\lim\limits_{i\in I} y_i =: y\in\mathbb{R}^n$ annehmen. Ferner besitzt das Netz $(E(y_i))_{i\in I}$ in der Graßmannschen Mannigfaltigkeit $G_d(\mathbb{R}^n)$ ein konvergentes Teilnetz, da $G_d(\mathbb{R}^n)$ kompakt ist; d. h. $\lim\limits_{i\in I} E(y_i) =: E\in G_d(\mathbb{R}^n)$. Folglich konvergiert $(\lambda_i)_{i\in I}$ in $\mathscr{K}'(\mathbb{R}^n)$, $\mathscr{D}'(\mathbb{R}^n)$ und $\mathscr{S}'(\mathbb{R}^n)$ gegen das Euklidische Maß $\tilde{\lambda}$ auf $y+E$, woraus $\lambda=\tilde{\lambda}$ und damit $\lambda\in\mathscr{M}_d$ folgt.

1.10. Zusammen mit Lemma 1.4 ergibt sich der

Satz. *$\mathscr{M}$ ist lokalkompakt mit abzählbarer Basis.*

1.11. **Theorem.** *Zu jedem G-invarianten Maß μ auf M existiert genau ein Maß ν auf $\mathscr{M}$ derart, daß für jedes $\varphi\in\mathscr{K}(M)$ die Funktion $\lambda\to\lambda(\varphi)$ auf $\mathscr{M}$ ν-integrierbar ist und $\mu(\varphi)=\int\limits_{\mathscr{M}}\lambda(\varphi)\,d\nu(\lambda)$ gilt.*

Beweis. Da $\tilde{M}_{d-1}$ abgeschlossen ist in M für alle $d=1,\dots,n$, kann $\mathscr{K}'(\tilde{M}_{d-1})$ als (abgeschlossener) Unterraum von $\mathscr{K}'(M)$ verstanden werden. Folglich hat $\bigcup\limits_{k<d}\mathscr{M}_k$ in $\mathscr{K}'(\tilde{M}_{d-1})$ dasselbe Erzeugnis wie in $\mathscr{K}'(M)$. Ebenso kann $\mathscr{K}'(\mathscr{M}_d)$ nach Lemma 1.9 als Unterraum von $\mathscr{K}'(\mathscr{M})$ verstanden werden. Ist d maximal mit $M_d\neq\emptyset$, so ist $M_d\subseteq M$ offen, und $\mathscr{K}(M_d)$ kann auf natürliche Weise in $\mathscr{K}(M)$ eingebettet werden.

Es genügt offenbar, den Satz für positive Maße μ zu beweisen. Sei d maximal mit $M_d\neq\emptyset$ und sei $\dot{\mu}_d$ das durch μ auf M_d induzierte Maß. Nach Korollar 1.7 existiert genau ein Maß $\nu_d\in\mathscr{K}'(\mathscr{M}_d)\subseteq\mathscr{K}'(\mathscr{M})$ mit $\dot{\mu}_d(\varphi)=\int\lambda(\varphi)\,d\nu_d(\lambda)$ für alle $\varphi\in\mathscr{K}(M_d)$. Wir sehen nun ein, daß die Funktionen $\tilde{\varphi}:\lambda\to\lambda(\varphi)$ auf $\mathscr{M}_d$ für alle $\varphi\in\mathscr{K}(M)$ integrierbar sind hinsichtlich des (positiven) Maßes ν_d; denn mit den in 1.3 konstruierten Funktionen β_i auf M_d gilt:

$$\begin{aligned}\int|\tilde{\varphi}(\lambda)|\,d\nu_d(\lambda)&\leqq\int\lambda(|\varphi|)\,d\nu_d(\lambda)=\sum_i\int\lambda(|\varphi|\cdot\beta_i)\,d\nu_d(\lambda)\\&=\sum_i\mu(|\varphi|\cdot\beta_i)\leqq\mu(|\varphi|)<\infty\,.\end{aligned}$$

Also wird durch $\mu_d(\varphi):=\int\lambda(\varphi)\,d\nu_d(\lambda)$ ein G-invariantes Maß auf M definiert.

Es ist klar, daß das Maß $\mu-\mu_d$ zu $\mathscr{K}'(\tilde{M}_{d-1})$ gehört. Nun wiederholen wir den soeben beschriebenen Prozeß, $\tilde{M}_{d-1}$ an die Stelle von M und $\mu-\mu_d$ an die Stelle von μ setzend. In dieser Weise fortfahrend, erhalten wir schließlich eine Zerlegung $\mu=\sum\limits_{d=0}^{n}\mu_d$, wobei die Maße μ_d triviale Fortsetzungen von Maßen auf M_d sind. (Natürlich setzen wir $\mu_d=0$, falls M_d leer ist.) Mit $\nu:=\sum\limits_{d=0}^{n}\nu_d$ ist dann die Behauptung des Satzes erfüllt.

1.12. *Bemerkung.* Wie der Beweis von Theorem 1.11 zeigt, kann das Desintegrationsmaß ν konstruktiv angegeben werden. Für $f\in\mathscr{K}(\mathscr{M}_d)$ gilt nämlich $\nu_d(f)=\mu(R_d f)$ (s. 1.5).

1.13. *Bemerkung.* Nicht für jedes Maß ν auf $\mathcal{M}$ wird durch die Gleichung $\mu(\varphi) = \int \lambda(\varphi)\, d\nu(\lambda)$ ein Maß μ auf M definiert (vgl. 1.8). Wirkt zum Beispiel $G = \mathbb{R}$ auf $M = \mathbb{R}^2$ vermöge $x \cdot t := (x_1, tx_1 + x_2)$, $x \in M$, $t \in G$, so gilt $\mathcal{M}_1 \cong S_1 = (\mathbb{R} \setminus \{0\}) \times \{0\}$, und für das Maß $\nu := \frac{1}{|x_1|} dx_1 \in \mathcal{K}'(\mathcal{M}_1) \subseteq \mathcal{K}'(\mathcal{M})$ konvergiert das Integral $\int \lambda(\varphi)\, d\nu(\lambda)$, $\varphi \in \mathcal{K}(M)$, im allgemeinen nicht.

2. Liesche Transformationsgruppen mit affinen Bahnen

2.1. Bei dem Bestreben, für Liesche Transformationsgruppen Resultate zu gewinnen, die den in Abschn. 1 für topologische Transformationsgruppen gewonnenen Ergebnissen entsprechen, wobei die Begriffe „Stetigkeit“ und „Maß“ durch die Begriffe „Glattheit“ und „Distribution“ zu ersetzen sind, treten sogleich erhebliche Schwierigkeiten auf:

a) Ist $M \subseteq \mathbb{R}^n$ eine (differenzierbare) Mannigfaltigkeit, so braucht M_d, $0 \leqq d \leqq n$, keine Mannigfaltigkeit mehr zu sein. Die in Abschn. 1 durchgeführten Reduktionen auf M_d sind also hier nicht mehr ohne weiteres durchführbar.

b) Im Beweis von Theorem 1.11 wurde die Zerlegbarkeit von Maßen in positive Maße verwendet, die man bei Distributionen nicht hat.

c) Im Beweis von Theorem 1.11 wurde die Lokalisierbarkeit von Maßen benutzt; und zwar durften wir schließen, daß ein von $\tilde{M}_{d-1}$ getragenes Maß schon zu $\mathcal{K}'(\tilde{M}_{d-1})$ gehört. Distributionen haben diese Eigenschaft nicht.

d) Die Menge $\mathcal{D}'_G(M)$ der G-invarianten Distributionen wird im allgemeinen nicht von $\mathcal{M}$ erzeugt [2, 4.5]; [4, 1.7]. Eine Desintegrationsformel wie in Theorem 1.11 kann also für G-invariante Distributionen nicht erwartet werden.

2.2. Um mit der Schwierigkeit 2.1 (a) zurechtzukommen, wollen wir M weiterhin lediglich als lokalkompakte Teilmenge von $\mathbb{R}^n$ statt als Untermannigfaltigkeit annehmen. Einen Glattheitsbegriff für Funktionen auf M erhalten wir einfach mittels Vererbung von $\mathbb{R}^n$.

Allgemeiner wollen wir einen Glattheitsbegriff auf einer lokalkompakten Teilmenge A einer beliebigen Mannigfaltigkeit $\mathcal{X}$ mittels Vererbung definieren; d. h. wir nennen eine Abbildung φ von A in eine Mannigfaltigkeit $\mathcal{Y}$ *glatt*, wenn sie lokal mit der Einschränkung einer glatten Abbildung von $\mathcal{X}$ nach $\mathcal{Y}$ übereinstimmt, wenn es also zu jedem Punkt $p \in A$ eine Umgebung $U \subseteq A$ und eine glatte Abbildung $h: \mathcal{X} \to \mathcal{Y}$ gibt mit $\varphi|_U = h|_U$. (Ist A eine Untermannigfaltigkeit von $\mathcal{X}$, dann stimmt selbstverständlich der so definierte Glattheitsbegriff mit dem klassischen Glattheitsbegriff überein.) Wir bezeichnen die Menge aller glatten (komplexwertigen) Funktionen auf A mit $\mathcal{E}(A)$ und die Menge aller glatten Funktionen mit kompaktem Träger mit $\mathcal{D}(A)$.

Es ist klar, daß jede Funktion aus $\mathcal{D}(A)$ sogar global auf A mit der Einschränkung einer Funktion aus $\mathcal{D}(\mathcal{X})$ übereinstimmt. Wir können somit auf natürliche Weise eine Topologie auf $\mathcal{D}(A)$ definieren, indem wir eine Folge (φ_n) in $\mathcal{D}(A)$ als gegen 0 konvergent erklären, wenn alle Funktionen φ_n von einer festen kompakten Menge getragen werden und wenn eine Nullfolge (h_n) in $\mathcal{D}(\mathcal{X})$ existiert mit $\varphi_n = h_n|_A$. [Ist A abgeschlossen, dann ist diese Topologie gerade die Quotiententopologie für die Restriktionsabbildung von $\mathcal{D}(\mathcal{X})$ auf $\mathcal{D}(A)$.] Den Dualraum $\mathcal{D}'(A)$ bezeichnen wir dann als den Raum der Distributionen auf A.

Für unseren zugrundeliegenden Raum $M \subseteqq \mathbb{R}^n$ gilt dann offenbar $\mathcal{M} \subseteqq \mathcal{D}'(M)$; auch trägt $\mathcal{M}$ die von $\mathcal{D}'(M)$ induzierte Topologie, wenn wir $\mathcal{D}'(M)$ mit der Topologie der beschränkten Konvergenz versehen. Dementsprechend bezeichnen wir – anders als in 1.1 – im vorliegenden Paragraphen mit $\mathcal{M}^{\perp}$ das Orthogonal von $\mathcal{M}$ in $\mathcal{D}(M)$ und mit $\mathcal{M}^{\perp\perp}$ das Orthogonal von $\mathcal{M}^{\perp}$ in $\mathcal{D}'(M)$, also den von $\mathcal{M}$ erzeugten abgeschlossenen Unterraum von $\mathcal{D}'(M)$.

Wir verlangen nun, daß G „glatt" auf M wirkt, d. h. daß die Abbildung $(x, a) \to x \cdot a$ von $M \times G$ nach $\mathbb{R}^n$ glatt ist.

2.3. Wir betrachten zunächst wieder (bis einschließlich 2.5) den Fall, daß die Dimension aller Bahnen in M konstant gleich d ist, $1 \leqq d \leqq n$. Dann sind die Abbildungen $B, \sigma, \Phi_U, \alpha_i$ und β_i glatt (s. 1.2 und 1.3). (Ist M eine Untermannigfaltigkeit von $\mathbb{R}^n$, so ist S eine Untermannigfaltigkeit von M und $\sigma: M \to S$ eine Submersion.) Die Zuordnung $\varphi \to \tilde{\varphi}$ (s. 1.5) definiert eine stetige surjektive Abbildung von $\mathcal{D}(M)$ nach $\mathcal{D}(S)$ [5, Theorem 1]. Ebenso definiert R eine stetige Abbildung von $\mathcal{D}(S)$ nach $\mathcal{D}(M)$, und P definiert eine Projektion in $\mathcal{D}(M)$. Analog zu 1.6 erhalten wir den

Satz. *Die zu $\varphi \to \tilde{\varphi}$ transponierte Abbildung liefert einen Isomorphismus von $\mathcal{D}'(S)$ auf $\mathcal{M}^{\perp\perp}$.*

2.4. Ungeachtet der Bemerkung 2.1 (d) haben wir in unserer hiesigen Situation konstanter Bahnendimension den

Satz. $\mathcal{M}^{\perp\perp} = \mathcal{D}'_G(M)$.

Beweis. Wir haben zu zeigen, daß $T \in \mathcal{D}'_G(M)$ zu $\mathcal{M}^{\perp\perp}$ gehört. Wegen $T = \sum_{i=1}^{\infty} \beta_i T$ dürfen wir $M = \sigma^{-1}(U)$ annehmen mit $U = U_i$ für ein $i \in \mathbb{N}$ (s. 1.3). Die G-Operation auf M überträgt sich dann vermöge Φ_U zu einer G-Operation auf $U \times \mathbb{R}^d$ mit den Bahnen $\{y\} \times \mathbb{R}^d$, $y \in U$. Wir haben also zu zeigen, daß $T' \in \mathcal{D}'_G(U \times \mathbb{R}^d)$ jede Funktion $\psi \in \mathcal{D}(U \times \mathbb{R}^d)$ annulliert, für die $\int \psi(y, z)\, dz = 0$ ist für alle $y \in U$. Nun sieht man mittels Induktion, daß ψ die Form $\sum_{j=1}^{d} \frac{\partial \psi_j}{\partial z_j}$ hat mit $\psi_j \in \mathcal{D}(U \times \mathbb{R}^d)$; wir brauchen also nur zu zeigen, daß T' jede Funktion der Form $\frac{\partial \psi}{\partial z_j}$ annulliert.

Für jedes Element X der Liealgebra $\mathfrak{g}$ von G wird durch $\tilde{X}\psi(y, z) := \frac{d}{dt} \psi((y, z) \cdot \operatorname{Exp} tX)|_{t=0}$ ein Vektorfeld definiert; es gilt $\tilde{X}\psi(y, z) = \sum_{j=1}^{d} a_j(y, z) \frac{\partial \psi}{\partial z_j}(y, z)$ mit glatten Funktionen $a_j(y, z)$. Da die G-Bahnen in $U \times \mathbb{R}^d$ die Gestalt $\{y\} \times \mathbb{R}^d$ haben, existieren zu jedem Punkt $(y_0, z_0) \in U \times \mathbb{R}^d$ Elemente $X^{(k)} \in \mathfrak{g}$, $k = 1, \ldots, d$, so daß die zugehörige Matrix $(a_j^{(k)}(y_0, z_0))_{j,k=1,\ldots,d}$ Maximalrang hat; dann haben aber auch die Matrizen $(a_j^{(k)}(y, z))_{j,k=1,\ldots,d}$ Maximalrang für alle Punkte (y, z) aus einer offenen Umgebung N_0 von (y_0, z_0), so daß $\frac{\partial \psi}{\partial z_j}(y, z) = \sum_{k=1}^{d} b_k^{(j)}(y, z) \tilde{X}^{(k)}\psi(y, z)$ für $(y, z) \in N_0$ gilt mit glatten Funktionen $b_k^{(j)}(y, z)$.

Besorgen wir uns für die Gesamtheit solcher Umgebungen N_0 eine lokalendliche Verfeinerung und dazu eine Partition der Eins, so erkennen wir, daß es genügt,

$$\left\langle T', \sum_{k=1}^{d} b_k^{(j)} \tilde{X}^{(k)} \psi \right\rangle = 0 \quad \text{für} \quad \psi \in \mathscr{D}(N_0)$$

zu beweisen.

Nun folgt für jedes $T' \in \mathscr{D}'_G(U \times \mathbb{R}^d)$ und für jedes $X \in \mathfrak{g}$ durch Differentiation der Gleichung $\langle T', \omega((y,z) \cdot \operatorname{Exp} tX) \rangle = \langle T', \omega \rangle$ nach t in $t=0$ die Beziehung $\langle T', \tilde{X}\omega \rangle = 0$, $\omega \in \mathscr{D}(U \times \mathbb{R}^d)$. Da die Euklidischen Maße auf den G-Bahnen $\{y\} \times \mathbb{R}^d$ invariant sind, folgt insbesondere die Beziehung $\int \tilde{X}\omega\, dz = 0$, woraus $\int (\tilde{X}\omega)\chi\, dz = -\int \omega(\tilde{X}\chi)\, dz$ hergeleitet werden kann, $\omega, \chi \in \mathscr{D}(U \times \mathbb{R}^d)$. Für $\psi \in \mathscr{D}(N_0)$ folgt also

$$0 = \int \frac{\partial \psi}{\partial z_j} dz = \sum_{k=1}^{d} \int b_k^{(j)} \tilde{X}^{(k)} \psi\, dz = -\int \sum_{k=1}^{d} (\tilde{X}^{(k)} b_k^{(j)}) \psi\, dz\,;$$

somit ergibt sich $\sum_{k=1}^{d} \tilde{X}^k b_k^{(j)} = 0$ auf N_0, also

$$\sum_{k=1}^{d} b_k^{(j)} \tilde{X}^{(k)} \psi = \sum_{k=1}^{d} b_k^{(j)} \tilde{X}^{(k)} \psi + \sum_{k=1}^{d} (\tilde{X}^{(k)} b_k^{(j)}) \psi = \sum_{k=1}^{d} \tilde{X}^{(k)} (b_k^{(j)} \psi)$$

und damit $\left\langle T', \sum_{k=1}^{d} b_k^{(j)} \tilde{X}^{(k)} \psi \right\rangle = 0$ für $\psi \in \mathscr{D}(N_0)$.

Bemerkung. Da wir M nicht als Mannigfaltigkeit vorausgesetzt haben, konnten wir [1, Theorem B] hier nicht anwenden.

2.5. Sei $\xi_0 \in \xi$ der Fußpunkt des Lotes vom Nullpunkt auf die d-Ebene $\xi \in G(d,n)$. Wegen $\bar{\sigma}(\xi) = \xi_0$ ist die Abbildung $\bar{\sigma}: \mathscr{B} \to S$ glatt, so daß die Räume $\mathscr{D}(\mathscr{B})$ und $\mathscr{D}'(\mathscr{B})$ mit den Räumen $\mathscr{D}(S)$ und $\mathscr{D}'(S)$ identifiziert werden können. Aus 2.3 und 2.4 erhalten wir also das

Korollar. *Die Abbildung $W \to \int_{\mathscr{B}} \lambda_\xi \, dW(\xi)$ von $\mathscr{D}'(\mathscr{B})$ nach $\mathscr{D}'(M)$ liefert eine Parametrisierung des Raumes der G-invarianten Distributionen auf M.*

2.6. Nun wenden wir uns dem allgemeinen Fall nicht notwendig konstanter Bahnendimension zu. Unser Ziel ist es, für Distributionen aus $\mathscr{M}^{\perp\perp}$ ein zu 1.11 analoges Desintegrationstheorem zu beweisen. Angesichts der in 2.1 genannten Schwierigkeiten werden sich weitere Voraussetzungen nicht vermeiden lassen.

Wir bezeichnen mit $\bar{\mathscr{B}}_d$ den Abschluß von $\mathscr{B}_d$ in $G(d,n)$.

Theorem. *M sei abgeschlossen in $\mathbb{R}^n$ und G operiere glatt auf M. Wir nehmen an, daß M_d dicht liegt in $\tilde{M}_d$ und daß sich die (glatte) kanonische Abbildung $B_d: M_d \to G(d,n)$ zu einer glatten Abbildung $\tilde{B}_d: \tilde{M}_d \to G(d,n)$ fortsetzen läßt, $0 \leqq d \leqq n$.*

Dann existieren zu jeder Distribution $T \in \mathscr{M}^{\perp\perp}$ Distributionen $W_d \in \mathscr{D}'(\bar{\mathscr{B}}_d)$, $0 \leqq d \leqq n$, so daß für alle $\varphi \in \mathscr{D}(M)$ gilt:

$$T(\varphi) = \sum_{d=0}^{n} \int_{\bar{\mathscr{B}}_d} \lambda_\xi(\varphi)\, dW_d(\xi)\,.$$

Beweis. Durch $\tilde{\sigma}_d(x) := (\tilde{B}_d(x))_0$ wird eine Fortsetzung von σ_d zu einer glatten Abbildung $\tilde{\sigma}_d : \tilde{M}_d \to \mathbb{R}^n$ definiert. Dementsprechend kann man $R_d : \mathscr{D}(S_d) \triangleq \mathscr{D}(\mathscr{B}_d) \to \mathscr{D}(M_d)$ zu einer stetigen Abbildung $\tilde{R}_d : \mathscr{D}(\bar{\mathscr{B}}_d) \to \mathscr{D}(\tilde{M}_d)$ fortsetzen, indem man $\tilde{R}_d f(x) := f(\tilde{B}_d(x)) \cdot \gamma(|x - \tilde{\sigma}_d(x)|)$ für $f \in \mathscr{D}(\bar{\mathscr{B}}_d)$ definiert. Die Abbildung $\varphi \to \varphi_d$ von $\mathscr{D}(\tilde{M}_d)$ nach $\mathscr{D}(\bar{\mathscr{B}}_d)$ mit $\varphi_d(\xi) := \lambda_\xi(\varphi)$ ist stetig, und für $f \in \mathscr{D}(\bar{\mathscr{B}}_d)$ gilt $(\tilde{R}_d f)_d = f$; daher wird durch $\tilde{P}_d : \varphi \to \tilde{R}_d \varphi_d$ eine stetige Fortsetzung der Projektion P_d in $\mathscr{D}(M_d)$ zu einer Projektion in $\mathscr{D}(\tilde{M}_d)$ definiert, deren Kern gerade $\mathscr{M}_d^\perp$ ist.

Ist nun d maximal mit $M_d \neq \emptyset$, so gilt $\tilde{M}_d = M$. Wir definieren $W_d \in \mathscr{D}'(\bar{\mathscr{B}}_d)$ durch $\langle W_d, f \rangle := \langle T, \tilde{R}_d f \rangle$ und $T_d \in \mathscr{D}'(M)$ durch $\langle T_d, \varphi \rangle := \langle T, \tilde{P}_d \varphi \rangle$. Dann gilt $\langle T_d, \varphi \rangle = \int_{\bar{\mathscr{B}}_d} \lambda_\xi(\varphi) \, dW_d(\xi)$, und es ist $T - T_d \in \mathscr{D}'(\tilde{M}_{d-1})$; denn für $\varphi \in \mathscr{D}(M)$ mit $\varphi|_{\tilde{M}_{d-1}} = 0$ gilt $\varphi - \tilde{P}_d \varphi \in \mathscr{M}^\perp$, da $\varphi - \tilde{P}_d \varphi \in \operatorname{Kern}(\tilde{P}_d) = \mathscr{M}_d^\perp$ und $(\varphi - \tilde{P}_d \varphi)|_{\tilde{M}_{d-1}} = 0$ ist. Nun zeigen wir, daß $T - T_d \in \tilde{\mathscr{M}}_{d-1}^{\perp\perp}$ ist mit $\tilde{\mathscr{M}}_{d-1} := \bigcup_{k<d} \mathscr{M}_k$. Dazu haben wir zu zeigen, daß $\varphi - \tilde{P}_d \varphi \in \mathscr{M}^\perp$ ist für $\varphi \in \tilde{\mathscr{M}}_{d-1}^\perp$. Wegen $\varphi - \tilde{P}_d \varphi \in \mathscr{M}_d^\perp$ bleibt $\lambda(\varphi - \tilde{P}_d \varphi) = 0$ zu zeigen für $\lambda \in \tilde{\mathscr{M}}_{d-1}$. Dazu genügt es, $(\tilde{P}_d \varphi)|_{\tilde{M}_{d-1}} = 0$ zu beweisen. Dies ist getan, wenn wir $\lambda_\xi(\varphi) = 0$ für $\xi \in \bar{\mathscr{B}}_d \setminus \mathscr{B}_d$ gezeigt haben. Nun ist λ_ξ für jedes $\xi \in \bar{\mathscr{B}}_d \setminus \mathscr{B}_d$ ein G-invariantes Maß auf $\tilde{M}_{d-1}$, annulliert also nach 1.1 jede Funktion aus $\tilde{\mathscr{M}}_{d-1}^\perp$.

Nun wiederholen wir den soeben durchgeführten Prozeß, wobei wir $\tilde{M}_{d-1}$ an die Stelle von M und $T - T_d$ an die Stelle von T setzen (vgl. 1.11). So fortfahrend erhalten wir schließlich die gewünschte Zerlegung $T = \sum_{d=0}^{n} T_d$.

2.7. *Bemerkung.* Wir können eine Desintegrationsformel wie in 2.6 auch in der folgenden allgemeineren Situation erhalten:

Sei wieder M abgeschlossen und die Wirkung von G auf M glatt. Sei eine Zerlegung von M in paarweise disjunkte G-invariante Mengen $M^{(k)}$, $k = 0, \ldots, m$, gegeben mit folgenden Eigenschaften:

(i) Die Dimension aller Bahnen in $M^{(k)}$ ist konstant gleich d_k, $k = 0, \ldots, m$;

(ii) $M^{(k)}$ ist offen in $\tilde{M}^{(k)} := \bigcup_{j \leq k} M^{(j)}$;

(iii) Es existiert eine stetige lineare Abbildung $E^{(k)} : \mathscr{D}(\bar{M}^{(k)}) \to \mathscr{D}(\tilde{M}^{(k)})$, so daß $E^{(k)}\varphi$ eine Fortsetzung von $\varphi \in \mathscr{D}(\bar{M}^{(k)})$ ist, für die $E^{(k)}\varphi = 0$ auf $\tilde{M}^{(k)} \setminus M^{(k)}$ gilt, falls $\varphi = 0$ auf $\bar{M}^{(k)} \setminus M^{(k)}$ gilt.

(Für die Zerlegung $M^{(k)} = M_k$, $k = 0, \ldots, n$, sind diese Eigenschaften natürlich erfüllt.)

Sei $\mathscr{B}^{(k)} \subseteq G(d_k, n)$ die Menge der G-Bahnen in $M^{(k)}$. Wir nehmen nun an, daß sich die (glatte) kanonische Abbildung $B^{(k)} : M^{(k)} \to \mathscr{B}^{(k)}$ zu einer glatten Abbildung $\bar{B}^{(k)} : \bar{M}^{(k)} \to \bar{\mathscr{B}}^{(k)}$ fortsetzen läßt für $k = 0, \ldots, m$.

Dann existieren zu jeder Distribution $T \in \mathscr{M}^{\perp\perp}$ Distributionen $W^{(k)} \in \mathscr{D}'(\bar{\mathscr{B}}^{(k)})$, $k = 0, \ldots, m$, so daß für alle $\varphi \in \mathscr{D}(M)$ gilt:

$$T(\varphi) = \sum_{k=0}^{m} \int_{\bar{\mathscr{B}}^{(k)}} \lambda_\xi(\varphi) \, dW^{(k)}(\xi).$$

Man beweist diese Aussage wie Theorem 2.6. Genauso wie im Beweis von Theorem 2.6 gewinnt man die Distributionen $W^{(k)}$ rekursiv; und zwar setzt man $\langle W^{(k)}, f \rangle := \langle T, E^{(k)} \tilde{R}^{(k)} f \rangle$ für $k = m$, wobei $\tilde{R}^{(k)} : \mathscr{D}(\bar{\mathscr{B}}^{(k)}) \to \mathscr{D}(\bar{M}^{(k)})$ durch

$\tilde{R}^{(k)}f(x) := f(\tilde{B}^{(k)}(x)) \cdot \gamma(|x - (\tilde{B}^{(k)}(x))_0|)$ definiert sei, und dann rekursiv $\langle W^{(k)}, f \rangle := \left\langle T - \sum_{j=k+1}^{m} \int_{\mathscr{B}^{(k)}} \lambda_\xi \, dW^{(k)}(\xi), \, E^{(k)}\tilde{R}^{(k)} f \right\rangle$ für $k = m-1, m-2, \ldots, 0$.

2.8. **Korollar.** *Die Voraussetzungen von Theorem 2.6 seien für das größte d mit $M_d \neq \emptyset$ erfüllt. Dann gibt es zu jeder Distribution $T \in \mathscr{D}'_G(M)$ eine Distribution $T_d \in \mathscr{M}^{\perp\perp}$, so daß der Träger von $T - T_d$ in $M \backslash M_d$ enthalten ist.*

Beweis. Sei $\dot{T}_d \in \mathscr{D}'_G(M_d)$ die Einschränkung von T auf M_d. Nach Satz 2.4 ist $\langle \dot{T}_d, \varphi - P_d\varphi \rangle = 0$ für alle $\varphi \in \mathscr{D}(M_d)$. Nun sei $T_d \in \mathscr{D}'(M)$ definiert wie in 2.6. Dann gilt für $\varphi \in \mathscr{D}(M_d)$:

$$\begin{aligned}\langle T_d, \varphi \rangle &= \langle T, \tilde{P}_d\varphi \rangle = \langle T, P_d\varphi \rangle \\ &= \langle \dot{T}_d, P_d\varphi \rangle = \langle \dot{T}_d, \varphi \rangle = \langle T, \varphi \rangle .\end{aligned}$$

3. Unipotente Gruppenoperationen

3.1. Sei V ein endlichdimensionaler reeller Vektorraum und G eine analytische Untergruppe von $GL(V)$. Die Liealgebra $\mathfrak{g}$ von G sei realisiert als Liesche Unteralgebra der Liealgebra $\mathfrak{gl}(V)$ aller V-Endomorphismen.

G wirkt in natürlicher Weise als Liesche Transformationsgruppe auf V. Die G-Bahn $B(p)$ eines Punktes $p \in V$ ist eine Untermannigfaltigkeit von V, und der Tangentialraum an $B(p)$ in p identifiziert sich mit dem Unterraum $\mathfrak{g}p$ von V. Also ist $B(p)$ genau dann affin, wenn $B(p) = p + \mathfrak{g}p$ gilt. In diesem Fall ist $\mathfrak{g}p$ invariant unter G und folglich auch unter $\mathfrak{g}$.

3.2. **Satz.** *Sind alle G-Bahnen affin, so ist G unipotent; d. h. G besteht aus unipotenten V-Endomorphismen.*

Beweis. Wir führen den Beweis durch Induktion nach der Dimension von V. Ist V eindimensional, so ist die Aussage trivial. Sei nun $\dim V > 1$ und existiere ein $p \in V$ mit $\dim \mathfrak{g}p > 0$. Dann sind $\mathfrak{g}p$ und der Quotientenraum $V/\mathfrak{g}p$ Vektorräume niedrigerer Dimension. Nach Induktionsvoraussetzung wirkt G unipotent auf $\mathfrak{g}p$. und G induziert eine unipotente Untergruppe $\bar{G}$ von $GL(V/\mathfrak{g}p)$; d. h. die Einschränkung von $\mathfrak{g}$ auf den Unterraum $\mathfrak{g}p$ von V sowie die zu $\bar{G}$ gehörige Lieunteralgebra $\bar{\mathfrak{g}}$ von $\mathfrak{gl}(V/\mathfrak{g}p)$ bestehen aus nilpotenten Endomorphismen. Für $X \in \mathfrak{g}$ ist also $X^m|_{\mathfrak{g}p} = 0$ und $\bar{X}^k = 0$ für hinreichend große natürliche Zahlen, also $X^{m+k} = 0$.

3.3. *Bemerkung.* Ist G unipotent, so ist die Bahn $B(p)$ eines Punktes $p \in V$ bereits dann affin, wenn $\mathfrak{g}p$ invariant ist unter $\mathfrak{g}$.

Daß aus der $\mathfrak{g}$-Invarianz von $\mathfrak{g}p$ die Beziehung $B(p) \subseteq p + \mathfrak{g}p$ folgt, ist wegen $(\operatorname{Exp} X)p = p + \sum_{k \geq 1} \frac{1}{k!} X^k p$ klar. Nun ist $B(p)$ aus Dimensionsgründen offen in $p + \mathfrak{g}p$, andererseits aber als Bahn einer unipotenten Gruppe auch abgeschlossen also folgt $B(p) = p + \mathfrak{g}p$.

3.4. Sei G unipotent und sei G^t die Gruppe der zu den V-Endomorphismen aus G transponierten Endomorphismen des Dualraums V^* von V. Die Liealgebra $\mathfrak{g}^t$ von

G^t realisiert sich dann als die Menge der zu den Endomorphismen aus $\mathfrak{g}$ transponierten Endomorphismen.

Satz. *Genau dann sind sämtliche G-Bahnen in V affin, wenn sämtliche G^t-Bahnen in V^* affin sind.*

Beweis. Nach 3.3 genügt es, unter Voraussetzung der $\mathfrak{g}$-Invarianz von $\mathfrak{g}p$ für alle $p \in V$ die $\mathfrak{g}^t$-Invarianz von $\mathfrak{g}^t f$ für $f \in V^*$ zu beweisen. Wäre $\mathfrak{g}^t f$ nicht X^t-invariant für ein $X^t \in \mathfrak{g}^t$, so gäbe es ein $p \in V$ mit $(\mathfrak{g}^t f)p = 0$ und $X^t(\mathfrak{g}^t f)p \neq 0$. Dann wäre $f(\mathfrak{g}p) = 0$, aber $f(\mathfrak{g}Xp) \neq 0$; das ist wegen $\mathfrak{g}Xp \subsetneqq \mathfrak{g}p$ nicht möglich.

3.5. Im folgenden nehmen wir an, daß V (und damit auch V^*) Euklidisch ist und daß alle G-Bahnen in V affin sind. Nach 3.2 ist G dann unipotent. Folglich sind die Euklidischen Maße auf den G-Bahnen G-invariant [6, S. 54]. Um die Abhängigkeit von G zum Ausdruck zu bringen, bezeichnen wir die Menge der Euklidischen Maße auf den G-Bahnen nun mit $\mathcal{M}_G$.

Die Fouriertransformation $\mathcal{F}$ ist ein Isomorphismus von $\mathcal{S}(V)$ auf $\mathcal{S}(V^*)$ bzw. von $\mathcal{S}'(V^*)$ auf $\mathcal{S}'(V)$. Wegen $\mathcal{F}(\varphi \circ a) = \mathcal{F}\varphi \circ (a^{-1})^t$, $\varphi \in \mathcal{S}(V)$, $a \in G$, führt $\mathcal{F}$ die Menge $\mathcal{S}'_{G^t}(V^*)$ der G^t-invarianten temperierten Distributionen auf V^* in die Menge $\mathcal{S}'_G(V)$ über. Für $\lambda \in \mathcal{M}_{G^t}$ ist natürlich $\mathcal{F}\lambda \in \mathcal{S}'_G(V)$ ein Maß langsamen Wachstums. Bezeichnen wir in Anlehnung an 1.1 und 2.2 für eine Teilmenge $\mathcal{A}$ von $\mathcal{S}'(V)$ mit $\mathcal{A}^\perp$ das Orthogonal von $\mathcal{A}$ in $\mathcal{S}(V)$ und mit $\mathcal{A}^{\perp\perp}$ den von $\mathcal{A}$ erzeugten abgeschlossenen Unterraum von $\mathcal{S}'(V)$, so folgt unter Beachtung von 1.1 der

Satz. $(\mathcal{F}\mathcal{M}_{G^t})^{\perp\perp} = \mathcal{M}_G^{\perp\perp}$.

3.6. Sei N eine nilpotente einfach zusammenhängende Liegruppe mit der Liealgebra $\mathfrak{n}$, die wir mit einem Skalarprodukt versehen. Ist G die adjungierte Gruppe von N, so ist G^t die koadjungierte Gruppe. Nach Satz 3.4 sind also genau dann alle adjungierten Bahnen affin, wenn alle koadjungierten Bahnen affin sind. Diese Situation wollen wir im folgenden voraussetzen. Sei $\mathcal{C}$ die Menge der Charaktere irreduzibler Darstellungen von N. Identifizieren wir $\mathfrak{n}$ und N vermöge der Exponentialabbildung, so liefert Satz 3.5 zusammen mit Kirillovs Charakterformel das

Korollar. $\mathcal{C}^{\perp\perp} = \mathcal{M}_G^{\perp\perp}$.

3.7. In der Situation von 3.6 betrachten wir für $p \in \mathfrak{n}^* =: M$ die alternierende Bilinearform $(Z, Z') \mapsto \frac{1}{2\pi} p([Z, Z'])$ auf $\mathfrak{n}$, die durch Quotientenbildung eine nichtausgeartete alternierende Bilinearform β_p auf dem Tangentialraum $\mathfrak{g}^t p$ an $\xi := B(p)$ definiert. Die 2-Form $q \mapsto \beta_q$ auf ξ ist G^t-invariant, so daß durch das m-fache äußere Produkt $\omega := \frac{1}{m!} \beta \wedge \ldots \wedge \beta$ eine G^t-invariante Volumenform und damit ein G^t-invariantes Maß κ_p auf ξ definiert wird ($2m := \dim \xi$). Das Maß κ_p heißt Kostantmaß, und $\mathcal{F}\kappa_p \in \mathcal{S}'(\mathfrak{n}) \triangleq \mathcal{S}'(N)$ ist der Charakter der zur Kirillovbahn $B(p)$ gehörigen irreduziblen Darstellung von N [7]. Da das Euklidische Maß λ_p auf $B(p)$ ebenfalls G^t-invariant ist, existiert ein Proportionalitätsfaktor $c(p) > 0$ mit $\lambda_p = c(p) \cdot \kappa_p$.

Für $d \in \{0, \ldots, n\}$ sei M_d die Menge der Punkte aus $M = \mathfrak{n}^*$ mit d-dimensionaler Bahn.

Lemma. *Die Funktion $p \to c(p)$ ist stetig auf M_d.*

Beweis. Für $p \in M_d$ finden wir gemäß 1.2 eine Umgebung U und eine stetig von $q \in U$ abhängende Orthonormalbasis $e_1(q), \ldots, e_d(q)$ von $\mathfrak{g}^t q$; die Orientierung sei so gewählt, daß $\omega_q(e_1(q) \wedge \ldots \wedge e_d(q)) > 0$ ist. Dann ist $\det(\beta_q(e_i(q), e_j(q)))_{1 \leqq i,j \leqq d} > 0$ und $\omega_q = [\det(\beta_q(e_i(q), e_j(q)))_{1 \leqq i,j \leqq d}]^{1/2} \cdot \lambda_q$, wobei λ_q hier das kanonische Volumenelement auf $B(q)$ bezeichne. Also ist $c(q) = [\det(\beta_q(e_i(q), e_j(q)))_{1 \leqq i,j \leqq d}]^{1/2}$ stetig auf U.

3.8. Sei $\mathcal{N}$ die Menge aller Kostantmaße und $\mathcal{N}_d$ die Menge der Kostantmaße auf den d-dimensionalen Bahnen, $0 \leqq d \leqq n$. Ähnlich wie in 1.9 beweist man:

Lemma. *$\mathcal{N}_d$ ist offen und abgeschlossen in $\mathcal{N}$.*

3.9. Aus 1.9, 3.8, 1.4, 3.7 und 1.10 ergibt sich der

Satz. *Die kanonische Bijektion zwischen $\mathcal{M}_{G^t}$ und $\mathcal{N}$ ist ein Homöomorphismus. Insbesondere ist $\mathcal{N}$ (und damit auch $\mathcal{C}$) ein lokalkompakter Raum mit abzählbarer Basis.*

3.10. Sei T eine zentrale (d. h. unter den inneren Automorphismen invariante) Distribution auf N. Dann ist T als Distribution auf $\mathfrak{n}$ invariant unter G. Nach [9] ist T genau dann als Distribution auf N positiv definit (hinsichtlich der Gruppenmultiplikation auf N), wenn T als Distribution auf $\mathfrak{n}$ positiv definit ist (hinsichtlich der Addition auf $\mathfrak{n}$).

Theorem. *Zu jeder zentralen positiv definiten Distribution T auf N existiert genau ein positives Radonmaß ν auf $\mathcal{C}$ derart, daß für jedes $\varphi \in \mathcal{S}(N)$ die Funktion $\chi \to \chi(\varphi)$ auf $\mathcal{C}$ ν-integrierbar ist und $T(\varphi) = \int_{\mathcal{C}} \chi(\varphi)\, d\nu(\chi)$ gilt.*

Beweis. Nach dem Bochner-Schwartz-Theorem ist T Fouriertransformierte eines positiven temperierten G^t-invarianten Maßes μ auf $\mathfrak{n}^*$. Nach Theorem 1.11 existiert dann genau ein positives Radonmaß ν' auf $\mathcal{M}_{G^t}$ mit $\mu = \int_{\mathcal{M}_{G^t}} \lambda\, d\nu'(\lambda)$. Nach 1.9, 1.4 und 3.7 ist die Funktion $\lambda \to c(\lambda)$ auf $\mathcal{M}_{G^t}$ stetig, wobei $c(\lambda) := c(p)$ für $\lambda = \lambda_p$ definiert sei. Also ist das Maß $c(\lambda)\, d\nu'(\lambda)$ und damit auch dessen Bildmaß ν'' unter der kanonischen Abbildung $\mathcal{M}_{G^t} \to \mathcal{N}$ ein positives Radonmaß. Mittels der Konvergenzsätze der Maßtheorie sehen wir, daß die "a priori" zunächst nur für $\varphi \in \mathcal{D}(\mathfrak{n}^*)$ geltende Gleichung $\mu(\varphi) = \int_{\mathcal{N}} \kappa(\varphi)\, d\nu''(\kappa)$ für alle $\varphi \in \mathcal{S}(\mathfrak{n}^*)$ richtig ist. Die Aussage des Theorems wird nun durch das Bildmaß ν von ν'' unter der Abbildung $\mathcal{F}: \mathcal{N} \to \mathcal{C}$ erfüllt.

3.11. *Bemerkung.* Ist speziell T die Diracsche Distribution im Einselement, so liefert Theorem 3.10 gerade die Plancherelformel [7].

Allgemeiner gilt die Formel 3.10 sinngemäß auch für die „orbitalen Integrale" $T = \lambda_Z \in \mathcal{M}_G$, $Z \in \mathfrak{n}$, da diese sich aus positiv definiten Distributionen linear kombinieren lassen. In diesem Fall läßt sich die Formel jedoch auch direkt aus der Plancherelformel für die Faktorgruppe $N/\mathrm{Exp}(\mathfrak{g}Z)$ herleiten.

3.12. Nun wollen wir noch eine Klasse unipotenter Matrizengruppen angeben, für die die Voraussetzungen von Theorem 2.6 bzw. von Bemerkung 2.7 erfüllt sind Da wir vornehmlich an Desintegrationen in Dualräumen interessiert sind (s. 3.7

bis 3.11), wollen wir die operierende Matrizengruppe als Gruppe G^t transponierter Matrizen verstehen.

Sei E eine Teilmenge der Menge aller Paare (i,j) natürlicher Zahlen mit $1 \leqq j < i \leqq n$.

Der Matrizenbereich

$$G^t := \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ a_{ij} & & 1 \end{pmatrix} \middle| \begin{matrix} a_{ij} \in \mathbb{R} & \text{für} & (i,j) \in E \\ a_{ij} = 0 & \text{für} & (i,j) \notin E \end{matrix} \right\}$$

bildet genau dann eine Gruppe, wenn mit (i,j) und (j,k) auch stets $(i,k) \in E$ ist. Dies wollen wir im folgenden annehmen.

Für die natürliche Wirkung von G^t auf $M := \mathbb{R}^n$ konstruieren wir nun eine Zerlegung gemäß Bemerkung 2.7.

Sei $e_1, \ldots, e_n$ die Einheitsbasis in $\mathbb{R}^n$. Für eine Teilmenge $\tau \subseteqq \{1, \ldots, n\}$ bezeichne H_τ das lineare Erzeugnis der Vektoren e_i, $i \in \tau$; ferner sei $N_\tau := \{x \in \mathbb{R}^n | x_i = 0$ für alle $i \in \tau\}$ und N^τ das Komplement von N_τ in $\mathbb{R}^n$.

Für $i \in \{1, \ldots, n\}$ sei $E_i := \{j | (i,j) \in E\}$. Dann ist N_{E_i} (und damit auch N^{E_i}) G^t-invariant. Zum Nachweis dieser Behauptung haben wir zu zeigen, daß für $x \in N_{E_i}$ und $j \in E_i$ stets $\sum_{k=1}^{j-1} a_{jk} x_k = 0$ ist. Ist nun $a_{jk} \neq 0$, so ist $(j,k) \in E$, wegen $(i,j) \in E$ also auch $(i,k) \in E$, folglich $x_k = 0$.

Für $\tau \subseteqq \{1, \ldots, n\}$ setzen wir nun $\tilde{\tau} := \bigcup_{i \notin \tau} E_i$ und definieren die G^t-invariante Menge $M^\tau := \left(\bigcap_{i \in \tau} N^{E_i}\right) \cap \left(\bigcap_{i \notin \tau} N_{E_i}\right) = \left(\bigcap_{i \in \tau} N^{E_i}\right) \cap N_{\tilde{\tau}}$. Für $x \in M^\tau$ ist $B(x) = x + H_\tau$. Ist $M^\tau \neq \emptyset$, so gilt $\bar{M}_\tau = N_{\tilde{\tau}}$, da M^τ Zariski-offen in $N_{\tilde{\tau}}$ ist, sowie $\tau \cap \tilde{\tau} = \emptyset$. Um letzteres einzusehen, beachten wir, daß $B(x) = x + H_\tau \subseteqq M^\tau \subseteqq N_{\tilde{\tau}}$ gilt für $x \in M^\tau$, woraus $H_\tau \subseteqq N_{\tilde{\tau}}$ und damit $\tau \cap \tilde{\tau} = \emptyset$ folgt.

Nun wählen wir eine Aufzählung $\tau(k)$, $k = 0, \ldots, m$, aller $\tau \subseteqq \{1, \ldots, n\}$ mit $M^\tau \neq \emptyset$ derart, daß für $k' < k$ stets $|\tau(k')| \leqq |\tau(k)|$ gilt, und setzen $M^{(k)} := M^{\tau(k)}$; dann ist $M = \bigcup_k M^{(k)}$ eine Zerlegung von M in paarweise disjunkte G^t-invariante Mengen.

Wegen $B(x) = x + H_\tau$ für $x \in M^\tau$ gilt 2.7 (i). Zum Nachweis von 2.7 (ii) genügt es, $M^{(k)} = \tilde{M}^{(k)} \cap \left(\bigcap_{i \in \tau(k)} N^{E_i}\right)$ zu zeigen. Ist also $x \in \tilde{M}^{(k)} \cap \left(\bigcap_{i \in \tau(k)} N^{E_i}\right)$, so ist $x \in M^\tau$ für ein τ mit $|\tau| \leqq |\tau(k)|$, also $x \in \bigcap_{i \notin \tau} N_{E_i}$; wegen $x \in \bigcap_{i \in \tau(k)} N^{E_i}$ folgt $\tau(k) \subseteqq \tau$, also $\tau(k) = \tau$.

Es bleibt noch 2.7 (iii) zu zeigen. Sei $p_k : \mathbb{R}^n \to \bar{M}^{(k)}$ die kanonische Projektion. Wir wählen $\alpha \in \mathscr{D}(\mathbb{R}^n)$ mit $\alpha(0) = 1$ und setzen $E^{(k)}\varphi(x) := \varphi(p_k(x))\,\alpha(x - p_k(x))$, $x \in \tilde{M}^{(k)}$, für $\varphi \in \mathscr{D}(\tilde{M}^{(k)})$. Nun folgt 2.7 (iii), wenn $\tilde{M}^{(k)} \setminus M^{(k)} \subseteqq p_k^{-1}(\tilde{M}^{(k)} \setminus M^{(k)})$ gezeigt ist. Wäre also $p_k(x) \in M^{(k)}$ für ein $x \in \tilde{M}^{(k)} \setminus M^{(k)}$, so wäre $p_k(x) \in \bigcap_{i \in \tau(k)} N^{E_i}$, also $x \in \left(\bigcap_{i \in \tau(k)} N^{E_i}\right) \cap \tilde{M}^{(k)} = M^{(k)}$.

3.13. Für $\tau \subseteqq \{1, \ldots, n\}$ mit $M^\tau \neq \emptyset$ sei $V^\tau := N_{\tau \cup \tilde{\tau}}$. Für $y \in V^\tau$ sei $\lambda^{(y)}$ das Euklidische Maß auf $y + H_\tau$. (Liegt y in der nichtleeren Zariski-offenen Teilmenge $V^\tau \cap M^\tau$ von V^τ, dann ist natürlich $\lambda^{(y)} = \lambda_y$.)

Theorem. *Zu jeder G^t-invarianten Distribution T auf $\mathbb{R}^n$ existieren Distributionen $W^\tau \in \mathscr{D}'(V^\tau)$, wobei τ alle Teilmengen von $\{1, \ldots, n\}$ mit $M^\tau \neq \emptyset$ durchlaufe, so daß für alle $\varphi \in \mathscr{D}(\mathbb{R}^n)$ gilt:*

$$T(\varphi) = \sum_\tau \int_{V^\tau} \lambda^{(y)}(\varphi)\, dW^\tau(y).$$

Beweis. Wir können $V^{\tau(k)}$ mit $\bar{\mathscr{B}}^{(k)}$ (Notation von 2.7) identifizieren vermöge $y \to y + H_{\tau(k)}$. Die Behauptung des Theorems folgt nun mit 2.7, da $T \in \mathscr{M}_{G^t}^{\perp\perp}$ ist nach [3, Bemerkung 1.7]. (Man überzeugt sich leicht davon, daß Bemerkung 1.7 in [3] für beliebige – d. h. nicht notwendig temperierte – Distributionen sinngemäß richtig bleibt.)

Bemerkung. Wegen $\tau \cap \tilde{\tau} = \emptyset$ identifiziert sich $\mathbb{R}^n$ mit dem kartesischen Produkt $V^\tau \times H_\tau \times H_{\tilde{\tau}}$, so daß die Desintegrationsformel in der übersichtlichen Form

$$T = \sum_\tau W^\tau \otimes 1_{H_\tau} \otimes \delta_{H_{\tilde{\tau}}}$$

geschrieben werden kann.

3.14. Ist die Distribution T aus Theorem 3.13 temperiert, so können auch die Distributionen W^τ temperiert gewählt werden. (Man beachte dazu die Definition von $W^\tau := W^{(k)}$ in 2.7.) In diesem Fall können wir zur Fouriertransformierten übergehen und unsere Desintegrationsformel als Entwicklung invarianter Distributionen in Termen invarianter Eigendistributionen verstehen. Die Distributionen $\mathscr{F}\lambda^{(y)}$ sind nämlich G-invariante Eigendistributionen der G-invarianten Differentialoperatoren mit konstanten Koeffizienten. Genauer gilt das

Korollar. *Zu jeder G-invarianten temperierten Distribution T auf $\mathbb{R}^n$ existieren Distributionen $W^\tau \in \mathscr{S}'(V^\tau)$ mit*

$$T = \sum_\tau \mathscr{F}\lambda^{(y)}\, dW^\tau(y).$$

Bemerkung. Ist N ein Produkt von Heisenberggruppen, so hat die koadjungierte Gruppe G^t von N (hinsichtlich einer geeigneten Basis von $\mathfrak{n}^*$) die Form 3.12. Das Korollar liefert in diesem Fall gerade die Rothschild-Wolf-Formel [8, Theorem 4.2].

3.15. Sei $P(D) = \sum_\alpha a_\alpha D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $D_j = -\sqrt{-1}\,\frac{\partial}{\partial x_j}$, ein G-invarianter Differentialoperator mit konstanten Koeffizienten auf $\mathbb{R}^n$. Dann ist das Polynom $P(x) = \sum_\alpha a_\alpha x^\alpha$ invariant unter G^t, also ein Polynom nur in den Variablen x_i mit $E_i = \emptyset$.

Wir fragen nun, wann die Gleichung $P(D)u = f$ für $f \in \mathscr{S}'_G$ eine Lösung $u \in \mathscr{S}'_G$ hat. (Man vergleiche hierzu [10].) Im allgemeinen ist das nicht der Fall [12, 1.5].

Im folgenden beachten wir, daß für eine abgeschlossene Teilmenge $A \subsetneqq \mathbb{R}^n$ der Raum $\mathscr{D}'(A)$ der Distributionen auf A (Notation von 2.2) identifiziert werden kann mit dem Raum derjenigen Distributionen auf $\mathbb{R}^n$, die alle auf A verschwindenden Testfunktionen annullieren.

Theorem. *Sei $\mathscr{T}$ die Menge derjenigen $\tau \subsetneqq \{1, \ldots, n\}$ mit $M^\tau \neq \emptyset$, für die das Polynom $P(x)$ auf M^τ nicht identisch verschwindet. Sei $f \in \mathscr{S}'_G$.*

Die Gleichung $P(D)u=f$ hat genau dann eine Lösung $u\in\mathscr{S}'_G$, wenn $f\in\mathscr{D}'\left(\bigcup_{\tau\in\mathscr{T}} N_\tau\right)$ ist.

Ist insbesondere $P|_{M^\tau}\neq 0$ für $\tau=\emptyset$, so hat die Gleichung $P(D)u=f$ immer eine G-invariante Lösung u.

Beweis. Sei $P(D)u=f$ für ein $u\in\mathscr{S}'_G$ und sei $\mathscr{F}^{-1}u=\sum_\tau W^\tau\otimes 1_{H_\tau}\otimes\delta_{H_{\tilde\tau}}$ gemäß 3.13. Dann ist $\mathscr{F}^{-1}f=P\cdot\mathscr{F}^{-1}u=\sum_{\tau\in\mathscr{T}}(P^\tau W^\tau)\otimes 1_{H_\tau}\otimes\delta_{H_{\tilde\tau}}$ mit $P^\tau:=P|_{V^\tau}$, also $f=(2\pi)^{|\tau|}\sum_{\tau\in\mathscr{T}}\mathscr{F}_{V^\tau}(P^\tau W^\tau)\otimes\delta_{H_\tau}\otimes 1_{H_{\tilde\tau}}\in\mathscr{D}'\left(\bigcup_{\tau\in\mathscr{T}}N_\tau\right)$.

Umgekehrt folgt für den Fall, daß $f\in\mathscr{D}'\left(\bigcup_{\tau\in\mathscr{T}}N_\tau\right)$ ist, die Existenz einer G-invarianten Lösung u aus den beiden folgenden Lemmata.

3.16. **Lemma.** *Für $\tau\in\mathscr{T}$ sei $f\in\mathscr{S}'_G(N_\tau)$. Dann existiert ein $u\in\mathscr{S}'_G$ mit $P(D)u=f$.*

Beweis. Auf N_τ wirkt die Untergruppe G^t_τ aller Matrizen $(a_{ij})\in G^t$ mit $a_{ij}=0$ für $i\in\tau, j<i$. Für $T\in\mathscr{D}'(N_\tau)$ ist die Distribution $T\otimes 1_{H_\tau}$ auf $\mathbb{R}^n$ genau dann invariant unter G^t, wenn T invariant unter G^t_τ ist.

Wir schreiben nun f in der Form $f=g\otimes\delta_{H_\tau}$ mit $g\in\mathscr{S}'(N_\tau)$; dann gilt $\mathscr{F}^{-1}f=(\mathscr{F}_{N_\tau}^{-1}g)\otimes 1_{H_\tau}$ und folglich ist $\mathscr{F}_{N_\tau}^{-1}g\in\mathscr{S}'_{G^t_\tau}(N_\tau)$. Nach der Desintegrationsformel aus 3.13 zerfällt $\mathscr{F}_{N_\tau}^{-1}g$ in der Form

$$\mathscr{F}_{N_\tau}^{-1}g=\sum_{\tau'}W^{\tau'}\otimes 1_{H_{\tau'}}\otimes\delta_{H_{(\tau\cup\tau')^\sim}}$$

mit $W^{\tau'}\in\mathscr{S}'(V^{\tau\cup\tau'})$, wobei nur über solche $\tau'\subsetneqq\{1,\dots,n\}$ zu summieren ist, für die $\tau'\cap\tau=\emptyset$ und $\left(\bigcap_{i\in\tau'}N^{E_i}\right)\cap N_\tau\cap N_{(\tau\cup\tau')^\sim}\neq\emptyset$ ist. Da P nur von den Variablen x_i mit $E_i=\emptyset$ abhängt, hat $P|_{N_{(\tau\cup\tau')^\sim}}$ die Form $Q^\tau\otimes 1_{H_{\tau\cup\tau'}}$ mit einem Polynom $Q^{\tau'}$ auf $V^{\tau\cup\tau'}$; wegen $M^\tau\subsetneqq N_{\tilde\tau}\subsetneqq N_{(\tau\cup\tau')^\sim}$ und $\tau\in\mathscr{T}$ ist $Q^{\tau'}\neq 0$. Nach Hörmanders Divisionstheorem ([11]) existiert also eine Distribution $U^{\tau'}\in\mathscr{S}'(V^{\tau\cup\tau'})$ mit $Q^{\tau'}U^{\tau'}=W^{\tau'}$. Da $U^{\tau'}\otimes 1_{H_{\tau'}}\otimes\delta_{H_{(\tau\cup\tau')^\sim}}\in\mathscr{S}'_{G^t_\tau}(N_\tau)$ ist, wird durch $u:=\sum_{\tau'}\mathscr{F}(U^{\tau'}\otimes 1_{H_{\tau\cup\tau'}}\otimes\delta_{H_{(\tau\cup\tau')^\sim}})$ eine G-invariante Distribution definiert, für die offenbar $P\cdot\mathscr{F}^{-1}u=\mathscr{F}^{-1}f$ gilt.

3.17. **Lemma.** *Seien $\tau_1,\dots,\tau_r\subsetneqq\{1,\dots,n\}$ gegeben derart, daß die Unterräume N_{τ_i}, $i=1,\dots,r$, G-invariant sind. Sei $f\in\mathscr{S}'_G$ gegeben mit der Eigenschaft, daß $f\in\mathscr{D}'\left(\bigcup_{i=1}^r N_{\tau_i}\right)$ gilt. Dann zerfällt f in der Form $f=\sum_{i=1}^r f_i$ mit $f_i\in\mathscr{S}'_G(N_{\tau_i})$.*

Beweis. Wir führen den Beweis durch Induktion nach n. Ist G trivial, so folgt die Behauptung mit elementaren distributionentheoretischen Methoden durch Induktion nach r. Sei also G nichttrivial und habe G (nach einer Permutation der Basisvektoren) als Matrizengruppe eine Form wie G^t in 3.12, so daß $E_n\neq\emptyset$ ist. (E sei gemäß 3.12 hinsichtlich G erklärt.) Ist G' die Gruppe derjenigen Matrizen $(a_{ij})\in G$ mit $a_{nj}=0$ für $j=1,\dots,n-1$, so ist $N_{\{n\}}$ invariant unter G', und durch $\tilde f(\psi):=f(\psi\otimes\alpha)$, $\psi\in\mathscr{S}(N_{\{n\}})$, wird eine G'-invariante temperierte Distribution $\tilde f$ auf $N_{\{n\}}$ definiert mit $\tilde f\in\mathscr{D}'\left(\bigcup_{i=1}^r N_{\tau_i\cup\{n\}}\right)$, wobei α eine Testfunktion auf $\mathbb{R}e_n$ sei mit

$\alpha(0)=1$ und $\int \alpha(te_n)\,dt=1$. Nach Induktionsvoraussetzung zerfällt $\bar{f}$ in der Form $\bar{f}=\sum_{i=1}^{r}\bar{f}_i$ mit $\bar{f}_i\in\mathcal{S}'_{G'}(N_{\tau_i\cup\{n\}})$. Für diejenigen i mit $n\in\tau_i$ ist $\bar{f}_i\otimes\delta_{\mathbb{R}e_n}\in\mathcal{S}'_G(N_{\tau_i})$, so daß wir f durch $f-\sum_{\substack{i=1\\ n\in\tau_i}}^{r}\bar{f}_i\otimes\delta_{\mathbb{R}e_n}$ ersetzen können und demnach annehmen dürfen, daß $n\notin\tau_i$ ist für alle $i=1,\ldots,r$. Dann ist aber $\bar{f}_i\otimes 1_{\mathbb{R}e_n}\in\mathcal{S}'_G(N_{\tau_i})$, so daß wir nur noch für $g:=f-\sum_{i=1}^{r}\bar{f}_i\otimes 1_{\mathbb{R}e_n}\in\mathcal{D}'\left(\bigcup_{i=1}^{r}N_{\tau_i}\right)$ die Behauptung des Lemmas zu beweisen brauchen, die jedoch nach Induktionsvoraussetzung sogleich folgt, wenn wir $g\in\mathcal{S}'(N_{E_n})$ zeigen können.

Dazu beachten wir zunächst, daß für alle $\varphi\in\mathcal{D}(\mathbb{R}^n)$ die Gleichung $g(\varphi)=f(\varphi-\bar{\varphi}\otimes\alpha)$ gilt mit $\bar{\varphi}(y):=\int\varphi(y+te_n)\,dt$, $y\in N_{\{n\}}$. Wegen $\int(\varphi-\bar{\varphi}\otimes\alpha)(x+te_n)\,dt=0$ für alle $x\in\mathbb{R}^n$ existiert nun eine Funktion $\psi\in\mathcal{D}(\mathbb{R}^n)$ mit $\varphi-\bar{\varphi}\otimes\alpha=\frac{\partial\psi}{\partial x_n}$; also gilt $\langle x_j g,\varphi\rangle=\left\langle f, x_j\frac{\partial\psi}{\partial x_n}\right\rangle$ für alle $j\in E_n$. Nun schreibt sich aber $x_j\frac{\partial\psi}{\partial x_n}$ in der Form $x_j\frac{\partial\psi}{\partial x_n}=\frac{d}{dt}\psi(a(t)x)|_{t=0}$ mit einer passenden Matrix $a(t)\in G$; unter Beachtung der G-Invarianz von f ergibt sich dann $x_j g=0$ für alle $j\in E_n$, woraus $g\in\mathcal{S}'(N_{E_n})$ folgt.

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Inversion of Abelian Integrals on Small Genus Curves

Kevin R. Coombes and Robert J. Fisher, Jr.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
Department of Mathematics, University of Oklahoma, 601 Elm Avenue, Norman, OK 73019, USA

Let M be a smooth projective curve of genus $g > 0$. Let J be the Jacobian of M and let $M^{(i)}$ be the i^{th} symmetric product. Fix a base point $P \in M$ and define a map $\phi_i : M^{(i)} \to J$ by $\phi_i(D) = D - iP$. Mattuck [6, 7] has shown that, if $i \geqq 2g-1$, then ϕ_i is a $\mathbb{P}^{i-g}$-bundle. The most interesting case occurs when $i = 2g-1$, since in [1] the bundles for larger i are determined by this one.

The inversion of abelian integrals problem asks: What is an explicit description of the transition functions of the bundle ϕ_i? For an excellent introduction to the problem, see Kempf's article [5]. Until now, the only complete answer was given in genus 1 by the Abel Inversion Theorem, the Riemann-Roch Theorem, and Riemann's approach through theta functions. Even without the transition functions, Gunning [1, 2] and Kempf [3, 4] were able to extract a great deal of information about M from the bundles ϕ_i.

In this paper, we present a solution to the inversion of abelian integrals problem in the cases

(i) curves of genus 2

(ii) non-hyperelliptic curves of genus 3.

The techniques we use are very geometric but essentially elementary. We strongly emphasize the role played by effective divisors and by the Riemann-Roch Theorem. To build sections, we use the commutative diagram

$$\begin{tikzcd} M^{(g)} \arrow[r, "+D_0"] \arrow[d, "\phi_g"'] & M^{(2g-1)} \arrow[d, "\phi_{2g-1}"] \\ J \arrow[r, "+D_0-(g-1)P"] & J \end{tikzcd} .$$

where D_0 is a fixed effective divisor of degree $g-1$. The Jacobi Inversion Theorem says that ϕ_g is a birational surjection; hence, it has an inverse on an open set. It is then necessary to choose enough different D_0's to trivialize the bundle. The geometry first enters when we must describe the open set over which a given collection of D_0's determines a trivialization. We then use explicit knowledge of the canonical embedding of M to describe the transition functions. For precise statements, see Theorems B and C,

Remarks. (i) Based on our results, it seems reasonable to expect that a trivializing atlas of $\phi_{2g-1}: M^{(2g-1)} \to J$ can be obtained using $g+1$ Zariski open sets, each of which has complement equal to a union of translates of the theta divisor. It should be of particular interest to interpret the geometry of such an explicit trivialization in terms of theta functions.

(ii) All our results apply equally well to the linearized inversion of abelian integrals problem [5]. It is only necessary to supply the correct technical means for translating the section $D_0 + \phi_g^{-1}(\quad)$ into a section of the linearized bundle.

(iii) The critical geometric fact in the genus 3 case is the isomorphism between the space $\Gamma(M, \mathscr{L}(2K))$ of quadratic differentials and the space $\Gamma(\mathbb{P}^2, \mathcal{O}(2)]$ of conics. For curves of higher genus, there is merely a surjection

$$\Gamma(\mathbb{P}^{g-1}, \mathcal{O}(2)) \twoheadrightarrow \Gamma(M, \mathscr{L}(2K)).$$

More care will therefore be needed to exploit this geometry. It is intriguing to note that

$$\dim \Gamma(\mathbb{P}^{g-1}, \mathcal{O}(2)) = \frac{g^2+g}{2} = \dim \mathscr{A}_g$$

$$\dim \Gamma(M, \mathscr{L}(2K)) = 3g-3 = \dim \mathscr{M}_g$$

where $\mathscr{A}_g$ is the moduli space of principally polarized abelian varieties of dimension g and $\mathscr{M}_g$ is the moduli space of curves of genus g. One might hope that an explicit description from this point of view would provide further insight into the Schottky problem.

1. Geometric Preliminaries

We begin with a brief review of the problem, emphasizing divisors rather than line bundles. Let M be a smooth projective curve of positive genus. A divisor on M is a finite collection $D = \Sigma n_i P_i$ of points $P_i \in M$ with multiplicities $n_i \in \mathbb{Z}$. The degree of D is $\deg(D) = \Sigma n_i$. If all $n_i \geqq 0$, then D is called effective.

To each divisor D is associated a locally free sheaf $\mathscr{L}(D)$ of rank one on M. On small open sets $\mathscr{U}_i$, $D \cap \mathscr{U}_i$ is the divisor of zeros and poles of some rational function f_i on $\mathscr{U}_i$. The transition functions of $\mathscr{L}(D)$ are given by multiplication by f_i/f_j, which is invertible on $\mathscr{U}_i \cap \mathscr{U}_j$. We will write $\ell(D) = \dim \Gamma(M, \mathscr{L}(D))$.

Two divisors are called linearly equivalent, written $D \sim D'$, provided there exists a rational function f on M with divisor $(f) = D - D'$. If $s \in \Gamma(M, \mathscr{L}(D))$, then its divisor of zeros $(s)_0$ determines s up to a constant. So, the projectivization $\mathbb{P}(D) := \mathbb{P}(\Gamma(M, \mathscr{L}(D)))$ is the space of effective divisors on M which are linearly equivalent to D.

Let J denote the Jacobian of M, which parametrizes linear equivalence classes of divisors of degree zero. Let $M^{(i)}$ be the i^{th} symmetric product. A point $D \in M^{(i)}$ is an effective divisor of degree i. Choose a base point $P \in M$ and define

$$\phi_i: M^{(i)} \to J \qquad \phi_i(D) = D - iP.$$

Moreover, let W_i denote $\phi_i(M^{(i)})$ in J. Mattuck [6] showed that, if $i \geqq 2g-1$, then ϕ_i is a projective bundle whose fiber over $R \in J$ is $\mathbb{P}^{i-g} = \mathbb{P}(R+iP)$. The problem of inverting abelian integrals on M asks:

What is an explicit description of the transition functions of ϕ_i?

Gunning [1, Theorem 11] reduces the problem to the case $i = 2g-1$. From now on, we restrict our attention to this case.

We approach the problem from a very geometric point of view. Firstly, recall the following fundamental facts.

Theorem (Riemann-Roch). *Let K be the canonical divisor on M. Let D be any divisor. Then*

$$\ell(D) = \ell(K-D) + \deg(D) + 1 - g.$$

Theorem (Jacobi Inversion). *The map $\phi_g : M^{(g)} \to J$ is a surjective birational equivalence.*

If $\deg(D) \geqq 2g-1$, then $\ell(K-D) = 0$. The Riemann-Roch Theorem gives a computation of $\ell(D)$ that depends only on the degree; this is the first indication that ϕ_i should be a bundle. If $0 < \deg(D) < 2g-1$ and $\ell(K-D) > 0$, then we say that D is special. Most divisors are non-special; the Jacobi Inversion Theorem reflects this fact for divisors of degree g. Write $B \subset J$ for the closed set on whose complement ϕ_g is an isomorphism.

We now describe our basic technique to get a section of ϕ_{2g-1} over a comprehensible open subset of J.

Proposition 1. *Fix an effective divisor D_0 on M of degree $g-1$. Let B_0 be the translate in J of B by $D_0 - (g-1)P$. Then D_0 determines a section*

$$\sigma_0 : J - B_0 \to M^{(2g-1)}$$

of ϕ_{2g-1}, which is characterized by the fact that the divisor $\sigma_0(R) - D_0$ is effective.

Proof. There is a commutative diagram

$$\begin{tikzcd} M^{(g)} \arrow[r, "+D_0"] \arrow[d, "\phi_g"'] \arrow[dr, dashed, "f_0"] & M^{(2g-1)} \arrow[d, "\phi_{2g-1}"] \\ J \arrow[r, "+D_0-(g-1)P"'] & J \end{tikzcd}.$$

The composite f_0, by the Jacobi Inversion Theorem, has an inverse on $J - B_0$. Take

$$\sigma_0(R) = f_0^{-1}(R) + D_0.$$

It is clear that $\sigma_0(\mathrm{R}) - D_0$ is effective of degree g, and must be the unique effective representative of its divisor class when $R \in J - B_0$. $\square$

An explicit description of the transition function still requires two steps. Firstly, describe how to choose a collection $D_0^{(1)}, \ldots, D_0^{(g+1)}$ of effective divisors of degree $g-1$ in such a way that the corresponding sections trivialize $M^{(2g-1)}$ on a describable open set. Secondly, give an explicit description of how the sections change when the collection of effective divisors is changed.

We illustrate the usefulness of emphasizing divisors with the following result, which we will need later. Define $f: M \to J$ by $f(X) = X + K - (2g-1)P$. Notice that f is the translation of the usual embedding ϕ_1 by the degree zero divisor class $K-(2g-2)P$; that is, $f(M) = W_1 + K - (2g-2)P$.

Theorem A. *The pullback $f^*(\phi_{2g-1})$ is a trivial bundle on M, with fibre naturally isomorphic to $\mathbb{P}(K)$.*

Proof. Let $X \in M$. The fibre over X is given by projectivizing

$$\Gamma(M, \mathscr{L}(f(X) + (2g-1)P)) = \Gamma(M, \mathscr{L}(K+X)).$$

But $\ell(K+X) = \ell(K) = g$ and there is a natural inclusion

$$\Gamma(M, \mathscr{L}(K)) \subset \Gamma(M, \mathscr{L}(K+X)).$$

By dimension counting, this inclusion is an isomorphism. The fibre is therefore independent of the point X and the pullback bundle is trivial. □

2. Curves of Genus Two

Let M be a curve of genus 2. In this section we study the $\mathbb{P}^1$-bundle $\phi_3 : M^{(3)} \to J$. As in Proposition 1, the choice of a single point $Q_0 \in M$ determines a section σ_0 of ϕ_3, which is characterized by the fact that

$$\sigma_0(R) = E_0(R) + Q_0$$

where $E_0(R)$ is the unique effective divisor linearly equivalent to $R + 3P - Q_0$. The section is defined on an open set $J - B_0$ for a translate B_0 of B, the image under ϕ_2 of the locus of special divisors of degree 2.

Lemma 2. *B_0 is a single point of J; namely, $B_0 = [K + Q_0 - 3P]$.*

Proof. The canonical class is the unique special divisor class of degree 2 on the hyperelliptic curve M. □

Now let $Q = \{Q_0, Q_1, Q_\infty\}$ be an ordered triple of points of M which lie in distinct fibres of the canonical hyperelliptic map of M to $\mathbb{P}^1$, and which are also not ramification points of this map. There are three corresponding sections of ϕ_3 on some open set of J. A trivialization of ϕ_3 is determined over the open set where these sections take on distinct values.

Proposition 3. *Let $\mathscr{U}$ be the maximal open set where the sections corresponding to Q trivialize ϕ_3. Then $J - \mathscr{U}$ consists of the union of three translates of the theta divisor; namely,*

$$\begin{aligned} &W_1 + [Q_0 + Q_1 - 2P] \\ &W_1 + [Q_0 + Q_\infty - 2P] \\ &W_1 + [Q_1 + Q_\infty - 2P]. \end{aligned}$$

Proof. By symmetry, it is enough to determine the locus where $\sigma_0(R) = \sigma_1(R)$. These sections are equal if and only if there is an equality of divisors

$$E_0(R) + Q_0 = E_1(R) + Q_1.$$

Since $Q_0 \neq Q_1$, the latter equality implies

$$E_0(R) = Q_1 + X$$
$$E_1(R) = Q_0 + X$$

for some $X \in M$. Hence the dependency locus is

$$\{R \in J : R \sim Q_0 + Q_1 + X - 3P \quad \text{for some} \quad X \in M\} = W_1 + [Q_0 + Q_1 - 2P]. \quad \square$$

Proposition 4. *Let Q and $\bar{Q}$ be two triples of points which determine trivializations of ϕ_3 on open sets $\mathscr{U}$ and $\bar{\mathscr{U}}$. Assume all 6 points of $Q \cup \bar{Q}$ have canonically distinct images. Then $J - (\mathscr{U} \cup \bar{\mathscr{U}})$ consists of exactly 18 points.*

Proof. This is a special case of the Poincaré relation. It suffices to check that

$$I = W_1 + [Q_0 + Q_1 - 2P] \cap W_1 + [\bar{Q}_0 + \bar{Q}_1 - 2P]$$

contains exactly 2 points. If $R \in I$, then there exist $X, Y \in M$ such that

$$R + 3P \sim X + Q_0 + Q_1 \sim Y + \bar{Q}_0 + \bar{Q}_1 .$$

Write * for the hyperelliptic involution on M. Then

$$X + Y^* \sim (Y + \bar{Q}_0 + \bar{Q}_1) - (Q_0 + Q_1) + Y^*$$
$$\sim K + (\bar{Q}_0 + \bar{Q}_1 - Q_0 - Q_1) .$$

By the assumption on Q and $\bar{Q}$, the latter is a nontrivial translate of the canonical class, hence nonspecial. So, it has a unique effective representative $S + T$. Thus, the only 2 points of I occur when

(i) $X = S \quad Y = T^*$

(ii) $X = T \quad Y = S^* .$ $\square$

Proposition 5. *There exist 3 triples $Q, \bar{Q}, \bar{\bar{Q}}$ such that the union of the corresponding trivializing neighborhoods is*

$$\mathscr{U} \cup \bar{\mathscr{U}} \cup \bar{\bar{\mathscr{U}}} = J .$$

Proof. Let Q and $\bar{Q}$ be any 2 triples as in Proposition 4. It suffices to choose $\bar{\bar{Q}}$ so that the translates of the theta divisor by pairs of points in $\bar{\bar{Q}}$ miss the 18 points of the complement of $\mathscr{U} \cup \bar{\mathscr{U}}$. This is an open condition. $\square$

Finally, we address the question of how to describe the transition functions. Fix a model of M of the form

$$M : y^2 = h(x)$$

where $h(x)$ is a monic polynomial of degree 6 with distinct zeroes. Fix a triple Q. A general point $R \in J$ determines a unique pair $X, Y \in M$ with

$$X + Y \sim R + 3P - Q_0 .$$

Functions on J will be described by symmetric functions in X and Y.

Lemma 6. *Suppose X, Y, Q_0 and Q_∞^* lie in distinct fibres of the canonical map. Then there is a unique cubic C of the form*

$$C : y = f(x)$$

such that C passes through all 4 points.

Proof. If all 4 points lie in the affine (x, y)-plane, the result is clear. The potential difficulties arise if one of the points lies over the singularity at infinity in the planar model. Homogenize the equations of M and C in the standard way and pass to the affine $y=1$ where

$$M: z^4 = \prod_{i=1}^{6} (x - a_i z)$$

$$C: z^2 = d \prod_{i=1}^{3} (x - b_i z).$$

The successive blow-ups which desingularize both curves can be summarized by the substitutions

$$z = tu^3 \qquad x = tu^2$$

whence

$$M: 1 = t^2 \prod (1 - a_i u)$$

$$C: 1 = dt \prod (1 - b_i u).$$

The exceptional points over ∞ occur where $u=0$. Thus, M meets C at a point over ∞ if and only if $d = \pm 1$. Since this imposes exactly one condition on C, the result follows. $\square$

The curves M and C intersect in points which are determined by the zeroes of

$$h(x) - f(x)^2 = 0.$$

This is a sextic equation, with four zeroes given by the x-coordinates of X, Y, Q_0, and Q_∞^*. Factor these out and let $q(x)$ denote the monic residual quadratic. Define a rational function in M by

$$\gamma = \gamma_{Q,R}(x, y) = (y - f(x))/(x - x_\infty) q(x),$$

where $Q_\infty = (x_\infty, y_\infty)$. For fixed Q and R, $\gamma_{Q,R}$ is a degree 3 map to $\mathbb{P}^1$ such that

$$(\gamma_{Q,R})_0 = X + Y + Q_0 = \sigma_0(R)$$

$$(\gamma_{Q,R})_\infty = \sigma_\infty(R).$$

In order to normalize this function correctly, let

$$c_{Q,R} = \gamma_{Q,R}(Q_1).$$

If $R \in \mathscr{U}$ is a point where the hypotheses of Lemma 6 hold, then $c_{Q,R}$ is a non-zero constant. Define

$$\psi_{Q,R} = \gamma_{Q,R}/c_{Q,R}.$$

So, $\psi_{Q,R}$ determines an isomorphism

$$\mathbb{P}(\Gamma(M, \mathscr{L}(R + 3P))) \approx \mathbb{P}^1.$$

As R varies, the trivialization of ϕ_3 made explicit by ψ agrees with the one defined abstractly as corresponding to Q.

Unfortunately, the definition of ψ does not make sense for all $R \in \mathscr{U}$. The difficulty occurs when 2 of the 4 points X, Y, Q_0, Q_∞^* lie in the same canonical fibre.

If 2 such points are equal, then there is no problem since one has merely specified the tangent direction to C at a point. If $X = Y^*$, then the point

$$R = [X + Y + Q_0 - 3P] = [K + Q_0 - 3P] \notin \mathscr{U},$$

so there is again no problem. Finally, what happens when $X = Q_0^*$? Then $R + 3P \sim K + Y(R)$ for uniquely determined $Y(R)$ and the appropriate definitions are

$$\gamma_{Q,R} = (x - x_0)/(x - x_\infty)$$
$$\psi_{Q,R} = \gamma_{Q,R}/\gamma_{Q,R}(Q_1).$$

In any case, let $\psi_{Q,R}^{-1}(z)$ also denote the divisor naturally defined by the fibre of $\psi_{Q,R}$ at $z \in \mathbb{P}^1$. For most $R \in \mathscr{U}$, $\psi_{Q,R}^{-1}(z)$ has degree 3 and is linearly equivalent to $R + 3P$. When it does not, it has degree 2; and this case was described above. So explicitly a trivialization on $\mathscr{U}$ is given as follows:

$$\psi^{-1} : \mathscr{U} \times P^1 \to \phi_3^{-1}(\mathscr{U})$$

is defined by

$$\psi^{-1}(R, z) = \begin{cases} \psi_{Q,R}^{-1}(z) & \text{if } \psi \text{ is a degree 3 map to } \mathbb{P}^1 \\ \psi_{Q,R}^{-1}(z) + Y(R) & \text{otherwise.} \end{cases}$$

One can check, essentially by l'Hôpital's Rule, that $\psi_{Q,R}^{-1}$ varies continuously with $R \in \mathscr{U}$. By construction, ψ is the trivialization determined by Q.

Theorem B. *Let Q and $\bar{Q}$ be two triples of points of M which determine trivializations of ϕ_3 on open sets $\mathscr{U}$ and $\bar{\mathscr{U}}$. Then the transition function $\tau = \psi \circ \bar{\psi}^{-1}$ is completely determined by*

$$\tau(0) = \psi(\bar{Q}_0).$$
$$\tau(1) = \psi(\bar{Q}_1).$$
$$\tau(\infty) = \psi(\bar{Q}_\infty).$$

Proof. The equations hold by construction. Since τ is an automorphism of a $\mathbb{P}^1$-bundle, it is determined by its values on the three points 0, 1, and ∞. □

3. Non-Hyperelliptic Curves of Genus Three

Let M be a non-hyperelliptic curve of genus 3. The canonical model of M is a degree 4 plane curve; a canonical divisor is a 4-tuple of collinear points. Fix a base point $P \in M$ to define the $\mathbb{P}^2$-bundle $\phi_5 : M^{(5)} \to J$. In order to describe sections, we need to understand special divisors.

Lemma 7. *Every special divisor of degree 3 on M is linearly equivalent to $K - X$ for some $X \in M$.*

Proof. If D is special, then $\ell(K - D) > 0$. Since $\deg(K - D) = 1$ and M is not a rational curve, an effective representative of $K - D$ is a uniquely determined point $X \in M$. Hence $D \sim K - X$. □

In the notation of Sect. 1, the lemma says that

$$B = -W_1 + [K - 4P].$$

Let D_0 be an effective divisor of degree 2 on M. By Proposition 1, D_0 determines a section

$$\sigma_0(R) = D_0 + E_0(R)$$

where $E_0(R)$ is the unique effective representative of the class $R + 5P - D_0$. The section σ_0 is defined on the open set $J - B_0$ where

$$B_0 = B + [D_0 - 2P] = -W_1 + [K + D_0 - 6P].$$

To trivialize ϕ_5, it will be necessary to specify four independent sections. Let $Q = \{Q_0, \ldots, Q_3\}$ be an ordered 4-tuple of points of M, no 3 of which are collinear. View the subscripts as elements of the cyclic group $\mathbb{Z}/4$. Now define four sections of ϕ_5 by

$$\sigma_i(R) = Q_i + Q_{i+1} + E_i(R).$$

Also, define $\Delta_i = W_2 + [Q_i + Q_{i+1} + Q_{i+2} - 3P]$.

Proposition 8. *Let $\mathscr{U}$ be the maximal open set in J over which the collection of sections $\{\sigma_i\}$ trivializes ϕ_5. Then*

$$J - \mathscr{U} = \bigcup_{i=0}^{3} \Delta_i.$$

Proof. Fix $R \in J$. Write $S_i = \sigma_i(R) \in \mathbb{P}(R + 5P)$. If no 3 of the S_i are collinear, then there is a unique isomorphism $F_R: \mathbb{P}(R+5P) \to \mathbb{P}^2$ such that

$$\begin{aligned} F_R(S_0) &= (1:0:0) \\ F_R(S_1) &= (0:1:0) \\ F_R(S_2) &= (0:0:1) \\ F_R(S_3) &= (1:1:1). \end{aligned}$$

So, it is enough to determine when 3 of the S_i are collinear. By symmetry under the action of the cyclic group, it suffices to determine this for S_1, S_2, S_3.

If $S_1 = S_2$, then there is an equality of divisors

$$Q_1 + Q_2 + E_1(R) = Q_2 + Q_3 + E_2(R).$$

Cancel the common term Q_2. Since $Q_1 \neq Q_3$, this equality forces Q_3 to lie in the support of $E_1(R)$. Therefore

$$R \sim Q_1 + Q_2 + E_1(R) - 5P \sim Q_1 + Q_2 + Q_3 + X + Y - 5P$$

for some $X, Y \in M$. So, $S_1 = S_2$ if and only if $R \in \Delta_1$.

Suppose $S_1 \neq S_2$. The line spanned by these two points in $\mathbb{P}(R + 5P)$ is just $L = \mathbb{P}(R + 5P - Q_2)$. So, $S_3 \in L$ if and only if $S_3 - Q_2$ is effective. Since the Q_i are distinct, this can only occur if $E_3(R) - Q_2$ is effective. But then

$$R \sim Q_0 + Q_3 + E_3(R) - 5P \sim Q_0 + Q_3 + Q_2 + X + Y - 5P$$

and $R \in \Delta_2$. So S_1, S_2, S_3 are collinear if and only if $R \in \Delta_1 \cup \Delta_2$. The result now follows by symmetry. $\square$

Proposition 9. *Let $Q^j = \{Q_0^j, \ldots, Q_3^j\}$ be a 4-tuple on M which determines a trivialization of ϕ_5 on a maximal open set $\mathscr{U}^j$. A complete trivializing atlas for ϕ_5 can be obtained by choosing 4 such general collections Q^j.*

Proof. Choose Q^1 arbitrarily. Define

$$Y^j = J - \mathscr{U}^1 - \ldots - \mathscr{U}^j .$$

Assume inductively that the first $j-1$ collections have been chosen so that Y^{j-1} has codimension $j-1$ in J. Write

$$Y^{j-1} = \bigcup_s Z_s$$

as a union of irreducible components. For each s, pick a point $R_s \in Z_s$.

Choose Q_0^j, Q_1^j to be points general enough and not lying in an earlier collection so that $R_s + 5P - Q_0^j - Q_1^j$ has a unique effective representative, which is denoted A_s. Then $A = \bigcup_s \operatorname{supp}(A_s)$ is finite. Choose $Q_2^j \in M \setminus A$ general enough so that $R_s + 5P - Q_i^j - Q_2^j$ has a unique effective representative $B_s^i (i=0,1)$. Also let $B^i = \bigcup_s \operatorname{supp}(B_s^i)$ for $i = 0, 1$. Let $C = A \cup B^0 \cup B^1$. Then C is also finite. Choose $Q_3^j \in M - C$. By construction, $R_s \in \mathscr{U}^j$ for each s. Therefore,

$$\operatorname{codim}(Y^j, J) = j .$$

Since J is 3-dimensional, $Y^4 = \emptyset$. □

In order to describe the transition functions, we must first develop a better geometric interpretation of the fibre $\mathbb{P}(R+5P)$ of ϕ_5 over $R \in J$. Let $D(R)$ be an effective representative of the class $2K - R - 5P$. By Proposition 1, $D(R)$ exists and is generically unique. Non-uniqueness occurs when $D(R)$ is special; i.e., when

$$2K - R - 5P \sim K - X \qquad R \sim K + X - 5P .$$

Let $M_0 = W_1 + [K - 4P]$, $V_0 = J - M_0$. So $D(R)$ is uniquely determined if and only if $R \in V_0$.

Proposition 10. *If $R \in V_0$, then $\Gamma(M, \mathscr{L}(R+5P))$ is naturally isomorphic to the vector space $\mathscr{C}(D(R))$ of all conics in $\mathbb{P}^2$ through the points $D(R)$.*

Proof. As usual, multiple points of $D(R)$ should be thought of as infinitely near points corresponding to the tangent directions of M. Since $R \in V_0$, the points of $D(R)$ are non-collinear and the vector space $\mathscr{C}(D(R))$ has the correct dimension.

Since M is canonically embedded, the space $\Gamma(M, \mathscr{L}(2K))$ is naturally identified with the space of all conics in $\mathbb{P}^2$. Under this identification, the subspace $\mathscr{C}(D(R))$ of conics with specified base points $D(R)$ is the image of

$$\Gamma(M, \mathscr{L}(R+5P)) = \Gamma(M, \mathscr{L}(2K - D(R))) \subset \Gamma(M, \mathscr{L}(2K)) . \quad \square$$

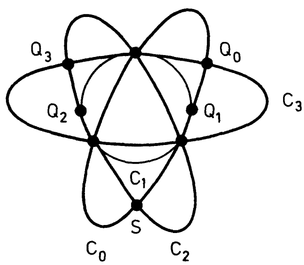
This interpretation can be extended over all of J. By Theorem A, ϕ_5 restricts to the trivial bundle on M with fibre $\mathbb{P}(K) =$ space of lines in $\mathbb{P}^2$. For $R \in M_0, D(R)$ consists of three points on a fixed line L_X through $X \sim K - D(R)$. So, $\mathscr{C}(D(R))$ consists of the reducible conics $L \cdot L_X$ for varying L.

Corollary 11. *The $\mathbb{P}^2$-bundle $\phi_5 : M^{(5)} \to J$ is isomorphic to the bundle whose fibre over $R \in J$ is $P(\mathscr{C}(D(R)))$.*

Let Q be a 4-tuple on M defining a trivialization of ϕ_5 on the open set $\mathscr{U}$. The conic $C_i \in \mathbb{P}(\mathscr{C}(D(R)))$ which corresponds to the section $\sigma_i(R)$ is the unique conic through the five points

$$Q_i + Q_{i+1} + D(R).$$

Pick arbitrary equations $\gamma_i \in \mathscr{C}(D(R))$ whose locus of zeroes is C_i. These equations need to be normalized correctly. For each $R \in \mathscr{U}$, we have



The common unlabelled intersection points form the support of $D(R)$. The point S is defined to be the unique fourth intersection point of C_0 and C_2. Write $\gamma = (\gamma_0 : \gamma_1 : \gamma_2)$. Then γ is a quadratic transformation of $\mathbb{P}^2$ which, when restricted, determines a morphism $\gamma_M : M \to \mathbb{P}^2$.

$$\gamma(Q_1) = (0:0:1)$$
$$\gamma(S) = (0:1:0)$$
$$\gamma(Q_2) = (1:0:0).$$

Normalize the choice of the γ_i so that

$$\frac{\gamma_1}{\gamma_2}(Q_0) = -1 \qquad \frac{\gamma_0}{\gamma_1}(Q_3) = -1.$$

Then the normalized γ is completely determined by the fixed 4-tuple Q and the variable $R \in \mathscr{U}$.

Proposition 12. *Let Q be a 4-tuple of points of M determining a trivializing neighborhood $\mathscr{U}$ for ϕ_5. Let $R \in \mathscr{U}$ and $F_R : \mathbb{P}(R + 5P) \to \mathbb{P}^2$ be the isomorphism described abstractly in Proposition 8. Then F_R is induced by the quadratic transformation γ given above.*

Proof. Let $\pi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ be a minimal resolution of the indeterminacy of γ. So, π is a regular birational map and there exists a regular map $\tilde{\gamma} : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ yielding a commutative diagram

$$\begin{tikzcd} & \tilde{\mathbb{P}}^2 \arrow[dl, "\pi"'] \arrow[dr, "\tilde{\gamma}"] & \\ \mathbb{P}^2 \arrow[rr, "\gamma"] & & \mathbb{P}^2. \end{tikzcd}$$

Let E be the exceptional divisor in $\tilde{\mathbb{P}}^2$. For any divisor D in $\mathbb{P}^2$, let $\tilde{D}$ denote its proper transform. Notice that

$$\tilde{C}_i \cdot \tilde{M} = C_i \cdot M - D(R).$$

Take coordinates $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$. Let H_i be the hyperplane $x_i = 0$ and let H be the hyperplane $x_0 + x_1 + x_2 = 0$. Then $\gamma^* (x_i) = \gamma_i$ for $i = 0, 1, 2$. So

$$\tilde{\gamma}^* (H_i) = \tilde{C}_i .$$

Therefore

$$\gamma_M^* (H_i) = \tilde{C}_i \cdot \tilde{M} = C_i \cdot M - D(R) = \sigma_i(R)$$

for $i = 0, 1, 2$. In particular, $\gamma_M^* \mathcal{O}_{\mathbb{P}^2}(1) = \mathscr{L}(R + 5P)$ and hence γ_M induces an isomorphism

$$(\gamma_M^*)^{-1} : \mathbb{P}(R + 5P) \to \mathbb{P}^2 = \mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}(1))).$$

The previous computation shows that F_R and $(\gamma_M^*)^{-1}$ agree on the three points C_0, C_1, C_2. But, because of the normalization,

$$h = \gamma^* (x_0 + x_1 + x_2) = \gamma_0 + \gamma_1 + \gamma_2$$

is such that $h(Q_0) = h(Q_3) = 0$. Thus

$$\hat{\gamma}^* (H) = \tilde{C}_3$$

and

$$\gamma_M^* (H) = \sigma_3 (R).$$

Therefore, $F_R = (\gamma_M^*)^{-1}$. □

Theorem C. *Let $Q, \bar{Q}$ be two 4-tuples on M which define trivializations of ϕ_5 on $\mathscr{U}, \bar{\mathscr{U}}$, respectively. Let $\tau = F \circ \bar{F}^{-1}$ be the transition function. Then τ is completely determined by*

$$\begin{aligned} \tau(0:0:1) &= \gamma(Q_1) \\ \tau(1:0:0) &= \gamma(\bar{Q}_2) \\ \tau(1:-1:0) &= \gamma(\bar{Q}_3) \\ \tau(0:1:-1) &= \gamma(\bar{Q}_0). \end{aligned}$$

Proof. The equalities hold by the construction of γ and by Proposition 12. The 4 specified points of $\mathbb{P}^2$ are in general position. An automorphism of $\mathbb{P}^2$ is determined by its action on 4 points in general position. □

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On the Proportionality of Covolumes of Discrete Subgroups

G. A. Margulis[1] and J. Rohlfs[2]

[1] Institute of Problems of Information Transmission, nl. Ermolova 19, SU-103051 Moscow, USSR
[2] Mathematisch-Geographische Fakultät, Katholische Universität, Ostenstrasse 26–28, D-8078 Eichstätt, Federal Republic of Germany

Introduction

Before we can state our main result, we have to fix some notation.

Let $S=\{v\}_{v\in S}$ be a finite set. For each $v\in S$ let $\mathscr{k}_v$ be a fixed locally compact field of characteristic zero. Suppose that for each $v\in S$ there is given an algebraic semisimple group $G_v|\mathscr{k}_v$ defined over $\mathscr{k}_v$ and write $G_S:=\prod_{v\in S} G_v(\mathscr{k}_v)$. We do not require that some $\mathscr{k}_v$, $v\in S$ is archimedean. Endowed with the topology which is induced by the locally compact topology of the $\mathscr{k}_v$, $v\in S$ the group G_S is locally compact and carries an invariant measure ω, which is unique up to a positive constant.

We recall that a lattice in G_S is a discrete subgroup Γ of G_S with finite covolume

$$\int_{G_S/\Gamma}\omega<\infty .$$

Two lattices Γ and Γ' are said to be commensurable if their intersection is of finite index in both groups. We say as usually that G_S has no compact factors if G_S does not contain a normal compact subgroup of positive dimension.

Our main result is as follows.

Theorem 1. *Assume that G_S is as above without compact factors and that $\Gamma\subset G_S$ is a lattice. Then there exists an invariant measure ω on G_S such that all lattices commensurable to Γ have integral covolume with respect to ω.*

In the assumptions of Theorem 1 we may replace G_S by a group H commensurable to G_S and we can replace G_S by a discret central extension $\tilde{H}$ of H and the conclusion of Theorem 1 holds for lattices in $\tilde{H}$. In particular if we call a real Lie group G semisimple if its Lie algebra is semisimple and say that G has no compact factors if its Lie algebra has no non trivial compact factors we have the following:

Corollary 1. *Let G be a real semi-simple Lie group without compact factors and suppose that $\Gamma\subset G$ is a lattice. Then there exists an invariant measure ω on G such that all lattices commensurable to Γ have integral covolume with respect to ω.*

The main result extends under some natural additional assumptions to lattices in groups over fields of positive characteristic. We will not pursue this here.

We point out that if G_S has compact factors or a non discrete center or a non-trivial unipotent radical then the conclusion of the theorem becomes false in general.

If G_S has compact factors we have to divide them out and have to apply the theorem to the resulting situation taking in account that extensions of finite index are irrelevant. If all k_v, $v \in S$ are archimedean this exactly means that instead of Γ we have to work with the induced group of motions of the associated symmetric space of maximal compact subgroups of G_S.

Our proof of Theorem 1 gives a certain uniquely determined measure depending on the commensurability class of a fixed lattice. An a priori construction of this measure is desirable but unknown except in the following situation:

Assume that $G|k$ is an absolutely simple algebraic group defined over a numberfield k and suppose that $G_\infty := G\left(k \bigotimes_{\mathbb{Q}} \mathbb{R}\right)$ is not compact and contains a compact Cartan subgroup. Then from Harder's Gauß-Bonnet theorem [7] one deduces the existence of an invariant measure ω, on G_∞ which is an integral multiple of the Euler-Poincaré measure [11], such that all arithmetic subgroups of G_∞ have integral covolume

$$\int_{G_\infty/\Gamma} \omega \in \mathbb{N}$$

with respect to ω.

Of course rather often such a G_∞ does not contain a compact Cartan subgroup, the Euler-Poincaré measure is zero and the above argument breaks down. Moreover, in that situation there is known no other non trivial index in the sense of Atiyah and Singer which could be used as a substitute of the Euler-Poincaré characteristic.

We now give an outline of the proof of Theorem 1. In the first paragraph we observe that for non arithmetic lattices the main result is an easy consequence of results of the first autor [8] and show that it suffices to prove the theorem for S-arithmetic subgroups $\Gamma \subset G(k)$ of an absolutely simple adjoint algebraic group G/k defined over a numberfield k. We then fix such an arithmetic group Γ^0 and show that the generalized index $[\Gamma^0 : \Gamma] = [\Gamma^0 : \Gamma^0 \cap \Gamma]/[\Gamma : \Gamma^0 \cap \Gamma]$ has bounded denominator for all S-arithmetic subgroups Γ. Of course it suffices to understand these indices for all the Γ's which are the representatives of the countably many $G(k)$-conjugacy classes of maximal S-arithmetic subgroups of $G(k)$. In Sect. 2 we prove that the maximal S-arithmetic groups are among the normalizers in $G(k)$ of globally parahoric subgroups of $\tilde{G}(k)$, where $\tilde{G}$ is the simply connected covering of G, and classify the conjugacy classes of parahoric and maximal arithmetic groups. These results are easy generalisations of the corresponding ones for split groups which are proved in [10]. In Sect. 3 we reduce the proof of the main theorem to a local result, Proposition 3.2, which says that over the completion k_v of k at a prime v which does not divide the order of the center of $\tilde{G}(k_v)$ the volume of a "hyperspecial" maximal subgroup of $G(k_v)$ is an integral multiple of the volume of each other maximal compact subgroup of $G(k_v)$. Originally we proved this result using the classification of local Lie groups. We thank J. Tits very much for explaining to us the short argument given in this paper.

For lattices in forms of PGL_2 the main result of this paper has been proved, along the same lines by Borel [3]. The work described in this paper has been done up to some details in summer 1979 when both autors were members of the Sonderforschungsbereich Theoretische Mathematik at Bonn. We want to express our thanks for the hospitality provided by this institution.

1. The Non Arithmetic Case and Reduction to the Arithmetic Case

In this paragraph we prove Theorem 1 for non arithmetic lattices and reduce its proof to the case of S-arithmetic irreducible lattices in absolutely simple adjoint algebraic groups over numberfields.

1.1. We recall that a lattice $\Gamma \subset G_S$ is reducible if there exist semi-simple algebraic groups $H_v | k_v$, $v \in S$ and surjective group homomorphisms $\varphi_v : G_v \to H_v$ defined over k_v – here we allow some but not all H_v, $v \in S$ to be the trivial group – such that for the induced map $\varphi : G_S \to H_S$ the group $\varphi(\Gamma)$ is a lattice in H_S and the kernel of φ is non compact and of positive dimension. The lattice Γ is said to be irreducible if it is not reducible.

1.2. **Lemma.** *It suffices to prove Theorem 1 for irreducible lattices in groups* $G_S = \prod_{v \in S} G_v(k_v)$, *where each* $G_v | k_v$ *is adjoint and connected.*

Proof. By induction after $\dim G_S$ and by integration in a fibration we can easily reduce to the case of irreducible lattices. Since $C(k_v)$, the center of $G_v(k_v)$, is finite and since $G_v(k_v)/C(k_v) \hookrightarrow \overline{G_v}(k_v)$ is of finite index, where $\overline{G_v}$ is the adjoint group belonging to G_v, the reduction to adjoint groups follows. Since $G_v^0(k_v) \hookrightarrow G_v(k_v)$ is of finite index, where G_v^0 is the connected component of G_v, we finally get the reduction to the case of connected groups. q.e.d.

1.3. We recall the notions of S-arithmetic subgroups and arithmetic lattices.

(i) Let k be a numberfield and S a finite set of places of k. Denote by $H | k$ a semisimple algebraic group defined over k and consider H as a subgroup defined over k of some GL_n, n bigg enough. Let $\mathcal{O}_S$ be the ring of elements of k which are integral at all finite places $v \notin S$. Then a subgroup $\Gamma \subset H(k)$ which is commensurable to the stabilizer of $\Lambda := \mathcal{O}_S^n \hookrightarrow k^n$ in $H(k)$ is called S-arithmetic.

It is well known that an S-arithmetic Γ is a lattice in $\prod_{v \in S \cup S_\infty} G(k_v)$, where S_∞ denotes the set of infinite places of k. Moreover, the notion of S-arithmeticity does not depend on the embedding $H \hookrightarrow \mathrm{GL}_n$ over k.

(ii) Let $G_S = \prod_{v \in S} G_v(k_v)$ as in the introduction. A lattice $\Gamma \subset G_S$ is said to be arithmetic if there exists a numberfield ℓ, a finite set of places w_v, $v \in S$ of ℓ and isomorphisms $\ell_{w_v} \xrightarrow{\sim} k_v$, where ℓ_{w_v} is the completion of ℓ with respect to w_v, an S-arithmetic subgroup $\Lambda \subset H(\ell)$ and surjective group homomorphisms $\varphi_v : H \times \ell_{w_v} \to G_v$ defined over k_v such that $\varphi(\Lambda)$ is commensurable to Γ and $\ker \varphi$ is compact, where $\varphi : \prod_{v \in S \cup S_\infty} H(\ell_{w_v}) \to G_S$ is induced by the φ_v, $v \in S$ and trivial maps φ_v for $v \in S_\infty - S$.

The definition implies that S contains all archimedean places w of ℓ such that $H(\ell_w)$ is not compact. Hence we can view Λ also as a lattice in $\prod_{v\in S} H(\ell_{w_v})$. We call a lattice non-arithmetic if Γ is not arithmetic.

1.4. **Proposition.** *If $\Gamma \subset G_S$ is an irreducible non-arithmetic lattice the conclusion of Theorem 1 holds.*

Proof. Let $\mathscr{A}(\Gamma) := \{h \in G_S \mid h^{-1}\Gamma h$ is commensurable to $\Gamma\}$ be the commensurability group of Γ. Then $\mathscr{A}(\Gamma)$ contains Γ and according to [8, Theorem 9] either $[\mathscr{A}(\Gamma):\Gamma] < \infty$ or Γ is arithmetic. Since Γ is non-arithmetic $\mathscr{A}(\Gamma)$ is a lattice. We choose an invariant measure ω such that $\int_{G_S/\mathscr{A}(\Gamma)} \omega = 1$ and we are done. q.e.d.

Next we reduce to the case of S-arithmetic subgroups. We need the following simple Lemma, which should be well known.

1.5. **Lemma.** *Let $G \mid \mathbb{Q}$ be a semi-simple algebraic group defined over $\mathbb{Q}$ and S a finite set of places of $\mathbb{Q}$. Then the orders of all finite subgroups of all S-arithmetic subgroups of $G(\mathbb{Q})$ are bounded.*

Proof. For some $n \in \mathbb{N}$ we have an embedding $G \hookrightarrow \mathrm{GL}_n$ defined over $\mathbb{Q}$. Let $\Gamma \subset G(\mathbb{Q})$ be an S-arithmetic subgroup. Then there is a lattice $\Lambda \subset \mathbb{Q}^n$; $\Lambda \simeq \mathbb{Z}_S^n$ such that Γ stabilizes Λ. Here $\mathbb{Z}_S$ consists of S-integral rational numbers. If $2 \neq p \notin S$ is a prime, then according to Minkowski the group $\mathscr{U}(p) := \ker(\mathrm{GL}_n(\Lambda) \xrightarrow{\psi} \mathrm{GL}_n(\Lambda/p\Lambda))$ is torsion free, where ψ is the map induced by reduction mod p. Hence every torsion free subgroup of Γ maps injectively to $\mathrm{GL}_n(\Lambda/p\Lambda)$. But the order of $\mathrm{GL}_n(\Lambda/p\Lambda)$ does not depend on Γ and the result follows. q.e.d.

We now show that to prove Theorem 1 it suffices to establish the following:

1.6. **Proposition.** *Let k be a numberfield and S a finite set of finite places of k. Assume that G/k is an absolutely simple adjoint algebraic group such that $G_{S \cup S_\infty} = \prod_{v \in S \cup S_\infty} G(k_v)$ is not compact. Let $\Gamma \subset G(k)$ be an S-arithmetic subgroup. Then there exists an invariant measure ω on $G_{S \cup S_\infty}$ such that all lattices in $G_{S_\infty \cup S}$ which are commensurable to Γ have integral covolume with respect to ω.*

We will prove Proposition 1.6 in Sect. 3. Now we assume that 1.6 holds and show that then Theorem 1 is true.

Using 1.2 and 1.4 we have to prove Theorem 1 for irreducible arithmetic lattices $\Gamma \subset G_S = \prod_{v \in S} G_v(k_v)$. We use the notation given in 1.3 (ii). Since Γ is an arithmetic lattice we have a map

$$\varphi: \prod_{v \in S \cup S_\infty} H(\ell_{w_v}) \to G_S$$

with compact kernel and an S-arithmetic subgroup $\Gamma_\ell \subset H(\ell)$ such that $\varphi(\Gamma_\ell)$ is commensurable with Γ.

We now show that we can choose $H \mid \ell$ to be absolutely simple. According to [4, 6.21] there are absolutely simple groups $H_i \mid \ell_i$ where ℓ_i is a finite extension of ℓ such that

$$H \mid \ell = \prod_{i=1}^{r} \mathrm{res}_{\ell_i/\ell} H_i$$

where $\mathrm{res}_{\ell_i|\ell}$ denotes the Weil-restriction. Let S_i be the set of places of ℓ_i lying over S and let $p_i: H \to \mathrm{res}_{\ell_i|\ell} H_i$ be the natural projection. Then $p_i(\Gamma_\ell) =: \Gamma_i$ is S_i-arithmetic in $H_i(\ell_i)$, i.e. if Γ_ℓ is irreducible then $r=1$. But Γ_ℓ is irreducible if and only if Γ is irreducible. Therefore $H = \mathrm{res}_{\ell_1|\ell} H_1$ where H_1 is absolutely simple. Since the S_1-arithmetic subgroups of H_1 and the S-arithmetic subgroups of H coincide we therefore can assume that $H|\ell$ is absolutely simple.

Using 1.5 for the group $\mathrm{res}_{\ell|\mathbb{Q}} H$ and using that $\ker\varphi$ is compact, we see that $\ker\varphi \cap \Gamma_\ell$ is of bounded order for all S-arithmetic subgroups Γ_ℓ. Hence Theorem 1 holds for $\Gamma \subset G_S$ and the claim preceding Proposition 1.6 holds.

2. Parahoric Subgroups

We assume in this chapter throughout that G/k is an absolutely simple connected and simply connected algebraic group defined over a numberfield k. We denote by S a finite set of non archimedean places of k and suppose that $\prod_{v \in S_\infty \cup S} G(k_v)$ is not compact. Let $\bar{G}$ be the adjoint group of G. In this chapter we show that the maximal S-arithmetic subgroups of $\bar{G}(k)$ are normalizers of parahoric subgroups of $G(k)$ and classify these groups up to $G(k)$-conjugacy.

2.1. We recall some known facts and fix our notation.

(i) If $\Gamma \subset G(k)$ is an S-arithmetic group, then the completion Γ_v of Γ in $G(k_v)$, $v \notin S$, is an open and compact subgroup.

(ii) If Γ^1 and Γ^2 are S-arithmetic subgroups of $G(k)$ then for almost all places v of k we have $\Gamma_v^1 = \Gamma_v^2$.

(iii) Suppose that $P_v \subset G(k_v)$, $v \notin S$, is an open and compact subgroup such that $P_v = \Gamma_v$ for some arithmetic subgroup Γ and almost all $v \notin S$. Then $\bigcap_{v \notin S} P_v \cap G(k) =: P$ is an S-arithmetic subgroup and P_v is the closure of P in $G(k_v)$ for all $v \notin S$.

(iv) Let ℓ be a finite extension of k such that $G \times \ell$ is split over ℓ. Denote by $\mathcal{O}_{S_\ell}$ the integral closure of $\mathcal{O}_S$ in ℓ and by $\mathcal{G}/\mathcal{O}_\ell$ the Chevalley scheme with generic fibre $G \times \ell$. Then $\mathcal{G}(\mathcal{O}_{S_\ell}) \subset G(\ell)$ is an S_ℓ-arithmetic subgroup and for all $w \in S_\ell$ the group $\mathcal{G}(\mathcal{O}_w)$ is a maximal compact subgroup of $G(\ell_w)$ which is special [14, 1.9, 3.8]. If $\Gamma \subset G(k)$ is an S-arithmetic group then using (ii) we see that for almost all $v \notin S$ we have $\Gamma_v = \mathcal{G}(\mathcal{O}_w) \cap G(k_v)$. Here $\mathcal{O}_w \cap k_v = \mathcal{O}_v$.

(v) For all $v \notin S$ let T_v be maximal k_v-split torus in $G \times k_v$. Denote by $\Phi(T_v, k_v) := \Phi_v$ the corresponding affine root system. We choose a chamber C_v in the apartment $A(G, T_v, k_v)$ and denote the corresponding local Dynkin diagram by Δ_v, [14, 1.8]. Often we identify Δ_v with its set of vertices or with the vertices of C_v.

We recall that to each facet F_{X_v} of C_v corresponding to a subset $X_v \subsetneq \Delta_v$ there is an open compact subgroup P_{X_v} of $G(k_v)$. All $G(k_v)$-conjugate subgroups P_v of P_{X_v} are called parahoric subgroups of type $X_v = \tau(P_v)$, [14, 3.1].

Now we use the notation introduced in (iv). For almost all places v of k the group $\mathcal{G}(\mathcal{O}_w) \cap G(k_v)$ is parahoric of some type X_v^0 such that $\Delta_v - X_v^0$ consists of one hyperspecial point. Here we use [14, 3.9] and the fact that ℓ/k is unramified for almost all $v \notin S$. After conjugation with elements of k_v we may assume that for almost all $v \notin S$ the torus T_v and the chamber C_v has been choosen in such a way that $\mathcal{G}(\mathcal{O}_w) \cap G(k_v) = P_{X_v^0}$.

2.2. *Definition.* (i) An S-arithmetic subgroup $\Gamma \subset G(k)$ is called parahoric if for all $v \notin S$ the completion Γ_v of Γ is a parahoric subgroup of $G(k_v)$.

(ii) If $\Gamma \subset G(k)$ is parahoric then the family $\{\tau(\Gamma_v)\}_{v \notin S}$ is called the type $\tau(\Gamma)$ of Γ.

(iii) A family $X = \{X_v\}_{v \in S}$, $X_v \subsetneq \Delta_v$, $X_v = X_v^0$ for almost all $v \notin S$, is called a type.

(iv) If $X = \{X_v\}_{v \notin S}$ is a type we denote by P_X the subgroup $G(k) \cap \bigcap_{v \notin S} P_{X_v}$ and by Γ_X the normalizer of P_X in $G(\bar{k})$. Here $\bar{k}$ denotes a fixed algebraic closure of k.

Using strong approximation [9] and well known properties of local parahoric subgroups we have.

2.3. **Proposition.** (i) *If X is a type then P_X is a parahoric subgroup of $G(k)$ of type $\tau(P_X) = X$.*

(ii) *If P is a parahoric subgroup of type $\tau(P) = X$ then there is a $g \in G(k)$ such that $g^{-1}Pg = P_X$.*

(iii) *Every parahoric subgroup P of $G(k)$ is its own normalizer in $G(k)$.*

(iv) *If $X = \{X_v\}_{v \in S}$ is a type and if Γ is a S-arithmetic subgroup of $G(k)$ containing P_X then Γ is parahoric. Suppose that $\tau(\Gamma_v) = Y_v$, $v \notin S$. Then $X_v \subset Y_v$ for all $v \notin S$ and $\Gamma = P_Y$, $Y = \{Y_v\}_{v \notin S}$.*

Denote by $\bar{G}$ the adjoint group corresponding to G, by $p: G \to \bar{G}$ the natural projection and by μ the kernel of p. The group $\bar{G}(k_v)$ acts by conjugation on the parahoric subgroups of $G(k_v)$ and thus induces an action on the local Dynkin diagram Δ_v. We get an homomorphism $\xi_v: \bar{G}(k_v) \to \mathrm{Aut}(\Delta_v)$ which is trivial on $p(G(k_v))$. Abbreviate $\xi_v(\bar{G}(k_v)) =: \Xi_v$, [14, 2.5].

We recall that a local type X_v is called maximal if $\Delta_v - X_v$ is the full orbit of a subgroup of Ξ_v, [5, 3.3.5].

2.4. *Definition.* A type $X = \{X_v\}_{v \in S}$ is called maximal if all X_v, $v \notin S$, are maximal.

We want to understand maximal arithmetic subgroups $\bar{\Gamma} \subset \bar{G}(k)$. For this we consider the preimage $\Gamma := p^{-1}(\bar{\Gamma})$ of $\bar{\Gamma}$ in $G(\bar{k})$ under the map $p: G(\bar{k}) \to \bar{G}(\bar{k})$. In general Γ is not contained in $G(k)$ but $[\Gamma : \Gamma \cap G(k)] < \infty$. Therefore we enlarge our notion of an S-arithmetic groups and say that a subgroup $\Gamma \subset G(\bar{k})$ is S-arithmetic if $[\Gamma : \Gamma \cap G(k)] < \infty$ and if $\Gamma \cap G(k)$ is S-arithmetic. By a generalisation of results of Borel to S-arithmetic groups we have for such a Γ that $p(\Gamma) \subset \bar{G}(k)$, [1, Theorems 3, 6], and that maximal S-arithmetic subgroups [in $G(\bar{k})$] do exist, [1, Theorem 4].

2.5. **Proposition.** *Suppose that Γ is a maximal S-arithmetic subgroup. Then $\Gamma \cap G(k) := P$ is parahoric of some maximal type X and Γ is $G(k)$-conjugate to Γ_X.*

Proof. The argument given in [10, 3.5 (i)] extends without difficulties to our situation. q.e.d.

In order to understand Γ_X/P_X we have to introduce some notions.

Let $\Xi := \{\{a_v\}_{v \notin S} \mid a_v \in \Xi_v,\ a_v = 1 \text{ for almost all } v\}$. Then Ξ acts on $\prod_{v \notin S} \Delta_v =: \Delta$. Let $X = \{X_v\}_{v \notin S}$ be a typ. We identify X with $\prod_{v \notin S} X_v \subset \prod_{v \notin S} \Delta_v$. Denote by Ξ_X the stabilizer of X in Ξ. The local maps ξ_v induce a map $\xi: \bar{G}(k) \to \Xi$. For this we remark that if $g \in \bar{G}(k)$ and if P is a parahoric subgroup then $g^{-1}P_v g = P_v$ for almost all $v \notin S$.

Consider the exact galois cohomology sequence

$$1 \to \mu(k) \to G(k) \to \bar{G}(k) \xrightarrow{\delta} H^1(k, \mu).$$

The map ξ induces a map

$$\xi : \mathrm{im}(\delta) \to \mathrm{Aut}(\Delta).$$

Given a typ X we abbreviate

$$H^1(k, \mu)_X := \{a \in \mathrm{im}(\delta) \mid \xi(a) \in \Xi_X\}.$$

Then ξ induces a map $\xi_X : H^1(k, \mu)_X \to \Xi_X$.

2.6. **Proposition.** *Suppose that $X = \{X_v\}_{v \in S}$ is a type with corresponding parahoric subgroup P_X and normalizer Γ_X of P_X. Then the following hold:*

(i) *We have an exact sequence*

$$1 \to \mu(\bar{k})/\mu(k) \to \Gamma_X/P_X \xrightarrow{d} H^1(k, \mu)_X \to 1.$$

Here d is induced by the composition of maps

$$\Gamma_X \xrightarrow{p} \bar{G}(k) \xrightarrow{\delta} H^1(k, \mu).$$

(ii) *The kernel of ξ_X is independent of X.*

Proof. The claim (ii) is obvious. For split groups over number fields a proof of (i) has been given in [10, 2.6 (ii)]. This proof holds verbatim in our situation. q.e.d.

For the application we have in mind it is not necessary to find out which of the groups Γ_X, X a maximal type, are really maximal. To be complete, we include a solution, which is again proved as in the split case [10].

2.7. *Definition.* A type $X = \{X_v\}_{v \notin S}$ is called $\mathcal{O}_S$-maximal if X is a maximal type and if for every maximal typ $X' \supsetneq X$ we have $H^1(k, \mu)_{X'} \subsetneq H^1(k, \mu)_X$.

2.8. **Proposition.** *Let $\Gamma \subset G(\bar{k})$ be an S-arithmetic group. Then Γ is maximal if and only if Γ is $G(k)$-conjugate to Γ_X where X is an $\mathcal{O}_S$-maximal type.*

Proof. See [10]. q.e.d.

3. Computation of Local Indices

In this paragraph we prove Proposition 1.6. We use the notation established in Sect. 2. In particular G/k now is simply connected. We choose a parahoric subgroup $P^0 \subset G(k)$ with type $X^0 = \{X_v^0\}_{v \notin S}$ which is special at all and hyperspecial at almost all $v \notin S$. We show that for all maximal S-arithmetic subgroups $\Gamma \subset G(\bar{k})$ the generalized index $[\Gamma^0 : \Gamma]$ has bounded denominator, where $\Gamma^0 = \Gamma_{X^0}$.

3.1. **Lemma.** *Let Γ be a maximal S-arithmetic subgroup of G such that $\Gamma \cap G(k)$ is parahoric of typ $X = \{X_v\}_{v \notin S}$. Then $[\Gamma^0 : \Gamma_X]$ is an integral multiple of*

$$\prod_{v \notin S} [P_v^0 : P_{X_v}]/|\Xi_{X_v}|.$$

Proof. For almost all $v \notin S$ we have $X_v = X_v^0$, $P_v^0 = P_{X_v}$, $|\Xi_{X_v}| = 1$. Hence the infinite product makes sense. There is a $g \in G(k)$ such that $\Gamma = g^{-1}\Gamma_X g$. Here we use Proposition 2.5. From Proposition 2.6 (i) and (ii) we deduce that

$$[\Gamma^0 : \Gamma] = [P^0 : g^{-1}P_X g]/|\mathrm{im}(\xi_X)|.$$

Choose a Tamagawa measure $\prod_v \omega_v$ on $G(A_k)$. Here A_k is the adele group for k, [13]. Then $\omega = \prod_{v \notin S} \omega_v$ is an invariant measure on $\prod_{v \notin S} G(k_v) =: G_S$ and

$$[P^0 : g^{-1}P_X g] = \mathrm{vol}_\omega(G_S/P^0)^{-1}/\mathrm{vol}_\omega(G_S/g^{-1}P_X g)^{-1} = [P^0 : P_X].$$

Using strong approximation [9] we see that

$$[P^0 : P_X] = \prod_{v \notin S} \mathrm{vol}_{\omega_v}(P_v^0)/\mathrm{vol}_{\omega_v}(P_{X_v}) = \prod_{v \notin S} [P_v^0 : P_{X_v}].$$

Because $|\mathrm{im}\,\xi_X|$ divides $|\Xi_X| = \prod_{v \notin S} |\Xi_{X_v}|$ we arrive at the statement of Lemma 3.1.

The parahoric subgroup $P^0 \subset G(k)$ can bee choosen in such a way that P_v^0 is hyperspecial at all places $v \notin S$ which are unramified in some splitting field ℓ for G. Therefore using Lemma 3.1 we can deduce the main result from the following:

3.2. **Proposition.** *Suppose that P_v^0 is hyperspecial and that X_v is a maximal type. Then, if $|\Xi_{X_v}|$ is prime to the characteristic of the residue field of k_v, the number $[P_v^0 : P_{X_v}]/|\Xi_{X_v}|$ is an integer.*

Proof. Since P_v^0 is hyperspecial there exists a finite unramified galois extension ℓ_w/k_v such that $G \times \ell_w$ splits over ℓ_w. Denote by S_v a maximal torus defined over k_v in $G \times k_v$ and by $T_v \subset S_v$ a subtorus which is maximal split in $G \times k_v$. Let Δ_w be the affine Dynkin diagram corresponding to the split group and denote by $X_w \subset \Delta_w$ the roots in Δ_w whose restriction to T_v lie in X_v. Let R_w be the (ordinary) root lattice corresponding to $(G \times \ell_w,\ S_v \times \ell_w)$ and introduce $R_{X_w} := \left\{\beta = \sum_{\alpha \in X_w} m_\alpha \alpha \in R_w / m_\alpha \in \mathbb{Z}\right\}$, the sublattice of R_w generated by roots in X_w.

For every $X_v \subsetneq \Delta_v$ there exists a uniquely determined group scheme $\mathcal{G}_{X_v}/\mathcal{O}_v$ with generic fibre $G \times k_v$ such that $P_{X_v} = \mathcal{G}_{X_v}(\mathcal{O}_v)$, [14, 3.4.1]. Since ℓ_w/k_v is unramified we have $P_{X_w} = \mathcal{G}_{X_v}(\mathcal{O}_w)$ and $P_{X_v} = P_{X_w}^{g_w}$ where $g_w = \mathrm{gal}(\ell_w/k_v)$ is the Galois group of ℓ_w/k_v.

Denote by $\bar{G}_{X_v} = \mathcal{G}_{X_v} \times \mathcal{O}_v/p_v$ the reduction $\bmod p_v$ of $\mathcal{G}_{X_v}$. Then $\bar{G}_{X_v}$ is connected over $\mathbb{F} = \mathbb{F}_{q_v} := \mathcal{O}_v/p_v\mathcal{O}_v$, [14, 3.5.2], and contains a uniquely determined Levi subgroup $\bar{G}_{X_v}^{\mathrm{red}}$. Since $G \times k_v$ is residually quasisplit [14, 1.11, 3.5.2] the group $\bar{G}_{X_v}$ resp. $\bar{G}_{X_v}^{\mathrm{red}}$ contain Borel subgroups $\bar{B}_{X_v}$ resp. $\bar{B}_{X_v}^{\mathrm{red}}$. Denote by $P_\emptyset$ the Iwahori subgroup of $G(k_v)$ corresponding to the empty set $\emptyset = X_v \subset \Delta_v$. Then, using [14, 3.7] we get

$$[P_{X_v} : P_\emptyset] = [\bar{G}_{X_v}^{\mathrm{red}}(\mathbb{F}) : \bar{B}_{X_v}^{\mathrm{red}}(\mathbb{F})].$$

The Bruhat decomposition shows that this number is congruent to $1 \bmod q_v$. In [14, 3.5.2] it is explained how the index can be computed and that the index is determined by the root system of $\bar{G}_{X_v}^{\mathrm{red}}$.

If $X_v^0 \subsetneq \Delta_v$ determines P_v^0 then X_w^0 is the complement of a hyperspecial point in Δ_w and the root system $R_{X_w^0}$ is identified with R_w and R_w is the natural root system associated to $\bar{G}_{X_v^0}^{\mathrm{red}}$ over the residue field of ℓ_w.

Denote by $\bar{S}_v$ the torus in $\bar{G}^{\text{red}}_{X^0_v}$ given by S_v [14, 3.5] and by $\bar{U}_\alpha$, $\alpha \in R_{X_w}$ the one parameter subgroup of $\bar{G}^{\text{red}}_{X^0_v}$ corresponding to α. Then the $\bar{U}_\alpha$, $\alpha \in R_{X_w}$, and $\bar{S}_v$ generate a connected reductive subgroup $\bar{H}_{X_w}$ of $\bar{G}^{\text{red}}_{X^0}$ with rootsystem R_{X_w}, see [6, Chap. VIII, Sect. 3], which is defined over $\mathbb{F}$. If $\bar{B}_H$ denotes a Borel subgroup of $\bar{H}_{X_w}$ we have

$$\begin{aligned}[P_{X^0_v}:P_{X_v}] &= [\bar{G}^{\text{red}}_{X^0_v}(\mathbb{F}):\bar{B}^{\text{red}}_{X^0_v}(\mathbb{F})]/[\bar{G}^{\text{red}}_{X_v}(\mathbb{F}):\bar{B}^{\text{red}}_{X_v}(\mathbb{F})]\\ &= [\bar{G}^{\text{red}}_{X^0_v}(\mathbb{F}):\bar{B}^{\text{red}}_{X^0_v}(\mathbb{F})]/[\bar{H}_{X_w}(\mathbb{F}):\bar{B}_H(\mathbb{F})] = [\bar{G}^{\text{red}}_{X^0_v}(\mathbb{F}):\bar{H}_{X_w}(\mathbb{F})]\cdot q_v^{-j} =: m\end{aligned}$$

where j is the difference of the dimensions of the unipotent radicals of $\bar{B}^{\text{red}}_{X^0_v}$ and $\bar{B}_H$. Here we use, that $\bar{H}_{X_w}$ and $\bar{G}^{\text{red}}_{X_v}$ have by construction the same root system. Since by a previous remark $m \equiv 1 \bmod q_v$ we deduce that m is an integer.

Let $\hat{W}$ ($= W'_a$ in the notation of [6, Chap. IV, Sect. 2.3]) be the "bigg" affine Weyl group of $G \times \ell_w$. Corresponding to $\mathcal{G}_{X^0_v}(\mathcal{O}_w)$ there is a point in the apartment given by S_v whose stabilizer W_0 in $\hat{W}$ is the finite Weyl group given by the root system R_w. We can view $\Xi_{X_v} \subset (\Xi_{X_w})^{g_w} \subset \hat{W}$ as a subgroup stabilizing the facett corresponding to X_w. We have a natural surjection $\hat{W} \to W_0$ which identifies Ξ_{X_v} with a subgroup again denoted by Ξ_{X_v} of W_0 and this subgroup stabilizes R_{X_w}, see [6, Chap. IV, Sect. 2.3, Proposition 6]. Hence Ξ_{X_v} can be considered as a "subgroup" of $\bar{G}^{\text{red}}_{X^0_v}(\mathbb{F})$ stabilizing $\bar{H}_{X_w}(\mathbb{F})$ and $|\bar{G}^{\text{red}}_{X^0_v}(\mathbb{F})/|\Xi_{X_v}|\cdot|\bar{H}_{X_w}(\mathbb{F})| \in \mathbb{N}$. Since $|\Xi_{X_v}|$ is prime to q_v by assumption we see that $[P_{X^0_v}:P_{X_v}]/|\Xi_{X_v}|$ is an integer. q.e.d.

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Untervarietäten der Siegelschen Modulmannigfaltigkeiten von allgemeinem Typ

R. Weissauer

Mathematisches Institut der Universität, Im Neuenheimer Feld 288, D-6900 Heidelberg, Bundesrepublik Deutschland

Einleitung

In [6] wurde von Tai gezeigt, daß für $g \geqq 9$ die Modulmannigfaltigkeit A_g der hauptpolarisierten abelschen Varietäten von allgemeinem Typ ist. Die Schranke wurde verbessert von Freitag [2] und Mumford [5] auf $g \geqq 8$ bzw. $g \geqq 7$. Es stellt sich nun natürlich die Frage, inwieweit generell Untervarietäten von A_g von allgemeinem Typ sind. Diese Frage wurde für den Fall der Divisoren von Freitag [3] untersucht. In der dortigen Arbeit wurde gezeigt, daß es eine Konstante g_0 gibt derart, daß für $g \geqq g_0$ jeder Divisor von A_g vom Tyg G ist. (Es handelt sich hierbei um eine Abschwächung des Begriffs allgemeiner Typ.) In der vorliegenden Arbeit wird etwas schärfer gezeigt, daß jeder Divisor von allgemeinem Typ ist für $g \geqq 10$. Dies ist insofern von Interesse da es wie in [3] zur Folgerung hat, daß jeder birationale Automorphismus von A_g für $g \geqq 10$ die Identität ist.

Im allgemeinen ist klar, daß es eine Schranke s gibt derart, daß alle Untervarietäten bis zur Kodimension s in A_g von allgemeinem Typ sind. Diese Konstante $s = s(g)$ hängt natürlich vom Geschlecht g ab. Die kleinste dem Autor bekannte Kodimension für welche es auch für großes Geschlecht g Untervarietäten von A_g gibt, welche nicht von allgemeinem Typ sind, ist die Kodimension $g-1$. Hier hat man unabhängig von g immer die Untervarietät $A_1 \times A_{g-1}$ in A_g als Gegenbeispiel, welche sogar nur die Kodairadimension $-\infty$ besitzt. Somit ergibt sich die obere Schranke $s(g) \leqq g-2$. Man könnte aber durchaus vermuten, daß diese Schranke scharf ist, falls nur g genügend groß ist, d. h. größer ist als eine gewisse Konstante g_0.

In dieser Arbeit wird gezeigt, daß wenigstens $s(g) \geqq g-13$ gilt. Mit anderen Worten: Jede Untervarietät von A_g der Kodimension $\leqq g-13$ ist von allgemeinem Typ für $g \geqq 13$.

1. Konstruktion von Tensoren

Es sei $g > 2$ und

$$A_g = \mathbb{H}_g / \Gamma_g, \qquad \Gamma_g = \mathrm{Sp}(2g, \mathbb{Z})$$

die Siegelsche Modulmannigfaltigkeit der hauptpolarisierten abelschen Varietäten vom Geschlecht g. Es bezeichne A_g^0 den regulären Ort von A_g. Uns interessiert die Konstruktion von holomorphen Tensoren

$$T \in \Omega^d(A_g^0)^{\otimes l}$$

und deren Fortsetzung zu holomorphen Tensoren auf ein singularitätenfreies vollständiges Modell $\tilde{A}_g$. Von besonderem Interesse ist die Einschränkung solcher Tensoren auf Untervarietäten Y von A_g. Es sei Y eine solche Untervarietät, P ein glatter Punkt von Y, welcher in A_g^0 liegt. Solche Punkte existieren immer für kleine Kodimensionen $\mathrm{cod}(Y: A_g) \leqq g-2$, denn bekanntlich ist $\mathrm{cod}(A_g^{\mathrm{sing}}: A_g) \geqq g-1$.

Der Tangentialraum T_pY von Y im Punkt P definiert ein nicht verschwindendes Element

$$\eta_Y(P) \in \Lambda^d T_p A_g, \qquad d = \dim Y.$$

eindeutig bis auf skalare Abänderung.

Punkte in A_g sind repräsentiert durch symmetrische komplexwertige Matrizen Z mit positivem Imaginärteil. Die Komponenten $Z_{\nu\mu}$ $(1 \leqq \nu \leqq \mu \leqq g)$ von Z definieren globale Koordinaten der Siegelschen oberen Halbebene $\mathbb{H}_g$. Für Teilmengen I der Menge aller Paare (ν, μ) mit $1 \leqq \nu \leqq \mu \leqq g$ setzen wir

$$dZ_I = \bigwedge_{(\nu,\mu)\in I} dZ_{\nu\mu}$$

bei geeignet gewählter Anordnung. Tensoren aus $\Omega^d A_g^0$ schreiben sich dann in der Form

$$\omega = \sum f_I dZ_I$$

mit gewissen Funktionen f_I auf $\mathbb{H}_g$. Die symplektische Gruppe operiert auf $\mathbb{H}_g$ mittels der bekannten Formel $Z \to MZ = (AZ+B)(CZ+D)^{-1}$ für $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$. Die natürliche Operation auf den Basisdifferentialen ist gegeben durch

$$dMZ = (CZ+D)'^{-1} dZ (CZ+D),$$

wobei dZ die Matrix der Differentiale $(dZ_{\nu\mu})$ bezeichne. Tensoren aus $\Omega^d(A_g^0)^{\otimes l}$ definieren $\mathrm{Sp}(2g, \mathbb{Z})$ invariante Tensoren auf einer offenen dichten Teilmenge von $\mathbb{H}_g$. Die $\mathrm{Sp}(2g, \mathbb{Z})$ Invarianz von

$$T = \sum f_{I_1, \ldots, I_l} dZ_{I_1} \otimes \ldots \otimes dZ_{I_l}$$

impliziert für die vektorwertige Funktion $f = (f_{I_1, \ldots, I_l})$ das Transformationsverhalten

$$f(MZ) = \varrho^{\otimes l}(CZ+D) f(Z), \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}),$$

wobei ϱ eine Darstellung von $\mathrm{Gl}(g, \mathbb{R})$ isomorph zu $V_\varrho = \Lambda^d \,\mathrm{Symm}^2 \mathbb{C}^g$ ist und f als Funktion mit Werten in V_ϱ aufgefaßt wird. Umgekehrt definiert eine Funktion f mit diesem Transformationsverhalten auf $\mathbb{H}_g$ einen Tensor auf A_g^0, da $\mathbb{H}_g \to A_g$ außerhalb des singulären Ortes von A_g unverzweigt ist für $g > 2$.

Mittels der Basen $dZ_{I_1}\otimes\ldots\otimes dZ_{I_l}$ fassen wir insbesondere alle Tensoren in $(\Omega^d(\mathbb{H}_g))^{\otimes l}$ als $\bigotimes^l V_\varrho$ wertige Funktionen auf $\mathbb{H}_g$ auf. Es sei

$$\bigotimes_1^l V_\varrho = \bigoplus_i V^{(i)}$$

eine Zerlegung der Darstellung $\varrho^{\otimes l}$ in irreduzible Komponenten $\varrho^{(i)}$. Dies induziert eine Zerlegung der Tensoren

$$T = \sum_i T^{(i)}.$$

Sei $V^{(i)}$ eine Komponente mit

$$\eta_Y^{\otimes l}(P)V^{(i)} \neq 0. \tag{1}$$

Hierbei wird $\eta_Y^{\otimes l}$ als Linearform auf $V^{(i)}$ aufgefaßt. Für jede irreduzible Darstellung $\varrho^{(i)}$ von $\mathrm{Gl}(g,\mathbb{R})$ auf $V^{(i)}$ ist das Gewicht $k^{(i)} = k(\varrho^{(i)})$ erklärt als die größte ganze Zahl k derart, daß $\varrho^{(i)}\otimes\det^{-k}$ eine polynominale Darstellung von $\mathrm{Gl}(g,\mathbb{R})$ ist. Das Maximum der Gewichte $k^{(i)}$ aller $V^{(i)}$ mit Eigenschaft (1) für $l\to\infty$ sei

$$k_Y = \sup_{i,l}\{k^{(i)}/l \colon \eta_Y^{\otimes l}(P)V^{(i)} \neq 0\}$$

wobei P ein generischer Punkt der Untervarietät Y sei (genaugenommen einer seiner Repräsentanten Z_0 in $\mathbb{H}_g$).

Lemma 1. *Es gilt* $k_Y \geqq g - \mathrm{cod}(Y\colon A_g)$.

Beweis. Es sei $\tilde{D}$ die Menge aller ganzen Zahlenpaare (i,j) mit der Eigenschaft $1\leqq i\leqq j\leqq g$. Eine Teilmenge D von $\tilde{D}$ heißt saturiert, falls für jedes Paar (i,j) in D auch alle Paare (i',j') aus $\tilde{D}$ mit $i'\leqq i$ und $j'\leqq j$ in D enthalten sind. Nach [4] entsprechen die irreduziblen Komponenten der Darstellung V_ϱ genau den d elementigen saturierten Teilmengen D von $\tilde{D}$. Das Gewicht $k(D)$ der zur Teilmenge D gehörigen irreduziblen Darstellung ist die Anzahl der Punkte (i,j) in D mit $j=g$. Der Punkt (g,g) wird doppelt gezählt. Für das l-fache Tensorprodukt von V gilt

$$\min_i k^{(i)}/l \geqq \min_D k(D).$$

Daraus ergibt sich sofort die Behauptung, da für mindestens eine Komponente $V^{(i)}$ die Behauptung (1) erfüllt ist.

Annahme. Y sei gegeben mit $k_Y > 0$.

Die Annahme ist nach Lemma 1 immer erfüllt für Y mit $\mathrm{cod}(Y\colon A_g)\leqq g-1$.

Wir wollen $\mathrm{Sp}(2g,\mathbb{Z})$ invariante holomorphe Tensoren konstruieren, welche auf Y nicht verschwinden. Spezielle Tensoren dieser Eigenschaft erhält man in der Form

$$T = f\cdot T_0$$

mit einer holomorphen Modulform f vom Gewicht $k = pq(k_Y - q^{-1})$ und einer vektorwertigen holomorphen Modulform

$$T_0\colon \mathbb{H}_g \to \bigotimes_1^p V_{\varrho(i)} \subseteq \bigotimes_1^p \bigotimes_1^q V_\varrho$$

des Transformationsverhaltens

$$T_0(MZ)=\varrho_0^{\otimes p}(CZ+D)f(Z), \qquad M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in \mathrm{Sp}(2g,\mathbb{Z}). \tag{2}$$

Hierbei sei wie bisher V_ϱ der von der Basisdifferentialen dZ_I aufgespannte $\mathbb{C}$-Vektorraum und

$$\varrho_0(g)=\varrho^{(i)}(g)\det(g)^{1-k_Y q}$$

eine Darstellung mit positivem Gewicht $k(\varrho_0)$. Dabei sind q und $V^{(i)}$, $\varrho^{(i)}$

$$V^{(i)}\subseteq \bigotimes_1^q V_\varrho$$

so gewählt, daß (1) für $q=l$ erfüllt ist. Die Zahl q kann dabei beliebig groß gewählt werden.

Bemerkung. Jeder solche Tensor $T=T_0 f$ definiert einen holomorphen Tensor auf A_g^0 vom Typ $\Omega^d(A_g^0)^{\otimes l}$ mit $l=pq$.

Das Element $\eta_Y(P)^{\otimes l}$ definiert eine Linearform auf $V_\varrho^{\otimes l}$ und damit eine Linearform auf $\bigotimes_1^p V_{\varrho(i)}$. Wir wollen uns davon überzeugen, daß es genügend Formen T_0 gibt, welche auf Y nicht verschwinden, d. h.

$$T_0(Z_0)\eta_Y(Z_0)^{\otimes l}\neq 0$$

für einen Repräsentanten Z_0 von P in $\mathbb{H}_g$.

Die obere Halbebene $\mathbb{H}_g$ besitzt als beschränktes Modell den verallgemeinerten Einheitskreis $\mathbb{E}_g=\{\zeta=\zeta^{(g)}=\zeta'\colon E-\zeta\bar{\zeta}>0\}$. Wegen $\varphi:\mathbb{E}_g \xrightarrow{\sim} \mathbb{H}_g$ genügt es die Konstruktion von T_0 im beschränkten Modell durchzuführen. Die transformierte Gruppe von $\mathrm{Sp}(2g,\mathbb{Z})$ definiert eine Automorphismengruppe Γ von $\mathbb{E}_g$. Analog erhält man einen transformierten Punkt $\zeta_0=\varphi^{-1}Z_0$ und eine Linearform $\eta_Y(\zeta_0)=\varphi_*^{-1}\eta_Y(Z_0)$ auf V_ϱ. Der Darstellung ϱ_0 von $\mathrm{Gl}(g,\mathbb{R})$ ist ein Automorphiefaktor

$$J_{\varrho_0}(M,\zeta)\in \mathrm{End}(V_{\varrho(i)}), \qquad M\in\Gamma, \qquad \zeta\in\mathbb{E}_g$$

zugeordnet. Formen T_0 auf $\mathbb{H}_g$ mit (2) entsprechen Formen T_0 auf $\mathbb{E}_g$ mit

$$T_0(M\zeta)=J_{\varrho_0}(M,\zeta)^{\otimes p}T_0(\zeta), \qquad M\in\Gamma.$$

Ist das Gewicht von $\varrho_0^{\otimes p}$ größer als $2g$, also insbesondere für $p>2g$, dann konvergiert die Poincaréreihe

$$P_p(\zeta)=\sum_{M\in\Gamma} J_{\varrho_0^{\otimes p}}(M,\zeta)^{-1}v^{\otimes p}, \qquad v\in V^{(i)}$$

absolut und gleichmäßig auf Kompakta von $\mathbb{E}_g$ und definiert eine Form T_0 mit der gewünschten Transformationseigenschaft auf $\mathbb{E}_g$. Wegen Lemma 2 verschwinden für $p_0>2g$ bei geeigneter Wahl von $v\in V^{(i)}$ nicht alle Werte $(\nu=1,2,\dots)$

$$P_{\nu p_0}(\zeta_P)\circ\eta_Y(\zeta_0)^{\otimes \nu p_0 q}=\sum_{M\in\Gamma}(J_{\varrho_0}(M,\zeta_0)^{-1}v\circ\eta_Y(\zeta_0)^{\otimes q})^{p_0\cdot\nu}.$$

Lemma 2. *Seien a_μ komplexe Zahlen und $\sum_{\mu=1}^{\infty} a_\mu$ absolut konvergent. Ist $\sum_{\mu=1}^{\infty} a_\mu^\nu = 0$ für alle $\nu = 1, 2, \ldots$ dann gilt $a_\mu = 0$ für alle μ.*

Siehe [2].

Man erhält als

Folgerung. *Für jedes l_0 gibt es ein $l > l_0$ und einen Tensor $T = f T_0$ in $\Omega^d(A_g^0)^{\otimes l}$, dessen Pullback auf Y im Punkt P genau dann verschwindet, wenn $f(Z_0) = 0$ ist.*

Hierbei ist f eine Modulform vom Gewicht $k = l(k_Y - \varepsilon)$ und $\varepsilon > 0$ kann beliebig klein gewählt werden. Auch die Wahl von f (bei festem Gewicht) unterliegt keinen Einschränkungen.

An dieser Stelle sei erwähnt, daß man durch Symmetrisieren von T_0 immer annehmen kann $T \in \mathrm{Symm}^l \Omega^d(A_g^0)$.

2. Fortsetzung auf ein singularitätenfreies Modell

Sei $\tilde{A}_g$ ein vollständiges glattes Modell von A_g. Ohne Einschränkung können wir annehmen $A_g^0 \subseteq \tilde{A}_g$ mit $\tilde{A}_g \backslash A_g^0$ als Divisor mit normalen Kreuzungen. Es sei $N = \dim A_g$ und $E = \{q \in \mathbb{C}\colon |q| < 1\}$ der komplexe Einheitskreis sowie $E^* = E \backslash \{0\}$. Wir nehmen im folgenden an, daß $\tilde{A}_g$ als Auflösung der Quotientensingularitäten einer toroidalen Kompaktifizierung A_g^+ (s. [6]) von A_g gegeben ist. Ein holomorpher Tensor T auf A_g^0 erweitert sich holomorph auf $\tilde{A}_g$ genau dann, wenn für jede holomorphe Abbildung ψ

$$\begin{array}{ccc} E^* \times E^{N-1} & \xrightarrow{\psi} & A_g^0 \\ \cap\!\!\mid & & \downarrow \\ E^N & \longrightarrow & \overline{\mathbb{H}_g/\Gamma_g} \quad \text{(Satakekompaktifizierung)} \end{array}$$

der Pullback $\psi^* T$ sich holomorph auf E^N erweitern läßt. Man erhält via universelle Überlagerung von A_g^0 eine Abbildung Ψ

$$\begin{array}{cccc} (z, q_2, \ldots, q_N) & \in \mathbb{H} \times E^{N-1} & \xrightarrow{\Psi} & \mathbb{H}_g \\ \downarrow & \downarrow & & \downarrow \\ (\exp 2\pi i z, q_2, \ldots, q_N) & \in E^* \times E^{N-1} & \xrightarrow{\psi} & A_g\,. \end{array}$$

Bekanntlich ist die Abbildung Ψ nach geeigneter Transformation unter $\mathrm{Sp}(2g, \mathbb{Z})$ von der Gestalt

$$\Psi(z, q_2, \ldots, q_N) = S_0 z + \Psi_0(q_1^{1/m}, q_2, \ldots, q_N)$$

für $q_1 = \exp(2\pi i z)$ und $z \in \mathbb{H}$ sowie $q_2, \ldots, q_N$ aus E^{N-1}. Hierbei ist Ψ_0 holomorph auf ganz E^N. Die Matrix S_0 ist symmetrisch, semipositiv und von der Form

$$S_0 = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}, \qquad S = S^{(r)} = S' > 0\,.$$

Es gilt

$$\Psi(z+1, q_2, \ldots, q_N) = M \Psi(z, q_2, \ldots, q_N)$$

mit einer Matrix M in $\mathrm{Sp}(2g, \mathbb{Z})$, welche im Stabilisator der Randkomponente $\mathbb{H}_{g-r}/\mathrm{Sp}(2g-2r, \mathbb{Z})$ der Satakekompaktifizierung liegt. Durch Transformation dieser „Randkomponente $\mathbb{H}_{g-r}$" in das beschränkte Modell $\mathbb{E}_{g-r}$ kann man M von der Form

$$M = \begin{pmatrix} E & S_0 \\ 0 & E \end{pmatrix} \begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix}$$

annehmen. Hierbei ist $U \in \mathrm{Gl}(g, \mathbb{Z})$ ein Element endlicher Ordnung

$$U^h = E.$$

Fall 1. Sei $\psi(0) \in (A_g^+)^{\mathrm{reg}}$ ein regulärer Punkt des Randdivisors $A_g^+ \backslash A_g$, dann ist

$$U = \pm E, \qquad S_0 \neq 0 \quad \text{ganzzahlig}$$

und ohne Einschränkung ist $m = 1$ sowie

$$\Psi(q_1, \ldots, q_N) = S_0 \frac{\log q_1}{2\pi i} + \Psi_0(q_1, \ldots, q_N).$$

Der Pullback des Tensors

$$T = f_{I_1, \ldots, I_l} dZ_{I_1} \otimes \ldots \otimes dZ_{I_l}$$

ist

$$\psi^* T = f_{I_1, \ldots, I_l}\left(S_0 \frac{\log q_1}{2\pi i} + \psi_0\right) \psi^* T',$$

$$\psi^* T' = \bigotimes_{a=1}^{l} \bigwedge_{a'=1}^{d} \left(S_0 \frac{dq_1}{q_1} + \omega_{aa'}\right)$$

für gewisse auf E^N holomorphe Differentiale $\omega_{aa'}$. Die Polordnung des Faktors $\psi^* T'$ ist somit höchstens q_1^{-l}. Andererseits besitzen die Koeffizienten $f_{I_1, \ldots, I_l}$ Fourierentwicklungen

$$f_{I_1, \ldots, I_l}(Z) = \sum_{H = H' \geqq 0} a_{I_1, \ldots, I_l}(H) e^{2\pi i \,\mathrm{Spur}(HZ)} \quad (H \text{ halbganz}),$$

wenn T eine Komponente eines $\mathrm{Sp}(2g, \mathbb{Z})$ invarianten Tensors auf $\mathbb{H}_g$ ist. Die holomorphe Fortsetzbarkeit eines solchen Tensors auf $q_1 = 0$ ist somit garantiert falls

$$a_{I_1, \ldots, I_l}(H) \neq 0 \;\Rightarrow\; \mathrm{Spur}(HS_0) \geqq l$$

für alle semipositiven ganzen Matrizen $S_0 \neq 0$ gilt. Das Minimum für alle solchen S_0 von $\mathrm{Spur}(HS_0)$ ist nach Barnes und Cohn [1]

$$\min_{S_0} \mathrm{Spur}(HS_0) = \min\{\tilde{H}_{nn} \colon \tilde{H} = U' H U \text{ für } U \in \mathrm{Sl}(g, \mathbb{Z})\}$$

für halbganze $H \geqq 0$.

Definition. Die Verschwindungsordnung $o(f)$ einer Modulform f vom Gewicht k

$$f(Z) = \sum_{\substack{H \\ \text{halbganz}}} a(H) e^{2\pi i \,\mathrm{Spur}(HZ)}$$

ist

$$o(f) = m/k,$$

wobei m die kleinste ganze Zahl mit $a\begin{pmatrix} * & * \\ * & m \end{pmatrix} \neq 0$ ist.

Aus dem Satz von Barnes und Cohn ergibt sich

Lemma 3. *Die Tensoren $T = T_0 f$ in $\Omega^d(A_g^0)^{\otimes l}$ sind fortsetzbar im Fall 1, falls $o(f) \geqq (k_Y - \varepsilon)^{-1}$ ist. (Bezeichnungen wie im vorigen Abschnitt.)*

Fall 2. $S_0 = 0$.

In diesem Fall handelt es sich um die Fortsetzung des Tensors auf eine Desingularisierung der elliptischen Fixpunkte von $\mathrm{Sp}(2g, \mathbb{Z})$ in $\mathbb{H}_g$. Durch eine modifizierte Cayley Transformation kann ohne Einschränkung angenommen werden, daß es sich bei dem Fixpunkt um den Nullpunkt des verallgemeinerten Einheitskreises $\mathbb{E}_g$ handelt und daß die Matrix U von der Gestalt

$$U = \begin{pmatrix} \zeta^{a_1} & & \\ & \ddots & \\ & & \zeta^{a_g} \end{pmatrix}$$

ist. Hierbei ist ζ eine primitive m-te Einheitswurzel ($m \geqq 2$) und obdA $U \neq \pm E$. Die Operation von M auf $\mathbb{E}_g$ ist dann gegeben durch

$$W \mapsto U'WU.$$

Für geeignete lokale Koordinaten Z_{ij} $(1 \leqq i \leqq j \leqq N)$ gilt also

$$MZ_{ij} = \zeta^{a_i + a_j} Z_{ij}.$$

Da die Einheitswurzeln $\zeta^{a_1}, \ldots, \zeta^{a_g}, \zeta^{-a_1}, \ldots, \zeta^{-a_g}$ Eigenwerte einer ganzzahligen Matrix M sind, ist unter den Zahlen

$$a_1, \ldots, a_g \in (\mathbb{Z}/m\mathbb{Z})$$

ein vollständiges Halbsystem von primitiven Elementen enthalten, d. h.: Für $x \in (\mathbb{Z}/m\mathbb{Z})^*$ liegt entweder x oder $-x$ in $\{a_1, \ldots, a_g\}$.

Lemma 4. *Es sei T ein holomorpher Tensor auf E^N vom Typ $\Omega^d(E^N)^{\otimes l}$. Es sei M eine lineare Substitution $MZ_i = \zeta^{b_i} Z_i$ $(i = 1, \ldots, N)$ mit einer primitiven m-ten Einheitswurzel ζ und $0 \leqq b_i < m$. Ist T invariant unter M, $\tilde{X}$ eine Desingularisierung des Quotienten $X = E^N/\langle M \rangle$ und $E^N \to X$ unverzweigt außerhalb einer Menge der* cod $\geqq 2$, *dann läßt sich T holomorph auf $\tilde{X}$ fortsetzen falls*

$$\sum_{i \in I} b_i \geqq m$$

bei beliebiger Wahl von ζ für alle Teilmengen $I \subseteq \{1, \ldots, N\}$ mit d Elementen gilt.

Es handelt sich hierbei um eine naheliegende Verallgemeinerung des Kriteriums von Tai-Ried im Fall $d = N$.

Beweis. Es genügt wieder die Fortsetzbarkeit von $\psi^* T$ auf E^N für jede holomorphe Abbildung

$$E^* \times E^{N-1} \xrightarrow{\psi} (E^N/\langle M \rangle)^{\mathrm{reg}}$$

zu zeigen. Via universelle Überlagerung erhält man eine Abbildung

$$\begin{array}{ccccc} (q_1^{1/m}, q_2, \ldots, q_N) & \in & E^N & \xrightarrow{\Psi} & E^N \backslash \text{Fixpkt.} \\ \downarrow & & \downarrow & & \downarrow \\ (q_1, q_2, \ldots, q_N) & \in & E^* \times E^{N-1} & \xrightarrow{\Psi} & (E^N/\langle M \rangle)^{\mathrm{reg}}. \end{array}$$

Für die Komponenten $\Psi_1, \ldots, \Psi_N$ der holomorphen Abbildung Ψ gilt notwendigerweise

$$\Psi_i(\zeta \tilde{q}_1, q_2, \ldots, q_N) = \zeta^{b_i} \Psi_i(\tilde{q}_1, \ldots, q_N), \qquad \tilde{q}_1 = q_1^{1/m}$$

also

$$\Psi_i(\tilde{q}_1, q_2, \ldots, q_N) = \tilde{q}_1^{b_i} \tilde{\Psi}_i(q_1, \ldots, q_N)$$

mit holomorphen Funktionen $\tilde{\Psi}_1, \ldots, \tilde{\Psi}_N$. Der Pullback eines holomorphen Tensors

$$T = f_{I_1, \ldots, I_l}(Z_1, \ldots, Z_N) \bigotimes_{\nu=1}^{l} dZ_{I_\nu}$$

ist somit

$$\Psi^* T = f_{I_1, \ldots, I_l}(\tilde{q}_1, q_2, \ldots, q_N) \Psi^* T'$$

mit

$$\begin{aligned} \Psi^* T' &= \bigotimes_{a=1}^{l} \bigwedge_{a'=1}^{d} \Psi^* Z_{i(aa')}(d \log Z_{i(aa')}) \\ &= \bigotimes_{a=1}^{l} \left[\bigwedge_{a'=1}^{d} \tilde{q}_1^{b_{i(aa')}} \tilde{\Psi}_{i(aa')} \left(\frac{b_{i(aa')}}{m} \frac{dq_1}{q_1} + \omega_{aa'} \right) \right] \end{aligned}$$

und gewissen in $q_1, \ldots, q_N$ holomorphen Differentialen $\omega_{aa'}$. Falls für jede der Summen

$$\sum_{a'=1}^{d} b_{i(aa')} \geqq m$$

gilt, dann läßt sich jede der Tensoren in den eckigen Klammern auf $q_1 = 0$ holomorph fortsetzen und somit auch T selbst.

Es bezeichne $[x]$ für $x \in \mathbb{R}$ diejenige Zahl $0 \leqq [x] < 1$ mit $x - [x] \in \mathbb{Z}$.

Zusammenfassung. *Die Tensoren $T = f \cdot T_0$ in $\Omega^d(A_g^0)^{\otimes l}$ besitzen die Fortsetzungseigenschaft für Desingularisierungen der elliptischen Fixpunkte, falls für jede Menge von Zahlen $a_1, \ldots, a_g$ mit $0 \leqq a_i < m$, welche ein vollständiges primitives Halbsystem enthält, gilt*

$$\sum_{(i,j) \in I} \left[\frac{a_i + a_j}{m} \right] \geqq 1 .$$

Hierbei sei I eine beliebige d elementige Teilmenge der Tupel (i, j) mit $1 \leqq i \leqq j \leqq g$.

Bemerkung. Ohne Einschränkung ist außerdem $m \geqq 2$ und $(a_1, \ldots, a_g) \neq (1, \ldots, 1)$ im Fall $m = 2$.

Fall 3. Es genügt abschließend die Fortsetzung des Tensors über die elliptischen Fixpunkte von A_g^+ zu behandeln. Dies entspricht einer Mischung der beiden Fälle 1 und 2:

$$U \neq \pm E, \qquad S_0 \neq 0.$$

Sind jedoch die im Fall 1 und 2 formulierten Bedingungen erfüllt, dann besitzt der Tensor auch im Fall 3 die gewünschte Fortsetzungseigenschaft. Dies folgt analog wie in [3] oder [6].

3. Ein numerisches Kriterium

Es sei $m \geqq 2$ und $a_1, \ldots, a_g$ ein Tupel von Zahlen in $\mathbb{Z}/m\mathbb{Z}$, welche ein vollständiges Halbsystem von primitiven Restklassen enthalten. Im folgenden seien i und j aus einem Repräsentantensystem $0 \leqq i, j \leqq m/2$ von $(\mathbb{Z}/m\mathbb{Z})/\pm 1$. Es bezeichne $A(i)$ die Anzahl der a_ν $(\nu = 1, \ldots, g)$ mit $a_\nu = \pm i$. Weiterhin seien $A_+(i)$, $A_-(i)$ die Anzahlen der $a_\nu = i$ bzw. $a_\nu = -i$. Man hat offensichtlich

$$A_+(i) + A_-(i) = A(i)$$

sowie

$$\sum_{i=0}^{m/2} A(i) = g. \tag{3}$$

Wir betrachten die Summen

$$S = \sum \left[\frac{a_\nu + a_\mu}{m} \right].$$

Annahme. *Die Summation erfolge über $\frac{1}{2}g(g+1) - g + \gamma$ verschiedene Tupel (ν, μ) mit $1 \leqq \nu \leqq \mu \leqq g$.*

(Hierbei sei γ eine Zahl $\geqq 2$.)

Wir wollen zeigen, daß im allgemeinen $S \geqq 1$ gilt. Wir machen daher die

Annahme. $S < 1$

und diskutieren die Ausnahmefälle.

Eine grobe Abschätzung in der jeder Summand von S entweder durch $1/m$ oder durch null abgeschätzt wird zeigt

$$S \geqq \frac{1}{2m} \sum_{i \neq j} A(i)A(j) + \frac{1}{4m} \sum_{i \neq 0, m/2} A(i)^2 - \frac{g-\gamma}{m}. \tag{4}$$

Die Summationen erfolgen dabei über alle i resp. j in $(\mathbb{Z}/m\mathbb{Z})/\pm 1$. Der zweite Summand ergibt sich wie folgt: Für $i \neq 0, m/2$ liefern die Paare $(a_\nu, a_\mu) = \pm(i, i)$ mod m mindestens den Beitrag

$$\begin{aligned} S' &\geqq \frac{1}{2m} A_+(i)(A_+(i)+1) + \frac{1}{2m} A_-(i)(A_-(i)+1) \\ &\geqq \frac{1}{4m} A(i)^2. \end{aligned}$$

Folgerung. Es gilt entweder

a) $$A(0)+A\left(\frac{m}{2}\right)\geqq g-2$$

oder

b) $$m\geqq\tfrac{1}{2}g,$$

denn es ist

$$\left(A(0)+A\left(\frac{m}{2}\right)\right)^2\geqq A(0)^2+A\left(\frac{m}{2}\right)^2\geqq g^2-4g+4\gamma-4m$$
$$>(g-3)^2 \tag{5}$$

für $m<\frac{1}{2}g$ wegen (3) und (4).

Im Fall a) hat somit ein vollständiges Halbsystem höchstens 2 verschiedene Elemente, da 0 und $\frac{m}{2}$ (außer für $m=2$) nicht primitiv sind. Es folgt $\varphi(m)\leqq 4$ und somit $m\in\{1,2,3,4,5,6,8,10,12\}$.

Annahme. $g\geqq 8$ *und* $\gamma\geqq 7$.

Diese Bedingung garantiert im Fall a) die Ungleichung $S\geqq 1$. Dieser Fall tritt also gar nicht auf. Beachte dazu im Fall a)

$$S\geqq\frac{2}{m}+\frac{g-2}{m}+\frac{g-2}{m}-\frac{g-\gamma}{m}\geqq\frac{13}{12},$$

falls $A(0)+A\left(\frac{m}{2}\right)=g-2$ und

$$S\geqq\frac{1}{m}+\frac{g-1}{m}-\frac{g-\gamma}{m}\geqq\frac{7}{6},$$

falls $A(0)+A\left(\frac{m}{2}\right)=g-1$ ist. Dann ist nämlich sogar $\varphi(m)\leqq 2$ und $m\leqq 6$. Ist schließlich $A(0)+A\left(\frac{m}{2}\right)=g$ dann ist notwendigerweise $m=2$ und

$$S\geqq\frac{g-1}{2}-\frac{g-\gamma}{2}\geqq 3,$$

da wir annehmen können

Annahme. $A(0)\neq g$ *und* $A\left(\frac{m}{2}\right)\neq g$.

Es bleibt also bei obigen Annahmen nur der Fall b)

$$\tfrac{1}{2}\varphi(m)\leqq g\leqq 2m$$

zu behandeln. Nun verwenden wir eine etwas bessere Abschätzung für S. In der Summe S liefert jedes a_ν $(\nu=1,\dots,g)$ mindestens $\frac{1}{2}\varphi(m)$ verschiedene Beiträge $\left[\frac{a_\nu+a_\mu}{m}\right]$, wobei höchstens $g-\gamma$ Beiträge wegfallen. Also gilt

$$S \geqq \frac{g}{m}\sum_{\nu=1}^{\varphi(m)/2}(\nu-1)-\frac{g-\gamma}{m}\left(\tfrac{1}{2}\varphi(m)-1\right). \tag{6}$$

Es können also wegen $g\geqq\frac{1}{2}\varphi(m)$ alle Fälle mit

$$\frac{\varphi(m)}{4m}\left(\tfrac{1}{2}\varphi(m)-1\right)\left(\tfrac{1}{2}\varphi(m)-2\right)+\frac{\gamma}{m}\left(\tfrac{1}{2}\varphi(m)-1\right)\geqq 1 \tag{6'}$$

eliminiert werden. Die Abschätzung

$$\varphi(m)=m\prod_{p|m}(1-p^{-1})\geqq m^{1-\varepsilon}C_\varepsilon,$$
$$C_\varepsilon=\prod_p \min(p^\varepsilon-p^{-1+\varepsilon},1)$$

für $\varepsilon=1/3$ und $C_\varepsilon\leqq 0.605$ ergibt $m\leqq 81$. Wegen $\varphi(m)\geqq\frac{4}{15}m$ für $m\leqq 81$ scheiden erneut wegen (6′) alle Fälle mit $\varphi(m)\geqq 11$ aus. Aus $\varphi(m)\leqq 10$ und $m\leqq\frac{15}{4}\varphi(m)$ folgt dann $m\leqq 38$. Setzt man für die verbleibenden Fälle die genauen Werte von $\varphi(m)$ in (6′) ein, dann entfallen alle m mit $\varphi(m)>4$ bis auf $m=18$. Letzterer Wert entfällt aber wegen (6). Es bleiben also höchstens die Werte $m\leqq 12$. Wegen der Annahme $g\geqq 8$ gilt dann $4m\leqq 6g$ und (5) liefert

$$\left(A(0)+A\left(\frac{m}{2}\right)\right)^2\geqq(g-5)^2+4\gamma-25.$$

Wegen $\gamma\geqq 7$ ist daher

$$A(0)+A\left(\frac{m}{2}\right)=g-r \tag{7}$$

für $r\leqq 4$. Die Fälle $r=0,1,2$ wurden bereits eliminiert.

Fall c). $A(0)\neq 0$ und $A\left(\frac{m}{2}\right)\neq 0$.

Man erhält aus den wechselseitigen Summen

$$S\geqq\tfrac{1}{2}A(0)A\left(\frac{m}{2}\right)-\tfrac{1}{2}(g-\gamma).$$

Aus $A(0)+A\left(\frac{m}{2}\right)\geqq g-4$ folgt außerdem

$$A(0)A\left(\frac{m}{2}\right)\geqq\min\left(g-5,\left(\frac{g-4}{2}\right)^2\right)\geqq g-5$$

und somit $S\geqq 1$ wegen $\gamma\geqq 7$.

Fall d). $A(0)=0$ oder $A\left(\frac{m}{2}\right)=0$.

Wir können wie oben gezeigt uns auf die Fälle $m \leqq 12$ mit $r=3,4$ in (7) beschränken. Für $r=4$ ist

$$S \geqq \frac{4(g-4)}{12} - \frac{g-\gamma}{12} \geqq \frac{3g+\gamma-16}{12} \geqq 1\,.$$

Für $r=3$ ist

$$S \geqq \frac{3(g-3)}{12} - \frac{g-\gamma}{12} \geqq \frac{2g+\gamma-9}{12} \geqq 1$$

auf Grund der Annahmen.

Zusammenfassung. *Für $\gamma \geqq 7$ und $g \geqq 8$ ist das numerische Kriterium $S \geqq 1$ erfüllt. Jeder Tensor in $\Omega^d(A_g^0)^{\otimes l}$ mit $d \geqq \dim(A_g) - g + \gamma$ läßt sich auf eine Desingularisierung der elliptischen Fixpunkte holomorph fortsetzen.*

4. Konstruktion von Modulformen

Sei $m=(m'm'') \in \mathbb{R}^{2g}$ und $(Z,z) \in \mathbb{H}_g \times \mathbb{C}^g$. Wir erinnern an einige Eigenschaften der Thetareihen

$$\theta_m(Z,z) = \sum_{h \in \mathbb{Z}^g} \exp(\pi i Z[h+m'] + 2\pi i\, {}^t(h+m')(z+m''))\,.$$

Es gilt

$$\theta_{m+h}(Z,z) = \exp(2\pi i\, {}^t m' h'')\theta_m(Z,z) \tag{8}$$

für $h=(h'h'')$ in $\mathbb{Z}^{2g}$ und

$$\theta_{m+h}(Z,z) = \exp(\pi i(Z[h'] + 2\,{}^t h'(z+h'') + 2\,{}^t h' m''))\theta_m(Z, z + {}^t h' Z + h'') \tag{9}$$

für $h=(h'h'')$ in $\mathbb{R}^{2g}$. Die Thetanullwerte $\theta_m(Z) = \theta_m(Z,0)$ verschwinden identisch dann und nur dann wenn $m \in (\frac{1}{2}\mathbb{Z})^{2g}$ ist und $2\,{}^t m' m'' \equiv \frac{1}{2} \bmod \mathbb{Z}$.

Für $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ definiert

$$M(Z,z) = ((AZ+B)(CZ+D)^{-1}, z(CZ+D)^{-1})\,,$$

$$m^M = m \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} + \tfrac{1}{2}((CD)_0 (AB')_0) \tag{10}$$

eine Operation von $\mathrm{Sp}(2g,\mathbb{Z})$ auf $\mathbb{H}_g \times \mathbb{C}^g$ beziehungsweise auf $\mathbb{R}^{2g} \bmod \mathbb{Z}^{2g}$. Hierbei bezeichne T_0 für eine symmetrische Matrix T den Diagonalenvektor. Es gilt

$$\theta_{m^M}(M(Z,z)) = u \det(CZ+D)^{1/2} \exp(\pi i (CZ+D)^{-1} C[z])\theta_m(Z,z) \tag{11}$$

für ein $u \in \mathbb{C}^*$ unabhängig von (Z,z).

Sei $l=2p$ und p prim sowie $\mathcal{M}$ in $\left(\frac{1}{l}\mathbb{Z}/\mathbb{Z}\right)^{2g}$ die Menge aller Charakteristiken $m=(m_p, m_2)$ in $\left(\frac{1}{p}\mathbb{Z}/\mathbb{Z}\right)^{2g} \oplus (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}$ mit $m_p \notin \mathbb{Z}$. $\mathrm{Sp}(2g,\mathbb{Z})$ operiert auf $\mathcal{M}$ mit 2 Orbiten. Es sei $\varepsilon > 0$ gegeben und $\tilde{\mathcal{M}}$ eine Teilmenge von $\mathcal{M}$ mit

$$\# \tilde{\mathcal{M}} < \varepsilon \# \mathcal{M}\,. \tag{12}$$

Wir setzen dann $\theta_{\mathcal{M},\tilde{\mathcal{M}}}(Z)=\prod_{m\in\mathcal{M}\setminus\tilde{\mathcal{M}}}\theta_m(Z,0)^l$ und

$$F_r(Z)=\sum_{M\in\Gamma_g/\Gamma_g[2p]}\theta_{\mathcal{M},\tilde{\mathcal{M}}}(Z)^r|M$$

mit gewissen Repräsentanten M sowie

$$\theta_{\mathcal{M},\tilde{\mathcal{M}}}(Z)|M=\det(CZ+D)^{(\#\tilde{\mathcal{M}}-\#\mathcal{M})/2}\theta_{\mathcal{M},\tilde{\mathcal{M}}}(MZ).$$

Hierbei sei $\Gamma_g[2p]$ die Hauptkongruenzuntergruppe aller M in $\mathrm{Sp}(2g,\mathbb{Z})$ mit $M\equiv E \bmod 2p$. Wegen (8), (10) und (11) definiert

$$\chi(M)=\theta^2_{\mathcal{M},\tilde{\mathcal{M}}}(Z)|M\theta^{-2}_{\mathcal{M},\tilde{\mathcal{M}}}(Z)$$

einen Charakter von $\Gamma_g[2p]$. Dieser hat endliche Ordnung und $F_r(Z)$ ist für alle Vielfachen r einer geeigneten Zahl r_0 eine Modulform zur vollen Modulgruppe.

Sei nun Z_0 ein fest gewählter Punkt in $\mathbb{H}_g$.

Behauptung. *Sei $\varepsilon>0$ gegeben, dann gilt für große p bei geeigneter Wahl einer Menge $\tilde{\mathcal{M}}$ in $\mathcal{M}$ mit* (12), *daß für mindestens ein r $F_{rr_0}(Z_0)$ nicht verschwindet.*

Wäre die Behauptung falsch, dann folgt aus Lemma 2

$$\theta_{\mathcal{M},\tilde{\mathcal{M}}}(Z_0)=0.$$

Wegen (9) bedeutet dies, daß der Thetadivisor $D_0=\{z\in A^{(0)}\colon \theta_0(Z_0,z)=0\}$ der abelschen Varietät $A^{(0)}=\mathbb{C}^g/\mathbb{Z}^g+Z_0\mathbb{Z}^g$ einen l-Torsionspunkt $m\in A_l^{(0)}$ mit $m\in\mathcal{M}\setminus\tilde{\mathcal{M}}$ enthält. Ist D die Vereinigung der Translate $m+D_0$ mit $m\in A_2^{(0)}$, dann gibt es eine Konstante $c(D)$ mit

$$\#\{m\in A_p^{(0)}\colon m\in D\}\leqq c(D)p^{2g-2}.$$

Für große p erfüllt daher

$$\tilde{\mathcal{M}}=\{m\in A_l^{(0)}\colon m=(m_p,m_2),\ m_p\in D\}$$

die Ungleichung (12). Für diese Wahl von $\tilde{\mathcal{M}}$ ist $\theta_{\mathcal{M},\tilde{\mathcal{M}}}(Z_0)\neq 0$ und die Behauptung folgt.

Jeder Summand von $F_r(Z)$ ist von der Form $\theta^r_{\mathcal{M},\tilde{\mathcal{M}}'}$ mit $\tilde{\mathcal{M}}'=M\tilde{\mathcal{M}}$. Um die Verschwindungsordnung von $F_r(Z)$ abzuschätzen genügt es daher die Verschwindungsordnung aller $\theta_{\mathcal{M},\tilde{\mathcal{M}}'}$ abzuschätzen mit einer Schranke, welche nur von $\#\tilde{\mathcal{M}}'/\#\mathcal{M}$ abhängt.

Die Verschwindungsordnung einer einzelnen Thetacharakteristik ist

$$o(\theta_m)=(m_g')^2,$$

falls die letzte Komponente m_g' von $m'\in\mathbb{R}^g$ die Ungleichung $-\frac{1}{2}\leqq m_g'\leqq\frac{1}{2}$ erfüllt. Dies soll vorausgesetzt werden. Es ergibt sich

$$o(\theta_{\mathcal{M},\tilde{\mathcal{M}}})=[\#\mathcal{M}\setminus\#\tilde{\mathcal{M}}]^{-1}\sum_{m\in\mathcal{M}\setminus\tilde{\mathcal{M}}}(m_g')^2$$

bei dieser Normierung der m. Für $\varepsilon\to 0$ und $p\to\infty$ erhält man für beliebige $\tilde{\mathcal{M}}$ in $\mathcal{M}$, welche die Ungleichung (12) erfüllen

$$\lim_{\varepsilon\to 0}\lim_{p\to\infty}\min_{\tilde{\mathcal{M}}} o(\theta_{\mathcal{M},\tilde{\mathcal{M}}})=\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2}x_g^2dx_1\dots dx_{2g}=\tfrac{1}{12}.$$

Folgerung. *Für jedes* $\varepsilon > 0$ *und* $Z_0 \in \mathbb{H}_g$ *gibt es eine Modulform* $F(Z)$ *zur vollen Modulgruppe mit* $F(Z_0) \neq 0$ *und* $o(F) \geqq \frac{1}{12+\varepsilon}$.

Dies ist in gewisser Weise bestmöglich, da für $o(F) > \frac{1}{12}$ immer $F(Z_0) = 0$ für $Z_0 = iE$ gilt.

Ist h eine beliebige Modulform, dann gilt $o(hf^t) \to o(f)$ für $t \to \infty$. Es gibt somit für jeden Punkt Z_0 in Y in diesem Punkt nicht verschwindende Modulformen $f_0, \ldots, f_d$ vorgegebenen Gewichtes $k \gg 0$ so, daß die meromorphen Funktionen f_ν / f_0 $(\nu = 1, \ldots, d)$ algebraisch unabhängig auf Y sind. Bei geeigneter Wahl von k sind die Tensoren $T_\nu = f_\nu T_0$ aus $\Omega^d(A_g^0)^{\otimes l}$ wegen Lemma 1 und Lemma 3 und den Überlegungen des vorigen Abschnittes für $\operatorname{cod}(Y: A_g) \leqq g - 13$ auf ein singularitätenfreies vollständiges Modell $\tilde{A}_g$ fortsetzbar. Mittels eingebetteter Desingularisierung kann der Abschluß $\bar{Y}$ von Y in $\tilde{A}_g$ glatt angenommen werden. Die Pullbacks der T_ν definieren somit plurikanonische Differentialformen auf $\bar{Y}$.

Korollar. *Jede Untervarietät* Y *von* A_g *mit* $\operatorname{cod}(Y: A_g) \leqq g - 13$ *ist von allgemeinem Typ.*

Bemerkung. Da Modulformen auf A_g Punkte trennen, zeigt unsere Konstruktion sogar, daß bei geeigneter Wahl die meromorphen Funktionen T_i / T_0 auf Y in einer Zariski offenen dichten Teilmenge von Y Punkte trennen. Der Körper der meromorphen Funktionen von Y wird somit im Fall $\operatorname{cod}(Y: A_g) \leqq g - 13$ von Quotienten von plurikanonischen Differentialformen von Y erzeugt.

Mit der Konstruktion der Modulformen in [3] ergibt sich das zum obigen Korollar analoge Resultat

Korollar. *Jeder Divisor von* A_g *ist von allgemeinem Typ für* $g \geqq 10$.

Korollar. *Jeder birationale Automorphismus von* A_g *ist die Identität für* $g \geqq 10$.

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Mathematische Annalen

Studies on the Painlevé Equations

III. Second and Fourth Painlevé Equations, P_{II} and P_{IV}

Kazuo Okamoto*

Université Louis Pasteur, Institut de Recherche Mathématique Avancée, 10, rue du Général Zimmer, F-67084 Strasbourg Cedex, France

Contents

* Address after September 20, 1986: Department of Mathematics, University of Tokyo, Komaba, Meguro, Tokyo, Japan

0. Introduction

The present article, the third part of the series of papers [14], deals with the second Painlevé equation and the fourth one. They are relatively known in the six Painlevé equations and investigated in many articles from several viewpoints (cf., for example, [1, 3, 5–10, 16]). We will study in what follows birational canonical transformations of the Painlevé system associated with each of the equations.

First of all, consider the fourth equation of Painlevé:

$$P_{\mathrm{IV}} \qquad \frac{d^2q}{dt^2}=\frac{1}{2q}\left(\frac{dq}{dt}\right)^2+\frac{3}{2}q^3+4tq^2+2(t^2-\alpha)q+\frac{\beta}{q},$$

α, β being complex constants. It is equivalent to the Hamiltonian system of differential equations:

$$\frac{dq}{dt}=\frac{\partial H}{\partial p}, \qquad \frac{dp}{dt}=-\frac{\partial H}{\partial q}, \tag{0.1}$$

with the Hamiltonian:

$$H_{\mathrm{IV}} \qquad 2qp^2-\{q^2+2tq+2\kappa_0\}p+\theta_\infty q,$$

where

$$\alpha=-\kappa_0+2\theta_\infty+1, \qquad \beta=-2\kappa_0^2.$$

We associated with P_{IV} the quartet:

$$\mathscr{H}_{\mathrm{IV}}=(q, p, H, t),$$

called the *Painlevé system associated with* P_{IV}. For a solution $(q,p)=(q(t),p(t))$ of (0.1) we define by

$$H(t)=H(t;\, q(t), p(t)) \tag{0.2}$$

the Hamiltonian function of $\mathscr{H}_{\mathrm{IV}}$ and by

$$H(t)=\frac{d}{dt}\log\tau(t) \tag{0.3}$$

the τ-function $\tau=\tau(t)$ of $\mathscr{H}_{\mathrm{IV}}$ related to (q,p). For the sake of simplification of presentation, we say (q,p) (resp. $H(t), \tau(t)$) is a solution (resp. a Hamiltonian function, a τ-function) of the Painlevé system $\mathscr{H}$.

A solution (q,p) of $\mathscr{H}_{\mathrm{IV}}$ is meromorphic on the whole complex plane $\mathbb{C}$. Moreover it is known:

Proposition 0.1 [13]. *If $t=t_0$ is a pole of (q,p), then it is a simple pole of $q(t)$ and*

$$\operatorname*{Res}_{t=t_0} q(t)=\pm 1\,.$$

The meromorphic function $H(t)$ admits a simple pole at t_0 if and only if the residue of $q(t)$ at $t=t_0$ is -1. The τ-function $\tau(t)$ is entire on $\mathbb{C}$.

We say the Hamiltonian $H_{\mathrm{IV}}=H(t;q,p)$ corresponds to the residue -1. There exists the other Hamiltonian corresponding to the residue 1. In fact, we have:

$$\bar{H}(t;q,\bar{p})=2q\bar{p}^2+(q^2+2tq-2\kappa_0)\bar{p}+(\theta_\infty-\kappa_0+1)q\,. \tag{0.4}$$

The Hamiltonian system

$$\bar{\mathscr{H}}_{\mathrm{IV}}=(q,\bar{p},\bar{H},t) \tag{0.5}$$

is associated with P_{IV}, so we can adopt it as the Painlevé system instead of $\mathscr{H}_{\mathrm{IV}}$. The τ-functions $\bar{\tau}=\bar{\tau}(t)$ defined by

$$\bar{H}(t)=\frac{d}{dt}\log\bar{\tau}(t)\,, \tag{0.6}$$

is entire function on $\mathbb{C}$.

$\bar{\mathscr{H}}_{\mathrm{IV}}$ is connected to the Painlevé system $\mathscr{H}$ through the canonical transformation:

$$\pi:\mathscr{H}_{\mathrm{IV}}\to\bar{\mathscr{H}}_{\mathrm{IV}} \tag{0.7}$$

such that

$$\bar{p}=p-\tfrac{1}{2}q-t\,, \tag{0.8}$$

$$\bar{H}=H+q+2\kappa_0 t\,. \tag{0.9}$$

In fact, we can verify easily:

$$pdq-Hdt=\bar{p}dq-\bar{H}dt+dK\,,$$

$$K=\tfrac{1}{4}q^2+tq+\kappa_0 t^2\,,$$

by the use of (0.4), (0.8), and (0.9). Every result obtained below with respect to $\mathscr{H}_{\mathrm{IV}}$ can be translated into what concerns $\bar{\mathscr{H}}_{\mathrm{IV}}$.

The Hamiltonian $H_{\mathrm{IV}}(t;q,p)$ contains the two parameter κ_0, θ_∞, while we consider the vector $\mathbf{v}=(v_1,v_2,v_3)$ in the three dimensional complex vector space $\mathbb{C}^3$ such that

$$v_1=-\tfrac{1}{3}(\kappa_0+\theta_\infty)\,,\qquad v_2=\tfrac{1}{3}(2\kappa_0-\theta_\infty)\,,\qquad v_3=\tfrac{1}{3}(-\kappa_0+2\theta_\infty)\,. \tag{0.10}$$

It suits us to regard $\mathbf{v}$ as parameters of the Painlevé system; namely, the space V_{IV} of parameters of $\mathscr{H}_{\mathrm{IV}}$ is by definition the hyperplane:

$$v_1+v_2+v_3=0 \tag{0.11}$$

in $\mathbb{C}^3$; a point $\mathbf{v}$ of V parametrizes $\mathscr{H}_{\mathrm{IV}}$ through (0.10). When considering the Painlevé system at an arbitrary fixed value v of parameters, we denote by $\mathscr{H}_{\mathrm{IV}}(\mathbf{v})$

(resp. $\mathscr{H}(\mathbf{v}), (q(\mathbf{v}), p(\mathbf{v})), \tau(\mathbf{v})$) the Painlevé system (resp. the Hamiltonian or a Hamiltonian function, a solution, a τ-function) at $\mathbf{v}$.

The key to the method of birational canonical transformations is the nonlinear differential equation satisfied by the Hamiltonian function. In fact we can show the

Proposition 0.2. *A Hamiltonian function H satisfies the differential equation:*

$$\left(\frac{d^2H}{dt^2}\right)^2 - 4\left(t\frac{dH}{dt} - H\right)^2 + 4\frac{dH}{dt}\left(\frac{dH}{dt} + 2\kappa_0\right)\left(\frac{dH}{dt} + 2\theta_\infty\right) = 0. \tag{0.12}$$

Moreover, if we define by

$$h = H - 2v_1 t, \tag{0.13}$$

the *auxiliary Hamiltonian function*, by taking (0.10) into consideration, then (0.12) can be written as:

$$\left(\frac{d^2h}{dt^2}\right)^2 - 4\left(t\frac{dh}{dt} - h\right)^2 + 4\Pi_{k=1}^3\left(\frac{dh}{dt} + 2v_k\right) = 0.$$

We denote it by E_{IV} or by $E_{\mathrm{IV}}(\mathbf{v})$ when considering the Painlevé system at $\mathbf{v}$. Inversely, given a particular solution h of E_{IV}, we obtain the solution (q, p) of $\mathscr{H}_{\mathrm{IV}}$ as rational function of h and its derivatives. The correspondence from E_{IV} to $\mathscr{H}$, thus established, is denoted by

$$\Gamma(h) = (q, p) \tag{0.14}$$

throughout this series of papers. As for the proof of Proposition 0.2, see Proposition 3.1: cf. also Proposition 1.1.

Let σ be a canonical transformation of the Painlevé system $\mathscr{H} = \mathscr{H}_{\mathrm{IV}}$:

$$\sigma : (q, p, H, t) \to (q', p', H', t'). \tag{0.15}$$

We say σ is *of the first kind*, if $t' = t$, in (0.15). The Painlevé system H is provided with the structure of fiber space over the affine space V with the fiber $\mathscr{H}(\mathbf{v})$ on $\mathbf{v}$. We denote also by σ the restriction $\sigma_{\mathbf{v}}$ of σ to $\mathscr{H}(\mathbf{v})$. The transformation (0.15) is said *birational* if it is defined by a birational map from (q, p, t) to (q', p', t') and t' is rational in t. We consider in this series of papers mainly birational canonical transformations of $\mathscr{H}$.

We will see there exists, for each of the Poincaré systems, $\mathscr{H}_{\mathrm{II}}$ and $\mathscr{H}_{\mathrm{IV}}$, a birational canonical transformation ψ, different from the identity, such that each fiber $\mathscr{H}(\mathbf{v})$ of $\mathscr{H}$ remains invariant under ψ: see Propositions 2.2 and 4.3. The transformation ψ is not of the first kind. In general, if, for any $\mathbf{v}$ of V, we have

$$\psi : \mathscr{H}(\mathbf{v}) \to \mathscr{H}(\mathbf{v}),$$

then $\mathscr{H}$ is said *stable* under ψ. We conjecture [14] that if $\mathscr{H}$ is stable with respect to a transformation ψ of the first kind, then ψ is the identity.

A canonical transformation σ of the Painlevé system $\mathscr{H}$ induces in the canonical manner a correspondence between τ-functions, which is determined uniquely up to multiplicative constants of the τ-functions. We denote this correspondence also by σ; therefore, if $\sigma\mathscr{H}(\mathbf{v}) = \mathscr{H}(\mathbf{v}')$, we write $\sigma \cdot \tau(\mathbf{v}) = \tau(\mathbf{v}')$ in the meaning described just above. Let g be a transformation of the affine space V. We denote by g_* the birational canonical transformation:

$$\sigma : \mathscr{H}(\mathbf{v}) \to \mathscr{H}(g(\mathbf{v}))$$

for any $\mathbf{v}$. Such $\sigma = g_*$, if it does exist, is said to be associated with g. We obtain from this the correspondence of the τ-functions: $g_*\tau(\mathbf{v}) = \tau(g(\mathbf{v}))$. Let G be a subgroup of the group $\mathscr{A}(V)$ of affine motion of V and suppose that, for each g of G, there exists the canonical transformation g_* of $\mathscr{H}$, associated with g and that we have the homomorphism:

$$\varrho : G \to G_* . \tag{0.16}$$

G_* denoting the totality of g_*. We call (0.16) the *representation of G on the Painlevé system $\mathscr{H}$*; we may say for short g_* is the representation of g as a birational canonical transformation of $\mathscr{H}$.

In this paper we associated with each of the Painlevé systems, $\mathscr{H}_{\mathrm{II}}$ and $\mathscr{H}_{\mathrm{IV}}$, the group G and construct the representation of G on the Painlevé system. To put it more precisely, we have, for $\mathscr{H}_{\mathrm{II}}$, the affine Weyl group of the root system of the type A_1 and, for $\mathscr{H}_{\mathrm{IV}}$, the semi-direct product of the affine Weyl group of the type A_2 and the cyclic group of the order three: see Theorem 1 and Theorem 3 respectively. A transformation g_* of G_* is of the first kind. We consider also a birational canonical transformation σ, which is not of the first kind, as for the Painlevé system $\mathscr{H}_{\mathrm{IV}}$: see Sects. 4.3 and 4.4.

Let $\bar{H} = \bar{H}(\mathbf{v})$ be the Hamiltonian function of $\bar{\mathscr{H}}_{\mathrm{IV}} = \pi\mathscr{H}_{\mathrm{IV}}$. We can verify $\bar{H}$ satisfies the equation:

$$\left(\frac{d^2\bar{H}}{dt^2}\right)^2 - 4\left(t\frac{d\bar{H}}{dt}\right) - \bar{H}\Big)^2 + 4\frac{d\bar{H}}{dt}\left(\frac{d\bar{H}}{dt} - 2\kappa_0\right)\left(\frac{d\bar{H}}{dt} + 2(\theta_\infty + 1 - \kappa_0)\right) = 0 . \tag{0.17}$$

Comparing this with (0.12), we obtain from (0.9) the following relations:

$$\bar{H}(\mathbf{v}) = H(\ell(\mathbf{v})) + 2\kappa_0 t , \tag{0.18}$$

$$H(\ell(\mathbf{v})) = H(\mathbf{v}) + q(\mathbf{v}) , \tag{0.19}$$

where ℓ is the parallel transformation:

$$\ell(\mathbf{v}) = \mathbf{v} + \tfrac{1}{3}(-1, -1, 2) , \tag{0.20}$$

that is, the change of the constants:

$$\theta_\infty \mapsto \theta_\infty + 1 ,$$

in the Hamiltonian of $\mathscr{H}_{\mathrm{IV}}$. We deduce from (0.17), (0.18) the birational canonical transformation ℓ_* associated with ℓ: see Sect. 3.3. Moreover we have the sequence of τ-functions:

$$\mathfrak{T}(\ell) = \{\tau_m ;\ m \in \mathbb{Z}\} , \tag{0.21}$$

called for short a *τ-sequence* with respect to ℓ, such that

$$\ell_*\tau_m = \tau_{m+1} .$$

Given the τ-sequence (0.21), we have from (0.19) the sequence of solutions:

$$q_m = \frac{d}{dt} \log \frac{\tau_{m+1}}{\tau_m}$$

of the Painlevé system $\mathscr{H}_m = \mathscr{H}(\ell^m(\mathbf{v}))$.

A canonical transformation from (q, p, H, t) to (q', p', H', t') is said *trivial* if

$$q' = q, \quad p' = p, \quad t' = t,$$
$$H' = H + \phi(t).$$

Two τ-functions τ_1, τ_2 are said *equivalent* each other if they are connected mutually by means of the trivial canonical transformation. By replacing in $\mathfrak{T}(\ell)$ τ_m by $\tilde{\tau}_m$, equivalent to τ_m, we show $\tilde{\tau}_m$ satisfy the Toda equation

$$\frac{d^2}{dt^2} \log \tilde{\tau}_m = \frac{\tilde{\tau}_{m-1}\tilde{\tau}_{m+1}}{\tilde{\tau}_m^2}: \tag{0.22}$$

cf. Theorem 2.

One of the main subjects of this paper concerns a family of particular solutions of the Painlevé system $\mathscr{H}_{\mathrm{IV}}$, written in terms of the classical transcendental functions or elementary functions. We call such a particular solution a *classical solution* of $\mathscr{H}$. We consider in V_{IV} the set of lines, called *walls* of Weyl chambers of the affine Weyl group, and verify that, if $\mathbf{v}$ is contained in walls, then the Painlevé system $\mathscr{H}_{\mathrm{IV}}(\mathbf{v})$ at $\mathbf{v}$ admits classical solutions (Theorem 4). There appears the Hermite function as a τ-function related to a classical solution of $\mathscr{H}_{\mathrm{IV}}$. As for the Painlevé system $\mathscr{H}_{\mathrm{II}}$, a wall of a Weyl chamber is reduced to a point, where we have a family of classical solutions written by the use of the Airy function.

It is known that the Toda equation (0.22) admits a solution $\{\tilde{\tau}_m\}$ of the form

$$\tilde{\tau}_m = P_m(t) \exp(-\tfrac{1}{14}t^3),$$

where $P_m(t)$ is polynomial in t. We prove this fact by virtue of the method of birational canonical transformations of $\mathscr{H}_{\mathrm{II}}$. $P_m(t)$ is called the Yablonskii-Vorob'ev polynomials [1, 6, 16]. Moreover we obtain from the Painlevé system $\mathscr{H}_{\mathrm{IV}}$ the solutions of (0.22):

$$\tilde{\tau}_m = T_m(t) \exp(\tfrac{1}{27}t^4 - \tfrac{1}{3}(m-1)t^2),$$
$$\tilde{\tau}'_m = T'_m(t) \exp(\tfrac{1}{27}t^4 - \tfrac{1}{3}(m-1)t^2),$$

such that $T_m(t), T'_m(t)$ are polynomials in t. They are a generalization of the Yablonskii-Vorob'ev polynomials: cf. [7, 10].

In Sect. 1, we explain again the method of birational canonical transformations with respect to the Painlevé system $\mathscr{H}_{\mathrm{II}}$ associated with the second Painlevé equation. We construct the representation of the affine Weyl group of the type A_1 on $\mathscr{H}_{\mathrm{II}}$, and then obtain the Toda equation for the τ-sequence $\mathfrak{T}(s^{(1)})$ (Proposition 1.8). Classical solutions and rational function solutions of $\mathscr{H}_{\mathrm{II}}$ are considered. The proofs of the results stated in Sect. 1 are given in Sect. 2. The first two sections deal with $\mathscr{H}_{\mathrm{II}}$, while the rest of this paper concerns the fourth system $\mathscr{H}_{\mathrm{IV}}$.

In Sect. 3, we obtain the differential equation E_{IV} satisfied by the auxiliary Hamiltonian function. The representation of the parallel transformation (0.20) is given in Sect. 3.3. Using the canonical transformation ℓ_* we show the τ-sequence $\mathfrak{T}(\ell)$ satisfies the Toda equation.

Section 4 concerns the representation of the group $G=G_{IV}$ on the Painlevé system. We have in particular the group $\tilde{W}_*$ of birational canonical transformations of the first kind; $\tilde{W}_*$ is isomorphic to the affine Weyl group of the type A_2.
We consider also transformations causing a change of the independent variable.

Classical solutions and rational function solutions are the subject of Sect. 5. We obtain the sequence of polynomials from the τ-functions related to rational solutions of $\mathscr{H}_{IV}$.

1. Second Painlevé Equation

1.1 Hamiltonian Function

We consider firstly the second equation of Painlevé:

$$P_{II} \qquad \frac{d^2q}{dt^2}=2q^3+tq+\alpha,$$

and recapitulate the method of birational canonical transformations with respect to this equation. The Hamiltonian associated with $P=P_{II}$ is of the form:

$$H_{II} \qquad \tfrac{1}{2}p^2-(q^2+\tfrac{1}{2}t)p-(\alpha+\tfrac{1}{2})q\,.$$

Let $(q,p)=(q(t),p(t))$ be a solution of the Painlevé system $\mathscr{H}_{II}=(q,p,H,t)$ with the Hamiltonian $H=H_{II}(t;q,p)$ and $H(t)$ the Hamiltonian function. We will prove the following proposition:

Proposition 1.1. *$h=H(t)$ satisfies the non-linear differential equation:*

$$E_{II} \qquad \left(\frac{d^2h}{dt^2}\right)^2+4\left(\frac{dh}{dt}\right)^3+2\frac{dh}{dt}\left(t\frac{dh}{dt}-h\right)-\frac{1}{4}b^2=0, \qquad b=\alpha+\frac{1}{2}.$$

Inversely a pair of function (q,p) defined by

$$q=\frac{1}{4\dfrac{dh}{dt}}\left(2\frac{d^2h}{dt^2}+b\right), \qquad p=-2\frac{dh}{dt} \tag{1.1}$$

is a solution of the Painlevé system $\mathscr{H}=\mathscr{H}_{II}$, provided that h is the general solution of $E=E_{II}$.

Remark 1.1. The equation E admits singular solutions of the form

$$h=\lambda t+\mu,$$

$$4\lambda^3-2\lambda\mu-\left(\frac{b}{2}\right)^2=0.$$

Remark 1.2. Consider the canonical transformation:

$$\bar{\pi}: \mathscr{H} \to \bar{\mathscr{H}} = (\bar{q}, \bar{p}, \bar{H}, \bar{t}) \tag{1.2}$$

such that

$$\bar{q} = q, \quad \bar{p} = p - 2q^2 - t, \quad \bar{H} = H + q, \quad \bar{t} = t.$$

We can adopt $\bar{\mathscr{H}}$ as the Painlevé system associated with P. In particular, the Hamiltonian of $\bar{\mathscr{H}}$ is:

$$\bar{H} = \tfrac{1}{2}\bar{p}^2 - (q^2 + \tfrac{1}{2}t)\bar{p} - (\alpha - \tfrac{1}{2})q. \tag{1.3}$$

1.2 Parameter Space of $\mathscr{H}$

Let V be the hyperplane:

$$X_1 + X_2 = 0$$

in the two dimensional complex linear space $\mathbb{C}^2$. We regard it as the parameter space V_{II} of $\mathscr{H}$ by means of the correspondence:

$$b \mapsto (b, -b);$$

for a point $\mathbf{v} = (v_1, v_2)$ of V, we denote by $h(\mathbf{v})$ or $h[b]$ a solution $h = h(t; \mathbf{v})$ of the differential equation E with the parameter:

$$b = v_1 = -v_2.$$

The solution (q, p) of the Painlevé system, given from h by the correspondence (1.1) is written as $(q(\mathbf{v}), p(\mathbf{v}))$ or $(q[b], p[b])$. When considering the Painlevé system with the parameter, we denote it by $\mathscr{H}(\mathbf{v})$ or by $\mathscr{H}[b]$, and by $\mathscr{H}(\mathbf{v})$ or by $\mathscr{H}[b]$ the Hamiltonian $H(t; q, p; \mathbf{v})$ of the Painlevé system.

1.3 Invariance of Differential Equation E

We obtain from Proposition 1.1 the correspondence

$$\Gamma: E(\mathbf{v}) \to \mathscr{H}(\mathbf{v}), \tag{1.4}$$

that is, for the general solution $h = h(\mathbf{v})$ of the equation E with the parameter $\mathbf{v}$, we have the solution $(q, p) = \Gamma(h)$ of $\mathscr{H}(\mathbf{v})$. We will show the

Proposition 1.2. *If (q, p) is a solution of $\mathscr{H}(\mathbf{v})$, then (q_1, p_1) defined by*

$$q^1 = q + \frac{b}{p}, \quad p^1 = p \tag{1.5}$$

is that of $\mathscr{H}(\mathbf{v}^1)$, where for $\mathbf{v} = (v_1, v_2)$, we put

$$\mathbf{v}^1 = (v_2, v_1). \tag{1.6}$$

Remark 1.3. Here we assume, for the Hamiltonian function $h = H(t)$,

$$\frac{d^2h}{dt^2} \not\equiv 0. \tag{1.7}$$

In the proof of Proposition 1.1 we will see (1.7) is not established if and only if $b=0$ and

$$p\equiv 0\,,\qquad \frac{dq}{dt}=-q^2-\frac{1}{2}t \tag{1.8}$$

for (q,p). In this case, $v^1=v$ and (1.5) is reduced to the identity transformation. Proposition 1.2 is verified by the use of the fact that E is invariant under the transformation:

$$s_1:\mathbf{v}\mapsto\mathbf{v}^1 \tag{1.9}$$

of V: note $s_1[b]=-b$. We obtain (1.5) from the following schema:

$$\begin{array}{ccc} E(\mathbf{v}) & = & E(\mathbf{v}^1)\\ \big\downarrow r & & \big\downarrow r\\ \mathscr{H}(\mathbf{v}) & \dashrightarrow & \mathscr{H}(\mathbf{v}^1)\,. \end{array}$$

Note that, by combining (1.5) with $H(\mathbf{v}^1)=H(\mathbf{v})$ we have the canonical transformation:

$$(s_1)_*:\mathscr{H}(\mathbf{v})\to\mathscr{H}(v^1)$$

of the Painlevé system: see Sect. 2.1.

1.4 Auxiliary Hamiltonian Function

Let $h=h(v)$ be a Hamiltonian function of $\mathscr{H}(\mathbf{v})$ and $(q,p)=\Gamma(h)$ the solution of $\mathscr{H}(\mathbf{v})$. Define by

$$h_{(1)}=h+q \tag{1.10}$$

the auxiliary Hamiltonian function $h_{(1)}$, which is meromorphic on $\mathbb{C}$. By using the results of [13], we can show, if $t=t_0$ is a pole of $h_{(1)}$, then it is a simple pole with the residue 1. Moreover we will prove:

Proposition 1.3. *$h_{(1)}$ satisfies the equation*

$$\left(\frac{d^2h_{(1)}}{dt^2}\right)^2+4\left(\frac{dh_{(1)}}{dt}\right)^3+2\frac{dh_{(1)}}{dt}\left(t\frac{dh_{(1)}}{dt}-h_{(1)}\right)-\frac{1}{4}(b-1)^2=0\,. \tag{1.11}$$

We deduce from (1.11):

$$h_{(1)}=h(\mathbf{v}^{(1)})\,, \tag{1.12}$$

where

$$\begin{aligned}\mathbf{v}^{(1)}&=s^{(1)}(\mathbf{v})\\ &=\mathbf{v}+(-1,1)\,.\end{aligned}$$

Then we obtain from (1.12) the birational canonical transformation

$$s_*^{(1)}:\mathscr{H}(\mathbf{v})\to\mathscr{H}(\mathbf{v}^{(1)})$$

associated with the parallel transformation $s^{(1)}$ of V. In particular, we have the

Proposition 1.4. $s_*^{(1)}$ *is given by*

$$q^{(1)} = -q + \frac{b-1}{p-2q^2-t}, \tag{1.13$_1$}$$

$$p^{(1)} = -p + 2q^2 + t, \tag{1.13$_2$}$$

$$H^{(1)} = H + q, \tag{1.13$_3$}$$

where we write $\mathscr{H}(\mathbf{v}^{(1)}) = (q^{(1)}, p^{(1)}, H^{(1)}, t)$.

Propositions 1.3 and 1.4 will be verified in Sect. 2.2. We will apply the birational canonical transformations $(s_1)_*$, $s_*^{(1)}$ successively to a solution of $\mathscr{H}(\mathbf{v})$, and examine classical solutions of the second Painlevé equation. Before discussing particular solutions of H, we associate explicitly the certain group $G = G_{\mathrm{II}}$ with $\mathscr{H}$.

1.5 Transformation Group of $\mathscr{H}$

Let $\mathbf{e}_1, \mathbf{e}_2$ be the canonical basis of $\mathbb{C}^2$ and consider in V the vector

$$\mathbf{a}_1 = \mathbf{e}_1 - \mathbf{e}_2 .$$

Then the affine transformation s_1 is the reflection:

$$s_1(\mathbf{v}) = \mathbf{v} - 2\frac{(\mathbf{v}|\mathbf{a}_1)}{(\mathbf{a}_1|\mathbf{a}_1)}\mathbf{a}_1$$

with respect to $\mathbf{a}_1$, where $(\mathbf{v}|\mathbf{v}')$ denotes the symmetric bilinear form in $\mathbb{C}^2$ such that $(\mathbf{e}_i|\mathbf{e}_j) = (\mathbf{e}_j|\mathbf{e}_i) = \delta_{ij}$. On the other hand, put

$$s_0 = s^{(1)} s_1 , \tag{1.14}$$

which is the reflection of V with respect to the hyperplane:

$$v_1 - v_2 + 1 = 0;$$

we have

$$s_0(\mathbf{v}) = s_1(\mathbf{v}) - \mathbf{a}_1 = (v_2 - 1, v_1 + 1).$$

The transformations s_0, s_1 generate the subgroup G of the group $\mathscr{A}(V)$ of affine motion; G is isomorphic to the affine Weyl group $W_a(R)$ of the root system $R = R_{\mathrm{II}}$ of the type A_1 [2]. The cyclic group W of the order two generated by s_1 is the Weyl group $W(R)$. We deduce from Propositions 1.3 and 1.4 the following theorem:

Theorem 1. *There exists the representation*

$$\varrho : G \to G_*$$

on the Painlevé system $\mathscr{H} = \mathscr{H}_{\mathrm{II}}$.

In fact, we have the birational canonical transformations $(s_1)_*, s_*^{(1)}$, which generate G_*. For g of G, we write the transformation $\varrho(g)$ of G_* as g_* unless there is a risk of confusion.

1.6 Classical Solution

If $\mathbf{v}=(0,0)$, that is, $\mathbf{v}=s_1(\mathbf{v})$, then $H(\mathbf{v})$ admits a particular solution of the form (1.8). The Riccati equation satisfied by q can be linearized by means of the function $\tau=\tau(t)$ given by

$$q=\frac{d}{dt}\log\tau. \tag{1.15}$$

In fact, we have for τ the linear equation

$$\frac{d^2\tau}{dt^2}+\frac{1}{2}t\tau=0, \tag{1.16}$$

hence, for example,

$$\tau=A_i\left(-\sqrt[3]{\frac{1}{2}}t\right),$$

A_i denoting the Airy function. We will prove in Sect. 2.2 the

Proposition 1.5. *The Painlevé system $\mathscr{H}_m=\mathscr{H}(\mathbf{v}_m^0)$ at $\mathbf{v}_m^0=(-m\cdot m)$ admits a one-parameter family of particular solutions represented as rational functions of a solution of* (1.16) *and its derivatives, m being an integer.*

1.7 Rational Solutions

We consider now the Painlevé system $\mathscr{H}(\mathbf{v}^{(0)})$ at $\mathbf{v}^{(0)}=(-\frac{1}{2},\frac{1}{2})$: note $\mathbf{v}^{(0)}$ is the fixed point of the transformation s_0, namely $s_0(\mathbf{v}^{(0)})=\mathbf{v}^{(0)}$. It is easy to see $\mathscr{H}(\mathbf{v}^{(0)})$ possesses the rational solution

$$(q,p)=\left(\frac{1}{t},\frac{1}{2}t\right). \tag{1.17}$$

For $m\in\mathbb{Z}$, put $\mathbf{v}^{(m)}=(-\frac{1}{2}-m,\frac{1}{2}+m)$. Then we can show the following proposition:

Proposition 1.6. *$\mathscr{H}(\mathbf{v}^{(m)})$ has a rational function solution.*

For example, we obtain easily

$$(q,p)=(0,\tfrac{1}{2}t)$$

as a solution of $\mathscr{H}(\mathbf{v}^{(-1)})$: see Sect. 1.10.

1.8 τ-Function

Let $\tau=\tau(\mathbf{v})$ the τ-function of $\mathscr{H}(\mathbf{v})$ related to a solution $(q(\mathbf{v}),p(\mathbf{v}))$. We have:

Proposition 1.7. *For* $\mathbf{v}^{(1)}=s^{(1)}(\mathbf{v})$,

$$q(\mathbf{v})=\frac{d}{dt}\log\frac{\tau(\mathbf{v}^{(1)})}{\tau(\mathbf{v})}. \tag{1.18}$$

In fact, (1.18) is an immediate consequence of (1.10), (1.12), and the definition of the τ-function.

For $\mathbf{v}$ of V, arbitrary fixed, put

$$\mathbf{v}^{(0)}=\mathbf{v}, \qquad \mathbf{v}^{(m)}=(s^{(1)})^m(\mathbf{v}).$$

By applying the canonical transformation $s_*^{(1)}$ successively to $\tau_0=\tau(\mathbf{v}^{(0)})$ of $\mathscr{H}(\mathbf{v}^{(0)})$, we have the τ-sequence

$$\mathfrak{T}(s^{(1)})=\{\tau_m;\ m\in\mathbb{Z}\}$$

such that $\tau_m=\tau(\mathbf{v}^{(m)})=(s_*^{(1)})^m\tau_0$. We will prove:

Proposition 1.8. *The τ-functions of $T(s^{(1)})$ satisfy the Toda equation:*

$$\left(\frac{d}{dt}\right)^2\log\tau_m=c(m)\frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2}, \tag{1.19}$$

$c(m)$ being non-zero constants.

1.9 Examples of τ-Functions

Consider again the particular solutions (1.8) of the Painlevé system at $\mathbf{v}=(0,0)$. Since

$$h(\mathbf{v})=H(t;\mathbf{v})=0$$

for a solution of the form (1.8), we can put

$$\tau_0=\tau(\mathbf{v})\equiv 1.$$

It follows from (1.15) and (1.18) that τ_1 is a solution of (1.16). We have (1.19) for $m\geqq 1$ with respect to the semi-sequence

$$\mathfrak{T}^+(s^{(1)})=\{\tau_m;\ m\geqq 0\}$$

with $\tau_0=1$. Moreover if we choose multiplicative constants of τ_m ($m\geqq 2$) such that $c(m)=1$ for $m\geqq 1$ in (1.19), τ_m ($m\geqq 2$) can be given by the formula of G. Darboux:

$$\tau_m=\begin{vmatrix} \tau, & \delta\tau, & \dots, & \delta^{m-1}\tau \\ \delta\tau, & \delta^2\tau, & \dots, & \delta^m\tau \\ \dots & & \dots, & \\ \delta^{m-1}\tau, & \delta^m\tau, & \dots, & \delta^{2m-2}\tau \end{vmatrix} \tag{1.20}$$

with $\tau=\tau_1$, $\delta=\dfrac{d}{dt}$ [4, 15]. *We will see:*

Proposition 1.9. *The formula* (1.20) *stands also for the semi-sequence*

$$\mathfrak{T}^-(s^{(1)})=\{\tau_m;\ m\leqq 0\}$$

with $\tau_0=1$, $\tau=\tau_{-1}=\tau_1$.

1.10 Yablonskii-Vorob'ev Polynomials

The other example of the τ-sequence concerns the rational solution (1.17) of $\mathscr{H}(\mathbf{v}_0)$ at $\mathbf{v}_0=(-\frac{1}{2},\frac{1}{2})$, the fixed point of s_0. We have for (1.17)

$$h(\mathbf{v}_0)=-\tfrac{1}{8}t^2$$

hence,

$$\tau(\mathbf{v}_0)=\exp(-\tfrac{1}{24}t^3). \tag{1.21}$$

On the other hand, since $q(\mathbf{v}_{-1})=0$, we obtain from (1.18) $\tau_{-1}=\tau_0$. For the τ-sequence

$$\mathfrak{T}(s^{(1)})=\{\tau_m=\tau(\mathbf{v}_m);\ \mathbf{v}_m=(-\tfrac{1}{2}-m,\tfrac{1}{2}+m),\ m\in\mathbb{Z}\},$$

we have hence

$$\tau_m=\tau_{-m-1}$$

and furthermore for $m>0$,

$$\tau_m=P_m(t)\tau_0. \tag{1.22}$$

Then $p_m(t)$ are reduced to polynomials of t, called, the *Yablonskii-Vorob'ev polynomials* [16].

The Painlevé system $\mathscr{H}(\mathbf{v}_m)$ admits a rational solution of the form:

$$q_m=\frac{d}{dt}\log\frac{P_{m+1}}{P_m},$$

which has been studied in [1, 10]. By choosing the constants $c(m)$ such that $P_m(t)$ are monic in t, we obtain, for example,

$$P_0=1, \qquad P_1=t, \qquad P_2=t^3+4t,$$

and

$$\deg P_m(t)=\tfrac{1}{2}m(m+1). \tag{1.23}$$

2. Painlevé System $\mathscr{H}_{\mathrm{II}}$

2.1 Differential Equation E

In this section we will prove the results stated in the last section. First of all, we verify Proposition 1.1. In fact, since

$$\frac{d}{dt}H(t)=\frac{\partial}{\partial t}H(t;q,p)|_{(q,p)=(q(t),p(t))},$$

we have for $h=H(t)$,

$$\frac{dh}{dt}=-\frac{1}{2}p(t), \tag*{(2.1)$_1$}$$

$$\frac{d^2h}{dt^2}=-q(t)p(t)-\frac{1}{2}b, \tag*{(2.1)$_2$}$$

from which we obtain (1.1) and then the differential equation E_{II}. Inversely we define (q, p) by (1.1) for a solution of E_{II} such that $\dfrac{d^2h}{dt^2} \neq 0$. By taking (1.1) into consideration, we have from E_{II},

$$\begin{aligned} h &= \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}t)p - bq \\ &= H(t; q, p) \end{aligned} \tag{2.2}$$

and then by $(2.1)_1$,

$$\frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} = 0\,.$$

On the other hand, we can verify

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

by using (2.1), (2.2), hence

$$\frac{d^2h}{dt^2}\left(\frac{dq}{dt} - \frac{\partial H}{\partial p}\right) = 0\,,$$

which shows that (q, p) is actually a solution of the Hamiltonian system with the Hamiltonian (2.2).

Now it is easy to verify Proposition 1.2. In fact, by putting for $\mathbf{v}^1 = s_1(\mathbf{v})$

$$\mathscr{H}(\mathbf{v}^1) = (q^1, p^1, H^1, t)$$

and $h^1 = h(\mathbf{v}^1)$, we have from (1.1)

$$q = \frac{2\dfrac{d^2h^1}{dt^2} - b}{4\dfrac{dh^1}{dt}}\,, \qquad p = -2\frac{dh^1}{dt}\,. \tag{2.3}$$

Since h^1 satisfies the exactly same equation as h, we obtain from (1.1) and (2.3) the expression (1.5) by putting $h = h^1$. The proofs of Propositions 1.1 and 1.2 are completed.

2.2 Verification of Propositions 1.3, 1.4, and 1.5

We prove at first Proposition 1.3. It is easy to verify:

$$\frac{dh_{(1)}}{dt} = \frac{1}{2}p - q^2 - \frac{1}{2}t\,, \tag*{$(2.4)_1$}$$

$$\frac{d^2h_{(2)}}{dt^2} = -qp + 2q^3 + tq + \frac{1}{2}(b-1) \tag*{$(2.4)_2$}$$

by the use of (2.1) and (1.10), and then we obtain

$$q=-\frac{1}{4\dfrac{dh_{(1)}}{dt}}\left(2\frac{d^2h_{(1)}}{dt^2}-b+1\right), \tag{2.5$_1$}$$

$$p=2\frac{dh_{(1)}}{dt}+2q^2+t. \tag{2.5$_2$}$$

On the other hand, it follows from (2.4), (1.10) that:

$$\begin{aligned} t\frac{dh_{(1)}}{dt}-h_{(1)}+2\left(\frac{dh_{(1)}}{dt}\right)^2 &= -q^2p+2q^4+tq^2+(b-1)q \\ &= q\left(\frac{d^2h_{(1)}}{dt^2}+\frac{1}{2}(b-1)\right). \end{aligned}$$

By substituting $(2.5)_1$ into this, we obtain the differential equation (1.11).

Proof of Proposition 1.4. By (1.12), we have from (1.1) the following expressions:

$$q^{(1)}=\frac{1}{4\dfrac{dh_{(1)}}{dt}}\left(2\frac{d^2h_{(1)}}{dt^2}+b-1\right), \quad p^{(1)}=-2\frac{dh_{(1)}}{dt},$$

from which $(1.13)_{1,2,3}$ follow by means of (2.5).

Remark 2.1. Let $\bar{\pi}$ be the canonical transformation (1.2), and write $\bar{\pi}_1=s_*^{(1)}\cdot\bar{\pi}^{-1}$:

$$\begin{tikzcd} s_*^{(1)}:\mathscr{H} \arrow[r] \arrow[dr, "\bar{\pi}"'] & \mathscr{H}^{(1)}=\mathscr{H}(\mathrm{v}^{(1)}) \\ & \bar{\mathscr{H}} \arrow[u, dashed] \end{tikzcd}$$

Then $\bar{\pi}_1$ is given as follows:

$$q=-q^{(1)}-\frac{b-1}{p^{(1)}}, \tag{2.6$_1$}$$

$$\bar{p}=-p^{(1)}, \tag{2.6$_2$}$$

where we put $\bar{\mathscr{H}}=\bar{\pi}\mathscr{H}=(q,\bar{p},\bar{H},t)$ (see (1.3)).

Remark 2.2. We have obtained (1.13) under the assumption that $\dfrac{d^2h_{(1)}}{dt^2}\not\equiv 0$. Now we suppose the contrary: $\dfrac{d^2h_{(1)}}{dt^2}\equiv 0$. It occurs actually when

$$\bar{p}=p-2q^2-t\equiv 0$$

and $b-1=0$. In this case we mean by (2.6) the following transformation:

$$q^{(1)}=-q, \quad p^{(1)}=-\bar{p}. \tag{2.7}$$

Definitively, the canonical transformation $s_*^{(1)}$ is well-defined in any case.

Here we give the explicit form of the transformation $(s_0)_*$ associated with (1.14).

Proposition 2.1. *If we put*

$$\mathscr{H}(s_0(\mathbf{v}))=(s_0)_*\mathscr{H}(\mathbf{v})=(q^0,p^0,H^0,t),$$

then

$$\begin{aligned} q^0&=-q'-\frac{b+1}{\bar{p}'},\\ p^0&=-\bar{p}',\\ H^0&=H+q', \end{aligned}$$

where q' and $\bar{p}'$ are:

$$\begin{aligned} q'&=q+\frac{b}{p},\\ \bar{p}'&=p-2(q')^2-t. \end{aligned}$$

This proposition can be verified by the use of $(s_0)_*=(s_*^{(1)})\circ(s_1)_*$. We do not enter into details of computation.

Verification of Proposition 1.5. The proposition is an immediate consequence of Remark 2.2. Apply the canonical transformation $s_*^{(1)}$ successively to the solution (1.15) of the Painlevé system $\mathscr{H}_0$ at $\mathbf{v}=(0,0)$.

2.3 Toda Equation

We prove in the following Proposition 1.8. For the sake of simplicity, put

$$h_m=\frac{d}{dt}\log\tau_m,$$

$$(q_m,p_m)=\Gamma(h_m).$$

We obtain from (1.1)

$$q_m=\frac{2\dfrac{d^2h_m}{dt^2}+b-m}{4\dfrac{dh_m}{dt}}$$

and from $(2.5)_1$ and (1.12)

$$q_{m-1}=-\frac{2\dfrac{d^2h_m}{dt^2}-b+m}{4\dfrac{dh_m}{dt}};$$

it follows that

$$q_m - q_{m-1} = \frac{d}{dt}\log\frac{dh_m}{dt}. \tag{2.8}$$

On the other hand, since

$$q_m = \frac{d}{dt}\log\frac{\tau_{m+1}}{\tau_m}$$

by virtue of (1.18), we have

$$q_m - q_{m-1} = \frac{d}{dt}\log\frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2}. \tag{2.9}$$

The Toda equation (1.19) is an immediate consequence of (2.8), (2.9).

Remark 2.3. We obtain besides (2.8)

$$q_m + q_{m-1} = \frac{1}{2}(b-m)\left(\frac{dh_m}{dt}\right)^{-1},$$

from which

$$\frac{d}{dt}\log\frac{\tau_{m+1}}{\tau_{m-1}} = \frac{b-m}{2c(m)}\frac{\tau_m^2}{\tau_{m-1}\tau_{m+1}}. \tag{2.10}$$

2.4 Verification of Proposition 1.9

As we have discussed in Remark 2.2, the birational canonical transformation $(s_*^{(1)})^{-1}$ can be applied also to the solution (1.8). In fact, we obtain from (2.7), q_{-1} is a solution of the Riccati equation:

$$\frac{dq}{dt} - q^2 - \tfrac{1}{2}t = 0.$$

On the other hand, since, by (1.18),

$$q_- = -\frac{d}{dt}\log\tau_-,$$

the τ-function τ_- is a constant multiple of τ_1. The proof of Proposition 1.9 is completed.

2.5 Change of the Independent Variable

All transformations of $\mathscr{H}$, considered above, are of the first kind. On the other hand, it is known the Painlevé equation P is invariant under the change of variables:

$$q \to \omega q, \qquad t \to \omega^2 t,$$

where $\omega^3 = 1$. Namely, we have:

Proposition 2.2. *The Painlevé system associated with the second Painlevé equation remains invariant under the canonical transformation:*

$$\psi:(q,p,H,t)\to(q'',p'',H'',t'')$$

such that

$$q''=\omega q\,,\qquad p''=\omega^2 p\,,$$
$$H''=\omega H\,,\qquad t''=\omega^2 t\,.$$

3. Fourth Painlevé Equation

3.1 Auxiliary Hamiltonian Function

The following of the present article concerns the fourth Painlevé equation P_{IV}. The Painlevé system associated with $P=P_{\mathrm{IV}}$ depends on parameters $\mathbf{v}=(v_1,v_2,v_3)$ such thaat

$$v_1=-\tfrac{1}{3}(\kappa_0+\theta_\infty)\,,\qquad v_2=\tfrac{1}{3}(2\kappa_0-\theta_\infty)\,,\qquad v_3=\tfrac{1}{3}(-\kappa_0+2\theta_\infty)\,. \tag{3.1}$$

The Painlevé system at $\mathbf{v}$ is written as $\mathscr{H}(\mathbf{v})$. Let $H(\mathbf{v})=H(t;\mathbf{v})$ be a Hamiltonian function of $\mathscr{H}(\mathbf{v})$, related to a solution $(q,p)=(q(\mathbf{v}),p(\mathbf{v}))$ of $\mathscr{H}(\mathbf{v})$. We define the auxiliary Hamiltonian function $h(\mathbf{v})=h(t;\mathbf{v})$ by:

$$h(\mathbf{v})=H(\mathbf{v})-2v_1 t\,. \tag{3.2}$$

First of all we prove the

Proposition 3.1. *$h=h(\mathbf{v})$ satisfies the equation*

$$E_{\mathrm{IV}}\qquad \left(\frac{d^2h}{dt^2}\right)^2-4\left(t\frac{dh}{dt}-h\right)^2+4\Pi_{k=1}^{3}\left(\frac{dh}{dt}+2v_k\right)=0\,.$$

There exists the one-to-one correspondence from a general solution of $E(\mathbf{v})$ to a solution of $\mathscr{H}(\mathbf{v})$.

We denote the correspondence introduced above by

$$\Gamma(h)=(q,p)\,. \tag{3.3}$$

Here $E(\mathbf{v})$ is the equation E_{IV} with the parameter $\mathbf{v}$.

Remark 3.1. A singular solution of $E(\mathbf{v})$ is of the form:

$$h=\lambda t+\mu \tag{3.4}$$

such that

$$\mu^2=(\lambda+2v_1)(\lambda+2v_2)(\lambda+2v_3)\,.$$

Proof of Proposition 3.1. Since by the definition

$$\begin{aligned}h&=2qp^2-\{q^2+2tq+2\kappa_0\}p+\theta_\infty q-2v_1 t\\&=(2qp-2v_2+2v_1)p-(qp-v_3+v_1)q-(2qp+2v_1)t\,,\end{aligned}$$

we obtain, by differentiating it with respect to t,

$$\frac{dh}{dt} = -2qp - 2v_1,$$

$$\frac{d^2h}{dt^2} = -4(qp - v_2 + v_1)p - 2(qp - v_3 + v_1)q.$$

It follows that

$$q = \frac{1}{2A_3}\left[\frac{d^2h}{dt^2} - 2\left(t\frac{dh}{dt} - h\right)\right], \qquad (3.5)_1$$

$$p = \frac{1}{4A_2}\left[\frac{d^2h}{dt^2} + 2\left(t\frac{dh}{dt} - h\right)\right], \qquad (3.5)_2$$

where we put for $k = 1, 2, 3$,

$$A_k = \frac{dh}{dt} + 2v_k;$$

note in particular:

$$qp = -\tfrac{1}{2}A_1. \qquad (3.6)$$

By eliminating (q, p) from (3.5), (3.6), we obtain the differential equation $E(v)$. Inversely, we determine the pair of functions (q, p) by (3.5). Then it is a solution of the Painlevé system $\mathscr{H}(\mathbf{v})$. We can verify this fact easily by using the equality:

$$\frac{d^3h}{dt^3} - 4t\left(t\frac{dh}{dt} - h\right) + 2\Sigma_{k=1}^3 \Pi_{l(\neq k)}\left(\frac{dh}{dt} + 2v_l\right) = 0,$$

obtained from $E(\mathbf{v})$.

3.2 Differential Equations $E(\mathbf{v})$ and $E(\ell(\mathbf{v}))$

Let $h = h(\mathbf{v})$ be an auxiliary Hamiltonian function of $\mathscr{H}(\mathbf{v})$ and (q, p) a solution of $\mathscr{H}(\mathbf{v})$ given by (3.3). We define the function h_1 by:

$$h_1 = h(\mathbf{v}) + q + \tfrac{2}{3}t, \qquad (3.7)$$

and prove the following proposition:

Proposition 3.2. *(q, p) can be written as rational functions of h_1 and its derivatives. Moreover h_1 satisfies the differential equation $E(\bar{\mathbf{v}})$ with $\bar{\mathbf{v}} = (v_1 - \frac{1}{3}, v_2 - \frac{1}{3}, v_3 + \frac{2}{3})$.*

Proof. It is convenient for us to consider the other Hamiltonian $\bar{H}(t; q, \bar{p})$ given by (0.4). In fact, we have from (0.9)

$$h_1 = \bar{H} - 2(v_2 - \tfrac{1}{3})t, \qquad (3.8)$$

where $\bar{H} = \bar{H}(t; \mathbf{v})$ is the Hamiltonian function of $\bar{\mathscr{H}} = \pi\mathscr{H}$. By putting

$$\bar{v}_1 = v_1 - \tfrac{1}{3}, \quad \bar{v}_2 = v_2 - \tfrac{1}{3}, \quad \bar{v}_3 = v_3 + \tfrac{2}{3}$$

for the sake of simplicity, we deduce from (3.8):

$$\frac{dh_1}{dt}=2q\bar{p}-2\bar{v}_2,$$

$$\frac{d^2h_1}{dt^2}=4(q\bar{p}-\bar{v}_2+\bar{v}_1)\bar{p}-2(q\bar{p}-\bar{v}_2+\bar{v}_3)q,$$

$$t\frac{dh_1}{dt}-h_1=-2(q\bar{p}-\bar{v}_2+\bar{v}_1)\bar{p}-(q\bar{p}-\bar{v}_2+\bar{v}_3)q.$$

Therefore we obtain:

$$q=-\frac{1}{2\bar{A}_3}\left[\frac{d^2h_1}{dt^2}+2\left(t\frac{dh_1}{dt}-h_1\right)\right], \tag{3.9$_1$}$$

$$\bar{p}=\frac{1}{4\bar{A}_1}\left[\frac{d^2h_1}{dt^2}-2\left(t\frac{dh_1}{dt}-h_1\right)\right], \tag{3.9$_2$}$$

where for $k=1,2,3$,

$$\bar{A}_k=\frac{dh_1}{dt}+2\bar{v}_k.$$

Recall

$$\bar{p}=p-\tfrac{1}{2}q-t; \tag{3.10}$$

hence (q,p) can be represented as rational functions of h_1, $\frac{dh_1}{dt}$ and $\frac{d^2h_1}{dt^2}$. Taking

$$q\bar{p}=\tfrac{1}{2}\bar{A}_2 \tag{3.11}$$

into consideration, we arrive at the following differential equation:

$$\left(\frac{d^2h_1}{dt^2}\right)^2-4\left(t\frac{dh_1}{dt}-h_1\right)^2+4\bar{A}_1\bar{A}_2\bar{A}_3=0,$$

which establishes Proposition 3.2.

Remark 3.2. Define the parallel transformation ℓ by $\bar{\mathbf{v}}=\ell(\mathbf{v})$, that is,

$$\ell(\mathbf{v})=\mathbf{v}+\tfrac{1}{3}(-1,-1,2). \tag{3.12}$$

Then h_1 satisfies the equation $E(\ell(\mathbf{v}))$ and so $h_1=h(\ell(\mathbf{v}))$. We have from (3.2) and (3.7):

$$H(\ell(\mathbf{v}))=H(\mathbf{v})+q(\mathbf{v}), \tag{3.13}$$

$H(\mathbf{v})$ and $H(\ell(\mathbf{v}))$ being the Hamiltonian functions. It results from (3.13) and the definition of the τ-function that:

Proposition 3.3. *The Painlevé transcendental function $q(\mathbf{v})=q(t;\mathbf{v})$ is the difference of logarithmic derivatives of the two τ-functions $\tau(\mathbf{v})=\tau(t;\mathbf{v})$ and $\tau(\ell(\mathbf{v}))=\tau(t;\ell(\mathbf{v}))$:*

$$q(\mathbf{v})=\frac{d}{dt}\log\frac{\tau(\ell(\mathbf{v}))}{\tau(\mathbf{v})}. \tag{3.14}$$

3.3 Representation of the Parallel Transformation ℓ

Let ℓ be the parallel transformation (3.12) in V. We deduce from Propositions 3.1 and 3.2 the following proposition:

Proposition 3.4. *There exists the birational canonical transformation ℓ_* from $\mathscr{H}(\mathbf{v})$ to $\mathscr{H}(\ell(\mathbf{v}))$.*

Proof. Write $\ell(\mathbf{v})=\bar{\mathbf{v}}=(\bar{v}_1,\bar{v}_2,\bar{v}_3)$ and $h_1=h(\bar{\mathbf{v}})$. As we have seen in the proof of Proposition 3.2, h_1 and its derivatives are written as polynomials of $(q,\bar{p})$ (see (3.10)). Since $(q(\bar{\mathbf{v}}),p(\bar{\mathbf{v}}))=\Gamma(h_1)$ with respect to the Painlevé system $\mathscr{H}(\bar{\mathbf{v}})$, a solution $(q(\bar{\mathbf{v}}),p(\bar{\mathbf{v}}))$ of $\mathscr{H}(\bar{\mathbf{v}})$ is rational with respect to a solution $(q(\mathbf{v}),\bar{p}(\mathbf{v}))$ of $\bar{\mathscr{H}}(\mathbf{v})$. Inversely regarding h_1 as the auxiliary Hamiltonian function of $\mathscr{H}(\bar{\mathbf{v}})$, we obtain from (3.9) the expression of $(q(\mathbf{v}),\bar{p}(\mathbf{v}))$ as a pair of rational functions of $(q(\bar{\mathbf{v}}),p(\bar{\mathbf{v}}))$. It is easy to see the birational transformation from $(q(\mathbf{v}),\bar{p}(\mathbf{v}))$ to $(q(\bar{\mathbf{v}}),p(\bar{\mathbf{v}}))$ extends to the canonical transformation:

$$\pi_1:\bar{\mathscr{H}}(\mathbf{v})\to\mathscr{H}(\bar{\mathbf{v}}),$$

with the relation of the Hamiltonians:

$$H(\bar{\mathbf{v}})=\bar{H}(\mathbf{v})-2(v_2-v_1)t.$$

The transformation ℓ_* is given by the composition of π and π_1:

$$\begin{tikzcd}
\mathscr{H}(\mathbf{v}) \arrow[rr, "\ell_*"] \arrow[dr, "\pi"'] & & \mathscr{H}(\bar{\mathbf{v}}) \\
& \bar{\mathscr{H}}(\mathbf{v}) \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

We determine the explicit form of π_1. Taking (3.11) and (3.6) into consideration, we obtain in fact,

$$q(\bar{\mathbf{v}})=2\bar{p}(\mathbf{v})\frac{\bar{A}(\mathbf{v})-v_2+v_1}{\bar{A}(\mathbf{v})-v_2+v_3+1}, \tag{3.15$_1$}$$

$$p(\bar{\mathbf{v}})=-\frac{1}{2}q(\mathbf{v})\frac{\bar{A}(\mathbf{v})-v_2+v_3+1}{\bar{A}(\mathbf{v})}, \tag{3.15$_2$}$$

or equivalently,

$$q(\mathbf{v})=-2p(\bar{\mathbf{v}})\frac{A(\bar{\mathbf{v}})-\bar{v}_2+\bar{v}_1}{A(\bar{\mathbf{v}})-\bar{v}_3+\bar{v}_1}, \tag{3.16$_1$}$$

$$\bar{p}(\mathbf{v})=\frac{1}{2}q(\bar{\mathbf{v}})\frac{A(\bar{\mathbf{v}})-\bar{v}_3+\bar{v}_1}{A(\bar{\mathbf{v}})}. \tag{3.16$_2$}$$

Here we write $A(\mathbf{v})=q(\mathbf{v})p(\mathbf{v})$ and $\bar{A}(\mathbf{v})=q(\mathbf{v})\bar{p}(\mathbf{v})$: note $\bar{A}(\mathbf{v})+A(\bar{\mathbf{v}})=v_2-v_1$. Moreover we have:

$$\begin{aligned}
&p(\bar{\mathbf{v}})dq(\bar{\mathbf{v}})-\bar{p}(\mathbf{v})dq(\mathbf{v})\\
&\quad=-d[\bar{A}(\mathbf{v})+(v_3-v_1+1)\log(\bar{A}(\mathbf{v})-v_2+v_3+1)+(v_2-v_1)\log\bar{p}].
\end{aligned}$$

3.4 τ-Sequence

For an arbitrary fixed point $\mathbf{v}$ of V, we consider the sequence $\{\mathbf{v}_m;\ m \in \mathbb{Z}\}$ of V such that $\mathbf{v}_0 = \mathbf{v}$,

$$\mathbf{v}_m = \ell^m(\mathbf{v}) = \mathbf{v} + \frac{m}{3}(-1, -1, 2),$$

and the sequence $\{\mathcal{H}_m;\ m \in \mathbb{Z}\}$ of the Painlevé systems:

$$\mathcal{H}_0 = \mathcal{H}(\mathbf{v}),$$
$$\mathcal{H}_m = \ell_*^m \mathcal{H}_0 = \mathcal{H}(\mathbf{v}_m),$$

connected each other through the canonical transformation ℓ_*. Put moreover

$$\mathcal{H}_m = (q_m, p_m, H_m, t)$$
$$\bar{\mathcal{H}}_m = \pi \mathcal{H}_m = (q_m, \bar{p}_m, \bar{H}_m, t).$$

We have the sequence of τ-functions:

$$\mathfrak{T}(\ell) = \{\tau_m;\ m \in \mathbb{Z}\},$$

called the *τ-sequence*, such that

$$H_m = \frac{d}{dt} \log \tau_m, \tag{3.17}$$

H_m being the Hamiltonian function of $\mathcal{H}_m$. We will show

Proposition 3.5. *$\mathfrak{T}(\ell)$ is subject to the constraint:*

$$\frac{d^2}{dt^2} \log \tau_m + 2(v_3 - v_1 + m) = c(m) \frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2}. \tag{3.18}$$

It is easy to see the functions $\tilde{\tau}_m$ $(m \in \mathbb{Z})$ defined by

$$\tilde{\tau}_m = c_m \cdot \tau_m \exp\{(v_3 - v_1 + m)t^2\} \tag{3.19}$$

satisfy the Toda equation

$$\frac{d^2}{dt^2} \log \tilde{\tau}_m = \frac{\tilde{\tau}_{m-1}\tilde{\tau}_{m+1}}{\tilde{\tau}_m^2}, \tag{3.20}$$

with the suitable choice of non-zero constants c_m. Since the canonical transformation corresponding to (3.19) is trivial, we can state the result of Proposition 3.5 as follows:

Theorem 2. *The τ-sequence $\mathfrak{T}(\ell)$ with respect to the parallel transformation ℓ satisfies the Toda equation* (3.20).

3.5 Proof of Proposition 3.5

Let h_m be the auxiliary Hamiltonian function of $\mathscr{H}_m$, given by (3.2). We have from (3.9)

$$q_{m-1} = -\frac{1}{2A_3(m)}\left[\frac{d^2h_m}{dt^2} + 2\left(t\frac{dh_m}{dt} - h_m\right)\right], \tag{3.21}$$

and from (3.5)

$$q_m = \frac{1}{2A_3(m)}\left[\frac{d^2h_m}{dt^2} - 2\left(t\frac{dh_m}{dt} - h_m\right)\right], \tag{3.22}$$

where

$$A_3(m) = \frac{dh_m}{dt} + 2\left(v_3 + \frac{2}{3}m\right).$$

It follows from (3.21), (3.22) that

$$q_m - q_{m-1} = \frac{d}{dt}\log A_3(m). \tag{3.23}$$

On the other hand, we have

$$q_m - q_{m-1} = \frac{d}{dt}\log\frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2} \tag{3.24}$$

since, by (3.14),

$$q_m = \frac{d}{dt}\log\frac{\tau_{m+1}}{\tau_m}. \tag{3.25}$$

Hence we obtain from (3.23), (3.24)

$$A_3(m) = c(m)\frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2},$$

and consequently (3.18), by taking

$$\frac{dh_m}{dt} = \frac{dH_m}{dt} - 2\left(v_1 - \frac{m}{3}\right)$$

into consideration

Remark 3.3. If we define the τ-function $\bar{\tau}_m$ of $\bar{\mathscr{H}}_m$ by:

$$\bar{H}_m = \frac{d}{dt}\log\bar{\tau}_m,$$

then

$$\bar{\tau}_m = \tau_{m+1}\exp\{(v_2 - v_1)t^2\}.$$

Here $\bar{H}_m$ denotes the Hamiltonian function of $\bar{\mathscr{H}}_m$.

3.6 Toda Equation

Note the parallel transformation ℓ yields the change of constants:

$$\theta_\infty \to \theta_\infty + 1$$

in the Hamiltonian of $\mathscr{H}(\mathbf{v})$. Consider now the transformation ℓ' of V such that

$$\kappa_0 \to \kappa_0 + 1$$

with respect to the constants of the Hamiltonian. We have:

$$\ell' = s_2 \ell s_2 .$$

Since $E(\mathbf{v}) = E(s_2(\mathbf{v}))$ and $h = H - 2v_1 t$, we can put $h(\mathbf{v}) = h(s_2(\mathbf{v}))$ and then

$$\tau(s_2(\mathbf{v})) = \tau(\mathbf{v}) ,$$
$$\tau(\ell'(\mathbf{v})) = \tau(\ell s_2(\mathbf{v})) .$$

It follows from (3.14) that

$$q(s_2(\mathbf{v})) = \frac{d}{dt} \log \frac{\tau(\ell'(\mathbf{v}))}{\tau(\mathbf{v})} . \tag{3.26}$$

Moreover by replacing in (3.18) $\mathbf{v}$ by $s_2(\mathbf{v})$, and by putting $\tau'_m = \tau((\ell')^m \mathbf{v}) = \tau(\ell^m s_2(\mathbf{v}))$, we obtain the following proposition:

Proposition 3.6. *The τ-sequence*

$$\mathfrak{T}(\ell') = \{\tau'_m ;\ m \in \mathbb{Z}\}$$

satisfies the equation:

$$\frac{d^2}{dt^2} \log \tau'_m + 2(v_2 - v_1 + m) = c(m) \frac{\tau'_{m-1} \tau'_{m+1}}{\tau'^2_m} . \tag{3.26}$$

Remark 3.4. For $\mathfrak{T}(\ell')$, we have from (3.26)

$$q(s_2(\ell')^m(v)) = \frac{d}{dt} \log \frac{\tau'_{m+1}}{\tau'_m} .$$

4. Transformation Group of Painlevé System $\mathscr{H}$

4.1 Affine Weyl Group

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the canonical basis in $\mathbb{C}^3$ and $(\mathbf{v}|\mathbf{v}')$ the symmetric bilinear form such that $(\mathbf{e}_i|\mathbf{e}_j) = (\mathbf{e}_j|\mathbf{e}_i) = \delta_{ij}$. The space V of parameters of the Painlevé system $\mathscr{H} = \mathscr{H}_{\mathrm{IV}}$ is the totality of the vectors of $\mathbb{C}^3$:

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3$$

such that

$$v_1 + v_2 + v_3 = 0 :$$

see (0.10). Consider in V the vectors:

$$\mathbf{a}_1=\mathbf{e}_1-\mathbf{e}_2, \quad \mathbf{a}_2=\mathbf{e}_2-\mathbf{e}_3, \quad \tilde{\mathbf{a}}=\mathbf{e}_1-\mathbf{e}_3;$$

let s_i $(i=1,2)$ be the reflection of V with respect to the hyperplane $(\mathbf{a}_i|\mathbf{v})=0$ and $\tilde{s}$ the reflection with respect to $(\tilde{\mathbf{a}}|\mathbf{v})=1$. We denote by $\tilde{W}$ the subgroup of the group $\mathscr{A}(V)$ of affine motion, generated by s_1, s_2, and $\tilde{s}$. $\mathbf{a}_1$, $\mathbf{a}_2$ constitute the fundamental roots of the root system $R=R_{\mathrm{IV}}$ of the type A_2 and $\tilde{\mathbf{a}}$ is the highest root [2]. $\tilde{W}$ is a realization of the affine Weyl group $W_a(R)$; we have for $\mathbf{v}=(v_1, v_2, v_3)$:

$$s_1: \mathbf{v} \mapsto (v_2, v_1, v_3),$$
$$s_2: \mathbf{v} \mapsto (v_1, v_3, v_2),$$
$$\tilde{s}: \mathbf{v} \mapsto (v_3+1, v_2, v_1-1).$$

We verify the

Proposition 4.1. *$\tilde{W}$ is represented on the Painlevé system $\mathscr{H}$ as the group $\tilde{W}_*$ of birational canonical transformations.*

Proof. We determine the canonical transformations $(s_i)_*$ $(i=1,2)$ associated with s_i, by virtue of the invariance of the differential equation E under the transformations s_i. Let $h=h(\mathbf{v})$ be a solution of $E(\mathbf{v})$, $h_i=h(s_i(\mathbf{v}))$ that of $E(s_i(\mathbf{v}))$ and $(q, p)=\Gamma(h)$ (resp. $(q_i, p_i)=\Gamma(h_i)$) a solution of $\mathscr{H}(\mathbf{v})$ (resp. $\mathscr{H}(s_i(\mathbf{v}))$: see Proposition 3.1. Since $E(\mathbf{v})=E(s_i(\mathbf{v}))$, we put $h=h_i$. By the use of the schema:

$$\begin{array}{ccc} E(\mathbf{v}) & = & E(s_i(\mathbf{v})) \\ {\scriptstyle \Gamma}\downarrow & & \downarrow{\scriptstyle \Gamma} \\ (s_i)_*: \mathscr{H}(\mathbf{v}) & \to & \mathscr{H}(s_i(\mathbf{v})), \end{array}$$

we obtain from (3.5):

$$q_i p_i = -\tfrac{1}{2}A_{i(1)}, \tag{4.1}$$

$$q_i = \frac{1}{2A_{i(3)}}\left[\frac{d^2h}{dt^2} - 2\left(t\frac{dh}{dt} - h\right)\right], \tag{4.2$_1$}$$

$$p_i = \frac{1}{2A_{i(2)}}\left[\frac{d^2h}{dt^2} + 2\left(t\frac{dh}{dt} - h\right)\right], \tag{4.2$_2$}$$

where

$$A_{i(j)} = \frac{dh}{dt} + 2v_{s_i(j)}.$$

It follows that

$$q_i = q \cdot \frac{A_3}{A_{i(3)}}, \tag{4.3$_1$}$$

$$p_i = p \cdot \frac{A_2}{A_{i(2)}}. \tag{4.3$_2$}$$

To compute the explicit form of the canonical transformation $\tilde{s}_*$, we consider again the other Hamiltonian system $\bar{\mathscr{H}}(\mathbf{v})$. It suffices to determine the explicit form of:

$$\tilde{\pi}=\pi\cdot\tilde{s}_*\cdot\pi^{-1}:\bar{\mathscr{H}}(\mathbf{v})\rightarrow\bar{\mathscr{H}}(\tilde{s}(\mathbf{v}))\,.$$

Since, for the parallel transformation ℓ given by (3.12),

$$\tilde{s}=\ell^{-1}s_1s_2s_1\ell\,,$$

we obtain, by using the results of Proposition 3.4, the following equalities:

$$q_s=q\frac{\bar{A}(\mathbf{v})-v_2+v_3+1}{\bar{A}(\mathbf{v})-v_2+v_1}, \tag{4.4$_1$}$$

$$\bar{p}_s=\bar{p}\frac{\bar{A}(\mathbf{v})-v_2+v_1}{\bar{A}(\mathbf{v})-v_2+v_3+1}, \tag{4.4$_2$}$$

$$\bar{H}(\mathbf{v})=\bar{H}(\tilde{s}(\mathbf{v}))\,, \tag{4.5}$$

where we write $\bar{\mathscr{H}}(\mathbf{v})=(q,\bar{p},\bar{H}(\mathbf{v}),t)$, $\bar{\mathscr{H}}(\tilde{s}(\mathbf{v}))=(q_s,\bar{p}_s,\bar{H}(\tilde{s}(\mathbf{v})),t)$ and

$$\bar{A}(\mathbf{v})=q\bar{p}=q_s\bar{p}_s\,.$$

Remark 4.1. For $i=1,2$, we obtain from (4.3) the canonical transformations:

$$q_1=q\,,\qquad p_1=p+\frac{v_1-v_2}{q}\,,$$

$$H(s_1(\mathbf{v}))=H(\mathbf{v})+2(v_2-v_1)t\,,$$

and

$$q_2=q\frac{A+v_1-v_3}{A+v_1-v_2}\,,\qquad p_2=p\frac{A+v_1-v_2}{A+v_1-v_3}\,,$$

$$H(s_2(\mathbf{v}))=H(\mathbf{v})\,,$$

$$A=qp=q_2p_2\,.$$

Remark 4.2. We obtain from (4.4), (4.5) the canonical transformation $\tilde{s}_*$; in particular the relation of the Hamiltonian is:

$$H(\tilde{s}(\mathbf{v}))=H(\mathbf{v})+(v_1-v_3-1)\left[\frac{q}{\bar{A}(\mathbf{v})+v_1-v_2}-2t\right].$$

4.2 Transformation Group

Let G be the subgroup of $\mathscr{A}(V)$, generated by s_1, s_2, and ℓ. By summing up the results obtained above, we arrive at the

Theorem 3. *There exists the non-linear representation*

$$\varrho:G\rightarrow G_*$$

on the Painlevé system associated with the fourth Painlevé equation.

Moreover we show:

Proposition 4.2. *G is the semi-direct product of $\tilde{W}$ and the cyclic group of the order three.*

Proof. Let R be the root system of the type A_2, $\mathbf{Q}(R)$ the root lattice and $\mathbf{P}(R)$ the weight lattice of R. The fundamental weights ϖ_i $(i=1,2)$ are:

$$\varpi_1=\tfrac{2}{3}\mathbf{e}_1-\tfrac{1}{3}\mathbf{e}_2-\tfrac{1}{3}\mathbf{e}_3,$$
$$\varpi_2=\tfrac{1}{3}\mathbf{e}_1+\tfrac{1}{3}\mathbf{e}_2-\tfrac{2}{3}\mathbf{e}_3,$$

and we have

$$\mathbf{P}(R)=\mathbf{Q}(R)+\varpi_1\mathbb{Z}$$

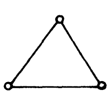
(see [2]). If we denote by $\ell(\varpi_i)$ the parallel transformation related to ϖ_i, then, by (3.12),

$$\ell=\ell(\varpi_2)^{-1}.$$

The quotient group $\mathbf{P}(R)/\mathbf{Q}(R)$ is generated by $\tilde{\ell}=s_1s_2\ell$:

$$\tilde{\ell}(\mathbf{v})=(v_3+\tfrac{2}{3}, v_1-\tfrac{1}{3}, v_2-\tfrac{1}{3}),$$

which is an automorphism of the extended Dynkin diagram of R:



G contains $\tilde{W}$ as a normal subgroup and is generated by $\tilde{W}$ and $\tilde{\ell}$.

4.3 Change of the Independent Variable of P

A canonical transformation g_* of $\mathscr{H}$ associated with G is of the first kind. On the other hand, P remains invariant under the change of variables:

$$(q,t)\to(-q,-t), \tag{4.6}$$

which extends to the canonical transformation of $\mathscr{H}$. In fact it is easy to see:

Proposition 4.3. *The Painlevé system $\mathscr{H}$ is invariant under the canonical transformation:*

$$\psi:(q,p,H,t)\to(-q,-p,-H,-t).$$

Consider in V the involution:

$$y: v\to -v.$$

We show:

Proposition 4.4. *The birational canonical transformation*

$$y_*:\mathscr{H}(\mathbf{v})\to\mathscr{H}(-\mathbf{v})=(q',p',H',t')$$

is given by the following expressions:

$$q' = -\sqrt{-4p}\,\frac{A-v_2+v_1}{A-v_3+v_1}, \tag{4.7$_1$}$$

$$p' = -\frac{\sqrt{-1}}{2}\,q\,\frac{A-v_3+v_1}{A-v_2+v_1}, \tag{4.7$_2$}$$

$$H' = -\sqrt{-1}H, \tag{4.8$_1$}$$

$$t' = \sqrt{-1}t, \tag{4.8$_2$}$$

where $A = qp = -q'p'$.

Proof. It is known that the change of variables

$$(q, t) \to (\sqrt{-1}q, \sqrt{-1}t)$$

yields in P only the alternation of the constant

$$\alpha \to -\alpha,$$

which is corresponding to the change of the constants:

$$\kappa_0 \to -\kappa_0, \qquad \theta_\infty \to \kappa_0 - \theta_\infty - 1,$$

or to the transformation of V:

$$x : \mathbf{v} \mapsto (\tfrac{1}{3} - v_2, \tfrac{1}{3} - v_1, -\tfrac{2}{3} - v_3).$$

We construct the transformation y_* by the use of the equality: $y = \ell s_1 x$. Consider firstly the canonical transformation π'' defined by:

$$q = \sqrt{-1}q'', \qquad p = -\sqrt{-1}\bar{p}'',$$
$$H = -\sqrt{-1}\bar{H}'', \quad t = \sqrt{-1}t''.$$

We can see by computation

$$\bar{H}'' = 2q''(\bar{p}'')^2 + ((q'')^2 + 2t''q'' + 2(v_1 - v_2))\bar{p}'' + (v_1 - v_3)q,$$

which shows $\mathscr{H}'' = \mathscr{H}(x(\mathbf{v}))$ (confer (0.4)). Therefore we have $x_* = \pi^{-1} \circ \pi''$ and then obtain the explicit forms (4.7)–(4.8) of y_*.

4.4 *Automorphism of the Root System*

Let R be the root system of the type A_2. The group $A(R)$ of automorphisms of the set of all roots of R is generated by the Weyl group $W(R)$ and the involution y. We denote by $\tilde{G}$ the subgroup of $\mathscr{A}(V)$, generated by G and y. By Proposition 4.4 and Theorem 3, we have obtained the nonlinear representation of $\tilde{G}$ on the Painlevé system $\mathscr{H}$, associated with P_{IV}. $\tilde{G}$ is the semidirect product of $A(R)$ and the weight lattice group $\mathbf{P}(R)$ [2].

5. Classical Solution

5.1 Particular Solutions of $\mathscr{H}(\mathbf{v})$

Consider in V the line V' defined by:

$$v_1 - v_3 = 0. \tag{5.1}$$

If $\mathbf{v}$ is on V', then $\mathscr{H}(\mathbf{v})$ possesses particular solutions such that

$$\frac{dq}{dt} = -q^2 - 2tq - 2\kappa_0, \quad p \equiv 0, \quad \kappa_0 = v_2 - v_1. \tag{5.2}$$

In this case we have $H(\mathbf{v}) \equiv 0$ and then by (3.13)

$$H(\ell(\mathbf{v})) = q(\mathbf{v}),$$

$q(\mathbf{v})$ being a solution of the Riccati equation (5.2); hence $q(\mathbf{v}) - 2(v_1 - \frac{1}{3})t$ is a solution of $E(\ell(\mathbf{v}))$. Moreover since $\tau(\mathbf{v}) = 1$ for (5.2), the τ-function $\tau_1 = \tau(\ell(\mathbf{v})) = \ell^*\tau$ satisfies

$$\frac{d^2\tau_1}{dt^2} + 2t\frac{d\tau_1}{dt} + 2\kappa_0\tau_1 = 0, \tag{5.3}$$

which is easily seen by (3.14). By replacing in (5.3) t by $\frac{1}{\sqrt{-2}} t_1$ and putting $T(t_1) = \tau_1(t)$, we obtain the differential equation of the Hermite function:

$$\frac{d^2T}{dt_1^2} - t_1\frac{dT}{dt_1} - \kappa_0 T = 0.$$

Therefore it follows that:

Proposition 5.1. *If* $\kappa_0 = v_2 - v_1$ *is an negative integer besides* (5.1), *then* $\mathscr{H}(\mathbf{v})$ *has a solution of the form* $(q, 0)$, *such that* q *is rational in* t.

By applying the birational canonical transformation ℓ_* successively to $\mathscr{H}(\mathbf{v})$ at $\mathbf{v}$ of V', we obtain the following proposition.

Proposition 5.2. *If* $\theta_\infty = m$, *then the Painlevé system admits the one-parameter family of particular solutions represented by the Hermite functions,* m *denoting a non-negative integer.*

We deduce from this proposition the Toda equation (3.18) for the semi-sequence of τ-functions:

$$\mathfrak{T}^+(\ell) = \{\tau_m;\ m \geqq 0\}$$

with $\tau_0 = 1$. Hence if we define $\tilde{\tau}_m$ by (3.19) with $v_1 = v_3$, then $\tilde{\tau}_m$ $(m \geqq 2)$ is given by the formula of Darboux, (1.20), with $\tau = \tilde{\tau}_1$.

5.2 Transformation ℓ_*^{-1}

Consider also in this paragraph the Painlevé system $\mathscr{H}(\mathbf{v})$ at $\mathbf{v}$ of V', and the particular solution (5.2). We attempt to obtain particular solutions of $\mathscr{H}(\ell^{-1}(\mathbf{v}))$, by applying ℓ_*^{-1} to $\mathscr{H}(\mathbf{v})$. However we have obtained in Sect. 3.3 the explicit form of ℓ_*^{-1} under the assumption that the auxiliary Hamiltonian function is not linear in t. We show below that ℓ_*^{-1} can be defined for (5.2) even if $\frac{d^2h}{dt^2}\equiv 0$. Put

$$\begin{aligned}\pi\mathscr{H}(\ell^{-1}(\mathbf{v}))&=\bar{\mathscr{H}}(\ell^{-1}(\mathbf{v}))\\&=(q_-,\bar{p}_-,\bar{H}_-,t).\end{aligned}$$

Then we have from (3.16)

$$q_-=-2p+\frac{2(v_2-v_1)}{q},$$

$$\bar{p}_-=\tfrac{1}{2}q.$$

Taking the limit: $p\to 0$, we obtain

$$q_-=\frac{2\kappa_0}{q},\qquad p_-=\frac{1}{2}q+\frac{\kappa_0}{q}+t,\tag{5.4}$$

where we write $p_-=\bar{p}_-+\frac{1}{2}q_-+t$. We claim (5.4) defines the transformation ℓ_*^{-1} on the family of particular solutions (5.2). In fact, we can show:

Proposition 5.3. *If q satisfies the Riccati equation* (5.2), *then* (q_-,p_-), *given by* (5.4), *is a solution of* $\mathscr{H}(\ell^{-1}(\mathbf{v}))$.

Proof. By (5.4), q_- satisfies the Riccati equation:

$$\frac{dq_-}{dt}=q_-^2+2tq_-+2\kappa_0,\tag{5.5}$$

from which we obtain

$$\frac{dq_-}{dt}+q_-^2+2tq_-+2\kappa_0=4q_-p_-.$$

In order to verify the proposition, we show q_- is a solution of the Painlevé equation P. In general, consider the canonical transformation $\pi\cdot(s_1)_*$: put

$$\pi\cdot(s_1)_*\mathscr{H}(\mathbf{v})=(q^0,p^0,\bar{H}^0,t).$$

We have:

$$q^0=q,\qquad p^0=p-\frac{1}{2}q-\frac{v_2-v_1}{q}-t,$$

$$H^0=H+q;$$

in particular, H^0 is written as follows:

$$H^0=2q^0p^{02}+(q^{02}+2tq^0+2\kappa_0)p^0+(\theta_\infty+1)q^0.$$

Therefore, if $\theta_\infty=-1$, then $\pi\cdot(s_1)_*\mathscr{H}(\mathbf{v})$ admits the particular solutions

$$\frac{dq^0}{dt}=q^{02}+2tq^0+2\kappa_0,\qquad p_0\equiv 0,$$

which shows q_- is actually a solution of P. The proof of Proposition 5.3 is completed.

We can apply the transformation ℓ_*^{-1} to $\mathscr{H}(\mathbf{v})$ by virtue of Proposition 5.3 and then obtain the semi-sequence of τ-functions:

$$\mathfrak{T}^-(\ell)=\{\tau_{-m};\ m\geqq 0\},$$

such that $\tau_0=1$ and τ_{-1} is a solution of

$$\frac{d^2\tau_{-1}}{dt^2}-2t\frac{d\tau_{-1}}{dt}+2\kappa_0\tau_{-1}=0.$$

The τ-functions τ_{-m} $(m\geqq 2)$ is given again by the use of (1.20) with $\tau=\tau_{-1}$.

5.3 Example of τ-Functions

Consider the linear equations

$$\frac{d^2T_m}{dt^2}+2t\frac{dT_m}{dt}+2(\kappa_0+m)T_m=0,$$

and the sequence of solutions $\{T_m;\ m\in\mathbb{Z}\}$, such that

$$\frac{dT_m}{dt}=T_{m+1}.$$

So by the similar manner to [15], we can verify:

Proposition 5.4. *$\{T_m;\ m\in\mathbb{Z}\}$ satisfies*

$$\frac{d^2}{dt^2}\log T_m+2(\kappa_0+m)=c'(m)\frac{T_{m-1}T_{m+1}}{T_m^2}\tag{5.6}$$

$c'(m)$ being non-zero constants.

(5.6) is the same equation as (3.26), while we obtain from (5.6) the other sequence of solutions of the Painlevé systems. For a point of the form

$$\mathbf{v}^0=(-\tfrac{1}{3}\kappa_0,\tfrac{2}{3}\kappa_0,-\tfrac{1}{3}\kappa_0),$$

we put $\mathbf{v}_0^0=\mathbf{v}^0$ and

$$\mathbf{v}_m^0=(\ell')^m(\mathbf{v}^0)=\mathbf{v}^0+\frac{m}{3}(-1,2,-1).$$

Note every $\mathbf{v}_m^0$ is on the line V' of V. Since $q_m^0=\dfrac{d}{dt}\log T_m$ satisfies

$$\frac{dq}{dt}=-q^2-2tq-2(\kappa_0+m)$$

for each m, $(q,p)=(q_m^0,0)$ is a particular solution of $\mathscr{H}(\mathbf{v}_m^0)$.

5.4 Classical Solutions

The particular solutions discussed above are written in terms of the Hermite functions and then called the classical solution of the Painlevé equation. Now we prove the

Theorem 4. *Classical solutions of the Painlevé system $\mathscr{H}=\mathscr{H}_{\mathrm{IV}}$ appear at walls of a Weyl chamber $\mathfrak{C}$ of the affine Weyl group $\tilde{W}=W_a(R)$ of the root system of the type A_2.*

Proof. We can assume walls of $\mathfrak{C}$ are defined by:

$$v_1-v_2=0, \tag{5.7$_1$}$$

$$v_2-v_3=0, \tag{5.7$_2$}$$

$$v_3-v_1=-1. \tag{5.7$_3$}$$

In fact, for another $\mathfrak{C}'$, there is a w of $\tilde{W}$ such that $\mathfrak{C}'=w(\mathfrak{C})$; hence we obtain particular solutions at walls of $\mathfrak{C}'$ by applying the canonical transformation w_*.

Case (5.7)$_1$. Since $\kappa_0=v_2-v_1=0$, $\mathscr{H}(\mathbf{v})$ admits the solutions:

$$q\equiv 0, \quad \frac{dp}{dt}=-2p^2+2tp-\theta_\infty,$$

which corresponds to the singular solution of P. Here we write $\theta_\infty=v_3-v_1$ as before.

Case (5.7)$_2$. We write

$$\theta=\kappa_0=\theta_\infty;$$

$\mathscr{H}(\mathbf{v})$ possesses a solution of the form:

$$\frac{dq}{dt}=-q^2-2tq+2\theta, \qquad qp=\theta. \tag{5.8}$$

Case (5.7)$_3$. It is the case studied in Sect. 5.2. Since $\theta_\infty=-1$, we have a solution of $H(\mathbf{v})$ given by (5.4)–(5.5).

5.5 Rational Solutions

To Riccati equations obtained above are solved in terms of the Hermite functions. It is easy to see:

Proposition 5.5. *The Painlevé system has a rational function solution at vertices of a Weyl chamber.*

For example, the vertices of the Weyl chamber $\mathfrak{C}$ enclosed by (5.7) are:

$$O \text{ (origin)}, \quad \varpi_i \quad (i=1,2) \quad \text{(weight vectors)}.$$

If $\mathbf{v}=\mathbf{0}$, then $\mathscr{H}(\mathbf{v})$ admits the solution:

$$(q,p)=(0,0).$$

When $\mathbf{v}=\varpi_1=(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, we obtain from (5.8)

$$(q,p)=\left(-2t,\frac{1}{2t}\right): \tag{5.9}$$

note $\kappa_0=\theta_\infty=-1$. By means of (5.4),

$$(q,p)=\left(-\frac{1}{t}, -\frac{1}{2t}\right) \tag{5.10}$$

is a rational solution at $\ell^{-1}(\varpi_1)=(1,0,-1)$. The Painlevé system at $\mathbf{v}=\varpi_2=(\frac{1}{3},\frac{1}{3},-\frac{2}{3})$ has a solution of the form

$$q=0, \quad \frac{dp}{dt}=-2p^2+2tp+1,$$

hence the rational solution $(q,p)=(0,t)$. By applying G_* to such solutions, we obtain the family of rational function solutions represented by the Hermite polynomials.

An interesting rational function appears at the barycenter:

$$u=(\tfrac{1}{3},0,-\tfrac{1}{3})$$

of the Weyl chamber $\mathfrak{C}$; u is the fixed point of the transformation $\tilde{\ell}$ considered in Sect. 4.3. It is the case $\kappa_0=-\frac{1}{3}$, $\theta_\infty=-\frac{2}{3}$ and

$$(q,p)=\left(-\frac{2}{3}t,\frac{1}{3}t+\frac{1}{2t}\right) \tag{5.11}$$

is a solution of $\mathscr{H}(\mathbf{v})$. Moreover we have

$$(q,p)=\left(-\frac{2}{3}t+\frac{1}{t},\frac{1}{3}t\right) \tag{5.11$'$}$$

at the point:

$$\mathbf{u}'=s_2(\mathbf{u})=(\tfrac{1}{3},-\tfrac{1}{3},0).$$

The rational solutions (5.9), (5.10), (5.11), and (5.11)$'$ of the fourth Painlevé equation P_{IV} may be investigated for the first time in [1]: cf. also [10]. On the other hand, we obtain from (5.11) the following proposition, by means of the canonical transformation ℓ_*.

Proposition 5.6. *There exists the sequence $\{Q_m(X);\, m\in\mathbb{Z}\}$ of monic polynomials such that*

$$\frac{d^2}{dX^2}\log Q_m+X^2+2m-1=\frac{Q_{m-1}Q_{m+1}}{Q_m^2},$$

$$Q_0=Q_1=1, \quad Q_m(\sqrt{-1}X)=(-1)^{\frac{1}{2}d(m)}Q_{-m+1}(X),$$

$$d(m)=\deg Q_m(X)=m(m-1).$$

The polynomials $Q_m(X)$ can be regarded as a generalization of the Yablonskii-Vorob'ev polynomials; as for details investigation of these polynomials, as well as of $R_m(X)$ given in the next paragraph, cf. [7].

5.6 Proof of Proposition 5.6

The Hamiltonian function related to the solution (5.11) is:

$$H(t;\mathbf{u})=\tfrac{4}{27}t^3+\tfrac{2}{3}t,$$

and we have then

$$\tau(t;\mathbf{u})=\exp(\tfrac{1}{27}t^4+\tfrac{1}{3}t^2).$$

Consider the τ-sequence $\mathfrak{T}(\ell)=\{\tau_m;\ m\in\mathbb{Z}\}$ such that $\tau_0=\tau(t;\mathbf{u})$. We obtain from (3.25)

$$\tau_1=\exp(\tfrac{1}{27}t^4),$$

and then, by the use of the Toda equation (3.18).

$$\tau_2=\mathrm{const}\cdot(\tfrac{4}{9}t^2+\tfrac{2}{3})\exp(\tfrac{1}{27}t^4-\tfrac{1}{3}t^2),$$

$$\tau_{-1}=\mathrm{const}\cdot(\tfrac{4}{9}t^2-\tfrac{2}{3})\exp(\tfrac{1}{27}t^4+\tfrac{2}{3}t^2),$$

and so on. Put

$$\tau_m=T_m(t)\exp(\tfrac{1}{27}t^4-\tfrac{1}{3}(m-1)t^2).$$

We see $T_m(t)$ are necessarily polynomial in t. In fact, the τ-function of the Painlevé system has to be entire [13]. Define the polynomials $Q_m(X)$ by:

$$t=\sqrt{\tfrac{3}{2}}X,\qquad T_m(t)=Q_m(X).$$

The assertion of Proposition 5.6 can be verified by the induction with respect to m; we do not enter into details.

Moreover we obtain from (5.11)′ the

Proposition 5.6′. *There exists the sequence $\{R_m(X);\ m\in\mathbb{Z}\}$ of monic polynomials such that*

$$\frac{d^2}{dX^2}\log R_m+X^2+2m=\frac{R_{m-1}R_{m+1}}{R_m^2},$$

$$R_0=1,\qquad R_1=X,\qquad R_m(\sqrt{-1}X)=(-1)^{\frac{1}{2}d'(m)}R_{-m}(X),$$

$$d'(m)=\deg R_m(X)=m^2.$$

In fact, we obtain from (5.11)′ the τ-sequence $\mathfrak{T}'(\ell)=\{\tau'_m;\ m\in\mathbb{Z}\}$ such that

$$\tau'_0=\tau(t;\mathbf{u}')=\exp(\tfrac{1}{27}t^4+\tfrac{1}{3}t^2),$$

$$\tau'_m=T'_m(t)\exp(\tfrac{1}{27}t^4-\tfrac{1}{3}(m-1)t^2).$$

$T'_m(t)$ are polynomial in t. Define the polynomials $R_m(X)$ by

$$t=\sqrt{\tfrac{3}{2}}X,\qquad T'_m(t)=R_m(X).$$

Proposition 5.6′ is established for $R_m(X)$.

Remark 5.1. Put

$$\tilde{\tau}_m = Q_m(X)\exp\left(\tfrac{1}{12}X^4 + \tfrac{1}{2}(2m-1)X^2\right),$$
$$\tilde{\tau}'_m = R_m(X)\exp\left(\tfrac{1}{12}X^4 + mX^2\right).$$

Then $\{\tilde{\tau}_m;\ m \in \mathbb{Z}\}$ and $\{\tilde{\tau}'_m;\ m \in \mathbb{Z}\}$ satisfy the Toda equation:

$$\frac{d^2}{dX^2}\log\tilde{\tau}_m = \frac{\tilde{\tau}_{m-1}\tilde{\tau}_{m+1}}{\tilde{\tau}_m^2}.$$

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Immersions minimales, première valeur propre du laplacien et volume conforme

A. El Soufi et S. Ilias

Laboratoire de Mathématiques, associé au CNRS n° 188, B.P. 74, F-38402 St-Martin-d'Heres, France

Introduction

Dans un article récent, Li et Yau introduisent un nouvel invariant conforme pour les variétés riemanniennes compactes qu'ils appellent "volume conforme" [10]. Ils montrent l'intérêt de cet invariant en s'en servant pour retrouver et généraliser plusieurs résultats concernant la théorie des surfaces (surfaces minimales des sphères, première valeur propre du laplacien et conjecture de Willmore).

Pour toute variété riemannienne compacte (M, g) de dimension m nous notons respectivement $\lambda_1(M, g)$ et $V(M, g)$ la premiere valeur propre du laplacien et le volume de cette variété. Le volume conforme est noté $V_c(M)$ (pour une définition voir le Par. 1).

Li et Yau montrent que pour les surfaces minimales des sphères canoniques, le volume riemannien est minoré par le volume conforme. C'est ce résultat que nous généralisons *en toute dimension* dans le théorème 1.1 aux variétés qui s'immergent minimalement dans des sphères. Signalons que Li et Yau avaient énoncé dans [10] un résultat analogue sous les hypothèses supplémentaires d'homogénéité sur la variété et d'équivariance sur l'immersion minimale, mais leur démonstration semble comporter une erreur comme viennent de le signaler dans [11] Montiel et Ros.

Dans le Par. 2, au théorème 2.2, nous généralisons en toute dimension un autre résultat que Li et Yau n'avaient démontré qu'en dimension 2: le produit $\lambda_1(M, g)V(M, g)^{2/m}$ est uniformément majoré sur chaque classe conforme de métriques par la constante $mV_c(M)^{2/m}$ (notons que $\lambda_1(M, g)$ n'est pas invariant par les homothéties et qu'on ne peut le borner qu'en le multipliant par une quantité géométrique de comportement inverse vis à vis des homothéties telle que $V(M, g)^{2/m}$).

Afin de mieux situer ce résultat rappelons que si M est une surface orientable donnée, la quantité $\lambda_1(M, g)V(M, g)$ est bornée uniformément sur l'ensemble de toutes les métriques g définies sur M par une constante qui ne dépend que du genre de la surface: c'est un résultat de Hersch [8] pour la sphère $\mathbf{S}^2$ et de Yang et Yau [16] dans le cas général. Berger s'était demandé dans [3] s'il existait une majoration analogue de $\lambda_1(M,g)\,V(M,g)^{2/m}$ en toute dimension. Les travaux de Urakawa [15], Bérard-Bergery et Bourguignon [2, 5], Tanno [14] et Muto [12] ont prouvé qu'il existe sur la sphère $\mathbb{S}^m$, pour tout m supérieur ou égal à 3, une suite de métriques g_t (non conformes entre elles) telles que $\lambda_1(\mathbb{S}^m, g_t)\,V(\mathbb{S}^m, g_t)^{2/m}$ tende vers l'infini avec t. En dimension supérieure on égale à 3, on ne peut donc espérer obtenir une borne supérieure uniforme qu'en se placant (par exemple) dans une classe conforme fixée, c'est précisément le résultat que nous obtenons en 2.2.

Dans le Par. 3, nous traduisons les théorèmes 1.1 et 2.2 dans le cas où la classe conforme considérée sur une variété M contient une métrique g_0 telle que (M, g_0) s'immerge minimalement dans une sphère par ses premières fonctions propres. Remarquons qu'il s'agit là d'une classe assez large puisque tout espace homogène irréductible ainsi que toute variété fortement harmonique s'immergent minimalement par leurs premières fonctions propres dans une sphère [4, 9]. Dans ce cas les théorèmes 1.1 et 2.2 deviennent (cf. 2.3, 3.1 et 3.2):

$$(*)\qquad V(M, g_0) = V_c(M)$$

$$(**)\qquad \lambda_1(M,g)\,V(M,g)^{2/m} \leqq \lambda_1(M, g_0)\,V(M, g_0)^{2/m}$$

pour toute métrique g conforme à g_0.

L'égalité (*) donne un calcul explicite du volume conforme pour toutes les variétés qui s'immergent minimalement par leurs premières fonctions propres dans des sphères (notons que le calcul de $V_c(M)$ en dimension supérieure ou égale à 3 n'était jusqu'ici disponible que pour les sphères canoniques ($\mathbb{S}^m$, *can*)). D'autre part, notons que dans le résultat déjà cité de Hersch le maximum de $\lambda_1(\mathbb{S}^2, g)\,V(\mathbb{S}^2, g)$ est atteint lorsque g est la métrique canonique de $\mathbb{S}^2$ et que dans [3], Berger avait obtenu, pour tout m, une borne supérieure (non optimale) de $\lambda_1(\mathbb{S}^m, g)\,V(\mathbb{S}^m, g)^{2/m}$ sur la classe conforme canonique de $\mathbb{S}^m$ conjecturant que le maximum est atteint lorsque $g = can$. La réponse à cette question est un cas particulier de (**).

Nous montrons ensuite (cf. 3.1) que, dans l'inégalité (**), l'égalité est atteinte si et seulement si il existe une constante k telle que g soit isométrique à kg_0 (si (M, g_0) n'est pas isométrique à ($\mathbb{S}^m$, *can*) ceci implique en fait que $g = kg_0$). Ceci prouve que (cf. 3.3), dans une classe conforme donnée, il ne peut exister plus d'une métrique admettant des immersions minimales composées de premières fonctions propres dans des sphères. Ce résultat d'unicité a été démontré en dimension 2 de manière indépendante par Montiel et Ros dans [11].

Enfin, nous montrons (cf. 3.5) que, parmi toutes les métriques minimales d'une même classe conforme, la métrique qui admet des immersions minimales par ses premières fonctions propres est caractérisée, quand elle existe, par la valeur de son volume.

1. Volume conforme d'une immersion minimale

Soit (M, g) une variété riemannienne compacte de dimension m. On notera dv_g son élément de volume riemannien et $V(M, g)$ son volume. Soit $\mathbb{S}^n$ la sphère unité munie de sa métrique canonique notée can. On désigne par G le groupe des difféomorphismes conformes de $\mathbb{S}^n$. (Une étude détaillée de ce groupe est faite dans [1]).

Définition. On appelle *n-volume conforme* d'une immersion $\varphi: M \to \mathbb{S}^n$ la quantité $V_c(n, \varphi)$ définie par:

$$V_c(n, \varphi) = \sup_{\gamma \in G} V(M, (\gamma \circ \varphi)^* can).$$

L'infimum sur toutes les immersions conformes de M dans $\mathbb{S}^n$ de $V_c(n, \varphi)$ est dit *n-volume conforme* de M, on le note $V_c(n, M)$. (Notons que le théorème de Nash-Moser assure l'existence-via une projection stéréographique-d'immersions conformes dans $\mathbb{S}^n$ pour n assez grand).

Vue la décroissance en n de $V_c(n, M)$, on définit le *volume conforme* de M par:

$$V_c(M) = \lim_{n \to \infty} V_c(n, M).$$

Pour les propriétés de cet invariant conforme, on peut consulter [7] ou [10].

Soit a un vecteur unitaire de $\mathbb{R}^{n+1}$ et notons A le champ de vecteurs sur $\mathbb{S}^n$ obtenu en projetant le champ constant a sur le fibré tangent à $\mathbb{S}^n$. Le champ A s'identifie au champ gradient de la fonction $u = \langle \cdot, a \rangle$ qui est une première harmonique sphérique (où $\langle \cdot, \cdot \rangle$ désigne le produit scalaire euclidien). Le flot $(\gamma_t^a)_t$ du champ A constitue un sous-groupe de G; on notera G' la réunion des différents flots $(\gamma_t^a)_t$ quand a parcourt la sphère unité de $\mathbb{R}^{n+1}$.

On a alors le lemme suivant:

Lemme. *Soit $\gamma \in G$, il existe $r \in O(n+1)$ et $\gamma_t^a \in G'$ tels que: $\gamma = r \circ \gamma_t^a$.*

Preuve. Soit $\gamma \in G$, il existe $x_0 \in \mathbb{S}^n$ tel que $\gamma(-x_0) = -\gamma(x_0)$ (en fait, par un argument topologique simple on peut montrer l'existence d'un tel x_0 pour tout difféomorphisme de $\mathbb{S}^n$ (voir, par exemple, [6, exercice 25, p. 59])). Soit r_1 une isométrie de $\mathbb{S}^n$ telle que $r_1(\gamma(x_0)) = x_0$. La transformation $r_1 \circ \gamma$ est conforme et elle laisse fixés les pointts x_0 et $-x_0$. La classification donnée dans [1] (théorème 3.5.1) et le fait que $G' = \{\pi_a^{-1} \circ H_k \circ \pi_a,\ a \in \mathbb{S}^n$ et $k \in \mathbb{R}\}$ (où π_a est la projection stéréographique de pôle a et H_k l'homothétie de $\mathbb{R}^{n+1}$ de centre 0 et de rapport k) nous permettent de déduire qu'il existe $r_2 \in O(n+1)$ et $\gamma_t^a \in G'$ tels que: $r_1 \circ \gamma = r_2 \circ \gamma_t^a$. D'où le résultat. □

Nous pouvons donc nous restreindre dans la définition du volume conforme d'une immersion aux éléments de G':

$$V_c(n, \varphi) = \sup\{V(M, (\gamma_t^a \circ \varphi)^* can);\ a \in \mathbb{S}^n \text{ et } t \geqq 0\}.$$

1.1. **Théorème.** *Soit (M, g) une variété riemannienne compacte de dimension m et supposons qu'il existe une immersion isométrique minimale φ de (M, g) dans $\mathbb{S}^n$. Alors*

$$V(M, g) = V_c(n, \varphi) \geqq V_c(n, M).$$

De plus, si (M, g) n'est pas isométrique à $(\mathbb{S}^m, can)$, alors $V(M, g) > V(M, (\gamma \circ \varphi)^* can)$ pour tout $\gamma \in G \backslash O(n+1)$.

Preuve. Pour simplifier nous allons supposer que M est une sous-variété minimale de $\mathbb{S}^n$. D'après les remarques ci-dessus il suffira de montrer que:

$$V(\gamma_t^a(M)) \leqq V(M) \quad \text{pour tout} \quad a \in \mathbb{S}^n \quad \text{et} \quad t \geqq 0 .$$

Fixons-nous un vecteur $a \in \mathbb{S}^n$ et soit $\gamma = \gamma_{t_0}^a$. La formule de la variation première donne [9]:

$$\left.\frac{d}{dt}\right|_{t=t_0} V(\gamma_t^a(M)) = - \int_{\gamma(M)} \langle H_x^{\gamma(M)}, A_x \rangle dv_{can} = - \int_M \langle H_{\gamma(x)}^{\gamma(M)}, A_{\gamma(x)} \rangle e^{mf} dv_{can},$$

où $H^{\gamma(M)}$ est la courbure moyenne dans $\mathbb{S}^n$ de la sous-variété $\gamma(M)$ et où on a posé $\gamma^* can = e^{2f} \cdot can$. Par un calcul direct nous montrons qu'en tout point x de M on a:

$$H_{\gamma(x)}^{\gamma(M)} = e^{-2f(x)} d\gamma(H_x^M - m(\nabla f)_x^{\perp}) = -m e^{-2f(x)} d\gamma((\nabla f^{\perp})_x),$$

où $H^M = 0$ est la courbure moyenne de M dans $\mathbb{S}^n$ et où $(\nabla f)^{\perp}$ est la composante normale à M du gradient de f. D'autre part, on a $A_{\gamma(x)} = d\gamma(A_x)$, il vient donc

$$\langle H_{\gamma(x)}^{\gamma(M)}, A_{\gamma(x)} \rangle = -m \langle (\nabla f)_x^{\perp}, A_x \rangle .$$

Pour calculer ∇f nous remarquons que $e^{2f} = |A_{\gamma(x)}|^2 / |A_x|^2$ et que $|A_x|^2 = 1 - u^2(x)$ où $A = \nabla u$. La différenciation de $e^{2f} = (1 - u^2 \circ \gamma)/(1 - u^2)$ donne:

$$df = -\frac{u \circ \gamma}{1 - u^2 \circ \gamma} du \circ d\gamma + \frac{u}{1 - u^2} du .$$

Pour tout $X \in T\mathbb{S}^n$ on a:

$$(du \circ d\gamma)_x(X) = \langle \nabla u_{\gamma(x)}, d\gamma_x(X) \rangle = \langle d\gamma_x(\nabla u_x), d\gamma_x(X) \rangle = e^{2f} \langle \nabla u_x, X \rangle$$

on en déduit que $\nabla f = \dfrac{u - u \circ \gamma}{1 - u^2} \nabla u$. D'où finalement

$$\left.\frac{d}{dt}\right|_{t=t_0} V(\gamma_t^a(M)) = m \int_M \frac{u - u \circ \gamma}{1 - u^2} |A^{\perp}|^2 e^{mf} dv_{can} \leqq 0 .$$

(car on a pour tout $x : u(x) \leqq u(\gamma(x))$). Ceci prouve que pour tout à la fonction $t \mapsto V(\gamma_t^a(M))$ est décroissante et est donc majorée par $V(\gamma_0^a(M)) = V(M)$.

Supposons qu'il existe $\gamma = \gamma_{t_0}^a$ avec $t_0 > 0$ tel que $V(\gamma_{t_0}^a(M)) = V(M)$; ceci implique d'après ce qui précède que $\dfrac{d}{dt} V(\gamma_t^a(M)) = 0$ pour tout $t \leqq t_0$ et donc que $A^{\perp} = 0$ sur M (on a en fait $u < u \circ \gamma$ si $\gamma \neq \mathrm{Id}$). La restriction de A à M nous donne un champ de vecteurs sur M dont les courbes intégrales sont des grands cercles passant par les points a et $-a$. Donc $a \in M$ et les géodésiques issues de a sont toutes des grands cercles. Par conséquent, M est l'image par l'exponentielle de $\mathbb{S}^n$ de son propre espace tangent en a; c'est donc une sous-sphère totalement géodésique de dimension m dans $\mathbb{S}^n$. □

1.2. *Remarque*. Le théorème précédent nous montre en particulier que si M est une sous-variété minimale non totalement géodésique de $\mathbb{S}^m$ alors, quelle que soit la transformation $\gamma \in G \backslash O(n+1)$, la sous-variété $\gamma(M)$ n'est pas minimale.

1.3. **Corollaire.** *Soit (M, g) une variété riemannienne compacte de dimension m. On suppose qu'il existe une immersion isométrique minimale φ de (M, g) dans une sphère, et on pose $i(\varphi) = \operatorname{Sup}\{\#\varphi^{-1}(\varphi(m))/m \in M\}$. On a alors:*

$$i(\varphi) \leqq \frac{V(M, g)}{V(\mathbf{S}^m, can)}.$$

En particulier, si $V(M, g) < 2V(\mathbf{S}^m, can)$, alors toute immersion isométrique minimale de (M, g) dans une sphère est un plongement.

Preuve. On sait d'après [10] que si pour une immersion φ il existe un point de $\mathbf{S}^m$ ayant k antécédents dans M alors $V_c(n, \varphi) \geqq kV(\mathbf{S}^m, can)$. Le résultat découle alors directement de 1.1. □

2. Volume conforme et premiere valeur propre du laplacien

L'objet essentiel de ce paragraphe est de donner une généralisation en toute dimension d'un résultat que Li et Yau n'avaient obtenu qu'en dimension 2 (théorème 1 de [10]). Bien que plusieurs des arguments souvent évoqués brièvement dans [10] restent valables en dimension quelconque, nous avons jugé utile d'en donner ici une démonstration plus complète tout en y ajoutant les arguments qui permettent la généralisation du résultat en dimension quelconque.

2.1. **Lemme.** *Soient (M, g) une variété riemannienne compacte et φ une immersion de M dans $\mathbf{S}^m$. Il existe $\gamma \in G'$ tel que $\psi = \gamma \circ \varphi = (\psi_1, \ldots, \psi_{n+1})$ vérifie:*

$$\int_M \psi_i dv_g = 0 \quad \textit{pour tout } i.$$

Pour une preuve voir [7] ou [10].

2.2. **Théorème.** *Soit (M, g) une variété riemannienne compacte de dimension m. Alors, pour tout n tel que le n-volume conforme de M soit défini, on a:*

$$\lambda_1(M, g) V(M, g)^{2/m} \leqq m V_c(n, M)^{2/m}.$$

L'égalité a lieu si et seulement si (M, g) admet, à une homothétie près, une immersion isométrique minimale dans $\mathbf{S}^n$ donnée par ses premières fonctions propres.

Preuve. Soit $\varphi = (\varphi_1, \ldots, \varphi_{n+1})$ une immersion conforme de M dans $\mathbf{S}^n$. D'après le lemme précédent, on peut supposer que $\int_M \varphi_i dv_g = 0$. Par le principe du minimax, on a:

$$\lambda_1 \leqq \frac{\sum_i \int_M |d\varphi_i|_g^2 dv_g}{\sum_i \int_M \varphi_i^2 dv_g} = \frac{\int_M \sum_i |d\varphi_i|_g^2 dv_g}{V(M)} \leqq \frac{\left[\int_M \left(\sum_i |d\varphi_i|_g^2\right)^{m/2} dv_g\right]^{2/m}}{V(M)^{2/m}}.$$

D'autre part, on a $\varphi^* can = \left(\frac{1}{m} \sum_i |d\varphi_i|_g^2\right) g$. On en déduit que

$$\int_M \left(\sum_i |d\varphi_i|_g^2\right)^{m/2} dv_g = m^{m/2} V(M, \varphi^* can).$$

D'où:

$$\lambda_1 \leqq m \frac{V(\varphi(M))^{2/m}}{V(M)^{2/m}}.$$

On en déduit immédiatement l'inégalité annoncée.

Il est clair que si (M, g) admet une immersion minimale par ses premières fonctions propres on a alors l'égalité dans 2.2, car d'une part on a $\lambda_1(M, g) = m$ et d'autre part on a, d'après le théorème 1.1, $V(M, g) \geqq V_c(n, M)$. Réciproquement, supposons qu'on ait l'égalité dans 2.2. Via une homothétie, on peut supposer $\lambda_1 = m$ pour avoir $V(M) = V_c(n, M)$. On considère alors une suite $(\varphi^k)_k$ d'immersions conformes de M dans $\mathbb{S}^n$ telles que $V_c(n, \varphi^k) \to V(M)$ et on suppose que $\int_M \varphi_i^k dv_g = 0$ pour tout k et i. En appliquant d'une part le principe du minimax, d'autre part l'inégalité de Hölder on obtient pour tout k

$$(*) \quad \lambda_1 \sum_i \int_M (\varphi_i^k)^2 dv_g \leqq \sum_i \int_M |d\varphi_i^k|^2 dv_g \leqq m V_c(n, \varphi^k)^{2/m} V(M)^{1-2/m}.$$

Ces inégalités montrent que pour chaque i, la suite $(\varphi_i^k)_k$ est bornée dans $H_1^2(M)$. Donc, par le théorème de Rellich (compacité de l'inclusion $H_1^2 \hookrightarrow L^2$) et quitte à prendre une sous-suite, $(\varphi_i^k)_k$ converge fortement dans L^2 vers une limite φ_i. Comme la convergence forte dans L^2 implique la convergence ponctuelle presque partout on a $\sum_i \varphi_i^2 = 1$ $p \cdot p$.

La première inégalité dans $(*)$ étant valable pour chaque i, elle nous permet d'avoir par passage à la limite:

$$\lim_{k \to \infty} \int_M |d\varphi_i^k|^2 dv_g = m \int_M \varphi_i^2 dv_g.$$

De ceci nous allons déduire que φ_i appartient au premier espace propre E_1 de M et que $(\varphi_i^k)_k$ converge vers φ_i au sens H_1^2. En effet, notons p la projection orthogonale sur E_1 et $p^\perp$ la projection sur $E_1^\perp$. On a alors

$$\|\varphi_i^k\|_{L^2}^2 = \|p\varphi_i^k\|_{L^2}^2 + \|p^\perp \varphi_i^k\|_{L^2}^2$$

et

$$\|d\varphi_i^k\|_{L^2}^2 = \|dp\varphi_i^k\|_{L^2}^2 + \|dp^\perp \varphi_i^k\|_{L^2}^2.$$

Du fait que $(\|d\varphi_i^k\|_{L^2}^2 - \lambda_1 \|\varphi_i^k\|_{L^2}^2) \xrightarrow[k \to \infty]{} 0$ il vient que

$$(\|dp^\perp \varphi_i^k\|_{L^2}^2 - \lambda_1 \|p^\perp \varphi_i^k\|_{L^2}^2) \xrightarrow[k \to \infty]{} 0 \quad \text{et comme} \quad \|dp^\perp \varphi_i^k\|_{L^2}^2 \geqq \lambda_2 \|p^\perp \varphi_i^k\|_{L^2}^2,$$

on a alors $(\lambda_2 - \lambda_1) \|p^\perp \varphi_i^k\|_{L^2}^2 \xrightarrow[k \to \infty]{} 0$. Ceci implique que $p^\perp \varphi_i = 0$, donc que $\varphi_i \in E_1$ et que φ_i est $\mathscr{C}^\infty$.

La convergence forte de $(\varphi_i^k)_k$ vers φ_i dans H_1^2 découle du fait que:

$$\begin{aligned} \|d\varphi_i^k - d\varphi_i\|_{L^2}^2 &= \|d\varphi_i^k\|_{L^2}^2 + \|d\varphi_i\|_{L^2}^2 - 2\langle d\varphi_i^k, d\varphi_i \rangle \\ &= \|d\varphi_i^k\|_{L^2}^2 + \lambda_1 \|\varphi_i\|_{L^2}^2 - 2\lambda_1 \langle \varphi_i^k, \varphi_i \rangle \end{aligned}$$

qui tend vers 0 quand $k \to \infty$.

Reste à montrer que l'application $\varphi=(\varphi_1,\ldots,\varphi_{n+1})$ est une immersion isométrique de M dans $\mathbb{S}^n$. En effet, la convergence forte dans H_1^2 de $(\varphi_i^k)_k$ prouve que pour tout $x\in M$ et pour tout $X\in T_xM$, on a:

$$\varphi^*can(X,X)=\sum_i |d\varphi_i(X)|^2=\lim_k \sum_i |d\varphi_i^k(X)|^2=\lim_k (\varphi^k)^*can(X,X).$$

Si on pose $(\varphi^k)^*can=f_kg$, ces dernières égalités impliquent alors que la suite $(f_k)_k$ tend presque partout vers $\frac{1}{m}\sum_i |d\varphi_i|^2$ et que l'on a (comme φ est $\mathscr{C}^\infty$) $\varphi^*can=\left(\frac{1}{m}\sum_i |d\varphi_i|^2\right)g$ en tout point. Il suffit d'appliquer le laplacien à la fonction $\sum_i \varphi_i^2=1$ pour voir que $\sum_i |d\varphi_i|^2=m$ et que $\varphi^*can=g$. □

Une conséquence immédiate des théorèmes 1.1 et 2.2 est le calcul du volume conforme des variétés qui s'immergent isométriquement et minimalement par leurs premières fonctions propres dans des sphères:

2.3. **Corollaire.** *Soit (M,g) une variété riemannienne compacte qui admet une immersion isométrique minimale $\varphi=(\varphi_1,\ldots,\varphi_{n+1})$ dans $\mathbb{S}^n$ telle que les composantes φ_i appartiennent au premier espace propre de (M,g). Alors:*

$$V_c(M)=V_c(n,M)=V_c(n,\varphi)=V(M,g).$$

Des exemples d'application sont donnés dans 3.6.

2.4. *Remarques.* (i) Dans [10], Li et Yau montrent que pour toute variété riemannienne compacte (M,g) de dimension m et pour tout n tel que $V_c(n,M)$ soit défini on a:

$$V_c(n,M)\geqq V(\mathbb{S}^m, can).$$

Ils se posent alors la question de savoir si l'égalité caractérise, à un difféomorphisme conforme près, la sphère $(\mathbb{S}^m, can)$. Le corollaire 2.3 donne en fait la réponse partielle suivante:

Soit (M,g) une variété riemannienne compacte de dimension m. Si la classe conforme de g contient une métrique g_0 telle que (M,g_0) s'immerge minimalement par ses premières fonctions propres dans une sphère alors, l'égalité

$$V_c(n,M)=V(\mathbb{S}^m, can),$$

n'a lieu que s'il existe un difféomorphisme conforme de (M,g) sur $(\mathbb{S}^m, can)$. (En effet, d'après (2.3) on a $V_c(n,M)=V(M,g_0)=V(\mathbb{S}^m, can)$. Or, parmi les sous-variétés minimales des sphères, $(\mathbb{S}^m, can)$ est caractérisée par la valeur de son volume).

(ii) Li et Yau font remarquer dans [10] que le volume conforme d'une surface de genre g est majoré par une constante qui ne dépend que de g. Cependant, le théorème 2.2 montre qu'en dimension supérieure ou égale à 3, il est en général impossible de majorer le volume conforme par une constante qui ne dépend que de la topologie de la variété. Pour cela il suffit d'appliquer 2.2 aux contre-exemples de [2, 5, 12, 14, 15] cités dans l'introduction de cet article.

3. Un théorème d'unicite pour les variétés qui s'immergent minimalement dans des sphères par leurs premières fonctions propres et une généralisation d'un résultat de Hersch

La majoration 2.2 prend une forme explicite (grâce à 1.1) dans le cas où la structure conforme considérée sur une variété M de dimension m contient une métrique g telle que (M, g) s'immerge minimalement dans une sphère: dans ce cas on a pour toute métrique $\tilde{g}$ conforme à g

$$\lambda_1(M, \tilde{g}) V(M, \tilde{g})^{2/m} \leqq m V(M, g)^{2/m}.$$

L'examen du cas d'égalité donne lieu à la proposition suivante dont découleront en particulier une généralisation d'un résultat connu de Hersch [8] et un théorème d'unicité pour les métriques g telles que (M, g) s'immerge minimalement par ses premières fonctions propres dans une sphère.

3.1. **Proposition.** (i) *Soit (M, g) une variété de dimension m non isométrique à $(\mathbb{S}^m, can)$. On suppose que (M, g) s'immerge isométriquement et minimalement dans une sphère S^n. Alors pour toute métrique $\tilde{g}$ conforme à g on a:*

$$\lambda_1(M, \tilde{g}) V(M, \tilde{g})^{2/m} \leqq m V(M, g)^{2/m}$$

où l'égalité n'a lieu que si $\tilde{g}$ est proportionnelle à g et si l'immersion minimale de (M, g) dans $\mathbb{S}^n$ est donnée par les premières fonctions propres.

(ii) *Pour toute métrique g sur $\mathbb{S}^m$ conforme à la métrique canonique on a*

$$\lambda_1(\mathbb{S}^m, g) V(\mathbb{S}^m, g)^{2/m} \leqq \lambda_1(\mathbb{S}^m, can) V(\mathbb{S}^m, can)^{2/m}$$

où l'égalité n'a lieu que si il existe une constante $k > 0$ telle que $(\mathbb{S}^m, g)$ soit isométrique à $(\mathbb{S}^m, k\,can)$.

3.2. *Remarque.* Dans le cas particulier où la variété (M, g) s'immerge isométriquement, et minimalement par ses premières fonctions propres dans une sphère, l'inégalité 3.1 (i) s'énonce de la manière suivante: pour toute métrique $\tilde{g}$ conforme à g on a

$$\lambda_1(M, \tilde{g}) V(M, \tilde{g})^{2/m} \leqq \lambda_1(M, g) V(M, g)^{2/m}.$$

De plus, l'égalité est atteinte si et seulement si $\tilde{g} = kg$.

Pour des exemples d'application voir 3.6.

Preuve de 3.1. L'assertion (ii) découle directement du théorème 2.2. Pour montrer (i) il nous reste à examiner le cas d'égalité. Celle-ci est évidemment vérifiée (d'après 2.2) sous les conditions de l'énoncé. Réciproquement, supposons l'égalité 3.1 vérifiée; ceci implique (via une homothétie) que $\lambda_1(M, \tilde{g}) = m$ et $V(M, \tilde{g}) = V(M, g)$. Notons $\varphi = (\varphi_1, \ldots, \varphi_{n+1})$ l'immersion isométrique minimale de (M, g) dans $\mathbb{S}^n$ et soit $\gamma \in G'$ tel que $\psi = \gamma \circ \varphi$ vérifie $\int_M \psi_i dv_{\tilde{g}} = 0$ pour tout i. Nous appliquons alors successivement le principe du minimax, l'inégalité de Hölder et le théorème 1.1 pour obtenir:

$$m=\lambda_1(M,\tilde g)\underset{(1)}{\leqq}\frac{\int\limits_M\sum\limits_i|d\psi_i|_{\tilde g}^2\,dv_{\tilde g}}{\int\limits_M\sum\limits_i\psi_i^2\,dv_{\tilde g}}=\frac{\int\limits_M\sum\limits_i|d\psi_i|_{\tilde g}^2\,dv_{\tilde g}}{V(M,\tilde g)}$$

$$\underset{(2)}{\leqq}\left(\int\limits_M\left(\sum_i|d\psi_i|_{\tilde g}^2\right)^{m/2}dv_{\tilde g}\right)^{2/m}V(M,\tilde g)^{-2/m}$$

$$=mV(M,\psi^*can)^{2/m}\,V(M,\tilde g)^{-2/m}$$

$$\underset{(3)}{\leqq}mV(M,g)^{2/m}\,V(M,\tilde g)^{-2/m}=m.$$

Ces trois inégalités sont donc en fait des égalités. La première implique qu'on a $\Delta^{\tilde g}\psi_i=m\psi_i$ pour tout i. La deuxieme implique pour $m\geqq 3$ que la fonction $\sum\limits_i|d\psi_i|_{\tilde g}^2$ est constante et ne peut qu'être égale à m $\Big($on peut obtenir aussi $\sum\limits_i|d\psi_i|_{\tilde g}^2=m$ pour tout $m\geqq 2$, en appliquant $\Delta^{\tilde g}$ à la fonction $\sum\limits_i\psi_i^2=1\Big)$.

Enfin, l'égalité (3) implique d'après 1.1 que γ est égale à l'identité puisque (M,g) n'est pas isométrique à $(\mathbb{S}^m, can)$. Il vient que $\tilde g=\psi^*can=\varphi^*can=g$ et que $\lambda_1(g)=\lambda_1(\tilde g)=m$. □

Une conséquence du corollaire 3.3 est le résultat d'unicité suivant:

3.3. **Corollaire.** (i) *Soit (M,g) une variété compacte de dimension m non isométrique à $(\mathbb{S}^m, can)$. S'il existe dans la classe conforme de g une métrique g_0 telle que (M,g_0) s'immerge isométriquement et minimalement par ses premières fonctions propres dans une sphère alors, cette métrique est unique.*

(ii) *Si g est une métrique appartenant à la classe conforme canonique de $\mathbb{S}^m$ et telle que $(\mathbb{S}^m,g)$ s'immerge isométriquement et minimalement par ses premières fonctions propres dans une sphère alors, il existe un difféomorphisme conforme γ de $(\mathbb{S}^m, can)$ telle que $g=\gamma^*can$. (La variété $(\mathbb{S}^m,g)$ est alors isométrique à $(\mathbb{S}^m, can)$.)*

Preuve. L'assertion (ii) provient du fait que $V(\mathbb{S}^m,g)=V_c(\mathbb{S}^m,g)=V_c(\mathbb{S}^m, can)$. Pour (i), supposons qu'il existe une métrique $\tilde g$ conforme à g_0 et vérifiant la même propriété de minimalité que g_0. L'égalité est alors atteinte dans 3.2 ce qui implique que $\tilde g$ est proportionnelle (et donc égale) à g_0. □

3.4. *Remarques.* (i) Le résultat 3.3 a été démontré en dimension 2 de manière indépendante par Montiel et Ros [11]. Ces derniers montrent de plus que dans certaines classes conformes du tore $\mathbb{T}^2$ il n'existe aucune métrique qui s'immerge minimalement par ses premières fonctions propres dans une sphère. Ceci prouve qu'un théorème d'existence pour ce type de métriques n'est pas possible en général.

(ii) Le corollaire 3.3 montre en particulier que parmi toutes les variétés (M,g) de dimension m qui s'immergent minimalement dans des sphères par leurs premières fonctions propres, la sphère $(\mathbb{S}^m, can)$ est caractérisée par le fait qu'elle admet des difféomorphismes conformes qui ne sont pas des isométries. En effet, si α est un difféomorphisme conforme de (M,g) (où (M,g) est non isométrique à $(\mathbb{S}^m, can)$) alors la métrique $\tilde g=\alpha^*g$ vérifie la même condition de minimalité que g et elle lui est donc égale.

3.5. **Corollaire.** *Soit (M,g) une variété qui s'immerge isométriquement et minimalement par ses premières fonctions propres dans une sphère. Alors, pour toute métrique $\tilde{g}$ conforme à g et telle que $(M,\tilde{g})$ s'immerge isométriquement et minimalement dans une sphère on a:*

$$V(M,\tilde{g}) \geqq V(M,g).$$

De plus, si (M,g) est non isométrique à une sphère canonique, l'égalité n'a lieu que si $\tilde{g}=g$. (Par ailleurs, il est bien connu que dans le cas de $(\mathbb{S}^m, can)$, l'égalité ci-dessus n'a lieu que si $(\mathbb{S}^m, g)$ et $(\mathbb{S}^m, can)$ sont isométriques).

3.6. *Applications et exemples.* Parmi les variétés (M,g) qui s'immergent minimalement par leurs premières fonctions propres dans des sphères nous connaissons les variétés homogènes irréductibles (résultat dù à Takahashi, [9]), les variétés fortement harmoniques [4], et le tore de Clifford généralisé $\mathbb{S}^p\left(\sqrt{\frac{p}{p+q}}\right) \times \mathbb{S}^q\left(\sqrt{\frac{q}{p+q}}\right)$. D'autres exemples sont donnés dans [13]. Les espaces symétriques compacts de rang 1 sont à la fois homogènes irréductibles et fortement harmoniques et nous fournissent donc avec le tore plat équilatéral T^2_{eq} et les tores de Clifford, des exemples pour lesquels nos résultats deviennent explicites. Le cas de $(\mathbb{S}^m, can)$ ayant déjà été abordé nous considérons, à titre d'exemples, M égale à l'une des variétés *de dimension m* suivantes: $\mathbb{R}P^m$, $\mathbb{C}P^d$, $\mathbb{H}P^d$, $\mathbb{C}aP^2$, $\mathbb{T}^2_{eq}$, $\mathbb{S}^p\left(\sqrt{\frac{p}{p+q}}\right) \times \mathbb{S}^q\left(\sqrt{\frac{q}{p+q}}\right)$ munies de leurs métriques canoniques. Nous avons alors:

(i) $V_c(n,M) = V\left(M, \frac{\lambda_1(can)}{m}\, can\right)$ dès que $n+1$ est supérieur ou égal à la multiplicité de λ_1.

(ii) Pour toute métrique g appartenant à la classe conforme canonique on a:

$$\lambda_1(M,g)\, V(M,g)^{2/m} \leqq \lambda_1(M,can)\, V(M,can)^{2/m}$$

où l'égalité n'a lieu que si $g = k\, can$.

(iii) La métrique canonique est dans sa classe conforme l'unique métrique qui admet des immersions isométriques minimales composées de premières fonctions propres dans des sphères.

(iv) Tout difféomorphisme conforme de (M, can) est une isométrie.

(v) Si une métrique g conforme à can s'immerge minimalement dans une sphère alors

$$V(M,g) \geqq V\left(M, \frac{\lambda_1(can)}{m}\, can\right)$$

où l'égalité n'a lieu que si $g = \frac{\lambda_1(can)}{m}\, can$.

Pour ces variétés on a le tableau suivant:

	$\mathbf{S}^m$	$\mathbf{R}P^m$	$\mathbf{C}P^d$	$\mathbf{H}P^d$	$\mathbf{C}_aP^2$	$\mathbf{S}^p(\sqrt{p/m})\times\mathbf{S}^q(\sqrt{q/m})$	$\mathbf{T}^2_{eq}$
can)	m	$2(m+1)$	$4(d+1)$	$8(d+1)$	48	$m=p+q$	$\frac{16\pi^2}{3}$
can)	ω_m	$\frac{\omega_m}{2}$	$\frac{\pi^d}{d!}$	$\frac{\pi^{2d}}{(2d+1)!}$	$\frac{6\pi^8}{11!}$	$\left(\frac{p^pq^q}{m^m}\right)^{1/2}\omega_p\omega_q$	$\frac{\sqrt{3}}{2}$
	ω_m	$\left(\frac{2(m+1)}{m}\right)^{m/2}\frac{\omega_m}{2}$	$\left(\frac{2(d+1)}{d}\right)^d\frac{\pi^d}{d!}$	$\left(\frac{2(d+1)}{d}\right)^{2d}\frac{\pi^{2d}}{(2d+1)!}$	$6\frac{(3\pi)^8}{11!}$	$\left(\frac{p^pq^q}{m^m}\right)^{1/2}\omega_p\omega_q$	$\frac{4\sqrt{3}}{3}\pi^2$

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Mathematische Annalen

Über polynomiale, insbesondere Riccatische, Differentialgleichungen mit Fundamentallösungen

Sebastian Walcher

Mathematisches Institut der Technischen Universität München, Arcisstraße 21, D-8000 München 2, Bundesrepublik Deutschland

1. Einleitung

Um die allgemeine Lösung einer linearen gewöhnlichen Differentialgleichung zu bestimmen, genügen endlich viele Lösungen. Auch die Riccatische Differentialgleichung in $\mathbb{R}$ besitzt diese Eigenschaft, wie aus der Konstanz des Doppelverhältnisses von vier Lösungen zu ersehen ist. Allgemeiner gilt dies für die Matrix-Riccati-Gleichung in $\mathbb{R}^{(n,n)}$; s. hierzu Levin [6], Reid [10] oder die neuere Arbeit von Harnad et al. [2].

Die genannten Differentialgleichungen besitzen also die „Fundamentallösungs-Eigenschaft“: Aus endlich vielen, „allgemein gelegenen“ Lösungen läßt sich (fast) jede weitere bestimmen.

Für eine beliebige Riccati-Differentialgleichung auf dem $\mathbb{R}^n$ $(n>1)$ gibt es kein entsprechendes Ergebnis; warum dies so ist, kommt als ein Resultat aus dieser Arbeit, die sich mit polynomialen Differentialgleichungen mit Fundamentallösungen befaßt.

Spezialisierung eines Satzes von Lie [7] auf diesen Fall zeigt, daß die polynomialen Differentialgleichungen mit Fundamentallösungen auf einem endlichdimensionalen $\mathbb{R}$-Vektorraum V gerade den endlichdimensionalen Unteralgebren der Lie-Algebra $\mathcal{P}o\ell\, V$ aller Polynomabbildungen von V in sich entsprechen.

Für $\dim V = 1$ lassen sich letztere vollständig bestimmen; für $\dim V > 1$ erscheint dieses Vorhaben ziemlich aussichtslos. Immerhin erhalten wir unter Benutzung von Resultaten von Kantor [3], Koecher [4] und Meyberg [9] Aussagen über spezielle Klassen von Unteralgebren und insbesondere eine Charakterisierung einer Klasse, der „transitiven“ Unteralgebren. Die hierzu gehörenden Differentialgleichungen lassen sich weitgehend analog zur Riccati-Gleichung in $\mathbb{R}$ behandeln. Dies wird im Schlußabschnitt ausgeführt.

2. Gewöhnliche Differentialgleichungen mit Fundamentallösungen

Zunächst zur Definition: Es sei V ein endlichdimensionaler $\mathbb{R}$-Vektorraum, $U \subset V$ offen und nichtleer, $I \subset \mathbb{R}$ offenes, nichtleeres Intervall.

$F: I \times U \to V$, $(t, x) \mapsto F(t, x)$ sei stetig und für jedes $t_0 \in I$ sei $x \mapsto F(t_0, x)$ analytisch. (Insbesondere genügt F damit einer lokalen Lipschitz-Bedingung bezüglich x.) Vorgelegt sei nun die Differentialgleichung $\dot{x} = F(t, x)$ auf U.

Wir nennen sie *Differentialgleichung mit Fundamentallösungen*, wenn Folgendes erfüllt ist:

Es gibt ein $m \in \mathbb{N}$, offene, nichtleere $G, G_1, \ldots, G_m \subset U$, $G' \subset V$, und eine analytische Abbildung $\Phi: G_1 \times \ldots \times G_m \times G' \to V$, so daß gilt:

Sind $z_1, \ldots, z_m$ Lösungen der Differentialgleichung auf einem offenen Intervall $I_1, t_0 \in I_1$, die $z_i(t_0)$ paarweise verschieden, $z_i(t) \in G_i$ für alle $t \in I_1$, so gibt es zu jedem $y \in G$ ein $b \in G'$ mit:

$\Phi(z_1(t), \ldots, z_m(t), b)$ ist die Lösung der Differentialgleichung zum Anfangswert y bei $t = t_0$.

Vor der Formulierung des Satzes von Lie, der die Differentialgleichungen mit Fundamentallösungen charakterisiert, noch etwas Terminologie:

Es sei $\mathscr{A}(U, V)$ der Vektorraum aller analytischen Abbildungen von U in V. Für $f, g \in \mathscr{A}(U, V)$ wird durch

$$[f, g](x) := Dg(x) \cdot f(x) - Df(x) \cdot g(x)$$

ein Element $[f, g]$ von $\mathscr{A}(U, V)$ erklärt. Bekanntlich ist $(\mathscr{A}(U, V), [\cdot, \cdot])$ eine Lie-Algebra.

Satz 1 (Lie [7]). *Voraussetzungen an I, U, F wie oben.*

Die Differentialgleichung $\dot{x} = F(t, x)$ besitzt Fundamentallösungen genau dann, wenn gilt:

Es gibt ein $r \in \mathbb{N}$, $f_1, \ldots, f_r \in \mathscr{A}(U, V)$, $\lambda_1, \ldots, \lambda_r \in C^0(I, \mathbb{R})$, so daß auf $I \times U$

$$F(t, x) = \sum_{i=1}^{r} \lambda_i(t) f_i(x)$$

erfüllt ist und $f_1, \ldots, f_r$ eine endlichdimensionale Unteralgebra von $\mathscr{A}(U, V)$ erzeugen. □

Die Polynomabbildungen von V in sich bilden offenbar eine Unteralgebra $\mathscr{P}\!ol\, V$ von $\mathscr{A}(V, V)$. Damit folgt das

Korollar. *Ist $F(t_0, x)$ Polynom in x für alle $t_0 \in I$, so besitzt die Differentialgleichung Fundamentallösungen genau dann, wenn $f_1, \ldots, f_r$ eine endlichdimensionale Unteralgebra von $\mathscr{P}\!ol\, V$ erzeugen.* □

Bemerkung 1. Jedem $p \in \mathscr{A}(U, V)$ ist (bezüglich einer Basis) das analytische Vektorfeld $\hat{p} = \sum_{i=1}^{n} p_i \frac{\partial}{\partial x_i}$ zugeordnet. Die Bedingung in Satz 1 besagt gerade, daß $\hat{f}_1, \ldots, \hat{f}_r$ eine endlichdimensionale Lie-Algebra $\mathscr{L}$ von analytischen Vektorfeldern erzeugen. Nach dem dritten Lieschen Satz (s. etwa [11]) gibt es dann eine lokale Transformationsgruppe auf U mit Lie-Algebra $\mathscr{L}$.

Bemerkung 2. Nach Satz 1 besitzt insbesondere jede autonome Gleichung $\dot{x} = F(x)$ Fundamentallösungen. Ein Blick auf den Beweis zeigt, warum dies so ist: Das Produktsystem

$$\begin{aligned} \dot{x} &= F(x) \\ \dot{y} &= F(y) \end{aligned} \quad \text{auf} \quad U \times U \subset V \times V$$

besitzt auf einem offenen $W \subset U \times U$ $2n-1$ unabhängige analytische erste Integrale $\psi_1, \ldots, \psi_{2n-1}$ ($n = \dim V$), von denen angenommen werden darf, daß

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial x_1} \cdots \frac{\partial \psi_1}{\partial x_n} \\ \vdots \qquad \vdots \\ \frac{\partial \psi_n}{\partial x_1} \cdots \frac{\partial \psi_n}{\partial x_n} \end{pmatrix}$$

auf $W' \subset W$ ($\emptyset \neq W'$ offen) invertierbar ist.

Die Fundamentallösungs-Eigenschaft folgt nun mit Hilfe des Satzes über implizite Funktionen (für Details s. [12]). Wir sehen, daß der praktische Wert dieser Erkenntnis gering ist. (Man bedenke vor allem, wie man an die ersten Integrale kommt!)

Zurück zum Problem der endlichdimensionalen Unteralgebren von $\mathscr{P}ol\, V$, wobei die eindimensionalen trivialerweise bekannt sind. Wir wollen es wenigstens für $V = \mathbb{R}$ vollständig lösen: Zunächst bilden offenbar die Polynome vom Grad $\leqq 2$ eine dreidimensionale Unteralgebra von $\mathscr{P}ol\, \mathbb{R}$.

Es sei nun $\mathscr{L}$ eine Unteralgebra von $\mathscr{P}ol\, \mathbb{R}$, $1 < \dim \mathscr{L} < \infty$, und $\max\{\mathrm{Grad} f \mid f \in \mathscr{L}\} =: m > 2$. Es sei $f \in \mathscr{L}$ normiert vom Grad m, weiter $g \in \mathscr{L}$ normiert, $g \notin \mathbb{R}f$, $\mathrm{Grad}\, g =: r \leqq m$. Dann gilt: $[g, f] = (m-r)x^{m+r-1} + \ldots \in \mathscr{L}$, damit $r \in \{0, 1, m\}$. $\mathrm{Grad}\, g = 0$ hat $\mathrm{Grad}[f, [f, g]] = 2m-2 > m$ zur Folge; Widerspruch. Für $\mathrm{Grad}\, g = m$ ist $\mathrm{Grad}(f-g) < m$, also $\mathrm{Grad}(f-g) = 1$. $\mathscr{L}$ enthält also ein (und nur ein) normiertes Grad-1-Polynom h und $\mathscr{L} = \mathbb{R}f + \mathbb{R}h$. Speziell ist $[h, f] = (m-1)(f + \sigma h)$ mit einem $\sigma \in \mathbb{R}$. Ersetze f durch $f + \sigma h$; dies ergibt $[h, f] = (m-1)f$.

Wir werten dies aus: Sei $f(x) = x^m + \sum_{i=1}^{m} \alpha_{m-i} x^{m-i}$, $h(x) = x + \gamma$. Dann folgt

$$(m-1)f = [h, f] = (m-1)x^m + \sum_{i=1}^{m} ((m-i-1)\alpha_{m-i} + (m-i+1)\alpha_{m-i+1}\gamma)x^{m-i}$$

(dabei $\alpha_m := 1$). Koeffizientenvergleich ergibt für $1 \leqq i \leqq m$: $i\alpha_{m-i} = (m-i+1)\alpha_{m-i+1}\gamma$.

Nun folgt leicht mit Induktion $\alpha_{m-i} = \binom{m}{i}\gamma^i$ und $f = (x+\gamma)^m$. Als Ergebnis halten wir fest:

Satz 2. *Die einzigen (nichttrivialen) polynomialen Differentialgleichungen auf $\mathbb{R}$ mit Fundamentallösungen haben die Form*

$$\dot{x} = \lambda_1(t)x^2 + \lambda_2(t)x + \lambda_3(t)$$

(Riccatische incl. lineare Differentialgleichung)

oder mit $m > 2$

$$\dot{x} = \mu_1(t)(x+\gamma)^m + \mu_2(t)(x+\gamma)$$

(Bernoullische Differentialgleichung)

(Dabei $\gamma \in \mathbb{R}$, λ_i, μ_j stetig auf einem Intervall I.) □

Es handelt sich also um alte Bekannte. Für $\dim V > 1$ werden die Verhältnisse komplizierter.

3. Endlichdimensionale, graduierte Unteralgebren von $\mathscr{P}o\ell\, V$

3.1. Allgemeines

Für diesen Abschnitt sei K ein Körper der Charakteristik Null (die Einschränkung $K=\mathbb{R}$ brächte keinen Vorteil), V ein endlichdimensionaler K-Vektorraum und $\mathscr{P}o\ell\, V$ der Vektorraum aller Polynomabbildungen von V in sich.

Bezeichnet man den Unterraum aller homogenen Polynome vom Grad $k+1$ mit $\mathscr{P}_k$ $(k\geqq -1)$ und setzt $\mathscr{P}_k:=\{0\}$ für $k<-1$, so gilt $\mathscr{P}o\ell\, V=\bigoplus\limits_{k\in\mathbb{Z}}\mathscr{P}_k$.

Speziell ist $\mathscr{P}_{-1}$ die Menge aller konstanten Abbildungen, kann also mit V identifiziert werden, und es ist $\mathscr{P}_0=\mathrm{Hom}(V,V)$.

Für $p,q\in\mathscr{P}o\ell\, V$ ist durch

$$[p,q](x):=Dq(x)\cdot p(x)-Dp(x)\cdot q(x)$$

ein Element $[p,q]\in\mathscr{P}o\ell\, V$ definiert und $(\mathscr{P}o\ell\, V,[\cdot,\cdot])$ ist eine Lie-Algebra. Wir können die Klammerverknüpfung für homogene Polynome etwas genauer beschreiben: Ist p homogen vom Grad $k\geqq 1$, so gibt es genau eine k-fach lineare, symmetrische Abbildung $\tilde{p}: V^k\to V$ mit $\tilde{p}(x,\dots,x)=p(x)$ für alle $x\in V$. Wir werden p und $\tilde{p}$ in der Bezeichnung nicht unterscheiden.

Für $p\in\mathscr{P}_l$, $q\in\mathscr{P}_m$ und $l,m\geqq 0$ ist dann

$$[p,q](x)=(m+1)q(x,\dots,x,p(x))-(l+1)p(x,\dots,x,q(x)).$$

Für $a\in V$ gilt

$$[a,p](x)=(l+1)p(x,\dots,x,a).$$

Also ist für alle i,j $[\mathscr{P}_i,\mathscr{P}_j]\subset\mathscr{P}_{i+j}$ und $\mathscr{P}o\ell\, V$ ist eine graduierte Lie-Algebra. Es geht nun um die Bestimmung der endlichdimensionalen Unteralgebren von $\mathscr{P}o\ell\, V$. Für $\dim V>1$ erscheint eine vollständige Bestimmung ziemlich aussichtslos; wir werden deshalb Einschränkungen vornehmen, deren erste die folgende ist:

Die endlichdimensionale Unteralgebra $\mathscr{L}\neq\{0\}$ von $\mathscr{P}o\ell\, V$ erbe die Graduierung von $\mathscr{P}o\ell\, V$, also

$$\mathscr{L}=\mathscr{L}_{-1}\oplus\dots\oplus\mathscr{L}_m\quad(\mathscr{L}_i\subset\mathscr{P}_i,\mathscr{L}_m\neq\{0\}).$$

Abkürzend werde gesagt, daß $\mathscr{L}$ graduiert ist; m heiße der Grad von $\mathscr{L}$.

Ist $\mathscr{I}$ Ideal von $\mathscr{L}$, so ist $\mathscr{I}$ nicht notwendig graduiert. Es gilt jedoch das

Lemma. *Ist $\mathscr{I}$ Ideal von $\mathscr{L}$, $\{0\}\neq\mathscr{I}\neq\mathscr{L}$, so enthält $\mathscr{L}$ ein graduiertes Ideal $\tilde{\mathscr{I}}$, $\{0\}\neq\tilde{\mathscr{I}}\neq\mathscr{L}$.*

Beweis. Setze

$$\tilde{\mathscr{I}}_k:=\{g\in\mathscr{L}_k\,|\,\exists p=p_k+\dots+p_m\in\mathscr{I}\text{ mit }g=p_k\}$$

für $-1\leqq k\leqq m$ und $\tilde{\mathscr{I}}:=\tilde{\mathscr{I}}_{-1}\oplus\dots\oplus\tilde{\mathscr{I}}_m$. □

Es sollen noch die halbeinfachen graduierten Unteralgebren vom Grad >0 charakterisiert werden; sowohl der folgende Satz wie sein Beweis sind bekannt:

Satz 3. *$\mathcal{L}=\mathcal{L}_{-1}\oplus \dots \oplus\mathcal{L}_m$ sei halbeinfach und $m>0$. Dann ist $m=1$.*

Beweis. Es sei $m\geqq 2$, $p\in\mathcal{L}_m$. Für $q\in\mathcal{L}_k$ $(-1\leqq k\leqq m)$ bildet $ad p\, ad q\, \mathcal{L}_i$ in $\mathcal{L}_{i+m+k}$ ab; wegen $m+k>0$ ist $ad p\, ad q$ nilpotent. Für die Killingform κ heißt dies: $\kappa(p,q) = \mathrm{Spur}\, ad p\, ad q=0$. Also $\kappa(\mathcal{L}_m, \mathcal{L})=0$, κ ist ausgeartet und $\mathcal{L}$ nicht halbeinfach. □

3.2. *Reduzibilität*

Es sei $\mathcal{L}=\mathcal{L}_{-1}\oplus \dots \oplus\mathcal{L}_m$ graduierte Unteralgebra von $\mathcal{P}ol\, V$. $\mathcal{L}$ heißt *reduzibel*, wenn es einen Unterraum U von $\mathcal{L}_{-1}$, $\{0\}\neq U\neq V$, gibt, so daß für alle k, $0\leqq k\leqq m$, und alle $p\in\mathcal{L}_k$ gilt: $p(V,\dots,V,U)\subset U$.

U ist also Ideal für jede k-fache, symmetrische Verknüpfung, die durch die $p\in\mathcal{L}_{k-1}$ gegeben ist (bzw. invarianter Unterraum für $k=0$).

Für den Fall $\mathcal{L}_{-1}=V$ stammt diese Definition von Kantor [3]. Hier reicht es, $[\mathcal{L}_0, U]\subset U$ zu fordern.

Lemma. *Es sei $\mathcal{L}$ reduzibel, U wie oben. Dann läßt sich die kanonische Projektion $V\to V/U$ fortsetzen zu einem Homomorphismus $f:\mathcal{L}\to\mathcal{P}ol(V/U)$ durch $f(p)=\bar p$ mit $\bar p(x+U):=p(x)+U$.*

Beweis. Die Wohldefiniertheit von f folgt aus der Definition der Reduzibilität. Die Linearität ist klar, deshalb reicht es, $[\bar p,\bar q]=\overline{[p,q]}$ für homogene Polynome zu zeigen. Dies aber rechnet man unmittelbar nach. □

3.3. *Transitive Unteralgebren*

Eine graduierte Unteralgebra $\mathcal{L}$ von $\mathcal{P}ol\, V$ heißt transitiv, wenn $\mathcal{L}_{-1}=V$ (s. Kantor [3]). In der Folge werden endlichdimensionale transitive Unteralgebren von $\mathcal{P}ol\, V$ betrachtet; zur Motivation hierfür sei auf den eindimensionalen Fall verwiesen.

Daß nicht jede Unteralgebra $\mathcal{B}$ von $\mathcal{P}ol\, V$ mit $V\subset\mathcal{B}$ graduiert ist, zeigt

Beispiel 1. Es sei $V=K^2$, $p\in\mathcal{P}_1$ bzw. $C\in\mathcal{P}_0$ durch $p(x):=\begin{pmatrix}x_1^2\\0\end{pmatrix}$ bzw. $Cx:=\begin{pmatrix}0\\x_1\end{pmatrix}$ definiert. Dann erzeugen $C+p$ und V eine endlichdimensionale Unteralgebra von $\mathcal{P}ol\, V$, wie aus $[[a,p],C+p]=2a_1(C+p)$ für alle $a\in V$ folgt.

Sie ist nicht graduiert; darüber hinaus gibt es auch keine endlichdimensionale, graduierte Unteralgebra von $\mathcal{P}ol\, V$, die C und p enthält. Dies ergibt sich aus $((ad\, p)^k C)(x)=(k+1)\begin{pmatrix}0\\x_1^k\end{pmatrix}$ für alle $k\geqq 1$.

Daß es für $\dim V>1$ transitive Unteralgebren vom Grad $m>1$ gibt, illustriert

Beispiel 2. Auf V sei eine kommutative Algebra vom Nilindex $m+1$ gegeben (jedes Produkt von $m+1$ Faktoren – unabhängig von der Klammerung – ist also Null). Nun sei $\mathcal{L}_{-1}=V$ und für $0\leqq k<m$ werde $\mathcal{L}_k$ aufgespannt von allen Abbildungen der Gestalt $x\mapsto F(x,\dots,x)$ bzw. $x\mapsto G(x,\dots,x,a_1,\dots,a_r)$, wobei F und G irgendein $(k+1)$- bzw. $(k+1+r)$-faches Produkt in der Algebra symbolisieren, $k+r<m$ und $a_1,\dots,a_r\in V$.

Für $\mathscr{L}:=\mathscr{L}_{-1}\oplus\ldots\oplus\mathscr{L}_{m-1}$ ist dann klar: $[\mathscr{L}_i,\mathscr{L}_j]\subset\mathscr{L}_{i+j}$ für $i+j<m$ bzw. $[\mathscr{L}_i,\mathscr{L}_j]=\{0\}$ für $i+j\geqq m$, denn für $p\in\mathscr{L}_i$, $q\in\mathscr{L}_j$ gilt im zweiten Fall schon

$$p(x,\ldots,x,q(x))=q(x,\ldots,x,p(x))=0\,.$$

Im allgemeinen wird auch $\mathscr{L}_{-1}\oplus\mathscr{L}_0\oplus\mathscr{L}_1$ keine Unteralgebra von $\mathscr{L}$ sein: Es seien $p,q\in\mathscr{L}_1$ mit $p(x)=x^2$, $q(x)=x(xa)$ $(a\in V)$. Dann ist

$$[q,p](x)=2x(x(xa))-x(x^2a)-x^2(xa)\,.$$

Aus $[q,p]=0$ folgt insbesondere $a^4=a^2a^2$. Ist die Algebra also nicht potenzassoziativ, so ist $\mathscr{L}_{-1}\oplus\mathscr{L}_0\oplus\mathscr{L}_1$ keine Unteralgebra von $\mathscr{L}$.

Allerdings sind transitive Unteralgebren „im wesentlichen" vom Grad 1. Vor dem entsprechenden Satz noch ein

Lemma (Koecher [4]). *$\mathscr{L}$ sei transitive Unteralgebra von $\mathscr{P}o\ell\,V$, $\mathscr{I}=\mathscr{I}_{-1}\oplus\ldots\oplus\mathscr{I}_m$ ein graduiertes Ideal. Ist $\mathscr{I}\neq\{0\}$, so auch $\mathscr{I}_{-1}\neq\{0\}$.*

Beweis. Angenommen, $\mathscr{I}_{-1}=\{0\}$. Setze $k:=\min\{l|\mathscr{I}_l\neq\{0\}\}$. Dann ist

$$[\mathscr{L}_{-1},\mathscr{I}_k]=\{0\}\ \Rightarrow\ p(x,\ldots,a)=0$$

für alle $p\in\mathscr{I}_k$, $x,a\in V$ $\Rightarrow$ $\mathscr{I}_k=\{0\}$ und Widerspruch. □

Satz 4 (Kantor [3]). *$\mathscr{L}$ sei transitiv,* Grad $\mathscr{L}>1$. *Dann ist $\mathscr{L}$ reduzibel.*

Beweis. Nach Satz 3 ist $\mathscr{L}$ nicht halbeinfach, besitzt also ein von Null verschiedenes abelsches Ideal. Das hieraus gemäß 3.1, Lemma, gewonnene Ideal $\mathscr{I}=\mathscr{I}_{-1}\oplus\ldots\oplus\mathscr{I}_m$ ist dann ebenfalls abelsch und nicht Null, also $\mathscr{I}_{-1}\neq\{0\}$, und $[\mathscr{L}_0,\mathscr{I}_{-1}]\subset\mathscr{I}_{-1}$. Die Annahme, daß $\mathscr{L}$ irreduzibel ist, führt auf $\mathscr{I}_{-1}=V$. Dann ist aber für $k\geqq1$ und alle $p\in\mathscr{L}_k$, $a,b\in V$

$$[b,[a,p]]\in[\mathscr{I}_{-1},[\mathscr{I}_{-1},\mathscr{L}_k]]\subset[\mathscr{I}_{-1},\mathscr{I}_{k-1}]=\{0\}$$

oder $p(x,\ldots,x,a,b)=0$ für alle $x,a,b\in V$, somit $p=0$ und Widerspruch. □

Bemerkung. Kantor gibt in [3] keinen Beweis für diesen Satz an.

Es bleiben also die transitiven Unteralgebren vom Grad 1 zu behandeln. Es wird sich zeigen, daß diese aus Jordan-Paaren kommen.

Zunächst zur Definition (vgl. Loos [8]).

Es seien V^+,V^- K-Vektorräume, $Q_+:V^+\to\mathrm{Hom}(V^-,V^+)$ und $Q_-:V^-\to\mathrm{Hom}(V^+,V^-)$ quadratische Abbildungen; weiter setze für $\sigma\in\{+,-\}$, $x,y\in V^\sigma$, $z\in V^{-\sigma}$.

$$Q_\sigma(x,y):=(Q_\sigma(x+y)-Q_\sigma(x)-Q_\sigma(y))\,;\qquad D_\sigma(x,z)y:=Q_\sigma(x,y)z\,.$$

$(V,Q):=(V^+,Q_+;V^-,Q_-)$ heißt *Jordan-Paar*, wenn gilt

$$D_\sigma(x,y)Q_\sigma(x)=Q_\sigma(x)D_{-\sigma}(y,x)\qquad\text{(JP 1)}$$

$$D_\sigma(Q_\sigma(x)y,y)=D_\sigma(x,Q_{-\sigma}(y)x)\qquad\text{(JP 2)}$$

$$Q_\sigma(Q_\sigma(x)y)=Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x)\qquad\text{(JP 3)}$$

für $\sigma\in\{+,-\}$ und alle $x\in V^\sigma$, $y\in V^{-\sigma}$.

Aus den definierenden Identitäten folgen unter anderem

$$D_\sigma(x,y)Q_\sigma(x)=Q_\sigma(x,Q_\sigma(x)y) \tag{JP 4}$$

$$Q_\sigma(x,D_\sigma(y,z)x)=D_\sigma(y,z)Q_\sigma(x)+Q_\sigma(x)D_{-\sigma}(z,y) \tag{JP 12}$$

Wichtig ist, daß in unserem Fall ($\operatorname{char} K \notin \{2,3\}$) aus (JP 12) [bzw. dessen linearisierter Version (JP 14)] schon (JP 1, 2, 3) folgen. (Zum Beweis s. [8], Ch. I, Sect. 2.) Weiter definieren wir

$$\mathscr{H}(V^\sigma,Q_\sigma)=\{B\in \operatorname{Hom}(V^\sigma,V^\sigma)|\exists C\in \operatorname{Hom}(V^{-\sigma},V^{-\sigma}) \text{ mit}$$
$$Q_\sigma(x,Bx)=BQ_\sigma(x)+Q_\sigma(x)C\}$$

$\mathscr{H}(V^\sigma,Q_\sigma)$ ist eine Lie-Algebra; aus (JP 12) folgt $D_\sigma(y,z)\in\mathscr{H}(V^\sigma,Q_\sigma)$ für alle y,z. Schließlich setzen wir $q_a(x):=Q_+(x)a$ ($x\in V^+, a\in V^-$). Nun gilt

Satz 5. *Es sei ein endlichdimensionales Jordan-Paar gegeben. Setze* $\mathscr{L}_{-1}:=V^+$, $\mathscr{L}_0:=\mathscr{H}(V^+,Q_+)$, $\mathscr{L}_1:=\{q_a|a\in V^-\}$. *Dann ist* $\mathscr{L}:=\mathscr{L}_{-1}\oplus\mathscr{L}_0\oplus\mathscr{L}_1$ *transitive Unteralgebra von* $\mathscr{P}\!\mathit{ol}\, V^+$.

Beweis. Es genügt, $[\mathscr{L}_i,\mathscr{L}_j]\subset\mathscr{L}_{i+j}$ für $0\leqq i+j\leqq 1$ und $[\mathscr{L}_1,\mathscr{L}_1]=\{0\}$ zu zeigen. Letzteres folgt direkt aus (JP 4); $[\mathscr{L}_0,\mathscr{L}_0]\subset\mathscr{L}_0$ folgt aus der Vorbemerkung, $[\mathscr{L}_0,\mathscr{L}_1]\subset\mathscr{L}_1$ aus der Definition von $\mathscr{H}(V^+,Q_+)$ und $[\mathscr{L}_{-1},\mathscr{L}_1]\subset\mathscr{L}_0$ mit (JP 12). □

Satz 6. $\mathscr{L}$ *sei transitive Unteralgebra von* $\mathscr{P}\!\mathit{ol}\, V$, $\operatorname{Grad}\mathscr{L}=1$. *Dann ist* $\mathscr{L}$ *Unteralgebra einer Algebra des in Satz 5 angegebenen Typs.*

Beweis. Definiere trilineare Verknüpfungen

$$\mathscr{L}_{-1}\times\mathscr{L}_1\times\mathscr{L}_{-1}\to\mathscr{L}_{-1};\ (a,p,b)\mapsto\langle apb\rangle:=[a,[b,p]],$$

$$\mathscr{L}_1\times\mathscr{L}_{-1}\times\mathscr{L}_1\to\mathscr{L}_1;\ (p,a,q)\mapsto\langle paq\rangle:=[p,[q,a]].$$

Diese sind jeweils im ersten und dritten Argument symmetrisch. Der entscheidende Teil des Beweises stammt von Meyberg [9]:

$$\langle b\langle paq\rangle c\rangle=\langle bq\langle apc\rangle\rangle+\langle cq\langle apb\rangle\rangle-\langle ap\langle bqc\rangle\rangle \tag{*}$$

für alle $a,b,c\in\mathscr{L}_{-1}$, $p,q\in\mathscr{L}_1$ und ebenso mit vertauschten Rollen. (Dies ist in beliebigen Grad-1-Unteralgebren richtig.) Nun sei $V^+:=V$, $V^-:=K^s$ mit $s:=\dim\mathscr{L}_1$ und $\psi: V^-\to\mathscr{L}_1$ ein Vektorraum-Isomorphismus.

Definiere nun trilineare Verknüpfungen

$$V^+\times V^-\times V^+\to V^+;\ (a,u,b)\mapsto\{aub\}:=\langle a,\psi(u),b\rangle,$$

$$V^-\times V^+\times V^-\to V^-;\ (u,a,v)\mapsto\{uav\}:=\psi^{-1}\langle\psi(u),a,\psi(v)\rangle.$$

Setzen wir $Q_+(x)u:=\{xux\}$, $Q_-(u)x:=\{uxu\}$ für $x\in V^+, u\in V^-$, so gilt wegen (*) die Identität (JP 14) aus [8]; also ist (V,Q) Jordan-Paar und $p(x)=Q_+(x)\psi^{-1}(p)$ für alle $p\in\mathscr{L}_1$, daher $\mathscr{L}_1=\{q_a|a\in V^-\}$.

Weil ψ Isomorphismus ist, gilt: $q_a=0\Rightarrow a=0$. Ist $B\in\mathscr{L}_0$, so folgt deshalb wegen $[\mathscr{L}_0,\mathscr{L}_1]\subset\mathscr{L}_1$: Für alle $a\in V^-$ gibt es genau ein $b=Ca\in V^-$ mit $Q_+(x,Bx)a-BQ_+(x)a=Q_+(x)b$ für alle $x\in V^+$. $C: V^-\to V^-$ is linear (ebenfalls wegen: $q_a=0\Rightarrow a=0$); also $Q_+(x,Bx)=BQ_+(x)+Q_+(x)C$ und $B\in\mathscr{H}(V^+,Q_+)$. □

Bemerkung 1. Der letzte Teil des Beweises stammt sinngemäß von Koecher [4].

Bemerkung 2. Der Beweis zeigt, daß man Q_+ auf simple Weise erhält: Ist $q_1, \dots, q_s$ eine K-Basis von $\mathcal{L}_1$, so gilt

$$Q_+(x)\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_s\end{pmatrix} = \sum_{i=1}^{s} \alpha_i q_i(x).$$

3.4. Der Fall der Dimension 2 *(K* $=\mathbb{R}$ *oder* $\mathbb{C}$, Grad $\mathcal{L} > 0$*)*

Die einfachen (und damit die halbeinfachen) Jordan-Paare über einem algebraisch abgeschlossenen Körper wurden von Loos in [8] klassifiziert. Nun ergibt sich aus den Definitionen leicht, daß ein Jordan-Paar genau dann (halb-) einfach ist, wenn dies für die nach Satz 5 daraus gewonnene transitive Lie-Algebra zutrifft. Für $K=\mathbb{C}$ sind die halbeinfachen transitiven endlichdimensionalen Lie-Algebren damit bekannt; für $K=\mathbb{R}$ erhält man sie als Reellifizierung oder reelle Form der komplexen (s. [11]; man beachte auch, daß für einen reellen Vektorraum V und $\mathcal{L} \subset \mathcal{P}ol\, V$ die Algebra $\mathbb{C} \otimes \mathcal{L}$ als Unteralgebra von $\mathcal{P}ol\,(C \otimes V)$ aufgefaßt werden kann.) Es bleiben also die nicht halbeinfachen transitiven Algebren zu bestimmen, die auch nicht Unteralgebra einer halbeinfachen sind.

Ist $m := \operatorname{Grad} \mathcal{L} > 1$ und sind $p \in \mathcal{L}_m$, $q \in \mathcal{L}_k$ $(k>1)$, so gilt für alle $a, x \in V$:

$$0 = [[a,p],q] \;\Rightarrow\; (k+1)q(x,\dots,x,p(x,\dots,x,a)) = mp(x,\dots,x,q(x),a).$$

Andererseits

$$0 = [p,q] \;\Rightarrow\; (k+1)q(x,\dots,x,p(x)) = (m+1)p(x,\dots,x,q(x)).$$

Einsetzen von $a=x$ in die erste Identität und Vergleich zeigt

$$p(x,\dots,x,q(x)) = q(x,\dots,x,p(x)) = 0;$$

speziell $p(x,\dots,x,p(x)) = 0$.

Es sollen nun für $\dim V = 2$ ($K=\mathbb{R}$ oder $\mathbb{C}$) alle maximalen transitiven Unteralgebren von $\mathcal{P}ol\, V$ von einem festen Grad m bestimmt werden. Dazu wird ein Resultat von Dorfmeister und Heinze [13] herangezogen:

Ist $p \in \mathcal{P}ol\, V$ homogen vom Grad $l \geqq 1$ und $p(x,\dots,x,p(x)) = 0$, so gilt $p(x) = \lambda(x)^l a$ $[a \in V,\ \lambda \in V^*$ mit $\lambda(a) = 0]$.

Es sei nun $m > 1$ und $p \in \mathcal{L}_m \backslash \{0\}$. Dann gilt $p(x) = \lambda(x)^{m+1} a$ (λ, a wie oben). Elementare Rechnung zeigt $\mathcal{L}_m = \langle p \rangle$, $\mathcal{L}_i = \langle p_i \rangle$ $(1 \leqq i < m)$, $\mathcal{L}_0 = \langle Id, p_0 \rangle$ und $\mathcal{L}_{-1} = V$. [Hierbei gilt $p_i(x) = \lambda(x)^{i+1} a$ für $0 \leqq i < m$.]

Für $m=1$, $\mathcal{L}$ nicht halbeinfach, besitzt $\mathcal{L}$ ein abelsches Ideal

$$\mathcal{I} = \mathcal{I}_{-1} \oplus \mathcal{I}_0 \oplus \mathcal{I}_1 \neq \{0\}.$$

Für $\mathcal{I}_1 \neq \{0\}$, $p \in \mathcal{I}_1 \backslash \{0\}$, folgt aus $[[b,p],p] = 0$ für alle $b \in V$ wieder $p(x, p(x)) = 0$ und $p(x) = \lambda(x)^2 a$ (λ, a wie oben). Aus $r \in \mathcal{P}_1$, $[p,r] = 0$ ergibt direkte Rechnung $r \in \langle p, q \rangle$; dabei $q(x) = \lambda(x)^2 c + 2\lambda(x)\mu(x) a$ $[c \in V,\ \mu \in V^*$ mit $\mu(c) = 0$, $\lambda(c) = \mu(a) = 1]$. Die durch q definierte kommutative Algebra auf V ist isomorph zur Algebra $K[\varepsilon]$ der Dualzahlen über K. Mit $\mathcal{L}_1 = \langle p, q \rangle$ ist $\mathcal{L} = V \oplus \mathcal{L}_0 \oplus \mathcal{L}_1$; $\mathcal{L}_0$ bestimmt sich aus $[\mathcal{L}_0, \mathcal{L}_1] \subset \mathcal{L}_1$. Eine elementare Rechnung (unter Benutzung der Tatsache, daß jedes $p \in \mathscr{L}_1$ eine Jordan-Algebra auf V definiert) zeigt, daß $\mathscr{L}$ im Fall $\mathscr{I}_1 = \{0\}$ nicht maximal ist.

Es bleiben die halbeinfachen transitiven Algebren:

Für $K = \mathbb{C}$ gibt es nur 2 Typen, die gemäß Satz 5 aus den Jordan-Paaren des Typs $I_{1,1} \oplus I_{1,1}$ bzw. $I_{1,2}$ kommen (vgl. [8]).

Für $K = \mathbb{R}$ treten als reelle Formen der o. g. Typen wieder nur die Algebren auf, die aus Jordan-Paaren des Typs $I_{1,1} \oplus I_{1,1}$ bzw. $I_{1,2}$ über $\mathbb{R}$ entstehen; dazu kommt die Reellifizierung der Lie-Algebra des eindimensionalen komplexen einfachen Jordan-Paars.

4. Eine Klasse von Differentialgleichungen mit Fundamentallösungen

Nun sei V wieder ein endlichdimensionaler Vektorraum über $\mathbb{R}$. Wir wollen die Differentialgleichungen mit Fundamentallösungen auf V näher betrachten, die transitiven Unteralgebren von $\mathscr{P}\!ol\, V$ entsprechen. (Zur Motivation bedenke man den eindimensionalen Fall.)

Es seien also $p_1, \dots, p_r$ homogene Polynome, so daß

$$\dot{x} = v(t) + \sum_{i=1}^{r} \lambda_i(t) p_i(x) \tag{1}$$

für alle stetigen $v: I \to V$ und $\lambda_1, \dots, \lambda_r: I \to \mathbb{R}$ die Fundamentallösungs-Eigenschaft besitzt. (I bezeichnet hier und in der Folge ein offenes Intervall, $I \neq \emptyset$.)

4.1. *Reduzibilität*

Es sei $\mathscr{L}$ die von V und den p_i erzeugte (transitive) Unteralgebra von $\mathscr{P}\!ol\, V$. Wie gesehen, ist $\operatorname{Grad} \mathscr{L} > 1$ möglich; allerdings ist $\mathscr{L}$ dann reduzibel. Diese Reduzibilität läßt sich auch für die Differentialgleichung auswerten. Seien $q_1, \dots, q_m \in \mathscr{P}\!ol\, V$ homogen vom $\operatorname{Grad} \geqq 1$, weiter U ein nichttrivialer Unterraum von V mit $q_i(V, \dots, V, U) \subset U$ $(1 \leqq i \leqq m)$. Dann ist $\bar{q}_i(\bar{x}) := q_i(x) + U$ für $\bar{x} = x + U \in V/U$ wohldefiniert. Weiter sei W ein zu U komplementärer Unterraum von V und $\psi: V/U \to W$ der natürliche Isomorphismus.

Satz 7. *Man erhält alle Lösungen von*

$$\dot{x} = v(t) + \sum_{i=1}^{m} \mu_i(t) q_i(x) \quad (v: I \to V,\ \mu_i: I \to \mathbb{R} \text{ stetig})$$

durch sukzessives Lösen von

$$\dot{\bar{x}} = \bar{v}(t) + \sum_{i=1}^{m} \mu_i(t) \bar{q}_i(\bar{x}) \text{ in } V/U$$

und

$$\dot{u} = \sum \mu_i(t) \sum_{k=1}^{\operatorname{Grad} q_i} \binom{\operatorname{Grad} q_i}{k} q_i(\underbrace{u, \dots u}_{k\text{-mal}}, w, \dots, w)$$

$$+ \left(\sum \mu_i(t) q_i(w) + v(t) - \dot{w}\right) \text{ in } U .$$

Dabei ist $w(t) := \psi(\bar{x}(t))$ für eine Lösung der Gleichung in V/U.

Beweis. Die Zerlegung $x = u + w \in U \oplus W$ liefert

$$\dot{x} = \dot{u} + \dot{w} = \sum_{i=1}^{m} \mu_i(t) \sum_{k=1}^{\mathrm{Grad}\, q_i} \binom{\mathrm{Grad}\, q_i}{k} q_i(\underbrace{u, \dots, u}_{k\text{-mal}}, w, \dots, w)$$

$$+ \sum_{i=1}^{m} \mu_i(t) q_i(w) + v(t)$$

Ist nun $w(t) = \psi(\bar{x}(t))$, so ist $\sum_{i=1}^{m} \mu_i(t) q_i(w) + v(t) - \dot{w} \in U$ (ebenso wie die restlichen Terme), und die Behauptung ist abzulesen. □

Bemerkung. Eine erste Version dieses Satzes stammt von Kunde [5]. Man beachte, daß es hier keine Rolle spielt, ob die q_i in einer endlichdimensionalen Unteralgebra von $\mathscr{P}\!ol\, V$ liegen.

Korollar. *Sei* Grad $\mathscr{L} > 1$. *Alle Lösungen von* (1) *erhält man durch sukzessives Lösen von linearen Differentialgleichungen und einer Gleichung in einer transitiven Unteralgebra vom* Grad $\leqq 1$.

Beweis. $\mathscr{L}$ ist nicht halbeinfach (Satz 3), besitzt also ein abelsches Ideal $\mathscr{I} \neq \{0\}$, das (nach 3.1., Lemma) als graduiert angenommen werden darf:
$\mathscr{I} = \mathscr{I}_{-1} \oplus \dots \oplus \mathscr{I}_m$, $\mathscr{I}_{-1} \neq \{0\}$ (3.1., Lemma). Setze $U := \mathscr{I}_{-1}$. Dann gilt $p_i(U, V, \dots, V) \subset U$ für alle i und $p_i(U, U, V, \dots, V) = \{0\}$ für Grad $p_i \geqq 2$. Also bleibt bei der Reduktion in Satz 7 nur eine lineare Differentialgleichung in U, und V/U, $\bar{p}_1, \dots, \bar{p}_r$ erzeugen eine transitive Unteralgebra $\tilde{\mathscr{L}}$ von $\mathscr{P}\!ol\,(V/U)$ (vgl. 3.2). Hat $\tilde{\mathscr{L}}$ noch Grad > 1, so wiederhole die Reduktion. Aus Dimensionsgründen hat dies nach endlich vielen Schritten ein Ende. □

Bemerkung. Der Beweis zeigt, daß entscheidend für diesen Schluß die Nicht-Halbeinfachheit von $\mathscr{L}$ ist (unabhängig vom Grad). So läßt sich etwa im Fall, daß $\mathscr{L}$ eine Algebra wie in 3.3, Beispiel 2, ist, das Problem vollständig auf lineare Gleichungen reduzieren, denn $\mathscr{L}$ ist auflösbar.
Auch für halbeinfaches, nicht einfaches $\mathscr{L}$ liefert Satz 7 wertvolle Hilfe: Ist $\mathscr{I} = \mathscr{I}_{-1} \oplus \mathscr{I}_0 \oplus \mathscr{I}_1$ Ideal mit komplementärem Ideal $\mathscr{K} = \mathscr{K}_{-1} \oplus \mathscr{K}_0 \oplus \mathscr{K}_1$, so setze $U := \mathscr{I}_{-1}$ und wähle als Komplement $W := \mathscr{K}_{-1}$. Dies liefert zwei separierte Gleichungen. (Ist $\mathscr{L}$ halbeinfach, so ist jedes Ideal von $\mathscr{L}$ graduiert; dies folgt aus $\mathrm{Id} \in \mathscr{L}$ und $[\mathrm{Id}, p] = kp$ für alle $p \in \mathscr{L}_k$; s. auch Koecher [4].)
Auf jeden Fall genügt es im Folgenden, sich um Grad $\mathscr{L} = 1$ zu kümmern.

4.2. Riccati-Gleichungen auf Jordan-Paaren

Es bleibt also noch zu behandeln (s. Satz 6, 7):

$$\dot{x} = Q_+(x)a + Bx + c \tag{2}$$

(Dabei $a: I \to V^-$, $c: I \to V^+$, $B: I \to \mathscr{H}(V^+, Q_+)$ stetig.)

Diese Gleichung wurde von Braun [1] untersucht. Wir folgen sinngemäß dieser Arbeit.

Es sei $z_0(t)$ eine Lösung von (2) auf I (dies läßt sich nach evtl. Verkleinerung von I annehmen) und $z(t)$ eine weitere. Dann ist $z-z_0$ Lösung von

$$\dot{x}=Q_+(x)a+Cx \tag{3}$$

mit $C=B+D_+(z_0,a):I\to\mathscr{H}(V^+,Q_+)$ stetig.

Nun sei $D:I\to\mathrm{Hom}(V^-,V^-)$ so, daß

$$Q_+(x,Cx)=CQ_+(x)+Q_+(x)D$$

für alle $x\in V$, $t\in I$.

Wähle ein $t_0\in V$ und definiere $T:I\to\mathrm{Gl}(V^+)$, $S:I\to\mathrm{Gl}(V^-)$ durch $\dot{T}=CT$, $T(t_0)=\mathrm{Id}$; $\dot{S}=SD$, $S(t_0)=\mathrm{Id}$. Dann gilt $Q_+(Tx)=TQ_+(x)S$ für alle $x\in V$, $t\in I$; denn für $t=t_0$ ist dies richtig, und Differenzieren ergibt die Behauptung. Ist $z(t)$ Lösung von (3), so ist $T^{-1}z$ Lösung von

$$\dot{x}=Q_+(x)\tilde{a} \tag{4}$$

mit $\tilde{a}=Sa$.

Für die Lösung $z(t)$ von (4) zum Anfangswert u bei $t=t_0$ gilt die explizite Formel

$$z(t)=(\mathrm{Id}-D_+(u,y(t))+Q_+(u)Q_-(y(t)))^{-1}(u-Q_+(u)y(t))$$

Dabei $y(t):=\int_{t_0}^{t}\tilde{a}(\tau)d\tau$.

Man beachte, daß z gerade das Quasi-Inverse von u in der Jordan-Algebra V_y^+ ist.

Bemerkung. Die Reduktion auf die reinquadratische Gleichung läßt sich ohne Mühe auf den Fall übertragen, daß $\mathscr{L}$ eine beliebige graduierte Unteralgebra vom Grad 1 ist. (Man beginne mit einer speziellen Lösung in $\mathscr{L}_{-1}$.)

4.3. Schlußbemerkungen

Für Riccati-Gleichungen auf Jordan-Paaren läßt sich die Fundamentallösungs-Eigenschaft direkt aus der Lösungsformel einsehen: Für Gleichungen des Typs (3) (eine unwesentliche Einschränkung) ist die allgemeine Lösung eine rationale Funktion im Anfangswert, und ihr Grad in „Zähler" und „Nenner" ist beschränkt. Nun ist aber eine solche rationale Funktion durch endlich viele, „allgemein gelegene" Punkt-Bildpunkt-Paare schon eindeutig bestimmt.

Enthält das Jordan-Paar invertierbare Elemente, gibt es also ein $x\in V^+$, so daß $Q_+(x)$ invertierbar ist, so lassen sich die Arbeiten von Levin [6] und Reid [10] zur Matrix-Riccati-Gleichung in $\mathbb{R}^{(n,n)}$ fast wörtlich übertragen; die Herleitung wird sogar etwas einfacher; s. hierzu [12]. Die allgemeine Riccati-Gleichung auf V besitzt für $\dim V>1$ keine Fundamentallösungen, denn es liegt bekanntlich $\mathscr{P}_{-1}\oplus\mathscr{P}_0\oplus\mathscr{P}_1$ in keiner endlichdimensionalen Unteralgebra von $\mathscr{P}\!ol\,V$. Man entnimmt dies auch direkt dem letzten Abschnitt von Beispiel 1 in 3.3.

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Bilinear Mappings and Trace Class Ideals

Andrew M. Tonge

Department of Mathematics and Statistics, Brunel University, Uxbridge, UB8 3PH, UK, and Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA

Introduction

In a recent paper [14], Redheffer and Volkmann investigated the norms of bilinear maps $\ell_2^N(C_p)\times\ell_2^N(C_q)\to C_r$. They had been motivated by previous work [12, 13] on a classical result of Schur concerning a special bilinear form on Hilbert space.

The purpose of this paper is to sharpen the main results in [14] and to resolve the questions left open in [14]. The new ingredients we use to tackle these problems are interpolation theory for trace class ideals, Nehari's theorem on Hankel operators, the Rudin-Shapiro polynomials and a factorisation result of Pisier.

1. Bilinear Maps with C_p-Valued Variables

Let T be a compact (linear) operator on a (complex) Hilbert space H. The operator T^*T is positive and compact, and so has a unique positive square root $(T^*T)^{1/2}$. Since $(T^*T)^{1/2}$ is compact, we can enumerate its positive eigenvalues in decreasing order:

$$\mu_1(T)\geqq\mu_2(T)\geqq\ldots\geqq 0.$$

For $1\leqq p<\infty$ we define the C_p-norm of T to be

$$\|T\|_p=(\sum|\mu_n(T)|^p)^{1/p}=[\operatorname{trace}(T^*T)^{p/2}]^{1/p}.$$

We set $\|T\|_\infty$ to be the operator norm of T.

For each $1\leqq p\leqq\infty$, the collection of all compact operators T with $\|T\|_p<\infty$ forms a Banach space C_p under the norm $\|\cdot\|_p$. We refer to the paper of McCarthy [9] for basic facts concerning the C_p-spaces.

It is important to note that the C_p-spaces behave just like the L_p-spaces under complex interpolation. Indeed, if $1\leqq p<q\leqq\infty$ and $0\leqq t\leqq 1$, then the complex intermediate space $[C_p, C_q]_t$ can be identified with C_r where $1/r=t/p+(1-t)/q$. Consequently there is a natural analogue of the Riesz-Thorin theorem for operators between C_p-spaces. We refer to [1] for standard information about interpolation theory and to [4] for more details on interpolation of C_p-spaces.

The main result in [14] is an inequality of the following type.

If $A=(a_{mn})$ is a complex $N\times N$ matrix, and if $S_1,\dots,S_N$, $T_1,\dots,T_N$ are compact operators on Hilbert space, then for some values of p, q, and r

$$\left\|\sum_{m,n=1}^{N} a_{mn}S_mT_n\right\|_r \leqq \|A\|_\infty\left(\sum_{m=1}^{N}\|S_m\|_p^2\right)^{1/2}\left(\sum_{n=1}^{N}\|T_n\|_q^2\right)^{1/2}.$$

Note that $\|A\|_\infty$ denotes the norm of A as an operator on N-dimensional Hilbert space. For precise statements, the reader should refer to (7) and (8) in [14].

We begin by reinterpreting this type of inequality in a form which makes it more amenable to our approach. If E is a Banach space, we write $\ell_2^N(E)$ for the Banach space $E\times\dots\times E$ (N copies) under the norm

$$\|(e_1,\dots,e_N)\| = \left(\sum_{n=1}^{N}\|e_n\|^2\right)^{1/2}.$$

The inequalities of Redheffer and Volkmann may now be stated in the following equivalent way.

If $A=(a_{mn})$ is a complex $N\times N$ matrix, then the bilinear map

$$B_A:\ell_2^N(C_p)\times\ell_2^N(C_q)\to C_r$$

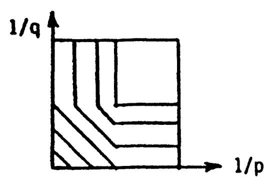
$$(S_m)_{m=1}^N\times(T_n)_{n=1}^N\to\sum_{m,n=1}^{N} a_{mn}S_mT_n$$

is bounded by $\|A\|_\infty$ for suitable values of p, q, and r.

Notation. Let $1\leqq p,q\leqq\infty$. Then $r(p,q)$ is defined by

$$1/r(p,q):=\begin{cases}1/p+1/q & (2\leqq p\leqq\infty,\ 2\leqq q\leqq\infty)\\ 1/p+1/2 & (2\leqq p\leqq\infty,\ 1\leqq q\leqq 2)\\ 1/2+1/q & (1\leqq p\leqq 2,\ 2\leqq q\leqq\infty)\\ 1 & (1\leqq p\leqq 2,\ 1\leqq q\leqq 2).\end{cases}$$

The level lines of $1/r(p,q)$ are depicted below.



Our first theorem sharpens the main result in [14].

Theorem 1. *Let $1\leqq p,q\leqq\infty$ and let $A=(a_{mn})$ be an $N\times N$ matrix. If $r\geqq r(p,q)$ then we can define a bilinear map*

$$B_A:\ell_2^N(C_p)\times\ell_2^N(C_q)\to C_r$$

$$(S_m)\times(T_n)\to\sum a_{mn}S_mT_n$$

which is bounded by $\|A\|_\infty$.

Note that (7) in [14] corresponds to the case $q=2$ of our theorem. Our results improve those of Redheffer and Volkmann. However, (8) of [14] corresponds to the case $2 \leqq p \leqq \infty$, $q=\infty$ of our theorem, and in this case Redheffer and Volkmann obtain the same results as the ones we prove.

Our next theorem shows that Theorem 1 is best possible in a rather strong sense.

Theorem 2. *Let $1 \leqq p, q \leqq \infty$ and let $K>0$. If $1 \leqq r < r(p,q)$ then there is a positive integer N and an $N \times N$ matrix A for which the map B_A defined above is not bounded by $K\|A\|_\infty$.*

Note that Theorem 2 improves a result in [14]. Redheffer and Volkmann constructed a matrix A with the property that for $1 \leqq p = r < \log 4/\log(2+8\pi^{-2}) < 2$ and $q=\infty$, the map B_A is not bounded by $\|A\|_\infty$.

The proof of Theorem 1 makes heavy use of a special case of the main lemma of [14]. This result was proved in [8].

Lemma. *Let $A=(a_{mn})$ be an $N \times N$ matrix and let $x_1, \ldots, x_N, y_1, \ldots, y_N$ be elements of Hilbert space. Then*

$$\left|\sum_{m,n=1}^{N} a_{mn}\langle x_m, y_n\rangle\right| \leqq \|A\|_\infty \left(\sum_{m=1}^{N} \|x_m\|^2\right)^{1/2} \left(\sum_{n=1}^{N} \|y_n\|^2\right)^{1/2}.$$

This lemma may be regarded as a "trivial" version of Grothendieck's inequality. We shall pursue this point later.

Proof of Theorem 1. We use the lemma to establish the cases $(p,q)=(\infty,\infty)$, $(\infty,2)$, $(2,\infty)$, and $(2,2)$. The remaining cases will then follow from an argument which uses interpolation and the monotonicity of the C_p-norms.

Assume henceforth that (S_m) and (T_n) are in the unit balls of $\ell_2^N(C_p)$ and $\ell_2^N(C_q)$ respectively.

Case 1. $p=q=\infty$; $r(p,q)=\infty$.

$$\begin{aligned}
\|\sum a_{mn}S_mT_n\|_\infty &= \sup\{|\sum a_{mn}\langle S_mT_nh, k\rangle| : \|h\| \leqq 1,\ \|k\| \leqq 1\}\\
&= \sup\{|\sum a_{mn}\langle T_nh, S_m^*k\rangle| : \|h\| \leqq 1,\ \|k\| \leqq 1\}\\
&\leqq \sup\{|\sum a_{mn}\langle y_n, x_m\rangle| : \sum\|x_m\|^2 \leqq 1,\ \sum\|y_n\|^2 \leqq 1\}\\
&\quad\ \text{[by the assumption on } (S_m) \text{ and } (T_n)]\\
&\leqq \|\bar{A}\|_\infty\\
&\quad\ \text{[by the lemma, with } \bar{A}=(\bar{a}_{mn})]\\
&= \|A\|_\infty .
\end{aligned}$$

Case 2(a). $p=\infty$, $q=2$; $r(p,q)=2$.

Using [9, Theorem 4.2] we can assert that

$$\|\sum a_{mn}S_mT_n\|_2 = \sup\{|\operatorname{trace} U \sum a_{mn}S_mT_n| : \|U\|_2 \leqq 1\}.$$

Now $V_m := US_m$ is in C_2 and $\sum\|V_m\|_2^2 \leqq \sum\|U\|_2^2\|S_m\|_\infty^2 \leqq 1$. Consequently, if we note that C_2 is a Hilbert space with inner product

$$\langle S, T\rangle = \operatorname{trace} S\bar{T}' \quad \text{(where } T' \text{ is the transpose of } T)$$

we can use the lemma to obtain

$$\|\sum a_{mn}S_mT_n\|_2 \leqq \sup\{|\sum a_{mn}\operatorname{trace} V_mT_n| : \sum \|V_m\|_2^2 \leqq 1\}$$
$$\leqq \|A\|_\infty .$$

Case 2(b). $p=2$, $q=\infty$; $r(p,q)=2$.

This is similar to case 2(a). This time one should post-multiply by U.

Case 2(c). $p=2$, $q=2$; $r(p,q)=1$.

Using [4, p. 132] we can assert that

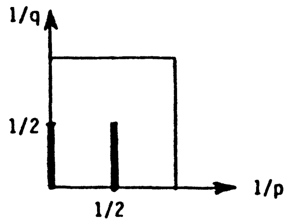
$$\|\sum a_{mn}S_mT_n\|_1 = \sup\{|\operatorname{trace}\sum a_{mn}S_mT_nU| : \|U\|_\infty \leqq 1\}.$$

Now $W_n := T_nU$ is in C_2 and $\sum \|W_n\|_2^2 \leqq \sum \|T_n\|_2^2 \|U\|_\infty^2 \leqq 1$. Consequently, as before

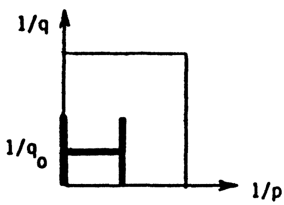
$$\|\sum a_{mn}S_mT_n\|_1 \leqq \sup\{|\sum a_{mn}\operatorname{trace} S_mW_n| : \sum \|W_n\|_2^2 \leqq 1\}$$
$$\leqq \|A\|_\infty .$$

The main work is over. We now use vector-valued interpolation results [1, p. 107].

The cases $p=\infty$, $2\leqq q\leqq\infty$ can be obtained by interpolating between the results when $p=\infty$, $q=2$ and when $p=\infty$, $q=\infty$. The theorem may also be proved when $p=2$, $2\leqq q\leqq\infty$ by interpolating between the results when $p=2$, $q=2$ and when $p=2$, $q=\infty$.



Next, given a fixed q_0 in the range $2\leqq q\leqq\infty$, we can establish our theorem for the cases $2\leqq p\leqq\infty$, $q=q_0$ by interpolating between the results when $p=2$, $q=q_0$ and when $p=\infty$, $q=q_0$.

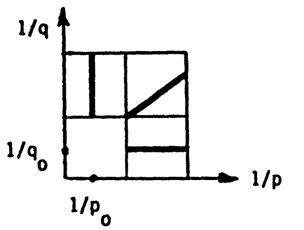


This proves what was wanted in the range $2\leqq p, q\leqq\infty$. The proof of the remaining cases requires only the monotonicity of the C_p-norms:

$$\text{if}\quad p_1\geqq p_2 \quad\text{then}\quad \|T\|_{p_1}\leqq \|T\|_{p_2} \quad \text{for any } T.$$

Thus, if we fix q_0 in the range $2\leqq q\leqq\infty$, all the cases $1\leqq p\leqq 2$, $q=q_0$ follow from the case $p=2$, $q=q_0$. If we fix p_0 in the range $2\leqq p\leqq\infty$, all the cases $p=p_0$,

$1 \leqq q \leqq 2$ follow from the case $p = p_0$, $q = 2$. Finally, all cases in the range $1 \leqq p$, $q \leqq 2$ follow from the single case $p = q = 2$.



We now proceed to the proof of Theorem 2. The basic idea can be traced back to [14], but the tools we use are radically different.

Proof of Theorem 2. First, let us dispose of the easy cases. When $1 \leqq p, q \leqq 2$, there is nothing to prove, since $r(p, q) = 1$. When $2 \leqq p, q \leqq \infty$ and $1 \leqq r < r(p, q)$, the map B_A is not well-defined, since the product of a C_p-operator with a C_q-operator need not be in C_r. To see this well-known fact, one simply has to consider diagonal operators.

The cases $1 \leqq p \leqq 2$, $2 \leqq q \leqq \infty$ are more delicate.

Fix a positive integer N and choose a vector $(\varepsilon_2, \ldots, \varepsilon_{2N})$ with each coefficient equal to ± 1. Define an $N \times N$ matrix $A = (a_{mn})$ by setting $a_{mn} = \varepsilon_{m+n}$.

Now construct diagonal operators $S_1, \ldots, S_N$, $T_1, \ldots, T_N$ as follows:

$$S_m = \mathrm{diag}(0, \ldots, 0, 1, 0 \ldots) \quad (1 \text{ is in the } m^{\text{th}} \text{ place});$$

$$T_n = \mathrm{diag}(\varepsilon_{1+n}, \ldots, \varepsilon_{N+n}, 0, \ldots).$$

Next, observe that $\|S_m\|_p = 1$ and $\|T_n\|_q = N^{1/q}$, so

$$\left(\sum_{m=1}^{N} \|S_m\|_p^2\right)^{1/2} = N^{1/2} \quad \text{and} \quad \left(\sum_{n=1}^{N} \|T_n\|_q^2\right)^{1/2} = N^{1/2+1/q}.$$

A computation shows that

$$\sum_{m,n=1}^{N} a_{mn} S_m T_n = \mathrm{diag}(N, \ldots, N, 0, \ldots)$$

and it follows that

$$\left\| \sum_{m,n=1}^{N} a_{mn} S_m T_n \right\|_r = N^{1+1/r}.$$

Let K be a fixed positive constant. If the operator B_A is to be bounded by $K\|A\|_\infty$ for every N and every $N \times N$ matrix, then we shall certainly require

$$N^{1+1/r} \leqq K \|A\|_\infty N^{1/2} N^{1/2+1/q}$$

for every N and every $N \times N$ matrix $A = (\varepsilon_{m+n})$.

However, by taking projections we get

$$\left\|\begin{array}{ccccc} \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \cdots & \varepsilon_N \\ \varepsilon_3 & \varepsilon_4 & & & \cdot \\ \varepsilon_4 & & & & \cdot \\ \vdots & & & & \vdots \\ \varepsilon_N & \cdot & \cdot & \cdots & \varepsilon_{2N} \end{array}\right\|_\infty \leqq \left\|\begin{array}{ccccccc} \varepsilon_2 & \cdots & \varepsilon_N & \cdots & \varepsilon_{2N} & 0 & \cdots \\ \vdots & & & & & & \\ \varepsilon_N & & & & & & \\ \vdots & & & & & & \\ \varepsilon_{2N} & & & & & & \\ 0 & & & & 0 & & \\ \vdots & & & & & & \end{array}\right\|_\infty .$$

But the norm of the infinite Hankel matrix thus produced is, by Nehari's theorem [see 12], at most

$$\max_{0 \leqq t \leqq 2\pi} \left| \sum_{k=2}^{2N} \varepsilon_k e^{ikt} \right| .$$

The Rudin-Shapiro polynomials [7, p. 75] guarantee that there is a choice of ± 1's which makes this last quantity at most $100N^{1/2}$.

We shall therefore need to require that for every N

$$N^{1+1/r} \leqq 100KN^{3/2+1/q} .$$

This is only possible when $1/r \leqq 1/2 + 1/q = 1/r(p,q)$.

The remaining cases in Theorem 2 may be treated similarly.

We now return to a remark we made previously, and indicate how to obtain results similar to, but considerably deeper than Theorems 1 and 2.

If $A = (a_{mn})$ is an $N \times N$ matrix, we set

$$\|A\|_{1,1} = \sup\{|\sum a_{mn} s_m t_n| : |s_m| \leqq 1,\ |t_n| \leqq 1,\ 1 \leqq m, n \leqq N\} .$$

The notorious inequality of Grothendieck [5] asserts that if $x_1, \ldots, x_N$, $y_1, \ldots, y_N$ are elements of the unit ball of Hilbert space, then

$$|\sum a_{mn} \langle x_m, y_n \rangle| \leqq 2\|A\|_{1,1} .$$

In fact, the constant 2 can be improved, but this is of little significance here.

A straightforward modification of the proofs of Theorems 1 and 2, in which Grothendieck's inequality is substituted for the lemma, enables us to show that the bilinear mapping

$$B_A : \ell_\infty^N(C_p) \times \ell_\infty^N(C_q) \to C_r$$
$$(S_m) \times (T_n) \to \sum a_{mn} S_m T_n$$

is bounded by $2\|A\|_{1,1}$ for exactly the same values p, q, r as before, and need not be bounded by $K\|A\|_{1,1}$ for all other values.

Note that we have used the notation $\ell_\infty^N(C_p)$ to denote the Banach space $C_p \times \ldots \times C_p$ (N copies) under the norm

$$\|(T_1, \ldots, T_N)\| = \max\{\|T_n\|_p : 1 \leqq n \leqq N\} .$$

When proving the analogue of Theorem 2, it is necessary to note that when A is an $N \times N$ matrix we always have

$$\|A\|_{1,1} \leqq N \|A\|_{\infty}.$$

It is also worth remarking that in all our results the C_p-spaces may be replaced by ℓ_p-spaces with only trivial modifications to the proofs.

2. A Relationship with H'-Algebras

In [14], Redheffer and Volkmann ask how to characterise the following class of Banach algebras.

Definition. Let R be a Banach algebra. We say that R is an RV-algebra if given any positive integer N and any $N \times N$ matrix $A = (a_{mn})$, the bilinear map

$$M_A : \ell_2^N(R) \times \ell_2^N(R) \to R$$

$$(x_m)_{m=1}^N \times (y_n)_{n=1}^N \to \sum_{m,n=1}^N a_{mn} x_m y_n$$

has norm at most $\|A\|_{\infty}$.

Note that this condition is equivalent to the requirement that

$$\left\| \sum_{m,n=1}^N a_{mn} x_m y_n \right\| \leqq \|A\|_{\infty} \left(\sum_{m=1}^N \|x_m\|^2 \right)^{1/2} \left(\sum_{n=1}^N \|y_n\|^2 \right)^{1/2}$$

for every N-tuple $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N)$ in R^N.

Redheffer and Volkmann give an example to show that not every Banach algebra is an RV-algebra.

In this section we show that RV-algebras can be characterised in terms of the continuity of the multiplication map $M : R \otimes R \to R$ when the tensor product is equipped with a suitable norm. We also give a naturally occurring example of a non-RV-algebra.

The main result is an easy consequence of a theorem of Pisier.

Theorem (Pisier [10, Theorem 2.4]). *Let $f : X \to Y$ be a map between Banach spaces. The following statements are equivalent:*

(i) *The map f factors through a Hilbert space H*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & & Y \\ & H \arrow[ur, "v"'] & \end{tikzcd}$$

in such a way that $f = v \circ u$ and $\|u\| \cdot \|v\| \leqq C$.

(ii) *For every positive integer N and every $N \times N$ matrix $A = (a_{mn})$*

$$\left(\sum_{m=1}^N \left\| \sum_{n=1}^N a_{mn} f(x_n) \right\|^2 \right)^{1/2} \leqq C \|A\|_{\infty} \left(\sum_{n=1}^N \|x_n\|^2 \right)^{1/2}.$$

Now if φ is an element of the dual R' of a Banach algebra R, we may associate with it a linear map $\Phi: R \to R'$ defined by

$$\langle x, \Phi(y) \rangle = \langle xy, \varphi \rangle .$$

Note that $\|\Phi\| \leqq \|\varphi\|$, and that for unital Banach algebras $\|\Phi\| = \|\varphi\|$.

Theorem 3. *A Banach algebra R is an RV-algebra if and only if for each $\varphi \in R'$ the map $\Phi: R \to R'$ factors through a Hilbert space H*

$$\begin{tikzcd} R \arrow[rr, "\Phi"] \arrow[dr, "u"'] & & R' \\ & H \arrow[ur, "v"'] & \end{tikzcd}$$

in such a way that $\Phi = v \circ u$ and $\|u\| \cdot \|v\| \leqq \|\varphi\|$.

Proof. Suppose first that each Φ factors as described. Choose a natural number N, elements $x_1, \ldots, x_N, y_1, \ldots, y_N$ of R satisfying

$$\sum_{m=1}^{N} \|x_m\|^2 \leqq 1 \quad \text{and} \quad \sum_{n=1}^{N} \|y_n\|^2 \leqq 1 ,$$

and an $N \times N$ matrix $A = (a_{mn})$. Then

$$\begin{aligned} \left\|\sum a_{mn} x_m y_n\right\| &= \sup\left\{\left|\left\langle \sum a_{mn} x_m y_n, \varphi \right\rangle\right| : \|\varphi\| \leqq 1\right\} \\ &= \sup\left\{\left|\sum_m \left\langle x_m, \sum_n a_{mn} \Phi(y_n) \right\rangle\right| : \|\varphi\| \leqq 1\right\} \\ &\leqq \sup\left\{\left(\sum_m \|x_m\|^2\right)^{1/2} \left(\sum_m \left\|\sum_n a_{mn} \Phi(y_n)\right\|^2\right)^{1/2} : \|\varphi\| \leqq 1\right\} \\ &\leqq \|A\|_\infty \quad \text{by Pisier's theorem.} \end{aligned}$$

Conversely, assume that R is an RV-algebra. Then if $A = (a_{mn})$ is an $N \times N$ matrix, $y_1, \ldots, y_N$ are elements of R and $\varphi \in R'$ we have

$$\begin{aligned} \left(\sum_{m=1}^{N} \left\|\sum_{n=1}^{N} a_{mn} \Phi(y_n)\right\|^2\right)^{1/2} &= \sup\left\{\left|\sum_{m=1}^{N} \left\langle x_m, \sum_{n=1}^{N} a_{mn} \Phi(y_n) \right\rangle\right| : \sum_{m=1}^{N} \|x_m\|^2 = 1\right\} \\ &= \sup\left\{\left|\left\langle \sum_{m,n=1}^{N} a_{mn} x_m y_n, \varphi \right\rangle\right| : \sum \|x_m\|^2 = 1\right\} \\ &\leqq \|\varphi\| \; \|A\|_\infty \left(\sum_{n=1}^{N} \|y_n\|^2\right)^{1/2} . \end{aligned}$$

Now apply Pisier's theorem to obtain the desired conclusion.

We remark that Banach algebras with the property that each Φ factors through Hilbert space were first considered by Charpentier [2] and were subsequently studied in [3, 7, 15]. Such algebras are known [2] to be characterised by the condition that the multiplication operator $R \otimes R \to R$ is continuous when the tensor product is given the H'-norm of Grothendieck [5]. This gives an immediate reformulation of Theorem 3 for unital Banach algebras.

Theorem 3′. *A unital Banach algebra R is an RV-algebra if and only if the multiplication $R\otimes_{H'}R\to R$ is a contraction.*

This reformulation tells us that RV-algebras are in some sense very close to being (non-self-adjoint) subalgebras of C^*-algebras. For further elucidation of this point, the reader should consult [2, 3, 15].

We now give an example of a Banach algebra which occurs naturally in harmonic analysis, but which is not an RV-algebra.

If f is a complex-valued function on the circle group T we write

$$\|f\|_1=\sum_{n=-\infty}^{\infty}|\hat{f}(n)|,$$

where $\hat{f}(n)$ is the n^{th} Fourier coefficient of f. The collection $A(T)$ of all such functions f for which $\|f\|_1<\infty$ forms a Banach algebra under the norm $\|\cdot\|_1$ and pointwise multiplication.

Example. *$A(T)$ is not an RV-algebra.*

Proof. Fix an integer N and set $x_n(t)=y_n(t)=e^{int}$ $(1\leqq n\leqq N)$. Note that $\sum_{n=1}^{N}\|x_n\|_1^2 = \sum_{n=1}^{N}\|y_n\|_1^2=N$. If $A=(a_{mn})$ is an $N\times N$ matrix,

$$\left\|\sum_{m,n=1}^{N}a_{mn}x_my_n\right\|_1=\left\|\sum_{m,n=1}^{N}a_{mn}e^{i(m+n)t}\right\|_1 =\sum_{k=2}^{2N}\left|\sum_{m+n=k}a_{mn}\right|.$$

Choose $(a_{mn})=(\varepsilon_{m+n})$. Then we obtain

$$\left\|\sum_{m,n=1}^{N}a_{mn}x_my_n\right\|_1=\sum_{k=2}^{2N}(k-1)|\varepsilon_k|.$$

As in the previous section, we can select $\varepsilon_k=\pm 1$ $(2\leqq k\leqq 2N)$ so that $\|A\|_\infty\leqq 100N^{1/2}$.

It follows that for $A(T)$ to be an RV-algebra it is necessary that for each N

$$N(2N-1)\leqq 100N^{1/2}\cdot N.$$

This is clearly impossible.

To conclude, we remark that if $M:E\times F\to G$ is a bilinear map between Banach spaces and if $A=(a_{mn})$ is an $N\times N$ matrix, then the bilinear map

$$M_A:\ell_2^N(E)\times\ell_2^N(F)\to G$$
$$(e_m)\times(f_n)\to\sum a_{mn}M(e_m,f_n)$$

is necessarily bounded by $\|A\|_\infty$ if and only if for each $\varphi \in G'$ the map $\Phi: E \to F'$ given by $\langle \varphi, M(e,f) \rangle = \langle \Phi(e), f \rangle$ factors through a Hilbert space H

$$\begin{tikzcd} E \arrow[rr, "\Phi"] \arrow[dr, "u"'] & & F' \\ & H \arrow[ur, "v"'] & \end{tikzcd}$$

in such a way that $\Phi = v \circ u$ and $\|u\| \cdot \|v\| \leqq \|\varphi\|$. The proof of this is an easy adaptation of the proof of Theorem 3.

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The Strong Rigidity of Locally Symmetric Complex Manifolds of Rank One and Finite Volume*

J. Jost and S.-T. Yau[2]

[1] Institut für Mathematik, Universität, Postfach 102148, D-4630 Bochum, Federal Republic of Germany

[2] Department of Mathematics, UCSD, La Jolla, CA 92093, USA

Rigidity question have attracted much interest in the past. In the compact case, we have the famous work of Calabi and Vesentini [3] and Mostow [17]. Whereas Calabi and Vesentini proved a local version, namely that compact quotients of bounded symmetric domains admit no nontrivial deformations in case the domain is irreducible and of complex dimension at least 2, Mostow proved a global rigidity result, at the expense, however, of working only within the class of quotients of symmetric domains. Mostow's work is based on quasiconformal mappings. A different analytic approach was recently undertaken by Siu [22]. If M is a compact Kähler manifold diffeomorphic (or, more generally, homotopically equivalent) to a quotient N of an irreducible bounded symmetric domain, he studied a harmonic homotopy equivalence the existence of which is assured by the theorem of Eells and Sampson, and demonstrated that this map has to be a biholomorphic diffeomorphism itself, thus in particular obtaining the global rigidity of N within the class of Kähler manifolds and thereby generalizing the results of Calabi-Vesentini as well as of Mostow for quotients of bounded symmetric domains. A corresponding result for irreducible compact quotients of products of upper half planes, a case not covered by Siu's arguments, was then obtained through the work of Jost and Yau [11, 12], and Mok [15], thereby completting Siu's approach. On the other hand, Mostow's results were extended to noncompact locally symmetric spaces of finite volume by Prasad [19] and Margulis [14].

In the present work, we start an extension of Siu's results to the noncompact case. We study locally symmetric varieties of rank one, thereby generalizing some of Prasad's results, as well as irreducible quotients of products of upper half planes.

We shall prove:

Theorem 1. *Let D be an irreducible bounded symmetric domain in $\mathbb{C}^n$, $n \geqq 2$, of rank 1, i.e. the unit ball in complex space of dimension at least 2. Let N be a quotient of finite volume by a discrete torsion free subgroup Γ of* $\mathrm{Aut}(D)$.

Suppose $\bar{M}$ is a compact Kähler manifold, S a subvariety of $\bar{M}$ with normal crossings (i.e. S has possibly self intersections that locally look like the intersection of

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coordinate hyperplanes in $\mathbb{C}^n$ and is otherwise regular) and suppose there exists a proper homotopy equivalence between $M := \bar{M} \setminus S$ and N.

Then M is $\pm$ biholomorphically equivalent to N.

Note. A special case was obtained in [24].

Remark. The assumptions of the theorem are in particular satisfied if M is a quasiprojective manifold, properly homotopically equivalent to N. Here, "quasiprojective" means that M is a Zariski dense open subset of a projective algebraic variety. By Hironaka's theorem [8], the possible singularities of this projective algebraic variety can be resolved, i.e. replaced by complex hyposurfaces with normal crossings without changing the variety away from the singularities. The same remark applies to our next result.

Theorem 2. *Let $H := \{x + iy \in \mathbb{C}: y > 0\}$ denote the upper halfplane. Let $N = H^n/\Gamma$, where Γ is a discrete irreducible torsionfree subgroup of* $\mathrm{Aut}(H^n)$, $n \geqq 2$.

Let M again be properly homotopically equivalent to N and admit a Kähler compactification $\bar{M}$ as in Theorem 1.

Then, there exists a diffeomorphism $f: M \to N$ with the property that for its lifting $F = (F_1, \ldots, F_n): \tilde{M} \to H^n$ to universal coverings, each F_i is $\pm$ holomorphic.

As Siu did, we shall make strong use of harmonic maps and their properties. As a general reference for harmonic maps, one can use [10]. Because of the noncommpactness of the manifolds considered, we have to overcome several new technical difficulties compared to the compact case.

We first have to construct a proper homotopy equivalence of finite energy. In order to achieve this, we have to use the existence of suitable compactifications of M and N, in particular, the existence of a smooth compactification $\bar{M}$ of M with $\bar{M} \setminus M$ consisting of a union of smooth divisors with normal crossings. This special structure will also be important for showing that a harmonic map of finite energy (into which we deform the original map of finite energy) has maximal rank. On the other hand, we do not really need the locally symmetric structure of N but only a suitable decay of its metric towards the ends or cusps.

In the rank one case, this follows from Lemma 8 of [23]. The result of this Lemma already holds if N is a complete Kähler manifold whose sectional curvature is bounded between two negative constants. In addition, we shall also need strong negativity of the curvature of N in the sense of [22]. In our arguments, we need the more special assumption that N is locally symmetric of rank one only in the case where N has only one end. In this situation, we appeal to a result of Selberg that the fundamental group of N is residually finite, allowing us to reduce this case to the case where the image has more than one end by passing to a suitable finite cover. Apparently, a generalization of Selberg's result to negatively curved manifolds of finite volume is not known.

In the case of Theorem 2, we note that by a result of Selberg (cf. [9, p. 277]), Γ is commensurable with the Hilbert modular group $\tilde{\Gamma}$ of some totally real field K with $[K: \mathbb{Q}] = n$ (hence in particular arithmetic). Therefore, for our purposes, it is sufficient to know the behavior of the metric of $H^n/\tilde{\Gamma}$ near the cusps.

We also note that, concerning Theorem 2, our contribution lies in the construction of a proper harmonic homotopy equivalence of finite energy and maximal rank. The remaining arguments needed to prove this theorem are due to Mok [16].

1. The Metrics on Domain and Image

a) The Domain M

By assumption, M admits a compactification as a smooth Kähler manifold $\bar{M}$, with the property that $\bar{M}\setminus M$ consists of a union of complex hypersurfaces with normal crossings. The Kähler metric of $\bar{M}$ restricts to a Kähler metric ω on M.

Terminology. We call these hypersurface compactifying divisors, or short, cd's. Also, we call the boundary of a neighborhood of a compactifying divisor a bn.

We let $s_i = s_i^l \otimes \ldots \otimes s_i^{m_i}$ be local sections of the normal bundles of the cd's. In particular, s_i vanishes on the corresponding cd. We then take

$$g := \sum_i \partial\bar{\partial}(\varphi(|s_i|)\log(|\log|s_i^1|^2|\cdot\ldots\cdot|\log|s_i^{m_i}|^2|) + c\omega,$$

where φ is a suitable cut-off functions so that $\varphi(|s_i|)$ is identical one near the corresponding cd and vanishes outside a neighborhood of the cd, in particular for, say, $|s_i| \geqq 3/4$, and where $c > 0$ is chosen sufficiently large to make g positive definite. g then is a Kähler metric with the following properties.

(K1) g is complete

(K2) (M, g) has finite volume

(K3) If D is a disk normal to a cd, i.e. a local complex curve in M intersecting the cd. at $0 \in D$, then the restriction of g to D behaves asymptotically (i.e. when approaching 0) like the Poincaré metric on the punctured disk

$$\partial\bar{\partial}\log(|\log|z|^2|) = \frac{1}{|z|^2(\log|z|^2)^2}|dz|^2$$

(K4) A cd metrically looks like a complex hypersurface in the following sense: if $\mathscr{C}$ is the collection of disks normal to the cd, then $\mathscr{C}$ has finite nonzero $(2n-2)$ dimensional Hausdorff measure, where $n = \dim_{\mathbb{C}} M$. (This is due to the part $c\omega$ in the definition of g.)

b) The Metric of Complete Negatively Curved Kähler Manifolds of Finite Volume

Let N be a complete Kähler manifold of finite volume with curvature bounded between two negative constants. Given an end of N, we choose a geodesic ray γ going into that end, an arbitrary initial point $t = 0$ on γ and a parametrization $\gamma(t)$ by arclength.

If γ' is another ray going into the same end, then we can choose $\gamma'(0)$ in such a way that for the parametrization $\gamma'(t)$ by geodesic distance

$$\lim_{t\to\infty} d(\gamma(t), \gamma'(t)) = 0.$$

We do this for every geodesic ray extending into the given cusp and put

$$\varrho := \exp(-\exp t). \tag{1.1}$$

We put $N_{\varrho_0} := \{x \in N : \varrho(x) \leqq \varrho_0\}$. As N is negatively curved and of finite topological type (this follows, e.g., from [23]), though each each point of N_{ϱ_0} there is precisely one geodesic ray going into the given cusp, provided $\varrho_0 > 0$ is sufficiently small.

Hence, the level sets $\varrho = \text{const} \leqq \varrho_0$ are smooth and diffeomorphic to each other, and the diffeomorphism is explicitly obtained by moving along the geodesic rays.

Hence, in N_{ϱ_0}, we can choose local coordinates in such a way that the coordinates on the level sets $\varrho = \text{const}$ are independent of ϱ, i.e. invariant under moving along geodesic rays.

It then follows from Lemma 8 of [23] that the corresponding metric tensor of the hypersurfaces $\varrho = \text{const}$ is bounded by $\frac{l}{|\log \varrho|}$. In other words, if this hypersurface is described by the coordinates $u_i (i = 1, \ldots, 2n-1,\ n = \dim_{\mathbb{C}} N)$ and the metric tensor is denoted by $g_{u_i u_j}$, then

$$|g_{u_i u_j}(u_1, \ldots, u_{2n-1}, \varrho)| \leqq \frac{c}{|\log \varrho|}. \tag{1.2}$$

Here c is a fixed constant, depending only on the geometry of N.

In the case of locally symmetric varieties of rank one, we can also give a more explicit description of the metric as follows (cf. [1]):

The complex unit ball

$$\left\{ z \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 < 1 \right\}$$

can be represented as a Siegel domain of genus 2, namely

$$\left\{ (w, u_1, \ldots, u_{n-1}) \in \mathbb{C}^n : \operatorname{Im} w - \sum_{i=1}^{n-1} |u_i|^2 > 0 \right\},$$

via the transformation

$$z_1 = \frac{w-i}{w+i}, \qquad z_{j+1} = \frac{u_j \cdot \sqrt{2}}{w+i} \ (j = 1, \ldots, n-1).$$

In this representation, the metric becomes

$$\partial \bar{\partial} \left(-\log (\operatorname{Im} w - \sum_{i=1}^{n-1} (|u_i|^2) \right). \tag{1.3}$$

If we consider a quotient of the unit ball by a discrete arithmetic group of isometries which has finite volume then near the so called cusps, i.e. the points where the fundamental domain reaches the boundary of the unit ball, this quotient is obtained by just putting

$$z = e^{2\pi i a w}, \qquad a \in \mathbb{R},$$

and then making the appropriate identifications in the u_i-directions. If we represent z in polar coordinates (ϱ, φ), then in particular

$$\operatorname{Im} w = \frac{-1}{2\pi a} \log \varrho$$

$\varrho = 0$ corresponds to the cusp.

In the unit ball, we can look at the disk

$$\{|z_1|<1,\ z_2=\ldots=z_n=0\}.$$

Subject to the operation of $S(U(n-1)\times U(1))$, this disk sweeps out the whole unit ball. In the Siegel domain representation, this disk becomes the half plane

$$\{\operatorname{Im} w>0,\ u_1=\ldots=u_{n-1}=0\}.$$

Since, as mentioned, the whole space can be recovered from the images of this half plane under isometries, it suffices to evaluate the metric on this half plane. In the quotient, we therefore have to consider

$$\partial\bar{\partial}\left(-\log\left(\frac{-1}{2\pi a}\log\varrho-\sum|u_i|^2\right)\right)$$

only where

$$\sum_{i=1}^{n-1}|u_i|^2=0. \tag{1.4}$$

Now

$$\frac{\partial^2}{\partial z\partial\bar{z}}\left(-\log\left(-\frac{1}{2\pi a}\log\varrho\right)\right)=\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho\frac{\partial}{\partial\varrho}\left(-\log\left(-\frac{1}{2\pi a}\log\varrho\right)\right)\right)=\frac{1}{\varrho^2(\log\varrho)^2}$$

and

$$\frac{\partial^2}{\partial u_i\partial\bar{u}_i}\left(-\log\left(-\frac{1}{2\pi a}\log\varrho-\sum|u_i|^2\right)\right)=-\frac{4\pi a}{\log\varrho}$$

because of (1.4).

c) The Metric of Hilbert Modular Varieties

These are irreducible noncompact quotients of finite volume of H^n, where $H=\{x+iy\in\mathbb{C}: y>0\}$ is the upper half plane. Apart from quotient singularities which disappear by passing to a finite covering and hence can be neglected for our purpose, these varieties again have cusp singularities. The kernel function now is

$$-\log(\operatorname{Im} w_1\ldots\operatorname{Im} w_n)$$

for $(w_1\ldots,w_n)\in H^n$, and near the cusp, one passes to the quotient by putting

$$\frac{i}{2}w_j=\sum_{k=1}^{n}a_{jk}\log z_k$$

where the a_{jk} are positive integers.

Putting $z_k=\varrho_k e^{i\varphi_k}$ with $\varrho_k>0$, the kernel function becomes

$$-\sum_{j=1}^{n}\log\left(\sum_{k=1}^{n}\left(-2a_{jk}\log\varrho_k\right)\right).$$

In order to compute the metric, we have to take $\partial\bar{\partial}$ of the kernel function, hence evaluate

$$-\frac{1}{\varrho_i}\frac{\partial}{\partial\varrho_i}\left(\varrho_i\frac{\partial}{\partial\varrho_i}\sum_j\log\left(\sum_k(-2a_{jk}\log\varrho_k)\right)\right)=\sum_{j=1}^n\frac{4a_{ji}^2}{\varrho_i^2}\frac{1}{\left(\sum_{k=1}^n(-2a_{jk}\log\varrho_k)\right)^2}.$$

Thus, if e.g. ϱ_ℓ tends to zero, in the z_ℓ-direction, the metric again behaves like the Poincaré metric on the punctured disk, whereas in the orthogonal directions, we get a decay of order

$$\frac{1}{(\log\varrho_\ell)^2}.$$

For more details on the preceding construction, we refer to Hirzebruch [9; particularly p. 193f., and p. 204ff.] as well as Ash et al. [1].

2. Construction of a Finite Energy Map

On the domain, we let (r,θ) be polar coordinates on a disk transversal to the cd so that $r=0$ lies on the cd. We denote the coordinates in the other directions by x. The volume form then behaves like

$$\frac{1}{r(\log r)^2}\,dr\,d\theta\,dx. \tag{2.1}$$

On the image, we take the same coordinates ϱ, $u(=(u_1,\dots,u_{2n-1})$ locally) as in 1b. (Similarly, in the case of 1c.) We then choose a fixed differentiable homotopy equivalence from the CR-hypersurface $r=\text{const}$ onto the CR-hypersurface $\varrho=\text{const}$, and we let r and $\varrho=\varrho(r)$ correspond via

$$\log\varrho(r)=-(\log r)^2. \tag{2.2}$$

We denote the map constructed in this way by h. We can control the components of the inverse metric tensor of the domain by 1a) in the following way[1]

$$\gamma^{rr}\sim r^2(\log r)^2,\qquad \gamma^{\theta\theta}\sim(\log r)^2\qquad\text{by (K3)} \tag{2.3}$$

and

$$\gamma^{xx}\sim\text{const}\qquad\text{by (K4)}. \tag{2.4}$$

Likewise, by 1b), for the metric tensor of the image

$$g_{\varrho\varrho}=\frac{1}{\varrho^2(\log\varrho)^2}\qquad\text{by (1.1)} \tag{2.5}$$

and

$$g_{u_iu_j}\leqq\frac{c_0}{|\log\varrho|}\qquad(c_0=\text{const})\qquad\text{by (1.2)} \tag{2.6}$$

Thus, the energy of h is controlled via

$$E(h)\leqq c_1\int\gamma^{rr}g_{\varrho\varrho}\left(\frac{\partial\varrho}{\partial r}\right)^2+\gamma^{\theta\theta}g_{u_iu_j}\frac{\partial u_i}{\partial\varphi}\left\{\frac{\partial u_j}{\partial\varphi}+\gamma^{xx}c_2\right\}\frac{1}{r(\log r)^2}\,dr\,d\varphi\,dx,$$

[1] Super- and subscripts denote the corresponding coordinate directions

where c_1 and c_2 again are finite constants.

From (2.2), we derive

$$\frac{1}{\varrho}\frac{\partial \varrho}{\partial r} = -\frac{2}{r}\log r, \tag{2.7}$$

and then from (2.3)–(2.7)

$$E(h) \leqq c_3 < \infty .$$

We apply the same construction near each cusp, and we then extend these maps to the bounded parts of domain and image to get a proper homotopy equivalence, again called h, of finite energy. In a similar way, we can construct a proper homotopy equivalence of finite energy if the image is a Hilbert modular variety with a metric as in 1c).

3. The Harmonic Map and Its Properties

a) Existence

Since the image has nonpositive curvature, we can use the argument of [20][2], to deform the map h of finite energy, constructed in the preceding section, into a harmonic map f of finite energy. f then is a smooth map. We want to verify that f is also a proper homotopy equivalence.

b) f is Pluriharmonic

We again make use of the domain metric of 1a). Near each cusp, we choose a family φ_ε of cut-off functions depending only on $|s_i|$ where s_i is again the section of the normal bundle of the corresponding cd. We let $\varphi_\varepsilon \equiv 1$ outside a neighborhood of the cd, $\varphi_\varepsilon \to 1$ as $\varepsilon \to 0$. $\varphi_\varepsilon(|s_i|) \equiv 0$ for $|s_i| \log |s_i| \leqq \varepsilon$, $0 \leqq \varphi_\varepsilon(|s_i|) \leqq 1$ everywhere, and finally, we let $t^2 (\log t)^2 \varphi_\varepsilon''(t)$ be bounded independently of ε as $\varepsilon \to 0$. Thus,

$$|\Delta \varphi_\varepsilon| \sim \big| |s_i|^2 (\log |s_i|)^2 \varphi_\varepsilon'' \big| \leqq c, \tag{3.1}$$

where c is independent of ε.

We then use the well-known formula (cf. e.g. [10, 1.6]) for the energy density $e(f) = |df|^2$

$$\Delta e(f) = |\Delta df|^2 + \langle df \cdot \mathrm{Ric}^M(e_i), df \cdot e_i \rangle - \langle R^N(df \cdot e_i, df \cdot e_j)\, df \cdot e_i, df \cdot e_j \rangle, \tag{3.2}$$

where (e_i) is an orthonormal frame on M, Ric^M is the Ricci tensor of M, and R^N the curvature tensor of N. On the other hand, by (3.1) and since f has finite energy,

$$\int_M \varphi_\varepsilon \Delta e(f) = \int_M \Delta \varphi_\varepsilon \cdot e(f) \tag{3.3}$$

is bounded independently of ε, and hence

$$\int_M \Delta e(f) < \infty . \tag{3.4}$$

[2] The noncompactness of the image presents no obstacle, since Hamilton's theorem [7] still holds

Since our domain metric has bounded Ricci curvature (cf. [4, Proposition (3.5)]),

$$\int \langle df \cdot \mathrm{Ric}^M(e_i),\ df \cdot e_i \rangle \leqq cE(f). \tag{3.5}$$

Since the sectional curvature of N is nonpositive, we obtain from (3.2), (3.4), (3.5)

$$\int_M |\nabla df|^2 < \infty. \tag{3.6}$$

For a smooth compact subset K of M, we then obtain

$$\left|\int_K \Delta e(f)\right| = \left|\int_{\partial K} \frac{\partial}{\partial n} e(f)\right| \leqq \left|\int_{\partial K} e(f)\right| \cdot \left|\int_{\partial K} |\nabla df|^2\right|. \tag{3.7}$$

Letting K run through a suitable exhaustion of M, (3.6), (3.7), and $E(f) < \infty$, imply

$$\int \Delta e(f) = 0. \tag{3.8}$$

As in [22], we put ($f = (f^1, \ldots, f^n)$ in local coordinates)

$$\bar\partial f^\alpha = \frac{\partial f^\alpha}{\partial \bar z^i} d\bar z^i,$$

$$D\bar\partial f^\alpha = \partial\bar\partial f^\alpha + \Gamma^\alpha_{\beta\gamma}\partial f^\beta \wedge \bar\partial f^\gamma, \text{ etc.}$$

employing in addition, however, a summation convention. Of course, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the image N, and we denote the metric tensor by $(g_{\alpha\beta})$.

If ω is the Kähler form of M, then the argument leading to (3.8) also gives

$$\int \partial\bar\partial (g_{\alpha\beta}\bar\partial f^\alpha \wedge \partial \bar f^\beta) \wedge \omega^{n-2} = 0. \tag{3.9}$$

On the other hand, we have Siu's Bochner type identity [22, Sect. 3] (in integrated form)

$$\int_M \partial\bar\partial (g_{\alpha\beta}\bar\partial f^\alpha \wedge \partial \bar f^\beta) \wedge \omega^{n-2} = \int_M \{R_{\alpha\beta\gamma\delta}\bar\partial f^\alpha \wedge \partial \bar f^\beta \wedge \partial f^\gamma \wedge \bar\partial \bar f^\delta \wedge \omega^{n-2}$$
$$- g_{\alpha\beta} D\bar\partial f^\alpha \wedge \bar D \partial \bar f^\beta \wedge \omega^{n-2}\}. \tag{3.10}$$

As in [22, Sect. 4], since f is harmonic and the metric of the unit ball is strongly negative, both terms in the integral on the right hand side are pointwise nonpositive, hence

$$D\bar\partial f^\alpha \equiv \bar D \partial \bar f^\beta \equiv 0 \tag{3.11}$$

and

$$R_{\alpha\beta\gamma\delta}\bar\partial f^\alpha \wedge \partial \bar f^\beta \wedge \partial f^\gamma \wedge \bar\partial \bar f^\delta \equiv 0. \tag{3.12}$$

Lemma 1. *f is pluriharmonic, i.e. the restriction of f to any local complex curve in M is harmonic.*

c) Global Behavior

A neighborhood of a cd can be locally fibered by local holomorphic curves of the type of the unit disk that are transversal to the cd and that have the property that their intersection with a bn is a homotopically nontrivial curve in the bn. This follows from the assumption that M and N are homotopically equivalent and from Lemma 3 below. On each subdisk, we let again its center O correspond to the intersection point with the cd. Because of (K 4) and $E(f) < \infty$, the restriction of f to almost every such disk has finite energy (and is harmonic as established in b). We now need

Lemma 2. *Let M be a manifold with (smooth) boundary and of finite volume, and let N be complete with nonpositive sectional curvature. Let $g\colon \partial M \to N$ be a smooth map. Let $f\colon M \to N$ be a harmonic map of finite energy with $f|\partial M = g$. Then there is no other harmonic map with finite energy and the same boundary values homotopic to f.*

Proof. We can take over the arguments of [21, Sect. 2], if we note that the function $\varphi = (\varrho^2+1)^{1/2}$ defined on p. 369 of [21] satisfies $\nabla\varphi = 0$ on ∂M in our case, so that we can still perform the integration by part used in the proofs of Lemmas 1 and 2 of [21]. Q.E.D.

In particular, under the assumptions of Lemma 1, any such harmonic map is necessarily energy minimizing. This will be essential for our subsequent reasoning.

Lemma 3. *Let X be a complete (noncompact) manifold of nonpositive curvature. Let U be a neighborhood of an end of X. Let γ be a curve in U which is homotopically nontrivial in U but which can be homotoped in U into arbitrarily short curves. Then γ is homotopically nontrivial in X.*

Proof. We take some δ-neighborhood V_δ of ∂U. There exists $\varepsilon > 0$ with

$$\operatorname{length}(\tau) > \varepsilon \tag{3.13}$$

for any homotopically nontrivial curve τ in V_δ. Let us assume also

$$\delta > \varepsilon . \tag{3.14}$$

Let U' be the component of $X \setminus V_\delta$ with

$$U' \cap U \neq \emptyset .$$

By assumption, we can move γ into U' and achieve

$$\operatorname{length}(\gamma) < \varepsilon . \tag{3.15}$$

Let $\gamma(t)$ denote the images of $\gamma = \gamma(0)$ under the heat flow ($t \geqq 0$). If the initial curve γ was parametrized proportional to arclength,

$$\operatorname{length}(\gamma(t)) \leqq \operatorname{length}(\gamma) \text{ for all } t \geqq 0 . \tag{3.16}$$

For background information on the heat flow, see [10, Chap. 3].

By (3.13)–(3.16), and since γ and hence $\gamma(t)$ is homotopically nontrivial in U, no $\gamma(t)$ can be contained in V_δ, and hence

$$\gamma(t) \subset U$$

for all $t \geqq 0$.

Since on the other hand, γ can be homotoped to arbitrarily short curves by moving towards the end, $\gamma(t)$ cannot converge to a closed geodesic (such a geodesic would have to realize the minimal length in its homotopy class) and hence has to move towards the end as well.

If γ would be homotopic to a point p, then

$$p(t) \equiv p$$

would be another homotopic solution of the heat flow, and by the stability lemma of Hartman (cf. [10, 3.4])

$$\operatorname{dist}(p(t), \gamma(t))$$

had to be nonincreasing contradicting that $\gamma(t)$ moves towards the end. Q.E.D.

A similar argument yields

Lemma 4. *Let X be a complete, nonpositively curved manifold with at least two ends. Let U be a neighborhood of one end, and let γ be a homotopically nontrivial curve in U that can be homotoped into arbitrarily small curves by moving towards the corresponding end. Then it is not possible to shrink γ by moving towards a different end as well.*

Let D be a disk, i.e. a local holomorphic curve, transversal to a cd and having the property that its intersection with a bn is homotopically nontrivial in this bn. By Lemma 1, $f|D$ is harmonic. Since the energy of $f|D$ and the fact that $f|D$ is harmonic do not depend on the metric of D but only on the conformal structure (cf. [10, 1.3]), we can use the standard flat metric on each such D. Again, we let $O \in D$ be the intersection of D with the cd.

Lemma 5. *If the energy of $f|D$ is finite, then $f|D$ is proper, i.e. for any compact $K \subset N$ there is a neighborhood U_K of $O \in D$ with*

$$f^{-1}(K) \cap D \cap U_K = \emptyset .$$

Proof. We write $u := f|D$. Let $D_\varrho := \{(r, \varphi) \in D \colon 0 \leqq r \leqq \varrho\}$, where we use standard polar coordinates on D.

Since $E(u) < \infty$, for the energy of u on D_ϱ we have

$$E_\varrho(u) = \int_{r=0}^{\varrho} \int_{\varphi=0}^{2\pi} \left(u_r^2 + \frac{1}{r^2} u_\varphi^2\right) \mathrm{r d}\varphi\, dr \to 0 \text{ as } l \to 0. \tag{3.17}$$

For $0 \leqq r_1 \leqq r_2$, we can find $\varphi_0 \in [0, 2\pi]$ with

$$\begin{aligned} d(u(r_1, \varphi_0), u(r_2, \varphi_0)) &\leqq \int_{r=r_1}^{r_2} \int_{\varphi=0}^{2\pi} |u_r|\, dr\, d\varphi \\ &\leqq \left(\int_{r_1}^{r_2} \int_{\varphi} \frac{1}{r} dr\, d\varphi\right)^{1/2} \cdot \left(\int_{r_1}^{r_2} \int_{\varphi} |u_r|^2 r dr d\varphi\right)^{1/2} \\ &\leqq \left(\log \frac{r_2}{r_1}\right)^{1/2} \cdot E_{r_2}(u)^{1/2} . \end{aligned} \tag{3.18}$$

Likewise, given $0 \leqq \varrho_1 < \varrho_2$, we can find $r_0 \in [\varrho_1, \varrho_2]$ with the property that for all $\varphi_1, \varphi_2 \in [0, 2\pi]$

$$\begin{aligned} d(u(r_0,\varphi_1), u(r_0,\varphi_2)) &\leqq \int_{\varphi=0}^{2\pi} |u_\varphi|\, d\varphi \leqq 2\pi \left(\int_0^{2\pi} |u_\varphi|^2\, d\varphi \right)^{1/2} \\ &\leqq 2\pi (E_{\varrho_2}(u))^{1/2} \cdot \left(\int_{\varrho_1}^{\varrho_2} \frac{1}{r}\, dr \right)^{-1/2} \\ &= 2\pi \left(\log \frac{\varrho_2}{\varrho_1} \right)^{-1/2} E_{\varrho_2}(u)^{1/2}. \end{aligned} \tag{3.19}$$

Let now an end of M be given. This end is mapped under the map h constructed in Sect. 2 to an end of N. Let U be a neighborhood of this end of N, and let D intersect the given end of M. As h is a homotopy equivalence, $h(\partial D)$ is (homotopic to) a curve satisfying the assumption of Lemma 3.

Using Lemmas 3 and 4 and (3.17)–(3.19) given $\varepsilon > 0$, we can find some sufficiently small $R_0 > 0$, some $R_1 \in \left[\frac{R_0}{2}, R_0\right]$, $R_2 \in \left[\frac{R_0}{8}, \frac{R_0}{4}\right]$, and $\varphi_0 \in [0, 2\pi]$ so that $u(\partial D_{R_1})$, $u(\partial D_{R_2})$, and $u([R_2, R_1] \times \{\varphi_0\})$ are all contained in U and of length at most $\varepsilon/4$.

We then look at the curve ℓ in D obtained in the following way: first, keep $r = R_1$ fixed and let φ run from φ_0 to $\varphi_0 + 2\pi$. Then, keep $\varphi = \varphi_0$ (identified with $\varphi_0 + 2\pi$) fixed and let r run from R_1 and R_2. Then keep $r = R_2$ fixed and let φ again run from φ_0 to $\varphi_0 + 2\pi$. Finally, keep again $\varphi = \varphi_0$ fixed and let r run from R_2 to R_1. ℓ then is a homotopically trivial curve in $D \setminus \{0\} \subset M$ with $u(\ell) \subset U$ and

$$\operatorname{length}(u(\ell)) < \varepsilon. \tag{3.20}$$

We then look at the lifts

$$\tilde{u}: D \setminus \{0\} \to \tilde{N}$$

to universal covers. The lift $\tilde{\ell}$ of ℓ is a closed curve and bounds a region B. We then look at the harmonic extension

$$\begin{gathered} \bar{u}: B \to \tilde{N} \\ \bar{u}|\tilde{\ell} = \tilde{u}|\tilde{\ell}. \end{gathered}$$

(3.20) implies that there is a ball $B(p, \varepsilon) \subset \tilde{N}$ of radius ε with

$$\bar{u}(\tilde{\ell}) \subset B(p, \varepsilon).$$

Since $\tilde{N}$ is simply connected and nonpositively curved, it follows from the maximum principle [applied to $\tilde{d}^2(p, \bar{u}(\cdot))$, where $\tilde{d}(\cdot, \cdot)$ is the distance function on $\tilde{N}$] that

$$\bar{u}(B) \subset B(p, \varepsilon). \tag{3.21}$$

On the other hand, by uniqueness (Lemma 2),

$$\bar{u} = \tilde{u}|B|.$$

This, together with (3.21), shows that u maps the whole annulus $R_2 \leqq r \leqq R_1$ into an ε-neighborhood of U.

In this way, we can cover D by smaller and smaller annuli that are mapped under u further and further towards the end, and we deduce that u is proper as claimed. Q.E.D.

We now let $\mathcal{D}$ be the collection of all disks of a local fibration of a neighborhood of a cd. by transversal disks with the above homotopy property. We want to control the energy of f on all disks in $\mathcal{D}$. We start with a disk $D \in \mathcal{D}$ on which the energy of f is finite. (3.17) and the chain of inequalities in (3.19) imply that we can always find some sufficiently small $r_0 > 0$ for which

$$\int_{\varphi=0}^{2\pi} |u_\varphi(r_0, \varphi)|^2 \, d\varphi := \delta(r_0, D)$$

becomes arbitrarily small. (Again, we put $u := f|D$.) Since f is smooth, we can find a neighborhood $\mathcal{U}$ of D in $\mathcal{D}$ such that for all $D' \in \mathcal{U}$

$$\delta(r_0, D') \leqq 2\delta(r_0, D).$$

On D'_{r_0}, we then construct a comparison map $v = v_{D'}$ as follows: for $r = r_0$, we put $v(r_0, \varphi) = f|D'(r_0, \varphi)$, and for $r = \frac{r_0}{2}$, $v\left(\frac{r_0}{2}, \varphi\right) = u(r_0, \varphi)$. For $r_0/2 \leqq r \leqq r_0$, we let v be the harmonic extension of its boundary values, and for $r \leqq r_0/2$, we define

$$v(r, \varphi) = u(2r, \varphi).$$

We see that there is a fixed constant K, independent of $D' \in \mathcal{U}$, that bounds the energy of $v_{D'}$ on D'_{r_0}. We note that $v_{D'}$ has the same boundary values as $f|D'$ on $\partial D'_{r_0}$.

Since harmonic maps with finite energy are unique by Lemma 2, the restriction $f|D'_{r_0}$ for $D' \in \mathcal{U}$ has to coincide with the energy minimizing map with the same boundary values. First, this follows for those D' where we already know that the energy of f is finite. Since these disks are dense in $\mathcal{U}$, and since we now have uniform bounds for the energy on these disks, and since f is smooth, the result follows for all disks in $\mathcal{U}$. On the other hand, $\mathcal{D}$ is compact (since the cd.'s are compact) and hence is covered by a finite number of such neighborhoods. Therefore, we can apply the argument of the proof of Lemma 5 uniformly to all $D \in \mathcal{D}$ and deduce

Lemma 6. *Let (γ_n) be a sequence of (in M) homotopically nontrivial curves shrinking to a point in a cd of $\bar{M}$. Then the sequence $f(\gamma_n)$ of image curves is not contained in any compact subset of N. Moreover, this establishes a well-defined correspondence between the ends of M and N.*

d) f has Maximal Rank at Some Point

α) We first treat the case where M (and N) have at least two ends.

From Lemma 6 we see that we can control the mapping of the ends. Namely, f maps a given end in the domain to the same end in the image as the original proper homotopy equivalence h of Sect. 2 does.

On the other hand, since we have more than one end, a bn. of an end defines a nontrivial homology class of real dimension $2n-1$, and the image under f again is a nontrivial homology class.

This together with Lemma 6 implies that the functional determinant of f cannot vanish identically.

β) The case where M and N have only one end can be reduced to the preceding one as follows:

After a suitable conjugation, we can assume that the given cusp in N corresponds to the point at infinity in the universal cover $\tilde{N}$ (i.e. to $\operatorname{Im} w = \infty$ resp. $\operatorname{Im} w_1 = \dots \operatorname{Im} w_n = \infty$ in the representations of 1b) resp. 1c)). The isotropy group of the cusp then has a subgroup of finite index leaving the level sets of the Bergmann kernel function invariant. The quotient of $\tilde{N}$ by this subgroup and hence also by the isotropy group has infinite volume, because geodesic rays running into the cusp are permuted and the distance between these rays increases when moving away from the cusp because of the nonpositivity of the sectional curvature. Hence $\pi_1(N)$ must contain an element γ not fixing the cusp. Since $\pi_1(N)$ is residually finite by a result of Selberg (cf. [2, p. 39]), it contains a normal subgroup Γ of finite index not containing γ. Then $\tilde{N}/\Gamma$ has more than one end. Therefore, by the argument of α), the lift of f to some finite coverings and hence also f has maximal rank somewhere.

For more details on the isotropy group of a cusp, cf. Eberlein [5] and Hirzebruch [9] (for the Hilbert modular varieties).

4. Proof of the Theorems

Theorem 2 follows from the argument of Mok [16] in conjunction with the preceding construction of a harmonic homotopy equivalence which has maximal rank somewhere, as established in 3d).

For Theorem 1, it first follows from Siu's arguments [22], using (3.12) and that the harmonic map f has maximal rank at some point, that f is $\pm$ holomorphic. N can be compactified as an algebraic variety by adding a simple point to each cusp. Note that this does not depend on the locally symmetric structure of N, but only on the fact that the sectional curvature of N is bounded between two negative constants, cf. [23]. The Schwarz lemma of [25] then implies in conjunction with Lemma 6 that f is proper.

If $\bar{M}$ is Hironaka's nonsingular compactification of M, then f can be extended as a holomorphic map $\bar{f}$: $\bar{M} \to \bar{N}$, cf. e.g. [13, Corollary 3.7, p. 100]. Also, the extension $\bar{f}$ is a map of degree ± 1. This is seen as follows: First, since $\pm$ holomorphic of maximal rank, it cannot have degree 0. Let g: $N \to M$ be a homotopy equivalence so that $h = f \circ g$ is homotopic to the identity of N. Let v_N be the volume form of N. We use our preceding construction (modifying h into a finite energy map and then use harmonic replacement on an increasing set of domains) to convert h into a proper harmonic self-map h' of N. It is clear from the construction that $\deg h = \int h * v_N)(\operatorname{vol} N)^{-1}$ remains unchanged, since integer valued. Namely, one just choses a domain D_n with $\left|\int_{D_n} h * v_N\right| \geqq (\deg h - \frac{1}{2}) \operatorname{vol} N$, observes that harmonic replacement on D_n does not change this number and then concludes that the degree can no longer jump to a smaller integer as the domain increases.

As above h' is $\pm$ holomorphic of maximal rank. The Schwarz lemma [25] then implies

$$|\deg f \cdot \deg g| = |\deg h'| \leqq 1$$

so that f is of degree ± 1.

In order to show that f is bijective on M, we can now proceed as in [22, p. 110f.], namely show that the set V of those points in M, where f is not locally homeomorphic, is empty, since otherwise V would be a complex hypersurface in M, extending into $\bar{M}$, whereas $f(V)$ would have codimension at least 2. The preimage of a generic point in $f(V)$ is a nontrivial compact analytic subvariety of M (since f is proper). This contradicts the fact that f is a homotopy equivalence.

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The Stable Adjunction Map on Gorenstein Varieties

Marco Andreatta
Dipartimento di Matematica, Università di Trento, I-38050 Povo (TN), Italy

Introduction

Let X be a normal irreducible n-dimensional projective variety whose local rings are Cohen Macaulay and whose dualizing sheaf, K_X, is invertible. We will deal with such varieties and call them Gorenstein n-folds; moreover, in order to use vanishing theorems for cohomology, we suppose there are only a finite number of non rational points.

Let L be an ample line bundle on X.

A powerful tool in the classification problems is the adjoint system $|K_X \otimes L^{n-1}|$ and the morphism that it eventually defines on X.

The study of this problem date back to the classical period of the Italian school, mainly in the works of Castelnuovo and Enriques. Sommese has set up the problem in modern language, clarifying and improving it [10]. If L is spanned by global sections or X is smooth it is known that $K_X \otimes L^{n-1}$ is numerically effective and big except for a few well understood cases [5, 6, 14].

Assume now $K_X \otimes L^{n-1}$ numerically effective and big. Suppose also the existence of $n-2$ divisors of $|L|$ intersecting scheme theoretically in a smooth surface S; this happens, for instance, if L is spanned and the singular locus has codimension at least 3. Then we will show that the set on which $K_X \otimes L^{n-1}$ is not ample is the disjoint union of Cartier divisors that can be blown down to give a new Gorenstein n-fold X', $\pi : X \to X'$, and an ample line bundle L' on X' such that $K_{X'} \otimes L'^{n-1}$ is ample and

$$\pi^*(K_{X'} \otimes L'^{n-1}) = K_X \otimes L^{n-1}.$$

Actually the map associated to a sufficiently high power of the adjoint system $|K_X \otimes L^{n-1}|$, that will be called the stable adjunction mapping associated to (X, L), gives the blow down $\pi : X \to X'$.

If moreover $h^0(K_X \otimes L^{n-2})^N \neq 0$ for some $N > 0$, we also prove that $K_{X'} \otimes L'^{n-2}$ is nef and that all the smooth surfaces S intersection of members of $|L|$ have Kodaira dimension $\geqq 0$ and $\pi_S : S \to S' = \pi(S)$ is the map of S onto its minimal model.

For $n=3$ the result has been proved in [11] under the additional hypothesis that X is smooth and with non negative Kodaira dimension and in [3] under the hypothesis that the logarithmic Kodaira dimension of X is bigger or equal then zero.

A similar result is proved in the smooth case in [5, 6].

0. Notation and Statement of the Result

0.0. We shall work over the field of complex number. All spaces are complex analytic and all maps are holomorphic. By *variety* we mean an irreducible and reduced complex analytic space. If X is a complex analytic space we denote its holomorphic structure sheaf by $\mathcal{O}_X$ and its dualizing sheaf by K_X. We denote by $\operatorname{reg}(X)$, $\operatorname{Sing}(X)$, $\operatorname{Irr}(X)$ respectively the set of smooth points, of singular points, of non rational points of X.

Let X an analytic space and S an analytic subspace of X. If E is a vector bundle on X we denote by E_S its restriction to S, with the exception $T_{X,S}$, denoting the restriction of the tangent space of X to S. $N_{S/X}$ will denote, when defined, the normal bundle of S in X.

0.1. Our notation will not distinguish between a holomorphic vector bundle on a complex space X and its sheaf of germs of holomorphic sections. Given a coherent analytic sheaf of abelian groups on X, $\mathcal{S}$, we denote: the set of the global section of $\mathcal{S}$ by $\Gamma(\mathcal{S})$ or $\Gamma(\mathcal{S}, X)$; the complex dimension of $H^i(X, \mathcal{S})$ by $h^i(\mathcal{S}) = h^i(X, \mathcal{S})$.

0.2. Let X be an irreducible projective variety and D be an effective Cartier divisor on X. Let $[D]$ be the holomorphic line bundle associated to D. If L is a holomorphic line bundle on X let $|L|$ denote the linear system of all Cartier divisors associated to L. We often let L denote either the homology class associated to a $S \in |L|$ or its Poincaré dual $c_1(L)$, e.g. given a curve $C \subset X$ then $L \cdot X = D \cdot X = c_1(L)[C]$.

0.3. A normal complex analytic space X is called *Gorenstein* if its local rings of holomorphic function are Cohen-Macaulay and K_X is invertible. Let L a line bundle on a Gorenstein space, X, and $A \in |L|$, A normal, we have

$$(K_X \otimes L)_A \cong K_A \quad \text{(see for instance [1])}.$$

0.4. A line bundle L on a projective variety X is said to be *big* if $c_1(L)^{\dim X} > 0$ and to be *numerically effective* (or *nef* for short) if $c_1(L)[C] \geqq 0$ for all effective curves on X. If L is nef then it is big if and only if $K(L) = \dim X$, where $K(L)$ is the dimension of the image of the map associated to L^N for $N \gg 0$. If there is a N such that L^N is spanned by global sections then L is said to be *semi-ample*. If L is semi-ample and the map associated $\Gamma(L^N)$ for $N \gg 0$ is an embedding then L is said to be *ample*.

0.5. What follows is a powerful technical result due to Sommese [14].

Theorem. *Let X be a Gorenstein variety with* $\operatorname{Irr}(X)$ *finite. Let L be an ample line bundle on X and let $A \in |L|$ be a Gorenstein variety. Let τ be an integer $\geqq 2$. If $K_A \otimes L_A^{\tau-1}$ is semi ample (nef), then $K_X \otimes L^\tau$ is semi ample (nef).*

0.6. We shall consider the set $\mathscr{P}=\{(X,L)\}$ of pair (X,L) where X is a Gorenstein n-fold, i.e. a Gorenstein variety of dimension n, with $\mathrm{Irr}(X)$ finite and L is an ample line on X. $(X,L)\in\mathscr{P}$ is often called a polarized variety.

Definition. A *reduction* of the pair $(X,L)\in\mathscr{P}$ is a pair $(X',L')\in\mathscr{P}$ such that

a) there is a surjective map $\pi: X\to X'$ expressing X as X' with a finite set $F\subset \mathrm{reg}(X')$ blown up,

b) $L\cong \pi^*L'\otimes[\pi^{-1}(F)]^{-1}$ or equivalently $K_X\otimes L^{n-1}\cong\pi(K_{X'}\otimes L'^{n-1})$.

Note that there is a one to one correspondence between the $A\in|L|$ that are smooth in a neighborhood of $\pi^{-1}(F)$ and the $A'\in|L'|$ such that $F\subset\mathrm{reg}(A')$ got by sending such an A' to its proper transform A. If there exists an $A\in|L|$ which is smooth in a neighborhood of $\pi^{-1}(F)$ then $L'=[\pi(A)]$ (see 2.4).

0.7. To achieve our goal we have to make the following assumption for the pair $(X,L)\in\mathscr{P}$.

We suppose the existence of $n-2$ divisor, A_i, belonging to $|L|$, which intersect scheme theoretically in a smooth surface S.

Observation. Call $D_0=X$ and $D_i=A_1\cap\ldots\cap A_i$ for $i=1,\ldots,n-2$. Our hypothesis will imply, using the Serre's criterion for normality and the amplenes of L, that each D_i is normal and Gorenstein.

Our assumption is fulfilled if, for instance, L is spanned and $\mathrm{Cod}\,\mathrm{Sing}(X)\geqq 3$; this follows by applying a general Bertini's theorem for which we refer to Sect. 0.4 of [14].

0.8. **Proposition.** *Let (X,L) be as in* 0.6 *and* 0.7 *and suppose $K_X\otimes L^{n-1}$ is nef. Then $K_X\otimes L^{n-1}$ is semi ample.*

Proof. By the hypothesis, using the adjunction formula in 0.3 we see that $K_{D_{n-2}}\otimes L_{D_{n-2}}$ is nef. By Theorem 2.6 in [7], $K_{D_{n-2}}\otimes L_{D_{n-2}}$ is semi ample. The conclusion follows from Theorem 0.5.

We can then give the following

0.8.1. *Definition.* Let $(X,L)\in\mathscr{P}$. Assume that $K_X\otimes L^{n-1}$ is numerically effective. Choose $t>0$ such that $(K_X\otimes L^{n-1})^t$ is spanned at all points by global sections and the map $\Phi: X\to\Phi(X)\subset\mathbb{P}_{\mathbb{C}}$ associated to $\Gamma[(K_X\otimes L^{n-1})^t]$ has connected fibres and a normal image. Then $X\to\Phi(X)$ is called the *stable adjunction mapping* associated to (X,L). (This definition is given in [13].)

0.9. We conclude the section stating the

Main Theorem. *Let X be a Gorenstein n-fold such that* $\mathrm{Irr}(X)$ *is a finite set and L be an ample line bundle on X. Suppose there exist a smooth surface S belonging to the system $|L|$ (see* 0.7*).*

If $K_X\otimes L^{n-1}$ is nef and big then there exist a reduction (X',L') of (X,L) (0.6) *such that $K_{X'}\otimes L'^{n-1}$ is ample.*

If $h^0(K_X\otimes L^{n-2})^N\neq 0$ for some $N>0$, all the smooth surfaces S in the intersection of members of $|L|$ have Kodaira dimension $\geqq 0$ and if $\pi: X\to X'$ is the map giving the reduction then $\pi_S: S\to S'=\pi(S)$ is the map of S onto its minimal model. Furthermore $K_{X'}\otimes L'^{n-2}$ is nef.

1. Some Technical Results

In this section we collect a number of technical lemmas, which will be used to prove the main theorem.

1.1. **Lemma.** *Let X be a normal variety of dimension $n \geqq 3$. Suppose it contains an ample Cartier divisor D biholomorphic to $\mathbb{P}^{n-1}$ and such that if $L=[D]$ then $L|_{\mathbb{P}^{n-1}} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Then $X \cong \mathbb{P}^n$ and $L=\mathcal{O}_{\mathbb{P}^n}(1)$.*

Proof. By the long exact sequence associated to the exact sequence

$$0 \to L^{-1} \to \mathcal{O}_X \to \mathcal{O}_{X|\mathbb{P}^{n-1}} \to 0 \quad \text{and using} \quad H^1(L^{-1})=0=H^1(\mathcal{O}_{\mathbb{P}^{n-1}})$$

it follows that $H^1(\mathcal{O}_X)=0$. Using now the long exact sequence associated to

$$0 \to \mathcal{O}_X \to L \to L_{\mathbb{P}^{n-1}} \to 0 \quad \text{we have that the map} \quad \Gamma(L) \to \Gamma(L_{\mathbb{P}^{n-1}})$$

is surjective. By this and the hypothesis it follows that $h^0(\mathcal{O}(L)) \geqq n+1$. It is well known that in this case (X, L) is biholomorphic to $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

1.2. **Lemma.** *Let $(X, L) \in \mathcal{P}$ a polarized variety and assume that* $\mathrm{Irr}(X)$ *is a finite set. Then $H^j(K_X^a \otimes L^{(n-1)(a-1)+r})=0$ for all $a, r, j \in \mathbb{N}^+$ and $a, j > 1$. Moreover if $A_i \in |L|$ and $D_i = A_1 \cap \ldots \cap A_i$ are as in* 0.7 *we have $H^j(K_{D_i}^a \otimes L^{(n-i-1)(a-1)+r})=0$ for all $a, r, j \in \mathbb{N}^+$ and $a, j > 1$.*

Proof. $K_X^{a-1} \otimes L^{(n-1)(a-1)} \otimes L^r$ is nef and big because so is $K_X \otimes L^{n-1}$ and L^r is ample. Applying the Kodaira-Kawamata-Viehweg vanishing theorem (see [9]) the result follows. The second part of the lemma follows by induction using the long exact sequence associated to the following exact sequence. Let $A \in |L|$ normal, we have

$$0 \to K_X^a \otimes L^{(n-1)(a-1)} \otimes L^r \to K_X^a \otimes L^{(n-1)(a-1)} \otimes L^{r+1} \to K_A^a \otimes L^{(n-2)(a-1)} \otimes L^r \to 0$$

obtained by tensoring with $K_X^{a-1} \otimes L^{(n-1)(a-1)} \otimes L^r$ the adjoint exact sequence $0 \to K_X \to K_X \otimes L \to K_A \to 0$.

A more elaborate version of the following Lemma 1.3.1 is given in [12]. Since [12] is not published we will give a short proof suggested by Sommese; it uses Lemma 1.3 due to Bombieri ([2]; see also Lemma 7.57 in [9]).

1.3. **Lemma.** *Let $\mathcal{L}$ be a line bundle on a smooth surface such that $\mathcal{L} \cdot \mathcal{L} \geqq 10$. $h^0(\mathcal{L}) \geqq 7$ and all $C \in \mathcal{L}$ are numerically 3-connected (i.e. for every splitting $C = C_1 + C_2$, with C_i effective, the inequality $C_1 \cdot C_2 \geqq 3$ holds).*

Then $K_S \otimes \mathcal{L}$ is very ample.

1.3.1. **Lemma.** *Let S be a smooth surface and L an ample line bundle on S such that $K_S \otimes L$ is nef and big.*

Then for all N large enough $K_S^N \otimes L^{N+1}$ is very ample.

Proof. It is sufficient to prove that $\mathcal{L} = K_X^N \otimes L^{N+2}$ verifies the hypotheses of Lemma 1.3.

$$\mathcal{L} \cdot \mathcal{L} = N^2(K_X \otimes L) \cdot (K_X \otimes L) + 4L \cdot L + 4N(K_X \otimes L) \cdot L \geqq 10 \ (\text{for } N \geqq 2).$$

By the Kodaira-Kawamata-Viehweg vanishing theorem and the Riemann-Roch theorem we have

$$h^0(K_X^N \otimes L^N \otimes L^2) = N^2/2(K_X \otimes L)\cdot(K_X \otimes L) + \text{terms of lower degree in } N,$$

and so $\mathscr{L}$ has more then 7 sections for N large enough.

$\mathscr{L}$ is ample and therefore 1-connected. Suppose there exists $C = C_1 + C_2 \in |\mathscr{L}|$ such that C_1 and C_2 are effective and $C_1 \cdot C_2 = 1$.

For $i = 1, 2$ we have the equality

$$C_i \cdot C_i = \mathscr{L} \cdot C_i - C_1 \cdot C_2 = NC_i \cdot (K_X \otimes L) + 2C_i \cdot L - C_1 \cdot C_2. \qquad (*)$$

This implies $C_i \cdot C_i > 0$; by the hypotheses on $K_X \otimes L$ and the Hodge Index theorem we have that $C_i \cdot (K_X \otimes L) > 0$.

Hodge Index theorem also tells us that $(C_1 \cdot C_2)^2 \geqq (C_1 \cdot C_1)(C_2 \cdot C_2)$, i.e. $C_i \cdot C_i = 1$. Using $(*)$ we have the contradiction $1 \geqq N + 2 - 1 > 1$.

Suppose now $C_1 \cdot C_2 = 2$, by $(*)$ we have that $C_i \cdot C_i \geqq 0$. We use again the index theorem to have $(K_X \otimes L) \cdot C_i > 0$ and $4 = (C_1 \cdot C_2)^2 \geqq (C_1 \cdot C_1)(C_2 \cdot C_2)$.

Therefore there is at least one i such that $2 \geqq C_i \cdot C_i$.

Going back to $(*)$ we have a contradiction for $N > 2$.

Thus $\mathscr{L}$ is 3-connected (for $N > 2$).

The next lemmas are special cases of Sommese's Lemmas 0.7.2 and 0.7.3 in [11] and they come from the foundational existence theorem and the properties of the Hilbert scheme.

1.4. **Lemma.** *Let E a smooth rational curve contained in the smooth points of an n-fold X such that $H^1(E, N_E) = 0$. If $\Gamma(N_E)$ spans N_E then the union of the smooth deformations of E in X is a dense subset of X.*

1.4.1. **Lemma.** *Let E as in* 1.4.

Assume that $T_{X,E}$ splits holomorphically as $F \oplus G$ where

a) *the composition of $T_E \to T_{X,E}$ with $T_{X,E} \to F$ is the* 0*-map and the composition of $T_E \to T_{X,E}$ with $T_{X,E} \to G$ has a subbundle H as image.*

b) *$\Gamma(F) = 0$ and $\Gamma(G/H)$ spans G/H.*

Call P the union of the smooth deformations of E in X and assume $\dim P = \operatorname{rank} G$. *Then the small deformations of E in X form a submanifold, V, of a neighborhood of E in the smooth points of X. Moreover G/H is the normal bundle of E in V.*

In order to apply the above result we will need the following

1.5. **Lemma.** *Take (X, L) as in* 0.7 *and suppose that D_{n-2} contains a smooth rational curve such that $E \cdot E = -1$ on D_{n-2} and $L \cdot E = a > 0$. Suppose that N_{E/D_i} is not spanned by global sections for all $i = 0, \ldots, n-2$.*

Then for $i = 0, \ldots, n-2$

$$N_{E/D_i} = N_{E/D_{i+1}} \oplus i^* N_{D_{i+1}/D_i} = \mathcal{O}(-1) \oplus \mathcal{O}(a)^{\oplus k} \oplus \mathcal{O}(a),$$

where $k = n - i - 3$ and $i: E \to D_{i+1}$ is the inclusion.

Proof. We have the exact sequences

$$0 \to N_{E/D_{i+1}} \to N_{E/D_i} \to i^* N_{D_{i+1}/D_i} \to 0. \tag{$*$}$$

For $i=n-3$, by the hypothesis, $(*)$ becomes

$$0 \to \mathcal{O}(-1) \to N_{E/D_{n-3}} \to \mathcal{O}(a) \to 0. \tag{$**$}$$

It splits: in fact by the Grothendieck's theorem $N_{E/D_{n-3}} = \mathcal{O}(r) \oplus \mathcal{O}(s)$ with $r+s = a-1 \geqq 0$. By hypothesis $N_{E/D_{n-3}}$ is not spanned and so we can suppose $r<0$ and $s \geqq 0$. Then $h^0(N_{E/D_{n-3}}) = s+1$. The long exact sequence associated to $(**)$ implies that $h^0(N_{E/D_{n-3}}) = a+1$ and therefore $s=a$ and $r=-1$. The splitting of $(**)$ is our thesis for $i=n-2$.

Suppose now $i<n-2$. By inductive hypothesis $(*)$ becomes

$$0 \to \mathcal{O}(-1) \oplus \mathcal{O}(a)^{\oplus k} \to N_{E/D_i} \to \mathcal{O}(a) \to 0. \tag{$***$}$$

Using Grothendieck's decomposition we can write

$$N_{E/D_i} = \mathcal{O}(b_1) \oplus \dots \oplus \mathcal{O}(b_{k+2}) \quad \text{with} \quad b_1 \leqq \dots \leqq b_{k+2}.$$

We have first of all $b_1 \geqq -1$. In fact if $b_1 < -1$ then $H^1(E) \neq 0$; but this is impossible from the long exact sequences associated to $(***)$.

By hypothesis we have that at least $b_1 = -1$. It follows from $(***)$ that for each $x \in E$ there are $k+1$ non constant global sections generating a subspace of codimension 1 in E_x, and thus that $b_2 > 0$.

We have then the following sequences

$$0 \to \mathcal{O}(-1) \to \mathcal{O}(-1) \oplus \mathcal{O}(b_i)^{\oplus} \to \mathcal{O}(b_i)^{\oplus} \to 0$$

and

$$0 \to N_{E/D_{n-2}} \to N_{E/D_{i+1}} \to i^* N_{D_{n-2}/D_{i+1}} \to 0$$

that is

$$0 \to \mathcal{O}(-1) \to \mathcal{O}(-1) \oplus \mathcal{O}(b_i)^{\oplus} \to \mathcal{O}(a)^{\oplus} \to 0.$$

They imply $\mathcal{O}(b_i)^{\oplus} = \mathcal{O}(a)^{\oplus}$ and hence the thesis.

The following trivial lemma is the starting point of our construction;

1.6. **Lemma.** *Let (X, L) a polarized smooth surface and suppose $K_X \otimes L$ big and nef. Then there is only a finite number of positive dimensional fibres of the stable adjunction map ϕ associated to (X, L) (see 0.8.1), say $E_1, \dots, E_r$, and each E_i is an irreducible curve, such that $E_i \cong \mathbb{P}^1$, $E_i \cdot E_i = -1$ and $L \cdot E_i = 1$.*

2. Proof of the Main Theorem

2.0. The proof of the theorem will be done by induction on the dimension of X in three different steps. In the first one we construct a finite set of contractible divisors of X. In the second step we show that these divisors are isomorphic to $\mathbb{P}^{n-1}$, are contained in the smooth points of X and are then Cartier. The last step proves that $K_X \otimes L^{n-1}$ is not ample only on these divisors and gives the conclusions.

2.1. In the notation in 0.7 we have, by the hypothesis and using the adjunction formula in 0.3, that $K_{D_i}\otimes L_{D_i}^{n-i-1}=K_{D_i}\otimes L_{D_i}^{\dim D_i-1}$ is nef and big for each D_i.

The following exact sequences, for $i=1,\ldots,n-2$ and $a>1$,

$$0\to K_{D_i}^a\otimes L_{D_i}^{(n-i-1)a-1}\to K_{D_i}^a\otimes L_{D_i}^{(n-i-1)a}\to K_{D_{i+1}}^a\otimes L_{D_{i+1}}^{(n-i-2)a}\to 0$$

give us, by means of Lemma 1.2, the surjectivity:

$$\Gamma(K_{D_i}^a\otimes L_{D_i}^{(n-i-1)a})\to\Gamma(K_{D_{i+1}}^a\otimes L_{D_{i+1}}^{(n-i-2)a})\to 0\,.$$

This two facts imply that, for N big enough, the stable adjunction map for (X,L), that is the map associated to $\Gamma(K_X\otimes L^{n-1})^N$ (see the definition in 0.8), restricts to the stable adjunction map for (D_i, L_{D_i}) for each i. Let us call this map Φ and Φ_{D_i} its restriction to D_i.

The hypothesis of bigness implies moreover that the Φ_{D_i} have $(n-i)$ dimensional images for each i.

2.2. *Part 1.* Let $n\geqq 3$. Consider $\Phi_{D_{n-2}}$; by the above it is the stable adjunction map for $(D_{n-2}, L_{D_{n-2}})$. By Lemma 1.6 the fibres of $\Phi_{D_{n-2}}$ are of finite number, say $E_1,\ldots,E_r$ (eventually $r=0$), each of them is a (-1) curve, that is a rational curve with self intersection -1, and $L\cdot E_s=1$ for each $s=1,\ldots,r$. Let E be one of these E_s.

2.2.1. **Proposition.** *The closure of the smooth deformation of E in X gives a divisor P on X meeting D_{n-2} transversally in E. Denoting with P' the normalization of P, $p:P'\to P$, then P' is biholomorphic to $\mathbb{P}^{n-1}$, $p^*[A_i]_{P'}=\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ for every i and* $\deg(N_{\mathrm{reg}P/E})=-1$.

Proof. We first rule out the possibility that all the deformations of E were dense in X. Simply note that $\Phi(E)$ is a point and by the maximum principle this must be true for all nearly deformations of E. Therefore, since $\dim\Phi(X)=n$, the deformations of E cannot be dense.

For $n=3$ the proposition can be proved exactly as in [11, 0.7.3 and 1.4]. The following proof works in fact also for $n=3$ but some extra work is needed because of the lack of Lemma 1.1 in dimension <3.

Let now $n>3$. By Lemmas 1.4 and 1.5.

$$\square\qquad N_{E/X}=N_{E/D_1}\oplus i^*N_{D_1/D_0}=\mathcal{O}(-1)\oplus\mathcal{O}(1)^{\oplus n-3}\oplus\mathcal{O}(1)\,.$$

Therefore $H^1(E,N_{E/X})=0=H^1(E,N_{E/D_1})$ and consequently there is no obstruction in deforming E in X and in D_1. By inductive hypothesis deforming E in D_1 we obtain a divisor P_1 meeting D_{n-2} transversally and such that, if $p_1:P_1'\to P_1$ is the normalization, $P_1'\cong\mathbb{P}^{n-2}$,

$$p^*[A_i\cap A_1]_{P_1'}=\mathcal{O}_{\mathbb{P}^{n-2}}(1)\quad\text{and}\quad\deg(N_{\mathrm{reg}P_1/D_1|E})=-1\,.$$

However, we know from the general theory of Hilbert scheme, that the global sections of $\Gamma(N_{E/X})$ give the tangent space of the base space of the deformations of E in X. Thus the decomposition $\square$ shows us that, deforming E in X, we can also move in the direction normal to D_1 in X, being $N_{D_1/D_0|E}=\mathcal{O}(1)$. So the closure of all smooth deformations of E in X has dimension $n-1$ and gives rise to a divisor, P, containing P_1. Let us see that $P\cap D_1=P\cap A_1=P_1$, and thus, by induction, that

$P \cap D_{n-2} = E$. The following decomposition is equivalent to $\square$

$$T_{X,E} = T_E \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1)^{\oplus n-3} \oplus \mathcal{O}(1);$$

this and Lemma 1.4.1 show that the small deformations of E in X form a submanifold V of a neighborhood of E in the smooth points of X and

$$N_{E/V} = \mathcal{O}(1)^{n-3} \oplus \mathcal{O}(1) = N_{D_{n-2}/D_1|E} \oplus N_{D_1/X|E}.$$

It follows by our construction that $N_{D_1 \cap V/X|P_1}$ = normal bundle of $\operatorname{reg} P_1$ in V and $N_{D_1/X|\operatorname{reg} P_1|E} = N_{D_1/X|E} = \mathcal{O}(1)$.

This implies, eventually shrinking V sufficiently, that $V \cap D_1 = P_1$. If $P \cap D_1 \neq P_1$ then, since $V \cap D_1 = P_1$, $P \cap D = P_1 + R$ where R is a divisor on P that contains no components equal to P_1 and the pullbacks of P_1 and R under the holomorphic map $p: P' \to P$ do not meet. But this is impossible since p is a finite to one map and effective ample divisors must be connected on a normal space of dimension at least two. Take now the normalization $p: P' \to P$ such that $p|_{P_1}: P'_1 \to P_1$ is the normalization of P_1.

Applying Lemma 1.1 at $P' \supset P'_1 \cong \mathbb{P}^{n-2}$ [note that $N_{P'_1/P'} = \mathcal{O}(1)$ follows from $p^*L|_{P'_1} = \mathcal{O}(1)$] we find that $P' \cong \mathbb{P}^{n-1}$ and P'_1 is a hyperplane section. By induction $p^*[A_i]|_{P'_1} = \mathcal{O}(1)$ and so $p^*[A_i]|_{P'} = \mathcal{O}(1)$. From $E \cdot E = -1$ it follows that $\deg N_{\operatorname{reg} P|E} = -1$.

Using the proposition we can now construct a finite number of divisors, P_s, one for each E_s, with the properties listed above; they are contracted to points by Φ and, since $\Phi(E_s) \neq \Phi(E_t)$ for $s \neq t$, they are disjoint. We will show in the next section that they are in fact Cartier divisors.

2.2.2. **Corollary.** *If E is a smooth projective rational curve on a smooth surface $S \in |L|$ such that $E \cdot E = -1$ on S, then $L \cdot E = 1$.*

Proof. Use $P \cdot D_{n-2} = E$ and $p^*[A_i]|_{P'} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

2.3. *Part 2.* Let P one of the divisors constructed in Part 1, $p: \mathbb{P}^{n-1} \cong P' \to P$ the normalization. Using the Proposition 2.2.1 it is easily checked that

$$p^*(K_X^{N-1} \otimes L^{(n-1)N-n+2}) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1).$$

We want to show that $(K_X^{N-1} \otimes L^{(n-1)N-n+2}) =: H$ is spanned in a neighborhood of P for some $N \gg 0$. We can carry the problem to $n=2$ by the following observation. (Remember $\dim D_i = n-i$.) If $K_{D_i}^{N-1} \otimes L_{D_i}^{(n-i-1)N-n+i+2}$ is spanned in a neighborhood of $P_i = P \cap D_i$ then $K_{D_{i-1}}^{N-1} \otimes L_{D_{i-1}}^{(n-i)N-n+i+1} =: B$ is spanned in a neighborhood of P_{i-1}.

In fact by the hypothesis, using $D_i \cap P_{i-1} = P_i$ and $H^1(K_{D_i}^{N-1} \otimes L^{(n-i)(N-1)}) = 0$ (Lemma 1.2), B is spanned in a neighborhood of P_i. We are able to conclude if we find, for every point $x \in P_{i-1} \backslash P_i$ a section of B not vanishing in x. This is easy: take $s \in ((K_{D_{i-1}} \otimes L^{(n-i)})^{N-1})$ not zero in x and tensor with the tantological section s_i of $[A_i]_{D_{i-1}}$. Now we only need to prove that for a smooth polarized surface (S, L) with $K_S \otimes L$ nef and big the line bundle $(K_S^{N-1} \otimes L^N)$ is spanned in a neighborhood of each fibre of the stable adjunction map Φ associated to (S, L) for sufficiently large N. By Lemma 1.3.1 we can actually find sufficiently large N for which the line bundle $(K_S^{N-1} \otimes L^N)$ is spanned at every points of S.

Since H is spanned by global sections in a neighborhood of each P_i and $p_i^*(H) = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ we can conclude that the p_i are biholomorphisms.

Let now $P = P_s$ for some s. We are in the following situation. There is a subvariety $P \subset X$, biholomorphic to $\mathbb{P}^{n-1}$ and such that $K_{X|P} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-n-1)$, and a holomorphic map $p: X^* \to A$ from a Zarisky neighborhood of P in X, X, to an affine variety, A, such that $p(P)$ is a single point and $p: X^* \backslash P \to A \backslash p(P)$ is a biholomorphism. Suppose now we prove that X is locally a hypersurface at any $x \in P \cap \operatorname{sing}(X)$, then we can apply the following result of Lipman-Sommese [8, Theorem 2.1].

Theorem. *In the above hypothesis either $P \cap \operatorname{sing} X$ is empty or it is of pure dimension $n-2$.*

Moreover remember that, by construction, $P \cap D_{n-2} = E \subset \operatorname{reg} X$. Thus, being L ample and $D_{n-2} \in |L|$, the dimension of the singular points of X contained in P is less then $n-2$.

We conclude that P is contained in the set of regular points of X. So that P is a Cartier divisor biholomorphic to $\mathbb{P}^{n-1}$ contracted by Φ to a smooth point of $\Phi(X)$.

The proof that X is locally a hypersurface at any $x \in P \cap \operatorname{sing} X$ can be given using the same inductive trick in the proof of Corollary 2.3 in [8] in the case $\dim X = 3$; however, we need an ample line bundle H on X such that 1) $H|_P = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, 2) for each $x \in P \cap \operatorname{sing} X$ there exists a normal Gorenstein irreducible divisor $D \in |H - x|$. Take $H = K_X^{N-1} \otimes L^{(n-1)N-n+2}$. We have already proved that $H|_P = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

Using the appropriate exact sequences, the fact that $K_{D_{n-2}}^{N-1} \otimes L^N$ is spanned and $H^1(K_{D_{n-i}}^{N-1} \otimes L^{(n-i-1)(N-1)}) = 0$ for each $i = 1, \ldots, n-2$ we can prove that H is spanned in a neighborhood of $D_{n-2} = A_1 \cap \ldots \cap A_{n-2}$. Take now a section $t \in (K_X \otimes L^{(n-1)})^{N-1}$ not identically zero on $A_i \cap P$; denote by s the tautological section of $[A_i]$. The zero set of $s \otimes t$ is a Cartier divisor $A_i + P_i \in |H|$ with P_i not containing P; using that $A_{1|P} = \mathcal{O}(1)$ and $H_P = \mathcal{O}(1)$ we can moreover suppose that $P_i \cap P = \phi$. Let $|H - x|$ denote the set of $D \in |H|$ containing $x \in P \cap \operatorname{sing} X$. The base locus of $|H - x|$ does not meet D_{n-2}: indeed, if it did, then since H is spanned in a neighborhood of D_{n-2} it would follow that $|H - x| = |H - y|$, where $y \in D_{n-2}$. Since $y \in A_i + P_i$ but not x, from $P_i \cap P = \phi$, this is absurd. Thus by Bertini's theorem there is an irreducible divisor $D \in |H - x|$ with D meeting D_{n-2} transversally in a smooth ample curve. This implies that $\operatorname{sing}(D)$ has codimension in D at least 2. By the Serre's criterion D is normal [1].

2.4. *Part 3.* In this last step we will show the two following claims:

1) There are no other irreducible positive dimensional varieties V such that $\Phi_N(V)$ is a point except the divisors P_s constructed before.

2) Therefore, if $X' = \Phi(X)$, $L' = [\Phi(A_1)]$ (note A_1 is smooth in a neighborhood of P_s by the inductive construction, $A_1 \cap P = P_1$ and $P_1 \subset \operatorname{reg} A_1$), $F = \Phi(U_s P_s)$ then (X', L') is a reduction of (X, L); F is the finite set of smooth points in X' to be blown up to obtain X; $K_{X'} \otimes L'^{n-1}$ is ample. That means (X', L') satisfies the thesis of the first part of the theorem. $1)_n$ and $2)_n$ will denote that the claim 1), respectively the claim 2), are true for $\dim X = n$.

Note that $2)_n$ follows from $1)_n$ by the first two steps; it remains to show that L' is ample. This follows easily noting that every section of L^n gives rise to a section of L'^n by Hartog's theorem and therefore, L^n being very ample for $n \gg 0$, $\Gamma((L')^n)$ span $(L')^n$ off F and the map associated to $\Gamma((L')^n)$ gives an embedding off F. By the Zariski-Fuijta theorem [4] $((L')^n)^m$ is spanned by global sections for some $m>0$. Since the holomorphic map associated to $\Gamma((L')^n)^m$ is an embedding off F, it follows that it has finite fibres. Therefore L' is ample.

It is immediate that $1)_2$ comes from Lemma 1.6.

Let us see that $2)_{n-1} \Rightarrow 1)_n$. Suppose there exist a variety V such that $\Phi(V)$ is a point. Being $A_1 = D_1$ an ample divisor we have that $V \cap A_1 \neq \phi$. Let us show that $x \in V \cap A_1$ implies that $x \in \cup P_s^1$ where $P_s^1 = P_s \cap A_1$. If $x \in A_1 - \cup P_s^1$ then by $2)_{n-1}$ there is a neighborhood around x of points y in A_1 such that $\Phi(x) \neq \Phi(y)$. Suppose that for each $y \notin A_1$ there is a $D \in |(K_X \otimes L^{n-1})^N|$ such that $x \in D$ but $y \notin D$; this will tell us that $\Phi(x) \neq \Phi(y)$ for all $y \notin A_1$ and conclude that $x \notin V \cap A_1$. Such a D exists if we can produce a section of the form $s \otimes t$ where t is a section of $K_X^N \otimes L^{(n-1)N-1}$ with $t(x) \neq 0$ and s is the tautological section of $[A_1] = L$. To obtain t we just use Lemma 1.2 and the following applied at the long exact sequence associated to

$$0 \to K_X \otimes K_X^{N-1} \otimes L^{(n-1)(N-1)} \otimes L^{n-3} \to K_X^N \otimes L^{(n-1)N-1} \to K_{D_1}^N \otimes L^{(n-2)N-1} \to 0.$$

Lemma. *There exists a large N such that $K_{D_1}^N \otimes L^{(n-2)N-1} \otimes (\otimes_s [P_s^1]^{-1})$ is spanned by global sections.*

Proof. By $2)_{n-1}$ there exist an ample line bundle $\mathscr{L}$ on $\Phi|_{D_1}(D_1) = D_1'$ such that $\Phi_{D_1}^*(\mathscr{L}) = K_{D_1} \otimes L^{(n-2)}$ and Φ_{D_1} is the map contracting the divisors P_s^1. Then

$$K_{D_i}^N \otimes L^{(n-2)N-1} \otimes (\otimes_s [P_s^1]^{-1}) = \Phi_{D_1}^*(K_{D1} \otimes \mathscr{L}^{N-1})$$

is spanned for large N.

Take $x \in V \cap A_1$, then $x \in P_s^1$ for some s; using again Lemma 1.2 and the above, we can choose t such that t_{A_1} vanishes only to the first order on P_s^1 in a neighborhood of x in A_1. By the implicit function theorem t vanishes only on a manifold in a neighborhood of x in X.

Since $s \otimes t$ vanishes on V it follows that $V \subset A_1 \cup P_s$. This proves $1)_n$.

The last part of the theorem remains to be proved.

We first show that the hypothesis $h^0(K_X \otimes L^{n-2})^N \neq 0$ for some $N>0$ will imply $h^0(K_{D_i} \otimes L^{n-i-2})^N \neq 0$ for some $N>0$ and for all $i=0, \dots, n-2$ (in the notation of 0.7). We can do this by induction if we show that our hypothesis implies that for all $A \in |L|$, A normal, $h^0(K_A \otimes L^{n-3})^N \neq 0$ for some $N>0$. Take $D \in |(K_X \otimes L^{n-2})^N|$. If A is not a component of D we are done by the adjunction formula. We can assume then $D = rA + P$ with A not a component of P and $r>0$. It follows that

$$(K_A \otimes L^{n-3})^N = L_A' \otimes [P \cap A]$$

that is the product of an ample divisor and an effective divisor. It follows that there exist an N' such that $h^0(K_A \otimes L^{n-3})^{N'} \neq 0$.

This fact proves that each S, smooth surface in the linear system $|L|$ (see 0.7), is of Kodaira dimension >0. To see that $\Phi(S)$ is the minimal model of S and that $(K_X \otimes L^{n-2})$ is nef note first that we can assume $n=3$; in fact S is contained in a 3-fold satisfying the hypothesis, D_{n-3}, and Theorem 0.5 applies. But

$h^0(K_{D_{n-3}} \otimes L)^N \neq 0$ for some $N > 0$ implies that the pair $(D_{n-3}, L_{D_{n-3}})$ has logarithmic Kodaira dimension $\geqq 0$ and the theorem has been proved in this hypothesis in [3].

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Configurations of -2 Rational Curves on Sectional Surfaces of n-Folds

M.L. Fania
Department of Mathematics, University of L'Aquila, I-67100 L'Aquila, Italy

Let X be an n-dimensional projective manifold. Let L be a line bundle on X which is ample and spanned by global sections. Assume that $K_X \otimes L^{n-1}$ is nef and big. Then there exists a reduction (X', L') of (X, L) such that $K_{X'} \otimes L'^{n-1}$ is ample, see [9, (4.5)]. Let π denote the reduction morphism. Moreover if $h^0((K_X \otimes L^{n-2})^N) \neq 0$ for some $N > 0$ it follows that $K_{X'} \otimes L'^{n-2}$ is semiample and any smooth S' which is the intersection of $n-2$ general members of $|L'|$ is a minimal model. We will assume throughout this paper that $\kappa(X, K_X \otimes L^{n-2}) = n$. Under this additional assumption it follows that S', the intersection of $n-2$ general members of $|L'|$, is of general type. Let ϕ_n denote the morphism associated to the linear system $|(K_{X'} \otimes L'^{n-2})^N|$ for $N \gg 0$, such that ϕ_n has connected fibres and normal image. We will classify the positive dimensional fibres of ϕ_n in order to get information about the positive dimensional fibres of $\phi_{n|S'}$. It is an easy consequence of the Kawamata-Viehweg vanishing theorem that $\phi_{n|S'}$ is the map associated to the linear system $|K_{S'}^N|$. Hence knowing the positive dimensional fibres of ϕ_n we get information about the maximal connected reduced curves C on S' satisfying

$$K_{S'} \cdot C = 0\,.$$

Such C are classified by the Dynkin diagram A_n for $n \geqq 1$, D_n for $n \geqq 4$, E_6, E_7, E_8. With the above assumptions and $\dim X = n \geqq 4$ we see that only one type can occur. To be more precise we have the following theorem.

Theorem. *Let X, L be as above. Assume that* $\dim X = n \geqq 4$. *Let S be a smooth surface gotten by the intersection of $n-2$ general members of the linear system $|L|$. Let $\pi_S\colon S \to S'$ be the map form S onto its minimal model. Let C be a maximal connected reduced curve on S' such that $K_{S'} \cdot C = 0$. Then the only possible Dynkin diagram associated to C is A_n with $n = 1, 2$.*

It was shown in [8] that when $\dim X = 3$ only A_n with $n = 1, 2, 3$ and D_n with $n = 4$ occur.

The paper is organized as follows.

In Sect. 0 we fix our notation and collect some results needed.

In Sect. 1 we state two variants of a theorem contained in [8] which will be used very often throughout the paper and prove the two statements not contained in [8].

In Sect. 2 we classify the positive-dimensional fibres of the morphism $\phi_{n\geqq 4}$. The configuration theorem stated earlier will be an easy consequence of the above classification.

0. Notation and Background Material

Throughout this paper we let X denote a pure dimensional complex projective manifold of dimension n.

(0.1) Let L be a line bundle on X. We say that L is big if $c_1(L)^{\dim X} > 0$. We say that L is numerically effective, or nef, for short if $c_1(L)[C] \geqq 0$ for all effective curves C on X. We say thet L is semiample if there exists an $m > 0$ such that L^m is spanned by global sections.

(0.2) Let L be an ample line bundle on X. We call the pair (X, L) a polarized manifold.

(0.3) By a quadric we mean a pair $(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, where $\mathcal{Q}$ is a degree two hypersurface in $\mathbb{P}^{n+1}$ and $\mathcal{O}_{\mathcal{Q}}(1) = \mathcal{O}_{\mathbb{P}^{n+1}}(1)|_{\mathcal{Q}}$.

(0.4) **Lemma.** *Let (Y, L) be a polarized normal irreducible projective variety with* $\dim \operatorname{Irr}(Y) \leqq 0$, *where* $\operatorname{Irr}(Y)$ *is the set of all non rational-singularities of Y. Let K_Y denote the dualizing sheaf of Y. If $K_Y \otimes L^{\dim Y} = \mathcal{O}_Y$, then Y is a quadric in $\mathbb{P}^{\dim Y+1}$ and $L = \mathcal{O}_{\mathbb{P}^{\dim Y+1}}(1)|_Y$.*

We omit the proof since it is standard.

(0.5) Let (X, L) be a polarized manifold. We say that (X, L) is a scroll over C if there exists a surjective holomorphic map $\phi: X \to C$ such that $K_X \otimes L^{\dim X - \dim C + 1} \simeq \phi * \mathcal{L}$ for an ample line bundle $\mathcal{L}$ on C. It follows that for every $c \in C$, $(\phi^{-1}(c), L_{|\phi^{-1}(c)}) \simeq (\mathbb{P}^{\dim X - \dim C}, \mathcal{O}(1))$.

(0.6) By F_r with $r \geqq 0$ we denote the rth Hirzebruch surface, i.e., the unique $\mathbb{P}^1$ bundle $p: F_r \to \mathbb{P}^1$ over $\mathbb{P}^1$ with a section E satisfying $E \cdot E = -r$. For $r \geqq 1$ we let $\tilde{F}_r$ denote the normal surface obtained from F_r by contracting E. Given a line bundle L on $\tilde{F}_r$, the pullback of L to F_r is of the form $([E] \otimes [f]^r)^a$ for some integer a. We denote by G the line bundle on $\tilde{F}_r$ that pulls back to $[E] \otimes [f]^r$.

(0.7) Let S' be a smooth projective surface which is a minimal model and of general type. Letting E be an irreducible curve on S' it follows that $K_{S'} \cdot E = 0$ if and only if E is a smooth rational curve satisfying $E \cdot E = -2$. A maximal reduced connected curve C satisfying $K_{S'} \cdot C = 0$ is well known to be a union of distinct smooth rational curve E_i, $C = \bigcup_{i=1}^{n} E_i$ satisfying:

a) $E_i \cdot E_i = -2$ for all i,
b) the intersection of any three of the irreducible components of C is empty,
c) if $E_i \cap E_j \neq \emptyset$ for $i \neq j$ then they intersect transversely in at most one point.

Since the intersection form of C is negative-definite, it must be one of the forms described by the Dynkin diagrams A_n with $n \geqq 1$, D_n with $n \geqq 4$ or E_6, E_7, E_8. Hence such Dynkin diagrams are the dual graphs of our curve C.

For more details see [3].

(0.8) Let L be an ample and spanned line bundle on a n-dimensional manifold X. Let $A_1, \dots, A_{n-2}$ be general members of the linear system $|L|$. We let $X_i = \bigcap_{j=i-1}^{n-2} A_j$ and often we denote X_2 by S. Hence we have the following chain $X \supseteq X_{n-1} \supseteq \dots \supseteq X_2 = S$. Note that $X_i \in |L_{X_{i+1}}|$.

(0.9) Let (X, L) be a polarized manifold. A reduction of (X, L) is a polarized manifold (X', L') such that:

a) there exists a morphism $\pi: X \to X'$ expressing X' as X with a finite set $F \subseteq X'$ blown up,

b) $L = \pi^* L' \otimes [\pi^{-1}(F)]^{-1}$ or equivalently $K_X \otimes L^{n-1} = \pi^*(K_{X'} \otimes L'^{n-1})$.

We state the following theorem for smooth varieties but it holds for varieties with moderate singularities as well, see [9, (4.5)].

(0.10). **Theorem.** *Let L be an ample and spanned line bundle on an n-dimensional projective manifold X. Assume that $K_X \otimes L^{n-1}$ is* nef *and* big. *Then there exists a reduction (X', L') of (X, L) such that $K_{X'} \otimes L'^{n-1}$ is ample. If $h^0((K_X \otimes L^{n-2})^N) \neq 0$ for some $N > 0$, then it follows that any smooth surface S' which is the intersection of $(n-2)$ general members of $|L'|$ is a minimal model of non-negative Kodaira dimension and $K_{X'} \otimes L'^{n-2}$ is* nef.

If $\kappa(X, K_X \otimes L^{n-2}) = n$ then the above S' is of general type, where $\kappa(X, K_X \otimes L^{n-2})$ denotes the $(K_X \otimes L^{n-2})$-dimension of the invertible sheaf $K_X \otimes L^{n-2}$.

(0.11) With the same assumption as in (0.10) the following are equivalent:

1) $K_{X'} \otimes L'^{n-2}$ nef

2) $K_{X'} \otimes L'^{n-2}$ semiample.

For a proof see [9, (5.1)].

(0.12) Let X' and L' be as in (0.10). Let $\phi: X' \to \mathbb{P}_{\mathbb{C}}$ be the map associated to $\Gamma((K_{X'} \otimes L'^{n-2})^N)$ for $N \geqq n_0$ for some n_0.

It is a standard check to see that we can choose n_0 so that ϕ has connected fibres and a normal image. It also is an immediate consequence of the Kawamata-Viehweg vanishing theorem and the fact that $K_{X'_i} \otimes L'^{i-2}$ is semiample and big for smooth $X'_i \in |L'_{X'_{i+1}}|$ that $\phi_{|X'_i}$ is the map associated to $\Gamma((K_{X'_i} \otimes L'^{n-2})^N)$. Hence $\phi_{S'}$ is the map associated to $\Gamma(K_{S'}^N)$ for $N \geqq 2$. The n_0 can therefore be chosen so that for $N \geqq n_0$, $\phi_{X'_i}$ has connected fibres and normal image. For simplicity, from now on we denote $\phi_{X'_i}$ by ϕ_i and ϕ by ϕ_n.

1. Positive Dimensional Fibres of ϕ_3

The following theorem but for the last two assertions in a) and for 5) was proved in [8]. We will state it for the convenience of the reader and prove the above mentioned parts which are not included in [8].

(1.0) **Theorem.** *Let X be a smooth projective threefold and let L be an ample line bundle on X. Assume that $\kappa(X, K_X \otimes L) = 3$. Let (X', L') be a reduction of X with $\pi: X \to X'$ the reduction map. Let $\phi_3: X' \to \mathbb{P}_{\mathbb{C}}$ be the map associated to the linear system $|(K_{X'} \otimes L')^N|$ for $N \geqq n_0$ for some $n_0 > 0$ such that ϕ_3 has connected fibres and a normal image. Let $x \in \phi_3(X')$ be so that* $\dim \phi_3^{-1}(x) > 0$.

a) *If $\phi_3^{-1}(x)$ contains a curve ℓ as a component then $\ell \cong \mathbb{P}^1$ and $L' \cdot \ell = 1$. Further there exists an irreducible divisor $D \subseteq X'$ such that $\ell \subseteq D$,* $\dim \phi_3(D) = 1$ *and D is a $\mathbb{P}^1$ bundle over $\phi_3(D)$, a smooth curve. Moreover if $\phi_3^{-1}(\phi_3(D)) = D \cup D_1$ then $\phi_3(D_1) = y \in \phi_3(D)$ and D_1 is biholomorphic to F_1.*

b) *If $\phi_3^{-1}(x)$ is a divisor, D, then it is reduced and either*

1) *D is biholomorphic to $\mathbb{P}^2$, $L'_D = \mathcal{O}_{\mathbb{P}^2}(1)$ and $N_{D/X'} = \mathcal{O}_{\mathbb{P}^2}(-2)$, or*
2) *D is biholomorphic to $\mathbb{P}^2$, $L'_D = \mathcal{O}_{\mathbb{P}^2}(2)$ and $N_{D/X'} = \mathcal{O}_{\mathbb{P}^2}(-1)$, or*
3) *D is biholomorphic to $\tilde{F}_2$, $L'_D = G$ and $N_{D/X'} = G^{-1}$, where G is as in* (0.6), *or*
4) *D is biholomorphic to F_0, $L'_D = [E] \otimes [f]$ and $N_{D/X'} = ([E] \otimes [f])^{-1}$, or*
5) *D is biholomorphic to F_1. Moreover there exists an irreducible divisor D_1 such that* $\dim \phi_3(D_1) = 1$ *and $\phi_3: D_1 \to \phi_3(D_1)$ is a $\mathbb{P}^1$ bundle over $\phi_3(D_1)$. Furthermore $D \cap D_1 = C$, such intersection is transverse and C is a smooth rational curve which is a fibre of $D_1 \to \phi_3(D_1)$ and is the unique curve of self-intersection -1 on F_1, or*
6) *$D = D_1 \cup D_2$ where D_1 is biholomorphic to F_2 and D_2 is biholomorphic to $\mathbb{P}^2$. Moreover $D_1 \cap D_2 = C$, such intersection is transverse and C is a smooth rational curve which is a line in $\mathbb{P}^2$ and is the unique curve of self-intersection -2 on F_2.*

Proof. We only need to show some of the statements in a) and in 5). For the rest of the proof see [8, (1.0.1)]. To prove a) let D be the irreducible divisor as in [8, (1.1)].

Let ℓ denote the general fibre of $\phi_{3|D}$. Note that ℓ is a smooth rational curve satisfying $L' \cdot \ell = 1$, see [8, (1.1)]. Moreover

$$\ell \cdot D = -1 . \tag{1}$$

Hence ℓ is not numerically effective.

If we knew that $\mathbb{R}_+[\ell]$ was an extremal ray then using [6, (3.3)] it follows that $\phi_{3|D}: D \to \phi_3(D)$ is a $\mathbb{P}^1$ bundle over a smooth curve.

If $\mathbb{R}_+[\ell]$ is not an extremal ray it follows (see [6, (1.4)]) that for an arbitrary positive ε, there exist a finite number r of rational curves $\ell_1, \dots, \ell_r$ in X' such that

$$\ell = \sum_{i=1}^{r} a_i \ell_i + C \tag{2}$$

where a_i are positive real numbers and $C \in \overline{NE}_\varepsilon(X') = \overline{NE}_\varepsilon(X', L')$.

We refer to [6] for details about the notation. We can choose $\varepsilon < 1$. Since $K_{X'} \otimes L'$ is nef and since $(K_{X'} \otimes L') \cdot \ell = 0$ by assumption it follows from (2) that

$$\ell = \sum_{i=1}^{r} a_i \ell_i . \tag{3}$$

Using (1) and (3) we conclude that $D \cdot \ell_i < 0$ for some i. Let $\mathrm{cont}_R: X' \to Y$ denote the contraction morphism associated to the extremal ray $\mathbb{R}_+[\ell_i]$, see [6, (3.1)]. Using the fact that ℓ_i is not numerically effective we conclude from [6, (3.3)] that $\dim Y = 3$ and that $\dim \mathrm{cont}_R(D) = 0$ or 1.

If $\dim \mathrm{cont}_R(D) = 0$ then from the list in [6, (3.3)] it follows that $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\{s\} \times \mathbb{P}^1$ is algebraically equivalent to $\mathbb{P}^1 \times \{t\}$. This is impossible since $\dim \phi_3(D) = 1$ and $\phi_3(X')$ is projective.

If $\dim \mathrm{cont}_R(D) = 1$ then either ℓ_i is a fibre of ϕ_3 or $\phi_3(\ell_i) = \phi_3(D)$. If ℓ_i is a fibre of ϕ_3 then D is the divisor associated to the extremal ray $\mathbb{R}_+[\ell_i]$. Hence we are done. If $\phi_3(\ell_i) = \phi_3(D)$ then from [6, (3.3)] we know that D is a $\mathbb{P}^1$ bundle over the smooth curve $\mathrm{cont}_R(D)$. It is easy to see that $\mathrm{cont}_R(D) \simeq \mathbb{P}^1$. Hence $D \simeq F_r$ with $r \geqq 0$, where F_r is as in (0.6). An easy numerical computation shows that $r = 0$. Hence $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{cont}_{R|D}$ is the projection onto a factor. Now we use the fact that $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and that the general fibre of $\phi_{3|D}$ is isomorphic to $\mathbb{P}^1$ to conclude that $\Phi_{3|D}$ is the projection onto the other factor.

If $\phi_3^{-1}(\phi_3(D)) = D \cup D_1$ then $\phi_3(D_1) = y \in \phi_3(D)$. Therefore D_1 is a fibre of $\phi_3 : X' \to \phi_3(X')$. Hence D_1 is biholomorphic to F_r with $r = 0$ or 1, or $\mathbb{P}^2$, or $\tilde{F}_2$ with the invariants stated in b). If $D_1 \cong F_r$ with $r = 0$ then $N_{F_0/X'} = ([E] \otimes [f])^{-1}$ is negative. Therefore F_0 can be contracted to a point. Let $\sigma : X' \to X$ be such contraction. Let $\eta : \tilde{D} \to D$ be a normalization of D. Note that $\tilde{D}$ is a smooth $\mathbb{P}^1$ bundle. Moreover $\dim(\sigma_{|D} \circ \eta)(\tilde{D}) = 2$. Since $\tilde{D}$ is a $\mathbb{P}^1$ bundle and a fibre gets contracted to a point then all nearby fibres should be contracted to a point, i.e., $\dim(\sigma_{|D} \circ \eta)(\tilde{D}) = 1$. This contradiction implies that $D_1 \cong F_0$ is not possible. Now note that $N_{\mathbb{P}^2/X'} = \mathcal{O}(-e)$ with $e = 1$ or 2 and $N_{\tilde{F}_2/X'} = L'^{-1}_{\tilde{F}_2}$. They are negative. Hence the same reasoning as above, rules out the case $D_1 \simeq \mathbb{P}^2$ or $\tilde{F}_2$. Hence $D_1 \simeq F_1$.

As for the proof of 5) note that if a fibre of ϕ_3 is biholomorphic to F_1 then $L'_{F_1} = [E] \otimes [f]^2$ and $N_{F_1/X'} = ([E] \otimes [f])^{-1}$. It can be easily seen that $N_{E/X'} = \mathcal{O}_E \oplus \mathcal{O}_E(-1)$. Hence $h^0(E, N_{E/X'}) > 0$ and $h^1(E, N_{E/X'}) = 0$. Therefore there are non-trivial smooth deformations of E in X'. From the above, using a basic result on Hilbert schemes it follows that there exist irreducible projective varieties $\mathcal{W}$ and $\mathcal{T}$ with the following properties:

1) $\mathcal{W} \subseteq \mathcal{T} \times X$ and the map $p : \mathcal{W} \to \mathcal{T}$ induced by the product projection is flat,

2) there exists a smooth $t \in \mathcal{T}$ such that $\mathcal{W}_t = p^{-1}(t)$ is identified with E via q, where $q : \mathcal{W} \to X$ is the map induced by the product projection.

Note that $\dim \mathcal{T} = 1$. Hence $\dim q(\mathcal{W}) = 2$. Denote $q(\mathcal{W})$ by D_1. Note that $\dim \phi_3(D_1) \leqq 1$. If $\dim \phi_3(D_1) = 0$, i.e., D_1 is a fibre of ϕ_3, then from [8, (1.0.1)] it follows that D_1 is biholomorphic to F_0, or F_1, or $\mathbb{P}^2$ or $\tilde{F}_2$. Using [8, (1.2), (1.4), and (1.5)] we see that none of these cases can occur. Hence $\dim \phi_3(D_1) = 1$. Now the same argument as above shows that D_1 is a $\mathbb{P}^1$ bundle over $\phi_3(D_1)$.

To finish the proof of 5) note that E, the unique curve in F_1 of self-intersection -1, is certainly contained in $D \cap F_1$. Moreover it is a straightforward check to see that $D \cap F_1 = E$. Hence we are done. □

2. Positive Dimensional Fibres of ϕ_n

(2.0) Let L, L', X and X' be as in (0.10). Assume that $\kappa(X, K_X \otimes L^{n-2}) = n$. Let $\phi_n : X' \to \mathbb{P}_{\mathbb{C}}$ be the map associated to $\Gamma((K_{X'} \otimes L'^{n-2})^N)$ for N so that ϕ_n has connected fibres and a normal image.

(2.1) **Lemma.** *$d\phi_n$ has rank n on $T_{X',x}$ for $x \in X'_{n-1} - \cup \mathscr{E}_i$, where $\cup \mathscr{E}_i$ is the union of the positive dimensional fibres of ϕ_{n-1}.*

Proof. Note that $\operatorname{rk} d\phi_{n-1} = n-1$ on $T_{X'_{n-1},x}$ for $x \in X'_{n-1} - \cup \mathscr{E}_i$.

Hence $d\phi_n$ will have rank n on $T_{X',x}$ if we can find $s \in \Gamma((K_{X'} \otimes L'^{n-2})^{NN'})$ for some N', of the form $s_1 \otimes s_2$ where $s_1 \in \Gamma(L')$ is the tautological section of $[X'_{n-1}] = L'$ and $s_2 \in \Gamma(K_{X'}^{NN'-1})$ with $s_2(x) \neq 0$.

From

$$0 \to K_{X'} \otimes (K_{X'} \otimes L'^{n-2})^{NN'-1} \otimes L'^{n-4} \to K_{X'}^{NN'} \otimes L'^{(n-2)NN'-1} \to$$
$$\to K_{X'_{n-1}}^{NN'} \otimes L'^{(n-3)NN'-1} \to 0,$$

the vanishing of $H^1(X', K_{X'} \otimes (K_{X'} \otimes L'^{n-2})^{NN'-1} \otimes L'^{n-4})$ and the fact that $K_{X'_{n-1}}^{NN'} \otimes L'^{(n-3)NN'-1}$ is spanned off $\cup \mathscr{E}_i$ for some N', it follows that there is $s_2 \in \Gamma(K_{X'}^{NN'} \otimes L'^{(n-2)NN'-1})$ with $s_2(x) \neq 0$. □

(2.2) **Lemma.** *Let $x \in \phi_n(X')$. Assume that $0 < \dim \phi_n^{-1}(x) < n-1$. Then $\dim \phi_n^{-1}(x) = n-2$.*

Proof. Note that the fibres of ϕ_n are pure dimensional. For a proof in the case $n=3$ see [8, (1.2)]. The higher dimensional case follows from this by slicing X' with general members of $|L'|$.

To prove the lemma it is enough to show that for $n \geqq 4$, ϕ_n has no 1-dimensional fibres. In fact if there is a fibre F with $1 < \dim F \leqq n-2$ then after slicing X' with a sufficient number of general members $A'_j \in |L'|$ we would be in the case $\dim \geqq 4$ and 1-dimensional fibres. We prove the lemma for $n=4$. The same proof yields also the general case.

Let F be a positive dimensionsl fibre of ϕ_4. Note that $F \cap X'_3 \neq \emptyset$. Moreover $F \cap X'_3 \subseteq \cup \mathscr{E}_i$. This last fact follows from (2.1). The same argument as in [4, (2.2)] shows that $F \subseteq X'_3 \cup \phi_4^{-1}(x_i)$, where $x_i = \phi_3(\mathscr{E}_i)$. Hence $\phi_4^{-1}(x_i)$ are the only positive dimensional fibres of ϕ_4. Note that $\dim \phi_4^{-1}(x_i) \geqq 2$. Therefore ϕ_4 has no 1-dimensional fibres.

(2.3) **Theorem.** *Let $x \in \phi_n(X')$. Assume that $\dim \phi_n^{-1}(x) > 0$. Then either $\dim \phi_n^{-1}(x) = n-1$ or $\dim \phi_n^{-1}(x) = n-2$. If $\dim \phi_n^{-1}(x) = n-1$,* i.e., *$\phi_n^{-1}(x)$ is a divisor, D, then either*

i) *$(D, L') \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ and $N_{D/X'} = \mathcal{O}_{\mathbb{P}^{n-1}}(-2)$ or*
ii) *$(D, L') \cong (\mathbb{Q}, \mathcal{O}_{\mathbb{Q}}(1))$ where $\mathbb{Q}$ is a hyperquadric in $\mathbb{P}^n$ and $\mathcal{O}_{\mathbb{Q}}(1) = \mathcal{O}_{\mathbb{P}^n}(1)|_{\mathbb{Q}}$.*

If $\dim \phi_n^{-1}(x) = n-2$ then there exists an irreducible divisor D containing $\phi_n^{-1}(x)$ such that D is a scroll over $\phi_n(D)$.

Proof. We prove the theorem for $n=4$. The general case will be done by induction.

Let $x \in \phi_n(X')$ be such that $\dim \phi_n^{-1}(x) > 0$. From (2.2) it follows that $\dim \phi_n^{-1}(x)$ can be either 2 or 3.

Assume that $\dim \phi_n^{-1}(x) = 3$. Let $D = \phi_n^{-1}(x)$. Choose a general $A' \in |L'|$. Then A' meets D transversely along $\phi_3^{-1}(x)$. For simplicity we let $Y = \phi_3^{-1}(x)$. From (1.0) we have that Y is biholomorphic to either

1) $\mathbb{P}^2$ with $L'_{\mathbb{P}^2}=\mathcal{O}_{\mathbb{P}^2}(2)$ and $N_{\mathbb{P}^2/X'_3}=\mathcal{O}_{\mathbb{P}^2}(-1)$, or
2) $\mathbb{P}^2$ with $L'_{\mathbb{P}^2}=\mathcal{O}_{\mathbb{P}^2}(1)$ and $N_{\mathbb{P}^2/X'_3}=\mathcal{O}_{\mathbb{P}^2}(-2)$, or
3) F_0 or 4) $\tilde{F}_2$ or 5) $\mathbb{P}^2\cup F_2$ or
6) $F_1\cup Y_1$, where Y_1 is a $\mathbb{P}^1$ bundle over a curve.

If Y is as in 1) then $N_{Y/D}=\mathcal{O}_{\mathbb{P}^2}(2)$. Hence D is the cone over $i_2(\mathbb{P}^2)$, where i_2 is the 2-fold Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$, see [1, Theorem 1]. The embedding dimension of the vertex of such cone is $>\dim X'$. Hence such D cannot exist.

If Y is as in 2) then $D\cong\mathbb{P}^3$ and Y is a hyperplane in $\mathbb{P}^3$. Therefore $N_{D/X'}=\mathcal{O}_{\mathbb{P}^3}(-2)$.

If Y is as in 3) and 4) we claim that D is a quadric. Note that in both cases $N_{Y/X'_3}=L'^{-1}_Y$, see [8, (0.9.1)] for details. Since $X'_3\cap D=Y$ and such intersection is transverse in X' we have that $N_{D/X',Y}=N_{Y/X'_3}=L'^{-1}_Y$. Hence $N_{D/X'}=L'^{-1}_D$ since $\operatorname{Pic}(D)$ injects into $\operatorname{Pic}(Y)$. We also have that $(K_{X'}\otimes L'^2)|_D=\mathcal{O}_D$. These last two facts together give $K_D\otimes L'_D=\mathcal{O}_D$. Therefore D is a hyperquadric, see (0.2).

If Y is as in 5) let X'_3 be a member of the linear system $|L'|$. Choose X'_3 general enough so that $X'_3\cap D=\mathbb{P}^2\cup F_2$ and such intersection is transverse. Hence $D=D_1\cup D_2$. Say that $\mathbb{P}^2\subseteq D_1$ and $F_2\subseteq D_2$. Note that $\mathbb{P}^2$ and F_2 are ample in D_1 and D_2 respectively. Hence it follows that $D_1\simeq\mathbb{P}^3$ and (see [2]) that D_2 is isomorphic to a $\mathbb{P}^2$ bundle $\bar{p}: D_2\to\mathbb{P}^1$. Here $\bar{p}$ denotes the extension of $p: F_2\to\mathbb{P}^1$ to D_2, where p is as in (0.6).

Note that D_1 and D_2 meet on a surface $\mathcal{S}$. Such $\mathcal{S}$ must contain $\ell=\mathbb{P}^2\cap F_2$, a linear $\mathbb{P}^1$, as ample divisor. Hence $\mathcal{S}\cong\mathbb{P}^2$. Since $\mathcal{S}\subseteq D_2$ it follows that $\mathcal{S}$ must be a fibre of $D_2\to\mathbb{P}^1$. Also $\mathcal{S}\supseteq\ell=$ exceptional curve of F_2 and ℓ cannot be contracted by $p: F_2\to\mathbb{P}^1$. Hence there are no such D's.

If Y is as in 6), i.e., $Y=F_1\cup Y_1$, let $\bar{D}=\phi_4^{-1}(\phi_3(Y_1))$. Note that $\bar{D}=\bar{D}_1\cup\bar{D}_2$ since $\phi_3^{-1}(Y_1)=F_1\cup Y_1$. Assume, for example, that $Y_1\subset\bar{D}_1$ and $F_1\subset\bar{D}_2$. We know that $\bar{D}_2$ is a $\mathbb{P}^2$ bundle $\bar{p}:\bar{D}_2\to\mathbb{P}^1$ over $\mathbb{P}^1$ and $\bar{D}_1$ is a $\mathbb{P}^2$ bundle over $\phi_4(Y_1)$, see [2]. In such a situation we have seen earlier that $\bar{D}_1\cap\bar{D}_2=\mathcal{S}=\mathbb{P}^2$. Since $\mathcal{S}\subset\bar{D}_2$ and $\bar{p}:\bar{D}_2\to\mathbb{P}^1$ is a $\mathbb{P}^2$ bundle it follows that $\mathcal{S}$ is a fibre of $\bar{p}$. On the other hand $E\subset\mathcal{S}^2$, where E is the exceptional curve of F_1. Hence $\dim\bar{p}(E)=0$, and this is impossible.

Assume that $\dim\phi_4^{-1}(x)=2$. Let $X'_3\in|L'|$ be a general element such that X'_3 meets $\phi_4^{-1}(x)$ transversely in X'_4. Note that $X'_3\cap\phi_4^{-1}(x)=\phi_3^{-1}(x)$. Hence $\dim\phi_3^{-1}(x)=1$. By (1.0) there exists a divisor Y in X'_3 containing $\ell=\phi_3^{-1}(x)$, with Y a $\mathbb{P}^1$ bundle over $\phi_3(Y)$. We claim that $D=\phi_4^{-1}(\phi_3(Y))$ is a scroll over $\phi_4(D)$ $=\phi_3(Y)$. To see this let f' and F denote the general fibres of ϕ_3 and ϕ_4 respectively. Note that $F\cap Y=f'$ and that $F\cap X'_3=f'$. Hence $f'\in|L'_F|$. This last fact together with the fact that $L'_{f'}=\mathcal{O}_{f'}(1)$ and that $F\simeq F_r$ with $r\geqq 0$ or $\tilde{F}_r$ with $r\geqq 1$ implies that $F\simeq\mathbb{P}^2$ and $L'_F=\mathcal{O}_F(1)$.

We will now use induction to prove the theorem for all n. Assume that the theorem is true for $\dim X'\leqq n-1$. Let $x\in\phi_n(X')$ be such that $\dim\phi_n^{-1}(x)>0$. From (2.2) it follows that either $\dim\phi_n^{-1}(x)=n-1$ or $\dim\phi_n^{-1}(x)=n-2$. Assume that $\dim\phi_n^{-1}(x)=n-1$ and let X'_{n-1} be a general member of the linear system $|L'|$. Let $\phi_n^{-1}(x)=D$. We have that $X'_{n-1}\cap D=\phi_{n-1}^{-1}(x)$ and such intersection is transverse in X'. Note that $D_1 = \phi_{n-1}^{-1}(x)$ is an $(n-2)$-dimensional fibres of ϕ_{n-1}. Hence we have either

i) $(D_1, L') \cong (\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}^{n-2}}(1))$ and $N_{\mathbb{P}^{n-2}/X'_{n-1}} = \mathcal{O}_{\mathbb{P}^{n-2}}(2)$ or
ii) $(D_1, L') \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$ where $\mathcal{Q}$ is a hyperquadric in $\mathbb{P}^{n-1}$ and $N_{\mathcal{Q}/X'} = L_{\mathcal{Q}}^{-1}$.

If we are in case i) since D_1 is an ample divisor on D and $N_{D_1/D} = L'_{D_1} = \mathcal{O}_{D_1}(1)$ it follows that $D \cong \mathbb{P}^{n-1}$ and $L'_D = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ and that $N_{D/X'} = \mathcal{O}_D(-2)$. If we are in the case ii) the same argument as in the case $n=4$ gives that D is a hyperquadric in $\mathbb{P}^n$ with $L'_D = \mathcal{O}_D(1)$ and $N_{D/X'} = L_D'^{-1}$.

Assume that $\dim \phi_n^{-1}(x) = n-2$ and let X'_{n-1} be a general member of the linear system $|L'|$ such that $X'_{n-1} \cap \phi_n^{-1}(x) = \phi_{n-1}^{-1}(x)$. Hence $\dim \phi_{n-1}^{-1}(x) = n-3$. Therefore by induction there exists a divisor $D_1 \subseteq X'_{n-1}$ such that $\phi_{n-1}^{-1}(x) \subseteq D_1$ and D_1 is a scroll over $\phi_{n-1}(D_1)$. We claim that $D = \phi_n^{-1}(\phi_{n-1}(D_1))$ is a scroll over $\phi_n(D)$. The proof of the claim is standard and left to the reader. □

(2.4) **Theorem.** *Let X and L be as in* (0.10). *Assume that* $\dim X = n \geqq 4$. *Moreover assume that $\kappa(X, K_X \otimes L^{n-2}) = n$. Let S be a smooth surface gotten by the intersection of $n-2$ general members of $|L|$. Let $\pi_S: S \to S'$ be the map from S onto its minimal model. Let C be a maximal connected reduced curve on S' such that $K_{S'} \cdot C = 0$. Then the only possible Dynkin diagram associated to C is A_n with $n = 1, 2$.*

Proof. We only need to check how S' can meet the fibres F of ϕ_n. From (2.2) it follows that $\dim F = n-1$ or $n-2$. If $\dim F = n-2$ or equivalently C is a fibre of ϕ_3 then as in [8] the Dynkin diagram is A_1 and $L' \cdot C = 1$. If $\dim F = n-1$ then either $F \cong \mathbb{P}^{n-1}$ or $F \cong \mathcal{Q}$. Since S' was the intersection of $n-2$ general member of $|L'|$ then $C = S' \cap F$ is a smooth curve. Such a curve C is contained either in $\mathbb{P}^2$ or in $\mathcal{Q} \cong F_0$ or $\tilde{F}_2$. Now the same argument as in [8, (1.8)] gives that the Dynkin diagrams are either A_1 or A_2. Moreover $L' \cdot C = 1$ in the case $C \subseteq \mathbb{P}^2$ and $L' \cdot C = 2$ in the case $C \subseteq F_0$ or $\tilde{F}_2$. Hence we get in Table 1.

Table 1

D	Dynkin diagram
Scroll	A_1
$\mathbb{P}^{n-1}$	A_1
$\mathcal{Q}$ = hyperquadric in $\mathbb{P}^n$	A_1 A_2

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Every Finite Complex has the Homology of a Duality Group

J.-C. Hausmann

Department of Mathematics, University of Geneva, CH-1211 Geneva 24, Switzerland

Introduction

A group G belongs to the class $\mathbf{D}(n)$ if, by definition:

a) There exists a G-module D and a class $e \in H_n(G;D)$ such that the cap product with e:

$$H^i(G;B) \to H_{n-i}(G;B \otimes_{\mathbf{Z}} D)$$

is an isomorphism for all i and all G-modules B.

b) The Eilenberg-McLane space $BG = K(G,1)$ is homotopy equivalent to a finite complex of dimension n.

Part a) of the above definition is the classical definition of a duality group, given by Bieri and Eckmann [BE1]. There is no known example of a group satisfying a) but not b), but it is not known whether a) implies b). Denote by $\mathbf{D}$ the union of the classes $\mathbf{D}(n)$, for all $n \geqq 0$.

The classical list of examples of groups in $\mathbf{D}$ (fundamental group of aspherical manifolds, free groups, cohomology dimension 2 groups with one end, torsion-free arithmetic groups, etc, see [Bn, VII 10]) was recently enriched by new examples: braid groups [Sq], mapping class groups [Hr]. This suggests that the class $\mathbf{D}$ is larger than previously expected and inspired the main result of this paper, Theorem A below.

The statement of Theorem A requires two definitions. A map $f: X \to Y$ is called *acyclic* if its homotopy-theoretical fibre is an acyclic space, or, equivalently, if the homomorphism $f_*: H_*(X;B) \to H_*(Y;B)$ is an isomorphism for any $\pi_1(Y)$-module B (see [HH] for a survey about acyclic maps). A group P is called *locally perfect* if any finitely generated subgroup of P is contained in a finitely generated perfect subgroup of P. Acyclic maps f with $\ker(\pi_1 f)$ locally perfect enjoy interesting geometric properties (see, for instance [HV]).

Theorem A. *Let X be a finite complex. Then, there is a group $G \in \mathbf{D}$ and an acyclic map $BG \to X$. Moreover,* $\ker(G \to \pi_1 X))$ *is locally perfect.*

As the homology of BG coincides with the Eilenberg-McLane homology of the group G, we obtain the result stated in the title of this paper.

Corollary. *Let X be a finite complex. Then, there is a group $G \in \mathbf{D}$ and an epimorphism $h: G \to \pi_1(X)$ such that, for all $\pi_1(X)$-modules B regarded as G-modules via h, there are isomorphisms $H_*(G;B) \cong H_*(X;B)$. In particular, these isomorphisms hold for all triivial coefficient modules B.*

In particular, the fact that a group G is in $\mathbf{D}$ implies nothing about its homology with integral coefficients, except being finitely generated.

Theorem A has well known predecessors: given a complex X, Kan and Thurston [KT] were the first to show the existence of an acyclic map $BG \to X$, for some (very large) group C. In [BDH], Baumslag et al., showed that, if X is finite, G can be taken so that BG is a finite complex. A simple proof of these facts given by Maunder [Ma] makes it possible to have $\dim BG = \dim X$. All these constructions enjoy strong functoriality properties.

In contrast, although it uses the principle of [Ma], our construction of the group G of Theorem A is not functorial. We strongly suspect that there is no such functorial construction. Also, the following problem remains open:

Problem. Let X be a finite complex. What is the minimal integer $r(X)$ such that there is a group $G \in \mathbf{D}(r(X))$ and an acyclic map $BG \to X$?

In this respect, the construction given in Sect. 3 below to prove Theorem A is not very efficient. It just gives the inequality $r(X) \leqq 10\,m(X) - 7$, where $m(X)$ is the minimal number of simplices of positive dimension of a polyhedron homotopy equivalent to X [see Remark (3.3)]. For instance: $1 = r(S^1) \leqq 23$. So far, there is no counter-example to the possible conjecture that $r(X)$ is equal to the homotopy dimension of X.

On the other hand, given a finite complex X, there are groups $G \in \mathbf{D}(r)$ with acyclic maps $BG \to X$ for r arbitrarily large [see Remark (3.4)].

1. Preliminary Results

The first three lemmas of this section consist of elementary properties of the classes $\mathbf{D}(n)$. Lemma (1.4) is a criterion for recognizing the injectivity of a homomorphism between amalgamated products. These results are used in Sect. 3 for the proof of Theorem A.

(1.1) **Lemma.** *Let A be a subgroup of the groups B and C. Suppose that B and C are in $\mathbf{D}(n)$ and that A is in $\mathbf{D}(n-1)$. Then the amalgamated product $B *_A C$ is in $\mathbf{D}(n)$.*

Proof. Condition a) of the definition of $\mathbf{D}(n)$ is fulfilled by [BE2, Theorem 3.2]. Condition b) comes from the fact that the space $B(B *_A C)$ is the union over BA of BB and BC [Bn, Theorem 7.3]. □

(1.2) **Lemma.** *Let $G \in \mathbf{D}(m)$ and $H \in \mathbf{D}(n)$. Then $G \times H \in \mathbf{D}(m+n)$.*

Proof. Condition a) is classical (see [BE1, Theorem 3.5]). Condition 2) is obvious, since $B(G \times H) \cong BG \times BH$. □

(1.3) **Lemma.** *Let G be a one-relator group $G = \langle a_1, a_2, \dots a_k | r\rangle$. Suppose that:*

1) *G is not cyclic*
2) *r is not a proper power*
3) *G is not a non-trivial free product.*

Then $G \in \mathbf{D}(2)$.

Proof. Condition 2) together with [DV, Theorem 2.1] implies that the 2-complex associated with the given presentation of G is homotopy equivalent to BG. This guarantees Condition b) of the definition of $\mathbf{D}(2)$, and implies that G is of cohomology dimension 2. By [BE1, Theorem 5.2], G is then a free product of duality groups of dimension 1 and 2. But Conditions 1) and 3) then imply that G is a duality group of dimension 2 and thus satisfies Condition a) for $n=2$. □

For the last lemma of this section, let us consider a homomorphism f between the following diagrams of groups and subgroups

$$\begin{pmatrix} C \subset A \subset A_1 \\ \cap \\ B \end{pmatrix} \xrightarrow{f} \begin{pmatrix} C' \subset A' \subset A_1' \\ \cap \\ B' \end{pmatrix}$$

which induces homomorphisms

$$f\colon D = A *_C B \to D' = A' *_{C'} B'$$

and

$$f\colon D_1 = A_1 *_C B \to D_1' = A_1' *_{C'} B'.$$

(1.4) **Lemma.** *Suppose that $f|A_1$ and $f|B$ are injective. Suppose also that $C' \cap f(A_1) = C' \cap f(B) = f(C)$. Then*
 a) *$f\colon D_1 \to D_1'$ is injective and*
 b) *$D' \cap f(D_1) = f(D)$.*

Proof. Recall the results about the unique writings for the elements of an amalgamated product $G = E *_H F$. Choose sets $\bar{E}$ and $\bar{F}$ or representatives for the right cosets $H \backslash E$ and $H \backslash F$, with 1 as a representative of H. Then any element g of G can be uniquely written as $g = he_1 f_1 \dots e_k f_k$, with $h \in H$, $e_i \in \bar{E}$ and $e_i \neq 1$ if $i = 1$, $f_i \in \bar{F}$ and $f_i \neq 1$ if $i < k$.

The condition $C' \cap f(A_1) = C' \cap f(B) = f(C)$ implies that the corresponding sets of representatives $\bar{A}_1'$ and $\bar{B}'$ for $C' \backslash A_1'$ and $C' \backslash B'$ can be chosen in the form

$$\bar{A}_1' = \bar{A} \amalg T \amalg \hat{A}_1', \qquad \bar{B}' = \bar{B} \amalg \hat{B}$$

where

$$\begin{aligned} \bar{A} &= \text{set of representatives for} \quad C \backslash A \\ A \amalg T &= \text{set of representatives for} \quad C \backslash A_1 \\ \bar{B} &= \text{set of representatives for} \quad C \backslash B. \end{aligned}$$

Using the above remarks about uniqueness of writings, it is easy to deduce Conditions a) and b). □

2. Some Acyclic Groups in D

The notation $H_*(X)$, for a space X, is used for $H_*(X\colon \mathbf{Z})$, where the integers $\mathbf{Z}$ are endowed with the trivial $\pi_1(X)$-action. Also $H_*(G)$, for a group G, will denote the Eilenberg-McLane homology $H_*(G;\mathbf{Z})$, where $\mathbf{Z}$ is endowed with the trivial G-

action. Of course, this is isomorphic to the homology $H_*(BG)$ of the space BG. This section is devoted to the proof of the following proposition:

(2.1) **Proposition.** *For $r \geqq 2$, there is a group $\Omega^r \in \mathbf{D}(r)$, such that:*
a) *$H_*(B\Omega^r) = H_*(pt)$. We say that Ω^r is an acyclic group.*
b) *$\Omega^r \times \mathbf{Z}$ is a subgroup of Ω^{r+1}.*

Proof. Define a group R by the presentation:

$$R = \langle z, t \mid tzt^{-1} = z^2 \rangle.$$

The group R is not a non-trivial free product. Indeed, as $R/[R,R]$ is infinite cyclic, one of the free summand would be a non-trivial perfect group. But R is solvable, because it is the semi-direct product of $\mathbf{Z}$ by $\mathbf{Z}[1/2]$. Hence, by Lemma (1.3), $R \in \mathbf{D}(2)$.

Let R_i $(i = 1,2,3,4)$ be copies of R, with generators t_i and z_i. Form the amalgamated products:

$$S = R_1 *_{(z_1 = t_2)} R_2$$
$$T = R_3 *_{(t_3 = z_4)} R_4.$$

By Lemma (1.1), the groups S and T are in $\mathbf{D}(2)$. The free group $F = \langle u, v \rangle$ admits monomorphisms $j_S \colon F \to S$ and $j_T \colon F \to T$ characterized by

$$j_S(u) = t_1 \qquad j_T(u) = z_3$$
$$j_S(v) = z_2 \qquad j_T(v) = t_4$$

Define $\Omega^2 = S *_F T$. As $F \in \mathbf{D}(1)$, the group Ω^2 is in $\mathbf{D}(2)$ by Lemma (1.1). One checks by Mayer-Vietoris sequences that Ω^2 is acyclic. (The reader may have recognized Ω^2 as the group invented by Higman [Hi].)

The group

$$A = \langle a, b, c, d \mid [a,b]\,[c,d]\,d^{-1} = 1 \rangle$$

is not a non-trivial free product. Indeed, the relator $[a,b]\,[c,d]\,d^{-1}$ is the product of the disjoint minimal word $[a,b]$ and $[c,d]\,d^{-1}$. The conclusion then follows from [Sh, Theorems 1 and 2]. Therefore, $A \in \mathbf{D}(2)$, by Lemma (1.3).

Let $1 \neq w \in \Omega^2$. Define

$$B = A * \Omega^2 / \{e = w\}$$
$$C_a = B * \Omega^2 / \{b = w\}$$
$$C_b = B * \Omega^2 / \{a = w\}$$

one has

a) B, C_a and C_b are groups in $\mathbf{D}(2)$
b) $H_1(B) = \mathbf{Z} \oplus \mathbf{Z}$, generated by a and b
$H_1(C_a) = \mathbf{Z}$, generated by a
$H_1(C_b) = \mathbf{Z}$, generated by b
c) $H_2(B) = H_2(C_a) = H_2(C_b) = 0$.

Thus, C_a and C_b are homology circles in $\mathbf{D}(2)$. They can be embedded in homology circles $D_a, D_b \in \mathbf{D}(3)$ by forming the push-out diagram

$$\begin{tikzcd}
\langle u\rangle \times \langle v\rangle \arrow[r] \arrow[d] & \Omega^2 \times \langle v\rangle \arrow[d] \\
\langle u\rangle \times C_a \arrow[r] \arrow[ur, phantom, "\Omega"] & D_a
\end{tikzcd}$$

where, where the left vertical arrow sends v to a. The same construction is used for D_b. The inclusions $C_a \to D_a$ and $C_b \to D_b$ are $\mathbf{Z}$-homology isomorphisms. Consider the two composed inclusions $B \to C_a \to D_a$ and $B \to C_b \to D_b$. Define Ω^3 by the psuh-out diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & D_a \arrow[d] \\
D_b \arrow[r] & \Omega^3
\end{tikzcd}$$

The properties of Ω^3 are the following:

1) $\Omega^3 \in \mathbf{D}(3)$, by Lemma (1.1)
2) Ω^3 is acyclic (use the Mayer-Vietoris sequence)
3) One has $\mathbf{Z} \times \Omega^2 \to D_a \to \Omega^3$.

Suppose now by induction that Ω^i is defined for $2 \leqq i \leqq n-1 \geqq 3$. Define Ω^n by the push-out diagram

$$\begin{tikzcd}
\langle w\rangle \times \Omega^{n-2} \arrow[r] \arrow[d] & \langle w\rangle \times \Omega^{n-1} \arrow[d] \\
\Omega^2 \times \Omega^{n-2} \arrow[r] & \Omega^n.
\end{tikzcd}$$

It is easily checked that this sequence of Ω^i's enjoys all the properties of Proposition (2.1). □

3. Proof of Theorem A

If X is a (simplical) polyhedron, we denote by $\mathbf{S}(X)$ the category whose objects are the subpolyedra of X and whose morphisms are the inclusions. We denote by m_X the number of simplices of X of dimension $\geqq 1$.

For each integer $r \geqq 0$, we consider the following category $\mathbf{C}(r)$: an object of $\mathbf{C}(r)$ is a polyhedron U such that each connected component U_i of U has the homotopy type of $B\pi_1(U_i)$ and $\pi_1(U_i)$ is in $\mathbf{D}(r)$. We shall write $U \in \mathbf{C}(r)$ to say that U is an object of $\mathbf{C}(r)$. A morphism of $\mathbf{C}(r)$ between U and V is an inclusion of U as a subpolyhedron of V such that the induced homomorphism from $\pi_1(U,u)$ to $\pi_1(V,u)$ is injective for all $u \in U$.

Theorem A is a direct consequence of the following proposition:

(3.1) **Proposition.** *Let X be a polyhedron. Then there exists an integer r and covariant functors*

$$L : \mathbf{S}(X) \to \mathbf{C}(r)$$
$$M : \mathbf{S}(X) \to \mathbf{C}(r)$$

satisfying the following properties:

a) *For each connected subpolyhedron Y of X, there is an acyclic map β_Y: $L(Y) \to Y$. The group* $\ker(\pi_1(L(Y)) \to \pi_1(Y))$ *is locally perfect. Moreover, if $Y \subset Y'$ is an inclusion of connected subpolyhedra of X, the following diagram*

$$\begin{tikzcd} L(Y) \arrow[r] \arrow[d, "\beta_Y"] & L(Y') \arrow[d, "\beta_{Y'}"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

is commutative.

b) *for any subpolyhedron Y of X, $M(Y)$ is an acyclic space (i.e. $H_*(M(Y);\mathbf{Z}) \cong H_*(pt;\mathbf{Z})$).*

c) *there is a natural transformation of functors $L \to M$. This means that there is an inclusion $L(Y) \subset M(Y)$ which is a morphism of $\mathbf{C}(r)$ and, if $Y \subset Y'$, the following diagram*

$$\begin{tikzcd} L(Y) \arrow[r] \arrow[d] & L(Y') \arrow[d] \\ M(Y) \arrow[r] & M(Y') \end{tikzcd}$$

is commutative

d) *For each inclusion $Y \subset Y'$ of subpolyhedra of X and for any $y \in L(Y)$, one has:*

$$\pi_1(M(Y), y) \cap \pi_1(L(Y'), y) = \pi_1(L(Y), y).$$

Proof. The proof is by induction on m_X. The induction starts with the case $m_X = 0$ (i.e. X is discrete), where we set $L(Y) = Y$, $\beta_Y = \mathrm{id}_Y$ and $M(Y) = cY$, the cone on Y.

The induction step consists of the following: suppose that (r, L, β_Y, M) as above is constructed for a polyhedron X. We then construct $(\bar{r}, \bar{L}, \bar{\beta}_Y, \bar{M})$ for a polyhedron $\bar{X} = X \cup e$, where e is a simplex of dimension $\geqq 1$. This will be done in several steps.

Step 1. For Y a subpolyhedron of X, define

$$L_1(Y) = \begin{cases} L(Y) \times B\Omega^3 & \text{if } Y \subset X \\ [L(Y \cap X) \times B\Omega^3] \cup_{L(\partial e) \times B\Omega^2} [M(\partial e) \times B\Omega^3] & \text{otherwise} \end{cases}$$

To simplify the notation we set

$$L_1(\mathring{e}) = M(\partial e) \times \Omega^3 \qquad \text{and} \qquad \hat{L}_1(\partial e) = L(\partial e) \times B\Omega^2.$$

If Y contains e, we can then write:

$$L_1(Y) = [L(Y \cap X) \times B\Omega^3] \cup_{\hat{L}_1(\partial e)} L_1(\mathring{e}).$$

It follows from the results of Sect. 1 that L_1 is a covariant functor from $\mathbf{S}(X)$ to $\mathbf{C}(r+3)$. If Y is a connected subpolyhedron of X, the acyclic map β_Y^1: $L_1(Y) \to Y$ is defined using the composition

$$L(Y \cap X) \times B\Omega^3 \longrightarrow L(Y \cap X) \xrightarrow{\amalg \beta_Z} Y$$

(Z = connected component of $Y \cap X$) which can be extended to the part $L_1(\mathring{e})$, because e is contractible. The group $\ker(\pi_1 \beta_Y^1)$ is normally generated by $\ker(\pi_1 \beta_Y)$, copies of Ω^3 and $\pi_1(M(\partial e))$. It is therefore locally perfect. The functor L_1 is covariant and satisfies Condition a) of (3.1). At this stage, the functor M will be defined only on $\mathbf{S}(X)$ by $M_1(Y) = M(Y) \times B\Omega^3$ for $Y \subset X$.

Step 2. One defines a space $\check{M}_2(e)$ as follows:

$$\check{M}_2(e) = [\hat{L}_1(\partial e) \times B\Omega^3] \cup_{\hat{L}_1(\partial e) \times B\Omega^2} [L_1(\mathring{e}) \times B\Omega^2].$$

It follows from Sect. 1 that $\check{M}_2(e) \in \mathbf{C}(r+5)$. By the Mayer-Vietoris sequence, the space $\check{M}_2(e)$ is acyclic.

To keep some coherence in the notation, write:

$$\begin{aligned}
L_2(Y) &= L_1(Y) \times B\Omega^2, && \text{for } Y \subset X, && L_2(Y) \in \mathbf{C}(r+5)\\
M_2(Y) &= M_1(Y) \times B\Omega^2, && \text{for } Y \subset X, && M_2(Y) \in \mathbf{C}(r+5)\\
\hat{L}_2(\partial e) &= \hat{L}_1(\partial e) \times B\Omega^2 \in \mathbf{C}(r+4) && && \\
L_2(\mathring{e}) &= L_1(\mathring{e}) \times B\Omega^2 \in \mathbf{C}(r+5). && &&
\end{aligned}$$

The inclusion $L_2(\mathring{e}) \subset \check{M}_2(e)$ induces a monomorphism

$$j_L: \pi_1(L_2(\mathring{e})) \to \pi_1(\check{M}_2(e)).$$

On the other hand, define a monomorphism

$$j_M: \pi_1(M_2(\partial e)) \to \pi_1(\check{M}_2(e))$$

by sending $(x, v) \in \pi_1(M_1(\partial e)) \times \Omega^2 = \pi_1(M_2(\partial e))$ to $(1, a)\overline{(x, v)}(1, a^{-1})$, where

i) $\overline{(x, v)}$ is the image of (x, v) under the monomorphism $\pi_1(M_1(\partial e)) \times \Omega^2 = \pi_1(L_2(\mathring{e})) \to \pi_1(\check{M}_2(e))$

ii) $(1, a) \in \pi_1(\hat{L}_1(\partial e)) \times \Omega^3$, with a an element of Ω^3 which commutes with those of $\Omega^2 \subset \Omega^3$. Such elements exist, because $\mathbf{Z} \times \Omega^2$ is a subgroup of Ω^3 by Proposition (2.1).

The choice of a implies that j_L and j_M coincide on $\pi_1(\hat{L}_2(\partial e))$, and this produces a homomorphism

$$j: \pi_1(K) \to \pi_1(\check{M}_2(e)),$$

where K is the space

$$K = \check{M}_2(\partial e) \cap_{\hat{L}_2(\partial e)} L_2(\mathring{e}).$$

Since a commutes with the elements of $\Omega^2 \subset \Omega^3$, one checks, using the unicity of the reduced writing in an amalgamated product, that j is injective. The monomorphism j can be realized by an inclusion of K as a subpolyhedron of a polyhedron $M_2(e)$ having the same homotopy type as $\check{M}_2(e)$. This inclusion is then a morphism of $\mathbf{C}(r+5)$.

Step 3. Define, for each subpolyhedron Y of X

$$M_3(Y)=\begin{cases} M_2(Y)\times B\Omega^3, & \text{if } Y\subset X\\ [M_2(Y\cap X)\times B\Omega^3]\cup_{M_2(\partial e)\times B\Omega^2}[M_2(e)\times B\Omega^3], & \text{otherwise,}\end{cases}$$

where the inclusion of $M_2(\partial e)$ into $M_2(e)$ is the one defined in Step 2, via the space K. The space $M_3(Y)$ contains the space $L_3(Y)=L_2(Y)\times B\Omega^2$. This is obvious when $Y\subset X$. When Y contains e, one has

$$L_3(Y)=[L_2(Y\cap X)\times B\Omega^2]\cup_{L_2(\partial e)\times B\Omega^2}[L_2(\mathring{e})\times B\Omega^2]$$

and therefore there is an inclusion $L_3(Y)\subset M_3(Y)$ inducing the obvious inclusion $L_2(Y\cap X)\times B\Omega^2\subset M_2(Y\cap X)\times B\Omega^3$ as well as the inclusion $L_2(\mathring{e})\subset M_2(e)$ defined in Step 2. One checks easily that

$$[\pi_1(M_2(\partial e))\times\Omega^2]\cap[\pi_1(L_2(\mathring{e}))\times\Omega^2]=\pi_1(L_2(\partial e))\times\Omega^2 .$$

On the other hand, one has

$$[\pi_1(M_2(\partial e))\times\Omega^2]\cap[\pi_1(L_2(Y\cap X))\times\Omega^2]=\pi_1(L_2(\partial e))\times\Omega^2 .$$

by Condition d). By Lemma (1.4), this implies that the inclusion $L_3(Y)\subset M_3(Y)$ induces a monomorphism on the fundamental groups. Also, if $Y\subset Y'$ are subpolyhedra of X, one has a commutative diagram

$$\begin{array}{ccc} L_3(Y) & \longrightarrow & L_3(Y')\\ \downarrow & & \downarrow\\ M_3(Y) & \longrightarrow & M_3(Y')\end{array}$$

The inclusion of the L_3's is a morphism of $\mathbf{C}(r+7)$ and the inclusion of the M_3's is a morphism of $\mathbf{C}(r+8)$. By Part b) of Lemma (1.4), one deduces that Condition d) is verified for (M_3, L_3). But we need a last adaptation of our construction in order to have $\bar{L}(Y)$ and $\bar{M}(Y)$ in the same class $\mathbf{C}(r)$.

Step 4. For Y a subpolyhedron of X, define $\bar{L}(Y)=L_3(Y)\times B\Omega^3$ and define $\bar{M}(Y)$ by the push-out diagram

$$\begin{array}{ccc} L_3(Y)\times B\Omega^2 & \longrightarrow & \bar{L}(Y)\\ \downarrow & & \downarrow\\ M_3(Y)\times B\Omega^2 & \longrightarrow & \bar{M}(Y)\end{array}$$

It follows from Sect. 1 that the inclusion $\bar{L}(Y)\subset\bar{M}(Y)$ is a morphism of $\mathbf{C}(r+10)$. Set $\bar{r}=r+10$. Condition b) is checked using the Mayer-Vietoris sequence. If $Y\subset Y'$ are subpolyhedra of X, there is a commutative diagram of inclusions as in Condition c). The following formula

$$[\pi_1(L_3(Y'))\times\Omega^2]\cap[\pi_1(L_3(Y))\times\Omega^3]=\pi_1(L_3(Y))\times\Omega^2$$

is obvious and the following formula

$$[\pi_1(M_3(Y))\times\Omega^2]\cap[\pi_1(L_3(Y'))\times\Omega^2]=\pi_1(L_3(Y))\times\Omega^2$$

comes from Condition d) for (M_3, L_3) which was established in Step 3. Therefore, using Part a) of Lemma (1.4), one deduces that the inclusion $\bar{M}(Y) \subset \bar{M}(Y')$ is a morphism of $\mathbf{C}(r)$. Part b) of Lemma (1.4) permits us to check Condition d) for $(\bar{M}, \bar{L})$. Observe that $L(Y) = L_1(Y) \times B\Omega^2 \times B\Omega^2 \times B\Omega^3$, and therefore there is a natural projection $L(Y) \to L_1(Y)$. The composition of this projection with β_Y^1, for Y connected, gives the required acyclic map $\bar{\beta}_Y: L(Y) \to Y$. Therefore $(\bar{r}, \bar{L}, \bar{\beta}_Y, \bar{M})$ is constructed, which completes the proof of Proposition (3.1). □

Remarks

(3.2) The principle of the proof of Proposition (3.1) is essentially the same as in [Ma] but a much stronger control of the successive amalgamations is required in order to stay in the classes $\mathbf{C}(r)$.

(3.3) The proof of Proposition (3.1) gives a (presumably very weak) upper bound of the integer $r(X)$ defined in the problem stated in the introduction. Suppose that X is a polyhedron such that $n_X = m(X)$, the minimal number of simplices of positive dimension of any polyhedron in the homotopy type of X. Write $X^{(0)} = X_0 \subset X_1 \subset \ldots \subset X_{m(X)} = X$, where $X_i = X_{i-1} \cup e$, with e a simplex of positive dimension. Then the spaces $L(X_i)$ given by the proof of (3.1) are in $\mathbf{C}(10i)$. Observe that for $L(X_{m(X)})$, one may stop the construction after Step 1, since there is no need for the space $M(X)$. Therefore, the proof of (3.1) produces a group $G \in \mathbf{D}(r)$ with an acyclic map $BG \to X$ with $r = 10(m(X) - 1) + 3 = 10\,m(X) - 7$. This gives the inequality $r(X) \leqq 10\,m(X) - 7$.

(3.4) For a connected CW-complex X, let $P(X)$ be the subset of all integers r such that there exists $G_r \in \mathbf{D}(r)$ with an acyclic map $BG_r \to X$. One has:

(3.4.1) $P(pt) = \{0\} \cup [2, \infty[$, by Lemma (2.1) and the fact that $\mathbf{D}(1)$ is the class of finitely generated free groups and therefore contains no acyclic group.

(3.4.2) If $\{r, r+1\} \subset P(X)$ and G_r is a subgroup of G_{r+1}, then $[r, \infty[\subset P(X)$. This comes from the last argument of the proof of Proposition (2.1).

(3.3.3) $P(S^1) = [1, \infty[$ by (3.3.2) and the prood of Proposition (2.1).

Problem. If $X \neq pt$, is $P(X) = [r(X), \infty[$?

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Mathematische Annalen

On a Problem of Bellenot and Dubinsky

Ehrhard Behrends[1], Susanne Dierolf[2], and Peter Harmand[3]

[1] I. Mathematisches Institut der Freien Universität, Arnimallee 3, D-1000 Berlin 33
[2] Fachbereich IV, Abteilung Mathematik der Universität, Postfach 3825, D-5500 Trier, Federal Republic of Germany
[3] Fachbereich 6 der Universität, Carl-von-Ossietzky-Strasse, D-2900 Oldenburg, Federal Republic of Germany

Dedicated to Professor Elmar Thoma on the occasion of his 60th birthday

Introduction

In this note we give an example of a separable Fréchet space which is not a Banach space, admitting a continuous norm but no nuclear Köthe quotient. This example gives a negative answer to a question of Bellenot and Dubinsky in [2]. The construction of our counterexample is based on a characterization of those Fréchet spaces whose strong dual has a representation as a strict LB-space, given by the second author and Moscatelli in [3].

For a dual pair (X, Y) we denote by $\beta(X, Y)$ and $\beta(Y, X)$ the corresponding strong topologies on X and Y, respectively, whereas the weak topologies are denoted by $\sigma(X, Y)$ and $\sigma(Y, X)$. By G' we mean the topological dual of a locally convex space G, and by $G'' := (G', \beta(G', G))'$ its bidual.

In [2] Bellenot and Dubinsky were concerned with the determination of those Fréchet spaces which admit a nuclear Köthe quotient, i.e. an infinite-dimensional nuclear quotient having a continuous norm and a basis. They proved [2, Theorem 2] that a separable Fréchet space F has no nuclear Köthe quotient if and only if $(F', \beta(F', F''))$ is a strict LB-space or – equivalently – if $F'^{\times} := (F', \beta(F', F''))'$ (i.e. the space of all bounded linear forms on the strong dual of F) provided with the strong topology $\beta(F'^{\times}, F')$ is a so-called quojection. Quojections are by definition those Fréchet spaces which are the projective limit of a projective sequence $((F_n)_{n\in\mathbb{N}}, (p_n)_{n\geq 2})$ of Banach spaces such that all the maps $p_n: F_n \to F_{n-1}$ are surjective. A convenient characterization of quojections is given by the condition that every quotient with a continuous norm must be Banach. Since a Fréchet space F for which $F'^{\times}$ is a quojection seems to be close to being a quojection itself, it was natural for Bellenot and Dubinsky to ask whether a separable Fréchet space without nuclear Köthe quotients must be a quojection or – equivalently – whether a separable non-Banach Fréchet space with a continuous norm must have a nuclear Köthe quotient.

Returning for a moment to arbitrary quojections (without a separability hypothesis) we recall from [3] that for a Fréchet space F one has the following string of implications:

$$\begin{aligned} &F \text{ is a quojection} \\ &\quad \Rightarrow (F', \beta(F',F)) \text{ is a strict } LB\text{-space} \\ &\quad \Rightarrow (F'^{\times}, \beta(F'^{\times}, F')) \text{ is a quojection}. \end{aligned}$$

In order to provide a negative answer to the above question it therefore suffices to give an example of a separable non-Banach Fréchet space F such that

i) F admits a continuous norm

ii) $(F', \beta(F',F))$ is a strict LB-space.

We mention that because of ii) such a space F will automatically be quasinormable hence distinguished and therefore its strong bidual $F''=F'^{\times}$ will be a quojection.

It follows from [3, Theorem 3 and Corollary 1], that there exists a non-Banach Fréchet space F satisfying i) and ii) if and only if there exists a dual Banach space X' (with unit ball B) containing a strictly increasing sequence $(H_n)_{n\in\mathbb{N}}$ of $\beta(X',X)$-closed, $\sigma(X',X)$-dense linear subspaces H_n such that

$$\overline{B\cap H_n}^{\sigma(X',X)} \subset H_{n+1} \quad \text{for all} \quad n\in\mathbb{N}.$$

Moreover, one obtains from the construction in [3, Sect. 4] that a separable Banach space X will lead to a separable Fréchet space F. In fact, given X and $(H_n)_{n\in\mathbb{N}}$, the space F is obtained as the projective limit of a sequence $(F_n)_{n\in\mathbb{N}}$ of Banach spaces such that, for each n, there is a natural continuous injection $X\to F_n$ with dense range.

Thus it remains to prove the following

Theorem. *There exists a separable Banach space X with the following property: its dual space X' (whose unit ball is denoted by B) contains a strictly increasing sequence $(H_n)_{n\in\mathbb{N}}$ of $\beta(X',X)$-closed, $\sigma(X',X)$-dense linear subspaces H_n such that the $\sigma(X',X)$-closure of $B\cap H_n$ is contained in H_{n+1} for every n.*

In order to prepare the proof of the theorem we remind the reader of a definition which can be traced back to Banach's book [1, p. 208]. For a subspace G of the dual X' of a Banach space X the *derivation $G_{(n)}$ of order n* of G is defined by induction as $G_{(n+1)}:=(G_{(n)})_{(1)}$, where $G_{(1)}$ means the collection of all limits of $\sigma(X',X)$-convergent sequences in G. If X is separable, $G_{(1)}$ is just the linear span of the $\sigma(X',X)$-closure of the unit ball of G, i.e. $G_{(1)}=\operatorname{lin}(\overline{G\cap B}^{\sigma(X',X)})$; one only has to note that $\sigma(X',X)$-convergent sequences are bounded and that B is $\sigma(X',X)$-metrizable for separable X.

In Théorème 1 of the Annexe [1], Banach has shown that for every $n\in\mathbb{N}$ there is a subspace G_n of $l^1=(c_0)'$ such that $(G_n)_{(n)}$ is not $\sigma(X',X)$-closed which, by the Krein-Smulian theorem, yields that

$$G_n \subsetneqq (G_n)_{(1)} \subsetneqq (G_n)_{(2)} \subsetneqq \dots \subsetneqq (G_n)_{(n)}.$$

Gluing together countably many copies of c_0 by the c_0-product, we obtain a space Z which has the l^1-sum of countably many copies of l^1 as its dual space and for which $G:=(\prod G_n)\cap Z'$ is a linear subspace of Z' such that $G\subsetneqq G_{(1)}\subsetneqq G_{(2)}\subsetneqq \dots$; by suitable identification we may regard G as a subspace of $l^1=(c_0)'$.

Since we will need a subspace G which is in addition $\sigma(X', X)$-dense, we form the polar G^0 of G in c_0. Then the dual of the quotient space $X := c_0/G^0$ can be canonically identified with the $\sigma(l^1, c_0)$-closure of G, such that $\sigma(l^1, c_0)$ induces the topology $\sigma(X', X)$ on X'. Therefore G is $\sigma(X', X)$-dense in X' and still has the property that the $G_{(n)}$ are strictly increasing.

We now turn to the proof of the theorem.

Let X and $G \subset X'$ be as above. Then for every $n \in \mathbb{N}$ the subspace $H_n := \overline{G_{(2n)}}^{\beta(X',X)}$ is $\beta(X', X)$-closed, $\sigma(X', X)$-dense and the sequence $(H_n)_{n\in\mathbb{N}}$ is increasing. Moreover, given any linear subspace H in X', one has that $B \cap \bar{H}^{\beta(X',X)} = \overline{B \cap H}^{\beta(X',X)}$, hence the $\sigma(X', X)$-closures of $B \cap H$ and $B \cap \bar{H}^{\beta(X',X)}$ coincide. Applying this fact to our spaces $G_{(2n)}$ we obtain that

$$\overline{B \cap H_n}^{\sigma(X',X)} \subset (H_n)_{(1)} = (G_{(2n)})_{(1)} = G_{(2n+1)} \subsetneqq G_{2n+2} \subset H_{n+1}$$

which also yields that $H_n \subsetneqq H_{n+1}$. This completes the proof.

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Der Primidealraum der $C^\star$-Algebra und die unzerlegbaren Charaktere der Gruppe $\mathrm{Gl}(\infty, q)$

·Wilfried Hauenschild

Fachbereich 17 Mathematik, Universität, Postfach 1621,
D-4790 Paderborn, Bundesrepublik Deutschland

Professor Elmar Thoma gewidmet zu seinem 60. Geburtstag

Für eine separable Gruppe G liefert jeder unzerlegbare Charakter α (positiv-definite, stetige Klassenfunktion) eine Faktordarstellung π_α, deren C^*-Kern, $\ker\pi_\alpha$, ein primitives Ideal ist. Man hat also eine kanonische Abbildung p von der Menge $E(G)$ der unzerlegbaren Charaktere in den Raum $\mathrm{Prim}\, C^*(G): p(\alpha) = \ker\pi_\alpha$. Nachdem sich diese Abbildung für den Fall $G = \mathfrak{S}_\infty$ in [4] als weder injektiv noch surjektiv erwiesen hatte, ist es vielleicht verwunderlich, daß sie für $G = \mathrm{Gl}(\infty, q) = \bigcup_{n\in\mathbb{N}} \mathrm{Gl}(n, q)$ eine Bijektion ist. Mit Methoden aus [3] erhalten wir unter Rückgriff auf Ergebnisse aus [1] und [7] für $\mathrm{Prim}\, C^*(\mathrm{Gl}(\infty, q))$ die gleiche Parameterisierung, wie sie Skudlarek in [5] für $E(\mathrm{Gl}(\infty, q))$ gezeigt hat, nämlich $(\hat{\mathbb{F}}_q^\times \times \mathbb{N}_0) \cup \{\infty\}$. Die entsprechende Identifizierung ist gerade die kanonische Abbildung p.

In einem ersten Abschnitt werden wir einige Ergebnisse aus [1], die Charaktere der $\mathrm{Gl}(n, q)$ betreffend, zusammenstellen und damit das Hauptresultat aus [7] über deren Einschränkung auf $\mathrm{Gl}(n-1, q)$ erläutern. Beides führt dann mit der Methode aus [3] zur Beschreibung von $\mathrm{Prim}\, C^*(\mathrm{Gl}(\infty, q))$ einschließlich der Hüllen-Kern-Topologie (Satz 1), und zwar dank der Vorarbeiten in [1] und [7], ohne auf die Theorie der uniformen Funktionen und Basischaraktere aus [1] eingehen zu müssen.

Dies ist erst im zweiten Teil nötig: Um $\ker\pi_\alpha$ zu beschreiben, ist es erforderlich, Skalarprodukte von $\alpha \mid \mathrm{Gl}(n, q)$ mit den Charakteren zu berechnen. Dies geschieht, indem zur Einschränkung $\alpha_{l,k}^{(n)}: \mathrm{Gl}(n, q) \to \mathbb{C}$, definiert durch $\alpha_{l,k}^{(n)}(A) = \theta^l(\det A) q^{-k\,\mathrm{rg}(A-E)}$ für $A \in \mathrm{Gl}(n, q)$ [5], wo θ ein Erzeuger von $\hat{\mathbb{F}}_q^\times$ ist, die sogenannten ϱ-Teile bestimmt werden, was u.a. mit Hilfe einer Kombination zweier Formeln geschieht, welche die Green-Polynome betreffen [8, 10.12, S. 128]; [1, Lemma 5.2]. Wir werden für die zweite Formel einen Beweis ohne Hopf-Algebren-Theorie mit Methoden aus [1] skizzieren; dieser benutzt Ergebnisse über Hall-Polynome, welche wohl bekannt sein dürften (vgl. unveröffentlichte Ergebnisse von Ph. Hall: Abelian groups and related modules, welche dem Autor nicht zugänglich waren).

Kennt man nun die ϱ-Teile einer uniformen Funktion, so kann man mit [1, Theorem 11], Skalarprodukte ausrechnen und damit die Bijektivität der betrachteten Abbildung zeigen, was im dritten Abschnitt geschieht (Satz 3).

1. Prim $C^*(\mathrm{Gl}(\infty, q))$

Sei q eine Primzahlpotenz und $1 \leqq s \leqq n$. Unter einem s-Simplex g verstehen wir eine Menge der Form $\{k, kq, \ldots, kq^{s-1}\} \subseteq \mathbb{Z}$ mit $0 \leqq k < q^s - 1$ derart, daß sämtliche Reste $\bmod(q^s - 1)$ der kq^j verschieden sind. Die kq^j heißen Wurzeln des Simplexes, und wir bezeichnen mit G die Menge aller Simplizes. Ist ε ein Erzeuger der multiplikativen Gruppe $\mathbb{F}_{q^{n!}}^x$, so ist $\varepsilon_s = \varepsilon^{\frac{q^{n!}-1}{q^s-1}}$ ein Erzeuger von $\mathbb{F}_{q^s}^x$, und $\varepsilon_s^k, \ldots, \varepsilon_s^{kq^{s-1}}$ sind die Wurzeln in $\mathbb{F}_{q^s}$ eines irreduziblen Polynoms $f \in \mathbb{F}_q[T]$ vom Grade s genau dann, wenn $\{k, \ldots, kq^{s-1}\}$ ein s-Simplex ist. Auf diese Weise erhält man eine Bijektion zwischen G und $F = \{f \in \mathbb{F}_q[T]; f \text{ irreduzibel mit } f \neq T\}$ (vgl. [1, S. 439]).

Sei nun $\Lambda = \bigcup_{n \in \mathbb{N}} \Lambda_n$, wo Λ_n die Menge der Partitionen $\lambda = \{1^{l_1} 2^{l_2} \ldots\}$ von n ist. Es ist also $|\lambda| = \sum_j j \cdot l_j = n$, und wir setzen $r(\lambda) = \sum_j l_j$. Man kann Λ mit der Menge aller Young-Diagramme identifizieren; $r(\lambda)$ ist dann die Anzahl der Zeilen. Wir schreiben $(\lambda)_k$ für die Länge der k-ten Zeile und $\lambda \leqq \mu$ für $\lambda, \mu \in \Lambda$, falls für alle $k \in \mathbb{N}$ gilt $(\lambda)_k \leqq (\mu)_k$. Nach [1, Theorem 14] kann man die Menge der Charaktere $\mathrm{GL}(n, q)\hat{}$ mit der Menge $\mathfrak{G}_n = \{\nu : G \to \Lambda;\ \sum_{g \in G} |\nu(g)| d(g) = n\}$ identifizieren. Satz 2 aus [7] liefert dann eine Beschreibung des Trägers der Einschränkung ν_0 des Charakters $\nu \in \mathfrak{G}_n$ auf $\mathrm{Gl}(n-1, q)$:

$$\begin{aligned}&\text{Für } \mu \in \mathfrak{G}_{n-1} \text{ gilt } (\nu_0, \mu) > 0, \text{ d.h. } \mu \in \operatorname{supp} \nu_0, \text{ genau dann,}\\ &\text{wenn } |(\mu(g))_k - (\nu(g))_k| \leqq 1 \text{ gilt für alle } g \in G \text{ und } k \in \mathbb{N}.\end{aligned} \tag{1}$$

In [3] nennen wir für eine abzählbare, lokal-endliche Gruppe $H_\infty = \bigcup_{n \in \mathbb{N}} H_n$ eine Menge $K = \{\nu^{(n)};\ n \in \mathbb{N}\} \subseteq X = \bigcup_{n \in \mathbb{N}} \hat{H}_n$ eine Kette in X, wenn $\nu^{(n)} \in \hat{H}_n$ und $\nu^{(n)} \in \operatorname{supp} \nu^{(n+1)} | H_n$ gilt für alle $n \in \mathbb{N}$. Zu einer Kette K definieren wir $A_K^{(n)} = \bigcup_{m \geqq n} \operatorname{supp} \nu^{(m)} | H_n \subseteq \hat{H}_n$ und $A_K = \bigcup_{n \in \mathbb{N}} A_K^{(n)} \subseteq X$. Weiter erhält man ein eindeutig bestimmtes Ideal $J_K \in \operatorname{Prim} C^*(H_\infty)$ mit $J_K \cap C^*(H_n) = \bigcap_{\varrho \in A_K^{(n)}} \ker \varrho$. Dies liefert eine Bijektion zwischen $\{A_K;\ K \text{ Kette in } X\}$ und $\operatorname{Prim} C^*(H)$ [3, Lemma 1 und Satz].

Kehren wir zurück zu $H_\infty = \mathrm{Gl}(\infty, q)$. Zu $0 \leqq l < q-1$ sei g_l der 1-Simplex $\{l\} \in G$; für $\nu \in \mathfrak{G} = \bigcup_{n \in \mathbb{N}} \mathfrak{G}_n$ setzen wir $|\nu| = \sum_{g \in G} |\nu(g)|\, d(g)$ und definieren für $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$:

$$\begin{aligned}&A_{l,k} = \{\nu \in \mathfrak{G};\ |\nu| - (\nu(g_l))_1 \leqq k\},\\ \text{d.h.}\qquad &A_{l,k}^{(n)} = \{\nu \in \mathfrak{G}_n;\ n - (\nu(g_l))_1 \leqq k\}.\end{aligned} \tag{2}$$

Definieren wir noch $J_{l,k} \subseteq C^*(\mathrm{Gl}(\infty, q))$ für $0 \leqq l < q-1$ und $k \in \mathbb{N}_0$ durch $J_{l,k} \cap C^*(\mathrm{Gl}(n, q)) = \bigcap_{\varrho \in A_{l,k}^{(n)}} \ker \varrho$ und $J_\infty = \{0\}$, so können wir unser erstes Ergebnis formulieren:

Satz 1. *Die Abbildung* $J : (\mathbb{F}_q^x \times \mathbb{N}_0) \cup \{\infty\} \to \operatorname{Prim} C^*(\mathrm{Gl}(\infty, q))$, *definiert durch* $J(\theta^l, k) = J_{l,k}$ bzw. $J(\infty) = J_\infty$ *(θ ein Erzeuger von $\mathbb{F}_q^x$), ist ein Homöomorphismus, falls man auf* $(\mathbb{F}_q^x \times \mathbb{N}_0) \cup \{\infty\}$ *die folgende Topologie definiert: V ist offen genau dann, wenn es eine Abbildung $h_V : \mathbb{F}_q^x \to \mathbb{N}_0$ gibt derart, daß* $V = \{(\theta^l, k);\ k \geqq h_V(\theta^l)\} \cup \{\infty\}$.

Beweis. Wir beweisen zunächst:

$$\operatorname{Prim} C^*(\mathrm{Gl}(\infty, q)) = \{J_{l,k}; 0 \leqq l < q-1, k \in \mathbb{N}_0\} \cup \{J_\infty\}.$$

Wegen der Resultate aus [3] haben wir zu zeigen:
Für jede Kette $K = \{\nu^{(n)}; n \in \mathbb{N}\}$ in $\mathfrak{G}$ mit $A_K \neq \mathfrak{G}$ gibt es $0 \leqq l < q-1$ und $k \in \mathbb{N}_0$ mit $A_K^{(n)} = A_{l,k}^{(n)}$ für alle $n \in \mathbb{N}$. Umgekehrt gibt es zu l, k eine Kette mit $A_K^{(n)} = A_{l,k}^{(n)}$.

Zu $\nu \in \mathfrak{G}$ definieren wir $I_\nu = \{(g,k) \in G \times \mathbb{N};\ k \leqq r(\nu(g))\}$ und zu einer Kette $K = \{\nu^{(n)};\ n \in \mathbb{N}\}$ in $\mathfrak{G}$: $I_K = \bigcup_{n \in \mathbb{N}} I_{\nu^{(n)}}$. Wir betrachten zunächst den Fall $\sup_{n \in \mathbb{N}} \sum_{(g,k) \in I_{\nu^{(n)}}} d(g) = \infty$: Zu gegebenem $m \in \mathbb{N}$ wählen wir n_0 so, daß $\sum_{(g,k) \in I_{\nu^{(n_0)}}} d(g) \geqq 2m+1$ gilt. Da $\sum_{(g,h) \in I_{\nu^{(n_0)}}} d(g)$ die mit dem Grad des Simplexes gewichtete Anzahl aller in den $\nu(g)$ vorkommenden Zeilen ist, gibt es nach (1) ein $\mu \in \operatorname{supp}(\nu^{(n_0)})_0$ mit $(\mu(g_0))_{2m} \geqq 1$. Für $\lambda \in \mathfrak{G}_{2m}$, definiert durch $\lambda(g_0) = \{1^{2m}\}$, gilt einmal $\lambda \in \operatorname{supp} \mu \mid G_{2m}$ und zum anderen $\hat{G}_m = \operatorname{supp} \lambda \mid G_m$. Zusammen gilt

$$\hat{G}_m = \operatorname{supp} \lambda \mid G_m \subseteq \operatorname{supp} \mu \mid G_m \subseteq \operatorname{supp} \nu^{(n_0)} \mid G_m,$$

d.h. $A_K = \mathfrak{G}$.

Sei nun $\sup_{n \in \mathbb{N}} \sum_{(g,k) \in I_{\nu^{(n)}}} d(g) = M < \infty$. Dann ist zunächst $d(g) \leqq M$ für alle $g \in G$ mit $(g,1) \in I_K$ und dann auch $k \leqq M$ für alle k mit $(g,k) \in I_K$. Das bedeutet aber $|I_K| < \infty$, und deshalb $J = \{(g,k);\ \sup_{n \in \mathbb{N}} (\nu^{(n)}(g))_k = \infty\} \neq \emptyset$. Nun unterscheiden wir wieder zwei Fälle: Sei zuerst $\sum_{(g,k) \in J} d(g) \geqq 2$. Dann gibt es $g \in G$ mit $d(g) \geqq 2$ und $(g,1) \in J$, oder $0 \leqq i_1 < i_2 < q-1$ mit $(g_{i_h}, 1) \in J$ $(h = 1, 2, g_{i_h} = \{i_h\} \in G)$, oder $0 \leqq i < q-1$ mit $(g_i, 2) \in J$. Zu gegebenem $m \in \mathbb{N}$ wählt man $n_0 \in \mathbb{N}$ so, daß $(\nu^{(n_0)}(g))_1 \geqq 2m$, $(\nu^{(n_0)}(g_{i_h}))_1 \geqq 2m (h = 1, 2)$ oder $(\nu^{(n_0)}(g_i))_2 \geqq 2m$. In allen drei Situationen gibt es ein $\mu \in \operatorname{supp} \nu^{(n_0)} \mid G_{n_0 - 2m}$ mit $(\mu(g_0))_{2m} \geqq 1$, und es folgt wie oben: $A_K = \mathfrak{G}$.

Im Falle $\sum_{(g,h) \in J} d(g) = 1$ gibt es $0 \leqq l < q-1$ mit $J = \{(g_l, 1)\}$. Insbesondere gibt es ein $N \in \mathbb{N}$ so, daß für alle $(g,k) \in I'_K = I_K \backslash \{(g_l, 1)\}$ gilt: $\sup_{n \in \mathbb{N}} (\nu^{(n)}(g))_k \leqq N$. Für

$$k_n = n - (\nu^{(n)}(g_l))_1 = \sum_{(g,k) \in I'_K} (\nu^{(n)}(g))_k d(g) \leqq |I_K| \cdot N \cdot M < \infty$$

gilt mit (1):
$$k_n = n - (\nu^{(n)}(g_l))_1 \leqq n + 1 - (\nu^{(n+1)}(g_l))_1 = k_{n+1}.$$

Also gibt es ein kleinstes $n_0 \in \mathbb{N}$ mit $k = k_{n_0} = \sup_{n \in \mathbb{N}} k_n$, und das bedeutet $(\nu^{(n)}(g_l))_1 = n - k$ für alle $n \geqq n_0$. Wir zeigen jetzt: $A_K^{(n)} = A_{l,k}^{(n)}$. Man sieht, daß es reicht, die Behauptung für $n > k$ zu zeigen. Für $\mu \in A_K^{(n)}$ existiert ein $m \geqq n$ mit $\mu \in \operatorname{supp} \nu^{(m)} \mid G_n$. Aus $(\nu^{(m)}(g_l))_1 = m - k$ folgt dann $(\mu(g_l))_1 \geqq m - k - (m - n) = n - k$. Ist umgekehrt $\mu \in \mathfrak{G}_n$ mit $(\mu(g_l))_1 \geqq n - k$, so gilt für $\lambda \in \mathfrak{G}_{n+k}$, definiert durch $\lambda(g_l) = \{1^k n^1\}$: $\lambda \in \operatorname{supp} \nu^{(n+2k)} \mid G_{n+k}$, und: $\mu \in \operatorname{supp} \lambda \mid G_n$, zusammen also $\mu \in \operatorname{supp} \nu^{(n+2k)} \mid G_n$.

Sind schließlich l und k gegeben, so definiert $\nu^{(n)} \in \mathfrak{G}_n$ mit $\nu^{(n)}(g_l) = \{1^k (n-k)^1\}$ eine Kette mit $A_K^{(n)} = A_{l,k}^{(n)}$ für alle $n \in \mathbb{N}$.

Nach [3] bilden die $U_\nu = \{J_{l,k};\ \nu \in A_{l,k}\} \cup \{J_\infty\}$ mit $\nu \in \mathfrak{G}$ eine Basis der Hüllen-Kern-Topologie auf $\operatorname{Prim} C^*(\mathrm{Gl}(\infty, q))$. Also ist $J_{l,k} \in U_\nu$ genau dann, wenn $k \geqq |\nu| - (\nu(g_l))_1$ für $0 \leqq l < q-1$ gilt. Daraus folgt, daß eine offene Menge V gegeben ist durch eine Abbildung h_V: $\{0, \ldots, q-2\} \to \mathbb{N}_0$: $J_{l,k} \in V \Leftrightarrow k \geqq h_V(l)$. Umgekehrt liefert jede solche Abbildung h: $\{0, \ldots, q-2\} \to \mathbb{N}_0$ eine offene Menge V_h mit: $J_{l,k} \in V_h \Leftrightarrow k \geqq h(l)$. $\square$

2. Die ϱ-Teile von $\alpha_{0,k}^{(n)}$

In [5] hat Skudlarek die Menge der unzerlegbaren Charaktere (positiv-definite Klassenfunktionen) $E(\mathrm{Gl}(\infty, q))$ beschrieben:

$$E(\mathrm{Gl}(\infty, q)) = \{\alpha_{\mathrm{reg}}\} \cup \{\alpha_{l,k}; 0 \leqq l < q-1, k \in \mathbb{N}_0\};$$

dabei ist $\alpha_{\mathrm{reg}}(A) = \delta_{A,E}$ und

$$\alpha_{l,k}(A) = \theta^l(\det A) \cdot q^{-k \operatorname{rg}(A-E)}$$

mit einem Erzeuger θ von $\hat{\mathbb{F}}_q^x$, der dualen Gruppe von $\mathbb{F}_q^x$. $E(\mathrm{Gl}(\infty, q))$ hat also die gleiche Parametrisierung wie $\operatorname{Prim} C^*(\mathrm{Gl}(\infty, q))$, nämlich $(\hat{\mathbb{F}}_q^x \times \mathbb{N}_0) \cup \{\infty\}$; dabei handelt es sich topologisch hier um die Einpunktkompaktifizierung des diskreten Raumes $\hat{\mathbb{F}}_q^x \times \mathbb{N}_0$. Unser Ziel wird es sein (Satz 3), zu zeigen, daß die entsprechende Identifizierung eine Bijektion zwischen $E(\mathrm{Gl}(\infty, q))$ und $\operatorname{Prim} C^*(\mathrm{Gl}(\infty, q))$ definiert.

Wir bezeichnen die Einschränkung von $\alpha_{l,k}$ auf $\mathrm{Gl}(n, q)$ mit $\alpha_{l,k}^{(n)}$. Bevor wir nun die ϱ-Teile der $\alpha_{l,k}^{(n)}$ angeben können, müssen wir erst einige Resultate aus der Theorie der sogenannten uniformen Funktionen aus [1] rekapitulieren.

Die Konjugationsklassen in $\mathrm{Gl}(n, q)$ stehen in ein-eindeutiger Beziehung zur Menge $\mathfrak{F}_n = \{\mu: F \to \Lambda; \sum_{f \in F} |\mu(f)| \, d(f) = n\}$: Ist $\mu(f) = \{1^{m_1} 2^{m_2} \ldots\}$, so ist m_j die Anzahl der Jordan-Kästchen der Länge $j \cdot d(f)$ zum irreduziblen Polynom $f \in F$. Wir identifizieren im folgenden μ mit der zugehörigen Konjugationsklasse. Zu $\varrho \in \Lambda_n$ definieren wir $(\mathbb{F}_q^x)^\varrho = (\mathbb{F}_q^x)^{r_1} \times \ldots \times (\mathbb{F}_{q^n}^x)^{r_n}$ und $c_\varrho(q) = |(\mathbb{F}_q^x)^\varrho|$. Eine ϱ-Funktion U_ϱ ist eine Abbildung U_ϱ: $(\mathbb{F}_q^x)^\varrho \to \mathbb{C}$ mit folgenden Eigenschaften: $U_\varrho(\gamma^\varrho) = U_\varrho(\gamma_{11}, \ldots, \gamma_{1r_1}, \gamma_{21}, \ldots, \gamma_{2r_2}, \ldots)$ ist invariant gegenüber Permutationen der $\gamma_{d1}, \ldots, \gamma_{dr_d}$ und gegenüber Ersetzen einer Wurzel γ_{dj} eines $f \in F$ durch eine andere. Ein $U: \mathfrak{F}_n \to \mathbb{C}$ heißt uniforme Funktion, falls es zu jedem $\varrho \in \Lambda_n$ eine ϱ-Funktion U_ϱ (den ϱ-Teil von U) gibt derart, daß

$$U(\mu) = \sum_\varrho \sum_m Q(m, \mu) U_\varrho(x^\varrho m) \tag{3}$$

gilt (vgl. [1, Definition 4.12]); dabei ist $x^\varrho m = \gamma^\varrho$, wo γ_{dj} eine Wurzel von $x_{dj}\alpha$ ist. Die $Q(m, \mu)$ sind mit Hilfe der Green-Polynome $Q_\varrho^\lambda(q)$ definiert. Diese können rekursiv aus den Hall-Polynomen $g_{\mu\nu}^\lambda(q)$ gewonnen werden [1, 4.2, 4.4]; [8, 10.13].

Wir verwenden noch die beiden Formeln auf S. 128 in [8] (vgl. auch [1, Lemma 5.2]): Für alle $\lambda \in \Lambda_n$ gelten:

$$\sum_{\varrho \in \Lambda_n} z_\varrho^{-1} Q_\varrho^\lambda(q) = 1 \tag{4}$$

und

$$\sum_{\varrho \in \Lambda_n} \operatorname{sgn} \varrho \, z_\varrho^{-1} Q_\varrho^\lambda(q) = q^{\binom{n}{2}} \delta_{\lambda, \{1^n\}}; \tag{5}$$

dabei ist für $\varrho=\{1^{r_1}2^{r_2}\dots\}$: $z_\varrho=\prod_j (j^{r_j}\cdot r_j!)$ und

$$\operatorname{sgn}\varrho=(-1)^{\Sigma(j-1)r_j}=(-1)^{n-r(\varrho)}.$$

Da der Beweis von (5) in [8] über die Theorie der Hopf-Algebren läuft, wollen wir hier kurz andeuten, wie er mit Methoden ähnlich denen aus [1] geführt werden kann. Dazu benötigen wir zunächst einige Formeln, welche die Hall-Polynome betreffen. Zunächst schreiben wir abkürzend für die Anzahl der s-dimensionalen Teilräume von $(\mathbb{F}_q)^r$:

$$N_s^r(q)=g^{\{1^r\}}_{\{1^{r-s}\}\{1^s\}}(q).$$

Für $N_s^r(q)$ gilt nach [2, (1.41)]:

$$N_s^r(q)=N_{r-s}^r(q) \quad \text{und} \quad N_s^r(q)=N_s^{r-1}(q)+q^{r-s}N_{s-1}^{r-1}(q), \tag{6}$$

woraus sofort

$$N_s^r(q)=q^sN_s^{r-1}(q)+N_{s-1}^{r-1}(q) \tag{7}$$

folgt.

Mit $\psi_{r,s}(t)=\prod_{j=0}^{s-1}(t^{r-j}-1)$ und $\psi_r=\psi_{r,r}$ gelten nun die folgenden Formeln:

$$\sum_{s=0}^{r-1} q^{\binom{s}{2}}\psi_{r-s-1}(q)N_s^r(q)=rq^{\binom{r}{2}} \quad \text{für alle } r\in\mathbb{N}, \tag{8}$$

und

$$\sum_{s=0}^{r} q^{\binom{s}{2}}\psi_{k,s}(q)N_s^r(q)=q^{rk} \quad \text{für alle } r,k\in\mathbb{N}_0. \tag{9}$$

Diese Formeln beweist man natürlich durch Induktion, indem man für (8) bzw. (9) die Rekursion (7) bzw. (6) benutzt, z.B. für (9):

$$\begin{aligned}\sum_{s=0}^{r+1} q^{\binom{s}{2}}\psi_{k,s}(q)N_s^{r+1}(q)&=\sum_{s=0}^{r+1} q^{\binom{s}{2}}\psi_{k,s}(q)\,(N_s^r(q)+q^{r-s+1}N_{s-1}^r(q))\\ &=\sum_{s=0}^{r} q^{\binom{s}{2}}\psi_{k,s}(q)N_s^r(q)+\sum_{s=1}^{r+1} q^rq^{\binom{s-1}{2}}\psi_{k-1,s-1}(q)\,(q^k-1)N_{s-1}^r(q)\\ &=q^{rk}+q^r(q^k-1)q^{r(k-1)}=q^{(r+1)k}.\end{aligned}$$

Wir benötigen noch eine Verallgemeinerung von (7) für Hall-Polynome:

$$g^\lambda_{\mu\{1^s\}}(q)=q^sg^{\lambda(r)}_{\mu(r)\{1^s\}}(q)+g^{\lambda(r)}_{\mu(r-1)\{1^{s-1}\}}(q); \tag{10}$$

hier ist $r=(\lambda)_1$ die Länge der ersten Zeile von λ und für $\mu=\{1^{m_1}2^{m_2}\dots\}$ und $t\in\mathbb{N}$ mit $m_t>0$: $\mu(t)=\{1^{m_1}\dots t^{m_t-1}\dots\}$. Die Formel (10) sieht man genauso ein wie (7), wenn man $g^\lambda_{\{1^s\}\mu}(q)$ betrachtet. Schließlich erhält man [1, (27) auf S. 428]:

$$\sum_{\nu\in\Lambda_{n-s}} g^\lambda_{\nu\{1^s\}}(q)=N_s^{r(\lambda)}(q). \tag{11}$$

Beim Beweis von (5) geht man nun vor wie in [1, Lemma 5.2]. Aus

$$Q^\lambda_\nu=\sum_{\substack{\tau\in\Lambda_{n-s}\\ \varrho\in\Lambda_s}} g^\lambda_{\tau\varrho}Q^\tau_{\nu(s)}Q^\varrho_{\{s\}}$$

[1, Lemma 4.4] folgt durch Summation über alle $\nu \in A_n$:

$$\sum_{\nu \in A_n} (s \cdot n_s) \operatorname{sgn} \nu z_\nu^{-1} Q_\nu^\lambda$$
$$= (-1)^{s-1} \sum_{\tau, \varrho} g_{\tau\varrho}^\lambda Q_{\{s\}}^\varrho \sum_{\nu' \in A_{n-s}} \operatorname{sgn} \nu' z_{\nu'}^{-1} Q_\nu^\tau ,$$

Summation über alle $1 \leqq s \leqq n$ liefert wegen $Q_{\{s\}}^\varrho = (-1)^{r(\varrho)-1} \psi_{r(\varrho)-1}$ [1, S. 441] mittels Induktion:

$$n \cdot \sum_{\nu \in A_n} \operatorname{sgn} \nu z_\nu^{-1} Q_\nu^\lambda(q)$$
$$= \sum_{s=1}^{n} (-1)^{s-1} \sum_{\varrho \in A_s} g_{\varrho\{1^{n-s}\}}^\lambda(q) (-1)^{r(\varrho)-1} \psi_{r(\varrho)-1}(q) q^{\binom{n-s}{2}}$$
$$= \sum_{0 < |\varrho| \leqq n} \operatorname{sgn} \varrho q^{\binom{n-|\varrho|}{2}} \psi_{r(\varrho)-1}(q) g_{\varrho\{1^{n-|\varrho|}\}}^\lambda(q) .$$

(5) ist also bewiesen, falls wir zeigen:

$$\sum_{0 < |\varrho| \leqq n} \operatorname{sgn} \varrho q^{\binom{n-|\varrho|}{2}} \psi_{r(\varrho)-1}(q) g_{\varrho\{1^{n-|\varrho|}\}}^\lambda(q) = n \cdot q^{\binom{n}{2}} \delta_{\lambda, \{1^n\}} \tag{12}$$

(vgl. [1, Lemma 5.3]).

Zum Beweis von (12) sei zunächst $\lambda = \{1^n\}$. Dann ist $g_{\varrho\{1^{n-|\varrho|}\}}^\lambda(q) \neq 0$, nur wenn $\varrho = \{1^{|\varrho|}\}$ ist.

Es gilt dann:

$$\sum_{s=1}^{n} q^{\binom{n-s}{2}} \psi_{s-1}(q) g_{\{1^s\}\{1^{n-s}\}}^{\{1^n\}}(q)$$
$$= \sum_{s=0}^{n-1} q^{\binom{s}{2}} \psi_{n-s-1}(q) N_s^n(q) = n \cdot q^{\binom{n}{2}} \quad \text{[nach (8)]}.$$

Ist aber $\lambda \neq \{1^n\}$, d.h. $(\lambda)_1 = r \geqq 2$, so gilt: Linke Seite von (12) $=$

$$\sum_{0 < |\varrho| \leqq n} \operatorname{sgn} \varrho q^{\binom{n-|\varrho|}{2}} \psi_{r(\varrho)-1}(q)$$
$$\times \left(q^{n-|\varrho|} g_{\varrho(r)\{1^{n-|\varrho|}\}}^{\lambda(r)}(q) + g_{\varrho(r-1)\{1^{n-|\varrho|-1}\}}^{\lambda(r)}(q)\right)$$
$$= \sum_{0 < |\varrho| \leqq n} \operatorname{sgn} \varrho q^{\binom{n-|\varrho|+1}{2}} \psi_{r(\varrho)-1}(q) g_{\varrho(r)\{1^{n-|\varrho|}\}}^{\lambda(r)}(q)$$
$$+ \sum_{0 < |\varrho'| \leqq n} \operatorname{sgn} \varrho' q^{\binom{n-|\varrho'|}{2}} \psi_{r(\varrho')-1}(q) g_{\varrho'(r-1)\{1^{n-|\varrho'|-1}\}}^{\lambda(r)}(q) .$$

$\varrho = \{1^{r_1} \ldots\} \to \varrho' = \{1^{r_1} \ldots (r-1)^{r_{r-1}+1} r^{r_r - 1} \ldots\}$ liefert eine Bijektion zwischen den Summanden beider Summen, und wegen $|\varrho'| = |\varrho| - 1$, $r(\varrho') = r(\varrho)$, $\operatorname{sgn} \varrho' = -\operatorname{sgn} \varrho$ und $\varrho(r) = \varrho'(r-1)$ heben sich die entsprechenden Summanden gegenseitig auf.

Wir definieren für $\mu = \{1^{m_1} \ldots\}$ und $\tau = \{1^{t_1} \ldots\}$ einen Multinomialkoeffizienten $\binom{\mu}{\tau} = \prod_j \binom{m_j}{t_j}$, und können nun eine Verallgemeinerung von [1, Lemma 5.2] beweisen:

Lemma 1. *Für n, $l \in \mathbb{N}_0$ und $\lambda \in \Lambda_n$ gilt:*

$$\sum_{\mu \in \Lambda_n} \left(\sum_{\tau \in \Lambda_l} \operatorname{sgn} \tau \binom{\mu}{\tau} \right) z_\mu^{-1} Q_\mu^\lambda(q) = q^{\binom{l}{2}} N_l^{r(\lambda)}(q).$$

Beweis.
$$\begin{aligned}
l.S. &= \sum_{\substack{\mu, \tau \\ \tau \leqq \mu}} \operatorname{sgn} \tau (z_\tau z_{\mu-\tau})^{-1} Q_\mu^\lambda(q) \\
&= \sum_{\substack{\tau \in \Lambda_l \\ \sigma \in \Lambda_{n-l}}} \operatorname{sgn} \tau (z_\tau z_\sigma)^{-1} Q_{\sigma+\tau}^\lambda(q) \\
&= \sum_{\sigma, \tau} \operatorname{sgn} \tau (z_\tau z_\sigma)^{-1} \sum_{\mu, \nu} g_{\mu\nu}^\lambda(q) Q_\sigma^\mu(q) Q_\tau^\nu(q) \quad [1, \text{Lemma } 4.4] \\
&= \sum_{\mu, \nu} g_{\mu\nu}^\lambda(q) \left(\sum_\sigma z_\sigma^{-1} Q_\sigma^\mu(q) \right) \left(\sum_\tau \operatorname{sgn} \tau z_\tau^{-1} Q_\tau^\nu(q) \right) \\
&= \sum_{\mu \in \Lambda_{n-l}} g_{\mu\{1^l\}}^\lambda(q) \cdot q^{\binom{l}{2}} \quad [\text{nach (4) und (5)}] \\
&= q^{\binom{l}{2}} N_l^{r(\lambda)}(q) \quad [\text{nach (11)}]. \quad \square
\end{aligned}$$

Sei jetzt $\varrho \in \Lambda_n$; für $\gamma^\varrho \in (\mathbb{F}_q^x)^\varrho$ sei $e_j(\gamma^\varrho) = |\{k; \gamma_{jk} = 1\}|$ und $\varepsilon(\gamma^\varrho) = \{1^{e_1(\gamma^\varrho)} 2^{e_2(\gamma^\varrho)} \ldots\} \in \Lambda$. Dann ist für jedes $\tau \in \Lambda$ die Funktion $(E_\tau)_\varrho : (\mathbb{F}_q^x)^\varrho \to \mathbb{C}$, definiert durch $(E_\tau)_\varrho(\gamma^\varrho) = \operatorname{sgn} \tau \binom{\varepsilon(\gamma^\varrho)}{\tau}$, offensichtlich eine ϱ-Funktion [1, Definition 4.9]. Mit Hilfe der zugehörigen uniformen Funktion E_τ mit

$$E_\tau(\mu) = \sum_{\varrho, m} Q(m, \mu) (E_\tau)_\varrho (x^\varrho m)$$

erhalten wir eine Darstellung der $\alpha_{0,k}^{(n)}$.

Proposition 1. *Für $n \in \mathbb{N}$ und $k \in \mathbb{N}_0$ gilt:*

$$\alpha_{0,k}^{(n)} = q^{-nk} \sum_{0 \leqq |\tau| \leqq n} \psi_{k,|\tau|}(q) E_\tau .$$

Beweis. Für einen "mode" m der ϱ-Variablen X^ϱ gilt mit den Bezeichnungen aus [1, Definition 4.6], falls man $f_0 \in F$ durch $f_0(t) = t - 1$ definiert: $\varepsilon(x^\varrho m) = \varrho(m, f_0)$. Es folgt somit: $(E_\tau)_\varrho(x^\varrho m) = \operatorname{sgn} \tau \binom{\varrho(m, f_0)}{\tau}$. Nun ist jeder "mode" von X^ϱ in die Klasse $\mu \in \mathfrak{F}_n$ [1, Definition 4.10] eindeutig bestimmt durch $\sigma = \varrho(m, f_0)$ und einen "mode" m' von $X^{\varrho - \sigma}$ in die Klasse μ' definiert durch $\mu'(f) = \mu(f)$ für $f \neq f_0$ und $\mu'(f_0) = (0)$. Also ergibt sich für $\mu \in \mathfrak{F}_n$ mit $|\mu(f_0)| = s$ und $0 \leqq |\tau| \leqq n$:

$$\begin{aligned}
E_\tau(\mu) &= \sum_{\varrho, m} Q(m, \mu) \operatorname{sgn} \tau \binom{\varrho(m, f_0)}{\tau} \\
&= \sum_{\sigma \in \Lambda_s} \operatorname{sgn} \tau \binom{\sigma}{\tau} z_\sigma^{-1} Q_\sigma^{\mu(f_0)}(q) \sum_{\varrho', m'} Q(m', \mu') \\
&= \sum_{\sigma \in \Lambda_s} \operatorname{sgn} \tau \binom{\sigma}{\tau} z_\sigma^{-1} Q_\sigma^{\mu(f_0)}(q) \quad [1, (26), \text{S. } 427].
\end{aligned}$$

Schließlich erhalten wir für $A \in \mu \in \mathfrak{F}_n$ (wie oben):

$$\begin{aligned}
&q^{-nk} \sum_{0 \leqq |\tau| \leqq n} \psi_{k,|\tau|}(q) E_\tau(\mu) \\
&= q^{-nk} \sum_{l=0}^{n} \psi_{k,l}(q) \sum_{\substack{\tau \in \Lambda_l \\ \sigma \in \Lambda_s}} \operatorname{sgn}\tau \binom{\sigma}{\tau} z_\tau^{-1} Q_\sigma^{\mu(f_0)}(q) \\
&= q^{-nk} \sum_{l=0}^{n} \psi_{k,l}(q) q^{\binom{l}{2}} N_l^{r(\mu(f_0))}(q) \qquad \text{(Lemma 1)} \\
&= q^{-k(n-r(\mu(f_0)))} \qquad [\text{nach (9)}] \\
&= q^{-k\,\mathrm{rg}(A-E)} = \alpha_{0,k}^{(n)}(\mu). \qquad \square
\end{aligned}$$

Damit sind nun auch die ϱ-Teile der $\alpha_{0,k}^{(n)}$ bekannt:

$$(\alpha_{0,k}^{(n)})_\varrho(\gamma^\varrho) = q^{-nk} \sum_{0 \leqq |\tau| \leqq n} \psi_{k,|\tau|}(q) \operatorname{sgn}\tau \binom{\varepsilon(\gamma^\varrho)}{\tau}.$$

3. $\ker \pi_{\alpha_{l,k}} = J_{l,k}$

Da $\alpha_{l,k}^{(n)}$ eine positiv-definite Klassenfunktion auf $\mathrm{Gl}(n,q)$ ist, gilt für alle $\nu \in \mathfrak{G}_n$: $(\nu, \alpha_{l,k}^{(n)}) \geqq 0$. Wir setzen: $\operatorname{supp}\alpha_{l,k}^{(n)} = \{\nu \in \mathfrak{G}_n;\ (\nu, \alpha_{l,k}^{(n)}) > 0\}$. Es gilt dann:

$$\ker \pi_{\alpha_{l,k}} \cap C^*(\mathrm{Gl}(n,q)) = \ker \pi_{\alpha_{l,k}^{(n)}} = \bigcap_{\nu \in \operatorname{supp}\alpha_{l,k}^{(n)}} \ker \nu .$$

Um unser Ziel zu erreichen, haben wir also $\operatorname{supp}\alpha_{l,k}^{(n)} = \mathrm{A}_{l,k}^{(n)}$ zu zeigen. Da nach [1, Theorem 14], jeder Charakter $\nu \in \mathfrak{G}_n$ eine Linearkombination der sogenannten Basis-Charaktere vom Typ ϱ ist [1, Definition 6.2], und zwar

$$\nu = \operatorname{sgn}\nu \sum_{\varrho, m} \chi(m, \nu) B^\varrho(h^\varrho m),$$

wobei wir $\operatorname{sgn}\nu = (-1)^{n - \Sigma|\nu(g)|}$ gesetzt haben, ist die Berechnung der folgenden Skalarprodukte ein wesentlicher Schritt.

Lemma 2. *Für $\varrho \in \Lambda_n$, $0 \leqq |\tau| \leqq n$ und einen "mode" m der dualen ϱ-Variablen Y^ϱ [1, vor Definition 8.1]) gilt:*

$$(B^\varrho(h^\varrho m), E_\tau) = \operatorname{sgn}\tau c_\tau(q)^{-1} \binom{\varrho(m, g_0)}{\varrho - \tau}.$$

Beweis. Da $B^\varrho(h^\varrho m)$ vom Typ ϱ ist, gilt für das Skalarprodukt nach [1, Theorem 11]:

$$(B^\varrho(h^\varrho m), E_\tau) = \operatorname{sgn}\tau (z_\varrho c_\varrho(q))^{-1} \sum_{\gamma^\varrho \in (\mathbb{F}_q^\times)^\varrho} B_\varrho(h^\varrho m : \gamma^\varrho) \binom{\varepsilon(\gamma^\varrho)}{\tau}.$$

Wegen $\varepsilon(\gamma^\varrho) \leqq \varrho$ für alle γ^ϱ ist dies nur dann ungleich Null, falls $\tau \leqq \varrho$ gilt. In dem Falle setzen wir $\sigma = \varrho - \tau$.

$Q \subseteq X^\varrho$ nennen wir σ-Teilmenge, falls Q für jedes d genau s_d der r_d ϱ-Variablen $x_{d_1}, \ldots, x_{dr_d}$ enthält. Ist nun α eine Substitution von X^ϱ, so ist durch die kanonische Abbildung von X^σ auf Q eine Substitution α_Q von X^σ definiert. Für die in der Definition von $B^\varrho(h^\varrho m)$ auftretenden Ausdrücke $S_d(k:\eta)$ gilt: $S_d(k:1) = S_d(0:\eta) = d$

für alle $k \in \mathbb{Z}$ und $\eta \in \mathbb{F}_{q^n!}^{\times}$. Beachtet man noch, daß $(B^\sigma(h^\sigma m), B^\sigma(h^\sigma m')) = z_\sigma \delta_{mm'}$ gilt [1, Beweis von Lemma 8.3], so erhält man, falls man $\mathbb{N}_k = \{1, \ldots, k\}$ setzt:

$$\begin{aligned}
&\sum_{\gamma^\varrho \in (\mathbb{F}_q^\times)^\varrho} B_\varrho(h^\varrho m : \gamma^\varrho) \binom{\varepsilon(\gamma^\varrho)}{\tau} \\
&= \prod_d \sum_{\gamma_d \in (\mathbb{F}_{q^d}^\times)^{r_d}} \binom{e_d(\gamma^\varrho)}{t_d} \sum_{\pi \in \mathfrak{S}_{r_d}} \prod_{l=1}^{r_d} S_d(h_{d\pi(l)}\alpha : \gamma_{dl}) \\
&= \prod_d \sum_{\substack{P_d \subseteq \mathbb{N}_{r_d} \\ |P_d| = s_d}} \sum_{\substack{\gamma_d \in (\mathbb{F}_{q^d}^\times)^{r_d} \\ \gamma_{dj} = 1, \text{falls } j \notin P_d}} t_d! \\
&\quad \times \sum_{\substack{\pi : P_d \to \mathbb{N}_{r_d} \\ \pi \text{ injektiv}}} \left(\prod_{l \in P_d} S_d(h_{d\pi(l)}\alpha : \gamma_{dl}) \right) \cdot d^{t_d} \\
&= \left(\prod_d \binom{r_d}{t_d} d^{t_d} t_d! \right) \left(\prod_d \sum_{\substack{Q_d \subseteq \mathbb{N}_{r_d} \\ |Q_d| = s_d}} \sum_{\gamma_d \in (\mathbb{F}_{q^d}^\times)^{s_d}} \sum_{\substack{\pi : \mathbb{N}_{s_d} \to Q_d \\ \pi \text{ bijektiv}}} \prod_{l=1}^{s_d} S_d(h_{d\pi(l)}\alpha : \gamma_{dl}) \right) \\
&= \binom{\varrho}{\tau} z_\tau \sum_{\substack{Q\sigma\text{-Teilm.} \\ \text{von } X^\varrho}} \prod_d \sum_{\gamma_d} \sum_{\pi \in \mathfrak{S}_{s_d}} \prod_{l=1}^{s_d} S_d(h_{d\pi(l)}\alpha_Q : \gamma_{dl}) \\
&= \binom{\varrho}{\tau} z_\tau \sum_Q \sum_{\gamma^\varrho \in (\mathbb{F}_q^\times)^\sigma} \prod_d (d^{s_d} s_d!)^{-1} \\
&\quad \times \sum_{\pi, \pi' \in \mathfrak{S}_{s_d}} \prod_{l=1}^{s_d} (S_d(h_{dl}\alpha_Q : \gamma_{d\pi(l)}) \overline{S_d(0 : \gamma_{d\pi'(l)})}) \\
&= \binom{\varrho}{\tau} z_\tau c_\sigma(q) \sum_Q (B^\sigma(h^\sigma \alpha_Q), B^\sigma(0)) \quad \text{(vgl. [1, S. 434])} \\
&= \binom{\varrho}{\tau} z_\tau c_\sigma(q) \binom{\varrho(m, g_0)}{\sigma} z_\sigma = z_\varrho c_\sigma(q) \binom{\varrho(m, g_0)}{\sigma}.
\end{aligned}$$

Also folgt die Behauptung, da $\binom{\varrho(m, g_0)}{\varrho - \tau} = 0$ gilt, falls nicht $\tau \leqq \varrho$. $\square$

Damit können wir nun unser nächstes Ergebnis beweisen.

Proposition 2. *Sei $\nu \in \mathfrak{G}_{k+1}$ und $\alpha_{0,k}^{(k+1)}$ wie oben. Dann gilt*
(i) $(\nu, \alpha_{0,k}^{(k+1)}) = 0$, *falls* $\nu(g_0) = (0)$, *und*
(ii) $(\nu, \alpha_{0,k}^{(k+1)}) > 0$, *falls* $\nu(g_0) = \{1^{k+1}\}$ *ist.*

Beweis. Wir rechnen zunächst ein Stück weit das folgende Skalarprodukt aus (mit $n = k+1$):

$$\begin{aligned}
(\nu, \alpha_{0,k}^{(n)}) &= q^{-nk} \operatorname{sgn} \nu \sum_{\varrho, m} \chi(m, \nu) \sum_{0 \leqq |\tau| \leqq n} \psi_{k,|\tau|}(q) (B^\varrho(h^\varrho m), E_\tau) \\
&= q^{-nk} \operatorname{sgn} \nu \sum_{\varrho, m} \chi(m, \nu) \sum_{0 \leqq |\tau| \leqq n} \psi_{k,|\tau|}(q) \operatorname{sgn} \tau \\
&\quad \times c_\tau(q)^{-1} \binom{\varrho(m, g_0)}{\varrho - \tau}.
\end{aligned}$$

Zu (i): Aus $v(g_0)=(0)$ folgt $\varrho(m,g_0)=(0)$. $\binom{\varrho(m,g_0)}{\varrho-\tau}$ ist also nur $\neq 0$, falls $\tau=\varrho$ gilt, d.h. $|\tau|=|\varrho|=k+1$. Dann ist aber $\psi_{k,|\tau|}(q)=0$, zusammen also $(v,\alpha_{0,k}^{(k+1)})=0$; zu (ii):

$$\begin{aligned}(v,\alpha_{0,k}^{(n)})&=q^{-nk}\sum_{\varrho\in\Lambda_n} z_\varrho^{-1}\chi_\varrho^{\{1^n\}}\sum_{0\leqq|\tau|\leqq n}\psi_{k,|\tau|}(q)\,\mathrm{sgn}\,\tau\\&\quad\times c_\tau(q)^{-1}\binom{\varrho(m,g_0)}{\varrho-\tau}\\&=q^{-nk}\sum_{l=0}^{n}\psi_{k,l}(q)\sum_{\varrho\in\Lambda_n}\sum_{\tau\in\Lambda_l} z_\varrho^{-1}\,\mathrm{sgn}\,\varrho\,\mathrm{sgn}\,\tau\, c_\tau(q)^{-1}\binom{\varrho}{\varrho-\tau}\\&=q^{-nk}\sum_{l=0}^{n}\psi_{k,l}(q)\sum_{\tau\in\Lambda_l}\sum_{\sigma\in\Lambda_{n-l}} z_\tau^{-1}z_\sigma^{-1}\,\mathrm{sgn}\,\sigma\, c_\tau(q)^{-1}.\end{aligned}$$

Nun gilt nach (5):

$$\sum_{\sigma\in\Lambda_{n-l}}\mathrm{sgn}\,\sigma\, z_\sigma^{-1}=\sum_{\sigma\in\Lambda_{n-l}}\mathrm{sgn}\,\sigma\, z_\sigma^{-1}Q_\sigma^{\{n-l\}}=\begin{cases}1 & \text{für } l=n-1,n\\ 0 & \text{sonst}\end{cases}$$

und wegen $c_\tau(q)^{-1}=\mathrm{sgn}\,\tau\,\psi_l(q)^{-1}Q_\tau^{\{1^n\}}(q)$ [1, S. 445] auch:

$$\sum_{\tau\in\Lambda_l}(z_\tau c_\tau(q))^{-1}=\psi_l(q)^{-1}\sum_{\tau\in\Lambda_l}\mathrm{sgn}\,\tau\, z_\tau^{-1}Q_\tau^{\{1^n\}}(q)=\psi_l(q)^{-1}q^{\binom{l}{2}}.$$

Zusammen folgt:

$$\begin{aligned}(v,\alpha_{0,k}^{(k+1)})&=q^{-k(k+1)}\sum_{l=k}^{k+1}\psi_{k,l}(q)\psi_l(q)^{-1}q^{\binom{l}{2}}\\&=q^{-k(k+1)}q^{\binom{k}{2}}>0.\quad\square\end{aligned}$$

Mit Hilfe dieser Proposition erhalten wir

Satz 2. *Es gilt* $\mathrm{supp}\,\alpha_{0,k}^{(n)}=A_{0,k}^{(n)}$ *für* $k\in\mathbb{N}_0$ *und* $n\in\mathbb{N}$.

Beweis. Da $\ker\pi_{\alpha_{0,k}}\in\mathrm{Prim}\,C^*(\mathrm{Gl}(\infty,q))$ ist, gibt es $0\leqq l<q-1$ und $m\in\mathbb{N}_0$ mit $\ker\pi_{\alpha_{0,k}}=J_{l,m}$ (J_∞ ist nicht möglich, da $\mathrm{supp}\,\alpha_{0,k}^{(k+1)}\neq\mathfrak{G}_{k+1}$ gilt). Es gilt also $\mathrm{supp}\,\alpha_{0,k}^{(n)}=A_{l,m}^{(n)}$ für alle $n\in\mathbb{N}$. Aus Proposition 2 (i) folgt: $\mathrm{supp}\,\alpha_{0,k}^{(k+1)}\subseteqq A_{0,k}^{(k+1)}$ und aus (ii): $\mathrm{supp}\,\alpha_{0,k}^{(k+1)}\not\subset A_{0,k-1}^{(k+1)}$. Für $1\leqq l<q-1$ und $v\in\mathfrak{G}_{k+1}$, definiert durch $v(g_l)=\{k+1\}$, gilt: $v\in A_{l,m}^{(k+1)}\setminus A_{0,k}^{(k+1)}$. Deshalb folgt aus $A_{l,m}^{(k+1)}\subseteqq A_{0,k}^{(k+1)}$ sofort $l=0$, d.h. $\mathrm{supp}\,\alpha_{0,k}^{(n)}=A_{0,m}^{(n)}$. Wegen $A_{0,j}^{(n)}\subsetneqq A_{0,j+1}^{(n)}$ für alle j liefert Proposition 2 (i): $m\leqq k$ und (ii): $m\geqq k$. $\square$

Als letztes kann man nun die Multiplikation der $v\in\mathfrak{G}_n$ mit den linearen Charakteren $\alpha_{l,0}^{(n)}$ durch eine Abbildung $T_l: G\to G$ beschreiben. Wir definieren für einen s-Simplex $g=\{k,kq,\dots,kq^{s-1}\}$:

$$T_l(g)=\left\{k+l\frac{q^s-1}{q-1},\left(k+l\frac{q^s-1}{q-1}\right)q,\dots,\left(k+l\frac{q^s-1}{q-1}\right)q^{s-1}\right\}.$$

Man sieht sofort, daß $T_l(g)$ wieder ein s-Simplex ist, und damit ist T_l eine Bijektion von G auf G mit $T_l(g_0)=g_l$. Für ein $v\in\mathfrak{G}_n$ setzen wir $v_l(g)=v(T_l^{-1}(g))$. Dann rechnet man nach: $v_l = v \cdot \alpha_{l,0}^{(n)}$, woraus sich $A_{0,k}^{(n)} \cdot \alpha_{l,0}^{(n)} = A_{l,k}^{(n)}$ ergibt und schließlich:

$$\begin{aligned}\operatorname{supp} \alpha_{l,k}^{(n)} &= \operatorname{supp}(\alpha_{0,k}^{(n)} \cdot \alpha_{l,0}^{(n)}) = (\operatorname{supp} \alpha_{0,k}^{(n)}) \cdot \alpha_{l,0}^{(n)} \\ &= A_{0,k}^{(n)} \cdot \alpha_{l,0}^{(n)} = A_{l,k}^{(n)}.\end{aligned}$$

Damit erhalten wir unseren Hauptsatz.

Satz 3. *Die kanonische Abbildung p von* $E(\mathrm{Gl}(\infty, q))$ *in* $\operatorname{Prim} C^*(\mathrm{Gl}(\infty, q))$ *ist eine Bijektion. Sie ist gegeben durch* $p(\alpha_{l,k}) = J_{l,k}$ *und* $p(\alpha_{\mathrm{reg}}) = J_\infty$.

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Weighted Versions of Beurling's Tauberian Theorem

H. G. Feichtinger[1] and H. J. Schmeißer[2]

[1] Institut für Mathematik, Universität, Strudlhofgasse 4, A-1090 Wien, Austria
[2] Sektion Mathematik, Universität, Universitätshochhaus, DDR-6900 Jena, German Democratic Republic

Dedicted to Professor E. Thoma on the occasion of his 60th birthday

Introduction

Wiener's well known Tauberian Theorem is equivalent to the following assertion. For $f \in L^\infty(\mathbb{R})$ the relation $\lim_{x\to\infty} K * f(x) = 0$ holds true for the convolution with any $K \in L^1$ whenever, it holds true for some $K_1 \in L^1(\mathbb{R})$ having a Fourier transform which does not vanish at any point. Beurling suggested such an implication to be true if convolution is replaced by certain related methods of taking moving averages (with variable band widths; cf. [5, 6] for details). Appropriately reformulated results and proofs (as given in [6]) even hold true in the m-dimensional setting, $m \geqq 1$.

In the present note weighted (=quantitative) versions are to be discussed, showing that a certain speed of convergence for a sufficiently "smooth" kernel K_1 with nonvanishing Fourier transform implies the same order of convergence for all kernels satisfying an appropriate decay condition. For an illustrating set of equivalent conditions (for $m=1$) the reader is referred to Theorem 3 below.

1. Notations and Preliminaries

Let us start with few definitions of basic operators defined on various function spaces on the Euclidean m-space $\mathbb{R}^m$, $m \geqq 1$. For a (class of) Lebesgue-measurable function(s) f on $\mathbb{R}^m$ we define

$$
\begin{aligned}
{}^\vee &: \check{f}(x) := f(-x); \\
L_y &: L_y f(x) := f(x-y), \qquad y \in \mathbb{R}^m; \\
M_\varrho &: M_\varrho f(x) := \varrho^m f(\varrho x), \qquad \varrho > 0; \\
D_\varrho &: D_\varrho f(x) := f(x/\varrho), \qquad \varrho > 0.
\end{aligned}
$$

Writing

$$\langle f, g \rangle := \int_{\mathbb{R}^m} f(x) g(x)\, dx$$

whenever they are "in duality" one has

$$\langle L_y f, g\rangle = \langle f, L_{-y} g\rangle \quad \text{and} \quad \langle M_\varrho f, g\rangle = \langle f, D_\varrho g\rangle . \tag{1}$$

Furthermore it is useful to note that

$$L_y M_\varrho = M_\varrho L_{\varrho y} \quad \text{or} \quad M_\varrho L_y = L_{y/\varrho} M_\varrho . \tag{2}$$

and that each of the three families of operators forms a commutative group under composition.

Let us put

$$g * f(x) := \int_{\mathbb{R}^m} f(x-y) g(y) dy , \qquad x \in \mathbb{R}^m ,$$

for the convolution of two functions f, g if the right-hand side makes sense for all $x \in \mathbb{R}^m$.

We shall use the weighted L^p-spaces $L^p_b(\mathbb{R}^m) := \{f \mid f w_b \in L^p\}$, where $w_b(x) := (1+|x|)^b$, $b \in \mathbb{R}$, and where L^p, $\|\ \|_p$, $1 \leqq p \leqq \infty$, denotes the usual Lebesgue space of p-integrable functions. The spaces L^p_b are endowed with their natural norm

$$\|f\|_{p,b} := \|f w_b\|_p = \Big(\int_{\mathbb{R}^m} |f(x) w_b(x)|^p\Big)^{\frac{1}{p}} .$$

(modification if $p = \infty$). Since the weights w_b satisfy

$$w_b(x+y) \leqq w_{|b|}(x) w_b(y) \quad \text{for} \quad x, y \in \mathbb{R}^m \tag{3}$$

one has

$$L^1_{|b|} * L^p_b \subseteqq L^p_b \quad \text{for} \quad b \in \mathbb{R}, \quad 1 \leqq p \leqq \infty \tag{4}$$

and

$$\|g * f\|_{p,b} \leqq \|g\|_{1,|b|} \|f\|_{p,b} \quad \text{for} \quad g \in L^1_{|b|}, \quad f \in L^p_b . \tag{5}$$

In particular, L^1_a is a Banach convolution algebra for $a \geqq 0$, called *Beurling algebra* (cf. [7, Chap. I, Sect. 6]). In (4) one may also replace L^∞_b by $C^0_b := \Big\{f \mid f$ continuous, $\lim_{x \to \infty} f(x) w_b(x) = 0\Big\}$.

The following invariance properties for these spaces are relevant for our arguments (cf. [7, Chap. 1], [2]):

Lemma 1. a) *For* $1 \leqq p \leqq \infty$, $b \in \mathbb{R}$ *one has*

$$L_y L^p_b \subseteqq L^p_b \quad \textit{and} \quad \|L_y f\|_{p,b} \leqq w_{|b|}(y) \|f\|_{p,b} \quad \textit{for} \quad y \in \mathbb{R}^m \tag{6}$$

and for $b \geqq 0$ *one has (with* $1/q = 1 - 1/p$*):*

$$M_\varrho L^p_b \subseteqq L^p_b \quad \textit{and} \quad \|M_\varrho f\|_{p,b} \leqq \varrho^{-b+m/q} \|f\|_{p,b} \quad \textit{for} \quad \varrho \leqq 1 \tag{7}$$

b) *For* $1 \leqq p < \infty$, $b \in \mathbb{R}$, *one has the following continuity properties:*

$$\lim_{y \to 0} \|L_y f - f\|_{p,b} = 0 \tag{8a}$$

$$\textit{for} \quad f \in L^p_b$$

$$\lim_{\varrho \to 1} \|M_\varrho f - f\|_{p,b} = 0 \tag{8b}$$

c) *For* $h \in L^1_{|b|}$ *with* $\int h(x)dx = 1$ *one has for* $1 \leqq p < \infty$, $b \in \mathbb{R}$

$$\lim_{\varrho \to \infty} \|M_\varrho h * f - f\|_{p,b} = 0 \quad \textit{for} \quad f \in L^p_b. \tag{9}$$

For $p = \infty$ relations (8) and (9) remain true for $f \in C^0_b$. We omit a detailed proof.

In order to describe the amount of smoothness required for certain kernels below the most convenient way will be to use certain (weighted) Lipschitz spaces. For our purposes the following ones ($p=1$ and $p=\infty$) will suffice ($e \in \mathbb{R}$, $1 \leqq p \leqq \infty$):

$$\mathrm{Lip}^p_e := \left\{ f \mid f \in L^p_e,\ \Lambda^q_e(f) := \|f\|_{p,e} + \sup_{|y| \leqq 1} |y|^{-1} \|L_y f - f\|_{p,e} < \infty \right\}. \tag{10}$$

Clearly, Lip^p_e is a Banach space endowed with the norm $\Lambda^p_e(\cdot)$. For simplicity we write $\mathrm{Lip}_e := \mathrm{Lip}^\infty_e$, $\Lambda_e(f) := \Lambda^\infty_e(f)$ and $\mathrm{Lip} := \mathrm{Lip}_0$, $\Lambda(f) := \Lambda_0(f)$.

Using (4) and (5) it is easily checked that one has for $1 \leqq p \leqq \infty$, $e \in \mathbb{R}$:

$$L^1_a * \mathrm{Lip}^p_e \subseteqq \mathrm{Lip}^p_e \quad \text{for} \quad a \geqq |e|, \tag{11}$$

and

$$\mathrm{Lip} \cdot \mathrm{Lip}^p_e \subseteqq \mathrm{Lip}^p_e, \tag{12}$$

together with the corresponding norm estimates. In particular, one has $\chi_t f \in \mathrm{Lip}^p_e$ for $f \in \mathrm{Lip}^p_e$, where for $t \in \mathbb{R}^m$ the character χ_t is given as $\chi_t(x) := \exp\left(i \sum_{j=1}^m x_j t_j\right) = \exp(ixt)$.

For a description of "moving averages" (with variable bandwidth) generalizing the usual convolution on $\mathbb{R}^m$ the following family of functions will be important.

Definition 1. A continuous function $s: \mathbb{R}^m \to [1, \infty)$ is called *self-neglecting* if it satisfies the following two conditions:

$S\,1)$ $\lim_{|x| \to \infty} s(x)/|x| = 0$,

$S\,2)$ $\lim_{|x| \to \infty} \dfrac{s(x + s(x)z)}{s(x)} = 1$ for each $z \in \mathbb{R}^m$.

Definition 2. Let $s(x)$ be a self-neglecting function. The *s-convolution* of $g \in L^1$, $f \in L^\infty$ is given by

$$\begin{aligned} g *_s f(x) &:= \int_{\mathbb{R}^m} f(x - s(x)y) g(y) dy \\ &= \int_{\mathbb{R}^m} f(y) g\left(\frac{x-y}{s(x)}\right) \frac{dy}{s(x)^m}. \end{aligned}$$

If $s(x) = w_\alpha(x)$ for some $\alpha \in [0,1]$ we write shortly $g *_\alpha f$ instead of $g *_{w_\alpha} f$ and speak of *α-convolution*.

Remark 1. It is clear that s-convolution is a bilinear mapping which reduces to ordinary convolution for $s \equiv 1$.

$$\begin{aligned} &\|g *_s f\|_\infty \leqq \|g\|_1 \|f\|_\infty \quad \text{for} \quad g \in L^1, f \in L^\infty \quad \text{and} \\ &g *_s f(x) \equiv \gamma \cdot \int g(y) dy \quad \text{if} \quad f(x) \equiv \gamma. \end{aligned} \tag{13}$$

Remark 2. Obviously, $g *_s f$ is a continuous function whenever $g \in L^p_b$ and $f \in L^q_{-b}$, $1/p + 1/q = 1$. Furthermore it is useful to keep in mind that $(r(x) = 1/s(x))$

$$g *_s f(x) = \langle L_x M_{r(x)} g, f \rangle = \langle M_{r(x)} g, L_{-x} f \rangle = \langle g, D_{r(x)} L_{-x} f \rangle . \tag{14}$$

Finally, let us introduce the (ordinary) Fourier transform by

$$\hat{f}(t) := \langle f, \chi_{-t} \rangle = \int_{\mathbb{R}^m} f(x) \exp(-ixt) dx , \qquad t \in \mathbb{R}^m .$$

Note that the letters $C, C_1, C_2, \dots$ will be reserved to generic positive constants. For example, one has

$$C_b^{-1} w_b(x) \leqq w_b(y) \leqq C_b w_b(x) \tag{15}$$

whenever $|x|/2 \leqq |y| \leqq 2|x|$.

2. The Main Result

For reference let us recall the modified version of Wiener's general Tauberian theorem which goes back to Beurling (we state the basic assertion involving convergence to zero).

Theorem 1 (Moh [5], Peterson [6]). *Let s be a self-neglecting function on $\mathbb{R}$, $K_1 \in L^1(\mathbb{R})$, $f \in L^\infty(\mathbb{R})$ be given. Assume that $\hat{K}_1(t) \neq 0$ for all $t \in \mathbb{R}$, and that*

$$\lim_{x \to \infty} K_1 *_s f(x) = 0 .$$

Then one has

$$\lim_{x \to \infty} K *_s f(x) = 0$$

for any $K \in L^1(\mathbb{R})$.

It will be our task to describe appropriate conditions on f and K_1 that will allow to conclude from a given "speed of convergence" for $K_1 *_s f(x)$ (for $x \to \infty$) that the same mode of convergence holds true for a broad class of kernels K. In order to keep the technical problems limited we restrict our attention to α-convolution (but work in $\mathbb{R}^m$). The main result obtained in this note will be the following theorem:

Theorem 2. *Let $\alpha \in (0,1)$ and $0 \leqq b \leqq m$ with $b < (1-\alpha)/\alpha$. Assume that one has for $f \in L^\infty_b$ and $K_1 \in L^1_a \cap \mathrm{Lip}_e$ for some $a > m$ and some $e > b + m + 1$*

$$\lim_{x \to \infty} [K_1 *_\alpha f(x)] \, |x|^b = 0 . \tag{$*$}$$

If K_1 has nonvanishing Fourier transform, i.e. if $\hat{K}_1(t) \neq 0$ for all $t \in \mathbb{R}^m$ then $(*)$ holds true for any $K \in L^\infty_a$.

Remarks on the Proof. The proof of this theorem will be given in several steps. As in the classical situation the main steps concern a proof that $(*)$ (equivalent to the assertion $K_1 *_\alpha f \in C^0_b$) holds true for $K_2 = L_t K_1$, $t \in \mathbb{R}^m$, hence for linear combinations, and by an approximation argument for more general kernels K_2, including those in L^∞_a. However, since the generalized α-convolution does *not* commute with translation nor defines an (associative or commutative) operation on the space of test functions much more refined estimates are required than in the case of ordinary convolution.

3. Some Lemmata

Lemma 2. *Let $\alpha\in[0,1)$, $b\in[0,m]$ be given. Then one has for any $a>m$*

$$w_{-a}*_\alpha w_{-b}(x)\sim w_{-b}(x) \quad \textit{for} \quad x\to\infty\,. \tag{16}$$

Proof. Splitting the integral defining the α-convolution into two parts over $D_x:=\{y\,|\,|y|\geqq|x|/2\}$ and over $B_x:=\{y\,|\,|y|<|x|/2\}$ one has

$$w_{-a}*_\alpha w_{-b}(x)=\int_{D_x\cup B_x} L_xM_{w_{-\alpha}(x)}w_{-a}(y)w_{-b}(y)dy=:I_{D_x}+I_{B_x}\,.$$

It is clear that

$$I_{D_x}\leqq\left[\sup_{y\in D_x} w_{-b}(y)\right]\|L_xM_{w_{-\alpha}(x)}w_{-a}\|_1\leqq C_b w_{-b}(x)\,\|w_{-a}\|_1$$

because w_{-b} is nonincreasing for $b\geqq0$. On the other hand

$$I_{B_x}\leqq\left[\int_{B_x} w_{-b}(y)dy\right]\sup_{y\in B_x}|L_xM_{w_{-\alpha}(x)}w_{-a}(y)|\,.$$

The simple estimate $w_{-b}(y)\leqq|y|^{-b}$ for (the relevant part where) $1\leqq|y|\leqq|x|/2$ allows to reduce the estimate for the first factor to an elementary expression of order $|x|^{-b+m}$ (respectively of order $\log|x|$ in the case $b=m$). Using the monotonicity of $w_{-\alpha}$ and inequalities (3), (10) one obtains ($y\in B_x$):

$$\begin{aligned}L_xM_{w_{-\alpha}(x)}w_{-a}(y)&=O(w_{-\alpha m}(x)w_{-a}(x/2)w_{-a}(w_{-\alpha}(x)\cdot x)\\&=O(w_{(\alpha-1)a-\alpha m}(x)) \quad \text{for} \quad x\to\infty\,.\end{aligned}$$

Combining these estimates one obtains for $b<m$:

$$\begin{aligned}I_{B_x}&=O(w_{m-b}(x)w_{(1-\alpha)a-\alpha m}(x))=O(w_{-b}(x)w_{(a-m)(\alpha-1)}(x))\\&=o(w_{-b}(x)) \quad \text{for} \quad x\to\infty\,.\end{aligned}$$

A slight modification shows that one has for $b=m$ still the same estimate: $I_{B_x}=o(w_{-b}(x))$ for $x\to\infty$, and thus

$$w_{-a}*_\alpha w_{-b}(x)=O(w_{-b}(x)) \quad \text{for} \quad x\to\infty\,.$$

In order to prove the converse note that one has $w_\alpha(x)\leqq|x|/2$ if $|x|\geqq n_\alpha>0$, and $D_x\supseteqq D'_x:=\{y\,|\,|y-x|<|x|/2\}$, thus

$$\begin{aligned}I_{D_x}&\geqq\left(\int_{D'_x} L_xM_{w_{-\alpha}(x)}w_{-a}(y)dy\right)\left(\inf_{y\in D'_x} w_{-b}(y)\right)\\&\geqq\left(\int_{B_x} M_{w_{-\alpha}(x)}w_{-a}(z)dz\right)C_b^{-1}w_{-b}(x)\\&\geqq\left(\int_{\{z\,|\,|z|<w_{-\alpha}(x)\}} M_{w_{-\alpha}(x)}w_{-a}(z)dz\right)C_b^{-1}w_{-b}(x)\\&=\left(\int_{\{y\,|\,|y|<1\}} w_{-a}(y)dy\right)C_b^{-1}w_{-b}(x)=C_{a,b}w_{-b}(x) \quad \text{for} \quad x\to\infty\,,\end{aligned}$$

and the proof is complete.

Remark 3. A more refined estimate would reveal that one even has in the above situation:

$$\lim_{x\to\infty} (w_{-a} *_\alpha w_{-b}(x) - \|w_{-a}\|_1 w_{-b}(x)) = 0 .$$

As an immediate consequence of Lemma 2 (and its proof) we obtain the following

Corollary 1. a) *For α, a, b as above one has*

$$L_a^\infty *_\alpha L_b^\infty \subseteqq L_b^\infty \tag{17a}$$

with the natural norm estimate for $g \in L_a^\infty$, $f \in L_b^\infty$

$$\|g *_\alpha f\|_{\infty,b} \leqq C \|g\|_{\infty,a} \|f\|_{\infty,b} . \tag{17b}$$

In particular, one has

$$\|g_n *_\alpha f - g *_\alpha f\|_{\infty,b} \to 0 \quad \textit{whenever} \quad \|g_n - g\|_{\infty,a} \to 0 \quad \textit{for} \quad n\to\infty .$$

b) *In contrast, one has $g *_\alpha f \notin C_b^0$ (or equivalently $g *_\alpha f(x)$ is not $o(w_{-b}(x))$ for $x\to\infty$) for $f = w_{-b} \in L_b^\infty$ and any nontrivial, positive $g \in L_a^\infty$.*

Remark 4. Part b) indicates that there are always non-trivial examples of functions $f \in L_b^\infty$ and $K \in L_a^\infty$ such that (*) does not hold true for all $K \in L_a^\infty$. This shows of course the relevance of the assumption (*) in the theorem, as condition on f, for some "nice" K_1.

The proof of Lemma 2 contains additional information concerning the continuity of α-convolution with respect to the first variable. In fact, much less than norm convergence in L_a^∞ is required. One has in the above situation:

Lemma 3. *Assume that $\alpha \in [0,1)$, $0 \leqq b \leqq m$, $a > m$.*

a) *Let $(g_n)_{n=1}^\infty$ be a bounded sequence in L_a^∞ which is convergent to $g \in L_a^\infty$ with respect to the L^1-norm. Then one has for any $f \in L_b^\infty$:*

$$\|g_n *_\alpha f - g *_\alpha f\|_{\infty,b} \to 0 \quad \textit{for} \quad n\to\infty . \tag{18}$$

b) *In particular, if $h \in L_a^1$ with $\int h(x)dx = 1$ is given, then one has for $g \in L_a^\infty$:*

$$\lim_{\varrho\to\infty} \|(M_\varrho h * g) *_\alpha f - g *_\alpha f\|_{\infty,b} = 0 . \tag{19}$$

Proof. a) Of course it suffices to verify (18) for $g=0$, i.e. for $\lim_{n\to\infty} \|g_n\|_1 = 0$. Splitting the α-convolution

$$|g_n *_\alpha f(x)| \leqq \|f\|_{\infty,b} |g_n *_\alpha w_{-b}(x)|$$

as in the proof of Lemma 2 to $I_{D_x}^n \cup I_{B_x}^n$ one obtains:

$$I_{D_x}^n \leqq \|f\|_{\infty,b} C_b w_{-b}(x) \|g_n\|_1 < \varepsilon w_{-b}(x) \quad \text{for} \quad n \geqq n_1(\varepsilon)$$

and

$$\begin{aligned} I_{B_x}^n &\leqq \|f\|_{\infty,b} \Big(\sup_{n\geqq 1} \|g_n\|_{\infty,a}\Big) \int_{B_x} L_x M_{w_{-\alpha}(x)} w_{-a}(y) w_{-b}(y) dy \\ &= o(w_{-b}(x)) \quad \text{for} \quad |x|\to\infty, \quad \text{i.e.} \\ I_{B_x}^n &< \varepsilon w_{-b}(x) \quad \text{for} \quad |x| \geqq C > 0 \quad \text{and all} \quad n \geqq 1 . \end{aligned}$$

Recalling now that $\|g *_\alpha f\|_\infty \leqq \|g\|_1 \|f\|_\infty$ one observes that one has of course uniform convergence in our case and consequently there exists $n_2(\varepsilon) \geqq n_1(\varepsilon)$ such that

$$|g_n *_\alpha f(x)| < \varepsilon\Big(\inf_{\{y \mid |y| \leqq C\}} w_{-b}(y)\Big) \leqq \varepsilon w_{-b}(x) \quad \text{for} \quad |x| \leqq C \quad \text{and all} \quad n \geqq n_2,$$

and thus altogether

$$|g_n *_\alpha f(x)| < \varepsilon w_{-b}(x) \quad \text{for all} \quad x \in \mathbb{R}^m, \quad \text{and} \quad n \geqq n_2,$$

or

$$\|g_n *_\alpha f\|_{\infty, b} < \varepsilon \quad \text{for} \quad n \geqq n_2,$$

i.e. (18) is proved.

b) Assertion (19) follows immediately from a), because (4) together with the boundedness of $(M_\varrho h)_{\varrho \in [1,\infty)}$ in L^1_a for any $h \in L^1_a$ implies that for any sequence $(\varrho_n)_{n \geqq 1}$ the sequence $(M_{\varrho_n} h *_\alpha g)_{n \geqq 1}$ is bounded in L^∞_a. On the other hand one has, due to the normalization of h the L^1-convergence to g for $\varrho_n \to \infty$ (cf. [7, Chap. I, Sect. 2.2]).

Before we can prove one of the main steps, i.e. that our smoothness requirements suffice in order to replace K_1 by $L_t K_1$ for any $t \in \mathbb{R}^m$ let us verify the following assertion:

Lemma 4. *Let $e > b + m + 1$ be given. Then one has for $K \in \mathrm{Lip}_e$:*

$$\lim_{\varrho \to 1} |\varrho - 1|^{-d} \|M_\varrho K - K\|_{1,b} = 0 \quad \textit{for any} \quad d < 1. \tag{20}$$

Proof. By the assumption one has

$$|K(y+x) - K(x)| \leqq C_K |y| w_{-e}(x) \quad \text{for} \quad x, y \in \mathbb{R}^m.$$

This implies (by setting $y := \varrho x - x$).

$$\|K(\varrho\,\cdot) - K\|_{\infty, e-1} \leqq C_K |\varrho - 1|$$

and further if, say $\varrho \in [\frac{1}{2}, 2]$,

$$\begin{aligned}\|M_\varrho K - K\|_{\infty, e-1} &\leqq \|M_\varrho K - K(\varrho\,\cdot)\|_{\infty, e-1} + C_K |\varrho - 1| \\ &\leqq |\varrho^m - 1|\, \|K\|_{\infty, e-1} + C_K |\varrho - 1| \leqq C_1 |\varrho - 1|.\end{aligned}$$

For $d < 1$ and $b < e - m - 1$ this clearly implies

$$|1 - \varrho|^{-d} \|M_\varrho K - K\|_{1,b} \leqq C_2 |\varrho - 1|^{1-d} \to 0 \quad \text{for} \quad \varrho \to 1.$$

Remark 5. It is obvious that a differentiable function $K \in L^\infty_e$ belongs to Lip_e if $|\nabla K| \in L^\infty_e$. Thus, clearly $\mathscr{S}(\mathbb{R}^m) \subseteqq \mathrm{Lip}_e$ for any $e \geqq 0$. Here $\mathscr{S}(\mathbb{R}^m)$ denotes the Schwartz space of rapidly decreasing functions.

Lemma 5. *Under the assumptions of Theorem 2 one has*

$$L_t K_1 *_\alpha f \in C^0_b \quad \textit{for any} \quad t \in \mathbb{R}^m.$$

Proof. For t, x, $y \in \mathbb{R}^m$ one has

$$L_t K_1 *_\alpha f(x) = \int_{\mathbb{R}^m} f(x - w_\alpha(x)y) K_1(y-t)dy .$$

Writing shortly $z := x - w_\alpha(x)t$, and $\varrho(x) = w_\alpha(z) w_{-\alpha}(x)$ one splits as follows (assuming $\|f\|_{\infty, b} \leqq 1$):

$$|L_t K_1 *_\alpha f(x)| \leqq |K_1 *_\alpha f(z)| + \int_{\mathbb{R}^m} w_{-b}(z - w_\alpha(z)y) \, |M_{\varrho(x)} K_1 - K_1|(y)dy .$$

Since $w_\alpha(x)$ is self-neglecting one can estimate the first term as follows:

$$\lim_{x \to \infty} |K_1 *_\alpha f(z)| w_b(x) = \lim_{x \to \infty} |K_1 *_\alpha f(z)| w_b(z) \lim_{x \to \infty} w_b(x)/w_b(z) = 0 .$$

With the help of the $L^\infty_{-b} - L^1_b$ duality and (3) the second term can be estimated by

$$\|M_{\varrho(x)} K_1 - K_1\|_{1,b} \, \|w_{-b}(z - w_\alpha(z)y)\|_{\infty, -b} \leqq \|M_{\varrho(x)} K_1 - K_1\|_{1,b} w_{-b}(x) w_{\alpha b}(x) .$$

Thus it is apparent that the order of convergence for $\varrho(x) \to 1$ has some relevance (for $x \to \infty$). One has by the mean value theorem and by the properties of w_α, $0 < \alpha < 1$ (for large $|x|$)

$$w_\alpha(x) |1 - \varrho(x)| = |w_\alpha(x - w_\alpha(x)t) - w_\alpha(x)| \leqq C \cdot w_\alpha(x) \, |t| \cdot w_{\alpha - 1}(x)$$

or $|1 - \varrho(x)| = 0(w_{\alpha-1}(x))$ for $x \to \infty$. Hence we have by the assumptions of Theorem 2, combined with Lemma 4, for any d with $\dfrac{\alpha b}{1-\alpha} < d < 1$ that

$$\begin{aligned} L_t K_1 *_\alpha f(x) &= O(w_{-b}(x)) + O(w_{(\alpha-1)d}(x)) O(w_{\alpha b}(x)) w_{-b}(x) \\ &= O(w_{-b}(x)) \end{aligned}$$

for $x \to \infty$. Thus, the proof is complete.

Remark 5. It is clear from the proof that the relevant condition on K_1 for the validity of Lemma 5 is the following one:

$$\lim_{\varrho \to 1} |1 - \varrho|^d \, \|M_\varrho K_1 - K_1\|_{1,b} = 0 \quad \text{for} \quad d = \alpha b/(1-\alpha) . \tag{21}$$

It is, of course, not only true for "smooth" kernels, but also for step functions, for example.

Proof of Theorem 2

Recall that we have to show (∗): $K *_\alpha f \in C^0_b$ for any $K \in L^\infty_a$.

Step 1. First of all we observe that the theorem holds for $K := g * K_1$ for any $g \in L^1_a$. In fact, it follows from Lemma 5 that it holds for kernels of the form $K_n = \sum_{i=1}^n a_i L_{y_i} K_1$. Since the usual L^1-approximation of the convolution $g * K_1$ by such linear combinations K_n satisfies in addition $\sup_n \|K_n\|_{\infty, a} < \infty$ in case $g \in L^1_a$ (cf. [7, Chap. I, Sect. 4.2]) it follows from Lemma 3a) that $(g * K_1) *_\alpha f \in C^0_b$.

Step 2. We show now that (*) holds true for $K \in L^1_a \cap C^0_a$. In view of (9) (Lemma 1c) one has $K = \lim_{\varrho \to \infty} M_\varrho h * K$ in L^∞_a for any $h \in L^1_a$ with $\int h(x)dx = 1$. Due to Corollary 1a) it therefore suffices to verify that $(M_\varrho h * K) *_\alpha f \in C^0_b$ for all $\varrho \geqq 1$, for a suitable function $h \in \mathscr{S}(\mathbb{R}^m) \subseteqq L^1_a$. We may assume that supp$\hat{h}$ is compact. Fixing $\varrho > 0$ it follows from the Theorem of Wiener-Levy for Beurling algebras (cf. [7, Chap. I, Sect. 6.5]) that there exists $g \in L^1_a$ such that $\hat{g}(t) = 1/\hat{K}_1(t)$ on supp$(M_\varrho h)^\wedge$ $= \varrho(\text{supp}\hat{h})$. It follows $M_\varrho h = g * K_1 * M_\varrho h$, and therefore

$$[K * M_\varrho h] *_\alpha f = [(K * g * M_\varrho h) * K_1] *_\alpha f \in C^0_b \quad \text{by Step 1.}$$

Step 3. The norm density of $L^1_a \cap C^0_a$ in C^0_a allows to conclude by means of Corollary 1a) that one has $K *_\alpha f \in C^0_b$ for $K \in C^0_a$.

Step 4. Let now $K \in L^\infty_a$ be given. Applying Step 3 we know that $g *_\alpha f \in C^0_b$ whenever $g \in C^0_{a'}$, for any $a' > m$. Choosing now $a' \in (m, a)$ and $h \in \mathscr{K}$, i.e. a continuous function h with compact support and with $\int h(x)dx = 1$ one verifies that $g := M_\varrho h * K \in C^0_a$ and consequently $(M_\varrho h * K) *_\alpha f \in C^0_b$ for all $\varrho \geqq 1$. Another application of Lemma 3b) then leads to the desired conclusion.

5. Corollaries

Corollary 2. *If one replaces (in the assumptions for Theorem 2) the single kernel K_1 by a family of kernels $(K_i)_{i \in I}$ in $L^1_a \cap \mathrm{Lip}_e$, and each of them satisfying* (*), *the conclusion of Theorem 2 persists to be true if the cospectrum of the family, i.e. cosp* $:= \{t \mid \hat{K}_1(t) = 0$ *for all* $i \in I\}$ *is empty.*

Proof. Assume first that one has two such kernels; K_1, K_2. It then suffices to modify the argument near the end of step 2 as follows (cf. also the localization principle discussed in [7, Chap. I, Sect. 3.4]): It is possible to find $g_1, g_2 \in L^1_a$ such that

$$M_\varrho h = (g_1 * K_1 + g_2 * K_2) * M_\varrho h$$

(recall that $\mathscr{S} \subseteqq \{\hat{g} \mid g \in L^1_a\}$ for any $a \geqq 0$ and therefore the required partition of unity is available). In the general case a compactness argument (for fixed $\varrho > 0$) allows to reduce the problem to the case of a finite family which is treated in a similar way.

Corollary 3. *Let* $K_1 \in \mathscr{S}(\mathbb{R}^m)$, $K_1 \neq 0$, *and* $f \in L^\infty_b$ *be given. Assume that for each* $t \in \mathbb{R}^m$ *one has*

$$\lim_{x \to \infty} [\chi_t K_1 *_\alpha f(x)] w_b(x) = 0\,.$$

Then

$$\lim_{x \to \infty} [K *_\alpha f(x)] w_b(x) = 0 \quad \textit{for any} \quad K \in L^\infty_a\,, \qquad a > m\,.$$

Proof. It is clear that $\mathscr{S}(\mathbb{R}^m) \subseteqq L^1_a \cap C^0_a \cap \mathrm{Lip}_e$ for all $a, e > 0$ and that $\{\chi_t K_1, t \in \mathbb{R}^m\}$ has empty cospectrum.

At this point it should be mentioned that the particular choice of the selfneglecting function is actually irrelevant for the validity of the main result. For example, it might have been replaced by $\tilde{w}_\alpha(x) := (1 + |x|^2)^{\alpha/2}$. In fact, one has

Corollary 4. *Given a selfneglecting function $s(x)$ which is equivalent to $w_\alpha(x)$ (i.e. with $s(x) \sim w_\alpha(x)$ for $x \to \infty$) one has for α, a, b as in Theorem 2:*

$$K *_s f \in C_b^0 \quad \textit{for all} \quad K \in C_a^0 \tag{22}$$

if and only if

$$K *_\alpha f \in C_b^0 \quad \textit{for all} \quad K \in C_a^0 . \tag{23}$$

Proof. If (22) holds true it follows from the closed graph theorem that f induces a bounded linear transformation from C_a^0 to C_b^0. Consequently, compact sets in C_a^0 are mapped into compact subsets of C_b^0. For $K \in C_a^0$ the mapping $\varrho \to M_\varrho K$ is continuous from $\mathbb{R}^m$ to C_a^0. Consequently for any $C > 0$ we see that $M_\varrho K *_s f(x)$ tends to zero (for $x \to \infty$) uniformly for $\varrho \in [C^{-1}, C]$. Since for some $C > 0$:

$$\varrho(x) := s(x)/w_\alpha(x) \in [C^{-1}, C] \quad \text{for all} \quad x \in \mathbb{R}^m,$$

it follows that for $|x| \geqq N$ (suitably chosen)

$$\begin{aligned} K *_\alpha f(x) &= \langle L_x M_{w_\alpha(x)} \check{K}, f \rangle = \langle L_x M_{s(x)}^{-1} M_{\varrho(x)} \check{K}, f \rangle \\ &= M_{\varrho(x)} K *_s f(x) < \varepsilon w_b(x), \end{aligned}$$

giving the desired assertion. The converse follows by symmetry.

Remark 6. It is clear from the main result that it suffices to assume (22) for all $K \in \mathscr{S}(\mathbb{R}^m)$ or all $K \in \mathscr{K}(\mathbb{R}^m)$, for example.

As another, more explicit consequence of Corollary 4 let us state the following one-dimensional result:

Corollary 5. *Let $0 \leqq \alpha \leqq 1/2$, $b \in [0, 1)$ and $a > 1$ be given. Let $f \in L_b^\infty(\mathbb{R})$ and $K_1 \in L_a^1 \cap \mathrm{Lip}_e(\mathbb{R})$, for some $e > b + 2$* satisfying $\int_{\mathbb{R}} K_1(x)dx \neq 0$ be given. Then one has*

$$\lim_{x \to +\infty} |x|^{b-\alpha} \int_{\mathbb{R}} f(y) K_1(\varrho |x|^{-\alpha}(x-y)) dy = 0$$

for all $\varrho > 0$ if and only if

$$\lim_{x \to \infty} |x|^{b-\alpha} \int_{\mathbb{R}} f(y) g(|x|^{-\alpha}(x-y)) dy = 0 \quad \textit{for all} \quad g \in L_a^\infty .$$

Proof. It is clear that the assumptions required for an application of Corollary 2 are satisfied in this case (in the case of a step function one readily verifies that (20) is actually verified). Since $\hat{K}_1(t) \neq 0$ near $t = 0$ it is clear that $\{(M_\varrho K_1)^\wedge, \varrho > 0\} = \{D_\varrho \hat{K}_1, \varrho > 0\}$ has empty cospectrum, and the proof of Corollary 2 therefore applies.

6. An Example

We conclude this note by the following fairly explicit result which is of some interest in connection with the central limit theorem as pointed out by Bingham [1]. It is obtained from the above results through the choice $m = 1$, $\alpha = 1/2$:

* (or a step function)

Theorem 3. *For $f \in L_b^\infty$, $b \in [0,1)$ the following conditions are equivalent:*

i) $$\lim_{x\to\infty} |x|^{b-1/2} \int_{-\infty}^{\infty} f(y)\exp(-t(x-y)^2/x)dy = 0$$

for some (respectively all) $t>0$;

ii) $$\lim_{x\to\infty} |x|^{b-1/2} \int_{x}^{x+t\sqrt{x}} f(y)dy = 0$$

for all $t>0$ (respectively for t_1, $t_2>0$ with t_1/t_2 irrational);

iii) $$\lim_{x\to\infty} |x|^{b+1/2} \int_{-\infty}^{\infty} f(y)/x + t(x-y)^2 dy = 0$$

for some (respectively all) $t>0$;

iv) $$\lim_{x\to\infty} |x|^{b-1/2} \int_{-\infty}^{\infty} f(y) g\left(\frac{x-y}{\sqrt{x}}\right) dy = 0$$

for all $g \in \mathcal{S}(\mathbb{R})$ (respectively all $g \in \mathcal{K}(\mathbb{R})$; respectively all $g \in L_a^\infty$ for some $a>1$);

v) $$\lim_{x\to\infty} |x|^{b-1/2} \int_{-\infty}^{\infty} f(y) g\left(\frac{x-y}{t\sqrt{y}}\right) dy = 0$$

some $g \neq 0$ with $\int g(x)dx \neq 0$ and for all $t>0$.

Proof. These results are special cases $m=1$, $\alpha=1/2$ of our main result. In fact, the conditions on b are then automatically satisfied for $b \in [0,1)$. For i) choose $K_1(x) := \exp(-tx^2)$, which belongs to $\mathcal{S}$ and has nonvanishing Fourier transform. For ii) one considers the characteristic functions of intervals $[0,t]$ which clearly fulfills (21). If one considers two such intervals with irrational quotient of their lengths, then their Fourier transforms do not have joint zeros and Corollary 2 applies. For iii) one considers $K_1(x) = (1+tx^2)^{-1}$. It is wellknown that this kernel has nonvanishing Fourier transform. A direct calculation reveals that $K_1 \in \mathrm{Lip}_e$ for $0 \leqq e < 1$ which is sufficient for our purpose here. Assertions iv) and v) are clear in view of Corollaries 2 and 5.

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Stable Trace Formula: Elliptic Singular Terms

Robert E. Kottwitz

Department of Mathematics, University of Washington, Seattle, WA 98195, USA

Let F be a local field of characteristic 0. For any torus T over F Tate-Nakayama duality yields a canonical isomorphism

$$H^1(F,T) \tilde{\rightarrow} \pi_0(\hat{T}^\Gamma)^D,$$

where $\hat{T}$ is the connected Langlands dual group of T, Γ is the absolute Galois group of F, $\pi_0(\ )$ denotes group of connected components, and $(\ )^D$ denotes Pontryagin dual.

If F is p-adic, this isomorphism can be generalized to a canonical bijection

$$H^1(F,G) \tilde{\rightarrow} \pi_0(Z(\hat{G})^\Gamma)^D,$$

where $Z(\hat{G})$ denotes the center of $\hat{G}$. This was done in [K3], but the proof given there does not extend to the case $F=\mathbb{R}$. In fact, in the real case all we get is a canonical map

$$H^1(\mathbb{R},G) \rightarrow \pi_0(Z(\hat{G})^\Gamma)^D.$$

In 1.2 we construct this map and determine its kernel and image. The construction works for p-adic fields as well. In Sect. 1 we establish a number of basic properties of the map.

Now let F be a number field. For any torus T over F Tate-Nakayama duality has the following three consequences. First, there exists a canonical isomorphism

$$\ker^1(F,T) \tilde{\rightarrow} \ker^1(F,\hat{T})^D,$$

where $\ker^i(F,*)$ denotes the kernel of

$$H^i(F,*) \rightarrow \prod_v H^i(F_v,*)$$

(the product is taken over all places v of F). Second, there exists a canonical isomorphism

$$H^1(F,T(\mathbb{A})/T(\bar{F})) \tilde{\rightarrow} \pi_0(\hat{T}^\Gamma)^D,$$

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where $\bar{F}$ is an algebraic closure of F and $\mathbb{A}$ is its adele ring. Third, the composition

$$H^1(F, T(\mathbb{A})) \to H^1(F, T(\mathbb{A})/T(\bar{F})) \to \pi_0(\hat{T}^\Gamma)^D$$

is obtained by combining the local isomorphisms

$$H^1(F_v, T(\bar{F}_v)) \tilde{\to} \pi_0(\hat{T}^{\Gamma(v)})^D$$

[$\Gamma(v)$ denotes $\mathrm{Gal}(\bar{F}_v/F_v)$] with the homomorphisms

$$\pi_0(\hat{T}^{\Gamma(v)})^D \to \pi_0(\hat{T}^\Gamma)^D$$

dual to the inclusions $\hat{T}^\Gamma \subset \hat{T}^{\Gamma(v)}$.

All three of these results can be generalized to connected reductive groups G over F. The first result was generalized in [K3, Sect. 4] to a canonical bijection

$$\ker^1(F, G) \tilde{\to} \ker^1(F, Z(\hat{G}))^D.$$

Because the Hasse principle is not known for E_8, it was possible to prove this only for G having no E_8 factors.

The second result generalizes to a canonical map

$$H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to \pi_0(Z(\hat{G})^\Gamma)^D,$$

where Z_G is the center of G. Its kernel is equal to the image of $H^1(F, G_{\mathrm{ad}})$.

The generalization of the third result is that the composition

$$H^1(F, G(\mathbb{A})) \to H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to \pi_0(Z(\hat{G})^\Gamma)^D$$

is obtained by putting together the local maps

$$H^1(F, G(\bar{F}_v)) \to \pi_0(Z(\hat{G})^{\Gamma(v)})^D$$

(see 2.5 for a precise statement) and that its kernel is the image of $H^1(F, G)$ (see 2.6). In Sect. 2 we prove all this, along with some related facts. In Sect. 10 we review some facts about elliptic and fundamental tori that are needed in Sects. 1–2.

These local and global cohomological results can be applied to harmonic analysis on reductive groups. Consider the local case first. Let G be a connected reductive group over F. For simplicity we assume that the derived group G_{der} is simply connected. Let γ be a semi-simple element of $G(F)$ and write I for the centralizer G_γ of γ. Since G_{der} is simply connected, I is a connected reductive F-group. The conjugacy classes within the stable conjugacy class of γ [i.e., the $G(\bar{F})$-conjugacy class in this case] are in 1–1 correspondence with the elements of

$$\ker[H^1(F, I) \to H^1(F, G)].$$

If γ is regular, then I is a torus and $H^1(F, I)$ is a group. In [L2, S2] it is explained how to use characters on this group to form κ-orbital integrals, which are used to match functions on G with functions on endoscopic groups of G.

If γ is not regular, then I is not a torus and $H^1(F, I)$ is not a group. Nevertheless we have a map

$$H^1(F, I) \to \pi_0(Z(\hat{I})^\Gamma)^D,$$

and $\pi_0(Z(\hat{I})^\Gamma)^D$ *is* a group. We use characters of this group – namely, elements of $\pi_0(Z(\hat{I})^\Gamma)$ – to define κ-orbital integrals. In Sect. 5 we conjecture that these

κ-orbital integrals can be matched with stable orbital integrals of semi-simple elements of endoscopic groups.

There are two ways in which the situation is more complicated than in the regular case. First, it is necessary to take into account the signs $e(I)$ of [K2]. Second, in the matching theorem for an endoscopic group H of G, some semi-simple elements of $H(F)$ will be matched with non-semi-simple elements of $G(F)$. We conjecture that the semi-simple elements of $H(F)$ which are matched with semi-simple elements of $G(F)$ are precisely those we call (G, H)-regular: a semi-simple $\gamma_H \in H(F)$ is (G, H)-*regular* if $\alpha(\gamma_H) \neq 1$ for every root α of G that does not come from H. This definition is explained in detail in Sect. 3. If $\gamma_H \in H(F)$ is (G, H)-regular semi-simple and is matched with the semi-simple element $\gamma \in G(F)$, then the (connected) centralizer I_H of γ_H in H is an inner form of the (connected) centralizer I of γ in G.

Now we come to the application of the global cohomological results. It turns out that all three global results are needed for the stabilization of the part of the trace formula for G indexed by elliptic semi-simple conjugacy classes in $G(F)$. The application of the second global result is the most interesting. It can be best understood by considering the case in which G is a simply connected semi-simple group. Choose an inner twisting $\psi: G_0 \to G$ with G_0 quasi-split. Let γ_0 be a semi-simple element of $G_0(F)$, and let I_0 denote the (connected) centralizer of γ_0 in G_0. Let γ be an element of $G(\mathbb{A})$ that is conjugate to $\psi(\gamma_0)$ under $G(\bar{\mathbb{A}})$. Then in Sect. 6 we construct from γ and γ_0 an element of $H^1(F, I_0(\bar{\mathbb{A}})/Z_{I_0}(\bar{F}))$. The second global result then produces an element

$$\operatorname{obs}(\gamma) \in \pi_0(Z(\hat{I}_0)^\Gamma)^D .$$

Assuming that G has no E_8 factors, we show that $\operatorname{obs}(\gamma)$ is trivial if and only if the $G(\mathbb{A})$-conjugacy class of γ contains an element of $G(F)$.

If we assume only that G_{der} is simply connected, the situation becomes slightly more complicated: $\operatorname{obs}(\gamma) \in \mathfrak{K}(I_0/F)^D$, where $\mathfrak{K}(I_0/F)$ is the subgroup of $\pi_0([Z(\hat{I}_0)/Z(\hat{G})]^\Gamma)$ defined in Sect. 4. The global conjecture on transfer factors in Sect. 6 involves $\operatorname{obs}(\gamma)$. For regular elements γ_0 all this was done by Langlands [L2, Ch. VII] and was reinterpreted in [K3].

The stabilization of the elliptic semi-simple part of the trace formula for G is given in Sect. 9. It is necessary to make a number of assumptions (see 9.3). The manipulation of the terms is done in essentially the same way as in [L2]; however, we cannot use Langlands's convergence argument because of the signs $e(I)$ occurring in the singular κ-orbital integrals, and so we are forced to prove the finiteness results of Sects. 7–8. These are enough to show that all the sums we consider have only a finite number of non-zero terms.

I am indebted to J. Arthur for several very helpful conversations about these finiteness questions, which he has already resolved for the ordinary trace formula [A]. The stable trace formula requires slightly more detailed results, which are unfortunately quite technical.

In 7.5 we establish the hypothesis made in [L2, p. 185]. Although the other finiteness results make this one unnecessary for the stabilization in Sect. 9, it may have other applications.

Much of the notation used in the paper has already appeared in this introduction. Other unexplained notation and terminology is taken from [K3]. Throughout the paper we have a field F (sometimes local, sometimes global) and an algebraic closure $\bar{F}$ of F, and we write Γ for $\mathrm{Gal}(\bar{F}/F)$. We often use $A(G)$ as an abbreviation for $\pi_0(Z(\hat{G})^\Gamma)^D$.

1. Local Cohomological Results

In this section F is a local field of characteristic 0.

1.1. For any torus T over F we have a canonical isomorphism

$$H^1(F,T) \xrightarrow{\sim} \pi_0(\hat{T}^\Gamma)^D, \tag{1.1.1}$$

obtained from the Tate-Nakayama isomorphism

$$H^1(F,T) \xrightarrow{\sim} H^1(F, X^*(T))^D$$

and the isomorphism

$$\pi_0(\hat{T}^\Gamma) \xrightarrow{\sim} H^1(F, X_*(\hat{T}))$$

induced by the connecting homomorphism for the exponential sequence

$$1 \to X_*(\hat{T}) \to \mathrm{Lie}(\hat{T}) \to \hat{T} \to 1\,.$$

For any connected reductive group G over F we write $A(G)$ for the finite abelian group $\pi_0(Z(\hat{G})^\Gamma)^D$. Now consider the functors $G \mapsto H^1(F,G)$ and $G \mapsto A(G)$ from the category of connected reductive F-groups and normal homomorphisms to the category of pointed sets.

1.2. Theorem. *There is a unique extension of* (1.1.1) *to a morphism of functors*

$$\alpha_G : H^1(F,G) \to A(G)\,. \tag{1.2.1}$$

For any maximal torus T of G the diagram

$$\begin{array}{ccc} H^1(F,T) & \longrightarrow & H^1(F,G) \\ \downarrow & & \downarrow \\ A(T) & \longrightarrow & A(G) \end{array} \tag{1.2.2}$$

commutes, where $A(T) \to A(G)$ is induced by $Z(\hat{G}) \hookrightarrow \hat{T}$. If F is p-adic, then (1.2.1) *is an isomorphism of functors. If $F = \mathbb{R}$, then*

$$\ker(\alpha_G) = \mathrm{im}[H^1(\mathbb{R}, G_{sc}) \to H^1(\mathbb{R}, G)]$$

and

$$\mathrm{im}(\alpha_G) = \ker[\pi_0(Z(\hat{G})^\Gamma)^D \to \pi_0(Z(\hat{G}))^D]\,,$$

where $\pi_0(Z(\hat{G})^\Gamma)^D \to \pi_0(Z(\hat{G}))^D$ is induced by the norm homomorphism $Z(\hat{G}) \to Z(\hat{G})^\Gamma$.

We extend the morphism of functors in two stages. At the first stage we extend it to groups whose derived group is simply connected. Consider such a group G.

and let $D=G/G_{\mathrm{der}}$. Since (1.2.1) must be functorial for the normal homomorphism $G\to D$, and since $Z(\hat{G})=\hat{D}$, we are forced to define α_G as the composed map

$$H^1(F,G)\to H^1(F,D)\tilde{\to}A(D)=A(G).$$

It is easy to check that this map is functorial in G.

At the second state we extend (1.2.1) to all groups G. Let $g\in H^1(F,G)$. We claim that there exist a z-extension $H\to G$ and an element $h\in H^1(F,H)$ such that $h\mapsto g$. The first step in finding H, h is to choose a finite Galois extension K/F such that K splits G and the image of g in $H^1(K,G)$ is trivial. Next we choose any z-extension $H\to G$ whose kernel Z is isomorphic to $R_{K/F}(Z_0)$, where Z_0 is a split K-torus [we use $R_{K/F}(\ )$ to denote restriction of scalars]. Consider the commutative diagram

$$\begin{array}{ccccccc}
\ldots\longrightarrow & H^1(F,H) & \longrightarrow & H^1(F,G) & \longrightarrow & H^2(F,Z) \\
 & \downarrow & & \downarrow & & \downarrow \\
\ldots\longrightarrow & H^1(K,H) & \longrightarrow & H^1(K,G) & \longrightarrow & H^2(K,Z).
\end{array}$$

Because Z is induced from Z_0, the restriction map $H^2(F,Z)\to H^2(K,Z)$ is injective, and the diagram shows that g lies in the image of $H^1(F,H)$.

Now it is clear how to define $\alpha_G(g)$. Since (1.2.1) must be functorial for the normal homomorphism $H\to G$, we are forced to define $\alpha_G(g)$ to be the image of $\alpha_H(h)$ under $A(H)\to A(G)$. There is no ambiguity in h, because $H^1(F,Z)$ is trivial, but we do need to show that our definition is independent of the choice of z-extension. Suppose that $H_1\to G$, $H_2\to G$ are two z-extensions of G and that $h_1\mapsto g$, $h_2\mapsto g$ ($h_i\in H^1(F,H_i)$ for $i=1,2$). Let H_3 be the fiber product of H_1, H_2 over G. Then $H_3\to G$ is a z-extension with kernel $Z_3=Z_1\times Z_2$, where $Z_i=\ker(H_i\to G)$ for $i=1,2$. For $i=1,2,3$ let z_i be the image of g under the connecting homomorphism $H^1(F,G)\to H^2(F,Z_i)$. Then $z_3=(z_1,z_2)$, and since z_1,z_2 are trivial, it follows that there exists $h_3\in H^1(F,H_3)$ such that $h_3\mapsto g$. Applying the functoriality in the first stage to the canonical projections $H_3\to H_1$, $H_3\to H_2$, we see that H_1, h_1 and H_2, h_2 yield the same $\alpha_G(g)$ (the uniqueness of h_1,h_2,h_3 implies that $h_3\mapsto h_1$, $h_3\mapsto h_2$).

We still need to show that α_G is functorial in G. Consider a normal homomorphism $G_1\to G_2$. Let $g_1\in H^1(F,G_1)$ and let g_2 be the image of g_1 in $H^1(F,G_2)$. For $i=1,2$ choose z-extensions $H_i\to G_i$ such that g_i belongs to the image of $H^1(F,H_i)$. Let H_3 be the fiber product of H_1 and H_2 over G_2. Then the diagram

$$\begin{array}{ccc}
H_3 & \longrightarrow & H_2 \\
\downarrow & & \downarrow \\
G_1 & \longrightarrow & G_2
\end{array}$$

commutes. Furthermore $H_3\to H_2$ is a normal homomorphism, and $H_3\to G_1$ is a z-extension with kernel $Z_1\times Z_2$, where $Z_i=\ker(H_i\to G_i)$ for $i=1,2$. In the same way as before we see that g_1 belongs to the image of $H^1(F,H_3)$. Applying the functoriality in the first stage to $H_3\to H_2$, we find that $\alpha_{G_1}(g_1)$ maps to $\alpha_{G_2}(g_2)$ under $A(G_1)\to A(G_2)$.

The second statement of the theorem does not follow from the functoriality of α, since $T\to G$ is not a normal homomorphism unless G is a torus. However, if G_{der}

is simply connected, we let $D=G/G_{\mathrm{der}}$, and obtain the desired commutativity from the functoriality of α for $T\to D$ and $G\to D$. Then, in the general case, we use z-extensions $H\to G$ to reduce to the case just treated.

Now we consider the remaining statements of the theorem. First we show that

$$\ker(\alpha_G)=\mathrm{im}[H^1(F,G_{\mathrm{SC}})\to H^1(F,G)].$$

If G_{der} is simply connected, this equality is obvious from the definition of α_G. In the general case the functoriality of α for $G_{\mathrm{SC}}\to G$ shows that $\mathrm{im}[H^1(F,G_{\mathrm{SC}})\to H^1(F,G)]$ is contained in $\ker(\alpha_G)$. Now let $g\in\ker(\alpha_G)$ and choose a z-extension

$$1\to Z\to H\to G\to 1$$

for which there exists $h\in H^1(F,H)$ with $h\mapsto g$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccc} H^1(F,Z) & \longrightarrow & H^1(F,H) & \longrightarrow & H^1(F,G) \\ \downarrow & & \downarrow & & \downarrow \\ A(Z) & \longrightarrow & A(H) & \longrightarrow & A(G). \end{array}$$

Since Z is an induced torus, $H^1(F,Z)$ and $A(Z)$ are trivial. Therefore $\alpha_H(h)=1$, and

$$h\in\mathrm{im}[H^1(F,H_{\mathrm{SC}})\to H^1(F,H)]$$

by the case already treated; since $H_{\mathrm{SC}}=G_{\mathrm{SC}}$, this shows that $g\in\mathrm{im}[H^1(F,G_{\mathrm{SC}})\to H^1(F,G)]$. Thus we have proved the statement about $\ker(\alpha_G)$ in case $F=\mathbb{R}$. If F is p-adic, Kneser's vanishing theorem for $H^1(F,G_{\mathrm{SC}})$ shows that $\ker(\alpha_G)$ is trivial, and a twisting argument (see 1.4) shows that α_G is injective.

Our last task is to prove the statements about $\mathrm{im}(\alpha_G)$ in the real and p-adic cases.

If F is real (respectively p-adic), let T be a fundamental (respectively elliptic) maximal F-torus of G. Then

$$H^1(F,T)\to H^1(F,G)$$

is surjective (see Sect. 10) and α_T is an isomorphism; therefore the second statement of the theorem implies that $\mathrm{im}(\alpha_G)$ is equal to $\mathrm{im}[A(T)\to A(G)]$. Let U denote the inverse image of T in G_{SC}. The exact sequence

$$1\to Z(\hat{G})\to\hat{T}\to\hat{U}\to 1$$

yields [K3, 2.3] an exact sequence

$$\ldots\to X_*(\hat{U})^{\Gamma}\to A(G)^D\to A(T)^D\to A(U)^D\to\ldots.$$

In the p-adic case $X_*(\hat{U})^{\Gamma}$ is trivial, and therefore $A(T)\to A(G)$ is surjective, which in turn implies that α_G is surjective. In the real case we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \ldots\longrightarrow & X_*(\hat{U}) & \longrightarrow & \pi_0(Z(\hat{G})) & \longrightarrow & \pi_0(\hat{T}) & \longrightarrow\ldots \\ & \downarrow & & \downarrow & & \downarrow & \\ \ldots\longrightarrow & X_*(\hat{U})^{\Gamma} & \longrightarrow & \pi_0(Z(\hat{G})^{\Gamma}) & \longrightarrow & \pi_0(\hat{T}^{\Gamma}) & \longrightarrow\ldots, \end{array}$$

in which all three vertical arrows are norm homomorphisms. In proving the commutativity of the left square one uses the following description of the connecting homomorphism

$$X_*(D_3)^\Gamma \to \pi_0(D_1^\Gamma) \tag{1.2.3}$$

of [K3, 2.3].

Let $\mu \in X_*(D_3)^\Gamma = \mathrm{Hom}_\Gamma(X^*(D_3), \mathbb{Z})$. Let μ_1 be the image of μ in $\mathrm{Hom}_\Gamma(X^*(D_3), \mathbb{C})$ and choose

$$\mu_2 \in \mathrm{Hom}_\Gamma(X^*(D_2), \mathbb{C})$$

such that $\mu_2 \mapsto \mu_1$. Let μ_3 be the image of μ_2 in $\mathrm{Hom}_\Gamma(X^*(D_2), \mathbb{C}^\times)$; then μ_3 comes from a (unique) element $\mu_4 \in \mathrm{Hom}_\Gamma(X^*(D_1), \mathbb{C}^\times) = D_1^\Gamma$. Changing μ_2 by an element of $\mathrm{Hom}_\Gamma(X^*(D_1), \mathbb{C})$ changes μ_4 by an element of $(D_1^\Gamma)^0$. The image of μ under (1.2.3) is equal to the image of μ_4 in $\pi_0(D_1^\Gamma)$.

Now we return to our commutative diagram. The map $X_*(\hat{U}) \to \pi_0(Z(\hat{G}))$ is surjective, since $\pi_0(\hat{T}) = \{1\}$. Furthermore the norm map $X_*(\hat{U}) \to X_*(\hat{U})^\Gamma$ is surjective [in other words, $\hat{H}^0(\mathbb{R}, X_*(\hat{U})) = \{0\}$] since U is a fundamental torus in a simply connected semi-simple group, and such a torus is necessarily of the form $T_a \times T_i$ with T_a anisotropic and T_i induced from $\mathbb{C}$ (see Sect. 10). At this point we know that $X_*(\hat{U})^\Gamma \to \pi_0(Z(\hat{G})^\Gamma)$ and the norm map $\pi_0(Z(\hat{G})) \to \pi_0(Z(\hat{G})^\Gamma)$ have the same image in $\pi_0(Z(\hat{G})^\Gamma)$. We also know that an element of $A(G)$ [i.e., a character on $\pi_0(Z(\hat{G})^\Gamma)$] is in the image of α_G if and only if it vanishes on

$$\mathrm{im}[X_*(\hat{U})^\Gamma \to \pi_0(Z(\hat{G})^\Gamma)].$$

The last statement of the theorem is now clear.

1.3. Let a_σ be a 1-cocycle of Γ in $G(\bar{F})$ and let $^*G$ be the inner twist of G by a_σ:

$$^*G(\bar{F}) = G(\bar{F}),$$

$$^*\sigma = \mathrm{Int}(a_\sigma) \circ \sigma.$$

The center of the dual group of $^*G$ is the same as that of G, which allows us to identify $A(^*G)$ with $A(G)$ and to regard the map (1.2.1) for $^*G$ as a map

$$H^1(F, {}^*G) \to A(G).$$

Let b_σ be a 1-cocycle of Γ in $^*G(\bar{F})$. Then $c_\sigma := b_\sigma a_\sigma$ is a 1-cocycle of Γ in $G(\bar{F})$. Denote by a', b', c' the images of $a_\sigma, b_\sigma, c_\sigma$ in $A(G)$.

1.4. Lemma. *The element c' is the product of a' and b'.*

If G_{der} is simply connected, the result is easy to prove, using that $A(G) = H^1(F, D)$, where $D = G/G_{\mathrm{der}} = {}^*G/{}^*G_{\mathrm{der}}$.

Now we do the general case. Choose a finite Galois extension K/F such that K splits G and $a_\sigma, b_\sigma, c_\sigma$ have trivial restriction to $\mathrm{Gal}(\bar{F}/K)$. Choose a z-extension $H \to G$ whose kernel is of the form $R_{K/F}(Z_0)$ for a split K-torus Z_0.

Choose a 1-cocycle A_σ of Γ in $H(\bar{F})$ such that $A_\sigma \mapsto a_\sigma$. Use A_σ to get $^*H$. Choose a 1-cocycle B_σ of Γ in $^*H(\bar{F})$ such that $B_\sigma \mapsto b_\sigma$. Let $C_\sigma = B_\sigma A_\sigma$; then $C_\sigma \mapsto c_\sigma$. The general case now follows from the special case, applied to $A_\sigma, B_\sigma, C_\sigma$.

1.5. Langlands [L1] (see also [B, Sect. 10]) associates to any element of $H^1(F, Z(\hat{G}))$ a character on $G(F)$ [more generally, he associates to any element of $H^1(W_F, Z(\hat{G}))$ a quasicharacter on $G(F)$]. Equivalently, there is a homomorphism

$$G(F) \to H^1(F, Z(\hat{G}))^D.$$

Let $1 \to G \to H \to I \to 1$ be an exact sequence of connected reductive F-groups. We get a diagram

$$\begin{array}{ccccccccccccc} 1 & \longrightarrow & G(F) & \longrightarrow & H(F) & \longrightarrow & I(F) & \longrightarrow & H^1(G) & \longrightarrow & H^1(H) & \longrightarrow & H^1(I) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & C(G) & \longrightarrow & C(H) & \longrightarrow & C(I) & \longrightarrow & A(G) & \longrightarrow & A(H) & \longrightarrow & A(I) \end{array}$$

in which we have used $H^1(G)$ [respectively $C(G)$] as an abbreviation for $H^1(F, G)$ [respectively $H^1(F, Z(\hat{G}))^D$]. The bottom row comes from applying $(\ )^D$ to the long exact sequence [K3, 2.3] attached to

$$1 \to Z(\hat{I}) \to Z(\hat{H}) \to Z(\hat{G}) \to 1.$$

1.6. Lemma. *The diagram above is commutative.*

The only non-obvious point is the commutativity of

$$\begin{array}{ccc} I(F) & \longrightarrow & H^1(F, G) \\ \downarrow & & \downarrow \\ H^1(F, Z(\hat{I}))^D & \longrightarrow & A(G). \end{array}$$

If G, H, I are tori, the bottom row of the commutative diagram can be rewritten as part of the dual of the long exact cohomology sequence for

$$1 \to X^*(I) \to X^*(H) \to X^*(G) \to 1$$

(see [K3, 2.2]), and the vertical maps are then induced by the Tate-Nakayama pairings

$$H^i(F, T) \times H^{2-i}(F, X^*(T)) \to \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$$

(see [K3, 3.1–3.3]). Therefore the square above is commutative or anticommutative, depending on the order in which the cup-product is taken for $i = 1$. In this lemma we normalize the pairing by requiring that the square be commutative.

If the derived groups of G, H, I are simply connected, we write $B = G/G_{\mathrm{der}}$, $C = H/H_{\mathrm{der}}$, $D = I/I_{\mathrm{der}}$, and reduce to the case of tori by considering the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 1. \end{array}$$

In the next step we assume only that G_{der} is simply connected. Choose a z-extension $J \to I$ and let K be the fiber product of H and J over I. We get a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & G \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & 1\,. \end{tikzcd}$$

The derived groups of G, J are simply connected; it follows that the same is true of K. Since $J(F) \to I(F)$ is surjective, the result follows from the previous step, applied to

$$1 \to G \to K \to J \to 1\,.$$

Finally we do the general case. Choose a z-extension $J \to H$, and let K be the inverse image of G under $J \to H$. Then K_{der} is simply connected (it is a normal subgroup of J_{der}), and we have a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & 1\,. \end{tikzcd}$$

We finish by applying the previous step to the top row of the diagram.

1.7. Suppose that G is semi-simple, and let C denote the kernel of $G_{\mathrm{SC}} \to G$. The exact sequence

$$1 \to C \to G_{\mathrm{SC}} \to G \to 1$$

induces a map

$$H^1(F, G) \to H^2(F, C)\,.$$

Duality for finite groups gives us an isomorphism

$$H^2(F, C) \to H^0(F, X^*(C))^D\,.$$

Furthermore $X^*(C)$ is equal to $Z(\hat{G})$. Putting all this together, we get a map

$$H^1(F, G) \to A(G)\,.$$

1.8. Lemma. *This map is equal to* α_G.

The proof uses z-extensions as in the proof of Theorem 1.2 and then proceeds in the same way as the proof of Remark 6.5 in [K3].

2. Global Cohomological Results

In this section F is a number field.

2.1. For any torus T over F we have a canonical isomorphism

$$H^1(F, T(\mathbb{A})/T(\bar{F})) \tilde{\to} \pi_0(\hat{T}^{\Gamma})^D\,, \tag{2.1.1}$$

obtained from the Tate-Nakayama isomorphism

$$H^1(F, T(\mathbb{A})/T(\bar{F})) \tilde{\to} H^1(F, X^*(T))^D$$

and the isomorphism $\pi_0(\hat{T}^\Gamma) \tilde{\to} H^1(F, X_*(\hat{T}))$ mentioned in 1.1.

Let G be a connected reductive group over F. As in Sect. 1 we write $A(G)$ instead of $\pi_0(Z(\hat{G})^\Gamma)^D$. We write Z_G for the center of G. Consider the two functors $G \mapsto H^1(F, G(\mathbb{A})/Z_G(\bar{F}))$ and $G \mapsto A(G)$ from the category of connected reductive F-groups and normal homomorphisms to the category of pointed sets.

2.2. Theorem. *There is a unique extension of* (2.1.1) *to a morphism of functors*

$$\beta_G : H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to A(G). \tag{2.2.1}$$

The kernel of β_G *is equal to the image of*

$$H^1(F, G(\bar{F})/Z_G(\bar{F})) \to H^1(F, G(\mathbb{A})/Z_G(\bar{F})).$$

The proof of existence and uniqueness of (2.2.1) is analogous to the proof of the corresponding part of Theorem 1.2. If G_{der} is simply connected, we write D for G/G_{der} and obtain β_G as the composition

$$H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to H^1(F, D(\mathbb{A})/D(\bar{F})) \tilde{\to} A(D) = A(G).$$

In the general case we need to use z-extensions

$$1 \to Z \to H \to G \to 1.$$

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \to & Z(\mathbb{A}) & \to & H(\mathbb{A}) & \to & G(\mathbb{A}) & \to & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \to & Z(\bar{F}) & \to & Z_H(\bar{F}) & \to & Z_G(\bar{F}) & \to & 1. \end{array}$$

The surjectivity of $H(\mathbb{A}) \to G(\mathbb{A})$ follows from the discussion of unramified z-extensions in the proof of Proposition 7.1. We get from the exact sequence

$$1 \to Z(\mathbb{A})/Z(\bar{F}) \to H(\mathbb{A})/Z_H(\bar{F}) \to G(\mathbb{A})/Z_G(\bar{F}) \to 1$$

an exact cohomology sequence, and because Z is an induced torus, the last part of this sequence reduces to

$$1 \to H^1(F, H(\mathbb{A})/Z_H(\bar{F})) \to H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to H^2(F, Z(\mathbb{A})/Z(\bar{F})).$$

Let $g \in H^1(F, G(\mathbb{A})/Z_G(\bar{F}))$. Just as in the proof of Theorem 1.2 we can choose the z-extension so that there exists a (unique) $h \in H^1(F, H(\mathbb{A})/Z_H(\bar{F}))$ such that $h \mapsto g$. We are forced to define $\beta_G(g)$ to be the image of $\beta_H(h)$ under $A(H) \to A(G)$. The proof that this definition is independent of the choice of z-extension and is functorial in G goes the same way as for Theorem 1.2.

We still need to determine the kernel of β_G. We start with the special case in which G_{der} is simply connected. Let $D = G/G_{\text{der}}$. The kernel of β_G is equal to the kernel of

$$H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to H^1(F, D(\mathbb{A})/D(\bar{F})), \tag{2.2.2}$$

and it is clear that this set contains the image of $H^1(F, G(\bar{F})/Z_G(\bar{F}))$. To prove the opposite inclusion we consider an element g in the kernel of (2.2.2). Since $G(\mathbb{A}) \to D(\mathbb{A})$ and $Z_G(\bar{F}) \to D(\bar{F})$ are surjective, the sequence

$$1 \to G_{SC}(\mathbb{A})/Z_{G_{SC}}(\bar{F}) \to G(\mathbb{A})/Z_G(\bar{F}) \to D(\mathbb{A})/D(\bar{F}) \to 1$$

is exact, and we see that there exists $g_{SC} \in H^1(F, G_{SC}(\mathbb{A})/Z_{G_{SC}}(\bar{F}))$ such that $g_{SC} \mapsto g$. This reduces us to the case in which G is semi-simple and simply connected. We must show that

$$H^1(F, G(\bar{F})/Z_G(\bar{F})) \to H^1(F, G(\mathbb{A})/Z_G(\bar{F}))$$

is surjective in this case.

To simplify the notation we temporarily write Z instead of Z_G. The commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(\bar{F}) & \longrightarrow & G(\mathbb{A}) & \longrightarrow & G(\mathbb{A})/Z(\bar{F}) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & Z(\bar{F}) & \longrightarrow & G(\bar{F}) & \longrightarrow & G(\bar{F})/Z(\bar{F}) & \longrightarrow & 1 \end{array}$$

gives us a commutative diagram with exact rows

$$\begin{array}{ccccc} \ldots \longrightarrow H^1(F, G(\mathbb{A})) & \longrightarrow & H^1(F, G(\mathbb{A})/Z(\bar{F})) & \longrightarrow & H^2(F, Z) \\ \uparrow & & \uparrow & & \| \\ \ldots \longrightarrow \quad H^1(F, G) & \longrightarrow & H^1(F, G_{ad}) & \longrightarrow & H^2(F, Z). \end{array}$$

We claim that $H^1(F, G_{ad}) \to H^2(F, Z)$ is surjective. Let $z \in H^2(F, Z)$. At any finite place of F for which some 2-cocycle representing z has trivial restriction to the inertia subgroup at v, the image of z in $H^2(F_v, Z)$ is trivial. Therefore there exists a finite set V of places of F outside of which z is locally trivial. There exists a maximal F-torus T of G such that T is anisotropic at every finite place in V and fundamental at every real place in V; for this T we have $H^2(F_v, T) = \{0\}$ for all $v \in V$ (see Sect. 10) and also $\ker^2(F, T) = \{0\}$ (so long as V is non-empty, which we may as well assume). Using all this, we see that the image of z in $H^2(F, T)$ is trivial. Therefore $z \in \operatorname{im}[H^1(F, T_{ad}) \to H^2(F, Z)]$, from which it is obvious that z belongs to the image of $H^1(F, G_{ad})$.

Now we continue with the proof that

$$H^1(F, G_{ad}) \to H^1(F, G(\mathbb{A})/Z(\bar{F}))$$

is surjective. Let c_σ be a 1-cocycle of Γ in $G(\mathbb{A})/Z(\bar{F})$. Now look at the last commutative diagram we wrote down. The claim that we just proved shows that there exists a 1-cocycle g_σ of Γ in $G_{ad}(\bar{F})$ such that c_σ, g_σ have the same image in $H^2(F, Z)$. We can use g_σ to twist G and $G(\mathbb{A})$, obtaining $*G$ and $*(G(\mathbb{A}))$. Note that $(*G)(\mathbb{A}) = *(G(\mathbb{A}))$. There exists a 1-cocycle h_σ of Γ in $(*G)(\mathbb{A})$ such that $c_\sigma = \bar{h}_\sigma g_\sigma$, where $\bar{h}_\sigma$ denotes the image of h_σ in $(*G)(\mathbb{A})/Z(\bar{F})$. Since $*G$ is simply connected, the map

$$H^1(F, *G) \to H^1(F, (*G)(\mathbb{A}))$$

is surjective. Therefore c_σ is cohomologous to a 1-cocycle of the form $h'_\sigma g_\sigma$, where h'_σ is a 1-cocycle in $^*G_{\mathrm{ad}}(\bar{F})$. This proves that the class of c_σ lies in the image of $H^1(F, G_{\mathrm{ad}})$.

Now we need to determine $\ker(\beta_G)$ in the general case. Let $g \in H^1(F, G(\bar{\mathbb{A}})/Z_G(\bar{F}))$. As before we choose a z-extension $H \to G$ for which there exists a (unique) $h \in H^1(F, H(\bar{\mathbb{A}})/Z_H(\bar{F}))$ such that $h \mapsto g$. Suppose that $g \in \ker(\beta_G)$. The homomorphism $A(H) \to A(G)$ is injective because Z is an induced torus; therefore $h \in \ker(\beta_H)$. By what we have already proved h belongs to the image of $H^1(F, H_{\mathrm{ad}})$. Therefore g belongs to the image of $H^1(F, G_{\mathrm{ad}})$.

Now suppose that g belongs to the image of $H^1(F, G_{\mathrm{ad}})$. Then h belongs to the image of $H^1(F, H_{\mathrm{ad}}) = H^1(F, G_{\mathrm{ad}})$. By what we have already proved h belongs to $\ker(\beta_H)$. Therefore g belongs to $\ker(\beta_G)$.

2.3. Let a_σ be a 1-cocycle of Γ in $G(\bar{\mathbb{A}})/Z_G(\bar{F})$. Let d_σ be the image of a_σ under $G(\bar{\mathbb{A}})/Z_G(\bar{F}) \to G_{\mathrm{ad}}(\bar{\mathbb{A}})$. As in 1.3 we get a twist $^*(G(\bar{\mathbb{A}}))$ of $G(\bar{\mathbb{A}})$. For each place v of F choose a place w of $\bar{F}$ over v, and let $d_\sigma(v)$ be the 1-cocycle of $\Gamma(v) := \mathrm{Gal}(\bar{F}_w/F_v) \subset \Gamma$ in $G_{\mathrm{ad}}(\bar{F}_w)$ obtained by taking the component of d_σ at w. Let $^*G_v$ be the twist of $G_v := G_{F_v}$ by $d_\sigma(v)$. Then

$$H^1(F, {}^*(G(\bar{\mathbb{A}}))) \tilde{\to} \bigoplus_v H^1(F_v, {}^*G_v),$$

where $\bigoplus_v$ denotes the subset of the direct product consisting of (x_v) such that $x_v = 1$ for all but a finite number of v. The local maps of Sect. 1 give us

$$\bigoplus_v H^1(F_v, {}^*G_v) \to \bigoplus_v A({}^*G_v).$$

The center of the dual group of $^*G_v$ is $Z(\hat{G})$, and thus the obvious inclusion $Z(\hat{G})^{\Gamma(v)} \supset Z(\hat{G})^\Gamma$ induces a homomorphism $A({}^*G_v) \to A(G)$ for each v. The sum of these homomorphisms is a homomorphism

$$\bigoplus_v A({}^*G_v) \to A(G).$$

Putting all this together, we get a map

$$H^1(F, {}^*(G(\bar{\mathbb{A}}))) \to A(G). \tag{2.3.1}$$

Let b_σ be a 1-cocycle of Γ in $^*(G(\bar{\mathbb{A}}))$. Write $\bar{b}_\sigma$ for the image of b_σ in $^*(G(\bar{\mathbb{A}})/Z_G(\bar{F}))$, and let $c_\sigma = \bar{b}_\sigma a_\sigma$; then c_σ is 1-cocycle of Γ in $G(\bar{\mathbb{A}})/Z_G(\bar{F})$. Denote by a', b', c' the images of $a_\sigma, b_\sigma, c_\sigma$ in $A(G)$ [apply (2.2.1) to a_σ, c_σ and (2.3.1) to b_σ].

2.4. Lemma. *The element c' is the product of a' and b'.*

If G is a torus, the result is part of the global Tate-Nakayama theory (see [K3, 3.4.3]). The rest of the proof parallels that of Lemma 1.4.

2.5. Corollary. *The composition*

$$H^1(F, G(\bar{\mathbb{A}})) \to H^1(F, G(\bar{\mathbb{A}})/Z_G(\bar{F})) \to A(G)$$

is equal to the composition of

(i) $H^1(F, G(\mathbb{A})) \xrightarrow{\sim} \bigoplus_v H^1(F_v, G)$,

(ii) $\bigoplus_v H^1(F_v, G) \to \bigoplus_v A(G_v)$,

(iii) $\bigoplus_v A(G_v) \to A(G)$.

To prove this take $a_\sigma = 1$ in previous lemma.

2.6. Proposition. *The kernel of the map $H^1(F, G(\mathbb{A})) \to A(G)$ of Corollary 2.5 is equal to the image of*

$$H^1(F, G) \to H^1(F, G(\mathbb{A})).$$

We temporarily write Z instead of Z_G. We have a commutative diagram with exact rows

$$\begin{tikzcd} \dots \arrow[r] & H^1(F,Z) \arrow[r] & H^1(F,G(\mathbb{A})) \arrow[r] & H^1(F,G(\mathbb{A})/Z(\bar{F})) \arrow[r] & H^2(F,Z) \\ \dots \arrow[r] & H^1(F,Z) \arrow[r] \arrow[u, equal] & H^1(F,G) \arrow[r] \arrow[u] & H^1(F,G_{\mathrm{ad}}) \arrow[r] \arrow[u] & H^2(F,Z), \arrow[u, equal] \end{tikzcd}$$

which we used in the proof of Theorem 2.2. Let $x \in H^1(F, G(\mathbb{A}))$ and let y denote the image of x in $H^1(F, G(\mathbb{A})/Z(\bar{F}))$. In view of the statement about $\ker(\beta_G)$ given in Theorem 2.2, what we must show is that x belongs to the image of $H^1(F, G)$ if and only if y belongs to the image of $H^1(F, G_{\mathrm{ad}})$. This is obvious from the diagram.

2.7. Let a_σ be a 1-cocycle of Γ in $G(\mathbb{A})/Z_G(\bar{F})$ and let b_σ be a 1-cocycle of Γ in $G(\bar{F})/Z_G(\bar{F})$. Then $c_\sigma = a_\sigma b_\sigma^{-1}$ is a 1-cocycle of Γ in ${}^*G(\mathbb{A})/Z({}^*G)(\bar{F})$, where ${}^*G$ is the inner twist of G by b_σ:

$${}^*G(\bar{F}) = G(\bar{F}),$$

$${}^*\sigma = \mathrm{Int}(b_\sigma) \circ \sigma.$$

We can identify $A({}^*G)$ with $A(G)$. Denote by a', c' the images of a_σ, c_σ in $A(G)$.

2.8. Lemma. *The elements a', c' are equal.*

The proof follows the usual pattern. If G_{der} is simply connected, we use $D = G/G_{\mathrm{der}}$. Then we use z-extensions to reduce the general case to the case just treated.

3. (G, H)-Regular Elements in an Endoscopic Group H for G

In this section F is a local or global field of characteristic 0, G is a connected reductive group over F, and (H, s, η) is an endoscopic triple for G [K3, Sect. 7].

3.1. For the moment we work over $\bar{F}$. Let γ_H be a semi-simple element of H. Choose a maximal torus T_H of H containing γ_H. There is a canonical G-conjugacy class of embeddings $j: T_H \to G$; choose one and let $\gamma = j(\gamma_H)$. The conjugacy class of γ is independent of the choice of T_H and j; thus $\gamma_H \mapsto \gamma$ induces a Γ-equivariant map from the set of semi-simple conjugacy classes in H to the set of semi-simple conjugacy classes in G.

Let $T=j(T_H)$ and use j to identify T_H, T. Let R (respectively R_H) denote the set of roots of T in G (respectively H). We have $R_H \subset R \subset X^*(T)$. We say that γ_H is (G,H)-regular if $\alpha(\gamma_H) \neq 1$ for every root α of G that does not come from a root of H. The (G,H)-regularity of γ_H depends only on γ_H, not on the choice of T_H and j.

Let I (respectively I_H) denote the identity component of the centralizer of γ in G (respectively, γ_H in H). The set $R(\gamma)$ of roots of T in I is equal to $\{\alpha \in R|\alpha(\gamma)=1\}$. The set $R_H(\gamma)$ of roots of T in I_H is equal to $\{\alpha \in R_H|\alpha(\gamma)=1\}$. Therefore $R_H(\gamma) \subset R(\gamma)$ and the two sets are equal if and only if γ_H is (G,H)-regular.

Now assume that γ_H is (G,H)-regular. Then $R_H(\gamma)=R(\gamma)$ and the same is true for the coroots of T in I_H, I. The theory of root data for reductive groups shows that $j: T_H \tilde{\to} T$ extends to an isomorphism $j_1: I_H \to I$, unique up to inner automorphisms coming from T. If $\gamma_H \in H(F)$ and $\gamma \in G(F)$, then I_H, I are defined over F and j_1 is an inner twisting. In particular we have $Z(\hat{I}_H)=Z(\hat{I})$.

3.2. Lemma. *Let γ_H, γ be as above and assume that γ_H is (G,H)-regular. If the centralizer of γ in G is connected, then so is the centralizer of γ_H in H.*

Let h belong to the centralizer of γ_H in H. Let T_H, j, T, I, I_H be as above. We want to show that $h \in I_H$. Using the conjugacy of maximal tori in I_H, we may modify h by an element of I_H and assume that h normalizes T_H. Let ω be the corresponding element of the Weyl group $\Omega(T_H, H)$. The isomorphism j allows us to regard $\Omega(T_H, H)$ as a subgroup of $\Omega(T, G)$. Since G_γ is connected and ω fixes γ, we must have $\omega \in \Omega(T, I) \subset \Omega(T, G)$. We also have $\Omega(T, I)=\Omega(T_H, I_H)$, because γ is (G,H)-regular. Therefore $\omega \in \Omega(T_H, I_H)$, which implies that $h \in I_H$.

4. Definitions of $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{K}$

In this section F is a local or global field of characteristic 0 and G is a connected reductive group over F. Let γ be a semi-simple element of $G(F)$, and let I denote the identity component of G_γ. As before we write $A(G)$ instead of $\pi_0(Z(\hat{G})^\Gamma)^D$.

4.1. We define $\mathfrak{D}(I/F)$ to be $\ker[H^1(F,I) \to H^1(F,G)]$. If G_γ is connected, then $\mathfrak{D}(I/F)$ is in 1–1 correspondence with the conjugacy classes in the stable conjugacy class of γ. The 1–1 correspondence arises as follows. Let $\gamma' \in G(F)$ be a stable conjugate of γ. Then $\{g \in G(\bar{F})|g\gamma g^{-1}=\gamma'\}$ is an F-torsor under I.

4.2. There is a canonical Γ-equivariant injection of $Z(\hat{G})$ into $Z(\hat{I})$. To construct it we choose a maximal F-torus T of I. Then T is also a maximal F-torus of G. We can regard $Z(\hat{G})$, $Z(\hat{I})$ as subgroups of $\hat{T}$. It is easy to see that $Z(\hat{G}) \subset Z(\hat{I})$ and that this injection does not depend on the choice of T.

4.3. Lemma. *Assume that F is local. Then the diagram*

$$\begin{array}{ccc} H^1(F,I) & \longrightarrow & H^1(F,G) \\ \downarrow & & \downarrow \\ A(I) & \longrightarrow & A(G) \end{array}$$

commutes, where the bottom arrow is induced by the injection $Z(\hat{G}) \to Z(\hat{I})$ of 4.2.

Since $I \to G$ is not a normal homomorphism, we cannot apply the functoriality of $H^1(F, G) \to A(G)$ directly. If F is complex, the result is trivial. If F is p-adic (respectively real), we choose an elliptic (respectively fundamental) maximal F-torus T of I. Now consider the diagram

$$\begin{array}{ccccc} H^1(F,T) & \longrightarrow & H^1(F,I) & \longrightarrow & H^1(F,G) \\ \downarrow & & \downarrow & & \downarrow \\ A(T) & \longrightarrow & A(I) & \longrightarrow & A(G). \end{array}$$

The lemma follows from the surjectivity of $H^1(F,T) \to H^1(F,I)$ (see Sect. 10) and the second statement of Theorem 1.2.

4.4. Assume that F is local. We define $\mathfrak{E}(I/F)$ to be the finite abelian group $\ker[A(I) \to A(G)]$. Lemma 4.3 gives us a canonical map $\mathfrak{D}(I/F) \to \mathfrak{E}(I/F)$, which is bijective in the p-adic case.

4.5. Assume that F is global. We define $\mathfrak{E}(I/\mathbb{A})$ to be $\bigoplus_v \mathfrak{E}(I/F_v)$, where v runs through the set of places of F.

4.6. The exact sequence

$$1 \to Z(\hat{G}) \to Z(\hat{I}) \to Z(\hat{I})/Z(\hat{G}) \to 1$$

gives us a homomorphism [K3, Corollary 2.3]

$$\pi_0([Z(\hat{I})/Z(\hat{G})]^\Gamma) \to H^1(F, Z(\hat{G})).$$

We define $\mathfrak{K}(I/F)$ to be the subgroup of $\pi_0([Z(\hat{I})/Z(\hat{G})]^\Gamma)$ consisting of all elements whose image in $H^1(F, Z(\hat{G}))$ is

(a) trivial if F is local,

(b) locally trivial if F is global.

If F is local, then

$$\mathfrak{K}(I/F) = \operatorname{cok}[A(G)^D \to A(I)^D] = \mathfrak{E}(I/F)^D.$$

5. Local Conjectures (Orbital Integrals)

In this section F is a local field of characteristic 0 and G is a connected reductive group over F. To keep the discussion as simple as possible we assume that G_{der} is simply connected, so that G_γ is connected for all semi-simple $\gamma \in G$, and stable conjugacy and $G(\bar{F})$-conjugacy are the same [K1]. The general case is no harder, but leads to more awkward statements.

5.1. We need to recall that there is a sign $e(I) = \pm 1$ attached to any connected reductive group I over F [K2].

5.2. Let γ be a semi-simple element of $G(F)$ and let $I = G_\gamma = G_\gamma^0$. Choose Haar measures dg, di on $G(F), I(F)$ respectively, and let O_γ denote the linear form on

$C_c^\infty(G(F))$ given by

$$O_\gamma(f) = \int_{I(F)\backslash G(F)} f(g^{-1}\gamma g)\frac{dg}{di}.$$

For any stable conjugate $\gamma' \in G(F)$ of γ the group $I' = G_{\gamma'}$ is an inner twist of I and in the usual way di gives us a Haar measure di' on $I'(F)$. We use dg, di' to form $O_{\gamma'}$, and we define a linear form SO_γ on $C_c^\infty(G(F))$ by the formula

$$SO_\gamma = \sum_{\gamma'} e(I')O_{\gamma'},$$

where γ' runs over a set of representatives for the conjugacy classes in the stable conjugacy class of γ. For G such that G_{der} is not simply connected it is necessary to modify the definition of SO_γ by including the factor $|\ker[H^1(F, I') \to H^1(F, G_{\gamma'})]|$ in the summand indexed by γ' [as usual, $I' = (G_{\gamma'})^0$].

5.3. Conjecture. *The distribution SO_γ is stable (see* [L2] *for the definition of stable distribution).*

5.4. Let (H, s, η) be an endoscopic triple for G. Choose an extension of $\eta: \hat{H} \to \hat{G}$ to an L-homomorphism $\eta': {}^LH \to {}^LG$. One expects to have a correspondence (f, f^H) [L2, S2] between functions $f \in C_c^\infty(G(F))$, $f^H \in C_c^\infty(H(F))$, such that

$$SO_{\gamma_H}(f^H) = \sum_\gamma \Delta(\gamma_H, \gamma)O_\gamma(f)$$

for every G-regular semi-simple element $\gamma_H \in H(F)$. The sum runs over a set of representatives for the conjugacy classes in $G(F)$ belonging to the $G(\bar{F})$-conjugacy class associated to γ_H (3.1); if this $G(\bar{F})$-conjugacy class contains no element of $G(F)$, than the sum is empty and the right side of the equation is 0. The complex numbers $\Delta(\gamma_H, \gamma)$ are called transfer factors; at the moment these have a complete definition only in the archimedean case [S2]. Presumably the final definition, whatever it turns out to be, will only specify the function $\Delta(\cdot,\cdot)$ up to multiplication by a non-zero scalar. Of course changing $\Delta(\cdot,\cdot)$ by a scalar causes the correspondence (f, f^H) to change by that scalar. The correspondence and the transfer factors depend on η'.

5.5. Conjecture. The function $\Delta(\cdot,\cdot)$ can be extended (continuously, in all likelihood) to all pairs (γ_H, γ) consisting of a (G, H)-regular semi-simple element $\gamma_H \in H(F)$ and a corresponding element $\gamma \in G(F)$, in such a way that

$$SO_{\gamma_H}(f^H) = \sum_\gamma \Delta(\gamma_H, \gamma)e(G_\gamma)O_\gamma(f).$$

We are again using compatible measures on I_{γ_H}, G_γ as in 5.2 [the two groups are inner twists of each other since γ_H is (G, H)-regular].

5.6. Note that $e(T) = 1$ for any torus T, so that 5.5 is compatible with 5.4 in case γ_H is G-regular.

The relation between $\Delta(\gamma_H, \gamma)$, $\Delta(\gamma_H, \gamma')$ for $\gamma' \in G(F)$ in the stable conjugacy class of γ should be

$$\Delta(\gamma_H, \gamma') = \Delta(\gamma_H, \gamma)\langle \mathrm{inv}(\gamma, \gamma'), \kappa\rangle.$$

Here $\mathrm{inv}(\gamma, \gamma')$ is the element of $\mathfrak{E}(I/F)$ obtained as the image under $\mathfrak{D}(I/F) \to \mathfrak{E}(I/F)$ of the element of $\mathfrak{D}(I/F)$ that measures the difference between γ, γ' (see 4.1). The element $\kappa \in \mathfrak{K}(I/F)$ comes from s *via*

$$Z(\hat{H}) \hookrightarrow Z(\hat{I}_H) \xrightarrow{\sim} Z(\hat{I}),$$

and $\langle \cdot , \cdot \rangle$ denotes the duality pairing between $\mathfrak{E}(I/F)$ and $\mathfrak{K}(I/F)$ (see 4.6). Thus we could also write the equation in 5.5 as

$$SO_{\gamma_H}(f^H) = \Delta(\gamma_H, \gamma) O_\gamma^\kappa(f),$$

where

$$O_\gamma^\kappa(f) = \sum_{\gamma'} \langle \mathrm{inv}(\gamma, \gamma'), \kappa \rangle e(G_{\gamma'}) O_{\gamma'}(f).$$

6. Global Conjecture

In this section F is a number field and G is a connected reductive group over F. We assume that G_{der} is simply connected. Consider an inner twisting $\psi: G_0 \to G$ with G_0 quasi-split. Let γ_0 be a semi-simple element of $G_0(F)$, and let I_0 denote the centralizer of γ_0 in G_0.

6.1. Let γ be an element of $G(\mathbb{A})$ that is conjugate to $\psi(\gamma_0)$ under $G(\bar{\mathbb{A}})$. Our aim is to construct an obstruction $\mathrm{obs}(\gamma) \in \mathfrak{K}(I_0/F)^D$ to the existence of an element of $G(F)$ in the $G(\mathbb{A})$-conjugacy class of γ. We must assume that G_{SC} has no E_8 factors, since the construction uses the Hasse principle for G_{SC}. For regular semi-simple γ_0 such an obstruction was obtained by Langlands in [L2, Chap. VII]. The method used here, however, is based on [K3, Sect. 9].

6.2. The first step is to get an obstruction $\mathrm{obs}_1(\gamma)$ to the existence of an element of $G(F)$ in the $G_{\mathrm{SC}}(\mathbb{A})$-conjugacy class of γ. This obstruction lies in $A(I_1)$, where I_1 is the centralizer of γ_0 in $(G_0)_{\mathrm{SC}}$ [with $A(\cdot)$ as in Sect. 2]. Let X_0 denote the set of pairs (i, g) satisfying

(a) $g \in G_{\mathrm{SC}}(\mathbb{A})$,
(b) $i: I_0 \to G$ is conjugate to $\psi|_{I_0}$ under $G(\bar{F})$,
(c) $i(\gamma_0) = g\gamma g^{-1}$.

It is not hard to see that X_0 is non-empty. We take i to be $\psi|_{I_0}$ and look for $g \in G_{\mathrm{SC}}(\bar{\mathbb{A}})$ such that $\psi(\gamma_0) = g\gamma g^{-1}$. By hypothesis such a g exists in $G(\bar{\mathbb{A}})$. Now we just need to note that

$$(G_0)_{\mathrm{SC}}(\bar{\mathbb{A}}) \cdot I_0(\bar{\mathbb{A}}) = G_0(\bar{\mathbb{A}}),$$

the point being that

$$(G_0)_{\mathrm{SC}}(\mathfrak{o}_{E_w}) \cdot I_0(\mathfrak{o}_{E_w}) = G_0(\mathfrak{o}_{E_w})$$

for any finite extension E of F that splits G_0, I_0 and any finite place w of E for which $I_0 \to G_0$ is defined over $\mathfrak{o}_{E_w}$ (see [K4, 3.3.4]).

The three groups Γ, $G_{\mathrm{SC}}(\bar{F})$, $I_1(\bar{\mathbb{A}})$ all act on X_0. Let $(i, g) \in X_0$, $\sigma \in \Gamma$, $h \in G_{\mathrm{SC}}(\bar{F})$, $t \in I_1(\bar{\mathbb{A}})$. The actions are given by:

(i) $\sigma(i,g)=(\sigma(i),\sigma(g))$,
(ii) $h\cdot(i,g)=(\mathrm{Int}(h)\circ i, hg)$,
(iii) $(i,g)\cdot t=(i, i(t^{-1})g)$.

The actions of $G_{SC}(\bar{F})$ and $I_1(\mathbb{A})$ commute, and for $x\in X_0$, $h\in G_{SC}(\bar{F})$, $t\in I_1(\mathbb{A})$ we have $\sigma(h\cdot x)=\sigma(h)\cdot\sigma(x)$ and $\sigma(x\cdot t)=\sigma(x)\cdot\sigma(t)$.

Let X denote the quotient of X_0 by $G_{SC}(\bar{F})$. It is not hard to check that X is an F-torsor under $I_1(\mathbb{A})/Z(I_1)(\bar{F})$. Assuming that G_{SC} has no E_8 factors, so that it satisfies the Hasse principle, one sees easily that γ is $G_{SC}(\mathbb{A})$-conjugate to an element of $G(F)$ if and only if $[X/I_1(\bar{F})]^\Gamma$ is non-empty. Let c be the element of $H^1(F, I_1(\mathbb{A})/Z(I_1)(\bar{F}))$ determined by X. It is immediate that $[X/I_1(\bar{F})]^\Gamma$ is non-empty if and only if c lies in the image of $H^1(F, I_1(\bar{F})/Z(I_1)(\bar{F}))$. Define $\mathrm{obs}_1(\gamma)$ to be the image of c under β_{I_1} (see 2.2). Theorem 2.2 gives us the following result.

6.3. Lemma. *Assume that G_{SC} has no E_8 factors. Then γ is $G_{SC}(\mathbb{A})$-conjugate to an element of $G(F)$ if and only if* $\mathrm{obs}_1(\gamma)$ *is trivial.*

6.4. Let $\gamma'\in G(\mathbb{A})$ and suppose that γ' is $G(\bar{\mathbb{A}})$-conjugate to γ. Let $Y=\{h\in G_{SC}(\bar{\mathbb{A}})\,|\,h\gamma h^{-1}=\gamma'\}$. As in 6.2 we see that Y is non-empty; therefore Y is an F-torsor under $G_{SC}(\bar{\mathbb{A}})_\gamma$. Let X (respectively X') denote the F-torsor under $I_1(\mathbb{A})/Z(I_1)(\bar{F})$ obtained from γ (respectively γ'). By considering the map $Y\times X\to X'$ defined by $(h,(i,g))\mapsto(i, gh^{-1})$, we see that we are in the situation of 2.3, 2.4. Therefore $\mathrm{obs}_1(\gamma')=\mathrm{obs}_1(\gamma)\cdot\mathrm{inv}_1(\gamma,\gamma')$, where $\mathrm{inv}_1(\gamma,\gamma')$ denotes the image under

$$H^1(F, G_{SC}(\bar{\mathbb{A}})_\gamma)\to A(I_1) \tag{6.4.1}$$

of the class of Y [(6.4.1) comes from (2.3.1)].

6.5. Now we get $\mathrm{obs}(\gamma)$ from $\mathrm{obs}_1(\gamma)$. Let $D=G/G_{\mathrm{der}}$. Then we have exact sequences

$$1\to G_{SC}\to G\to D\to 1\,,$$

$$1\to (G_0)_{SC}\to G_0\to D\to 1\,,$$

$$1\to I_1\to I_0\to D\to 1\,.$$

In particular $Z(\hat{I}_1)$ is equal to $Z(\hat{I}_0)/Z(\hat{G})$, which implies that $\mathfrak{K}(I_0/F)$ can be regarded as a subgroup of $\pi_0(Z(\hat{I}_1)^\Gamma)$. By duality we get a homomorphism

$$A(I_1)\to\mathfrak{K}(I_0/F)^D\,, \tag{6.5.1}$$

and we define $\mathrm{obs}(\gamma)$ to be the image of $\mathrm{obs}_1(\gamma)$ under (6.5.1).

6.6. Theorem. *Assume that G_{SC} has no E_8 factors. Then γ is $G(\mathbb{A})$-conjugate to an element of $G(F)$ if and only if* $\mathrm{obs}(\gamma)$ *is trivial.*

For each place v of F we choose a place w of $\bar{F}$ lying over F and let $\Gamma(v)=\mathrm{Gal}(\bar{F}_w/F_v)\subset\Gamma$. Consider the obvious maps

$$\lambda:\bigoplus_v \pi_0(Z(\hat{I}_1)^{\Gamma(v)})^D\to\pi_0(Z(\hat{I}_1)^\Gamma)^D$$

$$\mu:\bigoplus_v \pi_0(Z(\hat{I}_1)^{\Gamma(v)})^D\to\bigoplus_v \pi_0(Z(\hat{I}_0)^{\Gamma(v)})^D$$

as well as their duals λ^D,μ^D. On the one hand, by duality and the definition of $\mathfrak{K}(I_0/F)$, the obstruction $\mathrm{obs}(\gamma)$ is trivial if and only if $\mathrm{obs}_1(\gamma)$ belongs to the subgroup

$$\lambda(\ker(\mu)) \tag{6.6.1}$$

of $A(I_1)$. On the other hand, by 6.3 and 6.4 the element γ is $G(\mathbb{A})$-conjugate to an element of $G(F)$ if and only if $\mathrm{obs}_1(\gamma)^{-1}$ belongs to the image under (6.4.1) of the subset S of $H^1(F, G_{\mathrm{SC}}(\bar{\mathbb{A}})_\gamma)$ obtained by intersecting the kernels of

$$H^1(F, G_{\mathrm{SC}}(\bar{\mathbb{A}})_\gamma) \to H^1(F, G(\bar{\mathbb{A}})_\gamma), \tag{6.6.2}$$

$$H^1(F, G_{\mathrm{SC}}(\bar{\mathbb{A}})_\gamma) \to H^1(F, G_{\mathrm{SC}}(\bar{\mathbb{A}})). \tag{6.6.3}$$

To finish the proof it is enough to show that the image S' of S under (6.4.1) is equal to the subgroup (6.6.1). It is obvious that S' is contained in the subgroup (6.6.1). We now verify the opposite inclusion. To simplify notation we assume that $F=\mathbb{Q}$, so that there is only one infinite place, denoted ∞. The general case is no harder. Since $H^1(F, G_{\mathrm{SC}}(\bar{\mathbb{A}})) \tilde{\to} H^1(\mathbb{R}, G_{\mathrm{SC}}(\mathbb{R}))$ by Kneser's vanishing theorem, the set S contains the set

$$T = \bigoplus_p \ker[H^1(\mathbb{Q}_p, G_{\mathrm{SC}}(\bar{\mathbb{Q}}_p)_{\gamma_p}) \to H^1(\mathbb{Q}_p, G(\bar{\mathbb{Q}}_p)_{\gamma_p})]$$

of $H^1(\mathbb{Q}, G_{\mathrm{SC}}(\bar{\mathbb{A}})_\gamma)$, where p runs over the set of finite places of $\mathbb{Q}$ and γ_p is the component of γ at p. Let $\mu(v)$ denote the homomorphism

$$\pi_0(Z(\hat{I}_1)^{\Gamma(v)})^D \to \pi_0(Z(\hat{I}_0)^{\Gamma(v)})^D.$$

Using the bijectivity of (1.2.1) in the p-adic case, we see that S' contains

$$T' = \lambda\Big(\bigoplus_p \ker \mu(p)\Big).$$

It is clear that μ is equal to $\bigoplus_v \mu(v)$, where the sum is over all places v of $\mathbb{Q}$.

To finish the proof it is enough to show that T' is equal to the subgroup (6.6.1), which is equivalent to showing that the natural map

$$\ker \mu \cap \ker \lambda \to \ker \mu(\infty)$$

is surjective. This in turn follows from the surjectivity of

$$\ker[H^1(\mathbb{Q}, I_1) \to H^1(\mathbb{Q}, I_0)] \to \ker \mu(\infty)$$

(use Proposition 2.6); it remains to verify this surjectivity.

Recall that D denotes G/G_{der}. From 1.6, 1.7 and the exact sequence

$$1 \to I_1 \to I_0 \to D \to 1$$

we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & D(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q}, I_1) & \longrightarrow & H^1(\mathbb{Q}, I_0) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & D(\mathbb{R}) & \longrightarrow & H^1(\mathbb{R}, I_1) & \longrightarrow & H^1(\mathbb{R}, I_0) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & C(D) & \longrightarrow & A(I_{1,\infty}) & \longrightarrow & A(I_{0,\infty}) & \longrightarrow \cdots,
\end{array}$$

where $C(D)=H^1(F,\hat{D})^D=D(\mathbb{R})/N_{\mathbb{C}/\mathbb{R}}D(\mathbb{C})$. Since $D(\mathbb{Q})$ is dense in $D(\mathbb{R})$, it maps onto $D(\mathbb{R})/N_{\mathbb{C}/\mathbb{R}}D(\mathbb{C})$. Therefore

$$\ker[H^1(\mathbb{Q},I_1)\to H^1(\mathbb{Q},I_0)]$$

maps onto

$$\ker[A(I_{1,\infty})\to A(I_{0,\infty})]=\ker\mu(\infty).$$

This finishes the proof.

6.7. Let γ', $\operatorname{inv}_1(\gamma,\gamma')$ be as in 6.4. Let $\operatorname{inv}(\gamma,\gamma')$ be the image of $\operatorname{inv}_1(\gamma,\gamma')$ under the homomorphism (6.5.1)

$$A(I_1)\to\mathfrak{K}(I_0/F)^D.$$

Then $\operatorname{obs}(\gamma')=\operatorname{obs}(\gamma)\cdot\operatorname{inv}(\gamma,\gamma')$.

6.8. Let $\gamma_0'\in G_0(F)$ and suppose that γ_0' is stably conjugate to γ_0. Using γ_0' rather than γ_0, we get

$$\operatorname{obs}(\gamma)'\in\mathfrak{K}(I_0'/F)^D,$$

where I_0' is the centralizer of γ_0' in G_0. There is an inner twist $I_0'\to I_0$, canonical up to conjugation by an element of $I_0(\bar{F})$. This allows us to identify $\mathfrak{K}(I_0'/F)^D$ with $\mathfrak{K}(I_0/F)^D$; with this identification we have $\operatorname{obs}(\gamma)'=\operatorname{obs}(\gamma)$. This follows from Lemma 2.8.

6.9. Now we are ready to state a global conjecture, which generalizes the global hypothesis in [L2, Chap. VII, Sect. 7]. Let (H,s,η) be an endoscopic triple for G, and choose an extension of $\eta:\hat{H}\to\hat{G}$ to an L-homomorphism $\eta':{}^LH\to{}^LG$. We assume that the local Conjecture 5.5 holds at every place of F. We write $\Delta_v(\cdot,\cdot)$ for the transfer factors at the place v. Recall that $\Delta_v(\cdot,\cdot)$ can be replaced by $c\cdot\Delta_v(\cdot,\cdot)$ for any $c\in\mathbb{C}^\times$.

6.10. Conjecture. For a suitable normalization of the local transfer factors the following statements hold.

(a) For any (G,H)-regular semi-simple $\gamma_H\in H(F)$ and any $\gamma\in G(\mathbb{A})$ coming from γ_H the expression $\prod_v\Delta_v(\gamma_H,\gamma)$ has only a finite number of terms $\neq 1$ and hence has a well-defined product, which we denote by $\Delta(\gamma_H,\gamma)$.

(b) Let γ_H,γ be as in (a). Choose an inner twisting $\psi:G_0\to G$ and $\gamma_0\in G_0(F)$ such that γ_0 comes from γ_H (γ_0 exists by [K1, Theorem 4.4]). Let I_0 denote the centralizer of γ_0 in G_0, and let $\operatorname{obs}(\gamma)\in\mathfrak{K}(I_0/F)^D$ be the obstruction of 6.5. Then

$$\Delta(\gamma_H,\gamma)=\langle\operatorname{obs}(\gamma),\kappa\rangle, \tag{6.10.1}$$

where $\kappa\in\mathfrak{K}(I_0/F)$ is obtained from s via

$$Z(\hat{H})\hookrightarrow Z(\hat{I}_H)\xrightarrow{\sim}Z(\hat{I}_0)$$

[as usual, $I_H=H_{\gamma_H}=(H_{\gamma_H})^0$].

6.11. Remark. We see from 6.8 that the right side of (6.10.1) is independent of the choice of γ_0. We see from 6.7 that the left and right sides are multiplied by the same factor if γ is replaced by a $G(\mathbb{A})$-conjugate γ'.

7. Local Finiteness Results

In this section F is a p-adic field, $\mathfrak{o}$ is the valuation ring of F, k is the residue field of $\mathfrak{o}$, $\bar{k}$ is an algebraic closure of k, and G is an unramified connected reductive group over F. Let x_0 be a hyperspecial point in the building of G, and let $\mathbf{G}$ be the corresponding extension of G to a group scheme over $\mathfrak{o}$ [T]. We write K for the hyperspecial maximal compact subgroup $\mathbf{G}(\mathfrak{o}) = \mathrm{Stab}_{G(F)}(x_0)$ of $G(F)$.

7.1. Proposition. *Let γ be a semi-simple element of K such that $1-\alpha(\gamma) \in \mathfrak{o}_{\bar{F}}$ is either 0 or a unit for every root α of G, and let $I = G_\gamma^0$. Then I is unramified and $I(F) \cap K$ is a hyperspecial maximal compact subgroup of $I(F)$. Furthermore,*

$$\ker[H^1(F, I) \to H^1(F, G_\gamma)]$$

is trivial. Finally, if $\gamma' \in K$ is stably conjugate to γ, then γ' is conjugate to γ under K.

During the first part of the proof we assume that G_{der} is simply connected, so that $I = G_\gamma$. For the moment we consider the following special case. We assume that G is split over F; that $\gamma \in \mathbf{A}(\mathfrak{o})$, where $\mathbf{A}$ is a split maximal $\mathfrak{o}$-torus of $\mathbf{G}$; and that γ' is conjugate to γ under $G(F)$.

Let $\mathbf{G}_\gamma$ be the closed subgroup scheme of $\mathbf{G}$ whose points in any $\mathfrak{o}$-algebra R are given by

$$\mathbf{G}_\gamma(R) = \{g \in \mathbf{G}(R) | g\gamma g^{-1} = \gamma\} .$$

The image of γ in $\mathbf{G}(k)$ is semi-simple (it belongs to $\mathbf{A}(k)$), and the derived group of $\mathbf{G}_k$ is simply connected; thus we see that the special fiber of $\mathbf{G}_\gamma$ is a connected reductive group. The hypothesis about the values at γ of the roots α of G implies that special and general fibers of $\mathbf{G}_\gamma$ have the same dimension. Therefore $\mathbf{G}_\gamma$ is smooth over $\mathfrak{o}$ [SGA 3 VI_B 4.4] with connected reductive fibers. In particular G_γ is unramified and $\mathbf{G}_\gamma(\mathfrak{o}) = G_\gamma(F) \cap K$ is a hyperspecial maximal compact subgroup of $G_\gamma(F)$ [T].

The statement about γ' can be proved in the same way as Lemma 19 of [H]. We have assumed that γ' is conjugate to γ under $G(F)$. Choose a Borel $\mathfrak{o}$-subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{A}$ and let $\mathbf{N}$ be the unipotent radical of $\mathbf{B}$. Then $G(F) = K \cdot N(F) \cdot A(F)$, and without loss of generality we may assume that $\gamma' = n\gamma n^{-1}$ for some $n \in N(F)$.

To finish this step of the proof it is enough to show that $n \in N(\mathfrak{o}) N_\gamma(F)$, where $N_\gamma = G_\gamma \cap N$. Choose a total order $\alpha_1 < \ldots < \alpha_r$ of the usual kind on the set Δ of B-positive roots of A, and use this order to identify $\mathbf{N}$ with $\prod_{i=1}^{r} \mathbb{G}_a$ over $\mathfrak{o}$. This identification does not respect the multiplication law for $\mathbf{N}$; nevertheless $\prod_{i=j}^{r} \mathbb{G}_a$ is identified with a subgroup $\mathbf{N}_j$ of $\mathbf{N}$ ($1 \leqq j \leqq r$), and the projection $\mathbf{N}_j \to \mathbb{G}_a$ on the factor indexed by j is a homomorphism. Write n as $(x_1, \ldots, x_r) \in \prod_{i=1}^{r} \mathbb{G}_a$. Then

$n\gamma^{-1}n^{-1}\gamma^{-1}$ belongs to $\mathbf{N}(\mathfrak{o})$ and its first coordinate is $(1-\alpha_1(\gamma))x_1$. If $1-\alpha_1(\gamma)=0$, then α_1 is a root of G_γ, and by multiplying n on the right by an element of $N_\gamma(F)$, we may assume that $x_1=0$. If $1-\alpha_1(\gamma)\neq 0$, then by hypothesis $1-\alpha_1(\gamma)$ is a unit, and therefore $x_1\in\mathfrak{o}$. Multiplying n on the left by an element of $\mathbf{N}(\mathfrak{o})$, we may again assume that $x_1=0$. Then $n\in N_2$ and the same argument can be applied to x_2. Continuing with this until all the positive roots have been used, we see that $n\in\mathbf{N}(\mathfrak{o})\cdot N_\gamma(F)$.

Now we proceed with the next step. We continue to assume that G_{der} is simply connected, but drop the rest of the assumptions. Choose a maximal F-torus T of G containing γ, and choose a finite Galois extension E/F such that

(i) T splits over E,

(ii) γ,γ' are conjugate under $G(E)$.

Choose a maximal split $\mathfrak{o}_E$-torus $\mathbf{A}$ of $\mathbf{G}$ [(i) implies that G is split over E], and choose an element $\gamma''\in\mathbf{A}(\mathfrak{o}_E)$ that is conjugate under $G(E)$ to γ and γ'. Our previous work shows that γ,γ',γ'' are all conjugate under $G(\mathfrak{o}_E)$ and that $\mathbf{G}_{\gamma''}$ is smooth over $\mathfrak{o}_E$. Since $(\mathbf{G}_\gamma)_{\mathfrak{o}_E}$ is isomorphic to $\mathbf{G}_{\gamma''}$ and $\mathfrak{o}_E$ is faithfully flat over $\mathfrak{o}$, the group $\mathbf{G}_\gamma$ is smooth over $\mathfrak{o}$ and its fibers are connected reductive groups. In particular G_γ is unramified and $\mathbf{G}_\gamma(\mathfrak{o})=G_\gamma(F)\cap K$ is a hyperspecial maximal compact subgroup of $G_\gamma(F)$.

Now consider the closed subscheme $\mathbf{Y}$ of $\mathbf{G}$ whose points in any $\mathfrak{o}$-algebra R are given by

$$\mathbf{Y}(R)=\{g\in\mathbf{G}(R)\mid g\gamma g^{-1}=\gamma'\}\,.$$

Of course $\mathbf{Y}$ is smooth over $\mathfrak{o}$, since it becomes isomorphic to $\mathbf{G}_\gamma$ over $\mathfrak{o}_E$. Let $\tilde{\gamma},\tilde{\gamma}'$ denote the images of γ,γ' in $\mathbf{G}(k)$. The conjugacy of γ,γ' in $\mathbf{G}(\mathfrak{o}_E)$ implies the stable conjugacy of $\tilde{\gamma},\tilde{\gamma}'$. Furthermore we have seen that the special fiber of $\mathbf{G}_\gamma$ is connected. Therefore $\tilde{\gamma},\tilde{\gamma}'$ are conjugate under $\mathbf{G}(k)$. In other words, $\mathbf{Y}(k)$ is non-empty, and now the smoothness of $\mathbf{Y}$ over $\mathfrak{o}$ implies that $\mathbf{Y}(\mathfrak{o})$ is non-empty. This finishes the proof in the case that G_{der} is simply connected.

Now we consider the general case. We choose an unramified z-extension $\alpha: H\to G$ of G. The kernel Z of α is an unramified torus lying in the center of H. The building of H is the product of the buildings of H and Z. Let y_0 be a hyperspecial point in the building of H that projects to x_0 and let $\mathbf{H}$ be the corresponding extension of H to a group scheme over $\mathfrak{o}$. Let $\mathbf{Z}$ be the unique (up to isomorphism) extension of Z to a torus over $\mathfrak{o}$. There is an exact sequence

$$1\to\mathbf{Z}\to\mathbf{H}\to\mathbf{G}\to 1$$

and the morphism $\mathbf{H}\to\mathbf{G}$ is smooth. Furthermore, $\mathbf{H}(k)\to\mathbf{G}(k)$ is surjective, since $\mathbf{Z}_k$ is connected. Therefore $\mathbf{H}(\mathfrak{o})\to\mathbf{G}(\mathfrak{o})$ is surjective. Choose $\delta\in\mathbf{H}(\mathfrak{o})$ such that $\delta\mapsto\gamma$. Let $\mathbf{I}$ be the quotient of $\mathbf{H}_\delta$ by $\mathbf{Z}$ [SGA 3 XXII 4.3.2]. Then $\mathbf{I}$ is an extension of I to a smooth group scheme over $\mathfrak{o}$ with connected reductive fibers. Using that $\mathbf{H}(\mathfrak{o})\to\mathbf{G}(\mathfrak{o})$ and $\mathbf{H}_\delta(\mathfrak{o})\to\mathbf{I}(\mathfrak{o})$ are surjective, we see that $I(F)\cap K$ is equal to $\mathbf{I}(\mathfrak{o})$ and is therefore a hyperspecial maximal compact subgroup of $I(F)$.

Now we prove the statement about γ'. Since γ',γ are stably conjugate there exist $\delta'\in H(F)$ and $h\in H(\bar{F})$ such that $\delta'\mapsto\gamma'$ and $\delta'=h\delta h^{-1}$ [K1]. We claim that $\delta'\in\mathbf{H}(\mathfrak{o})$. Since $\delta'\mapsto\gamma'\in\mathbf{G}(\mathfrak{o})$, it is enough to check that δ' acts trivially on the building of Z, and this is obvious from the equation $\delta'=h\delta h^{-1}$. Our previous work shows that δ,δ' are conjugate under $\mathbf{H}(\mathfrak{o})$. Therefore γ,γ' are conjugate under $\mathbf{G}(\mathfrak{o})$.

Finally, we prove that

$$\ker[H^1(F, I) \to H^1(F, G_\gamma)]$$

is trivial. It is easy to translate this into the following statement about H [with $\delta \in \mathbf{H}(\mathfrak{o})$, $\delta \mapsto \gamma$ as above]: if $\delta' \in H(F)$ is stably conjugate to δ and if $\delta' \mapsto \gamma$, then δ' is conjugate to δ. Just as above we see that $\delta' \in \mathbf{H}(\mathfrak{o})$ and hence that δ' is indeed conjugate to δ.

7.2. Assume that $\gamma \in K$ satisfies the hypothesis of 7.1. Let f_K denote the characteristic function of the subset K of $G(F)$. Let dg (respectively di) be a Haar measure on $G(F)$ [respectively $I(F)$] that gives measure 1 to K [respectively $I(F) \cap K$]. Let γ' be a stable conjugate of γ and form the orbital integral $O_{\gamma'}(f_K)$ using dg/di', where di' is the measure on $I'(F)$ obtained from di.

7.3. Corollary. *The orbital integral $O_{\gamma'}(f_K)$ vanishes unless γ' is conjugate to γ, in which case it equals* 1.

If γ' is not conjugate to γ, then by 7.1 the orbit of γ' does not meet K, and clearly $O_{\gamma'}(f_K) = 0$. It remains to show that $O_\gamma(f_K) = 1$. We have $O_\gamma(f_K) = \operatorname{meas} I(F)\backslash X$ where

$$X = \{g \in G(F) \mid g^{-1}\gamma g \in K\}.$$

If G_{der} is simply connected, 7.1 shows directly that $X = I(F) \cdot K$. Using an unramified z-extension $H \to G$ as in the proof of 7.1, we see that $X = I(F) \cdot K$ in general. Therefore $O_\gamma(f_K) = \operatorname{meas}(K) \cdot \operatorname{meas}(I(F) \cap K)^{-1} = 1$.

7.4. For simplicity we assume that G_{der} is simply connected for the rest of Sect. 7. Let (H, s, η) be an endoscopic triple for G. Let γ_H be a semi-simple (G, H)-regular element of $H(F)$ and let γ be a corresponding element of $G(F)$, as in 3.1. Let $I = G_\gamma$ and let κ be the element of $\mathfrak{K}(I/F)$ obtained from s as in 5.6. Then we can form κ-orbital integrals O_γ^κ (again see 5.6).

7.5. Proposition. *Assume that H is not an unramified group. Let f belong to the Hecke algebra $\mathfrak{H}(G(F), K)$. Then $O_\gamma^\kappa(f) = 0$.*

Let Γ_0 denote the inertia subgroup of Γ. Since G is unramified, Γ_0 acts trivially on $X^*(Z(G))$, and therefore $Z(G)$ can be embedded in an unramified F-torus C'. Let $C = C'/Z(G)$. We form an unramified group G_1 by taking the quotient of $G \times C'$ by $Z(G)$, with $Z(G)$ embedded diagonally in $G \times C'$. The center of G_1 is connected (it is isomorphic to C') and there is an obvious exact sequence

$$1 \to G \to G_1 \to C \to 1,$$

which yields a dual exact sequence

$$1 \to \hat{C} \to \hat{G}_1 \to \hat{G} \to 1.$$

Pulling back *via* $\eta: \hat{H} \to \hat{G}$, we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{C} & \longrightarrow & \hat{H}_1 & \longrightarrow & \hat{H} & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta} & & \\ 1 & \longrightarrow & \hat{C} & \longrightarrow & \hat{G}_1 & \longrightarrow & \hat{G} & \longrightarrow & 1, \end{array}$$

where $\hat{H}_1$ denotes the fiber product of $\hat{G}_1, \hat{H}$ over $\hat{G}$. Note that $\hat{H}_1$ is a connected reductive group over $\mathbb{C}$.

We now define an action of Γ on $\hat{H}_1$. The fiber product construction does not automatically give such an action, since η need not be a Γ-map. However, the $\hat{G}$-conjugacy class of η is fixed by Γ, which means that for each $\sigma \in \Gamma$ there exists $g_\sigma \in \hat{G}$ such that

$$\eta \circ \sigma = \operatorname{Int}(g_\sigma) \circ \sigma \circ \eta . \tag{7.5.1}$$

For each $\sigma \in \Gamma$ choose $x_\sigma \in \hat{G}_1$ such that $x_\sigma \mapsto g_\sigma$. Then the restriction of $\operatorname{Int}(x_\sigma)$ to $\sigma(\eta_1(\hat{H}_1))$ is independent of the choices of g_σ and x_σ. We let Γ act on $\hat{H}_1$ in the unique way for which

$$\eta_1 \circ \sigma = \operatorname{Int}(x_\sigma) \circ \sigma \circ \eta_1 . \tag{7.5.2}$$

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{C} & \longrightarrow & Z(\hat{H}_1) & \longrightarrow & Z(\hat{H}) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \hat{C} & \longrightarrow & Z(\hat{G}_1) & \longrightarrow & Z(\hat{G}) & \longrightarrow & 1 . \end{array}$$

Our next step is to show that s does not belong to the image of $Z(\hat{H}_1)^{\Gamma_0}$. Let s_1 be any element of $Z(\hat{H}_1)$ that maps to s. Since H is not unramified, we may choose $\sigma \in \Gamma_0$ such that σ acts non-trivially on $\hat{H}$. The group G, however, is unramified, and thus σ acts trivially on $\hat{G}$; (7.5.1) now shows that $g_\sigma \notin \eta(\hat{H})$ (otherwise σ would act on $\hat{H}$ by an inner automorphism, which would necessarily be trivial since σ preserves some splitting of $\hat{H}$). Therefore $x_\sigma \notin \eta_1(\hat{H}_1)$. But $\eta_1(\hat{H}_1)$ is the identity component of the centralizer of s_1 in $\hat{G}_1$, and we have arranged that the derived group of $\hat{G}_1$ be simply connected, so that this centralizer is in fact connected. We conclude that x_σ does not centralize s_1 and hence that $\sigma(s_1) \neq s_1$.

Because F is local there is no harm in assuming that $s \in Z(\hat{H})^\Gamma$. The connecting homomorphism for the exact sequence

$$1 \to \hat{C} \to Z(\hat{H}_1) \to Z(\hat{H}) \to 1$$

sends s to an element $\alpha \in H^1(F, \hat{C})$, which is the Langlands parameter for a character χ of $C(F)$. We have seen that s does not belong to the image of $Z(\hat{H}_1)^{\Gamma_0}$. Therefore α is a ramified Langlands parameter for the unramified torus C, which shows that χ is non-trivial on $\mathbf{C}(\mathfrak{o})$ for the unique extension $\mathbf{C}$ of C to a torus over $\mathfrak{o}$.

Let H_1 be a quasi-split connected reductive group over F whose dual is $\hat{H}_1$. Choose an embedding $H \to H_1$ over F dual to $\hat{H}_1 \to H_1$. Although (H_1, s_1, η_1) is not an endoscopic triple for G_1 [since $s_1 \notin Z(\hat{G}_1) \cdot Z(\hat{H}_1)^\Gamma$], the results of Sect. 3 extend to H_1, G_1. Let I_1 (respectively I_{H_1}) denote the connected centralizer of γ in G_1 (respectively γ_H in H_1). Then γ is (G_1, H_1)-regular and we have

$$Z(\hat{H}_1) \hookrightarrow Z(\hat{I}_{H_1}) \xrightarrow{\sim} Z(\hat{I}_1) .$$

We also have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{C} & \longrightarrow & Z(\hat{H}_1) & \longrightarrow & Z(\hat{H}) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \hat{C} & \longrightarrow & Z(\hat{I}_1) & \longrightarrow & Z(\hat{I}) & \longrightarrow & 1 . \end{array}$$

Let $g_1 \in G_1(F)$, let c be the image of g_1 in $C(F)$, and let $\gamma' = g_1 \gamma g_1^{-1}$. Then $\gamma' \in G(F)$ is a stable conjugate of γ. We claim that

$$\langle \mathrm{inv}(\gamma, \gamma'), \kappa \rangle = \chi(c^{-1}). \tag{7.5.3}$$

Let κ_0 denote the image of s in $Z(\hat{I})^\Gamma$. From the commutative diagram above we see that the connecting homomorphism for

$$1 \to \hat{C} \to Z(\hat{I}_1) \to Z(\hat{I}) \to 1$$

maps κ_0 to α. Lemma 1.6 tells us that $\chi(c^{-1}) = \langle d, \kappa_0 \rangle$, where d is the image of c^{-1} under the connecting homomorphism

$$C(F) \to H^1(F, I)$$

coming from

$$1 \to I \to I_1 \to C \to 1.$$

To finish proving the claim it is enough to check that $d = \mathrm{inv}(\gamma, \gamma')$. Choose $g_2 \in I_1(\bar{F})$ such that $g_2 \mapsto c^{-1}$. Then $g_1 = g g_2^{-1}$ for some $g \in G(\bar{F})$. It is clear that the 1-cocycle

$$(g^{-1} \cdot \tau(g)) = (g_2^{-1} \cdot \tau(g_2)) \quad (\tau \in \Gamma)$$

represents the classes of both $\mathrm{inv}(\gamma, \gamma')$ and d.

With g_1, c as above we use g_1 and f to obtain $f_1 \in C_c^\infty(G(F))$, where

$$f_1(x) = f(g_1 x g_1^{-1}).$$

From (7.5.3) we see that

$$O_\gamma^\kappa(f_1) = \chi(c) \cdot O_\gamma^\kappa(f).$$

Here we used the fact that $e(G_\gamma) = e(G_{g_1 \gamma g_1^{-1}})$, which is obvious since $\mathrm{Int}(g_1)$ defines an F-isomorphism between the two groups.

Choose a maximal F-torus T in G whose split component is as large as possible and whose apartment contains x_0. We have an exact sequence of unramified tori

$$1 \to T \to T_1 \to C \to 1,$$

where T_1 is the centralizer of T in G_1. Let $\mathbf{T}, \mathbf{T}_1$ denote the unique extensions of T, T_1 to tori over $\mathfrak{o}$. Then

$$1 \to \mathbf{T}(\mathfrak{o}) \to \mathbf{T}_1(\mathfrak{o}) \to \mathbf{C}(\mathfrak{o}) \to 1$$

is exact.

We have seen that χ is non-trivial on $\mathbf{C}(\mathfrak{o})$. Thus we may choose $g_1 \in \mathbf{T}_1(\mathfrak{o})$ such that $\chi(c) \neq 1$, where c denotes the image of g_1 in $\mathbf{C}(\mathfrak{o})$. With f_1 as before we have

$$O_\gamma^\kappa(f_1) = \chi(c) \cdot O_\gamma^\kappa(f).$$

We claim that $f_1 = f$. Since $f \in \mathfrak{H}(G(F), K)$, it is enough to check that $g_1^{-1} K a K g_1 = K a K$ for any $a \in T(F)$. This is obvious, since g_1 normalizes K and commutes with a. We conclude that

$$O_\gamma^\kappa(f) = \chi(c) \cdot O_\gamma^\kappa(f),$$

and hence that $O_\gamma^\kappa(f) = 0$.

8. Global Finiteness Result

In this section F is a number field, G is a connected reductive group over F, and $\psi: G_0 \to G$ is an inner twisting with G_0 quasi-split over F.

8.1. The inner twisting ψ induces maps

$$\{\text{stable classes in } G(F_v)\} \to \{\text{stable classes in } G_0(F_v)\} \tag{8.1.1}$$

(see [K1]) for each place v of F. We say that the $G(\mathbb{A})$-conjugacy class of $\gamma \in G(\mathbb{A})$ *comes from* $\gamma_0 \in G_0(F)$ if every local component of γ maps under (8.1.1) to the stable class of γ_0.

8.2. Proposition. *Let C be a compact subset of $G(\mathbb{A})$. Then there are only finitely many $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ that meet C and come from some semi-simple element of $G_0(F)$.*

First we do the case in which G_{der} is simply connected. Choose a faithful representation $G \to GL_n$. The coefficients of the characteristic polynomial of an $n \times n$ matrix give us a continuous map $GL_n(\mathbb{A}) \to \mathbb{A}^n$. Let C' be the image of C under the composed map

$$G(\mathbb{A}) \to GL_n(\mathbb{A}) \to \mathbb{A}^n.$$

Then $C' \cap F^n$ is both discrete and compact and is therefore finite. Let γ be a semi-simple element of $G(\mathbb{A})$ whose $G(\mathbb{A})$-conjugacy class meets C and comes from some semi-simple element of $G_0(F)$. Then the image of γ in $\mathbb{A}^n$ belongs to $C' \cap \bar{F}^n \cap \mathbb{A}^n = C' \cap F^n$. Thus, in proving the finiteness statement it is enough to consider conjugacy classes in $G(\mathbb{A})$ mapping to a fixed element of $\mathbb{A}^n$. Furthermore, there are only a finite number of semi-simple conjugacy classes in $G_0(\bar{F})$ whose image under

$$G_0(\bar{F}) \xrightarrow{\psi} G(\bar{F}) \to GL_n(\bar{F}) \to \bar{F}^n$$

is a fixed element of $\bar{F}^n$, which means that it is enough to consider conjugacy classes in $G(\mathbb{A})$ that come from a fixed semi-simple $\gamma_0 \in G_0(F)$.

We may as well assume that there exists $\gamma \in G(\mathbb{A})$ whose conjugacy class comes from γ_0 (otherwise there is nothing to prove). Choose a compact open subgroup K of $G(\mathbb{A}_f)$. Choose a finite set V of places of F, including all infinite places, such that

(a) for all $v \notin V$ the group G is unramified at v and K can be written as $K_v K^v$, where K_v is a hyperspecial maximal compact subgroup of $G(F_v)$ and K^v is a compact open subgroup of $G(\mathbb{A}_f^v)$ ($\mathbb{A}_f^v$ is the ring of finite adeles with trivial v-component),

(b) for all $v \notin V$ the v-component γ_v of γ belongs to K_v and $1-\alpha(\gamma_v)$ is either 0 or a unit in $\mathfrak{o}_{\bar{F}_v}$ for every root α of G,

(c) C is contained in

$$\prod_{v \in V} G(F_v) \cdot \prod_{v \notin V} K_v.$$

It is possible to achieve (b) since γ comes from γ_0 and $1-\alpha(\gamma_0) \in \bar{F}$ is either 0 or a unit locally almost everywhere.

Let X be the set of $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ that meet C and come from γ_0. We want to show that X is finite. At any place v of F the stable conjugacy class of γ_v contains only finitely many conjugacy classes. Therefore it is enough to show that any conjugacy class in X contains an element γ' such that $\gamma'_v = \gamma_v$ for all $v \notin V$, and this follows immediately from 7.1.

Now we do the general case. Choose a z-extension $\alpha: H \to G$ with kernel Z. Let Y be the set of $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ that meet C and come from some semi-simple element of $G_0(F)$. We want to prove that Y is finite. Let $G(\mathbb{A})^1$ denote the intersection of the kernels of the characters $|\lambda|$, where λ runs through the set of homomorphisms $G \to \mathbb{G}_m$ over F. Any conjugacy class in Y is contained in $G(\mathbb{A})^1$, and thus by replacing C by $C \cap G(\mathbb{A})^1$, we may as well suppose that C is contained in $G(\mathbb{A})^1$. It is easy to see that α maps $H(\mathbb{A})^1$ onto $G(\mathbb{A})^1$. Because $H(\mathbb{A})^1$ is locally compact, we can find a compact subset C_H of $H(\mathbb{A})^1$ such that $\alpha(C_H) = C$. Furthermore, the compactness of $Z(F) \backslash Z(\mathbb{A})^1$ implies the existence of a compact subset C_Z of $Z(\mathbb{A})^1$ such that $Z(F) \cdot C_Z = Z(\mathbb{A})^1$. Replacing C_H by $C_Z \cdot C_H$, we may assume that $Z(F) \cdot C_H = \alpha^{-1}(C) \cap H(\mathbb{A})^1$.

We twist the z-extension $H \to G$ as in Sect. 5 of [K1], obtaining a commutative diagram

$$\begin{array}{ccc} H_0 & \xrightarrow{\psi_H} & H \\ {\scriptstyle \alpha_0}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G_0 & \xrightarrow{\psi} & G, \end{array}$$

where $\alpha_0: H_0 \to G_0$ is a z-extension with kernel Z and ψ_H is an inner twisting. Let Y_H be the set of $H(\mathbb{A})$-conjugacy classes in $H(\mathbb{A})$ that meet C_H and come from some semi-simple element of $H_0(F)$. From our previous work we know that Y_H is finite, and we will finish the proof by showing that the natural map $Y_H \to Y$ is surjective.

Consider a conjugacy class in Y and choose an element $\gamma \in C$ in that conjugacy class. Choose a semi-simple element $\gamma_0 \in G_0(F)$ such that γ comes from γ_0. Choose $\delta_0 \in H_0(F)$ such that $\delta_0 \mapsto \gamma_0$. For every place v of F there exists $\delta_v \in H(F_v)$ such that δ_v, δ_0 are stably conjugate and $\delta_v \mapsto \gamma_v$, where γ_v denotes the v-component of γ. Let δ be the element of $\prod_v H(F_v)$ whose v-components are the elements δ_v. It is easy to see that $\delta \in H(\mathbb{A})$ (see the part of the proof of Proposition 7.1 that uses $H \to G$). Since $\delta \in H(\mathbb{A})^1$ and $\alpha(\delta) = \gamma \in C$, we have $\delta \in zC_H$ for some $z \in Z(F)$. Then the $H(\mathbb{A})$-conjugacy class of $z^{-1}\delta$ belongs to Y_H and maps to the $G(\mathbb{A})$-conjugacy class of γ.

9. Stabilization of the Elliptic Terms in the Trace Formula

In this section F is a number field and G is a connected reductive group over F.

9.1. We use the canonical Haar measure dg on $G(\mathbb{A})$ (the one used in the definition of the Tamagawa number of G). For $f \in C_c^\infty(G(\mathbb{A}))$ we write $T_e(f)$ for the elliptic part of the trace formula [A] for f:

$$T_e(f) = \sum_{\gamma \in E} |G_\gamma(F)/I(F)|^{-1} \tau(I) O_\gamma(f),$$

where E is a set of representatives for the elliptic semi-simple conjugacy classes in $G(F)$, I is the identity component of G_γ, $\tau(I)$ is the Tamagawa number of I, and $O_\gamma(f)$ is the adelic orbital integral

$$\int_{I(A)\backslash G(A)} f(g^{-1}\gamma g)\frac{dg}{di}$$

[di is the canonical Haar measure on $I(\mathbb{A})$].

What does it mean for a semi-simple $\gamma \in G(F)$ to be elliptic? Over a general field F there are two reasonable definitions. The first is that γ is elliptic if and only if γ belongs to some elliptic maximal F-torus of G. The second is that γ is elliptic if and only if $Z(I)^0/Z(G)^0$ is anisotropic. If every connected reductive F-group possesses elliptic maximal F-tori (e.g., if F is a number field or a p-adic field), then the two definitions are equivalent, but in general they are not (e.g., $F=\mathbb{R}$). In this section we are working over a number field, and it makes no difference which definition we use.

9.2. For quasi-split G we define the stable analogue $ST_e(f)$ of $T_e(f)$ by the formula

$$ST_e(f)=\sum_{\gamma\in E_{st}} |(G_\gamma/I)(F)|^{-1}\tau(G)SO_\gamma(f),$$

where E_{st} is a set of representatives for the elliptic semi-simple stable conjugacy classes in $G(F)$, and $SO_\gamma(f)$ is the stable adelic orbital integral

$$SO_\gamma(f)=\sum_i e(\gamma_i)O_{\gamma_i}(f).$$

Here the sum runs over

$$i\in\ker[H^1(F,I(\mathbb{A}))\to H^1(F,G(\mathbb{A}))];$$

as usual i determines an element $\gamma_i\in G(\mathbb{A})$ [up to $G(\mathbb{A})$-conjugacy] whose local components are all stably conjugate to γ. The number $e(\gamma_i)$ is defined by

$$e(\gamma_i)=\prod_v e(I_{i,v}),$$

where $I_{i,v}$ is the connected centralizer in G_v of the component of γ_i at the place v of F (see 5.1). Since we always use canonical measures in defining our adelic orbital integrals O_{γ_i}, it is automatic that these measures satisfy the usual consistency requirement.

The finiteness results of Sects. 7 and 8 show that the sums defining $SO_\gamma(f)$ and $ST_e(f)$ have only a finite number of non-zero terms. Corollary 7.3 shows that the integral defining $O_{\gamma_i}(f)$ is convergent. It is easy to see that $SO_\gamma(f)$ and $|(G_\gamma/I)(F)|$ depend only on the stable conjugacy class of γ in $G(F)$. Thus the definition of $ST_e(f)$ makes sense.

9.3. From now on we simplify the discussion by assuming that G_{der} is simply connected. We choose an inner twisting $\psi: G_0\to G$ with G_0 quasi-split over F. We also choose a set $\mathfrak{E}$ of representatives for the isomorphism classes of elliptic endoscopic triples (H,s,η) for G [K3], and for each $(H,s,\eta)\in\mathfrak{E}$ we choose an L-homomorphism $\eta': {}^LH\to{}^LG$ extending η. Our goal is to stabilize $T_e(f)$, but to do this we must make some assumptions. First of all we assume the local and global conjectures in Sects. 5 and 6. We also assume that the "fundamental lemma" [L2, Chap. III] on spherical functions holds for G, H at places v where both groups are unramified (for our purposes it is enough to have the fundamental lemma for the unit element in the Hecke algebra of G). With these assumptions the local function correspondences of Sect. 5 yield functions $f^H \in C_c^\infty(H(\mathbb{A}))$ (as usual we write f^H even though f^H also depends on s, η), and these functions have the property that

$$SO_{\gamma_H}(f^H) = \sum_{\gamma} \langle \mathrm{obs}(\gamma), \kappa \rangle e(\gamma) O_\gamma(f) \tag{9.3.1}$$

for any (G, H)-regular semi-simple $\gamma_H \in H(F)$. In the sum γ runs over a set of representatives for the $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ that come from γ_H. The number $\langle \mathrm{obs}(\gamma), \kappa \rangle$ is explained in 6.9, and the number $e(\gamma)$ is given by

$$e(\gamma) = \prod_v e(I_v),$$

where I_v is the (connected) centralizer in G_v of the v-component of γ.

There are still more assumptions. We will need the Hasse principle and therefore we assume that G has no E_8 factors. Finally, we assume that Weil's conjecture on Tamagawa numbers is true for all groups I whose dimension is less than that of G. Recall that the conjecture for the group I states that $\tau(I_{\mathrm{SC}}) = 1$. This conjecture is true for quasi-split groups [La], but it remains to be shown that $\tau(I)$ is invariant under inner twists. However, one hopes [J-L, L3] that this last statement can be proved during the stabilization of the full trace formula for I, and thus our last assumption is quite reasonable, since the stabilization of the full trace formula for G will undoubtedly be done by induction on $\dim G$.

9.4. To any elliptic endoscopic triple (H, s, η) of G there is associated a number $\iota(G, H)$ [L2]. We need to recall the following formula for $\iota(G, H)$, given in Sect. 8 of [K3]:

$$\iota(G, H) = \tau_1(G) \cdot \tau_1(H)^{-1} \cdot \lambda^{-1}.$$

Here $\tau_1(G)$ denotes the relative Tamagawa number $\tau(G)/\tau(G_{\mathrm{SC}})$ of G, and λ denotes the cardinality of the group

$$\mathrm{Aut}(H, s, \eta)/H_{\mathrm{ad}}(F)$$

(see [K3, 7.5] for the definition of $\mathrm{Aut}(H, s, \eta)$).

9.5. We define $T_e^*(f)$ by the sum used to define $T_e(f)$, omitting all terms indexed by central elements γ of $G(F)$. Similarly we define $ST_e^{**}(f^H)$ by omitting certain terms from the sum defining $ST_e(f^H)$. If H is a quasi-split inner form of G, we omit all terms indexed by central elements γ_H of $H(F)$. If H is not a quasi-split inner form of G, then we omit all terms indexed by elements $\gamma_H \in H(F)$ that are not (G, H)-regular.

9.6. Theorem. *Under the assumptions of 9.3 we have*

$$T_e^*(f) = \sum_{(H, s, \eta) \in \mathfrak{E}} \iota(G, H) ST_e^{**}(f^H).$$

Since we have assumed that G_{der} is simply connected, we have $G_\gamma = I$. By Lemma 3.2 we also have $H_{\gamma_H} = I_H$ for (G, H)-regular semi-simple $\gamma_H \in H(F)$, where I_H denotes the identity component of H_{γ_H}. Therefore the numbers $|G_\gamma(F)/I(F)|$ and $|(H_{\gamma_H}/I_H)(F)|$ appearing in the sums defining $T_e^*(f)$ and $ST_e^{**}(f^H)$ are equal to 1.

Choose a set E_0^* of representatives for the non-central elliptic semi-simple stable conjugacy classes in $G_0(F)$. Then

$$T_e^*(f) = \sum_{\gamma_0 \in E_0^*} \tau(I_0) \sum_{\gamma} O_\gamma(f) \tag{9.6.1}$$

where I_0 is the (connected) centralizer of γ_0 in G_0 and the second sum runs over a set of representatives γ for the $G(F)$-conjugacy classes in $G(F)$ contained in the $G(\bar{F})$-conjugacy class of $\psi(\gamma_0)$. Note that I_0 is an inner twist of I and that $\dim(I) < \dim(G)$, so that our assumption on Tamagawa numbers implies that $\tau(I) = \tau(I_0)$ (this follows from Weil's conjecture for I, since one knows that the relative Tamagawa numbers of I, I_0 are equal).

The orbital integral $O_\gamma(f)$ depends only on the $G(\mathbb{A})$-conjugacy class of γ in $G(\mathbb{A})$. Consider a fixed γ contributing to the sum (9.6.1). Let I be the connected centralizer of γ in G. Then the number of terms in the second sum in (9.6.1) that are indexed by $G(\mathbb{A})$-conjugates of γ is equal to

$$|\ker[\ker^1(F, I) \to \ker^1(F, G)]| \tag{9.6.2}$$

with $\ker^1(F, *)$ as in Sect. 4 of [K3]. We are assuming that G has no E_8 factors. The same is then true of I. From Sect. 4 of [K3] we get a diagram

$$\begin{array}{ccc} \ker^1(F, I) & \longrightarrow & \ker^1(F, G) \\ \downarrow & & \downarrow \\ \ker^1(F, Z(\hat{I}))^D & \longrightarrow & \ker^1(F, Z(\hat{G}))^D \end{array}$$

in which the vertical maps are bijections and the bottom horizontal map is induced by the natural injection $Z(\hat{G}) \to Z(\hat{I})$ of 4.2.

This diagram commutes. We are assuming that G_{der} is simply connected; to prove the commutativity in the general case we would use a z-extension of G. Let $D = G/G_{\mathrm{der}}$. Then $\ker^1(F, G) \tilde{\to} \ker^1(F, D)$ and $\ker^1(F, Z(\hat{G}))^D \tilde{\to} \ker^1(F, \hat{D})^D$ by Lemma 4.3.1 of [K3]. Therefore the desired commutativity follows from the functoriality of

$$\ker^1(F, G) \to \ker^1(F, Z(\hat{G}))^D$$

for the normal homomorphisms $G \to D$ and $I \to D$.

Using that $Z(\hat{I}_0) = Z(\hat{I})$, we now see that the number of terms in the second sum in (9.6.1) that are indexed by $G(\mathbb{A})$-conjugates of γ is equal to

$$|\mathrm{cok}[\ker^1(F, Z(\hat{G})) \to \ker^1(F, Z(\hat{I}_0))]| . \tag{9.6.3}$$

Now consider an element $\gamma_0 \in E_0^*$ and an element $\gamma \in G(\mathbb{A})$ in the $G(\bar{\mathbb{A}})$-conjugacy class of $\psi(\gamma_0)$. The construction of 6.5 gives us

$$\mathrm{obs}(\gamma) \in \mathfrak{K}(I_0/F)^D .$$

Theorem 6.6 implies that

$$|\mathfrak{K}(I_0/F)|^{-1}\sum_{\kappa}\langle \mathrm{obs}(\gamma),\kappa\rangle, \tag{9.6.4}$$

where κ runs over $\mathfrak{K}(I_0/F)$, is equal to 1 if the $G(\mathbb{A})$-conjugacy class of γ contains an element of $G(F)$ and is equal to 0 otherwise.

We define $e(\gamma)$ by the formula

$$e(\gamma)=\prod_v e(I_v),$$

where I_v is the (connected) centralizer in G_v of the v-component of γ. If γ is $G(\mathbb{A})$-conjugate to an element of $G(F)$, then the last result in [K2] says that $e(\gamma)=1$.

Before continuing with the manipulation of the elliptic terms, we need to observe that the quotient of the number (9.6.3) by $|\mathfrak{K}(I_0/F)|$ is equal to $\tau_1(G)\cdot\tau_1(I_0)^{-1}$. This can be proved the same way as Lemma 8.3.2 of [K3], using the exact sequence

$$1\to\pi_0(Z(\hat{G})^{\Gamma})\to\pi_0(Z(\hat{I}_0)^{\Gamma})\to\mathfrak{K}(I_0/F)\to\ker^1(F,Z(\hat{G}))\to\ker^1(F,Z(\hat{I}_0)).$$

The 1 at the beginning of this sequence is there because γ_0 is elliptic, which means that $X_*(Z(\hat{I}_0)/Z(\hat{G}))^{\Gamma}$ is trivial.

These remarks show that $T_e^*(f)$ is given by

$$\sum_{\gamma_0\in E_0^*}\tau_1(G)\sum_{\gamma}\sum_{\kappa}\langle \mathrm{obs}(\gamma),\kappa\rangle e(\gamma)O_\gamma(f), \tag{9.6.5}$$

where γ now runs over a set of representatives for the $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ contained in the $G(\bar{\mathbb{A}})$-conjugacy class of $\psi(\gamma_0)$, and κ runs over $\mathfrak{K}(I_0/F)$. The finiteness results of Sects. 7–8 show that this triple sum has only finitely many non-zero terms and hence that it can be rearranged in any way we like.

Now we look at the right side of the equality stated in Theorem 9.6. Since the groups H are quasi-split, we have $\tau(H)=\tau_1(H)$, and therefore

$$\sum_{\mathfrak{E}}\iota(G,H)ST_e^{**}(f^H)$$

is equal to

$$\sum_{\mathfrak{E}}\tau_1(G)\cdot\lambda^{-1}\sum_{\gamma_H\in E_H^{**}}SO_{\gamma_H}(f^H),$$

where E_H^{**} is as follows: if H is a quasi-split inner form of G, then E_H^{**} is a set of representatives for the non-central elliptic semi-simple stable conjugacy classes in $H(F)$, and if H is not a quasi-split inner form of G, then E_H^{**} is a set of representatives for the (G,H)-regular elliptic semi-simple stable conjugacy classes in $H(F)$.

From (9.3.1) we now see that the right side of the equality stated in the theorem is equal to

$$\sum_{\mathfrak{E}}\tau_1(G)\cdot\lambda^{-1}\sum_{\gamma_H\in E_H^{**}}\sum_{\gamma}\langle \mathrm{obs}(\gamma),\kappa\rangle e(\gamma)O_\gamma(f), \tag{9.6.6}$$

where γ runs over a set of representatives for the $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ that come from γ_H.

We need to prove that (9.6.5) and (9.6.6) are equal. Given $(H,s,\eta)\in\mathfrak{E}$ and $\gamma_H\in E_H^{**}$, we get $\gamma_0\in G_0(F)$ (up to stable conjugacy) and $\kappa\in\mathfrak{K}(I_0/F)$, where $I_0=(G_0)_{\gamma_0}$ (see 6.10 for the construction of γ_0,κ). To indicate that (γ_0,κ) are obtained from (H,s,η,γ_H) in this way we write $(H,s,\eta,\gamma_H)\to(\gamma_0,\kappa)$. The following lemma finishes the proof of the theorem.

9.7. Lemma. *Let γ_0 be any elliptic semi-simple element of $G_0(F)$ and let $\kappa\in\mathfrak{K}(I_0/F)$, where $I_0=(G_0)_{\gamma_0}$. Then there exist $(H,s,\eta)\in\mathfrak{E}$ and a (G,H)-regular semi-simple element γ_H of $H(F)$ such that $(H,s,\eta,\gamma_H)\to(\gamma_0,\kappa)$. Moreover, $(H_1,s_1,\eta_1,\gamma_{H_1})\to(\gamma_0,\kappa)$ also holds if and only if there exists an isomorphism*

$$(H,s,\eta)\to(H_1,s_1,\eta_1)$$

carrying γ_H into a stable conjugate of γ_{H_1}, and such an isomorphism is unique up to composition with an element of $H_{\mathrm{ad}}(F)$.

Since we will use only the quasi-split form of G in the proof, we may as well simplify notation by dropping the subscripts from G_0,γ_0,I_0.

First we prove the existence of (H,s,η) and γ_H such that $(H,s,\eta,\gamma_H)\to(\gamma,\kappa)$. Choose an elliptic maximal F-torus T of G containing γ. Then T is also an elliptic maximal F-torus in I, so that we have a canonical embedding $Z(\hat{I})\to\hat{T}$. Choose an embedding $\hat{T}\to\hat{G}$ in the canonical $\hat{G}$-conjugacy class. Using

$$Z(\hat{I})\to\hat{T}\to\hat{G},$$

we get from κ an element $s\in\hat{G}$, well-defined up to $Z(\hat{G})$. Let $\hat{H}=(\hat{G})_s^0$. Since $s\in[\hat{T}/Z(\hat{G})]^\Gamma$, the Galois action on $\hat{T}$ preserves the root system of $\hat{H}$ and gives us a homomorphism $\varrho:\Gamma\to\mathrm{Out}(\hat{H})$. Let $\sigma\in\Gamma$. The $\hat{G}$-conjugacy class of $\hat{T}\to\hat{G}$ is fixed by σ, which means that there exists $g\in{}^LG$, projecting onto $\sigma\in\Gamma$, such that $\mathrm{Int}(g)$ preserves $\hat{T}$ and acts by σ_T on that group (we write σ_T to distinguish the action of σ on $\hat{T}$ from the action of σ on $\hat{G}$). Using this one checks easily that (s,ϱ) is an elliptic endoscopic datum for G [K3]. Consider the corresponding elliptic endoscopic triple $(H,s,\eta)\in\mathfrak{E}$.

There exists a maximal F-torus T_H of H such that T_H transfers to T. We choose an F-isomorphism $j:T_H\tilde{\to}T$ such that $T_H\xrightarrow{j}T\hookrightarrow G$ belongs to the canonical $G(\bar{F})$-conjugacy class. Let $\gamma_H=j^{-1}(\gamma)$ and let I_H be the connected centralizer of γ_H in H. We claim that γ_H is (G,H)-regular. Use j to identify T_H and T. What we need to check is that the set $R(\gamma)$ of roots of T in I is equal to the set $R_H(\gamma)$ of roots of T in I_H. Let R be the set of roots of T in G. Then we have

$$R(\gamma)=\{\alpha\in R\mid\alpha(\gamma)=1\},$$
$$R_H(\gamma)=\{\alpha\in R\mid\alpha(\gamma)=1 \text{ and } \alpha^\vee(s)=1\}.$$

But if $\alpha(\gamma)=1$, then $\alpha^\vee$ is a root of $\hat{T}$ in $\hat{I}$, and therefore $\alpha^\vee(s)=1$ since $s\in Z(\hat{I})$. This shows that γ_H is indeed (G,H)-regular. It is obvious that $(H,s,\eta,\gamma_H)\to(\gamma,\kappa)$.

Next we suppose that we also have $(H_1,s_1,\eta_1,\gamma_{H_1})\to(\gamma,\kappa)$, and show that there exists an isomorphism $(H,s,\eta)\to(H_1,s_1,\eta_1)$ carrying γ_H into a stable conjugate of γ_{H_1} (the converse is obvious).

Choose a maximal F-torus T_{H_1} of H_1 containing γ_{H_1} and choose an embedding $j_1:T_{H_1}\to G$ in the canonical $G(\bar{F})$-conjugacy class, such that $j_1(\gamma_{H_1})=\gamma$. After conjugating j_1 by an element of $I(\bar{F})$, we may suppose that $j_1(T_{H_1})=T$ and use j_1 to identify T_{H_1} with T over $\bar{F}$. The roots and coroots of T in H_1 are determined by κ, and are therefore the same as for H. Therefore there exists an isomorphism $\alpha_0: H \to H_1$ over $\bar{F}$ which extends $j_1^{-1} \circ j: T_H \to T_{H_1}$. Let $\sigma \in \Gamma$. Then $\sigma(j_1)=\mathrm{Int}(g) \circ j_1$ for some $g \in G(\bar{F})$, and since $\sigma(j_1)(\gamma_{H_1})=\gamma$ we have $g \in I(\bar{F})$. Now using that γ_{H_1} is (G, H_1)-regular, we see that there exists $h \in I_{H_1}(\bar{F})$ such that $\sigma(j_1)=j_1 \circ \mathrm{Int}(h)$. Therefore $\sigma(j_1^{-1} \circ j)$ is conjugate to $j_1^{-1} \circ j$ under $I_{H_1}(\bar{F})$, and we conclude that the $H_1(\bar{F})$-conjugacy class of $\alpha_0: H \to H_1$ is defined over F. Since H, H_1 are quasi-split there exists an $H_1(\bar{F})$-conjugate α of α_0 such that α is defined over F. It is not hard to see that $\alpha: H \to H_1$ defines an isomorphism $(H, s, \eta) \to (H_1, s_1, \eta_1)$. We have assumed that G_{der} is simply connected, and thus 3.2 guarantees that stable conjugacy and $H_1(\bar{F})$-conjugacy coincide for γ_{H_1}. It is now clear that $\alpha(\gamma_H)$ is stably conjugate to γ_{H_1}.

Finally we need to prove the uniqueness assertion about α. In other words, we need to show that if $\alpha \in \mathrm{Aut}(H, s, \eta)$ carries γ_H into a stable conjugate of itself, then $\alpha \in H_{\mathrm{ad}}(F) \subset \mathrm{Aut}(H, s, \eta)$. Choose $h \in H(\bar{F})$ such that $\alpha_0 := \mathrm{Int}(h) \circ \alpha$ carries γ_H into itself. Then α_0 preserves I_H and hence carries T_H into an $I_H(\bar{F})$-conjugate of itself. Thus we may as well suppose that $\alpha_0(T_H)=T_H$. As before we use $j: T_H \to T$ to identify T_H with T, and we also use j to identify the Weyl group $\Omega(T_H, H)$ with a subgroup of $\Omega(T, G)$. The restriction of α_0 to T_H is given by some $\omega \in \Omega(T, G)$. Since G_γ is connected and ω fixes γ, we have

$$\omega \in \Omega(T, I)=\Omega(T_H, I_H) \subset \Omega(T_H, H).$$

Choose a representative h_1 for ω^{-1} in $\mathrm{Norm}_H(T_H)$. Then $\mathrm{Int}(h_1) \circ \alpha_0$ fixes T_H pointwise and must therefore be inner. This shows that α is inner (over $\bar{F}$).

10. Elliptic and Fundamental Tori (Review)

In this section F is either p-adic or real, and G is a connected reductive group over F. In the real case a maximal $\mathbb{R}$-torus T of G is said to be *fundamental* if the dimension of its split component is as small as possible. In order to have uniform statements we adopt the same terminology in the p-adic case; in this case fundamental tori are elliptic [Kn; II, p. 271].

10.1. Lemma. *Let T be a maximal F-torus of G. If T transfers to every inner form of G, then $H^1(F, T) \to H^1(F, G)$ is surjective. If G is an adjoint group, then the converse is true.*

This lemma is true for any field F. For adjoint groups G, the lemma is almost a tautology. The first statement for general G follows from the adjoint case and the diagram

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & H^1(Z) & \longrightarrow & H^1(G) & \longrightarrow & H^1(G_{\mathrm{ad}}) & \longrightarrow & H^2(Z) \\
& \Vert & & \uparrow & & \uparrow & & \Vert \\
\ldots \longrightarrow & H^1(Z) & \longrightarrow & H^1(T) & \longrightarrow & H^1(T_{\mathrm{ad}}) & \longrightarrow & H^2(Z).
\end{array}$$

10.2. Lemma. *Let T be a fundamental torus of G. Then $H^1(F, T) \to H^1(F, G)$ is surjective.*

It follows from Lemma 2.8 of [S1] that in the real case T transfers to the fundamental torus of any inner form G, and we are done by the previous lemma. In the p-adic case we use the previous lemma to reduce to the adjoint case. Let C be the kernel of $G_{SC} \to G$; then Kneser [Kn] has shown that $H^1(F, G) \to H^2(F, C)$ is an isomorphism, and thus it suffices to prove that $H^1(F, T) \to H^2(F, C)$ is surjective. This follows from the triviality of $H^2(F, T_{SC})$ (use Tate-Nakayama duality; T_{SC} is anisotropic).

10.3. The following description of the fundamental torus in a quasi-split real group G will be used in 10.4. Let B_0 be a Borel subgroup of G defined over $\mathbb{R}$ and let T_0 be a maximal $\mathbb{R}$-torus of B_0. Let ω_0 be the longest element of the Weyl group Ω of T_0. We have $\omega_0 \in \Omega(\mathbb{R})$, since B_0 is defined over $\mathbb{R}$, and we also have $\omega_0^2 = 1$. Therefore ω_0 can be used to twist T_0. The twisted torus T appears in G since G is quasi-split. It is clear that T has no real roots; therefore T is fundamental.

10.4. Lemma. *Assume that G is a simply connected semi-simple group. Let T be a fundamental torus of G. Then $H^2(F, T)$ is trivial.*

In the p-adic case T is anisotropic and the result follows from Tate-Nakayama duality. Now consider the real case. We claim that T is a product $T_a \times T_i$, where T_a is anisotropic and T_i is of the form $R_{\mathbb{C}/\mathbb{R}}(S)$ for a $\mathbb{C}$-torus S.

In proving the claim we may as well transfer T to the quasi-split form and thus reduce to the quasi-split case. It is enough to find a basis for $X_*(T)$ that is permuted by $-\sigma$, where σ denotes complex conjugation. By 10.3 this is the same as finding a basis for $X_*(T_0)$ that is permuted by $(-\omega_0) \circ \sigma$. The set of simple coroots of T_0 does the job.

It is clear that $H^2(\mathbb{R}, T_a)$ and $H^2(\mathbb{R}, T_i)$ are both trivial. This finishes the proof.

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Determination of the Intertwining Operators for Holomorphically Induced Representations of $SU(p,q)$

Brian D. Boe[1] and Thomas J. Enright[2,]*

[1] Department of Mathematics, University of California, Berkeley, CA 94720, USA

[2] Department of Mathematics, University of California, San Diego, La Jolla, CA 92093, USA

1. Introduction

Let X and Y be two holomorphically induced representations for $SU(p,q)$ with integral highest weights. Then $\mathrm{Hom}(X,Y)$ equals either $\mathbb{C}$ or zero [1,2]. In this article we give formulas for $\mathrm{Hom}(X,Y)$ when the infinitesimal character of X and Y is integral and regular. One formula is given in terms of the highest weight of X and the other in terms of the highest weight of Y.

This article complements the results of [2] which include formulas for the composition factors of the modules X and Y above. With this in mind we shall suppose the reader is familiar with the notation and results of [2] especially sections eight, ten and twelve of that article. For references to other work on the description of $\mathrm{Hom}(X,Y)$ the reader should consult the introduction and bibliography in [1] and [2].

2. The Main Theorem

Let $\mathfrak{g}$ denote the Lie algebra $\mathrm{sl}(n+1,\mathbb{C})$ and let p and q be integers with $n+1 = p+q$, $0<p\leqq q$. Let $\mathfrak{h}$ be a Cartan subalgebra, Δ the set of roots of $(\mathfrak{g},\mathfrak{h})$ and Δ^+ a positive system of roots. Let $\alpha_1,\ldots,\alpha_n$ be the simple roots corresponding to the Dynkin diagram: $\bullet\underset{\alpha_1}{-}\bullet\ \ldots\ \underset{\alpha_n}{-}\bullet$. Let $\mathfrak{p}=\mathfrak{m}\oplus\mathfrak{u}$ denote the standard maximal parabolic subalgebra with complementary simple root α_p. Let all undefined notation be as in [2] (especially section eight).

Fix $x\in\mathscr{W}^{\mathfrak{m}}$ and set $\Psi^+=\Delta(\mathfrak{u})\cap x\Delta^+$ and $\Psi^-=\Delta(\mathfrak{u})\cap -x\Delta^+$. Then $\Delta(\mathfrak{u}) = \Psi^+\cup\Psi^-$. For $\gamma\in\Delta(\mathfrak{u})$ write $\hat{\gamma}=\gamma$ or $-\gamma$ depending as $\gamma\in\Psi^+$ or not. For any set A of orthogonal roots set $A^+=A\cap x\Delta^+$ and $A^-=A\cap -x\Delta^+$. If $A^+=\{\gamma_1,\ldots,\gamma_t\}$ then define $r_A\in\mathscr{W}$ by $r_A=s_{\gamma_1}\ldots s_{\gamma_t}$. For any set $A\subset\Delta(\mathfrak{u})$, let $A\hat{}=\{\hat{\gamma}\mid\gamma\in A\}$. So $A\hat{}\subsetneqq x\Delta^+$. For $y\in\mathscr{W}$, let $\overline{y\varrho}$ denote the unique element in the $\mathscr{W}_{\mathfrak{m}}$-orbit of $y\varrho$ which is $\Delta^+(\mathfrak{m})$-dominant. Define a partial order $\prec_x$ on $x\Delta^+$ by: $x\gamma\prec_x x\zeta$ if and only if

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$\zeta-\gamma$ is a nonnegative integral combination of elements in Δ^+ (i.e., $\gamma \leqq \zeta$ in the usual ordering).

Now associate to $x \in \mathscr{W}^{\mathrm{m}}$ a collection $\mathscr{S}_x$ of distinguished orthogonal subsets of $\Delta(\mathrm{u})$. A subset Ω of $\Delta(\mathrm{u})$ lies in $\mathscr{S}_x$ when it satisfies:

(i) Ω is a set of orthogonal roots.
(ii) Let $\beta \in \Omega$ and $\zeta \in \Delta(\mathrm{u})$ with $\hat{\beta} \neq \hat{\zeta}$ and $\hat{\zeta} \prec_x \hat{\beta}$. Then there exists $\gamma \in \Omega$ with $\langle \gamma, \zeta \rangle \neq 0$, $\hat{\gamma} \neq \hat{\beta}$ and $\hat{\gamma} \prec_x \hat{\beta}$. (2.1)

For $\Omega \in \mathscr{S}_x$, we say Ω is positive (respectively negative) if $\Omega \subset \Psi^+$ (respectively Ψ^-). We call a set Ω in $\mathscr{S}_x$ essential if for any $\beta \in \Omega$ there exists $\gamma \in \Omega \cap \Psi^+$ with $\hat{\beta} \prec_x \gamma$. Let $\mathscr{E}_x$ denote the collection of essential sets in $\mathscr{S}_x$. Note that any positive set in $\mathscr{S}_x$ is essential.

Lemma [2, Proposition 9.7]. *Fix $x \in \mathscr{W}^{\mathrm{m}}$. The mappings $\Omega \mapsto \Omega^+$ and $\Omega \mapsto r_\Omega x \varrho$ are injective when restricted to $\mathscr{E}_x$.*

Theorem 1 [2, Theorem 8.4]. *Set $\lambda = x\varrho$. The composition factors of $N(\lambda)$ are parameterized by $\mathscr{E}_x$. More precisely, $L(\xi)$ is a composition factor of $N(\lambda)$ if and only if $\xi = \overline{r_\Omega \lambda}$ for some Ω in $\mathscr{E}_x$.*

Our main result is the following:

Theorem 2. (i) *Set $\lambda = x\varrho$ and let $\xi \in \mathfrak{h}^*$. Then* $\operatorname{Hom}(N(\xi), N(\lambda)) \simeq \mathbb{C}$ *or zero depending as $\xi = \overline{r_\Omega \lambda}$ for some positive set Ω in $\mathscr{S}_x$ or not.*

(ii) *Set $\xi = y\varrho$ for some $y \in \mathscr{W}^{\mathrm{m}}$ and let $\lambda \in \mathfrak{h}^*$. Then* $\operatorname{Hom}(N(\xi), N(\lambda)) \simeq \mathbb{C}$ *or zero depending as $\lambda = \overline{r_{\hat{\Omega}} \xi}$ for some negative set Ω in $\mathscr{S}_y$ or not.*

Note. Before reading the proof the reader may wish to orient himself by means of the example at the end of this article. The notation introduced and used there is especially convenient in verifying some of the Bruhat relations given in the proof below.

Proof. We prove (i). The equivalence of (i) and (ii) is left as an exercise for the reader. $\operatorname{Hom}(N(\xi), N(\lambda)) = 0$ unless $L(\xi)$ occurs as a composition factor of $N(\lambda)$; and so, by Theorem 1 we may assume $\xi = \overline{r_\Omega \lambda}$ for some set Ω in $\mathscr{E}_x$.

If $x^{-1}\Omega$ is a set of simple roots in Δ^+ then by [1, Lemma 2.1], the standard map of $N(\xi)$ into $N(\lambda)$ is nonzero. So $\operatorname{Hom}(N(\xi), N(\lambda)) \simeq \mathbb{C}$ in this case. Now suppose $x^{-1}\Omega$ is not a set of simple roots and suppose Ω is a positive set. Choose any $\beta \in \Omega$ with $x^{-1}\beta$ not simple. Write $x^{-1}\beta = \alpha_i + \alpha_{i+1} + \ldots + \alpha_k$ and let j be the minimal index, $i<j<k$ with $\gamma = x\alpha_j \in \Omega$. Since Ω is a positive set in $\mathscr{S}_x$, γ exists. Now letting $\prec$ denote the Bruhat order on $\mathscr{W}^{\mathrm{m}}$ with 1 the maximal element we have the following relations: Set $\delta = x\alpha_{j-1}$. Then $s_{s_\gamma \delta} s_\gamma x \prec s_\gamma x \prec x$ and $r_\Omega \delta \in \Delta(\mathrm{m})$. These are precisely the conditions necessary for the reduction of [1, Theorem 2.4]. Using the notation presented there, $\operatorname{Hom}(N(\xi), N(s_\gamma \lambda)) = 0$ and thus

$$\operatorname{Hom}(N(\xi), N(\lambda)) \simeq \operatorname{Hom}(N(\xi), \varphi_\alpha \psi_\alpha N(\lambda))$$

where $\alpha = \alpha_j$. By the adjointness of φ_α and ψ_α,

$$\operatorname{Hom}(N(\xi), N(\lambda)) \simeq \operatorname{Hom}(\psi_\alpha N(\xi), \psi_\alpha N(\lambda)).$$

Let notation be as in the proof of [2, Proposition 11.2]. Then with $'$ denoting the corresponding object when $\mathfrak{g}$ is replaced by $\mathfrak{g}' \simeq \mathrm{sl}(n-1, \mathbb{C})$,

$$\mathrm{Hom}(N(\xi), N(\lambda)) \simeq \mathrm{Hom}(\psi_\alpha N(\xi), \psi_\alpha N(\lambda)) \simeq \mathrm{Hom}(N'(\xi'), N'(\lambda')).$$

This second isomorphism comes from the equivalence of categories in [2, Proposition 11.2]. Now $\xi' = \overline{r_{\Omega'}\lambda'}$ with Ω' a positive set in $\mathscr{E}'_{x'}$. Then the induction hypothesis given $\mathrm{Hom}(N'(\xi'), N'(\lambda')) \simeq \mathbb{C}$. This proves (i) if Ω is a positive set.

Now suppose $\Omega \in \mathscr{E}_x$ but Ω is not positive. Since Ω is essential choose $\zeta \in \Omega^+$ with $x^{-1}\zeta$ not a simple root. Write $x^{-1}\zeta = \alpha_i + \alpha_{i+1} + \ldots + \alpha_k$ and choose j minimal, $i<j<k$ with $\pm x\alpha_j \in \Omega$. Now by replacing ζ be a smaller positive root if necessary we may assume: if $\alpha_j \lneqq \beta \lneqq x^{-1}\zeta$ and $\pm x\beta \in \Omega$ then $-x\beta \in \Omega$. We now treat two cases separately.

Case I. $\gamma = x\alpha_j \in \Omega$. Let $\delta = x\alpha_i$ then we have the relations: $s_{s_\gamma\delta}s_\gamma x \prec s_\gamma x \prec x$ and $r_\Omega\delta \in \Delta(\mathfrak{m})$. These are precisely the relations applied in the first half of the argument. As we obtained there: $\mathrm{Hom}(N(\xi), N(s_\gamma\lambda)) = 0$ and so

$$\mathrm{Hom}(N(\xi), N(\lambda)) \simeq \mathrm{Hom}(N'(\xi'), N'(\lambda'))$$

where $\xi' = \overline{r_{\Omega'}\lambda'}$. Since γ is positive and Ω is not positive, Ω' is not positive. Thus by the induction hypothesis $\mathrm{Hom}(N'(\xi'), N'(\lambda')) = 0$. This completes the proof in Case I.

Case II. $-\gamma = -x\alpha_j \in \Omega$. Let $\beta = x\alpha_i$. Then the hypotheses of [1, Lemma 2.3] hold; i.e., $s_\beta x \prec x$ and $r_\Omega\beta \in \Delta(\mathfrak{m})$. So $\mathrm{Hom}(N(\xi), N(\lambda)) = 0$. This completes the proof.

3. An Example

We conclude with an example to illustrate a convenient language to calculate the collections of sets $\mathscr{S}_x$ and $\mathscr{E}_x$ involved in Theorems 1 and 2. We begin with the Lascoux and Schützenberger [3] parameterization of $\mathscr{W}^{\mathfrak{m}}$ in terms of monomials in two variables. Represent ϱ by the $n+1$-tuple $(n+1, n, \ldots, 1)$ with $n+1 = p+q$, $0 < p \leqq q$. The elements in $\mathscr{W}^{\mathfrak{m}} \cdot \varrho$ are $n+1$-tuples $\underset{\sim}{a} = (a_{n+1}, \ldots, a_1)$ which are permutations of the coordinates of ϱ and such that $a_{n+1} > a_n > \ldots > a_{q+1}$ and $a_q > \ldots > a_1$. Associate to $\underset{\sim}{a}$ a monomial $x_{n+1}x_n \ldots x_1$ defined by the conditions: $x_i = \alpha$ if $a_j = i$ for some $j > q$ and $x_i = \beta$ otherwise. Then ϱ corresponds to the monomial $\alpha\alpha \ldots \alpha\, \beta\beta \ldots \beta$ with p α's and q β's. The elements of $\mathscr{W}^{\mathfrak{m}} \cdot \varrho$ correspond to all monomials in α and β where α occurs p-times and β occurs q-times.

Let $\gamma \in \Delta^+$ and write $\gamma = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$, $i \leqq j$. Let $x \in \mathscr{W}^{\mathfrak{m}}$ and let $m = x_{n+1} \ldots x_1$ be the monomial associated to $x\varrho$ as above. Put $a = n+2-i$ and $b = n+1-j$. Then $x\gamma \in \Delta(\mathfrak{u})$ [respectively $\Delta(\bar{\mathfrak{u}}), \Delta(\mathfrak{m})$] precisely when $x_a = \alpha$ and $x_b = \beta$ (respectively $x_a = \beta$ and $x_b = \alpha$, either $x_a = x_b = \alpha$ or $x_a = x_b = \beta$).

For our example we set $p = q = 4$ and let $\lambda = x\varrho$ correspond to the monomial $m = \alpha\alpha\beta\alpha\alpha\beta\beta\beta$. Let $\mathscr{P}_x$ denote the positive sets in $\mathscr{S}_x$. Then $x^{-1}\mathscr{P}_x$ is comprised of the following seven sets: ϕ, $\{\alpha_2\}$, $\{\alpha_5\}$, $\{\alpha_2, \alpha_5\}$, $\{\alpha_5, \alpha_4+\alpha_5+\alpha_6\}$, $\{\alpha_2, \alpha_5, \alpha_4+\alpha_5+\alpha_6\}$,

$$\{\alpha_2, \alpha_5, \alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7\}.$$

The collection of essential sets $\mathscr{E}_x$ can be described by:

$$x^{-1}\mathscr{E}_x = x^{-1}\mathscr{P}_x \cup \{\{-\alpha_3, \alpha_5, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6\},$$
$$\{-\alpha_3, \alpha_5, \alpha_2+\alpha_3+\ldots+\alpha_6, \alpha_1+\alpha_2+\ldots+\alpha_7\}\}.$$

By Theorems 1 and 2 the composition factors of $N(\lambda)$ are given by $\mathscr{E}_x$. In this example the composition factors $L(\xi)$ with $\xi = y\varrho$ are given by the nine monomials corresponding to the nine sets above: $\alpha\alpha\beta\alpha\alpha\beta\beta\beta$, $\alpha\beta\alpha\alpha\alpha\beta\beta\beta$, $\alpha\alpha\beta\alpha\beta\alpha\beta\beta$, $\alpha\beta\alpha\alpha\beta\alpha\beta\beta$, $\alpha\alpha\beta\beta\beta\alpha\alpha\beta$, $\alpha\beta\alpha\beta\beta\alpha\alpha\beta$, $\beta\beta\alpha\beta\beta\alpha\alpha\alpha$, $\alpha\beta\beta\alpha\beta\alpha\alpha\beta$, $\beta\beta\beta\alpha\beta\alpha\alpha\alpha$. From this list only the last two composition factors do *not* correspond to nonzero homomorphisms $N(\xi) \to N(\lambda)$.

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Fibred Knots and Disks with Clasps

C. McA. Gordon[1,*] and José María Montesinos[2,**]

[1] Department of Mathematics, University of Texas, Austin, TX 78712, USA
[2] Facultad de Ciencias, Universidad de Zaragoza, E-50009 Zaragoza, Spain

A null-homotopic knot K in a 3-manifold M bounds a singular disk Δ whose only singularities are clasps. Taking a regular neighborhood of Δ shows that K is contractible in a handlebody in M. Also, every closed, orientable 3-manifold contains a fibred knot [G, My1]. The present paper contains some remarks motivated by these facts.

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We work in the PL category, and all our manifolds are orientable. A closed surface S in a 3-manifold M is *compressible* in M if either there exists a *compressing disk* for S in M (that is, a disk D such that $D \cap S = \partial D$ and ∂D is essential in S), or S has a 2-sphere component which bounds a 3-ball in M.

Lemma 1. *Let K be a fibred knot in a rational homology 3-sphere M. Let X be a compact submanifold of M which contains K and in which K is null-homologous. Then each component of ∂X is compressible in $M-K$.*

Proof. By hypothesis, $M-K$ fibres over S^1 with fibre $\operatorname{int} F$, say, where F is a surface in M with $\partial F = K$. Since K is null-homologous in X, there is a surface G in X with $\partial G = K$. Let x be any 1-cycle in ∂X. The intersection numbers $x \cdot G$ and $x \cdot (F \cup G)$ are zero – the first since $x \cap G = \emptyset$ and the second since $F \cup G$ is a 2-cycle in M and $H_2(M)=0$. Therefore $x \cdot F = 0$. It follows that the inclusion of ∂X in $M-K$ lifts to an embedding j of ∂X into the infinite cyclic covering $\operatorname{int} F \times R^1$ of $M-K$. Let S be a component of ∂X. If S is a 2-sphere, then, since $\operatorname{int} F \times R^1$ is irreducible, $j(S)$ bounds a 3-ball in $\operatorname{int} F \times R^1$, and hence S bounds a 3-ball in $M-K$. If S has positive genus, then $j_* : \pi_1(S) \to \pi_1(\operatorname{int} F \times R^1)$ is not injective, since $\pi_1(\operatorname{int} F \times R^1)$ is free. It follows that the map $\pi_1(S) \to \pi_1(M-K)$ induced by inclusion is not injective, and hence that S is compressible in $M-K$. □

Remark. The same argument shows that $M - \operatorname{int} X$ embeds in $\operatorname{int} F \times R^1$, which in turn embeds in S^3. If X is connected, then, since M is a rational homology sphere,

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each component of $M - \mathrm{int}X$ has connected boundary and is therefore a cube-with-knotted-holes [F].

Corollary 1.1. *If K is a fibred knot in a rational homology 3-sphere M which bounds a disk-with-clasps Δ in M, then the boundary of a regular neighborhood $N(\Delta)$ is compressible in $M - \mathrm{int}N(\Delta)$.*

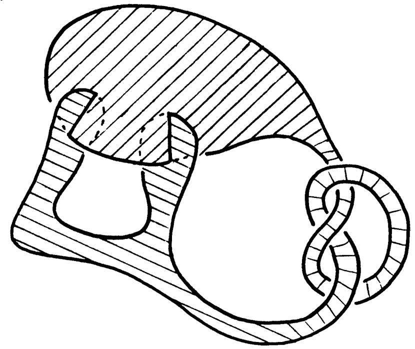
Proof. Let $p: \tilde{N}(\Delta) \to N(\Delta)$ be the universal covering of $N(\Delta)$, and consider the link $p^{-1}(K)$ in $\tilde{N}(\Delta)$. The components of $p^{-1}(K)$ correspond to the vertices of an infinite tree in such a way that the two components corresponding to the end points of an edge are homologically linked. From this it is easy to show that $\partial\tilde{N}(\Delta)$ is incompressible in $\tilde{N}(\Delta) - p^{-1}(K)$, which implies that $\partial N(\Delta)$ is incompressible in $N(\Delta) - K$. Since $\partial N(\Delta)$ is compressible in $M - K$ by Lemma 1, it must be compressible in $M - \mathrm{int}N(\Delta)$. □

Corollary 1.2. *If K is a fibred knot in a homology 3-sphere which is also a doubled knot (that is, K bounds a disk with one clasp), then K is a double of the trivial knot in S^3.*

Proof. Let Δ be the disk with one clasp bounded by K. Since the torus $\partial N(\Delta)$ is compressible in $M - \mathrm{int}N(\Delta)$ and $M - K$ is irreducible, $M - \mathrm{int}N(\Delta)$ is a solid torus. □

Remarks. (1) Corollary 1.2 is valid if the operation of doubling is replaced by taking any (non-trivial) satellite with zero winding number.

(2) Under the hypothesis of Corollary 1.1, it is not in general true without restriction on Δ that $M - \mathrm{int}N(\Delta)$ is a handlebody, as is shown by the following example of A. Casson.



However, we ask the following question

Question. *If K is a fibred knot in a rational homology 3-sphere M which bounds a disk-with-clasps Δ in M with the minimal number of clasps, is $M - \mathrm{int}N(\Delta)$ a handlebody?*

The *Heegaard genus* $g(X)$ of a compact 3-manifold X is the minimal number of 1-handles in any handle decomposition of X.

Lemma 2. *Let K be a fibred knot in a rational homology 3-sphere M. Let X be a compact submanifold of M which contains K and in which K is null-homologous. Then $g(M) \leqq g(X)$.*

Remarks. (1) It will follow from the proof that in fact $g(M)<g(X)$ unless $M-\operatorname{int}X$ is a disjoint union of handlebodies.

(2) The lemma can be generalized to say that if $X\cong \mathop{\#}\limits_{i=1}^{n} X_i$, then $M\cong \mathop{\#}\limits_{i=1}^{n} M_i$ where $g(M_i)\leqq g(X_i)$, $i=1,\dots,n$. We omit the details.

Proof. We prove the lemma by induction on the complexity $c(\partial X)$, which we define to be the sum of the squares of the genera of the components of ∂X.

Since any 2-sphere in ∂X bounds a 3-ball in $M-K$, we may suppose that ∂X has no 2-sphere components.

If $c(\partial X)=0$, then $\partial X=\emptyset$ and so $X=M$.

Suppose $c(\partial X)>0$. By Lemma 1, ∂X is compressible in $M-K$. We distinguish two cases.

(1) *∂X is compressible in $M-\operatorname{int}X$.* Let D be a compressing disk for ∂X in $M-\operatorname{int}X$, and let $X'=N(X\cup D)$. Then $g(X')\leqq g(X)$ and $c(\partial X')<c(\partial X)$, so we are done by induction.

(2) *∂X is compressible in $X-K$.* Let D be a compressing disk for ∂X in $X-K$. Note that K is null-homologous in $X-D$ (for if G is a surface in X bounded by K, then cutting G along $G\cap D$ and capping off the resulting boundary components with parallel copies of subdisks of D gives a surface in $X-D$ bounded by K). If D separates X, then it decomposes X as a non-trivial boundary connected sum $X_1 \#_\partial X_2$. If D does not separate X, then, since $H_2(X)=0$, ∂D does not separate ∂X, and we have $X\cong X_1 \#_\partial S^1\times D^2$. Hence in either case, $X\cong X_1 \#_\partial X_2$ with K null-homologous in (say) X_1. Haken's theorem [H] that Heegaard genus is additive with respect to connected sum extends to give additivity with respect to boundary connected sum [CG], so $g(X_1)<g(X)$. Since $c(\partial X_1)<c(\partial X)$, we are again done by induction. □

Corollary 2.1. *Let M be a homotopy 3-sphere containing a fibred knot which is null-homologous in a handlebody of genus 2. Then M is homeomorphic to S^3.*

Proof. By Lemma 2, $g(M)\leqq 2$, and therefore M is homeomorphic to S^3 by the Generalized Smith Conjecture [MB]. □

Remark. Since every closed 3-manifold contains a fibred knot [G], [My1], Corollary 2.1 implies that if every knot in a closed 3-manifold is contractible in a handlebody of genus 2, then M is homeomorphic to S^3. This generalizes Bing's result [B] that the same conclusion holds if every knot in M is contained in a 3-ball. (See also [My1].) Using Remark (2) after the statement of Lemma 2, the handlebody of genus 2 (both here and in Corollary 2.1) can be replaced by a punctured connected sum of handlebodies of genus 2. This in turn generalizes McMillan's result [Mc] that M is S^3 if every knot in M is contractible in a punctured connected sum of solid tori. (See also [My2], which shows that M is S^3 if every knot in M is contractible in a punctured connected sum of graphmanifolds.)

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Multi-Dimensional Spectral Theory of Bounded Linear Operators in Locally Convex Spaces

Volker Wrobel

Mathematisches Seminar der Universität, Olshausenstrasse 40–60, D-2300 Kiel 1, Federal Republic of Germany

1. Introduction

The basic concept for a multi-dimensional spectral theory is a suitable notion of joint spectrum. Therefore in Sect. 2 we start with a discussion of several concepts which are well-known in the case of Banach algebras (compare Zelasko [12] for a survey and an axiomatic approach). We mention the left and the right spectrum, the commutant and the bicommutant spectrum as well as the joint spectrum of Taylor [7]. In Sect. 3 we introduce the notion of related n-tuples of linear operators, which in the case $n=1$ has turned out to be useful in spectral theory (Pietsch [5], Wrobel [9, 10]). It turns out that the five spectra mentioned above are (almost) invariant under related n-tuples of linear operators. In specializing to n-tuples of commuting bounded linear operators on a Mackey-complete locally convex space in Sect. 4, we obtain that the five joint spectra have the same properties as they have in the Banach space setting. Especially the union of the left and the right spectrum, the commutant spectrum, the bicommutant spectrum and the joint spectrum of J. L. Taylor are non-empty, compact subsets of $\mathbb{C}^n$ for an n-tuple of pairwise commuting, bounded linear operators on a Mackey-complete locally convex space. Moreover, the concept of related n-tuples of linear operators allows one to establish Taylor's functional calculus [8] in the situation under consideration without any locally convex arguments.

2. Basic Concepts of Joint Spectra

Let A denote an algebra over the complex numbers $\mathbb{C}$ with unit element e. If $B \subset A$ is a subset of A and $a \in A$, we let

$$\mathrm{Sp}(a; B) := \{z \in \mathbb{C}\colon\ ze - a \textit{ has no inverse in } B\}$$

denote the *spectrum* of a with respect to B. Such a notion is useful in the setting of a locally convex algebra A, where B may be the set of (spectrally)-bounded elements of A (Allan [1], Neubauer [4]) in order to guarantee regularity properties of the resolvent $z \mapsto (ze-a)^{-1}$ outside this kind of spectrum. If $\mathbf{a} = (a_1, \ldots, a_n)$ is an

n-tuple of elements from A, we consider as a *joint spectrum* of $\mathbf{a}$ with respect to B the set

$$\mathrm{Sp}(\mathbf{a}; B) := \mathrm{Sp}_{\mathrm{left}}(\mathbf{a}; B) \cup \mathrm{Sp}_{\mathrm{right}}(\mathbf{a}; B),$$

where

$$\mathrm{Sp}_{\mathrm{left}}(\mathbf{a}; B) := \left\{ \mathbf{z} \in \mathbb{C}^n : e \notin \sum_{j=1}^{n} B(a_j - z_j e) \right\}$$

and

$$\mathrm{Sp}_{\mathrm{right}}(\mathbf{a}; B) := \left\{ \mathbf{z} \in \mathbb{C}^n : e \notin \sum_{j=1}^{n} (a_j - z_j e) B \right\}$$

denote the *left* and the *right spectrum*, respectively. These notions have been introduced by Bonsall and Duncan [2] for $B = A$ in the Banach algebra setting. Throughout this paper we shall restrict our attention to the algebra $A = \mathbf{L}(X)$ of all continuous linear operators on a locally convex space X, though some considerations will be of a purely algebraic nature. For a given n-tuple $(a_1, \ldots, a_n)$ of pairwise commuting operators from $\mathbf{L}(X)$, the set B will either be the *commutant*

$$\{\mathbf{a}\}^c := \{ b \in L(X) : ba_j = a_j b \ \textit{for}\ 1 \leqq j \leqq n \},$$

the *bicommutant*

$$\{\mathbf{a}\}^{cc} :: = \{ b \in \mathbf{L}(X) : bc = cb \ \textit{for all}\ c \in \{\mathbf{a}\}^c \}$$

or a suitable subset of them. In this situation, we have

$$\mathrm{Sp}(a; B) = \mathrm{Sp}_{\mathrm{left}}(a; B) = \mathrm{Sp}_{\mathrm{right}}(a; B).$$

Of course, these spectra heavily depend on the subset B and they rather reflect algebraic properties of a in B than spatial or mapping properties as one should expect from an n-tuple of linear operators. So the question arises, whether these joint spectra are natural choices at all for operators. In the case of Banach spaces, Taylor [7] however gave another notion of a joint spectrum, which is a subset of the commutant spectrum and gives rise to an analytic functional calculus, which in general admits more functions than the classical analytic functional calculus in commutative Banach algebras.

We recall Taylors concept for an n-tuple of pairwise commuting operators from $\mathbf{L}(X)$, where X is a locally convex space.

Let

$$\wedge [s_1, \ldots, s_n] := \bigoplus_{p=0}^{\infty} \wedge^p [s_1, \ldots, s_n]$$

denote the *exterior algebra* over $\mathbb{C}$ generated by the indeterminates $s_1, \ldots, s_n$ and $\wedge^p[s_1, \ldots, s_n]$ the space of elements $z \cdot s_{j_1} \wedge s_{j_2} \wedge \ldots \wedge s_{j_p}$ of degree p, where $\wedge$ denotes exterior multiplication. Given a locally convex space X, we let $\wedge[s_1, \ldots, s_n; X]$ and $\wedge^p[s_1, \ldots, s_n; X]$ denote the tensor products $X \otimes \wedge[s_1, \ldots, s_n]$ and $X \otimes \wedge^p[s_1, \ldots, s_n]$, respectively. If $\mathbf{a} = (a_1, \ldots, a_n)$ is an n-tuple of pairwise commuting operators $a_i \in \mathbf{L}(X)$, we consider $\tilde{\mathbf{a}} := a_1 s_1 + \ldots + a_n s_n$

acting as a graded module homomorphism of degree 1:

$$\tilde{\mathbf{a}}: \wedge^p[s_1, \ldots, s_n; X] \to \wedge^{p+1}[s_1, \ldots, s_n; X] \quad \text{for each } p,$$

$$\tilde{\mathbf{a}}(x_{j_1 \ldots j_p} \otimes s_{j_1} \wedge \ldots \wedge s_{j_p}) := \sum_{k=1}^{n} a_k(x_{j_1 \ldots j_p}) \otimes s_k \wedge s_{j_1} \wedge \ldots \wedge s_{j_p}.$$

As $a_i a_j = a_j a_i$ by assumption, we have $\tilde{\mathbf{a}} \wedge \tilde{\mathbf{a}} := \sum_{i<j} (a_i a_j - a_j a_i) s_i \wedge s_j = 0$. So $\tilde{\mathbf{a}}$ acts as a coboundary operator on $\wedge[s_1, \ldots, s_n; X]$. We say that the n-tuple $\mathbf{a}$ is *non-singular (singular)*, if the associated sequence $\mathbf{F}(X, \mathbf{a})$:

$$0 \to X \cong \wedge^0[s_1, \ldots, s_n; X] \xrightarrow{\tilde{\mathbf{a}}} \wedge^1[s_1, \ldots, s_n; X] \xrightarrow{\tilde{\mathbf{a}}} \ldots$$

$$\ldots \to \wedge^{n-1}[s_1, \ldots, s_n; X] \xrightarrow{\tilde{\mathbf{a}}} \wedge^n[s_1, \ldots, s_n; X] \cong X \to 0$$

is exact (not exact).

2.1. *Definition.* Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of pairwise commuting operators $a_i \in \mathbf{L}(X)$ and $\tilde{\mathbf{a}} = a_1 s_1 + \ldots + a_n s_n$. An element $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ belongs to the *Taylor spectrum* $\sigma(\mathbf{a}; X)$, if $\mathbf{z} - \mathbf{a} := (z_1 e - a_1, \ldots, z_n e - a_n)$ is singular.

We recall that

$$\sigma(\mathbf{a}; X) \subseteqq \mathrm{Sp}(\mathbf{a}; \{\mathbf{a}\}^c) \subseteqq \mathrm{Sp}(\mathbf{a}; \{\mathbf{a}\}^{cc}) \subseteqq \prod_{i=1}^{n} \mathrm{Sp}(a_i; \mathbf{L}(X))$$

and each of these inclusions is strict for $n > 1$ in general (see Taylor [7]).

For a moment consider the case of a single operator $\mathbf{a} = a_1$. We have $z \notin \sigma(a_1; X)$ if and only if $ze - a_1$ is one-to-one and onto, but $ze - a_1$ need not be open. So in general $\sigma(a_1; X)$ will be a proper subset of $\mathrm{Sp}(a_1; \{a_1\}^c) = \mathrm{Sp}(a_1; \mathbf{L}(X))$, whereas we have $\sigma(a_1; X) = \mathrm{Sp}(a_1; \mathbf{L}(X))$ if X is a Fréchet space (or more generally a barrelled, B_r-complete locally convex space [6]). Finally we mention that even for a nice operator a_1 on a Fréchet space X we may have $\mathrm{Sp}(a_1; \mathbf{L}(X)) = \emptyset$, so that in a general locally convex setting one has to modify Definition 2.1.

3. Systems of Related Linear Operators

Let X and Y denote topological vector spaces, and let $\mathbf{L}(X, Y)$ denote the space of all continuous linear operators from X to Y. The following notion will turn out to be useful in the remainder of the paper.

3.1. *Definition.* Let $q \in \mathbf{L}(Y, X)$, $p_i \in \mathbf{L}(X, Y)$, $a_i := q p_i \in \mathbf{L}(X)$, and $b_i := p_i q$ $(1 \leqq i \leqq n)$. The systems $\mathbf{a} := (a_1, \ldots, a_n)$ and $\mathbf{b} := (b_1, \ldots, b_n)$ are said to be *related*.

For $n = 1$ this notion is due to Pietsch [5].

Related systems of pairwise commuting operators have in common almost all spectral properties as we shall show. We start with

3.2. Lemma. *Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$ *denote related systems of operators in* $\mathbf{L}(X)$ *and* $\mathbf{L}(Y)$, *respectively. Moreover, assume that* $a_i a_j = a_j a_i$ *and* $b_i b_j = b_j b_i$ $(1 \leqq i, j \leqq n)$. *Then we have*

(1) $$\mathrm{Sp}_{\mathrm{left}}(\mathbf{a}; \mathbf{L}(X)) \cup \{0\} = \mathrm{Sp}_{\mathrm{left}}(\mathbf{b}; \mathbf{L}(Y)) \cup \{0\},$$

$$\mathrm{Sp}_{\mathrm{right}}(\mathbf{a}; \mathbf{L}(X)) \cup \{0\} = \mathrm{Sp}_{\mathrm{right}}(\mathbf{b}; \mathbf{L}(Y)) \cup \{0\},$$

and consequently

$$\mathrm{Sp}(\mathbf{a};\, \mathbf{L}(X)) \cup \{0\} = \mathrm{Sp}(\mathbf{b};\, \mathbf{L}(Y)) \cup \{0\}.$$

(2) *In the notation of 3.1 assume that q is one-to-one. Then*

$$\mathrm{Sp}(\mathbf{a};\, \{\mathbf{a}\}^c) \cup \{0\} = \mathrm{Sp}(\mathbf{b};\, \{\mathbf{b}\}^c) \cup \{0\},$$

and

$$\mathrm{Sp}(\mathbf{a};\, \{\mathbf{a}\}^{cc}) \cup \{0\} = \mathrm{Sp}(\mathbf{b};\, \{\mathbf{b}\}^{cc} \cup \{0\}.$$

Proof. In order to avoid a mix-up, we denote the unit elements of $\mathbf{L}(X)$ and $\mathbf{L}(Y)$ by I_X and I_Y, respectively. Let $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \mathrm{Sp}_{\mathrm{left}}(\mathbf{a};\, \mathbf{L}(X))$ and $d_1, \ldots, d_n \in \mathbf{L}(X)$ such that

$$(+) \qquad \sum_{i=1}^{n} d_i(a_i - z_i I_X) = I_X.$$

In order to prove (1), we may assume that $z_n \neq 0$. Then

$$\sum_{i=1}^{n-1} p_n d_i q(p_i q - z_i I_Y) + (p_n d_n q - I_Y)(p_n q - z_n I_Y) = z_n I_Y$$

and consequently $\mathbf{z} \in \mathbb{C}^n \setminus \mathrm{Sp}_{\mathrm{left}}(\mathbf{b};\, \mathbf{L}(Y))$. So let $\mathbf{z} \in \mathbb{C}^n \setminus \mathrm{Sp}_{\mathrm{left}}(\mathbf{b};\, \mathbf{L}(Y))$ such that $z_n \neq 0$ and $\check{d}_1, \ldots, \check{d}_n \in \mathbf{L}(Y)$ such that

$$(++) \qquad \sum_{i=1}^{n} \check{d}_i(b_i - z_i I_Y) = I_Y.$$

Then

$$\sum_{i=1}^{n-1} q\check{d}_i p_n(q p_i - z_i I_X) + (q\check{d}_n p_n - I_X)(q p_n - z_n I_X)$$
$$= z_n I_X + \sum_{i=1}^{n-1} q\check{d}_i(p_n q p_i - p_i q p_n).$$

But since $p_n q p_i q = p_i q p_n q$, we have $\left(\sum_{i=1}^{n-1} q\check{d}_i(p_n q p_i - p_i q p_n)\right)^2 = 0$ and thus the right-hand side is invertible and therefore $\mathbf{z} \in \mathbb{C}^n \setminus \mathrm{Sp}(\mathbf{a};\, \mathbf{L}(X))$. The second part of (1) is done by the same procedure.

In order to prove (2) let us first assume that $(+)$ is true for $d_1, \ldots, d_n \in \{a_1, \ldots, a_n\}^c$. For $1 \leqq i \leqq n-1$ define $\check{d}_i := z_n^{-1} p_n d_i q$ and $\check{d}_n := z_n^{-1}(p_n d_n q - I_Y)$. Then $(++)$ is fulfilled as we have seen in the first part of the proof. By the injectivity of q, we have $p_j q p_i = p_i q p_j$ since $a_j a_i = a_i a_j$. Consequently, $p_n d_i q b_j = p_n d_i q p_j q = p_n q p_j d_i q = p_j q p_n d_i q = b_j \check{d}_i z_n$ gives $\check{d}_1, \ldots, \check{d}_n \in \{b_1, \ldots, b_n\}^c$, i.e. $\mathbf{z} \in \mathbb{C}^n \setminus \mathrm{Sp}(\mathbf{b};\, \{\mathbf{b}\}^c)$.

Next suppose $(+)$ is true for $d_1, \ldots, d_n \in \{a_1, \ldots, a_n\}^{cc}$.

Let $c \in \{b_1, \ldots, b_n\}^c$. Then $qcp_n \in \{a_1, \ldots, a_n\}^c$ and thus $d_j qcp_n = qcp_n d_j$ and hence $qc\check{d}_j = qcp_n d_j q z_n^{-1} = d_j qcp_n q z_n^{-1} = d_j q p_n q c z_n^{-1} = q p_n d_j q c z_n^{-1} = q\check{d}_j c$ for $1 \leqq j \leqq n-1$ and similarly for $\check{d}_n$ as defined above. Hence $\check{d}_1, \ldots, \check{d}_n \in \{b_1, \ldots, b_n\}^{cc}$ by the injectivity of q. Consequently $\mathbf{z} \in \mathbb{C}^n \setminus \mathrm{Sp}(\mathbf{b};\, \{\mathbf{b}\}^{cc})$.

Next suppose $(++)$ is true with $\check{d}_1, \dots, \check{d}_n \in \{b_1, \dots, b_n\}^c$. Since $q\check{d}_i p_n a_j = q\check{d}_i p_n q p_j = q\check{d}_i p_j q p_n = q p_j q \check{d}_i p_n = a_j q \check{d}_i p_n$ by the injectivity of q, $(+)$ can be fulfilled by the first step of the proof with $d_1, \dots, d_n$ chosen in $\{a_1, \dots, a_n\}^c$. The last inclusion needs a little bit more care. Again assume that $(++)$ is fulfilled with $\check{d}_1, \dots, \check{d}_n \in \{b_1, \dots, b_n\}^{cc}$. Define $d_i := z_n^{-1} q\check{d}_i p_n$ for $1 \leqq i \leqq n-1$ and $d_n := z_n^{-1}(q\check{d}_n p_n - I_X)$. Then $(+)$ is fulfilled, but in general $d_i \in \{a_1, \dots, a_n\}^c \backslash \{a_1, \dots, a_n\}^{cc}$. So we repeat this procedure letting $\check{\check{d}}_i := z_n^{-1} p_n d_i q$ for $1 \leqq i \leqq n-1$ and $\check{\check{d}}_n := z_n^{-1}(p_n d_n q - I_Y)$ and finish with $\tilde{d}_i := z_n^{-1} q\check{\check{d}}_i p_n$ for $1 \leqq i \leqq n-1$ and $\tilde{d}_n := z_n^{-1}(q\check{\check{d}}_n p_n - I_X)$. Now each $\tilde{d}_i$ belongs to $\{a_1, \dots, a_n\}^{cc}$ and $(+)$ is fulfilled with $\tilde{d}_i$ instead of d_i.

Next we consider Taylor's joint spectrum.

3.3. Lemma. *Let* $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{L}(X)^n$ *and* $\mathbf{b} = (b_1, \dots, b_n) \in \mathbf{L}(Y)^n$ *fulfill the assumptions of 3.2. Then*

$$\sigma(\mathbf{a}; X) \cup \{0\} = \sigma(\mathbf{b}; Y) \cup \{0\}.$$

Proof. For $\mathbf{z} \in \mathbb{C}^n$ let $\tilde{\mathbf{a}} - \mathbf{z} := \sum_{i=1}^{n} (a_i - z_i I_X) s_i$ and $\tilde{\mathbf{b}} - \mathbf{z} := \sum_{i=1}^{n} (b_i - z_i I_Y) s_i$. If $a_i = q p_i$ and $b_i = p_i q$ (see 3.1), then

$$q(\tilde{\mathbf{b}} - \mathbf{z}) := \sum_{i=1}^{n} q(b_i - z_i I_Y) s_i = \sum_{i=1}^{n} (a_i - z_i I_X) q s_i =: (\tilde{\mathbf{a}} - \mathbf{z}) q.$$

Suppose $\mathbf{z} \in \mathbb{C}^n \backslash \sigma(\mathbf{a}; X)$ such that $z_n \neq 0$ and let $\xi \in \Lambda^p[s_1, \dots, s_n; Y]$ such that $0 = (\tilde{\mathbf{b}} - \mathbf{z})\xi$. Then we have $0 = q(\tilde{\mathbf{b}} - \mathbf{z})\xi = (\tilde{\mathbf{a}} - \mathbf{z}) q\xi$. By the exactness of $\mathbf{F}(X, \mathbf{a})$ we find $\eta \in \Lambda^{p-1}[s_1, \dots, s_n; X]$ such that $q\xi = (\tilde{\mathbf{a}} - \mathbf{z})\eta$. Consequently, $p_n q \xi = p_n(\tilde{\mathbf{a}} - \mathbf{z})\eta = \sum_{i=1}^{n} p_n(q p_i - z_i I_X) s_i \wedge \eta$. Since $a_i a_j = a_j a_i$ by assumption, we have $(p_n q p_i - p_i q p_n) X \subseteqq \operatorname{kern} q$. Now let $\eta = \sum x_{j_1 \dots j_{p-1}} \otimes s_{j_1} \wedge \dots \wedge s_{j_{p-1}}$ be a representation of η, where summation is taken over all multi-indices of length $p-1$ such that $1 \leqq j_1 < j_2 < \dots < j_{p-1} \leqq n$. Then

$$p_n q p_i x_{j_1 \dots j_{p-1}} = p_i q p_n x_{j_1 \dots j_{p-1}} + y^{(i)}_{j_1 \dots j_{p-1}},$$

where $y^{(i)}_{j_1 \dots j_{p-1}} \in \operatorname{kern} q$. Let

$$\xi_0 := \sum \sum y^{(i)}_{j_1 \dots j_{p-1}} s_i \wedge s_{j_1} \wedge \dots \wedge s_{j_{p-1}} \in \Lambda^p[s_1, \dots, s_n; Y],$$

where summation is taken over $1 \leqq i \leqq n$ and the same multi-indices as above. It is obvious that there is a decomposition

$$\Lambda^p[s_1, \dots, s_n; Y] = \Lambda^p[s_1, \dots, s_{n-1}; Y] \oplus s_n \wedge \Lambda^{p-1}[s_1, \dots, s_{n-1}; Y].$$

Given $\gamma \in \Lambda^p[s_1, \dots, s_n; Y]$ let γ' and $s_n \wedge \gamma''$ the uniquely determined direct summands in $\Lambda^p[s_1, \dots, s_{n-1}; Y]$ and $s_n \wedge \Lambda^{p-1}[s_1, \dots, s_{n-1}; Y]$, respectively. By the remarks above, we have

$$p_n q \xi = (\tilde{\mathbf{b}} - \mathbf{z}) p_n \eta + \xi_0,$$

and addition of $z_n \xi - p_n q \xi$ yields

$$z_n \xi = (\tilde{\mathbf{b}} - \mathbf{z}) p_n \eta + (z_n I_Y - b_n)\xi + \xi_0.$$

We are going to prove that the right-hand side is of the form $(\tilde{\mathbf{b}}-\mathbf{z})\gamma$. In order to do so decompose $\xi=\xi'\oplus s_n\wedge\xi''$. Since $(\tilde{\mathbf{b}}-\mathbf{z})\xi=0$ by assumption, we get

$$\begin{aligned} 0&=(b_n-z_nI_Y)s_n\wedge(\xi'+s_n\wedge\xi'')+\sum_{i=1}^{n-1}(b_i-z_iI_Y)s_i\wedge(\xi'+s_n\wedge\xi'')\\ &=(b_n-z_nI_Y)s_n\wedge\xi'+\sum_{i=1}^{n-1}(b_i-z_iI_Y)s_i\wedge s_n\wedge\xi'' \end{aligned}$$

and hence

$$(b_n-z_nI_Y)\xi'=\sum_{i=1}^{n-1}(b_i-z_iI_Y)s_i\wedge\xi''$$

by the properties of exterior multiplication. Since $0=p_nq(\tilde{\mathbf{b}}-\mathbf{z})\xi=(\tilde{\mathbf{b}}-\mathbf{z})p_nq\xi$ $=(\tilde{\mathbf{b}}-\mathbf{z})\wedge(\tilde{\mathbf{b}}-\mathbf{z})p_n\xi+(\tilde{\mathbf{b}}-\mathbf{z})\xi_0$ and $(\tilde{\mathbf{b}}-\mathbf{z})\wedge(\tilde{\mathbf{b}}-\mathbf{z})=0$, we have $(\tilde{\mathbf{b}}-\mathbf{z})\xi_0=0$ and by the same considerations with ξ replaced by ξ_0, we have

$$(b_n-z_nI_Y)\xi_0'=\sum_{i=1}^{n-1}(b_i-z_iI_Y)s_i\wedge\xi_0''.$$

Putting this together (observe that $b_n\xi_0=p_nq\xi_0=0$)

$$\begin{aligned} z_n\xi&=(\tilde{\mathbf{b}}-\mathbf{z})p_n\eta+(z_nI_Y-b_n)(\xi'+s_n\wedge\xi''+z_n^{-1}(\xi_0'+s_n\wedge\xi_0''))\\ &=\sum_{i=1}^{n-1}(b_i-z_iI_Y)s_i\wedge(p_n\eta-\xi''-z_n^{-1}\xi_0'')\\ &\quad+(b_n-z_nI_Y)s_n\wedge(p_n\eta-\xi''-z_n^{-1}\xi_0'') \end{aligned}$$

and therefore this establishes exactness of $\mathbf{F}(Y;\mathbf{b}-\mathbf{z})$ in all but the first and the last place. So let $(\tilde{\mathbf{b}}-\mathbf{z})x=0$. Then $0=q(b_i-z_iI_Y)x=(a_i-z_iI_X)qx$ for $1\leqq i\leqq n$. By the exactness of $\mathbf{F}(X,\mathbf{a}-\mathbf{z})$, we get $qx=0$ and hence $0=(p_nq-z_nI_Y)x=-z_nx$ implies $x=0$, since $z_n\neq0$ by assumption.

Finally, let $y\in Y$. Then $qy\in X$ and by the exactness of $\mathbf{F}(X,\mathbf{a}-\mathbf{z})$ we find $\xi\in\Lambda^{n-1}[s_1,\dots,s_n;X]$ such that $qy=(\tilde{\mathbf{a}}-\mathbf{z})\xi$. Consequently

$$p_nqy=p_n(\tilde{\mathbf{a}}-\mathbf{z})\xi=\sum_{i=1}^{n}(p_iqp_n-z_ip_n)s_i\wedge\xi+\xi_0,$$

where $\xi_0\in\ker q$ (we are identifying X and $\Lambda^n[s_1,\dots,s_n;X]$ and use that $(p_nqp_i-p_iqp_n)X\subset\ker q$). We now add z_ny-p_nqy and get

$$\begin{aligned} z_ny&=\sum_{i=1}^{n}(p_iq-z_iI_Y)s_i\wedge p_n\xi+(p_nq-z_nI_Y)s_n\wedge(-1)^n(y'+z_n^{-1}\xi_0')\\ &=\sum_{i=1}^{n}(p_iq-z_iI_Y)s_i\wedge(p_n\xi+(-1)^n(y'+z_n^{-1}\xi_0')), \end{aligned}$$

where $y'=ys_1\wedge\dots\wedge s_{n-1}$, $\xi_0'=\xi_0s_1\wedge\dots\wedge s_{n-1}$. This proves $\mathbf{z}\in\mathbb{C}^n\backslash\sigma(\mathbf{b};X)$.

For the reverse inclusion let $\mathbf{z}\in\mathbb{C}^n\backslash\sigma(\mathbf{b};Y)$ such that $z_n\neq0$ and $\xi\in\Lambda^p[s_1,\dots,s_n;X]$ such that $0=(\tilde{\mathbf{a}}-\mathbf{z})\xi$. Then

$$0=p_n(\tilde{\mathbf{a}}-\mathbf{z})\xi=(\tilde{\mathbf{b}}-\mathbf{z})p_n\xi+\xi_0,$$

where $\xi_0\in\Lambda^{p+1}[s_1,\dots,s_n;Y]$ has only coefficients belonging to $\ker q$ because $(p_iqp_j-p_jqp_i)X\subset\ker q$. But as $(\tilde{\mathbf{b}}-\mathbf{z})\wedge(\tilde{\mathbf{b}}-\mathbf{z})=0$, we also have $0=(\tilde{\mathbf{b}}-\mathbf{z})\xi_0$. By the exactness of $\mathbf{F}(Y,\mathbf{b}-\mathbf{z})$ we have $\xi_1\in\Lambda^p[s_1,\dots,s_n;Y]$ such that $\xi_0=(\tilde{\mathbf{b}}-\mathbf{z})\xi_1$.

Moreover $0=q\xi_0=q(\tilde{\mathbf{b}}-\mathbf{z})\xi_1=(\tilde{\mathbf{a}}-\mathbf{z})q\xi_1$, and $0=(\tilde{\mathbf{b}}-\mathbf{z})(p_n\xi+\xi_1)$. Again the exactness of $\mathbf{F}(Y,\mathbf{b}-\mathbf{z})$ implies the existence of $\eta\in\wedge^{p-1}[s_1,\dots,s_n;Y]$ such that

$$p_n\xi+\xi_1=(\tilde{\mathbf{b}}-\mathbf{z})\eta$$

and thus

$$qp_n\xi=q(\tilde{\mathbf{b}}-\mathbf{z})\eta-q\xi_1=(\tilde{\mathbf{a}}-\mathbf{z})q\eta-q\xi_1\,.$$

Now add $z_n\xi-qp_n\xi$, then

$$z_n\xi=(\tilde{\mathbf{a}}-\mathbf{z})q\eta-q\xi_1+(z_nI_X-qp_n)\xi\,.$$

Since $0=q\xi_0=q(\tilde{\mathbf{b}}-\mathbf{z})\xi_1=(\tilde{\mathbf{a}}-\mathbf{z})q\xi_1$, we get

$$0=p_nq\xi_0=p_nq(\tilde{\mathbf{b}}-\mathbf{z})\xi_1=(\tilde{\mathbf{b}}-\mathbf{z})p_nq\xi_1\,.$$

Then the exactness of $\mathbf{F}(Y,\mathbf{b}-\mathbf{z})$ implies the existence of a $\gamma\in\wedge^{p-1}[s_1,\dots,s_n;Y]$ such that $p_nq\xi_1=(\tilde{\mathbf{b}}-\mathbf{z})\eta$.

Using the same notations and an analogue decomposition of $\wedge^p[s_1,\dots,s_n;X]$, we have

$$q\xi_1=q\xi_1'+s_n\wedge q\xi_1''\quad\text{and}\quad \xi=\xi'+s_n\wedge\xi''$$

and consequently

$$(a_n-z_nI_X)\xi'=\sum_{i=1}^{n-1}(a_i-z_iI_X)s_i\wedge\xi''$$

$$(a_n-z_nI_X)q\xi_1'=\sum_{i=1}^{n-1}(a_i-z_iI_X)s_i\wedge q\xi_1''\,.$$

Now putting together

$$\begin{aligned}z_n\xi&=(\tilde{\mathbf{a}}-\mathbf{z})q\eta+(z_nI_X-qp_n)(\xi-z_n{}^{-1}q\xi_1)-z_n{}^{-1}qp_nq\xi_1\\&=(\tilde{\mathbf{a}}-\mathbf{z})(q\eta-z_n{}^{-1}q\gamma)+(z_nI_X-qp_n)(\xi-z_n{}^{-1}q\xi_1)\\&\quad(\text{we used } qp_nq\xi_1=q(\tilde{\mathbf{b}}-\mathbf{z})\gamma=(\tilde{\mathbf{a}}-\mathbf{z})q\gamma)\\&=\sum_{i=1}^{n-1}(a_i-z_iI_X)s_i\wedge(q\eta-z_n{}^{-1}q\gamma-\xi''-z_n{}^{-1}q\xi_1'')\\&\quad+(a_n-z_nI_X)s_n\wedge(q\eta-z_n{}^{-1}q\gamma-\xi''-z_n{}^{-1}q\xi_1'')\,.\end{aligned}$$

This proves the exactness of $\mathbf{F}(X,\mathbf{a}-\mathbf{z})$ in all but the first and the last place. Thus assume

$$0=(\tilde{\mathbf{a}}-\mathbf{z})x\,,$$

then

$$0=p_nqp_n(\tilde{\mathbf{a}}-\mathbf{z})x=p_n(\tilde{\mathbf{a}}-\mathbf{z})qp_nx=p_nq(\tilde{\mathbf{b}}-\mathbf{z})p_nx=(\tilde{\mathbf{b}}-\mathbf{z})p_nqp_nx$$

and hence $p_nqp_nx=0$ by the exactness of $\mathbf{F}(Y,\mathbf{b}-\mathbf{z})$. Then $0=p_n(z_nI_X-qp_n)x$ implies $p_nx=0$ because $z_n\neq0$ and thus $0=z_nx-qp_nx$ gives $x=0$. Let $x\in X$. For p_nx we find $\xi\in\wedge^{n-1}[s_1,\dots,s_n;Y]$ such that $p_nx=(\tilde{\mathbf{b}}-\mathbf{z})\xi$ and therefore $qp_nx=(\hat{\mathbf{a}}-\mathbf{z})q\xi$ and finally

$$\begin{aligned}z_nx&=(\tilde{\mathbf{a}}-\mathbf{z})q\xi+(z_nI_X-a_n)x\\&=(\tilde{\mathbf{a}}-\mathbf{z})(q\xi+(-1)^nxs_1\wedge\dots\wedge s_{n-1})\,.\end{aligned}$$

This proves the lemma.

In the results above the point 0 plays an exceptional role. This is clear already for $n=1$ and matrix products $a=\begin{pmatrix}1\\0\end{pmatrix}(1\ \ 0)$, $b=(1\ \ 0)\begin{pmatrix}1\\0\end{pmatrix}$, where b is invertible whereas $z\mapsto(z-a)^{-1}$ has a pole of order 1 in 0. Often it is desirable to have equality of the spectra under consideration in order to simplify calculations. This can be achieved by a simple trick which we describe in the case of Taylor's joint spectrum. Without loss of generality assume that $0\in\sigma(\mathbf{a};X)$ and let $\mathbf{b}\in\mathbf{L}(Y)^n$ be a related system. Instead of Y consider $\hat{Y}:=Y\oplus\mathbb{C}$, $\hat{p}_i x:=p_i x\oplus 0$, $\hat{q}(x\oplus z):=qx$, and $\hat{b}_i:=\hat{p}_i\hat{q}_i$ $(1\leqq i\leqq n)$. Obviously $a_i=\hat{q}\hat{p}_i$ and consequently $(a_1,\dots,a_n)$ and $(\hat{b}_1,\dots,\hat{b}_n)$ are related systems. Since $x\in\bigcap_{i=1}^{n}\operatorname{kern}\hat{b}_i$, the sequence $\mathbf{F}(Y,\hat{\mathbf{b}})$ is not exact, i.e. $0\in\sigma(\hat{\mathbf{b}};Y)$.

Our next concern will be a functional calculus of analytic functions. We restate that

$$\sigma(\mathbf{a};X)\cup\operatorname{Sp}(\mathbf{a};L(X))\subseteqq\operatorname{Sp}(\mathbf{a};\{\mathbf{a}\}^c)\subseteqq\operatorname{Sp}(\mathbf{a};\{\mathbf{a}\}^{cc}),$$

whereas it seems to be unknown whether $\operatorname{Sp}(\mathbf{a};\mathbf{L}(X))\subseteqq\sigma(\mathbf{a};X)$ is true for Banach spaces X. There are examples showing that $\operatorname{Sp}(\mathbf{a};\mathbf{L}(X))$ may be a proper subset of $\sigma(\mathbf{a};X)$. On the other hand it follows from a result in [11] that the boundary $\partial\sigma((a_1,a_2);X)$ is always contained in $\operatorname{Sp}((a_1,a_2);\mathbf{L}(X))$ for Banach spaces X.

Given a non-empty compact subset $K\subset\mathbb{C}^n$ let $\mathbf{A}(K)$ denote a subalgebra of the algebra $\mathcal{O}(K)$ of *germs of functions being analytic* in some open neighbourhood of K, and suppose that $\mathbf{A}(K)$ has the following properties:

(i) if $f\in\mathbf{A}(K)$ and $1/f\in\mathcal{O}(K)$, then $1/f\in\mathbf{A}(K)$;

(ii) if $f\in A(K)$ and $\mathbf{z}_0\in K$, then there exist $g_1,\dots,g_n\in A(K)$ such that

$$f(z)=f(z_0)+\sum_{i=1}^{n}(z_i-z_{0i})g_i(z).$$

For example $\mathbf{A}(K)$ may be the (germs of) rational functions on K or uniform limits of the latter.

3.4. Lemma. *Using notations of 3.1, let* $\mathbf{a}=(a_1,\dots,a_n)\in\mathbf{L}(X)^n$ *and* $\mathbf{b}=(b_1,\dots,b_n)\in\mathbf{L}(Y)^n$ *denote related systems of pairwise commuting linear operators on locally convex spaces X and Y. Moreover assume that* $\sigma(\mathbf{a};X)=\sigma(\mathbf{b};Y)$ *and* $\operatorname{Sp}(\mathbf{a};\mathbf{L}(X))=\operatorname{Sp}(\mathbf{b};\mathbf{L}(Y))$ *are non-empty compact subsets of* $\mathbb{C}^n$. *Let K be either* $\sigma(\mathbf{a};X)$ *or* $\operatorname{Sp}(\mathbf{a};\mathbf{L}(X))$ *and* $\mathbf{A}(K)$ *as above, and let* $0\in K$.

Then there exists an algebra homomorphism $\Phi:A(K+\mathbf{z}_0)\to L(X)$ *such that* $\Phi(\mathbf{z}\mapsto z_i)=z_{0i}I_X+a_i$ *and* $\Phi(\mathbf{z}\mapsto 1)=I_X$ *if and only if there exists an algebra homomorphism* $\psi:A(K+\mathbf{z}_0)\to L(Y)$ *such that* $\psi(\mathbf{z}\mapsto z_i)=z_{0i}I_Y+b_i$ *and* $\psi(\mathbf{z}\mapsto 1)=I_Y$.

Proof. Suppose there exists an algebra homomorphism $\Phi:\mathbf{A}(K+\mathbf{z}_0)\to\mathbf{L}(X)$ with the properties above. Instead of $\Phi(f)$ for $f\in\mathbf{A}(K+\mathbf{z}_0)$ given, write $f(\mathbf{z}_0+\mathbf{a})$. Fix $f\in\mathbf{A}(K+\mathbf{z}_0)$. By assumption we find $g_1,\dots,g_n\in\mathbf{A}(K+\mathbf{z}_0)$ such that

$$(+)\qquad f(\mathbf{z})=f(\mathbf{z}_0)+\sum_{i=1}^{n}(z_i-z_{0i})g_i(\mathbf{z}).$$

Then let

$$(++)\qquad f(\mathbf{z}_0+\mathbf{b}):=\psi(f):=f(\mathbf{z}_0)I_Y+\sum_{i=1}^{n} p_i g_i(\mathbf{z}_0+\mathbf{a})q\,.$$

Obviously, we have $\psi(\mathbf{z}\mapsto z_i)=z_{0i}I_Y+b_i$ and $\psi(\mathbf{z}\mapsto 1)=I_Y$. On the other hand, let

$$g(\mathbf{z})=g(\mathbf{z}_0)+\sum_{i=1}^{n}(z_i-z_{0i})h_i(\mathbf{z})\in \mathbf{A}(K+\mathbf{z}_0)\,.$$

Then

$$\begin{aligned}\psi(fg)&=\psi\left(f(\mathbf{z}_0)g(\mathbf{z}_0)+\sum_{i=1}^{n}(z_i-z_{0i})g_i(\mathbf{z})\left(g(\mathbf{z}_0)+\sum_{j=1}^{n}(z_j-z_{0j})h_j(\mathbf{z})\right)\right)\\&=f(\mathbf{z}_0)g(\mathbf{z}_0)I_Y+\sum_{i=1}^{n}p_i g_i(\mathbf{z}_0+\mathbf{a})\left(g(\mathbf{z}_0)I_X+\sum_{j=1}^{n}a_j h_j(\mathbf{z}_0+\mathbf{a})\right)q\\&=f(\mathbf{z}_0)g(\mathbf{z}_0)I_Y+g(\mathbf{z}_0)\sum_{i=1}^{n}p_i g_i(\mathbf{z}_0+\mathbf{a})q\\&\quad+\sum_{i=1}^{n}\sum_{j=1}^{n}p_i g_i(\mathbf{z}_0+\mathbf{a})qp_jh_j(\mathbf{z}_0+\mathbf{a})q\\&=\psi(f)\psi(g)\,,\end{aligned}$$

showing that $(++)$ defines an algebra homomorphism from $\mathbf{A}(K+\mathbf{z}_0)$ into $\mathbf{L}(Y)$. Vice versa assume an algebra homomorphism $\psi:\mathbf{A}(K+\mathbf{z}_0)\to\mathbf{L}(Y)$ is given and f of the form $(+)$. Then

$$\Phi(f):=f(\mathbf{z}_0)I_X+\sum_{i=1}^{n}qg_i(\mathbf{z}_0+\mathbf{a})p_i$$

defines an algebra homomorphism with the desired properties.

3.5. Corollary. *Assume that one of the equivalent statements in 3.4 is true. Then*

(i) $\sigma(f(\mathbf{z}_0+\mathbf{a});X)=f(\sigma(\mathbf{a};X)+z_0)$ *if* $K=\sigma(\mathbf{a};X)$.

(ii) $\mathrm{Sp}(f(\mathbf{z}_0+\mathbf{a});\mathbf{L}(X))=f(\mathrm{Sp}(\mathbf{a};L(X))+\mathbf{z}_0)$ *if* $K=\mathrm{Sp}(\mathbf{a};\mathbf{L}(X))$.

(iii) *The range of Φ is contained in $\{\mathbf{a}\}^{cc}$ if and only if the range of ψ as defined by $(++)$ is contained in $\{\mathbf{b}\}^{cc}$.*

Proof. The proof of (i) and (ii) follows from the fact that Φ is an algebra homomorphism and the properties of $\mathbf{A}(K+\mathbf{z}_0)$ by a standard argument. Next suppose that range $(\Phi)\subseteq\{\mathbf{a}\}^{cc}$, and let f, g_i, h_{ij}, $k_{ijl}\in\mathbf{A}(K+\mathbf{z}_0)$ such that

$$\begin{aligned}f(\mathbf{z})&=f(\mathbf{z}_0)+\sum_{i=1}^{n}(z_i-z_{0i})g_i(\mathbf{z}_0)+\sum_{i=1}^{n}\sum_{j=1}^{n}(z_i-z_{0i})(z_j-z_{0j})h_{ij}(\mathbf{z}_0)\\&\quad+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}(z_i-z_{0i})(z_j-z_{0j})(z_l-z_{0l})k_{ijl}(\mathbf{z})\,.\end{aligned}$$

Then

$$\begin{aligned}\psi(f)&=f(\mathbf{z}_0)I_Y+\sum_{i=1}^{n}g_i(\mathbf{z}_0)b_i+\sum_{i=1}^{n}\sum_{j=1}^{n}p_ia_jqh_{ij}(\mathbf{z}_0)\\&\quad+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}p_ia_ja_lk_{ijl}(\mathbf{z}_0+\mathbf{a})q\,.\end{aligned}$$

We show that each summand belongs to $\{\mathbf{b}\}^{cc}$. Since $p_i a_j q = b_i b_j$ this is clear for all summands with the exception of those in the triple sum. But if $d \in \{\mathbf{b}\}^c$, then $a_j q d p_i \in \{\mathbf{a}\}^c$ and thus

$$\begin{aligned} dp_i a_j a_l k_{ijl}(\mathbf{z}_0+\mathbf{a})q &= db_i b_j p_l k_{ijl}(\mathbf{z}_0+\mathbf{a})q = p_i a_j q d p_l k_{ijl}(\mathbf{z}_0+\mathbf{a})q \\ &= p_i k_{ijl}(\mathbf{z}_0+\mathbf{a}) a_j q d p_l q \\ &= p_i a_j k_{ijl}(\mathbf{z}_0+\mathbf{a}) q p_l q d = p_i a_j a_l k_{ijl}(\mathbf{z}_0+\mathbf{a}) q d . \end{aligned}$$

Hence $\psi(f) \in \{\mathbf{b}\}^{cc}$. In order to prove the reverse implication, observe that given $g \in \mathbf{A}(K+\mathbf{z}_0)$ we have $\Phi(g)q = q\psi(g)$. Hence

$$\begin{aligned} \Phi(f) = f(\mathbf{z}_0) I_X &+ \sum_{i=1}^{n} g_i(\mathbf{z}_0) a_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j h_{ij}(\mathbf{z}_0) \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} q b_i b_j \psi(k_{ijl}) p_l . \end{aligned}$$

Let $d \in \{\mathbf{a}\}^c$. Then $b_j p_i d q \in \{\mathbf{b}\}^c$ and consequently

$$\begin{aligned} q b_i b_j \psi(k_{ijl}) p_l d &= q\psi(k_{ijl}) b_i b_j p_l d = q\psi(k_{ijl}) b_i p_j d q p_l \\ &= a_i a_j d q \psi(k_{ijl}) p_l = d q b_i b_j \psi(k_{ijl}) p_l \end{aligned}$$

and thus proving that $\Phi(f) \in \{\mathbf{a}\}^{cc}$.

Given an algebra of (germs of) analytic functions $\mathbf{A}(K)$ let $\mathbf{A}(\mathbf{a})$ and $\mathbf{A}(\mathbf{b})$ denote the closure of $\{f(\mathbf{a}) : f \in \mathbf{A}(K)\}$ and $\{f(\mathbf{b}) : f \in \mathbf{A}(K)\}$ in $\mathbf{L}_b(X)$ and $\mathbf{L}_b(Y)$, respectively, where subscript "b" denotes the topology of uniform convergence on bounded subsets.

3.6. Lemma. *Let the assumptions of 3.5 be fulfilled. Then*

$$\mathrm{Sp}(\mathbf{a}; \mathbf{A}(\mathbf{a})) \cup \{0\} = \mathrm{Sp}(\mathbf{b}; \mathbf{A}(\mathbf{b})) \cup \{0\} .$$

Moreover,

$$\mathrm{Sp}(\mathbf{a}; \langle \mathbf{a} \rangle) \cup \{0\} = \mathrm{Sp}(\mathbf{b}; \langle \mathbf{b} \rangle) \cup \{0\} ,$$

where $\langle \mathbf{a} \rangle$ and $\langle \mathbf{b} \rangle$ denote the smallest closed subalgebras of $\mathbf{L}_b(X)$ and $\mathbf{L}_b(Y)$ containing $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively.

Proof. We analyze the proof of 3.2. Consequently, let $\mathbf{z} \in \mathbb{C}^n \backslash \mathrm{Sp}(\mathbf{a}; \mathbf{A}(\mathbf{a}))$ such that $z_n \neq 0$ and $d_1, \ldots, d_n \in \mathbf{A}(\mathbf{a})$ such that

$$\sum_{i=1}^{n} d_i (a_i - z_i I_X) = I_X .$$

But $d_i = \lim f_{i\lambda}(\mathbf{a})$ for a suitable net $(f_{i\lambda}(\mathbf{a}))_\lambda$. Letting $\check{d}_i := p_n d_i q$ for $1 \leqq i \leqq n-1$ and $\check{d}_n := (p_n d_n q - I_Y)$, we have

$$\check{d}_i = p_n \left(\lim_\lambda f_{i\lambda}(\mathbf{a}) \right) q = \lim_\lambda p_n f_{i\lambda}(\mathbf{a}) q = \lim_\lambda p_n q f_{i\lambda}(\mathbf{b}) \in \mathbf{A}(\mathbf{b})$$

for $1 \leqq i \leqq n-1$ and also $\check{d}_n \in \mathbf{A}(\mathbf{b})$ (compare $(++)$ in the proof of 3.4). So as in the proof of 3.2 we conclude $z \in \mathbb{C}^n \backslash \mathrm{Sp}(\mathbf{b}; \mathbf{A}(\mathbf{b}))$. In 3.2 the converse inclusion needed a little bit more care and therefore we prove it here, too. So let $z \in \mathbb{C}^n \backslash \mathrm{Sp}(\mathbf{b}; \mathbf{A}(\mathbf{b}))$

such that $z_n \neq 0$, $\check{d}_1, \ldots, \check{d}_n \in \mathbf{A}(\mathbf{b})$ such that

$$\sum_{i=1}^{n} \check{d}_i(b_i - z_i I_Y) = I_Y.$$

Again, $\check{d}_i = \lim_\lambda g_{i\lambda}(\mathbf{b})$ for suitable nets $(g_{i\lambda}(\mathbf{b}))_\lambda$. For $1 \leqq i \leqq n-1$ we have

$$d_i := q\check{d}_i p_n = q\left(\lim_\lambda g_{i\lambda}(\mathbf{b})\right) p_n = \lim_\lambda q g_{i\lambda}(\mathbf{b}) p_n = \lim_\lambda g_{i\lambda}(\mathbf{a}) q p_n$$

and $d_n = q\check{d}_n p_n$ and therefore

$$\begin{aligned} d_i a_i &= \left(\lim_\lambda g_{i\lambda}(\mathbf{a}) a_n\right) a_i = \lim_\lambda g_{i\lambda}(\mathbf{a}) a_i a_n = \lim_\lambda a_i g_{i\lambda}(\mathbf{a}) q p_n \\ &= \lim_\lambda a_i q g_{i\lambda}(\mathbf{b}) p_n = a_i d_i \end{aligned}$$

for $1 \leqq i \leqq n-1$ and $d_n a_n = a_n d_n$. Thus $\sum_{i=1}^{n} d_i(a_i - z_i I_X) = z_n I_X$. This proves the first part, but also the second taking $\langle \mathbf{a} \rangle$ and $\langle \mathbf{b} \rangle$ instead of $\mathbf{A}(\mathbf{a})$ and $\mathbf{A}(\mathbf{b})$.

4. Systems of Bounded Linear Operators

Recall that a linear operator $d: X \to Y$ on locally convex spaces is said to be *bounded*, if there exists a neighbourhood of zero U in X such that $d(U)$ is a bounded subset of Y. If U is supposed to be absolutely convex, then we may consider the span $Y_0 := \bigcup_{n \in \mathbb{N}} nB$, where B denotes the closure $\overline{d(U)}$ of $d(U)$ in Y. If provided with the *Minkowski-norm* m_B relative to B, the operator d decomposes into a product qp, where $p \in \mathbf{L}(X, Y_0)$ is given by $px := dx$, and $q \in \mathbf{L}(Y_0, Y)$ is the canonical embedding. If Y is *Mackey-complete*, then (Y_0, m_B) is Banach. Next suppose a pairwise commuting system $(a_1, \ldots, a_n) \in \mathbf{L}(X)^n$ of bounded linear operators is given. Let $U_1, \ldots, U_n$ be absolutely convex neighbourhoods of zero such that $a_i(U_i)$ is bounded for $1 \leqq i \leqq n$. Then take $U := \bigcap_{i=1}^{n} U_i$, B the closed hull of $a_1(U) + \ldots + a_n(U)$, $Y := \bigcup_{n \in \mathbb{N}} nB$ provided with the Minkowski-norm, $p_i \in \mathbf{L}(X, Y)$ given by $p_i x = a_i x$ for $1 \leqq i \leqq n$, and $q \in \mathbf{L}(Y, X)$ the canonical embedding. Let $b_i \in \mathbf{L}(Y)$ be $p_i q$ for $1 \leqq i \leqq n$. Then $\mathbf{a} := (a_1, \ldots, a_n)$ and $\mathbf{b} := (b_1, \ldots, b_n)$ are related systems in the sense of 3.1, and as q is one-to-one and $a_i a_j = a_j a_i$ we have $b_i b_j = b_j b_i$.

4.1. Theorem. *Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{L}(X)^n$ denote a commuting system of bounded linear operators on a Mackey-complete locally convex space. Then*

1) *$\sigma(\mathbf{a}; X)$ is a non-empty compact subset of $\mathbb{C}^n$.*

(2) *$\mathrm{Sp}(\mathbf{a}; \mathbf{A})$ is a non-empty compact subset of $\mathbb{C}^n$, where $\mathbf{A}$ is one of the algebras $\mathbf{L}(X)$, $\{\mathbf{a}\}^c$, or $\{\mathbf{a}\}^{cc}$.*

If X is non-normable, then $0 \in \sigma(\mathbf{a}; X) \cap \mathrm{Sp}(\mathbf{a}; \mathbf{A})$.

Proof. Let Y and $\mathbf{b} = (b_1, \ldots, b_n)$ be as above. Since $\mathbf{b}$ is a commuting system of Banach space operators, the spectra $\sigma(\mathbf{b}; Y)$ and $\mathrm{Sp}(\mathbf{b}; \mathbf{B})$ are non-empty compact subsets of $\mathbb{C}^n$ by results of Taylor [7, 3.1, 3.2] and Harte [3, 1.5, 4.3] respectively. Here $\mathbf{B}$ denotes, of course, one of the algebras $\mathbf{L}(Y)$, $\{\mathbf{b}\}^c$, or $\{\mathbf{b}\}^{cc}$. If X is non-normable, we are done by 3.3 and 3.2, if we show that $0 \in \sigma(\mathbf{a}; X) \cap \mathrm{Sp}(\mathbf{a}; \mathbf{L}(X))$.

But $0 \in \mathbb{C}^n \backslash \mathrm{Sp}(\mathbf{a}; \mathbf{L}(X))$ means that I_X can be written as a sum of bounded linear operators, and consequently X had to be normable by a well-known normability criterium, a contradiction. Suppose $0 \in \mathbb{C}^n \backslash \sigma(\mathbf{a}; X)$. As $\mathbf{F}(X, \mathbf{a})$ especially is exact in the last place, q is a surjection from Y onto X. Next let $y \in Y$ such that $0 = \sum_{i=1}^n b_i(y)s_i$. Then $0 = \sum_{i=1}^n qb_i(y)s_i = \sum_{i=1}^n a_i(qy)s_i$ and thus $qy = 0$ by the exactness of $\mathbf{F}(X, \mathbf{a})$ and finally $y = 0$ as q is one-to-one by construction. If $\xi \in \wedge^1[s_1, \ldots, s_n; Y]$ such that $\sum_{i=1}^n b_i s_i \wedge \xi = 0$. Then $0 = \sum_{i=1}^n qb_i s_i \wedge \xi = \sum_{i=1}^n a_i s_i \wedge q\xi$. By the exactness of $\mathbf{F}(X, \mathbf{a})$ we find $x \in X$ such that $q\xi = \sum_{i=1}^n a_i(x)s_i = \sum_{i=1}^n qp_i(x)s_i$. Therefore keeping in mind that q was surjective there exists $y \in Y$ such that $x = qy$ and thus $\xi = \sum_{i=1}^n p_i(x)s_i = \sum_{i=1}^n p_i q(y)s_i$. This especially proves that $\tilde{\mathbf{b}} = \sum_{i=1}^n b_i s_i : Y \to Y^n$ is a topological monomorphism and so is

$$\sum_{i=1}^n p_i s_i : X \to Y^n \quad \text{as } q \text{ is one-to-one}.$$

Consequently X is a Banach space. This proves the theorem. Indeed, one can show that $0 \in \mathbb{C}^n \backslash \sigma(\mathbf{a}; X)$ implies $0 \in \mathbb{C}^n \backslash \sigma(\mathbf{b}; Y)$, if q is one-to-one.

If $\mathbf{B}(X)$ denotes the (two-sided) *ideal of bounded linear operators* in $\mathbf{L}(X)$, then the considerations above show that

$$\mathrm{Sp}(\mathbf{a}; \mathbf{A}) = \mathrm{Sp}(\mathbf{a}; \mathbf{A} \cap (\mathbb{C} I_X + \mathbf{B}(X)))$$

for a commuting system of bounded linear operators $\mathbf{a} = (a_1, \ldots, a_n)$. The remark following 3.3 and 4.1 justify the following

4.2. Assumption. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{L}(X)^n$ denote a commuting system of bounded linear operators on a non-normable Mackey-complete locally convex space X, and let $\mathbf{b} = (b_1, \ldots, b_n)$ denote the canonical associated system on a Banach space Y such that

$$0 \in \sigma(\mathbf{b}; Y) \cap \mathrm{Sp}(\mathbf{b}; \mathbf{L}(Y))$$

and consequently

$$\sigma(\mathbf{a}; X) = \sigma(\mathbf{b}; Y) \quad \text{and} \quad \mathrm{Sp}(\mathbf{a}; \mathbf{L}(X)) = \mathrm{Sp}(\mathbf{b}; \mathbf{L}(Y)).$$

Now we are in a position to state the analytic functional calculus.

4.3. Theorem. *Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{L}(X)^n$ denote a commuting system of bounded linear operators on a Mackey-complete locally convex space X. Then*

(1) *There exists an algebra homomorphism $\Phi : \mathbf{A}(\sigma(\mathbf{a}; X)) \to \{\mathbf{a}\}^{cc}$ such that $\Phi(\mathbf{z} \mapsto z_i) = a_i$ and $\Phi(\mathbf{z} \mapsto 1) = I_X$. Explicitly Φ is given by*

$$\Phi(f) = f(0)I_X + \sum_{i=1}^n q\psi(g_i)p_i$$

for $f(\mathbf{z}) = f(0) + \sum_{i=1}^n z_i g_i(\mathbf{z})$, where $\psi : \mathcal{O}(\sigma(\mathbf{b}; Y)) \to \{\mathbf{b}\}^{cc}$ denotes J. L. Taylor's functional calculus of [8].

(2) *There exists an algebra homomorphism* $\Lambda: \mathbf{R}(\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X)) \to \{\mathbf{a}\}^{cc}$ *from the algebra* $\mathbf{R}(\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X))$ *of rational functions on* $\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X))$ *such that* $\Lambda(\mathbf{z} \mapsto z_i) = a_i$ *and* $\Lambda(\mathbf{z} \mapsto 1) = I_X$. *If* $r = \frac{f}{g}$, *where* f *and* g *are polynomials such that* g *has no zeros on* $\mathrm{Sp}(\mathbf{a}; L(X))$, *then* $\Lambda(r) = f(\mathbf{a})g(\mathbf{a})^{-1}$.

Proof. Part (1) follows immediately from 3.4 and 4.2. In order to see that $\Lambda(r) = f(a)g(a)^{-1}$ is well-defined in (2), we observe that by 3.6 and 4.2 $\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X)) = \mathrm{Sp}(\mathbf{b}; \mathbf{L}(Y))$. But for a polynomial f we have $f(\mathrm{Sp}(\mathbf{b}; \mathbf{L}(Y))) = \mathrm{Sp}(f(\mathbf{b}); \mathbf{L}(Y))$ by Harte [3, 4.3]. Thus $f(\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X))) = \mathrm{Sp}(f(\mathbf{a}); \mathbf{L}(X))$ by 3.5. Thus Λ is well-defined by $\Lambda(r) = f(\mathbf{a})g(\mathbf{a})^{-1}$, if g has no zeros on $\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X))$.

As an immediate consequence we obtain

4.4. Corollary. *With the same assumptions and notations as in 4.3 we have*

$$(1) \qquad f(\sigma(\mathbf{a}; X)) = \sigma(f(\mathbf{a}); X) \quad \text{for} \quad f \in \mathbf{A}(\sigma(\mathbf{a}; X));$$

$$(2) \qquad r(\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X)) = \mathrm{Sp}(r(\mathbf{a}); \mathbf{L}(X)) \quad \text{for} \quad r \in \mathbf{R}(\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X))).$$

Especially

$$(3) \qquad f(\mathrm{Sp}(\mathbf{a}; \mathbf{A})) = \mathrm{Sp}(f(\mathbf{a}); \mathbf{A}) \quad \text{for} \quad f \in \mathbf{A}(\mathrm{Sp}(\mathbf{a}; \mathbf{A})),$$

where $\mathbf{A}$ *denotes one of the algebras* $\{\mathbf{a}\}^c$, $\{\mathbf{a}\}^{cc}$, *closure of range* (Φ) *in* $\mathbf{L}_b(X)$, *or* $\langle \mathbf{a} \rangle$.

Proof. (1) and (2) follow from 4.3 and 3.4. Since $\sigma(\mathbf{a}; X)$ is a subset of $\mathrm{Sp}(\mathbf{a}; \mathbf{A})$ for all four algebras under consideration in (3), $f(\mathbf{a})$ is well-defined via 4.3 and $f \mapsto f(\mathbf{a})$ from $\mathbf{A}(\mathrm{Sp}(\mathbf{a}; \mathbf{A})) \to$ closure of range $(\Phi) \subset \{\mathbf{a}\}^{cc}$ is an algebra homomorphism. Thus (3) holds true (see 3.5).

It seems to be unknown whether the rational functional calculus $\Lambda: \mathbf{R}(\mathrm{Sp}(\mathbf{b}; \mathbf{L}(Y))) \to \{\mathbf{b}\}^{cc}$ is continuous if $\mathbf{R}(\mathrm{Sp}(\mathbf{b}; \mathbf{L}(Y)))$ is equipped with the relative topology of $\mathcal{O}(\mathrm{Sp}(\mathbf{b}; \mathbf{L}(Y)))$, whereas $\psi: \mathcal{O}(\sigma(\mathbf{b}; Y)) \to \{\mathbf{b}\}^{cc}$ is continuous [8, 4.3]. However, if $n=2$ then $\partial\sigma(\mathbf{b}; Y) \subset \mathrm{Sp}(\mathbf{b}; \mathbf{L}(Y))$ by [11, 2.1] and consequently ψ is continuous. This is a motivation for the following consideration.

Let K be either $\sigma(\mathbf{a}; X)$ or $\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X))$ and assume that $\mathbf{A}(K)$ is a closed subalgebra of $\mathcal{O}(K)$ with the properties (i) and (ii) (compare 3.4). Moreover suppose that $f \mapsto f(\mathbf{b})$ is a continuous functional calculus from $\mathbf{A}(K)$ into $\mathbf{L}_b(Y)$. Then we obtain a slight variation of [8, 5.2].

4.5. Theorem. *Let* $\mathbf{A}(K)$ *be as above. Then* $\mathrm{Sp}(\mathbf{a}; \langle \mathbf{a} \rangle)$ *is the polynomialls convex hull of* K.

Proof. Since $\mathrm{Sp}(\mathbf{b}; \langle \mathbf{b} \rangle)$ is polynomially convex by classical Banach algebra theory, the polynomially convex hull $h(K)$ is contained in $\mathrm{Sp}(\mathbf{b}; \langle \mathbf{b} \rangle)$. Thus assume $\mathbf{z} \notin h(K)$. Consequently there exists a polynomial f such that $|f(\mathbf{z})| > \sup\{|f(\mathbf{w})|: \mathbf{w} \in K\}$. Therefore $f(\mathbf{z})I_Y - f(\mathbf{b}) = \sum_{i=1}^{n} (z_i I_Y - b_i)g_i(\mathbf{b})$ is invertible in $\{\mathbf{b}\}^{cc}$ by the spectral mapping Theorem 4.4. Here $g_1, \ldots, g_n$ denote suitable polynomials. Since $1/(f(\mathbf{z}) - f(\cdot)) \in \mathcal{O}(h(K))$ and $h(K)$ is polynomially convex, there exists a sequence $(f_n)_n$ of polynomials approximating $1/(f(\mathbf{z}) - f(\cdot))$ uniformly. As the functional calculus is assumed to be continuous $f_n(\mathbf{b})$

$\to(f(\mathbf{z})I_Y - f(\mathbf{b}))^{-1}$ in $\mathbf{L}_b(Y)$ as $n\to\infty$. Thus

$$I_Y = (f(\mathbf{z})I_Y - f(\mathbf{b}))^{-1} \sum_{i=1}^{n} (z_i I_Y - b_i) g_i(\mathbf{b})$$
$$= \sum_{i=1}^{n} (z_i I_Y - b_i) \lim_k f_k(\mathbf{b}) g_i(\mathbf{b})$$

and this proves $\mathbf{z} \in \mathbb{C}^n \setminus \mathrm{Sp}(\mathbf{b}; \langle \mathbf{b} \rangle)$. Since $\mathrm{Sp}(\mathbf{a}; \langle \mathbf{a} \rangle) = \mathrm{Sp}(\mathbf{b}; \langle \mathbf{b} \rangle)$ by 3.6 and 4.2 we are done.

Associate with $\mathbf{A}(K)$ the algebras $\mathbf{A}(\mathbf{a})$ and $\mathbf{A}(\mathbf{b})$ as in 3.6. We next want to relate the spectra $\sigma(\mathbf{a}; X)$ and $\mathrm{Sp}(\mathbf{a}; \mathbf{A}(\mathbf{a}))$.

4.6. *Definition.* The *$\mathbf{A}(K)$-convex hull of K* is the set of all $\mathbf{w} \in \mathbb{C}^n$ such that the equation

$$(+++) \qquad 1 = \sum_{i=1}^{n} (z_i - w_i) f_i(\mathbf{z})$$

has no solution $(f_1, \ldots, f_n) \in \mathbf{A}(K)^n$ in a neighbourhood of K. Taylor [8] considers $\mathbf{A}(K) = \mathcal{O}(K)$ and $K = \sigma(\mathbf{a}; X)$.

4.7. Theorem. *Let* $\mathbf{A}(K)$, $\mathbf{A}(\mathbf{a})$ *as above for a commuting system* $\mathbf{a}$ *of bounded linear operators on a Mackey-complete locally convex space X. Then* $\mathrm{Sp}(\mathbf{a}; \mathbf{A}(\mathbf{a}))$ *is the* $\mathbf{A}(K)$*-convex hull of K.*

Proof. As ever $\mathbf{b}$ denotes the canonical associated system of operators on the Banach space Y. We prove that the theorem is true for $\mathbf{b}$, $\mathbf{A}(\mathbf{b})$, and Y. Thus we are done by 3.6 and 4.2. Assume that $\mathbf{w} \in \mathbb{C}^n$ is such that $(+++)$ can be solved with $(f_1, \ldots, f_n) \in \mathbf{A}(K)^n$. Then $\mathbf{w} \in \mathbb{C}^n \setminus \mathrm{Sp}(\mathbf{b}; \mathbf{A}(\mathbf{b}))$ by the functional calculus 4.3 and the definition of $\mathrm{Sp}(\mathbf{b}; \mathbf{A}(\mathbf{b}))$. Conversely suppose that there exist $d_1, \ldots, d_n \in \mathbf{A}(\mathbf{b})$ such that $I_Y = \sum_{i=1}^{n} (b_i - w_i I_Y) d_i$. Then there exist functions $g_1, \ldots, g_n \in \mathbf{A}(K)$ such that $\left\| \sum_{i=1}^{n} (b_i - w_i I_Y)(d_i - g_i(\mathbf{b})) \right\| < 1$. Thus $h(\mathbf{b}) := \sum_{i=1}^{n} (b_i - w_i I_Y) g_i(\mathbf{b})$ is invertible in $\mathbf{A}(\mathbf{b})$ (Neumann series), and $h(\mathbf{z}) = (z_1 - w_1) g_1(\mathbf{z}) + \ldots + (z_n - w_n) g_n(\mathbf{z})$. By the spectral mapping Theorem 4.4 h has no zeros in K and thus $\frac{1}{h} \in \mathbf{A}(K)$. Hence $I_Y = \sum_{i=1}^{n} (b_i - w_i I_Y)\left(\frac{1}{h} g_i\right)(\mathbf{b})$ implies $\mathbf{w} \in \mathbb{C}^n \setminus \mathrm{Sp}(\mathbf{b}; \mathbf{A}(\mathbf{b}))$.

With the assumptions and notations of the preceding we have

4.8. Theorem. *If K is polynomially convex, then* $\mathrm{Sp}(\mathbf{a}; \mathbf{A}) = K$ *for any closed subalgebra* $\mathbf{A} \subset \mathbf{L}_b(X)$ *with* $a_1, \ldots, a_n \in$ *center* $(\mathbf{A})$.

If K is $\mathbf{A}(K)$*-convex, then* $\mathrm{Sp}(\mathbf{a}; \mathbf{A}) = K$ *for any closed subalgebra* $\mathbf{A} \subset \mathbf{L}_b(X)$ *such that* $\mathbf{A}(\mathbf{a}) \subset$ *center* $(\mathbf{A})$.

Proof. If $a_1, \ldots, a_n \in$ *center* $(\mathbf{A})$, then $\langle \mathbf{a} \rangle \subset \mathbf{A}$ and $K \subset \mathrm{Sp}(\mathbf{a}; \mathbf{A}) \subset \mathrm{Sp}(\mathbf{a}; \langle \mathbf{a} \rangle)$. Theorem 4.5 gives $K = \mathrm{Sp}(\mathbf{a}; \mathbf{A}) = \mathrm{Sp}(\mathbf{a}; \langle \mathbf{a} \rangle)$. If $\mathbf{A}(\mathbf{a}) \subset$ *center* $(\mathbf{A})$ then $K \subset \mathrm{Sp}(\mathbf{a}; \mathbf{A}) \subset \mathrm{Sp}(\mathbf{a}; \mathbf{A}(\mathbf{a}))$ and Theorem 4.7 gives $K = \mathrm{Sp}(\mathbf{a}; \mathbf{A}) = \mathrm{Sp}(\mathbf{a}; \mathbf{A}(\mathbf{a}))$ if K is $\mathbf{A}(K)$-convex.

If $K=\sigma(\mathbf{a}; X)$ and $\mathbf{A}(K)=\mathcal{O}(\sigma(\mathbf{a}; X))$ 4.5, 4.7, 4.8 are translations of Taylor's corresponding results [8, 5.2, 5.4, 5.5]. If $n=2$, $K=\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X))$ and $\mathbf{A}(K)$ the uniform closure of (the germs of) rational functions in $\mathcal{O}(\mathrm{Sp}(\mathbf{a}; \mathbf{L}(X)))$ the results 4.5, 4.7, and 4.8 seem to be new even in the setting of Banach spaces.

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Periods and Gauss-Manin Connection for Families of p-Adic Schottky Groups

L. Gerritzen

Institut für Mathematik, Ruhr-Universität Bochum, Postfach 102148, D-4630 Bochum 1, Federal Republic of Germany

The periods q_{ij} of a p-adic Schottky group Γ with respect to a basis $\gamma_1, \ldots, \gamma_g$ of Γ are defined as the factors of automorphy of non-vanishing automorphic forms u_i on the domain Z of discontinuity of Γ, see [MD], [GP, Chap. VI, Sect. 2], for which $\frac{du_i}{u_i}$ are differentials of the first kind on the Mumford curve $C = Z/\Gamma$ of Γ. The differentials $\frac{du}{d}$ on C form a lattice in the vector-space of differentials of the first kind.

Also on C there exist differentials $\beta_1, \ldots, \beta_g$ of the second kind which can be integrated on Z such that

$$\frac{du_1}{u_1}, \ldots, \frac{du_g}{u_g}, \beta_1, \ldots, \beta_g$$

form a basis for the de Rham cohomology group $H^1_{\mathrm{DR}}(C)$. The integrals ζ_i of β_i can be chosen in such a way that

$$\zeta_i - \zeta_i \circ \gamma_j = \begin{cases} 1: & i = j \\ 0: & i \neq j . \end{cases}$$

The aim of this article is to show how the periods q_{ij} are related to the Gauss-Manin connection ∇ of a family of p-adic Schottky groups. The main result says that

$$\nabla\left(\frac{du_i}{u_i}\right) = \sum_{j=1}^{g} \beta_j \otimes \frac{dq_{ij}}{q_{ij}}$$

$$\nabla(\beta_i) = 0 .$$

In Sect. 1 the basic notations are introduced among which are the p-adic Siegel half space $\mathscr{S}$, the canonical affine torus $\mathscr{T}$ over $\mathscr{S}$, the canonical lattice Λ and the abelian variety $\mathfrak{A} = \mathscr{T}/\Lambda$ over $\mathscr{S}$. In Sect. 2 principal theta functions on $\mathscr{T}$ are treated. It is shown that the complements of the set of zeroes of canonically chosen theta functions are a covering of $\mathcal{T}$. This covering is used in the computation of the relative de Rham cohomology modules and the Gauss-Manin connection for $\mathfrak{A} \to \mathcal{S}$ following the procedure of Katz-Oda, [KO], in Sect. 4. The essential point is the introduction of the Λ-invariant differentials

$$\omega_i = \frac{dz_i}{z_i} + \sum_{j=1}^{g} \zeta_j \frac{dq_{ij}}{q_{ij}}$$

in Subsect. (3.3).

In Sect. 5 the main result is derived.

In Sect. 6 we give a simple application and derive the p-adic analogue to the classical formula relating the period q and Legendre parameter λ of an elliptic Mumford (=Tate) curve. I have obtained similar formulas for Mumford curves given by the equation $y^n = \frac{x^m - \lambda}{x^m - 1}$ but will give the details in a subsequent paper.

1. Notations

(1.1) K denotes an algebraically closed field of characteristic 0 together with a complete non-archimedean non-trivial valuation.

If X is a rigid K-analytic variety, see i.e. [BGR, Chap. 9, (9.3.1)] for the basic definitions, then $\mathcal{O}_X$ denotes the sheaf of analytic functions on X. Ω_X denotes the $\mathcal{O}_X$-module of analytic differential forms of degree 1, see [BKKN] or [Ki].

(1.2) Let M denote a free abelian group of finite rank g. By $T(M)$ we understand the analytic variety which is associated to the K-affine torus with character group M.

As an abstract group $T(M)$ coincides with $\mathrm{Hom}(M, K^*)$, K^* multiplicative group of K, and the multiplication in $T(M)$ is given by

$$(a \cdot a')(m) = a(m) \cdot a'(m)$$

for $a, a' \in T(M)$. Any $m \in M$ gives rise to a regular function z^m on $T(M)$ which satisfies

$$z^m(a) := a(m).$$

(1.3) Let M_2 be the symmetric group of degree 2 over M. Thus M_2 can be identified with $M \otimes M/U$, where U is the subgroup generated by $m \otimes m' - m' \otimes m$, $m, m' \in M$. The class of $m \otimes m'$ in M_2 will be denoted by $m \cdot m'$.

We denote by q^n, $n \in M_2$, the analytic function on the affine torus $T(M_2)$ given by $q^n(b) := b(n)$ for any $b \in T(M_2) = \mathrm{Hom}(M_2, K^*)$. The subdomain $\mathcal{S}(M)$ of $T(M_2)$ given by $\mathcal{S}(M) := \{b \in T(M_2)\colon \exists \varrho < 1$ such that $|b(m^2)| < \varrho$ for all $m \in M$, $m \neq 0\}$ is a Stein subdomain of $T(M_2)$, see [G1, Sect. (4)].

It can be called Siegel half space with respect to M, see [G1, Sect. (1)]. If $e_1, \ldots, e_g$ is a basis of M, then $\{e_i e_j\colon 1 \leqq i \leqq j \leqq g\}$ is a basis of M_2 and $T(M_2)$ can be identified with the set of all symmetric $g \times g$-matrices $b = (b_{ij})$, $b_{ij} = b_{ji} \in K^*$, for which the associated real symmetric matrix $(-\log|b_{ij}|)$ is positive definite.

(1.4) Denote by $\mathcal{T}(M)$ the product $T(M) \times \mathcal{S}(M)$ and by $\pi: \mathcal{T}(M) \to \mathcal{S}(M)$ the projection onto the second factor. We keep M fixed and denote $\mathcal{T}(M), \mathcal{S}(M)$ by $\mathcal{T}, \mathcal{S}$ respectively. Let $\lambda_m, m \in M$, be the mapping $\mathcal{T}(M) \to \mathcal{T}(M)$ which sends (a, b) onto (a', b) where

$$a'(m') := a(m') \cdot b(m \cdot m')$$

λ_m is bianalytic and $\lambda_{m+m'} = \lambda_m \circ \lambda_{m'}$.

The group Λ of all the bianalytic mappings λ_m, $m \in M$, is canonically isomorphic to M. The homomorphism $m \to \lambda_m$ is refered to as the canonical polarization of the family $\mathcal{T} \xrightarrow{\pi} \mathcal{S}$. Let $\mathfrak{a} = \mathcal{T} \bmod \Lambda$ be the space of Λ-orbits of $\mathcal{T}$. $\mathfrak{A}$ can be endowed with a structure as in [G 3] or [FP] or [M], that $\mathfrak{A} \to \mathcal{S}$ is an analytic family of abelian varieties.

2. Principal Theta Functions

(2.1) Let $h: M \to M_2$ be a homomorphism which satisfies

$$h(m) = m^2 \bmod 2M_2$$

for all $m \in M$.

In order to show the existence of such homomorphism we fix a basis $e_1, \ldots, e_g$ of M and consider the homomorphism h for which $h(e_i) = e_i^2$.

Then $h\left(\sum_{i=1}^{g} r_i e_i\right) = \sum_{i=1}^{g} r_i e_i^2$ while

$$\left(\sum_{i=1}^{g} r_i e_i\right)^2 = \sum_{i=1}^{g} r_i^2 e_i^2 + 2 \sum_{i<j} r_i r_j e_i e_j .$$

As $r_i \equiv r_i^2 \bmod 2\mathbb{Z}$ one gets $h(m) \equiv m^2$ for all $m \in M$.

If h' is another homomorphism $M \to M_2$ for which $h'(m) \equiv m^2 \bmod 2M_2$ for all $m \in M$ then $h' - h = 2h_0$ and h_0 is a homomorphism $M \to M_2$. Conversely if $h' := h + 2h_0$ then $h'(m) \equiv m^2 \bmod 2M_2$ for all $m \in M$.

We call any such h a principal theta characteristic for M.

(2.2) Let

$$\theta_h = \sum_{m \in M} q^{\frac{1}{2}(m^2 + h(m))} z^m ,$$

where h is a principal theta characteristic for M. We call θ_h the principal theta functions with respect to h.

As $\frac{1}{2}(m^2 + h(m)) \in M_2$ the expressions $q^{\frac{1}{2}(m^2 + h(m))} \cdot z^m$ are analytic functions on $\mathcal{T}$.

Proposition 1. *θ_h is an analytic function $\mathcal{T}$.*
The functional equation for θ_h is

$$\theta_h = q^{\frac{1}{2}(k^2 + h(k))} z^k \cdot (\theta_h \circ \lambda_k) .$$

Proof. By straightforward computations as in [GP, Chap. VI, Sect. 3, p. 197]. One has to use the fact that $\mathcal{S} = \bigcup_{0 < \varrho < 1} \mathcal{S}_\varrho$ where

$$\mathcal{S}_\varrho = \{b \in \mathcal{S} \colon \ |b(m^2)| < \varrho^{\|m\|^2} \text{ for all } m \in M\} ,$$

where $\| \ \|$ denotes a euclidean norm on M, see [G 1, Sect. (4), p. 5].

Also use

$$z^m \circ \lambda_k = q^{km} z^m \quad \text{for} \quad m \in M, \qquad q^n \circ \lambda_k = q^n \quad \text{any} \quad n \in M_2.$$

(2.3) Any $c \in T(M)$ can be considered as a bianalytic transformation $\mathscr{T} \to \mathscr{T}$ which sends a point (a, b), $a \in T(M)$, $b \in \mathscr{S}(M)$ to $(a \cdot c, b)$ where $(a \cdot c)(m) = a(m) \cdot c(m)$ for $m \in M$.

Thus $T(M)$ is a transformation group of $\mathscr{T}$.

The function $\theta_h \circ c$ is analytic on $\mathscr{T}$ and satisfies the functional equation

$$(\theta_h \circ c) = q^{\frac{1}{2}(k^2 + h(k))} c(k) \cdot z^k \cdot (\theta_h \circ c \circ \lambda_k)$$

as $c \circ \lambda_k = \lambda_k \circ c$ for any $c \in T(M)$ and $z^k \circ c = c(k) z^k$.

(2.4) Let E be the group generated by the elements of order 2 in $T(M)$. E is a group of order 2^g.

For any $\varepsilon \in E$ let $\mathscr{T}_\varepsilon = \{(a, b) \in \mathscr{T}\colon \ (\theta_h \circ \varepsilon)(a, b) \neq 0\}$.

Then $\mathscr{T}_\varepsilon = \varepsilon^{-1}(\mathscr{T}_1)$ where 1 denotes the neutral element in E.

Proposition 2. *$\mathscr{T}_\varepsilon$ is Λ-invariant and* $\bigcup_{\varepsilon \in E} \mathscr{T}_\varepsilon = \mathscr{T}$.

Proof. The fact that $\mathscr{T}_\varepsilon$ is Λ-invariant is derived immediately from the functional equation for $\theta_h \circ \varepsilon$.

In order to prove the second statement we have to show that there does not exist a point $(a, b) \in \mathscr{T}$ such that $(\theta_h \circ \varepsilon)(a, b) = 0$ for all $\varepsilon \in E$.

Let us assume that (a, b) is a point such that

$$(\theta_h \circ \varepsilon)(a, b) = 0$$

for all $\varepsilon \in E$.

Now $\theta_h = \sum_{m \in M} q^{\frac{1}{2}(m^2 + h(m))} z^m$ and

$$(\theta_h \circ \varepsilon)(a, b) = \sum_{m \in M} b(\tfrac{1}{2}(m^2 + h(m)) \cdot \varepsilon(m) \cdot a(m).$$

Let $f(m) = b(\frac{1}{2}(m^2 + h(m)) \cdot a(m)$.

Fix an index m_0 such that

$$|f(m_0)| \geqq |f(n)|$$

for all $n \in M$.

We put $n = m_0 + k$, substitute and obtain

$$|f(m_0)| \geqq |f(m_0 + k)|$$

for all $k \in M$. Now

$$\begin{aligned} \frac{f(m+k)}{f(m)} &= \frac{b(\frac{1}{2}(m^2 + 2mk + k^2 + h(k) + h(m))) \cdot a(m) a(k)}{b(\frac{1}{2}(m^2 + h(m))) \cdot a(m)} \\ &= b(mk) \cdot b(\tfrac{1}{2}(k^2 + h(k)) \cdot a(k). \end{aligned}$$

If $k = 2r$ is even and $r \neq 0$ one obtains

$$\begin{aligned} \left| \frac{f(m_0 + 2r)}{f(m_0)} \right| &= |b(m_0 r)^2 \cdot b(\tfrac{1}{2}(r^2 + h(r)))^2 \cdot b(r^2) a(r)^2| \\ &= \left| \left(\frac{f(m_0 + r)}{f(m_0)} \right)^2 \right| \cdot |b(r^2)| < |b(r^2)| < 1. \end{aligned}$$

Now $\dfrac{1}{\varepsilon(m_0)} (\theta_h \circ \varepsilon)(a, b) = \sum_{m \in M} f(m) \cdot \dfrac{\varepsilon(m)}{\varepsilon(m_0)}$

and

$$\sum_{\varepsilon\in E} f(m)\varepsilon(m-m_0)=\begin{cases}0: & m-m_0 \not\equiv 0 \bmod 2M\\ 2^g f(m): & m-m_0\equiv 0 \bmod 2M.\end{cases}$$

Thus $0=\sum_{\varepsilon\in E}\frac{1}{\varepsilon(m_0)}(\theta_h\circ\varepsilon)(a,b)=2^g\sum_{k\in 2M} f(m_0+k)$ but $|f(m_0+k)|<|f(m_0)|$ for all $k\in 2M$, $k\neq 0$.

This is a contradiction to the assumption that $(a,b)\in\bigcap_{\varepsilon\in E}\mathscr{T}_\varepsilon$.

3. Invariant Differentials

(3.1) We fix a basis $e_1,\dots,e_g$ of M and denote z^{e_i} by z_i and $q^{e_ie_j}$ by $q_{ij}=q_{ji}$. $\Omega_{\mathscr{T}}$ is a free $\mathcal{O}_{\mathscr{T}}$-module with basis $\left\{\frac{dz_i}{z_i},\frac{dq_{ij}}{q_{ij}}: 1\leqq i\leqq j\leqq g\right\}$.

The action of Λ on $\Omega_{\mathscr{T}}$ is given by

$$\frac{dz_i}{z_i}\circ\lambda_{e_k}=\frac{dz_i}{z_i}+\frac{dq_{ik}}{q_{ik}}$$

$$\frac{dq_{ij}}{q_{ij}}\circ\lambda_{e_k}=\frac{dq_{ij}}{q_{ij}}$$

because $z_i\circ\lambda_{e_k}=q_{ik}\cdot z_k$, $q_{ij}\circ\lambda_{e_k}=q_{ij}$.

The action of E on $\Omega_{\mathscr{T}}$ is given by

$$\frac{dz_i}{z_i}\circ\varepsilon=\frac{dz_i}{z_i}$$

$$\frac{dq_{ij}}{q_{ij}}\circ\varepsilon=\frac{dq_{ij}}{q_{ij}}$$

because $z_i\circ\varepsilon=\varepsilon(e_i)\cdot z_i$, $q_{ij}\circ\varepsilon=q_{ij}$.

(3.2) Let ∂_i, D_{ij} be the vectorfields on $\mathscr{T}$ defined by

$$df=\sum_{i=1}^g \partial_i(f)\frac{dz_i}{z_i}+\sum_{1\leqq i\leqq j\leqq g} D_{ij}(f)\frac{dq_{ij}}{q_{ij}}$$

for $f\in\mathcal{O}_{\mathscr{T}}(U)$, U open subdomain of $\mathscr{T}$.

Using the fact $df\circ\lambda_{e_k}=d(f\circ\lambda_{e_k})$ and the formulas in (3.1) one obtains

$$\sum_{i=1}^g \partial_i(f\circ\lambda_{e_k})\frac{dz_i}{z_i}+\sum_{i\leqq j} D_{ij}(f\circ\lambda_{e_k})\frac{dq_{ij}}{q_{ij}}$$
$$=\sum_{i=1}^g(\partial_i(f)\circ\lambda_{e_k})\left(\frac{dz_i}{z_i}+\frac{dq_{ik}}{q_{ik}}\right)+\sum_{i\leqq j}(D_{ij}(f)\circ\lambda_{e_k})\frac{dq_{ij}}{q_{ij}}$$

which shows that

$$\partial_i(f)\circ\lambda_{e_k}=\partial_i(f\circ\lambda_{e_k}).$$

If f is Λ-invariant then $\partial_i(f)$ is also Λ-invariant.

In the same way one obtains

$$\partial_i(f) \circ \varepsilon = \partial_i(f \circ \varepsilon)$$

for any $\varepsilon \in E$.

(3.3) Next we introduce a basis of Λ-invariant analytic differentials on the domain $\mathscr{T}_1 = \{(a,b) \in \mathscr{T}:\ \theta_h(a,b) \neq 0\}$.

Let

$$\zeta_i := \frac{\partial_i(\theta)}{\theta},$$

where θ is a principal theta function with respect to a principal characteristic h of M.

We derive the functional equation for ζ_i:

$$\zeta_i \circ \lambda_{e_k} = \frac{\partial_i(\theta \circ \lambda_{e_k})}{\theta \circ \lambda_{e_k}} = \frac{\partial_i(q^{-\frac{1}{2}(e_k^2 + h(e_k))} z_k^{-1} \theta)}{q^{-\frac{1}{2}(e_k^2 + h(e_k))} z_k^{-1} \cdot \theta}$$

$$= \begin{cases} \zeta_i: & i \neq k \\ \zeta_i - 1: & i = k \end{cases}$$

because $\partial_i(z_k^{-1}) = \begin{cases} 0: & i \neq k \\ -z_k^{-1}: & i = k. \end{cases}$

Let $\omega_i = \dfrac{dz_i}{z_i} + \sum_{j=1}^{g} \zeta_j \dfrac{dq_{ij}}{q_{ij}}$.

Proposition 3. $\left\{\omega_i, \dfrac{dq_{ij}}{q_{ij}}:\ 1 \leqq i \leqq j \leqq g\right\}$ *is a basis of Λ-invariant differentials in* $\Omega_{\mathscr{T}} | \mathscr{T}_1$.

Proof.

$$\omega_i \circ \lambda_{e_k} = \left(\frac{dz_i}{z_i} + \frac{dq_{ik}}{q_{ik}}\right) + \sum_{j=1}^{g} (\zeta_i \circ \lambda_{e_k}) \frac{dq_{ij}}{q_{ij}}$$

$$= \frac{dz_i}{z_i} + \frac{dq_{ik}}{q_{ik}} + \sum_{\substack{j=1 \\ j \neq k}}^{g} \zeta_i \frac{dq_{ij}}{q_{ij}} + (\zeta_k - 1) \frac{dq_{ik}}{q_{ik}} = \omega_i.$$

(3.4) The result of (3.3) extends to the domains $\mathscr{T}_\varepsilon$.

Let $\omega_{\varepsilon i} := \omega_i \circ \varepsilon$, $\zeta_{\varepsilon i} = \zeta_i \circ \varepsilon$.

Then $\omega_{\varepsilon i} = \dfrac{dz_i}{z_i} + \sum_{j=1}^{g} \zeta_{\varepsilon i} \dfrac{dq_{ij}}{q_{ij}}$ and $\zeta_{\varepsilon i} = \dfrac{\partial_i(\theta \circ \varepsilon)}{\theta \circ \varepsilon}$ by (3.2).

Proposition 3′. $\left\{\omega_{\varepsilon i}, \dfrac{dq_{ij}}{q_{ij}}:\ 1 \leqq i \leqq j \leqq g\right\}$ *is a basis of Λ-invariant differentials in* $\Omega_{\mathscr{T}} | \mathscr{T}_\varepsilon$.

(3.5) **Proposition 4.** *The differential*

$$\beta_{\varepsilon i} := d\zeta_{\varepsilon i}$$

is Λ-invariant.

Proof.

$$\beta_{\varepsilon i} \circ \lambda_{e_k} = d(\zeta_{\varepsilon i} \circ \lambda_{e_k}) = d(\zeta_{\varepsilon i} - \delta_{ik}) = d\zeta_{\varepsilon i} = \beta_{\varepsilon i},$$

where

$$\delta_{ik} = \begin{cases} 0: & i \neq k \\ 1: & i = k. \end{cases}$$

(3.6) The relative de Rham cohomology sheaf $\mathscr{H}^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S})$ of $\pi: \mathfrak{A} \to \mathscr{S}$ is by definition $R^1\pi_*(\Omega^{\cdot}_{\mathfrak{A}/\mathscr{S}})$ where $\Omega^{\cdot}_{\mathfrak{A}/\mathscr{S}}$ denotes the complex of $\mathscr{S}$-differentials on $\mathfrak{A}$ and $R^1\pi_*$ is the first hyperderived functor of π_*, see [H, Chap. II, Sect. 2]. As $\mathscr{S}$ is a Stein manifold the module $H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S})$ of global sections of $\mathscr{H}^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S})$ on $\mathscr{S}$ can be computed by a Čech calculation with respect to the covering $\mathfrak{E} = \{\mathfrak{A}_\varepsilon : \varepsilon \in E\}$ where $\mathfrak{A}_\varepsilon = \mathscr{T}_\varepsilon \bmod \Lambda$, because $\mathfrak{A}_\varepsilon$ is affine over $\mathscr{S}$, see [KO, Sect. 3, p. 205] or [K 2, (1.4), p. 15].

(3.7) Let $\mathscr{G}^{\cdot}$ be a positive complex $0 \to \mathscr{G}^0 \xrightarrow{\partial^0} \mathscr{G}^1 \xrightarrow{\partial^1} \mathscr{G}^2 \to \dots$ of $\mathcal{O}_{\mathfrak{A}}$-modules (resp. of $\mathcal{O}_{\mathscr{T}}$-modules).

Let $\mathscr{C}^{\cdot}(\mathfrak{E}, \mathscr{G}^q)$ be the Čech-complex of alternating cochains of the sheaf $\mathscr{G}^q$ with respect to the covering $\mathfrak{E} = (\mathfrak{A}_\varepsilon)_{\varepsilon \in E}$ of $\mathfrak{A}$ (resp. with respect to the covering $\mathfrak{E} = (\mathscr{T}_\varepsilon)_{\varepsilon \in E}$ of $\mathscr{T}$. The coboundary operators of $\mathscr{C}^{\cdot}(\mathfrak{E}, \mathscr{G}^q)$ will be denoted by δ. The coboundary operators of $\mathscr{G}^{\cdot}$ induce coboundary operators $\mathscr{C}^{\cdot}(\mathfrak{E}, \mathscr{G}^q) \xrightarrow{\partial^q} \mathscr{C}^{\cdot}(\mathfrak{E}, \mathscr{G}^{q+1})$ of complexes. One obtains a double complex $\mathscr{C}^{\cdot}(\mathfrak{E}, \mathscr{G}^{\cdot})$. Let $\mathscr{K}^{\cdot}(\mathfrak{E}, \mathscr{G}^{\cdot})$ be the associated simple complex which is a complex of $\mathcal{O}_{\mathscr{S}}(\mathscr{S})$-modules.

By definition

$$\mathscr{K}^r(\mathfrak{E}, \mathscr{G}^{\cdot}) = \bigoplus_{p+q=r} \mathscr{C}^p(\mathfrak{E}, \mathscr{G}^q).$$

The coboundary operator $d^r: \mathscr{K}^r(\mathfrak{E}, \mathscr{G}^{\cdot}) \to \mathscr{K}^{r+1}(\mathfrak{E}, \mathscr{G}^{\cdot})$ is defined on $\mathscr{C}^p(\mathfrak{E}, \mathscr{G}^q)$ by $\partial^p + (-1)^q \delta^p$. If $\mathscr{G}^{\cdot} \to \mathscr{F}^{\cdot}$ is a mapping of complexes of modules sheaves, then there is an associated mapping $\mathscr{K}^{\cdot}(\mathfrak{E}, \mathscr{G}^{\cdot}) \to \mathscr{K}^{\cdot}(\mathfrak{E}, \mathscr{F}^{\cdot})$ of complexes of $\mathcal{O}_{\mathscr{S}}(\mathscr{S})$-modules.

(3.8) According to [KO, Sect. 3, p. 206], the hypercohomology group $H^q(\mathfrak{A}, \mathscr{G}^{\cdot})$ of a complex $\mathscr{G}^{\cdot}$ of $\mathcal{O}_{\mathfrak{A}}$-modules is the q^{th} cohomology group of the complex $\mathscr{K}^{\cdot}(\mathfrak{E}, \mathscr{G}^{\cdot})$.

Applying this to the de Rham complex

$$\Omega^{\cdot}_{\mathfrak{A}/\mathscr{S}}$$

one obtains the result that $H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S})$ is the first cohomology group of the complex

$$\mathscr{K}^{\cdot}(\mathfrak{E}, \Omega^{\cdot}_{\mathfrak{A}/\mathscr{S}}).$$

Let now $\mathscr{K}^q = \mathscr{K}^q(\mathfrak{E}, \Omega^{\cdot}_{\mathfrak{A}/\mathscr{S}})$ and $d^q: \mathscr{K}^q \to \mathscr{K}^{q+1}$ denote the coboundary operator. One has

$$H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S}) = \operatorname{Kernel} d^1 / d^0(\mathscr{K}^0).$$

Now

$$\mathscr{K}^0 = \mathscr{C}^0(\mathfrak{E}, \mathcal{O}_{\mathfrak{A}})$$

$$\mathscr{K}^1 = \mathscr{C}^0(\mathfrak{E}, \Omega_{\mathfrak{A}/\mathscr{S}}) \oplus \mathscr{C}^1(\mathfrak{E}, \mathcal{O}_{\mathfrak{A}})$$

$$\mathscr{K}^2 = \mathscr{C}^0(\mathfrak{E}, \Omega^2_{\mathfrak{A}/\mathscr{S}}) \oplus \mathscr{C}^1(\mathfrak{E}, \Omega_{\mathfrak{A}/\mathscr{S}}) \oplus \mathscr{C}^2(\mathfrak{E}, \mathcal{O}_{\mathfrak{A}})$$

and $d^0: \mathscr{K}^0 \to \mathscr{K}^1$ is given by

$$d^0(f) = \delta f \oplus (df_\varepsilon)_{\varepsilon \in E}$$

for $f = (f_\varepsilon)_{\varepsilon \in E} \in \mathscr{C}^0(\mathfrak{E}, \mathcal{O}_{\mathfrak{A}})$, $f_\varepsilon \in \mathcal{O}_{\mathfrak{A}}(\mathfrak{A}_\varepsilon)$.

Here δf is the Čech 1-cochain given by $(\delta f)_{\varepsilon,\varepsilon'}=(f_{\varepsilon'}-f_\varepsilon)|\mathfrak{A}_\varepsilon\cap\mathfrak{A}_{\varepsilon'}$.

Next we explicate the operator $d^1:\mathcal{K}^1\to\mathcal{K}^2$.

If $\omega=(\omega_\varepsilon)_{\varepsilon\in E}$ is an element of $\mathscr{C}^0(\mathfrak{E},\Omega_{\mathfrak{A}/\mathscr{S}})$, $\omega_\varepsilon\in\Omega_{\mathfrak{A}/\mathscr{S}}(\mathfrak{A}_\varepsilon)$, then

$$d^1\omega=\delta\omega\oplus(d\omega_\varepsilon)_{\varepsilon\in E},$$

where $\delta\omega$ is the Čech 1-cochain given by

$$(\delta\omega)_{\varepsilon,\varepsilon'}=(\omega_{\varepsilon'}-\omega_\varepsilon)|\mathfrak{A}_\varepsilon\cap\mathfrak{A}_{\varepsilon'},$$

and $d\omega_\varepsilon$ is the differential of ω_ε in $\Omega^2_{\mathfrak{A}/\mathscr{S}}(\mathfrak{A}_\varepsilon)$.

If $f=(f_{\varepsilon,\varepsilon'})$ is a 1-cochain in $\mathscr{C}^1(\mathfrak{E},\mathcal{O}_{\mathfrak{A}})$, $f_{\varepsilon,\varepsilon'}\in\mathcal{O}_{\mathfrak{A}}((\mathfrak{A}_\varepsilon\cap\mathfrak{A}_{\varepsilon'})$, then

$$d^1f=\delta f\oplus(df_{\varepsilon,\varepsilon'})_{\varepsilon,\varepsilon'\in E},$$

where δf is the Čech 2-cochain given by

$$(\delta f)_{\varepsilon,\varepsilon',\varepsilon''}=(f_{\varepsilon'\varepsilon''}-f_{\varepsilon\varepsilon''}+f_{\varepsilon\varepsilon'})|\mathfrak{A}_\varepsilon\cap\mathfrak{A}_{\varepsilon'}\cap\mathfrak{A}_{\varepsilon''}$$

and $df_{\varepsilon,\varepsilon'}$ is the differential of $f_{\varepsilon,\varepsilon'}$ in $\Omega^1_{\mathfrak{A}/\mathscr{S}}(\mathfrak{A}_\varepsilon\cap\mathfrak{A}_{\varepsilon'})$. We consider now $\frac{dz_i}{z_i}$ as element of $\mathscr{C}^0(\mathfrak{E},\Omega_{\mathfrak{A}/\mathscr{S}})$, namely as constant 0-cochain taking the same value $\frac{dz_i}{z_i}$ in all the domains $\mathfrak{A}_\varepsilon$. Then $\frac{dz_i}{z_i}$ is also an element α_i' of $\mathcal{K}^1$ as $\mathscr{C}^0(\mathfrak{E},\Omega_{\mathfrak{A}/\mathscr{S}})$ is a direct summand of $\mathcal{K}^1$. The differential $d\left(\frac{dz_i}{z_i}\right)=0$ in the complex $\Omega^{\cdot}_{\mathfrak{A}/\mathscr{S}}$ and thus

$$d^1(\alpha_i')=0.$$

Let α_i denote the cohomology class of α_i' in $H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S})=\mathrm{Kernel}\,d^1/d^0(\mathcal{K}^0)$.

The function $\zeta_{\varepsilon i}-\zeta_{\varepsilon' i}$ is Λ-invariant on $\mathscr{T}_\varepsilon\cap\mathscr{T}_{\varepsilon'}$, see (3.4), and thus an element of $\mathcal{O}_{\mathfrak{A}}(\mathfrak{A}_\varepsilon\cap\mathfrak{A}_{\varepsilon'})$. Therefore

$$\beta_i':=(\zeta_{\varepsilon i}-\zeta_{\varepsilon' i})_{\varepsilon,\varepsilon'\in E}$$

is an element of $\mathscr{C}^1(\mathfrak{E},\mathcal{O}_{\mathfrak{A}})$ with $\delta\beta_i'=0$.

Let β_i'' be the Čech 0-cochain in $\mathscr{C}^0(\mathfrak{E},\Omega_{\mathfrak{A}/\mathscr{S}})$ given by

$$(\beta_i'')_\varepsilon:=d\zeta_{\varepsilon i}.$$

Now $\beta_i'+\beta_i''$ is an element of $\mathcal{K}^2$.

Let β_i denote the cohomology class of $\beta_i'+\beta_i''$ in $H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S})$.

Proposition 5. *$H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S})$ is freely generated over $\mathcal{O}_{\mathscr{S}}(\mathscr{S})$ by $\{\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g\}$.*

Proof. Consider the canonical commutative diagram

$$\begin{array}{ccccc}
\mathcal{K}^0 & \xrightarrow{d^0} & \mathcal{K}^1 & \xrightarrow{d^1} & \mathcal{K}^2\\
\downarrow & & \downarrow & & \downarrow\\
\mathcal{K}^0(\mathscr{T}) & \xrightarrow{d^0_{\mathscr{T}}} & \mathcal{K}^1(\mathscr{T}) & \xrightarrow{d^1_{\mathscr{T}}} & \mathcal{K}^2(\mathscr{T}),
\end{array}$$

where the second row is a piece of the complex $\mathcal{K}^{\cdot}(\mathfrak{E},\Omega^{\cdot}_{\mathscr{T}/\mathscr{S}})$, see (3.7).

$\mathscr{T}$ is a Stein domain and thus any coherent sheaf on $\mathscr{T}$ is acyclic. Therefore

$$\operatorname{Kernel} d^1_{\mathscr{T}}/\operatorname{Image} d^0_{\mathscr{T}}$$

is freely generated over $\mathcal{O}_{\mathscr{S}}(\mathscr{S})$ by $\alpha_1, \ldots, \alpha_g$, α_i being the cohomology class of $\dfrac{dz_i}{z_i}$.

Let now $\kappa \in \mathscr{K}^1$ with $d^1\kappa = 0$. Then there are $s_i \in \mathcal{O}_{\mathscr{S}}(\mathscr{S})$ such that

$$\kappa' := \kappa - \sum_{i=1}^{g} s_i \frac{dz_i}{z_i}$$

is of the form $\kappa' = d^0_{\mathscr{T}}(\tau)$, $\tau \in \mathscr{K}^0(\mathscr{T})$.

One cannot expect τ to be in $\mathscr{K}^0$. However

$$\tau = (\tau_\varepsilon)_{\varepsilon \in E} \in \mathscr{C}^0(\mathfrak{E}, \mathcal{O}_{\mathscr{T}})$$
$$\tau_\varepsilon \in \mathcal{O}_{\mathscr{T}}(\mathscr{T}_\varepsilon).$$

As $d^0_{\mathscr{T}}(\tau) = \kappa' \in \mathscr{K}^1$ we get

$$(\tau_\varepsilon - \tau_{\varepsilon'}) | \mathscr{T}_\varepsilon \cap \mathscr{T}_{\varepsilon'} \text{ is } \Lambda\text{-invariant}$$
$$d\tau_\varepsilon \text{ is } \Lambda\text{-invariant}.$$

As $\operatorname{char} K = 0$ we conclude from

$$d(\tau_\varepsilon \circ \lambda_{e_k} - \tau_\varepsilon) = 0$$

that there is a constant $c_{\varepsilon,k} \in K$ with

$$\tau_\varepsilon \circ \lambda_{e_k} - \tau_\varepsilon = c_{\varepsilon,k}.$$

But

$$\begin{aligned} c_{\varepsilon,k} - c_{\varepsilon',k} &= (\tau_\varepsilon \circ \lambda_{e_k} - \tau_\varepsilon) - (\tau_{\varepsilon'} \circ \lambda_{e_k} - \tau_{\varepsilon'}) \\ &= (\tau_\varepsilon - \tau_{\varepsilon'}) \circ \lambda_{e_k} - (\tau_\varepsilon - \tau_{\varepsilon'}) \\ &= 0 \end{aligned}$$

as $\tau_\varepsilon - \tau_{\varepsilon'}$ is Λ-invariant.

Thus $c_k := c_{\varepsilon,k}$ does not depend on ε.

Let $\kappa'' := \kappa' + \sum_{k=1}^{g} c_k(\beta'_k + \beta''_k) \in \mathscr{K}^1$.

Using the functional equation for $\zeta_{\varepsilon i}$, see (3.4), one finds that $\kappa'' \in d^0(\mathscr{K}^0)$. Along similar lines one proves that $\alpha_1, \ldots, \beta_g$ is a basis.

(3.9) The elements β_i do not depend on the choice of the principal theta characteristic h, see (3.1). If h' is another such characteristic, then

$$h' - h = 2h_0, \quad h_0 \in \operatorname{Hom}(M, M_2).$$

Let $c: \mathscr{T} \to \mathscr{T}$ be the analytic mapping which sends $(a, b) \in \mathscr{T}$ into (a', b)

$$a'(m) := b(h_0(m)) \cdot a(m),$$

where $a \in T$, $b \in \mathscr{S}$.

Then $\pi \circ c = \pi$ and

$$\theta_{h'} = \sum_{m \in M} q^{\frac{1}{2}(h(m)+m^2)} q^{h_0(m)} \cdot z^m$$

is equal to $\theta_h \circ c$.

If $\zeta'_i = \frac{1}{\theta_{h'}} \cdot \partial_i(\theta_{h'})$, $\zeta'_{\varepsilon i} = \zeta'_i \circ \varepsilon$ and $\mathfrak{A}'_\varepsilon = \{\zeta'_{\varepsilon i} \neq 0\}$, $\mathfrak{E}' = \{\mathfrak{A}'_\varepsilon\}$, then

$$\zeta'_{\varepsilon i} - \zeta_{\varepsilon' i}$$

is Λ-invariant for all $\varepsilon, \varepsilon' \in E$.

If $\mathfrak{E}^*$ is a refinement of $\mathfrak{E}$ and of $\mathfrak{E}'$ one obtains canonical homomorphisms

$$\mathscr{K}^\cdot(\mathfrak{E}, \Omega^\cdot_{\mathfrak{A}/\mathscr{S}}) \to \mathscr{K}^\cdot(\mathfrak{E}^*, \Omega^\cdot_{\mathfrak{A}/\mathscr{S}})$$

$$\mathscr{K}^\cdot(\mathfrak{E}', \Omega^\cdot_{\mathfrak{A}/\mathscr{S}}) \to \mathscr{K}^\cdot(\mathfrak{E}^*, \Omega^\cdot_{\mathfrak{A}/\mathscr{S}})$$

which induce isomorphisms for the cohomology.

4. Gauss-Manin Connection for $\mathfrak{A} \to \mathscr{S}$

(4.1) The Gauss-Manin connection for the family $\pi: \mathfrak{A} \to \mathscr{S}$ of abelian varieties on the first De Rham cohomology group $H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S})$ which is a module over $\mathcal{O}_{\mathscr{S}}(\mathscr{S})$, is a connection

$$\nabla: H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S}) \to H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S}) \otimes \Omega_{\mathscr{S}}(\mathscr{S}),$$

the tensor product being taken over $\mathcal{O}_{\mathscr{S}}(\mathscr{S})$. $\Omega_{\mathscr{S}}(\mathscr{S})$ is a free module over $\mathcal{O}_{\mathscr{S}}(\mathscr{S})$, freely generated by $\left\{\frac{dq_{ij}}{q_{ij}}: 1 \leqq i \leqq j \leqq g\right\}$. We use the notation of Sect. (3.8).

Theorem 1. $\nabla(\beta_i) = 0$ *for all* i

$$\nabla(\alpha_i) = \sum_{j=1}^{g} \beta_j \otimes \frac{dq_{ij}}{q_{ij}} \quad \textit{for all } i.$$

(4.2) We consider two pieces of the Koszul filtration of the exact sequence of locally free sheaves on

$$0 \to \pi^*(\Omega^1_{\mathscr{S}}) \to \Omega^1_{\mathfrak{A}} \to \Omega^1_{\mathfrak{A}/\mathscr{S}} \to 0,$$

see [K 2, (1.2), p. 8].

Denote by $F^\cdot_1$ the subcomplex of $\Omega^\cdot_{\mathfrak{A}}$ of $\mathcal{O}_{\mathfrak{A}}$-modules generated by $\pi^*(\Omega_{\mathscr{S}})$. Thus F^k_1 is the module generated by sections of the form $\omega \wedge \frac{dq_{ij}}{q_{ij}}$, $\omega \in \Omega^{k-1}_{\mathfrak{A}}$. As z_i, q_{ij} are local coordinates for $\mathfrak{A}$, one gets $\Omega^k_{\mathfrak{A}}/F^k_1 = \Omega^k_{\mathfrak{A}/\mathscr{S}}$.

Denote by $F^\cdot_2$ the subcomplex of $\Omega^\cdot_{\mathfrak{A}}$ of $\mathcal{O}_{\mathfrak{A}}$-modules generated by $\pi^*(\Omega^2_{\mathscr{S}})$. We consider the exact sequence of complexes

$$0 \to F^\cdot_1/F^\cdot_2 \to \Omega^\cdot_{\mathfrak{A}}/F^\cdot_2 \to \Omega^\cdot_{\mathfrak{A}/\mathscr{S}} \to 0$$

and

$$0 \to \mathscr{K}^i(F^\cdot_1/F^\cdot_2) \to \mathscr{K}^i(\Omega^\cdot_{\mathfrak{A}}/F^\cdot_2) \to \mathscr{K}^i(\Omega^\cdot_{\mathfrak{A}/\mathscr{S}}) \to 0$$

for $i = 0, 1, 2$; see (3.7).

First, we observe that the second cohomology group of the complex $\mathscr{K}^{\cdot}(F_1^{\cdot}/F_2^{\cdot})$ is isomorphic to $H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S})\otimes\Omega_{\mathscr{S}}(\mathscr{S})$, see [KO, Sect. 3, p. 205–209]. Next we describe explicitly the connection ∇.

Let $\kappa\in H^1_{\mathrm{DR}}(\mathfrak{A}/\mathscr{S})$. Pick a representative ω of the cohomology class κ in $\mathscr{K}^1(\Omega^{\cdot}_{\mathfrak{a}/\mathscr{S}})$. As $\mathscr{K}^1(\Omega^{\cdot}_{\mathfrak{a}})\to\mathscr{K}^1(\Omega^{\cdot}_{\mathfrak{a}/\mathscr{S}})$ is surjective, one can find a representative $\omega'\in\mathscr{K}^1(\Omega^{\cdot}_{\mathfrak{a}}/F_2^{\cdot})$ of ω.

Now $d^1(\omega')\in\mathscr{K}^2(\Omega^{\cdot}_{\mathfrak{a}}/F_2)$. The image of $d^1(\omega')$ in $\mathscr{K}^2(\Omega^{\cdot}_{\mathfrak{a}/\mathscr{S}})$ is zero and thus $d^1(\omega)\in\mathscr{K}^2(F_2^{\cdot}/F_1^{\cdot})$. Also $d^2(d^1(\omega'))=0$. Therefore $d^1(\omega')$ gives rise to a cohomology class $\nabla(\omega)$ in $H^2(\mathscr{K}^{\cdot}(F_2^{\cdot}/F_1^{\cdot}))\cong H^1_{\mathrm{DR}}(\mathfrak{a}/\mathscr{S})\otimes\Omega_{\mathscr{S}}(\mathscr{S})$. It is standard to show that $\nabla(\omega)$ does not depend on the choice of ω and thus only depends on κ.

(4.3) The proof that $\nabla(\beta_i)=0$ is very simple.

We consider $(\zeta_{\varepsilon i})_{\varepsilon\in E}$ as element ω_i of $\mathscr{K}^0(\Omega^{\cdot}_{\mathfrak{a}})$ and let $\beta_i':=d^0{}_{\Omega^{\cdot}_{\mathfrak{a}}}(\omega_i)\in\mathscr{K}^1(\Omega^{\cdot}_{\mathfrak{a}})$. Then $d^1{}_{\Omega^{\cdot}_{\mathfrak{a}}}(\beta_i')=0$ and β_i' represents β_i which means that β_i' is mapped onto β_i by the canonical map $\mathscr{K}^1(\Omega^{\cdot}_{\mathfrak{a}})\to\mathscr{K}^1(\Omega^{\cdot}_{\mathfrak{a}/\mathscr{S}})$. Thus by (4.2) $\nabla(\beta_i)=0$.

(4.4) Next we prove $\nabla(\alpha_i)=\sum_{j=1}^{g}\beta_j\otimes\dfrac{dq_{ij}}{q_{ij}}$.

Let $\omega_{\varepsilon i}=\dfrac{dz_i}{z_i}+\sum_{j=1}^{g}\zeta_{\varepsilon j}\dfrac{dq_{ij}}{q_{ij}}$.

It was shown in (3.4) that $\omega_{\varepsilon i}\in\Omega_{\mathfrak{a}}(\mathfrak{a}_\varepsilon)$. We consider now $\omega_i:=(\omega_{\varepsilon i})_{\varepsilon\in E}$ as element of $\mathscr{K}^1(\Omega^{\cdot}_{\mathfrak{a}})$. Then ω_i represents α_i. We have to compute $d^1{}_{\Omega^{\cdot}_{\mathfrak{a}}}(\omega_i)$.

As

$$\omega_{\varepsilon i}-\omega_{\varepsilon' i}=\sum_{j=1}^{g}(\omega_{\varepsilon j}-\omega_{\varepsilon' j})\frac{dq_{ij}}{q_{ij}}$$

we get $(\omega_{\varepsilon i}-\omega_{\varepsilon' i})_{\varepsilon,\varepsilon'}=\sum_{j=1}^{g}\beta_j'\dfrac{dq_{ij}}{q_{ij}}$ with β_j' as defined in (3.8):

Also

$$d\omega_{\varepsilon i}=\sum_{j=1}^{g}d\zeta_{\varepsilon i}\wedge\frac{dq_{ij}}{q_{ij}}=\Sigma\beta_j''\wedge\frac{dq_{ij}}{q_{ij}}.$$

If one recalls the definition of d^1, one gets $\nabla\alpha_j=\sum_{j=1}^{g}\beta_i\otimes\dfrac{dq_{ij}}{q_{ij}}$.

5. Gauss-Manin Connection for Schottky Groups

(5.1) Let S be a rigid analytic space over K, $\mathbb{P}$ the projective line over K, and $\pi:\mathbb{P}\times S\to S$ the projection of the product space $\mathbb{P}\times S$ onto the second factor.

Denote by $\mathrm{Aut}_S(\mathbb{P}\times S)$ the group of those bianalytic mappings $\gamma:\mathbb{P}\times S\to\mathbb{P}\times S$ for which $\gamma\circ\pi=\pi$. One can prove that there is an admissible covering $\mathscr{S}=(S_i)_{i\in I}$ of S, see [BGR, 9.3], for the notion of admissible coverings, such that the restriction $\gamma|\mathbb{P}\times S_i$ is fractional-linear for each i which means that there is matrix

$$\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}\in\mathrm{GL}_2(\mathcal{O}_S(S_i))$$

such that

$$(\gamma|S_i)(z,s)=\left(\frac{a_i(s)z+b_i}{c_i(s)z+d_i},s\right).$$

For any point $s\in S$ one has a canonical restriction homomorphism

$$\mathrm{Aut}_S(\mathbb{P}\times S)\to \mathrm{PGL}_2(K)$$

by restricting $\gamma\in\mathrm{Aut}_S(\mathbb{P}\times S)$ to the fiber $\mathbb{P}\times\{s\}$ over s.

Definition. A subgroup $\Gamma\subset\mathrm{Aut}_S(\mathbb{P}\times S)$ is called a Schottky group over S (or a family of Schottky groups parametrized by S) if for any point $s\in S$ the corresponding restriction homomorphism gives an isomorphism between Γ and a Schottky subgroup Γ_s of $\mathrm{PGL}_2(K)$.

The following result is basic; a proof of it has been given in [P].

Proposition 5. *There is an admissible subdomain Z of $\mathbb{P}\times S$ such that for any $s\in S$ the intersection $Z\cap(\mathbb{P}\cap\{s\})$ is the domain of ordinary points for the Schottky group Γ_s. If S is affinoid, there is a subdomain F of Z such that*

$$\bigcup_{\gamma\in\Gamma}\gamma(F)=Z\,,\quad \{\gamma(F)\colon\gamma\in\Gamma\}\ \textit{is an admissible covering of}\ Z\,,$$

$$\gamma(F)\cap F\quad \textit{is empty for almost all}\ \gamma\in\Gamma\,.$$

There is an admissible covering (S_i) of S such that for any i, there is a basis $\gamma_{i1},\dots,\gamma_{ig}$ of Γ such that $\gamma_{i1}|\mathbb{P}\times\{s\},\dots,\gamma_{ig}|\mathbb{P}\times\{s\}$ is Schottky basis for each $s\in S_i$.

Corollary. *The quotient space $C=Z/\Gamma$ is an analytic space and the canonical mapping $C\to S$ is a family of Mumford curves of genus $g=\mathrm{rank}\,\Gamma$.*

The domain Z of Proposition 5 is called the domain of ordinary points of Γ or the domain of discontinuity of Γ.

(5.2) Let Γ be a Schottky group over S and Z the domain of discontinuity of Γ.

An analytic mapping $a\colon S\to Z$ is called an ordinary section von Γ, if $\pi\circ a=\mathrm{id}_S$. We need for the construction of functions ordinary sections.

Proposition 6. *There is an admissible covering (S_i) of S and for any i there is an ordinary section $a_i\colon S_i\to Z$ for Γ.*

Proof. We may assume that S is affinoid and $\gamma_1,\dots,\gamma_g$ is a Schottky basis for all the points of S, using Proposition 5. Moreover we may assume that there are matrices

$$\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}\in\mathrm{GL}_2(\mathcal{O}_S(S))$$

such that

$$\gamma_i(z,s)=\frac{a_i(s)z+b_i(s)}{c_i(s)z+d_i(s)}.$$

Let $\varphi_i(s)$ (resp. $\psi_i(s)=\varphi_{-i}(s)$) be the attracting fixed point of γ_i. Then φ_i,ψ_i are analytic mappings $S\to\mathbb{P}$. Fix $2g+1$ points $p_1,\dots,p_{2g+1}$ on $\mathbb{P}$. Then for any point $s\in S$ there is an index j such that $\varphi_i(s)\neq p_j$ for all $i\in\{\pm1,\dots,\pm g\}$.

Let $S_j := \{s \in S: \varphi_i(s) \neq p_j \text{ for all } i\}$. Then S_j is an admissible subdomain of S and $\{S_j\}_{1 \leqq j \leqq 2g+1}$ is an admissible covering of S. Without loss of generality we may assume that $S = S_1$, $\mathbb{P} = K \cup \{\infty\}$ and $p_1 = \infty$. z denotes the global coordinate on $\mathbb{P}$ with $z(\infty) = \infty$.

Let $w_i = \dfrac{z - \varphi_i}{z - \psi_i}$. Then $w_i \circ \gamma_i = \mu_i w_i$ and $\mu_i \in \mathcal{O}_S^*(S)$ = multiplicative group of invertible functions in $\mathcal{O}_S(S)$.

As $\gamma_1, \ldots, \gamma_g$ is a Schottky basis for all points S we get

$$\left|\frac{w_i(\varphi_j(s))}{w_i(\varphi_k(s))}\right| < |\mu_i(s)|$$

for all i, j, k with $j \neq \pm i$, $k \neq \pm i$, see [G 4, Sect. 2]. Let $c \in K$, $|c| < 1$, such that

$$\left|\frac{w_i(\varphi_j(s))}{w_i(\varphi_k(s))}\right| < |c|\,|\mu_i(s)|$$

for all i, j, k with $j \neq \pm i$, $k \neq \pm i$.

This is possible because for any analytic function f on S there is a point s_0 such that $|f(s)| \leqq |f(s_0)|$ for all $s \in S$.

There is an affinoid covering (S_i) of S such that $\dfrac{|\mu_i(s)|}{|\mu_i(s')|} \leqq |c|^{-1}$ for any $s, s' \in S_i$ and all i. Again we may assume $S = S_1$.

Then let $a: S \to \mathbb{P}$ be such that

$$|w_i(a)| < \min_{1 \neq j,\, s \in S} |w_1(\varphi_j(s))|$$

$$|w_1(a)| > |\sqrt{c}| \min_{1 \neq j,\, s \in S} |w_1(\varphi_j(s))|$$

An elementary computation shows that a is ordinary. a can be chosen such that $w_1(a)$ is constant.

Corollary. *There is an admissible covering (S_i) of S and for any i, there are ordinary sections $a_i, b_i, c_i: S_i \to Z$ which do not meet.*

Proof. The above proof also gives this result.

(5.3) Let Γ be a Schottky group of rank g over S and Z be the domain of discontinuity of Γ.

A function p on $\mathbb{P} \times S$ is called a global coordinate if there are sections $a, b, z_0: S \to \mathbb{P} \times S$ which do not meet (i.e. $a(s) \neq b(s) \neq z_0(s) \neq a(s)$ for all $s \in S$) such that

$$p(z) = \left(\frac{z-a}{z-b}\right) \Big/ \left(\frac{z_0-a}{z_0-b}\right).$$

Proposition 7. *Let $p(z)$ be a global coordinate for $\mathbb{P} \times S$ and assume that the zero section a and the pole section b and z_0 are ordinary for Γ.*

Then

$$f(z) := \prod_{\gamma \in \Gamma} \frac{p(\gamma z)}{p(\gamma z_0)}$$

converges to a meromorphic function on Z.

Proof. One uses Proposition 5. Then one can proceed in the same way as in the proof of the corresponding statement in [GP, Chap. II, (2.2), p. 47].

Corollary 1. *f is Γ-automorphic and the mapping $\gamma \to \frac{f \circ \gamma}{f} \in \mathcal{O}_S^*(S)$ is a homomorphism $\Gamma \to \mathcal{O}_S^*(S)$.*

Proof. As in [GP, Chap. II, (2.3.1)].

Corollary 2. *If $b = \alpha \circ a$, $\alpha \in \Gamma$, then the function f has no zeroes and no poles on Z. It does not depend on a, but only on α and z_0 and will be denoted by u_{α, z_0}.*

Moreover $u_{\alpha, z_0} = u_{\alpha', z_0}$ if and only if $\alpha'\alpha^{-1}$ lies in the commutator subgroup $[\Gamma, \Gamma]$ of Γ.

Proof. As in [GP, Chap. II, (2.3.4), p. 49].

(5.4) Let f be an analytic nowhere vanishing function on Z and assume that

$$\frac{f \circ \gamma}{f} \in \mathcal{O}_S^*(S)$$

for all $\gamma \in \Gamma$.

Let z_0 be an ordinary section for Γ on S.

Proposition 8. *$f(z) = f(z_0) \cdot u_{\alpha, z_0}$ for some $\alpha \in \Gamma$.*

Proof. Follows from [GP, Chap. II, Sect. 3, Theorem on p. 58].

Denote by A_S the subsheaf of $\pi_*(\mathcal{O}_Z^*)$ whose sections over a subdomain S' of S consist of all nowhere vanishing analytic functions f on $Z_{S'} = \pi^{-1}(S') \cap Z$ for which

$$\frac{f \circ \gamma}{f} \in \mathcal{O}_S^*(S')$$

for all $\gamma \in \Gamma$.

A_S is a sheaf of multiplicative groups. It is called the sheaf of non-vanishing analytic Γ-automorphic forms with factors of automorphy in $\mathcal{O}_S^*$.

Let $M := \Gamma/[\Gamma, \Gamma]$ be the commutator factor group of Γ which is a free abelian group of rank g.

If S' is connected there is a canonical mapping $\chi: A_S(S') \to M$ which sends $f \in A_S(S')$ to the class in M of $\alpha \in \Gamma$ for which

$$f(z) = f(z_0) \cdot u_{\alpha, z_0}.$$

Proposition 9. *Let S be connected and assume that there is an ordinary section z_0 for Γ over S. Then*

$$\chi: A_S(S) \to M$$

is surjective and the kernel of χ is $\mathcal{O}_S^*(S)$.

Proof. In the construction of the function u_{α, z_0} is used an ordinary parameter section a such that a, $\alpha \circ a$, z_0 do not meet. There is an admissible covering (S_i) of S such that there is an ordinary section $a_1: S_i \to Z$ such that a_i, $\alpha \circ \alpha_i$, z_0 do not meet. This leads to a $u_{\alpha, z_0}^{(i)}$ on $Z_{S_i} = Z \cap \pi^{-1}(S_i)$. But $u_{\alpha, z_0}^{(i)}(z_0) \equiv 1$ and thus $u_{\alpha, z_0}^{(i)} = u_{\alpha, z_0}^{(j)}$ on

$Z_{S_i} \cap Z_{S_j}$. Therefore one obtains by glueing the $u^{(i)}_{\alpha, z_0}$ a function $f \in A_S(S)$ with $\chi(f) \equiv \alpha \mod [\Gamma, \Gamma]$.

(5.5) Let Γ be a Schottky group of rank g over S and $M = \Gamma/[\Gamma, \Gamma]$.
The period mapping

$$q : S \to \mathscr{S}(M)$$

is constructed as follows:
There is an admissible covering (S_i) of S, S_i connected, and homomorphisms $v_i : M \to A_S(S_i)$ such that $\chi \circ v_i = \mathrm{id}$.
Pick $s_0 \in S$. Let $s_0 \in S_i$, then

$$(\gamma, \gamma') \to \frac{v_i(\bar{\gamma})}{v_i(\bar{\gamma}) \circ \gamma'}(s_0)$$

is a symmetric, positive definite bimultiplicative form with values in K^*, see [GP, Chap. VI, Sect. 2, p. 190], and can thus be considered as element $q(s_0)$ of $\mathscr{S}(M)$. The point $q(s_0)$ does not depend on i because if $s_0 \in S_j$, then

$$\frac{v_i(m)}{v_j(m)} \in \mathcal{O}^*_S(S_i \cap S_j) \quad \text{for any} \quad m \in M .$$

q is analytic because $\dfrac{v_i(\bar{\gamma})}{v_i(\bar{\gamma}) \circ \gamma'}$ is analytic on S_i.

(5.6) In this section we assume that S is connected and $\chi : A_S(S) \to M$ is surjective. We choose a homomorphism $v : M \to A_S(S)$ such that $\chi \circ v = \mathrm{id}$.
Define a mapping

$$\varphi : Z \to \mathscr{T}$$

as follows:
Pick $(z_0, s_0) \in Z$, $z_0 \in \mathbb{P}$, $s_0 \in S$, then $v(m) \in A_S(S)$ is analytic function on Z and

$$v(m)(z_0, s_0) \in K^* .$$

As v is a homomorphism, the map

$$m \to v(m)(z_0, s_0)$$

is a homomorphism $M \to K^*$ which is an element $a(z_0, s_0)$ of $T(M)$.
Put $\varphi(z_0, s_0) := (a(z_0, s_0), q(s_0)) \in \mathscr{T}(M)$.
Clearly φ is an analytic map as $v(m)$ is analytic on Z and the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & \mathscr{T} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S & \longrightarrow & \mathscr{S} \end{array} \quad \text{is commutative.}$$

We claim: The diagram

$$\begin{array}{ccc} Z & \xrightarrow{\gamma} & Z \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\ \mathscr{T} & \xrightarrow[\lambda_{\bar{\gamma}}]{} & \mathscr{T} \end{array}$$

is commutative for any $\gamma \in \Gamma$, where $\bar{\gamma}$ is the class of γ in M and $\lambda_{\bar{\gamma}}$ is the transformation of (1.4).

Proof. Straightforward.

(5.7) Let $C = Z/\Gamma \xrightarrow{\pi} S$ be the family of Mumford curves induced by $Z \xrightarrow{\pi} S$. We obtain a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\bar{\varphi}} & \mathfrak{A} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ S & \xrightarrow[q]{} & \mathscr{S}, \end{array}$$

where $\bar{\varphi}$ is the unique mapping which makes the canonical diagram

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & \mathscr{T} \\ \downarrow & & \downarrow \\ C & \xrightarrow[\bar{\varphi}]{} & \mathfrak{A} \end{array} \qquad \text{commutative.}$$

Now we fix a basis $e_1, \ldots, e_g$ of M, put $u_i = v(e_i) \circ \varphi$ and let α_i be the differential $\dfrac{du_i}{u_i}$ which is a differential of the first kind on C/S. Let $\xi_i := \zeta_i \circ \varphi$, ζ_i as defined in (3.2), then $d\xi_i$ is a differential of the second kind on C/S.

We claim: $\alpha_1, \ldots, \alpha_g, d\xi_1, \ldots, d\xi_g$ constitute a basis of the $\mathcal{O}_S(S)$-module $H^1_{\mathrm{DR}}(C/S)$.

Here we could take the classical definition of $H^1_{\mathrm{DR}}(C/S)$ – differentials of second kind modulo exact differentials. Equivalently one could define $H^1_{\mathrm{DR}}(C/S)$ as the hypercohomology of the de Rham complex $\Omega^{\cdot}_{C/S}$.

Let $C_\varepsilon := \bar{\varphi}^{-1}(\mathfrak{a}_\varepsilon)$ and $\mathfrak{C} := \{C_\varepsilon : \varepsilon \in E\}$, see (3.6). Then

$$H^1_{\mathrm{DR}}(C/S) = H^1(\mathscr{K}^{\cdot}(\mathfrak{C}, \Omega^{\cdot}_{C/S})).$$

Now $\mathscr{K}^1(\mathfrak{C}, \Omega^{\cdot}_{C/S}) = \mathscr{C}^0(\mathfrak{C}, \Omega^1_{C/S}) \otimes \mathscr{C}^1(\mathfrak{C}, \mathcal{O}_C)$.

We consider $\dfrac{du_i}{u_i}$ as 0-cochain in $\mathscr{C}^0(\mathfrak{C}, \Omega^{\cdot}_{C/S})$ and denote its cohomology class by α_i.

We consider

$$(d(\zeta_{\varepsilon i} \circ \varphi))_{\varepsilon \in E} \oplus (\zeta_{\varepsilon i} \circ \varphi - \zeta_{\varepsilon' i} \circ \varphi)$$

as element of $\mathscr{K}^1(\mathfrak{C}, \Omega_{C/S})$ and denote its cohomology class by β_i.

Proposition 10. *$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ is a basis of the $\mathcal{O}_S(S)$-module $H^1_{\mathrm{DR}}(C/S)$.*

Proof. Similar to the proof of Proposition 5, (3.6). See also [G 2, Satz 6].

(5.8) Let ∇ be the Gauss-Manin connection for the family C/S. Then ∇ is a connection

$$\nabla : H^1_{\mathrm{DR}}(C/S) \to H^1_{\mathrm{DR}}(C/S) \otimes_{\mathcal{O}_S(S)} \Omega_S(S).$$

We also denote by q_{ij} the functions on S given by the composition of q_{ij} on $\mathcal{S}$ with the period mapping q.

Theorem 2.
$$\nabla(\alpha_i)=\sum_{j=1}^{g}\beta_j\otimes\frac{dq_{ij}}{q_{ij}}$$
$$\nabla(\beta_i)=0\,.$$

Proof. 1) If there exists an ordinary section $z_0: S\to Z$ we use the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & \mathcal{T} \\ \downarrow & & \downarrow \\ C & \xrightarrow{\bar\varphi} & \mathfrak{a} \\ \downarrow & & \downarrow \\ S & \xrightarrow[q]{} & \mathcal{S} \end{array}$$

and we can repeat the proof of Theorem 1.

2) In general we choose an admissible covering (S_ν) of S such that there exists an ordinary section over each S_ν, Proposition 6. Then we define α_i^ν, $\beta_i^\nu\in H^1_{\mathrm{DR}}(C_\nu/S_\nu)$, $C_\nu=\pi^{-1}(S_\nu)$. Then we check that
$$\alpha_i^\nu=\alpha_i^\mu$$
$$\beta_i^\nu=\beta_i^\mu$$
on $C_\nu\cap C_\mu$ using (3.9).

The formulas of the theorem are true for α_i^ν, β_i^ν and thus also for α_i, β_i where α_i, β_i are defined by patching the α_i^ν, β_i^ν.

(5.9) The differentials $d\xi_i$, $\xi_i := \zeta_i\circ\varphi$ on C/S have integrals ξ_i on the "universal covers" Z. A different approach to these integrals is given in [G 2]. Recently R. Coleman has developed a beautiful theory of p-adic integrals in [C]. He however treats only the case of good reduction. A unified general theory of these integrals is to be desired.

One should also study the case where S is the Teichmüller space for Γ, see [He].

6. Tate Curve

(6.1) The case $g=\operatorname{rank} M=1$, $M=\mathbb{Z}\cdot e$, gives family of Tate curves which has been studied intensively, see for instance [R, Rb, K 1, DR].

We get
$$S=\{\pi\in K:\ 0<|\pi|<1\}$$
$$Z=\{(\pi,z)\in K^2:\ \pi\in S,\ z\neq 0\}\,.$$
The transformation group $\Gamma=\Lambda\cong M$ generated by
$$(\pi,z)\xrightarrow{\gamma}(\pi,qz)\,,\qquad q=\pi^2$$

acts on Z and the projection π onto the first factor $C=Z/\Gamma\to S$ is the canonical family of Tate curves.

The principal theta function θ with respect to the principal characteristic

$$h: M\to M_2$$

given by $h(e)=-e^2=-e\otimes e$ is

$$\theta=\theta(q,z)=\sum_{n\in\mathbf{Z}} q^{\frac{1}{2}(n^2-n)}z^n .$$

Now

$$\theta(q,-1)=\sum_{n\in\mathbf{Z}} q^{\frac{1}{2}(n^2-n)}(-1)^n=\sum_{n=1}^{\infty} q^{\frac{1}{2}(n^2-n)}(-1)^n$$

$$+\sum_{n=1}^{\infty} q^{\frac{1}{2}((-n+1)^2-(-n+1))}(-1)^{-n+1}=0$$

as

$$n^2-n=(-n+1)^2-(-n+1).$$

One has the classical product decomposition

$$\theta=\Phi(q)\cdot(1+z)\cdot\prod_{n=1}^{\infty}(1+q^n z)\left(1+\frac{q^n}{z}\right)$$

with $\Phi(q)=\prod_{n=1}^{\infty}(1-q^n)$.

Proof. Let for the moment

$$\vartheta=(1+z)\prod_{n=1}^{\infty}(1+q^n z)\left(1+\frac{q^n}{z}\right).$$

Then one has the functional equation

$$\vartheta(q,qz)=z^{-1}\vartheta(q,z).$$

As $\theta(q,qz)=z^{-1}\theta(q,z)$, the quotient $\frac{\theta}{\vartheta}$ is an analytic Γ-invariant function on Z. Thus $\frac{\theta}{\vartheta}$ must be constant as function of z and is therefore only a function of q.

In order to determine $\Phi(q)$ one can proceed as follows:

Observe that

$$\partial^2(\theta)-\partial(\theta)=\sum_{n\in\mathbf{Z}}(n^2-n)q^{\frac{1}{2}(n^2-n)}z^n$$

$$\pi\frac{\partial\theta}{\partial\pi}=\sum_{n\in\mathbf{Z}}(n^2-n)\pi^{n^2-n}\cdot z^n .$$

We conclude that

$$\pi\frac{\partial\theta}{\partial\pi}=(\partial^2-\partial)(\theta).$$

We derive a differential equation for ϑ and use $\theta=\Phi\cdot\vartheta$, $\partial(\theta)=\Phi\cdot\partial(\vartheta)$, $\partial^2(\theta)=\Phi\partial^2(\vartheta)$,

$$\pi\frac{\partial\theta}{\partial\pi}=\pi\frac{\partial\Phi}{\partial\pi}\cdot\vartheta+\Phi\cdot\pi\cdot\frac{\partial\vartheta}{\partial\pi}$$

and obtain

$$\Phi\cdot(\partial^2-\partial)(\vartheta)=\pi\frac{d\Phi}{d\pi}\cdot\vartheta+\pi\Phi\frac{\partial\vartheta}{\partial\pi}.$$

Now $\zeta:=\dfrac{\partial(\theta)}{\theta}=\dfrac{\partial(\vartheta)}{\vartheta}$ and $\partial(\vartheta)=\zeta\cdot\vartheta$ and

$$\partial^2(\vartheta)=\partial(\zeta)\cdot\vartheta+\zeta(\zeta\cdot\vartheta).$$

Thus

$$\partial(\zeta)\cdot\vartheta+\zeta^2\cdot\vartheta-\zeta\vartheta=\frac{\pi\dfrac{d\Phi}{d\pi}}{\Phi}\cdot\vartheta+\pi\frac{\partial\vartheta}{\partial\pi}$$

which we divide by ϑ and obtain

$$\frac{\left(\pi\dfrac{d\Phi}{d\pi}\right)}{\Phi}=\partial(\zeta)+\zeta^2-\zeta-\frac{\pi\dfrac{\partial\vartheta}{\partial\pi}}{\vartheta}.$$

For ζ we have the following partial fraction decomposition

$$\zeta=\frac{\partial\vartheta}{\vartheta}=\frac{z}{1+z}+\sum_{n=1}^{\infty}\left[\frac{q^nz}{1+q^nz}+\frac{(-1)q^nz^{-1}}{1+q^nz^{-1}}\right].$$

Let $\eta:=\zeta-\dfrac{z}{1+z}$. Use $\dfrac{q^nz}{1+q^nz}=\dfrac{1+q^nz-1}{1+q^nz}=1-\dfrac{1}{1+q^nz}$ to obtain

$$\eta=\sum_{n=1}^{\infty}\left[\frac{1}{1+q^nz^{-1}}-\frac{1}{1+q^nz}\right].$$

Now

$$\partial\zeta=\frac{z}{(1+z)^2}+\partial\eta$$

$$\zeta^2=\frac{z^2}{(1+z)^2}+\frac{2z}{1+z}\cdot\eta+\eta^2$$

and

$$\frac{\left(\pi\dfrac{d\Phi}{d\pi}\right)}{\Phi}=\frac{z}{(1+z)^2}+\partial\eta+\frac{z^2}{(1+z)^2}+\frac{2z}{1+z}\eta+\eta^2-\frac{z}{(1+z)^2}-\partial\eta-\frac{\pi\dfrac{\partial\vartheta}{\partial\pi}}{\vartheta}$$

$$=\frac{2z}{1+z}\eta+\partial\eta+\eta(\eta-1)-\frac{\pi\dfrac{\partial\vartheta}{\partial\pi}}{\vartheta},$$

where

$$\frac{\pi\dfrac{\partial\vartheta}{\partial\pi}}{\vartheta}=2\sum_{n=1}^{\infty}\left[\frac{nq^nz}{1+q^nz}+\frac{nq^nz^{-1}}{1+q^nz^{-1}}\right]$$

$$\partial\eta=\sum_{n=1}^{\infty}\left[\frac{q^nz^{-1}}{(1+q^nz^{-1})^2}+\frac{q^nz}{(1+q^nz)^2}\right].$$

We evaluate these functions at the point $z=-1$.

$$\frac{\left(\pi\dfrac{\partial\vartheta}{\partial\pi}\right)}{\vartheta}(q,-1)=(-4)\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$$

$$(\partial\eta)(q,-1)=(-2)\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}$$

$$\eta(q,-1)=0.$$

If one expands η into a power series around the point -1 one writes

$$\eta=\sum_{i=1}^{\infty}\eta_i(1+z)^i$$

$$\frac{\eta}{1+z}=\sum_{i=1}^{\infty}\eta_i(1+z)^{i-1}$$

$$\partial\eta=z\frac{\partial\eta}{z}=z\sum_{i=1}^{\infty}i\cdot\eta_i(1+z)^{i-1}.$$

Evaluate these functions at the point $z=-1$

$$\frac{\eta}{1+z}(q,-1)=\eta_1(q)$$

$$\partial\eta(q,-1)=-\eta_1(q)$$

$$\frac{2z}{1+z}\eta(q,-1)=(-2)\eta_1(q)=2\partial\eta(q,-1).$$

Collecting these formulas we have

$$\frac{\left(\pi\dfrac{d\Phi}{d\pi}\right)}{\Phi}=3\partial\eta(q,-1)-\frac{\pi\dfrac{\partial\vartheta}{\partial\pi}}{\vartheta}(q,-1)$$

$$=(-6)\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}+4\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

But $\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} = \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}$ as can be shown by expanding both sides into power series in q using

$$\frac{x}{1-x} = \sum_{\nu=1}^{\infty} x^\nu, \qquad \frac{x}{(1-x)^2} = \sum_{\nu=1}^{\infty} \nu x^\nu$$

which gives

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} = \sum_{\substack{n\geq 1\\ \nu\geq 1}} \nu q^{n\nu}$$

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \sum_{\substack{n\geq 1\\ \nu\geq 1}} n q^{n\nu}.$$

Thus finally

$$\frac{\pi \frac{d\Phi}{d\pi}}{\Phi} = (-2) \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}$$

which allows to conclude that

$$\Phi = \Phi_0 \cdot \prod_{n=1}^{\infty} (1-q^n)$$

with $\Phi_0 \in K$.

Now we substitute $q=0$ and obtain

$$\theta(0,z) = 1+z$$

$$\vartheta(0,z) = 1+z$$

and thus $\Phi_0 = 1$.

(6.2) Let now

$$\wp = \wp(q,z) := -(\partial\zeta)(q,-z)$$

$$\wp' = \partial\wp.$$

Then $\wp, \wp'$ are Γ-invariant and

$$\wp = \frac{z}{(1-z)^2} + \sum_{n=1}^{\infty} \left[\frac{q^n z}{(1-q^n z)^2} + \frac{q^n z^{-1}}{(1-q^n z^{-1})^2}\right]$$

$$\wp' = \frac{z+z^2}{(1-z)^3} + \sum_{n=1}^{\infty} \left[\frac{q^n z + q^{2n} z^2}{(1-q^n z)^3} - \frac{q^n z^{-1} + q^{2n} z^{-2}}{(1-q^n z^{-1})^3}\right].$$

Let

$$e_1 = \wp(q,-1)$$

$$e_2 = \wp(q,\pi)$$

$$e_3 = \wp(q,-\pi).$$

Then it comes out as in the classical case that

$$(\wp')^2=4(\wp-e_1)(\wp-e_2)(\wp-e_3).$$

Proof. $\frac{1}{4}(\wp')^2$ is a cubic unitary polynomial of $\wp$, see [R, (40), p. 29].
But $\wp'(q,z^{-1})=-\wp'(q,z)$ and therefore $\wp'(q,-1)=\wp'(q,\pi)=\wp'(q,-\pi)=0$.

Proposition 11. $\wp-e_1=\psi(q)^2\cdot\dfrac{\theta^2(q,z)}{\theta^2(q,-z)}$ *with* $\psi(q)=\dfrac{\Phi(q)^2}{\vartheta(q,1)}$.

Proof. $\wp-e_1$ and $\left(\dfrac{\theta(q,z)}{\theta(q,-z)}\right)^2$ are both Γ-invariant meromorphic functions on Z whose divisors on $C=Z/\Gamma$ agree; they are equal to $2(\Gamma(-1)-\Gamma(1))$. Thus their quotient is a function which only depends on q. Also $\dfrac{\theta(q,z)}{\theta(q,-z)}=\dfrac{\vartheta(q,z)}{\vartheta(q,-z)}$.

Let $f:=\vartheta^2(q,-z)\cdot(\wp-e_1)$.

Then $f(q,z)$ is analytic at the point $z=1$ and $f(q,1)=\psi(q)^2\cdot\vartheta^2(q,1)$. As $\vartheta(q,-1)=0$ one has $f(q,1)=g(q,1)$ for $g:=\vartheta^2(q,-z)\cdot\wp$. In order to compute $f(q,1)$ we use the formula

$$\partial(\zeta)=\frac{\pi\dfrac{d\Phi}{d\pi}}{\Phi}-\zeta^2+\zeta+\frac{\pi\dfrac{\partial\vartheta}{\partial\pi}}{\vartheta}$$

of Sect. (6.1).

$$g=\vartheta^2(q,-z)\cdot(-\partial(\zeta)(q,-z)).$$

Thus $f(q,1)$ is equal to the value at $z=1$ for

$$\zeta^2(q,-z)\cdot\vartheta^2(q,-z)$$

which is $\partial(\vartheta)^2(q,-z)$.

Thus $f(q,1)=\partial(\vartheta)^2(q,-1)$.

Let $\vartheta=(1+z)\cdot F$ with $F=\prod_{n\geq 1}(1+q^n z)(1+q^n z^{-1})$.

Then $\partial(\vartheta)=z\dfrac{\partial\vartheta}{\partial z}=(1+z)\cdot z\dfrac{\partial F}{\partial z}+zF$ and $\partial(\vartheta)(q,-1)=-F(q,-1)$ $=-\prod_{n=1}^{\infty}(1-q^n)^2$ and thus

$$f(q,1)=\prod_{n=1}^{\infty}(1-q^n)^4=\Phi(q)^4.$$

(6.3) Define

$$x:=\frac{\wp-e_1}{e_2-e_1}=\frac{\theta^2(q,z)\cdot\theta^2(q,-\pi)}{\theta^2(q,-z)\cdot\theta^2(q,\pi)}.$$

Then $x-1=\dfrac{\wp-e_2}{e_2-e_1}$.

Let $\lambda:=x(q,-\pi)=\dfrac{e_3-e_1}{e_2-e_1}$.

Then
$$x-\lambda=\frac{\wp-e_3}{e_2-e_1}$$

and $x(x-1)(x-\lambda)=\dfrac{(\wp-e_1)(\wp-e_2)(\wp-e_3)}{(e_2-e_1)^3}$.

Let $\sqrt{e_2-e_1}:=\psi(q)\cdot\dfrac{\theta(q,\pi)}{\theta(q,-\pi)}$.

Because of Proposition 10 the square of $\sqrt{e_2-e_1}$ is indeed $e_2-e_1 =\wp(q,\pi)-\wp(q,-1)$.

Define
$$y:=\frac{\wp'}{2(e_2-e_1)^{3/2}}.$$

Then the Legendre equation holds
$$y^2=x(x-1)(x-\lambda).$$

The parameter λ is an analytic function of π and has the representation
$$\lambda(\pi)=\frac{\theta^4(q,-\pi)}{\theta^4(q,\pi)}=\frac{\vartheta^4(q,-\pi)}{\vartheta^4(q,\pi)}$$

and $\lambda(-\pi)=\dfrac{1}{\lambda(\pi)}$.
$$\vartheta(q,\pi)=(1+\pi)\prod_{n=1}^{\infty}(1+\pi^{2n+1})(1+\pi^{2n-1})=\prod_{n=1}^{\infty}(1+\pi^{2n-1})^2$$
$$\vartheta(q,-\pi)=\prod_{n=1}^{\infty}(1-\pi^{2n-1})^2$$

and
$$\lambda(\pi)=\frac{\prod_{n=1}^{\infty}(1-\pi^{2n-1})^8}{\prod_{n=1}^{\infty}(1+\pi^{2n-1})^8}.$$

Proposition 12. $\lambda(\pi)=\prod_{n=1}^{\infty}(1-\pi^{2n-1})^{16}\cdot\prod_{n=1}^{\infty}(1+q^n)^8$.

Proof. If t is topologically nilpotent then
$$\frac{1}{1-t}=\prod_{\nu=0}^{\infty}(1+t^{2^\nu})$$

and thus
$$\frac{1}{1+t}=(1-t)\prod_{\nu=1}^{\infty}(1+t^{2^\nu}).$$

Thus
$$\frac{1}{1+\pi^{2n-1}}=(1-\pi^{2n-1})\prod_{\nu=1}^{\infty}(1+(\pi^{2n-1})^{2^\nu})$$
$$\frac{1}{\prod_{n=1}^{\infty}(1+\pi^{2n-1})}=\prod_{n=1}^{\infty}\left[(1-\pi^{2n-1})\prod_{\nu=1}^{\infty}(1+\pi^{2^\nu(2n-1)})\right].$$

Any even positive integer k has a unique representation of the form

$$k=2^{\nu}(2n-1).$$

Thus

$$\prod_{n=1}^{\infty}\prod_{\nu=1}^{\infty}(1+\pi^{(2n-1)2^{\nu}})=\prod_{k=1}^{\infty}(1+\pi^{2k})$$

and

$$\frac{1}{\prod_{n=1}^{\infty}(1+\pi^{2n-1})}=\prod_{n=1}^{\infty}(1-\pi^{2n-1})\cdot\prod_{n=1}^{\infty}(1+q^{n})$$

from which the proposition follows.

Corollary. *$\lambda(\pi)=\prod_{i=0}^{\infty}\lambda_i\pi^i$ with $\lambda_0=1$, $\lambda_1=-16$, $\lambda_2=2^7$, all $\lambda_i\in\mathbb{Z}$, $\lambda_i\equiv 0 \bmod 16$ for $i\geqq 1$.*

Proof. Let for the moment

$$f(\pi)=\prod_{n\geqq 1}(1-\pi^{2n-1})^2\prod_{n\geqq 1}(1+\pi^{2n})$$

such that $\lambda=f^8$.

We consider f and λ as elements of the formal power series ring $\mathbb{Z}[\![\pi]\!]$ in the variable π with integer coefficients.

We claim:

$$f(\pi)\equiv f(\pi^2) \bmod 2\mathbb{Z}[\![\pi]\!].$$

As

$$(1-\pi^{2n-1})^2\equiv 1-(\pi^2)^{2n-1} \bmod 2\mathbb{Z}[\![\pi]\!]$$

we get

$$\begin{aligned} f(\pi) &\equiv \prod_{n\geqq 1}(1-q^{2n-1})\cdot(1+q^n) \bmod 2\\ &\equiv \prod_{n\geqq 1}(1-q^{2n-1})^2(1+q^{2n}) \bmod 2\\ &\equiv f(\pi^2).\end{aligned}$$

From $f(\pi)\equiv f(\pi^2)$ one gets that

$$f\equiv 1 \bmod 2\mathbb{Z}[\![\pi]\!]$$

$$f=1+2\pi g\,,\qquad g=\prod_{i=0}^{\infty}g_i\pi^i\,,\qquad g_i\in\mathbb{Z}\,,\qquad g_0=-1$$

$$\begin{aligned} f^8=1+8\cdot 2\pi g+\binom{8}{2}4\pi^2g^2+\binom{8}{3}8\pi^2g^3+\binom{8}{4}2^4\pi^4g^4\\ +\binom{8}{5}2^5\pi^5g^5+\binom{8}{6}2^6\pi^6g^6+\binom{8}{7}2^7\pi^7g^7+2^8\pi^8g^8\,.\end{aligned}$$

This shows that $\lambda_i\equiv 0 \bmod 16$ for $i\geqq 1$.

Corollary. *$\lambda(\pi)$ gives a bianalytic mapping of*

$$S=\{z\in K\colon |z|<1\} \quad \textit{onto} \quad D=\{z\in K\colon |z-1|<|16|\}.$$

The inverse mapping $\pi(\lambda)$ is given by a power series in $(\lambda-1)$ with coefficients in $\mathbb{Z}[\frac{1}{2}]$: $\pi(\lambda)=\sum_{i=1}^{\infty}\pi_i(\lambda-1)^i$ with $\pi_1=-\frac{1}{16}$ which converges for $|\lambda|<|16|$.

(6.4) Let dx be the relative differential of x with respect to $C\xrightarrow{\pi} S$, $\frac{dx}{y}\in\Gamma(S,\Omega_{C/S})$ is a differential of the first kind.

$$\frac{dx}{y}=\frac{d\wp}{(e_2-e_1)}\Big/\left(\frac{\wp'}{2(e_2-e_1)^{3/2}}\right)=2\sqrt{e_2-e_1}\,\frac{dz}{z}$$

because $\wp'\frac{dz}{z}=d\wp$.

By definition

$$\begin{aligned}2\sqrt{e_2-e_1}&=2\psi(q)\frac{\theta(q,\pi)}{\theta(q,-\pi)}=2\psi(q)\frac{\vartheta(q,\pi)}{\vartheta(q,-\pi)}\\&=2\Phi(q)^2\frac{\vartheta(q,\pi)}{\vartheta(q,1)\vartheta(q,-\pi)}\\&=2\cdot\prod_{n\geq 1}(1-q^n)^2\frac{\prod_{n\geq 1}(1+\pi^{2n-1})^2}{2\prod_{n\geq 1}(1+q^n)^2\prod_{n\geq 1}(1-\pi^{2n-1})^2}.\end{aligned}$$

We use the procedure in the proof of Proposition 10 to obtain

$$\frac{1}{\prod_{n\geq 1}(1-\pi^{2n-1})}=\prod_{n\geq 1}(1+\pi^n).$$

Thus

$$2\sqrt{e_2-e_1}=\left[\frac{\prod_{n\geq 1}(1-q^n)\cdot(1+\pi^{2n-1})(1+\pi^n)}{\prod_{n\geq 1}(1+\pi^{2n})}\right]^2$$

$$2\sqrt{e_2-e_1}=\Big[\prod_{n\geq 1}(1-q^n)(1+\pi^{2n-1})^2\Big]^2.$$

Proposition 13. *$2\sqrt{e_2-e_1}$ is a power series in π with integral coefficients*

$$2\sqrt{e_2-e_1}=\sum_{i=0}^{\infty}a_i\pi^i$$

$$a_0=1\,,\qquad a_1=4\,,\qquad a_i\equiv 0 \bmod 4 \quad \textit{for} \quad i\geq 1$$

and $(2\sqrt{e_2-e_1})(-\pi)=\dfrac{1}{(2\sqrt{e_2-e_1})(\pi)}.$

Proof. As above.

(6.5) Let
$$\nabla : H^1_{DR}(C/S) \to H^1_{DR}(C/S) \otimes \Omega_S(S)$$
be the Gauss-Manin connection.

The $\mathcal{O}_S(S)$-module $\Omega_S(S)$ of analytic differentials on S is generated by $d\pi$ and also by $d\lambda = \frac{d\lambda}{d\pi} \cdot d\pi$.

Let $\nabla_\lambda : H^1_{DR}(C/S) \to H^1_{DR}(C/S)$ be the composition of ∇ with the $\mathcal{O}_S(S)$-linear mapping that sends $\alpha \otimes d\lambda$ onto α for any $\alpha \in H^1_{DR}(C/S)$. The ∇_λ is a derivation over $\frac{d}{d\lambda}$ which means that
$$\nabla_\lambda(f \cdot \alpha) = f\nabla_\lambda(\alpha) + \frac{df}{d\lambda} \cdot \alpha$$
for any $f \in \mathcal{O}_S(S)$, $\alpha \in H^1_{DR}(C/S)$.

Let $L = \lambda(1-\lambda)\nabla^2_\lambda + (1-2\lambda)\nabla_\lambda - \frac{1}{4}$ be the hypergeometric differential operator for the parameters $\frac{1}{2}, \frac{1}{2}, 1$.

It is well known that
$$L\left(\frac{dx}{y}\right) = 0$$
see for instance [Dw, Chap. I].

From Sect. (5.8) we know that
$$\nabla_\lambda\left(\frac{dz}{z}\right) = \frac{\dot{q}}{q} d\zeta$$
with $\zeta := \frac{\partial\theta}{\theta}$ and $\dot{q} = \frac{dq}{d\lambda}$.

Also
$$\nabla_\lambda(d\zeta) = 0.$$

Let $E := 2\sqrt{e_2 - e_1}$ such that
$$\frac{dx}{y} = E\frac{dz}{z}.$$

Now $\nabla_\lambda\left(E\frac{dz}{z}\right) = \dot{E}\frac{dz}{z} + E\frac{\dot{q}}{q} d\zeta$
$$\nabla^2_\lambda\left(E\frac{dz}{z}\right) = \ddot{E}\frac{dz}{z} + \left(\dot{E}\frac{\dot{q}}{q} + E\left(\frac{\dot{q}}{q}\right)^{\boldsymbol{\cdot}} + \dot{E}\frac{\dot{q}}{q}\right) d\zeta.$$

Let $t := \frac{\dot{q}}{q}$ and use $L\left(E\frac{dz}{z}\right) = 0$. Then

$$(*) \qquad \left(\lambda(1-\lambda)\left(\frac{d}{d\lambda}\right)^2 + (1-2\lambda)\frac{d}{d\lambda} - \frac{1}{4}\right)(E) = 0$$

and

$$(**) \qquad \lambda(1-\lambda)(2\dot{E}t+Et^{\cdot})+(1-2\lambda)Et=0.$$

Proposition 14.

$$2\sqrt{e_2-e_1}=F(\lambda):={}_2F_1(\tfrac{1}{2},\tfrac{1}{2},1;1-\lambda)$$
$$=\sum_{n=1}^{\infty}\left(\frac{(\frac{1}{2})_n}{n!}\right)^2(1-\lambda)^n \quad where \quad (\tfrac{1}{2})_n=\prod_{i=0}^{n-1}(\tfrac{1}{2}+i).$$

Proof. E is a power series in $(1-\lambda)$ which converges for $|1-\lambda|<|16|$ with $E(1)=1$.

$$E(\lambda)=\sum_{n=0}^{\infty}E_n(1-\lambda)^n.$$

From $(*)$ one obtains that

$$E_{n+1}=E_n\frac{(\frac{1}{2}+n-1)^2}{n^2}$$

for $n\geqq 0$ which proves the result.

Next we consider the equation $(**)$. We can transform it into

$$\frac{\dot{t}}{t}=-\frac{1-2\lambda}{\lambda(1-\lambda)}-2\frac{\dot{F}}{F}.$$

Claim. $t=\dfrac{2}{\lambda(\lambda-1)F^2}=\dfrac{\dot{q}}{q}.$

Proof. Let $G=\frac{t}{2}\lambda(\lambda-1)F^2$. Then

$$\frac{\dot{G}}{G}=\frac{\dot{t}}{t}+\frac{1}{\lambda}+\frac{1}{\lambda-1}+2\frac{\dot{F}}{F}$$
$$=\frac{\dot{t}}{t}+\frac{2\lambda-1}{\lambda(\lambda-1)}+2\frac{\dot{F}}{F}=0.$$

Thus $\dot{G}=0$ and G is a constant. But t is a Laurent series in $\lambda-1$.

$$t=\frac{\dot{q}}{q}=2\frac{\dot{\pi}}{\pi} \quad \text{and} \quad \pi=\sum_{i=1}^{\infty}\pi_i(\lambda-1)^i, \qquad \pi_1=-\tfrac{1}{16}.$$

Thus $\dfrac{\dot{\pi}}{\pi}=\dfrac{1}{\lambda-1}+$power series in $(\lambda-1)$ and $G\equiv 1$.

We are now in a position to determine the coefficients π_i of the expansion

$$\pi=\sum_{i=1}^{\infty}\pi_i(\lambda-1)^i$$

recursively.

Let $F_i=\left(\dfrac{(\frac{1}{2})_i}{i!}\right)^2$ and $G_i=\sum_{j+k=i}F_j\cdot F_k$.

Then $G_0=1$, $G_1=2F_1=\frac{1}{2}$.

As
$$\dot{\pi}(\lambda-1)((\lambda-1)+1)F^2=\pi$$
we conclude that
$$\Big(\sum_{i\geqq 1} i\pi_i(\lambda-1)^{i+1}+\sum i\pi_i(\lambda-1)^i\Big)\sum_{i\geqq 0} G_i(\lambda-1)^i=\sum_{i\geqq 1}\pi_i(\lambda-1)^i.$$
Thus $\pi_n=\sum_{i+j=n}[(i-1)\pi_{i-1}+i\pi_i]\cdot G_j$
$$\pi_n=-\pi_{n-1}+\frac{1}{1-n}\sum_{\substack{i+j=n\\ i<n}}[(i-1)\pi_{i-1}+i\pi_i]G_j,\qquad n>1.$$
Remark. In the classical complex case $q=e^{2\pi i\tau}$, τ a complex number in the upper half plane, then $\frac{\dot q}{q}=2\pi i\dot\tau$.

Thus
$$\tau=\frac{1}{2\pi i}\int\frac{2}{\lambda(\lambda-1)F^2}\,d\lambda$$
which is the quotient of two solutions of the hypergeometric equation $L=0$.

If we write
$$\frac{1}{\lambda(\lambda-1)F^2}=\frac{1}{\lambda-1}-\frac{1}{\lambda}+\frac{1-F^2}{\lambda(\lambda-1)F^2}.$$
Then $\frac{1}{\lambda}$ and $\frac{1-F^2}{\lambda(\lambda-1)F^2}$ are both power series in $(\lambda-1)$ and the primitives $\int\frac{d\lambda}{\lambda}$, $\int\frac{1-F^2}{\lambda(\lambda-1)F^2}d\lambda$ do exist in
$$D=\{\lambda\in K\colon |\lambda-1|<|16|\}.$$
Then
$$\pi(\lambda)=\frac{\lambda-1}{\lambda}\exp\left(\int\frac{1-F^2}{\lambda(\lambda-1)F^2}d\lambda\right).$$
This is the analogue to the classical formula, see [Rb, p. 65].

Proposition 15. $d\zeta=\frac{\lambda(\lambda-1)}{2}\left(-\dot F\frac{dx}{y}+F\nabla_\lambda\left(\frac{dx}{y}\right)\right).$

Proof. $\nabla_\lambda\left(\frac{dz}{z}\right)=\frac{\dot q}{q}d\zeta=\frac{2}{\lambda(\lambda-1)F^2}d\zeta$ and $\frac{dz}{z}=\frac{1}{F}\frac{dx}{y}$.

Thus $\nabla_\lambda\left(\frac{dz}{z}\right)=\left(-\frac{\dot F}{F^2}\right)\frac{dx}{y}+\frac{1}{F}\nabla_\lambda\left(\frac{dx}{y}\right)$
$$d\zeta=\frac{\lambda(\lambda-1)}{2}F^2\left(-\frac{\dot F}{F^2}\frac{dx}{y}+\frac{1}{F}\nabla_\lambda\left(\frac{dx}{y}\right)\right)$$
$$d\zeta=\frac{\lambda(\lambda-1)}{2}\left(-\dot F\frac{dx}{y}+F\nabla_\lambda\left(\frac{dx}{y}\right)\right).$$

Remark. $\nabla_\lambda\left(\frac{dx}{y}\right)=\frac{(-1)}{2(\lambda-1)}\frac{dx}{y}+\frac{1}{2\lambda(\lambda-1)}\frac{xdx}{y}$

$\dot{F}=(-\tfrac{1}{4})\,{}_2F_1(\tfrac{3}{2},\tfrac{3}{2};2;1-\lambda)$.

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Harmonic Maps of the Moduli Space of Compact Riemann Surfaces

Georg Schumacher[1]

Mathematisches Institut der Universität, Einsteinstrasse 62,
D-4400 Münster, Federal Republic of Germany

1. Introduction – Statement of Results

The investigation of harmonic maps of compact Kähler manifolds with negative curvature has been of considerable interest. Applications have been directed towards the study of smooth quotients of bounded symmetric domains. As for the class of moduli spaces of compact Riemann surfaces and their compactifications, a first difficulty arises from the singularities: A treatment as Kähler V-manifolds equipped with the intrinsicly determined Petersson-Weil metric, however, involves a degeneration towards the compactifying divisor. In this paper we want to study the asymptotic behavior of the curvature tensor and prove its strong negativity. This enables us to apply Siu's $\partial\bar{\partial}$-Bochner technique to harmonic maps.

Based on the Petersson scalar product of automorphic forms, Weil defined an hermitian product on the cotangent space of the Teichmüller space at a point s by

$$\langle \phi dz^2, \psi dz^2 \rangle = \int \frac{\phi\bar{\psi}}{g^2}\, g dz\overline{dz},$$

where ϕdz^2 and ψdz^2 are quadratic holomorphic differentials and $g dz\overline{dz}$ denotes the Poincaré metric on the fiber X_s of s. Alternatively, it can be described as the integral of the contraction of an harmonic Beltrami differential and its conjugate, which, by the Kodaira-Spencer map correspond to tangent vectors of the Teichmüller space. This approach shows that the Petersson-Weil metric is related to the variation of the Poincaré metrics in a universal family. By a result of Ahlfors [A], the Teichmüller spaces, and thus the moduli spaces, regarded as V-manifolds are in this way equipped with a natural Kähler structure with strictly negative holomorphic sectional and Ricci curvatures. Royden conjectured a negative upper bound for the holomorphic sectional curvature. Tromba [TR] and Wolpert [WO] proved the conjecture.

A curvature condition, which is somewhat weaker than the Nakano-positivity of the cotangent bundle, but stronger than negativity of the sectional curvature, was introduced by Siu [S].

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Theorem 1. *The Petersson-Weil metric on the Teichmüller and moduli spaces of compact Riemann surfaces of genus greater than one has strongly negative curvature in the sense of Siu.*

In particular moduli spaces of compact Riemann surfaces satisfy a stronger curvature condition than quotients of bounded symmetric domains do.

The degeneration of the Petersson-Weil metric towards the compactifying divisor $D \subset \bar{M}_g$ can be studied on local universal families of compact Riemann surfaces with nodes. To each singularity one assigns a parameter t_j of the base, which corresponds to the resolution. H. Mazur [M] gave an estimate for the metric tensor for $t_j \to 0$, which implied the non-completeness of M_g. We will prove estimates for the Christoffel symbols and the curvature tensor. As a first application we get:

Theorem 2. *The sectional, Ricci and scalar curvatures of M_g are asymptotically bounded by $-\sum \log|t_i|$ for $t_j \to 0$.*

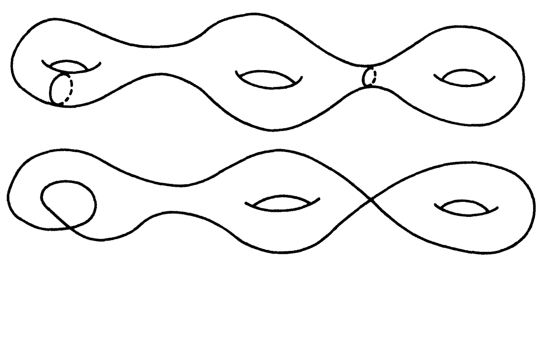
The estimates of the first and second order derivatives of the metric tensor and the strong negativity are substantial for the proof of

Theorem 3. *Let $f: \bar{M}_g \to \bar{M}_g$ be a harmonic map with $f(D) \subset D$, $f(M_g) \subset M_g$ and $rk_{\mathbb{R}} df \geqq 4$ at some point. Then f is holomorphic or antiholomorphic.*

This statement holds for any V-manifold as domain space with similar estimates of the metric towards a compactifying divisor and no curvature assumption. The above theorems will be used to prove a strong rigidity theorem.

2. The Poincaré Metric on Families of Stable Curves

(2.1) By a theorem of Bers [BE], on a family of compact hyperbolic Riemann surfaces, which degenerate to curves with ordinary double points, the Poincaré metrics tend continuously to the Poincaré metrics on the singular fibers with singularities removed. These metrics do not depend differentiably on the parameter, however, we need estimates for their partials with respect to the parameters towards the singular locus. We recall first some well-known facts: Assume that X_0 has singularities at points P_j, these have neighborhoods isomorphic to $\{(z, w) \in \mathbb{C}^2; |z|, |w| < 1, z \cdot w = 0\}$. The connected components Σ_j of $X_0 - \{P_1, \ldots, P_q\}$, $j = 1, \ldots, r$ are called parts of X_0. The parts are punctured Riemann surfaces, whose genus is denoted by p_j. One has to assume that the Σ_j are hyperbolic, i.e. one excludes the case of spheres with two or less punctures and compact tori. Topologically, from a nonsingular curve of genus p one obtains a Riemann surface of the above type with nodes by the contraction of a certain number of cycles:



As any contraction either increases the number of parts by one or decreases the genus of a part by one, the following equation holds:

$$p=p_1+\ldots+p_r+q+1-r. \tag{1}$$

Any part Σ_j possesses a $(3p_j-3+n_j)$-dimensional smooth universal family, where n_j is the number of punctures. Equation (1) yields

$$3p-3=\sum_{j=1}^{r}(3p_j-3+n_j)+q. \tag{2}$$

By an insertion of q points at the punctures one gets a family $\{X_\tau\}$ of Riemann surfaces with nodes, where τ ranges in an open subset of $\mathbb{C}^{3p-3-q}$. The resolution of these singularities can be simultaneously carried out for all X_τ, if one has started with Beltrami differentials on Σ_j with compact support: there exist neighborhoods N_j of all nodes p_j with $N_j \simeq \{(z_j, w_j); |z_j|, |w_j|<2, z_j\cdot w_j=0\}$, $N_j-\{p_j\}=N_j^1\cup N_j^2$ is a union of punctured discs. For each $(t_1,\ldots,t_q)$, $|t_j|<1$, the curve $X_{(t,\tau)}$ is constructed from $X_\tau-\cup\{p_j\}$ by removing the discs $\{z_j; 0<|z_j|\leqq 1\}$ and $\{w_j; 0<|w_j|\leqq 1\}$ and an insertion of an annulus $A_{j,t_j}=\{z_j;\ |t_j|\leqq|z_j|<1\}=\{w_j;\ |t_j|<|w_j|\leqq 1\}$ according to $z_j\cdot w_j=t_j$ (for $t_j=0$, X_τ is not changed at p_j.) The $X_{(t,\tau)}$ form a local, universal family over an open subset $S\subset\mathbb{C}^{3p-3}$. The universal covering of $S-\bigcup_{j=1}^{q}\{t_j=0\}$ is a domain of the Teichmüller space, whereas the quotient of S by the discrete group of automorphisms, whose orbits correspond to isomorphic fibers, represents an open subset of $\bar{M}_p$ – all such sets cover $\bar{M}_p-M_p$. (See Fay [FA] and Earle-Marden).

(2.2) We consider the family $f:\mathscr{X}\to S$ of compact Riemann surfaces $X_{(t,\tau)}$, equipped with the Poincaré metric $g(z,t,\tau)dz\overline{dz}$, which is in the singular case by definition the hyperbolic metric on the punctured surfaces. The families of annuli and unions of punctured discs, respectively, $A_{j,t_j}\subset X_{(t,\tau)}$ only depend on t_j, we denote by U_j the corresponding set in the total space. Let $g^*(z_j,t_j)dz_j\overline{dz_j}$ be the Poincaré metric on A_{j,t_j}. Fix a δ with $0<\delta<1$. Set

$$A_{j,t_j}^{1,\delta}:=\{z_j;|t_j|^{1/2}\leqq|z_j|<1-\delta\},\qquad A_{j,t_j}^{2,\delta}:=\{w_j;|t_j|^{1/2}\leqq|w_j|<1-\delta\}.$$

Lemma [M]. *There is a constant $c>0$, such that for small (t,τ) there are uniform estimates:*

$$(1/c)|z_j|^2\log^2|z_j|\leqq 1/g(z_j,t,\tau)\leqq c\cdot|z_j|^2\log^2|z_j| \quad \text{on} \quad A_{j,t_j}^{1,\delta}$$

with similar estimates on $A_{j,t_j}^{2,\delta}$.

(2.3) If $h(z,t,\tau)dz\overline{dz}$ is a family of auxiliary metrics on $X_{t,\tau}$ (with punctures, if some $t_j=0$), then, with respect to any fiber coordinate z:

$$g(z,t,\tau)=e^{\lambda(z,t,\tau)}\cdot h(z,t,\tau). \tag{3}$$

We construct h: Let $V_j\subset U_j\subset\mathscr{X}$ be the union of all $A_{j,t_j}^{1,\delta}\cup A_{j,t_j}^{2,\delta}$ for $j=1,\ldots,q$. Take the $g^*(z_j,t_j)$ on U_j, a C^∞-family of metrics on the fibers restricted to $\mathscr{X}-\bigcup_{k=1}^{q}V_j$, and define h as a convex sum of these such that h is a C^∞-family of metrics on $\mathscr{X}-\bigcup_{j=1}^{q}U_j$, and on V_j

$$h(z_j,t,\tau)=\pi^2/|z_j|^2\log^2|t_j|\cdot\sin^2(\pi\cdot\log|z_j|/\log|t_j|) \quad \text{for} \quad t_j\neq 0, \tag{4}$$

$$h(z_j,t,\tau)=1/|z_j|^2\log^2|z_j|, \quad \text{if} \quad t_j=0. \tag{5}$$

By lemma (2.2), the function $|\lambda|$ is bounded on $\mathscr{X}$.

(2.4) Denote by D the hermitian connection on all non-singular fibers $X_{(t,\tau)}$ induced by h. A norm on differentiable functions σ on $X_{(t,\tau)}$ is induced by

$$\int_{X_{(t,\tau)}} ((\sigma,\sigma)+(D\sigma,D\sigma)+(D^2\sigma,D^2\sigma))\,h\,dz\,\overline{dz}\,. \tag{6}$$

Remark. The above norms are equivalent [with constants independent of (t,τ)] to

$$\|\sigma\|^2_{(t,\tau)}=\|\sigma\|^2=\int_{X_{(t,\tau)}} (|\sigma|^2+h^{-2}|\partial^2\sigma/\partial z\,\partial\bar{z}|^2)\,h\cdot dz\,\overline{dz}\,. \tag{7}$$

For the proof, one uses the fact that the curvature of h on all fibers is uniformly bounded.

Definition. Let $H^2_{2,(t,\tau)}=H^2_2$ be the Sobolev space of functions on $X_{(t,\tau)}$, whose second derivatives exist in the distributional sense and are square-integrable, equipped with the norm (7).

(2.5) With respect to arbitrary holomorphic coordinates on the fibers the equation for constant negative Gauß curvature

$$\partial^2/\partial z\,\partial\bar{z}\log g=g/2 \tag{8}$$

reads in terms of the laplacian with respect to h:

$$\Delta_h\lambda-(e^\lambda-1)/2=1/2-\Delta_h\log h\,. \tag{9}$$

By construction, the right hand side of (9) vanishes on V_i.

Lemma. *The Sobolev norms $\|\lambda\|_{(t,\tau)}$ are bounded on $S-\bigcup\{t_j=0\}$.*

Proof. By the above reduction of the natural Sobolev norms to (7), we can use (9). As $|\lambda|$ is bounded by (2.3), the claim follows.

(2.6) We will estimate the derivatives of the Poincaré metrics g with respect to parameters towards the singular fibers.
Pick open sets $W_j\supsetneq U_j$, $j=1,\dots,q$ with $W_j\cap W_k=\emptyset$ for $j\neq k$, which still carry (z_j,t,τ)-coordinates and further W_σ, $\sigma\geqq q+1$ with $U_j\cap W_\sigma=\emptyset$, such that $\{W_\nu\}$ is an open covering of $\mathscr{X}$. Let v_j, $j=1,\dots,q$ be C^∞ vector fields on the complement of all singular fibers in $\mathscr{X}$, such that

$$f_*v_i=\partial/\partial t_i;\qquad i=1,\dots,q \tag{10}$$

$$v_i|W_i=\partial/\partial t_i\,,\quad \text{with respect to } (z_i,t,\tau)\text{-coordinates}. \tag{11}$$

Consider some $v=v_i$ and set $t=t_i$, then $v=\partial/\partial+a\partial/\partial z+b\partial/\partial\bar{z}$, where z is a coordinate on the fiber and a, b are suitable functions. If L_v denotes the Lie derivative, then $[\Delta_h,L_v]\lambda=(1/h)L_v(h)\cdot\Delta_h(\lambda)+M$, where M depends on the partials of λ with respect to z and $\bar{z}$ up to second order and vanishes on all V_j. So

$$\Delta_g(L_v(\lambda))-L_v(\lambda)/2=(1/h)L_v(h)((1/2)-e^{-\lambda}\Delta_h(\log h))+e^{-\lambda}L_v(\Delta_h(\log h))-M\,. \tag{12}$$

On any V_k, this is

$$\Delta_g(\partial\lambda/\partial t_i)-(\partial\lambda/\partial t_i)/2=(1-e^{-\lambda})(\partial(\log h)/\partial t_i)/2\,. \tag{13}$$

The norm of the right-hand side of (20) is determined up to a constant by $\|(L_v(h)/h)(1-e^{-\lambda})\|_{(t,\tau)}$ because λ as well as its derivatives with respect to fiber coordinates are uniformly continous on the complement of the V_j by Bers' result [BE]. By lemma (2.5), $\|1-e^{-\lambda}\|$ is bounded on $S-\bigcup\{t_k=0\}$, and from a direct calculation:

$$\|L_{v_i}(h)/h\|^2_{(t,\tau)}=O(-1/|t_i|^2\log^3|t_i|)$$

and

$$\|L_{v_{\bar{i}}}L_{v_i}(h)/h\|^2_{(t,\tau)}=O(-1/|t_i|^4/\log^5|t_i|).$$

Consider the operator $T: H^2_{2,(t,\tau)}\to H^2_{2,(t,\tau)}$; $T(\sigma)=-\Delta_g(\sigma)+\sigma/2$. Its smallest eigenvalue is 1/2, so we arrive at:

$$\|L_{v_i}(\lambda)\|^2=O(-1/|t_i|^2\log^3|t_i|), \tag{14}$$

$$\|L_{v_{\bar{j}}}L_{v_i}(\lambda)\|^2=O(1/|t_i|^2|t_j|^2\log^3|t_i|\log^3|t_j|), \quad \text{for} \quad i\neq j, \tag{15}$$

$$\|L_{v_{\bar{i}}}L_{v_i}(\lambda)\|^2=O(-1/|t_i|^4\log^5|t_i|). \tag{16}$$

(2.7) **Proposition.**

$$L_{v_i}(g)/g=O(-1/|t_i|\log|t_i|); \qquad i=1,\dots,q, \tag{17}$$

$$L_{v_{\bar{j}}}L_{v_i}(g)/g=O(1/|t_i|\,|t_j|\log|t_i|\log|t_j|); \qquad i,j=1,\dots,q. \tag{18}$$

Proof. We have to show these estimates for the derivatives of λ. In principle, the Sobolev embedding theorem for $H^2_2\to C^0$ is applicable to the given family of compact manifolds. However the best constants depend on some fractional power of the injectivity radius, i.e. of $-1/\log|t_i|$. To avoid this, we use it only on a complement of the neighborhoods of the singularities (with a uniform constant), interpret (12) as a Dirichlet problem on the rest and use the maximum principle. The boundary values satisfy estimates analogous to (14), whereas $\partial\log h/\partial t_i$ yield the weaker asymptotic bounds stated in the proposition. Second order derivatives are handled in an analogous way.

3. The Degeneration of the Petersson-Weil-Metric Towards the Compactifying Divisor

(3.1) In this section, we prove estimates for Christoffel symbols and the curvature tensor of the Petersson-Weil-metric on M_g, regarded as a V-manifold towards $\bar{M}_g-M_g$. This can be carried out on the base of a universal deformation $f:\mathscr{X}\to S$ of a compact Riemann surface with nodes. In particular, we prove estimates for first and second order derivatives of the metric tensor – the asymptotic behaviour of "diagonal terms" can be computed, these go faster to infinity than the general terms. However, for later applications, we need the full estimates rather than the weaker estimates suggested by the diagonal terms.

In the sequel we assume that all indices are different that are named by different letters, provided nothing else is stated; latin indices denote numbers $\leqq q$, and greek indices are $>q$.

(3.2) **Theorem.**

$$\partial G_{i\bar{j}}/\partial t_k = O((-1/|t_j| \cdot \log^3|t_j| \cdot |t_i| \cdot \log|t_i| \cdot |t_k| \cdot \log|t_k|) \times \min(1/\log^2|t_k|, 1/\log^2|t_i|)), \tag{19}$$

$$\partial G_{i\bar{j}}/\partial t_i = O(1/|t_j| \cdot \log^3|t_j| \cdot |t_i|^2 \cdot \log^3|t_i|), \tag{20}$$

$$\partial G_{i\bar{i}}/\partial t_i \simeq -1/|t_i|^3 \cdot \log^3|t_i|, \tag{21}$$

$$\partial G_{i\bar{\mu}}/\partial t_k = O((1/|t_i| \cdot \log|t_i| \cdot |t_k| \cdot \log|t_k|) \times \min(1/\log^2|t_i|, 1/\log^2|t_k|), \tag{22}$$

$$\partial G_{i\bar{\mu}}/\partial t_i = O(-1/|t_i|^2 \cdot \log^3|t_i|), \tag{23}$$

$$\partial G_{\nu\bar{\mu}}/\partial t_i = O(-1/|t_i| \cdot \log|t_i|). \tag{24}$$

Partial derivatives with respect to τ_ν satisfy the same estimates as the original functions. Second order estimates of the metric tensor satisfy the same estimates as the curvature tensor, namely

$$\frac{\partial^2 G_{ij}}{\partial t_k \partial t_l} = O((1/|t_i| \log|t_i| \cdot |t_j| \cdot \log|t_j| \cdot |t_k| \cdot \log|t_k| \cdot |t_l| \cdot \log|t_l|) \times \min(1/\log^2|t_i|, 1/\log^2|t_k|) \cdot (\min(1/\log^2|t_j|, 1/\log^2|t_l|) \tag{25}$$

if either $i=k$ or $j=l$, the above minima have to be replaced by $-1/\log|t_i|$ or $-1/\log|t_j|$, respectively,

$$\frac{\partial^2 G_{i\bar{j}}}{\partial t_i \partial t_{\bar{j}}} = O(1/|t_i|^2 \cdot \log^3|t_i| \cdot |t_j|^2 \cdot \log^3|t_j|), \tag{26}$$

$$\frac{\partial^2 G_{i\bar{i}}}{\partial t_i \partial t_{\bar{i}}} \simeq -1/(|t_i|^4 \cdot \log^3|t_i|), \tag{27}$$

$$\frac{\partial^2 G_{i\bar{j}}}{\partial t_k \partial t_{\bar{\mu}}} = O(-1/|t_i| \cdot \log|t_i| \cdot |t_j| \cdot \log|t_j| \cdot |t_k| \cdot \log|t_k|) \times \min(1/\log^2|t_i|, 1/\log^2|t_k|), \tag{28}$$

$$\frac{\partial^2 G_{i\bar{j}}}{\partial t_i \partial t_{\bar{\mu}}} = O(1/|t_j| \log|t_j| \cdot |t_i|^2 \log^3|t_i|), \tag{29}$$

$$\frac{\partial^2 G_{i\bar{j}}}{\partial t_\nu \partial t_{\bar{\mu}}} = O(1/|t_i| \log|t_i| \cdot |t_j| \log|t_j|), \tag{30}$$

$$\frac{\partial^2 G_{i\bar{i}}}{\partial t_\nu \partial t_{\bar{\mu}}} = O(-1/|t_i|^2 \log^3|t_i|), \tag{31}$$

$$\frac{\partial^2 G_{i\bar{\lambda}}}{\partial t_\nu \partial t_{\bar{\mu}}} = O(-1/|t_i| \log^3|t_i|), \tag{32}$$

$$\frac{\partial^2 G_{\kappa\bar{\lambda}}}{\partial t_\nu \partial t_{\bar{\mu}}} = O(1). \tag{33}$$

The same estimates hold for the curvature tensor [use also (19)–(24) and (39)–(46) below]. However, (27) will not be sufficient. In the next section we show by a different method

$$R_{i\bar{i}i\bar{i}}=O(-1/|t_i|^4\log^5|t_i|)\,. \tag{34}$$

Corollary 1.

$$\Gamma^i_{kl}=O((|t_i|/|t_k|\log|t_k|\cdot|t_l|\log|t_l|)\cdot\min(1/\log^2|t_k|,1/\log^2|t_l|)\,, \tag{35}$$

$$\Gamma^i_{kk}=O(-|t_i|/|t_k|^2\log^3|t_k|)\,, \tag{36}$$

$$\Gamma^i_{ii}\simeq 1/|t_i|\,. \tag{37}$$

Corollary 2.

$$\begin{aligned} R_{i\bar{j}}&=O(1/|t_i|\log|t_i|\,|t_j|\log|t_j|)\cdot\min(1/\log^2|t_i|,1/\log^2|t_j|)\,,\\ R_{i\bar{i}}&=O(1/|t_i|^2\log^2|t_i|)\,,\\ R_{i\bar{\mu}}&=O(-1/|t_i|\log^3|t_i|)\,,\\ R_{\nu\bar{\mu}}&=O(1)\,. \end{aligned} \tag{38}$$

Corollary 3. *The sectional, Ricci- and scalar curvature are asymptotically bounded by* $-\sum\limits_{i=1}^{q}\log|t_i|$.

(3.3) We need the estimates for the metric tensor itself by Mazur:

Theorem [M].

$$G_{i\bar{j}}=O(1/|t_i|\cdot\log^3|t_i|\cdot|t_j|\cdot\log^3|t_j|)\,, \tag{39}$$

$$G_{i\bar{i}}\simeq -1/|t_i|^2\cdot\log^3|t_i|\,, \tag{40}$$

$$G_{i\bar{\mu}}=O(-1/|t_i|\cdot\log^3|t_i|)\,, \tag{41}$$

$$G_{\nu\bar{\mu}}=O(1)\,, \tag{42}$$

$$G^{i\bar{j}}=O(|t_i|\,|t_j|)\,, \tag{43}$$

$$G^{i\bar{i}}\simeq -|t_i|^2\cdot\log^3|t_i|\,, \tag{44}$$

$$G^{i\bar{\mu}}=O(|t_i|)\,, \tag{45}$$

$$G^{\nu\bar{\mu}}=O(1)\,. \tag{46}$$

Proof of Corollary 3. We consider the sectional curvature. Given A^i and B^j. Then the sectional curvature is given by

$$\begin{aligned} s=R_{i\bar{j}k\bar{l}}(A^i\overline{B^j}-B^i\overline{A^j})(\overline{A^l}B^k-\overline{B^l}A^k)/G_{i\bar{j}}G_{k\bar{l}}[(A^i\overline{B^j}-B^i\overline{A^j})(\overline{A^l}B^k-\overline{B^l}A^k)\\ +(A^iB^k-B^iA^k)(\overline{A^j}\overline{B^l}-\overline{B^j}\overline{A^l})]\,. \end{aligned}$$

Asymptotically, these are determined by the diagonal terms, which are

$$R_{i\bar{j}i\bar{j}}(A^i\overline{B^j}-B^i\overline{A^j})(\overline{A^j}B^i-\overline{B^j}A^i)/G_{i\bar{i}}G_{j\bar{j}}(A^i\overline{B^j}-B^i\overline{A^j})(\overline{A^j}B^i-\overline{B^j}A^i)\,.$$

Its absolute value is smaller or equal to

$$\sum_{i,j}|R_{i\bar{j}i\bar{j}}|\Big/\Big(\sum_i G_{i\bar{i}}\cdot\sum_j G_{j\bar{j}}\Big)$$

which is by (26), (34), and (40) asymptotically bounded by

$$-\sum_i \log|t_i|.$$

(Equality holds for certain A^i, B^i.)

Remark. The Ricci-form η_{P-W} on M_g extends to a (1, 1)-current on $\bar{M}_g$, in particular $\int_{M_g} \eta^r_{P-W} \wedge \omega^s_{P-W}$ exist for $r+s=3p-3$.

(3.4) The proof of theorem (3.2) follows from estimates of partial derivatives of

$$G^{\alpha\bar{\beta}} = \int_{X(t,\tau)} \frac{\phi_\alpha \overline{\phi_\beta}}{g^2}\, g dz d\bar{z},$$

where $\{\phi_i dz^2, \phi_\nu dz^2\}$ is a basis of relative holomorphic quadratic differentials, dual to $\{\partial/\partial t_i, \partial/\partial \tau_\nu\}_{i=1,\ldots,q;\, \nu\geqq q+1}$. By [M, 5.1] on U_j^1:

$$\phi_i(z_j, t, \tau) = -\frac{t_i}{\pi}\left[\frac{\delta_{ij}}{z_j^2} + a_{-1}(z_j, t, \tau) + \frac{1}{z_j^2}\sum_{s=1}^{\infty}\left[\frac{t_j}{z_j}\right]^s \cdot t_j^{m(s)} \cdot a_s(t,\tau)\right] \tag{47}$$

$m(s)\geqq 0$, and a_{-1} with at most a simple pole at $z_j=0$ and a_s, $s\geqq 1$ holomorphic. For $\nu\geqq q+1$ on U_j^1:

$$\phi_\nu(z_j, t, \tau) = \phi_\nu(z_j, 0, 0) + \frac{1}{z_j^2}\sum_{s=1}^{\infty}\left[\frac{t_j}{z_j}\right]^s \cdot t_j^{\tilde{m}(s)} \cdot b_j(t,\tau) + \sum_{s=-1}^{\infty} (z_j)^s \cdot c_s(t,\tau), \tag{48}$$

where $\phi_\nu(z_j, 0, 0)$ has at most a simple pole and b_s, c_s are holomorphic around 0, $\tilde{m}(s)\geqq 0$. Similar equations hold on U_j^2 with respect to (w_j, t, τ)-coordinates. Immediately we get:

$$\begin{aligned} &\phi_i \sim |t_i|/|z_i|^2 \quad \text{on} \quad U_i, \\ &\phi_i = O(|t_i|\cdot(1/|z_j| + |t_j|/|z_j|^3)) \quad \text{on} \quad U_j, \end{aligned} \tag{49}$$

$\phi_i = O(|t_i|)$ on any compact set K that contains no singularity.

$$\begin{aligned} \partial\phi_i/\partial t_i &\sim 1/|z_i|^2 \quad \text{on} \quad U_i \\ &= O(1/|z_j| + |t_j|/|z_j|^3) \quad \text{on} \quad U_j \\ &= O(1) \quad \text{on} \quad K. \end{aligned} \tag{50}$$

$$\begin{aligned} \partial\phi_i/\partial t_j &= O(|t_i|\cdot(1/|z_i| + |t_i|/|z_i|^3)) \quad \text{on} \quad U_i \\ &= O(|t_i|\cdot(1/|z_j| + |t_j|/|z_j|^4)) \quad \text{on} \quad U_j \\ &= O(|t_i|\cdot(1/|z_k| + |t_k|/|z_k|^3)) \quad \text{on} \quad U_k \\ &= O(|t_i|) \quad \text{on} \quad K. \end{aligned} \tag{51}$$

$$\begin{aligned} \phi_\nu &= O(1/|z_i| + |t_i|/|z_i|^3) \quad \text{on} \quad U_i, \\ &= O(1) \quad \text{on} \quad K. \end{aligned} \tag{52}$$

$$\begin{aligned} \partial\phi_\nu/\partial t_i &= O(1/|z_i|^3) \quad \text{on} \quad U_i, \\ &= O(1/|z_j| + |t_j|/|z_j|^3) \quad \text{on} \quad U_j, \\ &= O(1) \quad \text{on} \quad K. \end{aligned} \tag{53}$$

Now with vector fields v_k on $\mathscr{X}$, whose projections to the base are $\partial/\partial t_k$, we have

$$\partial G^{i\bar{j}}(t,\tau)/\partial t^k = \int_{X_{(t,\tau)}} L_{v_k}((\phi_i \cdot \overline{\phi}_j)/g)dz\overline{dz}\,. \tag{54}$$

Estimates of the integral (54) by means of (17), (18), and (49)–(53) yield the following relations, which imply (3.2):

$$\begin{aligned}
&\partial G^{i\bar{j}}/\partial t_k = O(|t_i|\cdot|t_j|/|t_k|\cdot\log|t_k|)\,,\\
&\partial G^{i\bar{j}}/\partial t_i = O(|t_j|)\,,\\
&\partial G^{i\bar{i}}/\partial t_k = O((|t_i|^2\cdot\log^3|t_i|)/(|t_k|\cdot\log|t_k|))\,,\\
&\partial G^{i\bar{i}}/\partial t_i \sim -|t_i|\cdot\log^3|t_i|\,.
\end{aligned}\tag{55}$$

$$\begin{aligned}
&\partial G^{v\bar{j}}/\partial t_i = O(|t_j|/|t_i|\cdot\log|t_i|)\,,\\
&\partial G^{v\bar{j}}/\partial t_j = O(1)\,.
\end{aligned}\tag{56}$$

$$\partial G^{v\bar{\mu}}/\partial t_i = O(1/|t_i|\cdot\log|t_i|)\,. \tag{57}$$

$$\begin{aligned}
&\partial^2 G^{i\bar{j}}/\partial t_k\partial t_{\bar{l}} = O((|t_i|\cdot|t_j|)/(|t_k|\cdot\log|t_k|\cdot|t_l|\cdot\log|t_l|))\,,\\
&\partial^2 G^{i\bar{j}}/\partial t_i\partial t_{\bar{l}} = O(|t_j|/|t_l|\cdot\log|t_l|)\,,\\
&\partial^2 G^{i\bar{j}}/\partial t_i\partial t_{\bar{j}} = O(1)\,,\\
&\partial^2 G^{i\bar{i}}/\partial t_i\partial t_{\bar{i}} = O(\log^3|t_i|)\,.
\end{aligned}\tag{58}$$

(The remaining terms can be derived from the general formula.)

$$\begin{aligned}
&\partial^2 G^{v\bar{j}}/\partial t_k\partial t_{\bar{l}} = O(|t_j|\cdot(1/|t_k|\cdot\log|t_k|+1/|t_l|\cdot\log|t_l|))\,,\\
&\partial^2 G^{v\bar{j}}/\partial t_k\partial t_{\bar{j}} = O(|t_j|/|t_k|\cdot\log|t_k|)\,,\\
&\partial^2 G^{v\bar{j}}/\partial t_k\partial t_{\bar{k}} = O(|t_j|/|t_k|^2\cdot\log^2|t_k|)\,.
\end{aligned}\tag{59}$$

$$\begin{aligned}
&\partial^2 G^{v\bar{\mu}}/\partial t_k\partial t_{\bar{l}} = O(1/|t_k|\log|t_k|+1/|t_l|\log|t_l|)\,,\\
&\partial^2 G^{v\bar{\mu}}/\partial t_k\partial t_{\bar{k}} = O(1/|t_k|^2\cdot\log^2|t_k|)\,.
\end{aligned}\tag{60}$$

As an example we give the argument for (55), first formula: One performs the integration on U_i, U_j, U_k and all remaining U_l and their complement K. On U_k, e.g., the integrand can be estimated by (49), (51), (2.2), and (17) by

$$\begin{aligned}
&|t_i|\,|t_j|\,(1/|z_k|+|t_k|/|z_k|^3)^2|z_k|^2\cdot\log^2|z_k|\\
&\times(2+1/|t_k|\cdot\log|t_k|)\,|dz_k|^2\,.
\end{aligned}$$

The integral over $\{z_k;\,|t_k|^{(1/2)+\delta}\leqq|z_k|\leqq 1-\varepsilon\}$ yields the estimate.

We will prove (34) in the next section.

4. Strong Curvature of the Petersson-Weil Metric and Estimate of $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$

(4.1) The strongest curvature condition the Teichmüller space could satisfy might be the Nakano-positivity of the cotangent bundle. In his discussion of harmonic maps, Siu used a weakeded (dual) Nakano-negativity, which is still stronger than negative sectional curvature.

Definition. A Kähler manifold M has strongly seminegative curvature in the sense of Siu, if

$$R_{\alpha\bar\beta\gamma\bar\delta}(A^\alpha\overline{B^\beta}-C^\alpha\overline{D^\beta})\overline{(A^\delta\overline{B^\gamma}-C^\delta\overline{D^\gamma})}\geqq 0 \tag{61}$$

for all A^α, B^β, C^γ, D^δ. Its curvature is strongly negative in this sense, if equality holds only, if $A^\alpha\overline{B^\beta}=C^\alpha\overline{D^\beta}$.

(4.2) **Proposition.** *The Teichmüller space $\mathcal{T}_g$, $g\geqq 2$ is strictly negative in the sense of Siu.*

The proof follows from Wolpert's formula [WO]: Let $G_{\alpha\bar\beta}dt^\alpha dt^{\bar\beta}$ be the Petersson-Weil metric, then

$$R_{\alpha\bar\beta\gamma\bar\delta}(p)=\int_{X_p} L(\mu_\alpha\cdot\mu_{\bar\beta})\cdot(\mu_\gamma\cdot\mu_{\bar\delta})g\,dv$$

$$+\int_{X_p} L(\mu_\alpha\cdot\mu_{\bar\delta})\cdot(\mu_\gamma\cdot\mu_{\bar\beta})h\,dv\,, \tag{62}$$

where $g\,dv$ is the Poincaré metric, $L=-(\Delta-2)^{-1}$, and $\{\mu_\alpha\}$ is a basis of the harmonic Beltrami differentials corresponding to $\{\partial/\partial t_\alpha\}$ equipped with the cup-product, i.e., the contraction $\mu_\alpha\cdot\mu_\beta$. L is a positive definite, self-adjoint operator. Define

$$\mu_A:=A^\alpha\mu_\alpha\,,\qquad \mu_{\bar B}:=\overline{B^\beta}\mu_{\bar\beta},\,\ldots,\,R_{A\bar BC\bar D}=R_{\alpha\bar\beta\gamma\bar\delta}A^\alpha\overline{B^\beta}C^\gamma\overline{D^\delta}$$

and set

$$(A\cdot\bar B,C\cdot\bar D)=\int L(\mu_A\cdot\mu_{\bar B})\mu_C\cdot\mu_{\bar D}g\,dv\,. \tag{63}$$

Then by [WO, 4.3]

$$|(A\cdot\bar B,C\cdot\bar D)|\leqq\int L(|\mu_A|^2)^{1/2}L(|\mu_B|^2)^{1/2}|\mu_C|\,|\mu_D|g\,dv\,.$$

The latter is not greater than $(A\cdot\bar A,\ C\cdot\bar C)^{1/2}(B\cdot\bar B,\ D\cdot\bar D)^{1/2}$ and $(A\cdot\bar A,\ D\cdot\bar D)^{1/2}(B\cdot\bar B,\ C\cdot\bar C)^{1/2}$. We have to see that $R:=R_{A\bar BB\bar A}+R_{C\bar DD\bar C}-R_{A\bar BD\bar C}-R_{C\bar DB\bar A}\geqq 0$:

$$\begin{aligned}R&=(A\cdot\bar B,B\cdot\bar A)+(A\cdot\bar A,B\cdot\bar B)+(C\cdot\bar D,D\cdot\bar C)+(C\cdot\bar C,D\cdot\bar D)\\&\quad-(A\cdot\bar B,D\cdot\bar C)-(A\cdot\bar C,D\cdot\bar B)-(C\cdot\bar D,B\cdot\bar A)-(C\cdot\bar A,B\cdot\bar D)\\&=(A\cdot\bar B-C\cdot\bar D,B\cdot\bar A-D\cdot\bar C)+(A\cdot\bar A,B\cdot\bar B)+(C\cdot\bar C,D\cdot\bar D)-2\,\mathrm{Re}(A\cdot\bar C,D\cdot\bar B)\,.\end{aligned}$$

The first term is non-negative, whereas the rest is larger or equal to $[(A\cdot\bar A,B\cdot\bar B)^{1/2}-(C\cdot\bar C,\ D\cdot\bar D)^{1/2}]^2\geqq 0$. The relation $R=0$ only holds, if $A\cdot\bar B=C\cdot\bar D$, i.e. $A^\alpha\mu_\alpha\cdot B^{\bar\beta}\mu_{\bar\beta}=C^\alpha\mu_\alpha\cdot D^{\bar\beta}\mu_{\bar\beta}$. As the μ_α are harmonic, $g\cdot\overline{\mu_\alpha}$ are holomorphic quadratic differentials. This induces an equality of meromorphic and anti-meromorphic functions, from which $A^\alpha B^{\bar\beta}=C^\alpha D^{\bar\beta}$ follows. In particular M_g, as a V-manifold, has the same curvature property.

(4.3) *Remark.* If $\{\phi_i,\phi_v\}_{i\leqq q,\,v>q}$ is the dual basis of $\{\partial/\partial t_i,\partial/\partial\tau_v\}$, then the latter is in terms of harmonic Beltrami differentials

$$\mu_\alpha=\sum_\beta G_{\alpha\bar\beta}\cdot\phi_{\bar\beta}/g\,. \tag{64}$$

Proposition.

$$R_{\alpha\bar\alpha\alpha\bar\alpha}\leqq\int(\mu_\alpha\overline{\mu_\alpha})^2g\,dv\,. \tag{65}$$

Proof. Expand $\mu_\alpha\overline{\mu_\alpha}$ into a sum $\sum\psi_\varrho$ of orthogonal eigenfunctions of the Laplacian with eigenvalues $\lambda_\varrho \leqq 0$. Then $L\psi_\varrho = (1/(2-\lambda_\varrho))\psi_\varrho$ implies the estimate.

Now all integrals $\int((\phi_\alpha\phi_{\bar\beta}\phi_\gamma\phi_{\bar\delta})/g^4)\cdot g dv$ can be treated like in the third Section. Similar computations together with (64) give

$$R_{i\bar i i\bar i} = O(-1/|t_i|^4\cdot\log^5|t_i|) \quad \text{for} \quad i\leqq q.$$

For all off-diagonal terms this method yields weaker estimates than shown in the preceding section.

5. Harmonic Maps of $\bar M_g$

(5.1) Given a map $f: N\to M$ of Riemann manifolds with $ds_M^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ and $ds_N^2 = h_{ij}dy^i dy^j$, then its Dirichlet- or energy functional $E(f)$ is defined by

$$E(f) = 1/2 \int_N \operatorname{trace}(f^* ds_M^2) = 1/2\int h^{ij} g_{\alpha\beta}(\partial f^\alpha/\partial x^i)(\partial f^\beta/\partial x^j). \tag{66}$$

The critical points are called harmonic maps, whose Euler-Lagrange equation is

$$\Delta_N f^\alpha(y) + {}^M\Gamma^\alpha_{\beta\gamma}(f(y))(\partial f^\beta/\partial y^i)(\partial f^\gamma/\partial y^j)h^{ij}(y) = 0. \tag{67}$$

We will now extend Siu's theorem to the singular case of moduli spaces, where the Kähler metric degenerates. It states in its original form that a harmonic map f of compact Kähler manifolds with strong negative curvature is holomorphic or antiholomorphic, provided the rank over $\mathbb{R}$ of df is at least 4 at some point.

The notion of harmonic maps and Siu's theorem immediately carry over to compact Kähler V-manifolds. In the case of $\bar M_g$ the Kählerform ω_{P-W} is a (1,1)-current, which is d-closed in the sense of currents [W]. The singularities of the corresponding connection and the curvature around the compactifying divisor have been studied in Sect. 3. So by a harmonic map we mean a twice differentiable map with locally bounded derivatives, for which (67) holds with respect to local uniformizing systems, whereever it is defined.

(5.2) **Theorem.** *Let $f: \bar M_g \to \bar M_g$ be a harmonic map with $f(D)\subset D$, $f(M_g)\subset M_g$ and $\operatorname{rk}_{\mathbb{R}} df \geqq 4$ at some point. Then f is holomorphic or antiholomorphic.*

(5.3) We apply Siu's method to the above degenerate situation. The proof of his theorem is based on his $\partial\bar\partial$-Bochner formula for maps $f: N\to M$ of Kähler manifolds:

$$\partial\bar\partial\langle g, \bar\partial f\wedge\partial\bar f\rangle = \langle R, \bar\partial f\wedge\partial\bar f\wedge\partial f\wedge\overline{\partial f}\rangle - \langle g, D\bar\partial f\wedge\bar D\partial\bar f\rangle. \tag{68}$$

Here g and R, respectively, denote the metric and curvature tensors, respectively, the bracket is the contraction of tensors (with differential forms as values), and D is the exterior derivative of differential forms with values in $f^*T_M^{1,0}$ with respect to ds_N^2 and ds_M^2.

$$\partial\bar\partial(g_{\alpha\bar\beta}\bar\partial f^\alpha\wedge\partial f^{\bar\beta}) = R_{\alpha\bar\beta\gamma\bar\delta}\bar\partial f^\alpha\wedge\partial f^{\bar\beta}\wedge\bar\partial f^\gamma\wedge\bar\partial f^{\bar\delta} - g_{\alpha\bar\beta}D\bar\partial f^\alpha\wedge\bar D\partial f^{\bar\beta}. \tag{69}$$

Now (5.2) will follow, once we can show that

$$\int\partial\bar\partial\langle g, \partial f\wedge\partial\bar f\rangle\wedge\omega^{n-2} = 0, \qquad n = \dim(M).$$

(5.4) **Lemma.** *Under the assumptions of* (5.2) $\zeta = \partial\bar{\partial}\langle G, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{n-2}$ *is a d-exact current.*

The proof goes by a careful check of all separate terms of (69), one has to distinguish between "diagonal" terms of different levels and expand derivatives of functions f^{α} near the divisor at infinity, and on the boundary of small neighborhoods of D. Now by explicit integrations, one can see from (25)–(32) that ζ is in fact a current, and show its d-exactness by means of (19)–(24).

Remark. The first statement of the above theorem holds for any Kähler V-manifold as a domain space with similar estimates (39)–(46) of the metric towards a compactifying divisor with V-normal crossings and no curvature assumptions.

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Markov's Inequality and C^∞ Functions on Sets with Polynomial Cusps

W. Pawłucki and W. Pleśniak

Institute of Mathematics, Jagellonian University, Reymonta 4, PL-30 059 Kraków, Poland

Introduction

Markov's inequality is an important tool in the theory of polynomial approximation and therefore it is still the object of numerous investigations. As far as its theory is well developed in one-dimensional case (in which we refer the reader to the outstanding monograph [22]) incomparably less has been done in this subject in the case of several variables. So far, versions of Markov's inequality obtained in [17] and [26] by the method of Siciak's extremal function seem to be most satisfactory in multidimensional case. They include in particular earlier results of Baouendi and Goulaouic [1] for bounded convex domains in $\mathbb{R}^n$ or bounded domains with Lipschitz boundary. However, those results do not deal with sets admitting cusps and what is a matter of course such a case affords considerably more difficulties. A first attempt in this direction was made by Goetgheluck [9] who gave estimates of derivatives of polynomials on the set $E=\{(x,y)\in\mathbb{R}^2;\ 0\leqq x\leqq 1,\ 0\leqq y\leqq x^p\}$, where $p\geqq 1$. In [21] the second author proposed a method based on properties of the extremal function (see Sect. 1) that permitted to consider more general cases than that investigated in [9]. In particular there was made in [21] a first approach to Markov's inequality on subanalytic subsets of $\mathbb{R}^n$.

In this paper we introduce (in Sect. 2) uniformly polynomially cuspidal subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ (briefly, *UPC*) and state for them Markov's inequality (Theorem 3.1). This result includes known versions of Markov's inequality for subsets of $\mathbb{R}^n$ satisfying a parallelepiped property P (e.g. bounded convex domains or bounded Lipschitz domains). In Sect. 6, applying Hironaka's rectilinearization theorem and Łojasiewicz's inequality we prove (Corollary 6.6) that every bounded subanalytic subset E of $\mathbb{R}^n$ such that $\operatorname{int} E$ is dense in E is *UPC*. Hence we get in particular Markov's inequality for subanalytic sets announced earlier in [16].

Our second main result is Bernstein's theorem on polynomial approximation of C^∞ functions on *UPC* compact subsets of $\mathbb{R}^n$ (Theorem 5.1) which is proved by means of Markov's inequality as well as a Hölder continuity property of the extremal function of a *UPC* set stated in Sect. 4. This result essentially extends a version of Bernstein's theorem for subanalytic sets announced in [16]. The reason is that in order to obtain a C^∞ extension in the subanalytic case we can make use of a strong regularity property of the type of Whitney (Sect. 7). As is shown by Example 7.1, that property does not hold for all *UPC* subsets of $\mathbb{R}^n$.

As an application of Theorem 5.1 we extend to the case of *UPC* sets a Bernstein type criterion of C^∞ for quasianalytic functions (Remark 7.3) proved earlier in [17] for fat convex compact sets in $\mathbb{R}^n$ as well as a result of Siciak [26] on existence of "highly non-continuable" functions solving a problem posed by Globevnik and Stout, and Rudin (Remark 7.4).

Finally we note that the Hölder property of the extremal function of a subanalytic set yielded by Theorem 4.1 and Corollary 6.6 (involving Hironaka's rectilinearization theorem) implies L-regularity of (fat) subanalytic sets proved earlier in [20] by means of a curve selecting lemma and Puiseux's theorem (see Remark 6.7).

1. The Extremal Function

To every compact subset K of the space $\mathbb{C}^n$ there corresponds an *extremal function*

$$\Phi_K(x) = \sup\{|p(x)|^{1/\deg p};\ p \text{ is a polynomial from } \mathbb{C}^n \text{ to } \mathbb{C} \text{ of degree at least 1 with } \|p\|_K \leqq 1\}$$

for $x \in \mathbb{C}^n$, where $\|p\|_K := \sup|p|(K)$. The function Φ_K was introduced by Siciak [24] more than twenty years ago and through this period it appeared very useful in numerous problems of complex analysis, especially in the theory of polynomial approximation and interpolation. For references see [24, 25, 19, 12].

If $n=1$, the function Φ_K is equal to an *extremal function of Leja* [13]. In particular $\log\Phi_K$ is the *generalized Green's function* with a pole at infinity for the unbounded component of $\mathbb{C}\setminus K$. This implies a simple formula for Φ_K in case of $K=[-1,1]$, viz.

$$\Phi_{[-1,1]}(x) = |x+\sqrt{x^2-1}|, \quad \text{for} \quad x \in \mathbb{C}, \tag{1.1}$$

where the branch of the root is so chosen that $|x+\sqrt{x^2-1}| \geqq 1$ in the whole plane. Hence by an easy calculation, for each $t \in [0,1)$ we get

$$\Phi_{[t,1]}(0) = (1+\sqrt{t})/(1-\sqrt{t}). \tag{1.2}$$

From the definition of Φ_K we derive for each polynomial p, the inequality

$$|p(x)| \leqq \|p\|_K\,[\Phi_K(x)]^{\deg p}, \quad \text{for} \quad x \in \mathbb{C}^n, \tag{1.3}$$

called the *Bernstein-Walsh inequality*.

The extremal function may also be defined with the aid of plurisubharmonic functions. Following Zakharyuta [31] (see also [25]), for any subset E of $\mathbb{C}^n$ we define a function

$$V_E(x) = \sup\{u(x)\colon u \in L,\ u \leqq 0 \text{ on } E\}, \quad \text{for} \quad x \in \mathbb{C}^n,$$

where L is the *class of Lelong* of all plurisubharmonic functions on $\mathbb{C}^n$ such that $u(x)-\log(1+|x|)$ is bounded from above in $\mathbb{C}^n$.

It is known [31, 25] that if E is compact then $V_E = \log \Phi_E$. The upper semicontinuous regularization of V_E, V_E^* satisfies (in sense of currents) the *complex Monge-Ampère equation* $(dd^c V_E^*)^n = 0$ in $\mathbb{C}^n \setminus E$ [3]. Hence in particular the function V_E^* is a natural counterpart of the Green function.

A subset E of $\mathbb{C}^n$ is called (in honor of Leja) *L-regular at a point* $a \in \bar{E}$ if $V_E^*(a) = 0$. If E is L-regular at every point $x \in \bar{E}$, we say that E is *L-regular*. If E is compact, L-regularity of E is equivalent to continuity of V_E (or Φ_E) in $\mathbb{C}^n$ [31, 25].

2. Uniformly Polynomially Cuspidal Sets

Let E be a subset of $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We shall say that E is *uniformly polynomially cuspidal* (briefly, *UPC*) if there exist positive constants M and m, and a positive integer d such that for each point $x \in \bar{E}$, one may choose a polynomial map $h_x: \mathbb{K} \to \mathbb{K}^n$ of degree at most d satisfying the following conditions.

$$h_x((0,1]) \subset E \quad \text{and} \quad h_x(0) = x; \tag{2.1}$$

$$\operatorname{dist}(h_x(t), \mathbb{K}^n \setminus E) \geqq Mt^m \quad \text{for all } x \text{ in } \bar{E} \text{ and } t \in [0,1]. \tag{2.2}$$

One can readily show that every bounded set $E \subset \mathbb{R}^n$ satisfying the following *parallelepiped property P* (cf. [17, Lemma 3.1]; see also [26, Proposition 5.1]) is *UPC*.

(P). For each point $a \in \bar{E}$ there exists a non-singular affine map $h: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$a \in h(I^n) \subset E \cup \{a\}; \tag{2.3}$$

$$|\det h| \geqq c > 0, \tag{2.4}$$

where $I^n = \{x \in \mathbb{R}^n;\ |x_i| \leqq 1, i = 1, \ldots, n\}$, $\det h$ denotes the determinant of a matrix representing h and the constant c does not depend on the point a.

Every bounded convex set in $\mathbb{K}^n$ with non-void interior satisfies P (see [17, Corollary 3.1]), whence it is *UPC*. (Here we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$.) As observed in [1, p. 167], every bounded domain in $\mathbb{K}^n$ with Lipschitz boundary also satisfies P, and consequently it is *UPC*.

All these examples exclude sets with cusps. In Sect. 6 we prove that every fat subanalytic subset of $\mathbb{R}^n$ is *UPC*. Thus, we enlarge the above family of *UPC* sets in $\mathbb{R}^n$ to include sets with subanalytic cusps. For other examples of *UPC* sets see Sect. 7.

3. Markov's Inequality

Theorem 3.1. *Let E be a UPC subset of $\mathbb{K}^n$. Then there exists a constant $r > 0$ such that for each polynomial $p: \mathbb{K}^n \to \mathbb{K}$ of degree at most k and each $\alpha \in \mathbb{Z}_+^n$, we have*

$$\|D^\alpha p\|_E \leqq Ck^{r|\alpha|} \|p\|_E,$$

where the constant C depends only on E and α.

Proof. Fix $x \in \bar{E}$. Since E is *UPC* there exists a polynomial map h_x: $\mathbb{K} \to \mathbb{K}^n$ of degree at most d and positive constants M and m such that for each $t \in [0,1]$,

$$E_t(x) = \bigcup_{s \in [t,1]} B(h_x(s), Ms^m) \subset \bar{E},$$

the constants d, M, m being independent of x in $\bar{E}$, and $B(a,r) := \{x \in \mathbb{K}^n, |x-a| \leqq r\}$, where $|(u_1, \ldots, u_n)| = \max\{|u_1|, \ldots, |u_n|\}$ for $u \in \mathbb{K}^n$. Fix $t \in (0,1]$. By the definition of the extremal function, from (1.2) we get

$$\Phi_{\overline{E_t(x)}}(x) \leqq [\Phi_{[t,1]}(0)]^d = [(1+\sqrt{t})/(1-\sqrt{t})]^d.$$

If p: $\mathbb{K}^n \to \mathbb{K}$ is a polynomial of degree $\leqq k$, then for each $\alpha \in \mathbb{Z}^n_+$ we have

$$|D^\alpha p(x)| \leqq \|D^\alpha p\|_{E_t(x)} [\Phi_{\overline{E_t(x)}}(x)]^k,$$

whence by putting $t_k = 1/4k^2$ we get

$$|D^\alpha p(x)| \leqq e^{2d} \|D^\alpha p\|_{E_{t_k}} = e^{2d} |D^\alpha p(b)| \tag{3.1}$$

with a suitably chosen point $b \in \bar{E}_{t_k}$. Take a sequence $\{b^\nu\} \subset E_{t_k}(x)$ such that $b^\nu \to b$, as $\nu \to \infty$, and a sequence $\{s_\nu\} \subset [t_k, 1]$ such that $b^\nu \in B(h_x(s_\nu), Ms^m)$ for each ν. By the classical Markov (or Bernstein) inequality for the unit ball in $\mathbb{R}$ (or $\mathbb{C}$) we have for each ν,

$$|D^\alpha p(b^\nu)| \leqq [k^\omega/Ms_\nu^m]^{|\alpha|} \|p\|_{B(h_x(s_\nu), Ms^m)} \leqq [k^\omega/Mt_k^m]^{|\alpha|} \|p\|_E,$$

where $\omega = 2$, if $\mathbb{K} = \mathbb{R}$ and $\omega = 1$, if $\mathbb{K} = \mathbb{C}$. Now, by (3.1), the last inequalities give the assertion of the theorem.

Remark 3.2. Cauchy's formula and (1.3) yield another proof of Markov's inequality for compact subsets E of $\mathbb{K}^n$ the extremal function Φ_E of which satisfies the following Hölder condition.

$$\Phi_E(x) \leqq 1 + C\delta^\mu \quad \text{as} \quad \operatorname{dist}(x,E) \leqq \delta \leqq 1 \tag{3.2}$$

with some positive constants C and μ independent of δ. (See [17, Lemma 3.1] or [26, Lemma 3].) In Sect. 4 we shall prove that every UPC compact set in $\mathbb{K}^n$ satisfies (3.2).

If E is not *UPC* then Markov's inequality does not in general hold for E. To see this we adopt an example of Zerner [32].

Example 3.3. Define $E = \{(x,y) \in \mathbb{R}^2;\ 0 < x \leqq 1,\ 0 \leqq y \leqq \exp(-1/x)\} \cup \{(0,0)\}$ and consider the sequence of the polynomials $p_k(x,y) = y(1-x)^k$, $k = 1, 2, \ldots$. Then for all k, $(\partial p_k/\partial y)(0,0) = 1$ while $\|p_k\|_E = o(\exp(-\sqrt{k+1}))$, as $k \to \infty$. Thus, there is no $r > 0$ such that

$$|(\partial p_k/\partial y)(0,0)| \leqq (k+1)^r \|p_k\|_E \quad \text{for all } k.$$

4. Hölder Continuity Property of the Extremal Function

In the sequel, if $\mathbb{K} = \mathbb{R}$, the space $\mathbb{R}^n$ is identified with the subset $\mathbb{R}^n + i0$ of the space $\mathbb{C}^n$. Let h: $\mathbb{K} \to \mathbb{K}^n$ be a polynomial map of degree $\leqq d$. Let $M > 0$, m be a positive integer, and let

$$E=\{x\in \mathbb{K}^n;\ x=h(t)+M[(tu_1)^m, \ldots, (tu_n)^m],\ t\in[0,1],\ u_j\in\mathbb{K},$$
$$|u_j|\leqq 1, j=1, \ldots, n\}.$$

Then E is the image of the "pyramid"

$$S=\{(t, tu_1, \ldots, tu_n)\in\mathbb{K}^{n+1}; t\in[0,1], |u_j|\leqq 1, j=1, \ldots, n\}$$

by the polynomial map

$$p:\ \mathbb{C}\times\mathbb{C}^n\ni(t,v)\to h(t)+M(v_1^m, \ldots, v_n^m)\in\mathbb{C}^n$$

with $\deg p\leqq s:=\max(d,m)$. By the definition of the extremal function, we have

$$\Phi_E(p(t,v))\leqq[\Phi_S(t,v)]^s \quad \text{for} \quad (t,v)\in\mathbb{C}\times\mathbb{C}^n. \tag{4.1}$$

Since S is convex, the function Φ_S has the following Hölder property (see [26, Proposition 5.3]).

$$\Phi_S(x)\leqq 1+M_1\delta^{1/2}, \quad \text{as} \quad x\in S^\delta:=\{x\in\mathbb{C}^{n+1};\ \operatorname{dist}(x,S)\leqq\delta\} \tag{4.2}$$

with a positive constant M_1 independent of $\delta\leqq 1$. Observe that

$$P(\delta):=\{(t,v)\in\mathbb{C}\times\mathbb{C}^n;\ |t|\leqq\delta,\ |v_j|\leqq\delta, j=1, \ldots, n\}\subset S^\delta.$$

Hence we get

$$p(S^\delta)\supset p(P(\delta))\supset p(\{(0,\mathrm{v})\in\mathbb{C}\times\mathbb{C}^n;\ |v_j|\leqq\delta, j=1, \quad \ldots, n\})\supset B(h(0), M\delta^m).$$

Thus, by (4.1) and (4.2), we have

$$\Phi_E(x)\leqq 1+M_2\eta^{1/2m}, \quad \text{as} \quad \operatorname{dist}(x,h(0))\leqq\eta\leqq 1, \tag{4.3}$$

where the constant M_2 depends only on M, m and d. Using now (4.3) we derive an important estimate of the extremal function for subsets with uniform polynomial cusps, viz.

Theorem 4.1. *Suppose E is a UPC compact subset of $\mathbb{K}^n$. Then for $x\in\mathbb{C}^n$,*

$$\Phi_E(x)\leqq 1+C\delta^{1/2[m]}, \text{ as } \quad \operatorname{dist}(x,E)\leqq\delta\leqq 1,$$

where C is a positive constant depending only on the constants M, m and d of the definition of UPC, and $[m]:=k$, as $k-1<m\leqq k$, where $k\in\mathbb{Z}$.

Remark 4.2. It is known [26, Proposition 5.1] that if a compact subset E *of* $\mathbb{R}^n$ has the parallelepiped property P (Sect. 2) then the function Φ_E satisfies (3.2) with $\mu=1/2$. So far, there have been known no examples of sets with cusps satisfying (3.2) (with some positive μ). Theorem 4.1 together with Corollary 6.6 yield examples of such sets having polynomial cusps.

5. Polynomial Approximation of C^∞ Functions

Markov's inequality and Hölder continuity property of the extremal function allow us to prove the following version of a classical result of Bernstein [4].

Theorem 5.1. *Assume E is a UPC compact set in $\mathbb{R}^n$. A real-valued function f defined on E is the restriction to E of a C^∞ function f in $\mathbb{R}^n$ if and only if for each $r>0$,*

$$\lim_{k\to\infty} k^r \operatorname{dist}_E(f, \mathscr{P}_k)=0, \tag{5.1}$$

where $\mathscr{P}_k$ is the linear space of (the restrictions to E of) all polynomials from $\mathbb{R}^n$ to $\mathbb{R}$ of degree at most k, and

$$\operatorname{dist}_E(f, \mathscr{P}_k)=\inf\{\sup|f-p|(E): p\in\mathscr{P}_k\}.$$

Proof. The necessity of (5.1) (for any compact subset E of $\mathbb{R}^n$) in order that f be extendible to a C^∞ function follows easily from Jackson's theorem [29]. To prove the second part of the theorem observe that by (5.1) there exist polynomials $p_k\in\mathscr{P}_k$ $(k=1,2,\ldots)$ such that

$$f=\sum_{k=1}^{\infty} p_k, \tag{5.2}$$

and

$$\text{for each } r>0,\ \lim_{k\to\infty} k^r p_k=0 \text{ uniformly on } E. \tag{5.3}$$

Set $\varepsilon_k=(1/Ck)^{2[m]}$, for $k=1,2,\ldots$, where the constants C and m are defined in Theorem 4.1. By a known lemma [30, Chap. IV, Lemma 3.3] there are constants C_α (depending only on $\alpha\in\mathbb{Z}^n_+$) such that for any k, there exists a C^∞ function u_k on $\mathbb{R}^n$ satisfying

$$0\leqq u_k\leqq 1,\ u_k=1 \text{ in a neighbourhood of } E, \text{ and } u_k(x)=0 \text{ if } \operatorname{dist}(x,E)\geqq\varepsilon_k; \tag{5.4}$$

$$\text{for all } x\in\mathbb{R}^n \text{ and } \alpha\in\mathbb{Z}^n_+,\ |D^\alpha u_k(x)|\leqq C_\alpha\varepsilon_k^{-|\alpha|}. \tag{5.5}$$

Now define

$$\tilde{f}=\sum_{k=1}^{\infty} u_k p_k.$$

By (5.2) and (5.3) the series $\tilde{f}$ is uniformly convergent in $\mathbb{R}^n$, because by putting $E^k=\{x\in\mathbb{R}^n;\ \operatorname{dist}(x,E)\leqq\varepsilon_k\}$, from (5.4), (5.5), and Theorem 4.1 we derive

$$\sup_{\mathbb{R}^n}|u_k p_k|=\sup_{E^k}|u_k p_k|\leqq\sup_{E^k}|p_k|\leqq\|p_k\|_E\left(\sup_{E^k}\Phi_E\right)^k\leqq\|p_k\|_E(1+1/k)^k\leqq e\,\|p_k\|_E.$$

Moreover, by (5.4), (5.5) and Theorem 3.1, for each $\alpha\in\mathbb{Z}^n_+$,

$$\begin{aligned}\sup_{\mathbb{R}^n}|D^\alpha(u_k p_k)|&\leqq\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}\sup_{E^k}|D^{\alpha-\beta}u_k D^\beta p_k|\\&\leqq\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}C_{\alpha-\beta}\varepsilon_k^{-|\alpha-\beta|}\|D^\beta p_k\|_E(1+1/k)^k\leqq M_\alpha k^{s|\alpha|}\|p_k\|_E,\end{aligned}$$

where $s=\max(2[m],r)$, and M_α is independent of k. Hence by (5.3), $\tilde{f}$ is C^∞ on $\mathbb{R}^n$, and by (5.4), $\tilde{f}=f$ on E.

Remark 5.2. If E is not *UPC* then Theorem 5.1 does not in general hold. To see this take E to be the set of Example 3.2. Following Baouendi and Goulaouic [2, Remark II.2], consider the function $f(x,y)=y/x$ as $x\neq 0$, and $f(0,0)=0$. Then f

satisfies (5.1), since

$$\operatorname{dist}_E(f, \mathscr{P}_{k+1}) \leqq \sup_E \left| f(x,y) - y\sum_{l=0}^{k} (1-x)^l \right| \leqq \sqrt{k+1}\,\exp(-\sqrt{k+1}).$$

Nevertheless, f is not extendible to a C^∞ function because $\partial f/\partial y$ is not bounded on E.

6. Subanalytic Sets

In this section we shall show that the class of *UPC* subsets of $\mathbb{R}^n$ contains in particular a large family of fat subanalytic sets in $\mathbb{R}^n$. (We recall that $E \subset \mathbb{K}^n$ is said to be *fat* if $\overline{\operatorname{int} E} \supset E$.)

Let $\mathbb{X}$ be a real analytic manifold. We recall that a subset E of $\mathbb{X}$ is said to be *semianalytic* if for each point $x \in \mathbb{X}$, there exists an open neighborhood U of x in $\mathbb{X}$ and a finite number of real analytic functions f_{ij}, g_{ij} on U such that

$$E \cap U = \bigcup_i \bigcap_j \{f_{ij} = 0,\ g_{ij} > 0\}$$

[14]. The projection of a (relatively compact) semianalytic set need not be semianalytic [14, p. 133–135]. The class of sets obtained by enlarging the class of semianalytic sets to include images under proper analytic maps has been called class of subanalytic sets. Hironaka has developed the theory of such sets using his famous *desingularization theorem* which is one of most deep and most difficult theorems of analysis. Following Hironaka [10, 11], a subset E of a real analytic manifold $\mathbb{X}$ is said to be *subanalytic* if for each point $x \in \mathbb{X}$, there exists an open neighborhood U of x in $\mathbb{X}$ and a finite system of proper real analytic maps $f_{ij}\colon \mathbb{Y}_{ij} \to U$ ($j = 1, 2$), such that

$$E \cap U = \bigcup_i (\operatorname{Im} f_{i1} \setminus \operatorname{Im} f_{i2}).$$

(Here $\mathbb{Y}_{ij}$ is a real analytic manifold.)

Another (more elementary) approach to the subanalytic sets has been presented in [6] and subsequent papers by Łojasiewicz and his collaborators. Their definition of a subanalytic set (which is equivalent to the foregoing definition) is as follows. A subset E of $\mathbb{X}$ is *subanalytic* if for each point $x \in \mathbb{X}$, there exists an open neighborhood U of x in $\mathbb{X}$ such that $E \cap U$ is the projection of a relatively compact semianalytic subset A of $\mathbb{X} \times \mathbb{Y}$, where $\mathbb{Y}$ is a real analytic manifold.

If $\dim \mathbb{X} \geqq 3$, the class of all subanalytic sets is essentially larger than that of all semianalytic sets, both classes being identical if $\dim \mathbb{X} \leqq 2$ [14]. The union of a locally finite family and the intersection of a finite family of subanalytic sets is a subanalytic set. The complement of a subanalytic set is by a (non-trivial) result of Gabrielov [8] (see also [7]) a subanalytic set. The closure, interior and boundary of a subanalytic set is still subanalytic. The Cartesian product of subanalytic sets is a subanalytic set. For other properties of semianalytic and subanalytic sets we refer to [14, 10, 11, 6, 28].

An important tool in the theory of subanalytic sets is Hironaka's *rectilinearization theorem*.

Theorem 6.1 [11, Theorem 7.1]). *Let $\mathbb{N}$ be a real analytic manifold and E a subanalytic subset of $\mathbb{N}$. Let K be a compact subset of $\mathbb{N}$. Then there exists a finite number of real analytic maps π_j: $\mathbb{V}_j \to \mathbb{N}$ such that*

$$\text{each } \mathbb{V}_j \text{ is isomorphic to } \mathbb{R}^{n_j}, \text{ for some } n_j; \tag{6.1}$$

there exists a compact subset K_j of $\mathbb{V}_j$ such that

$$\bigcup_j \pi_j(K_j) \text{ is a neighborhood of } K \text{ in } \mathbb{N}; \tag{6.2}$$

$$\pi_j^{-1}(E) \text{ is union of quadrants in } \mathbb{R}^{n_j}. \tag{6.3}$$

(We recall that a subset Q of $\mathbb{R}^n$ is called a *quadrant* if there exists a disjoint partition $I_0 \cup I_+ \cup I_-$ of $\{1, \ldots, n\}$ such that Q is the set of points $x = (x_1, \ldots, x_n)$ satisfying $x_i = 0$, $i \in I_0$, $x_i > 0$, $i \in I_+$, and $x_i < 0$, $i \in I_-$.)

To our purpose we shall need the following corollary to the rectilinearization theorem.

Corollary 6.2. *Let $\mathbb{N}$ be an n-dimensional real analytic manifold. Let E be a relatively compact, subanalytic subset of $\mathbb{N}$ of pure dimension n. Then there exists a finite number of real analytic maps f_k: $\mathbb{R}^n \to \mathbb{N}$ such that for each k,*

$$f_k(J^n) \subset E \quad \text{and} \quad \bigcup_k f_k(I^n) = \bar{E},$$

where $J^n := \{x \in \mathbb{R}^n;\ |x_i| < 1, i = 1, \ldots, n\}$, and $I^n := \{x \in \mathbb{R}^n;\ |x_i| \leqq 1, i = 1, \ldots, n\}$.

(Here the dimension of a subanalytic set E is by definition the maximum of the dimensions of real analytic submanifolds of $\mathbb{N}$ contained in E. E is of pure dimension n if every non-void open subanalytic subset of E has dimension n.)

Proof. By [11, Remark 7.1.2], we may require that each π_j induces an open imbedding of an open dense subset of $\mathbb{V}_j$ into $\mathbb{N}$. Hence, since $\dim \mathbb{N} = n$, we may assume $\mathbb{V}_j = \mathbb{R}^n$, for each j. Choose, for each j, a bounded open cube T_j such that $T_j \supset K_j$. Then we have

$$E = \bigcup_j \pi_j(K_j) \cap E \subset \bigcup_j \pi_j(T_j) \cap E.$$

By (6.3),

$$\pi_j^{-1}(E) = \bigcup_\alpha Q_{j\alpha},$$

where each $Q_{j\alpha}$ is a quadrant in $\mathbb{R}^n$. We note that

$$\pi_j(T_j) \cap E \subset \pi_j(T_j \cap \pi_j^{-1}(E)) = \bigcup_\alpha \pi_j(T_j \cap Q_{j\alpha}) \subset E.$$

Hence

$$E \subset \bigcup_j \pi_j(T_j) \cap E \subset \bigcup_{j,\alpha} \pi_j(T_j \cap Q_{j\alpha}) \subset E. \tag{6.4}$$

Observe that the number of all possible quadrants in $\mathbb{R}^n$ is finite and each set $T_j \cap Q_{j\alpha}$ is an open cube in $\mathbb{R}^d$, where $0 \leqq d \leqq n$. Choose now from the (finite) family

$\{T_j \cap Q_{j\alpha}\}$ all these cubes which are of dimension n and denote them by $S_1, \ldots, S_r$. Then by (6.4), since E is of pure dimension n, there exist real analytic maps f_k: $\mathbb{R}^n \to \mathbb{N}$ $(k=1, \ldots, r)$ such that $f_k(S_k) \subset E$ and

$$\bar{E} = \overline{\bigcup_k f_k(S_k)} = \bigcup_k f_k(\bar{S}_k).$$

We may obviously assume that for each k, $S_k = J^n$. Then $\bar{S}_k = I^n$, and the proof is complete.

Proposition 6.3. *With the assumptions of Corollary* 6.2, *there exists a finite number of real analytic maps* φ_j: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{N}$ *such that*

$$\varphi_j(I^n \times (0,1]) \subset E, \textit{ for each } j; \tag{6.5}$$

$$\bigcup_j \varphi_j(I^n \times \{0\}) = \bar{E}. \tag{6.6}$$

Proof. By Corollary 6.2 it suffices to consider the case where $\mathbb{N} = \mathbb{R}^n$ and $E = J^n$, and this is an elementary exercise. For, take e.g. the functions $\alpha(u) = (1+u)/2$ and $\beta(u) = (1-u)/2$, $u \in \mathbb{R}$. For each $i \in \{1, \ldots, n\}$, we define

$$g_i(x_1, \ldots, x_n) = (\alpha(x_i)x_1, \ldots, \alpha(x_i)\, x_{i-1}, x_i, \alpha(x_i)\, x_{i+1}, \ldots, \alpha(x_i)\, x_n)$$

and

$$h_i(x_1, \ldots, x_n) = (\beta(x_i)x_1, \ldots, \beta(x_i)\, x_{i-1}, x_i, \beta(x_i)\, x_{i+1}, \ldots, \beta(x_i)\, x_n).$$

Each of the g_i's and h_i's transforms the cube I^n onto a pyramid contained in I^n the base of which is one of the faces of the cube I^n and the vertex of which is at the centre of the opposite face. We notice that for each i,

$$g_i(I^n \cap \{x_i \in (-1,1)\}) \subset J^n$$

and

$$h_i(I^n \cap \{x_i \in (-1,1)\}) \subset J^n.$$

Now we can take φ_j to be the following maps:

$$G_i(x_1, \ldots, x_n, t) = g_i(x_1, \ldots, x_{i-1}, (x_i - t + 1)/2, x_{i+1}, \ldots, x_n)$$

and

$$H_i(x_1, \ldots, x_n, t) = h_i(x_1, \ldots, x_{i-1}, (x_i + t - 1)/2, x_{i+1}, \ldots, x_n),$$

for $i = 1, \ldots, n$.

The following theorem asserts the *UPC* property of fat subanalytic sets. It can also be considered as a refinement of a *Bruhat-Cartan-Wallace curve selecting lemma* for subanalytic sets [6].

Theorem 6.4. *Let E be a bounded, open subanalytic set in* $\mathbb{R}^n$. *Then there exists a map* h: $\bar{E} \times \mathbb{R} \to \mathbb{R}^n$ *such that h is a polynomial with respect to the second variable*:

$$h(x,t) = \sum_{k=0}^{d} t^k a_k(x) \textit{ with the degree } d \textit{ independent of } x \textit{ in } \bar{E}; \tag{6.7}$$

$$h(x,0) = x \quad \textit{for each } x \textit{ in } \bar{E}; \tag{6.8}$$

$$h(\bar{E} \times (0,1]) \subset E; \tag{6.9}$$

there exist positive constants M and m such that

$$\operatorname{dist}(h(x,t), \mathbb{R}^n \setminus E) \geqq Mt^m \quad \textit{for all } x \in \bar{E} \textit{ and } t \in [0,1]. \tag{6.10}$$

Proof. Let φ_j ($j=1, \ldots, s$) be the maps defined in Propsosition 6.3. By a remark of Hörmander (see [14, p. 137]; [28, Lemma 3.1]) for each j, the (graph of the) map

$$I^n \times [0,1] \ni (y,t) \to \operatorname{dist}(\varphi_j(y,t), \mathbb{R}^n \setminus E)$$

is subanalytic. Hence by *Łojasiewicz's inequality* (see [11, Sect. 9], [28, Lemma 3.2]),

$$\operatorname{dist}(\varphi_j(y,t), \mathbb{R}^n \setminus E) \geqq M_j t^{m_j}$$

for $y \in I^n$ and $t \in [0,1]$, where M_j and m_j are positive constants independent of (y,t). Set $L = \min\{M_j\}$ and $m = \max\{m_j\}$. Then for each j,

$$\operatorname{dist}(\varphi_j(y,t), \mathbb{R}^n \setminus E) \geqq Lt^m \tag{6.11}$$

for all (y,t) in $I^n \times [0,1]$. Take a positive integer $d \geqq m$. For any fixed $y \in \mathbb{R}^n$, let $T_j(y, \cdot)$ be the Taylor polynomial at 0 of degree d of the function $\varphi_j(y, \cdot)$, i.e.

$$T_j(y,t) = \sum_{k=0}^{d} (t^k/k!)\,(\partial^k \varphi_j/\partial t^k)\,(y,0).$$

Then we have

$$\varphi_j(y,t) = T_j(y,t) + t^{d+1}\, Q_j(y,t) \tag{6.12}$$

for $(y,t) \in \mathbb{R}^n \times \mathbb{R}$, where Q_j: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are analytic maps. Choose $\delta \in (0,1]$ so that for each j, $|tQ_j(y,t)| \leqq L/2$ as $y \in I^n$ and $t \in [0,\delta]$. Then by (6.11) and (6.12),

$$\operatorname{dist}(T_j(y,t), \mathbb{R}^n \setminus E) \geqq Lt^m - (L/2)\,t^d \geqq (L/2)\,t^m$$

as $y \in I^n$ and $t \in [0,\delta]$, $j = 1, \ldots, s$. Hence by putting δt in place of t we get

$$\operatorname{dist}(T_j(y,\delta t), \mathbb{R}^n \setminus E) \geqq Mt^m$$

for $y \in I^n$, $t \in [0,1]$, $j = 1, \ldots, s$, where $M := L\delta^m/2$. This gives the assertion of the theorem, since

$$\bigcup_j T_j(I^n \times \{0\}) = \bigcup_j \varphi_j(I^n \times \{0\}) = \bar{E}.$$

Remark 6.5. We may require that the (graphs of the) coefficients a_k in (6.7) are subanalytic and bounded on E. For, if we put for each j, $h_j(y) = \varphi_j(y,0)$ for $y \in I^n$, and $E_j = h_j(I^n)$, then E_j is compact, subanalytic and for each $x \in E_j$, the fibre $h_j^{-1}(x)$ is a compact subanalytic subset of I^n. Hence (the graph of) the function $\alpha_j(x) := \sup h_j^{-1}(x)$, where the supremum is taken with respect to the lexicographical order in I^n, is subanalytic and so is (the graph of) the function $\alpha(x) = \alpha_j(x)$ as $x \in F_j$, where $F_1 = E_1$, and $F_j = E_j \setminus (E_1 \cup \ldots \cup E_{j-1})$ as $j > 1$.

From Theorem 6.4 we immediately derive

Corollary 6.6. *Every bounded subanalytic subset of* $\mathbb{R}^n$ *with* int E *dense in E is UPC.*

Remark 6.7. In [20] it was proved that every fat subanalytic subset of $\mathbb{R}^n$ is (locally) L-regular. The proof was based on a curve selecting lemma and *Puiseux's theorem*. For compact sets the result also follows from Corollary 6.2 and the fact that the image of an L-regular compact set by a non-degenerate holomorphic map is still L-regular [18]. By Corollary 6,6, Theorem 4.1 essentially improves this result.

7. Some Examples and Applications

Corollary 6.6 together with Theorems 3.1 and 5.1 yield Markov's inequality and Bernstein's theorem for subanalytic sets in $\mathbb{R}^n$. In such a case Bernstein's theorem can also be proved in another way [16] by making use of the fact that every closed subanalytic subset E of $\mathbb{R}^n$ such that $\operatorname{int} E$ is dense in E satisfies the following *strong regularity condition* of the type of Whitney [5, Theorem 6.17].

(*SRC*). For every compact subset K of E, there exists $M>0$ and an integer $m \geqq 1$ such that any two points $b, y \in K$ can be joined by a semianalytic arc σ in E such that

$$|\sigma| \leqq M|b-y|^{1/m}$$

and σ intersects the boundary of E in at most finitely many points.

The following example shows that there exist *UPC* compact subsets of $\mathbb{R}^n$ that do not satisfy *SRC*. Therefore Theorem 5.1 essentially extends Bernstein's theorem proved in [16].

Example 7.1. Define $U=\{(x,y)\in\mathbb{R}^2;\ 0<x\leqq 1,\ 0<y<\exp(-1/x)\}$ and take $E=[0,1]\times[-1,1]\setminus U$. Then E does not satisfy *SRC* and it does satisfy *UPC* and even the condition P (see Sect. 2). Actually, the set E has not *Whitney's extension property* (briefly, *WEP*), since the function $f(x,y)=\exp(-1/x)$ as $x>0$, $y>\exp(-1/x)$, and $f(x,y)=0$ otherwise, cannot be extended to a C^∞ function on $\mathbb{R}^2$. (We recall that a fat subset E of $\mathbb{R}^n$ has *WEP* if for every function $f\colon E\to\mathbb{R}$ which is uniformly continuous along with all its partial derivatives in $\operatorname{int} E$, there exists a C^∞ function $\tilde{f}$ on $\mathbb{R}^n$ such that $\tilde{f}=f$ on E.)

In the next example we show that *UPC* sets may exhibit very irregular behavior.

Example 7.2. Let $\{a_n\}$ and $\{\varepsilon_n\}$ be strictly decreasing sequences of positive numbers both tending to zero such that $a_n-a_{n+1}>\varepsilon_n+\varepsilon_{n+1}$. Define $E_n=\{(x,y)\in\mathbb{R}^2;\ 0\leqq x<y,\ |y-a_n|<\varepsilon_n\}$ and put $E=[0,1]\times[-1,1]\setminus\bigcup_{n=2}^{\infty}E_n$. Then the "comb" E is not subanalytic, since $E\cap\{x=0\}$ has infinitely many components. If we put $a_n=2^{-n}$ and $\varepsilon_n=(1/2)\exp(-n^2)$, then E does not satisfy *SRC* but it does satisfy P, whence E is *UPC*.

With the same ε_n's, by putting $a_n=1/n$ we get an example of a set which satisfies neither *SRC* nor P but is *UPC*. (Consider, for each $n\geqq 2$, the family of the parabolas $h_{1n}(x,y,t)=(x+t,y+t^2/4)$ as $(x,y)\in\{a_n+\varepsilon_n\leqq y\leqq(a_n+a_{n-1})/2,\ 0\leqq x<y$, and $h_{2n}(x,y,t)=(x+t,y-t^2/4)$ as $(x,y)\in\{(a_n+a_{n-1})/2<y\leqq a_{n-1}-\varepsilon_{n-1}$, $0\leqq x<y\}$.)

We close this paper by two remarks in which we apply our version of Bernstein's theorem to improve some earlier results.

Remark 7.3. Let E be a compact subset of $\mathbb{K}^n$. We say that a function $f\colon E\to\mathbb{K}$ is *quasianalytic* on E in the sense of Bernstein if there exists an increasing sequence $\{k_j\}$ of positive integers such that

$$\limsup_{j\to\infty}\,[\operatorname{dist}_E(f,\mathscr{P}_{k_j})]^{1/k_j}<1\,. \tag{7.1}$$

(For properties of quasianalytic functions we refer to [17].) If E is a compact interval in $\mathbb{R}$, Mazurkiewicz [15] proved that all quasianalytic functions form a residual subset of the Banach space $C(E)$ of all continuous functions on E. Hence he derived that there exist quasianalytic functions without derivatives at any point of E. Earlier Bernstein [4] showed that regularity properties of a quasianalytic function f depend on the speed of growth of the sequence $\{k_j\}$ with which the function f satisfies (7.1). In [17. Theorem 9.3] there was proved the following version of Bernstein's result.

Suppose E is a fat, convex compact subset of $\mathbb{R}^n$. *If*

$$\lim_{j\to\infty} (\log k_{j+1})/k_j = 0 \tag{7.2}$$

then every function f satisfying (7.1) *extends to a C^∞ function on* $\mathbb{R}^n$. *Conversely, if* (7.2) *does not hold, there exists a function f satisfying* (7.1) *which is not extendible to a C^∞ function on* $\mathbb{R}^n$.

By Theorem 5.1 we may replace E with any compact *UPC* subset of $\mathbb{R}^n$. Thus, we omit a WEP argument of the proof of [17, Theorem 9.3].

Remark 7.4. In connection with a result of Globevnik and Stout, Rudin [23, p. 415] asked whether it is possible to construct a holomorphic function h in the unit ball B in $\mathbb{C}^n$ with continuous or smooth boundary values, being "highly non-continuable" in the following sense.

(*HNCP*). For every complex line L the plane domain $L\cap B$ is the natural domain of existence of h restricted to $L\cap B$.

This question was answered in the affirmative by Siciak [26, 27] by the method of the extremal function, viz. if K is a polynomially convex compact subset of $\mathbb{C}^n$, the required function h: $K\to\mathbb{C}$ may be defined by the formula

$$h(x) = \sum_{j=1}^{\infty} a^{\sqrt{k_j}} p_j(x), \qquad \text{for } x\in K, \tag{7.3}$$

where $a\in(0,1)$, $\{k_j\}$ is an increasing sequence of positive integers and $\{p_j\}$ is a sequence of polynomials from $\mathbb{C}^n$ to $\mathbb{C}$ such that

$$\lim_{j\to\infty} \sqrt{k_{j+1}}/k_j = \infty; \tag{7.4}$$

$$\deg p_j \leqq k_j; \tag{7.5}$$

and

$$\Phi_K(x) = \sup_j |p_j(x)|^{1/k_j} = \limsup_{j\to\infty} |p_j(x)|^{1/k_j}, \qquad \text{for } x\in\mathbb{C}^n.$$

Then h is quasianalytic on K, holomorphic in the interior of K and cannot be continued analytically from K in a strong sense [26]. Moreover, *if D is a bounded convex domain in $\mathbb{C}^n$ or a bounded domain with Lipschitz boundary such that $K=\bar{D}$ is polynomially convex then h is extendible to a C^∞ function in $\mathbb{C}^n$ and has HNCP with B replaced by D* [*26, Theorem 3*].

By (7.3)–(7.5), it follows that for each $s>0$,

$$\lim_{k\to\infty} k^s \operatorname{dist}_K(h, \mathcal{P}_k) = 0.$$

Thus, by Theorem 5.1 we may extend Siciak's result by replacing D with any bounded *UPC* domain in $\mathbb{C}^n$ (treated as $\mathbb{R}^{2n}$) such that $K=\bar{D}$ is polynomially convex (in $\mathbb{C}^n$). We note in particular that contrary to Siciak's argument we do not need the *WEP* assumption on K what is essential because of Example 7.1.

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Stable Rank 2 Vector Bundles with Chern-Classes $c_1 = -1$, $c_2 = 4$

Wolfram Decker

Universität Kaiserslautern, Fachbereich Mathematik, Erwin-Schrödinger-Strasse, D-6750 Kaiserslautern, Federal Republic of Germany

0. Introduction

In 1973 Horrocks and Mumford [H-M] discovered a stable rank 2 vector bundle $\mathscr{F}_{\mathrm{HM}}$ on $\mathbb{P}_4 = \mathbb{P}_4(\mathbb{C})$. At the present state this is essentially the only known indecomposable rank 2 vector bundle on $\mathbb{P}_4$. $\mathscr{F}_{\mathrm{HM}}$ has Chern-classes $c_1 = -1$, $c_2 = 4$ and is the cohomology of a monad

$$5\mathcal{O}(-1) \to 2\Omega^2(2) \to 5\mathcal{O}.$$

Equivalently

$$H^2(\mathbb{P}_4, \mathscr{F}_{\mathrm{HM}}(-1)) = 0.$$

As a cohomology bundle of a monad of this type $\mathscr{F}_{\mathrm{HM}}$ is unique:

Theorem [D-S]. *A stable rank 2 vector bundle $\mathscr{F}$ on $\mathbb{P}_4$ with $c_1 = -1$, $c_2 = 4$ and*

$$H^2(\mathbb{P}_4, \mathscr{F}(-1)) = 0$$

is up to an automorphism of $\mathbb{P}_4$ isomorphic to the Horrocks-Mumford-bundle $\mathscr{F}_{\mathrm{HM}}$.

In this paper we prove:

Theorem 1. *Any stable rank 2 vector bundle $\mathscr{F}$ on $\mathbb{P}_4$ with $c_1 = -1$, $c_2 = 4$ satisfies*

$$H^2(\mathbb{P}_4, \mathscr{F}(-1)) = 0.$$

This result has first been announced but not yet published by Robert F. Lipshutz from Berkeley.

As a corollary of Theorem 1 we obtain:

Theorem 2. *There exists no stable rank 2 vector bundle on $\mathbb{P}_5$ with $c_1 = -1$, $c_2 = 4$.*

So $\mathscr{F}_{\mathrm{HM}}$ cannot be extended to $\mathbb{P}_5$. Switzer has shown (cf. [O-S-S, p. 116]) that such extensions exist topologically.

To prove Theorem 1 we apply Beilinson's spectral sequence [Bei] with E_1-terms

$$E_1^{ij} = H^j(\mathbb{P}_4, \mathscr{F}(i)) \otimes \Omega^{-i}(-i).$$

So we need information on the cohomology groups $H^i\mathscr{F}(m)$ in the range $-4 \leqq m \leqq 0$. The dimensions of these groups satisfy the following relations:

$i \backslash m$	-4	-3	-2	-1	0
4	0	0	0	0	0
3	$5+r$	q	p	0	0
2	r	q	$2+2p$	q	r
1	0	0	p	q	$5+r$
0	0	0	0	0	0.

We have to show that $\mathscr{F}$ is the cohomology of a monad

$$5\Omega^4(4) \to 2\Omega^2(2) \to 5\Omega^0(0),$$

i.e. that p, q, and r are zero or equivalently that q is zero. How to proceed in detail is suggested by another characterization of bundles with

$$H^2(\mathbb{P}_4, \mathscr{F}(-1)) = 0:$$

For *any* hyperplane $H \subset \mathbb{P}_4$ the holomorphic restriction $\mathscr{F}_H$ is stable and has minimal spectrum $(-1, -1, 0, 0)$ (see [B-E, Sect. 2] and [Ha 2, 7.] for the definition and the properties of the spectrum). Our proof goes along the line of showing this for an arbitrary $\mathscr{F}$:

$p=0$. Consequently for *any* hyperplane $H \subset \mathbb{P}_4$

$$H^0(H, \mathscr{F}_H(-1)) = 0,$$

hence the spectrum of $\mathscr{F}_H$ is defined. A priori the possible spectra are $(-1, -1, 0, 0)$, $(-2, -1, 0, 1)$ (which is the maximal one for stable bundles), $(-2, -2, 1, 1)$ or $(-3, -2, 1, 2)$.

$r=0$. Consequently $q \leqq 2$ and so the spectrum $(-3, -2, 1, 2)$ cannot occur. Moreover $\mathscr{F}$ is the cohomology of a monad

$$5\Omega^4(4) \oplus q\Omega^3(3) \to q\Omega^3(3) \oplus 2\Omega^2(2) \oplus q\Omega^1(1) \to q\Omega^1(1) \oplus 5\Omega^0(0)$$

and it remains to show that such a monad can only exist if $q=0$.

$q \leqq 1$. Consequently the spectrum $(-2, -2, 1, 1)$ cannot occur.

A basic tool for the proofs of Theorem 1 and Theorem 2 is the restriction theorem of Barth [Ba, Theorem 3].

1. Some Remarks on Beilinson's Spectral Sequence

In this section V is a fixed $(n+1)$-dimensional vector space over $\mathbb{C}$ and $\mathbb{P}_n = \mathbb{P}(V)$ the projective space of lines in V, so

$$H^0(\mathbb{P}_n, \mathcal{O}(1)) = V^*.$$

$\mathscr{T} = \mathscr{T}_{\mathbb{P}_n}$ denotes the holomorphic tangent bundle, $\Omega^i = \Lambda^i \mathscr{T}^*$.

The *Koszul-complex* on $\mathbb{P}(V)$ is the exact sequence

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \Lambda^{i+1}V^*\otimes\mathcal{O}(-1) \arrow[rr] \arrow[dr] & & \Lambda^{i}V^*\otimes\mathcal{O} \arrow[rr] & & \Lambda^{i-1}V^*\otimes\mathcal{O}(1) \arrow[r] & \cdots \\
& & \Omega^i(i) \arrow[ur] \arrow[dr] & & & & \\
& 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

defined by contraction with the tautological subbundle

$$\mathcal{O}(-1)\to V\otimes\mathcal{O}.$$

To describe a vector bundle $\mathcal{F}$ on $\mathbb{P}(V)$ we apply Beilinson's spectral sequence ([Bei] or [O-S-S, II, Sect. 3]) which converges to

$$E^i=\begin{cases}\mathcal{F} & \text{for} \quad i=0\\ 0 & \text{otherwise}.\end{cases}$$

The E_1-terms are

$$E_1^{ij}=H^j(\mathbb{P}(V),\mathcal{F}(i))\otimes\Omega^{-i}(-i).$$

As for the differentials we note:

Lemma 1 [Bei]. *There are canonical isomorphisms*

$$\operatorname{Hom}(\Omega^i(i),\Omega^j(j))\cong H^0(\mathbb{P}(V),(\Lambda^{i-j}\mathcal{T})(j-i))\cong\Lambda^{i-j}V$$

defined by contraction. The composition of morphisms coincides with multiplication in ΛV.

Proof. Use the Koszul-complex. $\square$

So conditions on a vector bundle on $\mathbb{P}(V)$ may be expressed as conditions on certain matrices with entries in the exterior algebra ΛV. For example notice:

Remark 1. Let

$$t\Omega^i(i)\xrightarrow{A}s\Omega^{i-1}(i-1)$$

be a vector bundle homomorphism, i.e. let A be a $s\times t$-matrix with entries in V. Then a necessary condition for A to be pointwise surjective is: If $(a_1,\dots,a_t)$ is a nontrivial linear combination of the rows of A, then

$$\dim\operatorname{span}(a_1,\dots,a_t)\geqq i+1:$$

A is pointwise surjective iff its dual map is pointwise injective iff

$$s\Lambda^{i-1}V\wedge x\xrightarrow{\wedge A}t\Lambda^iV\wedge x$$

is injective for *any* $\langle x\rangle\in\mathbb{P}(V)$. $\square$

To study the restriction of vector bundles to linear subspaces of $\mathbb{P}(V)$ we need:

Lemma 2. *For any hyperplane $H\subset\mathbb{P}_n$ the holomorphic restriction*

$$\Omega^i_{\mathbb{P}_n}(i)\otimes\mathcal{O}_H\cong\Omega^{i-1}_H(i-1)\oplus\Omega^i_H(i).$$

Proof. Let
$$U=\{x^*=0\}\subset V$$
be the subspace of V corresponding to H,
$$H=\mathbb{P}(U).$$
Choose a complement
$$V=\langle x\rangle\oplus U,$$
$x\in V$ with $x^*(x)=1$. Then ΛV decomposes
$$\Lambda V=\bigoplus_i(x\wedge(\Lambda^{i-1}U)\oplus\Lambda^iU).$$
The dual decomposition
$$\Lambda V^*=\bigoplus_i(x^*\wedge(\Lambda^{i-1}U^*)\oplus\Lambda^iU^*)$$
induces the isomorphisms above:

The Koszul-complex decomposes
$$\begin{array}{ccccccc}
\dots\to & x^*\wedge(\Lambda^iU^*)\otimes\mathcal{O}(-1) & \to & x^*\wedge(\Lambda^{i-1}U^*)\otimes\mathcal{O} & \to\dots \\
 & \downarrow & & \downarrow & \\
\dots\to & \Lambda^iU^*\otimes\mathcal{O} & \to & \Lambda^{i-1}U^*\otimes\mathcal{O}(1) & \to\dots
\end{array}$$
and the vertical maps are zero after restriction to $H=\{x^*=0\}$. □

Remark 2. Let
$$\Omega^i(i)\xrightarrow{a}\Omega^j(j)$$
be a vector bundle homomorphism. With
$$H=\mathbb{P}(U)$$
as above write
$$a=x\wedge a''+a'\in x\wedge\Lambda^{i-j-1}U\oplus\Lambda^{i-j}U.$$
Then
$$a\otimes\mathcal{O}_H=\left(\Omega_H^{i-1}(i-1)\oplus\Omega_H^i(i)\xrightarrow{\begin{pmatrix}\pm a' & 0\\ a'' & a'\end{pmatrix}}\Omega_H^{j-1}(j-1)\oplus\Omega_H^j(j)\right).\quad □$$

2. How to Obtain Information on the Cohomology Groups $H^i\mathcal{F}(m)$

Let $\mathcal{F}$ be a stable rank 2 vector bundle on the complex projective space $\mathbb{P}_n=\mathbb{P}(V)$ with Chern-classes $c_1=-1$, $c_2=4$.

Since $\mathcal{F}$ is normalized ($c_1=-1$) the stability simply means
$$H^0(\mathbb{P}_n,\mathcal{F})=0$$

[O-S-S, II, 1.2]. Consequently

$$(S) \qquad H^0(\mathbb{P}_n, \mathscr{F}(m))=0 \quad \text{for} \quad m \leqq 0.$$

A linear subspace $\mathbb{P}_k \subset \mathbb{P}_n$, $k \geqq 2$, is *stable* (otherwise *unstable*) *for* $\mathscr{F}$, if the holomorphic restriction

$$\mathscr{F}_{\mathbb{P}_k} = \mathscr{F} \otimes \mathscr{O}_{\mathbb{P}_k}$$

is stable.

As a 2-bundle $\mathscr{F}$ carries a self-duality with values in $\mathscr{O}(c_1)$:

$$\mathscr{F}^* \cong \mathscr{F}(1),$$

induced by the pairing

$$\mathscr{F} \otimes \mathscr{F} \xrightarrow{\wedge} \Lambda^2 \mathscr{F} \cong \mathscr{O}(-1).$$

Thus

$$(D) \qquad (H^i \mathscr{F}(m))^* \cong H^{n-i} \mathscr{F}^*(-m-n-1) \cong H^{n-i} \mathscr{F}(-m-n)$$

by Serre-duality. (We sometimes abbreviate $H^i \mathscr{S} = H^i(X, \mathscr{S})$ and $h^i \mathscr{S} = \dim H^i \mathscr{S}$.)

The Euler-characteristic

$$\chi \mathscr{F}(m) = \sum (-1)^i h^i \mathscr{F}(m)$$

equals

$$(R) \qquad \begin{array}{ll} m^2+2m-3 & n=2 \\ (2m^3+9m^2-11m-30)/6 & n=3 \\ (m^4+8m^3-m^2-68m-60)/12 & n=4 \\ (2m^5+25m^4+40m^3-325m^2-942m-480)/120 & n=5 \end{array} \quad \text{if}$$

by the theorem of Riemann-Roch.

Because of Beilinson's spectral sequence we are especially interested in the cohomology groups $H^i \mathscr{F}(m)$ in the range $-n \leqq m \leqq 0$.

Lemma 3. *If $n=2$, then the dimensions $h^i \mathscr{F}(m)$ in the range $-2 \leqq m \leqq 0$ are given by the table*

$i \backslash m$	-2	-1	0
2	0	0	0
1	3	4	3
0	0	0	0.

Proof. Use (*S*), (*D*), and (*R*). □

Results in case $n \geqq 3$ can be obtained by restricting $\mathscr{F}$ to linear subspaces of $\mathbb{P}_n$. For example use the exact sequences

$$S(H,m) \qquad \ldots \longrightarrow H^i \mathscr{F}(m-1) \xrightarrow{\cdot x^*} H^i \mathscr{F}(m) \longrightarrow H^i(H, \mathscr{F}_H(m)) \longrightarrow \ldots$$

corresponding to a hyperplane

$$H=\{x^*=0\}\subset \mathbb{P}(V).$$

Or blow up $\mathbb{P}_n$ along a fixed $\mathbb{P}_{n-2}\subset\mathbb{P}_n$:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \mathbb{P}_1 & & \mathbb{P}_n . \end{tikzcd}$$

Identify the hyperplanes in $\mathbb{P}_n$ containing $\mathbb{P}_{n-2}$ with the fibres of π:

$$H_{(\lambda:\mu)}\cong\pi^{-1}(\lambda:\mu).$$

Then

$$\mathscr{F}_{H_{(\lambda:\mu)}}(m)\cong\sigma^*\mathscr{F}(m)\otimes\mathcal{O}_{\pi^{-1}(\lambda:\mu)}.$$

The cohomology groups of the direct image sheaves

$$R^j\pi_*\sigma^*\mathscr{F}(m)$$

are related with those of $\mathscr{F}(m)$ by Leray's spectral sequences

$$H^i(\mathbb{P}_1, R^j\pi_*\sigma^*\mathscr{F}(m)) \;\Rightarrow\; H^{i+j}(\mathbb{P}_n, \mathscr{F}(m)) \tag{2.1}$$

and with those of $\mathscr{F}_{\mathbb{P}_{n-2}}(m)$ via the exact sequences

$$\ldots\to R^j\pi_*\sigma^*\mathscr{F}(m-1)\otimes\mathcal{O}_{\mathbb{P}_1}(1)\to R^j\pi_*\sigma^*\mathscr{F}(m)\to H^j(\mathbb{P}_{n-2}, \mathscr{F}_{\mathbb{P}_{n-2}}(m))\otimes\mathcal{O}_{\mathbb{P}_1}\to\ldots. \tag{2.2}$$

(For (2.2) recall the ideal sheaf of the exceptional divisor $D\subset X$:

$$0\to\sigma^*\mathcal{O}_{\mathbb{P}_n}(-1)\otimes\pi^*\mathcal{O}_{\mathbb{P}_1}(1)\to\mathcal{O}_X\to\mathcal{O}_D\to 0.)$$

If $n=3$, then blow up $\mathbb{P}_3$ along its *general* line L. By the theorem of Grauert-Mülich (cf. [Ba, Theorem 1])

$$\mathscr{F}_L\cong\mathcal{O}_L\oplus\mathcal{O}_L(-1).$$

So

$$h^0(P, \mathscr{F}_P(-1))=0$$

for any plane $P\supset L$. Use this and the base change theorem to obtain the *spectrum* $(a_1, \ldots, a_{c_2})$ of $\mathscr{F}$ [B-E, Ha 2]:

$$\mathscr{H}=R^1\pi_*\sigma^*\mathscr{F}(-1)$$

is locally free of rank $c_2=h^1(P, \mathscr{F}_P(-1))=4$, i.e.

$$\mathscr{H}\cong\bigoplus_{i=1}^{c_2}\mathcal{O}_{\mathbb{P}_1}(a_i)$$

with $(a_1, \ldots, a_{c_2})$ uniquely determined by

$$a_1\leqq\ldots\leqq a_{c_2}.$$

Moreover

$$h^1(\mathbb{P}_3, \mathscr{F}(m)) = h^0(\mathbb{P}_1, \mathscr{H}(m+1)) \quad \text{for} \quad m \leqq -1$$

and

$$h^2(\mathbb{P}_3, \mathscr{F}(m)) = h^1(\mathbb{P}_1, \mathscr{H}(m+1)) \quad \text{for} \quad m \geqq -2.$$

Lemma 4 [B-M]. *Let $n=3$. There are two possible cases for the spectrum of $\mathscr{F}$. The dimensions $h^i\mathscr{F}(m)$ in the range $-3 \leqq m \leqq 0$ are either given by the table*

$i \backslash m$	-3	-2	-1	0
3	0	0	0	0
2	5	3	1	0
1	0	1	3	5
0	0	0	0	0

(if $\mathscr{F}$ has maximal spectrum $(-2, -1, 0, 1)$) or by the table

$i \backslash m$	-3	-2	-1	0
3	0	0	0	0
2	5	2	0	0
1	0	0	2	5
0	0	0	0	0

(if $\mathscr{F}$ has minimal spectrum $(-1, -1, 0, 0)$). In both cases

$$h^1\mathscr{F}(m) = 0 \quad \textit{for} \quad m \leqq -3,$$

$$h^2\mathscr{F}(m) = 0 \quad \textit{for} \quad m \geqq 0.$$

Proof. Use the symmetry and the connectedness of the spectrum [Ha 2, 7.2, 7.5], (S), (D), and (R). □

Remark 3. There exist stable bundles with maximal spectrum and stable bundles with minimal spectrum (cf. [B-M]). □

If $n=4$, then blow up $\mathbb{P}_4$ along its *general* plane P. By the restriction theorem of Barth [Ba] $\mathscr{F}_P$ is stable. So $\mathscr{F}_H$ is stable for any hyperplane $H \supset P$.

Suppose that the spectrum of $\mathscr{F}_H$ is constant when varying those hyperplanes. Then the sheaves

$$R^j\pi_*\sigma^*\mathscr{F}(m), \quad j=0, \ldots, 3, \quad m=-3, \ldots, 0$$

are locally free, i.e. direct sums of line bundles with ranks given by Lemma 4 (apply the base change theorem). If the appropriate spectral sequences (2.1) degenerate, then the dimensions $h^i\mathscr{F}(m)$ have to satisfy "growth conditions" similar to those of line bundles on $\mathbb{P}_1$ (for example $h^0\mathcal{O}(m) \neq 0$ implies $h^0\mathcal{O}(m-1) < h^0\mathcal{O}(m)$).

3. Proof of Theorem 1

In this section V is a fixed 5-dimensional vector space over $\mathbb{C}$ and $\mathcal{F}$ is a stable rank 2 vector bundle on $\mathbb{P}_4 = \mathbb{P}(V)$ with $c_1 = -1$, $c_2 = 4$.

First information on the cohomology groups $H^i\mathcal{F}(m)$ can be obtained from the exact sequences $S(H, m)$ corresponding to the *general* hyperplane $H \subset \mathbb{P}_4$: Since H is stable for $\mathcal{F}$ [Ba]

$$\begin{aligned} H^1\mathcal{F}(m-1) = H^1\mathcal{F}(m) = 0 \quad &\text{for} \quad m \leqq -3 \\ H^3\mathcal{F}(m) = H^3\mathcal{F}(m+1) = 0 \quad &\text{for} \quad m \geqq -1 \end{aligned} \tag{3.1}$$

by Lemma 4, (S), (D), and Serre's Theorem B.

So the table

$i \backslash m$	-4	-3	-2	-1	0
4	0	0	0	0	0
3	$5+r$	q	p	0	0
2	r	q	$2+2p$	q	r
1	0	0	p	q	$5+r$
0	0	0	0	0	0

(3.2)

reflects the relations between the dimensions $h^i\mathcal{F}(m)$ given in the range $-4 \leqq m \leqq 0$ by (S), (D), and (R).

For further information we use the blown up of $\mathbb{P}_4$ along its *general* plane P as at the end of Sect. 2:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \mathbb{P}_1 & & \mathbb{P}_4. \end{tikzcd}$$

Lemma 5. $p = 0$.

Proof. Suppose not. Then by $S(H, -2)$, (3.1), and Lemma 4

(1) $$p = h^1\mathcal{F}(-2) = h^1(H, \mathcal{F}_H(-2)) = 1,$$

i.e. $\mathcal{F}_H$ has maximal spectrum for *any* hyperplane H which is stable for $\mathcal{F}$.

So the sheaves

$$R^j\pi_*\sigma^*\mathcal{F}(m), \quad j = 0, \ldots, 3, \quad m = -3, \ldots, 0$$

are direct sums of line bundles with ranks given by Lemma 4. The line bundle

(2) $$R^1\pi_*\sigma^*\mathcal{F}(-2) \cong \mathcal{O}_{\mathbb{P}_1}$$

since

$$h^0(\mathbb{P}_1, R^1\pi_*\sigma^*\mathcal{F}(-2)) = h^1\mathcal{F}(-2) = 1$$

by Leray's spectral sequence (2.1) and (1). Thus the nonzero part of (2.2)$\otimes\mathcal{O}_{\mathbb{P}_1}(-1)$ for $m=-2$ is

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathcal{O}_{\mathbb{P}_1}(-1) \arrow[r] & 3\mathcal{O}_{\mathbb{P}_1}(-1) \arrow[rr] \arrow[dr] & & R^2\pi_*\sigma^*\mathscr{F}(-3) \arrow[r] & R^2\pi_*\sigma^*\mathscr{F}(-2)\otimes\mathcal{O}_{\mathbb{P}_1}(-1) \arrow[r] & 0 \\
& & & \mathscr{A} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

(recall that

$$h^1(P,\mathscr{F}_P(-2))=3$$

by Lemma 3). Then

$$\mathscr{A}\cong 2\mathcal{O}_{\mathbb{P}_1}(-1),$$

hence

$$\begin{aligned}q=h^2\mathscr{F}(-3)&=h^0(\mathbb{P}_1,R^2\pi_*\sigma^*\mathscr{F}(-3))=h^0(\mathbb{P}_1,R^2\pi_*\sigma^*\mathscr{F}(-2)\otimes\mathcal{O}_{\mathbb{P}_1}(-1))\\ &<h^0(\mathbb{P}_1,R^2\pi_*\sigma^*\mathscr{F}(-2))=h^2\mathscr{F}(-2)=4\end{aligned}$$

by (2.1) (notice that

$$h^1(\mathbb{P}_1,R^1\pi_*\sigma^*\mathscr{F}(-2))=h^1\mathcal{O}_{\mathbb{P}_1}=0$$

by (2) and

$$h^2\mathscr{F}(-2)=4$$

by (3.2) and (1)).

In other words

(3) $$h^1\mathscr{F}(-1)=q<4.$$

To derive a contradiction we use the existence of a hyperplane which is unstable for $\mathscr{F}$ (of order 2 [Ha 2, p. 164]).

The kernel of the natural map

$$V^*\otimes H^1\mathscr{F}(-2)\to H^1\mathscr{F}(-1)$$

is nonzero by reasons of dimension. So we may choose a hyperplane

$$\tilde{H}=\{\tilde{x}^*=0\}$$

such that

$$H^1\mathscr{F}(-2)\xrightarrow{\cdot\tilde{x}^*}H^1\mathscr{F}(-1)$$

is zero. Then

$$h^0(\tilde{H},\mathscr{F}_{\tilde{H}}(-1))=h^1\mathscr{F}(-2)=1$$

and

$$h^0(\tilde{H},\mathscr{F}_{\tilde{H}}(-2))=0$$

(use the exact sequences $S(\tilde{H}, m)$). Hence $\mathscr{F}_{\tilde{H}}$ is an extension

$$0 \to \mathcal{O}_{\tilde{H}}(1) \to \mathscr{F}_{\tilde{H}} \to \mathscr{I}_Y(-2) \to 0$$

by Serre-correspondence (cf. [Ha 1, Sect. 1]). Consequently

$$h^1\mathscr{F}(-1) \geqq h^0(\tilde{H}, \mathscr{F}_{\tilde{H}}) = h^0(\tilde{H}, \mathcal{O}_{\tilde{H}}(1)) = 4$$

(use $S(\tilde{H}, 0)$), a contradiction to (3). □

Remark 4. By Lemma 5

$$h^0(H, \mathscr{F}_H(-1)) = 0$$

for *any* hyperplane $H \subset \mathbb{P}_4$ (use $S(H, -1)$ and (S)). So even if H is unstable for $\mathscr{F}$ the spectrum of $\mathscr{F}_H$ is defined and has nice properties [Ha 2]. The possible spectra and the corresponding dimensions $h^1(H, \mathscr{F}_H(-1))$ are:

spectrum	$h^1(H, \mathscr{F}_H(-1))$
$(-1, -1, 0, 0)$	2
$(-2, -1, 0, 1)$	3
$(-2, -2, 1, 1)$	4
$(-3, -2, 1, 2)$	5

. □

Lemma 6. $r = 0$.

Proof. Suppose not, i.e. suppose

$$H^2\mathscr{F} \neq 0,$$

and let $l \geqq 0$ be maximal with

$$H^2\mathscr{F}(l) \neq 0.$$

From Beilinson's spectral sequence applied to $\mathscr{F}(l+1)$ we obtain a vector bundle epimorphism

$$H^2\mathscr{F}(l-1) \otimes \Omega^2(2) \to H^2\mathscr{F}(l) \otimes \Omega^1(1)$$

(recall that

$$H^3\mathscr{F}(m) = 0 \quad \text{for} \quad m \geqq -2$$

by (3.1) and Lemma 5). Thus

$$h^2\mathscr{F}(l-1) \geqq 3$$

by Remark 1. Now use the exact sequences $S(H, m)$, Lemma 4, and Lemma 5 to deduce that

(1) $$q = h^2\mathscr{F}(-1) = \ldots = h^2\mathscr{F}(l-1) = 3$$

and that $\mathscr{F}_H$ has maximal spectrum for *any* hyperplane H which is stable for $\mathscr{F}$.

So – as in the preceeding proof – the sheaves

$$R^j\pi_*\sigma^*\mathscr{F}(m), \quad j = 0, \ldots, 3, \quad m = -3, \ldots, 0$$

are direct sums of line bundles with ranks given by Lemma 4. Write

$$\mathscr{H} = R^1\pi_*\sigma^*\mathscr{F}(-1),$$
$$\tilde{\mathscr{H}} = R^1\pi_*\sigma^*\mathscr{F}(-2).$$

Then

$$\operatorname{rank}\mathscr{H} = 3, \tag{2}$$
$$\operatorname{rank}\tilde{\mathscr{H}} = 1 \tag{3}$$

and moreover

$$h^0(\mathbb{P}_1, \mathscr{H}) = h^1\mathscr{F}(-1) = q = 3, \tag{4}$$
$$h^0(\mathbb{P}_1, \tilde{\mathscr{H}}) = h^1\mathscr{F}(-2) = p = 0 \tag{5}$$

by (2.1), (1), Lemma 5. The nonzero part of (2.2)$\otimes\mathcal{O}_{\mathbf{P}_1}(-1)$ for $m=-1$ is

$$0 \to \tilde{\mathscr{H}} \to \mathscr{H}(-1) \to H^1(P, \mathscr{F}_P(-1)) \otimes \mathcal{O}_{\mathbf{P}_1}(-1) \to \dots. \tag{6}$$

Then

$$h^0(\mathbb{P}_1, \mathscr{H}(-1)) = h^0(\mathbb{P}_1, \tilde{\mathscr{H}}) = 0$$

by (6), (5), consequently

$$\mathscr{H} \cong 3\mathcal{O}$$

by (2), (4). This implies

$$h^1(\mathbb{P}_1, \tilde{\mathscr{H}}) = 0$$

by (6) again, so

$$R^1\pi_*\sigma^*\mathscr{F}(-2) = \tilde{\mathscr{H}} \cong \mathcal{O}_{\mathbf{P}_1}(-1) \tag{7}$$

by (3), (5). Thus the nonzero part of (2.2)$\otimes\mathcal{O}_{\mathbf{P}_1}(-1)$ for $m=-2$ is

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathcal{O}_{\mathbf{P}_1}(-2) \arrow[r] & 3\mathcal{O}_{\mathbf{P}_1}(-1) \arrow[rr] \arrow[dr] & & R^2\pi_*\sigma^*\mathscr{F}(-3) \arrow[r] & R^2\pi_*\sigma^*\mathscr{F}(-2)\otimes\mathcal{O}_{\mathbf{P}_1}(-1) \arrow[r] & 0 \\
& & & \mathscr{A} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

(recall that

$$h^1(P, \mathscr{F}_P(-2)) = 3$$

by Lemma 3). Then

$$\mathscr{A} \cong \mathcal{O}_{\mathbf{P}_1}(-1) \oplus \mathcal{O}_{\mathbf{P}_1},$$

hence

$$\begin{aligned} q = h^2\mathscr{F}(-3) &= h^0(\mathbb{P}_1, R^2\pi_*\sigma^*\mathscr{F}(-3)) = 1 + h^0(\mathbb{P}_1, R^2\pi_*\sigma^*\mathscr{F}(-2)\otimes\mathcal{O}_{\mathbf{P}_1}(-1)) \\ &\le 1 + h^0(\mathbb{P}_1, R^2\pi_*\sigma^*\mathscr{F}(-2)) = 1 + h^2\mathscr{F}(-2) = 3 \end{aligned}$$

by (2.1) (notice that

$$h^1(\mathbb{P}_1, R^1\pi_*\sigma^*\mathscr{F}(-2)) = h^1\mathcal{O}_{\mathbb{P}_1}(-1) = 0$$

by (7) and that

$$h^2\mathscr{F}(-2) = 2$$

by (3.2) and Lemma 5). This is a contradiction to

(1) $$q = 3. \quad \square$$

By the last two results the Table (3.2) can be rewritten as follows:

$i \backslash m$	-4	-3	-2	-1	0
4	0	0	0	0	0
3	5	q	0	0	0
2	0	q	2	q	0
1	0	0	0	q	5
0	0	0	0	0	0.

(3.3)

Moreover

$$q = h^2\mathscr{F}(-3) \leqq 2: \tag{3.4}$$

Use (3.3), Lemma 3 and the exact sequence

$$0 \to \mathscr{F}(-4) \to 2\mathscr{F}(-3) \to \mathscr{F}(-2) \to \mathscr{F}_P(-2) \to 0$$

corresponding to a plane which is stable for $\mathscr{F}$ [Ba].

Remark 5. Let us mention that by Lemma 5 and (3.4)

$$h^1(H, \mathscr{F}_H(-1)) \leqq 4$$

for *any* hyperplane $H \subset \mathbb{P}_4$ (use $S(H, -1)$). So the spectrum of $\mathscr{F}_H$ cannot be $(-3, -2, 1, 2)$ (compare Remark 4). $\square$

As a consequence of (3.3) we prove that $\mathscr{F}$ is the cohomology of a suitable monad. Then it remains to show that such a monad can only exist if its type is that one of $\mathscr{F}_{\mathrm{HM}}$, i.e. if $q = 0$.

Lemma 7. *$\mathscr{F}$ is the cohomology of a monad*

$$\begin{array}{c} (H^3\mathscr{F}(-4)\otimes\Omega^4(4))\oplus(H^2\mathscr{F}(-3)\otimes\Omega^3(3)) \\ \Big\downarrow{\scriptstyle \varphi=(\varphi_{ij})} \\ (H^3\mathscr{F}(-3)\otimes\Omega^3(3))\oplus(H^2\mathscr{F}(-2)\otimes\Omega^2(2))\oplus(H^1\mathscr{F}(-1)\otimes\Omega^1(1)) \\ \Big\downarrow{\scriptstyle \psi=(\psi_{ij})} \\ (H^2\mathscr{F}(-1)\otimes\Omega^1(1))\oplus(H^1\mathscr{F}\otimes\Omega^0(0)) \end{array} \tag{M}$$

such that:

(i) $$H^2\mathscr{F}(-3)\otimes\Omega^3(3) \xrightarrow{\varphi_{12}} H^3\mathscr{F}(-3)\otimes\Omega^3(3)$$

and

$$H^1\mathscr{F}(-1)\otimes\Omega^1(1)\xrightarrow{\psi_{13}} H^2\mathscr{F}(-1)\otimes\Omega^1(1)$$

are zero.

(ii) φ_{22} *is "dual" to* ψ_{12}*:*

$$\begin{tikzcd}
H^2\mathscr{F}(-3)\otimes\Omega^3(3) \arrow[r, "\varphi_{22}"] \arrow[d, phantom, "\wr\Vert"] & H^2\mathscr{F}(-2)\otimes\Omega^2(2) \arrow[d, phantom, "\wr\Vert"] \\
(H^2\mathscr{F}(-1))^*\otimes\Omega^1(1)^*\otimes\mathcal{O}(-1) \arrow[r, "\psi_{12}^*\otimes\mathrm{id}_{\mathcal{O}(-1)}"'] & (H^2\mathscr{F}(-2))^*\otimes\Omega^2(2)^*\otimes\mathcal{O}(-1).
\end{tikzcd}$$

Remark 6. The vertical isomorphisms are defined by (D) together with the pairings

$$\Omega^i(i)\otimes\Omega^{4-i}(4-i)\xrightarrow{\wedge}\Omega^4(4)\cong\mathcal{O}(-1).$$

Notice that the pairing (D) on $H^2\mathscr{F}(-2)$ is skew:

$$\begin{tikzcd}
H^2\mathscr{F}(-2)\otimes H^2\mathscr{F}(-2) \arrow[r] \arrow[dr, "Q"'] & H^4(\mathscr{F}\otimes\mathscr{F}(-4)) \arrow[d, "\wedge"] \\
 & H^4(\Lambda^2\mathscr{F})(-4)\cong H^4\mathcal{O}(-5)\cong\mathbb{C}.
\end{tikzcd}$$ □

Proof of Lemma 7. First recall the Bott-formula (cf. [O-S-S, p. 8]) and apply Horrock's technique [Ho] to eliminate those cohomology groups $H^i\mathscr{F}(m)$ which appear in Beilinson's E_1-diagram outside the diagonal:

The universal extension

(1) $$0\to\mathscr{F}\to\mathscr{Q}'\to(H^1\mathscr{F})\otimes\mathcal{O}\to 0$$

eliminates in the range $-4\leqq m\leqq 0$ precisely $H^1\mathscr{F}$:

$$H^i\mathscr{Q}'(m)\cong\begin{cases}0 & i=1,\quad m=0\\ H^i\mathscr{F}(m) & \text{otherwise}.\end{cases}$$

So

$$\mathrm{Ext}^2(\mathcal{O}(1),\mathscr{Q}')\cong\mathrm{Ext}^1(\Omega^1(1),\mathscr{Q}')$$

by the Koszul-complex and Lemma 6 and the universal extension

(2) $$0\to\mathscr{Q}'\to\mathscr{Q}\to(H^2\mathscr{F}(-1))\otimes\Omega^1(1)\to 0$$

eliminates $H^2\mathscr{F}(-1)$. (1) and (2) define a (non-unique) extension

(3) $$0\to\mathscr{F}\to\mathscr{Q}\to(H^2\mathscr{F}(-1)\otimes\Omega^1(1))\oplus(H^1\mathscr{F}\otimes\mathcal{O})\to 0$$

(notice that

$$\mathrm{Ext}^1(\Omega^1(1),\mathcal{O})\cong H^1\Omega^3(4)=0).$$

Using Serre-duality we may similarly eliminate $H^3\mathscr{F}(-4)$ and $H^2\mathscr{F}(-3)$ by an extension

(4) $$0\to(H^3\mathscr{F}(-4)\otimes\mathcal{O}(-1))\oplus(H^2\mathscr{F}(-3)\otimes\Omega^3(3))\to\mathscr{P}\to\mathscr{Q}\to 0.$$

In the range $-4\leqq m\leqq 0$

$$H^3\mathscr{P}(-3)\cong H^3\mathscr{F}(-3),\quad H^2\mathscr{P}(-2)\cong H^2\mathscr{F}(-2),\quad H^1\mathscr{P}(-1)\cong H^1\mathscr{F}(-1)$$

and

$$H^i\mathscr{P}(m)=0 \quad \text{otherwise}.$$

So

$$\mathscr{P}\cong(H^3\mathscr{F}(-3)\otimes\Omega^3(3))\oplus(H^2\mathscr{F}(-2)\otimes\Omega^2(2))\oplus(H^1\mathscr{F}\otimes\Omega^1(1))$$

by Beilinson's spectral sequence since

$$\operatorname{Ext}^1(\Omega^i(i),\Omega^j(j))=0$$

for $0<i,j<4$ (cf. [Tr, 2.5]).

Finally complete (3) and (4) to the display of a monad of the desired type.

(i) By the Bott-formula

$$h^3\Omega^3=1=h^1\Omega^1$$

and

$$\dim\operatorname{coker}(H^2\mathscr{F}(-3)\otimes H^3\Omega^3 \xrightarrow{H^3(\varphi_{12})} H^3\mathscr{F}(-3)\otimes H^3\Omega^3)=h^3\mathscr{F}(-3),$$

$$\dim\ker(H^1\mathscr{F}(-1)\otimes H^1\Omega^1 \xrightarrow{H^1(\psi_{13})} H^2\mathscr{F}(-1)\otimes H^1\Omega^1)=h^1\mathscr{F}(-1)$$

(use the display of the monad). So

$$\varphi_{12}=0=\psi_{13}$$

(the only endomorphisms of Ω^i are its homotheties, cf. Lemma 1).

(ii) Let $((M)', (M)'')$ be any of the pairs

$$((M),(M)^*(-1)),\ ((M),(M)),\ ((M)^*(-1),(M)^*(-1)) \quad\text{or}\quad ((M)^*(-1),(M)).$$

Then every homomorphism of the cohomology bundles

$$\mathscr{F}'\to\mathscr{F}''$$

can be lifted to a homomorphism of monads which is

(5) uniquely determined up to a homotopy:

$$\begin{tikzcd} (M)' \arrow[d] & \mathscr{A}' \arrow[r] \arrow[d] & \mathscr{B}' \arrow[r] \arrow[d, "\varrho"] \arrow[dl, dashed] & \mathscr{C}' \arrow[d] \arrow[dl, dashed] \\ (M)'' & \mathscr{A}'' \arrow[r] & \mathscr{B}'' \arrow[r] & \mathscr{C}'' \end{tikzcd}$$

(use the Koszul-complex and the Bott-formula to compute the corresponding obstruction groups, compare [O-S-S, p. 276]. ϱ is not "too far away" from being unique: Recall the isomorphisms of Remark 6 and consider ϱ as a matrix

$$\varrho=(\varrho_{ij}):\bigoplus_{j=1}^{3}U_j\otimes\Omega^j(j)\to\bigoplus_{i=1}^{3}U_i\otimes\Omega^i(i).$$

Then

(6) ϱ is upper triangular

with uniquely determined diagonal: apply Lemma 1, (5), and (i).

As a consequence we obtain: The canonical isomorphism

$$\mathscr{F}\cong\mathscr{F}^*(-1)$$

is covered by an isomorphism of monads whose component

$$\varrho_{22}: H^2\mathscr{F}(-2)\otimes\Omega^2(2)\to(H^2\mathscr{F}(-2))^*\otimes(\Omega^2(2))^*\otimes\mathcal{O}(-1)$$

satisfies

$$\varrho_{22}^*\otimes\mathrm{id}_{\mathcal{O}(-1)}=-\varrho_{22}.$$

Let ι denote the isomorphism

$$\Omega^2(2)\cong\Omega^2(2)^*\otimes\mathcal{O}(-1)$$

from Remark 6. Then

$$\mathrm{Hom}(\Omega^2(2),\Omega^2(2)^*\otimes\mathcal{O}(-1))\cong\mathbb{C}\cdot\iota$$

(compare Lemma 1) and

$$\iota^*\otimes\mathrm{id}_{\mathcal{O}(-1)}=\iota.$$

Thus

$$\varrho_{22}=Q'\otimes\iota$$

for a symplectic form Q' on $H^2\mathscr{F}(-2)$. Then $Q'=\lambda\cdot Q$ for a nonzero scalar λ and because of (i) and (6) the assertion is true after a change of φ. □

After a choice of bases the monad has the form

$$(M)\ 5\Omega^4(4)\oplus q\Omega^3(3)\xrightarrow{B=(B_{ij})} q\Omega^3(3)\oplus2\Omega^2(2)\oplus q\Omega^1(1)\xrightarrow{A=(A_{ij})} q\Omega^1(1)\oplus5\Omega^0(0)$$

(recall (3.3)). By Lemma 1 A and B are matrices with entries in ΛV operating by contraction. The components with entries in $\Lambda^0V=\mathbb{C}$ vanish:

$$B_{12}=0=A_{13} \tag{3.5}$$

by Lemma 7, (i). For a symplectic basis of $H^2\mathscr{F}(-2)$ $\left(\text{then } Q \text{ is just the matrix } Q=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)$ and dual bases of $H^2\mathscr{F}(-3)\cong(H^2\mathscr{F}(-1))^*$ and $H^2\mathscr{F}(-1)$

$$B_{22}=Q\cdot{}^tA_{12} \tag{3.6}$$

by Lemma 7, (ii). That (M) is a complex implies that

$$A_{12}\wedge(Q\cdot{}^tA_{12})=0.$$

Consequently

$$\begin{aligned}A_{12}&=\begin{pmatrix}a&0\\b&0\end{pmatrix}\quad\text{if}\quad q=2,\\ A_{12}&=(a\quad 0)\quad\text{if}\quad q=1\end{aligned} \tag{3.7}$$

with $a,b\in V$ after a suitable choice of the symplectic basis of $H^2\mathscr{F}(-2)$. (Recall that $q\leqq2$ by (3.4).)

To show that $q=0$ we study again the restriction of $\mathscr{F}$ to suitable linear subspaces $\mathbb{P}_k\subset\mathbb{P}_4$. The dimensions $h^i(\mathbb{P}_k,\mathscr{F}_{\mathbb{P}_k}(m))$ in the range $-k\leqq m\leqq0$ are determined by the components of A and B with entries in Λ^jV, $j\leqq4-k$, and the equations of $\mathbb{P}_k$.

Consider e.g. a hyperplane

$$H=\{x^*=0\}\subset \mathbb{P}(V)$$

and write

$$x^*(C)=(x^*(c_{ij}))$$

if $C=(c_{ij})$ is a matrix with entries in V.

Use Lemma 2 and Remark 2 to restrict the monad:

$$\begin{array}{ll} & 5\Omega_H^3(3)\oplus q(\Omega_H^2(2)\oplus\Omega_H^3(3)) \\ & \qquad\Big\downarrow{\scriptstyle B\otimes\mathcal{O}_H} \\ (M)\otimes\mathcal{O}_H & q(\Omega_H^2(2)\oplus\Omega_H^3(3))\oplus 2(\Omega_H^1(1)\oplus\Omega_H^2(2))\oplus q(\Omega_H^0(0)\oplus\Omega_H^1(1)) \\ & \qquad\Big\downarrow{\scriptstyle A\otimes\mathcal{O}_H} \\ & q(\Omega_H^0(0)\oplus\Omega_H^1(1))\oplus 5\Omega_H^0(0)\,. \end{array}$$

Then

$$\mathscr{F}_H \text{ is } \textit{stable} \text{ iff} \tag{3.8}$$

$$\dim\operatorname{coker}\left(5H^3\Omega_H^3\oplus qH^3\Omega_H^3 \xrightarrow{(x^*(B_{11})\ \ 0)} qH^3\Omega_H^3\right)=h^3(H,\mathscr{F}_H(-3))=0$$

iff

$$\dim\ker\left(qH^0\Omega_H^0 \xrightarrow{{}^t(0\ \ x^*(A_{23}))} qH^0\Omega_H^0\oplus 5H^0\Omega_H^0\right)=h^0(H,\mathscr{F}_H)=0$$

and

$$\mathscr{F}_H \text{ has } \textit{minimal (maximal) spectrum} \text{ iff} \tag{3.9}$$

$$\dim\operatorname{coker}\left(qH^2\Omega_H^2 \xrightarrow{{}^t(0\ \ x^*(B_{22}))} qH^2\Omega_H^2\oplus 2H^2\Omega_H^2\right)=h^2(H,\mathscr{F}_H(-2))=2\ (=3)$$

iff

$$\dim\ker\left(2H^1\Omega_H^1\oplus qH^1\Omega_H^1 \xrightarrow{(x^*(A_{12})\ \ 0)} qH^1\Omega_H^1\right)=h^1(H,\mathscr{F}_H(-1))=2\ (=3):$$

Apply the Bott-formula (cf. [O-S-S, p. 8]) to the display of $(M)\otimes\mathcal{O}_{H'}$ identify

$$H^i(H,\Omega_H^i)\cong\mathbb{C}$$

and recall Remark 4 concerning the possible spectra.

Remark 7. Suppose that $\mathscr{F}_H$ is stable and has maximal spectrum. Then $q=1$ or $q=2$ and by (3.8), (3.9), and Remark 8 below we may reduce $(M)\otimes\mathcal{O}_H$ to a monad

$$(M)_H \qquad 5\Omega_H^3(3)\oplus\Omega_H^2(2)\xrightarrow{B_H}3\Omega_H^2(2)\oplus 3\Omega_H^1(1)\xrightarrow{A_H}\Omega_H^1(1)\oplus 5\Omega_H^0(0)$$

for $\mathscr{F}_H$ such that

$$\Omega_H^2(2)\to 3\Omega_H^2(2),\qquad 3\Omega_H^1(1)\to\Omega_H^1(1)$$

are zero. This is the type of monad which we expect for a stable 2-bundle with $c_1=-1$, $c_2=4$ and maximal spectrum on $\mathbb{P}_3$ by Lemma 4 (compare the proof of

Lemma 7). But since $(M)_H$ comes from a monad on $\mathbb{P}_4$ some further components of A_H and B_H have to be zero by (3.5) and (3.7). □

Remark 8. Let

$$\mathscr{A}_1 \oplus \mathscr{A}_2 \xrightarrow{\beta=(\beta_{ij})} \mathscr{A}_1 \oplus \mathscr{B} \oplus \mathscr{C}_2 \xrightarrow{\alpha=(\alpha_{ij})} \mathscr{C}_1 \oplus \mathscr{C}_2$$

be a monad such that

$$\beta_{11}=\mathrm{id}, \quad \alpha_{23}=\mathrm{id}.$$

Then

$$\mathscr{A}_2 \xrightarrow{\tilde{\beta}} \mathscr{B} \xrightarrow{\tilde{\alpha}} \mathscr{C}_1,$$

defined by

$$\tilde{\beta}=\beta_{22}-\beta_{21}\circ\beta_{12}, \quad \tilde{\alpha}=\alpha_{12}-\alpha_{13}\circ\alpha_{22},$$

is a monad, too, and there exists a canonical isomorphism

$$\ker\tilde{\alpha}/\mathrm{im}\,\tilde{\beta} \cong \ker\alpha/\mathrm{im}\,\beta. \quad \square$$

Lemma 8. $q \leqq 1$.

Proof. Suppose not. Then

$$q=2$$

by (3.4) and

$$A_{12}=\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$$

by (3.7). So by (3.9) there exists *no* hyperplane H such that $\mathscr{F}_H$ has minimal spectrum.

Choose a plane and hyperplane

$$P \subset H \subset \mathbb{P}_4$$

which are stable for $\mathscr{F}$ [Ba]. Then $\mathscr{F}_H$ has maximal spectrum (recall Lemma 4). Restrict (M) to H and reduce $(M)\otimes\mathcal{O}_H$ to $(M)_H$ as in Remark 7. Restrict $(M)_H$ to P and reduce similarly $(M)_H \otimes \mathcal{O}_P$ to a monad

$$(M)_P \qquad 3\Omega_P^2(2) \xrightarrow{B_P} 4\Omega_P^1(1) \xrightarrow{A_P} 3\Omega_P^0(0)$$

for $\mathscr{F}_P$ of that type we expect by Lemma 3 and Beilinson's spectral sequence. Write

$$A_P=(a_{ij}), \quad B_P=(b_{jk})$$

and observe that

$$a_{ij}=0 \quad \text{for} \quad 3\leqq j\leqq 4, \quad 1\leqq i\leqq 3,$$
$$b_{jk}=0 \quad \text{for} \quad 1\leqq j\leqq 2, \quad 1\leqq k\leqq 3$$

by Remark 7. So by Remark 9 below there exists a nontrivial linear combination of the rows of A_p whose components are linearly dependent. Consequently, a contradiction, A_P is not pointwise surjective by Remark 1. (Similarly B_P is not pointwise injective.) □

Remark 9. Let W be a k-dimensional vector space over $\mathbb{C}$ and $(e_1, \ldots, e_k), (f_1, \ldots, f_k) \in kW$ be two row vectors. Then there exists a nontrivial linear combination

$$\begin{pmatrix} e \\ f \end{pmatrix} \text{ of } \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}, \ldots, \begin{pmatrix} e_k \\ f_k \end{pmatrix} \in 2W$$

such that e, f are linearly dependent (cf. [D-S, Remark 4]). □

Lemma 9. $q=0$.

Proof. Suppose not. Then

$$q=1$$

by Lemma 8 and

$$A_{12}=(a \quad 0)$$

by (3.7). Moreover

$$a \neq 0:$$

Otherwise

$$\Omega^3(3) \xrightarrow{A_{11}} \Omega^1(1)$$

has to be pointwise surjective (recall that $A_{13}=0$ by (3.5)). This is impossible since we may write

$$A_{11}=\sum_{i=1}^{k} a_i \wedge b_i$$

with

$$k \leqq (\dim V-1)/2=2.$$

Let $P \subset \mathbb{P}_4$ be the *general* plane. Then P is stable for $\mathscr{F}$ [Ba] and there is a (unique) hyperplane $H \subset \mathbb{P}_4$ containing P and $\langle a \rangle$. So $\mathscr{F}_H$ is stable and has maximal spectrum by (3.9). Restrict (M) to H and reduce $(M) \otimes \mathcal{O}_H$ to $(M)_H$ as in Remark 7. Observe that

$$3\Omega_H^1(1) \to \Omega_H^1(1)$$

is zero and that

$$3\Omega_H^2(2) \to \Omega_H^1(1)$$

is given by at most two linearly independent vectors. So, a contradiction, A_H is not pointwise surjective by Remark 1. (Similarly B_H is not pointwise injective.) □

4. Proof of Theorem 2

Suppose that there exists a stable rank 2 vector bundle $\mathscr{F}$ on $\mathbb{P}_5 = \mathbb{P}(V)$ with $c_1=-1$, $c_2=4$.

Apply the exact sequences $S(H,m)$ corresponding to the *general* hyperplane $H\subset\mathbb{P}_5$. Since H is stable for $\mathscr{F}$ [Ba]

$$\begin{array}{llll} H^1\mathscr{F}(m-1)=H^1\mathscr{F}(m) & =0 & \text{for} & m\leqq -1 \\ H^2\mathscr{F}(m-1)=H^2\mathscr{F}(m) & =0 & \text{for} & m\leqq -3 \\ H^3\mathscr{F}(m) & =H^3\mathscr{F}(m+1)=0 & \text{for} & m\geqq -2 \\ H^4\mathscr{F}(m) & =H^4\mathscr{F}(m+1)=0 & \text{for} & m\geqq -4 \end{array}$$

by Theorem 1, (S), (D), and Serre's Theorem B.

So the table

$i \backslash m$	-5	-4	-3	-2	-1	0
5	0	0	0	0	0	0
4	$4+r$	0	0	0	0	0
3	r	1	1	0	0	0
2	0	0	0	1	1	r
1	0	0	0	0	0	$4+r$
0	0	0	0	0	0	0

reflects the relations between the dimensions $h^i\mathscr{F}(m)$ given in the range $-5\leqq m\leqq 0$ by (S), (D), and (R).

Since

$$h^1\mathscr{F}(-1)=0$$

any hyperplane $H\subset\mathbb{P}_5$ is stable for $\mathscr{F}$ (use $S(H,0)$). But the natural map

$$V^*\otimes H^2\mathscr{F}(-2)\to H^2\mathscr{F}(-1)$$

has a nonzero kernel by reasons of dimension. So there exists a hyperplane $H\subset\mathbb{P}_5$ such that

$$h^1(H,\mathscr{F}_H(-1))=h^2\mathscr{F}(-2)=1$$

(use $S(H,-1)$). This is a contradiction to Theorem 1.

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Regularity of the Complex Monge-Ampere Equation for Radially Symmetric Functions of the Unit Ball

David Monn

Institute for Defense Analyses, Thanet Road, Princeton, NJ 08540, USA

1. Introduction

Let M denote the complex Monge-Ampere operator in $\mathbb{C}^n$, which is defined for C^2 functions as the determinant of the complex Hessian matrix,

$$M = \det \left| \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \right| .$$

This operator can be viewed as a natural generalization to n variables of the Laplace operator in one complex variable because of the similar way it behaves under holomorphic changes of variables. Namely, if f is a holomorphic mapping,

$$M(u \circ f) = |J_{\mathbb{C}} f|^2 \cdot Mu \circ f$$

where $J_{\mathbb{C}}$ is the complex Jacobian determinant. Furthermore, when $n=1$ the Monge-Ampere operator is simply one-fourth of the Laplacian. This fact has led to the study of Monge-Ampere equations with the hope of developing a non-linear potential theory in $\mathbb{C}^n$.

Of primary importance in the study of the Monge-Ampere operator has been the following Dirichlet problem:

$$u \text{ plurisubharmonic}$$
$$Mu = f \quad \text{in} \quad \Omega$$
$$u = \phi \quad \text{on} \quad b\Omega$$

where $f \in C^\infty(\bar{\Omega})$ is non-negative, $\phi \in C^\infty(b\Omega)$ and Ω is a strongly pseudoconvex domain. The condition that u be plurisubharmonic is imposed for uniqueness (a proof of which can be found in [2, 6]); and as a result, the complex Hessian matrix will be positive semidefinite, forcing Mu to be non-negative. It has recently been shown by Caffarelli et al. [4] that there exists a C^∞-smooth solution to this problem *provided f is non-vanishing on $\bar{\Omega}$*. However, Bedford and Fornaess [1], and Gamelin and Sibony (see [7]) have constructed counterexamples that show that C^2 solutions need not exist if f is allowed to vanish in Ω.[1] To understand this phenomenon better we consider what happens in the special case where f is a radially symmetric function of the unit ball, using the observation of Kerzman [6] that an explicit solution can be obtained for this class of functions. Such results will hopefully illuminate the behavior of solutions of the Monge-Ampere equation when the right hand side vanishes, and may suggest what to look for in less symmetric cases.

The problem that will be considered here can be stated as follows.

Let B denote the unit ball in $\mathbb{C}^n$. Given $f \geqq 0$ on B, find a plurisubharmonic function $u \in C^2(B)$ such that

$$\begin{aligned} &\text{(i)} \quad Mu = f \quad \text{in} \quad B, \text{ and} \\ &\text{(ii)} \quad u = 0 \quad \text{on} \quad bB. \end{aligned} \tag{1.1}$$

A radial function u can be considered simply as a function of one real variable r. So in Sect. 2 we will compute Mu directly, obtaining a non-linear ordinary differential equation $Mu(r) = f(r)$. This equation is then solved by two integrations, giving u in terms of f. The result has been summarized up in Theorem 1.

Theorem 1. *Let u be a plurisubharmonic radial function in $C^2(B)$. Then u is given explicitly by the formula*

$$u(r) = \int_r^1 \frac{-2}{t} \left[2n \int_0^t x^{2n-1} f(x)\, dx \right]^{1/n} dt + u(1), \; r = |z|,$$

where $f(r) = Mu(r)$.

We will show that by placing certain conditions on $f(r)$ in the above formula, we can insure the smoothness of $u(r)$. And if $u \in C^2(B)$, then it is the unique solution to (1.1), so that regularity of the solution is simply determined by the regularity of the formula.

Theorem 2. *Let $f \in C^k(\overline{B})$ be a non-negative radially symmetric function of the unit ball. If $f(0) \neq 0$, then there exists a unique solution, $u \in C^{k+2}(\overline{B})$, to (1.1) given explicitly by*

$$u(r) = \int_r^1 \frac{-2}{t} \left[2n \int_0^t x^{2n-1} f(x)\, dx \right]^{1/n} dt, \; r = |z|, \tag{1.2}$$

Notice what the theorem is saying. It says that no matter how bad the zero set of f is away from the origin, as long as $f(0) \neq 0$, the solution is still in $C^{k+2}(\overline{B})$.

If $f(0) = 0$, the function u given in (1.2) is not always C^{k+2} at every point in B, as Examples 2.6 and 2.7 will show. However, the next two theorems show that certain added conditions on the function f will insure the regularity of u.

Theorem 3. *Let f be a non-negative radial function such that $f^{1/j} \in C^k(\overline{B})$ for some $j \geqq n(k+1)$, $k \geqq 1$. Then $u \in C^{k+2}(\overline{B})$.*

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[1] Bedford and Taylor [2] have developed the concept of a generalized solution to the Dirichlet problem, extending the class of functions for which $M = \text{const} \cdot (\partial\bar{\partial})^n$ is defined to all continuous plurisubharmonic functions. They then prove the existence and uniqueness of generalized solutions to the Dirichlet problem

Theorem 4. *Let f be a non-negative radial function in $C^k(\overline{B})$ such that $f(r)=o(r^k)$. Then $u\in C^p(\overline{B})$ where $p\in\mathbb{Z}^+$ and $\frac{k+1}{n}-1<p\leqq\frac{k+1}{n}$.*

Theorem 4 arises from an interesting real variable result concerning n^{th} roots of functions in $C^k[0,1]$. Glaeser [5] proved that if $\phi\in C^2[0,1]$, with $\phi(0)=\phi'(0)=\phi''(0)=0$ and $\phi(x)>0$ for $x>0$, then $\phi^{1/2}\in C^1[0,1]$. He also showed by counterexample that $\phi^{1/2}$ need not be in $C^2[0,1]$ even if $\phi\in C^\infty[0,1]$ with all derivatives vanishing at zero and $\phi(x)>0$ for $x>0$. However, he suggested that $\phi^{1/2}$ may be smooth if additional hypotheses were placed on ϕ, such as $\phi'\geqq 0$. We show in Proposition 5.3 that this conjecture is in fact true; and we use this result to prove Theorem 4.

2. The Integral Formula

Before we compute the Monge-Ampere operator for radial functions, we must first see how to get from a C^k radial function of n complex variables to a function of one real variable r, and vice versa. The relationship turns out to be a fairly simple one and is stated in theorem 1.4. The proof is straightforward and will be ommitted.

Proposition 2.1. *Let u be a radial function on $\overline{B}$, and let $\tilde{u}(r)=u(r,0,\dots,0)$. Then $u\in C^k(\overline{B})$ if and only if*

(i) $\qquad \tilde{u}\in C^k[0,1],$ *and*

(ii) $\qquad \tilde{u}^{(\ell)}(0)=0$ *for all $\ell\leqq k$, ℓ odd.*

For the remainder of this paper, u will be used to denote both $u(z)$ and $\tilde{u}(r)$. Its meaining should be clear from the context.

Let $u=u(r=|z|)$ be any radial function in $C^2(B)$. Then a straightforward computation yields

$$\frac{\partial^2 u}{\partial z_i\partial\bar{z}_j}=\frac{1}{4r^3}\,\bar{z}_i z_j[ru''-u']+\delta_{ij}\cdot\frac{u'}{2r}$$

where δ_{ij} is the Kronecker delta.

Now in order to compute the determinant we first factor out $(ru''-u')/4r^3$, which leads us to study the nxn matrix $A=(\bar{z}_i z_j)+\varphi\cdot I$, where I is the identity matrix and $\phi=\frac{2r^2u'}{ru''-u'}$.

Lemma 2.2. $\det A=\phi^{n-1}(r^2+\phi)$.

Proof. We first notice that ϕ is an eigenvalue of A:

$$\det(A-\phi I)=\det(\bar{z}_i z_j)=\sum_\sigma\left[\varepsilon(\sigma)\cdot\prod_{j=1}^n(\bar{z}_{\sigma(j)}z_j)\right]$$

the sum is taken over all permutations σ of $\{1,\dots,n\}$, and $\varepsilon(\sigma)$ is the sign of the permutation; each product containing one element from each row (hence, $\bar{z}_i$

appears for each i) and one element from each column (hence, z_j appears for each j). Thus,

$$\det(\bar{z}_i z_j) = \sum_{\sigma} \varepsilon(\sigma) \cdot |z_1 z_2 \cdots z_n|^2 = 0$$

since there are an even number ($n!$) of permutations σ. The rank of the matrix $(\bar{z}_i z_j) = {}^t\bar{z} \cdot z$ is one. Hence, the eigenspace associated with ϕ has dimension at least $n-1$. So ϕ is an eigenvalue of multiplicity (at least) $n-1$. The trace of A is

$$\sum_{j=1}^{n} |z_j|^2 + n\phi = r^2 + n\phi = (n-1)\phi + \lambda$$

where λ is the unknown eigenvalue. Thus, $\lambda = r^2 + \phi$. Therefore,

$$\det A = \phi^{n-1} \cdot (r^2 + \phi)$$

as claimed.

Proposition 2.3. *For a radial function $u \in C^2(B)$, $u = u(r)$,*

$$Mu = \frac{1}{4}\left[\frac{u'}{2r}\right]^{n-1} \cdot \left[u'' + \frac{1}{r}u'\right]$$

Proof. This follows from Lemma 2.2 and the fact that $Mu = \det(cA) = c^n \cdot \det A$ where $c = (ru'' - u')/4r^3$. Q.E.D.

Let $f(r) = Mu$. Then we have a non-linear O.D.E.,

$$ru'' \cdot (u')^{n-1} + (u')^n = 2^{n+1} \cdot r^n f$$

which can be solved for u by two integrations as follows.

$$\frac{r}{n} \cdot D_r[(u')^n] + (u')^n = 2^{n+1} r^n f$$

Letting $v = (u')^n$ the left hand side becomes

$$\frac{1}{nr^{n-1}}(r^n v' + nr^{n-1} v) = \frac{1}{nr^{n-1}} D_r[r^n v].$$

So that,

$$D_r[r^n v] = n2^{n+1} r^{2n-1} f$$

hence,

$$r^n v(r) = n2^{n+1} \int_0^r x^{2n-1} f(x)\, dx + \text{const.}$$

Since v is bounded on $[0,1]$ the constant must be zero. Hence,

$$v(r) = (u')^n = \frac{n2^{n+1}}{r^n} \int_0^r x^{2n-1} f(x)\, dx.$$

We now solve for u by taking the n^{th} root and integrating once again. However if n is even we get two solutions. So although we want to consider only real-valued functions for u and f, we temporarily introduce an n^{th} root of unity for convenience. So we have,

$$u(r)=\omega_n \int_r^1 \frac{-2}{t}\left[2n\int_0^t x^{2n-1} f(x)\,dx\right]^{1/n} dt + u(1) \tag{2.4}$$

where ω_n is a (real) n^{th} root of unity. This shows that a condition like plurisubharmonicity is needed for uniqueness. We now look at the solution for which $\omega_n = 1$.

Proposition 2.5. *Let $u \in C^2(\bar{B})$ be given by*

$$u(r)=\int_r^1 \frac{-2}{t}\left[2n\int_0^t x^{2n-1} f(x)\,dx\right]^{1/n} dt + u(1).$$

Then u is plurisubharmonic if and only if $f \geqq 0$.

Proof. If u is plurisubharmonic then $\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right)$ is positive semi-definite. Hence, all eigenvalues are non-negative, and thus $Mu = f \geqq 0$.

Conversely, suppose that $f(r) \geqq 0$. Then

$$u'(r)=\frac{2}{r}\left[2n\int_0^r x^{2n-1} f(x)\,dx\right]^{1/n} \geqq 0,$$

and thus u is increasing. Let p be any point in B and $r_0 = |p|$. There are two cases to consider – $u'(r_0) \neq 0$ and $u'(r_0) = 0$.

Case 1. Suppose $u'(r_0) \neq 0$. Then $u(r) - u(r_0)$ is a C^2 defining function for $B_{r_0} = \{|z| < r_0\}$. Since B_{r_0} is strictly pseudoconvex,

$$\sum_{i,j} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(p)\, w_i \bar{w}_j > 0,$$

for all $w \in T_p(B_{r_0}) = \left\{ v \in \mathbb{C}^n \,\middle|\, \sum_{j=1}^n \frac{\partial u}{\partial z_j}(p) \cdot v_j = 0 \right\}$. So the complex Hessian matrix is positive definite on the $(n-1)$-dimensional subspace $T_p(B_{r_0})$, hence it has (at least) $n-1$ positive eigenvalues. Since the determinant is non-negative, the remaining eigenvalue must be non-negative. So the quadratic form is positive semi-definite on the entire space. Hence, u is plurisubharmonic at p.

Case 2. Suppose $u'(r_0) = 0$. Then $u'(r) = 0$ for every $r \leqq r_0$; and since u is increasing, it achieves its minimum at p. If ℓ is any complex line through p, u restricted to ℓ achieves its minimum at p. Thus, u satisfies the sub-averaging property, and hence is subharmonic on ℓ at p. Since this is true for any ℓ, u is plurisubharmonic at p. Q.E.D.

Theorem 1 now follows as a corollary to Proposition 2.5 and the solution (2.4) to the differential equation.

Example 2.6. Let $f(r) = r^k$, $k \geqq 0$. Then

$$u(r) = \text{const} \cdot \left[r^{2+\frac{k}{n}} - 1 \right]$$

where the constant equals $[2n/(2n+k)]^{1+(1/n)}$.

If k is a multiple of $2n$, then $u \in C^{\infty}(\bar{B})$; otherwise, $u \in C^{p+2}(\bar{B})$ where p is the greatest integer less than k/n. In particular, if k is an even integer which is not a multiple of $2n$, then $f \in C^{\infty}(\bar{B})$ and $f \geqq 0$, but the solution u is not in $C^{\infty}(B)$. This example shows that even if f has only an isolated zero in the domain, it may not yield a smooth solution.

Example 2.7. Let $f(r) = (r - \frac{1}{2}) r^{1-2n}/n$ for $r > \frac{1}{2}$, and $f(r) = 0$ otherwise. Then $u'(r) = 2r^{-1}(r - \frac{1}{2})^{2/n}$ for $r > \frac{1}{2}$, and $u'(r) = 0$ for $r \leqq \frac{1}{2}$. In this case $f \in C^0(\bar{B})$, but $u \notin C^2(\bar{B})$.

3. Regularity of the Solution

Now in order to solve (1.1) it is necessary to show that formula (1.2) is in $C^2[0,1]$ with $u'(0) = 0$. This is not always true, however, as Example 2.7 shows. But certain things can be said in general about the formula, and are stated in the next proposition.

Proposition 3.1. *If $f \in C^k(\bar{B})$ is radial and non-negative, then*

$$u(z) = \int_{|z|}^{1} \frac{-2}{t} \left[2n \int_0^t x^{2n-1} f(x)\, dx \right]^{1/n} dt \in C^{k+2}(\bar{B} \setminus bB_\alpha) \cap C^1(B).$$

where $\alpha = \inf\{r : f(r) > 0\}$.

Proof. If $f \in C^k[0,1]$, then

$$g(r) \overset{\text{def}}{=} 2n \int_0^r x^{2n-1} f(x)\, dx$$

is in $C^{k+1}[0,1]$. Furthermore, if $\alpha = \inf\{x : f(x) > 0\}$, then $g(r) = 0$ for $r \leqq \alpha$ and $g(r) > 0$ for $r > \alpha$. Thus,

$$\frac{1}{r}[g(r)]^{1/n} \in C^{k+1}([0,1] \setminus \{\alpha\}),$$

hence, $u(r) \in C^{k+2}([0,1] \setminus \{\alpha\})$, so that $u(z) \in C^{k+2}(\bar{B} \setminus bB_\alpha)$. Since

$$u'(r) = \frac{2}{r} \left[2n \int_0^r x^{2n-1} f(x)\, dx \right]^{1/n} \leqq \text{const} \cdot r,$$

we see that u' is continuous and tends to zero as $r \to 0$. Hence, $u \in C^1(\bar{B})$ by Proposition 2.1. Q.E.D.

Suppose $f(0) \neq 0$ so that $\alpha = 0$. Then u is not only in $C^{k+2}(\bar{B} \setminus \{0\})$ by the above proposition, but in fact, $u \in C^{k+2}(\bar{B})$ as theorem 2 shows.

Proof of Theorem 2. Let $\phi(r) = \frac{1}{r^{2n}} \int_0^r x^{2n-1} f(x)\, dx$. We need to show that

$$r \cdot [\phi(r)]^{1/n} = \text{const} \cdot u'(r) \in C^{k+1}[0,1]$$

with all *even* derivatives vanishing at zero.

We may view f as an even function defined in $[-1,1]$; hence, both ϕ and $\phi^{1/n}$ are even functions of $[-1,1]$. Using the substitution $x = ry$,

$$\phi(r) = \int_0^1 y^{2n-1} f(ry)\,dy.$$

So that for any $\ell \leqq k$,

$$\phi^{(\ell)}(r) = \int_0^1 y^{2n+\ell-1} f^{(\ell)}(ry)\,dy = \frac{1}{r^{2n+\ell}} \int_0^r x^{2n+\ell-1} f^{(\ell)}(x)\,dx \tag{3.2}$$

and,

$$\phi^{(\ell)}(r) \to \frac{1}{2n+\ell} f^{(\ell)}(0) \quad \text{as} \quad r \to 0.$$

Hence, $\phi \in C^k[-1,1]$, and $\phi(r) > 0$ for all $-1 \leqq r \leqq 1$. So, $r \cdot [\phi(r)]^{1/n}$ is an odd function in $C^k[-1,1]$. In order to get $r \cdot [\phi(r)]^{1/n} \in C^{k+1}[-1,1]$, it is enough to show $r \cdot D^{k+1}(\phi^{1/n}) \to 0$ as $r \to 0$. Differentiating (3.2) directly and multiplying by r, we get

$$r \cdot \phi^{(k+1)}(r) = f^{(k)}(r) - (2n+k)\,\phi^{(k)}(r) \to 0 \quad \text{as} \quad r \to 0. \tag{3.3}$$

Since all terms of $D^{k+1}(\phi^{1/n})$ are bounded except possibly for the term

$$\text{const} \cdot \phi^{(k+1)}(r) \cdot [\phi(r)]^{(1/n)-1},$$

we have from (3.3) that $\lim_{r \to 0} r \cdot D^{k+1}(\phi^{1/n}) = 0$. Thus, $r \cdot [\phi(r)]^{1/n} \in C^{k+1}[-1,1]$. And since $r[\phi(r)]^{1/n}$ is an odd function, all even derivatives vanish at zero. So by Proposition 2.1,

$$u(r) = \text{const} \cdot \int_r^1 t \cdot [\phi(t)]^{1/n}\,dt \in C^{k+2}(\bar{B}).$$

4. Proof of Theorem 3

Let us first consider derivatives of a composition of two real-valued functions, $f \circ g$. Differentiating several times reveals an interesting pattern which can be stated as follows.

Proposition 4.1. *The k^{th} derivative of a composition of two real-valued functions, $f \circ g$, can be written as a sum of terms of the form*

$$f^{(\lambda)}(g) \cdot P(g', g'', \ldots, g^{(k+1-\lambda)})$$

where P is a monomial of degree $\lambda \leqq k$ and of weighted degree k.

Definition. Let P be a monomial in $y', \ldots, y^{(k)}$, $P = \text{const} \cdot \prod_{j=1}^{k} (y^{(j)})^{p_j}$. Then the *weighted degree* of P is given by

$$\text{wt.deg. } P = \sum_{j=1}^{k} j \cdot p_j$$

For example, wt. deg. $[y'(y'')^2]=5$.

Proof of Proposition 4.1. By induction on k.

Corollary 4.2. *The k^{th} derivative of $f^{1/n}$ can be written as a sum of terms of the form*

$$\frac{P(f',\ldots,f^{(k+1-\lambda)})}{f^{\lambda-(1/n)}}$$

where $\lambda\leqq k$, and P is a monomial of degree λ and weighted degree k.

Lemma 4.3. *If $f^{1/j}\in C^k[0,1]$ for some $j\geqq k+1$, $f\geqq 0$, then*

$$|f^{(\ell)}(x)|\leqq c[f(x)]^{1-(\ell/j)},\ 0\leqq \ell\leqq k,$$

where c is a constant which depends on j, k, and f, but is independent of x.

Proof. By induction on k. The case $k=0$ is trivial. Assume the lemma is true for $k-1$. Let $f^{1/j}\in C^k[0,1]$. Then for all $\ell\leqq k-1$

$$|f^{(\ell)}|\leqq \text{const}\cdot f^{1-(\ell/j)}$$

So by Corollary 4.2,

$$D^k(f^{1/j})=\sum_{\nu}\frac{P_\nu(f',\ldots,f^{(k+1-\lambda_\nu)})}{f^{\lambda_\nu-(1/j)}}+\frac{1}{j}\cdot\frac{f^{(k)}}{f^{1-(1/j)}}$$

where $\deg P_\nu=\lambda_\nu\geqq 2$ and wt. deg. $P_\nu=k$. Hence,

$$\begin{aligned}|f^{(k)}|&\leqq j\cdot f^{1-(1/j)}\cdot\left[c_1+\sum_\nu\left|\frac{P_\nu}{f^{\lambda_\nu-(1/j)}}\right|\right]\\&\leqq f^{1-(1/j)}\cdot[c_2+c_3\cdot f^{(1-k)/j}]\\&\leqq f^{1-(k/j)}\cdot[c_2\cdot f^{(k-1)/j}+c_3]\\&\leqq \text{const}\cdot f^{1-(k/j)}\end{aligned}$$

Lemma 4.4. *Let $f\in C^1[0,1]$, $f\geqq 0$, and $f(0)=0$. Then for any $0\leqq x_0\leqq 1$ such that $\int_0^{x_0} f>0$,*

$$\frac{f(x_0)}{\int_0^{x_0} f(t)\,dt}\leqq 2\cdot\frac{\sup_{x\leqq x_0}|f'(x)|}{\sup_{x\leqq x_0}|f(x)|}$$

Proof. Let $f(a)=\sup_{x\leqq x_0}|f(x)|$ and $L=\sup_{x\leqq x_0}|f'(x)|$. For $x\leqq a$,

$$f(a)-f(x)=\int_x^a f'(t)\,dt\leqq L\cdot(a-x).$$

So, $f(x)\geqq L\cdot(x-a)+f(a)$. If we let $x=0$, we see that $a-\frac{f(a)}{L}\geqq 0$. Hence,

$$\int_0^{x_0} f(x)\,dx\geqq\int_{a-\frac{f(a)}{L}}^{a} f(x)\,dx\geqq\int_{a-\frac{f(a)}{L}}^{a}[L\cdot(x-a)+f(a)]\,dx=\frac{1}{2L}(f(a))^2,$$

Therefore,

$$\frac{f(x_0)}{\int\limits_0^{x_0} f(t)\,dt} \leqq \frac{2L}{f(a)}$$

This completes the proof.

Remark. Suppose $g \in C^{mn}[0,1]$, $g(x)>0$ for $x>0$, and $g(x)=o(x^{mn})$. Then $g^{1/n}=o(x^m)$ and $g^{1/n} \in C^{mn}(0,1]$; but it is not necessarily true that $g^{1/n} \in C^m[0,1]$. So in order for $g^{1/n}$ to be in $C^m[0,1]$, it must be shown that $\lim\limits_{x\to 0} D^k g^{1/n}=0$ for each $k \leqq m$. This is the case in the proof of Theorem 3.

Proof of Theorem 3. As before, let $\alpha=\inf\{x: f(x)>0\}$. By Theorem 2 we may assume $f(0)=0$; and since $f^{1/j} \in C^1(\bar{B})$, $D(f^1/^j)(\alpha)=0$. Hence, $f(x+\alpha)=o(x^j)$. Let $g(x)=\int\limits_0^x t^{2n-1} f(t)\,dt$. Clearly, $g^{1/n} \in C^{k+1}(\alpha,1]$. To get smoothness at α, we will show that for $m \leqq k+1$,

$$D^m g^{1/n}(x)=o(x^{k+2-m}) \qquad \text{if} \quad \alpha=0, \tag{4.5}$$

and

$$D^m g^{1/n}(x+\alpha)=o(x^{k+1-m}) \qquad \text{if} \quad \alpha>0. \tag{4.6}$$

From these, we not only see that $g^{1/n} \in C^{k+1}[\alpha,1]$, but by the product rule we also get that $1/x \cdot [g(x)]^{1/n} \in C^{k+1}[0,1]$ with each derivative vanishing at α. This would then prove the theorem since

$$u'(r)=\text{const} \cdot \frac{1}{r}[g(r)]^{1/n}.$$

To prove (4.5) and (4.6), we first use Lemma 4.3 to obtain an estimate for derivatives of g. Notice that Lemma 4.3 doesn't depend on α.

$$\begin{aligned}
|g^{(\ell)}(x)| &\leqq |D_x^{\ell-1}[x2^{n-1}f(x)]| \\
&\leqq \sum_{\nu=0}^{\ell-1} c_\nu |f^{(\ell-1-\nu)}(x) \cdot D_x^\nu(x^{2n-1})| \\
&\leqq \sum_{\nu=0}^{2n-1} c'_\nu \cdot f^{1-(\ell-1-\nu)/j} \cdot x^{2n-1-\nu} \\
&\leqq x^{2n-1} \cdot f^{1-(\ell-1)/j} \cdot \left[\sum_{\nu=0}^{2n-1} c'_\nu \cdot x^{-\nu} f^{\nu/j}\right] \\
&\leqq \text{const} \cdot g'(x) \cdot [f(x)]^{(1-\ell)/j}
\end{aligned} \tag{4.7}$$

for all $1 \leqq \ell \leqq k+1$, where c_ν and c'_ν are constants. The last inequality follows from the fact that $f^{1/j}=o(x)$. More specifically, for $\ell=2$ we have

$$|g''(x)| \leqq \text{const} \cdot g' \cdot f^{(1-\ell)/j} \leqq \text{const} \cdot x^{(2n-1)/j}(g')^{1-(1/j)}. \tag{4.8}$$

Now by (4.7) and Corollary 4.2, for $m \leqq k+1$,

$$|D_x^m(g^{1/n})| \leqq \sum_\nu \frac{|P_\nu(g',\ldots,g^{(m+1-\lambda_\nu)})|}{g^{\lambda_\nu-(1/n)}} \leqq \sum_\nu c_\nu \cdot \frac{(g')^{\lambda_\nu} \cdot f^{(\lambda_\nu-m)/j}}{g^{\lambda_\nu-(1/n)}},$$

where $\deg P_\nu = \lambda_\nu \leqq m$, wt. deg. $P_\nu = m$, and c_ν are constants. But by Lemma 4.4, (4.8), and the fact that for $\beta > 0$, $\sup_{t \leqq x} |t^\beta f(t)| \leqq x^\beta \cdot \sup_{t \leqq x} |f(t)|$, we have

$$\frac{g'(x)}{g(x)} \leqq \frac{\sup_{t \leqq x} |g''(t)|}{\sup_{t \leqq x} |g'(t)|} \leqq c \cdot x^{(2n-1)/j} \left[\sup_{t \leqq x} |g'(t)|\right]^{-1/j}.$$

Therefore, letting $\beta = (\lambda/j) - (1/nj)$,

$$\begin{aligned} \frac{(g')^\lambda \cdot f^{(\lambda-m)/j}}{g^{\lambda-(1/n)}} &\leqq c \cdot x^{\beta(2n-1)} \left[\sup_{t \leqq x} |g'(t)|\right]^{-\beta} \cdot (g')^{1/n} \cdot f^{(\lambda-m)/j} \\ &\leqq cx \cdot \left[\sup_{t \leqq x} |f(t)|\right]^{(1/n)+(1/nj)-(m/j)} \\ &\leqq \text{const} \cdot x(x-\alpha)^{k+1-m+(1/n)}. \end{aligned}$$

The last inequality comes from the fact that $f(x+\alpha) = o(x^j)$ and $j \geqq n(k+1)$. Hence,

$$|D_x^m(g^{1/n})| \leqq \text{const} \cdot x(x-\alpha)^{k+1-m+(1/n)}.$$

This gives (4.5) and (4.6), and finishes the proof of the theorem.

5. Proof of Theorem 4

The key element in the proof is a lemma of Boas and Polya [3], which has sometimes been referred to as Hadamard's Inequality (see also [8]).[2]

Lemma 5.1 (Hadamard's Inequality). *Let $f \in C^k[0,1]$. Define*

$$\mathfrak{M}_n(f) = \sup_{0 \leqq x \leqq 1} |f^{(n)}(x)|$$

$$\mathfrak{M}_n^*(f) = \max\{\mathfrak{M}_n, n! \cdot \mathfrak{M}_0\}$$

Then for $0 < m < k$,

$$\mathfrak{M}_m \leqq c(m,k) \cdot \mathfrak{M}_0^{1-(m/k)} \cdot \mathfrak{M}_k^{*\,m/k}$$

where $c(m,k)$ is a constant depending on the integers m and k.

Corollary 5.2. *Let $f \in C^k[0,1]$, $f \geqq 0$, $f' \geqq 0$, and $f(x) = o(x^k)$. Then for $0 < m \leqq k$, $f^{(m)}(x) = o(f^{1-(m/k)})$.*

Proof. The case $m = k$ follows from Taylor's formula. For $m < k$, we let $f(x) = 0$ for $x < 0$, and define for fixed $0 \leqq x \leqq 1$,

$$f_x(t) = f(t + x - 1).$$

So, $f_x \in C^k[0,1]$ and $f_x' \geqq 0$. By the previous lemma,

$$|f^{(m)}(x)| \leqq \mathfrak{M}_m(f_x) \leqq c(m,k) \cdot [f(x)]^{1-(m/k)} \cdot [\mathfrak{M}_k^*(f_x)]^{m/k}.$$

[2] The author wishes to thank J. Mather for pointing out Hadamard's inequality and its proof by Sergeraert [8]

Since $f(x)=o(x^k)$, $\mathfrak{M}_k^*(f_x)\to 0$ as $x\to 0$. Hence,

$$\frac{f^{(m)}(x)}{[f(x)]^{1-(m/k)}}\to 0 \quad \text{as} \quad x\to 0.$$

Proposition 5.3. *Let* $f\in C^k[0,1]$, $f\geqq 0$, $f'\geqq 0$, *and* $f(x)=o(x^k)$. *Then* $f^{1/n}\leqq C^p[0,1]$ *where* $p\in\mathbb{Z}^+$ *such that* $(k/n)-1<p\leqq k/n$.

Proof. Follows from Corollaries 4.2 and 5.2 and the fact that $f(x)=o(x^k)$.

Proof of Theorem 4. By Proposition 5.3,

$$\phi(r)=\left[\int_0^r x^{2n-1}f(x)\,dx\right]^{1/n}\in C^p[0,1]$$

where $p\in\mathbb{Z}$ and $k+1/n-1<p\leqq k+1/n$. Also, $\phi(r)=o(r^p)$. So by Taylor's formula, $\phi^{(m)}(r)=o(r^{p-m})$. Hence,

$$u'(r)=\text{const}\cdot\frac{1}{r}\phi(r)\in C^{p-1}[0,1]$$

with all derivatives up to order $p-1$ vanishing at zero. Hence, $u(z)\in C^p(\bar{B})$.

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Einfache holomorphe Abbildungen

Guntram Bohnhorst

Mathematisches Institut, Universität, Postfach 4469, D-4500 Osnabrück, Bundesrepublik Deutschland

Einleitung

In [Ho, 1] hat Holmann eine Klasse holomorpher Blätterungen mit Singularitäten eingeführt und studiert (vgl. auch [Ho, 2]). Die lokalen Bausteine dieser Blätterungen sind offene, einfache holomorphe Abbildungen. Eine Abbildung $f: X \to Y$ heißt dabei einfach in $x \in X$, wenn x eine Umgebungsbasis $\mathscr{U}$ besitzt, so daß für jedes $U \in \mathscr{U}$ alle Fasern von $f|U$ zusammenhängend sind; f heißt einfach (lokaleinfach in [Ho, 1], [Re]), wenn f in jedem Punkt $x \in X$ einfach ist.

Zum Problem der Charakterisierung einfacher holomorpher Abbildungen wurde in [M–M] im Fall $\dim X = 2$ und in [Re] gezeigt, daß eine holomorphe Funktion $f: X \to \mathbb{C}$ auf einer Mannigfaltigkeit genau dann in $x \in X$ einfach ist, wenn die Exponenten $v_1, \ldots, v_r$ in der Primfaktorzerlegung von $f_x - f(x) \in \mathcal{O}_{X,x}$ teilerfremd sind. Insbesondere ist f genau dann einfach, wenn f reduziert ist, d.h. wenn alle Fasern $X_t = f^{-1}(t)$ reduziert sind. Diese Aussage läßt sich auf den Fall holomorpher Abbildungen verallgemeinern. Es gilt:

Theorem. *Es seien X ein reduzierter, lokal vollständiger Durchschnitt, Y eine komplexe Mannigfaltigkeit und $f: X \to Y$ eine offene holomorphe Abbildung. Dann ist f genau dann einfach, wenn f reduziert ist.*

Schumacher hat gezeigt, daß einfache Abbildungen wie im Theorem reduziert sind [Sch]. Wir zeigen in dieser Arbeit, daß auch die Umkehrung richtig ist. Wir merken an, daß sich die Reduziertheit von f in der Situation des Satzes auf einfache Weise prüfen läßt (2.3).

Eine allgemeinere punktale Aussage analog zum Satz von Reiffen-Mattei-Moussu läßt sich zwar formulieren, ist aber für Abbildungen i.a. falsch. Wir geben dazu ein Gegenbeispiel an, daß auch bisherige Vorstellungen über die Offenheit von Niveaumengenrelationen korrigiert.

Bei der vorliegenden Arbeit handelt es sich um den ersten Teil von [Bo, 2].

1

Wir beginnen mit einigen Vorbereitungen.

X sei ein komplexer Raum.

$S(X):=\{x\in X|X$ ist nahe x keine Mannigfaltigkeit$\}$ ist eine analytische Teilmenge von X, die Singularitätenmenge von X.

1.1. Definition. X heißt generisch reduziert, falls $S(X)$ nirgends dicht ist in X.

Diese Eigenschaft kann auch algebraisch beschrieben werden. Dazu sei A ein kommutativer noetherscher Ring mit 1.

1.2. Definition. A heißt generisch reduziert, wenn in einer Primärzerlegung des Nullideals von A alle zu isolierten Primidealen gehörigen Primärideale bereits prim sind.

1.3. Satz. *Folgende Aussagen sind äquivalent:*

(1) *X ist generisch reduziert.*

(2) *$\mathcal{O}_{X,x}$ ist generisch reduziert $\forall x\in X$.*

(3) *Auf jeder irreduziblen Komponenten X' von X_{red} liegt ein Punkt x, in dem $\mathcal{O}_{X,x}$ generisch reduziert ist.*

Beweis. (1) $\Rightarrow$ (2): Wir zeigen die Behauptung für einen Punkt $a\in X$ und dürfen nach event. Verkleinern von X von folgenden Gegebenheiten ausgehen:

$(0)=\bigcap_{i=1}^{s}\mathfrak{q}_i$ ist eine Primärzerlegung von (0) in $A=\mathcal{O}_{X,a}$; $\mathfrak{p}_i:=r(\mathfrak{q}_i)$.

$\mathfrak{p}_1,\dots,\mathfrak{p}_r$ sind isoliert, $\mathfrak{p}_{r+1},\dots,\mathfrak{p}_s$ sind eingebettet. $\mathcal{P}^i$, $\mathcal{Q}^i$ sind kohärente Idealgarben auf X mit $\mathcal{P}^i_a=\mathfrak{p}_i$, $\mathcal{Q}^i_a=\mathfrak{q}_i$, $\mathcal{P}^i=r(\mathcal{Q}^i)$, $\mathfrak{n}_X=\bigcap_{i=1}^{r}\mathcal{P}^i$; die Nullstellenmengen $X_i=\mathcal{N}(\mathcal{P}^i)$, $1\leqq i\leqq r$, sind die irreduziblen Komponenten von X_{red}.

Es sei nun $i\in\{1,\dots,r\}$.

Nach Voraussetzung ist $S:=\mathrm{Tr}(\mathcal{P}^i/\mathcal{Q}^i)=\mathcal{N}(\mathcal{A}n\,\mathcal{P}^i/\mathcal{Q}^i)$ eine nirgends dichte analytische Teilmenge von X_i. Für die Idealgarbe $\mathcal{J}$ von S gilt daher: $\mathcal{P}^i\subset\mathcal{J}$, $\mathcal{P}^i_x\neq\mathcal{J}_x\;\forall x\in X_i$.

Sei $k\in\mathbb{N}$ mit $\mathfrak{a}:=\mathcal{J}^k_a\subset An\,\mathfrak{p}_i/\mathfrak{q}_i$.

Wir wählen $f_i\in\mathfrak{a}\backslash\mathfrak{p}_i$ und $f_j\in\mathfrak{q}_j\backslash\mathfrak{p}_i$ für $j\neq i$. Für $f:=f_1\cdot\ldots\cdot f_s$ gilt dann: $f\cdot\mathfrak{p}_i\subset\bigcap_{j=1}^{s}\mathfrak{q}_j=(0)$.

$\Rightarrow\mathfrak{q}_i=\mathfrak{p}_i$ (vgl. z.B. [A–M]).

(3) $\Rightarrow$ (1): Wir haben zu zeigen, daß für jede irreduzible Komponente X' von X_{red} gilt: $X'\not\subset S(X)$. Wir wählen einen Punkt $a\in X'$ gemäß (3) und dürfen dann nahe a von der obigen Situation sowie $X'=X_1$ und $\mathcal{P}^i=\mathcal{Q}^i$ für $i=1,\dots,r$ ausgehen. Für $x\in X_1\backslash\left(\bigcup_{j=1}^{s}\mathcal{N}(\mathcal{P}^j)\right)$ gilt dann:

$\mathfrak{n}_{X,x}=\bigcap_{i=1}^{r}\mathcal{P}^i_x=\mathcal{P}^1_x=\bigcap_{j=1}^{s}\mathcal{Q}^j_x=(0)$. Ist zusätzlich $x\notin S(X_{\mathrm{red}})$, so ist $x\notin S(X)$. –

1.4. Korollar. *X sei generisch reduziert und Cohen-Macaulaysch. Dann ist X reduziert.*

Diese Aussage wurde mit anderen Methoden in [Sch] bewiesen.

2

Von nun an seien X ein reindimensionaler reduzierter komplexer Raum, Y eine komplexe Mannigfaltigkeit und $f: X\to Y$ eine offene holomorphe Abbildung.

X, Y seien zusammenhängend, $n := \dim X$, $q := \dim Y$, $p := n - q$. Die Offenheit von f bedeutet, daß alle Fasern $X_t = f^{-1}(t)$, $t \in f(X)$, rein p-dimensional sind. Im weiteren fassen wir die Fasern holomorpher Abbildungen stets als komplexe Räume auf, versehen mit der durch die Abbildung induzierten komplexen Struktur.

2.1. Definition. f heißt (generisch) reduziert, falls alle Fasern X_t (generisch) reduziert sind.

Mit $S(f)$ bezeichnen wir die Menge der Punkte von X, in denen f keine Mersion ist. $S(f)$ ist eine analytische Teilmenge von X (vgl. [Fi]). Auf $X \backslash S(X)$ wird $S(f)$ durch die q-Minoren der Jacobimatrix von f beschrieben. Aus [Fi, S. 102], folgt

2.2. Lemma. $S(f) = \bigcup\limits_{t \in Y} S(X_t)$.

Als Folgerung erhalten wir im Zusammenhang mit 1.4

2.3. Satz. *Äquivalent:*

(1) *f ist generisch reduziert.*

(2) $\dim S(f) \cap X_t < p \; \forall t \in Y$.

Ist X Cohen-Macaulaysch, so sind (1), (2) *äquivalent zu*

(3) *f ist reduziert.*

Beim Beweis von (1) $\Leftrightarrow$ (3) beachte man, daß auch die Fasern X_t wieder Cohen-Macaulaysch sind. –

Insbesondere ist f auf einer Umgebung von x generisch reduziert, falls $\mathcal{O}_{X_t, x}$, $t = f(x)$, generisch reduziert ist.

3

Das wesentliche Hilfsmittel beim Beweis des Theorems ist eine verallgemeinerte Version des Milnorschen Faserungssatzes, die wir nun herleiten. Wir benötigen dazu den

3.1. Faserungssatz von Ehresmann. *M, N seien C^∞-Mannigfaltigkeiten. M besitze einen Rand ∂M, ∂N sei leer und N sei zusammenhängend.*

$\varphi : M \to N$ sei eine eigentliche C^∞-Submersion, derart daß auch $\varphi | \partial M$ eine Submersion ist. Dann ist φ ein C^∞-Faserbündel. Insbesondere ist $\varphi | M \backslash \partial M$ ein C^∞-Faserbündel.

Man beachte, daß $\varphi | \partial M$ genau dann eine Submersion ist, falls die Fasern von φ den Rand ∂M transversal schneiden.

Im folgenden seien X ein Gebiet im $\mathbb{C}^n$ mit $0 \in X$ und $Y = \mathbb{C}^q$. Es gelte $f(0) = 0$. Für $\varepsilon, \delta \in \mathbb{R}_+^*$ setzen wir

$$B_\varepsilon := \{z \in \mathbb{C}^n \mid |z| < \varepsilon\}, \qquad S_\varepsilon := \{z \in \mathbb{C}^n \mid |z| = \varepsilon\},$$
$$P = P_\delta := \{w \in \mathbb{C}^q \mid |w_i| < \delta, i = 1, \ldots, q\}.$$

Ist $\varphi : M \to N$ eine C^∞-Abbildung zwischen C^∞-Mannigfaltigkeiten, so bestehe $S(\varphi)$ aus den Punkten von M, in denen $\operatorname{rg} d_x \varphi$ nicht lokal konstant ist.

3.2. Satz. *Sind $\varepsilon \leqq \varepsilon_0$ und $\delta \leqq \delta_0(\varepsilon)$ hinreichend klein, so existiert eine abgeschlossene, nirgends dichte, subanalytische Menge Δ in P, so daß für $G := f^{-1}(P_\delta) \cap B_\varepsilon$ und $\Sigma := f^{-1}(\Delta) \cap B_\varepsilon$ gilt:*

$$f : G \backslash \Sigma \longrightarrow P \backslash \Delta$$

ist über jeder der endlich vielen Zusammenhangskomponenten $P_1, \ldots, P_r$ von $P \backslash \Delta$ ein C^∞-Faserbündel.

Beweis. Wir wählen ε, δ so, daß gilt $\bar{B}_\varepsilon \subset X$, $P_\delta \subset f(B_\varepsilon)$ und setzen $\Delta := f(S(f|G) \cup S(f|D))$, $D := f^{-1}(P) \cap S_\varepsilon$. Dann ist Δ subanalytisch [Hi, Ha] und abgeschlossen in P. Nach dem Satz von Sard ist Δ nirgends dicht. Die Behauptung folgt nun, indem man 3.1 auf $M = f^{-1}(P_i) \cap \bar{B}_\varepsilon$, $\partial M = f^{-1}(P_i) \cap S_\varepsilon$, $N = P_i$ und $\varphi = f|M$ anwendet. –

Ist $q = 1$, so zeigen zwei Zusatzüberlegungen, daß bei geeigneten ε, δ $\Delta \subset \{0\}$ gilt [Mi, H]. Man erhält auf diese Weise den klassischen Milnorschen Faserungssatz. Insbesondere ist dann $r = 1$.

Ist $q \geqq 2$, so ist i.a. $r \geqq 2$. Dies zeigt das folgende wohlbekannte

3.3. Beispiel. $f : \mathbb{C}^3 \to \mathbb{C}^2 := (x, z^2 - xy)$.

$S(f) = f^{-1}(0)$ ist die Koordinatenachse $\mathbb{C}_y$. f ist offen, aber nicht reduziert.

Ist $\varepsilon \in \mathbb{R}^*_+$, so sind die Fasern $f^{-1}(s, t) \cap B_\varepsilon$ für kleine $s, t \in \mathbb{C}^*$ zusammenhängend, falls $\frac{t}{s^2}$ klein ist, und nicht zusammenhängend, falls $\frac{t}{s}$ groß ist. –

Insbesondere ist hier Δ reell 1-codimensional. Hierfür ist stets die Menge $f(S(f|D))$, nicht aber $f(S(f))$ verantwortlich. Wir zeigen an einem weiteren Beispiel, daß auch bei reduzierten Abbildungen Δ i.a. reell 1-codimensional ist.

3.4. Beispiel. $f : \mathbb{C}^4 \to \mathbb{C}^2 = (xy, x^2 + z^2 + w^2)$.

$S(f) = \mathbb{C}^2_{z,w} \cup \mathbb{C}_y$. f ist offen und reduziert.

Sei $\varepsilon > 0$. $R_\varepsilon(f) := S(f|S_\varepsilon)$ enthält die Menge $\mathbb{C}^2_{x,y} \cap S_\varepsilon$. $\Rightarrow \dim_{\mathbb{R}} f(R_\varepsilon(f)) \cap U = 3$ für jede Umgebung U von 0. –

Die Situation ändert sich bei reduzierten Abbildungen mit 1-dimensionalen Fasern.

3.5. Satz. *f sei reduziert und es gelte $p = 1$, d.h. $q = n - 1$. Dann gilt in* 3.2 *bei hinreichend kleinen ε, δ:*

(1) *Die Fasern X_t mit $t \in P_\delta$ schneiden S_ε transversal.*

(2) *Δ ist eine (komplex-) analytische Teilmenge von P_δ. Insbesondere ist $P \backslash \Delta$ zusammenhängend.*

Beweis. Zunächst kann ε so gewählt werden, daß $X_0 = f^{-1}(0)$ keinen von 0 verschieden singulären Punkt in $\bar{B}_\varepsilon$ besitzt und S_ε transversal schneidet (vgl. z.B. [Lo, S. 103]). Da f in jedem Punkt $x \in X_0 \cap S_\varepsilon$ regulär ist (2.2), gilt dies auch für alle Nachbarfasern. Weil $f|S(f)$ diskret ist, folgt dann auch (2). –

4

Wir beweisen nun die Rückrichtung des Theorems. $f : X \to Y$ sei also eine offene, reduzierte holomorphe Abbildung auf einem lokal vollständigen Durchschnitt X

in eine Mannigfaltigkeit Y. Wir zeigen, daß f in einem vorgegebenen Punkt $a \in X$ einfach ist. Dabei dürfen wir X, Y verkleinern und insbesondere $Y = \mathbb{C}^q$ sowie $f(a) = 0$ annehmen.

4.1. Es genügt, die Behauptung für Mannigfaltigkeiten X zu zeigen.

Beweis. O.E. existieren ein Gebiet $G \subset \mathbb{C}^N$ und eine offene holomorphe Abbildung $\varphi: G \to \mathbb{C}^m$ mit $X = \varphi^{-1}(0)$, und es gilt:

f läßt sich fortsetzen zu einer holomorphen Abbildung $\tilde{f}: G \to \mathbb{C}^q$;

$\Phi := (\tilde{f}, \varphi): G \to \mathbb{C}^{q+m}$ ist offen.

Nach Konstruktion ist $X_t = \Phi^{-1}(t, 0)$. Daher ist Φ nahe a reduziert (2.3) und somit nach Voraussetzung einfach in a. Dann ist auch f einfach in a. –

Der Beweis zeigt, daß wir für die Rückrichtung des Theorems die Reduziertheit von X nicht benötigen. In dem hier betrachteten Fall folgt jedoch die Reduziertheit von X aus der Reduziertheit von f (vgl. [Fi, S. 158]).

Im weiteren dürfen wir also annehmen, daß X ein Gebiet im $\mathbb{C}^n$ ist und $a = 0$ gilt. Da der Fall $p = 0$ trivial ist, sei ferner $p \geqq 1$.

4.2. Die Behauptung gilt im Fall $p = 1$.

Beweis. Wir wenden 3.5/3.2 an und haben dann für geeignete $\varepsilon, \delta > 0$:

(a) $\bar{B}_\varepsilon \subset X$.

(b) $X_0 \cap B_\varepsilon$ ist zusammenhängend (vgl. z.B. [Re]).

(c) Alle Fasern X_t, $t \in P_\delta$, schneiden S_ε transversal.

(d) $\exists$ analytische Menge $\Delta \subsetneqq P = P_\delta$, so daß

$$G \backslash \Sigma \xrightarrow{f} P \backslash \Delta$$

ein C^∞-Faserbündel mit typischer Faser F ist.

$$G = f^{-1}(P_\delta) \cap B_\varepsilon; \qquad \Sigma = f^{-1}(\Delta) \cap B_\varepsilon.$$

Wir setzen $\tilde{f} := G \xrightarrow{f} P$.

(1) G und $G \backslash \Sigma$ sind zusammenhängend.

Beweis. Jede Zusammenhangskomponente von $G \backslash \Sigma$ häuft sich wegen (d), (c) gegen $X_0 \cap B_\varepsilon$. Daher ist G und damit auch $G \backslash \Sigma$ zusammenhängend. –

Wir wählen einen regulären Punkt b aus $X_0 \cap B_\varepsilon$ und eine zu X_0 transversal verlaufende $(n-1)$-dimensionale komplexe Mannigfaltigkeit $T \subset B_\varepsilon$ durch b. O.E. ist $f: T \to P$ biholomorph.

Für $z \in G$ sei N_z die Zusammenhangskomponente von $\tilde{f}^{-1}\tilde{f}(z)$ mit $z \in N_z$; $\tilde{G} := \cup \{N_z | z \in T \backslash \Sigma\}$.

Aus (d) folgt

(2) $\tilde{G}$ ist offen und abgeschlossen in $G \backslash \Sigma$.

Ergebnis: $\tilde{G} = G \backslash \Sigma$. Jede Faser $\tilde{f}^{-1}(t)$ mit $t \in P \backslash \Sigma$ ist zusammenhängend.

(3) Alle Fasern $\tilde{f}^{-1}(t)$ mit $t \in \Delta$ sind zusammenhängend.

Beweis. Wir nehmen an, eine Faser $\tilde{f}^{-1}(t)$, $t \in \Delta$, zerfällt in Zusammenhangskomponenten $L_1, \ldots, L_r$, $r \geqq 2$. Wegen (c) gilt zunächst: $d(L_i, L_j) > 0$ für $i \neq j$; $d =$ euklidischer Abstand. Daher zerfallen auch alle Nachbarfasern in mindestens r Zusammenhangskomponenten. Widerspruch. –

Ist $p \geqq 2$, so versagt der Beweis, wie das Beispiel 3.4 zeigt. Dieser Fall kann jedoch auf den Fall $p=1$ zurückgeführt werden. Dazu benötigen wir den folgenden Hilfssatz aus [Sc–S]. Einen einfachen geometrischen Beweis findet man in [Bo, 2].

4.3. Lemma. $f:(\mathbb{C}^n,0)\to(\mathbb{C}^q,0)$ *sei ein offener, reduzierter holomorpher Abbildungskeim. Dann existiert ein holomorpher Abbildungskeim* $\varphi:(\mathbb{C}^n,0)\to(\mathbb{C}^{p-1},0)$, *so daß* $(f,\varphi):(\mathbb{C}^n,0)\to(\mathbb{C}^{n-1},0)$ *offen und reduziert ist.*

Abschluß des Beweises. Es sei $p \geqq 2$. Aufgrund von 4.3 dürfen wir annehmen, daß eine holomorphe Abbildung $\varphi: X\to\mathbb{C}^{p-1}$ existiert, derart daß $\Phi=(f,\varphi): X \to\mathbb{C}^{n-1}$ offen und reduziert ist. O.E. sind alle Fasern von Φ zusammenhängend (4.2) und $\Phi(X)=P\times Q$ mit zwei Polyzylindern $P\subset\mathbb{C}^q$, $Q\subset\mathbb{C}^{p-1}$. Dann ist für jedes $t\in P$ $\varphi_t: X_t \xrightarrow{\varphi} Q$ offen, surjektiv und hat zusammenhängende Fasern. Hieraus folgt sofort: X_t ist zusammenhängend. –

5

Abschließend betrachten wir noch einmal eine offene holomorphe Abbildung $f: X\to Y$ zwischen Mannigfaltigkeiten X, Y. Einige der folgenden Aussagen gelten auch in allgemeineren Situationen. n, p, q, X_t sind wie in 2–4; $\hat{X}_t := (X_t)_{\mathrm{red}}$. Für offene Teilmengen U von X sind U_t, $\hat{U}_t$ entsprechend definiert. Wir erinnern zunächst an eine bekannte Konstruktion.

5.1. Es sei $a\in X_t$.

(1) Es existieren eine zusammenhängende Umgebung U von a und eine Submersion $\pi: U\to P=\pi(U)\subset\mathbb{C}^p$ mit $X_t\cap\pi^{-1}\pi(a)=\{a\}$.

O.E. ist $(f,\pi): U\to V\times P$, $V:=f(U)$, eine verzweigte Überlagerung. Insbesondere ist $\pi_\tau: \hat{U}_\tau\to P$ für alle $\tau\in V$ eine verzweigte Überlagerung.

(2) Ist $a\notin S(\hat{X}_t)$, so kann π so gewählt werden, daß $\pi_t: \hat{U}_t\to P$ biholomorph ist.

f induziert auf jeder irreduziblen Komponenten F von $\hat{X}_t$ eine Multiplizität $\nu_f(F)$, die wie folgt beschrieben werden kann (vgl. [Tu, Sto]).

5.2. Wir wählen $a\in F\setminus S(\hat{X}_t)$ und $\pi: U\to P$ gemäß 5.1. s bzw. s_τ sei die Blätterzahl von (f,π) bzw. π_τ. Dann gilt:

$$\nu_f(F)=s=\limsup_{\tau\to t} s_\tau .$$

I.a. sind diese Multiplizitäten von den in [Sch] definierten Multiplizitäten verschieden.

f ist genau dann in $x\in F\setminus S(\hat{X}_t)$ regulär, wenn $\nu_f(F)=1$ gilt.

Ist $q=1$ und $f=f_1^{\nu_1}\dots f_r^{\nu_r}$ die Primfaktorzerlegung von $\tilde{f}=f_a-f(a)\in\mathcal{O}_{X,a}$, so entsprechen $f_1,\dots,f_r$ eindeutig den irreduziblen Komponenten $F_1,\dots,F_r$ des analytischen Mengenkeims $(\hat{X}_t,a)$, $t=f(a)$, und es ist $\nu_i=\nu_f(F_i)$.

Die hier auftauchende Frage, ob auch im Fall $q\geqq 2$ die Teilerfremdheit der Multiplizitäten $\nu_f(F_1),\dots,\nu_f(F_r)$ der irreduziblen Komponenten von $(\hat{X}_t,a)$ die Einfachheit von f in a impliziert, muß verneint werden.

5.3. Beispiel. $f:\mathbb{C}^3 \to \mathbb{C}^2 := (xy, (x+y)z)$.

f ist offen, aber nicht reduziert. $f^{-1}(0)$ besteht aus drei irreduziblen Komponenten, den Koordinatenachsen $\mathbb{C}_x$, $\mathbb{C}_y$, $\mathbb{C}_z$. Die Multiplizitäten sind 1, 1, 2.

f ist nicht einfach in 0, denn für $t \in \mathbb{C}^*$ besteht $f^{-1}(0,t)$ aus zwei Zusammenhangskomponenten $A_t \subset \mathbb{C}^2_{x,z}$ und $B_t \subset \mathbb{C}^2_{y,z}$. –

Eng verknüpft hiermit ist die Frage nach der Offenheit der Niveaumengenrelation N_f von f. Dabei versteht man unter den Niveaumengen von f die Zusammenhangskomponenten der Fasern von f, und N_f ist die durch die Niveaumengen induzierte Äquivalenzrelation auf X.

5.4. Äquivalent:

(1) N_f ist eine offene Äquivalenzrelation.

(2) Für jede Folge (L_ν) von N_f-saturierten Teilmengen von X ist $\lim L_\nu$ N_f-saturiert.

(3) Sind L_ν, L Niveaumengen von f mit $L \cap \lim L_\nu \neq \emptyset$, so ist $L \subset \lim L_\nu$.

Dazu vgl. man [Ka]. –

5.5. Ist $N_{f|U}$ offen für beliebig kleine Umgebungen U von a, so zeigt eine einfache Überlegung, daß f bereits dann einfach in a ist, falls für eine der obigen Multiplizitäten $\nu_f(F_i) = 1$ gilt. –

In [St] und [Ku] wurde behauptet, daß für offene holomorphe Abbildungen f zwischen Mannigfaltigkeiten N_f stets offen ist. Dieser "Niveaumengensatz" ist falsch, wie das Beispiel 5.3 zeigt. Es ist etwa

$$\lim_{t \to 0} A_t = \mathbb{C}_x \cup \mathbb{C}_z \subsetneqq f^{-1}(0)\,. -$$

Richtig ist hier lediglich die folgende Aussage (vgl. auch [St]).

5.6. Satz. *Für jede Folge* (L_ν) *von Niveaumengen von* f *mit* $f(L_\nu) \to t$ *ist* $\lim L_\nu$ *eine Vereinigung von irreduziblen Komponenten von* $\hat{X}_t$. *Insbesondere gilt* 5.4 (3) *für irreduzible Niveaumengen* L.

Beweis. $L := \lim L_\nu$, $S := S(\hat{X}_t)$. 5.1 zeigt

(1) $a \in L \Rightarrow \exists$ irreduzible Komponente F von $\hat{X}_t$ durch a mit $(L \cap F) \setminus S \neq \emptyset$.

(2) $L \setminus S$ ist Vereinigung von Zusammenhangskomponenten von $X_t \setminus S$.

Da L überdies abgeschlossen ist, folgt hieraus die Behauptung. –

Aus 5.6 und 5.5 folgt sofort unser Einfachheitssatz für Abbildungen $f: X \to Y$ mit $\dim S(f) \cap X_t \leqq p-2\ \forall t \in Y$, da in diesem Fall alle Fasern normal sind (vgl. [Ab, 45.16]).

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Mathematische Annalen

On Faithful Representations of the Holomorph of Lie Groups

Dong Hoon Lee and Ta-Sun Wu

Department of Mathematics and Statistics, Case Western Reserve University, University Circle, Cleveland, OH 44106, USA

1. Introduction

Let G be a Lie group and let $\mathrm{Aut}(G)$ denote the group of all automorphisms of the Lie group G, endowed with its natural structure of a Lie group. The natural semidirect product $G \rtimes \mathrm{Aut}(G)$ is called the *holomorph* of G. In [2], Hochschild has shown that if the nilradical (that is, the maximum nilpotent analytic normal subgroup) N of a real or complex analytic group G is simply connected, then the holomorph of G is faithfully representable. Moreover, one of the authors in [4] has given an intrinsic characterization of those complex analytic groups whose holomorph is faithfully representable. In this paper, we are interested in determining when the holomorph of a Lie group G is faithfully representable (that is, admits a faithful finite-dimensional continuous representation). Our first result (Theorem 1) answers this question when G is an f.c.c. Lie group. Also in this work, we extend the result of [4] to real analytic groups (Theorem 2). Our method is entirely different from that of [4] and may be adapted to cover the complex case as well.

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2. Preliminaries

Let G be a Lie group. We denote by G_0 the connected component of the identity element of G. We call a Lie group G an f.c.c. group if G/G_0 is finite. If G is an f.c.c. Lie group, then G contains a finite subgroup D such that $G = G_0 D$ ([3], Theorem 2.34). The Lie algebra of G (or rather G_0) is denoted by $\mathscr{L}(G)$. We denote by $\mathrm{Aut}(G)$ the group of all automorphisms of the Lie group G. $\mathrm{Aut}(G)$ itself becomes a Lie group in a natural manner, and its identity component is denoted by $\mathrm{Aut}_0(G)$ instead of $(\mathrm{Aut}(G))_0$. By a representation of G, we shall always mean a continuous finite-dimensional representation of G [that is, a continuous homomorphism of G into the full linear group $GL(V)$ of a finite-dimensional real linear space V].

If G is a subgroup of $GL(V)$, we denote by G^* the topologically connected component of the identity of the Zariski closure of G (i.e. the smallest algebraic group containing G). By the well known theorem of Whitney, G^* is of finite index in the Zariski closure of G.

For subgroups A and B of a group G, let $[A, B]$ denote the subgroup of G that is generated by the commutators $[a, b] = aba^{-1}b^{-1}$ with $a \in A$ and $b \in B$.

Definition. Let G be a Lie group and H a closed subgroup of G. We say that a representation σ of G is an extension of the representation ϱ of H if the representation space V of σ contains the representation space of ϱ as an H-stable subspace and σ coincides with ϱ on H.

We shall say that a Lie group G is *reductive* if G has a faithful representation and every representation of G is completely reducible. By a *nucleus* of a Lie group G, we mean a closed, simply connected, normal, solvable subgroup E of G such that G/E is reductive. If G is a faithfully representable analytic group (or more generally, f.c.c. Lie group), then G has a nucleus. Moreover, if E is any nucleus, then G is a semidirect product $G = EM$, where M is a closed subgroup (and, in fact, a maximal reductive subgroup) of G. (See [1, Chap. XVIII].) We also note that if N is the radical of the commutator subgroup of G, then by Lie's theorem every (analytic) representation of G is unipotent on N. Hence N is contained in every nucleus of G.

In later sections, we will refer to the following lemma and its corollary several times.

Lemma 1. *Let G be a Lie group and N a closed normal subgroup of G. If N admits a faithful representation σ which can be extended to a representation of G, and if the quotient group G/N is faithfully representable, then G is faithfully representable.*

Proof. Let ϱ be an extension of σ to G, and let ϕ be the representation of G which is the composition of the canonical map $G \to G/N$ and a faithful representation of G/N. Then the direct sum of ϱ and ϕ is a desired faithful representation of G.

Corollary. *Let H be a closed normal subgroup of a Lie group G such that G/H is finite. If H is faithfully representable, then so is G.*

Proof. Let ϱ be a faithful representation of H, and let σ denote the induced representation $\mathrm{Ind}_H^G(\varrho)$. Since G/H is a finite group, it is faithfully representable. Our assertion then follows from the lemma above.

3. Proof of Theorem 1

Theorem 1. *Let G be an* f.c.c. *and faithfully representable Lie group. Then $G \rtimes \mathrm{Aut}(G)$ is faithfully representable if and only if the following two conditions are satisfied:*

(i) $\mathrm{Aut}_0(G)$ *is of finite index in* $\mathrm{Aut}(G)$;

(ii) *There exists a nucleus E of G which is stable under* $\mathrm{Aut}_0(G)$.

In this section, we show that the conditions (i) and (ii) are sufficient, and the necessity part will be proven in the next section.

We start with the following lemma.

Lemma 2. *Let G be an* f.c.c. *and faithfully representable Lie group. If* $\mathrm{Aut}_0(G)$ *leaves a nucleus E of G stable, then there exists a faithful representation of G_0 which can be extended to a representation of* $G_0 \rtimes \mathrm{Aut}_0(G)$.

Proof. Let M be a maximal reductive subgroup of G so that we have the semidirect product decompositions $G = EM$ and $G_0 = EM_0$, and let N be the nilradical of E. Then N is an $\mathrm{Aut}_0(G)$-stable normal subgroup of G. By [1, Theorem 3.1, p. 219], there exists a faithful representation ϱ of E that is unipotent on N. The representation ϱ is easily seen to satisfy the condition of [1, Theorem 2.2, p. 215], so that ϱ can be extended to representation ϱ' of G whose restriction to N is still unipotent. Let ξ be the composition of the canonical projection from G_0 to M_0 and a faithful representation of M_0. The direct sum $\sigma = \varrho' \oplus \xi$ is a faithful representation of G_0 which is unipotent on N.

Now we claim that σ is extendable to a representation of $G_0 \rtimes \mathrm{Aut}_0(G)$. Let R denote the radical of G_0. Then $R = E(R \cap M_0)$ (semidirect), and the quotient group R/N is abelian because $[R, G_0] \subset N$. Thus we have the direct product decomposition $R/N = (E/N)C$, where C is the maximum compact subgroup of R/N. For any $x \in G_0$, let $\tilde{x}$ denote the coset $xN \in G_0/N$. The group $\mathrm{Aut}_0(G)$ acts on G_0/N by $\alpha \cdot \tilde{x} = \widetilde{\alpha(x)}$, and we form the semidirect product $(G_0/N) \rtimes \mathrm{Aut}(G)$. Define $\pi: G_0 \rtimes \mathrm{Aut}_0(G) \to (G_0/N) \rtimes \mathrm{Aut}_0(G)$ by $\pi(x, \alpha) = (\tilde{x}, \alpha)$. Let $x \in R$ and $\alpha \in \mathrm{Aut}_0(G)$. Then we have $\pi([(x,1),(1,\alpha)]) = [(\tilde{x},1),(1,\alpha)] = (\tilde{x}\alpha \cdot \tilde{x}^{-1}, 1) = \pi(x\alpha(x^{-1}), 1)$, and since the connected group $\mathrm{Aut}_0(G)$ acts on the torus C trivially, we have $\tilde{x}(\alpha \cdot \tilde{x}^{-1}) = \tilde{e}(\alpha \cdot \tilde{e}^{-1})$, where $\tilde{e}$ is the E/N-component of $\tilde{x}$ in the decomposition $R/N = (E/N)C$ with $e \in E$. Thus $\pi([x,1),(1,\alpha)]) = \pi(e(e^{-1}), 1) \in (E/N) \times 1$.

On the other hand, we know that if δ is any derivation of a solvable Lie algebra L, then $\delta(L)$ is contained in the nilradical of L. From this, we see that $[(x,1),(1,\alpha)] \in N' \times 1$, where N' is the nilradical of R, and hence $\pi([(x,1),(1,\alpha)]) \in (N'/N) \times 1 \subset C \times 1$.

Thus $\pi([x,1),(1,\alpha)]) \in (E/N) \times 1 \cap C \times 1 = (1)$, and $[(x,1),(1,\alpha)] \in \mathrm{Ker}(\pi) = N \times 1$ for all $x \in R$ and $\alpha \in \mathrm{Aut}_0(G)$. Thus by [1, Theorem 2.2, p. 215], the representation σ of G_0 can be extended to a representation $\tilde{\sigma}$ of $G_0 \rtimes \mathrm{Aut}_0(G)$.

Corollary 1. *Let G be as in the lemma, and let A denote the subgroup of* $\mathrm{Aut}(G_0)$ *obtained by restricting the elements of* $\mathrm{Aut}_0(G)$ *to G_0. Then $G_0 \rtimes A$ is faithfully representable.*

Proof. Proceeding as in the proof of Lemma 2 with $\mathrm{Aut}_0(G)$ replaced by A, we obtain a faithful representation σ of G_0 which can be extended to a representation of $G_0 \rtimes A$. The group A is faithfully representable, as we can view A as an automorphism group of the Lie algebra of G. Thus by Lemma 1, $G_0 \rtimes A$ is faithfully representable.

Corollary 2. *Let G and A be as in the above corollary. Then $(G_0 \times \ldots \times G_0) \rtimes A$ is faithfully representable.*

Now we are ready to Prove the sufficiency part of the theorem. Thus assume the conditions (i) and (ii) of Theorem 1. We first show that $\mathrm{Aut}_0(G)$ is faithfully representable. Choose a finite subgroup D of G such that $G = G_0 D$, and let A denote the subgroup of $\mathrm{Aut}(G_0)$ obtained by restricting the elements of $\mathrm{Aut}_0(G)$ to G_0. We define $\phi: \mathrm{Aut}_0(G) \to (G_0 \times \ldots \times G_0) \rtimes A$ by

$$\phi(\alpha) = (d_1 \cdot (d_1^{-1}), \ldots, d_n \cdot (d_n^{-1}), \alpha | G_0),$$

where $D = \{d_1, d_2, \ldots, d_n\}$. It is easy to show that ϕ is an injective morphism. Since $(G_0 \times \ldots \times G_0) \rtimes A$ is faithfully representable by the corollary above, $\mathrm{Aut}_0(G)$ is faithfully representable. By Lemma 2, G_0 admits a faithful representation which can be extended to $G_0 \rtimes \mathrm{Aut}_0(G)$. Since $\mathrm{Aut}_0(G)$ is faithfully representable, $G_0 \rtimes \mathrm{Aut}_0(G)$ is faithfully representable by Lemma 1. Since the normal subgroup $G_0 \rtimes \mathrm{Aut}_0(G)$ is of finite index in $G \rtimes \mathrm{Aut}(G)$, $G \rtimes \mathrm{Aut}(G)$ is faithfully representable by the corollary to Lemma 1.

4. Proof of Theorem 1 (continued)

Lemma 3. *Let H be a reductive* f.c.c. *Lie group contained as a Lie subgroup in a $GL(V)$ (and, consequently, being a closed subgroup of $GL(V)$). If N is a subgroup of $GL(V)$ that normalizes H, then N^* also normalizes H.*

Proof. Since H is reductive, the Lie algebra of H is the direct sum of a semisimple Lie algebra and the Lie algebra of a compact abelian subgroup of the the containing general linear group. It follows from the standard result that therefore the Lie algebra of H coincides with that of the Zariski closure $\bar{H}$ of H, whence $H^* = H_0$. Now N^* acts on H by conjugation, and this action induces an action of N^* on $\bar{H}/H^* = \bar{H}/H_0$. Since $\bar{H}/H^*$ is finite, the connected group N^* acts trivially on $\bar{H}/H_0$ (and hence on H/H_0). This implies that N^* leaves H stable, proving that N^* normalizes H.

Now we are ready to prove the necessity part of the theorem. Thus let ϱ be a faithful representation of $G \rtimes \mathrm{Aut}(G)$. For any subgroup S of G, we identify S $(= S \times 1)$ with its image $\varrho\ (S \times 1)$, and $\mathrm{Aut}(G)$ with $A = \varrho(1 \times Aut(G))$. Then $G \rtimes \mathrm{Aut}(G)$ is identified with $\mathrm{Im}(\varrho) = GA$, and A acts on G by conjugation. Let R denote the radical of G. R is a normal subgroup of GA, and hence R^* is in the radical of $(GA)^*$. Consequently, $[R^*, A^*]$ is in the unipotent radical U of the algebraic closure of GA. In particular, $[R, A^* \cap A]$ is contained in the simply connected nilpotent normal subgroup $(U \cap G)_0$ of G. Let $Q = (U \cap G)_0$. Since $[R, G_0] \subset Q$, R/Q is abelian, and hence we have the direct product decomposition $R/Q = VT$, where T is the maximum torus of R/Q and V is a vector subgroup of R/Q. Let E be the inverse image of V under the canonical map $G \to G/Q$. Clearly E is a simply connected, closed, solvable, normal subgroup of G. Since $(G_0/E)/(R/E) \cong G_0/R$ is semisimple, while $R/E \cong T$ is compact, G_0/E (and hence G/E) is reductive. Thus E is a nucleus of G.

Now we will show that $\mathrm{Aut}_0(G)$ leaves the nucleus E invariant and is of finite index in $\mathrm{Aut}(G)$. For any subgroup S of G, let $\mathrm{Aut}(G, S)$ denote the subgroup of $\mathrm{Aut}(G)$ consisting of all $\alpha \in \mathrm{Aut}(G)$ with $\alpha(S) = S$. Through the identification $\mathrm{Aut}(G) = A$ we made earlier, we have $\mathrm{Aut}(G, S) = A_S$, where A_S denotes the normalizer of S in A. Let $a \in A^* \cap A$ and $x \in R$. Then $[x, a] \in [R, A^* \cap A] \subset (U \cap G)_0$, showing $[x, a] \in Q \subset E$. This shows, in particular, that $aEa^{-1} = E$ for all $a \in A^* \cap A$.

Thus $A^* \cap A \subset A_E$. Since $A^* \cap A$ is of finite index in A, it follows that the index of A_E in A is finite. This shows that $\mathrm{Aut}(G, E)$ is of finite index in $\mathrm{Aut}(G)$, and hence in order to show that $\mathrm{Aut}_0(G)$ is of finite index $\mathrm{Aut}(G)$, it is enough to show that the index of $\mathrm{Aut}_0(G)$ in $\mathrm{Aut}(G, E)$ is finite. Pick a maximal reductive subgroup M of G, and let $\eta: \mathrm{Aut}(G, M) = A_M \to \mathrm{Aut}(M)$ be the restriction map. We first claim that $\mathrm{Im}(\eta)/\mathrm{Int}_M(M_0)$ is finite, where $\mathrm{Int}_H(S)$ for any subgroup S of a group H denotes the group of all inner automorphisms of H that are induced by the elements of S. By Lemma 3, A_M^* leaves M stable. We consider the subgroup $B = A_M^* \cdot A_M$. $B_0 = A_M^*$ is of finite index in B. Extend η to $\eta: B \to \mathrm{Aut}(M)$ canonically. Since M is reductive, $\mathrm{Aut}_0(M) = \mathrm{Int}_M(M_0)$, and thus $\eta^{-1}(\mathrm{Int}_M(M_0))$ is of finite index in B. We have the following isomorphisms of finite groups:

$$\begin{aligned}\eta^{-1}(\mathrm{Int}_M(M_0))A_M/\eta^{-1}(\mathrm{Int}_M(M_0)) &= A_M/(\eta^{-1}(\mathrm{Int}_M(M_0)) \cap A_M)\\ &= A_M/(\eta \mid A_M)^{-1}(\mathrm{Int}_M(M_0)) = \mathrm{Aut}(G, M)/\eta^{-1}(\mathrm{Int}_M(M_0)).\end{aligned}$$

Thus $\eta^{-1}(\mathrm{Int}_M(M_0)) = \mathrm{Int}_M(M_0)\,\mathrm{Ker}(\eta)$ is of finite index in $\mathrm{Aut}(G, M)$, and if $\mathscr{B} = \{\alpha \in \mathrm{Aut}(G, E): \alpha = 1 \text{ on } M\}$, then $\mathrm{Int}_M(M_0)\mathscr{B}$ is of finite index in $\mathrm{Aut}(G, E)$ $\cap \mathrm{Aut}(G, M)$. By conjugacy of maximal reductive subgroups of G, we have $\mathrm{Aut}(G, E) = \mathrm{Int}_G(G_0)(\mathrm{Aut}(G, E) \cap \mathrm{Aut}(G, M))$, and hence $\mathrm{Int}_G(G_0)\mathscr{B}$ is of finite index in $\mathrm{Aut}(G, E)$. Since E is simply connected, $\mathrm{Aut}(E) \cong \mathrm{Aut}(\mathscr{L}(E))$ is a real algebraic group. The restriction map $\alpha \to \alpha \mid E$ defines an injective homomorphism $\mathscr{B} \to \mathrm{Aut}(E)$. The image of $\mathscr{B}$ under this homomorphism consists of those $\beta \in \mathrm{Aut}(E)$ which commute with the action of M on E, and hence is an algebraic subgroup of $\mathrm{Aut}(E)$. Thus the index of $\mathscr{B}_0$ in $\mathscr{B}$ is finite. This shows that $\mathrm{Int}_G(G_0)\mathscr{B}_0$ is of finite index in $\mathrm{Int}_G(G_0)\mathscr{B}$ and hence in $\mathrm{Aut}(G, E)$. This proves that $\mathrm{Aut}_0(G) = \mathrm{Int}_G(G_0)\mathscr{B}_0$ is of finite index in $\mathrm{Aut}(G)$.

5. Proof of Theorem 2

Lemma 4. *Let T be a torus, and let A be a subgroup of* $\mathrm{Aut}(T)$. *If $T \rtimes A$ is faithfully representable, then A is finite.*

Proof. Let $\varrho: T \rtimes A \to GL(V)$ be a faithful representation. Replacing T and A by their images under ϱ, our assertion may be stated as follows: Let T be *a* toral subgroup $GL(V)$ and let N be a subgroup of $GL(V)$ which normalizes T. If the centralizer of T in N is trivial, then N is finite. Clearly N^* normalizes T. Since N^* is connected, N^* centralizes T, and hence $N^* \cap N$ is trivial. Since N^* is of finite index in the Zariski closure of N, it follows that N is finite.

Theorem 2. *Let G be a faithfully representable real analytic group. Then $G \rtimes$* $\mathrm{Aut}(G)$ *is faithfully representable if and only if exactly one of the following occurs:*

(i) *The nilradical N of G is simply connected;*
(ii) $G = G'$;
(iii) *G/G' is a 1-dimensional torus.*

Proof. We first assume that $G \rtimes \mathrm{Aut}(G)$ is faithfully representable. If N is trivial, then $G = G'$. Now suppose that N is nontrivial and is not simply connected. Thus the maximum torus K of N is not trivial. Choose a nucleus E of G, and let $H=[G,R]$. Since H is contained in the radical of the commutator group G', $H\subset E$. As a connected normal subgroup of the simply connected group E, H is simply connected and closed in E.

Let R denote the radical of G, and let S be a Levi factor of G. Since S is faithfully representable, the center of S is finite, and hence S is closed in G. The discrete subgroup $R\cap S$ is central in S and hence it is finite. As a finite subgroup of R, $R\cap S$ is contained in a maximal torus, say T, of R. Then R is the semidirect product ET.

We now show that $K=T$ and $\dim K=1$. Let T' be a subtorus of T such that $T=KT'$ (direct product). We have $R\cap S=T\cap S$. Let $S\cap T=\{s_1,s_2,\ldots,s_m\}$, and write each $s_i=k_it_i$ with $k_i\in K$, $t_i\in T'$. In what follows, we adopt the following convention: given two tori L and M and a closed subgroup F of L, $\mathrm{Hom}(L,M)_F$ denotes the group of all homomorphisms α of L into M with $\alpha(F)=1$, and $\mathrm{Aut}(L)_F$ denotes the group of all automorphisms α of L such that $\alpha=1$ on F. Note that $\mathrm{Hom}(L,M)_F$ is canonically isomorphic with $\mathrm{Hom}(L/F,M)$, and thus it is an infinite group whenever L/F is infinite. Returning to our proof, assume that T' is nontrivial and let $F=\{t_1,t_2,\ldots,t_m\}$, and for $\alpha\in\mathrm{Hom}(T',K)_F$, define $\tilde{\alpha}:T\to T$ by $\tilde{\alpha}(kt)=k\alpha(t)t$. Then $\tilde{\alpha}\in\mathrm{Aut}(T)_F$, and $\alpha\to\tilde{\alpha}$ defines an injective homomorphism from $\mathrm{Hom}(T',K)_F$ to $\mathrm{Aut}(T)_F$. This implies, in particular, that $\mathrm{Aut}(T)_F$ is an infinite group. On the other hand, for each $\theta\in\mathrm{Aut}(T)_F$, we define $\theta'\in\mathrm{Aut}(G)$ by $\theta'(xts)=x\theta(t)s$, for $(x,t,s)\in E\times T\times S$. The map $\theta\to\theta'$ is an injective homomorphism, and thus we have an injective homomorphism $T\rtimes\mathrm{Aut}(T)_F\to G\rtimes\mathrm{Aut}(G)$. Since $G\rtimes\mathrm{Aut}(G)$ is faithfully representable, we see that $T\rtimes\mathrm{Aut}(T)_F$ is faithfully representable. By Lemma 4, $\mathrm{Aut}(T)_F$ is finite, which is a contradiction. Therefore, T' must be trivial, and $T=K$ follows.

To show that K is 1-dimensional, suppose the contrary, and pick a circle subgroup K' and a complement K'' of K' in K so that $K=K'K''$ (direct product). By picking K' carefully, we may ensure that $K'\cap S=1$. Let $S\cap K=\{k_1,k_2,\ldots,k_m\}$, and write each $k_i=k_i'k_i''$ with $k_i'\in K'$, $k_i''\in K''$. Let $F=\{k_1',\ldots,k_m'\}$. Then each homomorphism in $\mathrm{Hom}(K',K'')$ defines an element of $\mathrm{Aut}(K)_F$ as before, and hence we see that $\mathrm{Aut}(K)_F$ is infinite. On the other hand, since $K\rtimes\mathrm{Aut}(K)_F$ is faithfully representable, $\mathrm{Aut}(K)_F$ is finite by Lemma 4. This contradiction shows that $\dim(K)=1$.

Now we want to show that $E=H$. Assume that E contains H properly. The closed subgroup KS is reductive, and since KS normalizes the subgroup $H=[R,G]$, we can find an $\mathrm{Ad}(KS)$-stable linear subspace V of $\mathscr{L}(E)$ such that $\mathscr{L}(E)=\mathscr{L}(H)\oplus V$. Because $[E,G]\subset H$, every element of V is fixed under the elements of $\mathrm{Ad}(KS)$. Pick a non-zero $X\in V$, and consider the 1-parameter subgroup $P=\{\exp(tX):t\in R\}$ of V. Let Q be a closed normal subgroup of E so that $E=QP$ (semidirect). Then we have the semidirect decomposition $G=Q(PKS)$. Now for any nontrivial homomorphism $\theta:P\to K$, define $\theta'\in\mathrm{Aut}(G)$ by $\theta'(qpx)=qp\,\theta(p)x$ for $(q,p,x)\in Q\times p\times KS$. Then in the group $G\rtimes\mathrm{Aut}(G)$, the commutator $c=[(p,1),(1,\theta')]=(p,1)(1,\theta')(p,1)^{-1}(1,\theta')^{-1}=(\theta(p^{-1}),1)$. Thus, since $\theta(p^{-1})\in K$, every representation of $G\rtimes\mathrm{Aut}(G)$ maps the commutator element c to a semisimple element. On the other hand, since c lies in the radical of the commutator subgroup of $G\rtimes\mathrm{Aut}(G)$, every representation of $G\rtimes\mathrm{Aut}(G)$ maps c to a unipotent element. Under our present assumptions, we can choose p so that

$c \neq 1$, which yields a contradiction. Consequently, we must have $E = H$. We have $R = N = KH = K[R, G]$, and therefore $G = KG'$, proving G/G' is a 1-dimensional torus.

Now we show that the conditions are sufficient. If the nilradical N of G is nontrivial and simply connected, then the assertion follows from the result of Hochschild [2]. Thus we may assume that N is not simply connected, and let K denote the maximum torus of N. Then K is the maximum torus of the center of G, and hence is a characteristic subgroup of G. If $G = G'$, then necessarily $K = 1$, and N is trivial (that is, G is semisimple). By [1, Theorem 2.2, p. 215], a faithful representation G can be extended to $G \rtimes \mathrm{Aut}_0(G)$. Since $\mathrm{Aut}_0(G)$ is of finite index in $\mathrm{Aut}(G)$, the assertion in this case follows from Lemma 1.

Now assume that G/G' is a 1-dimensional torus. In this case, $G = KG'$, $K \cap G'$ finite and K is 1-dimensional. Let $\mathscr{D} = \{\alpha \in \mathrm{Aut}(G)\colon\ \alpha = 1 \text{ on } K\}$. Clearly $\mathrm{Aut}_0(G) \subset \mathscr{D}$, and the index of $\mathscr{D}$ in $\mathrm{Aut}(G)$ is at most 2. The restriction morphism $\mathscr{D} \to \mathrm{Aut}(G')$ is injective and maps $\mathrm{Aut}_0(G)$ onto $\mathrm{Aut}_0(G')$. Since the maximum torus of the nilradical M of G' is trivial, M is simply connected. In this case, the holomorph of G' is faithfully representable, as we have seen above. Thus by Theorem 1, $\mathrm{Aut}_0(G')$ is of finite index in $\mathrm{Aut}(G')$. In particular, $\mathrm{Aut}_0(G)$ is of finite index in $\mathscr{D}$ and hence in $\mathrm{Aut}(G)$. By the corollary to Lemma 1, our conclusion would follow as soon as we show that $G \rtimes \mathrm{Aut}(G)$ is faithfully representable. Let ϱ be a faithful representation of G, and let R denote the radical of G. For $x \in R$ and $\alpha \in \mathscr{D}$, $[(x, 1), (1, \alpha)] = (x\alpha(x^{-1}), 1)$ belongs to the radical R_1 of G'. Thus ϱ is unipotent on $[R \times 1, 1 \times \mathrm{Aut}_0(G)]$, and again by [1, Theorem 2.2, p. 215], ϱ can be extended to a representation of $G \rtimes \mathrm{Aut}_0(G)$. Hence by Lemma 1, $G \rtimes \mathrm{Aut}_0(G)$ is faithfully representable.

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Special Values of Artin L-Series

David Goss* and Warren Sinnott**

Department of Mathematics, Ohio State University, 231 W. 18th Avenue, Columbus, OH 43210, USA

Introduction

Let k be a number field, and ϱ a complex representation of $\mathrm{Gal}(\bar{k}/k)$ with character χ. Let $L(\varrho, s) = L(\chi, s)$ be the Artin L-series of ϱ. Assume that the values of $L(\varrho, s)$ at the negative integers are not all zero. As is well known, this implies that k is totally real and that ϱ is totally-even or totally-odd (see Chap. 1). The purpose of this paper is to prove two results concerning $\{L(\varrho, -h), h = 0, 1, 2, \ldots\}$. Our first result asserts that this sequence determines $L(\varrho, s)$ as a Dirichlet series with a functional equation of Artin type. Our second result concerns a certain power series in $(T-1)$, $f(\varrho, T)$, which is canonically associated to $\{L(\varrho, -h), h = 0, 1, 2, \ldots\}$ and which has coefficients in the field $\mathbb{Q}(\chi)$ obtained by adjoining the values of χ to $\mathbb{Q}$. We show that, under the assumption of the *p-adic* Artin Conjecture and the vanishing of Iwasawa's μ-invariant, all the coefficients of $f(\varrho, T)$, except perhaps the first, are *integral* at all primes $\wp$ of $\mathbb{Q}(\chi)$ that lie over p. Conversely, the integrality of this power series at $\wp$ will imply the p-adic Artin Conjecture for ϱ and its twists by the Teichmüller character.

The formalism of Artin L-series allows us to reduce to having $k = \mathbb{Q}$, and it is in this context that we establish our results. In Chap. 1 we establish our first main result. We do this by showing, first of all, that one can read off the form of the functional equation for $L(\varrho, s)$ from $\{L(\varrho, -h), h = 0, 1, 2, \ldots\}$. Then this functional equation will give us the values of $L(\varrho, s)$ at either the even or odd *positive* integers. By standard results in complex function theory, we can then read off $L(\check{\varrho}, s)$ as an analytic function, and thus $L(\varrho, s) = \overline{L(\check{\varrho}, \bar{s})}$.

We note that the L-series of a representation ϱ of $\mathrm{Gal}(\bar{k}/k)$ determines the *induction* of ϱ to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ up to isomorphism.

In our second chapter, we study $f(\varrho, T)$. It turns out that, at each prime p, $f(\varrho, T)$ is the *unique* candidate for (the Fourier transform of) a p-adic measure whose moments are the values of $L(\varrho, s)$ at the non-positive integers. The proof of our second main result proceeds by first using the p-adic Artin Conjecture (applied

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to twists of ϱ by the Teichmüller character) to construct a measure on Z_p^* with the appropriate moments. This measure is then extended to a measure on Z_p in such a way as to put back the missing Euler p-factor. The proof is completed by identifying the Fourier transform of this measure with $f(\varrho, T)$.

In Chap. 3, we make some general remarks about the functions $f(\varrho, T)$ in the p-adic and complex analytic context. In general, one can show that $f(\varrho, T)$ is the asymptotic expansion at $T=1$ of the *formal* Mellin Transform of $L(\varrho, s)$. Furthermore, when ϱ is abelian, it is well known that $f(\varrho, T)$ is a rational function which plays a role in the analytic continuation of $L(\varrho, s)$. However, the proof of the main result of chapter one can easily be modified to show that, when $\deg(\varrho)\geqq 2$, $f(\varrho, T)$ has a trivial radius of convergence – thus it certainly does not define the germ of any analytic function. Still, it is reasonable to search for the algebraic (and complex-analytic) meaning of $f(\varrho, T)$ for arbitrary ϱ. Indeed, the main result of chapter one implies that, in some sense, $L(\varrho, s)$ is the "∞-adic" interpolation of the sequence $\{L(\varrho, -h), h=0,1,2,\}$.

1

Let $G=\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ equipped with its canonical topology. Let V be a finite dimensional C-vector space and let $\varrho: G\to GL(V)$ be a continuous representation; i.e., ϱ factors through a *finite* Galois extension of $\mathbb{Q}$. We let χ be the character of ϱ. As in [C–L1, T1], we let $L(\varrho, s)=L(\chi, s)$ be the Artin L-series of ϱ. By Brauer Induction, one knows that $L(\chi, s)$ has a meromorphic continuation to the complex plane and is *finite* at the non-positive integers. By Serre's variant of Brauer Induction [C–L1, appendix] and the results of Klingen and Siegel, these values lie in $\mathbb{Q}(\chi)$ (the field obtained by adjoining the values of χ to $\mathbb{Q}$).

Definition 1.1. a) We set $g(x)=g(\varrho, x)=\sum_{k=0}^{\infty}(L(\varrho, -k)/k!)x^k$.

b) Let $\log(T)=\sum_{n=1}^{\infty}(-1)^{n-1}(T-1)^n/n$, as usual. We put

$$f(T)=f(\varrho, T)=g(\varrho, \log(T)),$$

as a power series in (T–1).

It is trivial to see that the coefficients of both series lie in $\mathbb{Q}(\chi)$.

Let D be the operator $T(d/dT)$ applied to power series in $(T-1)$. It is a simple matter to see that $f(\varrho, T)$ is the unique power series in $(T-1)$ with

$$D^k f(\varrho, T)|_{T=1}=L(\varrho, -k), \qquad k=0,1,2,\ldots.$$

We are interested in those ϱ for which $g(\varrho, x)$ (and thus also $f(\varrho, T)$) is not a constant power series. By checking Γ-factors (as in [T1]), one finds readily that this forces ϱ to be either *totally-even* (i.e., $\varrho(\tau)=1_V$ for all complex conjugations τ) or *totally-odd* (i.e., $\varrho(\tau)=-1_V$ for all τ).

Theorem 1.2. *Assume that $g(\varrho, x)$ is not constant. Then $g(\varrho, x)$ (or $f(\varrho, T)$) determines $L(\varrho, s)$ as an Artin L-series; that is, $g(\varrho, x)=g(\varrho', x)$ if and only if $\varrho\simeq\varrho'$, i.e., $\chi=\chi'$.*

Proof. As above, we know that ϱ is totally-odd or totally-even. We begin by showing how one can use $\{L(\varrho, -k), k=0,1\ldots\}$ to read off all the information necessary to find the functional equation for $L(\varrho, s)$.

Set $d=\deg(\varrho)$ and $f=f(\varrho)=$ the conductor of ϱ. Put

$$a_1=a_1(\varrho)=d\,, \qquad a_2=a_2(\varrho)=0\,,$$

if ϱ is totally-even, and

$$a_1=a_1(\varrho)=0\,, \qquad a_2=a_2(\varrho)=d\,,$$

if ϱ is totally-odd. We note that $a_i(\varrho)=a_i(\check{\varrho})$, $i=1,2$, where $\check{\varrho}$ is the dual to ϱ.

Let

$$L_\infty(\varrho, s)=\pi^\alpha \Gamma(s/2)^{a_1}\Gamma((s+1)/2)^{a_2}$$

where $\alpha=-(a_2+sd)/2$; and put

$$\Lambda(\varrho, s)=f^{s/2}L_\infty(\varrho, s)L(\varrho, s)\,.$$

As in [T1], one has the functional equation

$$\Lambda(\varrho, 1-s)=W(\varrho)\Lambda(\check{\varrho}, s)\,,$$

where $\check{\varrho}$ is the dual of ϱ and $W(\varrho)$ is the *root-number* of ϱ. One has $|W(\varrho)|=1$.

Thus one finds that

$$\begin{aligned}L(\varrho, 1-s)=W(\varrho)f^{s-1/2}\pi^{d/2}\pi^{-sd}(\Gamma(s/2)/\Gamma((1-s)/2)^{a_1}\\ \times(\Gamma((1+s)/2)/\Gamma((2-s)/2))^{a_2}L(\check{\varrho}, s)\,.\end{aligned}$$

One knows that

$$\Gamma(s/2)/\Gamma((1-s)/2)=\Gamma(s)2^{1-s}\pi^{-1/2}\cos(s\pi/2)\,,$$

and

$$\Gamma((1+s)/2)/\Gamma((2-s)/2)=\Gamma(s)2^{1-s}\pi^{-1/2}\sin(s\pi/2)\,.$$

The function $\Gamma(s/2)$ has poles at non-positive even numbers and $\Gamma((s+1)/2)$ has poles at the negative odd numbers. Thus one can determine which of a_1 or a_2 is 0 from the class mod 2 of negative integers where L vanishes.

Assume now that $a_1\neq 0$; a similar argument will work when $a_2\neq 0$. Let k vary through positive *even* integers. One has

$$|L(\varrho, 1-k)|=f^{k-1/2}\pi^{d/2}\pi^{-dk}(\Gamma(k)2^{1-k}\pi^{-1/2})^{a_1}|L(\check{\varrho}, k)|\,. \tag{$*$}$$

Note that as $s\to\infty$, $L(\check{\varrho}, s)\to 1$. Further, by Stirling's formula $A^s/\Gamma(s)\to 0$ as $s\to\infty$ for any real $A>0$; it follows that $|L(\varrho, 1-k)|/(\Gamma(k)^x)$ approaches ∞ with k if $x<a_1$ and approaches 0 with k for $x>a_1$. Thus we can read off a_1. As $a_1=d$, we now also know d. Returning to $(*)$ we can now find f. Finally, removing the absolute value signs from $(*)$, we can find $W(\varrho)$. Consequently, we know the form of the functional equation for $L(\varrho, s)$.

From the functional equation, we can now read off $L(\check{\varrho}, k)$ for k positive and even. Dirichlet series are bounded for $\mathbf{Re}(s)\gg 0$. Therefore, by Carlson's theorem

[Ti1, p. 186] we know $L(\check{\varrho}, s)$ as an analytic function; and so, by the functional equation, we have determined $L(\varrho, s)$ as an analytic function. As $L(\varrho, s)$ determines χ, the result is established.

Remarks 1.3. a) If $L(\varrho, s)=0$ at *all* the negative integers, the conclusion of Theorem 1.2 is easily seen to be false.

b) Using Stirling's Formula as in the above proof, one can see that if $d \geqq 2$, then $g(\varrho, x)$ [and so $f(\varrho, T)$] has radius of convergence equal to 0.

2

Let ϱ be as before. Let p be a prime number and ω the Teichmüller character at p. Let σ be an injection of $\mathbb{Q}(\chi)$ into $\bar{\mathbb{Q}}_p$. As in ([Gr1]), one can define the p-adic L-series associated to $\varrho^\sigma \otimes \omega^i$, $1 \leqq i \leqq p-1$. In this chapter, we will show how the p-adic Artin Conjecture (applied to $\varrho^\sigma \otimes \omega^i$, all i), together with $\mu=0$ for totally real fields, implies that $f(\varrho, T)$ possesses the integrality properties stated in the introduction. We note that the p-adic Artin Conjecture is a consequence of Iwasawa's Main Conjecture [Gr1]. The reader may find it instructive to compare with [G2] where a function field analogue of our result is shown.

We begin by recalling some basic facts about p-adic measures. Let L be a finite extension of $\mathbb{Q}_p$, and let $\mathbb{O}$ be its ring of integers. There is an $\mathbb{O}$-algebra isomorphism between the convolution algebra of $\mathbb{O}$-valued measures on $\mathbb{Z}_p$ and the power series ring $\mathbb{O}$ [[T–1]], given by the "Fourier transform":

$$\hat{\lambda}(T)=\int_{\mathbb{Z}_p} T^x d\lambda(x)=\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p}\binom{x}{n} d\lambda(x)\right)(T-1)^n .$$

We need the following operations on measures. If $c \in \mathbb{Z}_p$, $c \neq 0$, and if λ is an $\mathbb{O}$-valued measure on $\mathbb{Z}_p$, we let $\lambda \circ c$ denote the measure on $\mathbb{Z}_p$ defined by

$$\lambda \circ c(A)=\lambda(cA),$$

for any compact open subset $A \subset \mathbb{Z}_p$. If $A \subset \mathbb{Z}_p$ is compact open, we denote by $\lambda|_A$ the measure on $\mathbb{Z}_p$ obtained by restricting λ to A and extending by 0. It is easy to see that

$$\lambda \circ c|_A=\lambda|_{cA} \circ c .$$

We also write $\int_{\mathbb{Z}_p} \ldots d\lambda(cx)$ for $\int_{\mathbb{Z}_p} \ldots d\lambda \circ c(x)$ and $\int_A \ldots d\lambda(x)$ for $\int_{\mathbb{Z}_p} \ldots d\lambda|_A(x)$.

Let W be the group of roots of unity in $\mathbb{Z}_p^*$, and let w be the order of W: then $w=p-1$ if $p>2$ and $w=2$ if $p=2$. Also, let $U=1+2p\mathbb{Z}_p$ be the group of "principal units" in $\mathbb{Z}_p^*$, so that $\mathbb{Z}_p^*=U \times W$. If β is an $\mathbb{O}$-valued measure on $\mathbb{Z}_p$ supported on U, consider

(2.1) $$f(s)=\int_U x^s d\beta(x), \qquad s \in \mathbb{Z}_p .$$

Then $f(s)$ is an *Iwasawa function over* $\mathbb{O}$, i.e., a uniform limit (on $\mathbb{Z}_p$) of finite $\mathbb{O}$-linear combinations of exponentials $s \to u^s$, $u \in U$. Conversely, any Iwasawa function over $\mathbb{O}$ may be represented as an integral as in (2.1), for some (unique) $\mathbb{O}$-valued measure β supported on U.

Next, we establish the following p-adic interpretation of $f(\varrho, T)$. Let L be the field of rationality of the trace of ϱ^σ; i.e. $L=\mathbb{Q}_p(\chi^\sigma)$; so $f^\sigma(\varrho, T)\in L[[T-1]]$. Let $\mathbb{O}$ be the ring of integers of L.

Proposition 2.2. *Assume that there exists an $\mathbb{O}$-valued measure λ so that*

$$\int_{\mathbb{Z}_p} x^k d\lambda(x) = L(\varrho, -k)^\sigma, \qquad k=0,1,2\dots.$$

Then

$$\hat{\lambda}(T)=f^\sigma(\varrho, T)\in \mathbb{O}[[T-1]].$$

Proof. By ([K1]) one knows that λ corresponds to the unique series $g(T)$ with $D^k g(T)|_{T=1}=L(\varrho,-k)^\sigma$ for $k=0,1,2\dots$. Hence $\hat{\lambda}(T)=f^\sigma(\varrho, T)$, by 1.1, and therefore $f^\sigma(\varrho, T)$ has coefficients in $\mathbb{O}$.

Proposition 2.3. *Let $\{f_i\}$, $1\leqq i\leqq w$, be arbitrary given Iwasawa functions over $\mathbb{O}$. Then there is a unique measure α on $\mathbb{Z}_p^*$ with values in $1/2\mathbb{O}$ so that*

$$f_i(s)=\int_{\mathbb{Z}_p^*} \langle x\rangle^s \omega^i(x)d\alpha(x), \tag{2.4}$$

for each i.

Proof. For any measure α on $\mathbb{Z}_p^*$, we may write

$$\int_{\mathbb{Z}_p^*} \langle x\rangle^s \omega^i(x)d\alpha(x)=\sum_{\eta\in W}\int_{\eta U}\langle x\rangle^s\omega^i(x)d\alpha(x).$$

If we make the change of variable $x\to\eta x$ in the integral on the right, we find

$$\begin{aligned}\sum_{\eta\in W}\int_{\eta U}\langle x\rangle^s\omega^i(x)d\alpha(x)&=\sum_{\eta\in W}\int_U\langle \eta x\rangle^s\omega^i(\eta x)d\alpha(\eta x)\\ &=\sum_{\eta\in W}\int_U x^s\eta^i\omega^i(x)d\alpha(\eta x)\\ &=\int_U x^s d\gamma_i(x);\end{aligned}$$

where

$$\gamma_i=\sum_{\eta\in W}\eta^i\alpha\circ\eta_U=\sum_{\eta\in W}\eta^i\alpha_{\eta U}\circ\eta,$$

for $i=1,\dots,w$. Let β_i be the $\mathbb{O}$-valued measure on U corresponding to f_i by 2.1. Thus we need a measure α so that $\gamma_i=\beta_i$, $i=1,\dots,w$ (γ_i as above).

To find α we set

$$\gamma_i=\sum_{\eta\in W}\eta^i\alpha_{\eta U}\circ\eta=\beta_i, \tag{2.6}$$

and "solve" for α; i.e., fix $\theta\in W$, multiply by θ^{-i}, and sum over i. We obtain

$$w\cdot\alpha|_{\theta U}\circ\theta=\sum_{i=1}^{w}\theta^{-i}\beta_i,$$

so

$$\alpha=\sum_{\theta\in W}\alpha_{\theta U}=1/w\sum_{\theta\in W}\sum_{i=1}^{w}\theta^{-i}\beta_i\circ\theta^{-1};$$

which has values in $(1/w)\mathbb{O}=(1/2)\mathbb{O}$.

Theorem 2.7. *Assume that the coefficients of $f^\sigma(\varrho,T)$, except perhaps the constant term, lie in $\mathbb{O}$;* i.e., $f^\sigma(\varrho,T)-f^\sigma(\varrho,1)\in\mathbb{O}[[T-1]]$. *Then for each $i=1,\dots,w$, the p-adic L-function $(1/2)L_p(\varrho^\sigma\otimes\omega^i,s)$ is an Iwasawa function over $\mathbb{O}$.*

Proof. Let

$$f_i(s)=\int_{\mathbb{Z}_p^*}\langle x\rangle^s\omega^i(x)d\alpha(x),$$

where α corresponds to $f^\sigma(\varrho,T)-f^\sigma(\varrho,1)$. Fix i with $1\leqq i\leqq w$, and let $s\in\mathbb{Z}_p$; if $p=2$, we also suppose that $i\equiv s(\operatorname{mod}2)$. Then there is a sequence n_j of positive integers satisfying (i) $n_j\to\infty$ in $\mathbb{R}$, (ii) $n_j\to s$ in $\mathbb{Z}_p$, and (iii) $n_j\equiv i(\operatorname{mod}w)$. Thus,

$$\begin{aligned}f_i(s)&=\lim_j\int_{\mathbb{Z}_p}x^{n_j}d\alpha(x)\\&=\lim_j L(\varrho,-n_j)^\sigma\\&=\lim_j E_p(\varrho^\sigma,p^{n_j})L_{(p)}(\varrho,-n_j)^\sigma;\end{aligned}$$

where

$$E_p(\varrho^\sigma,x)=\det(1-\varrho^\sigma(\mathrm{Frob}_p)x|V^{I_p})^{-1},$$

is the Euler factor at p (which belongs to $1+x\mathbb{O}[[x]]$ as $\mathbb{O}$ is p-adic), and $L_{(p)}$ means "remove Euler p-factor". Thus

$$f_i(s)=L_p(\varrho^\sigma\otimes\omega^{i+1},-s). \tag{2.8}$$

For $p>2$, this shows that$(1/2)L_p(\varrho\otimes\omega^i,s)$ is anIwasawa function over $\mathbb{O}$ as desired. When $p=2$, we first note that the condition $i\equiv s(\operatorname{mod}2)$ may be dropped in 2.8, as both sides are analytic in s. Also

$$\begin{aligned}f_i(s)&=\int_U x^s d\alpha(x)+\int_{-U}\langle x\rangle^s\cdot(-1)^i d\alpha(x),\\&=\int_U x^s d\alpha(x)+(-1)^i\int_U\langle x\rangle^s d\alpha(-x),\\&=\int_U x^s d\beta(x);\ \beta=\alpha+(-1)^i\alpha\circ(-1).\end{aligned}$$

Now we need the fact that α has a parity: $\alpha\circ(-1)=\pm\alpha$. Indeed, recall that for $k\geqq1$, $L(\varrho,-k)=0$ unless ϱ and k have opposite parity. This implies that $g(\varrho,x)-g(\varrho,0)$ is either an even or an odd function, and thus that $f(\varrho,T^{-1})-f(\varrho,1)=\pm(f(\varrho,T)-f(\varrho,1))$. Since $f^\sigma(\varrho,T^{-1})-f^\sigma(\varrho,1)$ is the Fourier transform of $\alpha\circ(-1)$, it follows that $\alpha\circ(-1)=\pm\alpha$ as desired. Thus β is either 0 or takes values in $2\mathbb{O}$, which concludes the proof.

We also have the converse result.

Theorem 2.9. *Suppose that* $(1/2)L_p(\varrho^\sigma \otimes \omega^i, s)$ *is an Iwasawa function over* $\mathfrak{O}$ *for each* $i=1, \ldots, w$. *Then*

$$f^\sigma(\varrho, T) - f^\sigma(\varrho, 1) \in \mathfrak{O}[[T-1]].$$

Proof. We apply 2.4 to $L_p(\varrho^\sigma \otimes \omega^i, -s)$, for $i=1, \ldots, w$. Let α^* be the resulting measure; so α^* takes values in $\mathfrak{O}$. Let $s=-k$, where $k \geqq 1$, and let $i \equiv k (\bmod w)$. We find

$$\begin{aligned} L_p(\varrho^\sigma \otimes \omega^i, -k) &= \int_{\mathbb{Z}_p} x^k d\alpha^*(x), \\ &= L_{(p)}(\varrho, -k)^\sigma, \\ &= E_p(\varrho^\sigma, p^k)^{-1} L(\varrho, -k)^\sigma. \end{aligned}$$

Write

$$E_p(\varrho^\sigma, x) = \sum_{m \geqq 0} c_m x^m, \qquad c_m \in \mathfrak{O}, \qquad c_0 = 1.$$

Then,

$$L(\varrho, -k)^\sigma = \sum_{m \geqq 0} c_m \int_{\mathbb{Z}_p^*} (p^m x)^k d\alpha^*(x).$$

Let α_m^* be the direct image of α^* under the map $p^m : \mathbb{Z}_p \to \mathbb{Z}_p$ – so α_m^* is *supported* on $p^m \mathbb{Z}_p^*$, and

$$\alpha_m^*(p^m Y) = \alpha^*(Y), \qquad (Y \subset \mathbb{Z}_p).$$

Thus

$$L(\varrho, -k)^\sigma = \sum_{m \geqq 0} c_m \int_{p^m \mathbb{Z}_p^*} x^k d\alpha_m^*(x).$$

This suggests defining a measure α on $\mathbb{Z}_p$ by

$$\alpha|_{p^m \mathbb{Z}_p^*} = c_m \alpha_m^*; \qquad m = 0, 1, 2, \ldots. \tag{2.10}$$

This will determine $\alpha(Y)$ for any compact-open $Y \subset \mathbb{Z}_p - \{0\}$. To determine α completely we need to specify, e.g., $\alpha(\mathbb{Z}_p)$. Define

$$\alpha(\mathbb{Z}_p) = L(\varrho, 0)^\sigma \tag{2.11}$$

which lies in L, but which is not known to be integral in general. However this measure satisfies

$$\int_{\mathbb{Z}_p} x^k d\alpha(x) = L(\varrho, -k)^\sigma;$$

for $k=0, 1, \ldots$. Hence by the discussion at the beginning of the chapter, the Fourier transform of α is $f^\sigma(\varrho, T)$. Let δ_0 denote the Dirac measure of mass 1 concentrated at $0 \in \mathbb{Z}_p$: the Fourier transform of δ_0 is the power series $T^0 = 1$. One sees that $\alpha - \alpha(\mathbb{Z}_p)\delta_0$ is $\mathfrak{O}$-valued (since it satisfies (2.10) but assigns $\mathbb{Z}_p$ measure 0). Thus,

$$f^\sigma(\varrho, T) - f^\sigma(\varrho, 1) \in \mathfrak{O}[[T-1]].$$

This completes the proof.

Finally, let ϱ be an absolutely irreducible representation of degree > 1. Then we have shown:

Theorem 2.12. *Assume the p-adic Artin Conjecture for all p, and $\mu=0$ for all p and all totally real fields. Then $f(\varrho, T)-f(\varrho, 1)$ has coefficients in the ring of integers of $\mathbb{Q}(\chi)$.*

Remark 2.13. The p-adic Artin Conjecture alone (without any assumption on the Iwasawa invariant) implies that $f(\varrho, T)-f(\varrho, 1)$ has $\wp$-adically bounded denominators for all finite primes $\wp$ of $\mathbb{Q}(\chi)$.

3

In this last chapter we will make some general remarks about the functions $f(\varrho, T)$. We begin by looking at the simplest case where ϱ is *abelian and non-trivial*. Let f be the conductor of ϱ. Then, as in [K1], one finds that

$$f(\varrho, T)=\left(\sum_{i=1}^{f-1} \varrho(i)T^i\right)\Big/(1-T^f).$$

Thus $f(\varrho, T)$ is a rational function which is integral at primes not dividing f. On the other hand, in our first chapter we pointed out that when $\deg(\varrho)>1$, $f(\varrho, T)$ has a trivial radius of convergence. Thus, $f(\varrho, T)$ does not define the germ of any analytic (let alone algebraic) function on $\mathbb{C}$. Still, it makes sense to inquire if $f(\varrho, T)$ has any sort of algebraic characteristics. For instance, one knows that for those representations ϱ which are induced from an abelian character of a totally real field, $f(\varrho, T)$ is, roughly speaking, the image under the norm of rational function measures in *several* variables (see [K2]).

Secondly, when ϱ is abelian and non-trivial, one can also find $f(\varrho, T)$ *complex-analytically*. Indeed, one has

$$\Gamma(s)L(\varrho, s)=\int_{R+} f(\varrho, e^{-t})t^{s-1}dt.$$

This integral converges for all s with *positive* real part. Moreover, integration by parts (for instance, see [G1]) will extend the definition of the integral holomorphically to the whole complex plane. Conversely, if one only knew $f(\varrho, T)$ from the *p-adic* theory one could prove the above equality as follows: Define a new function $\bar{L}(\varrho, s)$ as in the above equation. As before one can analytically continue $\bar{L}(\varrho, s)$ to the complex plane. Moreover, one sees directly that $\bar{L}(\varrho, -k)=L(\varrho, -k)$ for k a non-positive integer. Furthermore, Hurwitz's proof of the functional equation for partial zeta-functions immediately gives a functional equation to $\bar{L}(\varrho, s)$. Thus, one sees that L and $\bar{L}$ have the same values at some class mod 2 of *positive* integers. Finally, $\bar{L}$ is trivially seen to be bounded for $\mathrm{Re}(s)\gg 0$. Thus, again, Carlson's theorem tells us the two functions are equal.

Let $\sum a_n n^{-s}$ the Dirichlet series associated to $L(\varrho, s)$. Define the *formal Mellin Transform* $\hat{L}(\varrho, T)$ of $L(\varrho, s)$ by

$$\hat{L}(\varrho, T)=\sum a_n T^n.$$

From the definition of $L(\varrho, s)$ as an Euler product it follows that the a_n are integers in $\mathbb{Q}(\chi)$. [Note that $f(\varrho, T)$ also has coefficients in $\mathbb{Q}(\chi)$.] Hence, $\hat{L}^\sigma(\varrho, T)$ converges *p-adically* for $|T|_p<1$. Further, under the assumption of the p-adic Artin

Conjecture, we have shown that the power series $f^{\sigma}(\varrho, T)$ converges for all T with $|T-1|_p<1$. In the case ϱ abelian, it follows trivially that these two power series define the same global p-adic meromorphic function. It would be very interesting to understand the p-adic relationships between these functions in general: for example (to be optimistic) is there an analytic function $F(T)$ defined on some connected affinoid $\subset \mathbb{P}^1(\mathbb{C}_p)$ whose Taylor series at $T=0$ and $T=1$ are given respectively by $\hat{L}^{\sigma}(\varrho, T)$ and $f^{\sigma}(\varrho, T)$?

Finally, one can ask whether $f(\varrho, T)$ plays any role in the complex-analytic continuation of $L(\varrho, s)$ when ϱ is irreducible of higher degree. We note that if $L(\varrho, s)$ is entire, then $\hat{L}(\varrho, T)$ has $f(\varrho, T)$ as its asymptotic expansion as $T\uparrow 1$, $0<T<1$, (cf. [I1]).

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On the Plurigenera of Minimal Algebraic 3-Folds with $K \approx 0$

Yujiro Kawamata

Department of Mathematics, University of Tokyo, Hongo, Tokyo 113, Japan

If S is an algebraic surface over $\mathbb{C}$ with its Kodaira dimension $\kappa(S)=0$, then the classification theory of surfaces tells us that $P_{12}(S)\neq 0$. In this paper we shall extend this result to certain algebraic 3-folds. Our main result (Theorem 3.2) states: there exists a positive integer m_0, which is calculable but large, such that for an arbitrary projective 3-fold X over $\mathbb{C}$ having at most terminal singularities whose canonical divisor K_X is numerically trivial, we have

$$P_{m_0}(X) := \dim H^0(X, \mathcal{O}_X(m_0K_X)) \neq 0 .$$

(Those X's are nothing but good minimal 3-folds with $v(X)=0$ in the sense of [K3] and [KMM].) In order to prove this, we shall show that $0 \leqq \chi(X, \mathcal{O}_X) \leqq 4$ (Theorem 3.1) and that the value of $\chi(X, \mathcal{O}_X)$ is determined by the distribution of points of (local) index >1 on X. More precisely, we have the following formula

$$\chi(X, \mathcal{O}_X) = \tfrac{1}{24}\sum\left(r-\frac{1}{r}\right),$$

where the summation is taken over all (virtual) index r points counted with some multiplicities (Theorem 2.4, cf. Sect. 1). For example, if X is the quotient of a 3-dimensional abelian variety by an involution $x \mapsto -x$, then there are 64 index 2 points and $\chi(X, \mathcal{O}_X)=4$. We note that $\chi(X, \mathcal{O}_X)=0$ if X is Gorenstein.

Since the Kodaira dimension κ, plurigenera P_m, and $\chi(\mathcal{O})$ are birational invariants, we should obtain similar results for an arbitrary algebraic 3-fold with $\kappa=0$, once we proved the conjectural existence of a good minimal model [R2, K3, KMM]. We refer the reader to [HM] on some results for the case of 3-folds with $\kappa>0$. The following question is also related to our results: is there essentially only a finite number of families of 3-folds with $\kappa=0$?

1. Preliminaries

We essentially follow the notation of [K2] or [KMM]. Let V be a normal complex analytic space of dimension 3. Let $Z_2(V)$ and $\mathrm{Div}(V)$ denote the groups of Weil divisors and of Cartier divisors, respectively. By extending the coefficients to $\mathbb{Q}$, we also define the groups of $\mathbb{Q}$-*divisors* $Z_2(V)_{\mathbb{Q}}$ and of $\mathbb{Q}$-*Cartier divisors* $\mathrm{Div}(V)_{\mathbb{Q}}$. The reflexive sheaf of rank one corresponding to $D \in Z_2(V)$ is denoted by $\mathcal{O}_V(D)$. A Weil divisor D is called $\mathbb{Q}$-*Cartier* if it is in the image of the natural homomorphism $\mathrm{Div}(V)_{\mathbb{Q}} \to Z_2(V)_{\mathbb{Q}}$. The linear equivalence of Cartier divisors and the numerical equivalence of $\mathbb{Q}$-Cartier divisors are denoted by $\sim$ and $\approx$, respectively. The *canonical divisor* K_V is an element of $Z_2(V)$ which coincides with the usual one when restricted to $\mathrm{Reg}(V)$, the regular part of V. Let $f: W \to V$ be a proper bimeromorphic morphism. Then there are natural homomorphisms $f_*: Z_2(W)_{\mathbb{Q}} \to Z_2(V)_{\mathbb{Q}}$ and $f^*: \mathrm{Div}(V)_{\mathbb{Q}} \to \mathrm{Div}(W)_{\mathbb{Q}}$.

V is said to have *terminal singularities* if

(i) K_V is $\mathbb{Q}$-Cartier, and

(ii) $\mu: W \to V$ being a desingularization [HN], $K_W - \mu^* K_V$ is effective and its support contains all the exceptional divisors of μ.

In this case V has only isolated rational singularities [SB] or [E]. The *local index* $i(V)$ of V (or more precisely, the l.c.m. of the local indices) is defined by

$$i(V) = \min\{m \in \mathbb{N}; mK_V \in \mathrm{Div}(V)\}.$$

In case $mK_V \sim 0$ [i.e., $\mathcal{O}_X(mK_V) \cong \mathcal{O}_V$] for some positive integer m with $i(V)|m$, we also define the *global index* $I(V)$ of V by

$$I(V) = \min\{m \in \mathbb{N}; mK_V \sim 0\}.$$

Note that $i(V)|I(V)$ and the two indices coincide if V is small.

Let θ be a nowhere vanishing section of $\mathcal{O}_V(rK_V)$ with $r = I(V)$. Let

$$\mathcal{R} \cong \bigoplus_{j=0}^{r-1} \mathcal{O}_V(-jK_V)$$

be the sheaf of $\mathcal{O}_V$-algebras obtained as the quotient of $\bigoplus_{j \geqq 0} \mathcal{O}_V(-jK_V)$ by an ideal $(\theta - 1) \cdot \bigoplus_{j \geqq r} \mathcal{O}_V(-jK_V)$. Then an r-fold ramified covering $\pi: W = \mathrm{Specan}(\mathcal{R}) \to V$ is called a *canonical cover* of V. (We note that this construction depends on the choice of θ.) Since r is chosen to be minimum, W is irreducible. Since $\mathrm{div}(\theta) = 0$, π is etale outside $\mathrm{Sing}(V)$, the singular points of V. Thus W is normal, has at most terminal singularities, and $K_W \sim 0$ (cf. [R1, Corollary 1.9], [K2] or [KMM]).

Let (V, p) be a germ of a terminal singularity of dimension 3, let r be the index, and let $\pi: W \to V$ be a canonical cover with $\pi^{-1}(p) = q \in W$. Let $G \cong \mathbb{Z}/(r)$ be the Galois group $\mathrm{Gal}(W/V)$. Reid [R1] showed that either one of the following holds:

(i) (W, q) is non-singular, or

(ii) (W, q) is a cDV point, i.e., the total space of a one parameter deformation of a rational double point.

Let G act on $\mathbb{C}^3$ with coordinates $\{x_1, x_2, x_3\}$ by $g(x_k) = \zeta^{a_k} x_k$, where g and ζ are generators of G and $\boldsymbol{\mu}_r$. Then the quotient singularity $(\mathbb{C}^3/G, 0)$ is said to be of type $\frac{1}{r}(a_1, a_2, a_3)$. In case (i), [D2] and [MS] showed that (V, p) is a quotient singularity of type $\frac{1}{r}(1, -1, s)$ for some positive integers r and s such that $(r, s) = 1$.

The case (ii) is studied by Mori [M] in detail. In particular, it is shown that an arbitrary terminal singularity (V, p) of dimension 3 of type (ii) has a small deformation to n terminal quotient singularities (V_j, p_j) $(1 \leqq j \leqq n)$ with $n \geqq 2$. Moreover, we have $i(V_j) = i(V)$ for all j except in the case of [M, Theorem 12 (2)]. In this exceptional case, $i(V) = i(V_1) = 4$ and $i(V_j) = 2$ for $j \geqq 2$. We consider the singularity (V, p) as a composite of singularities with virtual indices $i(V_1), \ldots, i(V_n)$.

2. Computation of $(\Delta \cdot c_2)$

Let (V, p) be a germ of a terminal singularity of dimension 3, and let $\mu: W \to V$ be a desingularization such that $W - \mu^{-1}(p) \xrightarrow[\mu]{\cong} V - \{p\}$. We write $K_W = \mu^* K_V + \Delta$ and $\Delta = \sum_j a_j F_j$, where the F_j are exceptional divisors of μ. Let $c_2(T_W, D)$ be the second Chern form relative to a connection D on T_W. We define

$$(\Delta \cdot c_2(W)) = \sum_j a_j \int_{F_j} c_2(T_W, D).$$

It is well known that the above integrals do not depend on the choice of D.

Lemma 2.1. *$(\Delta \cdot c_2(W))$ does not depend on the choice of the resolution μ.*

Proof. Since (V, p) is an isolated singularity, by using [A2, Theorem 3.8] and [HR] (or [A1, Theorem 1.5]), we obtain an algebraic variety X with a point x such that $(X^h, x) \cong (V, p)$, where X^h is the underlying complex analytic space of X. We may assume that X is complete and the singular locus $\mathrm{Sing}(X)$ consists of one point $\{x\}$. Then the desingularization $\mu: W \to V$ is embedded into a proper bimeromorphic morphism $\tilde{\mu}: Y \to X$ from a compact complex manifold Y. Since X has at most rational singularities, m being a positive integer such that $mK_X \in \mathrm{Div}(X)$, the theorem of Riemann-Roch [HB] yields the following formula:

$$\begin{aligned}
\chi(X, \mathcal{O}_X(mK_X)) &= \chi(Y, \mathcal{O}_Y(m\tilde{\mu}^* K_X)) \\
&= \tfrac{1}{6}(m\tilde{\mu}^* K_X)^3 - \tfrac{1}{4}(m\tilde{\mu}^* K_X)^2(\tilde{\mu}^* K_X + \Delta) \\
&\quad + \tfrac{1}{12}(m\tilde{\mu}^* K_X)\left((\tilde{\mu}^* K_X + \Delta)^2 + c_2(Y)\right) + \chi(X, \mathcal{O}_X) \\
&= \tfrac{1}{12} m(m-1)(2m-1) K_X^3 + \tfrac{1}{12} m(\tilde{\mu}^* K_X \cdot c_2(Y)) + \chi(X, \mathcal{O}_X).
\end{aligned}$$

We also have

$$\begin{aligned}
\chi(X, \mathcal{O}_X) = \chi(Y, \mathcal{O}_Y) &= \tfrac{1}{24} c_1(Y) c_2(Y) \\
&= -\tfrac{1}{24}(\tilde{\mu}^* K_X \cdot c_2(Y)) - \tfrac{1}{24}(\Delta \cdot c_2(Y)).
\end{aligned}$$

Therefore, $(\Delta \cdot c_2(W)) = (\Delta \cdot c_2(Y))$ is expressed in terms of coefficients of $\chi(X, \mathcal{O}_X(mK_X))$. Q.E.D.

We define $\delta(V, p) := -(\Delta \cdot c_2(W))$. The following lemma is also obtained by M. Reid independently by a different method.

Lemma 2.2. *Let (V, p) be a germ of a quotient singularity of type $\frac{1}{r}(s, -s, 1)$ for some integers r and s with $0<s<r$ and $(r, s)=1$. Then*

$$\delta(V, p)=r-\frac{1}{r}.$$

Proof. We proceed by induction on r. (V, p) is represented by a toric variety X whose corresponding cone σ in $N_{\mathbb{R}}$ is a simplex spanned by $v_1=(1,0,0)$, $v_2=(0,1,0)$, and $v_3=(s, r-s, r)$ [D1]. Let us subdivide σ by adding a new vertex $v_4=(1,1,1)$:

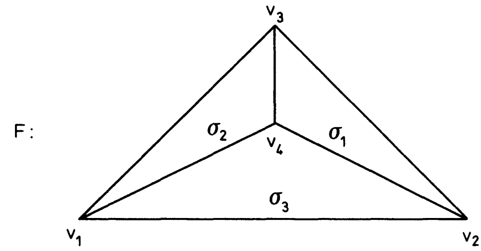

Fig. 1

Let X_1 be the toric variety corresponding to the fan F thus obtained and let $f: X_1 \to X$ be the natural proper birational morphism. X_1 has two singular points corresponding to the cones σ_1 and σ_2; these are quotient singularities of types $\frac{1}{r-s}(1, r, -r)$ and $\frac{1}{s}(1, r, -r)$, respectively. Let D_i $(i=1,2,3)$ be prime divisors on X which correspond to the v_i, let D'_i be the strict transforms of the D_i on X_1, and let E be the exceptional divisor of f corresponding to v_4. We consider the functions x^{m_i} on X corresponding to elements $m_i \in M$ $(i=1,2,3)$ given by $m_1=(r, 0, -s)$, $m_2=(0, r, -(r-s))$, and $m_3=(0,0,1)$. Then $\operatorname{div}(x^{m_i})=rD_i$ for all i and $\operatorname{div}(f^*x^{m_i})=rD'_i+k_iE$ with $k_1=r-s$, $k_2=s$, and $k_3=1$. Since $K_X \sim -\sum_i D_i$ and $K_{X_1} \sim -\sum_i D'_i - E$, we conclude that

$$K_{X_1}=f^*K_X+\frac{1}{r}E.$$

Let $\mu: Y \to X_1$ be a toric desingularization given by a suitable subdivision of F, and we write $K_Y=\mu^*K_{X_1}+\Delta_1$. Thus $K_Y=(f\circ\mu)^*K_X+\Delta$ with $\Delta=\Delta_1+\frac{1}{r}\mu^*E$. Let l_{ij} $(1\leqq i<j\leqq 4)$ be curves on X_1 which correspond to the $\overline{v_iv_j}$. By [D1, Corollary 11.5],

$$\mu_*c_2(Y)=\sum_{ij} l_{ij}.$$

Since $(E\cdot l_{12})=1$, $(E\cdot l_{13})=\frac{1}{s}$, $(E\cdot l_{14})=-\frac{r}{s}$, $(E\cdot l_{23})=\frac{1}{r-s}$, $(E\cdot l_{24})=-\frac{r}{r-s}$, and $(E\cdot l_{34})=-\frac{r}{s(r-s)}$, we have

$$(\mu^*E\cdot c_2(Y))=(E\cdot\mu_*c_2(Y))=1-\frac{r}{s}-\frac{r}{r-s}.$$

By induction, $(\Delta_1 \cdot c_2(Y)) = \frac{1}{r-s} + \frac{1}{s} - r$. Hence

$$(\Delta \cdot c_2(Y)) = (\Delta_1 \cdot c_2(Y)) + \frac{1}{r}(\mu^* E \cdot c_2(Y)) = \frac{1}{r} - r. \quad \text{Q.E.D.}$$

Lemma 2.3. *Let (V, p) be a germ of a terminal singularity of dimension 3 of index r whose canonical cover is a cVD point. Assume that (V, p) has a small deformation to n terminal quotient singularities (V_j, p_j) $(1 \leqq j \leqq n)$ with $n \geqq 2$. Then*

$$\delta(V, p) = \sum_j \delta(V_j, p_j).$$

Proof. Let $G = \mathbb{Z}/(r)$. By [R1] and [A1, A2, HR], there is a polynomial

$$\varphi \in R := \mathbb{C}[X_1, \ldots, X_4]$$

with an action of G on R through $GL(4, \mathbb{C})$ such that

(i) $Y := \{y \in \mathbb{C}^4; \varphi(y) = 0\}$ has an isolated singularity at $y = 0$,

(ii) φ is a semi-invariant under the action of G, and

(iii) $(Y/G, o) \cong (V, p)$, where o is the image of 0.

By [M], there is a linear form $l \in R$ such that $\varphi + l$ is also a semi-invariant. We set

$$\mathscr{Y} = \{(y, t) \in \mathbb{C}^4 \times \mathbb{C}; \varphi(y) + tl(y) = 0\}.$$

Let G act on $\mathscr{Y}$ by $g(y, t) = (g(y), t)$ for $g \in G$, and set $\mathscr{X} = \mathscr{Y}/G$. Let $\beta : \mathscr{Y} \to \mathbb{C}$ be the projection given by $\beta(y, t) = t$, and let $\alpha : \mathscr{X} \to \mathbb{C}$ be the induced morphism. There are a G-invariant open neighborhood U of $0 \in \mathbb{C}^4$ and a positive number ε such that

$$\mathscr{Y}_1 := \{(y, t) \in \mathscr{Y}; y \in U \text{ and } |t| \leqq \varepsilon\}$$

is non-singular outside $\{0\}$ and that the action of G is free outside a subvariety T which is finite over $\{t \in \mathbb{C}; |t| \leqq \varepsilon\}$ by β. We set $\mathscr{X}_1 = \mathscr{Y}_1/G$. Then $(\alpha^{-1}(0), o) \cong (V, p)$, and $\alpha^{-1}(t) \cap \mathscr{X}_1$ has n terminal quotient singularities for each t with $0 < |t| < \varepsilon$ [M]. Let $\bar{\mathscr{Y}}$ be the completion of $\mathscr{Y}$ in $\mathbb{P}^5$ and let $\bar{\mathscr{X}} = \bar{\mathscr{Y}}/G$. We let also

$$\mathscr{Y}_2 = \{(y, t) \in \mathscr{Y}; y \in U \text{ and } \varepsilon' \leqq |t| \leqq \varepsilon\}$$

for some $0 < \varepsilon' < \varepsilon$, and $\mathscr{X}_2 = \mathscr{Y}_2/G$.

Let $\mu : \mathscr{Z} \to \bar{\mathscr{X}} - \mathscr{X}_2$ be a projective bimeromorphic morphism of complex analytic spaces which is obtained by patching a desingularization of $\bar{\mathscr{X}} - \mathscr{X}_1$ and an isomorphism over $\mathscr{X}_1 - \mathscr{X}_2$. Thus $\mathrm{Sing}(\mathscr{Z})$ is finite over $\{t \in \mathbb{C}; |t| < \varepsilon'\}$. By Bertini's theorem, there are a positive number ε'' and a G-invariant polynomial $P \in \mathbb{C}[X_1, \ldots, X_4, T]$ which has sufficiently small coefficients and vanishes at $(0, 0)$ with sufficiently high multiplicity such that

(i) $Y_s := \{(y, t) \in \mathscr{Y}; t = P(y, t) + s\}$ does not intersect $\mathscr{Y}_2$ for $|s| < \varepsilon''$,

(ii) $\bar{Y}_s$ being the closure of Y_s in $\bar{\mathscr{Y}}$, the singular locus of $Z_s := \mu^{-1}(\bar{Y}_s/G)$ consists of $\mathrm{Sing}(\mathscr{Z}) \cap Z_s$ for $|s| < \varepsilon''$, and

(iii) letting $S = \{s \in \mathbb{C}; |s| < \varepsilon''\}$ and $\mathscr{Z}_S = \bigcup_{s \in S} Z_s \subset \mathscr{Z}$, the projection $f : \mathscr{Z}_S \to S$ is flat.

We set $\mathrm{Sing}(Z_0)=\{z_0\}$ and $\mathrm{Sing}(Z_s)=\{z_s^{(1)},\dots,z_s^{(n)}\}$ for $s\neq 0$. By [HR], if the degree of zero of P at $(0,0)$ is sufficiently high, then $(Z_0,z_0)\cong(V,p)$ and the $(Z_s,z_s^{(j)})$ are terminal quotient singularities for $s\neq 0$ and $1\leqq j\leqq n$. Then by the deformation invariance of χ, we have

$$\chi(Z_0,\mathcal{O}_{Z_0}(mrK_{Z_0}))=\chi(Z_s,\mathcal{O}_{Z_s}(mrK_{Z_s}))\quad\text{for all}\quad m\in\mathbb{N}.$$

By the proof of Lemma 2.1, we obtain

$$\delta(Z_0,z_0)=\sum_j\delta(Z_s,z_s^{(j)}).\quad\text{Q.E.D.}$$

The following theorem is an easy corollary of Lemmas 2.2 and 2.3.

Theorem 2.4. *Let X be a normal compact complex analytic space of dimension 3. Assume that X has at most terminal singularities and that $K_X\approx 0$. Then*

$$\chi(X,\mathcal{O}_X)=\tfrac{1}{24}\sum\left(r(x)-\frac{1}{r(x)}\right),$$

where the summation $\sum$ is taken over all virtual terminal quotient singular points x of indices $r(x)$ on X as explained at the end of Sect. 1.

3. Proof of the Main Result

Theorem 3.1. *Let X be a normal projective 3-fold over $\mathbb{C}$ having at most terminal singularities. Assume that $K_X\approx 0$. Then*

$$0\leqq\chi(X,\mathcal{O}_X)\leqq 4.$$

Proof. The first inequality is an easy consequence of Theorem 2.4. It follows from [U] (cf. [K1] and [K4, Theorem 8.2]) that there is a positive integer m such that $mK_X\in\mathrm{Div}(X)$ and $mK_X\sim 0$. Let r be the global index of X and let $\pi: Y\to X$ be a canonical cover. Since Y is Cohen-Macauley, we have

$$H^2(Y,\mathcal{O}_Y)^*\cong H^1(Y,\omega_Y)\cong H^1(Y,\mathcal{O}_Y),$$

by [HA]. On the other hand, we have a bound $\dim H^1(Y,\mathcal{O}_Y)\leqq 3$ by [U] (cf. [K1]). Since $\pi_*\mathcal{O}_Y\cong\bigoplus_{j=0}^{r-1}\mathcal{O}_X(-jK_X)$, we have $\dim H^2(X,\mathcal{O}_X)\leqq 3$ and we finish the proof. Q.E.D.

Theorem 3.2. *There exists a positive integer m_0 such that*

$$P_{m_0}(X):=\dim H^0(X,\mathcal{O}_X(m_0K_X))\neq 0$$

for all X which satisfy the conditions of Theorem 3.1.

Remark 3.3. A candidate for m_0 is the l.c.m. of m_1 and $12m_2$ defined below.

(1) Let m_1 be the l.c.m. of all the positive integers r such that $\varphi(r)\leqq 44$, where $\varphi(r)$ is the number of generators of $\boldsymbol{\mu}_r$.

(2) Let us consider an equation

$$\sum_{j=1}^{t}\left(r_j-\frac{1}{r_j}\right)=24n,$$

where t and the r_j are unknown positive integers and $n=1,2,3$ or 4. It is easy to see that there are only a finitely many solutions $\{r_1^{(k)}, \ldots, r_{t_k}^{(k)}\}$ $(1 \leqq k \leqq N)$. Let m_2 be the l.c.m. of all the $r_j^{(k)}$.

Proof. Let X be a 3-fold which satisfies the conditions of Theorem 3.1. We consider two cases separately.

Case (1): $\chi(X, \mathcal{O}_X)=0$.

We may assume that $H^3(X, \mathcal{O}_X) \cong H^0(X, \omega_X)^* = 0$. Then $H^1(X, \mathcal{O}_X) \neq 0$, since $H^0(X, \mathcal{O}_X) \neq 0$. By [U] and [K4, Theorem 8.3], the Albanese morphism $\alpha: X \to A$ is an etale fiber bundle over an abelian variety A. By assumption, $\dim A = 1$ or 2. Thus the fiber F of α is either an elliptic curve or a minimal surface with $\kappa=0$. In particular, X is non-singular, and there is a finite etale morphism $A \times F \to X$.

Let $\pi: Y \to X$ be a canonical cover, let $H = H^3(Y, \mathbb{Z})$/torsion, and let $G \cong \mathbb{Z}/(r)$ be the Galois group $\mathrm{Gal}(Y/X)$ with $r = I(X)$. Then there is either an abelian variety or a product of a $K3$ surface and an elliptic curve which is finite and etale over Y. Hence $\operatorname{rank} H \leqq 44$. The action of G on Y induces a representation $G \to GL(H)$. By assumption, G acts faithfully on $H^0(Y, \mathcal{O}_Y(K_Y)) \subset H_{\mathbb{C}}$; θ being a nowhere vanishing section of $\mathcal{O}_Y(K_Y)$, we have $g^*\theta = \zeta\theta$ for some generators g and ζ of G and $\boldsymbol{\mu}_r$, respectively. Thus the eigenvalue ζ of g has a minimal equation over $\mathbb{Q}$ of degree at most 44. Hence $\varphi(r) \leqq 44$ and $r | m_1$. This proves the first case.

Case (2): $\chi(X, \mathcal{O}_X) \neq 0$.

Let $i(X)$ and $I(X)$ be the local and global indices of X. By Theorem 2.4, $i(X)|m_2$. Suppose that $I(X) \nmid 12i(X)$. Then $H^0(X, \mathcal{O}_X(mi(X)K_X)) = 0$ for $m = 1$, 2, 3, and 4. Since $\chi(X, \mathcal{O}_X(mi(X)K_X)) = \chi(X, \mathcal{O}_X) > 0$ for $m \in \mathbb{Z}$, we have

$$\dim H^2(X, \mathcal{O}_X(mi(X)K_X)) > 0 \quad \text{for} \quad m = 1,\ 2,\ 3, \text{ and } 4.$$

Let $\pi: Y \to X$ be a canonical cover. Since $I(X) > 4i(X)$, we have

$$\sum_{m=1}^{4} \dim H^2(X, \mathcal{O}_X(mi(X)K_X)) \leqq \dim H^2(Y, \mathcal{O}_Y) \leqq 3,$$

as in the proof of Theorem 3.1, a contradiction. Thus $I(X)|12i(X)$, hence $P_{12m_2}(X) \neq 0$. Q.E.D.

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A Remark on Kawamata's Paper "On the Plurigenera of Minimal Algebraic 3-Folds with $K \approx 0$"

David R. Morrison*

Department of Mathematics, Duke University, Durham, NC 27706, USA

Let X be a complex projective threefold having at most terminal singularities whose canonical divisor is numerically trivial. Kawamata [4] has shown that there is a natural number m_0 (independent of X) such that the m_0^{th} plurigenus $P_{m_0}(X)$ is always nonzero. Kawamata's choice of m_0 can in principle be computed explicitly, although it is clear from his formulation that it would be quite unpleasant to do so.

In this note, we combine Kawamata's method with an analysis of certain finite group actions and prove:

Theorem 1. *If* $\chi(\mathcal{O}_X) \geqq 2$ *then* $P_{12}(X) \neq 0$.

Theorem 2. *If* $\chi(\mathcal{O}_X) = 1$ *then* $P_{120}(X) \neq 0$.

In the case that $\chi(\mathcal{O}_X) = 0$ and $p_g(X) = P_1(X) = 0$, Kawamata proves that X is nonsingular, and finds a number m_1 for which $P_{m_1}(X)$ is nonzero. A slight improvement of Kawamata's choice of m_1 has been obtained by Beauville [1, Proposition 8] (see also Ueno [8]): one may take

$$m_1 = \text{l.c.m.}\ \{m : \phi(m) \leqq 20\} = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$$

where $\phi(m)$ is the number of primitive m^{th} roots of unity. Since this number m_1 is divisible by 120, and $\chi(\mathcal{O}_X) \geqq 0$ for all X with terminal singularities and numerically trivial canonical divisor, we get the

Corollary. *If X is a complex projective threefold having at most terminal singularities whose canonical divisor is numerically trivial, then $P_{m_0}(X) \neq 0$, where $m_0 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ is Beauville's number.*

1. Irregular Canonical Covers

Throughout this paper, X will denote a complex projective threefold having at most terminal singularities and numerically trivial canonical divisor, and Y will be the *canonical cover* of X. Y is a threefold with trivial canonical divisor having at most

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Gorenstein terminal singularities, and $X = Y/G$ with G a cyclic group acting on Y; moreover, the induced action of G on the 1-dimensional space $H^{3,0}(Y)$ is faithful. The order of G is called the *index* of X, and denoted by $I(X)$. Note that $P_{I(X)}(X) \neq 0$.

If X is irregular, an earlier result of Kawamata [3] shows that the Albanese map $X \to \mathrm{Alb}(X)$ is an étale fiber bundle (whose fiber is a variety with numerically trivial canonical divisor). As Kawamata points out in [4], each component of the singular locus of X then has dimension at least $\dim \mathrm{Alb}(X)$. Since X has at most isolated singularities, it must in fact be smooth; this implies that $\chi(\mathcal{O}_X) = 0$.

In this section, we consider the case in which the canonical cover Y of X is irregular. The remarks in the preceding paragraph apply to Y as well, so that Y is smooth, and the Albanese map $Y \to \mathrm{Alb}(Y)$ is an étale fiber bundle.

Proposition 1. *If X is regular and the canonical cover Y of X is irregular, then $I(X)$ divides 12.*

Proof. Let g be a generator of the covering group G, and consider the action of g on the Albanese A of Y. Now $A \cong \mathbb{C}^k/\Lambda$, and the action of g is induced by an affine linear transformation $z \mapsto \alpha z + \beta$ on $\mathbb{C}^k$. Suppose that $g(w) \neq w$ for all $w \in A$. Then the linear equation $(\alpha - I)z = -\beta$ has no solution, which implies that $\alpha - I$ is singular and 1 is an eigenvalue of α. But in that case, Y has a G-invariant holomorphic 1-form which descends to a 1-form on X. This contradicts our hypothesis that X is regular; hence, the action of g has a fixed point w_0 on A.

If $\dim A = 3$ (so that $Y = A$) then X has a terminal quotient singularity at the image of w_0. By [2] and [7], the representation of G on the tangent space $T_{w_0}(Y) \cong H^{1,0}(Y)^*$ decomposes into 1-dimensional complex representations with characters χ, χ^{-1}, and χ^a for some primitive characters χ and χ^a of G. This implies that the representation of G on $H^1(Y) \cong H^{1,0}(Y) \oplus \overline{H^{1,0}(Y)}$ decomposes into representations with characters χ^{-1}, χ, χ^{-a}, χ, χ^{-1}, and χ^a. This latter representation is defined over $\mathbb{Q}$; moreover, since some characters are repeated in the decomposition over $\mathbb{C}$, the representation must be reducible over $\mathbb{Q}$. This implies that $\phi(I(X))$ divides $\dim H^1(Y)$ and is strictly less than $\dim H^1(Y)$. Since $\dim H^1(Y) = 6$, $\phi(I(X)) \leqq 3$ and hence $I(X)$ divides 12.

We next claim that it is not possible for the dimension of A to be 2. For in that case, $Y \cong (B \times E)/\Gamma$ for some abelian surface B, some elliptic curve E, and some abelian group Γ which acts trivially on $H^1(B)$ but non-trivially on $H^1(E)$. Γ then acts trivially on $H^{2,0}(B)$ and so acts non-trivially on $H^{2,0}(B) \otimes H^{1,0}(E) \cong H^{3,0}(B \times E)$. But since $H^{3,0}(B \times E)$ is 1-dimensional, $H^{3,0}(Y) \cong H^{3,0}(B \times E)^{\Gamma} = \{0\}$, a contradiction. (See also [8].)

Finally, if $\dim A = 1$, then g induces an automorphism of an elliptic curve A which has a fixed point w_0; as is well known, this implies that $I(X) = \mathrm{ord}(g)$ divides 12. Q.E.D.

Proof of Theorem 1. As noted at the beginning of the section, since $\chi(\mathcal{O}_X) \neq 0$, it follows from [3] that X is regular. Now if Y is regular, then $h^{1,0}(Y) = h^{2,0}(Y) = 0$; since holomorphic forms on X pull back to holomorphic forms on Y, we see that $h^{1,0}(X) = h^{2,0}(X) = 0$. But then $\chi(\mathcal{O}_X) = 1 - h^{3,0}(X) \leqq 1$, contrary to hypothesis. Thus, X is regular and Y is irregular, and the theorem follows from Proposition 1. Q.E.D.

2. Covers of Terminal Singularities

Let (V, P) be a germ of a three-dimensional terminal singularity of index r. The *local canonical cover* of (V, P) is a cyclic cover $(W, Q) \to (V, P)$ of degree r, with (W, Q) a Gorenstein terminal singularity. (W, Q) is isomorphic to the germ of a hypersurface $\{\Phi = 0\}$ in $\mathbb{C}^4$ at the origin, and the action of the covering group G extends to $\mathbb{C}^4$; we call (V, P) a *quotient singularity* if $\{\Phi = 0\}$ is smooth at the origin, an *ordinary singularity* if Φ may be chosen to be G-invariant, and an *exceptional singularity* if it is not ordinary. Mori's classification theorem [5] implies that every exceptional singularity has index 4, with $\Phi = x^2 + y^2 + f(z, u^2)$ and with a generator of G acting on $\mathbb{C}^4$ by $(x, y, z, u) \mapsto (ix, -iy, -z, iu)$, where $i = \sqrt{-1}$.

Each terminal singularity of index $r > 1$ can be deformed to a collection of terminal quotient singularities [5, 4]. In the ordinary case, these all have index r, and we call the number of singularities in such a deformation the *weight* of (V, P). In the exceptional case, the deformation has 1 quotient singularity of index 4, and w quotient singularities of index 2; we call w the *weight* in this case.

Proposition 2 (S. Mori). *Let (V, P) be a three-dimensional terminal singularity of index r and weight w. Suppose that $r = mn$ with $m > 1$ and $n > 1$. Let (W, Q) be the canonical cover with covering group G, and let (V', P') be the quotient of (W, Q) by $G' \cong \mathbb{Z}/n\mathbb{Z} \subset G$. Then*

(i) *(V', P') is an ordinary terminal singularity of index n.*
(ii) *If (V, P) is ordinary, then (V', P') has weight w.*
(iii) *If (V, P) is exceptional, then $m = n = 2$ and (V', P') has weight $2w + 1$.*

Proof. Suppose that (V, P) is ordinary. Since r is a composite number, Mori's classification [5] shows that $\Phi = xy + f(z, u^r)$ and a generator of G acts on $\mathbb{C}^4$ by $(x, y, z, u) \mapsto (\zeta x, \zeta^{-1} y, z, \zeta^a u)$ for some primitive r^{th} roots of unity ζ and ζ^a. After replacing z by $z + \lambda u^r$ for some constant λ, if necessary, a deformation giving quotient singularities is given by the same action on the hypersurface $\{\Phi = tz\}$. For $t \neq 0$, the fixed points of G on this hypersurface are given by $x = y = u = f(z, 0) = 0$; there are w of these. But then it is clear that the number w depends on the equation Φ and not on the group G, proving (i) and (ii) in this case.

If (V, P) is exceptional, then $r = 4$ so that $m = n = 2$. The description of Φ and the action of G given at the beginning of this section shows that a generator g of G sends Φ to $-\Phi$; hence, g^2 preserves Φ so that (V', P') is ordinary. Again after replacing z by $z + \lambda u^r$, a deformation giving quotient singularities is given by $\{\Phi = tz\}$, with the same action of G. Now g has a single fixed point (at the origin) on the hypersurface $\{\Phi = tz\}$ for $t \neq 0$, but g^2 has additional fixed points at $x = y = u = f(z, 0) = 0$; the number of these is $2w$ (since they are exchanged in pairs by the action of g). Thus, g^2 has a total of $2w + 1$ fixed points on $\{\Phi = tz\}$ for $t \neq 0$, so that (V', P') has weight $2w + 1$. Q.E.D.

3. Regular Canonical Covers

We now return to the global situation considered in Sect. 1: X has at worst terminal singularities and has numerically trivial canonical divisor, Y is the canonical cover of X, G is the covering group, and $I(X)=|G|$ is the index of X. For each $r>1$, we define

$\tau_r(X)$ = the sum of the weights of the ordinary singular points of X having index r,
$\tau_{\mathrm{exc}}(X)$ = the sum of the weights of the exceptional singular points of X,
$\tau'_{\mathrm{exc}}(X)$ = the number of exceptional singular points of X,

and also define

$$\sigma_r(X)=\begin{cases}\tau_2(X)+\tau_{\mathrm{exc}}(X) & \text{if } r=2\\ \tau_4(X)+\tau'_{\mathrm{exc}}(X) & \text{if } r=4\\ \tau_r(X) & \text{if } r\neq 2,4.\end{cases}$$

Then Kawamata's theorem [4] on the holomorphic Euler characteristic says:

$$\chi(\mathcal{O}_X)=\frac{1}{24}\sum_r (r-r^{-1})\,\sigma_r(X). \tag{*}$$

For each d dividing $I(X)$, we define $G_d\cong\mathbb{Z}/d\mathbb{Z}\subset G$, and let $X_d=Y/G_d$, so that $X=X_{I(X)}$ and $Y=X_1$.

Proposition 3. *Suppose that for each $d\neq 1$ dividing $I(X)$, we have $\chi(\mathcal{O}_{X_d})=1$.*

(i) *If $I(X)$ is a prime power, then $I(X)$ and the collection of nonzero $\sigma_r(X)$'s must be one of the following*:

(1) $I(X)=2$, $\sigma_2(X)=16$
(2) $I(X)=3$, $\sigma_3(X)=9$
(3) $I(X)=4$, $\sigma_2(X)=6$, $\sigma_4(X)=4$
(4) $I(X)=5$, $\sigma_5(X)=5$
(5) $I(X)=8$, $\sigma_2(X)=3$, $\sigma_4(X)=1$, $\sigma_8(X)=2$.

(ii) *$I(X)$ divides* 120.

(iii) *If more than one prime divides $I(X)$, then $I(X)$ and the collection of nonzero $\sigma_r(X)$'s must be one of the following*:

(6) $I(X)=6$, $\sigma_2(X)=5$, $\sigma_3(X)=4$, $\sigma_6(X)=1$
(7) $I(X)=10$, $\sigma_2(X)=3$, $\sigma_5(X)=2$, $\sigma_{10}(X)=1$
(8) $I(X)=12$, $\sigma_2(X)=2$, $\sigma_3(X)=2$, $\sigma_4(X)=1$, $\sigma_{12}(X)=1$.

Proof. We begin by associating to X a finite set $V(X)$ with a G-action, called the *virtual fixed point set*. We describe $V(X)$ by describing its G-orbits, which are determined by the orders of their stabilizers. To each ordinary singular point of X of index $r>1$ and weight w, we associate w G-orbits with stabilizer of order r; to each exceptional singular point of weight w we associate one G-orbit with stabilizer of order 4, and w G-orbits with stabilizer of order 2. Note that $\sigma_r(X)$ is then the number of G-orbits in $V(X)$ with stabilizer of order r.

Let $\pi\colon Y\to X$ and $\pi_d\colon X_d\to X$ be the natural maps. If P is a singular point of X of index r, then $\pi^{-1}(P)$ consists of $I(X)/r$ points which form an orbit for the action of G on Y. If $n=\gcd(d,r)$, then the action of G_d on this set of points breaks up into several G_d-orbits, each with stabilizer of order n. Thus, if P' is a singular point of X_d of index n, then $P=\pi_d(P')$ is a singular point of X of index r for some r such that $n=\gcd(d,r)$. Moreover, if (V',P') is the germ of X_d at P' and $V=\pi_d(V')$, then the induced map $\pi_d\colon(V',P')\to(V,P)$ is a cover of the type considered in Proposition 2.

Let P' be a singular point of X_d of index n, and let r be the index of $P=\pi_d(P')$. If P is ordinary, each of the G-orbits in $V(X)$ corresponding to P breaks up into a set of G_d-orbits which may be identified with the finite set $\pi_d^{-1}(P)$; the stabilizers of these G_d-orbits all have order n. Since the weights of P and P' are the same by Proposition 2, we may identify these G_d-orbits with the orbits associated to the points $\pi_d^{-1}(P)$ in $V(X_d)$. If P is exceptional and $n=4$, the map $(V', P') \to (V, P)$ is an isomorphism and the action of G_d on the G-orbits in $V(X)$ corresponding to P produces a set of G_d-orbits of the appropriate type for each point in $\pi_d^{-1}(P)$. Finally, if P is exceptional and $n=2$, the G-orbit with stabilizer of order 4 breaks into $I(X)/2d$ G_d-orbits with stabilizer of order 2 while each of the w G-orbits with stabilizer of order 2 breaks into $I(X)/d$ G_d-orbits with stabilizer of order 2. We get a total of $(2w+1)\, I(X)/2d\, G_d$-orbits, each with stabilizer of order 2, corresponding to $I(X)/2d$ points in $\pi_d^{-1}(P)$. Since the weight of each of these points is $2w+1$ by proposition 2, we may again identify these G_d-orbits with the orbits associated to the points $\pi_d^{-1}(P)$ in $V(X_d)$.

We conclude that the union of all G_d-orbits in $V(X_d)$ with stabilizer of order $n>1$ can be identified with the union of all G-orbits in $V(X)$ with stabilizer of order r for some r dividing $I(X)$ such that $gcd(d,r)=n$. Counting the number of points in these orbits, we find

$$\frac{d}{n}\sigma_n(X_d)=\sum \frac{I(X)}{r}\sigma_r(X), \qquad (**)$$

where the summation runs over all r dividing $I(X)$ such that $gcd(d,r)=n$.

To prove (i), suppose that $I(X)=p^k$ and let $\sigma_{i,j}=\sigma_{p^i}(X_{p^j})$. Applying equation (**) to $X_{p^{j+1}}$(instead of X) with $d=p^j$, we find:

$$\sigma_{i,j}=\begin{cases} p\,\sigma_{i,j+1} & \text{when } i<j \\ p\,\sigma_{i,j+1}+\sigma_{i+1,j+1} & \text{when } i=j. \end{cases}$$

Moreover, since $\chi(\mathcal{O}_{X_{p^j}})=1$ for all $j \geqq 1$, formula (*) implies

$$\sum_i (p^i-p^{-i})\,\sigma_{i,j}=24.$$

These equations suffice to determine $I(X)$ and $\{\sigma_r(X)\}$, as we will now show.

We have $(p-p^{-1})\sigma_{1,1}=24$ so that $\sigma_{1,1}=24p(p^2-1)^{-1}$. This is only an integer when p^2-1 divides 24, which implies $p=2$, 3, or 5. If $k=1$ then $\sigma_p(X)=\sigma_{1,1}$ is the only nonzero σ_r; this gives cases (1), (2), and (4).

If $k \geqq 2$, we have the equations

$$p\,\sigma_{1,2}+\sigma_{2,2}=\sigma_{1,1}=24p(p^2-1)^{-1}$$
$$(p-p^{-1})\sigma_{1,2}+(p^2-p^{-2})\sigma_{2,2}=24.$$

These have the unique solution

$$\sigma_{1,2}=\frac{24(p^2-p+1)}{p^2(p^2-1)}$$

$$\sigma_{2,2}=\frac{24}{p(p+1)}.$$

When $p=3$ (respectively $p=5$) we find that $\sigma_{1,2}=7/3$ (respectively $\sigma_{1,2}=21/25$), a contradiction. Thus, $k \geqq 2$ implies that $p=2$; when $I(X)=4$ we have $\sigma_2(X) = \sigma_{1,2}=6$ and $\sigma_4(X)=\sigma_{2,2}=4$, giving case (3).

If $k \geqq 3$ then $p=2$ and $\sigma_{1,k}=2^{-1}\sigma_{1,k-1}$. Thus, $\sigma_{1,k}=2^{2-k}\sigma_{1,2}=2^{-k}\cdot 24$, which implies that $k=3$ and $I(X)=8$. In this case, $\sigma_2(X)=\sigma_{1,3}=3$; the other values $\sigma_4(X)=\sigma_{2,3}$ and $\sigma_8(X)=\sigma_{3,3}$ are easily found as the unique solution to the equations

$$2\sigma_{2,3}+\sigma_{3,3}=\sigma_{2,2}=4$$

$$(3/2)\cdot 3+(15/4)\sigma_{2,3}+(63/8)\sigma_{3,3}=24.$$

The solution is $\sigma_{2,3}=1$, $\sigma_{3,3}=2$, giving case (5).

Part (ii) follows directly from part (i) by considering the covers X_{p^k} where p^{k+1} does not divide $I(X)$. We leave the proof of (iii) [which is similar to that of (i), using equations (*) and (**)] to the reader. Q.E.D.

We began our work on this note by attempting a machine computation of Kawamata's number m_0. The computer produced a complete set of solutions to the Diophantine equation

$$\sum_r (r-r^{-1})\sigma_r=24.$$

Eight of these solutions can be found in proposition 3, but there are a few solutions which do not come from the geometric situation which produced the equation. The "extra" solutions are:

(9) $\sigma_2=11$, $\sigma_4=2$
(10) $\sigma_2=1$, $\sigma_4=6$
(11) $\sigma_3=4$, $\sigma_4=2$, $\sigma_6=1$.

Further information about our computer computation can be found in [6].

Proof of Theorem 2. As noted at the beginning of Sect. 1, since $\chi(\mathcal{O}_X)\neq 0$ it follows from [3] that X is regular. If Y is irregular, the theorem follows from Proposition 1. On the other hand, if Y is regular, then $h^{1,0}(Y)=h^{2,0}(Y)=0$. Now for each $d\neq 1$ dividing $I(X)$, every holomorphic form on X_d pulls back to a holomorphic form on Y; this implies that $h^{1,0}(X_d)=h^{2,0}(X_d)=0$. Since $h^{3,0}(X_d)$ is also zero, we see that $\chi(\mathcal{O}_{X_d})=1$. The theorem now follows from Proposition 3 (ii). Q.E.D.

We close by giving an example to show that the factor of 5 occuring in Theorem 2 is really necessary. Let Y be the quintic hypersurface in $\mathbb{P}^4$ defined by the equation $x^4y+y^4z+z^4u+u^4v+v^4x=0$, and let $G\cong\mathbb{Z}/5\mathbb{Z}$ act on $\mathbb{P}^4$ by $(x,y,z,u,v)\mapsto(x,\zeta y,\zeta^2 z,\zeta^3 u,\zeta^4 v)$ where $\zeta=e^{2\pi i/5}$. Then $X=Y/G$ is a threefold with terminal singularities and numerically trivial canonical divisor, $\chi(\mathcal{O}_X)=1$ and $I(X)=5$.

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Note **added in proof.** A paper in preparation by S. Mori entitled "Flip conjecture and the existence of minimal models for threefolds" contains further results related to Proposition 2, using slightly different terminology. For ordinary terminal singularities, the notion of "axial multiplicity" in that paper coincides with our notion of "weight", but an exceptional terminal singularity of weight w has axial multiplicity $2w + 1$.

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Examples on Polynomial Invariants of Knots and Links

Taizo Kanenobu*

Department of Mathematics, Kyushu University 33, Fukuoka 812, Japan

In 1984, Jones [12] discovered a polynomial invariant of the isotopy type of an oriented knot or link in a 3-sphere from a representation of the braid group into certain finite dimensional von Neumann algebras. The Jones polynomial is also calculated by a recursive formula involving changing crossings in a projection of the knot or link until the unlink is obtained. Since then, some new polynomial invariants have been discovered. Generalizing Jones' method directly or using a recursive formula, Freyd et al. [7], and Przytycki and Traczyk [26] discovered at almost the same time independently a 2-variable Laurent polynomial invariant of the isotopy type of an oriented knot or link. This 2-variable Jones polynomial specializes to both the Jones and the classical reduced Alexander polynomials. The latter is also produced by the many valuable Alexander polynomial of a link. Adopting a more general recursive formula, Brandt et al. [2] and Ho [10] discovered a polynomial invariant of an unoriented knot or link, which is called the Q or absolute polynomial, see Sect. 2. And most recently Kauffman [15] discovered a 2-variable polynomial invariant which specializes to both the Jones polynomial and the Q polynomial [17]. The purpose of this paper is to consider the difference and the similarity of these polynomial invariants except for the Kauffman polynomial through several examples.

The 2-variable Jones polynomial, and therefore, the Jones and the reduced Alexander polynomials are skein invariant, see Sect. 1, but not the Q polynomial. In Sect. 4, we consider a family of ribbon knots $K(a, b)$, which is a generalization of the family of knots given by the author in [14]. This family can be completely classified up to skein equivalence by either the 2-variable Jones polynomial or the Jones and the Alexander polynomials. Each of the skein equivalent class contains infinitely many knots, and especially the knots $K(a+1, -a)$, $a=0, 1, 2, \ldots$, which are all skein equivalent, are completely classified by the Q polynomial (Theorem 3).

The V_∞ formula for the Jones polynomial (Corollaries 1.1 or 1.2) devised by Jones and Birman, is a special (or weak, in a sense) point compared with the reduced Alexander polynomial. In Sect. 5, using this formula, we give an example of

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arbitrarily many prime knots with the same Jones but distinct Alexander polynomials.

In Sect. 6, we consider 2-bridge knots and links which are classified by Schubert [28]. We construct arbitrarily many 2-bridge knots with the same Jones and Q polynomials (Theorem 6) and arbitrarily many skein equivalent fibered 2-bridge links with the same 2-variable Alexander and Q polynomials, and therefore, the same 2-variable Jones, Jones and reduced Alexander polynomials (Theorem 7). Also we give a pair of skein equivalent 2-bridge links with the same Q polynomial, and therefore, the same 2-variable Jones, Jones and reduced Alexander polynomials, but distinct 2-variable Alexander polynomials (Theorem 8). Lastly, we give an example of a fibered 2-bridge link which is transformed into a non-isotopic link by reversing the orientation of one component, but whose skein equivalent class and 2-variable Alexander polynomial do not change (Theorem 9).

1. Jones and 2-Variable Jones Polynomials

We use the word *tangle* to mean a pair (B, α) where B is a 3-ball and α is a set of arcs and zero or more loops properly embedded in B, see [4, 24, 25]. The rectangle labelled n stands for a 2-string integral tangle with n half twists as shown in Fig. 1.

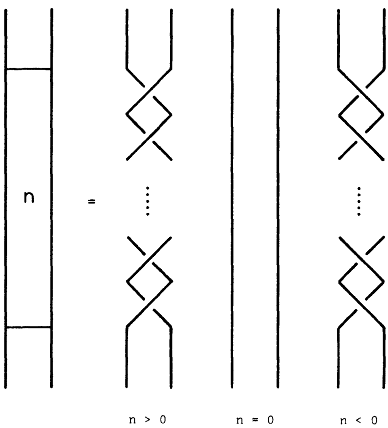


Fig. 1

The Jones polynomial $V(L; t)$ [12] is an invariant of the isotopoy type of an oriented knot or link L in a 3-sphere S^3, which is defined by the following formulas:

$$V(U) = 1 \quad \text{for the unknot } U; \tag{1.1}$$

$$t^{-1} V(L_-) - t V(L_+) = (t^{1/2} - t^{-1/2}) V(L_0), \tag{1.2}$$

where the L_i are identical except near one point where they are as in Fig. 2. We call (L_+, L_-, L_0) a *skein triple*.

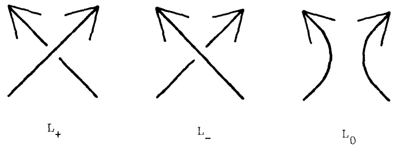


Fig. 2

The Jones polynomial was generalized to a 2-variable Laurent polynomial invariant [7], which we call the *2-variable Jones polynomial*. Here, we follow the definition of Lickorish and Millett [18]: The 2-variable Jones polynomial $P(L; l, m) \in Z[l^{\pm 1}, m^{\pm 1}]$ of an oriented knot or link L in S^3 is defined by

$$P(U) = 1 \quad \text{for the unknot } U; \tag{1.3}$$

$$lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0, \tag{1.4}$$

where (L_+, L_-, L_0) is a skein triple.

The classical reduced Alexander polynomial $\Delta(L; t)$ and the Jones polynomial $V(L; t)$ are given by the formulas:

$$P(L; i, i(t^{1/2} - t^{-1/2})) = \Delta(L; t), \tag{1.5}$$

$$P(L; it, i(t^{1/2} - t^{-1/2})) = V(L; t), \tag{1.6}$$

where $i = \sqrt{-1}$.

We denote by μ_V and μ_P the Jones and the 2-variable Jones polynomials of a trivial 2-component link, that is, $\mu_V = -t^{-1/2} - t^{1/2}$ and $\mu_P = -(l + l^{-1})m^{-1}$.

Proposition 1.1 [12, Theorems 6 and 3; 18, Propositions 9 and 10].

(1) $V(L_1 \# L_2) = V(L_1)V(L_2)$, $P(L_1 \# L_2) = P(L_1)P(L_2)$, *where* $L_1 \# L_2$ *is any connected sum of* L_1 *and* L_2.

(2) $V(rL; t) = V(L; t^{-1})$, $P(rL; l, m) = P(L; l^{-1}, m)$, *where* rL *is the mirror image of* L.

Skein equivalence [4, 8, 18] is the smallest equivalence relation '$\sim$' on the set of all oriented links in S^3 such that (i) if L and L' are ambient isotopic, which we denote by $L \# L'$, then $L \sim L'$;

(ii) if (L_+, L_-, L_0) and (L'_+, L'_-, L'_0) are skein triples then
(a) $L_+ \sim L'_+$ and $L_0 \sim L'_0$ implies $L_- \sim L'_-$, and
(b) $L_- \sim L'_-$ and $L_0 \sim L'_0$ implies $L_+ \sim L'_+$.

Proposition 1.2 [18, Proposition 15]. *The 2-variable Jones polynomial, and therefore, the Jones and the reduced Alexander polynomials are skein invariants.*

Any oriented link L is skein equivalent to the link obtained by reversing the orientation of every component of L. *Mutation*, see [8, 18], consists of removing a tangle with 2 arcs and zero or more loops from a link, rotating it through angle π, and replacing it. The skein equivalence class is unchanged by mutation. The 2-fold covering spaces of S^3 branched over two mutants are homeomorphic [21, 31].

Lemma 1.1. *Let $\tilde{L}_n$ be the link as shown in Fig. 3, where T is any tangle. Let $\tilde{V}_n$ be its Jones polynomial. Then*

$$\tilde{V}_{2k} = \mu_V^{-1}(1 - t^{2k}) V_\infty + t^{2k} \tilde{V}_0,$$

$$\tilde{V}_{2k+1} = \mu_V^{-1}(1 - t^{2k}) V_{-\infty} + t^{2k} \tilde{V}_1,$$

where $V_{\pm\infty}$ are the Jones polynomials of the links $L_{\pm\infty}$ as shown in Fig. 4.

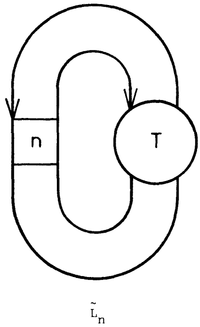


Fig. 3

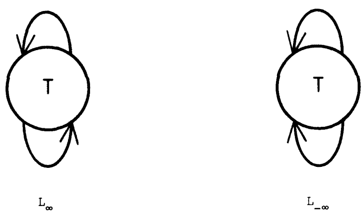


Fig. 4

Proof. From (1.2), $t^{-1}\tilde{V}_n - t\tilde{V}_{n-2} = (t^{1/2} - t^{-1/2}) V(-1)^n \infty$. Then

$$\begin{aligned} \tilde{V}_n - \mu_V^{-1} V(-1)^n \infty &= t^2 (\tilde{V}_{n-2} - \mu_V^{-1} V(-1)^n \infty) \\ &= \begin{cases} t^n (\tilde{V}_0 - \mu_V^{-1} V_\infty) & \text{if } n \text{ is even,} \\ t^{n-1} (\tilde{V}_1 - \mu_V^{-1} V_{-\infty}) & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and we obtain the formula. □

Theorem 1. *Let L_n be the link as shown in Fig. 5, where T is any tangle. Let V_n be its Jones polynomial. Let c_n be the number of the components of L_n. Let λ_n be the linking number of the component K_n with the remainder of L_n.*

Case 1. $c_0 > c_1$.

$$V_n = t^{-3(\lambda_0 + n/2)} (\mu_V^{-1}(1 - (-t)^n) V_\infty + (-t)^n \tilde{V}_0).$$

Case 2. $c_0 < c_1$.

$$V_n = t^{-3(\lambda_1 + n/2 - 1/2)} (\mu_V^{-1}(1 - (-t)^{n-1}) V_{-\infty} + (-t)^{n-1} \tilde{V}_1).$$

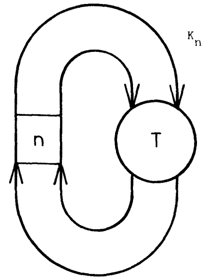


Fig. 5

Here $V_{\pm\infty}$, $\tilde{V}_0$, and $\tilde{V}_1$ are the Jones polynomials given in Lemma 1.1.

Proof. For Case 1, since $\lambda_{2k} = \lambda_0 + k$, by the Jones reversing result, see [19, 22], we have

$$\begin{aligned} V_{2k} &= t^{-3(\lambda_0+k)} \tilde{V}_{2k} \\ &= t^{-3(\lambda_0+k)} (\mu_V^{-1}(1-t^{2k}) V_\infty + t^{2k} \tilde{V}_0). \end{aligned}$$

From (1.2),

$$t^{-1} V_{2k} - t V_{2k+2} = (t^{1/2} - t^{-1/2}) V_{2k+1}.$$

Substituting the above formulas, we have

$$V_{2k+1} = t^{-3\lambda_0 - 3k - 1/2} (\mu_V^{-1} (t^{2k} + t^{-1}) V_\infty - t^{2k} \tilde{V}_0),$$

and we obtain the desired formula.

For case 2, we can prove similarly and the proof is complete. □

From Theorem 1, we can easily deduce the V_∞ formula discovered by Jones and Birman [1], see also [17, 20].

Corollary 1.1.

$$t^{1/2} V_1 - t^{-1/2} V_{-1} = \begin{cases} (t^{1/2} - t^{-1/2}) t^{-3\lambda_0} V_\infty & \text{for Case 1,} \\ (t^{1/2} - t^{-1/2}) t^{-3(\lambda_1 - 1/2)} V_{-\infty} & \text{for Case 2.} \end{cases}$$

Combining (1.2) and Corollary 1.1, we have

Corollary 1.2. *For Case 1,*

$$V_1 = -t^{-1/2} V_0 - t^{-3\lambda_1 - 1} V_\infty,$$

$$V_{-1} = -t^{1/2} V_0 - t^{-3\lambda_0 + 1} V_\infty.$$

For Case 2,

$$V_1 = -t^{-1/2} V_0 - t^{-3\lambda_1 + 1/2} V_{-\infty},$$

$$V_{-1} = -t^{1/2} V_0 - t^{-3\lambda_{-1} - 1/2} V_{-\infty}.$$

2. Q Polynomial

The Q polynomial or the absolute polynomial $Q(L;x) \in Z[x^{\pm 1}]$ [2, 10] is an invariant of the isotopy type of an unoriented knot or link in S^3, which is defined by the following formulas:

$$Q(U) = 1 \quad \text{for the unknot } U; \tag{2.1}$$

$$Q(L_+) + Q(L_-) = x(Q(L_0) + Q(L_\infty)), \tag{2.2}$$

where the L_i are identical except near one point where they are as in Fig. 6. The Q polynomial has the following property:

Proposition 2.1 [2, Property 1].
(1) $Q(L_1 \# L_2) = Q(L_1)\,Q(L_2)$.
(2) $Q(rL) = Q(L)$.
(3) *If L_2 is a mutant of L_1, then $Q(L_1) = Q(L_2)$.*

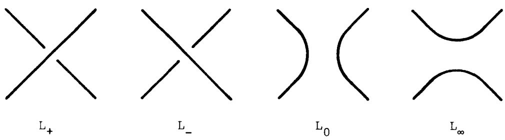


Fig. 6

We use the notation $\mu_Q = 2x^{-1} - 1$ for the Q polynomial of a trivial 2-component link. Let Q_n be the Q polynomial of the unoriented link of Figs. 4 and 5. Then from (2.2),

$$Q_{n-1} + Q_{n+1} = xQ_n + xQ_\infty .$$

Let $R_n = Q_n - \mu_Q^{-1} Q_\infty$. Then we have

$$R_{n+1} - xR_n + R_{n-1} = 0.$$

Putting $\alpha + \beta = x$ and $\alpha\beta = 1$, we have

$$R_{n+1} - \alpha R_n = \beta(R_n - \alpha R_{n-1}) = \beta^n (R_1 - \alpha R_0),$$

and

$$R_{n+1} - \beta R_n = \alpha(R_n - \beta R_{n-1}) = \alpha^n (R_1 - \beta R_0),$$

and therefore we have

$$R_n = \sigma_n R_1 - \sigma_{n-1} R_0,$$

where σ_n is a symmetric polynomial defined as follows:

$$\sigma_n = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\dfrac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} & \text{if } n < 0. \end{cases}$$

It is easy to see that $\sigma_{-n} = -\sigma_n$ and $x\sigma_n = \sigma_{n-1} + \sigma_{n+1}$. Thus we obtain

Proposition 2.2. $Q_n = \sigma_n Q_1 - \sigma_{n-1} Q_0 - \mu_Q^{-1}(\sigma_n - \sigma_{n-1} - 1) Q_\infty$.

3. Alexander Polynomial

The Alexander polynomial $\Delta(L; t_1, t_2, \ldots, t_\nu)$ of a ν-component link $L = L_1 \cup L_2 \cup \cdots \cup L_\nu$ in S^3 is an element of the polynomial ring $Z[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_\nu^{\pm 1}]$, and is determined only up to multiplication by a unit $\pm t_1^{n_1} t_2^{n_2} \cdots t_\nu^{n_\nu}$. The reduced Alexander polynomial $\Delta(L; t)$ is given by the following formula:

$$\Delta(L; t) = \begin{cases} (t-1)\,\Delta(L; t, t, \cdots, t) & \text{if } \nu \geqq 2, \\ \Delta(L; t) & \text{if } \nu = 1, \end{cases}$$

up to a unit $\pm t^n$.

Let $L_i = K_{i0} \cup K_{i1} \cup \cdots \cup K_{i\nu_i}$, $i = 1, 2$, be a $(\nu_i + 1)$-component link in S^3. Let $\Delta_i(t_{i0}, t_{i1}, \cdots, t_{i\nu_i})$ be its Alexander polynomial, where each t_{ij} corresponds to K_{ij}. Let L be the connected sum of L_1 and L_2 with K_{10} and K_{20} joined, which we present by $K \cup K_{11} \cup \cdots \cup K_{1\nu_1} \cup K_{21} \cup \cdots \cup K_{2\nu_2}$ using the same symbols except the joined component K. Let $\Delta(t, t_{11}, \cdots, t_{1\nu_1}, t_{21}, \cdots, t_{2\nu_2})$ be its Alexander polynomial, where t corresponds to K. Then we have

Proposition 3.1. *Suppose* $\nu_1 \geqq 1$. *Then*

$$\begin{aligned} &\Delta(t, t_{11}, \cdots, t_{1\nu_1}, t_{21}, \cdots, t_{2\nu_2}) \\ &= \begin{cases} (t-1)\,\Delta_1(t, t_{11}, \cdots, t_{1\nu_1})\,\Delta_2(t, t_{21}, \cdots, t_{2\nu_2}) & \text{if } \nu_2 \geqq 1, \\ \Delta_1(t, t_{11}, \cdots, t_{1\nu_1})\,\Delta_2(t) & \text{if } \nu_2 = 0. \end{cases} \end{aligned}$$

Proposition 3.2. *Let* $\tilde{L}_{2n} = \tilde{K}_{2n,1} \cup \tilde{K}_{2n,2} \cup \cdots \cup \tilde{K}_{2n,\nu}$ *and* $\tilde{L}_\infty = \tilde{K}_{\infty,0} \cup K_{\infty,1} \cup \cdots \cup \tilde{K}_{\infty,\nu}$ *be the links as shown in Fig. 7, where T is any tangle. Let* $\Delta_{2n}(t_1, t_2, \cdots, t_\nu)$ *and* $\Delta_\infty(t_0, t_1, \cdots, t_\nu)$ *be the Alexander polynomials of* $\tilde{L}_{2n}$ *and* $\tilde{L}_\infty$, *respectively, where* t_j *corresponds to* $\tilde{K}_{i,j}$. *Then they can be normalized so that*

$$\Delta_{2n}(t_1, t_2, \cdots, t_\nu) = n(t_1 - 1)\,\Delta_\infty(t_1, t_1, t_2, \cdots, t_\nu) + \Delta_0(t_1, t_2, \cdots, t_\nu).$$

The proofs of these propositions are standard, see [5, 30], so we omit them.

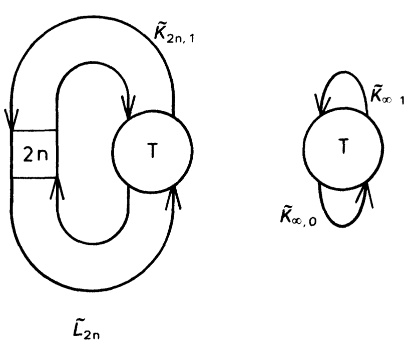


Fig. 7

4. $K(a,b)$

In this section we consider a family of knots $K(a,b)$ as shown in Fig. 8. The family of knots $K_{p,q}$ given in [14] is $K(-2p, -2q)$ in this notation. In the same way as in [14], we have

Proposition 4.1. *$K(a,b) \approx K(b,a)$ and $rK(a,b) \approx K(-a, -b)$.*

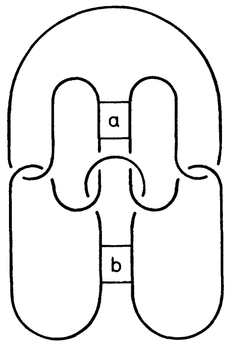


Fig. 8

The knots with small crossings [27, Appendix C] contained in this family are the following:

$$K(0,0)=4_1 \# 4_1,\ K(0,-1)=8_8,\ K(1,-1)=8_9,$$
$$K(2,-1)=10_{129},\ K(2,0)=10_{137},\ K(1,1)=10_{155},$$
$$K(2,-3)=13_{6714}.$$

For 13_{6714}, we refer [18, Fig. 37].

Professors J. Przytycki and P. Traczyk remarked the author that the two knots $K(2p,2q)$ and $K(2p',2q')$ with $p+q=p'+q'$ not only have the same 2-variable Jones polynomial but are skein equivalent. Here we have

Proposition 4.2. *Let $P(a,b)$, $V(a,b)$, and $\Delta(a,b)$ be the 2-variable Jones, Jones, and Alexander polynomials of $K(a,b)$, respectively. Let $a=2p+\varepsilon$ and $b=2q+\delta$, where ε, $\delta=0,1$. Then*

$$P(a,b)=(-l^2)^{p+q}(P(\varepsilon,\delta)-1)+1,$$

where

$$P(0,0)=((l^{-2}+1+l^2)+m^2)^2,$$
$$P(0,1)=P(1,0)=(-l^{-4}-l^{-2}+2+l^2)$$
$$+(l^{-4}+2l^{-2}-2-l^2)m^2+(-l^{-2}+1)m^4,$$
$$P(1,1)=(l^{-4}+4l^{-2}+3)+(-3l^{-4}-8l^{-2}-3)m^2$$
$$+(l^{-4}+5l^{-2}+1)m^4-l^{-2}m^6;$$
$$V(a,b)=(-t)^{a+b}(V(0,0)-1)+1,$$

where

$$V(0,0) = (t^{-2} - t^{-1} + 1 - t + t^2)^2;$$

$$\Delta(a,b) = \Delta(\varepsilon, \delta),$$

where

$$\Delta(0,0) = (t^{-1} - 3 + t)^2,$$

$$\Delta(0,1) = \Delta(1,0) = 2t^{-2} - 6t^{-1} + 9 - 6t + 2t^2,$$

$$\Delta(1,1) = -t^{-3} + 3t^{-2} - 5t^{-1} + 7 - 5t + 3t^2 - t^3.$$

Proof. For the 2-variable Jones polynomial, we calculate in the same way as in [14]. For the Jones polynomial, we use Corollary 1.2. □

Proposition 4.3. *$K(a,b) \sim K(a',b')$ iff $a+b=a'+b'$ and $a \equiv a' \pmod 2$ when $a+b \equiv 0 \pmod 2$.*

Proof. Consider a crossing in the integral tangles with a and b half twists in $K(a,b)$. Then it is easy to see that $(K(a,b),\ K(a-2,b),\ U^2)$ and $(K(a,b),\ K(a,b-2),\ U^2)$ are skein triples, where U^2 is a trivial 2-component link. Thus $K(a-2,b)$ and $K(a,b-2)$ are skein equivalent. Since $K(a,b) \approx K(b,a)$, if $a+b=a'+b'$ and $a \equiv a' \pmod 2$ when $a+b \equiv 0 \pmod 2$, then $K(a,b)$ and $K(a',b')$ are skein equivalent. Conversely, from the 2-variable Jones polynomial of $K(a,b)$, we see that the above condition is necessary for $K(a,b)$ and $K(a',b')$ to be skein equivalent. This completes the proof. □

Remarks. (1) The family $K(a,b)$ can be completely classified up to skein equivalence by either 2-variable Jones polynomial or the Jones and the Alexander polynomials.

(2) $4_1 \# 4_1$ and 8_9 may be the simplest example of knots with the same Jones polynomial.

In Example 16 of [18], 8_8, $r(10_{129})$, and 13_{6714} are given as an example of knots with the same 2-variable Jones polynomial and it is shown that 8_8 and 13_{6714} are skein equivalent and not mutant one another. It is also shown that 8_8 has unknotting number 2 and $r(10_{129})$ has 1, and so, the 2-variable Jones polynomial cannot give complete unknotting number information. Moreover, since we have shown that 8_8 and $r(10_{129})$ are skein equivalent, we can answer Question 10 of [18]:

Theorem 2. *There exist two knots with distinct unknotting numbers which are skein equivalent but not mutant one another.*

It is remarked in [14] that $K(-2p, -2q)$ is a generalized symmetric union [16] of the figure-eight knot. $K(a,0)$, where a is odd, is a symmetric skew union [16] of the figure-eight knot, and so, a ribbon knot. For other $K(a,b)$, since $(K(a,b),\ K(a-2,b),\ U^2)$ are skein triples, we have

Proposition 4.4. *$K(a,b)$ is a ribbon knot.*

Now we calculate $Q(a,b)$ the Q polynomial of $K(a,b)$. From (2.2),

$$Q(a,b) + Q(a-2,b) = xQ(a-1,b) + x\mu_Q.$$

Letting $R(a,b)=Q(a,b)-1$, we have

$$R(a,b)-xR(a-1,b)+R(a-2,b)=0. \tag{4.1}$$

In the same way as in Proposition 2.2, we have

$$\begin{aligned}R(a,b)&=\sigma_a R(1,b)-\sigma_{a-1}R(0,b)\\&=\sigma_a\sigma_{b+1}R(1,0)-\sigma_a\sigma_b R(1,-1)-\sigma_{a-1}\sigma_b R(0,1)+\sigma_{a-1}\sigma_{b-1}R(0,0).\end{aligned}$$

Since $R(1,0)=R(0,1)=R(0,-1)$, from (4.1), we have

$$xR(0,0)=2R(1,0).$$

Then we obtain

$$R(a,b)=-\sigma_a\sigma_b R(1,-1)+(\sigma_a\sigma_{b+1}-\sigma_{a-1}\sigma_b+2x^{-1}\sigma_{a-1}\sigma_{b-1})R(1,0)$$

By $\sigma_{a-1}+\sigma_{a+1}=x\sigma_a$, we have

Proposition 4.5. $Q(a,b)=-\sigma_a\sigma_b(Q(8_9)-1)+x^{-1}(\sigma_{a+1}\sigma_{b+1}+\sigma_{a-1}\sigma_{b-1})(Q(8_8)-1)+1$, *where* $Q(8_8)=1+4x+6x^2-10x^3-14x^4+4x^5+8x^6+2x^7$, $Q(8_9)=-7+4x+16x^2-10x^3-16x^4+4x^5+8x^6+2x^7$.

Corollary 4.1. $\deg Q(a,-a)=2a+5, a\geqq 1$, *and* $\deg Q(0,0)=6$, $\deg Q(a+1,-a)=2a+6, a\geqq 1$, *and* $\deg Q(1,0)=7$.

For $c\geqq 2$ and $a\geqq 0$, $Q(a+c,-a)$ cannot be distinguished by the degrees. Combining Proposition 4.2 and the above corollary, we have

Theorem 3. *There exist infinitely many knots which are skein equivalent and therefore they have the same 2-variable Jones, Jones and Alexander polynomials, but are distinguished by the Q polynomial.*

Questions. (1) Can the family $K(a,b)$ be completely classified by the 2-variable Jones and Q polynomials, or the Jones, Alexander, and Q polynomials?

(2) $K(a,b)\approx K(a',b')$ iff $(a,b)=(a',b')$ or (b',a')?

We give other properties of $K(a,b)$, which are shown in the same way as in [14]:

Proposition 4.6. *$K(a,b)$ are 3-bridge except for $K(1,0)$ and $K(1,-1)$.*

Proposition 4.7. *$K(a,b)$ are prime except for $K(0,0)$.*

5. $K(p_1, p_2, \cdots, p_n)$, $n\geqq 3$

In this section we consider a family of knots $K(p_1,p_2,\cdots,p_n)$, $n\geqq 3$, as shown in Fig. 9. The knots with small crossings [27, Appendix C] contained in this family are the following:

$$K(0,0,0)=9_{41},\ K(0,0,-1)=9_{27}.$$

It is easy to see

Proposition 5.1. $K(p_1,p_2,\cdots,p_n)\approx rK(-p_1-1,-p_2-1,\cdots,-p_n-1)$.

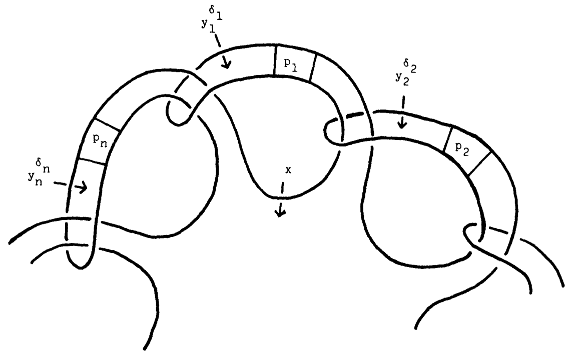


Fig. 9

Considering the integral tangle with p_i half twists in $K(p_1, \cdots, p_i, \cdots, p_n)$, we see that $(K(p_1, \cdots, p_i, \cdots, p_n),\ K(p_1, \cdots, p_i - 2, \cdots, p_n), U^2)$, $1 \leqq i \leqq n$, are skein triples; this also shows that $K(p_1, p_2, \cdots, p_n)$ is a ribbon knot. Hence $K(p_1, \cdots, p_i - 2, \cdots, p_n)$ and $K(p_1, \cdots, p_j - 2, \cdots, p_n)$ are skein equivalent. Since $K(p_1, \cdots, p_n)$ and $K(p_{\phi(1)}, \cdots, p_{\phi(n)})$ are isotopic, where ϕ is a cyclic permutation of $1, 2, \cdots, n$, we have

Proposition 5.2. *If $p_1 + p_2 + \cdots + p_n = q_1 + q_2 + \cdots + q_n$ and $p_i \equiv q_{\phi(i)} \pmod 2$ for all i, where ϕ is a cyclic permutation of $1, 2, \cdots, n$, then $K(p_1, p_2, \cdots, p_n)$ and $K(q_1, q_2, \cdots, q_n)$ are skein equivalent.*

Proposition 5.3. *Let $P(p_1, p_2, \cdots, p_n)$, $V(p_1, p_2, \cdots, p_n)$, and $\Delta(p_1, p_2, \cdots, p_n)$ be the 2-variable Jones, Jones, and Alexander polynomials of $K(p_1, p_2, \cdots, p_n)$. Let ε_i be 0 if p_i is even and 1 if p_i is odd. Let e be the number of 0 in $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$. Then*

$$P(p_1, p_2, \cdots, p_n) = (-l^2)^{\sum_{i=1}^{n}(p_i - \varepsilon_i)/2}(P(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) - 1) + 1,$$

$$V(p_1, p_2, \cdots, p_n) = (-t)^{\sum_{i=1}^{n} p_i}(V(0, 0, \cdots, 0) - 1) + 1,$$

$$\Delta(p_1, p_2, \cdots, p_n) = \Delta(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = f(t) f(t^{-1}),$$

where $f(t) = (-t)^e - (1-t)^n$.

Proof. For the 2-variable Jones polynomial, we can prove in the same way as in Proposition 4.2. For the Jones polynomial,

$$V(p_1, p_2, \cdots, p_n; t) = P(p_1, p_2, \cdots, p_n;\ it, i(t^{1/2} - t^{-1/2}))$$

$$= (-t)^{\sum_{i=1}^{n}(p_i - \varepsilon_i)}(V(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) - 1) + 1,$$

and by Corollary 1.2,

$$V(\varepsilon_1, \cdots, 1, \cdots, \varepsilon_n) = -t(V(\varepsilon_1, \cdots, 0, \cdots, \varepsilon_n) - 1) + 1,$$

and so we obtain the formula.

For the Alexander polynomial, by [9] or [11], $K(p_1, p_2, \cdots, p_n)$ and $K(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ have the same associated ribbon 2-knot, which we denote by

$K^2(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$. Using the meridians in Fig. 9, $\pi_1(S^4 - K^2(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n))$ has the following presentation [32]:

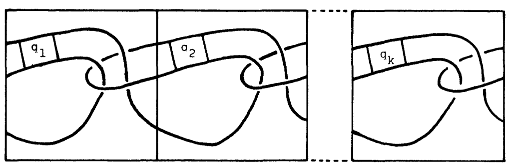
$$\langle x, y_1, y_2, \cdots, y_n; \ y_1 = y_2^{-\delta_2} x y_2^{\delta_2}, \ y_2 = y_3^{-\delta_3} x y_3^{\delta_3}, \cdots, y_n = y_1^{-\delta_1} x y_1^{\delta_1} \rangle,$$

where $\delta_j = 1 - 2\varepsilon_j$. By the free differential calculus [5], we obtain the Alexander polynomial of $K^2(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$:

$$1 - (1 - t^{-\delta_1})(1 - t^{-\delta_2}) \cdots (1 - t^{-\delta_n}),$$

which equals $f(t)$ up to $(-t)^e$. By [6], $f(t) f(t^{-1})$ is the Alexander polynomial of $K(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$. □

Proposition 5.4. *Suppose that $n \geqq 4$ and p_2, $p_j \neq 0, -1$ for some $j \geqq 4$. Then $K(p_1, p_2, \cdots, p_n)$ is a prime knot.*



$T(q_1, q_2, \cdots, q_k)$

Fig. 10

Proof. Let $T(q_1, q_2, \cdots, q_k)$ be the tangle as shown in Fig. 10. Then $K(p_1, p_2, \cdots, p_n)$ is a union of $T(p_1, p_2, \cdots, p_{j-1})$ and $T(p_j, p_{j+1}, \cdots, p_n)$. Let $(B, \alpha_1 \cup \alpha_2 \cup \alpha_3) = T(p_1, p_2)$. Since $p_2 \neq 0, -1$, there are one knotted arc, say α_1, and two unknotted arcs α_2 and α_3. The knotted factor is the 2-bridge knot with Conway's notation $C(2, p_2)$, see Sect. 6. $(B, \alpha_1 \cup \alpha_2)$ with ears, that is, the result of adding a trivial tangle to $(B, \alpha_1 \cup \alpha_2)$ on the outside of B can be a trivial knot, see Fig. 11a. $(B, \alpha_1 \cup \alpha_3)$ with ears can be a 2-bridge knot $C(2, p_2 + 2)$, which is not $C(2, p_2)$, see Fig. 11b. Then by Lemma 5.4 of [24], $(B, \alpha_1 \cup \alpha_2)$ and $(B, \alpha_1 \cup \alpha_3)$ are prime tangles. Therefore by Lemma 5.6 of [24], $(B, \alpha_1 \cup \alpha_2 \cup \alpha_3)$ is a prime tangle.

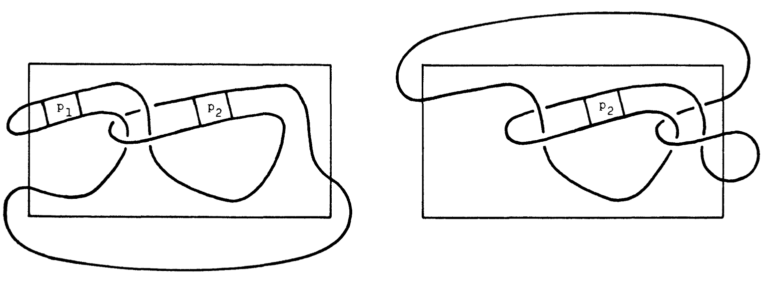


Fig. 11a, b

Applying Theorem 3 of [25], we see that $T(p_1, p_2, p_3)$ which is a union of $T(p_1, p_2)$ and $T(p_3)$, is a prime tangle. Continuing this, $T(p_1, p_2, \cdots, p_{j-1})$ is shown to be a prime tangle. Similarly, $T(p_j, p_{j+1}, \cdots, p_n)$ is also prime, and thus by Theorem 2 of [25] $K(p_1, p_2, \cdots, p_n)$ is a prime knot. This completes the proof. □

There exist a pair of knots with the same Jones polynomial but distinct Alexander polynomial. For example, $4_1 \# 4_1$ and 8_9 in Sect. 4 are such a pair. By composition, we get arbitrarily many knots with the same Jones polynomial but distinct Alexander polynomials. Combining Propositions 5.3 and 5.4, we have

Theorem 4. *There exist arbitrarily many prime knots with the same Jones polynomial but distinct Alexander polynomials.*

In [7], Lickorish and Millett gives $K = 11_{388}$ as an example of a knot such that $P(K) \neq P(rK)$, $V(K) = V(rK)$ and $\Delta(K) = \Delta(rK)$. So $K \# rK$ and $K \# K$ are a pair of knots such that one is not the mirror image of the other and that they have the same Jones and Alexander polynomials but distinct 2-variable Jones polynomials. Using $K(p_1, p_2, \cdots, p_n)$ we have

Theorem 5. *There exist a pair of prime knots such that one is not the mirror image of the other and that they have the same Alexander and Jones polynomials but distinct 2-variable Jones polynomials.*

Proof. $K(0, -3, 0, 2)$ and $K(0, -3, -1, 3)$ are such a pair. By Proposition 5.3, they have the same Alexander and Jones polynomials. For 2-variable Jones polynomials,

$$P(0, -3, 0, 2) = P(-1, 0, 0, 0)$$

and

$$\begin{aligned} P(0, -3, -1, 3) &= (-l^2)(P(0, -1, -1, -1) - 1) + 1 \\ &= (-l^2)(P(-1, 0, 0, 0; l^{-1}, m) - 1) + 1, \end{aligned}$$

since $K(0, -1, -1, -1) = rK(-1, 0, 0, 0)$. Now

$$\begin{aligned} P(-1,0,0,0) = {} & (l^{-2} + 5 + 6l^2 + 4l^4 + l^6) - (4l^{-2} + 12 + 13l^2 + 6l^4 + l^6)m^2 \\ & + (3l^{-2} + 12 + 10l^2 + 3l^4)m^4 - (l^{-2} + 5 + 3l^2)m^6 + m^8, \end{aligned}$$

and so $P(0, -3, 0, 2) \neq P(0, -3, -1, 3)$. □

6. 2-Bridge Knots and Links

According to Conway [4], one denotes a 2-bridge knot or link by $C(a_1, a_2, \cdots, a_n)$ which represents a knot or link of Fig. 12.

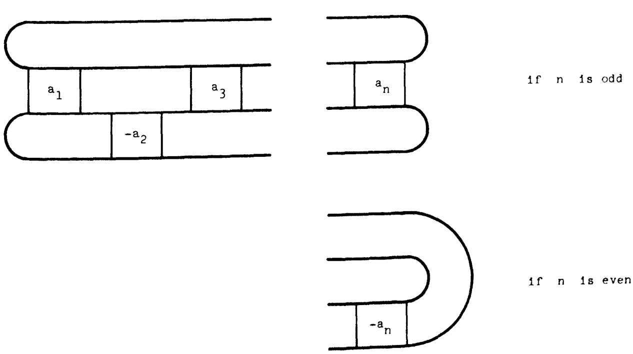


Fig. 12

Lemma 6.1. *Let f be the Q polynomial of a 2-bridge knot or link $C(a_1, a_2, \cdots, a_n)$ and Q_b that of $C(a_1, a_2, \cdots, a_n, b, -a_n, \cdots, -a_2, -a_1)$. Then*

$$Q_b = \tfrac{1}{2}(\mu_Q - \mu_Q^{-1} f^2)(\sigma_{b+1} - \sigma_{b-1}) + \mu^{-1} f^2.$$

Proof. $C(a_1, a_2, \cdots, a_n, b, -a_n, \cdots, -a_2, -a_1)$ is the mirror image of $C(-a_1, -a_2, \cdots, -a_n, -b, a_n, \cdots, a_2, a_1)$, which is isotopic to $C(a_1, a_2, \cdots, a_n, -b, -a_n, \cdots, -a_2, -a_1)$, and thus by Proposition 2.1, $Q_b = Q_{-b}$. Then applying Proposition 2.2, we have

$$Q_b = Q_{-b} = (\sigma_b Q_1 - \sigma_{b-1} Q_0) - (\sigma_b - \sigma_{b-1} - 1)\mu_Q^{-1} f^2.$$

From (2.2),

$$2Q_1 = Q_1 + Q_{-1} = xQ_0 + xf^2.$$

Then noting, $Q_0 = \mu_Q$ and $x\sigma_b = \sigma_{b+1} + \sigma_{b-1}$, we obtain the desired formula. □

Lemma 6.2. *Let $C(a_1, a_2, \cdots, a_n)$ be a 2-bridge knot and g its Jones polynomial. If b is odd, then $C(a_1, a_2, \cdots, a_n, b, -a_1, -a_2, \cdots, -a_n)$ is also a knot and its Jones polynomial is*

$$t^{-3\beta/2}(\mu^{-1}(1 + t^\beta) g(t) g(t^{-1}) - t^\beta \mu),$$

where $\beta = (-1)^n b$.

Proof. Apply Theorem 1.

Theorem 6. *There exist arbitrarily many 2-bridge knots with the same Jones and Q polynomials.*

Proof. Let $C(a_1, a_2, \cdots, a_n)$ and $C(b_1, b_2, \cdots, b_n)$ be the 2-bridge knots having the same Jones and Q polynomials with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} = \begin{pmatrix} s & q \\ r & p \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix} = \begin{pmatrix} v & t \\ u & p \end{pmatrix},$$

where $p \neq \pm 1$. Then the 2-fold branched covering spaces of S^3 branched over these 2-bridge knots are the lens spaces $L(p, q)$ and $L(p, t)$, respectively, see [28]. If d is odd, then by Lemmas 6.1 and 6.2, the following four 2-bridge knots have the same Q and Jones polynomials:

$$C(a_1, \cdots, a_n, d, -a_n, \cdots, -a_1),\ C(a_n, \cdots, a_1, d, -a_1, \cdots, -a_n),$$
$$C(b_1, \cdots, b_n, d, -b_n, \cdots, -b_1),\ C(b_n, \cdots, b_1, d, -b_1, \cdots, -b_n).$$

Since

$$\hat{A} = \begin{pmatrix} 0 & 1 \\ 1 & -a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -a_n \end{pmatrix} = (-1)^n \begin{pmatrix} s & -q \\ -r & p \end{pmatrix},$$

$$A \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} \hat{A}^T = (-1)^n \begin{pmatrix} -dq^2 & (-1)^n + dpq \\ (-1)^n - dpq & dp^2 \end{pmatrix},$$

and

$$A^T \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} \hat{A} = (-1)^n \begin{pmatrix} -dr^2 & (-1)^n + dpr \\ (-1)^n - dpr & dp^2 \end{pmatrix},$$

where A^T is the transposed matrix of A, the first and the second 2-bridge knots have lens spaces $L(dp^2, 1+dpq)$ and $L(dp^2, 1+dpr)$ as their 2-fold branched covering spaces. In the same way, the third and the fourth ones have $L(dp^2, 1+dpt)$ and $L(dp^2, 1+dpu)$. If $L(p,q)$ and $L(p,t)$ are not homeomorphic, that is, $\pm q \not\equiv t^{\pm 1} (\mathrm{mod}\, p)$, and $q \not\equiv \pm r$, $t \not\equiv \pm u$ $(\mathrm{mod}\, p)$, then these four lens spaces are not homeomorphic each other. Thus the four 2-bridge knots are neither isotopic nor mutant each other. Furthermore it holds that $(-1)^n + dpq \not\equiv \pm((-1)^n - dpq)$, $(-1)^n + dpr \not\equiv \pm((-1)^n - dpr)$, $(-1)^n + dpt \not\equiv \pm((-1)^n - dpt)$, and $(-1)^n + dpu \not\equiv \pm((-1)^n \pm dpu)$ $(\mathrm{mod}\, dp^2)$, and so we can continue this construction.

Beginning with $C(2,3)$ such that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 3 \\ 2 & 7 \end{smallmatrix}\right)$, we obtain 2^N distinct 2-bridge knots for any integer N with the same Jones and Q polynomials. This completes the proof. $\square$

Remarks. (1) The 2-bridge knot $C(a_1, \cdots, a_n, \pm 1, -a_n, \cdots, -a_1)$ is a symmetric skew union [16] of $C(a_1, \cdots, a_n)$.

(2) By the above constructin, $C(3, 2, 1, -2, -3) = 10_{22}$ and $C(2, 3, 1, -3, -2) = 10_{35}$ have the same Jones and Q polynomials. This may be the simplest example of a pair of 2-bridge knots with the same Jones and Q polynomials. But they have distinct Alexander polynomials, and therefore, distinct 2-variable Jones polynomials. In fact, they have distinct genera. Do the 2-bridge knots with the same Jones and Q polynomials constructed in the above proof have distinct Alexander polynomials? Do they have distinct genera? Does there exist an example of 2-bridge knots with the same 2-variable Jones polynomial?

Next, we consider an oriented 2-bridge knot or link putting in the form as shown in Fig. 13, which we denote by $D(c_1, c_2, \cdots, c_n)$, see [29]. $C(2c_1, 2c_2, \cdots, 2c_n)$ is isotopic to $D(c_1, c_2, \cdots, c_n)$ with its orientation ignored. Since a 2-bridge knot or link is alternating, it is fibered iff the leading coefficient of its reduced Alexander polynomial is ± 1 [23], which equals $c_1 c_2 \cdots c_n$ for $D(c_1, c_2, \cdots, c_n)$ [29]. Thus we have

Lemma 6.3. *$D(c_1, c_2, \cdots, c_n)$ is fibered iff $c_i = \pm 1$ for all i.*

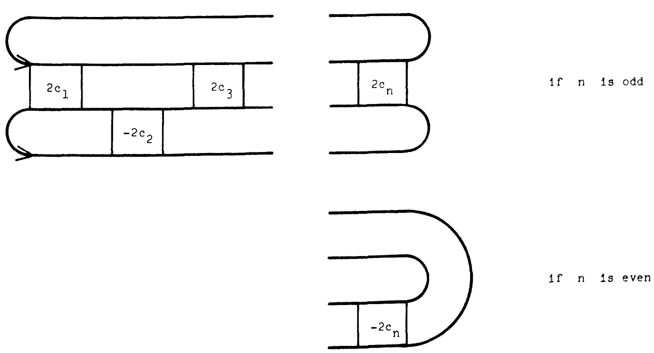


Fig. 13

Lemma 6.4. *Let $\Delta(a_1, a_2, \cdots, a_{2k+1}; t_1, t_2)$ be the 2-variable Alexander polynomial of the 2-bridge link $D(a_1, a_2, \cdots, a_{2k+1})$. Then*

$$\begin{aligned}&\Delta(c_1, c_2, \cdots, c_{2n+1}, d, -c_{2n+1}, \cdots, -c_2, -c_1; t_1, t_2)\\&= d(t_1-1)(t_2-1)\,\Delta(c_1, c_2, \cdots, c_{2n+1}; t_1, t_2)^2\end{aligned}$$

up to units.

Proof. Apply Propositions 3.1 and 3.2.

Lemma 6.5. *If $D(a_1, \cdots, a_m)$ and $D(b_1, \cdots, b_n)$ are skein equivalent, then so are $D(a_1, \cdots, a_m, (-1)^m d, -a_m, \cdots, -a_1)$ and $D(b_1, \cdots, b_n, (-1)^n d, -b_n, \cdots, -b_1)$.*

Proof. For $d=0$, $D(a_1, \cdots, a_m, 0, -a_m, \cdots, -a_1) \approx D(b_1, \cdots, b_n, 0, -b_n, \cdots, -b_1) \approx U^2$. It is easy to see that $(D(a_1, \cdots, a_m, d, -a_m, \cdots, -a_1), D(a_1, \cdots, a_m, d+(-1)^m, -a_m, \cdots, -a_1), D(a_1, \cdots, a_m) \# D(-a_1, \cdots, -a_m))$ is a skein triple. Since $D(a_1, \cdots, a_m) \# D(-a_1, \cdots, -a_m)$ and $D(b_1, \cdots, b_n) \# D(-b_1, \cdots, -b_n)$ are skein equivalent, by induction on d, the lemma follows. □

Proposition 6.1. *The 2-variable Jones polynomial of $D(a_1, a_2, \cdots, a_n, d, -a_n, \cdots, -a_2, -a_1)$ is*

$$(-l^2)^{(-1)^n}(\mu_P - \mu_P^{-1} h(l,m)\,h(l^{-1},m)) + \mu_P^{-1} h(l,m)\,h(l^{-1},m),$$

where $h(l,m)$ is the 2-variable Jones polynomial of $D(a_1, a_2, \cdots, a_n)$.

Proof. Let P_d be the 2-variable Jones polynomial of $D(a_1, a_2, \cdots, a_n, d, -a_n, \cdots, -a_2, -a_1)$. If n is even, then by (1.4), we have

$$l^{-1} P_d + l P_{d-1} + m h(l,m)\,h(l^{-1},m) = 0.$$

Note that $h(l^{-1},m)$ is the 2-variable Jones polynomial of $D(-a_1, -a_2, \cdots, -a_n)$. Then

$$\begin{aligned}P_d - \mu_P^{-1} h(l,m)\,h(l^{-1},m) &= -l^2\,(P_{d-1} - \mu_P^{-1} h(l,m)\,h(l^{-1},m))\\&= (-l^2)^d\,(P_0 - \mu_P^{-1} h(l,m)\,h(l^{-1},m))\\&= (-l^2)^d\,(\mu_P - \mu_P^{-1} h(l,m)\,h(l^{-1},m)),\end{aligned}$$

and so we have the desired formula.

If n is odd, then

$$l P_d + l^{-1} P_{d-1} + m h(l,m)\,h(l^{-1},m) = 0,$$

and in the same way, we obtain the formula. □

Theorem 7. *There exist arbitrarily many skein equivalent fibered 2-bridge links with the same 2-variable Alexander and Q polynomials, and therefore, the same 2-variable Jones, Jones and reduced Alexander polynomials.*

Proof. By Lemmas 6.1, 6.4 and 6.5, two 2-bridge links $D(c_1, c_2, \cdots, c_{2n+1}, 1, -c_{2n+1}, \cdots, -c_2, -c_1)$ and $D(c_{2n+1}, \cdots, c_2, c_1, 1, -c_1, -c_2, \cdots, -c_{2n+1})$ have the same Q and 2-variable Alexander polynomials, and are skein equivalent. In the same way as in Theorem 6, noting Lemma 6.3, we start with $D(1, -1, 1, 1, -1)$, which gives $\begin{pmatrix} -8 & 13 \\ -11 & 18 \end{pmatrix}$, then we can construct arbitrarily many skein equivalent fibered 2-bridge links with the same polynomial invariants. □

Theorem 8. *There exist a pair of skein equivalent 2-bridge links with the same Q polynomial, and therefore, the same 2-variable Jones, Jones and reduced Alexander polynomials, but distinct 2-variable Alexander polynomials.*

Proof. Let us consider $D(1,2,1,-2,-1)$ and $D(2,1,1,-1,-2)$. By Lemmas 6.1 and 6.5, they are skein equivalent and have the same Q polynomial. For the 2-variable Alexander polynomial, by Theorem 3 of [13], the former has t_1-degree 2 and the latter 4. This completes the proof. □

Let $D(c_1, c_2, \cdots, c_{2k+1})$ be a 2-bridge link with

$$B=\begin{pmatrix} 0 & 1 \\ 1 & 2c_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 2c_2 \end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ 1 & 2c_{2k+1} \end{pmatrix}=\begin{pmatrix} s & q \\ r & p \end{pmatrix}=\frac{p}{|p|}\begin{pmatrix} s' & q' \\ r' & p' \end{pmatrix}.$$

Then this is the 2-bridge link with Schubert's normal form of type (p', q') [28], which we denote by $S(p', q')$. Two 2-bridge links $S(\alpha, \beta)$ and $S(\alpha', \beta')$ are isotopic iff $\alpha=\alpha'$ and $\beta \equiv \beta' \pmod{2\alpha}$, and are isotopic with their orientations ignored iff $\alpha=\alpha'$ and $\beta^{\pm 1} \equiv \beta' \pmod{\alpha}$. Thus $S(p', q')$ and $S(p', r')$ are isotopic. Furthermore, $S(\alpha, \beta)$ is transformed into $S(\alpha, \alpha+\beta)$ by reversing the orientation of one component, see [29] or [3, Theorem 12.6]. Now it is easy to see

Lemma 6.6. *If $D(c_1, c_2, \cdots, c_{2k+1})$ is transformed into the isotopic link by reversing the orientation of one component, then $D(c_1, \cdots \cdots, c_{2k+1}, d, -c_{2k+1}, \cdots, -c_1)$ and $D(c_{2k+1}, \cdots, c_1, d, -c_1, \cdots, -c_{2k+1})$ are non-isotopic, but isotopic with their orientations ignored.*

Theorem 9. *There exists a fibered 2-bridge link which is transformed into a non-isotopic link by reversing the orientation of one component, but whose skein equivalent class and 2-variable Alexander polynomial do not change.*

Proof. Since $D(1,1,-1)$, which is $S(8,3)$ or $S(8,-5)$, is transformed into the isotopic link by reversing the orientation of one component, by Lemma 6.6, the two fibered 2-bridge links $D(1,1,-1,1,1,-1,-1)$ and $D(-1,1,1,1,-1,-1,1)$ are non-isotopic, but isotopic with their orientations ignored. Furthermore, they have the same 2-variable Alexander polynomial by Lemma 6.4 and are skein equivalent by Lemma 6.5. This completes the proof. □

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Note added in proof. Z. Iwase and H. Kiyoshi solved affirmatively the questions in Sect. 4 (to appear in Kobe J. Math.).

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A Note on Coherent G-Sheaves

Mark Roberts

Mathematics Institute, University of Warwick, Coventry CV4 7AL, England

Introduction

Sheaves defined on G-spaces and carrying actions of the group G which cover the actions on the base spaces have been studied by a number of authors [3, 6, 8], chiefly in the case when G is a finite group or acts properly discontinuously. In this note we consider sheaves on certain complex analytic G-spaces, X, where G is a reductive complex Lie group, for which complex analytic "categorical" quotient spaces X/G exist. These spaces are described in more detail in Sect. 3. The modules of sections of coherent analytic sheaves are Fréchet spaces and, as described in Sect. 2, the group actions on sheaves discussed in this paper are required to be holomorphic, in the sense that the induced actions on the spaces of sections are holomorphic. The point of this is that a general result concerning holomorphic group actions of reductive complex Lie groups on Fréchet spaces, given in Sect. 1, enables us to "average" sections of the G- sheaf over G. This is an important tool which is available for all G-sheaves (of abelian groups, say) when G is finite, but which requires extra, topological, structure on the sheaves for more general G.

The chief result of this paper is that the germs of invariant sections of a coherent analytic G-sheaf, $\mathscr{S}$, on an appropriate G-space X form a coherent analytic sheaf, denoted $\pi_*^G\mathscr{S}$, on X/G. The proof of this is given in Sect. 3. In Sect. 4 we identify the stalks of $\pi_*^G\mathscr{S}$, which by definition are germs of invariant sections of $\mathscr{S}$, but which we prove to correspond bijectively to invariant germs of sections of $\mathscr{S}$.

1. Fréchet Space Representations of Reductive Complex Lie Groups

Let F be a Fréchet space and $\mathscr{A}(F)$ the set of invertible linear endomorphisms of F. Composition of maps gives a group structure to $\mathscr{A}(F)$. Let G be a reductive complex Lie group.

Definition. A holomorphic representation of G on F is a group homomorphism $\tau: G \to \mathscr{A}(F)$ such that the associated map $\underset{(g,f)\to\tau(g)(f)}{G\times F\to F}$ is analytic.

For an account of analytic functions between Fréchet spaces see [2]. Note that if τ is a holomorphic representation of G on F then for each $f \in F$ the map $\underset{g \to \tau(g)(f)}{G \to F}$ is analytic.

Let $\tau: G \to \mathscr{A}(F)$ be a holomorphic representation of G and H a subgroup of G. An element f in F is said to be H finite if it is contained in a finite dimensional H stable subspace of F.

Theorem 1.1. (a) *The set of G finite elements is dense in F.*

(b) *There exists a unique continuous linear operator $L: F \to F$ satisfying*:

(i) $L(\tau(g)(f)) = L(f)$ *for all g in G*

(ii) $L^2 = L$

(iii) $L(f) = f$ *if and only if* $\tau(g)(f) = f$ *for all g in G.*

These properties imply that $\tau(g) L(f) = L(f)$ and so L is a projection from F onto the subspace of G invariant vectors, F^G.

Proof. (a) Let K be a maximal compact subgroup of G [9]. The representation τ restricts to a continuous representation of K on F and, by a result of Harish-Chandra (e.g. [4]), the set of K finite elements in F is dense. Thus it is sufficient to show that any K finite element is G finite.

Let f be K finite and let V be a K stable finite dimensional subspace of F containing f. The representation τ induces a continuous representation $K \to \mathrm{GL}(V)$ which extends to a unique holomorphic representation $\varrho: G \to \mathrm{GL}(V)$ because G is the "universal complexification" of K [9].

For each $v \in V$, the function $\alpha_v: g \to \tau(g)(v) - \varrho(g)(v)$ is an analytic function on G, taking values in F and vanishing on K. If $k \in K$, the function $\hat{\alpha}_v: \eta \to \alpha_v(\mathrm{k}(\exp_G \eta))$ is defined on the Lie algebra, $\mathfrak{g}$, of G and vanishes on the subalgebra $\mathfrak{k}$ given by K. Since $\mathfrak{g}$ is the complexification of $\mathfrak{k}$ it follows that $\hat{\alpha}_v$ vanishes on the whole of $\mathfrak{g}$ and so $\alpha_v = 0$ on kG_0 (where G_0 is the identity component of G) and hence on the whole of G. Thus, for all v in V, $\tau(g)(v) = \varrho(g)(v)$ and V is G stable under the action given by τ; that is f is G finite.

(b) Averaging over K defines a continuous linear endomorphism L of F [4, Sect. 3.3]:

$$L(f) = \int_K \tau(k) \cdot f dk .$$

This operator satisfies (ii) and (i) and (iii) with G replaced by K. The argument of (a) applied to the analytic functions $f - \tau(g) f$ and $L(f) - L(\tau g) f)$ on G shows that (ii) and (iii) also hold for G. Finally the uniqueness of L follows from the facts that its restriction to each finite dimensional G- stable subspace is unique [12] and that G-finite elements are dense in F. $\square$

The first part of the theorem is a straightforward generalization of a result of Richardson [14] who considered the case when F is the space of holomorphic functions on a complex manifold and τ is induced from an action of G on the manifold.

2. Analytic G-Sheaves

Let G be a reductive complex Lie group and X a complex G-space, by which we mean a complex space with a holomorphic action of G,

$$\sigma: G\times X\to X \qquad (g,x)\mapsto g\cdot x\,.$$

Let $\mathscr{U}$ denote the set of open subsets of X and $\mathscr{U}_G$ that of open subsets which are G-stable.

Definition. An analytic G-sheaf on X is a sheaf $\mathscr{S}$ with the following structure.

(i) For each $U\in\mathscr{U}$ the module of sections $\mathscr{S}(U)$ is a Fréchet space and for any pair $U,V\in\mathscr{U}$ with $U\subset V$ the restriction homomorphism $\varrho_{U,V}:\mathscr{S}(V)\to\mathscr{S}(U)$ is continuous.

(ii) For each $g\in G$ and $U\in\mathscr{U}$ there is a module homomorphism

$$\varphi_{g,U}:\mathscr{S}(U)\to\mathscr{S}(g\cdot U)\,.$$

If $g,h\in G$ then $\varphi_{g,hU}\circ\varphi_{h,U}=\varphi_{gh,U}$.

(iii) If $U\in\mathscr{U}$ satisfies $\sigma^{-1}(U)\supset\Gamma\times W$ where $W\in\mathscr{U}$ and Γ is an open subset of G, then the map

$$\Gamma^{-1}\times\mathscr{S}(U)\to S(W) \qquad (g,s)\mapsto \varrho_{W,g\cdot U}(g\cdot s)$$

is holomorphic. In particular this implies that if $U\in\mathscr{U}_G$ the representation of G on $\mathscr{S}(U)$ is holomorphic.

A homomorphism of analytic G-sheaves is a sheaf homomorphism $\alpha:\mathscr{S}_1\to\mathscr{S}_2$, between analytic G-sheaves, satisfying the following conditions for each $U\in\mathscr{U}$.

(i) The module homomorphism $\alpha(U):\mathscr{S}_1(U)\to\mathscr{S}_2(U)$ is continuous.

(ii) For each $g\in G$, $\alpha(gU)\circ\varphi_{g,U}=\psi_{g,U}\circ\alpha(U)$, where φ and ψ denote the group actions on $\mathscr{S}_1$ and $\mathscr{S}_2$ respectively.

Analytic G-sheaves together with their homomorphisms form a category. Every analytic G-sheaf is a Fréchet sheaf as defined in [11] and also a G-sheaf as in [8]. The extra requirement, the analyticity of the group action, seems natural in the context of analytic geometry.

Examples. (1) Sheaves of germs of sections of analytic G-bundles are analytic G-sheaves.

(2) Any coherent sheaf on X is a Fréchet sheaf [7]. If in addition it satisfies conditions (ii) and (iii) of the definition it will be called a coherent analytic G-sheaf. The kernel, image and cokernel of an analytic G-sheaf homomorphism between coherent analytic G-sheaves are coherent analytic G-sheaves. Conversely, the following result shows that every coherent analytic G-sheaf is locally presented by sheaves of germs of sections of analytic G-bundles, provided X satisfies a certain condition.

Proposition 2.1. *Let X be a complex G-space that is covered by G-stable open Stein subspaces and let $\mathscr{S}$ be a coherent analytic G-sheaf on X. Then for every point $x\in X$*

there exist a G-stable neighbourhood U of x, finite dimensional representations V, W of G and an exact sequence:

$$\mathcal{O}_X(U)\otimes W \to \mathcal{O}_X(U)\otimes V \to \mathcal{S}(U)\to 0$$

Proof. By restricting to a G-stable Stein neighbourhood of x, it can be supposed that X is itself Stein. Since $\mathcal{S}$ is coherent there exists a finite set of sections $\{s_j\}_{j=1}^r \subset \mathcal{S}(X)$ and a neighbourhood Y of x such that the set of restrictions $\{\varrho_{Y,X}(s_j)\}_{j=1}^r$ generates $\mathcal{S}(Y)$ as an $\mathcal{O}_X(Y)$ module. Since $\operatorname{Hom}(\mathcal{O}_X^r, \mathcal{O}_X^r)$ is a coherent sheaf the space of sections $\operatorname{Hom}(\mathcal{O}_X(Y)^r, \mathcal{O}_X(Y)^r)$ is a Fréchet space and so the map

$$\Phi : \operatorname{Hom}(\mathcal{O}_X(Y)^r, \mathcal{O}_X(Y)^r) \to \mathcal{S}(Y)^r$$

$$\alpha = (\alpha_{i,j}) \to \alpha \cdot s = \left(\sum_{j=1}^{r} \alpha_{ij}\, \varrho_{Y,X}(s_j)\right)$$

is a surjective homomorphism of Fréchet spaces which is therefore open by the Open Mapping Theorem. The subset $\operatorname{Aut}(\mathcal{O}_X(Y)^r)$ of invertible homomorphisms is open in the source and therefore its image under Φ is also open. Hence if $\varrho : \mathcal{S}(X)^r \to \mathcal{S}(Y)^r$ is the map given by restriction, $\varrho^{-1}(\Phi(\operatorname{Aut}(\mathcal{O}_X(Y)^r)))$ is an open subset of $\mathcal{S}(X)^r$ containing $(s_1, \ldots, s_r)$. This set consists of elements of $\mathcal{S}(X)^r$ whose components have restrictions forming generating sets for $\mathcal{S}(Y)$. The group G acts on $\mathcal{S}(X)^r$ by the diagonal action and by Theorem 1.1 (a) the set of G-finite vectors is dense. Thus there exists a finite set of G-finite global sections $\{s_j\}_{j=1}^r \subset \mathcal{S}(X)$ such that the restrictions $\{\varrho_{Y,X}(s_j)\}_{j=1}^r$ generate $\mathcal{S}(Y)$.

Let V be a finite dimensional G-stable subspace of $\mathcal{S}(X)$ containing $\{s_j\}_{j=1}^r$ and define a G-sheaf homomorphism

$$\varphi : \mathcal{O}_X \otimes V \to \mathcal{S}$$

at the level of stalks by

$$\varphi\left(\sum_{i=1}^{r} f_{i,y} \otimes v_i\right) = \sum_{i=1}^{r} f_{i,y}\, v_{i,y}$$

where $v_{i,y}$ is the germ of $v_i \in V \subset \mathcal{S}(X)$ at $y \in X$. By construction φ is surjective over Y. By the G invariance of V it will therefore be surjective on $U := G \cdot Y$. Thus there is a surjective equivariant module homomorphism

$$\varphi : \mathcal{O}_X(U) \otimes V \to \mathcal{S}(U) \to 0 .$$

The proof of the proposition is completed by repeating this construction with $\ker\varphi$ replacing $\mathcal{S}$. □

If $U \in \mathcal{U}_G$ then there is a holomorphic linear action of G on $\mathcal{S}(U)$. The fixed point set $\mathcal{S}(U)^G$ is the space of invariant sections of $\mathcal{S}$ over U. From Theorem 1.1 we immediately obtain the following result which formalises the idea of averaging a section of $\mathcal{S}$ over G.

Proposition 2.2. *Let X be a complex G-space and $\mathcal{S}$ an analytic G-sheaf on X. Then for each G-stable open subset U of X there exists a unique G-invariant projection L_U of $\mathcal{S}(U)$ onto $\mathcal{S}(U)^G$. The uniqueness of this operator implies*:

(i) *if* $U_1 \subset U_2$ *are two G-stable open subsets of X and* $\varrho_{1,2}: \mathscr{S}(U_2) \to \mathscr{S}(U_1)$ *is the restriction map, then*

$$L_U \circ \varrho_{1,2} = \varrho_{1,2} \circ L_U,$$

(ii) *if* $\alpha: \mathscr{S}_1 \to \mathscr{S}_2$ *is a G-sheaf homomorphism between coherent G-sheaves then, for each open G-stable subset U of X, the induced map*

$$\alpha(U): \mathscr{S}_1(U) \to \mathscr{S}_2(U)$$

commutes with the projections onto $\mathscr{S}_1(U)^G$ *and* $\mathscr{S}_2(U)^G$. □

3. Sheaves of Invariant Sections

This section contains the main result of this paper, which states that the germs of invariant sections of a coherent analytic G-sheaf on a complex G-space X form a coherent sheaf over the quotient space X/G, under conditions which ensure that a suitable quotient space exists. These conditions are the following.

C1. The G-space X has an open covering by G-stable Stein subspaces.
C2. The closure of any G-orbit in X contains a unique closed G-orbit.

Examples of spaces satisfying these conditions occur in the work of Bialynicki-Birula and Sommese on $\mathbb{C}^*$ actions on Kähler manifolds [1].

Let X be a complex G-space satisfying C1 and C2 and let Y be the set of closed G-orbits in X. By C2 there is a well defined map $\pi: X \to Y$ given by letting $\pi(x)$ equal the closed orbit in $\overline{G \cdot x}$. The set Y is given the quotient topology induced from X and is made into a ringed space by defining

$$\mathcal{O}_Y(U) := \mathcal{O}_X(\pi^{-1}(U))^G$$

for every open subset U of Y. It follows immediately from the work of Snow [16] that $(Y, \mathcal{O}_Y)$ is a complex space and π an analytic map with the following universal property.

If Z is a complex space and $\varrho: X \to Z$ is a G-invariant analytic map then there is an analytic map $\sigma: Y \to Z$ such that $\varrho = \sigma \circ \pi$.

Thus Y is the *categorical quotient* of X by G [12, 16], and will be denoted by X/G. If $\xi \in X/G$ then the unique closed orbit in $\pi^{-1}(\xi)$ will be denoted by $T(\xi)$.

Snow also proved a slice theorem for such G-spaces. If x is a point in X with a closed orbit then the isotropy subgroup G_x is a reductive complex Lie group and there exists a finite dimensional representation R of G_x, a G_x-stable locally closed subvariety, S, of R, containing 0, and an equivariant analytic map $\tau: S, 0 \to X, x$, satisfying:

(i) τ is an isomorphism onto its image,

(ii) $S \times_{G_x} G$ is isomorphic to a π saturated open neighbourhood of x.

Since reductive complex Lie groups have unique compatible reductive algebraic group structures and holomorphic representations of the Lie groups correspond bijectively to rational representations of the algebraic group [10], the slice S can be taken to be an open subset (in the transcendental topology) of an affine algebraic variety with a rational action of the reductive algebraic group.

A consequence of the slice theorem is that, in the neighbourhood of a closed orbit, a G-sheaf is completely determined by its restriction to a slice through the orbit. This can be seen by looking at the induced action of G on the sheaf space $\tilde{\mathcal{S}}$ of $\mathcal{S}$ given by

$$g \cdot \eta = \varrho_{g \cdot x, g \cdot U}(g \cdot s),$$

where $g \in G$, $x \in X$, $\eta \in \mathcal{S}_x$, U is a neighbourhood of x and s is an element of $\mathcal{S}(U)$ satisfying $\varrho_{x,U}(s) = \eta$. In particular there is an action of G_x on $\mathcal{S}_x$ for each x and if $T = G \cdot x$ is a closed orbit and $S, 0 \to X, x$ is a slice through T, then

$$\tilde{\mathcal{S}} |_{S \times_{G_x} G} \cong (\tilde{\mathcal{S}} |_S) \times_{G_x} G.$$

It follows from this that invariant sections of $\mathcal{S}$, and invariant germs of sections along T, are determined by their restrictions to S.

We continue to suppose that X satisfies C1 and C2 and so has a quotient space X/G.

Definitions. (1) If $\mathcal{S}$ is a G-sheaf on X define the sheaf $\pi_*^G \mathcal{S}$ on X/G by

$$\pi_*^G \mathcal{S}(U) := \mathcal{S}(\pi^{-1}(U))^G$$

for all open subsets U of X/G. The restriction homomorphisms of $\pi_*^G \mathcal{S}$ are those of $\mathcal{S}$ restricted to the spaces of invariant sections. Technically this only defines $\pi_*^G \mathcal{S}$ as a presheaf, but it is easily seen to be a sheaf.

(2) If $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ is a G-sheaf homomorphism then the sheaf homomorphism $\pi_*^G \varphi : \pi_*^G \mathcal{S}_1 \to \pi_*^G \mathcal{S}_2$ is defined by

$$\pi_*^G \varphi(U) := \varphi(\pi^{-1}(U)) |_{\mathcal{S}_1(\pi^{-1}(U))^G}.$$

With these definitions π_*^G is a functor from the category of G-sheaves to that of sheaves on X/G.

Theorem 3.1. *If X is a complex G-space satisfying* C1 *and* C2 *and $\mathcal{S}$ is a coherent analytic G-sheaf on X, then $\pi_*^G \mathcal{S}$ is a coherent sheaf on X/G.*

Proof. By Proposition 2.1, if $\xi \in X/G$ there exists a G-stable neighbourhood V of $T(\xi)$ over which $\mathcal{S}$ has a presentation by sheaves of germs of sections of G-bundles:

$$\mathcal{S}_2 \to \mathcal{S}_1 \to \mathcal{S} |_V \to 0.$$

For any neighbourbood U of ξ in X/G such that $\pi^{-1}(U) \subset V$ there is a commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{S}_2(\pi^{-1}(U)) & \to & \mathcal{S}_1(\pi^{-1}(U)) & \to & \mathcal{S}(\pi^{-1}(U)) & \to & 0 \\
\jmath \Big\downarrow L & & \jmath \Big\downarrow L & & \jmath \Big\downarrow L & & \\
\mathcal{S}_2(\pi^{-1}(U))^G & \to & \mathcal{S}_1(\pi^{-1}(U)^G & \to & \mathcal{S}(\pi^{-1}(U))^G & \to & 0
\end{array}$$

where L is the averaging operator. The top row of this diagram is exact and a diagram chase shows that this implies that the bottom row is also exact. It follows that the sequence of sheaves,

$$\pi_*^G \mathcal{S}_1 \to \pi_*^G S_2 \to \pi_*^G (\mathcal{S} |_V) \to 0$$

is exact at ξ and the coherence of $\pi_*^G\mathscr{S}$ at ξ will be proved if it can be shown that the $\pi_*^G\mathscr{S}_j$ are coherent. This is done in the following lemma, which is a complex analytic analogue of a result of Poenaru [13].

Lemma 3.2. *Let X be a complex G-space satisfying* C1 *and* C2 *and $\mathscr{S}$ the sheaf of germs of sections of an analytic G-bundle over X. Then $\pi_*^G\mathscr{S}$ is a coherent sheaf on X/G.*

Proof. To prove coherence at $\xi \in X/G$ it can be supposed, by the slice theorem, that $(X, T(\xi))$ is of the form $(G \times_H Y, G \times_H \{y_0\})$ where H is a reductive subgroup of G and Y is a rational H-space which has y_0 as a fixed point. Let $\mathscr{S}|_Y$ denote the H-bundle obtained by restricting $\mathscr{S}$ to Y. Then $\mathscr{S}$ is completely determined by $\mathscr{S}|_Y$ and $\pi_*^G\mathscr{S}$ is coherent at ξ if and only if $\pi_*^H(\mathscr{S}|_Y)$ is coherent at $\pi(y_0)$. The spaces of sections of $\pi_*^H(\mathscr{S}|_Y)$ over sufficiently small neighbourhoods, U, of $\pi(y_0)$ can be identified with the spaces of equivariant analytic mappings $C_H^\omega(\pi^{-1}(U), W)$, where W is the fibre of the original G-bundle at a point on $T(\xi)$. It is sufficient to show that these spaces are finitely generated over $\mathcal{O}_{Y/H}(U) = \mathcal{O}_Y(\pi^{-1}(U))^H$. The proof now follows closely that of Poenaru.

Let W^* denote the dual of W. For $f \in C_H^\omega(\pi^{-1}(U), W)$ define $\alpha f \in \mathcal{O}_{Y\times W}(\pi^{-1}(U) \times W^*)^H$ by

$$\alpha f(y, \eta) = \langle f(y), \eta\rangle \quad \text{for} \quad y \in \pi^{-1}(U),\ \eta \in W^*.$$

In the opposite direction, if $h \in \mathcal{O}_{Y\times W}(\pi^{-1}(U) \times W^*)^H$ define $\beta h \in C_H^\omega(\pi^{-1}(U), W)$ by

$$\beta h(y) = \left(\frac{\partial h}{\partial \eta_1}(y, 0), \ldots, \frac{\partial h}{\partial \eta_r}(y, 0)\right)$$

for $\eta = (\eta_1, \ldots, \eta_r) \in W^*$. Then $\beta\alpha f = f$ for all $f \in C_H^\omega(\pi^{-1}(u), W)$.

There exist polynomial germs $\sigma_1, \ldots, \sigma_k \in \mathcal{O}_{Y\times W}(Y \times W^*)^H$ such that, if $\sigma = (\sigma_1, \ldots, \sigma_k)\colon Y \times W^* \to \mathbb{C}^k$, then

$$\mathcal{O}_{Y\times W}(\pi^{-1}(U) \times W^*)^H \cong (\sigma^*\mathcal{O}_{\mathbb{C}^k}^k)(\pi^{-1}(U) \times W^*).$$

Let $h \in \mathcal{O}_{Y\times W}(\pi^{-1}(U) \times W^*)^H$ and choose $\tilde{h} \in \mathcal{O}_{\mathbb{C}^k}(\mathbb{C}^k)$ such that $h = \sigma^*\tilde{h}$. Then

$$\beta h = \sum_{i=1}^{k} \left(\frac{\partial \tilde{h}}{\partial z_j} \circ \sigma|_{\pi^{-1}(U)\times\{0\}}\right) \cdot \beta\sigma_i$$

(where the z_i are co-ordinates for $\mathbb{C}^k$) and it follows that the equivariant polynomial mappings $\{\beta\sigma_i\}_{i=1}^k$ generate $C_H^\omega(\pi^{-1}(U), W)$ over $\mathcal{O}_Y(\pi^{-1}(U))^H$. □

4. Germs of Invariant Sections

In this section we show that "invariant germs of sections" of a coherent analytic G-sheaf $\mathscr{S}$ on a G-space X can be identified with "germs of invariant sections" of $\mathscr{S}$. If T is a closed G-orbit in X then there is an induced action of G on the stalk $\mathscr{S}_T$ defined by

$$g \cdot \eta = \varrho_{T, g\cdot U}(g \cdot s),$$

where $g \in G$, $\eta \in \mathscr{S}_T$, U is a neighbourhood of T and s is an element of $\mathscr{S}(U)$ satisfying $\varrho_{T,U}(s) = \eta$. The fixed point set of this action, consisting of invariant germs of sections of $\mathscr{S}$ along T, is denoted $\mathscr{S}_T^G$.

Proposition 4.1. *Let $\mathscr{S}$ be a coherent analytic G-sheaf on a complex G-space X satisfying* C1 *and* C2 *and let $\xi \in X/G$. Then*

$$(\pi_*^G \mathscr{S})_\xi \cong (\mathscr{S}_{T(\xi)})^G.$$

Proof. Let $\mathscr{U}$ be a system of neighbourhoods of ξ. Then

$$(\pi_*^G \mathscr{S})_\xi = \lim_{\mathscr{U}} (\pi_*^G \mathscr{S})(U) = \lim_{\mathscr{U}} \mathscr{S}(\pi^{-1}(U))^G.$$

Restriction defines a module homomorphism,

$$\varphi : (\pi_*^G \mathscr{S})_\xi \to \mathscr{S}_{T(\xi)}{}^G.$$

Let $s_1, s_2 \in \mathscr{S}(\pi^{-1}(U))^G$ for some $U \in \mathscr{U}$ satisfy

$$\varrho_{T(\xi),\pi^{-1}(U)}(s_1) = \varrho_{T(\xi),\pi^{-1}(U)}(s_2).$$

Then there exists a neighbourhood W of $T(\xi)$, contained in $\pi^{-1}(U)$, such that

$$\varrho_{W,\pi^{-1}(U)}(s_1) = \varrho_{W,\pi^{-1}(U)}(s_2).$$

This implies that

$$\varrho_{G \cdot W,\pi^{-1}(U)}(s_1) = \varrho_{G \cdot W,\pi^{-1}(U)}(s_2)$$

and the injectivity of φ follows from the fact that any G-stable neighbourhood of $T(\xi)$ contains $\pi^{-1}(U)$ for some $U \in \mathscr{U}$.

The surjectivity of φ is given by the following lemma.

Lemma 4.2. *Let $\mathscr{S}$ and X be as above. If T is a closed orbit in X and $\eta \in \mathscr{S}_T^G$ then there exist a G-stable neighbourhood U of T and $s \in \mathscr{S}(U)^G$ such that $\varrho_{T,U}(s) = \eta$.*

Proof. By the slice theorem we can assume that T is a fixed point of the G action on X. We first show that the lemma is true if G is replaced by K, a maximal compact subgroup of G.

If U is any neighbourhood of T in X there is a K-stable neighbourhood V of T with compact closure $\bar{V} \subset U$. Using this and the definition of the group action on $\mathscr{S}_T$ it can be seen that for every $\eta \in \mathscr{S}_T^G$ and $g \in K$ there exist a K-stable neighbourhood U_g of T and $s \in \mathscr{S}(U_g)$ such that $\varrho_{T,U_g}(s) = \eta$ and $g \cdot s = s$.

Let $\{g_i\}_{i=1}^k$ be a finite set of elements of K generating a dense subgroup of K [5, 2.12, Example 4] and set $U = \bigcap_{i=1}^k U_{g_i}$. Then U is a K-stable open neighbourhood of T and there exist $s_i \in \mathscr{S}(U)$ such that $g_i \cdot s_i = s_i$ and $\varrho_{T,U}(s_1) = \ldots = \varrho_{T,U}(s_k) = \eta$. By shrinking U if necessary we can suppose $s_1 = \ldots = s_k = s$, say.

Clearly $g \cdot s = s$ for all g in the group generated by $g_1, \ldots, g_k$; but this is a dense subgroup of K and so, by the continuity of the action of K on $\mathscr{S}(U)$, $g \cdot s = s$ for all g in K.

Now we show that $s \in \mathscr{S}(U)$ extends naturally to a G-invariant element, s, of $\mathscr{S}(V)$, where $V = G \cdot U$, satisfying $\varrho_{g \cdot U, V}(s) = g \cdot s$ for all $g \in G$. This is true if and only if for any pair of elements $(g,h)\in G\times G$,

$$\varrho_{gU\cap hU,gU}(g\cdot s)=\varrho_{gU\cap hU,hU}(h\cdot s).$$

Since K intersects every component of G there is a continous family of pairs of group elements, $(g_t,h_t)\colon[0,1]\to G\times G$, with $(g_0,h_0)\in K\times K$ and $(g_1,h_1)=(g,h)$. For each t in $[0,1]$ let V_t be a K-stable open neighbourhood of T such that $\bar{V}_t$ is compact and is contained in $g_tU\cap h_tU$. Then $\bar{V}_t$ is contained in $g_sU\cap h_sU$ for all s in a neighbourhood of t and using the compactness of $[0,1]$ we can find a K-stable open neighbourhood V of T with compact closure $\bar{V}\subset g_tU\cap h_tU$ for all t in $[0,1]$.

Let $\Gamma=\{(g',h')\in G\times G\colon g'U\cap h'U\supset\bar{V}\}$. Then Γ is an open neighbourhood of $K\times K\cup\{(g_t,h_t)\colon t\in[0,1]\}$ in $G\times G$ and

$$\varphi\colon\Gamma\to\mathscr{S}(V)\qquad(g',h')\mapsto\varrho_{V,g'U}(g'\cdot s)-\varrho_{V,h'U}(h'\cdot s)$$

is a holomorphic map which is constant on $K\times K$ and hence also on the connected components of Γ intersecting $K\times K$. It follows that $\varrho_{V,gU}(g\cdot s)=\varrho_{V,hU}(h\cdot s)$ as required. □

Final Remark. Coherent algebraic G-sheaves on an algebraic variety with a rational action of a reductive algebraic group can also be treated by methods analogous to those used in this paper, the algebraic analogue of the Stein G-space being the affine G-space. Instead of holomorphic actions on the spaces of sections, $\mathscr{S}(U)$, there are "dual actions" in the sense of Mumford [12]. In particular, suppose X is an algebraic variety with a rational action of the reductive algebraic group G and let X_{ss} denote the set of semistable points in X with respect to some linearisation of the action and $\pi\colon X_{ss}\to X_{ss}/G$ its quotient space [12]. Then, if $\mathscr{S}$ is a coherent G-sheaf on X and $\mathscr{S}_{ss}$ is its restriction to X_{ss} we get that $\pi_*^G\mathscr{S}_{ss}$ is a coherent sheaf on X_{ss}/G.

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Characterisations of Finitely Determined Equivariant Map Germs

Mark Roberts

Mathematics Institute, University of Warwick, Coventry CV4 7AL, England

Introduction

One of the most important concepts in the singularity theory of smooth mappings is that of finite determinacy. The first thorough investigation of the main ideas was carried out by Mather [13] as part of his study of stable mappings and much of the work of later authors has been concerned with refining these ideas and adapting them to the study of the C^k $(0 \leqq k < \infty)$ analogues of the C^∞ equivalence relations studied by Mather. For a review of this work see Wall [22]. The study of singularities of invariant functions and equivariant mappings was first proposed by Thom [20] and since then has been taken up by a number of authors [1–3, 7, 10–12, 14, 17, 18, 21]. The present paper concentrates on the problem of characterising equivariant map germs (both smooth germs equivariant with respect to compact Lie group actions and holomorphic germs equivariant with respect to reductive complex Lie group actions) which are finitely determined with respect to equivariant groups of equivalences generalising the groups $\mathscr{R}$, $\mathscr{L}$, $\mathscr{A}$, $\mathscr{C}$, $\mathscr{K}$ studied by Mather. Three characterisations are given:

(1) necessary and sufficient "infinitesimal" algebraic conditions for a germ to be finitely determined (Theorem 1.1),

(2) "geometric criteria" for finite determinacy in terms of the stability of nearby germs (Theorem 2.1), and

(3) a characterisation of finitely determined equivariant germs by means of transversality conditions on the "G-jet" extension of a representative of the germ (Corollary 4.6).

All these results are equivariant generalisations of results proved by Mather [13] and Gaffney [6]. The proof of the infinitesimal criteria for equivariant germs has been given by Damon [5] and is not repeated here.

In addition to the three sections giving the characterisations there is a section outlining the construction of "G-jet" bundles; for further details the reader is referred to [23]. There is also a section discussing the existence of finitely determined germs for some particular examples of group actions. In another paper [16] the author has treated the question of when finite determinacy "holds in general" in spaces of equivariant map germs. For a recent survey of this and other work that has been done on the finite determinacy of equivariant map germs see Wall [24].

1. Infinitesimal Criteria for Finity Determinacy

Let G be a compact Lie group and let V and W be finite dimensional representations of G. The local rings of germs of smooth real valued functions on $(V, 0)$ and $(W, 0)$ are denoted by $\mathscr{E}(V)$ and $\mathscr{E}(W)$, their unique maximal ideals by $m(V)$ and $m(W)$ and the $\mathscr{E}(V)$ module of germs of mappings from $(V, 0)$ to $(W, 0)$ by $\mathscr{E}(V, W)$. The actions of G on V and W induce actions on $\mathscr{E}(V)$ and $\mathscr{E}(W)$, on $m(V)$ and $m(W)$, and on $\mathscr{E}(V, W)$. The fixed point sets of these actions, denoted by the addition of a subscript G, consist of either invariant function germs or equivariant map germs. If f is an element of $\mathscr{E}_G(V, W)$ there is a G-stable neighbourhood of the origin in V and an equivariant representative of f defined on that neighbourhood [15]; we will use the symbol "f" to denote either the germ or a suitable representative.

For $f \in \mathscr{E}(V, W)$, we use $\theta(f)$ to denote the $\mathscr{E}(V)$ module of vector fields along f; if f is the identity on V (resp. W) then this is replaced by $\theta(V)$ (resp. $\theta(W)$). If $f \in \mathscr{E}_G(V, W)$ there is an induced action of G on $\theta(f)$ and we again use a subscript G to denote the fixed point set of this action.

In this paper we are concerned with the finite determinacy of equivariant germs with respect to the groups of equivalences $\mathscr{B}_G$ and $\mathscr{B}_G^*$ where $\mathscr{B}$ is any one of the standard groups $\mathscr{R}, \mathscr{L}, \mathscr{A}, \mathscr{C}, \mathscr{K}$. The groups $\mathscr{B}_G$ are the natural equivariant analogues of the groups $\mathscr{B}$; their subgroups $\mathscr{B}_G^*$ are useful for essentially technical reasons, though occasionally they do take the place of the $\mathscr{B}_G$ in generalisations of non-equivariant results (for an example see [16]). For definitions and a more complete discussion, see [23].

Each of these groups has a "tangent space"; that for $\mathscr{B}_G$ is the invariant part of the tangent space for $\mathscr{B}$, under the induced action of G:

$$T\mathscr{B}_G(f) := (T\mathscr{B}(f))^G.$$

The tangent spaces for $\mathscr{B}_G^*$ are:

$$T\mathscr{R}_G^*(f) = tf(m_G(V) \cdot \theta_G(V)), \qquad T\mathscr{A}_G^*(f) = T\mathscr{R}_G^*(f) + T\mathscr{L}_G^*(f),$$
$$T\mathscr{L}_G^*(f) = \omega f(m_G(W) \cdot \theta_G(W)), \qquad T\mathscr{K}_G^*(f) = T\mathscr{R}_G^*(f) + T\mathscr{C}_G^*(f),$$
$$T\mathscr{C}_G^*(f) = f^* m_G(W) \cdot \theta_G(f),$$

where tf and ωf denote the usual module homomorphisms obtained by composing with the derivative of f and pulling back by f, respectively.

Extended tangent spaces, $T_e\mathscr{B}_G(f)$, are defined similarly; those for $\mathscr{B}_G^* = \mathscr{R}_G^*$, $\mathscr{L}_G^*$ and $\mathscr{A}_G^*$ are the same as for the corresponding $\mathscr{B}_G$, while $T_e\mathscr{C}_G^*(f) = T\mathscr{C}_G^*(f)$ and $T_e\mathscr{K}_G^*(f) = T_e\mathscr{R}_G^*(f) + T_e\mathscr{C}_G^*(f)$.

Finite determinacy of an equivariant map germ f with respect to any of these group of equivalences is defined exactly as usual and the following result can be proved using the machinery developed in [5].

Theorem 1.1. *For $f \in \mathscr{E}_G(V, W)$ the following are equivalent:*
(i) *the germ f is finitely $\mathscr{B}_G$ determined,*
(ii) *for some k $T\mathscr{B}_G(f) \supset (m(V)^k \theta(f))^G$,*
(iii) *for some k $T_e\mathscr{B}_G(f) \supset (m(V)^k \theta(f))^G$,*
(iv) *the quotient $(m(V) \cdot \theta(f))^G / T\mathscr{B}_G(f)$ is a finite dimensional real vector space,*
(v) *the quotient $\theta_G(f)/T_e\mathscr{B}_G(f)$ is a finite dimensional real vector space. The same statement holds with $\mathscr{B}_G^*$ replacing $\mathscr{B}_G$.* □

Remarks. (a) In [3] Bierstone proved that given V and W, there exist positive integers A and B such that for any ℓ,

$$m_G(V)^{\ell A+B}\,\theta_G(f) \subset (m(V)^{\ell A+B}\theta(f))^G \subset m_G(V)^{\ell}\,\theta_G(f).$$

It follows that in (ii) and (iii) of the theorem, $(m(V)^k \theta(f))^G$ can be replaced by $m_G(V)^k \theta_G(f)$.

(b) Using the same result it is also easily shown that for $\mathscr{B} = \mathscr{R}, \mathscr{L}$ or $\mathscr{A}$, the tangent space $T\mathscr{B}_G^*(f)$ has finite codimension in $T\mathscr{B}_G(f)$ and so f is finitely $\mathscr{B}_G^*$ determined if and only if it is finitely $\mathscr{B}_G$ determined. This is not true for $\mathscr{C}$ or $\mathscr{K}$ [16].

(c) Methods for estimating the degree of determinacy of an equivariant map germ are developed in [4].

Results completely analogous to these are true for complex analytic map germs between representations of reductive complex Lie groups. We shall now show that any real analytic map germ, equivariant with respect to actions of a compact Lie group, K, can be complexified to yield a complex analytic map germ which is equivariant with respect to actions of a reductive complex Lie group, G, in such a way that the real germ is $\mathscr{B}_K$ (resp. $\mathscr{B}_K^*$) finitely determined if and only if its complexification is $\mathscr{B}_G$ (resp. $\mathscr{B}_G^*$) finitely determined.

Recall, from [8] for example, that any compact Lie group K has a unique "universal complexification", G, which is a reductive complex Lie group containing K as a maximal compact subgoup. This group has the property that every finite dimensional continuous representation of K extends to a unique holomorphic representation of G [9].

Suppose V and W are real representations of K and $f\colon (V, 0) \to (W, 0)$ is an equivariant analytic map germ. Complexifying f gives a complex analytic germ $f_{\mathbf{C}}\colon (V \otimes \mathbf{C}, 0) \to (W \otimes \mathbf{C}, 0)$ which is equivariant with respect to the induced actions of K on $V \otimes \mathbf{C}$ and $W \otimes \mathbf{C}$. These actions extend uniquely to representations of the complexification G.

Lemma 1.2. *The germ $f_{\mathbf{C}}$ is equivariant with respect to the actions of G on $V \otimes \mathbf{C}$ and $W \otimes \mathbf{C}$.*

Proof. Let $x \in V \otimes \mathbf{C}$ and define $\varphi_x\colon G \to W \otimes \mathbf{C}$ by $\varphi_x(g) = g^{-1} \cdot f_{\mathbf{C}}(g \cdot x)$. Then φ_x is a holomorphic map on G which, by the K equivariance of $f_{\mathbf{C}}$, is constant on K. It follows that φ_x is constant on the whole of G and so $g^{-1} \cdot f_{\mathbf{C}}(g \cdot x) = f_{\mathbf{C}}(x)$ for all g in G. □

Proposition 1.3. *Let K be a compact Lie group and G its universal complexification. Suppose V and W are real representations of K and $f\colon (V, 0) \to (W, 0)$ is the germ of an* *equivariant analytic map. Then f is finitely $\mathscr{B}_K$ (resp. $\mathscr{B}_K^*$) determined if and only if its complexification $f_{\mathbb{C}}$ is finitely $\mathscr{B}_G$ (resp. $\mathscr{B}_G^*$) determined.*

Proof. The module $\theta(f_{\mathbb{C}})$ can be identified with $\theta(f)\otimes\mathbb{C}$ and the action of G on $\theta(f_{\mathbb{C}})$ extends the induced action of K on $\theta(f)\otimes\mathbb{C}$. Similarly the action of G on $T\mathscr{B}(f_{\mathbb{C}})\cong T\mathscr{B}(f)\otimes\mathbb{C}$ extends that of K and the actions of G on the finite dimensional spaces

$$\frac{T\mathscr{B}(f_{\mathbb{C}})}{T\mathscr{B}(f_{\mathbb{C}})\cap m(V\otimes\mathbb{C})^s\theta(f_{\mathbb{C}})}\cong\left(\frac{T\mathscr{B}(f)}{T\mathscr{B}(f)\cap m(V)^s\theta(f)}\right)\otimes\mathbb{C}$$

extend those of K, for all positive integers s. The uniqueness of these last extensions implies

$$\left(\frac{T\mathscr{B}(f_{\mathbb{C}})}{T\mathscr{B}(f_{\mathbb{C}})\cap m(V\otimes\mathbb{C})^s\theta(f_{\mathbb{C}})}\right)^G\cong\left(\frac{T\mathscr{B}(f)}{T\mathscr{B}(f)\cap m(V)^s\theta(f)}\right)^K\otimes\mathbb{C}$$

for all s, and so $T\mathscr{B}_G(f_{\mathbb{C}})\cong T\mathscr{B}_K(f)\otimes\mathbb{C}$. Similarly $(m(V\otimes\mathbb{C})^r\theta(f_{\mathbb{C}}))^G\cong(m(V)^r\theta(f))^K\otimes\mathbb{C}$ and the proposition follows from the real analytic and complex analytic versions of Theorem 1.1. An identical proof works for $\mathscr{B}_K^*$ finite determinacy. □

Finally we note that the same argument as for the non-equivariant case [22] shows that a real analytic map germ is $\mathscr{B}_K$ (resp. $\mathscr{B}_K^*$) finitely determined as an analytic germ if and only if it is $\mathscr{B}_K(\mathscr{B}_K^*)$ finitely determined as a smooth germ.

2. Geometric Criteria for Finite Determinacy

In this section we give the second characterisation of finitely determined equivariant map germs, restricting, necessarily, to complex analytic germs. We use sheaf theory methods, adapting to the equivariant context the treatment given in [22] and making use of the results of [15], to which we refer for definitions of the notation used here.

If $f:(V,0)\to(W,0)$ is a complex analytic mapping (defined on some neighbourhood of the origin) we denote by $\mathscr{T}\mathscr{B}(f)$ the "sheaf of extended $\mathscr{B}$ tangent spaces" of f. For $\mathscr{B}=\mathscr{R},\mathscr{C}$ or $\mathscr{K}$ this is a sheaf on V and the stalk at $x\in V$ is the extended $\mathscr{B}$ tangent space of the germ of f at x. For $\mathscr{B}=\mathscr{L}$ or $\mathscr{A}$ it is a sheaf on W and the stalk at a point y is the extended $\mathscr{B}$ tangent space of f along $f^{-1}(y)$. If G is a reductive complex Lie group acting linearly on V and W and f is equivariant then the $\mathscr{T}\mathscr{B}(f)$ become G-sheaves and we define $\mathscr{T}\mathscr{B}_G(f)$ to be $\pi_*^G\,\mathscr{T}\mathscr{B}(f)$, where π denotes either the quotient mapping $\pi:V\to V/G$, for $\mathscr{R}$, $\mathscr{C}$ and $\mathscr{K}$, or $\pi:W\to W/G$, for $\mathscr{L}$ and $\mathscr{A}$. Note that here V/G and W/G are categorical quotients [15, 19] and not, in general, orbit spaces.

When $\mathscr{B}=\mathscr{R},\mathscr{C}$ or $\mathscr{K}$ the sheaf $\mathscr{T}\mathscr{B}(f)$ is coherent and it follows from [15] that $\mathscr{T}\mathscr{B}_G(f)$ is also coherent and that the stalk $(\mathscr{T}\mathscr{B}_G(f))_\xi$, for $\xi\in V/G$, is isomorphic to the extended $\mathscr{B}_G$ tangent space, $T_e\mathscr{B}_G(f)_S$, of the germ of f along the unique closed orbit $S=T(\xi)$ in $\pi^{-1}(\xi)$. This is defined in a manner completely analogous to the definition for the germ of f at 0. As we shall see below, the corresponding results for $\mathscr{B}=\mathscr{L}$ or $\mathscr{A}$ are true if f is respectively $\mathscr{C}_G^*$ or $\mathscr{K}_G^*$ finitely determined.

For $\mathscr{B}=\mathscr{R},\mathscr{L}$ or $\mathscr{A}$ the sheaf $\mathscr{T}\mathscr{B}_G^*(f)$ is the same as $\mathscr{T}\mathscr{B}_G(f)$, since the extended tangent spaces are the same. The sheaf $\mathscr{T}\mathscr{C}_G^*(f)$ is defined by

$$\mathscr{T}\mathscr{C}_G^*(f)=\bar{f}^* m_G(W)\cdot \mathscr{V}_G(f),$$

where $\mathscr{V}_G(f)=\pi_G^*(\mathscr{V}(f))$ is the sheaf of germs of equivariant vector fields along f, $m_G(\mathrm{W})$ is the ideal sheaf of germs of functions vanishing at $\pi(0)$ in W/G and $\bar{f}: V/G\to W/G$ is the "quotient mapping" induced by f. For $\mathscr{K}_G^*$ we take $\mathscr{T}\mathscr{K}_G^*(f) = \mathscr{T}\mathscr{R}_G(f)+\mathscr{T}\mathscr{C}_G^*(f)$. If $\mathscr{B}$ is either $\mathscr{C}$ or $\mathscr{K}$ it is easily seen that $\mathscr{T}\mathscr{B}_G^*(f)$ is coherent and its stalk at $\xi\in V/G$ can be identified with the extended $\mathscr{B}_G^*$ tangent space of f along the closed orbit $T(\xi)$.

Definition. An equivariant complex analytic mapping $f: V\to W$ is said to be infinitesimally $\mathscr{B}_G$ (resp. $\mathscr{B}_G^*$) stable along a closed G orbit S in V if $T_e\mathscr{B}_G(f)_S =\theta_G(f)_S$ (resp. $T_e\mathscr{B}_G^*(f)_S=\theta_G(f)_S$).

More generally, S can be replaced in the definition by a finite union of G-orbits. We also define

$$\Sigma_G(f)=\{\xi\in V/G: f \text{ is not } \mathscr{R}_G \text{ stable along } T(\xi)\}.$$

The geometric characterisations of $\mathscr{B}_G$ and $\mathscr{B}_G^*$ finite determinacy are given by the following theorem.

Theorem 2.1. *The germ at* 0 *of an equivariant complex analytic mapping* $f: V,0\to W,0$ *is* $\mathscr{B}_G$ *(resp.* $\mathscr{B}_G^*$*) finitely determined if and only if*

(i) *for* $\mathscr{B}=\mathscr{R}$, $\mathscr{C}$ *or* $\mathscr{K}$, *there exists a G-stable neighbourhood* U *of* 0 *in* V *such that* f *is* $\mathscr{B}_G$ *(resp.* $\mathscr{B}_G^*$*) stable along all closed orbits in* $U\setminus\{0\}$

(ii) *for* $\mathscr{B}=\mathscr{L}$ *(resp.* $\mathscr{A}$*), there exists a neighbourhood* U *of* $\pi(0)\in V/G$ *such that* $\bar{f}|_U$ *(resp.* $\bar{f}|_{U\cap\Sigma_G(f)}$*) is a finite map and* f *is* $\mathscr{B}_G$ *stable* (*or, equivalently,* $\mathscr{B}_G^*$ *stable*) *along all finite unions of closed orbits in* $\pi^{-1}(U\setminus\pi(0))$.

Proof. of (i). For $\mathscr{B}=\mathscr{R}$, $\mathscr{C}$ or $\mathscr{K}$ define

$$\mathscr{S}\mathscr{B}_G(f)=\mathscr{V}_G(f)/\mathscr{T}\mathscr{B}_G(f),\qquad \mathscr{S}\mathscr{B}_G^*(f)=\mathscr{V}_G(f)/\mathscr{T}\mathscr{B}_G^*(f).$$

These are coherent sheaves on V/G. Their stalks at $\pi(0)$ are finite dimensional if and only if the germ of f at 0 is respectively $\mathscr{B}_G$ or $\mathscr{B}_G^*$ finitely determined. The stalks at $\xi\in V/G$ are trivial if and only if f is $\mathscr{B}_G$ or $\mathscr{B}_G^*$ stable along $T(\xi)$. Thus the theorem follows from the sheaf theoretic Nullstellensatz, which states that a coherent sheaf has a finite dimensional stalk at a point if and only if every other stalk in a neighbourhood of the point is zero. □

Before proving part (ii) of the theorem we give a more explicit description of $\mathscr{C}_G^*$ and $\mathscr{K}_G^*$ finitely determined germs.

Proposition 2.2. *Let* $f: V,0\to W,0$ *be an equivariant complex analytic mapping.*

(i) *Then* f *is* $\mathscr{C}_G^*$ *stable along a closed orbit* S *if and only if* $f(x)\neq 0$ *for all* x *in* S. *It is* $\mathscr{K}_G^*$ *stable along* S *if and only if it is either* $\mathscr{C}_G^*$ *stable or* $\mathscr{R}_G$ *stable.*

(ii) *The germ of* f *at* 0 *is* $\mathscr{C}_G^*$ *(resp.* $\mathscr{K}_G^*$*) finitely determined if and only if there exists a neighbourhood* U of $\pi(0)$ *in* V/G *such that* $\bar{f}|_U$ *(resp.* $\bar{f}|_{U\cap\Sigma_G(f)}$*) is a finite mapping.*

Proof. (i) If $f(x)=0$ for $x\in S$, then

$$\mathcal{T}\mathcal{C}_G^*(f)_{\pi(S)}=(\bar{f}^*m_G(W)\cdot\mathcal{V}_G(f))_{\pi(S)}\subset m_G(V)_S\cdot\mathcal{V}_G(f)_{\pi(S)}\neq\mathcal{V}_G(f)_{\pi(S)}$$

where $m_G(V)_S$ is the maximal ideal of the stalk of the structure sheaf of V/G at $\pi(S)$. Conversely, if $f(x)\neq 0$ then $\bar{f}^*m_G(W)$ is equal to the structure sheaf of V/G at $\pi(S)$ and so $\mathcal{T}\mathcal{C}_G^*(f)_{\pi(S)}=\mathcal{V}_G(f)_{\pi(S)}$. If f is either $\mathcal{C}_G^*$ stable or $\mathcal{R}_G$ stable along S then it is clearly $\mathcal{K}_G^*$ stable. Conversely suppose f is $\mathcal{K}_G^*$ stable, but not $\mathcal{C}_G^*$ stable, along S. Then $(\bar{f}^*m_G(W)\cdot\mathcal{V}_G(f))_{\pi(S)}\subset m_G(V)_S\cdot\mathcal{V}_G(f)_{\pi(S)}$ and so

$$(\mathcal{T}\mathcal{R}_G(f))_{\pi(S)}+m_G(V)_S\cdot\mathcal{V}_G(f)_{\pi(S)}=\mathcal{V}_G(f)_{\pi(S)},$$

and Nakayama's lemma implies $\mathcal{T}\mathcal{R}_G(f)_{\pi(S)}=\mathcal{V}_G(f)_{\pi(S)}$.

(ii) By Theorem 2.1 (i) and part (i) of this proposition, f is $\mathcal{C}_G^*$ finitely determined at 0 if and only if there exists a neighbourhood U of $\pi(0)$ in V/G such that $\bar{f}^{-1}(0)\cap U=\{\pi(0)\}$. This is equivalent to $\bar{f}|_U$ being finite.

Similarly f is $\mathcal{K}_G^*$ finitely determined at 0 if and only if there is a neighbourhood U of $\pi(0)$ in V/G such that $\bar{f}^{-1}(0)\cap U\cap\Sigma_G(f)=\pi(0)$ and so if and only if $\bar{f}|_{U\cap\Sigma_G(f)}$ is finite. □

In the proof of part (ii) of Theorem 2.1 we will also need the following easily proved lemma.

Lemma 2.3. *Let $f\colon V,0\to W,0$ be an equivariant complex analytic mapping.*

(i) *If f is $\mathcal{B}_G$ stable along the finite union of closed orbits $S=\bigcup_{i\in I}T(\xi_i)$, then for each subset J of I, the mapping f is $\mathcal{B}_G$ stable along $\bigcup_{i\in J}T(\xi_i)$.*

(ii) *Let $S_1=\bigcup_{i\in I}T(\xi_i)$ and $S_2=\bigcup_{j\in J}T(\xi_j)$ be two finite unions of closed orbits. For $\mathcal{B}=\mathcal{L}$ and $\mathcal{A}$ suppose in addition that $f(\xi_i)\neq f(\xi_j)$ for any $i\in I$ and $j\in J$. Then, if f is $\mathcal{B}_G$ stable along S_1 and along S_2, it is $\mathcal{B}_G$ stable along $S_1\cup S_2$.* □

Proof of Part (ii) of Theorem 2.1. For $\mathcal{B}=\mathcal{L}$ it can be assumed that f is $\mathcal{C}_G^*$ finitely determined and so that for some neighbourhood U of $\pi(0)$ in V/G the restriction $\bar{f}_U:=\bar{f}|_U$ is finite and $\bar{f}_U^{-1}(0)=\{\pi(0)\}$. By Grauert's Theorem $\bar{f}_{U*}\mathcal{V}_G(f)$ is a coherent sheaf on W/G and hence so is

$$\mathcal{S}\mathcal{L}_G(f):=\bar{f}_{U*}\mathcal{V}_G(f)/\mathcal{T}\mathcal{L}_G(f).$$

Let η be a point in W/G, let $\bar{f}_U^{-1}(\eta)=\{\xi_i\}_{i=1}^r$ and put $S=\bigcup_{i=1}^r T(\xi_i)$. Then $(\bar{f}_{U*}\mathcal{V}_G(f))_\eta\cong\theta_G(f)_S$ and $\mathcal{T}\mathcal{L}_G(f)_\eta\cong T\mathcal{L}_G(f)_S$. So f is $\mathcal{L}_G$ stable along S if and only if $\mathcal{S}\mathcal{L}_G(f)_\eta=0$ and is $\mathcal{L}_G$ (and $\mathcal{L}_G^*$) finitely determined at $\pi(0)$ if and only if $\mathcal{S}\mathcal{L}_G(f)_{\pi(0)}$ is finite dimensional. The theorem therefore follows from the Nullstellensatz and Lemma 2.3.

For $\mathcal{B}=\mathcal{A}$ it can be assumed that f is $\mathcal{K}_G^*$ finitely determined and so that for some neighbourhood U of $\pi(0)$ in V/G the restriction $\bar{f}_{U\cap\Sigma(f)}$ is finite and $\bar{f}_U^{-1}(0)=\{\pi(0)\}$. Since the support of $\mathcal{S}\mathcal{R}_G(f)$ is $\Sigma_G(f)$, Grauert's theorem implies that $\bar{f}_{U*}\mathcal{S}\mathcal{R}_G(f)$ is a coherent sheaf on W/G and hence so is

$$\mathcal{S}\mathcal{A}_G(f):=\bar{f}_{U*}\mathcal{S}\mathcal{R}_G(f)/\mathcal{T}\mathcal{L}_G(f).$$

The proof is now essentially the same as for L_G. □

3. Equivariant Jets

This section contains a brief discussion of "G-jets"; a more complete account is given in [23]. Let $C_G(V, W)$ denote the space of equivariant complex analytic maps from V to W and, for a closed orbit S in V, let $C_G(V, W)_S$ denote the space of germs along S.

Definition. If k is a non-negative integer, $I_G^k(V, W)$ is defined to be the set of equivalence classes of pairs (ξ, f), with $\xi \in V/G$ and $f \in C_G(V, W)$, under the relation $(\xi_1, f_1) \sim (\xi_2, f_2)$ if and only if $\xi_1 = \xi_2$ and the images $[f_1]$ and $[f_2]$ of the germs of f_1 and f_2 along $T(\xi_1)$ in $C_G(V, W)_{T(\xi)}/m_G(V)_{T(\xi)}^{k+1} C_G(V, W)_{T(\xi)}$ are equal.

Let $\{(V/G)_\lambda\}_{\lambda \in \Lambda}$ denote the stratification of V/G by slice representation type of points in closed orbits in V [19]. For each λ let G_λ denote a member of the conjugacy class of isotropy groups of points in closed orbits in $V_\lambda = \pi^{-1}(V/G)_\lambda$ and let R_λ be a representation of G_λ belonging to the isomorphism class of slice representations at these points. If $\xi \in (V/G)_\lambda$ then

$$\frac{C_G(V, W)_{T(\xi)}}{m_G(V)_{T(\xi)}^{k+1} C_G(V, W)_{T(\xi)}} \cong \frac{C_{G_\lambda}(R_\lambda, W)_0}{m_{G_\lambda}(R_\lambda)_0^{k+1} C_{G_\lambda}(R_\lambda, W)_0}$$

There is an obvious projection of $I_G^k(V, W)$ onto V/G; let $I_G^k(V, W)_\xi$ denote the fibre over the point $\xi \in V/G$ and similarly, if U is a subset of V/G, let $I_G^k(V, W)_U$ denote the inverse image of U under this projection. Define $I_G^k(V, W)_\lambda$ to be $I_G^k(V, W)_{(V/G)_\lambda}$

Proposition 3.1. *$I_G^k(V, W)_\lambda$ is an analytic bundle over $(V/G)_\lambda$ with fibre over ξ isomorphic to*

$$C_{G_\lambda}(R_\lambda, W)_0/m_{G_\lambda}(R_\lambda)_0^{k+1} C_{G_\lambda}(R_\lambda, W)_0$$

Proof. Let $\xi \in (V/G)_\lambda$ and $x \in T(\xi)$. By the slice theorem [19] there exists a neighbourhood U' of ξ in V/G and a map $\tau: R_\lambda, 0 \to (\pi^{-1}(U'), x)$ which induces an analytic isomorphism

$$\varphi: (R_\lambda^{G_\lambda}/N(G_\lambda), 0) \to (U, \xi)$$

where $U = U' \cap (V/G)_\lambda$ and $N(G_\lambda)$ is the normaliser of G_λ in G. Define a trivialisation of $I_G^k(V, W)_\lambda$, over U, by

$$\varphi: I_G^k(V, W)_\lambda \to (R_\lambda^{G_\lambda}, N(G_\lambda)) \times (C_{G_\lambda}(R_\lambda, W)_0/m_{G_\lambda}(R_\lambda)^{k+1} C_{G_\lambda}(R_\lambda, W)_0$$
$$[(\xi, f)] \to (\varphi^{-1}(\xi), [(f \circ \tau)^w]),$$

where w is a point in $R_\lambda^{G_\lambda}$ projecting down to $\varphi^{-1}(\xi)$ in $R_\lambda^{G_\lambda}/N(G_\lambda)$ and $(f \circ \tau)^w(y) = (f \circ \tau)(w + y)$. The map φ is independent of the choice of w and is bijective; the inverse being given by

$$\psi: (R_\lambda^{G_\lambda}/N(G_\lambda)) \times (C_{G_\lambda}(R_\lambda, W)_0/m_{G_\lambda}(R_\lambda)^{k+1} C_{G_\lambda}(R_\lambda, W)_0) \to I_G^k(V; W)_\lambda$$
$$(\xi, [f]) \to [\varphi(\xi), \tilde{f}],$$

where $\tilde{f}$ is defined on a neighbourhood of $T(\xi)$, isomorphic to $R_\lambda \times_{G_\lambda} G$, by $\tilde{f}[y, g] = g \cdot f(y)$.

If U_1 and U_2 are open subsets of $(V/G)_\lambda$ with $U_1 \cap U_2 = U \neq \emptyset$ and φ_1 and φ_2 are trivialisations of $I_G^k(V, W)_{U_1}$ and $I_G^k(V, W)_{U_2}$ of the above form, then the transition functions $\varphi_1|_U \circ \varphi_2^{-1}|_{\varphi_2(U)}$ are analytic, proving the proposition.

For $f \in C_G(\mathrm{V},\mathrm{W})$ define maps

$$i_G^k f_\lambda : (V/G)_\lambda \to I_G^k(V,W)_\lambda \qquad \xi \to [(\xi, f)]$$

for each $\lambda \in \Lambda$ and non-negative integer k, By composing these maps with the trivialisations of the previous proposition it can be seen that $i_G^k f_\lambda$ is an analytic section of $I_G^k(V,W)_\lambda$.

This construction can be extended in an obvious manner to obtain "multi-G-jet" spaces ${}_rI_G^k(V,W)$ which project down onto

$$(V/G)^{(r)} := \{\xi = (\xi_1, \dots, \xi_r) \in V/G^r : \xi_i \neq \xi_j ; i \neq j\}.$$

The orbit type stratification $(V/G)_{\lambda \in \Lambda}$ of V/G induces a stratification $\{(V/G^{(r)})_\gamma\}_{\gamma \in \Gamma}$ of $V/G^{(r)}$, where $\Gamma = \Lambda^r$ and $((V/G)^{(r)})_\gamma = (\Pi_i (V/G)_{\lambda_i}) \cap (V/G)^{(r)}$ for $\gamma = (\lambda_1, \dots, \lambda_r) \in \Lambda^r$.

As above, the restriction of ${}_rI_G^k(V,W)$ to $(V/G^{(r)})_\gamma$, denoted ${}_rI_G^k(V,W)_\gamma$, is an analytic vector bundle and multi-G-jet extensions can be defined as sections of these bundles.

4. Finite Determinacy and Transversality

The third characterisation of $\mathscr{B}_G$ and $\mathscr{B}_G^*$ finitely determined germs is obtained from the geometric criteria by replacing the condition that f is stable along (resp. finite unions of) closed orbits by transversality conditions on the G-jet (resp. multi-G-jet) extensions of f. For simplicity we shall restrict attention to the groups $\mathscr{B}_G$.

Let $\mathscr{R}_G^e$ denote the group of all equivariant diffeomorphisms of V (not necessarily preserving 0), $\mathscr{L}_G^e$ those of W and $\mathscr{A}_G^e = \mathscr{R}_G^e \times \mathscr{L}_G^e$. The group $\mathscr{C}_G^e$ is defined to be the same as $\mathscr{C}_G$ and $\mathscr{K}_G^e$ is a semi-direct product of $\mathscr{R}_G^e$ and $\mathscr{C}_G^e$, defined in the same way as $\mathscr{K}_G$. We will use $\mathscr{B}_G^e$ to denote any of these groups. The action of $\mathscr{B}_G$ on $C_G(V,W)$ extends, in an obvious manner, to an action of $\mathscr{B}_G^e$. The groups $\mathscr{B}_G^e$ also act on V, the actions of $\mathscr{L}_G^e$ and $\mathscr{C}_G^e$ being trivial and those of $\mathscr{A}_G^e$ and $\mathscr{K}_G^e$ factoring through the natural action of $\mathscr{R}_G^e$. Moreover, these actions on V commute with those of G and so induce actions on V/G. For the remainder of this section the notation $\mathscr{B}_G^e$ will be abbreviated to $\mathscr{B}$.

Let $\xi \in V/G$ and $x \in T(\xi)$ and let $\mathscr{B}_x$ (resp. $\mathscr{B}_\xi$) denote the subgroup of $\mathscr{B}$ leaving x (resp. ξ) fixed. If $\mathscr{B} = \mathscr{R}_G^e$, $\mathscr{A}_G^e$ or $\mathscr{K}_G^e$ and G_x is the isotropy group of x, G_x acts on $\mathscr{B}_x$, by $(g \cdot \alpha)(y) = g \cdot \alpha(y)$ for all $g \in G_x$, $\alpha \in \mathscr{B}_x$ and $y \in V$ if $\mathscr{B} = \mathscr{R}_G^e$, and by the trivial extension of this action if $\mathscr{B} = \mathscr{A}_G^e$ or $\mathscr{K}_G^e$. There is an isomorphism

$$\chi : \mathscr{B}_\xi \cong \mathscr{B}_x \times_{G_x} G$$

defined by

$$\chi(\alpha) = [\alpha', g],$$

where g is an element of G such that $g^{-1} \cdot \alpha(x) = x$ and $\alpha' = g^{-1} \circ \alpha$ (g exists since $\alpha \in \mathscr{B}_\xi$). Of course, if $\mathscr{B} = \mathscr{L}_G^e$ or $\mathscr{C}_G^e$, then $\mathscr{B}_x = \mathscr{B}_\xi = \mathscr{B}$.

The action of $\mathscr{B}$ on $C_G(V,W)$ induces actions on $I_G^k(V,W)$ for each k; these actions preserve the restrictions $I_G^k(V,W)_\gamma$ and restrict to actions of the groups $\mathscr{B}_x$ and $\mathscr{B}_\xi$ on $I_G^k(V,W)_\xi$, the action of $\mathscr{B}_\xi$ factoring through that of $\mathscr{B}_x$. In turn this action of $\mathscr{B}_x$ factors through that of $\mathscr{B}_x^k$, the group of k-G-jets at ξ of elements of $\mathscr{B}_x$.

Lemma 4.1. (i) *The action of $\mathscr{B}_x^k$ on $I_G^k(V,W)_\xi$ is algebraic.*

(ii) *For $\mathscr{B}=\mathscr{R}_G^e$, $\mathscr{A}_G^e$ and $\mathscr{K}_G^e$, the orbit $\mathscr{B}\cdot z$, $z\in I_G^k(V,W)_\xi$, is a fibre bundle over $(V/G)_\lambda$, with fibre $\mathscr{B}_x\cdot z$ over ξ.*

Proof. The proof of (i) is elementary. For (ii), let

$$\varphi:(R_\lambda^{G_\lambda}/N(G_\lambda),0)\to(U,\xi)$$

be the map defined in the proof of Proposition 3.1 and define a trivialisation

$$\varphi: I_G^k(V,W)_U\cap\mathscr{B}\cdot z\to R_\lambda^{G_\lambda}/N(G_\lambda)\times\mathscr{B}_x\cdot z$$

by

$$(\alpha\cdot z)\to(\varphi^{-1}(\alpha(\xi)),\ \alpha'\cdot z)$$

where α' is an element of $\mathscr{B}_x$ defined as follows: for $\mathscr{B}=\mathscr{R}_G^e$ define α' on a neighbourhood of $T(\xi)$ isomorphic to $R_\lambda\times_{G_\lambda}G$ by $\alpha'[(y,g)]=\alpha[(y,g)]+g(x-\alpha(x))$, and for $\mathscr{B}=\mathscr{A}_G^e$, $\mathscr{K}_G^e$ write α as $\alpha=\beta\cdot\gamma$, where $\beta\in\mathscr{R}_G^e$ and $\gamma\in\mathscr{C}_G$, and define α' by $\alpha'=\beta'\cdot\gamma\in\mathscr{B}_x$ where β' is as constructed for $\mathscr{R}_G^e$.

The transition functions associated with these trivialisations are analytic. □

Corollary 4.2. *The orbits $\mathscr{B}\cdot z$ are smooth submanifolds of $I_G^k(V,W)_\lambda$.* □

The above discussion is easily generalised to the induced actions of $\mathscr{B}$ on ${}_rI_G^k(V,W)$; again the orbits of these actions are smooth submanifolds of the restrictions ${}_rI_G^k(V,W)_\lambda$.

Theorem 4.3. *Let $\{\xi_i\}_{i=1}^r$ be a finite subset of V/G with $\xi_i\in(V/G)_{\lambda_i}$ and let $\gamma=(\lambda_1,\dots,\lambda_r)$ and $\xi=(\xi_1,\dots,\xi_r)\in(V/G^{(r)})_\gamma$. Then $f\in C_G(V,W)$ is infinitesimally $\mathscr{B}_G$ stable along $S=\bigcup_{i=1}^r T(\xi_i)$ if and only if*

$${}_ri_G^kf_\gamma:(V/G^{(r)})_\gamma\to{}_rI_G^k(V;W)_\gamma$$

is transversal at ξ to the orbits of the action of $\mathscr{B}_G^e$ on ${}_rI_G^k(V/W)_\lambda$, where, for $\mathscr{B}=\mathscr{R}$, $\mathscr{C}$, $\mathscr{K}$ we can take $k=0$, and for $B=\mathscr{L}$, $\mathscr{A}$, we take

$$k=\dim_{\mathbb{C}}\theta_G(W)_0/m_G(W)_0\,\theta_G(W)_0.$$

Proof. For simplicity (chiefly notational) the proof is given only for the $r=1$ case.

Let k be as defined in the statement of the theorem. Suppose $f\in C_G(V,W)$ and $\xi\in(V/G)_\lambda$ and let $z=i_G^kf(\xi)\in I_G^k(V,W)_\xi$. Then since $i_G^kf_\lambda$ is a section of $I_G^k(V,W)_\lambda$.

$$T_z(I_G^k(V,W)_\lambda)=D_\xi(i_G^kf_\lambda)\,(T_\xi(V/G)_\lambda)\oplus T_z(I_G^k(V,W)_\xi)\,.$$

Let p_f denote the induced projection:

$$p_f:T_z(I_G^k(V,W)_\lambda)\to T_z(I_G^k(V,W)_\xi)\,.$$

Lemma 4.4. (i) $T_z(I_G^k(V,W)_\xi)=\theta_G(f)_{T(\xi)}/m_G(V)_{T(\xi)}^{k+1}\,\theta_G(f)_{T(\xi)}$

(ii) $$p_f(T_z\mathscr{B}_G^e\cdot z)=\frac{T_e\mathscr{B}_G(f)_{T(\xi)}+m_G(V)_{T(\xi)}^{k+1}\,\theta_G(f)_{T(\xi)}}{m_G(V)_{T(\xi)}^{k+1}\,\theta_G(f)_{T(\xi)}}.$$ □

Lemma 4.5. *The map f is infinitesimally $\mathscr{B}_G$ stable along $T(\xi)$ for $\xi \in (V/G)_\lambda$ if and only if*

$$\frac{(T_e\mathscr{B}_G(f))_{T(\xi)} + m_G(V)^{k+1}_{T(\xi)}\,\theta_G(f)_{T\xi)}}{m_G(V)^{k+1}_{T(\xi)}\,\theta_G(f)_{T(\xi)}} = \frac{\theta_G(f)_{T(\xi)}}{m_G(V)^{k+1}_{T(\xi)}\,\theta_G(f)_{T(\xi)}}. \quad \square$$

The proofs of these lemmas are completely analogous to those in the non equivariant case [13].

The two lemmas imply that f is infinitesimally $\mathscr{B}_G$ stable along $T(\xi)$ for $\xi \in (V/G)$

$$\Leftrightarrow T_z I^k_G(V,W)_\xi = p_f(T_z\mathscr{B}^e_G \cdot z)$$

$$\Leftrightarrow T_z I^k_G(V,W)_\xi = T_z\mathscr{B}^e_G \cdot z + D_\xi(i^k_G f_\lambda)\,(T_\xi(V/G)_\lambda)$$

$\Leftrightarrow i^k_G f_\lambda$ is transversal to the orbit $\mathscr{B}^e_G \cdot z$ at ξ. $\square$

A result similar to this one was proved by Bierstone, [3], for infinitesimal $\mathscr{A}_G$ stability (the other cases could be treated similarly). Bierstone, however, proved that stability is equivalent to equivariant transversality conditions on the ordinary jet extension of f, rather than ordinary transversality conditions on the G-jet extension.

Finally, combining this theorem with the geometric characterisation of finite determinacy gives

Corollary 4.6. *The map $f \in C_G(V,W)$ is $\mathscr{B}_G$ finitely determined at* 0 *if and only if,*

(i) *for $\mathscr{R}_G$, $\mathscr{C}_G$, $\mathscr{K}_G$, there exists a neighbourhood U of $\pi(0) \in V/G$ such that for each λ*

$$i^0_G f_\lambda : (V/G)_\lambda \cap (U \setminus \{0\}) \to I^0_G(V,W)_\lambda$$

is transversal to the orbits of the action of $\mathscr{B}^e_G$ on $I^0_G(V,W)_\lambda$.

(ii) *for $\mathscr{L}_G$ (resp. $\mathscr{A}_G$) there exists a neighbourhood U of $\pi(0) \in V/G$ such that $\bar{f}\,|\,U$ (resp. $\bar{f}\,|\,U \cap \Sigma_G(f)$) is a finite map and for all r and $\gamma \in \Lambda^r$*

$${}_r i^k_G f_\gamma : (V/G^{(r)})_\gamma \cap (U \setminus \{0\})^r \to {}_r I^k_G(V,W)_\gamma$$

is transversal to the orbits of the action of $\mathscr{B}^e_G$ on ${}_r I^k_G(V,W)_\gamma$, where $k = \dim_{\mathbb{C}} \theta_G(W)_0 / m_G(W)_0\theta_G(W)_0$. $\square$

5. Examples

The discussion in this section uses some elementary consequences of the above characterisations of $\mathscr{B}_G$ finitely determined equivariant map germs in an attempt to determine necessary and sufficient conditions on V and W for finitely determined germs to exist. The examples discussed are very special, and only $\mathscr{R}_G$, $\mathscr{C}_G$, and $\mathscr{K}_G$ finite determinacy are considered, but this suffices to indicate the complexity of the problem.

Invariant Maps

Let W be a trivial representation of G, of dimension p say. Without loss of generality it can be assumed that the representation $\tau : G \to \mathrm{GL}(V)$ defining the action of G on

V has trivial kernel and so the categorical quotient space V/G is a geometric quotient if and only if G is finite. Let $Z(V)=\pi^{-1}\pi(0))$; then it is not difficult to see that the finiteness of G is equivalent to the condition $Z(V)=\{0\}$ and so also to $\dim Z(V)=0$.

Over each stratum of V/G the 0-G-jet space is given by

$$I_G^0(V,W)_\lambda \cong (V/G)_\lambda \times W$$

and the orbits of the actions of $\mathscr{R}_G^e$, $\mathscr{C}_G^e$ and $\mathscr{K}_G^e$ are of the form:

$$\mathscr{R}_G^e: (V/G)_\lambda \times \{y\} \qquad \text{for } y\in W$$

$$\mathscr{C}_G^e: \{\xi\}\times\{0\} \text{ and } \{\xi\}\times(W\setminus\{0\}) \qquad \text{for } \xi\in V/G$$

$$\mathscr{K}_G^e: (V/G)_\lambda\times\{0\} \text{ and } (V/G)_\lambda\times(\mathrm{W}\setminus\{0\}).$$

The map f is $\mathscr{B}_G$ stable along $T(\xi)$ for $\xi\in(V/G)_\lambda$ if and only if $i_G^0 f_\lambda$ is transversal at ξ to these orbits; this condition reduces to

f is $\mathscr{R}_G$ stable along $T(\xi)$ if and only if f is a submersion at $x\in T(\xi)$

f is $\mathscr{C}_G$ stable along $T(\xi)$ if and only if $f(x)\neq 0$, for $x\in T(\xi)$

f is $\mathscr{K}_G$ stable along $T(\xi)$ if and only if $f \pitchfork 0$ at $x\in T(\xi)$.

Notice also that f is $\mathscr{C}_G$ stable (resp. finitely determined) if and only if it is $\mathscr{C}_G^*$ stable (resp. finitely determined) and $\mathscr{K}_G$ stable (resp. finitely determined) if and only if it is $\mathscr{K}_G^*$ stable (resp. finitely determined).

An immediate consequence of this and the geometric characterisation of finite determinacy is the

Proposition 5.1. *If G is finite, then $f\in\mathscr{E}_G(V,W)$ is $\mathscr{R}_G$ (resp. $\mathscr{C}_G$, $\mathscr{K}_G$) finitely determined if and only if f is $\mathscr{R}$ (resp. $\mathscr{C}$, $\mathscr{K}$) finitely determined.* □

In fact one of the implications of the proposition is true much more generally; if f is any equivariant map which is k-$\mathscr{B}$-determined then it is k-$\mathscr{B}_G$-determined, for $\mathscr{B}=\mathscr{R}$, $\mathscr{L}$, $\mathscr{A}$, $\mathscr{C}$ or $\mathscr{K}$ [18, 21]. However, if G is not finite or acts non-trivially on W then equivariant map germs are hardly ever finitely determined in the space of all map germs.

Corollary 5.2. *Suppose G is finite.*

(i) (a) *If $f\in\mathscr{E}_G(V,W)$ is $\mathscr{R}_G$ finitely determined and not a submersion then $p\leqq 1$.*

(b) *If $p\leqq 1$ then f is $\mathscr{R}_G$ finitely determined if and only if it is $\mathscr{K}_G$ finitely determined.*

(ii) (a) *If $f\in\mathscr{E}_G(V,W)$ is $\mathscr{C}_G$ finitely determined then $p\geqq \dim V$.*

(b) *If $p\geqq\dim V$ then f is $\mathscr{C}_G$ finitely determined if and only if it is $\mathscr{K}_G$ finitely determined.*

The proof is immediate from Proposition 5.1 and the corresponding facts for $\mathscr{R}$, $\mathscr{C}$ and $\mathscr{K}$ finite determinacy [22]. □

In [16] the author proves that for any G and any representations V and W of arbitrary dimension, $\mathscr{K}_G$ finite determinacy holds in general in $\mathscr{E}_G(V,W)$; that is "most" invariant map germs are $\mathscr{K}_G$ finitely determined. This, when combined with Corollary 5.2, solves the problem of the existence of $\mathscr{R}_G$, $\mathscr{C}_G$ and $\mathscr{K}_G$ finitely determined invariant map germs for finite groups and also the $\mathcal{K}_G$ finite determinacy problem for arbitrary reductive complex (resp. compact smooth) Lie groups.

The general problem of the existence of $\mathcal{C}_G$ finitely determined invariant germs can be solved in a similar manner.

Proposition 5.3. (i) *If $f \in \mathcal{E}_G(V, W)$ is $\mathcal{C}_G$ finitely determined, then $p \geqq \dim V/G$*

(ii) *If $p \geqq \dim V/G$ and f is $\mathcal{K}_G$ finitely determined, then it is $\mathcal{C}_G$ finitely determined.*

Proof. (i) The map $f: V \to W$ induces $\bar{f}: V/G \to W$ which is an analytic map with p dimensional target and so $\operatorname{codim} \bar{f}^{-1}(0) \leqq p$; but if f is $\mathcal{C}_G$ finitely determined, $\operatorname{codim} f^{-1}(0) = \dim V/G$.

(ii) The map $f: V \to W$ can only be $\mathcal{R}_G$ stable along $T(\xi)$, if $\bar{f}|(V/G)_\lambda$ is a submersion at ξ. So if $p > \dim V/G$ then $\mathcal{K}_G$ stability along $T(\xi)$ is equivalent to $\mathcal{C}_G$ stability and hence $\mathcal{K}_G$ finite determinacy is equivalent to $\mathcal{C}_G$ finite determinacy. If $p = \dim V/G$, then $\mathcal{K}_G$ stability along $T(\xi)$ is again equivalent to $\mathcal{C}_G$ stability except for ξ in the top dimensional stratum, $(V/G)_p$ say, of V/G. Thus if f is $\mathcal{K}_G$ finitely determined, but not $\mathcal{C}_G$ finitely determined, $\bar{f}^{-1}(0)$ must have non-empty intersection with $(V/G)_p$ and by the curve selection lemma there will be an analytic curve Γ in V/G, passing through the origin, but otherwise lying in $(V/G)_p \cap \bar{f}^{-1}(0)$. The derivative Df must annihilate the tangent space to this curve and so f can not be a submersion at points of Γ. That is, the curve Γ consists of points ξ such that f is neither $\mathcal{R}_G$ nor $\mathcal{C}_G$ stable along $T(\xi)$, and hence not $\mathcal{K}_G$ stable. This contradicts the $\mathcal{K}_G$ finite determinacy of f. □

Note that if $f: V \to W$ is invariant and $f(0) = 0$, then $f^{-1}(0) \supset Z(V)$ and so, if G is not finite, no G invariant germ can be $\mathcal{C}$ finitely determined.

The next proposition gathers together some necessary conditions for an invariant map germ to be $\mathcal{R}_G$ finitely determined. Recall that, if $\{\pi_i\}_{i=1}^{\ell}$ is a generating set (of minimal cardinality) for the ring of invariant polynomials on V, then $\pi: V \to \mathbb{C}^\ell$ is an invariant map which embeds V/G in $\mathbb{C}^\ell$. Any invariant map $f: V \to W$ is induced by a map $\bar{f}: \mathbb{C}^\ell \to W$.

Proposition 5.4. *If $f \in \mathcal{E}_G(V, W)$ is $\mathcal{R}_G$ finitely determined, then*

(i) $p \leqq \min\{\dim(V/G)_\lambda : (V/G)_\lambda \neq \{\pi(0)\}\}$,

(ii) *either f is a submersion or, for each reductive subgroup H of G, $p \leqq \dim(Z(V) \cap V^H) + 1$ and*

(iii) *either $\bar{f}$ is a submersion or $p \leqq \ell - \dim V/G + 1$ (for any $\bar{f}$ inducing f).*

Proof. (i) This follows immediately from the fact that $f = \bar{f} \circ \pi$ is a submersion at $x \in T(\xi)$ for $\xi \in (V/G)_\lambda$, if and only if $\bar{f}|(V/G)_\lambda$ is a submersion at ξ.

(ii) For each subgroup H of G, the map f is a submersion at $x \in V^H$ if and only if $f|V^H$ is a submersion at x; if f is not a submersion at 0, then $f|V^H$ can not be a submersion at 0 and the set of all points where $f|V^H$ is not a submersion will have codimension greater than or equal to $\dim V^H - p + 1$ in V^H. For f to be $\mathcal{R}_G$ finitely determined this set must be contained in $Z(V) \cap V^H$ and so $\operatorname{codim}(Z(V) \cap V^H) < \dim V^H - p + 1$; i.e. $p \leqq \dim(Z(V) \cap V^H) + 1$.

(iii) If $\bar{f}$ is not a submersion at 0 then the set of points ξ where $\bar{f}$ is not a submersion will have codimension at least $\ell - p + 1$, as will the intersection of this set with V/G. However, for f to be $\mathscr{R}_G$ finitely determined this intersection has to contain $\pi(0)$ as an isolated point and so $\dim V/G \leqq \ell - p + 1$, which gives (iii). $\square$

To establish sufficient conditions for the existence of $\mathscr{R}_G$ finitely determined germs seems rather more difficult; here I shall be content to point out that if $p=1$ then any $\mathscr{R}_G$ finitely determined invariant function germ is $\mathscr{K}_G$ finitely determined (by the same argument as for functions without group actions [22]) and to give the following example of a non-submersive $\mathscr{R}_G$ finitely determined germ with $p=2$.

The representation V is given by the $\mathbb{C}^*$ action on $\mathbb{C}^3$:

$$t \cdot (x, y, z) = (t^{-1}x, ty, tz).$$

The ring of invariant polynomials of V is generated by $\{xy, xz\}$ and f is defined by:

$$f\colon V \to \mathbb{C}^2\ (x, y, z) \to (xy, xz).$$

Then f is $\mathscr{R}_G$ finitely determined and, checking the conditions of Proposition 7.4:

(i) $p = \min\{\dim(V/G)_\lambda : (V/G)_\lambda \neq \{\pi(0)\}\} = 2$

(ii) $p < \dim Z(V) + 1 = 3$

(iii) $\bar{f}$ is a submersion.

Finally note,

(i) in [16] the author proves that $\mathscr{R}_G$ finite determinacy holds in general in $\mathscr{E}_G(V, W)$ if and only if $p \leqq 1$, and

(ii) if $p > 1$ then no element of $\mathscr{E}_G(V, W)$ can be $\mathscr{R}$ finitely determined except for submersions. In fact this is almost true even for the case $p=1$, if G is not finite; Slodowy [18] proved that if $f\colon V \to \mathbb{C}$ is a G invariant function germ with $\dim G \geqq 1$ and $j^2 f(0) = 0$, then f can not be $\mathscr{R}$ finitely determined.

Some Equivariant Functions

The examples above showed, among other things, that the equivariant finite determinacy of invariant function germs behaves in a similar manner to finite determinacy when there is no group action present, especially if the group is finite. The second set of examples will show that if there is a non-trivial group action on the target of an equivariant function germ then the behaviour is more complicated.

Let G be the cyclic group of order r, identified with the set of r-th roots of unity in $\mathbb{C}$, and let ω be a generator of this group. Let s be an integer with $1 \leqq s \leqq r-1$. The representation V will be the $2+n$ dimensional representation of G given by

$$\omega \cdot (x_1, x_2, y_1, \ldots, y_n) = (\omega x_1, \omega x_2, y_1, \ldots, y_n).$$

and W the 1 dimensional representation given by

$$\omega \cdot z = \omega^s z\,.$$

Thus the examples are parameterised by the triple (r, n, s).

The algebra of G invariant polynomials on V is generated by the monomials $x_1^p x_2^q$, with $p+q=r$, and $y_1, \ldots, y_n$. The module $\mathscr{E}_G(V, W)$ is generated over $\mathscr{E}_G(V)$ by monomials $x_1^p x_2^q$ with $p+q=s$. The orbit type stratification of V consists of two strata $V_1 = V^G$ and $V_2 = V \setminus V^G$. The G-jet space $I_G^0(V, W)$ is given by

$$I_G^0(V, W)_{V_1} \cong V_1 \times M, \qquad \text{where } M = \mathscr{E}_G(V, W)/m_G(V)\,\mathscr{E}_G(V, W),$$

$$I_G^0(V, W)_{V_2} \cong V_2 \times \mathbb{C}\,.$$

The action of GL (2) on the subspace of V with coordinates x_1 and x_2 induces the action on M obtained by its identification with the space of binary forms of degree s and the action of $\mathscr{R}_G^e$ on $I_G^0(V,W)_{V_1}$ has orbits of the form $V_1 \times \mathrm{GL}(2) \cdot \varphi$, for $\varphi \in M$. The action of $\mathscr{R}_G^e$ on $I_G^0(V,W)_{V_2}$ has orbits $V_2 \times \{z\}$ where $z \in \mathbb{C}$.

Any $f \in C_G(V,W)$ can be written

$$f(x,y) = \sum_{|\alpha|=s} h_\alpha(x,y)\, x^\alpha$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index with $|\alpha| = \alpha_1 + \alpha_2$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$. For each α, h_α is an invariant function on V. The set $\{h_\alpha\}$ defines an invariant map $h: V \to M$ and the transversality criteria for the $\mathscr{R}_G$ finite determinacy of f at 0 translate into the pair of conditions

(i) $f|V_2 : U \cap V_2 \to W$ is a submersion, and

(ii) $h|V_1 : U \cap V_1 \to M$ is transversal to the orbits of GL (2) on M, for some punctured neighbourhood U of 0.

Condition (i) imposes no restrictions on (r,n,s) for the existence of $\mathscr{R}_G$ finitely determined germs. However condition (ii) means that either

(a) $\dim V_1 = 0$, i.e. $n=0$, or

(b) $\dim V_1$ is greater than or equal to the minimum codimension of the orbits of GL (2) on M.

The minimum codimension of the orbits of GL (2) on the space of binary forms of degree s is equal to

$$\max\{0, \dim M - \dim \mathrm{GL}(2)\} = \begin{cases} s-3 & \text{if } s \geqq 3 \\ 0 & \text{if } s = 1,2 \end{cases}$$

(since for $s \geqq 3$ the top dimensional orbit, consisting of non-singular binary forms, has finite isotropy group). Thus a necessary condition for $\mathscr{E}_G(V,W)$ to contain $\mathscr{R}_G$ finitely determined germs is

$$n = 0 \quad \text{or} \quad s \leqq 2 \quad \text{or} \quad n \geqq s-3\,.$$

In fact these conditions are also sufficient. If $n=0$ or $s=1$ or 2 let $f(x_1, x_2, y) = \sum_{|\alpha|=s} a_\alpha x^\alpha$, where the a_α are complex numbers chosen so that the resulting binary form is non-singular; then f is $\mathscr{R}_G$ finitely determined at 0 (in fact $\mathscr{R}$ finitely determined if $n=0$). For $s \geqq 3$ and $n \geqq s-3$, define $h: V_1 \to M$ so that $h(0)$ belongs to the stratum of non-singular binary forms and h is transversal to the orbit $\mathrm{GL}(2) \cdot h(0)$ at 0. Then $f(x,y) := \sum_{|\alpha|=s} h_\alpha(y)\, x^\alpha$ is an equivariant function which is $\mathscr{R}_G$ finitely determined at 0.

These examples are also studied in [16] where it is shown that $\mathscr{R}_G$ finite determinacy holds in general if and only if $n=0$ or $s \leqq 3$.

Finally, it is easily checked that the $\mathscr{K}_G^e$ orbits in $I_G^0(V,W)$ are the same as the $\mathscr{R}_G^e$ orbits and so a germ is $\mathscr{K}_G$ finitely determined if and only if it is $\mathscr{R}_G$ finitely determined.

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Singular Solutions of the p-Laplace Equation

Satyanad Kichenassamy[1] and Laurent Veron[2]

[1] Ecole Normale Supérieure, 45 rue d'Ulm, F-75230 Paris Cedex 05, France
[2] Département de Mathématiques, Université de Tours, Parc de Grandmont, F-37200 Tours, France

0. Introduction

Let Ω be an open subset of $\mathbb{R}^N$, $N>1$, and p a real number greater than 1, we shall say that a function u belonging to the Sobolev space $W^{1,p}(\Omega)$ is p-harmonic in Ω if

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx=0 \tag{0.1}$$

for any φ in the space $C_0^1(\Omega)$ of C^1 functions with compact support in Ω. This means that u is a weak solution of the "p-Laplace" equation in Ω

$$A_p u=-\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0, \tag{0.2}$$

which formally corresponds to the Euler-Lagrange equation for

$$\int_\Omega |\nabla u|^p\,dx. \tag{0.3}$$

The case $p=2$ gives rise to usual harmonic functions and when $p=N$ the functional (0.3) is conformally invariant. The regularity problem for a wider class of quasilinear elliptic equations in divergence form have first been studied in the sixties by Ladyzenskaya and Ural'Ceva [11] and Serrin [16, 17] who also got very important results concerning singularities of solutions of such equations. Concerning the equation (0.2) the main breakthrough is due to Ural'Ceva [22] who proved the $C^{1,\alpha}$ regularity of the solutions when $p\geqq 2$ and Lewis [13] when $1<p<2$ (see also [1, 6, 19, 21]).

The purpose of this paper is to continue the work of Serrin and to study the singularity problem associated to the equation (0.2): *assume* 0 *belongs to* Ω, $\Omega'=\Omega-\{0\}$ *and* u *is p-harmonic in* Ω, *then can we describe the behaviour of* u *near* 0 *and does* u *satisfy any equation in* Ω?

When $p=2$ the answers are wellknown as u admits a development in series of spherical harmonics but this is not generalizable to the nonlinear case. It is known (Serrin [16, 17]) that if $1<p\leqq N$ and u is p-harmonic, bounded below in Ω', then

1) there exists $C\in\mathbb{R}$ such that

$$A_p u=C\delta \tag{0.4}$$

in the sense of distributions in Ω, where δ is the Dirac measure at 0,

2) if μ denotes the radial solution of $A_p u = \delta$ given by

$$\mu(x) = \begin{cases} C(p,N)\,|x|^{(p-N)/(p-1)} & \text{for} \quad 1 < p < N, \\ C(N)\,\mathrm{Log}(1/|x|) & \text{for} \quad p = N, \end{cases} \tag{0.5}$$

where $C(p,N) = \dfrac{p-1}{N-p}(N\omega_N)^{-1/(p-1)}$, $C(N) = (N\omega_N)^{-1/(N-1)}$ (ω_N being the volume of the unit ball in $\mathbb{R}^N$) and if u is not bounded above, then there exists $K \in \mathbb{R}$ such that

$$\frac{1}{K}\mu \leqq u \leqq K\mu \tag{0.6}$$

in some neighbourhood of 0.

In this article we first improve estimate (0.6) for a wider class of p-harmonic functions and we prove the following: *Let $1 < p \leqq N$ and let u be p-harmonic in Ω' such that $u/\mu \in L^\infty_{\mathrm{loc}}(\Omega)$ when $p = N$ $u/\mu \in L^\infty(\Omega)$ should be read $u/\mu \in L^\infty(\Omega \cap B_{1/2}(0))$. Then there exists $\gamma \in \mathbb{R}$ such that*

$$u - \gamma\mu \in L^\infty_{\mathrm{loc}}(\Omega). \tag{0.7}$$

Moreover

$$\lim_{x \to 0} |x|^{(N-1)/(p-1)} \nabla(u - \gamma\mu)(x) = 0 \tag{0.8}$$

and

$$A_p u = \gamma|\gamma|^{p-2}\delta \tag{0.9}$$

in $\mathscr{D}'(\Omega)$.

Our method consists of a combination of scaling arguments and regularity estimates, together with a sharp maximum principle which makes it similar in some sense to the one used by Tolksdorf in [20]. p-harmonic functions of the above type do exist. Namely we can solve the Dirichlet problem for (0.4):

Let $1 < p \leqq N$, $g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, $\gamma \in \mathbb{R}$. Then there exists a unique $u \in C^1(\Omega')$ such that $|\nabla u|^{p-1} \in L^1_{\mathrm{loc}}(\Omega)$, $\nabla u \in L^p(\Omega - B_r(0))$, $r > 0$ small enough and

$$u/\mu \in L^\infty(\Omega), \tag{0.10}$$

satisfying

$$\begin{cases} A_p u = \gamma|\gamma|^{p-2}\delta & \text{in} \quad \mathscr{D}'(\Omega), \\ \quad u = g & \text{on} \quad \partial\Omega. \end{cases} \tag{0.11}$$

Moreover u satisfies (0.7) *and* (0.8).

A weaker form of the uniqueness part of this result for $p = N = 3$ has been recently announced by Chrusciel [3]. As a byproduct we prove that *if $1 < p \leqq N$ and u is p-harmonic in $\mathbb{R}^N - \{0\}$ such that*

$$|u| \leqq a|\mu| + b, \tag{0.12}$$

$a, b \in \mathbb{R}$; then there exist γ, $\lambda \in \mathbb{R}$ such that

$$u = \gamma\mu + \lambda. \tag{0.13}$$

As for nonisotropic singularities we may expect to find p-harmonic analogs of $\mathcal{Re}(z^{-\beta})$, $\beta>0$, $N=2$. This is indeed the case. If we write $x\in\mathbb{R}^2$ as $|x|e^{i\sigma}$ we prove:

Assume $N=2$ and $p>1$. Then for each positive integer k, there exist a $\beta_k>0$ and $\omega_k:\mathbb{R}\mapsto\mathbb{R}$ with least period $2\pi/k$ of class C^2 such that

$$u_k(x)=|x|^{-\beta_k}\omega_k(\sigma) \tag{0.14}$$

is p-harmonic in $\mathbb{R}^2-\{0\}$. β_k is the positive root of

$$(\beta+1)^2=\left(1+\frac{1}{k}\right)^2\left(\beta^2-\beta\frac{2-p}{p-1}\right). \tag{0.15}$$

(β_k,ω_k) is unique up to translation and homothety over ω_k.

All the solutions with $\beta<0$ have already been obtained by Krol and Mazja [10] by a similar method (see also [2, 5, 9, 12, 19] for some numerical calculations for $k=2$).

This article is organized as follows: Sect. 1: The isotropy theorem; Sect. 2: The singular Dirichlet problem; Sect. 3: Pseudo-radial singular solutions. Some of these results have been announced in [8].

1. The Isotropy Theorem

We assume that Ω is an open subset of $\mathbb{R}^N$, $N>1$, containing 0, $\Omega'=\Omega-\{0\}$, p is a real number and μ is defined by (0.5), so it satisfies (0.9) with $\gamma=1$. We shall frequently write $\mu(r)$ for $\mu(x)$ whenever $|x|=r$. Our main result is the following:

Theorem 1.1. *Assume $1<p\leqq N$ and u is a p-harmonic function in Ω' such that $u(x)/\mu(x)$ remains bounded in some neighbourhood of 0. Then there exists a real number γ such that*

$$u-\gamma\mu\in L^\infty_{\mathrm{loc}}(\Omega). \tag{1.1}$$

Moreover when $\gamma\neq 0$ the following relation holds

$$\lim_{x\to 0}|x|^{(N-p)/(p-1)+|\alpha|}D^\alpha(u-\gamma\mu)(x)=0 \tag{1.2}$$

for all multi-indices $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_N)$ with $|\alpha|=\alpha_1+\alpha_2+\ldots+\alpha_N\geqq 1$, $|\nabla u|^{p-1}\in L^1_{\mathrm{loc}}(\Omega)$ and u satisfies the following equation (even if $\gamma=0$)

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u)=\gamma|\gamma|^{p-2}\delta \tag{1.3}$$

in the sense of distributions in Ω.

Without any loss of generality we may assume $\Omega\supset\{x\in\mathbb{R}^N:|x|\leqq 1\}=\bar{B}_1(0)$. We first have the following elementary estimates:

Lemma 1.1. *Under the hypotheses of Theorem 1.1 there exist two constants $\alpha=\alpha(N,p)$ and $C=C(N,p,u)$, $C\geqq 0$ and $\alpha\in(0,1)$ such that for any x,x' satisfying $0<|x|\leqq|x'|\leqq 1$ we have*

$$|\nabla u(x)|\leqq C|x|^{-1}\mu(x), \tag{1.4}$$

$$|\nabla u(x)-\nabla u(x')|\leqq C|x-x'|^\alpha|x|^{-1-\alpha}\mu(x). \tag{1.5}$$

Proof. For $0<a<1/2$ and $0<|x|<1$ we set $x=ay/(1+a)$ and $v(y)=u(ay/(1+a))/\mu(a)$. The function v is p-harmonic and bounded in $\{y:\frac{1}{2}\leqq|y|\leqq 3\}$ so we deduce from the a priori estimates of [13] (see also [1, 6, 19, 22]) that

$$\|\nabla v\|_{C^{\alpha}\{y:1/2\leqq|y|\leqq 2\}}\leqq C, \tag{1.6}$$

where α and C depend on (N,p) and (N,p,u) respectively, which implies (1.4) and (1.5).

Remark 1.1. The estimates (1.5) and (1.6) are not optimal in the case $p=N$ as the coefficient $\mu(x)$ is not necessary in that case, but this cannot be proved by now as it requires a scaling over $u-\gamma\mu$ which we do not yet know to be bounded. Moreover the hypothesis "u/μ bounded in some neighbourhood of 0" can be replaced by the following weaker one

$$\int_{r<|x|<2r}|u|^p\,dx\leqq C\begin{cases} r^{(p^2-N)/(p-1)} & \text{for } 1<p<N,\\ (r\,\mathrm{Log}(1/r))^N & \text{for } p=N,\end{cases} \tag{1.7}$$

for $0<r<1/4$.

The following strict comparison principle due to Tolksdorf [20, Proposition 3.3.2] is important in the sequel:

Lemma 1.2. *Assume G is a connected open subset of $\mathbb{R}^N$, u_1, and u_2 are p-harmonic in G $(p>1)$ such that*

$$u_1\geqq u_2 \quad in \quad G. \tag{1.8}$$

If u_1 and u_2 are not identical in G then

$$u_1>u_2 \quad in \quad G. \tag{1.9}$$

As a consequence we have the following

Corollary 1.1. *Assume G is a connected open subset of $\mathbb{R}^N$ which does not contain 0 and u is p-harmonic in G $(p>1)$. If $v=u/\mu$ or $u-\mu$ achieves its maximum in G, v is constant.*

Proof of Theorem 1.1. Case $1:1<p<N$. We define γ^+ and γ^- by

$$\gamma^+=\limsup_{x\to 0}u(x)/\mu(x),\qquad \gamma^-=\liminf_{x\to 0}u(x)/\mu(x). \tag{1.10}$$

If $\gamma^+=\gamma^-=0$ then $\lim_{x\to 0}u(x)/\mu(x)=0$. So we can assume that $\gamma^+>0$ (or $\gamma^-<0$ in the same way) and let β be the supremum of $u(x)$ on S^{N-1}. So the function $u_\beta=u-\beta$ still satisfies $\limsup_{x\to 0}u_\beta(x)/\mu(x)=\gamma^+$ and $\sup_{x\in S^{N-1}}u_\beta(x)=0$ and for the sake of simplicity we still call it u. We now define a function $\tilde{\gamma}$ on $[0,1]$ by

$$\tilde{\gamma}(r)=\sup_{r\leqq|x|\leqq 1}u(x)/\mu(x). \tag{1.11}$$

The function $\tilde{\gamma}$ is nonnegative. Moreover if there exist some $r\in(0,1]$ and some y with $r<|y|<1$ such that $\tilde{\gamma}(r)=u(y)/\mu(y)$ then $u(x)=\tilde{\gamma}(r)\,\mu(x)$ for any x in $\Gamma_r=\{\xi:r\leqq|\xi|\leqq 1\}$; this is exactely Corollary 1.1 and as $\sup_{x\in S^{N-1}}u(x)=0$ we get $\tilde{\gamma}=0$

on $[r, 1]$ and $u=0$ in Γ_r. Hence the function $\tilde{\gamma}$ is nonincreasing and we have

$$\left.\begin{aligned} &\text{(i)} && \tilde{\gamma}(r)=\sup_{|x|=r} u(x)/\mu(x)\,, \\ &\text{(ii)} && \lim_{r\downarrow 0} \tilde{\gamma}(r)=\gamma^+\,, \end{aligned}\right\} \tag{1.12}$$

and there exists x_r such that $|x_r|=r$ and $\tilde{\gamma}(r)=u(x_r)/\mu(x_r)$. We now define the function u_r on $\Lambda_r=\{\xi:0<|\xi|<1/r\}$ by

$$u_r(\xi)=u(r\xi)/\mu(r)\,. \tag{1.13}$$

The function u_r is p-harmonic in Λ_r and we have from Lemma 1.1

$$\left.\begin{aligned} |u_r(\xi)|&\leqq C|\mu(r\xi)/\mu(r)|\leqq C\mu(\xi)\,, \\ |\nabla u_r(\xi)|&\leqq (r/\mu(r))|(\nabla u)(r\xi)|\leqq C|\xi|^{(1-N)/(p-1)}\,, \\ |\nabla u_r(\xi)-\nabla u_r(\xi')|&\leqq C|\xi-\xi'|^{\alpha}(\min(|\xi|,|\xi'|))^{(1-N)/(p-1)-\alpha}\,, \end{aligned}\right\} \tag{1.14}$$

where C does not depend on r. From the Arzela-Ascoli's theorem there exists a p-harmonic function v defined in $\mathbb{R}^N-\{0\}$ and a sequence $\{r_n\}\to 0$ such that $\{u_{r_n}\}$ converges to v in the C^1 topology of any compact subset of $\mathbb{R}^N-\{0\}$. Moreover we have

$$\frac{u_r(\xi)}{\mu(\xi)}=\frac{u(r\xi)}{\mu(r)\,\mu(\xi)}=\frac{u(r\xi)}{\mu(r\xi)}\,\frac{\mu(r\xi)}{\mu(r)\,\mu(\xi)}=\frac{u(r\xi)}{\mu(r\xi)\,\mu(1)}\,, \tag{1.15}$$

so $u_r(\xi)/\mu(\xi)\leqq\gamma^+/\mu(1)$; if we set $\xi_r=x_r/r$, then $u_r(\xi_r)/\mu(\xi_r)=\tilde{\gamma}(r)/\mu(1)$. From the compactness of S^{N-1} we can suppose that there exists ξ_0 in S^{N-1} such that $\lim_{r_n\to 0}\xi_{r_n}=\xi_0$, which yields

$$v(\xi_0)/\mu(\xi_0)=\gamma^+/\mu(1) \quad\text{and}\quad v(\xi)/\mu(\xi)\leqq\gamma^+/\mu(1)\,. \tag{1.16}$$

From Lemma 1.2 it implies that $v(\xi)=\gamma^+\,\mu(\xi)/\mu(1)=\lim_{r\to 0} u_r(\xi)$ uniformly on every compact subset of $\mathbb{R}^N-\{0\}$. In the particular case where $\xi\in S^{N-1}$ we have in setting $x=r\xi$:

$$\lim_{x\to 0} u(x)/\mu(x)=\gamma^+=\gamma\,. \tag{1.17}$$

In order to prove the boundedness of $u-\gamma\mu$ we consider, for $\varepsilon>0$, the following p-harmonic functions in $B_1(0)-\{0\}$

$$v_\varepsilon^+(x)=(\gamma+\varepsilon)\,\mu(x)-(\gamma+\varepsilon)\,\mu(1)+\sup_{x\in S^{N-1}} u(x)\,, \tag{1.18}$$

$$v_\varepsilon^-(x)=(\gamma-\varepsilon)\,\mu(x)-(\gamma-\varepsilon)\,\mu(1)+\inf_{x\in S^{N-1}} u(x)\,. \tag{1.19}$$

As $(u-v_\varepsilon^+)^+$ and $(v_\varepsilon^- -u)^-$ vanish an $\partial B_1(0)$ and in a neighbourhood of 0 we get $v_\varepsilon^-(x)\leqq u(x)\leqq u_\varepsilon^+(x)$ in $\bar{B}_1(0)-\{0\}$. Letting ε going to 0 implies

$$\inf_{x\in S^{N-1}} u(x)-\gamma\mu(1)\leqq u-\gamma\mu\leqq\sup_{x\in S^{N-1}} u(x)-\gamma\mu(1)\,. \tag{1.20}$$

We also have from (1.14) and the fact that $\lim_{r\to 0} u_r(\xi)=\gamma\mu(\xi)/\mu(1)$:

$$\lim_{r\to 0} \nabla u_r(\xi)=\gamma/\mu(1)\,\nabla\mu(\xi)\,, \tag{1.21}$$

uniformly on every compact subset of $\mathbb{R}^N-\{0\}$. If we take $|\xi|=1$ and set $x=r\xi$, we get (1.2) for any α such that $|\alpha|=1$. From (1.21) we also get that ∇u_r never vanishes on $\bar{\Gamma}=\{\xi:1/2\leqq|\xi|\leqq 2\}$ for $0<r\leqq r_0$ and therefore u_r satisfies a nondegenerate elliptic equation in Γ and is C^∞. Using the same device as in Lemma 1.1 we deduce

$$|D^\alpha u_r(\xi)|\leqq C|\xi|^{(p-N)/(p-1)-|\alpha|} \tag{1.22}$$

for any multi-indices α ($|\alpha|>2$) which implies (1.2). We also have from Green's formula

$$\int_{|x|>r} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx=-\int_{|x|=r}\varphi|\nabla u|^{p-2}u_\nu\,dS \tag{1.23}$$

for any $\varphi\in C_0^1(\Omega)$ and $0<r<1$. As

$$\nabla u(x)\underset{x\to 0}{\sim}-\gamma(N\omega_N)^{-1/(p-1)}|x|^{(1-N)/(p-1)}x/|x|\,,$$

we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx=\gamma|\gamma|^{p-2}\,\varphi(0)\,. \tag{1.24}$$

Case 2: $p=N$. We define in the same way as above γ^+ and γ^- and we may assume that $\gamma=\gamma^+=\limsup_{x\to 0} u(x)/\mu(x)>0$. Moreover u is modified in such a way that if $\tilde{\gamma}$ is defined by

$$\tilde{\gamma}(r)=\sup_{r\leqq|x|\leqq\frac{1}{2}} u(x)/\mu(x)\,, \tag{1.25}$$

then $\tilde{\gamma}(1/2)=0$, $\tilde{\gamma}$ is nonincreasing and $\lim_{r\to 0}\tilde{\gamma}(r)=\gamma$. We define again $u_r(\xi)=u(r\xi)/\mu(r)$ on $\Lambda_r=\{\xi:0<|\xi|<1/(2r)\}$. So u_r is N-harmonic in Λ_r and it satisfies the following a priori estimates from Lemma 1.1:

$$|u_r(\xi)|\leqq C(1+|\mathrm{Log}|\xi||/\mathrm{Log}(1/r))\,, \tag{1.26}$$

$$|\nabla u_r(\xi)|\leqq C(1+|\mathrm{Log}|\xi||/\mathrm{Log}(1/r))\,|\xi|^{-1}\,, \tag{1.27}$$

$$|\nabla u_r(\xi)-\nabla u_r(\xi')|\leqq C|\xi-\xi'|^\alpha(1+|\mathrm{Log}|\xi||/\mathrm{Log}(1/r))\,|\xi|^{-1-\alpha}\,, \tag{1.28}$$

for $0<|\xi|\leqq|\xi'|\leqq 1/(2r)$ and $r<1/2$; hence $1/\mathrm{Log}(1/r)\leqq 1/\mathrm{Log}2$ and there exists a N-harmonic function v and a sequence $\{r_n\}\to 0$ such that $\{u_{r_n}\}$ converges to v in the C^1 topology of any compact subset of $\mathbb{R}^N-\{0\}$. Moreover the function v clearly satisfies

$$|v(\xi)|\leqq C\,, \tag{1.29}$$

$$|\nabla v(\xi)|\leqq C|\xi|^{-1}\,, \tag{1.30}$$

$$|\nabla v(\xi)-\nabla v(\xi')|\leqq C|\xi-\xi'|^\alpha|\xi|^{-1-\alpha}\,, \tag{1.31}$$

for $0<|\xi|\leqq|\xi'|$. From a result of Serrin [16] the function v which is uniformly bounded in $\mathbb{R}^N-\{0\}$ can be extended to $\mathbb{R}^N$ as a N-harmonic function $\tilde{v}$ still satisfying (1.29) (the singularity at 0 is removable) and from a Liouville-type theorem due to Reshetniak [15] such a function $\tilde{v}$ is constant. In order to determine its value we know that there exists ξ_r on S^{N-1} such that $\tilde{\gamma}(r)=u(r\xi_r)/\mu(r)$. We can assume that $\{\xi_{r_n}\}$ converges to some $\xi_0\in S^{N-1}$ and then

$$\gamma=\lim_{n\to+\infty}\tilde{\gamma}(r_n)=v(\xi_0)=\tilde{v}(\xi). \tag{1.32}$$

Hence $\lim_{r\to0}u_r(\xi)=\gamma$ and in particular for $|\xi|=1$ and $|x|=r$ we get $\lim_{x\to0}u(x)/\mu(x)=\gamma$. The proof of the boundedness of $u-\gamma\mu$ is exactely the same as in the case $1<p<N$. For $0<r|\xi|\leqq1/2$ and $0<r\leqq1/2$ we set

$$v_r(\xi)=u(r\xi)-\gamma\mu(r). \tag{1.33}$$

The function $v_r(\xi)$ is N-harmonic and bounded on any compact subset of Λ_r as we have $\mu(r\xi)=\mu(r)+\mu(\xi)$ and

$$\gamma\mu(\xi)-|u(r\xi)-\gamma\mu(r\xi)|\leqq v_r(\xi)\leqq\gamma\mu(\xi)+|u(r\xi)-\gamma\mu(r\xi)|. \tag{1.34}$$

From $C^{1,\alpha}$ estimates we get

$$\|\nabla v_r\|_{C^\alpha(1/2\leqq|\xi|\leqq2)}\leqq C\|v_r\|_{L^\infty(1/4\leqq|\xi|\leqq3)}, \tag{1.35}$$

which implies the improvement of estimates (1.6) and (1.7):

$$|\nabla u(x)|\leqq C|x|^{-1}, \tag{1.36}$$

$$|\nabla u(x)-\nabla u(x')|\leqq C|x-x'|^\alpha|x|^{-1-\alpha}, \tag{1.37}$$

for $0<|x|\leqq|x'|\leqq1$. Returning to (1.33) we have

$$|\nabla v_r(\xi)|\leqq C|\xi|^{-1}, \tag{1.38}$$

$$|\nabla v_r(\xi)-\nabla v_r(\xi')|\leqq C|\xi-\xi'|^\alpha|\xi|^{-1-\alpha}, \tag{1.39}$$

for $0<|\xi|\leqq|\xi'|\leqq1/2r$. In order to prove (1.2) we look for the point where the bounded function $u-\gamma\mu$ achieves its supremum on $\bar{B}_{1/2}(0)$. If it achieves this supremum in $B_{1/2}(0)$ then it is constant from Lemma 1.2 and everything is done, so we suppose that it not constant and either this supremum is achieved for $|x|=1/2$, or it is achieved at 0 and

$$\sup_{x\in\bar{B}_{1/2}(0)}(u(x)-\gamma\mu(x))=\limsup_{x\to0}(u(x)-\gamma\mu(x)). \tag{1.40}$$

Case 1. We assume (1.40) and set λ the value of this supremum. We define $\lambda(r)=\sup_{r\leqq|x|\leqq1/2}(u(x)-\gamma\mu(x))=\sup_{|x|=r}(u(x)-\gamma\mu(x))$ and the function λ is nonincreasing. Moreover there exists x_r such that $|x_r|=r$ and $\lambda(r)=u(x_r)-\gamma\mu(x_r)$. We now consider the set of functions $\{v_r\}$ defined in (1.33) which is relatively compact in the C^1 topology of any compact subset of Λ_r by (1.38) and (1.39); so there exist $v\in C^{1,\alpha}(\mathbb{R}^N-\{0\})$ and a sequence $\{r_n\}\to0$ such that $\{v_{r_n}\}$ converges to v in this topology. Moreover we can assume that $\xi_{r_n}=x_{r_n}/r_n$ converges to some ξ_0 on S^{N-1}.

As we have $u(r\xi)-\gamma\mu(r\xi)\leqq\lambda$ and

$$u(r_n\xi_{r_n})-\gamma\mu(r_n\xi_{r_n})\xrightarrow[n\to+\infty]{}\lambda,$$

we deduce

$$v(\xi)\leqq\gamma\mu(\xi)+\lambda \quad\text{and}\quad v(\xi_0)=\gamma\mu(\xi_0)+\lambda. \tag{1.41}$$

From Corollary 1.1 it implies that $v=\gamma\mu+\lambda$ and v_r converges to $\gamma\mu+\lambda$ as r goes to 0 in the C^1 topology of any compact subset of $\mathbb{R}^N-\{0\}$. Returning to (1.33) it means

$$\lim_{x\to 0}(u(x)-\gamma\mu(x))=\lambda, \tag{1.42}$$

$$\lim_{x\to 0}|x|\,\nabla u(x)=\gamma\nabla\mu(x/|x|), \tag{1.43}$$

which is (1.2) for any α of length 1. For $|\alpha|\geqq 2$ the proof is as in the case $1<p<N$.

Case 2. We assume $\sup\limits_{x\in B_{1/2}(0)}(u-\gamma\mu)(x)=\sup\limits_{|x|=1/2}(u-\gamma\mu)(x)$. We perform the scaling transformation (1.33) and there exist a N-harmonic function v and a sequence $\{r_n\}\to 0$ such that $\{v_{r_n}\}$ converges to v in the C^1 topology of any compact subset of $\mathbb{R}^N-\{0\}$. Moreover, from (1.34), v satisfies

$$C_1\leqq v(\xi)-\gamma\mu(\xi)\leqq C_2, \tag{1.44}$$

in $\mathbb{R}^N-\{0\}$ where C_1 and C_2 are two constants. We look now at the points where $v-\gamma\mu$ achieves its supremum in $\mathbb{R}^N-\{0\}$. If this supremum is achieved at some ξ_0 then $v-\gamma\mu$ is equal to some constant λ. Returning to (1.33) it implies that

$$\lim_{r_n\to 0}(u(r_n\xi)-\gamma\mu(r_n\xi))=\lambda \tag{1.45}$$

in the $C^1_{\mathrm{loc}}(\mathbb{R}^N-\{0\})$ topology. For $\varepsilon>0$ fixed there exists n_0 such that for $n\geqq n_0$ we have

$$\gamma\mu(r_n\xi)+\lambda-\varepsilon\leqq u(r_n\xi)\leqq\gamma\mu(r_n\xi)+\lambda+\varepsilon$$

for $\xi\in S^{N-1}$. Applying the maximum principle in $\{x: r_n<|x|<r_{n_0}\}$ we deduce $\gamma\mu(x)+\lambda-\varepsilon\leqq u(x)\leqq\gamma\mu(x)+\lambda+\varepsilon$. As ε is arbitrary it implies again (1.42) and (1.43). So we are left with the case where the supremum of $v-\gamma\mu$ is achieved either at 0 or at infinity. In the second case we perform the inversion θ of $\mathbb{R}^N-\{0\}$ defined by $\theta(x)=x/|x|^2$ which leaves the equation invariant [15] and exchanges 0 and infinity. We defined $\tilde{v}$ to be v or $v\circ\theta$ and $\tilde{v}$ achieves its supremum ν at 0. If $\tilde{v}=v$ then as in case 1 applied to v instead of u, we have

$$\lim_{\xi\to 0}(v(\xi)-\gamma\mu(\xi))=\nu, \tag{1.46}$$

which implies

$$\lim_{\xi\to 0}\lim_{r_n\to 0}(u(r_n\xi)-\gamma\mu(r_n\xi))=\nu. \tag{1.47}$$

With the maximum principle as above we get (1.42) and (1.43). If $\tilde{v}=v\circ\theta$, then

$$\lim_{|\xi|\to+\infty}(v(\xi)-\gamma\mu(\xi))=\nu, \tag{1.48}$$

which means

$$\lim_{|\xi|\to+\infty}\ \lim_{r_n\to 0}\ (u(r_n\xi)-\gamma\mu(r_n\xi))=v\,. \tag{1.49}$$

For $\varepsilon>0$ there exists $K>0$ such that for $|\xi|=K$ we have

$$v-\varepsilon\leqq \lim_{r_n\to 0}\ (u(r_n\xi)-\gamma\mu(r_nK)\leqq v+\varepsilon \tag{1.50}$$

as $r_nK\to 0$ we deduce again as before (with the maximum principle) (1.42) and (1.43) [or (1.2)]. The proof of (1.3) is the same as in the case $1<p<N$.

Remark 1.4. We can notice that in the case $p=N$ we have used a stronger result that the boundedness of $u-\gamma\mu$: we have proved that $u(x)-\gamma\mu(x)$ admits a limit as x tends to 0. It is clear that this result is also true when $1<p<N$. An interesting result should be to prove that $\nabla u-\gamma\nabla\mu$ is bounded near 0 (this is true when $p=2$), or more generally $D^\alpha(u-\gamma\mu)$ for $|\alpha|\geqq 1$, if $\gamma\neq 0$.

Remark 1.5. The above method allows us also to deal with the asymptotic behaviour of a function u which is p-harmonic in the complementary of a compact subset K of $\mathbb{R}^N$ and such that u/μ is bounded in some neighbourhood of infinity. When $p=N$ we just have to perform an inversion which is conformal and changes $\mathbb{R}^N-K$ into a bounded subset containing 0. When $1<p<N$ the proof of Theorem 1.1 can be adapted. In both case (1.17) and (1.2) still hold with $x\to 0$ replaced by $|x|\to+\infty$.

Remark 1.6. The previous method can be adapted to the case $p>N$. It has been proved by Serrin [18] that any bounded p-harmonic function u in G' can be extended as a continuous function $\tilde u$ in G. There is no loss of generality to suppose $\tilde u(0)=0$ and we have the following isotropy result: *assume* $|\tilde u(x)|\leqq C|x|^{(p-N)/(p-1)}$ *for* $|x|\leqq 1$, *then there exists a real number* γ *such that* (1.17), (1.2), *and* (1.3) *hold.*

2. The Singular Dirichlet Problem

We assume here that Ω is a bounded open subset of $\mathbb{R}^N$, $N>1$, containing 0, with a regular boundary $\partial\Omega$, we set $\Omega'=\Omega-\{0\}$ and we consider the following problem: find a function u defined in Ω' such that

$$\left\{\begin{array}{ll}-\operatorname{div}(|\nabla u|^{p-2}\nabla u)=\lambda\delta & \text{in}\quad \mathscr{D}'(\Omega),\\ u=g & \text{on}\quad \partial\Omega,\end{array}\right\} \tag{2.1}$$

with $p>1$, $\lambda\in\mathbb{R}$, and $g\in L^\infty(\Omega)\cap W^{1,p}(\Omega)$. When $p>N$, $\delta\in W^{-1,p'}(\Omega)$ so it comes from the theory of monotone operators (see for instance [14]) that there exists a unique $u\in W^{1,p}(\Omega)$ satisfying (2.1). When $1<p\leqq N$ the situation is completely different and we prove the following result:

Theorem 2.1. *Assume* $1<p\leqq N$ *then there exists a unique function* $u\in C^{1,\alpha}(\Omega')$ *such that* $|\nabla u|^{p-1}\in L^1(\Omega)$, $\nabla u\in L^p(\Omega-B_r(0))$ *for* $r>0$ *small enough and*

$$u/\mu\in L^\infty(\Omega)\,, \tag{2.2}$$

satisfying (2.1). *Moreover the following estimates hold:*

$$u-|\lambda|^{1/(p-1)}\operatorname{sign}(\lambda)\mu\in L^\infty(\Omega), \tag{2.3}$$

$$\nabla(u-|\lambda|^{1/(p-1)}\operatorname{sign}(\lambda)\mu)=o(|x|^{(1-N)/(p-1)}), \tag{2.4}$$

$$D^\alpha(u-|\lambda|^{1/(p-1)}\operatorname{sign}(\lambda)\mu)=o|x|^{(p-N)/(p-1)-|\alpha|} \tag{2.5}$$

if $\lambda\neq 0$ *for any* α *with* $|\alpha|\geqq 1$.

Proof. Uniqueness. Set u_1 and u_2 two solutions of (2.1). From Theorem 1.1 they both satisfy

$$\lim_{x\to 0} u_i(x)/\mu(x)=|\lambda|^{1/(p-1)}\operatorname{sign}(\lambda). \tag{2.5}$$

Moreover they also satisfy (2.3) and (2.4), so we have

$$u_1-u_2\in L^\infty(G), \tag{2.6}$$

$$\nabla(u_1-u_2)=o(|x|^{(1-N)/(p-1)}) \quad \text{as} \quad x\to 0. \tag{2.7}$$

From the equation, we have, for all $r>0$ small enough,

$$\int_{\Omega-B_R(0)} (|\nabla u_1|^{p-2}\nabla u_1-|\nabla u_2|^{p-2}\nabla u_2)\cdot\nabla(u_1-u_2)\,dx$$
$$=-\int_{|x|=r}(u_1-u_2)\left(|\nabla u_1|^{p-2}\frac{\partial u_1}{\partial\nu}-|\nabla u_2|^{p-2}\frac{\partial u_2}{\partial\nu}\right)dS \tag{2.8}$$

and the right-hand side of (2.8) tends to 0 as $r\to 0$. As for the left-hand side it is greater than

$$\left\{\begin{array}{ll} C|\nabla(u_1-u_2)|^p & \text{if} \quad p\geqq 2, \\ C(1+|\nabla u_1|+|\nabla u_2|)^{p-2}\,|\nabla(u_1-u_2)|^2 & \text{if} \quad 1<p\leqq 2. \end{array}\right\} \tag{2.9}$$

So $\nabla(u_1-u_2)=0$ and $u_1=u_2$.

Existence. If $\lambda=0$ (2.1) is a classical Dirichlet problem, so we assume $\lambda\neq 0$ and $\lambda>0$ for example. For $\varepsilon>0$ small enough let u_ε be the solution of

$$\left.\begin{array}{lll} -\operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)=0 & \text{in} & \Omega-B_\varepsilon(0), \\ u_\varepsilon=\lambda^{1/(p-1)}\mu(\varepsilon) & \text{on} & \partial B_\varepsilon(0), \\ u_\varepsilon=g & \text{on} & \partial\Omega, \end{array}\right\} \tag{2.10}$$

$\Big($such a u_ε can be obtained in minimizing the Dirichlet integral $\int_{\Omega-B_\varepsilon(0)}|\nabla v|^p\,dx$ among the $v\in W^{1,p}(\Omega-B_\varepsilon(0)\Big)$ satisfying the two boundary conditions of (2.10).

From the maximum principle applied to u_ε and the two functions $\lambda^{1/(p-1)}\mu-K$ and $\lambda^{1/(p-1)}\mu+K$ where $K>0$ is defined by $K=\lambda^{1/(p-1)}\sup_{\partial\Omega}|\mu|+\sup_{\partial\Omega}|g|$, we have

$$\lambda^{1/(p-1)}\mu-K\leqq u_\varepsilon\leqq\lambda^{1/(p-1)}\mu+K, \tag{2.11}$$

in $\Omega - B_\varepsilon(0)$. If G is any compact subset of $\Omega - \{0\}$, then for $\eta > 0$ small enough, $G \subset \Omega - B_\eta(0)$. For $0 < \varepsilon < \frac{1}{2}\eta < \eta$, (2.11), (1.9), and [13] imply the following estimates

$$\|\nabla u_\varepsilon\|_{C^{1,\alpha}(G)} \leqq C, \tag{2.12}$$

$$\|\nabla u_\varepsilon\|_{L^p(\Omega - B_\eta(0))} \leqq C, \tag{2.13}$$

where C does not depend on ε. Then there exist a subsequence $\{\varepsilon_n\}$ going to 0 a p-harmonic function in $\Omega' u$ such that $\{u_{\varepsilon_n}\}$ converges to u in the $C^1_{\text{loc}}(\Omega - \{0\})$ topology and $\{\nabla u_{\varepsilon_n}\}$ converges weakly to ∇u in $L^p_{\text{loc}}(\bar{\Omega} - \{0\})$ so the boundary condition on $\partial\Omega$ is preserved (in fact this convergence is strong) and u also satisfies (2.13) in $\Omega - \{0\}$. From Theorem 1.1 it implies that

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda \delta, \tag{2.14}$$

holds in $\mathscr{D}'(\Omega)$. Hence u is the solution of (2.1) and $\{u_\varepsilon\}$ converges to u. Moreover the following properties hold [see (2.9)]:

$$(1 + |\nabla u| + |\nabla \mu|)^{p-2} |\nabla(u - |\lambda|^{1/(p-1)} \operatorname{sign}(\lambda)\mu|^2 \in L^1(\Omega), \quad 1 < p \leqq 2, \tag{2.15}$$

$$\nabla(u - |\lambda|^{1/(p-1)} \operatorname{sign}(\lambda)\mu) \in L^p(\Omega), \quad p \geqq 2. \tag{2.16}$$

Remark 2.1. A special case of Theorem 2.1 has been obtained by Chrusciel [3]; his construction of the solution is the same as ours. The more general problem of solving

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \nu \tag{2.17}$$

where ν is a bounded measure in Ω is also of great interest. A first, never published, result in that direction has been obtained by Gariepy and Pierre with $\nu \in L^1(\Omega)$, $u \in L^q(\Omega)\left(1 < q < \dfrac{N(p-1)}{N-p}\right)$ and $\nabla u \in L^r(\Omega)\left(1 < r < \dfrac{N(p-1)}{N-1}\right)$. Even in that case uniqueness is not known.

The following result is proved from the previous methods and results.

Theorem 2.2. *Assume $1 < p \leqq N$ and u is p-harmonic in $\mathbb{R}^N - \{0\}$ such that the following inequality holds in $\mathbb{R}^N - \{0\}$:*

$$|u(x)| \leqq a|\mu(x)| + b \tag{2.18}$$

$a, b \in \mathbb{R}$. Then $u(x) = \gamma\mu(x) + \lambda$ for $x \neq 0$, γ, and λ being constant.

Proof. Case 1. $1 < p < N$. The main point is to prove that $\lim\limits_{|x| \to +\infty} u(x)$ exists.

Using the same dilatation method and $C^{1,\alpha}$ a priori estimates as in Lemma 1.1 it is clear that there exist $C \geqq 0$ and $\alpha \in (0,1)$ such that

$$|\nabla u(x)| \leqq C|x|^{-1}, \tag{2.19}$$

$$|\nabla u(x) - \nabla u(x')| \leqq C|x - x'|^\alpha |x|^{-1-\alpha}, \tag{2.20}$$

for any $1 \leqq |x| \leqq |x'|$.

From Theorem 1.1 there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{x \to +\infty} u(x)/\mu(x) = \gamma, \tag{2.21}$$

$$u - \gamma\mu \in L^\infty(\mathbb{R}^N). \tag{2.22}$$

From Corollary 1.1 we have the following situation:

(i) either the maximum of $u - \gamma\mu$ is achieved at some interior point of $\mathbb{R}^N - \{0\}$ and the theorem is proved,

(ii) either this maximum is achieved at infinity,

(iii) or this maximum is achieved at 0.

In case (ii) we set for $r > 0$

$$\lambda(r) = \sup_{|x| \le r} (u - \gamma\mu)(x) \tag{2.23}$$

and

$$v_r(\xi) = u(r\xi) - \gamma\mu(r). \tag{2.24}$$

The function λ is increasing with limit $\lambda(\infty)$ as $r \to +\infty$ and for any $r > 0$ there exists $\xi_r \in S^{N-1}$ such that $v_r(\xi_r) = \lambda(r)$. As for v_r it is p-harmonic in $\mathbb{R}^N - \{0\}$ and satisfies

$$|\nabla v_r(\xi)| \le C|\xi|^{-1}, \tag{2.25}$$

$$|\nabla v_r(\xi) - \nabla v_r(\xi')| \le C|\xi|^{-1-\alpha}|\xi - \xi'|^\alpha, \tag{2.26}$$

for $\frac{1}{r} \le |\xi| \le |\xi'|$. Hence there exist a p-harmonic function v in $\mathbb{R}^N - \{0\}$ and a sequence $\{r_n\} \to +\infty$ such that $\{v_{r_n}\}$ converges to v in the $C^1_{\mathrm{loc}}(\mathbb{R}^N - \{0\})$ topology. Moreover ξ_{r_n} converges to some ξ_0 on S^{N-1}. As we have

$$v_r(\xi) = u(r\xi) - \gamma\mu(r) \le \lambda(\infty) + \gamma(\mu(r\xi) - \mu(r)), \tag{2.27}$$

we deduce $v(\xi) \le \lambda(\infty)$ and $v(\xi_0) = \lambda(\infty)$. Therefore $v = \lambda(\infty)$ and taking $|\xi| = 1$ in (2.24) implies

$$\lim_{|x| \to +\infty} (u(x) - \gamma\mu(x)) = \lim_{|x| \to +\infty} u(x) = \lambda(\infty). \tag{2.28}$$

In case (iii) we replace λ by $\tilde{\lambda}$ defined on $(0, +\infty)$ by

$$\tilde{\lambda}(r) = \sup_{|x| \ge r} (u - \gamma\mu)(x). \tag{2.29}$$

The function $\tilde{\lambda}$ decreases to some $\tilde{\lambda}(\infty)$ as $r \to +\infty$. Using the same notations as above we have

$$v_r(\xi) \le \tilde{\lambda}(r) + \gamma(\mu(r\xi) - \mu(r)) \tag{2.30}$$

for any $|\xi| > 0$, hence $v(\xi) \le \tilde{\lambda}(\infty)$ and $v(\xi_0) = \tilde{\lambda}(\infty)$ which implies again that $\lim_{|x| \to +\infty} u(x)$ exists.

In order to end the proof, we set $\lambda = \lim_{|x| \to +\infty} u(x)$ and we claim that

$$u(x) = \gamma\mu(x) + \lambda. \tag{2.31}$$

If $\gamma=0$ then u is bounded and constant. So we assume $\gamma \neq 0$, $\gamma>0$ for example, and we set $u_\gamma(x)=\gamma\mu(x)+\lambda$. For $\varepsilon>0$ we have

$$u(x) \leqq (1+\varepsilon)u_\gamma + |\lambda|\varepsilon, \tag{2.32}$$

near 0 and near infinity so as u_γ is p-harmonic the maximum principle implies $u(x) \leqq (1+\varepsilon)u_\gamma+|\lambda|\varepsilon$, $\forall x \neq 0$ and $u \leqq u_\gamma$. In the same way $u_\gamma \leqq u$ and $u=u_\gamma$.

Case 2. $p=N$. We shall use strongly the conformal invariance of the equation with respect to the transformation $\theta: x \mapsto x/|x|^2$. From Theorem 1.1 there exist γ and γ' such that

$$\lim_{x\to 0} u(x)/\mu(x)=\gamma, \qquad \lim_{|x|\to+\infty} u(x)/\mu(x)=\gamma'. \tag{2.33}$$

Moreover the following inequalities hold

$$\|u-\gamma\mu\|_{L^\infty(B_1(0))} \leqq K, \qquad \|u-\gamma'\mu\|_{L^\infty(\mathbb{R}^N-B_1(0))} \leqq K. \tag{2.34}$$

We can assume that at least one of the two numbers γ and γ' is not zero otherwise u is constant and, for example, $\gamma>0$. For $R>1$ and $\varepsilon>0$ small enough we set

$$v_\varepsilon(x)=(1-\varepsilon)\gamma\mu(x)+(\gamma'-(1-\varepsilon)\gamma)\,\mu(R)-K. \tag{2.35}$$

The function v is p-harmonic in $\{x: 0<|x|<R\}$. Moreover $u(x)>v_\varepsilon(x)$ in some neighbourhood of 0, and also for $|x|=R$ as $v_\varepsilon(R)=\gamma'\mu(R)-K$. So $u \geqq v_\varepsilon$ for $0<|x|<R$ and as $\varepsilon\to 0$

$$u(x) \geqq \gamma\mu(x)+(\gamma'-\gamma)\,\mu(R)-K, \tag{2.36}$$

also for $0<|x|<R$. In the same way we have

$$u(x) \leqq \gamma\mu(x)+(\gamma'-\gamma)\,\mu(R)+K, \tag{2.37}$$

in $\{x: 0<|x|<R\}$. Fixing x and letting $R\to+\infty$ in (2.36) and (2.37) implies $\gamma=\gamma'$. From Theorem 1.1 we know that $u(x)-\gamma\mu(x)$ admits a finite limit as x tends to 0 or equivalently as $|x|$ tends to infinity. Se we set

$$\lambda = \lim_{|x|\to+\infty} (u(x)-\gamma\mu(x)).$$

For $\varepsilon>0$, there exists $R=R(\varepsilon)$ such that for $|x| \geqq R$, $|u(x)-\gamma\mu(x)-\lambda|<\varepsilon$. So if we compare u and $u_\varepsilon=\gamma\mu+\lambda+\varepsilon$ being solutions of the singular Dirichlet problem

$$\left.\begin{aligned} -\operatorname{div}(|\nabla v|^{p-2}\nabla v) &= \gamma\delta \quad \text{in} \quad \mathscr{D}'(B_R(0)),\\ v &= g \qquad \text{for} \quad |x|=R, \end{aligned}\right\} \tag{2.38}$$

where $g=u$ or u_ε, we get $u \leqq u_\varepsilon$ in $B_R(0)$. In the same way $u \geqq \gamma\mu+\lambda-\varepsilon$. As $\varepsilon\to 0$ we get $u=\gamma\mu+\lambda$.

Remark 2.2. When $p=2$ the condition (2.18) is not optimal. Due to the development of harmonic functions in series of spherical harmonics, it is easy to prove that a necessary and sufficient condition in order to get $u=\gamma\mu+\lambda$ is that the following holds!

$$\lim_{x\to 0} |x|^{N-1}u(x)=0 \quad \text{and} \quad \lim_{|x|\to+\infty} u(x)/|x|=0. \tag{2.39}$$

We believed that such a type of conditions can be adapted to the nonlinear case (see Remark 3.1 in Sect. 3).

In the course of this proof we have shown the following result, which can also be proved directly:

Corollary 2.2. *Any nonnegative N-harmonic function in* $\mathbb{R}^N-\{0\}$ *is a constant.*

3. Pseudo-Radial Solutions

In this section we look for p-harmonic functions u in $\mathbb{R}^N-\{0\}$, $N>1$, which can be writen under the following pseudo-radial form

$$u(x)=|x|^{-\beta}\,\omega(x/|x|) \tag{3.1}$$

for $x\neq 0$. If we endow S^{N-1} with its standard Riemannian structure induced by the usual metric on $\mathbb{R}^N$ via the embedding $i:S^{N-1}\to\mathbb{R}^N$ any C^1 map $\omega:S^{N-1}\to\mathbb{R}$ admits a covariant derivative which corresponds through i to a vector tangent to S^{N-1}. We identify those two objects and denote by $\nabla_\tau\omega$ this gradient and $|\nabla_\tau\omega|$ its length (in S^{N-1} or in $\mathbb{R}^N$) and let $\operatorname{div}_{S^{N-1}}$ be the divergence operator acting on C^1 vector fields defined on S^{N-1}. If (r,σ) are the radial coordinates in $\mathbb{R}^N-\{0\}$ $(r>0$, $\sigma\in S^{N-1})$ the function u defined in (3.1) will be written as

$$u(x)=\tilde{u}(r,\sigma)=r^{-\beta}\omega(\sigma)\,, \tag{3.2}$$

and we have the following easy to prove result (see [20]):

Proposition 3.1. *Assume* $p>1$. *Then the function u under the form* (3.2) *is p-harmonic in* $\mathbb{R}^N-\{0\}$ *if and only if* $\omega\in W^{1,p}(S^{N-1})$ *and*

$$-\operatorname{div}_{S^{N-1}}((\beta^2\omega^2+|\nabla_\tau\omega|^2)^{(p-2)/2}\nabla_\tau\omega)+\beta\theta(\beta^2\omega^2+|\nabla_\tau\omega|^2)^{(p-2)/2}\omega=0 \tag{3.3}$$

in weak sense over S^{N-1}, *with* $\theta=N-1-(\beta+1)(p-1)$.

When $p=2$ (3.3) reduces to

$$-\Delta_{S^{N-1}}\omega=-\beta\theta\omega \tag{3.4}$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1}, which is just an eigenvalue problem. The set of values $\{-\beta_k\theta_k\}_{k\geqq 0}$ forms an increasing unbounded nonnegative sequence and the corresponding ω_k are the eigenfunctions or spherical harmonics. The first eigenvalue is 0 associated to the one-dimensional eigenspace of constant functions, the second eigenvalue is $N-1$ associated to a N-dimensional eigenspace of odd functions which keep a constant sign on an hemisphere of S^{N-1}. Thanks to Tolksdorf's result this result is easily generalizable to the case $p=N$ and we have:

Theorem 3.1. *Assume* $p=N$ *and S is an open hemisphere of* S^{N-1} *then there exists only one function* ω *which is positive on S, negative on* $S^{N-1}-\bar{S}$ *with* $\max\limits_{\sigma\in S}\omega(\sigma)=1$ *such that*

$$u(x)=r^{-1}\omega(\sigma) \tag{3.5}$$

is p-harmonic in $\mathbb{R}^N-\{0\}$. If $S=\{\sigma=(\sigma'\cos\theta,\sin\theta):\theta\in(-\frac{\pi}{2},\frac{\pi}{2}),\ \sigma'\in S^{N-2}\}$ then

$$\omega(\sigma)=\cos\theta\,. \tag{3.6}$$

Tolksdorf's result dealt with $u(x)=r\omega(\sigma)$ but due to the conformal invariance of the N-Laplace equation the result is the same for (3.5). Moreover from [20, Corollary 2.1 and Proposition 2.1.1] there exists no exponent $\beta\in(-1,1)-\{0\}$ such that a function u defined by (3.2) could be N-harmonic in $\mathbb{R}^N$ (or $\mathbb{R}^N-\{0\}$).

In a forthcoming work we shall extend this result to the case $p\neq N$ and prove the existence of many anisotropic singularities under the form (3.1).

When $N=2$ Eq. (3.3) reduces to

$$-((\beta^2\omega^2+\omega_\sigma^2)^{(p-2)/2}\omega_\sigma)_\sigma+(1-(\beta+1)(p-1))\,\beta(\beta^2\omega^2+\omega_\sigma^2)^{(p-2)/2}\omega=0\,. \tag{3.7}$$

Introducing the variable $Y=\omega_\sigma/\omega$, then Y satisfies

$$\left(\frac{\beta^2}{Y^2+\beta^2}-\frac{\beta+1}{Y^2+\beta(\beta-\beta_0)}\right)Y_\sigma=1\,, \tag{3.8}$$

where $\beta_0=(2-p)/(p-1)$. Equation (3.8) has been completely integrated by Krol and Mazja [10] for $\beta<0$: and they proved:

Theorem 3.2. *Assume $N=2$ and $p>1$. Then for each positive integer k there exist a $\beta_k<0$ and $\omega_k:\mathbb{R}\to\mathbb{R}$ with least period $2\pi/k$ of class C^∞ such that*

$$u_k(x)=|x|^{-\beta_k}\omega_k(x/|x|) \tag{3.9}$$

is p-harmonic in $\mathbb{R}^2$; β_k is the root $\leqq -1$ of

$$\left(1-\frac{1}{k}\right)^2(\beta^2-\beta\beta_0)=(\beta+1)^2\,. \tag{3.10}$$

(β_k,ω_k) is unique up to translation and homothety over ω_k.

Formula (3.10) is explicitely given in [5, 9, 12]. For $\beta>0$ the same method of integration provides us with all the singular pseudo-radial p-harmonic functions in $\mathbb{R}^2-\{0\}$.

Theorem 3.3. *Assume $N=2$ and $p>1$. Then for each positive integer k there exist a $\beta_k>0$ and $\omega_k:\mathbb{R}\to\mathbb{R}$ with least period $2\pi/k$ of class C^∞ such that*

$$u_k(x)=|x|^{-\beta_k}\,\omega_k(x/|x|) \tag{3.11}$$

is p-harmonic in $\mathbb{R}^2-\{0\}$; β_k is the positive root of

$$(\beta+1)^2=\left(1+\frac{1}{k}\right)^2(\beta^2-\beta\beta_0)\,. \tag{3.12}$$

(β_k,ω_k) is unique up to translation and homothety over ω_k.

Remark 3.1. A natural question concerning singular p-harmonic functions in $\mathbb{R}^2-\{0\}$ is the following: assume u is p-harmonic in $\mathbb{R}^2-\{0\}$ such that

$$\lim_{x\to 0}|x|^{\beta_1}u(x)=0 \tag{3.13}$$

where β_1 is the exponent defined in Theorem 3.2; then does it exist some constant C such that

$$|u(x)|/\mu(x) \leqq C \tag{3.14}$$

in some neighbourhood of 0 $\left(\text{which would imply that } \lim_{x\to 0} u(x)/\mu(x) \text{ exists}\right)$? In that direction it can be checked that there exists no singular p-harmonic function in $\mathbb{R}^2 - \{0\}$ under the following form

$$u(r,\sigma) = r^{-\beta}\omega(\sigma + \alpha \operatorname{Log} r) \tag{3.15}$$

unless $\alpha = 0$, although the function $Y = \omega_\sigma/\omega$ satisfies an algebraic differential equation, namely

$$\left(\frac{\alpha(\alpha^2+1)Y + \alpha^2(\beta_0 - \beta + 1) + \beta + 1}{(\alpha^2+1)Y^2 - \alpha(2\beta - \beta_0)Y + \beta(\beta - \beta_0)} + \frac{-\alpha(\alpha^2+1)Y + \beta(\alpha^2 - 1)}{(\alpha^2+1)Y^2 - 2\alpha\beta Y + \beta^2}\right) Y_\sigma = -1. \tag{3.16}$$

Equation (3.16) admits $2\pi/k$-periodic solutions if and only if

$$\left.\begin{array}{ll} \text{(i)} & 4\beta^2 - 4\beta\beta_0 - \alpha^2\beta_0^2 > 0, \\ \text{(ii)} & (\alpha^2\beta_0 + 2\alpha^2 + 2\beta + 2)^2 = \left(1 + \frac{1}{k}\right)^2 (4\beta^2 - 4\beta\beta_0 - \alpha\beta_0^2), \end{array}\right\} \tag{3.17}$$

but the $2\pi/k$-periodicity of ω needs $\lim_{\sigma\, 0} \int_\sigma^{\pi/k - \sigma} Y(\tau)\, d\tau = 0$ which implies $\alpha = 0$.

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The Primary Components of and Integral Closures of Ideals in 3-Dimensional Regular Local Rings

Craig Huneke*

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

In this paper we present several results which deal with the primary components and integral closures of ideals in a three dimensional regular local ring. In particular, we compare p^n, $(p^n)_a$, and $p^{(n)}$ where p is a height two prime in a three dimensional regular local ring. Here $(p^n)_a$ is the integral closure of p^n and $p^{(n)}$ is the n^{th} symbolic power of p. There is a containment $p^n \subseteq (p^n)_a \subseteq p^{(n)}$ since the powers p_p^n of R_p are integrally closed, being valuation ideals for the p-adic valuation of R_p.

It is only recently that examples have been given where $p^n \neq (p^n)_a$ for some n. See especially the paper [V] where this problem is closely studied. On the other hand it is not difficult to see that if $p^n = p^{(n)}$ for all large n or even $(p^n)_a = p^{(n)}$ for all large n (see [C–N] and [Hu1]) then p is a complete intersection. Hence in general $p^n \neq (p^n)_a$ and $(p^n)_a \neq p^{(n)}$ for large n provided p is not generated by a regular sequence. If $(p^n)_a = p^{(n)}$ for some n then $(p^i)_a = p^{(i)}$ for $i \leqq n$ by Ratliff [R].

Our first theorem states that $p^{(n)} \neq p^n$ *for any* $n \geqq 2$ unless p is a complete intersection. This result arises by a simple argument using intersection multiplicities. We are also able to show if R is a 4-dimensional regular local ring and p a height 3 prime such that R/p is Gorenstein but not a complete intersection, then $p^{(n)} \neq p^n$ for any $n \geqq 3$. In this case $p^{(2)} = p^2$ always holds by [He].

Our second theorem provides an upper bound for the maximal n such that $(p^n)_a = p^{(n)}$, in terms of invariants of R/p – for instance the multiplicity, $e(R/p)$, of R/p.

We give a sequence of examples to illustrate the latter theorem. We give height two primes p_n in a 3-dimensional regular local ring R such that the multiplicity, $e(R/p_n)$, is n^2+n+1, while $p_n^{(i)} = (p_n^i)_a$ for all $i \leqq n$. This compares with the statement of Corollary 2.17 which states that if $p^{(i)} = (p^i)_a$ then $i \leqq (8e-7)^{1/2} - 5$, where $e = e(R/p)$.

Next we turn to more explicit computations of $p^{(n)}/p^n$. For any prime p of height 2 in a 3-dimensional regular ring we can explicitly give $p^{(2)}/p^2$ as a module, while in the special case that p is generated by three elements we can explicitly give $p^{(n)}/p^n$ as a module; however it is not clear how useful this is, although we are able to use it to

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show two corollaries. First we show that if p is the defining ideal of a curve $k[[t^{n_1}, t^{n_2}, t^{n_3}]]$ where $(n_1, n_2, n_3) = 1$, then $p^{(2)}/p^2$ is a self-dual cyclic module. Secondly we are able to give examples of height two primes p generated by three elements such that $p^{(2)}$ requires arbitrarily many generators.

One of the motivations for this study came from the beautiful theory of Zariski [Z–S, II, Appendix 5], concerning the structure of integrally closed ideals in regular local rings of dimension 2. One theorem of Zariski's states that if I and J are integrally closed ideals in a 2-dimensional regular local ring, then IJ is also integrally closed. This is false in dimension three, and we give an example in a 3-dimensional regular local ring (R, m) of an integrally closed ideal I such that mI is not integrally closed. We also prove a result which gives a condition for $(m^n I)_a = m^n I_a$ for an ideal in a regular local ring of dimension at most three. We also give an example which shows a question posed by Krull has a negative solution.

2. Powers of Prime Ideals

We recall some basic notations and definitions. If I is an ideal in a ring R the integral closure of I, denoted I_a, is

$$\{r \in R \mid r \text{ satisfies an equation } r^n + a_1 r^{n-1} + \ldots + a_n = 0$$
$$a_i \in I^i,\ 1 \leqq i \leqq n\}.$$

If R is a local ring we abbreviate the property of being Cohen-Macaulay by saying R is C-M. We use the basic definitions of [N–R] concerning analytic spread and reduction without comment.

(2.1) **Theorem.** *Let S be a C-M local ring and let P be a prime ideal of S such that S/P is C-M and S_P is regular. Suppose $\underline{x} = x_1, \ldots, x_t$ is a set of elements of S which form a regular sequence on S and S/P. Further let "$-$" denote the map from S to $R = S/\underline{x}$ and assume $\bar{P} = p$ is a prime ideal of R such that R_p is regular. Then S/P^n is C-M if and only if R/p^n is C-M.*

Proof. Observe that both R and R/p are C-M since S and S/P are C-M and $\underline{x}$ is a regular sequence on S and S/P.

Since $S/P^n \otimes_S R \simeq R/p^n$, if S/P^n is C-M then necessarily $\underline{x}$ will be a regular sequence on S/P^n and so R/p^n will also be C-M.

Now assume R/p^n is C-M. Let $\dim R/p = d$ and choose $y'_1, \ldots, y'_d$ in R which form an s.o.p. in R/p. Let $y_1, \ldots, y_d$ be liftings of $y'_1, \ldots, y'_d$ to S. We use Serre's intersection multiplicity: if M, N are two S-modules with at least one of M or N having finite projective dimension, and $\ell(M \otimes_S N) < \infty$, then set

$$\chi(M, N) = \sum_{i=0}^{\infty} (-1)^i \ell(\operatorname{Tor}_i(M, N))$$

(see [Se]).

We compute $\chi(R/p^n, R/(y'_1, \ldots, y'_d))$ in two ways. As $y'_1, \ldots, y'_d$ is regular over R/p^n,

$$\chi(R/p^n, R/(\underline{y}')) = \ell(R/p^n \otimes R/(\underline{y}')). \tag{2.2}$$

On the other hand, filter R/p^n by $\{R/Q\}$ where $Q \in \mathrm{Spec}(R)$. For all such primes appearing in this filtration, $p \leqq Q$, so either $\dim R/Q < \dim R/p^n$ or $Q = p$. The number of copies of R/p appearing in any filtration is $\ell(R_p/p_p^n)$. Since $\chi(\_, R/(\underline{y}'))$ is additive and $\chi(R/Q, R/(\underline{y}'))=0$ if $p \lneqq Q$ by vanishing [Se], we have

$$\begin{aligned}(2.3)\qquad \chi(R/p^n, R/(\underline{y}')) &= \ell(R_p/p_p^n)\,\chi(R/p, R/(\underline{y}'))\\ &= \ell(R_p/p_p^n)\,\ell(R/p \otimes R/(\underline{y}'))\end{aligned}$$

where the last equality follows since R/p is C-M and $y_1', \ldots, y_d'$ form an s.o.p. for R/p.

Next we compute $\chi(S/P^n, S/(\underline{x}, \underline{y}))$. Using the same reasoning as above, we obtain

$$(2.4)\qquad \chi(S/P^n, S/(\underline{x}, \underline{y})) = \ell(S_P/P_P^n)\,\ell(S/P \otimes S/(\underline{x}, \underline{y})).$$

However, $\ell(S_P/P_P^n) = \ell(R_p/p_p^n)$ since both S_P and R_p are regular of the same dimension. Also, $\ell(S/P \otimes S/(\underline{x}, \underline{y})) = \ell(S/(P, \underline{x}) \otimes_{S/\underline{x}} S/(\underline{x}, \underline{y})) = \ell(R/p \otimes R/(\underline{y}'))$ and so comparing (2.3) and (2.4),

$$\begin{aligned}\chi(S/P^n, S/(\underline{x}, \underline{y})) &= \chi(R/p^n, R/(\underline{y}'))\\ &= \ell(R/p^n \otimes R/(\underline{y}'))\\ &= \ell((S/P^n \otimes S/\underline{x}) \otimes_{S/(\underline{x})} S/(\underline{x}, \underline{y}))\\ &= \ell(S/P^n \otimes S/(\underline{x}, \underline{y})).\end{aligned}$$

Hence $\chi(S/P^n, S/(\underline{x}, \underline{y})) = \ell(S/P^n \otimes S/(\underline{x}, \underline{y}))$. Thus S/P^n is C-M.

This theorem was discovered as a result of several conversations with J. Herzog.

(2.5) **Corollary.** *Let R be a regular 3-dimensional local ring and p a height two prime of R such that p requires at least three generators. Then $p^{(n)} \neq p^n$, $n \geqq 2$.*

Proof. Let $0 \to G \xrightarrow{A} F \to p \to 0$ be a minimal free resolution of p where $A = (a_{ij})$ is an $k \times (k+1)$ matrix. Let $S = R[x_1, \ldots, x_{kk+1}]_{(m, x_{ij})}$ and set $P = I_k(X)$ where $X = (x_{ij})$. Then S_P is regular, S/P is C-M and $\{x_{ij} - a_{ij}\}$ is a regular sequence such that $S/(x_{ij} - a_{ij}) = R$ and $S/P \otimes_S R = R/p$. We apply Theorem 2.1 to conclude that R/p^n is not C-M for $n \geqq 2$ provided S/P^n is not C-M for $n \geqq 2$. As R/p^n is C-M if and only if $p^{(n)} = p^n$, it suffices to show S/P^n is not C-M if $n \geqq 2$. Weyman [W] gave resolutions of these for all $n \geqq 2$, and none of them are C-M (using the Auslander Buchsbaum formula).

(2.6) **Corollary.** *Let R be a 4-dimensional regular local ring and $p < R$ a height 3 prime ideal such that R/p is Gorenstein, and such that p requires at least four generators. Then $p^{(2)} = p^2$ and $p^{(n)} \neq p^n$ if $n \geqq 3$.*

Proof. The proof is similar to the corollary above. By the Buchsbaum-Eisenbud theorem [B–E] p is defined by the $2n \times 2n$ Pfaffians of an alternating $(2n+1) \times (2n+1)$ matrix $A = (a_{ij})$. Let $X = (x_{ij})$ be a generic alternating matrix (so that $x_{ij} = -x_{ji}$, $x_{ii} = 0$).

Herzog [He, Satz 2.8], proved that p/p^2 is C-M and hence $p^{(2)} = p^2$.

In general let $S=R[x_{ij}]_{(m,x_{ij})}$ and let $P=Pf_{2n}(X)$, the ideal generated by the $2n\times 2n$ Pfaffians of X. Then S/P is Gorenstein, $S/(x_{ij}-a_{ij})\simeq R$, $S/P\otimes_S R=R/p$. We may apply Theorem 2.1 to conclude that if S/P^n is not C-M for $n\geqq 3$, then $p^{(n)}\neq p^n$ for $n\geqq 3$.

To prove S/P^n is not C-M for $n\geqq 3$ we compute a suitable intersection multiplicity. Let $m_R=(y_1,\dots,y_d)$ and let $f_1,\dots,f_\varepsilon$ be a s.o.p. of linear forms in $k[x_{ij}]_{(x_{ij})}/P$ where $k=R/m$ can be assumed to be infinite. Then $(y_1,\dots,y_d,f_1,\dots,f_\varepsilon)=J$ is an ideal generated by an S-sequence which is also a S/P sequence. Then

$$\begin{aligned}\chi(S/P^k,S/J)&=\ell(S_P/P^k_P)\,\chi(S/P,S/J)\\&=\binom{k+2}{3}\chi(S/P,S/J)=\binom{k+2}{3}((2n+1)(n+1)n/6)\end{aligned}$$

where $\chi(S/P,S/J)=(2n+1)(n+1)(n)/6$ follows from [He], for instance. On the other hand $\ell(S/P^k\otimes S/J)\geqq\ell(k[X,Y,Z]/(X,Y,Z)^{kn})$ as the elements of (P^k,J) are forms of degree kn in $k[x_{ij}]_{(x_{ij})}/(f_1,\dots,f_\varepsilon)\simeq k[X,Y,Z]_{(X,Y,Z)}$ so that we obtain that if S/P^k is C-M, then $\chi(S/P^k,S/J)=\ell(S/P^k\otimes S/J)$ or,

$$\left[\binom{k+2}{3}(2n+1)(n+1)n\right]\Big/6\geqq\binom{kn+2}{3}$$

or

$$(k+2)(k+1)k(2n+1)(n+1)n\geqq 6(kn+2)(kn+1)kn$$

or

$$(k+2)(k+1)(2n+1)(n+1)\geqq 6(kn+2)(kn+1)$$

or

$$\begin{aligned}[6k^2-2(k+2)(k+1)]n^2+[18k-3(k+2)(k+1)]n&\\+12-(k+2)(k+1)\leqq 0\,.&\end{aligned}$$

As $6k^2-2(k+2)(k+1)>0$ for $k\geqq 3$ and if $n=2$ the left-hand side is strictly positive, we see that for $n\geqq 2$ and $k\geqq 3$, this is impossible. Hence S/P^k is not C-M for $k\geqq 3$, $n\geqq 2$.

One can give a lower bound for the length of $p^{(2)}/p^2$ by using some results on linkage classes.

(2.7) **Theorem.** *Let p be a height 2 prime in a 3-dimensional regular local ring R, $1/2\in R$. Then $\ell(p^{(2)}/p^2)\geqq\binom{n-1}{2}$ where $n=\mu(p)$.*

Proof. In fact we show $\dim_{R/m}(\operatorname{soc}(p^{(2)}/p^2))=\binom{n-1}{2}$. If $\mu(p)=2$, $p^{(2)}=p^2$ so the formula is valid. Hence we may assume $\mu(p)\geqq 3$. As R/p is C-M and height $p=2$, p is in the linkage class of a complete intersection [P–S]. In particular, $H_1(p;R)$, the first Koszul homology on any set of generators of p, is a C-M R-module [Hu3]. There is an exact sequence, [S–V]

$$0\to\delta(p)\to H_1(p;R)\to(R/p)^n\to p/p^2\to 0\,,$$

where $n=\mu(p)$. Since $(\delta(p))_p$ is zero as p_p is a complete intersection, and $H_1(p;R)$ is C-M it follows that $\delta(p)=0$. However $\delta(p)$ is also the kernel of the map from $S_2(p)$ to p^2 where $S_2(p)$ = the second symmetric power of p [S–V]. Hence $S_2(p)\simeq p^2$. A resolution of $S_2(p)$ and hence of p^2 is given in [W]: if

$$0\to G\to F\to p\to 0$$

is a minimal resolution of p over R, then a minimal resolution of S_2p (and hence p^2) is given by

$$0\to \Lambda^2G\xrightarrow{\varphi_3} F\otimes G\xrightarrow{\varphi_2} S_2F\xrightarrow{\varphi_1} p^2\to 0\,.$$

This exact sequence identifies $\mathrm{Ext}^3(R/p^2,R)$ with $\mathrm{coker}\,\varphi_3^*$, so that $\mu(\mathrm{Ext}^3(R/p^2,R))=\mathrm{rank}(\Lambda^2G)=\binom{n-1}{2}$.

From the exact sequence

$$0\to p^{(2)}/p^2\to R/p^2\to R/p^{(2)}\to 0$$

and the fact that $R/p^{(2)}$ is C-M, we see $\mathrm{Ext}^3(R/p^2,R)=\mathrm{Ext}^3(p^{(2)}/p^2,R)$. As $\ell(p^{(2)}/p^2)<\infty$, we see that $\mu(\mathrm{Ext}^3_R(p^{(2)}/p^2,R))=\mu(\mathrm{Tor}^R_3(p^{(2)}/p^2,k))$ = (by computing using the Koszul complex of m) = $\mu(\mathrm{soc}(p^{(2)}/p^2))$. Hence

$$\binom{n-1}{2}=\dim_{R/m}\mathrm{soc}(p^{(2)}/p^2)$$

and the theorem is proved. □

Our results concerning $p^{(n)}$ can be expanded if p has 3 generators. Let $p=(a,b,c)$ and

$$0\to G\xrightarrow{\varphi} F\to p\to 0$$

be the minimal free resolution of p over R, where R is a 3-dimensional regular local ring, $\mathbb{Q}\subseteqq R$. Here $F\simeq R^3$ and $G\simeq R^2$. Consider as in [W] the complexes.

$$0\to \Lambda^2G\otimes S_{n-2}F\xrightarrow{\varphi_2} G\otimes S_{n-1}F\xrightarrow{\varphi_1} S_nF\to S_nP\to 0 \tag{2.8}$$

(2.9) **Proposition.** *In the notation above,*

$$p^{(n)}/p^n=(\mathrm{coker}\,\varphi_2^*)^v$$

where

$$*=\mathrm{Hom}_R(\ ,R)\quad and\quad v=\mathrm{Hom}_R(\ ,E),$$

E-injective hull of the residue field of R.

Proof. The criterion of [W], Theorem 1(b), shows that the complexes (4.2) are actually exact. Also since R_p is regular, $ht p=2$, and $\mu(p)=3$, p is generated by a d-sequence [Hu], and in particular $S_np\simeq p^n$ for all $n\geqq 1$, [Hu, V]. Hence (2.8) provides minimal resolutions of p^n for all n, and $\mathrm{coker}\,\varphi_2^*\simeq \mathrm{Ext}^2_R(p^n,R)$ $\simeq \mathrm{Ext}^3_R(R/p^n,R)$. By local duality, $(\mathrm{Ext}^3_R(R/p^n,R))^v\simeq H^0_m(R/p^n)$ as $\dim R=3$. However $H^0_m(R/p^n)=p^{(n)}/p^n$.

(2.10) **Corollary.** *Let p have resolution as in* (2.8). *Choose bases of G and F and write* $\varphi=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$. *Then* $p^{(2)}/p^2 \simeq (R/(a_{ij}))^v$.

Proof. We only need to observe that $R/(a_{ij}) \simeq \operatorname{coker} \varphi_2^*$, and this follows immediately from the definition of the map φ_2.

(2.11) **Corollary.** *Suppose p is the defining ideal for a monomial curve* $k[[t^{m_1}, t^{m_2}, t^{m_3}]]$ *where* $m_1 < m_2 < m_2$, $(m_1, m_2, m_3) = 1$. *Then $p^{(2)}/p^2$ is cyclic.*

Proof. By Herzog (see [K]) the ideal p is defined by the 2×2 minors of a matrix with the form

$$\begin{pmatrix} X^{i_1} & Y^{i_2} & Z^{i_3} \\ Z^{j_1} & X^{j_2} & Y^{j_3} \end{pmatrix}.$$

We may apply Corollary 4.2 to conclude that

$$p^{(2)}/p^2 \simeq (R/(X^{i_1}, Y^{i_2}, Z^{i_3}, Z^{j_1}, X^{j_2}, Y^{j_3}))^v \simeq (R/(X^{n_1}, Y^{n_2}, Z^{n_3}))^v$$

where $n_k = \min(i_k, j_k)$, and since $R/(X^{n_1}, Y^{n_2}, Z^{n_3})$ is Gorenstein, $(R/(X^{n_1}, Y^{n_2}, Z^{n_3}))^v \simeq R/(X^{n_1}, Y^{n_2}, Z^{n_3})$ and so $p^{(2)}/p^2$ is cyclic, and furthermore is self-dual.

We may use Corollary 2.10 to give examples of height two primes generated by three elements which require an arbitrarily large number of generators for $p^{(2)}$. It suffices to find $a_1, \ldots, a_6 \in m$ such that the socle of $R/(a_1, \ldots, a_6)$ is arbitrarily large and such that the 2×2 minors of the matrix,

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix},$$

generate a prime ideal. To accomplish this goal we first prove a general criterion for obtaining a prime ideal. This lemma is probably useful in other settings.

(2.12) **Proposition.** *Let* $R = k[[x, y, z]]$, *k a field, and suppose $I \subset R$ is an ideal primary to* (x, y, z). *Let* $I = (f_1, \ldots, f_n)$ *and let* $X = (x_{ij})$ *be a generic* $6 \times n$ *matrix.*

$$\begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix} = X \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

and put $S = R[X]_{mR[X]}$. *Put J = ideal generated by the 2×2 minors of* $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$. *Then J is prime.*

Proof. Consider the ideal Q generated by the 2×2 minors of A in $T = R[X]_{(m, X)}$. It suffices to show Q is prime. Now $m^N \subseteqq I$ for some $N \gg 0$. Choose $m, k > N$ such that $(m, k) = 1$. Write

$$(x^{k+1}, y^k, z^k, z^m, x^m, y^m)^t = B(f_1, \ldots, f_n)^t$$

where $B=(b_{ij})$ is a $6\times n$ matrix with coefficients in m. The 2×2 minors of

$$\begin{pmatrix} x^{k+1} & y^k & z^k \\ z^m & x^m & y^m \end{pmatrix}$$

generate a prime ideal q in R since this ideal defines the curve $k[[t^{k^2+m^2+mk}, t^{k^2+m^2+mk+k}, t^{k^2+m^2+km+k+m}]]$.

Now consider the sequence $x_{ij}-b_{ij}$ in T. Let P be the ideal they generate and φ the map from T to $T/P=R$. It is clear that $\varphi(Q)=q$ is prime. Hence $(Q, x_{ij}-b_{ij}) =(q, x_{ij}-b_{ij})$ is prime and $ht(Q, x_{ij}-b_{ij})=ht(q, x_{ij}-b_{ij})=2+6n$ so that $ht((q, x_{ij}-b_{ij})/Q)=6n$. It follows that Q must be prime.

I would like to thank M. Hochster for conversations regarding Proposition 2.12.

(2.13) *Example.* We construct primes with three generators having arbitrarily large numbers of generators for $p^{(2)}$. By using Proposition 2.12 it clearly suffices to find $a_1, \ldots, a_6 \in R=k[[x,y,z]]$ such that $ht(a_1, \ldots, a_6)=3$ and $\dim_k \operatorname{soc}(R/(a_1, \ldots, a_6))$ becomes arbitrarily large. However let X be a generic skew-symmetric matrix of size $2n+1\times 2n+1$. Then $P=Pf_{2n}(X)$ the ideal generated by the size $2n$ Pfaffians of X is a height 3 Gorenstein ideal minimally generated by $2n+1$ elements. By cutting down by moding out a suitable number of linear forms we can obtain an ideal $I\subseteq R$ which is minimally generated by $2n+1$ elements such that R/I is Gorenstein. Choose any regular sequence a, b, c in I and let $J=(a,b,c):I$. Since R/I is Gorenstein, J has four generators, say g_1, g_2, g_3, g_4. In addition, the canonical module of R/J is isomorphic to $I/(a,b,c)$ and hence requires at least $(2n+1)-3=2n-2$ generators. As the number of generators of the canonical module is equal to the number of generators of $\operatorname{soc}(R/J)$, we see that $\dim(\operatorname{soc}(R(J)))\geqq 2n-2$. Now let Y be a generic 6×4 matrix and let $(a_1, \ldots, a_6)^t =Y(g_1, g_2, g_3, g_4)^t$ in $S=R[Y]_{mR[Y]}$. Then $(a_1, \ldots, a_6)S=JS$ and so $\dim \operatorname{soc}(S/(a_1, \ldots, a_6)S)\geqq 2n-2$. Clearly $Q=I_2(A)$, $A=\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$ provides the desired example.

We now turn to the integral closures of powers of primes. The next lemma, although quite simple, seems not to appear in the literature. Compare with [Sa, Theorem 2.2], and comments afterwords.

(2.14) **Lemma.** *Let R, m be a local Noetherian ring with R/m infinite and let I be an ideal in R with minimal reduction J. Then $J^n\cap m(I^n)_a=mJ^n$ for all n.*

Proof. For $n=1$ this is well-known [N–R]. Consider $T=\bigoplus(I^n)_a$ and $S=R[Jt]$. Then T is integral over S and $S/mS\simeq\bigoplus J^n/mJ^n$ is a domain since J is generated by analytically independent elements [N–R]. By lying over there is a prime Q of T lying over mS. In particular $mS\subseteq mT\cap S\subseteq Q\cap S=mS$, so $mS=mT\cap S$. Hence $m(I^n)_a\cap J^n=mJ^n$.

This lemma answers a question by McAdam in [Mc, p. 31]. We can state:

Proposition. *Let (R, m) be a 2-dimensional local C-M domain with multiplicity e. If I is an ideal and $m\in \bar{A}^*(I)$ then $m\in \operatorname{Ass}(R/(I^n)_a)$ for $n\geqq e$.*

Proof. Here, $m \in \bar{A}^*(I)$ simply means $m \in \operatorname{Ass}(R/(I^n)_a)$ for $n \gg 0$. Suppose $m \notin \operatorname{Ass}(R/(I^n)_a)$ for some fixed n. Then if y is chosen in $m - m^2$, superficial for R [we may assume (R/m) is infinite] and also chosen to be a non-zero divisor on $R/(I^n)_a$, then

$$\mu((I^n)_a) = \mu(((I^n)_a, y)/(y)) \leqq e(R/(y))$$

(see [Sa, Chap. 3, Theorem 1.1]) $= e(R) = e$ as y is superficial.

On the other hand $m \in \bar{A}^*(I)$ implies $\ell(I) = 2$ (see [Mc, Proposition 4.6] and hence by Lemma 2.14, if J is a minimal reduction of I, then $\mu(J^n) \leqq \mu(I^n)_a)$ so that $n+1 = \mu(J^n) \leqq \mu((I^n)_a) \leqq e$, so $n \leqq e-1$. Hence if $n \geqq e$, $m \in \operatorname{Ass}(R/(I^n)_a)$.

It would be nice to know if Lemma 2.14 is true without the assumption of an infinite residue field. We can use Lemma 2.14 to prove the following theorem. We let $\ell(I)$ = analytic spread of I.

(2.15) **Theorem.** *Let R be a regular local ring of dimension 3 and let I be a height 2 ideal of R, such that $\ell(I) = 3$. Suppose $R/(I^n)_a$ is C-M. Then $(n+3)/2 \leqq \operatorname{ord}(I)$, where by definition $\operatorname{ord}(I) = d$ if $I \subset m^d$, $I \nsubseteq m^{d+1}$. In particular for $n > 2d-3$, $(I^n)_a$ cannot be unmixed.*

Proof. We may assume R/m is infinite. Let J be a minimal reduction of I. By Lemma 2.14, J^n/mJ^n is contained in $(I^n)_a/m(I^n)_a$ and so

$$\mu((I^n)_a) \geqq \mu(J^n) = \binom{n+2}{2}. \tag{2.16}$$

Suppose now $R/(I^n)_a$ is C-M and choose an $x \in m \backslash m^2$ such that x is not a zero divisor on $R/(I^n)_a$. Then $\mu((I^n)_a) = \mu((I^n)_a, x)/(x)$. Set $K = ((I^n)_a, x)/(x) \subseteq S = R/xR$. If x is chosen generally enough $\operatorname{ord}(K) = \operatorname{ord}((I^n)_a))$ which in turn $= \operatorname{ord}(I^n)$ since all powers of m are integrally closed. Choose $f \in K$ such that $\operatorname{ord}(f) = \operatorname{ord}(K) = nd$ where $d = \operatorname{ord}(I)$. Consider $K/(f) \subseteq S/(f)$. Then $K/(f)$ is a C-M module over $S/(f)$ which is a 1-dimensional C-M ring of multiplicity nd. Hence [Sa, Chap. 3, Theorem 1.1], $\mu(K/(f)) \leqq nd$ so that $\mu(K) \leqq nd+1$. Combining this with (2.16), we obtain

$$\binom{n+2}{2} \leqq nd+1 .$$

Simplifying gives $(n+3)/2 \leqq d$, and hence the theorem.

Remark. Exactly the same proof as in the theorem shows that if I is a height 2 ideal in a regular local ring of arbitrary dimension such that $\ell(I) = k$, then if $R/(I^n)_a$ is Cohen-Macaulay,

$$\binom{n+k-1}{k-1} \leqq n(\operatorname{ord}(I)) + 1 .$$

(2.17) **Corollary.** *Let R be 3-dimensional regular local ring, and let p be a height two prime ideal which is not a complete intersection. Set $e = e(R/p)$, e the multiplicity of R/p. Suppose $(p^n)_a = p^{(n)}$. Then $n \leqq (8e-7)^{1/2} - 5$.*

We first remark that our assumptions imply $\ell(p) = 3$.

The proof is a matter of applying the two main theorems. By Theorem 2.15 we know that $(n+3)/2 \leqq d$, where $d = \operatorname{ord}(p)$.

However we can do slightly better. We have shown that $p^n \neq p^{(n)}$, if $n \geqq 2$. If J is as in the proof of Theorem 2.15 – that is a minimal reduction of p – then $J^n \subseteqq p^n$ so that $J^n \neq p^{(n)}$. Hence $\mu(p^{(n)}) \geqq \mu(J^n)+1$. [Assuming $p^{(n)} = (p^n)_a$ and using Lemma 2.14.] Thus we obtain

$$1 + \binom{n+2}{2} \leqq nd+1 \quad \text{where} \quad d = \operatorname{ord}(p)$$

or $n \leqq 2d-4$.

We relate d to e. If y is a reduction of R/p (we may assume R/m is infinite) then $e = \ell(R/(p,y))$. Let $S = R/(y)$, $J = (p,y)/(y)$. Then $\operatorname{ord}(J) = d$ for general y. Let $q = m/(y)$. We claim $J \neq q^d$. Suppose $J = q^d$. Then $J^n = q^{dn}$ and so J^n is integrally closed. Thus $((p^n)_a, y)/(y) \subseteqq J^n$ so that $((p^n)_a, y) = (p^n, y)$. Therefore $(p^n)_a \subseteqq p^n + ((p^n)_a : y)y$ implies $(p^n)_a = p^n$ as by assumption, $(p^n)_a = p^{(n)}$ so that $((p^n)_a : y) = (p^n)_a$. Then $p^n = p^{(n)}$ also which contradicts Corollary 2.5. Thus $J \neq q^d$. Since $J \subsetneqq q^d$ it follows that $\ell(S/J) \geqq \ell(S/q^d)+1 = d(d+1)/2+1$. By above, $d \geqq (n+4)/2$. Combining $e \geqq d(d+1)/2+1$ with $d \geqq (n+4)/2$, yields the corollary.

(2.18) *Example.* Let p be the kernel of the map of $k[[x,y,z]] = R$ to $S = k[[t^{n^2+n+1}, t^{n^2+2n+1}, t^{n^2+2n+2}]]$. Then p is a height 2 prime of R with $e(R/p) = n^2+n+1$. The ideal p is defined by the 2×2 minors of

$$A = \begin{pmatrix} x^{n+1} & y^n & z^n \\ z & x & y \end{pmatrix}$$

(see [He]). Hence $\operatorname{ord}(p) = n+1$. We will show that $(p^i)_a = p^{(i)}$ for all $i \leqq n$ if $\operatorname{char}(k) = 0$. Corollary 2.17 shows that $(p^i)_a \neq p^{(i)}$ for $i \geqq (8(n^2+n+1)-7)^{1/2}-5 \approx 2\sqrt{2}n$. Theorem 2.15 (as improved for prime ideals in the proof of Corollary 2.17) shows that $2\operatorname{ord}(p)-4 \geqq i$ if $(p^i)_a = p^{(i)}$. Here $2\operatorname{ord}(p)-4 = 2n-2$. So both the bounds of Theorem 2.9 and Corollary 2.11, although not on the nose for these examples (except for $n=2$), are at least the correct size [linear on $\operatorname{ord}(p)$, square root of the multiplicity].

To prove that $(p^i)_a = p^{(i)}$ for $i \leqq n$, it is enough to prove that $(p^n)_a = p^{(n)}$ by Ratliff [R]. However in any case our proof that $(p^n)_a = p^{(n)}$ will show $(p^i)_a = p^{(i)}$. We will show that

$$m^{n-1} p^{(n)} \subseteqq m^{n-1} p^n. \tag{2.19}$$

It follows that $p^{(n)}$ is integral over p^n. Also from (2.19) it follows that

$$m^{n-1} p^{n-i} p^{(i)} \subseteqq m^{n-1} p^{(n)} = m^{n-1} p^n = m^{n-1} p^{n-i} p^i$$

so that

$$(m^{n-1} p^{n-i}) p^{(i)} = (m^{n-1} p^{n-i}) p^i \tag{2.20}$$

from which it follows that $(p^i)_a = p^{(i)}$ for $i \leqq n$.

Consider $\ell(R/p^{(n)}, x)$. As in the proof of Theorem 2.7, we obtain that

$$\begin{aligned} \ell(R/p^{(n)}, x) &= \chi(R/p^{(n)}, R/(x)) = \chi(R/p, R/(x)) \cdot \ell(R_p/p_p^n) \\ &= (n^2+n^2+1)\binom{n+1}{2}. \end{aligned}$$

Hence if we can find an ideal I such that $p^n \subseteqq I \subseteqq p^{(n)}$ and $\ell(R/(I,x)) = (n^2+n+1)\binom{n+1}{2}$, then it will follow that $(I,x)=(p^{(n)},x)$ and so $p^{(n)} \subseteqq I + x(p^{(n)}:x) = I + xp^{(n)}$ so that $I = p^{(n)}$.

We will show in Appendix 1 that there exist elements $h_{j,i}^{(k)} \in R$ which satisfy the following equations:

$$h_{j,i}^{(k)} \in p^{(k)}, \tag{2.21}$$

and

$$h_{j,i}^{(k)} \equiv (-1)^{(i+1)(j+1)+ik}\binom{k-1}{j-1} z^{jn+i} y^{(k-j+1)n-i} \operatorname{mod}(x). \tag{2.22}$$

(Notice the only time the sign is minus is if i,j are even or i,k are odd) for $1 \leqq i \leqq j-1$, $2 \leqq j \leqq k$, $2 \leqq k \leqq n$.

Let I be the ideal generated by p^n and $h_{j,i}^{(n)}$ for $1 \leqq i \leqq j-1, 2 \leqq j \leqq n$. We first claim $I = p^{(n)}$. By above it suffices to show $\ell(R/(I,x)) = (n^2+n+1)\binom{n+1}{2}$. Notice that $p \equiv (y^{n+1}, z^{n+1}, zy^n) \operatorname{mod}(x)$ so that $p^n \equiv (y^{n+1}, z^{n+1}, zy^n)^n \operatorname{mod}(x)$. Hence generators for $p^n(\operatorname{mod}(x))$ are monomials in y and z of total degree $n(n+1)$; $z^i y^j \in (y^{n+1}, z^{n+1}, zy^n)^n$ $(i+j=n(n+1))$ if and only if $z^i y^j = (zy^n)^a (z^{n+1})^b (y^{n+1})^c$ where $a+b+c=n$ if and only if

$$i = a + b(n+1) \quad \text{where} \quad a+b \leqq n,$$

if and only if $i = bn+e$ where $b \leqq e \leqq n$.

On the other hand (as $\operatorname{char} k = 0$), the ideal $I(\operatorname{mod}(x))$ contains $h_{j,i}^{(n)} \equiv z^{jn+i} y^{(n-j+1)n-i} \operatorname{mod}(x)$, up to a unit.

Here, $1 \leqq i \leqq j-1$ and $2 \leqq j \leqq n$. These are monomials of total degree $n(n+1)$ and are exactly the monomials of degree $n(n+1)$ not contained in $(y^{n+1}, z^{n+1}, zy^n)^n$.

Hence

$$(I,x) = (m^{n(n+1)}, x)$$

and

$$\ell(R/I,x) = \ell(R/m^{n(n+1)}, x) = \binom{n(n+1)+1}{2} = \frac{(n(n+1)+1)(n(n+1))}{2}$$
$$= (n^2+n+1)\binom{n+1}{2} = \ell(R/p^{(n)}, x).$$

We may conclude $I = p^{(n)}$.

It remains to see that $m^{n-1} I \subseteqq m^{n-1} p^n$, i.e. that

$$m^{n-1} h_{j,i}^{(n)} \in m^{n-1} p^n \quad \text{for} \quad 2 \leqq j \leqq n, 1 \leqq i \leqq j-1.$$

To show this we induct on k to show

$$m^{k-1} h_{j,i}^{(k)} \subseteqq m^{n-1} p^k,$$

and using the following sets of equations which give recursive formulas for the $h_{j,i}^{(k)}$, and which we will verify in the appendix. By convention we define,

$$h_{j,0}^{(k)}=(-1)^j\binom{k-1}{j-1}z^{n-k}b^{j-1}c^{k-j+1}, \tag{2.23}$$

$$h_{k+1,i}^{(k)}=-\binom{k-1}{i-1}x^{n-k+1}a^{k+1-i}c^{i-1}, \tag{2.24}$$

and

$$h_{j,j}^{(k)}=(-1)^{jk}\binom{k-1}{j-1}y^{n-k}a^{k-j}b^j. \tag{2.25}$$

Then, we will show in the appendix that the elements $h_{j,i}^{(k)}$ for $2\leqq k\leqq n$, $2\leqq j\leqq k$, $1\leqq j\leqq n$ satisfy the following equations and relations:

$$xh_{j,i}^{(k)}=ah_{j-1,i}^{(k-1)}+(-1)^{k-j+1}ch_{j-1,i-1}^{(k-1)} \tag{2.26}$$

$$yh_{j,i}^{(k)}=(-1)^i ah_{j,i}^{(k-1)}+(-1)^{k-j}bh_{j-1,i-1}^{(k-1)} \tag{2.27}$$

$$zh_{j,i}^{(k)}=(-1)^{i+1}ch_{j,i}^{(k-1)}-bh_{j-1,i}^{(k-1)}. \tag{2.28}$$

$$h_{j,i}^{(k)}\in p^{(k)}, \tag{2.29}$$

and

$$h_{j,i}^{(k)}\equiv(-1)^{(i+1)(j+1)+ik}\binom{k-1}{j-1}z^{jn+i}y^{(k-j+1)n-i}. \tag{2.30}$$

Here, $a=y^{n+1}-xz^n$, $b=z^{n+1}-x^{n+1}y$, and $c=x^{n+2}-zy^n$ are minimal generators of p.

These formulas give the existence and recursion for $h_{j,i}^{(k)}$. The proof of these formulas, while straightforward, is tedious and we leave it to the appendix.

These examples were quite complicated to verify, which leaves the question: is there some easier and more natural way to verify $(p^n)_a=p^{(n)}$? Right now I know of no way without actually computing generators.

We claim by induction from (2.23)–(2.30) that $m^{k-1}h_{j,i}^{(k)}\subseteqq m^{n-1}p^k$ for all allowed j, i, and for $2\leqq k\leqq n$. For $k=2$, (2.23) shows $mh_{2,1}^{(2)}\subseteqq m^{n-1}p^2$. In general (2.24)–(2.30) show that

$$\begin{aligned} m^{k-1}h_{j,i}^{(k)} &\subseteqq m^{k-2}(\textstyle\sum ph_{j,i}^{(k-1)}+m^{n-k+1}p^k)\\ &\subseteqq p\textstyle\sum m^{k-2}h_{j,i}^{(k-1)}+m^{n-1}p^k\\ &\subseteqq p(m^{n-1}p^{k-1})+m^{n-1}p^k\\ &=m^{n-1}p^k. \end{aligned}$$

For $k=n$ we finally obtain that

$$m^{n-1}(I)\subseteqq m^{n-1}p^n$$

as $I=(p^n,h_{j,i}^{(n)})$.

We should remark that although we believe this example has the same properties in any characteristic, the generators for the symbolic powers change in different characteristics – this is indicated by the appearance of the binomial coefficients in (2.22)–(2.30). For instance in any characteristic but 3, the elements $h^{(3)}_{2,1}, h^{(3)}_{3,1}, h^{(3)}_{3,2}$ minimally generate $p^{(3)}/p^3$, while in characteristic 3, $p^{(3)}/p^3$ is cyclic!

3. Integrally Closed Primary Ideals

We present related examples and theorems of products of integrally closed ideals in 3-dimensional regular local rings. If R is a 2-dimensional regular local ring, Zariski [Z–S, Appendix 5], showed that the product of any two integrally closed ideals is integrally closed. A special case of this is important: namely when one of the ideals is the maximal ideal. Unfortunately neither of these statements remain true if R is a 3-dimensional regular local ring. It was well-known that the product of 2 integrally closed ideals need not be integrally closed, (see Example 3.1 for easy examples) but as far as we know, the second statement was open. We give examples to show that it is false in general. Lipman [L] and later Rees [Re] have extended the 2-dimensional case to pseudo-rational rings. We also give a sufficient condition for $(mI)_a = mI_a$ (Theorem 3.5).

(3.1) *Example.* Let I, J be two integrally closed primary ideals in $k[x, y]_{(x,y)} = R$ such that $I+J$ is not integrally closed. [For instance take $I=(x^2+y^3, m^4)$, $J=(y^3, m^4)$.] Let $I_1=(I,z)$, $J_1=(J,z)$ in $S=R[z]_{(m,z)}$. Then I_1J_1 is not integrally closed, but I_1 and J_1 are integrally closed.

Proof. Let $b\in(I+J)_a$, $b\notin I+J$. Then as $Iz+Jz\subseteqq I_1J_1$, it is easily seen that $bz\in(I_1J_1)_a$. Suppose $bz\in I_1J_1$. Then $\exists i_1,\ldots,i_n\in I$, $j_1,\ldots,j_n\in J$ such that $bz=\sum_k (i_k+r_kz)(j_k+s_kz)$ where $r_k, s_k\in R$. Then,

$$\sum i_kj_k + z\left(\sum r_kj_k + i_ks_k - b\right) + z^2\sum r_ks_k = 0$$

and since the coefficients are in R, all must be zero: in particular

$$b=\sum r_kj_k + i_ks_k \in I+J\,.$$

The contradiction shows that $bz\notin I_1J_1$ and hence I_1J_1 is not integrally closed.

This procedure clearly fails to provide examples where one of the ideals is maximal. To obtain these we let

$$R=k[x,y,z]_{(x,y,z)},\; m=(x,y,z)R\,.$$

(3.2) *Example.* Let $I=(x^2+xy^2, z^4)$, $J=m^4$. Then $I=(I)_a$, $J=(J)_a$ but $IJ\neq(IJ)_a$.

Proof. Clearly $J=(J)_a$. To see $I=(I)_a$ note that

$$I=(x,z^4)\cap(x+y^2,z^4)\,,$$

and hence it is enough to show (x,z^4) and $(x+y^2,z^4)$ are integrally closed. It is enough to show this locally at all associated primes of each, that is it is enough to show $(x,z^4)_{(x,z)}$ and $(x+y^2,z^4)_{(x+y^2,z)}$ are integrally closed. Both of these claims are immediate as x (respectively $x+y^2$) is a regular parameter of $R_{(x,z)}$ (respectively $R_{(x+y^2,z)}$).

Next we claim that $xy^2z^4 \in (IJ)_a$ but $xy^2z^4 \notin IJ$. We prove the first statement. Observe that

$$(3.3) \qquad (xy^2z^4)^2 + (xy^2z^4)(y^4z^4) - (x^2+xy^2)y^4z^8 = 0$$

so that xy^2z^4 satisfies the equation

$$T^2 + T(y^4z^4) - (x^2+xy^2)y^4z^8 = 0$$

as $y^4z^4 \in IJ$ and $(x^2+xy^2)y^4z^8 \in (IJ)^2$ this equation shows that $xy^2z^4 \in (IJ)_a$.

To show $xy^2z^4 \notin I \cdot J$, we give a grading to $I \cdot J$ – namely set $\deg x = 2$, $\deg y = 1$, $\deg z = 1$. Then both I and J, and hence $I \cdot J$ are homogeneous and $\deg(xy^2z^4) = 8$, so it suffices to show $xy^2z^4 \notin (I \cdot J)_8 = \sum_{i+j=8} I_i J_j = I_4 \cdot J_4$ as $(I)_i = 0$ for $i \leqq 3$ and $J_j = 0$ for $j \leqq 3$. However

$$J_4 = \langle y^4, y^3z, y^2z^2, yz^3, z^4 \rangle$$

while $I_4 = \langle x^2+xy^2, z^4 \rangle$.

A simple check shows that $xy^2z^4 \notin I_4 \cdot J_4$.

It follows from this example that either $I_1 = I$, $I_2 = Im$, $I_3 = Im^2$, or $I_4 = Im^3$ has the property that $(I_j \cdot m)_a \neq (I_j \cdot m)$ but $(I_j)_a = I_j$.

This example has one drawback. The ideal I is not m-primary. However, we can modify I to obtain an m-primary integrally closed ideal K such that Km^4 is not integrally closed by setting $K = (I, m^n)_a$ for $n \gg 0$. To prove this we need a lemma which was pointed out to me by J. Lipman.

(3.4) **Lemma.** *Let R be a local domain, I an ideal of R. Then $(I)_a = (\cap IV) \cap R$, where the intersection is taken over all discrete valuation rings in the quotient field of R which contain R and have center m.*

Proof (due to Lipman). Suppose $x \notin (I)_a$. We will show there is a $DVR\ V$ dominating R such that $x \notin IV$.

Let $S = R[Ix^{-1}]$, $J = (Ix^{-1})S$. Then $JS \neq S$ since if $JS = S$, $x \in (I)_a$ contrary to assumption, hence $JS \neq S$. As $S/JS \cong R/J \cap R$, we obtain that $M = mS + JS$ is the unique maximal ideal containing JS. Set $T = S_M$, with maximal ideal M. Choose any $y \in M \backslash M^2$ in T and let $A = T[My^{-1}]$, and choose a minimal prime p over $y\bar{T}$, where $\bar{T}$ = integral closure of T. Then let $V = (\bar{T})_p$. Since $yT = MT \subsetneqq (\bar{T})_p$ we obtain that $m \subsetneqq mS \subset mT \subsetneqq MT \subsetneqq (\bar{T})_p$ so that $(\bar{T})_p$ is a DVR with center m on R. As $Ix^{-1} \subsetneqq M \subset p(\bar{T})_p$ we obtain that $x \notin I(\bar{T})_p$ as required.

Now let $K_n = (I, m^n)_a$. Then K_n is homogeneous with the grading given above. I claim that if $n \gg 0$, $[K_n]_4 = [I]_4$. To prove this consider $B = \bigcap_{n \geqq 0} K_n$. As $[K_n]_i \supseteqq [K_{n+1}]_i \supseteqq \dots$ for any given i, this chain must stabilize and the stable subspace if $[B]_i$. Thus it suffices to show that $[B]_4 = [I]_4$. However we claim $B = I$. By Lemma 3.4, it is enough to show $BV = IV$, if V is a DVR centered on m. Then for $N \gg 0$, $m^N V \subsetneqq IV$. Hence, as $I \subset B \subset (I + m^N)_a$ we obtain $IV \subsetneqq BV \subsetneqq (I + m^N)_a V = (I + m^N) V \subsetneqq IV$ so that $IV = BV$. Thus our claim is established. Now clearly if n is large enough so that $[K_n]_4 = [B]_4 = [I]_4$ then $xy^2z^4 \in (Im^4)_a \subsetneqq (K_n m^4)_a$, but $xy^2z^4 \notin K_n m^4$ since $xy^2z^4 \notin [K_n]_4 [m^4]_4$ by above. Thus either K_n, mK_n, m^2K_n, or

$m^3 K_n$ provides an example of an integrally closed ideal primary to m whose product with m is not integrally closed.

One can prove $m^n I$ is integrally closed under a variety of assumptions – we choose one here which generalizes the fact that if R is a 2-dimensional regular local ring and $I = (I)_a$, then $(m^n I)_a = m^n I$. We prove:

(3.5) **Theorem.** *Let I be an ideal in a regular local ring of dimension $\leqq 3$ with R/m infinite. Suppose for general $x \in m \backslash m^2$, $(I, x)/(x) \subsetneqq R/(x)$ is integrally closed. Then $(m^n I)_a = m^n I_a$ for all $n \geqq 1$.*

We remark that the assumption that $(I, x)/(x) \subsetneqq R/(x)$ is integrally closed always holds if $\dim R \leqq 2$ as then $R/(x)$ is either a DVR or a field.

Proof. Let $v_1, \ldots, v_n$ be the Rees valuations of I. (See [Re2] or [Mc, Chap. XI].) If $e_i = v_i(I) = \min\{v_i(x)\, x \in I\}$, then $x \in (I)_a$ if and only if $v_i(x) \geqq e_i$ for $i = 1, \ldots, n$ [Re2].

(3.6) **Lemma.** *Let $m = (x_1, \ldots, x_n)$. Then for general $(\alpha_1, \ldots, \alpha_n)$ units, $v_j\left(\sum_i \alpha_i x_i\right) = v_j(m)$, $1 \leqq j \leqq n$.*

Proof. It is enough to show this for one valuation, v. Call a subspace $W \subset m/m^2$ v-stable if whenever $\bar{x} \in W$, $\bar{x} \neq 0$, then $v(x) = v(m)$. Here "$-$" means image in m/m^2. Notice that if $\bar{y} = \bar{x} \in W$ then $v(x) = v(y)$ since $x - y \in m^2$ so that $v(x-y) > v(m)$. Such spaces exist since if $v(x) = v(m)$ then $k\bar{x} \subset m/m^2$ is v-stable. Let W be a maximal v-stable subspace, and let $x_1, \ldots, x_k \in m$ be chosen such that $\bar{x}_1, \ldots, \bar{x}_k$ is a basis of W.

We claim there is a basis of m/m^2, say $\bar{x}_1, \ldots, \bar{x}_k, \bar{x}_{k+1}, \ldots, \bar{x}_n$ such that $v(x_i) > v(m)$ for $i \geqq k+1$. Suppose $x_{k+1}, \ldots, x_i$ have been chosen such that $\bar{x}_1, \ldots, \bar{x}_i$ are linearly independent and $v(x_j) > v(m)$ for $k+1 \leqq j \leqq i$. Choose $\bar{x}_{i+1}$ not in the subspace spanned by $\bar{x}_1, \ldots, \bar{x}_i$, provided $i < n$. If $v(x_{i+1}) > v(m)$, we can continue. If not then $v(x_{i+1}) = v(m)$. However $\langle W, \bar{x}_{i+1} \rangle$ is not v-stable so there is an x'_{i+1} such that $\bar{x}'_{i+1} \in \langle W, \bar{x}_{i+1} \rangle$ and $v(x'_{i+1}) > v(m)$. As $\bar{x}_{i+1} \notin W$ since W is v-stable, it follows that $x'_{i+1} = \omega + \alpha x_{i+1}$, $\alpha \neq 0$, $\omega \in (x_1, \ldots, x_k)$. Then clearly $\bar{x}_1, \ldots, \bar{x}_i, \bar{x}'_{i+1}$ are linearly independent, which establishes our claim.

Now let U be the open set of k^n determined by $\bar{\alpha}_1 \neq 0$. We claim if $(\alpha_1, \ldots, \alpha_n) \in U$, then $v\left(\sum_j \alpha_j x_j\right) = v(m)$. For, as $v(x_j) > v(m)$ for $j > k$, we must have $v\left(\sum_{l=k+1}^{n} \alpha_l x_l\right) \geqq \min_{j=k+1}^{n} (v(x_j)) > v(m)$, and so

$$v\left(\sum_j \alpha_j x_j\right) = v\left(\sum_{j=1}^{k} \alpha_j x_j\right).$$

As $\bar{\alpha}_1 \neq 0$, $\sum \alpha_j \bar{x}_j$ is a non-zero element of W. Hence $v\left(\sum_{j=1}^{k} \alpha_j x_j\right) = v(m)$ which proves our lemma.

Now let $x \in m - m^2$ be chosen so that $v_j(x) = v_j(m)$ for $1 \leqq j \leqq n$, and such that $(I, x)/(x)$ is integrally closed. If $\dim R \leqq 2$, then $(m, x)(I, x)/(m) = (mI, x)/(x)$ is also integrally closed, while if $\dim R = 3$, this proof applied to the $\dim R = 2$ case shows

$(mI,x)/(x)$ is integrally closed. Then $((mI)_a,x)/(mI,x)=0$. However,

$$((mI)_a,x)/(mI,x)\simeq(mI)_a/(mI+(x)\cap(mI)_a)$$
$$=(mI)_a/(mI+x((mI)_\alpha:x)).$$

We claim $((mI)_a:x)=I_a$. Let $r\in((mI)_a:x)$ so that $rx\in(mI)_a$. Then $v_j(rx)\geqq v_j((mI)_a)=v_j(mI)\geqq v_j(m)+v_j(I)$ so that

$$v_j(r)+v_j(x)\geqq v_j(m)+v_j(I),$$

or $v_j(r)\geqq v_j(I)$ for $j=1,\dots,n$. Hence $r\in(I)_a$ as claimed. Thus,

$$0=(mI)_a/(mI+x(I)_a)\quad\text{so}\quad(mI)_a=mI_a\quad\text{as required}.$$

This proves the case $n=1$. The general case follows by induction: provided we can show $(m^{n-1}I,x)$ is integrally closed for general x, we obtain $(m(m^{n-1}I))_a=m(m^{n-I}I)_a=m^nI_a$. However if $\dim R\leqq 2$, then this follows as any ideal in a DVR or a field is integrally closed, and hence if $\dim R=2$, then $(m^nI)_a=m^nI_a$ for *all* ideals I. If $\dim R=3$, we apply the dimension two result above to conclude that

$$((m^nI,x)/(x))_a=((I,x)/(x))_a((m^n,x)/(x))=(Im^n,x)/(x).$$

(3.7) *Example*. We close this section with an example which gives an answer (in the negative) to a question posed by Krull in [K]. W. Heinzer pointed out this question to me. Krull asks if there exists an ideal I primary to a prime p such that $(I)_a$ has an embedded component. Let p be the defining ideal of $k[[t^3,t^4,t^5]]$ in $R=k[[x,y,z]]$ where $\operatorname{char}k=2$. The ideal p is generated by a,b,c where $a=x^3-yz$, $b=y^2-xz$, $c=z^2-x^2y$. Also $(p^2)_a=p^2$. (For example see [V].) In fact $(p^n)_a=p^n$ for all n.

Since R is a regular local ring of characteristic 2 the Frobenius is exact so that $\operatorname{Ass}(R/(a^2,b^2,c^2))=\operatorname{Ass}(R/(a,b,c))=\{p\}$. Hence $I=(a^2,b^2,c^2)$ is primary to p. However, $I\subset p^2\subset(I)_a$ so that $(I)_a=p^2$ since $(p^2)_a=p^2$. Since $p^2\neq p^{(2)}$ by Corollary 2.5, it follows $(I)_a$ has an embedded component.

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Appendix. Proof of Example 2.18

Recall $p=(a,b,c)$ where $a=y^{n+1}-xz^n$, $b=z^{n+1}-x^{n+1}y$, and $c=x^{n+2}-zy^n$. In particular $a\equiv y^{n+1}\operatorname{mod}(x)$, $b\equiv z^{n+1}\operatorname{mod}(x)$, and $c\equiv -zy^n\operatorname{mod}(x)$. Hence $z^{n-1}c^2-y^{n-1}ab\equiv 0\operatorname{mod}(x)$ so there is an element $h_{2,1}^{(2)}$ such that

(A.1) $$xh_{2,1}^{(2)}=z^{n-1}c^2-y^{n-1}ab.$$

This equation shows $h_{2,1}^{(2)}\in p^{(2)}$.

Multiply (A.1) by y and use

(A.2) $$za+xb+yc=0$$

to obtain,

$$xyh_{2,1}^{(2)}=z^{n-1}c(-za-xb)-y^nab$$
$$=-z^nac-xz^{n-1}bc-a(-x^{n+1}a-z^nc)\ \text{by}$$

(A.3) $$x^{n+1}a=y^nb+z^nc+0\,,$$
$$=x(x^na^2-z^{n-1}bc)\,.$$

Hence,

(A.4) $$yh_{2,1}^{(2)}=x^na^2-z^{n-1}bc\,.$$

Multiply (A.1) by z and use (A.2) and (A.3) to obtain

$$xzh_{2,1}^{(2)}=c(-x^{n+1}a-y^nb)-y^{n-1}b(-xb-yc)$$
$$=x(y^{n-1}b^2-x^nac)$$

so

(A.5) $$zh_{2,1}^{(2)}=y^{n-1}b^2-x^nac\,.$$

Thus (A.1), (A.4), and (A.5) give (2.26)–(2.28). Reading (A.5) mod(x) shows that

$$h_{2,1}^{(2)}\equiv z^{2n+1}y^{n-1}$$

which agrees with (2.30) for the values $j=2$, $i=1$, $k=2$.

We also observe that

$$h_{j,0}^{(k)}\equiv(-1)^k\binom{k-1}{j-1}z^{n-k}(z^{n+1})^{j-1}(-zy^n)^{k-j+1}\operatorname{mod}(x)$$
$$\equiv(-1)^{j+1}\binom{k-1}{j-1}z^{jn}y^{(k-j+1)n}\operatorname{mod}(x)$$

which agrees with formula (2.30), and also clearly $h_{j,0}^{(k)}\in p^k\subseteqq p^{(k)}$.

Also,

$$h_{j,j}^{(k)}\equiv(-1)^{jk}\binom{k-1}{j-1}y^{n-k}(y^{n+1})^{k-j}(z^{n+1})^j\operatorname{mod}(x)$$
$$\equiv(-1)^{jk}\binom{k-1}{j-1}y^{n(k-j+1)-j}z^{jn+j}\operatorname{mod}(x)$$

which also agrees with (2.30). Clearly $h_{i,k}^{(k)}\in p^k\subseteqq p^{(k)}$.

Finally, $h_{k+1,i}^{(k)}\equiv 0\operatorname{mod}(x)$, which agrees with (2.30) if we set $\binom{k-1}{k}=0$.

We now prove (2.26)–(2.30) by induction on $i+j+k$. The first case $k=2, j=2$, $i=1$ is above. By induction,

$$ah_{j-1,i}^{(k-1)}\equiv(y^{n+1})(-1)^{j(i+1)+(k-1)i}\binom{k-2}{j-2}z^{(j-1)n+i}y^{(k-j+1)n-i}$$
$$\equiv(-1)^{j(i+1)+(k-1)i}\binom{k-2}{j-2}z^{(j-1)n+i}y^{(k-j+2)n-i+1}\,,$$

while

$$\begin{aligned}
&(-1)^{k-j+1}ch^{(k-1)}_{j-1,i-1}\\
&\equiv(-1)^{k-j+1}(-zy^n)(-1)^{ji+(i-1)(k-1)}\binom{k-2}{j-2}z^{(j-1)n+i-1}y^{(k-j+1)n-i+1}\\
&\equiv(-1)^{k-j+2+ij+ik-i-k+1}\binom{k-2}{j-2}z^{(j-1)n+i}y^{(k-j+2)n-i+1}\\
&\equiv(-1)^{ik+ij+j-i+1}\binom{k-2}{j-2}z^{(j-1)n+i}y^{(k-j+2)n-i+1},
\end{aligned}$$

so that

$$ah^{(k-1)}_{j-1,i}+(-1)^{k-j+1}ch^{(k-1)}_{j-1,i-1}\equiv 0 \bmod (x).$$

Hence there is an element $h^{(k)}_{j,i}$ satisfying:

(A.6) $$xh^{(k)}_{j,i}=ah^{(k-1)}_{j-1,i}+(-1)^{k-j+1}ch^{(k-1)}_{j-1,i-1}.$$

Multiply (A.6) by y and use our induction and (A.2):

$$\begin{aligned}
xyh^{(k)}_{j,i}&=a((-1)^i ah^{(k-2)}_{j-1,i}+(-1)^{k-j}bh^{(k-2)}_{j-2,i-1})\\
&\quad+(-1)^{k-j+1}(-za-xb)h^{(k-1)}_{j-1,i-1}\\
&=(-1)^i a^2h^{(k-2)}_{j-1,i}+(-1)^{k-j}abh^{(k-2)}_{j-2,i-1}\\
&\quad+(-1)^{k-j}zah^{(k-1)}_{j-1,i-1}+(-1)^{k-j}xbh^{(k-1)}_{j-1,i-1}\\
&=(-1)^i a^2h^{(k-2)}_{j-1,i}+(-1)^{k-j}abh^{(k-2)}_{j-2,i-1}\\
&\quad+(-1)^{k-j}a((-1)^i ch^{(k-2)}_{j-1,i-1}-bh^{(k-2)}_{j-2,i-1})+(-1)^{k-j}xbh^{(k-1)}_{j-1,i-1}\\
&=(-1)^i a(ah^{(k-2)}_{j-1,i}+(-1)^{k-j}ch^{(k-2)}_{j-1,i-1})+(-1)^{k-j}xbh^{(k-1)}_{j-1,i-1}\\
&=(-1)^i a(xh^{(k-1)}_{j,i})+(-1)^{k-j}xbh^{(k-1)}_{j-1,i-1}
\end{aligned}$$

so that

(A.7) $$yh^{(k)}_{j,i}=(-1)^i ah^{(k-1)}_{j,i}+(-1)^{k-j}bh^{(k-1)}_{j-1,i-1}.$$

Now multiply (A.7) by z and use the induction and (A.2):

$$\begin{aligned}
zyh^{(k)}_{j,i}&=(-1)^i(-xb-yc)h^{(k-1)}_{j,i}+(-1)^{k-j}b((-1)^i ch^{(k-2)}_{j-1,i-1}-bh^{(k-2)}_{j-2,i-1})\\
&=(-1)^{i+1}ych^{(k-1)}_{j,i}+(-1)^{i+1}b(ah^{(k-2)}_{j-1,i}+(-1)^{k-j}ch^{(k-2)}_{j-1,i-1})\\
&\quad+(-1)^{k+i-j}bch^{(k-2)}_{j-1,i-1}+(-1)^{k-j+1}b^2h^{(k-2)}_{j-2,i-1}\\
&=(-1)^{i+1}ych^{(k-1)}_{j,i}-b((-1)^i ah^{(k-2)}_{j-1,i}+(-1)^{k-j}bh^{(k-2)}_{j-2,i-1})\\
&=(-1)^{i+1}ych^{(k-1)}_{j,i}-b(yh^{(k-1)}_{j-1,i})
\end{aligned}$$

so that

(A.8) $$zh^{(k)}_{j,i}=(-1)^{i+1}ch^{(k-1)}_{j,i}-bh^{(k-1)}_{j-1,i}.$$

Hence (A.6)–(A.8) prove (2.26)–(2.28), and by induction it is clear that $h^{(k)}_{j,i}\in p^{(k)}$ so that (2.29) is verified. Finally we prove (2.30). Read (A.7) mod(x) and use the induction hypothesis.

$$yh_{j,i}^{(k)} \equiv (-1)^i (y^{n+1})(-1)^{(j+1)(i+1)+i(k-1)} \binom{k-2}{j-1} z^{jn+i} y^{(k-j)n-i}$$

$$+(-1)^{k-j} z^{n+1} \binom{k-2}{j-2} (-1)^{ij+(i-1)(k-1)} z^{(j-1)n+i-1} y^{(k-j+1)n-i+1}$$

$$\equiv (-1)^{(j+1)(i+1)+ik} \left[\binom{k-2}{j-1} + \binom{k-2}{j-2} \right] z^{jn+i} y^{(k-j+1)n-i+1}$$

$$\equiv (-1)^{(j+1)(i+1)+ik} \binom{k-1}{j-1} z^{jn+i} y^{(k-j+1)n-i+1}$$

so that

$$h_{j,i}^{(k)} \equiv (-1)^{(j+1)(i+1)+ik} \binom{k-1}{j-1} z^{jn+i} y^{(k-j+1)n-i}$$

which verifies (2.30) and finishes the proof.

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Rigidity Properties of Compact Lie Groups Modulo Maximal Tori

Stefan Papadima

Department of Mathematics, INCREST, Bdul Pacii 220, RO-79622 Bucharest, Romania

1. Introduction and Statement of Main Results

When trying to understand the topological symmetry of homogenous spaces of the form $M = G/K$, where G and K are compact connected Lie groups of the same rank, one is quickly led to concentrate on the cohomological aspect, for several reasons. First of all, the cohomology algebras (for characteristic zero coefficients $\mathbb{F}$) have nice convenient descriptions in terms of invariants of Weyl groups [4]. Next, it is known that, for two such manifolds M and M', the set of homotopy classes of maps between their rationalizations is in natural bijection with the set of graded algebra morphisms between their rational cohomology algebras (see the proof of Theorem 1.1 [10]). Moreover since they are 1-connected finite formal complexes [24], it follows, again by [24], that the knowledge of $[M_0, M'_0]$, i.e. of rational cohomology morphisms, offers the homotopy classification of maps between M and M', up to finite ambiguity (see also [10, 22, 16]). Various kinds of applications are possible, see e.g. Corollary 1.3 and Theorems 1.4–1.6 below.

The main aim of this paper is to begin a systematic study of cohomology automorphisms of such M, both over $\mathbb{Q}$ and over $\mathbb{Z}$; here we shall be concerned only with the case $K = T$, a maximal torus, which is the most natural to start with. In the last decade much work has been done in the direction of determining the rational cohomology endomorphisms and/or automorphisms of complex flag manifolds $M = G/K = U(n_1 + \ldots + n_k) \bmod U(n_1) \times \ldots \times U(n_k)$, see e.g. [10, 19, 17] and their references. The methods were based more or less on direct computations using the special features of the cohomological structure of complex flags and complete results are available only in a few particular cases (up to our present knowledge). For $M = G/T$, G compact connected arbitrary, we obtain a complete and simple description of the cohomological symmetry (both rational and integral). Our method relies on the relationship between the invariants of the Weyl group and the geometry of the Stiefel diagram of G and it was inspired by the results for classifying spaces of [2]; it has the advantage of working uniformly and of minimizing the computational effort.

The proofs of the results on cohomology automorphisms occupy the next section. In more detail, recall that the Weyl group W acts on the Lie algebra V of T

and preserves the integral lattice Γ (more on notations may be found at the beginning of Sect. 2). The classical description by Borel [4] of $H^*(G/T;\mathbb{F})$ in terms of invariants of W in the polynomial graded algebra on $\Gamma\otimes\mathbb{F}$ implies that the graded algebra automorphisms of G/T over $\mathbb{F}$ may be identified with those $\mathbb{F}$-linear automorphisms of $\Gamma\otimes\mathbb{F}$ whose polynomial extension preserves the ideal generated by the positive degree invariants of W (Proposition 2.1). Obvious examples are the elements of the normalizer of W in $GL(\Gamma\otimes\mathbb{F})$; when $\mathbb{F}=\mathbb{Q}$, this normalizer coincides with the admissible automorphisms of [2]. Our first result establishes that there are no other cohomology automorphisms of G/T.

1.1. Theorem. *The group of graded algebra automorphisms of $H^*(G/T;\mathbb{F})$ is (anti)isomorphic to the normalizer of W in $GL(\Gamma\otimes\mathbb{F})$, $\mathbb{F}=\mathbb{R}$ or $\mathbb{Q}$.*

For convenience we first give the proof for $\mathbb{F}=\mathbb{R}$, and then deduce the result for $\mathbb{F}=\mathbb{Q}$ in a straightforward manner.

1.2. Theorem. *The group of graded algebra automorphisms of $H^*(G/T;\mathbb{Z})$ is (anti)isomorphic to the group of automorphisms of the root system of G.*

The result over $\mathbb{Z}$ is also derived from our knowledge of the picture over $\mathbb{R}$. See also Remark 2.12.

1.3. Corollary. *The group of homotopy classes of self-homotopy equivalences of G/T is finite.*

Proof. By the above theorem the group of integral cohomology automorphisms is finite. On the other hand we have seen that there are only finitely many homotopy classes of self-maps inducing the identity in rational cohomology.

We chose to say that $M=G/K$ (or $M=BG$) has the rigidity property with respect to some question related to its topological symmetry if the answer may be formulated in terms of the corresponding Lie theory. Examples: what is the structure of the group of self-homotopy equivalences of M_0? For $M=G/K$ a complex flag manifold, in all known cases this group turns out to be generated by grading automorphisms (which act on each $H^{2i}(M;\mathbb{Q})$ as $\lambda^i\cdot\mathrm{id}$, for some nonzero $\lambda\in\mathbb{Q}$, and which come from Frobenius self-maps in positive characteristic Lie theory, by [9]) together with the rational automorphisms coming from the action of the normalizer $N_G(K)$ on M. A subtler question was formulated, for $M=BG$, in [2]: which self-equivalences of M_0 are defined after finite localization (for a finite 1-connected complex M the answer is: all of them, see e.g. [16])? By [2] this subgroup of the rational automorphisms of $M=BG$ may be identified with $N_{GL(\Gamma\otimes\mathbb{Q})}(W)/W$.

The second aim of this paper is to improve the rather vaguely formulated definition of the rigidity properties. Theorems 1.1 and 1.2 above may be considered as typical examples in this direction. We shall next state in precise form three more examples of rigidity (Theorems 1.4–1.6 below) and later give the proofs as applications of our results on cohomology automorphisms (in Sects. 3–5).

The main application is devoted to geometry. Let M be a closed 1-connected Riemannian manifold and let f be an isometry of M. A geodesic curve c is called *f-invariant* if it is nonconstant and there exists a period t such that $f(c(x))=c(x+t)$, any x. When $f=\mathrm{id}$ one recovers the classical notion of closed geodesic.

The question of the existence and of the abundance of various kinds of geodesics is a central problem in Riemannian geometry. A major development in this area is contained in the paper [25]; they pointed out the relationship between the rational homotopy properties of M and the existence of closed geodesics on M. Further refinements of both Morse theory and rational homotopy theory involved here led to the conclusion that the nonexistence of (many) f-invariant geodesics imposes severe restrictions on the rational homotopy properties of f. The following result in this direction will be strong enough for our present purposes (subtler statements may be found in [12, 14], see also 3.4). Denoting by $[\pi_*(M)\otimes\mathbb{Q}]^f$ the fixed points of the obvious action of f, one has:

Theorem (see [12, 14, 13]). (i) *If there are no f-invariant geodesics then* $\dim[\pi_*^{\mathrm{odd}}(M)\otimes\mathbb{Q}]^f=0$.

(ii) *If there are only finitely many geometrically distinct f-invariant geodesics then* $\dim[\pi_*^{\mathrm{odd}}(M)\otimes\mathbb{Q}]^f\leqq 1$.

Using this approach, strong existence theorems were obtained in [13]: if M is odd-dimensional every isometry has an invariant geodesic; if $\dim\pi_*(M)\otimes\mathbb{Q}=\infty$, every isometry has infinitely many invariant geodesics. These leave still open the case $M=G/K$, K a closed connected subgroup of maximal rank. For $K=T$ we are able, by computing the rational homotopy fixed points of the self homotopy equivalences of M, to obtain a complete solution of the existence problem for invariant geodesics on M.

1.4. Theorem. *Let f be an isometry of $M=G/T$ (for an arbitrary metric).*

(i) $\dim[\pi_*^{\mathrm{odd}}(M)\otimes\mathbb{Q}]^f>0$.

(ii) $\dim[\pi_*^{\mathrm{odd}}(M)\otimes\mathbb{Q}]^f>1$, *unless $M=(SU(n)/T)^l$, $n=2$ or 3, and f^* equals the cohomology automorphism corresponding by the isomorphism of Theorem 1.2 to the root system automorphism given by*

$$a(v_1,\dots,v_{l-1},v_l)=(a_l(v_l),a_1(v_1),\dots,a_{l-1}(v_{l-1}))\,,$$

where a_j are automorphisms of the corresponding type A_1 or type A_2 root systems, and (in the A_2 case) they are subject to the condition that $a_1\cdot a_2\dots a_l$ induces the nontrivial automorphism of the Dynkin diagram; in all these cases $\dim[\pi_*^{\mathrm{odd}}(M)\otimes\mathbb{Q}]^f=1$.

This result is the best one can hope, see 3.4.

Section 4 deals with fixed point free self-maps. Let M be a compact connected manifold. We shall consider, following [18], *the noncoincidence index of M*, denoted by $n(M)$, defined as the maximum k for which there exist $k-1$ fixed point free self/maps of M, no two of which having a coincidence. In a similar manner, we may define *the free symmetry index of M*, denoted by $f(M)$, as the maximum of the cardinalities of the discrete groups which can act freely on M. For $M=G/T$, we shall determine $n(M)$ and $f(M)$, by using our knowledge of cohomology automorphisms together with a computation relating Lefschetz numbers to rational homotopy fixed points:

1.5. Theorem. *Set $G/T=M$.*

(i) *$f(M)=$ order of W.*

(ii) *$n(M)=\infty$ unless G is simple, and in this case $n(M)=f(M)$.*

By Sullivan [23] a homotopy type M may be described as the collection of its localizations $\{M_p | p \text{ a prime}\}$ together with the coherence information provided by the rationalization maps $\{M_p \to M_0\}$. The simplest situations arise when the collection of the localizations $\{M_p\}$ already determines M. Thus, a homotopy type M is called *generically rigid* [11] if the genus of M, defined as the set of homotopy types M' with the property that $M'_p \simeq M_p$, for all primes p, consists of M alone.

1.6. Theorem. *G/T is generically rigid, for any G.*

This result is entitled to be called a rigidity property (in our sense) by more than philological reasons. The method developed in [11] indicates that, for a 1-connected finite formal complex M, the generic rigidity is a consequence of the fact that the group of self-homotopy equivalences of M_0, to be denoted in the sequel by $E_0(M)$, is generated by rationalizations of self-homotopy equivalences which are defined after inverting at most one prime. When $M = G/K$, this in turn is a direct consequence, via etale homotopy theory [8, 9], of the rigidity of $E_0(M)$. We mean by this that $E_0(M)$ is generated by self-maps which come from the purely inseparable isogenies of the corresponding Lie theory. For $K = T$ this rigidity property follows from the detailed description of rational cohomology automorphisms of M given in 2.8 and 2.9 (see the proof of Proposition 5.1 and the remarks preceding its statement); the basic arguments for establishing the rigidity of $E_0(M)$ are extracted from [2, Propositions 2.13 and 2.15]. As far as generic rigidity is concerned, we follow [11], obtaining a little more (see Theorem 5.3).

2. Cohomology Automorphisms

Let G be a compact connected Lie group and let T be a maximal torus. By classical Lie theory [4], $G/T = G_1/T_1$, where in addition G_1 is 1-connected (and its Lie algebra equals the semisimple part of the Lie algebra of G). Therefore we may and we shall indeed from now on suppose that G is 1-connected. We can further write $G = \prod G_i$ as the product of its simple components. Denoting by V the Lie algebra of T and by Γ the kernel of the exponential map of T, there is a corresponding product splitting for T, V, and Γ.

The real vector space V is endowed with the euclidean metric coming from the Killing form of G. This choice of metric provides a canonical isomorphism $V \xrightarrow{\sim} V^*$ and an euclidean structure on the dual space V^* (everything being compatible with the splittings). Denote by $\Phi^* \subset V^*$ the root system (in the axiomatic sense of [21]) consisting of the roots of the adjoint representation of T in the Lie algebra of G [1]. The group $N_G(T)/T$ is canonically isomorphic to the Weyl group of this root system, and both will be denoted in the sequel by W [1]. The decomposition of G into simple components corresponds to the decomposition of Φ^*, $\Phi^* = \coprod \Phi_i^*$, into irreducible components; similarly, the Weyl group decomposes as a direct product. It will be convenient to normalize the metric on each component of $V (V = \prod V_i)$ in order to make all short roots of the corresponding Φ_i^* have length equal to $\sqrt{2}$ (for types A, D, and E we consider that all roots are short). Denoting by $(\ ,\ )$ the resulting metric on V and by $\tau: V \to V^*$ the corresponding isometry we obtain a root system $\Phi \subset V$ which is isometrically isomorphic to Φ^* and whose

Weyl group action on V corresponds to the adjoint action of the Weyl group of G [1].

Recall next from [4] that, with characteristic zero coefficients $\mathbb{F}$, the spectral sequence of the fibration $G/T \hookrightarrow BT \to BG$ gives the isomorphism $H^*(G/T;\mathbb{F}) = H^*(BT;\mathbb{F})/\mathrm{ideal}(H^+(BT;\mathbb{F})^W)$. Denoting by $\mathbb{F}[\Gamma\otimes\mathbb{F}]$ the graded $\mathbb{F}$-algebra of polynomial functions on $\Gamma\otimes\mathbb{F}$ (with degree of the generators $=1$), on which W naturally acts by $p^w = p\circ w$, for $p\in\mathbb{F}[\Gamma\otimes\mathbb{F}]$ and $w\in W$, the natural isomorphism $\Gamma\otimes\mathbb{F}\to H^2(BT;\mathbb{F})$ gives rise to an algebra isomorphism

$$\mathbb{F}[\Gamma\otimes\mathbb{F}]/\mathrm{ideal}(\mathbb{F}[\Gamma\otimes\mathbb{F}]^{+W}) \xrightarrow{\sim} H^*(G/T;\mathbb{F}) \tag{1}$$

which doubles the degrees.

Denote by $\mathfrak{E}\subset \mathrm{gl}(\Gamma\otimes\mathbb{F})$ the submonoid consisting of those $\mathbb{F}$-linear maps $b:\Gamma\otimes\mathbb{F}\to\Gamma\otimes\mathbb{F}$ with the property that, for any $p\in\mathbb{F}[\Gamma\otimes\mathbb{F}]^{+W}$, $p^b = p\circ b \in \mathrm{ideal}(\mathbb{F}[\Gamma\otimes\mathbb{F}]^{+W})$, and by $\mathfrak{A}$ the group of invertible elements of $\mathfrak{E}$.

2.1. Proposition. *The correspondence which associates to each $b\in\mathfrak{E}$ the induced graded algebra endomorphism of $H^*(G/T;\mathbb{F})$, via* (1), *establishes an antimultiplicative isomorphism onto the cohomology endomorphisms of G/T over $\mathbb{F}$, under which the cohomology automorphisms correspond to the elements of $\mathfrak{A}$.*

Proof. Since G is in particular semisimple each Weyl group W_i acts irreducibly on V_i [21]. It follows that there are no nonzero degree one invariants of the Weyl group W in $\mathbb{F}[\Gamma\otimes\mathbb{F}]$ for $\mathbb{F}=\mathbb{R}$ and consequently (see [6, p. 126]) for any $\mathbb{F}$. Hence the projection $\mathbb{F}[\Gamma\otimes\mathbb{F}]\to H^*(G/T;\mathbb{F})$ induces an isomorphism $\mathbb{F}[\Gamma\otimes\mathbb{F}]^1 \xrightarrow{\sim} H^2(G/T;\mathbb{F})$ at the level of indecomposable algebra generators. Using this fact, all the assertions of the proposition follow easily.

From now on we shall use for the inverse of the above isomorphism, $\mathrm{End}\,H^*(G/T;\mathbb{F})\xrightarrow{\sim}\mathfrak{E}$, the notation $f\to H_2(f;\mathbb{F})$.

We are now moving towards the proof of Theorem 1.1. Set $\mathbb{F}=\mathbb{R}$. Consider the subgroup $D\subset\prod GL(V_i)$, consisting of automorphisms which act as multiplication by some positive real number on each V_i. Since D centralizes W, we plainly have $D\subset\mathfrak{A}$.

2.2. Lemma. *For each $a\in\mathfrak{A}$ there exists $d\in D$ such that ad is an isometry.*

Proof. Assume $a\in\mathfrak{E}$. Since we just saw that there are no nonzero elements in $\mathbb{R}[V]^{1W}$ it follows from the definition of $\mathfrak{E}$ that the action of a on $\mathbb{R}[V]$ must preserve the linear subspace $\mathbb{R}[V]^{2W} = \bigoplus \mathbb{R}[V_i]^{2W_i}$. By irreducibility (see [6, p. 66]) each $\mathbb{R}[V_i]^{2W_i}$ is one dimensional, generated by the invariant quadratic form q_i defined by $q_i(x)=(x_i,x_i)$, for $x=\sum x_i\in V$. Hence $q_i^a=\sum_j A_{ij}q_j$, any i. Summation gives $(ax,ax)=\sum A_j q_j(x)$, for any $x\in V$. If $a\in\mathfrak{A}$ pick a nonzero $x_i\in V_i$ and deduce that $A_i>0$, for any i. Define then d by $d=\prod\lambda_i$, with $\lambda_i = 1/\sqrt{A_i}$.

The same method gives the following result, which will be useful in Sect. 4:

2.3. Lemma. *For any $a\in\mathfrak{E}$, $\ker a$ is W-invariant.*

Proof. If $ax=0$ then we may write, for any $w\in W$, $0=(ax,ax)=\sum q_i^a(x) = \sum q_i^{aw}(x) = (awx, awx)$.

2.4. Proof of Theorem 1.1 for $\mathbb{F}=\mathbb{R}$. We have to show that $\mathfrak{A}\subset N$, N being the normalizer of the Weyl group in $GL(V)$. By Lemma 2.2 it is enough to show that any isometric $a\in\mathfrak{A}$ normalizes W. For each $\alpha\in\Phi$ denote by H_α the hyperplane orthogonal to α, and set $H=\bigcup H_\alpha$. We claim that if suffices to prove that any such a leaves H invariant. Indeed, assuming that for any $\alpha\in\Phi$ there exists $\beta\in\Phi$ such that $a(H_\alpha)=H_\beta$, it is immediate to see, using the fact that a is isometric, that $as_\alpha a^{-1}=s_\beta$, where s_α and s_β are the reflections in the corresponding hyperplanes; since the reflections s_α generate W it follows that $a\in N$.

In order to prove that the action of a preserves H we proceed to the determination of the cohomology classes in $H^2(G/T;\mathbb{R})$ which have maximal height (a computation for complex flag manifolds may be found in [19]).

Denoting by r the number of positive roots (and recalling that $2n$ = dimension of G/T) we shall consider the following polynomial function $p\in\mathbb{R}[V]^n$, constructed by evaluating n-th powers of 2-dimensional cohomology classes of G/T on the fundamental class of G/T: $p(x)=\langle\tau(x)^n,[G/T]\rangle$, any $x\in V$. For any $w\in W$,

$$\begin{aligned}p^{w^{-1}}(x)&=\langle(\tau(x)^w)^n,[G/T]\rangle=\langle(\tau(x)^n)^w,[G/T]\rangle\\&=\langle\det(w\ )\cdot\tau(x)^n,[G/T]\rangle=\det(w)\cdot p(x)\,.\end{aligned}$$

It is now easy to infer that $p(x)=\lambda\cdot\prod_{\alpha\in\Phi^+}(\alpha,x)$, any $x\in V$, for some $\lambda\in\mathbb{R}$ ([6, p. 113]). Moreover, λ must be nonzero, for otherwise (use [21, p. 134]) we would have $H^{2n}(G/T;\mathbb{R})=0$.

Coming back to our given orthogonal $a\in\mathfrak{A}$, it is clear that a^* must preserve the zeroes of $p\circ\tau^{-1}$, which implies that a^{-1} preserves the zero set of p, which is just H.

Our next task is to describe the group structure of N. Let us choose a system of simple roots $S\subset\Phi$; if $\Phi=\coprod\Phi_i$ is the decomposition into irreducible components, there is a corresponding splitting $S=\coprod S_i$, with $S_i\subset\Phi_i$ a system of simple roots. We shall denote by Graphaut(S) the group of permutations of S which are automorphisms of the Coxeter graph structure. It contains the subgroup Dgraut(S), consisting of graph automorphisms which preserve short and long roots.

2.5. Proposition. *There is a split exact sequence*

$$1\to D\times W\to N\underset{\sigma}{\overset{\gamma}{\rightleftarrows}}\mathrm{Graphaut}(S)\to 1\,.$$

Proof. The proof is inspired by [2, Proposition 2.13]. The novelty consists in the construction of the splitting (in [2] the surjectivity of γ is verified by case-by-case checking); though perhaps known to Adams and Mahmud, this explicit construction will play a key role in what follows, therefore we shall treat this point carefully. As a word of caution, note that our graph-automorphisms are called in [2] diagram isomorphisms.

Denote by C the Weyl chamber corresponding to S. The arguments of 2.4 show that the elements of N act on Weyl chambers. Call the stability group of C, N_C. An element $a\in N_C$ permutes the walls of C and thus induces a permutation of S, denoted by $\gamma(a)$, characterized by: $a(H_\alpha)=H_{\gamma(a)\alpha}$, any $\alpha\in S$. It can be checked [2] that $\gamma(a)$ is a graph-automorphism and that we have an exact sequence $1\to D\to N_C\longrightarrow\mathrm{Graphaut}(S)$. Granting for the moment the existence of a splitting

σ: Graphaut$(S)\to N_C$, we can easily finish the proof. The transitivity of the Weyl group action on chambers provides a natural group surjection $N_C/D\to N/D\cdot W$, which is in fact an isomorphism (using the simple transitivity). The existence of γ and σ for N_C gives thus rise to the corresponding constructions for N and the asserted split exact sequence is established. It remains to construct the splitting for N_C.

Pick $g\in$ Graphaut(S). We claim that there exists uniquely $\mu: S\to\mathbb{R}_+$ such that defining $b\in GL(V)$ by $b(\alpha)=\mu(\alpha)\cdot g(\alpha)$, for any $\alpha\in S$, we have

$$bs_\alpha b^{-1}=s_{g(\alpha)}, \quad \text{for any} \quad \alpha\in S \tag{1}$$

and

$$\prod_{\beta\in S_i}\mu(\beta)=1, \quad \text{for any} \quad i \tag{2}$$

where $S=\coprod S_i$ is the splitting given by the irreducible components of Φ. Defering the proof, notice that, by (1), $b\in N$, and that in order to show $b\in N_C$ and $\gamma(b)=g$ it is harmless to assume (by Lemma 2.2) that b is an isometry, eventually changing $\mu(\alpha)$ to some other positive $v(\alpha)$. If $x\in C$, then, for any $\alpha\in S$, we have $0<(x,\alpha)=(b(x),b(\alpha))=v(\alpha)\cdot(b(x),g(\alpha))$, which shows that $b\in N_C$, and again by orthogonality $b(H_\alpha)=H_{g(\alpha)}$, any $\alpha\in S$, hence $\gamma(b)=g$. Put then $\sigma(g)=b$ and emphasize the dependence on g writing μ_g instead of μ. The fact that σ is a group morphism is equivalent to $\mu_{gg'}=\mu_{g'}\cdot(\mu_g\circ g')$, for any g, g' (3). Since an easy computation shows that conditions (1) are equivalent to

$$\mu(\beta)/\mu(\alpha)=\langle\beta,\alpha\rangle/\langle g(\beta),g(\alpha)\rangle, \text{ for any } \alpha,\beta\in S \text{ such that } \langle\beta,\alpha\rangle\neq 0 \tag{4}$$

(where $\langle\ ,\ \rangle$ denote Cartan integers, as in [21]) one may use (4) and (2) for a rapid proof of (3). Observing further that conditions (4) are involving independently the various irreducible components and that it is enough to check them only when $|\alpha|\leqq|\beta|$, it is clear how to prove the existence and the uniqueness of a solution μ of (1) with arbitrarily prescribed values at the "central nodes" of each component (specifically, we may choose the node 1 for each A_l, C_l, and for G_2, the node l for each B_l, the node $l-2$ for each D_l, the node 4 for each E_l and for F_4, in the notations of [21], p. 58). For each component Φ_i choose such a "central node" β_0 and, for any other $\beta\in S_i$, by choosing a string of nodes joining β_0 to β and iterating (4), deduce that $\mu(\beta)=\mu(\beta_0)\cdot c_\beta$, where the positive constant c_β is independent of μ; therefore $\prod_{\beta\in S_i}\mu(\beta)=\mu(\beta_0)^{|S_i|}\cdot c_i$, where c_i is positive and independent of μ. This helps to complete the proof.

2.6. Remark. Let us say that two Dynkin diagrams are *$\mathbb{Q}$-isomorphic* if the underlying graphs are isomorphic; among connected ones the only $\mathbb{Q}$-isomorphic but not isomorphic ones are B_l and C_l. Let us say that a diagram is *$\mathbb{Q}$-isotypic (isotypic)* if all its irreducible components are $\mathbb{Q}$-isomorphic (isomorphic); there are the obvious notions of decomposition into *$\mathbb{Q}$-isotypic components* and into *isotypic components*. A similar terminology applies to root systems, Lie algebras and Lie groups. These notions naturally arise in connection with graph (diagram) automorphism groups, which obviously split as direct products, according to the decomposition into $\mathbb{Q}$-isotypic (isotypic) components. As a byproduct of the above proof, if $G=\prod G_j$ is the $\mathbb{Q}$-isotypic decomposition, it follows that all the groups in the statement of 2.5 split as the direct product of the groups corresponding to each G_j, in a manner compatible with γ and σ.

2.7. Proof of Theorem 1.1 for $\mathbb{F}=\mathbb{Q}$. By extension of scalars the group of graded algebra automorphisms of $H^*(G/T;\mathbb{Q})$ is identified with the subgroup of graded algebra automorphisms of $H^*(G/T;\mathbb{R})$ which preserve the rational structure $H^2(G/T;\mathbb{Q})\subset H^2(G/T;\mathbb{R})$. Using the inverse of the isomorphism established in Proposition 2.1 we may further identify the rational cohomology automorphisms with $\mathfrak{A}\cap GL(\Gamma\otimes\mathbb{Q})$ which, by the result for $\mathbb{F}=\mathbb{R}$, is nothing else but the normalizer of W in $GL(\Gamma\otimes\mathbb{Q})$.

We shall denote in the sequel by $N_{\mathbb{Q}}$ this normalizer group; set also $D_{\mathbb{Q}}$ = group of automorphisms of V which act as rational positive scalars on each irreducible component V_i. As far as the group structure of $N_{\mathbb{Q}}$ is concerned, we have the following replica of Proposition 2.5:

2.8. Proposition. *The split exact sequence of Proposition 2.5 restricts to an exact sequence*

$$1\to D_{\mathbb{Q}}\times W\to N_{\mathbb{Q}}\xrightarrow{\gamma}\mathrm{Graphaut}(S)\to 1$$

which splits as the direct product of the analogous exact sequences corresponding to the $\mathbb{Q}$*-isotypic components of* V.

Proof. We only have to check that the restriction of γ to $N_{\mathbb{Q}}$ is still onto. Actually it can be shown that for any $g\in\mathrm{Graphaut}(S)$ there exists $d\in D$ such that $\sigma(g)d\in GL(\Gamma\otimes\mathbb{Q})$, using the construction of σ given in the proof of Proposition 2.5 and keeping in mind that $S\subset\Gamma$ (see 2.10). Here is an alternative proof, based on [2], Proposition 2.13, which disposes of the case when G is simple. For a $\mathbb{Q}$-isotypic G it is immediate to see that we have a split exact sequence (which will be useful again later on)

$$1\to\prod_{i=1}^{m}\mathrm{Graphaut}(S_i)\to\mathrm{Graphaut}(S)\overset{\pi_0}{\rightleftarrows}\Sigma_m\to 1\,, \tag{1}$$

where $\Phi=\coprod_{i=1}^{m}\Phi_i$ is the decomposition into irreducible components and $\pi_0(g)$ represents the permutation of the connected components of the Coxeter graph induced by $g\in\mathrm{Graphaut}(S)$. It is equally immediate to see that $\pi_0\circ\gamma:N_{\mathbb{Q}}\to\Sigma_m$ is onto, which implies the result for the $\mathbb{Q}$-isotypic case. The general case follows by Remark 2.6.

2.9. Remarks. Assume G is simple. In most cases, namely excepting $\Phi=B_2, F_4$ or G_2, we have equality between graph-automorphisms and diagram-automorphisms; in the exceptional cases there is only one automorphism of the diagram but there is one more exotic graph-automorphism, which turns the graph end for end. If $g\in D$ graut then g preserves Cartan integers hence, by (4) and (2) in the proof of Proposition 2.5, $\mu_g=1$; since $S\subset\Gamma$ (2.10), it follows that the splitting σ of Proposition 2.5 also splits the exact sequence of Proposition 2.8, when Graphaut(S) contains only diagram automorphisms. However it is not difficult to see that the exact sequence does not split, for $G=B_2, F_4$ or G_2 (due to the presence of a square root factor in the expression of μ_g, g being the nontrivial graph-automorphism). In the general case, one can see that $N_{\mathbb{Q}}$ is generated by direct products of grading automorphisms and the automorphisms of the root system, eventually together with the exotic admissible isomorphisms of [2], corresponding to the graph isomorphisms between irreducible Dynkin diagrams which do not respect the length of the roots (for more details see the proof of Proposition 5.1).

The following well-known lemma will be of great help. For the sake of completeness, we are going to include its proof.

2.10. Lemma. *Γ coincides with the free abelian group generated by $2\alpha/(\alpha,\alpha)$, $\alpha \in S$. In particular $S \subset \Gamma$.*

Proof. The simple connectivity of G implies that Γ is generated by $2\alpha/(\alpha,\alpha)$, $\alpha \in \Phi$, see [1, p. 129]. On the other hand, this in nothing else but the root lattice of the inverse root system $\Phi^{\vee}$ and the first assertion is then standard (see e.g. [21]). With our choice of metric it follows immediately that $S \subset \Gamma$.

2.11. Proof of Theorem 1.2. Since $H^*(G/T;\mathbb{Z})$ is torsion free (by Borel and Bott, see e.g. [5]), extension of scalars identifies $\operatorname{Aut} H^*(G/T;\mathbb{Z})$ with a certain subgroup of $\operatorname{Aut} H^*(G/T;\mathbb{R})$, which is plainly contained in the subgroup of those elements of $\operatorname{Aut} H^*(G/T;\mathbb{R})$ which induce unimodular self-maps of $H^2(G/T;\mathbb{Z})$. Due to the 1-connectedness assumption on G this latter group is identified, via the inverse of the isomorphism described in Proposition 2.1, with $N \cap GL(\Gamma)$. Our first aim is to show that $N \cap GL(\Gamma) = \operatorname{Aut}(\Phi)$.

The exact sequence of Proposition 2.5 easily describes $\operatorname{Aut}(\Phi)$ as $W \cdot \sigma(D\,\mathrm{graut}(S))$. Since all short roots have the same length, we infer from the previous lemma that $\operatorname{Aut}(\Phi) \subset N \cap GL(\Gamma)$.

In order to prove the other inclusion, we have to start with an element $a \in N \cap GL(\Gamma)$, of the form $a = d \cdot \sigma(g)$, with $d \in D$ and $g \in \mathrm{Graphaut}(S)$, and show that necessarily $g \in D\,\mathrm{graut}(S)$. [Since $D \cap GL(\Gamma) = \{1\}$ it will follow that $a \in \operatorname{Aut}(\Phi)$.]

Writing $d = \operatorname{diag}(d_\alpha)_{\alpha \in S}$, with $d_\alpha \in \mathbb{R}_+$, we know that $a(\alpha) = d_{g\alpha}\mu_g(\alpha) \cdot g(\alpha)$ for any $\alpha \in S$. By Lemma 2.10 the condition $a(\Gamma) \subset \Gamma$ simply means that for every $\alpha \in S$, $a\,(2\alpha/(\alpha,\alpha)) = n_\alpha \cdot (2g(\alpha)/(g(\alpha), g(\alpha)))$, for some $n_\alpha \in \mathbb{Z}$, which may be rewritten as

$$d_{g\alpha}\mu_g(\alpha) = [(\alpha,\alpha)/(g\alpha, g\alpha)] \cdot n_\alpha (n_\alpha \in \mathbb{Z}_+), \quad \text{any} \quad \alpha \in S. \tag{1}$$

Adding the condition that $\det(a) = \pm 1$ and recalling that $\det(\sigma(g)) = \pm 1$, for any $g \in \mathrm{Graphaut}(S)$ (by construction), and then multiplying the conditions (1), we find out that we must have $n_\alpha = 1$, for any $\alpha \in S$. Take any $\alpha, \beta \in S$ such that $\langle \alpha, \beta \rangle \neq 0$. Since g is a graph-automorphism and $d \in D$, we know that $d_{g\alpha} = d_{g\beta}$. Dividing the equality (1) corresponding to β by that corresponding to α and using the defining properties of μ_g [namely (4) in the proof of Proposition 2.5] we deduce that $\langle g\alpha, g\beta \rangle = \langle \alpha, \beta \rangle$, therefore $g \in D\,\mathrm{graut}(S)$ and the proof of our first claim is completed.

On the other hand, it follows by classical Lie theory that for any $g \in D\,\mathrm{graut}(S)$ there exists a group automorphism h of G which leaves T invariant and such that $\gamma(H_2(\bar{h}^*;\mathbb{R})) = g$, where $\bar{h}$ denotes the induced map on G/T (see [7, Sect. 33], and also [2, p. 14]). Therefore all automorphisms of the root system are induced, by extension of scalars, by automorphisms of $H^*(G/T;\mathbb{Z})$, q.e.d.

2.12. Remark. Using our choice of metric on V (see also Lemma 2.10) it is not hard to see that the subgroups $N_{GL(\Gamma\otimes\mathbb{F})}(W)$ and $\mathrm{Aut}(\Phi)$ are stabilized by taking adjoints in $GL(V)$, hence antiisomorphic with themselves.

3. Invariant Geodesics

As explained in the introduction, the results of Grove, Halperin, and Vigué reduce the geometric problem to the computation of rational homotopy fixed points of rational homotopy equivalences of G/T. These in turn correspond bijectively [10] to the automorphisms of the cohomology, which may be identified as in the previous section with the normalizer of the Weyl group. We are going to associate a number $F(a)$ to each $a\in N$ in such a way that $F(a)=\dim[\pi_*^{\mathrm{odd}}(G/T)\otimes\mathbb{Q}]^h$ whenever $a=H_2(h^*;\mathbb{R})$ and h is an isometry of G/T, and then proceed to the effective computation of these numbers.

Denote by I the subalgebra of the invariants of the Weyl group in $\mathbb{R}[V]$ (which is a commutative graded algebra freely generated by r elements, $r=\dim V$, see [6, p. 107]) and by I^+ the positive degree invariants. In the notations of Sect. 2 (which will be used throughout this paper) the natural action on $\mathbb{R}[V]$ of any $a\in N$ induces a linear map, denoted by $\pi(a): I^+/I^+\cdot I^+\to I^+/I^+\cdot I^+$. Define then: $F(a)=\dim(I^+/I^+\cdot I^+)^{\pi(a)}$.

3.1. Proposition. *For any rational homotopy equivalence $h: G/T\to G/T$ we have* $\dim[\pi_*^{\mathrm{odd}}(G/T)\otimes\mathbb{Q}]^h=F(H_2(h^*;\mathbb{R}))$.

Proof. The proof uses rational homotopy theory [24]. We may safely replace $\mathbb{Q}$ by $\mathbb{R}$ in the left hand side and then use minimal models over $\mathbb{R}$. Since G/T is formal [24] its minimal model coincides with the minimal model of its cohomology algebra considered with trivial differential. The minimal model $\varrho:(\mathfrak{M},d)\to(H^*(G/T),0)$ is constructed as follows. Pick homogeneous elements $p_1,\dots,p_r\in I^+$ which freely generate the graded algebra I. Construct $\mathfrak{M}$ as a free commutative graded algebra by setting $\mathfrak{M}=VZ_0\otimes\wedge Z_1$ (where V and $\wedge$ indicate symmetric, respective exterior algebras) with $Z_0=V^*$ and $\deg(z)=2$, for any $z\in Z_0$, and $Z_1=\mathbb{R}-\mathrm{span}\{y_1,\dots,y_r\}$ with $\deg(y_i)=\deg(p_i)-1$ [where $\deg(p_i)$ is considered by identifying $\mathbb{R}[V]$ and VZ_0 as *graded* algebras]. Set $dz=0$ for $z\in Z_0$ and $dy_i=p_i$ for any i. Since there are no nonzero invariants of W in V^* (remember that G is still assumed to be 1-connected), it follows that $(\mathfrak{M},d)$ is indeed minimal. Define $\varrho z=z$ for $z\in Z_0$ and $\varrho y_i=0$ for any i. Since $(p_1,\dots,p_r)$ is a regular sequence in $\mathbb{R}[V]$ (see [6, p. 115]), it follows from [24] that ϱ is a minimal model map. Given $h^*: H^*(G/T)\to H^*(G/T)$, it is easy to construct a differential graded algebra map $\hat{h}:(\mathfrak{M},d)\to(\mathfrak{M},d)$ such that $\varrho\hat{h}=h^*\varrho$. By [10] $\hat{h}$ will represent a minimal model of h, therefore, by rational homotopy theory, the dual of the action of h on $\pi_*^{\mathrm{odd}}(G/T)\otimes\mathbb{R}$ is identified with the action of $\hat{h}$ on the indecomposables of odd degree of $\mathfrak{M}$, i.e. with the linear part of the restriction $\hat{h}|Z_1$. Notice that $\hat{h}(VZ_0)\subset VZ_0$ and that the restriction $\hat{h}|VZ_0$ coincides with the action on $\mathbb{R}[V]$ of $a=H_2(h^*;\mathbb{R})$, by construction. Denoting by J the ideal of $\mathbb{R}[V]$ generated by I^+, i.e. $J=(VZ_0)\cdot dZ_1$ we infer that $\hat{h}(J)\subset J$. To be more precise, decompose $\hat{h}|Z_1$ as follows: $\hat{h}|Z_1=h_1+h_2$, where $h_1: Z_1\to(VZ_0)\cdot Z_1$ and $h_2: Z_1\to(VZ_0\otimes\wedge Z_1)$

$\cdot Z_1 \wedge Z_1$, write the commutation condition with d and find out that $\hat{h}d|Z_1 = dh_1$. Writing further $h_1 = \pi^{\mathrm{odd}}(\hat{h}) + h_1'$, where $\pi^{\mathrm{odd}}(\hat{h}): Z_1 \to Z_1$ and $h_1': Z_1 \to (V^+ Z_0) \cdot Z_1$ deduce that $\hat{h}d|Z_1 = d\pi^{\mathrm{odd}}(\hat{h})$, in $J/(V^+ Z_0) \cdot J$. All these considerations together imply that the action of h on $\pi_*^{\mathrm{odd}}(G/T) \otimes \mathbb{R}'$ may be identified with the action of a in $J/(V^+ Z_0) \cdot J$. Consider now the natural surjection: $I^+/I^+ \cdot I^+ \to J/(V^+ Z_0) \cdot J$, which is an isomorphism, due to the fact that $\mathbb{R}[V]$ is a free graded module over I [6, p. 105], and conclude the proof.

Since in the geometric applications h will be a self-homotopy equivalence and since we know in that case, by Theorem 1.2, that $H_2(h^*; \mathbb{R})$ must lie in $W \cdot \sigma(D\,\mathrm{graut}(S))$ and since obviously $F(wa) = F(a)$, for any $w \in W$ and $a \in N$, we could just compute $F(\sigma(g))$, $g \in D\,\mathrm{graut}(S)$, checking case by case, in order to give the proof of Theorem 1.4. However, we shall not pursue this way, but we choose to use the following general result, which provides both a very convenient new description of $F(\sigma(g))$, $g \in D\,\mathrm{graut}(S)$, and a useful information related to Lefschetz numbers (see the next section).

3.2. Lemma. *If $a \in N$ has finite order and leaves some Weyl chamber invariant, then $F(a) = \dim V^a$ and this number is positive.*

Proof. Assume $a(C) = C$, for some Weyl chamber C, and let $\mathrm{ord}(a) = m < \infty$. We first show that a has a fixed point in C. Indeed, starting with any $y \in C$, set $x = \sum_{i=1}^{m} a^i(y)$; then $x \in C$ and $a(x) = x$. In particular this proves the second assertion.

For any linear map b, denote by $m_1(b)$ the multiplicity of the eigenvalue 1. In order to compute $F(a) = m_1(\pi(a))$ choose homogeneous elements $p_1, \ldots, p_r \in I^+ (r = \dim V)$ which freely generate the algebra I and, writing that $p_i^a \in I$, for any i, deduce the existence of a polynomial function $A: \mathbb{R}^r \to \mathbb{R}^r$ with the property that $p \circ a = A \circ p\,(*)$, where $p: V \to \mathbb{R}^r$ has p_i as its i-th component; by the very definition of $\pi(a)$, it has the same characteristic polynomial as the linear part of A, hence $F(a) = m_1(D_0 A)$. By taking derivatives in $(*)$, in a point $x \in C$ which is fixed by a, and recalling that the jacobian of p is nonsingular in all points of C (see [6, p. 113]), we infer that $m_1(a) = m_1(D_{p(x)} A)$, which of course also equals $\dim V^a$. We know in fact that, for any $t \in \mathbb{R}_+$, $D_{p(tx)} A$ has the same characteristic polynomial as a (using tx in place of x), hence, letting t go to zero, we conclude that a and $D_0 A$ have the same characteristic polynomial, and this finishes the proof.

3.3. *Proof of Theorem 1.4.* If $f \in \mathrm{Isom}(G/T)$ then, by Theorem 1.2, $H_2(f^*; \mathbb{R}) = w \cdot \sigma(g)$, with $w \in W$ and $g \in D\,\mathrm{graut}(S)$, and, by Proposition 3.1, $\dim[\pi_*^{\mathrm{odd}}(G/T) \otimes \mathbb{Q}]^f = F(\sigma(g))$. Lemma 3.2 applies then to $\sigma(g)$ (see the construction of σ in Proposition 2.5) and clarifies the first assertion of the theorem. Moreover, recalling that $\mu_g = 1$, for any $g \in D\,\mathrm{graut}(S)$, the same lemma gives that $F(\sigma(g))$ equals the number of cycles of g, considered as a permutation of S. The fact that any diagram automorphism respects the isotypic components (see Remark 2.6) implies that $\dim[\pi_*^{\mathrm{odd}}(G/T) \otimes \mathbb{Q}]^f > 1$ unless Φ is isotypic, say $\Phi = \Phi_1 \coprod \ldots \coprod \Phi_l$, with $\Phi_1 = \ldots = \Phi_l$ = irreducible. For such an isotypic root system, the split exact sequence (1) constructed in the proof of Proposition 2.8 restricts to a similar split exact sequence, in which $D\,\mathrm{graut}$ replaces Graphaut. We thus see that $\dim[\pi_*^{\mathrm{odd}}(G/T)\otimes\mathbb{Q}]^f>1$ unless $\pi_0(g)$ is a cycle, say $(1\ 2\ldots l)$, and in this case g acts on S_i as $g_i: S_i\to S_{i+1}$ where g_i is a diagram isomorphism, for any i. Since it is clear that g acts as a cycle on S if and only if $g_l g_{l-1}\ldots g_1$ acts as a cycle on S_1 and since the only cyclic diagram automorphisms of the connected Dynkin diagrams are the identity of the type A_1 and the nontrivial diagram automorphism of the type A_2, the second assertion of our theorem follows [remember that the order of taking products in $D\,\mathrm{graut}(A_2)$ is irrelevant!].

3.4. Remarks. It is shown in [14] that the finiteness assumption on the number of f-invariant geodesics imposes a stronger restriction on f than the one we quoted in the Introduction, namely

$$\dim[\pi_*^{\mathrm{even}}(M)\otimes\mathbb{Q}]^f\leqq\dim[\pi_*^{\mathrm{odd}}(M)\otimes\mathbb{Q}]^f\leqq 1\,.$$

However, in our case the first inequality always holds.

This can be seen as follows: for any $f\in\mathrm{Isom}(M)$, $\pi_*(f)\otimes\mathbb{Q}$ has finite order, by Corollary 1.3.

The desired inequality may thus be rewritten as

$$m_1(\pi_{\mathrm{even}}(f)\otimes\mathbb{Q})\leqq m_1(\pi_{\mathrm{odd}}(f)\otimes\mathbb{Q})$$

which is a direct consequence of a result of Halperin, see [15, Theorem 3] [for a detailed statement of this result, see the proof of Theorem 4.1; the key fact here is that $\dim\pi_*(G/T)\otimes\mathbb{Q}<\infty$, see the construction of the minimal model of G/T given in the proof of Proposition 3.1].

We finally mention the existence of examples of isometries f having only finitely many invariant geodesics and with $H_2(f^*;\mathbb{R})$ as in the statement of our theorem (see also [13]).

4. Fixed Point Free Self-Maps

Before taking up the proof of Theorem 1.5, let us notice two simple general facts. Firstly, we always have $f(M)\leqq n(M)$, for a compact connected manifold M. Second, if $M=G/K$, where G is a compact connected Lie group and K is a closed subgroup, there is a natural free action of the group $N_G(K)/K$ on M which provides a lower bound for the free symmetry of M.

4.1. Proof of Theorem 1.5. If $G=G_1\times G_2$ then $M=M_1\times M_2$ and it is immediate to produce arbitrarily large families of fixed point free self-maps of M without coincidences. From now on we shall treat (i) and (ii) simultaneously, and assume that G is simple in (ii).

We have to show that, given a family $\mathfrak{S}$ consisting of k fixed point free homeomorphisms (respectively self-maps) without coincidences, we must have $k<$ order of W. Notice first that the family $\mathfrak{S}^*=\{H^*(\varphi;\mathbb{R})\,|\,\varphi\in\mathfrak{S}\}$ must consist only of automorphisms of $H^*(G/T;\mathbb{R})$. This follows from the fact that $H^*(\varphi;\mathbb{R})$ is induced by the action on $\mathbb{R}[V]$ of some $a\in\mathfrak{E}$, see Proposition 2.1; if a is not a linear isomorphism then the irreducibility of the Weyl group action on V implies, via Lemma 2.3, that $H^*(\varphi;\mathbb{R})$ is trivial, which contradicts the fact that φ has no fixed points, by the Lefschetz fixed point theorem. Observe next that $\mathfrak{S}^*$ has the same cardinality as $\mathfrak{S}$, by the Lefschetz coincidence theorem [26] (see also [18, Theorem 4.1 and Proposition 4.2]). Considering the family $\mathfrak{F}=\{H_2(\varphi^*;\mathbb{R})\,|\,\varphi\in\mathfrak{S}\}$, consisting of k distinct elements, we know that $L(a)=0$ for any $a\in\mathfrak{F}$ and $L(a,b)=0$ for any $a,b\in\mathfrak{F}$, $a\neq b$ (by the Lefschetz theorems), and $\mathfrak{F}\subset W\cdot\sigma(\text{Graphaut})$ (see the proof of Theorem 1.2), respectively $\mathfrak{F}\subset N_{GL(V)}(W)$ (by the previous remarks); we have denoted here by $L(a,b)$, for $a,b\in\mathfrak{E}$, the Lefschetz coincidence number [26, 18] of the endomorphisms of $H^*(G/T;\mathbb{R})$ which correspond to a and b by Proposition 2.1, and $L(a)=L(a,\text{id})$, for any $a\in\mathfrak{E}$, as usual. The key step of the proof is contained in the following:

Claim. Write, according to Proposition 2.5, $a=w\cdot\sigma(g)$, with $w\in W$ and $g\in\text{Graphaut}$ [respectively $a=\lambda\cdot w\cdot\sigma(g)$, with $\lambda\in\mathbb{R}_+$, $w\in W$, and $g\in\text{Graphaut}$]. If $L(a)=0$ then $w\neq\text{id}$ (respectively $\lambda=1$ and $w\neq\text{id}$).

Granting the claim, we are going to finish quickly the proof of the theorem. Since we know that $\mathfrak{F}\subset(W\backslash\{\text{id}\})\cdot\sigma(\text{Graphaut})$, we may write $\mathfrak{F}=\coprod_g W_g\cdot\sigma(g)$, where

$$W_g=\{w\in W, w\neq\text{id}\,|\,w\cdot\sigma(g)\in\mathfrak{F}\},\quad\text{for any}\quad g\in\text{Graphaut}.$$

If $w\in W_g\cap W_h$ then $L(w\sigma(g),w\sigma(h))=\pm L(\sigma(gh^{-1}))$ see e.g. [18, Proposition 4.2], and, since this number is nonzero by the previous claim, our hypotheses imply that $g=h$. This shows that $k<$ order of W.

The proof of the claim uses rational homotopy theory. We recall the following result, due to Halperin [15, Theorem 3], which relates Lefschetz numbers to rational homotopy fixed points: let X be a 1-connected rational space with the property that $\dim H_*(X)<\infty$ and $\dim\pi_*(X)<\infty$, and let $\varphi:X\to X$ be any map; then $m_1(\pi_{\text{odd}}(\varphi))\geqq m_1(\pi_{\text{even}}(\varphi))$ (compare with Remark 3.4), where m_1 denotes the multiplicity of the eigenvalue 1, and equality holds if and only if $L(\varphi^*)\neq 0$. Setting $X=(G/T)_0$, we may apply this result, taking φ to be the formal map which induces the cohomology automorphism corresponding to a (remember that G/T is a formal space by [24]). Denoting by $(\mathfrak{M},d)$ the minimal model of X (see the proof of Proposition 3.1) and by $\hat{\varphi}$ the minimal model of φ, we may compute m_1 using the induced map on de Rham homotopy, denoted by $\pi^*(\hat{\varphi})$. Recall from the proof of Proposition 3.1 that $\pi^{\text{even}}(\mathfrak{M})=V^*$ and $\pi^{\text{odd}}(\mathfrak{M})=I^+/I^+\cdot I^+$, and that $\pi^{\text{even}}(\hat{\varphi})=a^*$ and $\pi^{\text{odd}}(\hat{\varphi})=\pi(a)$. If $\lambda\neq 1$, it is easy to see that $m_1(a^*)=0$ and $m_1(\pi(a))=0$, due to the fact that $\text{ord}(w\sigma(g))<\infty$, which implies in turn that $L(a)\neq 0$. It remains to show that $L(\sigma(g))\neq 0$, for any $g\in\text{Graphaut}$. By the previous remarks, this is equivalent to $F(\sigma(g))=\dim V^{\sigma(g)}$, in the notations of Sect. 3, and it is a consequence of Lemma 3.2, which is available since $\sigma(g)$ leaves some Weyl chamber invariant by construction (see the proof of Proposition 2.5). The proof of Theorem 1.5 is now complete.

4.2. Remarks. Set $M=G/K$ (G compact connected and K a closed subgroup). If $\text{rk}\,K<\text{rk}\,G$ then it is well-known that $\chi(M)=0$ and consequently $n(M)=\infty$, by [20]. Even if $\text{rk}\,K=\text{rk}\,G$, it may happen that $n(M)=\infty$, as we have just seen (see also [20] for less trivial examples). Concerning assertion (i) of Theorem 1.5, the main point there is that we always have $f(G/T)<\infty$, which in this case easily follows, via Lefschetz theory, from the fact that $\chi(G/T)\neq 0$ and from the finiteness of integral cohomology automorphisms [the estimate $f(G/T)\leqq$ order of W is then a direct consequence of the well-known fact that the order of a finite group acting freely on a compact connected manifold M with $\chi(M) \neq 0$ must divide $\chi(M)$]; we have prefered the alternative proof presented in 4.1 since it simultaneously gives the estimate for $n(G/T)$, when G is simple.

Actually it can be shown that one has $f(M) \leqq |\chi(M)|$, for any compact connected manifold M with $\chi(M) \neq 0$. The finiteness of $f(M)$ is also related to fixed points of torus actions on M, as shown by the following simple observation: if $f(M) < \infty$ then any torus acting on M has a fixed point (just look at the action of a generator of the torus). In particular one has $f(G/K) = \infty$ whenever $\operatorname{rk} K < \operatorname{rk} G$, since in that case it is well-known [3] that G/K admits a circle action without fixed points. Summing up, the finiteness of $f(M)$ characterizes the equal rank situations among compact homogeneous spaces of the form $M = G/K$, or equivalently the situations when $\chi(M) \neq 0$. Actually it happens that in general whenever $\chi(M) = 0$ there is a free action of $\mathbb{Z}$ on M, hence $f(M) = \infty$.

5. Generic Rigidity

Our first task will be to clarify the assertions made in the Introduction in connection with the rigidity of $E_0(M)$ and its relationship with the amount of localization needed to construct generators for $E_0(M)$.

Given a set of primes P, we shall denote by $E_P(M)$ the subgroup of $E_0(M)$ consisting of rationalizations of self-homotopy equivalences of M_P, the localization of M at P; the same construction, applied to the complementary set of primes, will be denoted by $E_{1/P}(M)$; for notational convenience, $E_1(M)$ will stand for the rationalizations of self-homotopy equivalences of M. The isomorphism established in Sect. 2, given by $f \to H_2(f^*; \mathbb{Q})$, will serve to identify $E_0(M)$ and $N_{\mathbb{Q}}$, when $M = G/T$. In the notations of Proposition 2.8, which describes the group structure of $N_{\mathbb{Q}}$, we obviously have $W \subset E_1$ and, as far as the products of grading automorphisms are concerned, we know by [9] that, for any simple G, if $d \in D_{\mathbb{Q}}$ is a prime, then $d \in E_{1/d}$ and it is induced by the corresponding Frobenius isogeny. The generators of $N_{\mathbb{Q}}$ corresponding to graph-automorphisms are settled in the next proposition (plainly it is enough to consider the $\mathbb{Q}$-isotypic case).

5.1. Proposition. *If G is $\mathbb{Q}$-isotypic then $\gamma | E_{1/q}$ is still onto, where q denotes the maximum number of bonds appearing in the Dynkin diagram.*

Proof. The group structure of Graphaut(S) is described by the exact sequence (1) which was derived in the proof of Proposition 2.8. If Φ is not isotypic then all its irreducible components are of type B_r or C_r, for some $r \geqq 3$, and have no nontrivial graph-automorphisms, and our claim amounts to the existence of a homotopy equivalence between $(B_r/T)_{1/2}$ and $(C_r/T)_{1/2}$. This is provided by [9] and comes from the exceptional isogeny in characteristic 2 relating the orthogonal and symplectic groups. Since for an isotypic G the composition $\pi_0 \circ \gamma | E_1$ is plainly onto, we are reduced to the case when G is simple. Since moreover $D\,\mathrm{graut} \subset \gamma(E_1)$ by characteristic zero Lie theory (see the proof of Theorem 1.2), we are finally left with three cases: $\Phi = B_2$, F_4 or G_2, each one with a single graph-automorphism which does not respect the lengths. The first one follows by using the already mentioned homotopy equivalence $(\mathrm{SO}(5)/T)_{1/2} \xrightarrow{\sim} (\mathrm{Sp}(2)/T)_{1/2}$ and the other ones by recalling the self-homotopy equivalences of $(G/T)_{1/q}$ constructed in [8] for $G=F_4$ and G_2 from the exceptional isogenies in characteristic 2 (respectively 3) of these groups (see also [2, Proposition 2.15]).

5.2. Corollary. *For any $G, M=G/T$ has the following property:*

$$\text{given a set of primes } P, \text{ for any } f \in E_0(M) \text{ there exist}$$
$$f_1 \in E_P(M) \text{ and } f_2 \in E_{1/P}(M) \text{ such that } f=f_1 f_2. \qquad (*)$$

Proof. Reduce to the $\mathbb{Q}$-isotypic case, recall the structure of $E_0(M)=N_{\mathbb{Q}}$ and use Proposition 5.1 and the remarks preceding it.

Given the corollary, the theorem below readily implies Theorem 1.6.

5.3. Theorem. *Let M be a 1-connected finite formal complex. If M has the property $(*)$ stated in Corollary 5.2, then M is generically rigid.*

Proof. If $M=G/K$ is a complex flag manifold, the generic rigidity of M was derived in [11] from the assumption that $E_0(M)$ is generated by grading automorphisms together with $N_G(K)/K$; using [9] it is easy to see that this assumption implies the property $(*)$. Our contribution consists in observing that the arguments of [11] still work for an arbitrary 1-connected finite formal complex, provided the property $(*)$ holds. The generic rigidity of M follows immediately if M satisfies the hypotheses of Lemma 1.3 [11] and the conclusion of Lemma 2.2 [11]. The argument showing that Lemma 1.3 is available for our M is the same as in [11]. Writing $P=\{p_1, \ldots, p_n\}$, an easy induction which uses property $(*)$ shows that for any $f_1, \ldots, f_n \in E_0(M)$ there exists $f \in E_{1/P}(M)$ such that $ff_i \in E_{p_i}(M)$, for any i. In particular the conclusion of Lemma 2.2 [11] holds for M.

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Algebraicity Criteria for Compact Complex Manifolds

Thomas Peternell

Mathematisches Institut der Universität Bayreuth, Postfach 101251, D-8580 Bayreuth, Federal Republic of Germany

Introduction

This paper is concerned with the investigation of the obstructions of Moišezon manifolds to be projective, or – in algebraic language – of smooth algebraic spaces of finite type over $\mathbb{C}$ (for example smooth complete abstract varieties over $\mathbb{C}$) to be projective. Since the paper of Moišezon [10] one knows that a Moišezon manifold is projective if and only if it is Kähler. Hence the Kähler property kills all such obstructions. Such an obstruction one could think of is the existence of compact curves in X that are homologous to zero (clearly in a Kähler manifold those curves cannot exist, because integration of a Kähler form over a curve gives a strictly positive number). Indeed, there are 3-dimensional Moišezon manifolds constructed by Hironaka which are not projective; moreover the non-projectivity is shown by proving the existence of such a curve. By a curve we will always mean a finite linear combination $C=\sum n_i C_i$, $n_i \in \mathbb{N}$ and $C_i \subset X$ irreducible reduced curves.

In this paper it is proved that at least in dimension 3 the only obstructions of a Moišezon manifold to be projective are the curves and limits of curves which are homologous to zero, that is: a 3-dimension Moišezon manifold without (effective) curves homologous to zero and with the following property: {there is no effective curve C and no positive closed current $T=\lim_{j\to\infty} \sum_{i=1}^{n_j} \lambda_{ij} T_{C_{ij}}$ ($\lambda_{ij}>0$, $C_{ij}\subset X$ irreducible curves, $T_{C_{ij}}$ is integration over C_{ij}) such that $C+T\sim 0$} is projective. It turns out that the condition $C+T \nsim 0$ must only be verified for curves $C=\sum_{i=1}^{k} \lambda_i C_i$ where $C_1, \dots, C_m$ are finitely many irreducible curves such that $X \setminus \bigcup_{j=1}^{m} C_j$ is quasi-projective.

In the following the idea of the proof is sketched, this will also give an understanding of the difficulties in the higher dimensional case [note also that in dimension 2 any Moišezon manifold is already projective (Kodaira-Chow)]. First one proves a projectivity criterion which generalizes Moišezon's theorem mentioned above, namely one assumes only the existence of a positive (1,1)-form satisfying a weaker condition than closedness, see Sect. 1.

In Sect. 2 we prove a criterion for the existence of such a positive form, the form will exist if there are no positive currents on X of bi-dimension (1,1) which are limits of linear combinations of curves in the weak topology and which are exact. This is related to recent work of Harvey and Lawson [7].

Then we study positive currents which are limits of curves and cohomologous to 0. We prove:

Let $\pi: \hat{X} \to X$ be a blow-up with smooth center S in the Moišezon manifold X such that $\hat{X}$ is projective. If T is a positive current which is cohomologous to 0 and which vanishes on S, then $T=0$. (For the precise formulation see 2.4.) Proceeding further we need the assumption $\dim X=3$. The reason is the following. If $\dim X=3$, we can make X projective by blowing up smooth curves and points. By induction we need only to consider the case of one blow-up $\pi: \hat{X} \to X$ in the smooth curve S. If T is a limit as above, positive and $T \sim 0$, then we decompose $T=\chi_S T+\chi_{X \setminus S} T$. Since S is a curve, a theorem of Siu says that $\chi_S T=\lambda T_S$ with some $\lambda \geqq 0$ [clearly if $\dim S>1$, we have no such conclusion since $\chi_S T$ has not bidimension (1,1)]. We have $\lambda>0$ by Theorem 2.4 above. Now by our assumption we exclude $\lambda T_S+\chi_{X \setminus S} T \sim 0$.

It seems likely that there is a Moišezon manifold χ without effective curves ~ 0 but with a curve S and a positive current T as above such that $\lambda S+T \sim 0$, although no example is known.

In Sect. 4 we prove a generalization of this theorem in higher dimension but we need additional assumptions. Also in Sect. 2 one finds several applications and corollaries of the main result.

1. A Generalization of Moišezon's Projectivity Criterion

Let X be a compact complex manifold of dimension n. For $1 \leqq q \leqq n$ we denote by $A_q(X,k)$ the linear subspace of $H_{2q}(X,k)$ (over k) generated by the homology classes of q-dimensional reduced irreducible subspaces of X, where $k=\mathbb{Q}$ or $k=\mathbb{R}$. Furthermore we denote by $A^{2q}(X,k)$ the linear subspace of $H^{2q}(X,k)$ which is the image of $A_{2n-2q}(X,k)$ under the isomorphism

$$H_{2n-2q}(X,k) \to H^{2q}(X,k)$$

(Poincaré duality). Thus $A^{2q}(X,k)$ is the subspace generated by the fundamental classes of q-codimensional subspaces of X (over k). The group of divisors on X is denoted by $\operatorname{Div}(X)$ and the Picard group by $\operatorname{Pic}(X^+)=H^1(X,\mathcal{O}^*)$.

Lemma 1.1. *Let X be a Moišezon manifold of dimension n (hence there are n algebraically independent meromorphic functions on X, by definition). Then the canonical map*

$$\Phi: A^2(X,k) \to A_2(X,k)^*$$

given by $\Phi([\psi])([C])=\int_C \psi$, is surjective ($k=\mathbb{Q}, \mathbb{R}$).

Proof. It is sufficient to prove this over $\mathbb{Q}$. Let $\varphi: A_2(X,\mathbb{Q}) \to \mathbb{Q}$ be a linear map, or equivalently $\varphi: A^{2n-2}(X,\mathbb{Q}) \to \mathbb{Q}$. One has:

$$A^{2n-2}(X,\mathbb{Q}) \subset H^{n-1,n-1}(X) \cap H^{2n-2}(X,\mathbb{Q}),$$

here we are using the fact that on any Moišezon manifold one has Hodge decomposition. This can be shown using the Frölicher spectral sequence, see e.g. Ueno [16] [or Deligne, Publ. IHES 35, Proposition 5.3 in the case of schemes (that can be carried over immediately)]. By extending by zero, we get a linear map

$$\tilde{\varphi}\colon H^{n-1,n-1}(X) \cap H^{2n-2}(X, \mathbb{Q}) \to \mathbb{Q}.$$

[in fact one can show $A^{2n-2}(X, \mathbb{Q}) = H^{n-1,n-1}(X) \cap H^{2n-2}(X, \mathbb{Q})$]. Now consider for a moment the following general situation: let V_1, V_2 be finite-dimensional $\mathbb{Q}$-vector spaces with $V_1 \subset V_2$. Let $\alpha\colon V_1 \to \mathbb{Q}$ be linear. Extend α first to $V_1 \otimes_{\mathbb{Q}} \mathbb{R}$ by linearity and then to $V_2 \otimes_{\mathbb{Q}} \mathbb{R}$ by zero; we get $\alpha_1\colon V_2 \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}$. On the other hand, extend first α by zero to V_2 and then by linearity to $V_2 \otimes_{\mathbb{Q}} \mathbb{R}$ getting a linear map α_2. Then clearly: $\alpha_1 = \alpha_2$. We apply this in the case $V_1 = H^{n-1,n-1}(X) \cap H^{2n-2}(X, \mathbb{Q})$ and $V_2 = H^{2n-2}(X, \mathbb{Q})$. Then $V_2 \otimes_{\mathbb{Q}} \mathbb{R} = H^{2n-2}(X, \mathbb{R})$. Hence we get an extension $\hat{\varphi}\colon H^{2n-2}(X, \mathbb{R}) \to \mathbb{R}$ of φ which is rationally defined. Complexify $\hat{\varphi}$. Since we have

$$H^{2n-2}(X, \mathbb{C}) \simeq H^{n,n-2}(X) \otimes H^{n-1,n-1}(X) \otimes H^{n-2,n}(X)$$

and the complexified $\hat{\varphi}$ is zero on $H^{n,n-2}(X)$ and $H^{n-2,n}(X)$ by construction, $\hat{\varphi}$ is given by a real (1,1)-form $[\omega] \in H^{1,1}(X)$. On the other hand $\hat{\varphi}$ is rationally defined and hence $\hat{\varphi}$ is given by $[\omega'] \in H^2(X, \mathbb{Q})$. Thus $\gamma := [\omega] = [\omega'] \in H^{1,1}(X) \cap H^2(X, \mathbb{Q})$. Now on an arbitrary complex manifold any such γ is a rational multiple of the Chern class of a line bundle on X. Furthermore, $\mathrm{Pic}(X) = \mathrm{Div}(X)$ on any Moišezon manifold. Thus we see $\gamma \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, and the proof is finished.

Remark. Lemma 1.1 has been proved by another way by Moišezon [10]. We included a proof here in order to simplify matters and also to see what is really involved. Looking at the proof we see that instead of assuming "X Moišezon" we only need $\mathrm{Div}(X) = \mathrm{Pic}(X)$ and that we have a "natural" Hodge decomposition on X (see Deligne, Publ. IHES 35, 5.2.7 and 5.3.1).

Corollary 1.2. *Let X be a compact complex manifold with* $\mathrm{Pic}(X) = \mathrm{Div}(X)$ *such that Hodge decomposition holds on X. Then the conclusion of Lemma 1.1 still holds.*

In order to prove the main theorem of this section we first state:

Theorem 1.3. *Let X be a Moišezon manifold and suppose that there exists an $\varepsilon > 0$* and a linear map $\varphi\colon A_2(X, \mathbb{Q}) \to \mathbb{Q}$ *such that $\varphi([C]) \geqq \varepsilon m(C)$ for all irreducible curves $C \subset X$, where $m(C) := \sup_{p \in C} m_p(C)$, $m_p(C)$ denoting the multiplicity of C in p. Then X is projective.*

Proof. By Lemma 1.1 there exists $D \in A^2(X, \mathbb{Q})$ such that: $\varphi([C]) = (D \cdot C)$ for all $[C] \in A_2(X, \mathbb{Q})$. Multiplying φ by a suitable $m \in \mathbb{N}$ we may assume $D \in \mathrm{Div}(X)$. Then the ampleness of D and thus the projectivity of X follows from Seshadri's criterion, which goes as follows: let X be a Moišezon manifold and $D \in \mathrm{Div}(X)$ such that $(D \cdot C) \geqq \varepsilon m(C)$ for all curves $C \subset X$ with $\varepsilon > 0$ independent of C. Then D is ample. For a proof see Hartshorne [5]; although there only the case of a nonsingular complete scheme is treated, the proof for a Moišezon manifold is exactly the same.

Theorem 1.4. *Let X be a Moišezon manifold. Assume there exist a real* (1,1)*-form* ω *and a real* 2*-form* ψ *on* X such that:

1) ω *is positive definite,*
2) $d(\omega-\varphi)=0$,
3) $\int_C \varphi=0$ *for all curves* $C\subset X$.

Then X is projective.

Proof. We want to apply Theorem 1.3. So consider the linear map $\psi\colon A_2(X,\mathbb{R})\to\mathbb{R}$, $\psi([C]):=\int_C \omega$. Because of Conditions 2) and 3), ψ is well defined.

First we make ψ rational using a method of Moišezon [10]. By Lemma 1.1 there exists $D\in\mathrm{Div}(X)\otimes_{\mathbb{Z}}\mathbb{R}=A^2(X,\mathbb{R})$ such that $\psi([C])=(D\cdot C)$ for all curves $V\subset X$. Write $D=\sum r_i\cdot D_i$ with $r_i\in\mathbb{R}$ and $D_i\in\mathrm{Div}(X)$. For all i chose rational sequences $(r_{i\nu})_\nu$ converging to r_i. Let $D^{(\nu)}:=\sum r_{i\nu}D_i\in A_{2n-2}(X,\mathbb{Q})$ and $\alpha_\nu:=\sum(r_i-r_{i\nu})\,c_1(D_i)$. Here $c_1(D_i)$ means a (1,1)-form representing the Chern class of D_i.

Since $(\alpha_\nu)_\nu$ converges compactly to 0 there exists a ν_0 such that for all $\nu\geqq\nu_0$ the form $\omega-\alpha_\nu$ is positive definite. Take $\nu\geqq\nu_0$ and let $\tilde\omega:=\omega-\alpha_\nu$. Then $\tilde\omega$ has obviously the same properties as ω (i.e. conditions 1)–3) are valid), furthermore $\int_C\tilde\omega=(D\cdot C)$ for all C. Hence we may assume $D\in\mathrm{Div}(X)$ a priori.

In order to be able to apply theorem 1.3 we must show $\psi([C])\geqq\varepsilon\, m(C)$ for curves and a fixed $\varepsilon>0$.

We claim that there exists $\varepsilon>0$ such that the following holds: if $p\in X$ is arbitrary and $\pi_p\colon\hat X_p\to X$ is the blowing-up of X at p and α_p is a suitable representative of the first Chern class of the exceptional divisor $E_p=\pi^{-1}(p)$, then $\hat\omega_p:=\pi_p^*(\omega)-\varepsilon\alpha_p$ is a positive definite (1,1)-form.

For fixed p this is well known. In order to get ε independent of p, one puts the manifolds $\hat X_p$ into a complex analytic family parametrized by X (local families are sufficient) and concludes by means of continuity and compactness. One obtains the family as follows: take $X\times X$ and blow up the diagonal. Restricting to $X\times\{p\}$ one gets just $\hat X_p$.

Now choose such an ε, we may assume ε rational. Write $\varepsilon=k/m$, $k,m\in\mathbb{N}$. Let $C\subset X$ be an arbitrary irreducible reduced curve, $p\in C$ a point and $\hat C$ the strict transform of C under π_p. Since we have:

$$\int_{\hat C}\pi_p^*(\omega)=\int_C\omega=(D\cdot C)$$

and

$$(\hat C\cdot E_p)=m_p(C)$$

we get:

$$0<\int_{\hat C} m\hat\omega_p=\int_{\hat C}(m\pi_p^*(\omega)-k\alpha_p)=-\int_{\hat C}c_1(kE_p)+m\cdot(D\cdot C)=-km(C)+m(D\cdot C),$$

thus $m(D\cdot C)\geqq km_p(C)$ for all $p\in C$, i.e. $(D\cdot C)\geqq\varepsilon m(C)$, and the proof is complete applying Theorem 1.3.

Remark. Theorem 1.4 generalizes Moišezon's theorem: If a Moišezon manifold is Kähler then it is projective. In Sect. 4 we will prove a (partial) generalization of Theorem 1.4 avoiding the assumption "X Moišezon" (i.e. replacing it).

2. The Main Result

We begin by fixing notations. Let X be a compact complex manifold. (1) We let $\mathscr{E}^{p,q}(X)$ be the space of C^∞-forms of type (p,q) on X, $\mathscr{E}^r(X)$ the space of C^∞-forms of degree r. The index "$\mathbb{R}$" will always mean real forms. All these spaces are equipped with the usual Fréchet topologies. Define $\mathscr{D}^{p,q}(X) := \mathscr{E}^{p,q}(X)'$, the space of currents of bidimension (p,q) or, for short, (p,q)-currents. (2) $\varphi \in \mathscr{E}^{p,p}(X)$ is said to be positive, if for all $x \in X$ the vector $\varphi(x)$ is in the cone generated by $(i \cdot u_1 \wedge \bar{u}_1) \wedge \ldots \wedge (i \cdot u_p \wedge \bar{u}_p)$, $u_i \in \wedge^{1,0} T_x^*(X)$. (3) $T \in \mathscr{D}^{p,p}(X)$ is said to be positive, iff $T(\varphi) \geqq 0$ for all positive forms φ.

Since we are only interested in the cases $p = n-1$ and $p = 1$, where $n = \dim X$, we have not to worry about different notions of positivity [6, 8].

We denote by $\mathscr{P}^p(X)$ the cone of positive (p,p)-currents, by $\mathscr{P}_c^p(X)$ the cone of positive closed (p,p)-currents, by $\mathscr{P}_a^p(X)$ the cone of the positive currents which are weak limits $\lim_j \sum_{i=1}^{n_j} \lambda_{ij} T_{C_{ij}}$, where $\lambda_{ij} \in \mathbb{R}$ and C_{ij} are irreducible curves in X. Here we define for any analytic set $Z \subset X$ of pure dimension p the associated (p,p)-current T_Z by $T_Z(\varphi) = \int_Z \varphi$.

Finally we define $\mathscr{P}_b^p(X)$ to be the cone of those currents in $\mathscr{P}_a^p(X)$, where we can choose all λ_{ij} to be positive. The following space will be important in what follows: $B := \{T \in \mathscr{D}_{\mathbb{R}}^{1,1}(X) \mid \text{there exists } S \in \mathscr{D}^3(X) \text{ such that } T \text{ is the (1,1)-part of } dS\}$.

Of course, the index "$\mathbb{R}$" denotes real currents. The starting point in this section is

Theorem 2.1 (Harvey-Lawson). *Let X be a compact complex manifold. Then there exists $\omega \in \mathscr{E}_{\mathbb{R}}^2(X)$ such that*

1) $d\omega = 0$,
2) *the* (1,1)*-part* $\omega^{1,1}$ *of* ω *is positive definite,*
3) *there exists a form* α *with* $\omega^{2,0} = \partial\alpha$

if and only if the condition $\mathscr{P}_c^1(X) \cap B = \{0\}$ *is satisfied.*

Proof. Harvey-Lawson [7, Theorem 38].

For our purposes we need the following slightly changed version of Harvey and Lawson's theorem.

Theorem 2.2. *Let X be a compact manifold. Assume that $\mathscr{P}_a^1(X) \cap B = \{0\}$. Then there exist forms $\omega \in \mathscr{E}_{\mathbb{R}}^{1,1}(X)$ and $\varphi \in \mathscr{E}_{\mathbb{R}}^2(X)$ such that*

1) ω *is positive definite,*
2) $d(\omega - \varphi) = 0$,
3) $\int_C \varphi = 0$ *for all curves* $C \subset X$.

Proof. The proof will follow the main idea of that one in [7]. If $A \subset \mathcal{D}^{1,1}_{\mathbb{R}}(X)$ we define $A^\perp := \{\alpha \in \mathcal{E}^{1,1}_{\mathbb{R}}(X) \mid T(\alpha) = 0 \text{ for all } T \in A\}$. Let $Z := \{T \in \mathcal{D}^{1,1}_{\mathbb{R}}(X) \mid dT = 0, T = \lim\limits_j \sum\limits_{i=1}^{n_j} \lambda_{ij} T_{C_{ij}}$ in the weak topology with $\lambda_{ij} \in \mathbb{R}$ and irreducible curves $C_{ij} \subset X$. Our assumption says $\mathcal{P}^1(X) \cap B \cap Z = \{0\}$. Hence by Hahn-Banach (introduce a suitable compact set in order to be able to apply Hahn-Banach, see [7, Theorem 14]) we get a linear form on $\mathcal{D}^{1,1}_{\mathbb{R}}(X)$ which is positive on $\mathcal{P}^1(X)$ and 0 on $B \cap Z$ ($B \cap Z$ is closed). This implies the existence of $\omega \in \mathcal{E}^{1,1}_{\mathbb{R}}(X)$ which is positive definite such that $\omega \in (B \cap Z)^\perp$. If we can show:

$$(*) \quad (B \cap Z)^\perp = B^\perp + Z^\perp,$$

we may choose $\varphi \in Z^\perp$ such that $\omega - \varphi \in B^\perp$. Since $B^\perp = \{\psi \in \mathcal{E}^{1,1}_{\mathbb{R}}(X) \mid d\psi = 0\}$, we conclude $d(\omega - \varphi) = 0$. Obviously $\varphi \in Z^\perp$ implies $\int_C \varphi = 0$ for all curves C. So it remains only to show (*).

For this we have to verify that $B^\perp + Z^\perp$ is closed in $\mathcal{E}^{1,1}_{\mathbb{R}}(X)$. From now on our proof differs essentially form that one in [7]. Consider $F := \mathcal{E}^{1,1}_{\mathbb{R}}(X)/d\mathcal{E}^1_{\mathbb{R}}(X) \cap \mathcal{E}^{1,1}_{\mathbb{R}}(X)$. F is again a Fréchet space. Consider the quotient map κ: $\mathcal{E}^{1,1}_{\mathbb{R}}(X) \to F$. Defining $B' := \kappa(B^\perp)$ and $Z' := \kappa(Z^\perp)$, we obviously have $\kappa^{-1}(B') = B^\perp$ and $\kappa^{-1}(Z') = Z^\perp$. We may consider B' as subspace of $H^2(X, \mathbb{R})$, hence it is finite-dimensional. Since $Z^\perp$ is closed in $\mathcal{D}^{1,1}_{\mathbb{R}}(X)$ it is easily seen that Z' is closed too. But then it is clear that $B' + Z'$ must be closed in F. Observing $\kappa^{-1}(B' + Z') = \kappa^{-1}(B') + \kappa^{-1}(Z')$, the proof is finished.

Combining Theorem 2.2 and Theorem 1.4 we get:

Corollary 2.3. *Let X be a Moišezon manifold satisfying $\mathcal{P}^1_a(X) \cap d\mathcal{D}^3(X) = \{0\}$, then X is projective.*

Proof. We must only show that for any Moišezon manifold X the condition $\mathcal{P}^1_a(X) \cap d\mathcal{D}^3(X) = \{0\}$ implies $\mathcal{P}^1_a(X) \cap B = \{0\}$. Take a hypersurface $H \subset X$ and let η_H be the fundametal class of H. Take a current $T \in \mathcal{P}^1_a(X) \cap B$. Then: $T(\eta_H) = (\partial S + \bar{\partial} R)(\eta_H) = 0$, where S and R currents such that $T = \partial S + \bar{\partial} R$ (this is exactly the condition for T to be the (1,1)-component of dQ). It follows that $([T].\,\alpha) = 0$ for all $\alpha \in A^2(X) = \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, $[T]$ denoting the cohomology class of T in $H^{2n-2}(X, \mathbb{R})$. An application of Lemma 1.1 gives the conclusion.

Remark. Clearly the assumption of corollary 2.3 is also necessary for the projectivity of X: take a Kähler metric with (1,1)-form ω and let $T \in \mathcal{P}^1_a(X)$, $T = dQ$ with a current Q. Now $T(\omega) \geqq 0$ and $T(\omega) = 0$ iff $T = 0$; on the other hand $T(\omega) = dQ(\omega) = Q(d\omega) = 0$.

The first main result of this section is

Theorem 2.4. *Let X be a Moišezon manifold, π: $\hat{X} \to X$ a blowing-up of X with a smooth centers such that $\hat{X}$ is projective. Let $T \in \mathcal{P}^1_a(X) \cap d\mathcal{D}^3(X)$ with $\chi_S T = 0$. Then $T = 0$.*

Here the current $\chi_S T$ is defined as follows. Take C^∞-functions ϱ_ν: $X \to \mathbb{R}$ with $\operatorname{supp}(\varrho_\nu) \subset X \setminus S$ such that $0 \leqq \varrho_\nu \leqq 1$ for all ν and such that $(\varrho_\nu) \nearrow \chi_{X \setminus S}$, the characteristic function of $X \setminus S$. Then let $\chi_S T(\varphi) := \lim\limits_\nu T((1 - \varrho_\nu)\varphi)$ for all φ. Of course we define $\chi_{X \setminus S} T$ by $\chi_{X \setminus S} T := T - \chi_S T$.

Proof. We may assume that S is not a point, because otherwise X is projective and the assertion is clear.

We choose a Kähler metric on $\hat{X}$ with associated (1,1)-form λ. We identify λ with the current T_λ given by $T_\lambda(\varphi)=\int_X \lambda \wedge \varphi$; more generally we consider $\mathscr{E}_{\mathbb{R}}^{n-1,n-1}(X)$ as subspace of $\mathscr{D}_{\mathbb{R}}^{1,1}(X)$. Then $\tilde{\lambda}:=\pi_*(\lambda)$ is a positive closed current on X and can be considered as smooth (1,1)-form on $X\setminus S$, but of course $\tilde{\lambda}$ is not smooth everywhere.

Let T be a (1,1)-current on X. Choose a finite covering (U_i) of X by open Stein coordinate charts $U_i\subset X$. Choose an associated partition of unity (ϱ_i). For $0<\varepsilon<1$ let η_ε^i be the usual smoothing kernel on U_i (see e.g. [4, 9]) and let $\eta_\nu^i:=\eta_{1/\nu}^i$ for $\nu\in\mathbb{N}$. Let $\psi_\nu^i:=T*\eta_\nu^i$ be the "canonical" smoothing of T/U_i ($*$ denotes convolution). Define $\psi_\nu:=\sum_i \varrho_i\psi_\nu^i$. Then $\psi_\nu\to T$ in the weak topology; if T is positive, then all ψ_ν are positive as currents.

Define $T(\tilde{\lambda}):=\lim\int_X \psi_\nu\wedge\tilde{\lambda}$, if the limit exists and $T(\tilde{\lambda})=\mathscr{D}$ otherwise (clearly $\int_X \psi_\nu\wedge\tilde{\lambda}=\int_X \pi^*(\psi_\nu)\wedge\lambda<\infty$).

Choose C^∞-functions $g_\nu\colon X\to\mathbb{R}$ with $0\leqq g_\nu\leqq 1$ such that $g_\nu\nearrow\chi_{X\setminus S}$. Let $T'(\lambda):=\lim_\nu T(g_\nu\cdot\tilde{\lambda})$, if the limit exists and $T'(\lambda)=\infty$ otherwise (we are interested in the positive case, hence we can say that either the limit exists (after passing to a subsequence) or the limit is $+\infty$). We want to give another interpretation of $T'(\lambda)$. Put $T'':=(\pi|\hat{X}\setminus\hat{S})^{-1}{}_*(T|X\setminus S)$ (with $\hat{S}:=\pi^{-1}(S)$), then T'' is a current on $\hat{X}\setminus\hat{S}$. If T'' has locally finite volume near $\hat{S}$, the trivial extension T' of T'' on $\hat{X}$ exists and so we may form $T'(\lambda)$ which is clearly the same number as defined before. Now let T be positive. Then T' will already exist if we only know the existence of $\lim_\nu T(g_\nu\tilde{\lambda})$ (by [6, 1.23]). Now for all μ and all $\nu_0\in\mathbb{N}$ the following inequality holds:

$$\int_X \psi_\mu\wedge g_{\nu_0}\tilde{\lambda}\leqq \lim_\nu\int_X \psi_\mu\wedge g_\nu\tilde{\lambda}=\int_X \psi_\mu\wedge\tilde{\lambda},$$

hence we get: $T(g_\nu\tilde{\lambda})\leqq T(\tilde{\lambda})$ for all ν, hence $T(\tilde{\lambda})<\infty$ implies the existence of T' and the inequality $T'(\lambda)\leqq T(\tilde{\lambda})$ holds.

We introduce the following spaces and cones.

$$\tilde{\mathscr{D}}:=\tilde{\mathscr{D}}_{\mathbb{R}}^{1,1}(X):=\{T\subset\mathscr{D}=\mathscr{D}_{\mathbb{R}}^{1,1}(X)\,|\,\chi_S T=0,\ T(\tilde{\lambda})<\infty,\ T'(\lambda)<\infty\}$$

and $\tilde{\mathscr{P}}_a^1(X):=\mathscr{P}_a^1(X)\cap\tilde{\mathscr{D}}$.

Then obviously $\mathscr{E}_{\mathbb{R}}^{n-1,n-1}(X)\subset\tilde{\mathscr{D}}$ (one has for a smooth current T: $T'=T_{\pi^*(\psi)}$, if $T=T_\psi$).

Now the plan of the proof can be given as follows. In 1) we will show that for any irreducible curve C not contained in S, the associated current $T=T_C$ is in $\tilde{\mathscr{P}}_a^1(X)$ and satisfies $([T'].[\hat{S}])=T(\tilde{\lambda})-T'(\lambda)$.

In 2) we will prove $\tilde{\mathscr{P}}_a^1(X)\cap d\mathscr{D}^3(X)=\{0\}$; in 3) we derive from 1) and 2) the existence of $\omega\in\mathscr{E}_{\mathbb{R}}^{1,1}(X)$ and $\varphi\in\mathscr{E}_{\mathbb{R}}^2(X)$ such that ω is positive definite on $X\setminus S$, $d(\omega-\varphi)=0$ and $\int_C\varphi=0$ for all curves C not contained in S. Using 3) it will be easy to prove the theorem (4)).

1) Let $T\in\tilde{\mathscr{P}}_a^1(X)$. First we want to compute $\int_X\psi_\nu\wedge\tilde{\lambda}$. Since λ is a positive closed current there are plurisubharmonic functions $\varphi_i\colon U_i\to[-\infty,R_0)$ (with a suitable

R_0 independent of i) such that $\tilde{\lambda} = \sqrt{-1}\,\partial\bar{\partial}\tilde{\varphi}_i$ on U_i in the sense of currents [4]. The restrictions $\varphi_i | U_i \setminus S$ are smooth and $\varphi_i = -\infty$ on S. Write $\varphi_i = \log \tilde{\varphi}_i$ on U. Then $\tilde{\varphi}_i \geqq 0$ on U_i and furthermore $\tilde{\varphi}_i$ is smooth on U. This is because $\tilde{\lambda}$ has locally singularities like $1/(|z_1|^2 + \ldots + |z_r|^2)$ if S is given locally by $z_1 = \ldots = z_r = 0$. Hence φ_i is essentially $\log(|z_1|^2 + \ldots + |z_r|^2)$; the exact proof is just computation in local coordinates.

Let $0 < r < R < R_0$. Then we define the following sets:

$$\Omega_i(r) := \{z \in U_i \,|\, \tilde{\varphi}_i(z) < r\};$$

$$\Omega_i(r, R) := \{z \in U_i \,|\, r < \varphi_i(z) < R\}.$$

Now the following basic formula holds for any current Q of bidimension (1,1) which is of order 0 as well as dQ and $\partial\bar{\partial}Q$ and for which $\operatorname{supp}(Q) \cap \Omega_i(r) \subset\subset U_i$ for all i:

$$\int\limits_{\Omega_i(r,R)} Q \wedge \partial\bar{\partial}\log\tilde{\varphi}_i = R^{-1} \int\limits_{\Omega_i(R)} Q \wedge \partial\bar{\partial}\tilde{\varphi}_i - r^{-1} \int\limits_{\Omega_i(r)} Q \wedge \partial\bar{\partial}\tilde{\varphi}_i$$
$$- \int\limits_r^R (t^{-1} - R^{-1})\,dt \int\limits_{\Omega_i(t)} \partial\bar{\partial}Q - (r^{-1} - R^{-1}) \int\limits_0^r dt \int\limits_{\Omega_i(t)} \partial\bar{\partial}Q.$$

Here $\int\limits_{\Omega} Q \wedge \alpha$ is defined as $Q(\chi_\Omega \alpha)$. Formula (1 a) is a special case of Proposition 1 in Skoda [15], see also Demailly [2].

We are interested in the case $\operatorname{supp}(Q) \subset\subset \Omega_i(R)$. Then (1 a) has the following easy form:

$$\int\limits_{\Omega_i(r,R)} Q \wedge \partial\bar{\partial}\log\tilde{\varphi}_i = -r^{-1} \int\limits_{\Omega_i(r)} Q \wedge \partial\bar{\partial}\tilde{\varphi}_i - r^{-1} \int\limits_0^r dt \int\limits_{\Omega_i(t)} \partial\bar{\partial}Q - \int\limits_r^R t^{-1}\,dt \int\limits_{\Omega_i(t)} \partial\bar{\partial}Q. \tag{1b}$$

This follows at once from Skoda's proof.

Using (1 b) we get:

$$T(\tilde{\lambda}) = \lim_\nu \sum_i \int\limits_{U_i} \varrho_i \psi_\nu^i \wedge \tilde{\lambda} = \lim_\nu \lim_{r\to 0} \sum_i \int\limits_{\Omega_i(r,R)} \varrho_i \psi_\nu^i \wedge \tilde{\lambda}$$
$$= \alpha \lim_\nu \lim_r {}^x r^{-1} \sum_i \int\limits_{\Omega_i(r)} \varrho_i \psi_\nu^i \wedge \partial\bar{\partial}\tilde{\varphi}_i + r^{-1} \sum_i \int\limits_0^r dt \int\limits_{\Omega_i(t)} \partial\bar{\partial}\varrho_i \wedge \psi_\nu^i$$
$$\left. + \sum_i \int\limits_r^R t^{-1}\,dt \int\limits_{\Omega_i(t)} \partial\bar{\partial}\varrho_i \wedge \psi_\nu^i \right\}, \tag{1c}$$

where $\alpha = \sqrt{-1}$.

Since $\lim_r r^{-1} \operatorname{vol}(\Omega_i(r)) = \lim r^{-1} \int\limits_0^r dt \operatorname{vol}(\Omega_i(t)) = 0$ (the volume with respect to the euclidean metric), we get from (1 c) easily

$$T(\tilde{\lambda}) = \alpha \lim_\nu \lim_r \sum_i \int\limits_r^R t^{-1}\,dt \int\limits_{\Omega_i(t)} \partial\bar{\partial}\varrho_i \wedge \psi_\nu^i. \tag{1d}$$

Using $T'(\lambda) = \lim\limits_{r\to 0} \lim\limits_{\nu\to\infty} \sum\limits_i \int\limits_{\Omega_i(r,R)} \varrho_i \psi_\nu^i \wedge \tilde{\lambda}$, we get similarly

$$T'(\lambda) = \alpha \lim_r \lim_\nu \left\{ r^{-1} \sum_i \int\limits_{\Omega_i(r)} \varrho_i \psi_\nu^i \wedge \partial\bar{\partial}\tilde{\varphi}_i + \sum_i \int\limits_r^R t^{-1}\,dt \int\limits_{\Omega_i(t)} \partial\bar{\partial}\varrho_i \wedge \psi_\nu^i \right\}. \tag{1e}$$

Now we restrict to the case $T=T_C$ with an irreducible curve $C\subset X$. In this case, $T'=T_{\hat C}$, $\hat C$ denoting the strict transform of C in X; here we assume that the curve C is not contained in S.

Denoting by $\Phi:=\sum_j dz_j\wedge d\bar z_j$ the euclidean volume form on U_i we have the following estimate:

$$\left|\int_{\Omega_i(t)}\partial\bar\partial\varrho_i\wedge\psi_\nu^i\right|=\left|\int_X(\chi_{\Omega_i(t)}\partial\bar\partial\varrho_i*\eta_\nu^i)\wedge T\right|\leqq M\int_{\Omega_i(t)}\Phi\wedge T \tag{1f}$$

with a constant M independent of ν.

This estimate is based on an inequality which can be found in Skoda [15] (inequality (10)) or in Harvey [6].

Now $\int_{\Omega_i(t)}\Phi\wedge T=\int_{\Omega_i(t)\cap C}\Phi$, hence $\lim\int_r^R t^{-1}\,dt\int_{\Omega_i(t)}\Phi\wedge T<\infty$.

This is probably well-known. If C is smooth and intersecting S transversely it is trivial; otherwise one gets a very similar situation by some blowing up's (such that C becomes smooth and intersecting only transversely – but the map φ_i is more difficult). But also a direct proof in the "singular" case can be given. Note that the above limit does not exist if $C\subset S$. Note also that $\int_{\Omega_i(t)}\Phi\wedge T\sim 0(t^{1+\varepsilon})$ as $t\to 0$ with an $\varepsilon>0$ if $C\not\subset S$ (and $\varepsilon=0$ if $C\subset S$).

Now we know $T(\tilde\lambda)<\infty$, moreover from (1d) and (1e) we deduce

$$T(\tilde\lambda)-T'(\lambda)=\alpha\lim_r\lim_\nu\sum_i\int_{\Omega_i(r)}\varrho_i\psi_\nu^i\wedge\bar\partial\partial\tilde\varphi_i \tag{1g}$$

using the theorem of dominated convergence and (1f). So we have:

$$T(\tilde\lambda)-T'(\lambda)=\alpha\lim_r r^{-1}\sum_i\int_{\Omega_i(r)\cap C}\varrho_i\bar\partial\partial\tilde\varphi_i. \tag{1h}$$

Now we normalize the Kähler form λ such that $\int_F\lambda=1$ for all fibers F of $\pi|\hat S:\hat S\to S$. Then the last limit is nothing than $(\hat S\cdot\hat C)$, i.e. we claim:

$$T(\tilde\lambda)-T'(\lambda)=(\hat S\cdot\hat C). \tag{1i}$$

This formula generalizes the well-known formula for the Lelong number $\Theta(T_C,0)=m_C(0)=$ multiplicity of the curve C in the point 0 (note that in the case S being a point 0 we have $T(\tilde\lambda)-T'(\lambda)=\Theta(T_C,0)$), compare [4].

A proof of (1i) can be given as follows. First we note that the limit $T(\tilde\lambda)$ does not depend on the choice of the covering (U_i) and the partition of unity (ϱ_i) as one checks easily. Now we smooth T by closed forms: there are C^∞-forms ψ'_ν, $\nu\in\mathbb{N}$, such that $\bar\partial\psi'_\nu=0$ for all $\nu\in\mathbb{N}$, moreover $[\psi'_\nu]=[T]$ for all ν and such that $\psi'_\nu\to T$ weakly [4]. Let $\psi'^i_\nu:=\varrho_i\psi'_\nu$.

Then we have formula (1h) also for ψ'_ν instead of ψ_ν. All formal computations are exactly the same (the positivity of ψ_ν is never used), we have only to worry about the estimate (1f). But first we can choose the covering and the partition of unity appropriately: since $S\cap C$ is finite (if empty we have nothing to prove) we choose (U_i) and (ϱ_i) such that for all $p\in S\cap C$ there exists exactly one $i_0=i_0(p)$ with $p\in U_{i_0}$. Consequently $\varrho_{i_0}=1$ near p and $\varrho_i=0$ near p for $i\neq i_0$. Let $V:=\{z\,|\,\varrho_{i_0}(z)=1\}$. Then $\Omega_{i_0}(t)\cap C\subset\mathring V$ if t is small. Hence

$$\int_{\Omega_{i_0}(t)\cap C}\partial\bar\partial\varrho_{i_0}\wedge\psi'^{i_0}_\nu=0$$

if t is small.

So we get a constant M such that

$$\left| \int_{\Omega_{i_0}(t) \cap C} \partial\bar{\partial}\varrho_{i_0} \wedge \psi_v'^{i_0} \right| \leqq M \left| \int_{\Omega_{i_0}(t)} \Phi \wedge T \right|$$

for all t.

We proceed in this manner for all $p \in S \cap C$ getting the desired inequality (there is nothing to prove for those i with $S \cap C \cap U_i = \varphi$).

Letting $\tilde{T}(\tilde{\lambda}) = \lim_v \int_X \psi_v' \wedge \tilde{\lambda}$, we have proved:

$$\tilde{T}(\tilde{\lambda}) - T'(\lambda) = \alpha \lim_r r^{-1} \sum_i \int_{\Omega_i(r) \cap C} \varrho_i \bar{\partial}\partial\tilde{\varphi}_i. \tag{1h'}$$

Comparing with (1h) we conclude: $\tilde{T}(\tilde{\lambda}) = T(\tilde{\lambda})$. By construction we have $\int_X \psi_v' \wedge \tilde{\lambda} = ([\psi_v'].\ [\tilde{\lambda}]) = ([T].[\tilde{\lambda}])$ for all v, hence $\tilde{T}(\tilde{\lambda}) = ([T].[\tilde{\lambda}])$. Then a straight forward calculation of cohomology classes shows:

$$\tilde{T}(\tilde{\lambda}) - T'(\lambda) = ([C].[\tilde{\lambda}]) - ([\hat{C}].[\lambda]) = (\hat{S}.\hat{C}),$$

hence the same holds for $T(\tilde{\lambda})$ instead of $\tilde{T}(\tilde{\lambda})$.

This proves assertion (1i) and we see that $T = T_C \in \tilde{\mathscr{P}}_a^1(X)$ for all curves C not contained in S and $([T'].[\hat{S}]) = T(\tilde{\lambda}) - T'(\lambda)$. 2) Now we want to prove: $\tilde{\mathscr{P}}_a^1(X) \cap d\mathscr{D}^3(X) = \{0\}$.

Take $T \in \tilde{\mathscr{P}}_a^1(X) \cap d\mathscr{D}^3(X)$. Say $T = \lim_{j \to \infty} \sum_{i=1}^{n_j} \lambda_{ij} T_{C_{ij}}$ with $\lambda_{ij} \in \mathbb{R}$ and irreducible curves C_{ij}. Since $\chi_S T = 0$, we may assume that all C_{ij} are not contained in S.

Then $T' = \lim_{j \to \infty} \sum_{i=1}^{n_j} \lambda_{ij} T_{\hat{C}_{ij}}$, $\hat{C}_{ij}$ denoting as usual the strict transform of C_{ij} in $\hat{X}$.

For the cohomology classes (in $H^4(\hat{X}, \mathbb{R})$) we have: $[\hat{C}_{ij}] = \pi^*([C_{ij}]) + \kappa_{ij}[F]$, where F is an irreducible curve in a fiber of $\pi/\hat{S}$. All $\kappa_{ij} \leqq 0$, because $0 \leqq (\hat{C}_{ij} \cdot \hat{S}) = \kappa_{ij}(F \cdot \hat{S})$ and $(F \cdot \hat{S}) < 0$. Hence we get:

$$[T'] = \pi^*[T] + \left(\lim_j \sum_{i=1}^{n_j} \lambda_{ij}\kappa_{ij}\right)[F],$$

and denoting $\kappa := \lim_j \sum_i \lambda_{ij}\kappa_{ij}$ and using $T \sim 0$:

$$[T'] = \kappa[F]. \tag{2a}$$

The formula from a) yields: $([T'] \cdot [\hat{S}]) = T(\tilde{\lambda}) - T'(\lambda)$ (reduce the problem to the case that all $\lambda_{ij} > 0$). Then $T(\tilde{\lambda}) \geqq T'(\lambda)$ gives

$$T'(\eta_{\hat{S}}) = ([T'] \cdot [\hat{S}] \geqq 0, \tag{2b}$$

$\eta_{\hat{S}}$ denoting the fundamental class of $\hat{S}$.

Putting (2a) and (2b) together we get

$$T(\eta_{\hat{S}}) = \kappa(F.\hat{S}) \geqq 0, \text{ hence } \kappa \leqq 0, \text{ since } (F \cdot \hat{S}) < 0.$$

On the other hand (2a) yields:

$$0 \leqq T'(\lambda) = \kappa \int_F \lambda, \text{ hence } \kappa \geqq 0.$$

So we conclude $\kappa=0$, hence $T'\sim 0$, thus $T'=0$, because T' is a positive current on the projective manifold $\hat{X}$.

Since $T'=0$ implies $T=\kappa_S T$, we finally get $T=0$.

3) Next we claim that $\tilde{\mathscr{P}}_a^1(X)\cap d\mathscr{D}^3(X)=\{0\}$ implies the existence of $\omega\in\mathscr{E}_{\mathbb{R}}^{1,1}(X)$ and $\varphi\in\mathscr{E}_{\mathbb{R}}^2(X)$ such that

(3a) ω is positive definite on $X\setminus S$,

(3b) $d(\omega-\varphi)=0$,

(3c) $\int_C \varphi=0$ for all curves C which are not contained in S.

The proof is very similar of that one of 2.2. Let B and Z be as in 2.2. Define $\tilde{B}:=B\cap\tilde{\mathscr{D}}$, $\tilde{Z}:=Z\cap\tilde{\mathscr{D}}$.

Then $\tilde{B}\cap\tilde{Z}$ is closed in $\tilde{\mathscr{D}}$ and $\{0\}=\tilde{\mathscr{P}}_a^1(X)\cap\tilde{B}=\mathscr{P}^1(X)\cap\tilde{\mathscr{D}}\cap B\cap Z$, hence by Hahn-Banach we get a linear and continous map $\tilde{\Phi}:\tilde{\mathscr{D}}\to\mathbb{R}$ such that $\tilde{\Phi}\,|\,\tilde{B}\cap Z=0$ and $\tilde{\Phi}\,|\,\mathscr{P}^1(X)\cap\tilde{\mathscr{D}}$ is positive.

Here we need the existence of a compact set K in $\tilde{\mathscr{P}}_a^1(X)$ such that for all $T\in\tilde{\mathscr{P}}_a^1(X)$, $T\neq 0$, there is $\lambda>0$ such $\lambda T\in K$. This K is easily constructed. We already mentioned that $\mathscr{E}_{\mathbb{R}}^{n-1,n-1}(X)\subset\tilde{\mathscr{D}}$, hence $\tilde{\mathscr{D}}$ is weakly dense in $\mathscr{D}=\mathscr{D}_{\mathbb{R}}^{1,1}(X)$ because $\mathscr{E}_{\mathbb{R}}^{n-1,n-1}(X)$ is dense in $\mathscr{D}$. Let Φ be the continous extension of $\tilde{\Phi}$ to $\mathscr{D}$. Then Φ is given by $\omega\in\mathscr{E}_{\mathbb{R}}^{1,1}(X)$, especially $\Phi(T)=T(\omega)$ for all $T\in\tilde{\mathscr{D}}$. Let $\tilde{B}^{\perp}=\{\psi\in\tilde{\mathscr{D}}'\,|\,<\psi,T>0$ for all $T\in\tilde{B}\}$, analogously $\tilde{Z}^{\perp}$. Because of the density of $\tilde{\mathscr{D}}$ in $\mathscr{D}$ we have $\tilde{\mathscr{D}}'=\mathscr{D}'=\mathscr{E}_{\mathbb{R}}^{1,1}(X)$, hence $\tilde{B}^{\perp}=\{\psi\in\mathscr{E}_{\mathbb{R}}^{1,1}(X)\,|\,T(\psi)=0$ for all $T\}$, analogously for $\tilde{Z}^{\perp}$. So it does not matter where we take the annihilator.

We have $(\tilde{B}\cap\tilde{Z})^{\perp}=\tilde{B}^{\perp}+\tilde{Z}^{\perp}$ if the latter is closed. We see immediately that $\tilde{B}=B$. Hence by the same method as in 2.2 we get the closedness of $\tilde{B}^{\perp}+\tilde{Z}^{\perp}$. Now we may choose $\varphi\in\tilde{Z}^{\perp}$, such that $\omega-\varphi\in B^{\perp}$, hence $d(\omega-\varphi)=0$. Clearly $\varphi\in\tilde{Z}^{\perp}$ implies $\int_C\varphi=0$ for all irreducible curves C which are not contained in S, since by (1) we have $T_C\in\tilde{\mathscr{D}}$ for all curves $C\not\subset S$. Since $T(\omega)>0$ for all $T\in\mathscr{P}^1(X)\cap\tilde{\mathscr{D}}$, $T\neq 0$ and since all $T\in\mathscr{P}^1(X)$ with $\operatorname{supp}(T)\subset X\setminus S$ are in $\tilde{\mathscr{D}}$, we conclude [6]: ω is positive definite on $X\setminus S$. So ω and φ satisfy (3a)–(3c).

4) Now we prove the assertion of the theorem. Let $T\in\mathscr{P}_a^1(X)$, $T\sim 0$, $\chi_S T=0$. By (2) and (3) there are $\omega\in\mathscr{E}_{\mathbb{R}}^{1,1}(X)$ and $\varphi\in\mathscr{E}_{\mathbb{R}}^2(X)$ such that the conditions (3a, b, c) are satisfied.

Write $T=\lim\limits_j\sum\limits_{i=1}^{n_j}\lambda_{ij}T_{C_{ij}}$ with $\lambda_{ij}\in\mathbb{R}$ such that for all i,j the curves $C_{ij}\not\subset S$. Then we have:

$$T(\omega)=\lim_j\sum_i\lambda_{ij}\int_{C_{ij}}\omega=\lim_j\sum_i\lambda_{ij}\int_{C_{ij}}(\omega-\varphi)=T(\omega-\varphi)=0$$

because $d(\omega-\varphi)=0$ and $T\sim 0$.

Since ω is positive definite on $X\setminus S$, we conclude: $\operatorname{supp}(T)\subset S$ [6], hence $T=\chi_S T=0$.

Thus Theorem 2.4 is proved. In the 3-dimensional case Theorem 2.4 has the following application.

Theorem 2.5. *Let X be a Moišezon 3-fold. Assume that there is no $T\in\mathscr{P}_a^1(X)$ and no effective curve C such that $C+T\sim 0$. Then X is projective.*

Proof. There exists a finite sequence $\pi_\nu: X_\nu\to X_{\nu-1}$ of blow-up's, $1\leqq\nu\leqq r$, such that $\hat{X}:=X_r$ is projective and for all ν the center S_ν of $\pi_{\nu+1}$ is smooth. Let

$\pi := \pi_1 \circ \ldots \circ \pi_r$. Let r be minimal with these properties. First we reduce the problem to the case $r=1$, i.e. X can be made projective by one blow-up. So we may assume that 2.5 is valid for all X which can be projective by one blow-up.

We proceed by induction on $r=r(X)$. So assume that the claim is proved for all X with $r(X)=r-1$. Take an exact sequence $\hat{X} \to \ldots \to X_1 \to X_0 = X$ as above. If we can show that X_1 has neither curves ~ 0 nor a curve C and $T \in \mathscr{P}_a^1(X_1)$ with $C+T \sim 0$ then by induction hypothesis X_1 is projective and hence X is projective (2.5 for $r=1$).

Take an effective curve $C = \sum n_i c_i$ on X_1, $n_i \in \mathbb{N}$, and let $C \sim 0$. Then $\sum n_i \pi_1(C_i) \sim 0$, hence all $n_i = 0$ for which $\pi_1(C_i)$ is not a point since by assumption X has no effective curves ~ 0 (strictly speaking we should distinguish between the set-theoretical image of C_i by π_1 and the image of C_i as cycle). So we may assume that all C_i are contained in fibres of π_1.

Let $\check{S}_1 := \pi_1^{-1}(S_1)$. Then $\left(\sum\limits_i n_i C_i \cdot \check{S}_1\right) = 0$.

On the other hand $(C_i \cdot \check{S}_1) < 0$ since the normal bundle of $\check{S}_1$ is negative relative π_1. Hence all $n_i = 0$ and X_1 has no effective curves ~ 0.

Now assume that $C+T \sim 0$ for some effective curve C – we may assume C irreducible – and $T \in \mathscr{P}_a^1(X_1)$. Hence $\pi_1(C) + \pi_{1*}(T) \sim 0$. Since $\pi_{1*}(T) \in \mathscr{P}_a^1(X)$ we must have (by our assumption) dim $\pi_1(C) = 0$, i.e. $C \subset \check{S}_1$. Let S' be the degeneracy set of $\pi' : \hat{X} \to X_1$.

Since $\dim S' = 1$, also $\dim(S' \cap S_1) \leqq 1$. For any curve $C' \not\subset S'$ and any ample line bundle $\mathscr{L}$ on $\hat{X}$ we have

$$(c_1(\pi'_*(\mathscr{L})) \cdot C') > 0.$$

This is an easy computation.

So $(c_1(\pi'_*(\mathscr{L})) \cdot C') \leqq 0$ can hold for only finitely many curves $C' \subset X_1$. If S_1 is a point, then $\check{S}_1 = \mathbb{P}_2$ and it is clear that $(c_1(\pi'_*(\mathscr{L})) \cdot C) > 0$ (since all curves in S_1 are homologous modulo a positive multiple). Then $(T \cdot c_1(\pi'_*(\mathscr{L}))) < 0$, (*).

We may assume: $T = \lim\limits_j \sum\limits_i \lambda_{ij} T_{C_{ij}}$ with $\lambda_{ij} > 0$ (change T in its cohomology class using Lemma 4.2 below and the fact that we may assume the existence of the "strict transform" of T in $\hat{X}$ as in 2.4, 3)). So we conclude: $\chi_{S'} T \neq 0$ (otherwise (*) is violated).

If S_1 is a curve, $\check{S}_1$ is a ruled surface and C a fiber of $\check{S}_1 \to S_1$. Again we conclude $\chi_{S'} T = 0$.

In any case write $T = \sum\limits_{i=1}^{p} \lambda_i S'_i + \chi_{X_1 \setminus S'} T$, $\lambda_i > 0$, $p \geqq 1$, S'_i irreducible components of S', by Siu [14].

Thus $\pi_{1*}(T) = \sum\limits_{i=1}^{p} \lambda_i \pi_1(S'_i) + \pi_{1*}(\chi_{X_1 \setminus S'} T) \sim 0$. If $\dim \pi_1(S'_i) = 0$ then we see as above:

$$(S'_i \cdot c_1(\pi'_*(\mathscr{L}))) > 0,$$

and since

$$(\chi_{X_1 \setminus S'} T \cdot c_1 \pi'_*(\mathscr{L}))) \geqq 0$$

there must be at least one i_0 such that $\dim \pi_1(S'_{i_0}) = 1$ yielding a contradiction to our assumption.

Thus X_1 has no curve C and no $T \in \mathscr{P}_a^1(X)$ with $C + T \sim 0$; also no effective curve ~ 0. So X is projective.

So we may assume $r = 1$. Let $T \in \mathscr{P}_a^1(X) \cap d\mathscr{D}^3(X)$. We must show $T = 0$ (2.3). If $\chi_S T = 0$, this follows from 2.4. So assume $\chi_S T \neq 0$. By [14] we have: $\chi_S T = \kappa T_S$ with $\kappa > 0$, since T is positive; here we use essentially that S is a curve. Hence $\kappa S + \chi_{X \setminus S} T \sim 0$ which was excluded by our assumption.

Remark. We have seen that in 2.5 we have to check $C + T \not\sim 0$ only for those T with $T = \lim \sum \lambda_{ij} T_{C_{ij}}$, $\lambda_{ij} > 0$. Furthermore there are a priori only finitely many irreducible curves $c_1, \ldots, c_r$ for which an equation

$$\sum \varrho_j c_j + T \sim 0 \qquad (\varrho_j > 0)$$

is possible.

Corollary 2.6. *Let X be a Moišezon manifold of dimension 3. Let $K \subset A_2(X)$ be the cone generated by the classes of irreducible curves. Assume that X contains no irreducible curves ~ 0 and $K \cap -\bar{K} = \{0\}$. Then X is projective. ($\bar{K}$ is the closure of K in the finite-dimensional vector space $A_2(X)$ with the Euclidean topology).*

Proof. By 2.5 we have to show that there is no effective curve ~ 0 and that $C + T \sim 0$ is impossible (C a curve, $T \in {}_a^1(X)$). The first is clear. If $C + T \sim 0$ then $C \sim -T$. Since $[T] \in \bar{K}$ (use the remark following 2.5!) we have $[C] \in K \cap (-\bar{K})$ which is impossible.

Theorem 2.7. *Let X be a Moišezon 3-fold, $\pi: \hat{X} \to X$ a modification which is a sequence of r blow-up's with smooth centers such that $\hat{X}$ is projective. Assume that for any irreducible curve $C \subset X$ and some ample line bundle $\mathscr{F}$ on $\hat{X}$:*

$$(C \cdot c_1(\pi_*(\mathscr{F}))) > 0.$$

Then X is projective.

Proof. We may assume $r = 1$ (!). If $C = \sum r_i C_i \sim 0$, $r_i \geqq 0$, then by $(C_i \cdot c_1(\pi_*(\mathscr{F}))) > 0$, we conclude $r_i = 0$ for all i. If $C + T \sim 0$, $T \in \mathscr{P}_a^1(X)$, $C \neq 0$, we have $(C \cdot c_1(\pi_*(\mathscr{F}))) > 0$ and $(T \cdot c_1(\pi_*(\mathscr{F}))) \geqq 0$ (see the proof of 2.5), which is impossible. So $C = 0$ and $(T \cdot c_1(\pi_*(\mathscr{F}))) = 0$. Consequently $\chi_S T = 0$. Let T' be defined as in 2.4 (we may assume the existence of T', see proof of 2,5). It follows (cp. Remark 2.10):

$$(T' \cdot c_1(\mathscr{F})) \leqq (T \cdot c_1(\pi_*(\mathscr{F}))) = 0,$$

hence $T' = 0$ and $T = 0$.

So we can apply 2.5.

Remark 2.8. If we assume in 2.7

$$(C \cdot c_1(\pi_*(\mathscr{F}))) > 0$$

for all ample line bundles $\mathscr{F}$ on $\hat{X}$ then it is immediately seen that a priori $T \sim 0$ (in the proof of 2.7), hence we can directly apply 2.4.

Theorem 2.9. *Let X be a Moišezon 3-fold, $\mathscr{L}$ a line bundle on X, $c_1(\mathscr{L})^3 > 0$. Let $\mathscr{L}^\mu$ be globally generated for some μ and assume that the set where the associated* *map* $\psi: X \to \mathbb{P}(H^0(X, \mathscr{L}^\mu))$ *is not injective ("the degeneracy set of* ψ*") is of dimension* $\leqq 1$. *Assume that there are no effective curves* ~ 0 *in* X. *Then* X *is projective.*

Proof. Let $\phi: X \to Y$ be the Stein factorization of ψ. Y is a normal projective variety, $\dim Y = \dim X$ for μ sufficiently large (because $\mathscr{L}$ is numerically effective, $c_1(\mathscr{L})^3 > 0$, hence $\kappa(X, \mathscr{L}) = 3$, see e.g. [8], so ϕ is a modification).

Let $R \subset X$ be the degeneracy set of ϕ; $\dim R \leqq 1$. If $C + T \sim 0$, C an effective curve, $T \in \mathscr{P}_a^1(X)$, then $(C \cdot c_1(\mathscr{L})) \geqq 0$ and $(T \cdot c_1(\mathscr{L})) \geqq 0$ (we may assume $T = \lim \sum \lambda_{ij} T_{C_{ij}}, \lambda_{ij} > 0$!), hence $(C \cdot c_1(\mathscr{L})) = 0$ and $(T \cdot c_1(\mathscr{L})) = 0$. Consequently $C \subset R$ and $\operatorname{supp}(T) \subset R$ (to see $\operatorname{supp}(T) \subset R$, take an ample line bundle $\mathscr{F}$ on Y, then $(T \cdot c_1(\phi^*(\mathscr{F}))) = 0$ from which the claim follows by standard arguments in the theory of positive closed currents.)

Hence $T = \sum_{i=1}^{q} \lambda_i R_i$, $\lambda_i \geqq 0$, R_i the irreducible components of R of dimension 1. So $C + \sum \lambda_i R_i \sim 0$, contradiction.

Remark 2.10. Recall the definition of the cone $\mathscr{P}_b^1(X)$ from the beginning of this section. Then we will show in a moment in a very elementary way that in the situation of Theorem 2.4.

(1) $T \in \mathscr{P}_b^1(X) \cap d\mathscr{D}^3(X)$ and $\chi_S T = 0$ implies $T = 0$. But in order to get an alternate (and more natural) proof of 2.4 and thus of 2.5 we needed to know

(2) $\mathscr{P}_b^1(X) = \mathscr{P}_a^1(X)$.

Here we can formulate this only as a *conjecture*: If T is a positive closed current on a Moišezon manifold such that $T = \lim_{j \to \infty} \sum_{i=1}^{m_j} \lambda_{ij} T_{C_{ij}}$ in the weak topology with $\lambda_{ij} \in \mathbb{R}$ and irreducible curves $C_{ij} \subset X$ then one can also write: $T = \lim_{\mu \to \infty} \sum_{\nu=1}^{\nu_\mu} \lambda'_{\nu\mu} T_{D_{\nu\mu}}$ with $\lambda'_{\nu\mu} > 0$ and irreducible curves $D_{\nu\mu} \subset X$. Of course one should expect this in all dimensions.

Let us mention that the conjecture is a special case of the following conjecture of Demailly [1] (at least if X is projective): if T is a positive closed (p,p)-current such that $[T] \in (H^{n-p,n-p}(X) \cap H^{2n-2}(X, \mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{R}$ then $T = \lim_{j \to \infty} \sum_{i=1}^{n_j} \lambda_{ij} T_{Y_{ij}}$ with $\lambda_{ij} > 0$ and irreducible p-dimensional subsets $Y_{ij} \subset X$. But Demailly's conjecture is much stronger than ours; for example it implies the Hodge conjecture (while ours has nothing to do with the Hodge conjecture).

Now we come to the proof of (1). We assume the situation of 2.4. Let $T = \lim_j \sum_i \lambda_{ij} T_{C_{ij}}$ with $\lambda_{ij} > 0$ and irreducible curves C_{ij} not contained in S such that $T \sim 0$. Let λ be a Kähler form on $\hat{X}$. We define $T'(\lambda)$ as in 2.4 but define ad hoc:

$$T(\tilde{\lambda}) := ([T] \cdot [\tilde{\lambda}]), \ \tilde{\lambda} := \pi_*(\lambda).$$

All we have to prove is

(3) $T'(\lambda) \leqq T(\tilde{\lambda})$, because then $T \sim 0$ implies $T' = 0$, hence $T = \chi_S T = 0$.

Define $T_j := \sum_{i=1}^{n_j} \lambda_{ij} T_{C_{ij}}$ and $T'_j := \sum_{i=1}^{n_j} \lambda_{ij} T_{\hat{C}_{ij}}$, $\hat{C}_{ij}$ defining as usual the strict transform of C_{ij}.

Then: $[T_j'] = \pi^* [T_j] + \varrho_j [F]$ with an appropriate curve F in a fiber of π. Thus we get:

(4) $T_j'(\lambda) = (\pi^* [T_j] \cdot [\lambda]) + \varrho_j \int_F \lambda = ([T_j] \cdot [\tilde{\lambda}]) + \varrho_j \int_F \lambda.$

Let $\eta_{\hat{S}}$ be the fundamental class of $\hat{S}$ in $\hat{X}$. Then $T_j'(\eta_{\hat{S}}) = \varrho_j (F \cdot \hat{S})$ for all j. On the other hand: $T_j'(\eta_{\hat{S}}) = \sum_i \lambda_{ij} (\hat{C}_{ij} \cdot \hat{S}) \geqq 0$, since $\hat{C}_{ij}$ and $\hat{S}$ are meeting at most in a finite number of points, hence $(\hat{C}_{ij} \cdot \hat{S}) \geqq 0$.

Since $(\hat{F} \cdot \hat{S}) < 0$, we conclude $\varrho_j \leqq 0$ for all j. So (4) implies: $\lim_j T_j'(\lambda) \leqq ([T] \cdot [\tilde{\lambda}]) = T(\tilde{\lambda})$ (especially T' exists). So (3) and hence (1) are proved.

3. Study of the Cones $\mathscr{P}_a^1(X)$ and $\mathscr{P}_a^{n-1}(X)$

From the results of Sect. 2 it should be clear that one needs more detailed information on the cone $\mathscr{P}_a^1(X)$ in dimension >3. Here we want to study a bigger cone, namely the cone $\mathscr{P}_d^p(X)$ consisting of those positive (p,p)-currents T for which $[T] \in (H^{n-p,n-p}(X) \cap H^{2n-2p}(X, \mathbb{Q})) \otimes \mathbb{R}$. We are mainly interested in the cases $p=1$ and $p=n=\dim X$. Notice the inclusions $\mathscr{P}_b^p(X) \subset \mathscr{P}_a^p(X) \subset \mathscr{P}_d^p(X)$ (the above mentioned conjecture of Demailly (2.6) says that they are equal). Furthermore let $\mathscr{P}_c^p(X)$ be the cone of positive closed (p,p)-currents.

Theorem 3.1. *Let X be a projective manifold of dimension n. Fix a Hodge metric on X with associated (1,1)-form ω. Then there exists an injective map α: $\mathscr{P}_d^{n-p,n-p}(X) \to \mathscr{P}_d^p(X)$ given by $\alpha(T) = \omega^{n-2p} \wedge T (2p \leqq n)$. Furthermore, for all $T \in \mathscr{P}_d^p(X)$ there exists $\tilde{T} \in \mathscr{P}_d^{n-p}(X)$ such that $[\alpha(\tilde{T})] = [T]$ in $H^{2n-2p}(X)$ (hence α is surjective on the cohomology level).*

Proof. First we have to show that $T \in \mathscr{P}_d^{n-p}(X)$ implies $\alpha(T) \in \mathscr{P}_d^p(X)$. If φ is positive, then $\omega^{n-2p} \wedge \varphi$ is also positive [6, 9], hence $\alpha(T)$ is positive. Since $[\omega] \in H^2(X, \mathbb{Z})$ and $[T] \in (H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})) \otimes \mathbb{R}$, we see that $[\omega^{n-2p} \wedge T] \in (H^{2n-2p}(X, \mathbb{Q}) \cap H^{n-p,n-p}(X)) \otimes \mathbb{R}$, hence $\alpha(T) \in \mathscr{P}_d^p(X)$.

b) α is injective: if $\omega^{n-2p} \wedge T = 0$, then $T(\omega^p) = 0$, hence $T = 0$ (X is projective).

c) Now we prove the surjectivity on the cohomology level. We consider the map β: $V \to W$, where

$$V = \{\varphi \in \mathscr{E}_{\mathbb{R}}^{p,p}(X) \mid d\varphi = 0\} / d\mathscr{E}_{\mathbb{R}}^{2p-1}(X) \cap \mathscr{E}_{\mathbb{R}}^{p,p}(X)$$

and

$$W = \mathscr{E}_{\mathbb{R}}^{n-p,n-p}(X) / d\mathscr{E}_{\mathbb{R}}^{2n-2p-1}(X) \cap \mathscr{E}_{\mathbb{R}}^{n-p,n-p}(X),$$

given by $\beta([\varphi]) = [\omega^{n-2p} \wedge \varphi]$.

Since $d\varphi = 0$, β is well defined. β is injective: take $[\varphi] \in V$ with $[\omega^{n-2p} \wedge \varphi] = 0$, then by the hard Lefschetz theorem we have $[\varphi] = 0$.

We consider the usual Fréchet topology on $\mathscr{E}_{\mathbb{R}}^{p,p}(X)$. Then V and W are Fréchet spaces again. Now let $T \in \mathscr{P}_d^p(X)$. T defines a continous linear map T_0: $V \to \mathbb{R}$ since $dT = 0$. Consider V as subspace of W. By Hahn-Banach we get a continous linear map $\tilde{T}_0$: $W \to \mathbb{R}$ with $\tilde{T}_0 | V = T_0$. Letting $\tilde{T}(\varphi) = \tilde{T}_0([\varphi])$ we obtain a closed

$(n-p, n-p)$-current $\tilde{T}$, but in general $\tilde{T}$ will not be positive. So we have to choose $\tilde{T}$ more carefully.

Define $K := \{\gamma \in W$ there exists a positive form $\varphi \in \mathscr{E}_{\mathbb{R}}^{n-p,n-p}(X)$ with $\gamma = [\varphi]$ in $W\}$.

Then K is a convex cone with $0 \in K$ and such that $K \cap (-K) = \{0\}$. All these properties are trivial except for the last one. In order to verify it, take $\gamma \in K \cap (-K)$. Then there are φ, $\varphi \in \mathscr{E}_{\mathbb{R}}^{n-p,n-p}(X)$ such that φ and $-\varphi$ are positive forms (or currents) and $\gamma = [\varphi] = [\psi]$ in W. Hence $\varphi = \psi + d\varrho$ with an appropriate ϱ. Consequently $\int_X \varphi \wedge \omega^p$. On the other hand $\int_X \varphi \wedge \omega^p \geqq 0$ and $\int_X \psi \wedge \omega^p \leqq 0$, so $\int_X \varphi \wedge \omega^p = \int_X \psi \wedge \omega^p = 0$. Since φ and $-\psi$ are positive as currents we conclude $\varphi = \psi = 0$.

Defining $\alpha \leqq \beta$ iff $\beta - \alpha \in K$, we get an ordered vector space. Now a Hahn-Banach type theorem for positive linear forms ([13, p. 227]., the the closedness of the cone K is not necessary, or Bourbaki, Esp. vect. top., Chap. 2, Sect. 3, no. 4, Proposition 6), there exists a continous linear map $\tilde{T}_0: W \to \mathbb{R}$ with $\tilde{T}_0 | V = T_0$ and $T_0 | K \geqq 0$, if the condition $\mathring{K} \cap V \neq \phi$ is satisfied, where $\mathring{K}$ denotes the interior of K in W. But $[\omega^{n-p}] \in \mathring{K} \cap V$ [6], so $\tilde{T}_0$ exists. Now define $\tilde{T} \in \mathscr{P}_d^{n-p}(X)$ by $\tilde{T}(\varphi) = \tilde{T}_0([\varphi])$ for $\varphi \in \mathscr{E}_{\mathbb{R}}^{n-p,n-p}(X)$ and $\tilde{T}(\varphi) = \tilde{T}(\mathrm{Re}(\varphi)) + i\tilde{T}(\mathrm{Im}(\varphi))$ in general.

Then $\tilde{T}$ is positive and closed by construction. Furthermore $[\alpha(\tilde{T})] = [T]$ since $\omega^{n-2p} \wedge \tilde{T}(\varphi) = T(\varphi)$ for all closed $\varphi \in \mathscr{E}_{\mathbb{R}}^{p,p}(X)$ (observe the following: if T is a closed current and $T(\varphi) = 0$ for all closed φ then $[T] = 0$).

It remains to show $[T] \in (H^{2n-2p}(X, \mathbb{Q}) \cap H^{n-p,n-p}(X)) \otimes \mathbb{R}$. But this clear since $[\omega] \in (H^2(X, \mathbb{Z}) \cap H^{1,1}(X))$.

Remark. α is far from being surjective as shown in the following considerations. We claim that $T_C \notin \mathrm{Im}\, \alpha$ for all irreducible curves $C \subset X$. In fact, choose C^∞-functions $\eta_\nu: X \to \mathbb{R}$, $0 \leqq \eta_\nu \leqq 1$, $\mathrm{supp}(\eta_\nu) \subset\subset X \setminus C$ (C a fixed curve) such that for all $K \subset\subset X \setminus C$ there exists ν_0 with $\eta_\nu | K = 1$ for all $\nu \geqq \nu_0$.

Suppose now $T_C = \omega^{n-2} \wedge T$ with $T \in \mathscr{P}_d^{n-1}(X)$. Then $T_C(\eta_\nu \omega) = 0$ for all ν. On the other hand $T\eta_\nu \omega^{n-1}) > 0$ for $\nu \gg 0$, because otherwise $\mathrm{Supp}(T) \subset C$, hence $T = 0$ (by the so-called theorem of the support, see [6]).

The philosophy of 3.1 is the following: if one has proved a certain property for the cone $\mathscr{P}_d^{n-1}(X)$ (and this is by far the "easiest" case) then one has the same property also for $\mathscr{P}_d^1(X)$ at least on the cohomology level.

As in Sect. 2 we denote for a given (p,p)-current T by T' the uniquely determined current on $\hat{X}(\pi: \hat{X} \to X$ a modification) with $\pi_*(T') = \chi_{X \setminus S} T$ and $\chi_S T' = 0$, if existent. If T is positive, we know that T' exists iff $T'(\omega) < \infty$, where ω is a Kähler form on the projective manifold X.

Theorem 3.2. *Let X be a n-dimensional Moišezon manifold and $\pi: \hat{X} \to X$ a finite sequence of blowing up's with smooth centers such $\hat{X}$ is projective. Let S be the center of π and $\hat{S} := \pi^{-1}(S)$. Let ω be the (1,1)-form of a Hodge metric on $\hat{X}$ and $\alpha: \mathscr{P}_d^{n-1}(\hat{X}) \to \mathscr{P}_d^1(\hat{X})$ be the map from 3.1. Then:*

a) *we have maps*

$$\pi_*^{(1)}: \mathscr{P}_d^{n-1}(\hat{X}) \to \mathscr{P}_d^{n-1}(X)$$

and

$$\pi_*^{(2)}: \mathscr{P}_d^1(\hat{X}) \to \mathscr{P}_d^1(X)$$

given by pushing down currents;

b) $\operatorname{Ker} \pi_*^{(1)} = \{T \in \mathscr{P}_d^{n-1}(X) \mid \chi_S T\}$;

c) $\mathscr{P}_b^1(X) \subset \operatorname{Im} \pi_*^{(2)}$, $\mathscr{P}_b^{n-1}(X) \subset \operatorname{Im} \pi_*^{(1)}$;

moreover $\operatorname{Im} \pi_*^{(i)} = \{T \mid T' \text{ exists}\}$

d) *there exists a commuting diagram*

$$\begin{array}{ccc}
\mathscr{P}_d^{n-1}(\hat{X}) & \xrightarrow{\alpha} & \mathscr{P}_d^1(\hat{X}) \\
{\scriptstyle \pi_*^{(1)}} \downarrow & & \downarrow {\scriptstyle \pi_*^{(2)}} \\
\mathscr{P}_d^{n-1}(X) \cap \operatorname{Im} \pi_*^{(1)} & \xrightarrow{\beta} & \mathscr{P}_d^1(X)
\end{array}$$

with a suitable map β.

Hence any $T \in \mathscr{P}_d^1(X) \cap \operatorname{Im} \pi_*^{(2)}$ *is cohomologous to* $\beta(\tilde{T})$ *with a* $\tilde{T} \in \mathscr{P}_d^{n-1}(X) \cap \operatorname{Im} \pi_*^{(1)}$.

Proof. a) First we have to show that the push-down π_* maps $\mathscr{P}_d^p(\hat{X})$ to $\mathscr{P}_d^p(X)$ for $p = 1, n-1$. It is clear that π_* maps $\mathscr{P}_c^p(\hat{X})$ to $\mathscr{P}_c^p(X)$ (for all p) since $\pi^*(\varphi)$ is a positive form for any positive form φ on X. Since $A^{2p}(\hat{X}, \mathbb{Q}) = H^{p,p}(\hat{X}) \cap H^{2p}(\hat{X}, \mathbb{Q})$ in our cases we have to show that $[T] \in A^{2p}(\hat{X}, \mathbb{Q})$ implies $[\pi_*(T)] \in A^{2p}(X, \mathbb{Q})$.

Taking such a T, we have $T = \sum r_i T_{Z_i} + dQ$, where the Z_i are irreducible subspaces of $\hat{X}$ of dimension or codimension 1.
It follows: $\pi_*(T) = \sum r_i T_{\pi(Z_i)} + d\pi_*(Q)$, where we omit all summands for which $\pi(Z_i)$ is of lower dimension than Z_i.

Thus $[\pi_*(T)] \in A^{2p}(X, \mathbb{Q})$. The proof of b) is clear.

c) First we treat the case $p = n-1$. The fact that T' exists for any $T \in \mathscr{P}_b^{n-1}(X)$ can be shown in the same manner as the analogous fact in the (1,1)-case (see 2.6). Hence only the second claim must be proved. Let $T \in \mathscr{P}_d^{n-1}(X)$ such that T' exists. Since T' is the trivial extension of $(\pi \mid \hat{X} \setminus \hat{S})^{-1}{*}(T \mid X \setminus S)$, T' is a positive closed current by a theorem of Skoda [15]. It remains to show $[T'] \in A^2(\hat{X})$. We may restrict ourselves to one blowing up π with smooth center S.

Then it is well known that

$$A^2(\hat{X}, \mathbb{R}) \simeq \pi^*(A^2(X, \mathbb{R})) \oplus \mathbb{R}$$

and

$$H^2(\hat{X}, \mathbb{R}) \simeq \pi^*(H^2(X, \mathbb{R})) \oplus \mathbb{R}.$$

In this decomposition we have: $[T'] = \pi^*[T] + \alpha$, because S is of codimension at least 2, hence $\chi_S T = 0$ (since T is of bidimension $(n-1, n-1)$), hence $T = \chi_{X \setminus S} T$. Furthermore $\alpha \in A^2(\hat{X}, \mathbb{R})$ (the factor $\mathbb{R}$ in the above decomposition is generated by the cohomology class of the exceptional divisor $\hat{S}$). This proves the assertion in the case $p = n-1$. Now we treat the case $p = 1$. Again only the second statement has to be proved. Let $T \in \mathscr{P}_d^1(X)$. First suppose that $\chi_S T = 0$. Then we proceed in the same manner as before using

$$A^{2n-2}(\hat{X}, \mathbb{R}) \simeq \pi^*(A^{2n-2}(X, \mathbb{R})) \oplus \mathbb{R}$$

and

$$H^{2n-2}(\hat{X}, \mathbb{R}) \simeq \pi^*(H^{2n-2}(X, \mathbb{R})) \oplus \mathbb{R}.$$

Now let T be arbitrary. Considering the decomposition $T = \chi_S T + \chi_{X \setminus S} T$ with the positive closed currents $\chi_S T$ and $\chi_{X \setminus S} T$ we may restrict ourselves to the case

$T=\chi_S T$. Then there exists a positive closed current $\tilde{T}$ on S such that $i_*(\tilde{T})=T$, where $i\colon S\to X$ denotes the inclusion (see the proof of 4.1). Now $\hat{S}=\pi^{-1}(S)$ is a $\mathbb{P}_r$-bundle over S, thus Lemma 3.3 below tells us that there exists $\tilde{T}\in\mathscr{P}_c^1(\hat{S})$ such that $p_*(\tilde{T})=\tilde{T}$, where $p\colon\hat{S}\to S$ is the projection map. Letting $j\colon\hat{S}\to\hat{X}$ be the inclusion map, we define $\hat{T}:=j_*(\tilde{T})$. Then $\hat{T}\in\mathscr{P}_c^1(\hat{X})$ and $\pi_*(\hat{T})=T$. By analogous considerations as above we see $[\hat{T}]\in A^{2n-2}(\hat{X})$. Hence c) is proved (modulo Lemma 3.3).

d) The existence of β is shown as follows: let $T\in\mathscr{P}_d^{n-1}(X)\cap\operatorname{Im}\pi_*^{(1)}$ and let T' be the uniquely determined current in $\mathscr{P}_d^{n-1}(\hat{X})$ with $\pi_*(T')=T$ and $\chi_S T'=0$ (remember $\chi_S T=0$). Define $\beta(T)=\pi_*^{(2)}(\alpha(T'))$. Then by definition of β, the resulting diagram commutes. The other statement follows from 3.1.

Remark. The proof of 3.2 shows that all statements remain correct if we replace $\mathscr{P}_d^p(X)$ by $\mathscr{P}_c^p(X)$. Furthermore, statements a), b) and c) are also valid for $\mathscr{P}_a^p(X)$ or $\mathscr{P}_b^p(X)$ instead of $\mathscr{P}_d^p(X)$. An open problem is to determine the image of $\pi_*\colon\mathscr{P}_c^p(\hat{X})\to\mathscr{P}_c^p(X)$.

Lemma 3.3. *Let X be a Moišezon manifold and E a holomorphic vector bundle on X. Then $\pi_*\colon\mathscr{P}_c^p(\mathbb{P}(E))\to\mathscr{P}_c^p(X)$ is surjective for any p, π denoting the projection map $\mathbb{P}(E)\to X$.*

Proof. First note that indeed $\pi_*(\mathscr{P}_c^p(\mathbb{P}(E))\subset\mathscr{P}_c^p(X)$. Now let $T\in\mathscr{P}_c^p(X)$. T defines a linear continous map $T_0\colon V:=\mathscr{E}_{\mathbb{R}}^{p,p}(X)/\{d\mathscr{E}_{\mathbb{R}}^{2p-1}(X)\cap\mathscr{E}_{\mathbb{R}}^{p,p}(X)\}\to\mathbb{R}$. Define $W:=\mathscr{E}_{\mathbb{R}}^{p,p}(\mathbb{P}(E))/\{d\mathscr{E}_{\mathbb{R}}^{2p-1}(\mathbb{P}(E))\cap\mathscr{E}_{\mathbb{R}}^{p,p}(\mathbb{P}(E))\}$. We get a map $\gamma\colon V\to W$ by pulling back forms. We claim that γ is injective. To prove this, let $\varphi\in\mathscr{E}_{\mathbb{R}}^{p,p}(X)$ such that $\pi^*(\varphi)=d\alpha$ with $\alpha\in\mathscr{E}_{\mathbb{R}}^{p-1}(X)$. Consequently $\pi^*(d\varphi)=d\pi^*(\varphi)=0$, thus $d\varphi=0$ (because π is locally trivial). So we need only the injectivity of the pull back map $H^{p,p}(X)\to H^{p,p}(\mathbb{P}(E))$. But this is well known. Thus γ is injective. Equip V and W with the natural Fréchet topologies. Then V is a closed subspace of W via γ. Define a cone $K\subset W$ by $K:=\{\alpha\in W\,|\,\alpha$ is represented by a positive form $\varphi\}$. Define $\alpha\leqq\beta$ if and only if $\beta-\alpha\in K$. Then $(W,\leqq)$ is an ordered vector space (compare the proof of 3.1). The only non-trivial part to verify is that $\alpha\leqq\beta$ and $\beta\leqq\alpha$ imply $\alpha=\beta$, i.e. we must prove $K\cap(-K)=\{0\}$. Thus we have to consider two positive forms φ and $-\psi$ such that $\varphi=\psi+d\varrho$. Performing some blowing up's if necessary, we may assume that X is projective. Take a Kähler form ω on X. Then $\int_X\varphi\wedge\omega^{n-p}=\int_X\psi\wedge\omega^{n-p}$, hence we see (as in 3.1): $\int_X\varphi\wedge\omega^{n-p}=\int_X\psi\wedge\omega^{n-p}=0$ and conclude $\varphi=\psi=0$.

A Hahn-Banach type theorem says that we may extend T_0 to a positive linear form $T_1\colon W\to\mathbb{R}$ if the condition

$$T_0\,|\,V\cap(U-C)\leqq N$$

with a positive number N is satisfied with a suitable convex neighborhood U of 0 in W (see [13, Chap. 5, Theorem 5.4]; the closedness of the positive cone is unnecessary). Now this condition is easily verified and so we get $\tilde{T}\in\mathscr{P}_c^p(\mathbb{P}(E))$ with $\pi_*(\tilde{T})=T$ (in the same way as in 3.1).

4. A Generalization of Theorem 2.5

In this section we prove a partial generalization of the results of section 2 using the results of the previous section. As in 2.6 we define K to be the cone in

$A^{2n-2}(X) \simeq A_2(X)$ generated by the classes of irreducible curves in the n-dimensional compact manifold X.

Theorem 4.1. *Let X be a Moišezon manifold such that $K \cap (-\bar{K}) = \{0\}$. Assume that there is no irreducible curve ~ 0 in X and that there is no $T \in \mathscr{P}^1_a(X) \setminus \{0\}$ such that* $\chi_C T = 0$ for all curves *$C \subset X$ and such that $T \sim 0$. Then X is projective. Before we can prove Theorem 4.1 we need a lemma.*

Lemma 4.2. *Let X be a projective manifold and $T \in \mathscr{P}^1_d(X)$. Then T is cohomologous to a limit $\lim\limits_j \sum\limits_{i=1}^{n_j} r_{ij} T_{C_{ij}}$, $r_{ij} > 0$ and $C_{ij} \subset X$ irreducible curves, in the weak topology.*

Proof. Let $\tilde{T} \in \mathscr{P}^{n-1}_d(X)$, $n = \dim X$. Then T is cohomologous to a weak limit $\lim\limits_j \sum\limits_{i=1}^{n_j} s_{ij} T_{H_{ij}}$ with $s_{ij} > 0$ and hypersurfaces $H_{ij} \subset X$. This follows from [1].

Take a Hodge metric with form ω coming from the hyperplane section H. By Theorem 3.1 there exists $\tilde{T} \in \mathscr{P}^{n-1}_d(X)$ such that T is cohomologous to $\omega^{n-2} \wedge \tilde{T}$. Writing $\tilde{T}$ as above as a weak limit, we conclude that T is cohomologous to a weak limit $\lim\limits_j \sum\limits_{i=1}^{n_j} s_{ij} T_{(H_{ij} \cdot H^{n-2})}$ (this limit exists!). So it remains to show the following. Let H' be a hypersurface in X, then $(H' \cdot H^{n-2}) = \sum \lambda_i C_i$ for H generic, where all $\lambda_i > 0$ and the C_i are irreducible curves. To prove this, it is sufficient to verify the following. If $Y \subset X$ is irreducible and p-dimensional then $(Y \cdot H) = \sum \lambda_k \cdot Y_k$ with all $\lambda_k > 0$ and the Y_k irreducible and $(p-1)$-dimensional for H generic. To do this, we substitute H by a linearly equivalent hyperplane section $\tilde{H}$ such that $Y \not\subset \tilde{H}$ (if necessary). Then all irreducible components of $Y \cap \tilde{H}$ are $(p-1)$-dimensional (apply the projective dimension theorem in $\mathbb{P}_N$), hence Y and $\tilde{H}$ intersect properly and the claim follows from intersection theory.

Proof of 4.1. By 2.3 we must show $\mathscr{P}^1_a(X) \cap dD^3(X) = \{0\}$. Arguing similary as in 2.5 we may restrict to the case that there is one blow-up $\pi\colon \hat{X} \to X$ with smooth center $S \subset X$ such that $\hat{X}$ is projective. Let $T \in \mathscr{P}^1_a(X) \cap d\mathscr{D}^3(X)$, $T \neq 0$. By 2.4 we have $\chi_S T \neq 0$. Since we may assume the existence of the "strict transform" T' (proceed as in 2.5), we may assume $\chi_{X \setminus S} T \sim \lim \sum \lambda_{ij} T_{C_{ij}}$, $\lambda_{ij} > 0$; hence $[\chi_{X \setminus S} T] \in \bar{K}$. If we know:

$$(*) \qquad [\chi_S T] \in \bar{K}$$

we conclude as follows.

There is a curve $C \subset X$ such that either $\chi_S T = \lambda C + \tilde{T}$ or $\chi_{X \setminus S} T = \lambda C + \tilde{T}$, $\lambda > 0$ (by assumption). In both situations $\tilde{T}$ is again of the form $\lim \sum \lambda_{ij} T_{C_{ij}}$ with all $\lambda_{ij} > 0$ (in the first situation argue as in (**) below). Hence $[\tilde{T}] \in \bar{K}$. So we have a decomposition:

$$T = \lambda C + \tilde{T}, \ [\tilde{T}] \in \bar{K}.$$

It follows $[C] \in K \cap \bar{K} = \{0\}$, contradiction and T cannot exist. To prove (*) we change notations and show (**) if $T \in \mathscr{P}^1_a(X)$, $\operatorname{supp}(T) \subset S$, then

$$T \sim \lim \sum \lambda_{ij} T_{C_{ij}}, \qquad \lambda_{ij} > 0, \ \text{ in } \ X.$$

Proof of (**). Let $i\colon S \to X$ be the inclusion. By Harvey [6], Theorem 1.7(b) there is $\tilde{T} \in \mathscr{P}^1_a(S)$ such that $i_*(\tilde{T}) = T$ (it is clear that $\tilde{T}$ must be positive and closed and it is also easily seen that $\tilde{T}$ is a limit of linear combinations of curves).

By 4.2 we have $\tilde{T} \sim \lim \sum \lambda_{ij} T_{C_{ij}}$, $\lambda_{ij} > 0$, $C_{ij} \subset S$ curves, where T_C here means integration over C in S. We may apply 4.2 because $\hat{S} = \pi^{-1}(S)$ is projective, hence S

is projective since $\hat{S}$ is a $\mathbb{P}_r$-bundle over S. We conclude: $T = i_*(\tilde{T}) \sim \lim \sum \lambda_{ij} T_{C_{ij}}$, $T_{C_{ij}}$ being integration over C_{ij} in X. This proves (**).

Remark. Of course one expects a better result than 4.1; the last condition should be unnecessary. But it is difficult to deal with those currents, for which $\chi_C T = 0$ for all curves C. This information is equivalent to some informations on the Lelong numbers $\Theta(T, x)$, $x \in X$. To explain this let $E_c := \{x \in X \mid \Theta(T, x) \geqq c\}$ for $c > 0$. By a theorem of Siu [14] one knows that E_c is analytic (of dimension $\geqq k$ for a positive closed current of didimension $(n-k, n-k)$). Now E_c must be finite for any c, otherwise we could find a curve $C \subset E_c$ and by [14, Proposition 12.6], we would get:

$$\chi_C T = \lambda T_C \quad \text{with} \quad \lambda = \inf_{x \in C} \Theta(T, x) \geqq c > 0.$$

Hence $E := \{x \in X \mid \Theta(T, x) > 0\}$ is at most countable (but not closed unless finite). So in some sense those T are similar to smooth currents or more generally currents which are given by forms locally in L^1; these of course cannot have support in S. But there will be a lot of other currents having the above property.

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Local Cohomology Along Exceptional Sets

Ulrich Karras

Abteilung Mathematik, Universität, Postfach 500500, D-4600 Dortmund, Federal Republic of Germany

Introduction

Let Y be a closed analytic subset of a complex manifold X of dimension n. Let $\mathscr{F}$ be a locally free coherent sheaf on X, and let ω be the dualizing sheaf. Then the cup-product and natural mappings induce duality homomorphisms

$$\Psi_i : H^i_Y(X, \mathscr{F}) \to (H^{n-i}(Y, \mathscr{F}^* \otimes \omega))', \quad i \geqq 0.$$

Here $\mathscr{F}^*$ means the dual sheaf $\mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$ and $'$ denotes dual vector space. We are particularly interested in the case that Y is an exceptional set. Then Y can be blown down to isolated singular points and a spectral sequence argument shows that the local cohomology groups $H^i_Y(X, \mathscr{F})$ are finite-dimensional for $i \leqq n-1$. Thus one can show that the duality homomorphisms are essentially induced by Serre duality and that following stronger result is valid.

Theorem. *Let E be an exceptional subset of a reduced equi-dimensional complex space X of dimension n, and let $\mathscr{F}$ be a locally free coherent sheaf on X. If X is Cohen-Macaulay, then $H^i_E(X, \mathscr{F})$ is finite-dimensional and*

$$\Psi_i : H^i_E(X, \mathscr{F}) \to (H^{n-i}(E, \mathscr{F}^* \otimes \omega_X))' \text{ is an isomorphism} \quad \text{for} \quad i < n.$$

Moreover, if X is smooth near E, then Serre duality induces isomorphisms

$$H^i_E(X, \mathscr{F} \otimes_X \Omega^q_X) \cong (H^{n-i}(E, \mathscr{F}^* \otimes_X \Omega^{n-q}_X))' \quad \text{for} \quad i < n,$$

where Ω^q_X denotes the sheaf of holomorphic q-forms on X.

The purpose of this note is of two kinds. At first we wish to point out the remarkable fact that this result may be considered as a culmination point of a priori two quite different duality theorems, namely Serre duality and Grothendieck duality for algebraic local cohomology, compare Theorem 3.6. Secondly, although above result is more or less known to the experts, it might fill a gap of having a reference available.

Taking into account that $H^i_E(X, \mathscr{F})$ may be identified with cohomology of a strongly pseudoconvex neighborhood of E with compact support, see 2.3, the theorem above can be readily deduced from Serre duality, see Theorem 3.2.

The algebraic local cohomology groups $H^i_{[Y]}(X,\mathscr{F})$ are defined by $\varinjlim \operatorname{Ext}^i(X;\mathcal{O}_X/J^m,\mathscr{F})$ where J is an ideal sheaf defining the subspace Y. They were originally introduced by Grothendieck in order to compute local cohomology in algebraic geometry. However this method does not in general work in the analytic category. Of course, there also exists a morphism comparing algebraic local cohomology and local cohomology. But this morphism is in general neither injective nor surjective even if $\mathscr{F}$ is coherent, see [13, 14]. Our main result (1.6) in Sect. 1 says that in case of an exceptional subset E we have isomorphisms $H^i_{[E]}(X,\mathscr{F})\cong H^i_E(X,\mathscr{F})$ for $i<d$ where d is equal to the depth (=homological codimension) of $\mathscr{F}$ on X.

Now the crucial point is that the algebraic local cohomology functor provides a good tool to carry over methods of algebraic geometry to analytic geometry. In particular it enabled us to combine the massive machineries of local duality in algebraic geometry [7], and of relative duality in analytic geometry, [11]. As a result we obtain a rather general duality theorem, compare Theorem 3.6, from which we can easily deduce a second proof of the theorem we announced.

In Sect. 4 we finally give some applications concerning isolated singularities.

1. Comparison with Algebraic Local Cohomology

At first let us briefly recall some important facts about local cohomology, see [6, 7, 14].

1.1. Let Y be a closed subset of a topological space X, and let $\mathscr{F}$ be an abelian sheaf on X. Then $\Gamma_Y(X,\mathscr{F})$ denotes the subgroup of $\Gamma(X,\mathscr{F})$ consisting of all those sections of $\mathscr{F}$ whose support is contained in Y. By $\underline{\Gamma}_Y(\mathscr{F})$ we denote the sheaf associated to the presheaf $U\to\Gamma_{Y\cap U}(U,\mathscr{F}|U)$. Define $\mathbb{R}\Gamma_Y(X,\cdot)$, respectively $\mathbb{R}\underline{\Gamma}_Y(\cdot)$, to be the right derived functor of $\Gamma_Y(X,\cdot)$, respectively $\underline{\Gamma}_Y(\cdot)$, on the derived category. The i^{th} right derived functors are denoted by $H^i_Y(X,\mathscr{F})$, respectively $\mathscr{H}^i_Y(\mathscr{F})$, and are called local cohomology groups, respectively local cohomology sheaves, of $\mathscr{F}$ with supports in Y. There are exact sequences

$$\dots\to H^i_Y(X,\mathscr{F})\to H^i(X,\mathscr{F})\to H^i(X-Y,\mathscr{F})\to H^{i+1}_Y(X,\mathscr{F})\to\dots,$$

$$0\to\mathscr{H}^0_Y(\mathscr{F})\to\mathscr{F}\to j_*(\mathscr{F}|X-Y)\to\mathscr{H}^1_Y(\mathscr{F})\to 0$$

and

$$\mathscr{H}^{q+1}_Y(\mathscr{F})\cong R^q j_*(\mathscr{F}|X-Y),\qquad q>0.$$

Here j denotes the natural embedding $X-Y\to X$.

Furthermore, since $\Gamma_Y(X,\cdot)=\Gamma(X,\cdot)\circ\underline{\Gamma}_Y(\cdot)$, we have $\mathbb{R}\Gamma_Y(X,\cdot)=\mathbb{R}\Gamma(X,\cdot)\circ\mathbb{R}\underline{\Gamma}_Y(\cdot)$ or equivalently in terms of spectral sequences

$$E_2^{pq}=H^p(X,\mathscr{H}^q_Y(\mathscr{F}))\Rightarrow H^n_Y(X,\mathscr{F}),\qquad n=p+q.\tag{1.1.1}$$

1.2. Suppose that X is a complex space and Y a closed analytic subset. Let $\mathscr{F}$ be an analytic sheaf on X. Then $\mathscr{H}^i_Y(\mathscr{F})$, $i\geqq 0$, is an analytic sheaf whose support is contained in Y. However, it is in general neither an $\mathcal{O}_Y$-sheaf nor a quasi-coherent sheaf even if $\mathscr{F}$ is coherent.

To compute local cohomology, Grothendieck introduced algebraic local cohomology. Let J be an ideal sheaf defining Y. Then one defines

$$\Gamma_{[Y]}(X,\mathscr{F}):=\varinjlim_{m}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X/J^m,\mathscr{F}) \quad\text{and}\quad \underline{\Gamma}_{[Y]}(\mathscr{F}):=\varinjlim_{m}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X/J^m,\mathscr{F}).$$

This definition depends only on Y and is independent of the choice of J. One observes that $\underline{\Gamma}_{[Y]}(\mathscr{F})$ is an analytic subsheaf of $\mathscr{F}$ whose sections are locally annihilated by powers of J. Since both $\Gamma_{[Y]}(X,\cdot)$ and $\underline{\Gamma}_{[Y]}(\cdot)$ are left exact and covariant functors, it follows that the right derived functors $\mathbb{R}\Gamma_{[Y]}(X,\cdot)$ and $\mathbb{R}\underline{\Gamma}_{[Y]}(\cdot)$ and the i^{th}-right derived functors $H^i_{[Y]}(X,\mathscr{F}^\cdot)$ and $\mathscr{H}^i_{[Y]}(\mathscr{F}^\cdot)$, $i\geqq 0$, exist. Here $\mathscr{F}^\cdot$ denotes an analytic sheaf-complex with bounded cohomology. Since taking cohomology of complexes commutes with direct limits it follows that $H^i_{[Y]}(X,\mathscr{F}^\cdot)=\varinjlim \operatorname{Ext}^i(X;\mathcal{O}_X/J^m,\mathscr{F}^\cdot)$ and, of course, analogous for the case of algebraic local cohomology sheaves.

1.3. For each m one has a natural map

$$\operatorname{Hom}(\mathcal{O}_X/J^m,\mathscr{F})\to\Gamma_Y(X,\mathscr{F})$$

since an element of the first group is given by a global section of $\mathscr{F}$ which is annihilated by J^m, and hence has support on Y. Taking the direct limit as m varies and taking derived functors, one deduces a map of functors

$$\mathbb{R}\varrho:\mathbb{R}\Gamma_{[Y]}(X,\cdot)\to\mathbb{R}\Gamma_Y(X,\cdot).$$

Thus we obtain natural homomorphisms $\varrho_i: H^i_{[Y]}(X,\mathscr{F})\to H^i_Y(X,\mathscr{F})$, $i\geqq 0$, and by analogous arguments $\underline{\varrho}_i:\mathscr{H}^i_{[Y]}(\mathscr{F})\to\mathscr{H}^i_Y(\mathscr{F})$, $i\geqq 0$.

It follows from [14, Proposition 1.9] that, if $\mathscr{F}$ is a coherent analytic sheaf, ϱ_0 and $\underline{\varrho}_0$ are isomorphisms. However, since the category of coherent sheaves is not "thick enough", we cannot apply the lemma on way-out functors, [7, I.7.1] to deduce that the morphism $\mathbb{R}\varrho$ of derived functors is an isomorphism. Indeed, one has only following weaker results:

Suppose the sheaves $\mathscr{H}^i_Y(\mathscr{F})$ are coherent for $0\leqq i\leqq q$. Then

$$\underline{\varrho}_i \text{ is an isomorphism for } 0\leqq i\leqq q \text{ and injective for } i=q+1, \tag{1.3.1}$$

$$\varrho_i \text{ is an isomorphism for } 0\leqq i\leqq q. \tag{1.3.2}$$

The first part follows from [13, I.5.5]. Then, using the spectral sequence in (1.1.1), one can easily check the second assertion.

1.4. In [14, Theorem 3.5], there are given necessary and sufficient conditions under which local cohomology sheaves are coherent. Let $\mathscr{F}$ be a coherent analytic sheaf on X. Then the sets $S_m(\mathscr{F}):=\{x\in X\,\operatorname{depth}_x\mathscr{F}\leqq m\}$ are closed analytic subsets of X and are called the *singularity subvarieties* of $\mathscr{F}$. Now given a closed analytic subset A of X. Then for $q\geqq 0$ following conditions are equivalent:

(a) $\mathscr{H}^i_A(\mathscr{F})$ is coherent for $0\leqq i\leqq q$.

(b) $\dim_{\mathbb{C}}(A\cap\bar{S}_{k+q+1}(\mathscr{F}\,|\,X-A))\leqq k$ for all $k\in\mathbb{Z}$.

Examples. (1.4.1) Let $n=\inf\{\operatorname{depth}_y\mathscr{F}\,|\,y\in X-A\}$. Then the local cohomology sheaves $\mathscr{H}^i_A(\mathscr{F})$ are coherent for all $i<n-\dim A$.

(1.4.2) Let $\mathscr{F}$ be a locally free coherent sheaf on a resolution of an isolated surface singularity. Then the local cohomology sheaf $\mathscr{H}^1_E(\mathscr{F})$ along the exceptional set E is not coherent.

1.5. Now suppose that E is a connected exceptional subset of a complex space X. Then the Stein-factorization $\pi: X \to S$ with respect to E yields a complex space S having a distinguished singularity at $p := \pi(E)$.

$$\Gamma_p(S, \pi_*\mathscr{F}) \cong \Gamma_E(X, \mathscr{F}) \quad \text{and} \quad \Gamma_{[p]}(S, \pi_*\mathscr{F}) \cong \Gamma_{[E]}(X, \mathscr{F}). \tag{1.5.1}$$

The first identification is obvious. To check the second one, recall that $\Gamma_{[p]}(S, \pi_*\mathscr{F}) \cong \varinjlim \operatorname{Hom}(S; \mathcal{O}_S/\mathfrak{m}^n, \pi_*\mathscr{F})$ where $\mathfrak{m}$ is the maximal ideal sheaf at p. On the other hand we observe that $\operatorname{Hom}(S; \mathcal{O}_S/\mathfrak{m}^n, \pi_*\mathscr{F}) \cong \operatorname{Hom}(X; \mathcal{O}_X/\pi^*\mathfrak{m}^n \cdot \mathcal{O}_X, \mathscr{F})$. Now the assertion can be readily seen since $\pi^*\mathfrak{m}^n \cdot \mathcal{O}_X$ is an ideal sheaf defining the exceptional set E and $\mathcal{O}_S/\mathfrak{m}^n$ is supported on p.

As an application we immediately get following commutative diagram of derived functors:

$$\begin{tikzcd} \mathbb{R}\Gamma_{[p]}(S, \cdot) \circ \mathbb{R}\pi_* \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathbb{R}\Gamma_p(S, \cdot) \circ \mathbb{R}\pi_* \arrow[d, phantom, "\wr\|"] \\ \mathbb{R}\Gamma_{[E]}(X, \cdot) \arrow[r, "\mathbb{R}\varrho"] & \mathbb{R}\Gamma_E(X, \cdot) \end{tikzcd} \tag{1.5.2}$$

This yields a commutative diagram of spectral sequences:

$$\begin{tikzcd} H^i_{[p]}(S, R^k\pi_*\mathscr{F}) \arrow[r, "\varrho_{ik}"'] \arrow[d, Rightarrow] & H^i_p(S, R^k\pi_*\mathscr{F}) \arrow[d, Rightarrow] \\ H^m_{[E]}(X, \mathscr{F}) \arrow[r, "\varrho_m"'] & H^m_E(X, \mathscr{F}), \quad m = k+i. \end{tikzcd} \tag{1.5.3}$$

Remark. A good reference for local cohomology at isolated singular points is the paper of Greuel [4].

1.6. Proposition. *Let E be an exceptional subset of a complex space X of dimension n. Let $\mathscr{F}$ be a coherent analytic sheaf on X such that* $\operatorname{depth}_x \mathscr{F} \geqq d$ *for $x \in U - E$ where U is an open neighbourhood of E in X. Then the homomorphisms*

$$\varrho_k: H^m_{[E]}(X, \mathscr{F}) \to H^m_E(X, \mathscr{F})$$

are isomorphisms for $m < d$.

Proof. Without loss of generality we may assume that E is connected. Using (1.5.3) it suffices to show that the homomorphisms ϱ_{ik} are bijective for all values of i if $k > 0$ and for all $i < d$ if $k = 0$, see [5, Chap. 0, Proposition 11.15]. By hypothesis it follows from 1.4 that the sheaves $\mathscr{H}^i_p(\pi_*\mathscr{F})$ are coherent for $i < d$. Furthermore it is clear that the sheaves $\mathscr{H}^i_p(R^k\pi_*\mathscr{F})$ are coherent for all i if $k \geqq 1$ since the higher direct image sheaves are supported on p. So we can apply (1.3.2).

2. Cohomology with Compact Supports

2.1. Suppose that X is a paracompact Hausdorff space. Let the subscript c denote the family of compact subsets of X, and define $\Gamma_c(X, \mathscr{F})$ to be the group of global sections of $\mathscr{F}$ whose supports are in c. Define $\mathbb{R}\Gamma_c(X,\cdot)$ and $H^i_c(X,\mathscr{F})$ to be the right derived functors. There exists a natural exact sequence

$$\ldots\to H^i_c(X,\mathscr{F})\to H^i(X,\mathscr{F})\to H^i_\infty(X,\mathscr{F})\to\ldots \tag{2.1.1}$$

where $H^i_\infty(X,\mathscr{F})$ is called the cohomology at infinity. Take K to be a compact subset of X, then there are natural homomorphisms $H^i(X-K,\mathscr{F})\to H^i_\infty(X,\mathscr{F})$, $i\geqq 0$. This can be used to compute cohomology at infinity as follows. If $X=\bigcup K_\alpha$, $\alpha\in\Lambda$, where Λ is a directed set, K_α a subset of the interior of K_β if $\alpha<\beta$ and K_α a compact subset of X, then the natural homomorphisms above induce an isomorphism

$$\varinjlim H^i(X-K_\alpha,\mathscr{F})\cong H^i_\infty(X,\mathscr{F}) \quad \text{for each} \quad i\geqq 0,\ [9]. \tag{2.1.2}$$

Now let Y be a compact subset of X. Then we obtain following commutative diagram of exact cohomology sequences:

$$\begin{array}{ccccccccc}
..\to H^{i-1}(X,\mathscr{F}) & \to & H^{i-1}_\infty(X,\mathscr{F}) & \to & H^i_c(X,\mathscr{F}) & \to & H^i(X,\mathscr{F}) & \to & H^i_\infty(X,\mathscr{F})\to ..\\
\| & & \uparrow & & \uparrow\gamma_i & & \| & & \uparrow\\
..\to H^{i-1}(X,\mathscr{F}) & \to & H^{i-1}(X-Y,\mathscr{F}) & \to & H^i_Y(X,\mathscr{F}) & \to & H^i(X,\mathscr{F}) & \to & H^i(X-Y,\mathscr{F})\to ..
\end{array} \tag{2.1.3}$$

In general one does not know anything about the map γ_i but in the special case of exceptional subsets one can get much more precise informations.

2.2. Let E be an exceptional subset of a reduced complex space X. Then it is well known, [2, Satz 5], that there exists a strongly pseudoconvex neighborhood M of E in X and a non-negative exhaustion function ξ on M such that ξ is strongly plurisubharmonic on $M-E$ and $E=\{x\in M\,|\,\xi(x)=0\}$. Clearly, $H^k_E(X,\mathscr{F})=H^k_E(M,\mathscr{F})$ for $k\geqq 0$. Hence we have natural homomorphisms $\gamma_k\colon H^k_E(X,\mathscr{F})\to H^k_c(M,\mathscr{F})$, $k\geqq 0$.

2.3. Proposition. *If $\mathscr{F}$ is a coherent analytic sheaf on X such that* $\operatorname{depth}_x\mathscr{F}\geqq d$ *for $x\in M-E$, then*

$$\gamma_k\colon H^k_E(X,\mathscr{F})\to H^k_c(M,\mathscr{F})$$

is an isomorphism for $k<d$.

Proof. Applying the 5-lemma in (2.1.3) it suffices to show that

(*) $H^k(M-E,\mathscr{F})\to H^k_\infty(M,\mathscr{F})$ is bijective for $k<d-1$ and injective for $k=d-1$.

Let $B_r:=\{x\in M\,|\,\xi(x)<r\}$ and $\bar{B}_r:=\{\xi(x)\leqq r\}$, $r\geqq 0$, where ξ is an exhaustion function on M as in 2.2. Clearly $B_0=\emptyset$ and $\bar{B}_0=E$. By (2.1.2) we have $\varinjlim H^k(M-\bar{B}_r,\mathscr{F})\cong H^k_\infty(M,\mathscr{F})$. Now we set $M':=M-\bar{B}_r$ and $B'_s:=B_s\cap M'$ for $s>r\geqq 0$. Then it follows from [1; Théorème 15] that

$$H^k(M-\bar{B}_r,\mathscr{F})=H^k(M',\mathscr{F})\to H^k(M'-B'_s,\mathscr{F})=H^k(M-B_s,\mathscr{F})$$

is an isomorphism if $k<d-1$ and is injective if $k=d-1$. On the other hand note that the diagrams

$$\begin{array}{ccc}
H^k(M-\bar{B}_r,\mathscr{F}) & \longrightarrow & H^k(M-\bar{B}_s,\mathscr{F})\\
& \searrow & \downarrow\\
& & H^k(M-B_t,\mathscr{F})
\end{array}$$

commute for all $k \leqq d-1$ and $t > s > r \geqq 0$. Hence the assertion in (*) can be readily seen.

3. Duality

3.1. At first we recall some basic facts about duality theory and set some notations for what follows. Let X be an equi-dimensional reduced space of dimension n. By [12] there exists a *dualizing complex* $\omega_X^{\cdot} \in D_c^+(X)$ on X. Here $D_c^+(X)$ denotes the derived category of below bounded complexes whose cohomology sheaves are coherent. If X is Cohen-Macaulay, then it follows from [12, p. 89] that the cohomology of $\omega_X^{\cdot}$ is only non-trivial at the degree $-n$. The corresponding cohomology sheaf is denoted by ω_X and is called the *dualizing sheaf* on X. One may ω_X identify with the direct image of the sheaf of holomorphic n-forms on the smooth part of X if X is normal, too. Furthermore it is well known that X is Gorenstein if and only if the dualizing sheaf is invertible.

3.2. Theorem. a) *Let $\mathscr{F}$ be a locally free coherent sheaf on an equi-dimensional reduced complex space X of dimension n, and let E be an exceptional subset of X. If X is Cohen-Macaulay, then Serre duality induces isomorphisms*

$$\psi_i: H_E^i(X, \mathscr{F}) \to (H^{n-i}(E, \mathscr{F}^* \otimes \omega_X))' \quad \textit{for} \quad i < n.$$

b) *If there exists an open smooth neighborhood of E in X, then Serre duality induces isomorphisms*

$$\psi_{iq}: H_E^i(X, \mathscr{F} \otimes \Omega_X^q) \to (H^{n-i}(E, \mathscr{F}^* \otimes \Omega_X^{n-q}))' \quad \textit{for} \quad i < n.$$

Proof. At first we do the second part. By hypothesis, we can choose a smooth strongly pseudoconvex neighborhood, say M, of E in X. Taking into account that $\Omega_X^q | M = \Omega_M^q$, Serre duality, see [9], yields isomorphisms $H_c^i(M, \mathscr{F} \otimes \Omega_X^q) \cong (H^{n-i}(M, \mathscr{F}^* \otimes \Omega_X^{n-q}))'$ for $i < n$ since $H^k(M, \mathscr{G})$ is finite dimensional for $k \geqq 1$ if $\mathscr{G}$ is a coherent analytic sheaf on M, see [1, Theorem 11]. Putting Serre duality together with the isomorphisms $H^k(M, \mathscr{F}^* \otimes \Omega^{n-q}) \cong H^k(E, \mathscr{F}^* \otimes \Omega_X^{n-q})$ for $k \geqq 1$ after possibly shrinking of M, [1, Proposition 21], and $\gamma_i: H_E^i(X, \mathscr{F} \otimes \Omega_X^q) \to H_c^i(M, \mathscr{F} \otimes \Omega_X^q)$, $i < n$, see 2.3, we obtain the isomorphisms ψ_{iq} we are looking for.

Analogous arguments work in case a) where we use generalized Serre duality, see [12]. It is easy to check that ψ_i can be described as in the introduction.

3.3. Corollary. *Let E be an exceptional set of a n-dimensional complex manifold M, then*

$$H_E^i(M, \mathcal{O}_M) = 0 \quad \textit{and} \quad H_E^i(M, \omega_M) \cong H^{n-i}(E, \mathcal{O}_M) \quad \textit{for} \quad i < n.$$

The proof is a trivial consequence of the theorem above and the vanishing theorem of Grauert and Riemenschneider [3].

3.4. Let E be a connected exceptional subset of X, and let $\pi: X \to S$ denote the Stein-factorization of X with respect to E. We write A for the local ring of S at $p := \pi(E)$. Let $\omega_S^{\cdot}$ be the dualizing complex on S. Then it can be readily seen, [12], that $\omega_p^{\cdot} := (\omega_S^{\cdot})_p$ is a dualizing complex in $D_c^+(A)$, i.e. the derived category of complexes of A-modules whose cohomology modules are of finite type. We may also assume that $\omega_p^{\cdot}$ is normalized in the sense of [7, p. 276].

3.5. Let $\mathscr{F}^{\cdot} \in D^+(S)$. We write $\mathbb{R}\Gamma_{[p]}(\mathscr{F}^{\cdot})$ for the stalk of $\mathbb{R}\underline{\Gamma}_{[p]}(\mathscr{F}^{\cdot})$ at p. It is clear that $H^i_{[p]}(\mathscr{F}^{\cdot}) = \varinjlim \operatorname{Ext}^i_A(A/\mathfrak{m}^k, \mathscr{F}^{\cdot}_p)$ where $\mathfrak{m}$ is the maximal ideal of A. Thus $\mathbb{R}\Gamma_{[p]}(\mathscr{F}^{\cdot})$ is uniquely determined by $\mathscr{F}^{\cdot}_p$. On the other hand we observe that

$$\mathbb{R}\Gamma_{[p]}(S, \mathscr{F}^{\cdot}) \cong \mathbb{R}\Gamma_{[p]}(\mathscr{F}^{\cdot})$$

since $\mathbb{R}\Gamma_{[p]}(S, \cdot) = \mathbb{R}\Gamma(S, \cdot)\,\mathbb{R}\underline{\Gamma}_{[p]}(\cdot)$, see (1.1), and $\mathbb{R}\underline{\Gamma}_{[p]}(\mathscr{F}^{\cdot})$ is supported on p. So $\mathbb{R}\Gamma_{[p]}(S, \cdot)$ may be interpreted as the derived functor of $\Gamma_{[p]}(\cdot)$ on $D^+_c(A)$. Now take $\mathscr{F}^{\cdot} = \omega^{\cdot}_S$. Then referring to [7, pp. 276–277], we can deduce that

$$I := H^0_{[p]}(\omega^{\cdot}_S) \text{ is the injective hull (up to non-unique isomorphism) of the residue field } \mathbb{C} = A/\mathfrak{m} \text{ and } \mathbb{R}\Gamma_{[p]}(\omega^{\cdot}_S) \cong I \text{ in } D^+_c(A). \tag{3.5.1}$$

3.6. Theorem. *Let E be a connected exceptional subset of an equi-dimensional complex space X of dimension n, and let $\mathscr{F}$ be a coherent analytic sheaf on X. Then there exist (algebraic) isomorphisms*

$$\Phi_i \colon H^i_{[E]}(X, \mathscr{F}) \to \operatorname{Hom}_A(\operatorname{Ext}^{-i}_{\mathcal{O}_X}(E; \mathscr{F}, \omega^{\cdot}_X), I), \quad 0 \leqq i \leqq n.$$

Proof. As above A denotes the local ring of the space S at $p = \pi(E)$ one obtains by blowing down the exceptional set, $\pi \colon X \to S$. Referring to our discussion in (3.5), analogous arguments as in [7, p. 277] show that there is a natural morphism $\Gamma_{[p]}(\cdot) \to \operatorname{Hom}_A(\operatorname{Hom}_A(\cdot, \omega^{\cdot}_p), \Gamma_{[p]}(\omega^{\cdot}_S))$ which gives rise to a morphism of functors on $D^+(A)$,

$$\theta \colon \mathbb{R}\Gamma_{[p]}(\cdot) \to \operatorname{Hom}_A(\mathbb{R}\operatorname{Hom}_A((\cdot, \omega^{\cdot}_p), I).$$

Since π is proper we know by Grauert's coherence theorem that $\mathbb{R}\pi_*\mathscr{F} \in D^+_c(S)$. Thus we may apply Grothendieck's local duality theorem, [7, p. 278], which implies that θ induces an isomorphism if we replace $(\cdot)$ by $\mathbb{R}\pi_*\mathscr{F}$, respectively $(\mathbb{R}\pi_*\mathscr{F})_p$. On the other hand we observe that

$$\mathbb{R}\Gamma_{[p]}(\mathbb{R}\pi_*\mathscr{F}) \cong \mathbb{R}\Gamma_{[p]}(S, \mathbb{R}\pi_*\mathscr{F}) \cong \mathbb{R}\Gamma_{[E]}(X, \mathscr{F}),$$

compare 3.5 and (1.5.2). Consequently, the homomorphisms

$$\theta_i \colon H^i_{[E]}(X, \mathscr{F}) \to \operatorname{Hom}_A(\operatorname{Ext}^{-i}_A((\mathbb{R}\pi_*\mathscr{F})_p, \omega^{\cdot}_p), I)$$

induced by θ are isomorphisms for $i \geqq 0$. The second important ingredient we need is the relative duality, compare [9, p. 264], which yields a natural isomorphism

$$\tau \colon \mathbb{R}\pi_*\mathbb{R}\,\mathscr{H}om(\mathscr{F}, \omega^{\cdot}_X) \xrightarrow{\cong} \mathbb{R}\,\mathscr{H}om_{\mathcal{O}_S}(\mathbb{R}\pi_*\mathscr{F}, \omega^{\cdot}_S).$$

Hence τ induces isomorphisms

$$\tau_i \colon \operatorname{Ext}^{-i}_{\mathcal{O}_X}(E; \mathscr{F}, \omega^{\cdot}_X) \xrightarrow{\cong} \operatorname{Ext}^{-i}_A((\mathbb{R}\pi_*\mathscr{F})_p, \omega^{\cdot}_p), \quad i \geqq 0.$$

By putting together θ_i and τ_i^{-1}, we obtain the isomorphisms Φ_i, $i \geqq 0$, we are looking for.

3.7. Corollary. *If $\mathscr{F}$ is locally free and X is Cohen-Macaulay, then there are isomorphisms*

$$\Phi_i \colon H^i_{[E]}(X, \mathscr{F}) \to (H^{n-i}(E, \mathscr{F}^* \otimes \omega_X))', \quad 0 \leqq i \leqq n.$$

Proof. We note that there exists a canonical isomorphism

$$\operatorname{Hom}_A(N, I) \xrightarrow{\cong} \varinjlim \operatorname{Hom}_{\mathbb{C}}(N/\mathfrak{m}^k N, \mathbb{C})$$

for any A-module N, see [5, p. 63]. Thus $\mathrm{Hom}_A(N,I)\cong\mathrm{Hom}_{\mathbb{C}}(N,\mathbb{C})$ if N is artinian. On the other hand we have $\mathrm{Ext}_{\mathcal{O}_X}^{-i}(E;\mathcal{F},\omega_X^{\cdot})=\mathrm{Ext}_{\mathcal{O}_X}^{n-i}(E;\mathcal{F},\omega_X)$, compare 3.1, and $\mathrm{Ext}_{\mathcal{O}_X}^{n-i}(E;\mathcal{F},\omega_X)=H^{n-i}(E,\mathcal{F}^*\otimes\omega_X)$ since $\mathcal{F}$ is locally free.

3.8. Second proof of Theorem 3.2. By 3.7 and 1.6 the composition $\Phi_i\circ\varrho_i^{-1}$ defines a duality isomorphism for all $i<n$. Further it can be readily seen that ψ_i may be identified with $\Phi_i\circ\varrho_i^{-1}$, $i<n$.

4. Isolated Singularities

Let V be a reduced Stein space such that $U=V-\{p\}$ is a n-dimensional complex manifold. Then the space germ (V,p) is called a n-dimensional isolated singularity. The object of this section is to point out several results concerning invariants of isolated singularities. Although most of them are more or less known, it seems to be worthwhile to present proofs under the unifying viewpoint of applying local cohomology. Especially this method provides some considerable simplifications.

4.1. Suppose that $\pi:M\to V$ is a resolution of the singularity (V,p). Then $\Gamma(V,R^i\pi_*\mathcal{O}_M)$ are finite-dimensional $\mathbb{C}$-vector spaces for $i>0$, and it is well known that their dimensions are independent of the resolution, i.e. they are numerical invariants of the singularity (V,p) and will be denoted by $h^i(V,p)$, $i>0$. Also $\delta(V,p):=\dim_{\mathbb{C}}\Gamma(V,\pi_*\mathcal{O}_M/\mathcal{O}_V)$ turns out to be an invariant of (V,p) because $\pi_*\mathcal{O}_M$ can be identified with the normalization of $\mathcal{O}_V$. Hence $\delta(V,p)=\dim_{\mathbb{C}}\tilde{\mathcal{O}}_{V,p}/\mathcal{O}_{V,p}$. The *geometric genus* $p_g(V,p)$ of (V,p) is defined by

$$(-1)^{n+1}p_g(V,p):=\delta(V,p)+\sum_i(-1)^{i+1}h^i(V,p),\qquad 1\leqq i\leqq n-1\,.$$

Following result gives an intrinsic characterization of these invariants which is due to Yau [18].

4.2. Proposition. *Let (V,p) be a n-dimensional isolated singularity, $n\geqq 2$. Then*
(i) $h^i(V,p)=0$ *for* $i\geqq n$.
(ii) $h^i(V,p)=\dim_{\mathbb{C}}H^{i+1}_{\{p\}}(V;\mathcal{O}_V)$ *for* $1\leqq i\leqq n-2$.
(iii) $h^{n-1}(V,p)=\dim_{\mathbb{C}}(\Gamma(U,\Omega^n_U)/L^2(U,\Omega^n_U))$.
Here, $L^2(U,\Omega^n_U)$ means the subspace of n-forms on U which are square-integrable.

Proof. Part (i) is well known. To show the other statements, we first observe that, by the vanishing of $H^i_E(M;\mathcal{O}_M)$, 3.3, it follows from the long exact sequence of local cohomology that $H^i(M;\mathcal{O}_M)\cong H^i(M-E;\mathcal{O}_M)$ if $i\leqq n-2$. Furthermore, the long exact sequence of local cohomology with respect to the singular point p,

$$\to H^i(V;\mathcal{O}_V)\to H^i(U;\mathcal{O}_U)\to H^{i+1}_{\{p\}}(V;\mathcal{O}_V)\to H^{i+1}(V;\mathcal{O}_V)\to\,, \tag{4.2.1}$$

yields that $H^i(M;\mathcal{O}_M)\cong H^{i+1}_{\{p\}}(V;\mathcal{O}_V)$ for $1\leqq i\leqq n-2$ since $H^k(V;\mathcal{O}_V)=0$ for $k\geqq 1$ and $M-E\cong U$. So it remains to check the last statement. Let $\tilde{j}:M-E\to M$ be the natural embedding. Consider the exact sequence

$$0\to\omega_M\to\tilde{j}_*(\tilde{j}^*\omega_M)\to\mathcal{H}^1_E(\omega_M)\to 0\,,$$

compare 1.1 and note that $\mathcal{H}^0_E(\omega_M)=0$. Since further $R^1\pi_*\omega_M=0$, we get following exact sequence of analytic sheaves on V:

$$0\to\pi_*\omega_M\to\pi_*\tilde{j}_*(\tilde{j}^*\omega_M)\to\pi_*\mathcal{H}^1_E(\omega_M)\to 0\,. \tag{4.2.2}$$

Let $j: U \to V$ denote the inclusion of the smooth locus. Then we have $\pi_* \tilde{j}_*(\tilde{j}^* \omega_M) = j_* \pi_*(\tilde{j}^* \omega_M) = j_* \omega_U$. Thus (4.2.2) implies that

$$H^0(M; \mathscr{H}_E^1(\omega_M)) = \Gamma(V, \pi_* \mathscr{H}_E^1(\omega_M)) \cong \Gamma(V, j_* \omega_U)/\Gamma(V, \pi_* \omega_M).$$

On the other hand, $H_E^1(M; \omega_M) = H^0(M; \mathscr{H}_E^1(\omega_M))$, [14, 0.6]. Applying 3.3 and Laufer's description of $\pi_* \omega_M$, [8, Theorem 3.1], one can now easily finish the proof.

4.3. Corollary (Kempf). *Let (V, p) be an isolated singularity of dimension $n \geqq 2$. Then following conditions are equivalent:*

(a) *(V, p) is normal and $h^i(V, p) = 0$ for all $i > 0$.*

(b) *(V, p) is Cohen-Macaulay and $\pi_* \omega_M \cong \omega_V$, where $\pi: M \to V$ is a resolution and ω_V is the dualizing sheaf on V.*

Proof. Normality of (V, p) means that $H^0(V, \mathcal{O}_V) \cong H^0(U, \mathcal{O}_U)$. Thus it follows from Proposition 4.2 and (4.2.1) that $H^i_{\{p\}}(V, \mathcal{O}_V) = 0$ for $i < n$ if we assume (a). There exists a characterization of depth in terms of local cohomology which implies that $\operatorname{depth}(\mathcal{O}_{V,p}) \geqq r$ if and only if $H^i_{\{p\}}(V, \mathcal{O}_V) = 0$ for $i < r$. So (V, p) is Cohen-Macaulay. Moreover we observe that $\omega_V = j_* \omega_U$ since (V, p) is normal. Then the last statement in 4.2 is equivalent to say that $\omega_V \cong \pi_* \omega_M$. It is now also clear how to prove the conversion "(b) $\Rightarrow$ (a)".

Remark. The properties (a) and (b) characterize rational singularities.

4.4. Corollary. *If (V, p) is a Cohen-Macaulay singularity of dimension $n \geqq 2$, then*

$$p_g(V, p) = h^{n-1}(V, p) = \dim_{\mathbb{C}}(\omega_V / \pi_* \omega_M)_p.$$

Proof. Recall that Cohen-Macaulay singularities of dimension $n \geqq 2$ are automatically normal. Hence $h^i(V, p) = 0$ for $i \leqq n-2$ by previous result.

4.5. From now on let us assume that (V, p) is a normal surface singularity. Let $\pi: M \to V$ be a resolution. Denote by $D((M, E), -)$ the functor of deformations of the germ of M along the exceptional set E. There are natural subfunctors $ES((M, E), -) \subset \mathscr{B}((M, E), -) \subset D((M, E), -)$ where $\mathscr{B}((M, E), -)$ is the functor of those deformations of (M, E) which simultaneously blow down to deformations of (V, p) and $ES((M, E), -)$ is the functor of blowing down deformations to which all exceptional divisors on M lift in an equisingular way. If π is a good resolution, last condition is equivalent to saying deformations of M do not change the topology of the embedding of E into M. Denote by T_V^1 the finite-dimensional vector space of infinitesimal deformations of (V, p) which can be identified with $\operatorname{Ext}^1_{\mathcal{O}_V}(\Omega_V^1, \mathcal{O}_V)$. Then the blowing down condition defines following commutative diagram

$$\begin{tikzcd} \mathscr{B}((M, E), \mathbb{C}[\varepsilon]) \arrow[r, "\Phi"'] & T_V^1, \quad \varepsilon^2 = 0. \\ ES((M, E), \mathbb{C}[\varepsilon]) \arrow[u, hook] \arrow[ur, "\Phi_{ES}"'] & \end{tikzcd}$$

4.6. Theorem (Wahl). *Suppose that $\pi: M \to V$ is an equivariant resolution. Then the Kodaira-Spencer isomorphism $D((M, E), \mathbb{C}[\varepsilon]) \xrightarrow{\cong} H^1(M, \Theta_M)$ induces an isomorphism*

(i) $\operatorname{Ker}(\Phi) \xrightarrow{\cong} H_E^1(M, \Theta_M)$.

If π is the minimal good resolution, then one has:

(ii) $\operatorname{Ker}(\Phi_{ES}) \cong H_E^1(M, \Theta_M(\log E)) = 0$.

(iii) $\dim_{\mathbb{C}} \operatorname{Ker}(\Phi) = \#\{$*irreducible components* E_i *of* E *with* $E_i^2 = -2\}$ *if* (V, p) *is rational.*

Proof. If $\mathscr{F}$ is a locally free coherent sheaf on M, we know by 1.6 that $H_E^1(M, \mathscr{F}) \cong \varinjlim \operatorname{Ext}_M^1(\mathcal{O}_L, \mathscr{F})$ where L runs through the set of positive exceptional divisors on M. Thus we can argue as in [15, Lemma B.2] to show that

$$H_E^1(M, \mathscr{F}) \cong \varinjlim H^0(M, \mathscr{F} \otimes \mathcal{O}_L(L)).$$

Using this computation formula and the duality in 3.2, the same proof as in the algebraic case works, compare [15–17].

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A Künneth Formula for the Cyclic Cohomology of $\mathbb{Z}/2$-Graded Algebras

Christian Kassel

Départment de Mathématique, Université Louis Pasteur, 7, Rue R. Descartes, F-67084 Strasbourg, France

We define here Hochschild and cyclic (co)homology groups for $\mathbb{Z}/2$-graded algebras. A definition of cyclic cohomology of such algebras over the complex numbers has already be given by Kastler [13] who points out that a $\mathbb{Z}/2$-graded version of Connes's theory is likely to be fundamental in physics. Kastler uses "non-commutative differential forms" in the spirit of [5]. The definitions we give here are more algebraic. They are valid for $\mathbb{Z}/2$-graded algebras over any commutative ground ring k. Nevertheless, we assume in this paper that k is always a field in order to avoid lengthy, though elementary flatness conditions.

We denote by $HC^*(A)$ the *cyclic cohomology* groups of a $\mathbb{Z}/2$-graded algebra A over k. These groups coincide with Connes's groups $H^*_\lambda(A)$ when A is trivially graded and k is the field of complex numbers [5]. As in Connes's theory, $HC^*(A)$ has a natural module structure over $HC^*(k)=k[\sigma]$ where σ generates $HC^2(k)$. The action of σ on $HC^*(A)$ is a degree 2 endomorphism $S:HC^n(A)\to HC^{n+2}(A)$, which allows one to define a stabilized cohomology $H^*(A)=\varinjlim_n HC^{*+2n}(A)$ which we call *periodic cohomology* and whose geometric significance was first pointed out by Connes [5, Sect. 5]. Periodic cohomology behaves quite well when A satisfies what we call Property (P) in Sect. 2. This property implies finite dimensionality for $H^*(A)$ and is satisfied in most interesting examples.

The heart of the paper is the investigation of the behaviour of cyclic and periodic (co)homology with respect to the skew tensor product of algebras. In the cohomological case, the main results are the following.

Theorem 1. *Let A and A' be unital associative $\mathbb{Z}/2$-graded algebras over a field k of characteristic different from 2. Assume all vector spaces $HC^n(A')$ are finite dimensional. Then there exists a natural exact sequence*

$$\begin{aligned}0\to HC^*(A)\underset{HC^*(k)}{\otimes} HC^*(A')\to HC^*(A\otimes A')\\ \to \operatorname{Tor}_*^{HC^*(k)}(HC^*(A),HC^*(A'))[-1]\to 0\,.\end{aligned}$$

Theorem 2. *Let A and A' be unital associative $\mathbb{Z}/2$-graded algebras over a field k of characteristic different from 2. Assume that A' satisfies Property (P), then*

$$H^*(A\otimes A')\simeq H^*(A)\underset{H^*(k)}{\otimes} H^*(A')\,.$$

We use these results to compute the cyclic cohomology of *Clifford* and *Grassmann algebras*.

The paper is organized as follows. In Sect. 1, we set up the theory of $\mathbb{Z}/2$-graded vector spaces, complexes and algebras. Section 2 is devoted to the definition of Hochschild, cyclic and periodic homology groups for $\mathbb{Z}/2$-graded algebras and to the proof of their basic properties. The corresponding cohomology theories are defined in Sect. 3.

Künneth formulae for the Hochschild, cyclic and periodic homologies are stated and proved in Sect. 4. As immediate consequences, we derive Theorems 1 and 2 in Sect. 5 and show how they can be entended to locally convex topological algebras over the complex numbers.

We give also miscellaneous applications of the Künneth formulae in Sect. 5. But the most important one is the computation of the cyclic (co)homology of the Clifford algebra for any quadratic form. This is achieved in Sect. 6. Taking the null form, we get the cyclic cohomology of the Grassmann algebras which, according to Kastler, is of interest to physicists. Finally, we show how the homology of the total symmetric representation of the adjoint representation of the Lie algebra $\mathfrak{gl}(A) = \varinjlim_n \mathfrak{gl}(A)$ can be obtained from the Künneth formula for the Grassmann algebra Λ_1 on one generator.

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1. $\mathbb{Z}/2$-Graded Algebras

We denote by k a field of characteristic different from 2 and by ε an element of the set $\{+, -\}$ which we identify with $\{+1, -1\}$.

(1.1) A $\mathbb{Z}/2$-*graded vector space* can be defined in three equivalent different ways. First, it is a k-vector space V with a decomposition into two subspaces $V = V^+ \oplus V^-$. Elements of V^ε will be called homogeneous of degree $|v| = 0$ if $\varepsilon = +$ and $|v| = 1$ if $\varepsilon = -$. Secondly, it is a k-vector space V equipped with an involutory endomorphism θ (i.e. such that $\theta^2 = \mathrm{id}_V$). These two definitions are equivalent in view of the identity

$$v = \frac{v + \theta(v)}{2} + \frac{v - \theta(v)}{2}$$

and of: $V^\varepsilon = \{v \in V \mid \theta(v) = \varepsilon v\}$ for $\varepsilon \in \{+, -\}$.

Thirdly, a $\mathbb{Z}/2$-graded vector space can be defined as a $k[\mathbb{Z}/2]$-module V, the generator of $\mathbb{Z}/2$ acting on V by the involution θ. This makes sense since $\theta^2 = \mathrm{id}_V$. Besides, if we denote also the generator of $\mathbb{Z}/2$ by θ, we have $k[\mathbb{Z}/2] = k[\theta]/(\theta^2 - 1)$.

We say that a $\mathbb{Z}/2$-graded vector space V is *trivially graded* if, equivalently, $V = V^+$ or $\theta = \mathrm{id}_V$ or V is a trivial $k[\mathbb{Z}/2]$-module.

A map of $\mathbb{Z}/2$-graded vector spaces is a linear map $f: V \to W$ which, equivalently, sends V^ε into W^ε ($\varepsilon = \pm 1$) or commutes with θ or is a map of $k[\mathbb{Z}/2]$-modules.

The group algebra $k[\mathbb{Z}/2]$ has a natural structure of Hopf algebra (the coproduct sends the generator θ to $\theta\otimes\theta$). Therefore, given two $\mathbb{Z}/2$-graded vector spaces V and W, the tensor product $V\otimes W$ (taken over k) has a natural structure of $k[\mathbb{Z}/2]$-module. This means that the involution on $V\otimes W$ is given by $\theta\otimes\theta$. Therefore

$$(V\otimes W)^{\varepsilon}=(V^{+}\otimes W^{\varepsilon})\oplus(V^{-}\otimes W^{-\varepsilon}) \quad (\varepsilon=\pm 1).$$

With this grading, $V\otimes W$ is generally called the *skew tensor product* of V and W and is sometimes denoted by $V\hat{\otimes} W$ (we shall not use this notation).

(1.2) A positively graded complex of vector spaces $V_* = \left\{\ldots \to V_2 \xrightarrow{d} V_1 \xrightarrow{d} V_0 \to 0\right\}$ is a *$\mathbb{Z}/2$-graded complex* if all vector spaces V_i ($i\geqq 0$) are $\mathbb{Z}/2$-graded and if all differentials $d: V_i\to V_{i-1}$ are maps of $\mathbb{Z}/2$-graded spaces. We define the skew tensor product of two $\mathbb{Z}/2$-graded complexes V_* and W_* by

$$(V\otimes W)_n = \bigoplus_{p+q=n} V_p\otimes W_q$$

with $\mathbb{Z}/2$-grading given as above and with differential $d(v\otimes w)=d(v)\otimes w +(-1)^{|v|}v\otimes d(w)$ (for any homogeneous v).

Suppose now that V_* is a $\mathbb{Z}/2$-graded complex such that all V_i are finite dimensional vector spaces ($i\geqq 0$). Then one can define the *character* of V_* as the following power series with coefficients in the algebra $k[\theta]/(\theta^2-1)$

$$\mathrm{ch}(V_*)=\sum_{n\geqq 0}(\dim V_n^{+}+\theta\dim V_n^{-})t^n.$$

It is easy to check that if the character of V_* and of W_* are defined, then so it is for $V_*\otimes W_*$ and one has: $\mathrm{ch}(V_*\otimes W_*)=\mathrm{ch}(V_*)\cdot\mathrm{ch}(W_*)$ in $(k[\theta]/(\theta^2-1))[[t]]$. This can be applied to $(\mathbb{N}\times\mathbb{Z}/2)$-graded vector spaces which we view as $\mathbb{Z}/2$-graded complexes with zero differentials.

(1.3) We now define $\mathbb{Z}/2$-graded algebras. A *$\mathbb{Z}/2$-graded algebra* is an associative unital k-algebra A such that the multiplication is a map $A\otimes A\to A$ of $\mathbb{Z}/2$-graded vector spaces. Equivalently, (i) the involution θ is a homomorphism of algebras: $\theta(aa')=\theta(a)\theta(a')$ for all $a,a'\in A$, or (ii) $A^{\varepsilon}\cdot A^{\varepsilon'}\subset A^{\varepsilon\varepsilon'}$, $\varepsilon,\varepsilon\in\{\pm 1\}$, or (iii) A is an algebra over the Hopf algebra $k[\mathbb{Z}/2]$ in the sense of Steenrod [19]. Note that the unit 1 is in A^{+}.

As examples of $\mathbb{Z}/2$-graded algebras, let us mention the Clifford and Grassmann algebras we shall investigate in Sect. 6. Also, any $\mathbb{Z}$-graded algebra $A=\bigoplus_{n\in\mathbb{Z}} A_n$ gives rise to a $\mathbb{Z}/2$-graded algebra B defined by

$$B^{+}=\bigoplus_n A_{2n} \quad\text{and}\quad B^{-}=\bigoplus_n A_{2n+1}.$$

The skew tensor product of two $\mathbb{Z}/2$-graded algebras A and B has a canonical algebra structure given by

$$(a\otimes b)\cdot(a'\otimes b')=(-1)^{|b||a'|}aa'\otimes bb'.$$

(1.4) A left *$\mathbb{Z}/2$-graded A-module* is a A-module M such that the structural map $A\otimes M\to M$ is a map of $\mathbb{Z}/2$-graded spaces. Equivalently, $\theta(am)=\theta(a)\theta(m)$

($a \in A, m \in M$). Similarly, one defines right A-modules. A left A-module M can be turned into a right module over the *opposite algebra* A^0 by: $ma = (-1)^{|a||m|} am (a \in A, m \in M)$. The opposite algebra A^0 has the same underlying $\mathbb{Z}/2$-graded vector space as A, but the multiplication is given by: $(a, b) \mapsto (-1)^{|a||b|} ba\ (a, b \in A)$. One can also define $\mathbb{Z}/2$-graded A-bimodules. These are equivalent to left and right $\mathbb{Z}/2$-graded $A \otimes A^0$-modules. The equivalences are given by:

$$amb = (-1)^{|b||m|}(a \otimes b)m = (-1)^{|a|(|b|+|m|)} m(b \otimes a).$$

(1.5) Given a $\mathbb{Z}/2$-graded algebra A and a left (resp. right) $\mathbb{Z}/2$-graded A-module M (resp. N) with structural map $\mu_M : A \otimes M \to M$ (resp. $\mu_N : N \otimes A \to N$), we define the tensor product $N \underset{A}{\otimes} M$ of N and M over A as the cokernel of $N \otimes A \otimes M \xrightarrow{\mu_N \otimes \mathrm{id} - \mathrm{id} \otimes \mu_M} N \otimes M$.

In view of the formulae given in 1.4, it is an easy exercise to check that, if M is a $\mathbb{Z}/2$-graded A-bimodule, then

$$A \underset{A \otimes A^0}{\otimes} M \simeq M/[A, M]_{\mathrm{gr}},$$

where $[A, M]_{\mathrm{gr}}$ is the subspace spanned by all *graded commutators* $[a, m]_{\mathrm{gr}} = am - (-1)^{|a||m|} ma\ (a \in A, m \in M)$.

2. Hochschild and Cyclic Homologies

(2.1) The quickest way to define Hochschild and cyclic (co)homology groups for a $\mathbb{Z}/2$-graded algebra A begins by noticing that we can associate to any such A a *cyclic* $k[\mathbb{Z}/2]$-*module* $C(A)$, in the sense of Connes [6], by $C_n(A) = A \otimes \ldots \otimes A$ ($n+1$ times). Face maps and degeneracies are given by

$$d_i(a_0 \otimes \ldots \otimes a_n) = \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & \text{if } 0 \leqq i < n \\ (-1)^{|a_n|(|a_0| + \ldots + |a_{n-1}|)} a_n a_0 \otimes \ldots \otimes a_{n-1} & \text{if } i = n. \end{cases}$$

$$s_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes \ldots \otimes a_n \quad \text{for } 0 \leqq i \leqq n.$$

With the extra map $t(a_0 \otimes \ldots \otimes a_n) = (-1)^{|a_n|(|a_0| + \ldots + |a_{n-1}|)} a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$, one easily checks that $C(A)$ becomes a cyclic object in the category of $\mathbb{Z}/2$-graded vector spaces [8, Sect. 3]. Notice that the signs in the definition of d_n and t agree with the *standard sign commutation rule* (as defined e.g. in [14, p. 164]). The involution given on $C(A)$ by the $\mathbb{Z}/2$-grading is defined by $\theta(a_0 \otimes \ldots \otimes a_n) = \theta(a_0) \otimes \ldots \otimes \theta(a_n)$; it commutes with the structure maps d_i, s_i and t.

(2.2) The *Hochschild homology* groups $H_*(A, A)$ are by definition the homology groups of the complex $(C(A), b)$ where b is the differential given by $b = \sum_{i=0}^{n} d_i$ on $C_n(A)$. If A is trivially graded, then these groups coïncide with the ones defined in the ungraded situation. One always has $H_0(A, A) = A/[A, A]_{\mathrm{gr}}$ where $[A, A]_{\mathrm{gr}}$ is the subspace spanned by all graded commutators in A.

As in the ungraded case, the Hochschild homology of a $\mathbb{Z}/2$-graded algebra can be defined as a derived functor. Set $b' = \sum_{i=0}^{n-1} d_i$ on $C_n(A)$. According to [8],

$(C(A), b')$ has a contracting homotopy, hence it is acyclic. If we denote $C_{n+1}(A) = A \otimes A^{\otimes n} \otimes A$ by $C_n^0(A)$, we see that $(C^0(A), b')$ is a resolution of A by free A-bimodules. Define

$$\varphi : C_n(A) \to A \underset{A \otimes A^0}{\otimes} C_n^0(A) = C_n^0(A)/[A, C_n^0(A)]$$

by

$$\varphi(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes (1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1).$$

Lemma 1. *φ is an isomorphism of complexes between $(C(A), b)$ and $\left(A \underset{A \otimes A^0}{\otimes} C^0(A), \mathrm{id} \otimes b'\right)$.*

Proof. In view of the fact that $A \underset{A \otimes A^0}{\otimes} (A \otimes A^{\otimes n} \otimes A) \simeq A \otimes (A^{\otimes n})$, it is clear that φ is an isomorphism for all $n \geqq 0$. It suffices to prove that $(\mathrm{id} \otimes b')\varphi = \varphi b$

$$\begin{aligned}(\mathrm{id} \otimes b')\varphi(a_0 \otimes \ldots \otimes a_n) = {} & a_0 \otimes (a_1 \otimes \ldots \otimes a_n \otimes 1) \\ & + \sum_{i=1}^{n} (-1)^i a_0 \otimes (1 \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \otimes 1) \\ & + (-1)^n a_0 \otimes (1 \otimes a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n).\end{aligned}$$

Use now the formulae given at the end of Sect. 1 for the bimodule structure on $C_n^0(A)$. We get

$$a_0 \otimes (a_1 \otimes \ldots \otimes a_n \otimes 1) = a_0 a_1 \otimes (1 \otimes a_2 \otimes \ldots \otimes a_n \otimes 1)$$

and

$$a_0 \otimes (1 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^{|a_n|(|a_0| + \ldots |a_{n-1}|)} a_n a_0 \otimes (1 \otimes a_1 \otimes \ldots \otimes a_{n-1} \otimes 1).$$

Hence $(\mathrm{id} \otimes b')\varphi(a_0 \otimes \ldots \otimes a_n) = \varphi(b(a_0 \otimes \ldots \otimes a_n))$. $\square$

We have thus proved that

(2.3) $H_*(A, A) = \mathrm{Tor}_*^{A \otimes A^0}(A, A)$

for any unital $\mathbb{Z}/2$-graded algebra A.

(2.4) A cyclic module defines a map B as in [5, 8]. Therefore we get a mixed complex or a Λ-module $(C(A), b, B)$ in the sense of [12], i.e. a $\mathbb{Z}/2$-graded complex $(C(A), b)$ with a degree $+1$ map B such that $B^2 = Bb + bB = 0$. Let us recall from [12] how one defines cyclic homology from a mixed complex (M, b, B). It is the homology $HC_*(M)$ of a complex $B(M)$ with $B(M)_n = M_n \oplus M_{n-2} \oplus M_{n-4} \oplus \ldots$ and with differential $d(m_n, m_{n-2}, m_{n-4}, \ldots) = (bm_n + Bm_{n-2}, bm_{n-2} + Bm_{n-4}, \ldots)$. Such a complex comes with a natural projection $S(m_n, m_{n-2}, m_{n-4}, \ldots) = (m_{n-2}, m_{n-4}, \ldots)$ from $B(M)_n$ to $B(M)_{n-2}$, which gives $B(M)$ a structure of comodule over $HC_*(k)$ and which gives rise to a natural short exact sequence of $\mathbb{Z}/2$-graded complexes [12, Sect. 1].

(2.5) $0 \to M_n \xrightarrow{I} B(M)_n \xrightarrow{S} B(M)_{n-2} \to 0.$

We define the *cyclic homology* groups $HC_*(A)$ of a $\mathbb{Z}/2$-graded algebra A as the cyclic homology groups of the mixed complex $(C(A), b, B)$ defined above,

$$HC_*(A) = HC_*(C(A)) = H_*(B(C(A)), d).$$

From (2.5), it is clear that one gets immediately a Connes-type long exact sequence between Hochschild and cyclic groups

$$(2.6) \quad \ldots \to H_n(A, A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \to H_{n-1}(A, A) \to \ldots .$$

As a consequence of (2.6), one sees that $HC_0(A) = A/[A, A]_{\mathrm{gr}}$.

For later use, let us recall that Hochschild and cyclic homology can also be computed from the *normalized Hochschild mixed complex* $(\bar{C}(A), b, B)$. It is a quotient of the Hochschild complex $C(A)$: $\bar{C}(A) = C(A)/D(A)$ where $D(A)$ is the subcomplex generated by the images of the degeneracies s_i in $C(A)$. The map b is given by the same formula as before, namely

$$(2.7) \quad b(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$$
$$+ (-1)^{n + |a_n|(|a_0| + \ldots + |a_{n-1}|)} a_n a_0 \otimes \ldots \otimes a_{n-1} .$$

The map B can be made explicit on $\bar{C}(A)$ by

$$(2.8) \quad B(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni + (|a_i| + \ldots + |a_n|)(|a_0| + \ldots + |a_{i-1}|)}$$
$$1 \otimes a_i \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

and one has $H_*(A, A) = H_*(\bar{C}(A), b)$ and $HC_*(A) = HC_*(\bar{C}(A)) = H_*(B(\bar{C}(A)), d)$.

We give now an elementary result which is sometimes very useful for computational purposes. Let us recall the dependence of Hochschild and cyclic homology groups on the ground field k by writing

$$H_*(A, A) = H_*^{[k]}(A, A) \quad \text{and} \quad HC_*(A) = HC_*^{[k]}(A).$$

Lemma 2. *If K is a field containing k as a subfield then*

$$H_*^{[K]}\left(A \otimes_k K, A \otimes_k K\right) = H_*^{[k]}(A, A) \otimes_k K$$

and

$$HC_*^{[K]}\left(A \otimes_k K\right) = HC_*^{[k]}(A) \otimes_k K$$

for any $\mathbb{Z}/2$-graded k-algebra A.

Proof. It is a consequence of the natural isomorphism

$$\left(A \otimes_k K\right) \otimes_K \ldots \otimes_K \left(A \otimes_k K\right) \simeq \left(A \otimes_k \ldots \otimes_k A\right) \otimes_k K$$

and of the flatness of K over k. □

(2.9) We end this section by recalling that since we have a mixed complex $(C(A), b, B)$ for any $\mathbb{Z}/2$-graded algebra A, we can define *periodic cyclic homology* groups $HC_*^{\mathrm{per}}(A)$ as in [8, 12] by

$$HC_*^{\mathrm{per}}(A)=H_*\left[\varprojlim\left\{\to B(C(A))\xrightarrow{S} B(C(A))\to\dots\right\}\right].$$

We recall Property (P) for an algebra A [12, Sect. 3].

(P): $HC_*(A)$ is the sum of an extended comodule $HC_*(k)\otimes U_*$ and of a trivial comodule V_* (i.e. $S=0$ on V_*) where $U=\bigoplus_{n\geqq 0} U_n$ is a finite dimensional k-vector space.

Many interesting algebras (graded or not) have Property (P). For such algebras, one has (see [12, Sect. 3]).

$$HC_*^{\mathrm{per}}(A)=\varprojlim HC_{*+2n}(A)=\bigoplus_{n\equiv *(2)} U_n\,.$$

All homology groups defined above for $\mathbb{Z}/2$-graded algebras, are naturally $\mathbb{Z}/2$-graded. The corresponding homology groups defined in [8] for a positively graded algebra A coincide with the homology groups defined here for the $\mathbb{Z}/2$-graded algebra B associated to A in 1.3, because the signs introduced by the gradings depend only on the parity of the degree.

3. Dualizing

In this section we define Hochschild and cyclic cohomology groups by dualizing the complexes introduced in Sect. 2.

(3.1) Let $V=V^+\oplus V^-$ be a $\mathbb{Z}/2$-graded vector space over k. The *algebraic dual space* $V^*=\mathrm{Hom}_k(V,k)$ can be given a $\mathbb{Z}/2$-grading by $(V^*)^\varepsilon=(V^\varepsilon)^*\,(\varepsilon=\pm)$. The associated involution on V^* is the transpose map of the involution on V.

We can now dualize the $\mathbb{Z}/2$-graded complex $(C_*(A),b)$ of Sect. 2. Denote by $C^n(A,A^*)$ the dual space $\mathrm{Hom}_k(A^{\otimes(n+1)},k)$ of $C_n(A)$. Its elements can be viewed as multilinear forms on A. The transpose of the differential b on $C_*(A)$ is a degree $+1$ differential which we denote again by b. If $f\in C^{n-1}(A,A^*)$ is a n-linear form on A, $bf\in C^n(A,A^*)$ will be the $(n+1)$-linear form given by

$$\begin{aligned}(bf)(a_0\otimes\dots\otimes a_n)=&\sum_{i=0}^{n-1}(-1)^i f(a_0\otimes\dots\otimes a_ia_{i+1}\otimes\dots\otimes a_n)\\&+(-1)^{n+|a_n|(|a_0|+\dots+|a_{n-1}|)}f(a_na_0\otimes\dots\otimes a_{n-1})\,.\end{aligned}$$

By definition, the *Hochschild cohomology* groups $H^*(A,A^*)$ of A are the cohomology groups of the $\mathbb{Z}/2$-graded complex $(C^*(A,A^*),b)$. It is clear that $H^0(A,A^*)$ is the space of graded traces, i.e.

$$H^0(A,A^*)=\{\tau\in\mathrm{Hom}_k(A,k)|\tau=0\quad\text{on}\quad[A,A]_{\mathrm{gr}}\},$$

which is the dual space of $H_0(A,A)=A/[A,A]_{\mathrm{gr}}$. More generally, it is clear (since k is a field) that $H^n(A,A^*)$ is the dual space of $H_n(A,A)$.

$$H^*(A,A^*)=H_*(A,A)^*\,.$$

If A is trivially graded, we recover the usual expression for the coboundary b (see [4] and [5, Sect. 1]).

As in the ungraded case, the Hochschild cohomology groups of a unital $\mathbb{Z}/2$-graded algebra can be computed in terms of derived functors, namely

Lemma 3. $H^n(A, A^*) = \mathrm{Ext}^n_{A\otimes A^0}(A, A^*)$ *for all* $n \geqq 0$.

Here the algebraic dual space A^* is given the following A-bimodule structure. If $f \in A^*$ and $a, b \in A$, then afb is the linear form given by $c \mapsto (-1)^{|a|(|f|+|b|+|c|)} f(bca)$.

Proof. To compute the above Ext-terms, we use the resolution $(C^0_*(A), b')$ defined in Sect. 2. The lemma will be proved if we show that $\Psi: \mathrm{Hom}_{A\otimes A^0}(C^0_n(A), A^*) \to C^n(A, A^*) = \mathrm{Hom}_k(A^{\otimes(n+1)}, k)$ is an isomorphism of complexes, where

$$(\Psi F)(a_0 \otimes \ldots \otimes a_n) = (-1)^{|a_0|(|a_1|+\ldots+|a_n|)} F(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)(a_0),$$

for any $a_0, a_1, \ldots, a_n$ in A and any F in $\mathrm{Hom}_{A\otimes A^0}(C^0_n(A), A^*) = \mathrm{Hom}_{A\otimes A^0}(A \otimes A^{\otimes n} \otimes A, A^*)$. Clearly Ψ is an isomorphism since it is defined via the following classical sequence of obvious isomorphisms

$$\mathrm{Hom}_{A\otimes A^0}(A \otimes A^{\otimes n} \otimes A, A^*) \simeq \mathrm{Hom}_k(A^{\otimes n}, A^*) \simeq \mathrm{Hom}_k(A \otimes A^{\otimes n}, k).$$

We have to check the compatibility of Ψ with the coboundaries, namely we have to show that $\Psi b' = b\Psi$. Let F be an $A\otimes A^0$-linear map from $C^0_n(A)$ to A^*. We abbreviate $|a_0|(|a_1| + \ldots + |a_n|)$ by N. Then

$$\begin{aligned}(\Psi(b'F))(a_0 \otimes \ldots \otimes a_n) &= (-1)^N (b'F)(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)(a_0)\\ &= (-1)^N F(b'(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1))(a_0)\\ &= (-1)^N F(a_1 \otimes \ldots \otimes a_n \otimes 1)(a_0)\\ &\quad + \sum_{i=1}^{n-1} (-1)^{i+N} F(1 \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \otimes 1)(a_0)\\ &\quad + (-1)^{n+N} F(1 \otimes a_1 \otimes \ldots \otimes a_n)(a_0).\end{aligned}$$

Now since F is a map of $A\otimes A^0$-modules, we have

$$F(a_1 \otimes \ldots \otimes a_n \otimes 1)(a_0) = (-1)^{|a_1|(|a_0|+|a_2|+\ldots+|a_n|)} F(1 \otimes a_2 \otimes \ldots \otimes a_n)(a_0 a_1)$$

and

$$F(1 \otimes a_1 \otimes \ldots \otimes a_n)(a_0) = F(1 \otimes a_1 \otimes \ldots \otimes a_{n-1} \otimes 1)(a_n a_0).$$

Therefore

$$\begin{aligned}(\Psi(b'F))(a_0 \otimes \ldots \otimes a_n) &= (-1)^{|a_0 a_1|(|a_2|+\ldots+|a_n|)} F(1 \otimes a_2 \otimes \ldots \otimes a_n)(a_0 a_1)\\ &\quad + \sum_{i=1}^{n} (-1)^{i+N} F(1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes 1)(a_0)\\ &\quad + (-1)^{n+N} F(1 \otimes a_1 \otimes \ldots \otimes a_{n-1} \otimes 1)(a_n a_0)\\ &= b(\Psi F)(a_0 \otimes \ldots \otimes a_n). \quad \square\end{aligned}$$

Lemma 3 means that the useful fact that Hochschild cohomology can be computed with any resolution of A by free A-bimodules, extends to the graded case.

(3.2) We construct now the cyclic cohomology of a unital $\mathbb{Z}/2$-graded algebra. Connes shows in [5, Sect. 3] that $C^*(A, A^*)$ (where A is a ungraded algebra) possesses an endomorphism B of degree -1 such that

(3.3) $B^2 = Bb + bB = 0$.

This operator is the transpose of the degree $+1$ endomorphism B of cyclic homology. Therefore for a graded algebra, we define the $\mathbb{Z}/2$-graded map $B: C^n(A, A^*) \to C^{n-1}(A, A^*)$ as the transpose of the map $B: C_n(A) \to C_{n+1}(A)$ defined in Sect. 2.

We call $B^n(C^*(A, A^*))$ the dual complex of the complex $B(C(A))_n$ introduced in Sect. 2. Obviously,

$$B^n(C^*(A, A^*)) = C^n(A, A^*) \oplus C^{n-2}(A, A^*) \oplus C^{n-4}(A, A^*) \oplus \dots$$

and the coboundary d is given by

$$d(c^n, c^{n-2}, c^{n-4}, \dots) = (bc^n, Bc^n + bc^{n-2}, Bc^{n-2} + bc^{n-4}, \dots).$$

The cohomology groups of $(B^*(C^*(A, A^*)), d)$ are, by definition, the *cyclic cohomology* groups of the $\mathbb{Z}/2$-graded algebra A. We denote then by $HC^*(A)$.

Since (2.5) yields by duality the exact sequence of complexes

$$\text{(3.4)}\quad 0 \to B^{n-2}(C^*(A, A^*)) \xrightarrow{S} B^n(C^*(A, A^*)) \xrightarrow{I} C^n(A, A^*) \to 0,$$

we get a long exact sequence

$$\text{(3.5)}\quad \dots \to HC^{n-2}(A) \xrightarrow{S} HC^n(A) \xrightarrow{I} H^n(A, A^*) \to HC^{n-1}(A) \to \dots$$

relating the Hochschild and the cyclic cohomology groups of a $\mathbb{Z}/2$-graded algebra.

Notice also that cyclic cohomology is dual to cyclic homology

$$HC^n(A) = HC_n(A)^*.$$

When A is a trivially graded complex algebra, then we recover Connes's definition: $HC^n(A) = H^n_\lambda(A)$ in Connes's notation [5].

Let us make a few observations on the map S introduced in (3.4) as the kernel of the natural projection I. Since the cyclic cohomology $HC^*(k) \simeq k[\sigma]$ is a (trivially graded) algebra on a single generator $\sigma \in HC^2(k)$, it acts by $\sigma \mapsto S$ on $B^*(C(A, A^*))$ and an $HC^*(A)$ which both get $HC^*(k)$-module structures dual to the $HC_*(k)$-comodule structures on cyclic homology [12, Sect. 1]. The operator S also yields the following definition of *periodic cohomology*.

(3.6) $H^*(A) = \varinjlim HC^{*+2i}(A)$.

These groups are periodic of period 2. Since homology commutes with direct limits, $H^*(A)$ can also be defined as the cohomology groups of the complex

$$\varinjlim B^{*+2i}(C(A, A^*)) = \left\{ \to \bigoplus_{n\,\text{even}} C^n(A, A^*) \xrightarrow{b+B} \bigoplus_{n\,\text{odd}} C^n(A, A^*) \to \right\}.$$

The reader can easily check that if the algebra A has Property (P), as defined in Sect. 2, then $H^*(A)$ is dual to $HC^{\text{per}}_*(A)$.

4. Künneth Formulae

We recall that the Hochschild homology of a tensor product of (ungraded) unital algebras over a field is given by

(4.1) $H_*(A\otimes A', A\otimes A')\simeq H_*(A,A)\otimes H_*(A',A')$

[4, XI]. This isomorphism is realized by two maps of complexes $C(A)\otimes C(A') \underset{f}{\overset{g}{\rightleftarrows}} C(A\otimes A')$; g is the shuffle map and f is the Alexander-Whitney map: both fg and gf are homotopic to the identity. Moreover they both are defined on the normalized Hochschild complexes $\bar{C}(A)\otimes\bar{C}(A') \underset{f}{\overset{g}{\rightleftarrows}} \bar{C}(A\otimes A')$ where even $fg=$identity.

Now in the $\mathbb{Z}/2$-graded case, there exist graded versions of f and $g: C(A)\otimes C(A') \underset{f}{\overset{g}{\rightleftarrows}} C(A\otimes A')$ which are maps of $\mathbb{Z}/2$-graded complexes satisfying the same properties as in the classical case. For a definition of f and g, see [14, p. 313]. Actually the expressions for f and g are respectively obtained from the classical Alexander-Whitney and shuffle maps by formally applying the standard sign commutation rule. This makes sense because f and g are linear in all variables, which means for instance that $f(a_0\otimes a_0'\otimes a_1\otimes a_1'\otimes\ldots\otimes a_n\otimes a_n')$ is a sum of words in the variables $a_0,\ldots,a_n$, $a_0',\ldots,a_n'$ where each variable appears once and only once. Actually this linearity property is also shared by the maps b and B, which makes possible the application of the sign commutation rule. The graded map f has the following explicit form

$$f(a_0\otimes a_0'\otimes\ldots\otimes a_n\otimes a_n')$$
$$=\sum_{i=0}^{n}(-1)^{\varepsilon(i)}a_0a_na_{n-1}\ldots a_{i+1}\otimes a_1\otimes\ldots\otimes a_i\otimes a_0'\ldots a_i'\otimes a_{i+1}'\otimes\ldots\otimes a_n',$$

where

$$\varepsilon(i)=\sum_{0\leqq j<k\leqq n}|a_j'|\,|a_k|+\sum_{i+1\leqq j<k\leqq n}|a_j|\,|a_k|+\left(\sum_{j=1}^{i}|a_j|\right)\left(\sum_{k=i+1}^{n}|a_k|\right).$$

Since f and g are homotopy inverses in the graded case as well, then the isomorphism (4.1) is also valid for the Hochschild homology of $\mathbb{Z}/2$-graded algebras.

In [12, Sect. 2], we defined maps $G^{(i)}:\bar{C}(A)\otimes\bar{C}(A')\to\bar{C}(A\otimes A')$ for all $i\geqq 0$. The maps $G^{(i)}$ are of degree $2i$ and verify

(4.2) i) $G^{(0)}=g$

ii) $[G^{(i+1)},b]+[G^{(i)},B]=0$ for all $i\geqq 0$;

they are linear in all variables. Therefore as explained above, they give rise to $\mathbb{Z}/2$-graded maps $G^{(i)}:\bar{C}(A)\otimes\bar{C}(A')\to\bar{C}(A\otimes A')$. The graded maps $G^{(i)}$ satisfy also (4.2), according to the general principle stated in [8, p. 202]. This proves, as in [12, Proposition 2.3], the existence of a map G such that the following diagram is commutative

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "g"] & B(M) \arrow[r, "S"] \arrow[d, "G"] & B(M)[2] \arrow[r] \arrow[d, "G[2]"] & 0 \\
0 \arrow[r] & N \arrow[r] & B(M) \arrow[r, "S"] & B(M)[2] \arrow[r] & 0,
\end{tikzcd}
$$

where $M=\bar{C}(A)\otimes\bar{C}(A')$ and $N=\bar{C}(A\otimes A')$. The map G is given by

$$G(m_n, m_{n-2}, m_{n-4}, \ldots)=(G^{(0)}m_n+G^{(1)}m_{n-2}+\ldots, G^{(0)}m_{n-2}+G^{(1)}m_{n-4}+\ldots, \ldots).$$

Since g induces an isomorphism in homology, so does G:

(4.3) $HC_*(\bar{C}(A)\otimes\bar{C}(A'))\simeq HC_*(\bar{C}(A\otimes A'))\simeq HC_*(A\otimes A')$.

We now state and prove the following Künneth formula for cyclic homology.

Theorem 3. *Let A and A' be associative unital algebras over a field k of characteristic different from 2.*

a) *Then there exists a natural long exact sequence*

$$
\begin{aligned}
&\ldots\to HC_n(A\otimes A')\\
&\to \bigoplus_{p+q=n} HC_p(A)\otimes HC_q(A') \xrightarrow{S\otimes\mathrm{id}-\mathrm{id}\otimes S} \bigoplus_{p+q=n-2} HC_p(A)\otimes HC_q(A')\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to HC_{n-1}(A\otimes A')\to\ldots
\end{aligned}
$$

b) *If moreover A' has Property (P), i.e. $HC_*(A')=HC_*(k)\otimes U_*\oplus V_*$ as in Sect. 2, then for any algebra A,*

$$HC_n(A\otimes A')=\bigoplus_{p+q=n}[HC_p(A)\otimes U_q\oplus H_p(A,A)\otimes V_q]$$

and

$$HC_*^{\mathrm{per}}(A\otimes A')=HC_*^{\mathrm{per}}(A)\otimes_{HC_*^{\mathrm{per}}(k)}HC_*^{\mathrm{per}}(A').$$

Notice that the long exact sequence in Part (a) can be expressed in terms of the cotensor product (and of its first derived functor) of comodules over $HC_*(k)$ as in [12].

Proof of Theorem 3 (b). This is an immediate formal consequence of Part (a). See [12, Sect. 3] for details. Moreover, the proof of Theorem 3.10 in [12] shows that the map S on $HC_*(A\otimes A')$ decomposes as $S\otimes\mathrm{id}$ on $HC_*(A)\otimes U_*$ and as the zero map on $H_*(A,A)\otimes V_*$. This remark will be useful later when we compute the cyclic homology of Grassmann algebras. □

Proof of Theorem 3 (a). We proved in [12, Sect. 1] that if M and N are mixed complexes or Λ-modules, then the complex $B(M\otimes N)$ (introduced in Sect. 2) is isomorphic to the cotensor product of $B(M)$ and $B(N)$. Equivalently, this means that we have the following exact sequence of complexes

(4.4) $0\to B(M\otimes N)\to B(M)\otimes B(N)\xrightarrow{S\otimes\mathrm{id}-\mathrm{id}\otimes S} B(M)\otimes B(N)[2]\to 0$.

The surjectivity of $S \otimes \mathrm{id} - \mathrm{id} \otimes S$ results from the fact that S is surjective on $B(M)$. Apply this to $M = \bar{C}(A)$ and $N = \bar{C}(A)$ and take the long homology exact sequence associated to (4.4). Then the isomorphisms (4.3) produce the desired result. $\square$

5. Proofs of Theorems 1 and 2

In this section we prove Theorems 1 and 2 stated in the introduction by dualizing the results of Sect. 4. We also show how Theorems 1 and 2 can be adapted to complex algebras endowed with a Fréchet topology. We end the section by a few examples.

(5.1) *Proof of Theorem 1.* Apply duality to the long exact sequence of Theorem 3 (a). Since $HC^n(A) = HC_n(A)^*$ and since $HC^n(A')$ is finite dimensional, we have

$$[HC_p(A) \otimes HC_q(A')]^* \simeq HC_p(A)^* \otimes HC_q(A')^* .$$

We thus obtain the following long exact sequence

$$\begin{aligned} \ldots &\to HC^{n-1}(A \otimes A') \\ &\to \bigoplus_{p+q=n-2} HC^p(A) \otimes HC^q(A') \xrightarrow{S \otimes \mathrm{id} - \mathrm{id} \otimes S} \bigoplus_{p+q=n} HC^p(A) \otimes HC^q(A') \\ &\to HC^n(A \otimes A') \to \ldots . \end{aligned}$$

Now we recall that if P and Q are modules (via S) over $HC^*(k) = k[\sigma]$, then the tensor product (and its derived functor) of $k[\sigma]$-modules is defined by the following exact sequence

$$0 \to \mathrm{Tor}^{k[\sigma]}(P, Q)[-2] \to P \otimes Q[-2] \xrightarrow{S \otimes \mathrm{id} - \mathrm{id} \otimes S} P \otimes Q \to P \underset{k[\sigma]}{\otimes} Q \to 0 .$$

(Here $V[p]^*$ is the complex given by $V[p]^n = V^{n+p}$). This proves Theorem 1.

(5.2) *Proof of Theorem 2.* We suppose that A' satisfies Property (P). We have $HC_*(A') = HC_*(k) \otimes U_* \oplus V_*$. By Theorem 3 (b), for any algebra A, $HC_*(A \otimes A') = HC_*(A) \otimes U_* \oplus H_*(A, A') \otimes V_*$ and the map S is $S \otimes \mathrm{id}$ on the first summand and zero on the second one. Let us dualize. Then

$$HC^*(A \otimes A') = HC^*(A) \otimes U^* \oplus (H_*(A, A) \otimes V_*)^*$$

because U_* is finite-dimensional. We take now the direct limit under the map S. Since $S = 0$ on $[H_*(A, A) \otimes V_*]^*$, we have

$$\begin{aligned} H^*(A \otimes A') &= \varinjlim (HC^*(A) \otimes U^*) \\ &= \left(\varinjlim HC^*(A)\right) \otimes U^* \\ &= H^*(A) \otimes U^* . \end{aligned}$$

Let us take $A = k$. Then $H^*(A') = H^*(k) \otimes U^*$. Therefore $H^*(A) \otimes_{H^*(k)} H^*(A') = H^*(A) \otimes_{H^*(k)} (H^*(k) \otimes U^*) = H^*(A) \otimes U^* = H^*(A \otimes A')$. $\square$

(5.3) Before we give some applications of Theorems 1 and 2, we point out that they can be adapted to the following topological framework: $k = \mathbb{C}$ is the field of complex numbers and the $\mathbb{Z}/2$-graded algebras we consider are complex algebras endowed with a Fréchet space topology for which the product $A \times A \to A$ is continuous (as in [5, Sect. 5). We assume also that the involution θ giving the $\mathbb{Z}/2$-grading is continuous. One replaces the algebraic dual A^* of A by the topological dual and the space $C^n(A, A^*)$ (of Sect. 3) of $(n+1)$-linear forms on A by the space of continuous $(n+1)$-linear functionals as explained in [5, Sect. 5]. $C^*(A, A^*)$ is stable under b and B. Hochschild and cyclic cohomology of A are defined from the continuous $C^*(A, A^*)$ as in Sect. 3. They are related by the exact sequence (3.5). The Hochschild groups can be computed with the help of a canonical projective resolution of A over $A \hat{\otimes}_\pi A^0$ as in [5, 10]. Here A^0 is the opposite algebra of A as defined in 1.4 and $\hat{\otimes}_\pi$ means the completed projective topological tensor product of Grothendieck [9]. We recall also that if E and F are Fréchet spaces and if moreover F is nuclear [9], then we may drop the subscript π from $E \hat{\otimes}_\pi F$ because the completion $E \hat{\otimes} F$ of the tensor product is defined unambiguously. Besides, the functor $- \hat{\otimes} F$ is exact [7].

Assume now that A and A' are Fréchet algebras and that A' is nuclear. Then the discussion of Sect. 4 is valid in the topological situation since the polynomial maps f, g, $G^{(i)}$ are all continuous. To complete the proof of a topological Künneth formula, we need only compute the cohomology of a topological tensor product of two complexes of Fréchet spaces. This is possible under certain assumptions as shown by Karoubi [11]. The rest is straightforward. Therefore we have

Theorem 4. *Let A and A' be $\mathbb{Z}/2$-graded complex Fréchet algebras. Suppose that A' is nuclear, $HC^*(A)$ and $HC^*(A')$ are Hausdorff.*

a) *If $HC^*(A')$ is finite-dimensional, then there exists a natural exact sequence*

$$\ldots \to HC^{n-1}(A \hat{\otimes} A')$$

$$\to \bigoplus_{p+q=n-2} HC^p(A) \otimes HC^q(A') \xrightarrow{S \otimes \mathrm{id} - \mathrm{id} \otimes S} \bigoplus_{p+q=n} HC^p(A) \otimes HC^q(A')$$

$$\to HC^n(A \hat{\otimes} A') \to \ldots$$

b) *If A' has property (P), then*

$$H^*(A \hat{\otimes} A') = H^*(A) \otimes_{H^*(\mathbb{C})} H^*(A').$$

(5.4) We give a few applications of Theorems 1–4. First consider the case when $A' = \mathscr{S}(\mathbb{N})$ (resp. $\mathscr{S}(\mathbb{Z})$) which is a completion of $\mathbb{C}[X]$ (resp. $\mathbb{C}[X, X^{-1}]$). Then one has

(5.5) $HC^n(A \hat{\otimes} \mathscr{S}(\mathbb{N})) = HC^n(A) \oplus H^n(A, A^*)^\infty$

(5.6) $HC^n(A \hat{\otimes} \mathscr{S}(\mathbb{Z})) = HC^n(A) \oplus HC^{n-1}(A) \oplus H^n(A, A^*)^\infty$

(where the superscript ∞ denotes a direct product of infinitely many copies).

(5.7) $H^n(A \hat{\otimes} \mathscr{S}(\mathbb{Z})) = H^n(A)$ and $H^n(A \hat{\otimes} \mathscr{S}(\mathbb{Z})) = H^n(A) \oplus H^{n-1}(A)$

(these were also proved by Masuda [15]).

Let V be a compact smooth manifold and $C^\infty(V)$ the nuclear Fréchet algebra of smooth functions on V. Then one has

(5.8) $H^*(C^\infty(V) \hat{\otimes} A) = H_*(V, \mathbb{C}) \otimes_{H^*(\mathbb{C})} H^*(A)$

as a consequence of [5, Théorème 46].

If A_θ is an irrational rotation algebra, Connes's computations [5, Sect. 5] and Theorem 4 imply

(5.9) $H^n(A \hat{\otimes} A_\theta) = H^n(A)^2 \oplus H^{n-1}(A)^2$.

Finally, let F_N be the free group on N generators and $\mathcal{S}^\sharp(F_N)$ be the nuclear algebra considered in [16]. Then

(5.10) $HC^n(A \hat{\otimes} \mathcal{S}^\sharp(F_N)) = HC^n(A) \oplus HC^{n-1}(A)^N \oplus H^n(A, A^*)^\infty$,

(5.11) $H^n(A \hat{\otimes} \mathcal{S}^\sharp(F_N)) = H^n(A) \oplus H^{n-1}(A)^N$.

6. Clifford and Grassmann Algebras

(6.1) As an application of Theorems 1–3, we compute the cyclic (co)homology of the Clifford algebra $C(q)$ (considered as a $\mathbb{Z}/2$-graded algebra) associated to a quadratic form q over a field k of characteristic different from 2. We can always put q under the form $q(x) = \sum_{i=1}^{n} a_i x_i^2$, which means that $C(q)$ is the k-algebra generated by degree-one elements $e_1, \ldots, e_n$ such that $e_i^2 = a_i$ for all i and $e_i e_j + e_j e_i = 0$ if $i \neq j$. If q is the null form or equivalently all $a_i = 0$, then $C(q)$ is Λ_n the Grassmann (or exterior algebra) on n generators. It is well known that as a $\mathbb{Z}/2$-graded algebra, $C(q) = C\langle a_1 \rangle \otimes \ldots \otimes C\langle a_n \rangle$ where $C\langle a \rangle$ is the Clifford algebra generated by a degree-one element e of square a. We set $\Lambda = C\langle 0 \rangle$; $\Lambda = \Lambda_1$.

Proposition 1. *Let k be a field of characteristic $\neq 2$ and $C(q)$ be as above. If q is non degenerate, then $HC^*(C(q)) = HC^*(k)$ as a $HC^*(k)$-module.*

Proof. This will be a consequence of the fact that

$$H_*(C(q), C(q)) = \begin{cases} 0 & \text{if } * \geqq 1 \\ k & \text{if } * = 0. \end{cases}$$

A straightforward computation shows that if all a_i are invertible, then all elements of the basis $e_{i_1} \ldots e_{i_k}$ $(i_1 < \ldots < i_k)$, except $e_1 \ldots e_n$, belong to $[C(q), C(q)]_{\mathrm{gr}}$. Therefore

$$H_0(C(q), C(q)) = C(q)/[C(q), C(q)]_{\mathrm{gr}} = k e_1 \ldots e_n.$$

The vanishing of the higher Hochschild groups results from the fact (see [2, IV–V]) that $C(q)$ is a separable $\mathbb{Z}/2$-graded algebra i.e. $C(q)$ is a projective $C(q) \otimes C(q)^0$-module. Since by Lemma 1, $H_*(A, A) = \mathrm{Tor}_*^{A \otimes A^0}(A, A)$, we see that $H_i(A, A) = 0$ for $i \geqq 1$ and any separable $\mathbb{Z}/2$-graded algebra. This could also be proved by application of Theorem 3 to the computations on $C\langle a \rangle$ which will be done below.

Note that $HC^*(C(q))$ is generated by $\tau, S\tau, S^2\tau, \ldots$ where τ is the graded trace on $C(q)$ whose value on $e_1 \ldots e_n$ is 1 and 0 on the other elements of the canonical basis. $\square$

Remark. Let us denote by $|C(q)|$ the Clifford algebra $C(q)$ considered as a trivially graded algebra. Then it is well-known [1] that if $\bar{k}$ is an algebraic closure of k, then

$|C(q)|\otimes\bar{k}$ is isomorphic to a matrix algebra over $\bar{k}$ if n is even and over $\bar{k}\oplus\bar{k}$ if n is odd. By Morita invariance and by Lemma 2, we have

$$HC^*(|C(q)|)\simeq\begin{cases}HC^*(k) & \text{if } n \text{ even}\\ HC^*(k)^2 & \text{if } n \text{ odd}.\end{cases}$$

for any non-degenerate quadratic form $q(x)=\sum_{i=1}^{n} a_i x_i^2$.

(6.2) Proposition 1 allows us to construct an algebra for which all (co)homology theories considered here vanish.

Corollary 1. *Let A be the $\mathbb{Z}/2$-graded algebra generated over a field k of characteristic different from 2 by a infinite countable set of degree-one generators $e_1, e_2, \ldots$ subject to the relations: $e_ie_j+e_je_i=0$ for all $i\neq j$ and $e_i^2=1$ for all $i\in\mathbb{N}-\{0\}$. Then $H_*(A,A)=HC_*(A)=HC_*^{\mathrm{per}}(A)=0$.*

Proof. $A=\varinjlim C(q_n)$ where $q_n(x)=x_1^2+\ldots+x_n^2$. It is enough to prove that $H_*(A,A)=0$. Since homology commutes with direct limits, we have $H_i(A,A)=\varinjlim H_i(C(q_n),\ C(q_n))=0$ if $i\geqq 1$ by Proposition 1. Now $H_0(C(q_n),\ C(q_n))$ is generated by $e_1,\ldots,e_n$, but $e_1\ldots e_n\in[C(q_{n+1}),\ C(q_{n+1})]_{\mathrm{gr}}$. Therefore the natural embedding $C(q_n)\to C(q_{n+1})$ induces the zero map on Hochschild homology. □

(6.3) We now turn to the case when q is no longer non-degenerate. Let us assume that in the decomposition $q(x)=\sum_{i=1}^{n} a_i x_i^2$, exactly r elements of the set $\{a_1,\ldots,a_n\}$ are zero. In that case, $C(q)\simeq C(q')\otimes\Lambda_r$ where q' is a non-degenerate form. By Proposition 1 and Theorem 3, we have:

$$HC^*(C(q))=HC^*(\Lambda_r)$$

and

$$H^*(C(q))=H^*(\Lambda_r).$$

Therefore it is enough to compute the cyclic homology of the exterior algebra Λ_r on r generators.

Proposition 2. *If k is of characteristic 0, then Λ_r has Property (P). Moreover $HC^*(\Lambda_r)$ is the direct sum $HC^*(\Lambda_r)=HC^*(k)\oplus V_r^*$ of a free rank-one $HC^*(k)$-module and of a trivial $HC^*(k)$-module V_r^* such that*

$$\mathrm{ch}(V_r^*)=\frac{2^{r-1}(1+\theta)-(1-t)^r}{(1+t)(1-t)^r}.$$

Therefore, $H^*(\Lambda_r)=H^*(k)$.

Proof. We prove the proposition by applying Theorem 3 to $\Lambda_r=\Lambda\otimes\Lambda_{r-1}$. Let us first compute the Hochschild and cyclic homologies of Λ. The normalized Hochschild complex $\bar{C}(\Lambda)$ is generated in degree n by $x(n)=1\otimes e\otimes\ldots\otimes e$ (e appearing n times) and $y(n)=e\otimes e\otimes\ldots\otimes e$ (e appearing $(n+1)$-times). We easily check that $bx(n)=by(n)=0$. Therefore $H_n(\Lambda,\Lambda)=kx(n)\oplus ky(n)$ for all $n\geqq 0$. Since $x(n)$ and $y(n)$ are of different parity, we see that $\operatorname{ch}(H_*(\Lambda,\Lambda)) =(1+\theta)(1+t+t^2+\dots)=\frac{1+\theta}{1-t}$. By the Künneth formula for Hochschild homology and since $(1+\theta)^r=2^{r-1}(1+\theta)$, we have

$$\operatorname{ch}(H_*(\Lambda_r,\Lambda_r))=\frac{2^{r-1}(1+\theta)}{(1-t)^r}.$$

In order to calculate the cyclic homology of Λ, we have now to compute $Bx(n)$ and $By(n)$. This is given by formula (2.8). We get $Bx(n)=0$ and $By(n)=(n+1)x(n+1)$. Since k is of characteristic zero, we have $HC_*(\Lambda) =HC_*(k)\oplus V_*$ where $V_n=ky(n)$ and $Sy(n)=0$. Therefore $\Lambda=\Lambda_1$ has Property (P) and $HC_*^{\mathrm{per}}(\Lambda)=HC_*^{\mathrm{per}}(k)$. Clearly $\operatorname{ch}(V_*)=\frac{t+\theta}{1-t^2}$.

We now prove Proposition 2 by induction on r. We have proved it for $r=1$. Suppose it is true for $i=1,\dots,r-1$. Then applying Theorem 3 (b) to $\Lambda_r=\Lambda\otimes\Lambda_{r-1}$ we see that

$$\begin{aligned}HC_*(\Lambda_r)&=HC_*(\Lambda)\oplus H_*(\Lambda,\Lambda)\otimes V_{r-1}^{**}\\&=HC_*(k)\oplus V_*\oplus H_*(\Lambda,\Lambda)\otimes V_{r-1}^{**}.\end{aligned}$$

We see from the proof of Theorem 3 (b) that Λ_r satisfies Property (P). Therefore $HC^*(\Lambda_r)=HC^*(k)\oplus W^*$ where W^* is the trivial module $V_*^*\oplus H^*(\Lambda,\Lambda^*)\otimes V_{r-1}^*$. From this decomposition, it follows $H^*(\Lambda_r)=H^*(k)$.

To end the proof of the proposition, we have to check that $\operatorname{ch}(W^*)=\operatorname{ch}(V_r^*)$.

$$\begin{aligned}\operatorname{ch}(W^*)&=\frac{t+\theta}{1-t^2}+\frac{1+\theta}{1-t}\cdot\frac{[2^{r-2}(1+\theta)-(1-t)^{r-1}]}{(1+t)(1-t)^{r-1}}\\&=\frac{(t+\theta)(1-t)^{r-1}+2^{r-1}(1+\theta)-(1+\theta)(1-t)^{r-1}}{(1+t)(1-t)^r}\\&=\frac{2^{r-1}(1+\theta)-(1-t)^r}{(1+t)(1-t)^r}.\qquad\square\end{aligned}$$

(6.4) The above calculations can be used to compute the homology of $\mathrm{gl}(A)$ with coefficients in the symmetric algebra over its adjoint representation. Let us recall that, given a k-algebra A, $\mathrm{gl}_n(A)$ is the Lie algebra of $n\times n$-matrices with coefficients in A; $\mathrm{gl}_n(A)$ embeds into $\mathrm{gl}_{n+1}(A)$ in the usual way. Therefore one can consider $\mathrm{gl}(A)=\lim \mathrm{gl}_n(A)$. Let $S^*(\mathrm{gl}(A))=\oplus_{n\geq 0}S^n(\mathrm{gl}(A))$ be the total symmetric representation of the adjoint representation of $\mathrm{gl}(A)$. We compute its homology $H_*(\mathrm{gl}(A),S^*(\mathrm{gl}(A)))$.

Recall that Quillen [17] defined the homology of any graded (in particular $\mathbb{Z}/2$-graded) Lie algebra L as the homology of a differential graded coalgebra $\mathscr{C}(L)$. Let Λ be again the $\mathbb{Z}/2$-graded exterior algebra on a generator e of degree one. We consider now the $\mathbb{Z}/2$-graded Lie algebra $L=\mathrm{gl}(A\otimes\Lambda)=\mathrm{gl}(A)\oplus e\,\mathrm{gl}(A)$ (A is trivially graded). It is easy to check that $\mathscr{C}(\mathrm{gl}(A\otimes\Lambda))$ as defined in [17] is isomorphic to the standard Koszul complex $\Lambda^*(\mathfrak{gl}(A))\otimes S^*(\mathfrak{gl}(A))$ computing $H_*(\mathfrak{gl}(A), S^*(\mathfrak{gl}(A)))$. Hence $H_*(\mathfrak{gl}(A), S^*(\mathfrak{gl}(A)))=H_*(\mathfrak{gl}(A\otimes\Lambda), k)$.

Now by a result of Burghelea [3] and Staffeldt [18] extending Loday-Quillen's theorem,

$$\mathrm{Prim}_* H_*(\mathfrak{gl}(A\otimes\Lambda), k)=HC_{*-1}(A\otimes\Lambda)$$

when k is a field of characteristic zero. Theorem 3 and Proposition 2 immediately imply $HC_n(A\otimes\Lambda)=HC_n(A\otimes\Lambda)^+\oplus HC_n(A\otimes\Lambda)^-$ with

$$HC_n(A\otimes\Lambda)^+=HC_n(A)\oplus H_{n-1}(A,A)\oplus H_{n-3}(A,A)\oplus H_{n-5}(A,A)\oplus\dots$$
$$HC_n(A\otimes\Lambda)^-=H_n(A,A)\oplus H_{n-2}(A,A)\oplus H_{n-4}(A,A)\oplus\dots.$$

Since $HC_*(A)$ generates $H_*(\mathfrak{gl}(A), k)$, we have proved

Corollary 2. *Let A be an associative unital algebra over a field k of characteristic zero. Then $H_*(\mathfrak{gl}(A), S^*(\mathfrak{gl}(A)))\simeq H_*(\mathfrak{gl}(A), k)\otimes\Lambda V_*$ where ΛV_* is the commutative graded algebra generated by the graded vector space given in degree n by*

$$V_n=H_{n-1}(A,A)\oplus H_{n-2}(A,A)\oplus H_{n-3}(A,A)\oplus\dots.$$

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