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Isolated singularities of positive solutions of p-Laplacian type equations in $\mathbb{R}^d$

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ABSTRACT

We study the behavior of positive solutions of p-Laplacian type elliptic equations of the form

$$Q'(u) := -\Delta_p(u) + V|u|^{p-2}u = 0 \quad \text{in } \Omega \setminus \{\zeta\}$$

near an isolated singular point $\zeta \in \Omega \cup \{\infty\}$, where $1 < p < \infty$, and Ω is a domain in $\mathbb{R}^d$ with $d > 1$. We obtain removable singularity theorems for positive solutions near ζ. In particular, using a new three-spheres theorems for certain solutions of the above equation near ζ we prove that if V belongs to a certain Kato class near ζ and $p > d$ (respectively, $p < d$), then any positive solution u of the equation $Q'(u) = 0$ in a punctured neighborhood of $\zeta = 0$ (respectively, $\zeta = \infty$) is in fact continuous at ζ. Under further assumptions we find the asymptotic behavior of u near ζ.

1. Introduction

The aim of this paper is to study the behavior of positive solutions of the quasilinear elliptic equation

$$Q'_V(u) = Q'(u) := -\Delta_p(u) + V|u|^{p-2}u = 0 \quad \text{in } \Omega \setminus \{\zeta\} \tag{1.1}$$

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near an *isolated* singular point $\zeta \in \Omega \cup \{\infty\}$, where $\Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < \infty$, Ω is a domain in $\mathbb{R}^d$ with $d > 1$, and V is a real function satisfying

$$V \in L^\infty_{\mathrm{loc}}\big(\Omega \setminus \{\zeta\}\big). \tag{1.2}$$

We are interested in the local behavior of solutions near ζ, and in removable singularity theorems for such equations. Therefore, unless otherwise stated, we assume that either $\zeta = 0$ and $\Omega = B_R$ is the open ball of radius $R > 0$ centered at the origin, or $\zeta = \infty$ and Ω is the exterior domain $\Omega = B_R^* := \mathbb{R}^d \setminus \overline{B_R}$.

The different behaviors of positive solutions near isolated singularities in different cases are well illustrated when $V = 0$ and Q' is the p-Laplacian.

Example 1.1. Let $\Omega = \mathbb{R}^d \setminus \{0\}$, $d > 1$, and denote

$$\alpha^* := \frac{p-d}{p-1}. \tag{1.3}$$

Define the “fundamental solution” of the p-Laplacian by

$$v_{\alpha^*}(x) := \begin{cases} |x|^{\alpha^*}, & p \neq d, \\ |\log(|x|)|, & p = d,\ |x| \neq 1. \end{cases} \tag{1.4}$$

Then for any $A > 0$ and $B \geqslant 0$ the positive function $u(x) := A v_{\alpha^*}(x) + B$ is p-harmonic in punctured neighborhoods of $\zeta = 0$ and $\zeta = \infty$. Indeed, let v be a radial function, and denote $r := |x|$, $v' := \partial v/\partial r$; then the radial p-Laplacian is given by

$$-\Delta_p(v) = -\frac{1}{r^{d-1}}\big(r^{d-1}\big|v'\big|^{p-2}v'\big)' = -\big|v'\big|^{p-2}\left[(p-1)v'' + \frac{d-1}{r}v'\right]. \tag{1.5}$$

Consequently, for $v_\alpha(r) := r^\alpha$ with $\alpha \neq 0$ we have

$$-\Delta_p(v_\alpha) = -|\alpha|^{p-2}\alpha\big[\alpha(p-1) + d - p\big]r^{\alpha(p-1)-p}. \tag{1.6}$$

Substituting $\alpha = \alpha^*$, we obtain for $p \neq d$ that

$$-\Delta_p\big(u(x)\big) = -\Delta_p\big(A v_{\alpha^*}(x) + B\big) = 0 \quad \forall x \in \mathbb{R}^d \setminus \{0\}.$$

The case $p = d$ is similar (for $|x| \neq 0, 1$).

Consequently,

$$\lim_{x\to 0} u(x) = \begin{cases} \infty, & p \leqslant d, \\ B, & p > d, \end{cases} \qquad \lim_{x\to\infty} u(x) = \begin{cases} B, & p < d, \\ \infty, & p \geqslant d. \end{cases} \tag{1.7}$$

Definition 1.2. We call the cases $\zeta = 0$ with $p \leqslant d$, and $\zeta = \infty$ with $p \geqslant d$ the *classical cases*. The complementary cases (where ζ has a nonzero capacity with respect to the p-Laplacian [9, Example 2.22]) are called the *nonclassical cases*. Note that in the classical cases we have $\lim_{x\to\zeta} v_{\alpha^*}(x) = \infty$.

Our aim is to find an appropriate class of potentials for which positive solutions of the equation $Q'(u) = 0$ near ζ exhibit similar behavior to that of the p-Laplacian. We consider the following classes of singular potentials (cf. [6, Definitions 1.1 and 2.1]).

Definition 1.3. Let Ω be a domain, and let $\zeta \in \Omega \cup \{\infty\}$, where $\zeta = 0$ or $\zeta = \infty$. Let $V \in L^\infty_{\mathrm{loc}}(\Omega \setminus \{\zeta\})$. We say that (1.1) has a *Fuchsian type singularity at* ζ if there exists a positive constant C such that V satisfies

$$|x|^p \left|V(x)\right| \leqslant C \quad \text{near } \zeta. \tag{1.8}$$

For $R > 0$ let V_R be the *scaled potential* defined by

$$V_R(x) := R^p V(Rx), \quad x \in \Omega/R. \tag{1.9}$$

Let $\{R_n\} \subset \mathbb{R}_+$ be a sequence satisfying $R_n \to \zeta$ (where ζ is either 0 or ∞) such that

$$V_{R_n} \underset{n\to\infty}{\longrightarrow} W \quad \text{in the weak* topology of } L^\infty_{\mathrm{loc}}(Y), \tag{1.10}$$

where $Y := \lim_{n\to\infty} \Omega/R_n = \mathbb{R}^d \setminus \{0\}$. Define the *limiting dilated equation with respect to* (1.1) (and the sequence $\{R_n\}$) as

$$\mathcal{D}^{\{R_n\}}(Q)(w) := -\Delta_p(w) + W|w|^{p-2}w = 0 \quad \text{on } Y. \tag{1.11}$$

Definition 1.4. Let $V \in L^\infty_{\mathrm{loc}}(\Omega \setminus \{\zeta\})$. We say that V has a *weak Fuchsian singularity at* ζ if inequality (1.8) is satisfied, and in addition, there exist m sequences $\{R_n^{(i)}\}_{n=1}^\infty \subset \mathbb{R}_+$, $1 \leqslant i \leqslant m$, satisfying $R_n^{(i)} \to \zeta^{(i)}$, where $\zeta^{(1)} = \zeta$, and $\zeta^{(i)} = 0$ or $\zeta^{(i)} = \infty$ for $2 \leqslant i \leqslant m$, such that

$$\mathcal{D}^{\{R_n^{(m)}\}} \circ \cdots \circ \mathcal{D}^{\{R_n^{(1)}\}}(Q)(w) = -\Delta_p(w) \quad \text{on } Y. \tag{1.12}$$

Remark 1.5. Let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ and let $SC_V \subset L^\infty_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ be the set of the scaled potentials $SC_V := \{R^p V(Rx),\ R > 0\}$. If V is a weak Fuchsian potential, then $W = 0$ belongs to a weak* closure of SC_V. We do not know whether the reverse implication holds true.

Throughout the paper we assume that near ζ the function V is of the form:

$$\left|V(x)\right| \leqslant G\big(|x|\big) = \frac{g(|x|)}{|x|^p}, \tag{1.13}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive bounded continuous function which satisfies some further conditions.

For functions satisfying (1.13), we say that V belongs to *Kato class near* ζ if g satisfies the following integral condition

$$\left|\int_\zeta^1 t^{\alpha^*} \left|\int_a^t \frac{g(s)}{s^p} s^{d-1}\,\mathrm{d}s\right|^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}\right| = \left|\int_\zeta^1 \left|t^{1-d}\int_a^t \frac{g(s)}{s^p} s^{d-1}\,\mathrm{d}s\right|^{\frac{1}{p-1}} \mathrm{d}t\right| < \infty, \tag{C1}$$

where a is chosen to make the class of admissible singular potentials near ζ almost optimal. So, $a = \zeta$ in the classical cases and $a = 1$ in the nonclassical cases, i.e. a is given by

	$p < d$	$p > d$	$p = d$
$\zeta = 0$	$a = 0$	$a = 1$	$a = 0$
$\zeta = \infty$	$a = 1$	$a = \infty$	$a = \infty$

Remark 1.6. The quasilinear Kato class in Ω is defined and studied in [10,14–16] using the Wolff potential. We note that if $V \in L^\infty(\Omega \setminus \{\zeta\})$ belongs to Kato class near ζ, then V belongs to the Kato class in Ω.

Some of our results are valid under the following closely related *Dini condition* at ζ

$$\left| \int_\zeta^1 \frac{g(s)}{s} \, \mathrm{d}s \right| < \infty, \quad p \neq d,$$

$$\left| \int_\zeta^1 \frac{g(s)}{s} |\log s|^{d-1} \, \mathrm{d}s \right| < \infty, \quad p = d. \tag{C2}$$

In Section 2 we prove the following lemma.

Lemma 1.7. *If $V \in L^\infty_{\mathrm{loc}}(\Omega \setminus \{\zeta\})$ satisfies either condition* (C1) *or condition* (C2) *near ζ, then V has a weak Fuchsian singularity at ζ. Moreover,*

$$R^p V(Rx) \underset{R\to\zeta}{\longrightarrow} 0 \quad \text{in the weak* topology of } L^\infty_{\mathrm{loc}}\big(\mathbb{R}^d \setminus \{0\}\big). \tag{1.14}$$

Remarks 1.8. 1. Two prototypes of functions g that satisfy both conditions (C1) and (C2) are $g(|x|) = |x|^\varepsilon$ and $|\log(|x|)|^{-\beta}$ if $\zeta = 0$, or $g(|x|) = |x|^{-\varepsilon}$ and $|\log(|x|)|^{-\beta}$ if $\zeta = \infty$, where ε is any positive number and $\beta > (p-1)$ for $p \neq d$ or $\beta > p$ for $p = d$.

2. If V belongs to $L^q(\Omega \setminus \{\zeta\})$, then without loss of generality we may assume that $G \in L^q(\Omega \setminus \{\zeta\})$, where $G(s) = g(s)|s|^{-p}$ satisfies (1.13). Therefore, in the classical case $p \leqslant d$ and $\zeta = 0$ (respectively, $p \geqslant d$ and $\zeta = \infty$), if $V \in L^q(\Omega \setminus \{\zeta\})$ with $q > d/p$ (respectively, $q < d/p$), then conditions (C1) and (C2) are satisfied near ζ (cf. Theorem 1.10 and [25]).

3. In the nonclassical cases, the integrability of V near ζ clearly implies that V satisfies conditions (C1) and (C2) near ζ.

Under the above assumptions, positive solutions of the equation $Q'(u) = 0$ in $\Omega \setminus \{\zeta\}$ are $C^{1,\alpha}$-continuous and satisfy Harnack inequality in $\Omega \setminus \{\zeta\}$ [23, Theorem 7.4.1]. Moreover, the equation $Q'(u) = 0$ admits positive (weak) solutions near the singular point ζ (see Lemmas 2.3 and 3.1).

The present paper is a continuation of our recent paper [6]. In that paper we proved *ratio limit* theorems for *quotients* of two positive solutions of the equation $Q'_V(u) = 0$ near ζ if V satisfies condition (C1) or (C2) near ζ. More precisely, under the (weaker) assumption that V has a weak Fuchsian singularity at the isolated singular point ζ, it was shown that if u and v are two positive solutions of the equation $Q'(w) = 0$ in a punctured neighborhood of ζ, then

$$\lim_{x\to\zeta} \frac{u(x)}{v(x)} \quad \text{exists;}$$

the limit might be infinite. As a result, some positive Liouville theorems, and removable singularity theorems for the equation $Q'(w) = 0$ in certain domains were obtained.

The goal of the present paper is to study the precise behavior of a *single* positive solution u defined near ζ. One of the main results of the present work is that under assumption (C1)

$$\lim_{x\to\zeta} u(x) \quad \text{exists.}$$

Moreover, in the nonclassical cases (where either $\zeta = 0$ and $p > d$, or $\zeta = \infty$ and $p < d$) positive solutions of $Q'(u) = 0$ in $\Omega \setminus \{\zeta\}$ are in fact continuous at the singular point ζ (see Theorem 1.14).

Under further assumptions we obtain the exact asymptotic behavior for the case $\lim_{x\to\zeta} u(x) = 0$ (see Theorem 1.17). The particular case $V = 0$ has been studied in [11,26]. The case $p = \infty$, $\zeta = 0$, and $Q'(u) = -\Delta_\infty(u)$ has been studied in [3] and stimulated the present paper as well. We note that our result answers a question posed in [21, Section 5] for the case $\zeta = 0$ and $p > d$.

Remark 1.9. Suppose that V has a *Fuchsian type singularity* at ζ. Then even in the linear case ($p = 2$) positive solutions of $Q'(u) = 0$ in $\Omega \setminus \{\zeta\}$ might not admit a limit at ζ (see [19, Example 9.1]), albeit, a ratio limit theorem holds true near any Fuchsian singular point if $p = 2$ [19]. Such a ratio limit theorem should also be true for $p \neq 2$ if (1.8) is satisfied (see [6] for partial results).

Isolated singularities have been studied extensively in the past fifty years. Indeed, the case $p < d$ and $\zeta = \infty$ has been studied in [15,18, and the references therein]. On the other hand, *interior* singularities of p-Laplacian type equations for the case $1 < p \leqslant d$ have been studied by Serrin in [25] who proved the following removable singularity result.

Theorem 1.10. *Let $1 < p \leqslant d$, and assume that $V \in L^q_{\mathrm{loc}}(B_R)$ with $q > d/p$. Let u be a positive solution of $Q'(u) = 0$ in the punctured ball $B_R \setminus \{0\}$. Then either u has a removable singularity at the origin, or*

$$u(x) \asymp \begin{cases} |x|^{\alpha^*} & \text{if } p < d, \\ -\log|x| & \text{if } p = d. \end{cases} \tag{1.15}$$

We improve the above result and obtain the *asymptotic behavior* of positive solutions near $\zeta = 0$ (the proof is given in Section 7).

Theorem 1.11. *Let $1 < p \leqslant d$, and assume that V satisfies (1.13) near $\zeta = 0$ and that $V \in L^q_{\mathrm{loc}}(B_R)$, with $q > d/p$. Let u be a positive solution of $Q'(u) = 0$ in the punctured ball $B_R \setminus \{0\}$. Then either u has a removable singularity at the origin, or*

$$u(x) \underset{x\to 0}{\sim} \begin{cases} |x|^{\alpha^*} & \text{if } p < d, \\ -\log|x| & \text{if } p = d. \end{cases} \tag{1.16}$$

For bounded potentials the above result is due to [11,20]. The assumptions on V can be further weakened to nearly optimal classes. In particular, the removability of isolated singularity has been studied under the assumptions that $p \leqslant d$, and V belongs to certain (Kato) classes of functions or measures; see [8,10,14–16,18] and the references therein.

Let us mention the other classical case. In [6] we studied the asymptotic behavior of positive p-harmonic functions near ∞ for the case $p \geqslant d$ (cf. [26]). Using a modified Kelvin transform, we proved:

Theorem 1.12. *Let $p \geqslant d > 1$, and let u be a positive solution of the equation $-\Delta_p(u) = 0$ in a neighborhood of infinity in $\mathbb{R}^d$. Then either u has a removable singularity at ∞ (i.e., u admits a finite limit as $x \to \infty$), or*

$$u(x) \underset{x\to\infty}{\sim} \begin{cases} |x|^{\alpha^*} & \text{if } p > d, \\ \log|x| & \text{if } p = d. \end{cases}$$

Remark 1.13. The case $p = d$ in Theorem 1.12 clearly follows from Theorem 1.11 using the conformality of the d-Laplacian.

We continue with two new general theorems concerning removable singularity. The first result deals with the *existence* of the limit. We prove

Theorem 1.14. *Suppose that V satisfies condition* (C1) *near ζ. Let u be a positive solution of the equation $Q'(u) = 0$ in a punctured neighborhood of ζ. Then*

$$\lim_{x\to\zeta} u(x) = \ell, \tag{1.17}$$

where $0 \leqslant \ell \leqslant \infty$.

More precisely, for $p \neq d$ we have in the classical cases ($p < d$ and $\zeta = 0$, or $p > d$ and $\zeta = \infty$) that $0 < \ell \leqslant \infty$, and in the nonclassical cases ($p > d$ and $\zeta = 0$, or $p < d$ and $\zeta = \infty$) we have $0 \leqslant \ell < \infty$.

Assume further that V satisfies Dini's condition (C2) *near ζ, then in the classical cases there are positive solutions satisfying* (1.17) *with $0 < \ell < \infty$, and with $\ell = \infty$, while in the nonclassical cases there are positive solutions satisfying* (1.17) *with $0 < \ell < \infty$, and with $\ell = 0$.*

Remarks 1.15. 1. Serrin [26] proved Theorem 1.14 for equations in divergence form without a potential term.

2. If $p < d$ and $\zeta = 0$, then the finiteness of the limit implies that the singularity is removable (see Corollary 2.7). This statement is not true for $p > d$ as can be seen from the case of the p-Laplace equation and the "fundamental solution" v_{α^*} (see Example 1.1).

3. The proof of Theorem 1.14 in the nonclassical cases is based on a new general three-spheres theorem which is valid near ζ (see Theorem 5.1). For related three-spheres theorems for elliptic PDEs see for example [1,5,12,17,22,27] and the references therein.

Example 1.16. The statements of Theorem 1.14 do not hold true if V has a Fuchsian type singularity at ζ. Indeed, consider the Hardy potential $V(x) = \lambda|x|^{-p}$. So, V has Fuchsian type singularities at $\zeta = 0$ and $\zeta = \infty$. By Hardy's inequality (2.7), the equation

$$-\Delta_p(u) - \lambda\frac{|u|^{p-2}u}{|x|^p} = 0 \tag{1.18}$$

admits a positive solution near ζ (and also in $\mathbb{R}^d \setminus \{0\}$) if and only if

$$\lambda \leqslant c_H := \left|\frac{p-d}{p}\right|^p.$$

Moreover, for $\lambda = c_H$, Eq. (1.18) on $\mathbb{R}^d \setminus \{0\}$ admits a unique (up to a multiplicative constant) positive (super)solution which is given by $u(x) = |x|^{\gamma_*}$, where $\gamma_* := (p-d)/p$.

On the other hand, if $\lambda < c_H$, then (1.18) on $\mathbb{R}^d \setminus \{0\}$ has two positive (radial) solutions of the form $v_\pm(x) := |x|^{\gamma_\pm(\lambda)}$, where $\gamma_-(\lambda) < \gamma_* < \gamma_+(\lambda)$, and $\gamma_\pm(\lambda)$ are solutions of the transcendental equation (cf. (1.6))

$$-\gamma|\gamma|^{p-2}\big[\gamma(p-1) + d - p\big] = \lambda.$$

In particular, the statements of Theorem 1.14 do not hold true for (1.18) with $\lambda \neq 0$. On the other hand, since Hardy's potential is radially symmetric, the ratio limit theorem holds true for positive solutions of (1.18) near ζ [6].

Our next result extends the results of Theorems 1.11 and 1.12 and gives the *asymptotic behavior* of positive solutions near ζ in the *nonclassical cases* ($p > d$ and $\zeta = 0$, or $p < d$ and $\zeta = \infty$) under the additional assumption that the potential V is *integrable* near the singular point ζ. Recall (Remark 1.8) that conditions (C1) and (C2) hold under these assumptions.

Theorem 1.17. *Suppose that V has a Fuchsian type singularity and is integrable near ζ. Assume that u is a positive solution of the equation $Q'(u) = 0$ in the punctured ball $B_R \setminus \{0\}$ and $p > d$ (respectively, B_R^* and $p < d$) for some $R > 0$. Then*

$$\text{either} \quad 0 < \lim_{x \to \zeta} u(x) < \infty, \quad \text{or} \quad u(x) \underset{x \to \zeta}{\sim} |x|^{\alpha^*}.$$

Remarks 1.18. 1. Suppose that $\zeta = 0$ and $p > d$, then Theorems 1.14 and 1.17 seem to be new even for $V \in L^\infty(B_R)$.

2. The integrability assumption in Theorem 1.17 seems to be too restrictive; we believe that the assertion of the theorem is still valid under assumptions (C1) and (C2). This is indeed the case for example if $p = 2 < d$ and $\zeta = \infty$ (see for example [8,10,14,15,18]).

The outline of the paper is as follows. In the next section we recall some notions and results we need throughout the paper. In particular, we present a uniform Harnack inequality and a ratio limit theorem for positive solutions defined near the singularity, and we introduce Wolff's potential near ζ. In Section 3 we prove the existence of two classes of positive solutions of the equation $Q'(u) = 0$ near ζ, the small and the large one, with asymptotic behaviors similar to those of the p-Laplace equation. In Section 4 we obtain the asymptotic behavior of small solutions near ζ for the case $p \neq d$. The proof uses Lindqvist's method in [13]. In Section 5 we state and prove a three-spheres theorem that holds true for certain positive sub/supersolutions near isolated singular points. Using this three-spheres theorem and the asymptotic behavior of small solutions, we prove in Section 6 Theorem 1.14. Section 7 is devoted to the proof of the asymptotic behavior of singular solutions (Theorems 1.11 and 1.17). In particular, Theorem 1.17 follows from the asymptotic behavior of small solutions and our three-spheres theorem. We conclude the paper in Section 8 with some applications to a positive Liouville theorem in $\mathbb{R}^d$, and to the behavior near an interior isolated singularity of positive solutions of minimal growth in a neighborhood of infinity in Ω.

2. Preliminaries

The following notations and conventions will be used throughout the paper. We denote by $B_R(x_0)$ (respectively, $S_R(x_0)$) the open ball (respectively, the spheres) of radius R centered at x_0, and let $B_R := B_R(0)$, $S_R := S_R(0)$, and $B_R^* := \mathbb{R}^d \setminus \overline{B_R}$. We write $\Omega_1 \Subset \Omega_2$ if Ω_2 is open, $\overline{\Omega_1}$ is compact and $\overline{\Omega_1} \subset \Omega_2$.

For a function u defined in $\{x \mid 0 \leqslant R_1 < |x| < R_2 \leqslant \infty\}$, we use the notations

$$m_u(r) := \inf_{x \in S_r} u(x), \qquad M_u(r) := \sup_{x \in S_r} u(x) \quad \forall R_1 < r < R_2.$$

The subscript u in the notations m_u and M_u will be omitted when there is no danger of confusion. Let $f, g \in C(D)$ be nonnegative functions, we denote $f \asymp g$ on D if there exists a positive constant C such that

$$C^{-1} g(x) \leqslant f(x) \leqslant C g(x) \quad \text{for all } x \in D.$$

By $f \underset{x \to \zeta}{\sim} g$ we mean that

$$\lim_{x \to \zeta} \frac{f(x)}{g(x)} = C$$

for some positive constant C. Finally, for a radial function f defined in a punctured neighborhood of ζ, we use the notation $\tilde{f}(|x|) := f(x)$.

A function $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be a (*weak*) *solution* of the equation

$$Q'(u) = -\Delta_p(u) + V|u|^{p-2}u = 0$$

in a domain Ω if

$$\int_\Omega \left(|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi + V|v|^{p-2}v\varphi\right) \mathrm{d}x = 0 \quad \forall \varphi \in C_0^\infty(\Omega). \tag{2.1}$$

We say that a real function $v \in C^1_{\mathrm{loc}}(\Omega)$ is a *supersolution* (respectively, *subsolution*) of the equation $Q'(u) = 0$ in Ω if for every nonnegative $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \left(|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi + V|v|^{p-2}v\varphi\right) \mathrm{d}x \geqslant 0 \quad (\text{respectively, } \leqslant 0). \tag{2.2}$$

Recall the definition of the Wolff potential on $\mathbb{R}^d$ [9]. For $s > 1$, $\alpha > 0$ with $0 < \alpha s < d$, and a measure μ, the *Wolff potential* is defined for $0 < \rho \leqslant \infty$ by:

$$W^\rho_{\alpha,s}\mu(x) := \int_0^\rho \left(\frac{\mu(B(x,t))}{t^{d-\alpha s}}\right)^{1/(s-1)} \frac{\mathrm{d}t}{t}.$$

Suppose that $G : \mathbb{R}^+ \to \mathbb{R}^+$, $G(s) = g(s)|s|^{-p}$, satisfies the Kato type condition (C1). Motivated by the above definition (with $\alpha = 1$, and $s = p$), we define the *Wolff potential of G around ζ* by

$$W^\zeta_G(x) = W_G(x) := \frac{1}{|B_1|}\left|\int_\zeta^{|x|} t^{\alpha^*}\left(\int_{X_t} G(|y|)\,\mathrm{d}y\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}\right|, \tag{2.3}$$

where the d-dimensional domain X_t depends on p, d, and ζ and is given by

	$p < d$	$p > d$	$p = d$
$\zeta = 0$	$X_t = B_t$	$X_t = B_1 \setminus B_t$	$X_t = B_t$
$\zeta = \infty$	$X_t = B_t \setminus B_1$	$X_t = B^*_t$	$X_t = B^*_t$

By integration over the spherical variables, we get the expression

$$W_G(x) = \left|\int_\zeta^{|x|} t^{\alpha^*}\left|\int_a^t \frac{g(s)}{s^p} s^{d-1}\,\mathrm{d}s\right|^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}\right| = \left|\int_\zeta^{|x|}\left|t^{1-d}\int_a^t \frac{g(s)}{s^p} s^{d-1}\,\mathrm{d}s\right|^{\frac{1}{p-1}} \mathrm{d}t\right| < \infty, \tag{2.4}$$

where as above, $G(|x|) = g(|x|)|x|^{-p}$, and a is given as in condition (C1) by

	$p < d$	$p > d$	$p = d$
$\zeta = 0$	$a = 0$	$a = 1$	$a = 0$
$\zeta = \infty$	$a = 1$	$a = \infty$	$a = \infty$

In particular, W_G is well defined for G in Kato's class near ζ. In the sequel we use our convention $\tilde{W}_G(|x|) := W_G(x)$. It turns out that W_G solves the nonhomogeneous p-Laplace equation near ζ.

Lemma 2.1. *Suppose that G satisfies condition* (C1) *near ζ. Then near ζ we have*

$$-\Delta_p\big(W_G(x)\big) = \begin{cases} -G(|x|) & \text{in the classical cases}, \\ G(|x|) & \text{in the nonclassical cases}, \end{cases}$$

where $G(|x|) = \frac{g(|x|)}{|x|^p}$.

Moreover, W_G has the following properties

1. $\lim_{x\to\zeta} W_G(x) = 0$.
2. $\partial \tilde{W}_G(r)/\partial r > 0$ *if* $\zeta = 0$.
3. $\partial \tilde{W}_G(r)/\partial r < 0$ *if* $\zeta = \infty$.

Furthermore, in the nonclassical cases, if G is integrable, then

$$W_G(x) \underset{x\to\zeta}{\sim} v_{\alpha^*}(x).$$

Proof. The first statement is a straightforward computation (cf. [24]). By Lebesgue dominated convergence theorem we have $\lim_{x\to\zeta} W_G(x) = 0$. Assertions 2 and 3 are obtained by differentiation. Finally, the last assertion of the lemma follows from (2.4) by l'Hôpital's rule and the integrability assumption. □

We introduce another type of potential for functions G satisfying (C2):

$$U_G(x) := \begin{cases} \alpha^* \int_b^{v_{\alpha^*}(x)} \int_\zeta^{v_{\alpha^*}^{-1}(\tau)} \frac{g(s)}{s}\,\mathrm{d}s\,\mathrm{d}\tau, & p \neq d, \\ \int_1^{v_{\alpha^*}(x)} \int_\zeta^{v_{\alpha^*}^{-1}(\tau)} \frac{g(s)}{s} |\log s|^{d-1}\,\mathrm{d}s\,\mathrm{d}\tau, & p = d, \end{cases} \tag{2.5}$$

where b is given by

	$p < d$	$p > d$
$\zeta = 0$	$b = 1$	$b = 0$
$\zeta = \infty$	$b = 0$	$b = 1$

We give up of the positivity of U_G, it plays no role in the sequel. It can be verified easily that U_G satisfies the following properties.

Lemma 2.2. *Let G satisfy* (C2); *then U_G is well defined and*

$$\lim_{x\to\zeta} \frac{U_G(x)}{v_{\alpha^*}(x)} = \lim_{x\to\zeta} \partial_{v_{\alpha^*}} U_G(x) = 0.$$

Moreover,

$$\big(\partial^2_{v_{\alpha^*}} U_G\big)(x) = \begin{cases} \frac{g(|x|)}{v_{\alpha^*}(|x|)}, & p \neq d, \\ \frac{g(|x|)}{(v_{\alpha^*}(|x|))^{1-d}}, & p = d. \end{cases}$$

Proof. The second statement follows by a direct computation, while the first follows by l'Hôpital's rule and Lebesgue's dominated convergence theorem. □

We are now ready to prove Lemma 1.7 claiming that each of our assumptions (C1) and (C2) implies that V has a weak Fuchsian singularity at ζ.

Proof of Lemma 1.7. We first prove the claim assuming (C2). Note that if $p = d$, then (C2) implies that

$$\left| \int_{\zeta}^{1} \frac{g(s)}{s} \, \mathrm{d}s \right| < \infty.$$

Let $R_n \to \zeta$. It is sufficient to prove that

$$|R_n|^p \frac{g(R_n|x|)}{|R_n|^p |x|^p} = \frac{g(R_n|x|)}{|x|^p} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in the weak}^* \text{ topology of } L^\infty_{\mathrm{loc}}\big(\mathbb{R}^d \setminus \{0\}\big).$$

So, it suffices to prove that $g(R_n s) \to 0$ in the weak* topology of $L^\infty_{\mathrm{loc}}(\mathbb{R}_+ \setminus \{0\})$.

By compactness, there is $g_\infty \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+ \setminus \{0\})$ such that (up to a subsequence)

$$g(R_n x) \underset{n\to\infty}{\longrightarrow} g_\infty(x)$$

in the weak* topology of $L^\infty_{\mathrm{loc}}(\mathbb{R}_+ \setminus \{0\})$. So for any fixed $0 < a < b < \infty$ we have

$$\int_a^b \frac{g(R_n s)}{s} \, \mathrm{d}s \underset{n\to\infty}{\longrightarrow} \int_a^b \frac{g_\infty(s)}{s} \, \mathrm{d}s.$$

On the other hand,

$$\int_a^b \frac{g(R_n s)}{s} \, \mathrm{d}s = \int_{R_n a}^{R_n b} \frac{g(s)}{s} \, \mathrm{d}s \underset{n\to\infty}{\longrightarrow} 0$$

by Lebesgue's dominated convergence theorem. Hence $g_\infty = 0$.

Assume now that (C1) is satisfied. The structure of the proof is similar. Let $G_n(x) := R_n^p G(R_n x)$. We may suppose that

$$G_n(x) \underset{n\to\infty}{\longrightarrow} G_\infty(x)$$

in the weak* topology of $L^\infty_{\mathrm{loc}}(\mathbb{R}_+ \setminus \{0\})$. By Fatou's lemma we have

$$W_{G_\infty}(x) \leqslant \liminf_{n\to\infty} W_{G_n}(x).$$

On the other hand, by a direct computation we obtain

$$W_{G_n}(x) = \frac{1}{|B_1|} \left| \int_{\zeta}^{R_n|x|} t^{\alpha^*} \left(\int_{R_n X_{(t/R_n)}} G\big(|y|\big) \, \mathrm{d}y \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \right|. \tag{2.6}$$

We note that (both in the classical cases and in the nonclassical cases) if x is near ζ and t belongs to the interval of the integration in (2.6), then $R_n X_{(t/R_n)} \subset X_t$. Consequently, for such x we have

$$W_{G_n}(x) \leqslant \frac{1}{|B_1|}\left| \int_{\zeta}^{R_n|x|} t^{\alpha^*} \left(\int_{X_t} G\big(|y|\big)\, \mathrm{d}y \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \right|.$$

Therefore, our assumption (C1) and Lebesgue's dominated convergence theorem imply that $W_{G_n}(x) \to 0$. Hence, we arrived at

$$W_{G_\infty}(x) = 0$$

near ζ. Thus, $G_\infty(x) = 0$ near ζ. $\square$

Next, we present a simple sufficient condition ensuring the existence of positive solutions near ζ. We note that Lemma 3.1 implies that such positive solutions exist (for any $1 < p < \infty$) if either condition (C1) or (C2) is satisfied.

Lemma 2.3. *Suppose that $p \neq d$ and that V satisfies (1.13) with a continuous positive function g satisfying $\lim_{x\to\zeta} g(|x|) = 0$. Then the equation $Q'(u) = 0$ admits positive solutions in a punctured neighborhood of the singular point ζ.*

Proof. For $\zeta = 0$ (respectively, $\zeta = \infty$) and $p \neq d$, the lemma's assumption and Hardy's inequality

$$\int_{\mathbb{R}^d\setminus\{0\}} |\nabla u|^p \,\mathrm{d}x \geqslant \left|\frac{p-d}{p}\right|^p \int_{\mathbb{R}^d\setminus\{0\}} \frac{|u|^p}{|x|^p}\,\mathrm{d}x, \quad u \in C_0^\infty\big(\mathbb{R}^d \setminus \{0\}\big), \tag{2.7}$$

imply that for R small (respectively, large) enough the functional

$$Q(u) := \int_\Omega \big(|\nabla u|^p + V|u|^p\big)\,\mathrm{d}x$$

is nonnegative on $C_0^\infty(\Omega)$, where $\Omega = B_R \setminus \{0\}$ (respectively, $\Omega = B_R^*$), and therefore, by [20], the equation $Q'(u) = 0$ admits positive solutions in $B_R \setminus \{0\}$ (respectively, B_R^*). $\square$

We recall that positive super- and subsolutions satisfy the following weak comparison principle.

Theorem 2.4 *(Weak comparison principle). (See [7].) Let $V \in L^\infty_{\mathrm{loc}}(\Omega)$, and let Ω' be a bounded $C^{1,\alpha}$ subdomain of a domain Ω, such that $\Omega' \Subset \Omega$. Assume that the equation $Q'(w) = 0$ admits a positive solution in Ω and suppose that $u, v \in C^1(\Omega') \cap C(\overline{\Omega'})$, $u, v \geqslant 0$, satisfy the following inequalities*

$$\begin{aligned} Q'(u) &\leqslant 0 \quad \text{in } \Omega', \\ Q'(v) &\geqslant 0 \quad \text{in } \Omega', \\ u &\leqslant v \quad \text{on } \partial\Omega'. \end{aligned} \tag{2.8}$$

Then $u \leqslant v$ in Ω'.

Our results hinge on a uniform Harnack estimates for positive solutions near an isolated Fuchsian type singularity [4,21], and on a ratio limit theorem for the quotients of any such two solutions near a weak Fuchsian singularity [6, Theorem 2.6]. The following statements are formulated for the potentials under consideration.

Proposition 2.5. *Assume that V has a Fuchsian type singularity at ζ. Fix $0 < r_0 < R$ (respectively, $r_0 > R$). Then there exists a constant $C = C(r_0, R, V, d, p) > 0$ such that any positive solution u of the equation $Q'(u) = 0$ in $B_R \setminus \{0\}$ (respectively, B_R^*) satisfies the following uniform Harnack inequality*

$$M_u(r) \leqslant C m_u(r) \quad \textit{for all } 0 < r < r_0 \quad (\textit{respectively, } r > r_0). \tag{2.9}$$

The above uniform Harnack inequality together with a dilatation argument and standard elliptic estimates imply the following gradient estimates:

Lemma 2.6. *Assume that V has a Fuchsian type singularity at ζ. Fix $0 < r_0 < R$ (respectively, $r_0 > R$). Then there exists a constant $C = C(r_0, R, V, d, p) > 0$ such that for any positive solution u of the equation $Q'(u) = 0$ in $B_R \setminus \{0\}$ (respectively, B_R^*) we have*

$$\left|\nabla u(x)\right| \leqslant C\frac{u(x)}{|x|} \quad \textit{for all } 0 < |x| < r_0 \quad \big(\textit{respectively, } |x| > r_0\big). \tag{2.10}$$

Lemma 2.6 implies the following removable singularity result.

Corollary 2.7. *Suppose that V has a Fuchsian type singularity at the origin, and that $p < d$. Let u be a positive bounded solution of the equation $Q'(u) = 0$ in the punctured ball $B_R \setminus \{0\}$. Then u has a removable singularity at the origin.*

Proof. By Lemma 2.6, $u \in W^{1,p}(B_R)$, and consequently u is a positive solution in the ball B_R. □

In [6] we proved the following ratio limit theorem.

Proposition 2.8. *(See [6, Theorem 2.6].) Assume that V has a weak Fuchsian type singularity at ζ. Let u and v be two positive solutions of the equation $Q'(w) = 0$ in a punctured neighborhood of ζ. Then the limit*

$$\lim_{x \to \zeta} \frac{u(x)}{v(x)} \quad \textit{exists};$$

the limit might be infinite.

We conclude this section with the following elementary lemma whose proof can be easily verified.

Lemma 2.9. *Let u be a positive solution of the equation $Q_V'(u) = 0$ in Ω, and $f \in C^2(\mathbb{R}_+)$ be a positive function. Then*

$$Q_V'\big(f(u)\big) = -(p-1)\left|f'(u)\right|^{p-2} f''(u)|\nabla u|^p - \left|f'(u)\right|^{p-2} f'(u)V|u|^{p-1} + V f(u)^{p-1}. \tag{2.11}$$

3. Existence of the desired solutions

In this section we prove the existence of positive subsolutions, supersolutions, and solutions of the equation $Q'(u) = 0$ near ζ with the desired asymptotic behavior. We present explicit formulas for such sub- and supersolutions in terms of the potentials W_G and U_G (see (2.3) and (2.5)). The existence of the appropriate solutions follows using Perron's method (cf. [7], in particular the discussion around (3.1) therein).

Lemma 3.1. *Suppose that V satisfies condition* (C1) (*respectively,* (C2)) *near ζ. Denote*

$$u_\pm(x) := \begin{cases} 1 \mp CW_G(x) & \text{in the classical cases}, \\ 1 \pm CW_G(x) & \text{in the nonclassical cases}, \end{cases}$$
$$v_\pm(x) := v_{\alpha^*}(x) \mp CU_G(x), \tag{3.1}$$

where W_G, U_G and $v_{\alpha^*}(x)$ are defined by (2.3), (2.5) *and* (1.4), *respectively, and C is a large positive constant. Then u_+ (respectively, v_+) is a positive supersolution and u_- (respectively, v_-) is a positive subsolution of the equation $Q'(u) = 0$ near the singular point ζ. Moreover,*

$$u_\pm(x) \underset{x\to\zeta}{\sim} 1 \quad \bigl(\text{respectively, } v_\pm(x) \underset{x\to\zeta}{\sim} v_{\alpha^*}(x)\bigr).$$

Proof. In light of Lemma 2.1 we have that $u_\pm \sim 1$ near ζ, and $-\Delta_p u_\pm = \pm C^{p-1}G(|x|)$. Therefore, near ζ we have

$$Q'_V(u_\pm) = \pm C^{p-1}G\bigl(|x|\bigr) + u_\pm^{p-1}V(x) \gtreqless \pm C^{p-1}G\bigl(|x|\bigr) \mp (1+\varepsilon)\bigl|V(x)\bigr| \gtreqless 0. \tag{3.2}$$

Similarly, Lemma 2.2 implies that $v_\pm \sim v_{\alpha^*}$ near ζ. Using Lemmas 2.9 and 2.2, we obtain

$$-\Delta_p v_\pm \sim \pm C^{p-1}\bigl(\partial^2_{v_{\alpha^*}} U_G\bigr)(x)\bigl|\nabla v_{\alpha^*}(x)\bigr|^p \sim \pm C^{p-1}\frac{g(|x|)}{|x|^p} v_{\alpha^*}^{p-1}.$$

Therefore, near ζ we have

$$\begin{aligned} Q'_V(v_\pm) &\sim \pm C^{p-1}G\bigl(|x|\bigr)v_{\alpha^*}^{p-1}(x) + V(x)v_{\alpha^*}^{p-1}(x) \\ &\gtreqless \bigl(\pm C^{p-1}G\bigl(|x|\bigr) \mp (1+\varepsilon)\bigl|V(x)\bigr|\bigr)v_{\alpha^*}^{p-1}(x) \gtreqless 0. \qquad \square \end{aligned} \tag{3.3}$$

Remark 3.2. The existence of positive supersolutions of the equation $Q'(u) = 0$ in a domain D implies the existence of positive solutions of the equation $Q'(u) = 0$ in D. Hence, Lemma 3.1 provides us with a proof of the claim of Lemma 2.3 under different assumptions, and in particular, covers the missing case $p = d$.

We conclude this section with a lemma that claims that as in the case of the p-Laplacian (see Example 1.1), there exist two positive solutions u_{small} and u_{large} of the equation $Q'(u) = 0$ near ζ, each of which behaves asymptotically either as the constant function or as the "fundamental solution" $v_{\alpha^*}(x)$ (see (1.4)) of the p-Laplacian.

Lemma 3.3. *Suppose that V satisfies conditions* (C1) *and* (C2) *near ζ. Then there exist two positive solutions $u^{(\zeta)}_{\mathrm{small}}$ and $u^{(\zeta)}_{\mathrm{large}}$ of the equation $Q'(u) = 0$ defined in a punctured neighborhood of ζ and satisfying*

$$\lim_{x\to\zeta} \frac{u^{(\zeta)}_{\mathrm{small}}(x)}{u^{(\zeta)}_{\mathrm{large}}(x)} = 0.$$

More precisely, we have

$$u^{(\zeta)}_{\mathrm{small}}(x) \underset{x\to\zeta}{\sim} \min\bigl\{1, v_{\alpha^*}(x)\bigr\}$$

and

$$u^{(\zeta)}_{\text{large}}(x) \underset{x\to\zeta}{\sim} \max\{1, v_{\alpha^*}(x)\}.$$

Proof. By Lemma 3.1 we have positive subsolutions and supersolutions which behave as the desired solutions. Therefore, using a standard sub/supersolution argument (based on the weak comparison principle, cf. [7], in particular the discussion around (3.1) therein), it follows that there exist positive solutions with the desired asymptotic behavior. □

4. The asymptotics of 'small' solutions near ζ

Throughout this section we assume that $p \neq d$. The aim of this section is to prove that all positive solutions of the equation $Q'(u) = 0$ near ζ that are bounded by the small positive solution of the p-Laplace equation near ζ, have the same asymptotic behavior. Hence, if V satisfies conditions (C1) and (C2) near ζ, and if a solution u near ζ satisfies $u \leqslant Cu^{(\zeta)}_{\text{small}}$, then $u \sim u^{(\zeta)}_{\text{small}}$.

Theorem 4.1. *Suppose that V has a weak Fuchsian type singularity at ζ. Let v_1, v_2 be two distinct positive solutions of the equation $Q'(u) = 0$ in a punctured neighborhood Ω of ζ satisfying*

$$v_i(x) \leqslant C \min\{1, |x|^{\alpha^*}\} \quad \textit{near } \zeta,$$

where C is a positive constant and $i = 1, 2$. Then

$$v_1 \underset{x\to\zeta}{\sim} v_2.$$

Proof. The proof is based on Lindqvist's method [13]. Without loss of generality we assume that $\partial\Omega \setminus \{\zeta\} = S_R$ for some $R > 0$, and $v_i \in C(\overline{\Omega} \setminus \{\zeta\})$, $v_i > 0$, $i = 1, 2$, on S_R. Then for some $C_1 > 0$ small $C_1 v_i \leqslant 1 \leqslant C_1^{-1} v_i$ on S_R. By Perron's sub/supersolution (cf. [7], in particular the discussion around (3.1) therein) method, there exist positive solutions u_i of the Dirichlet problem

$$Q'(u_i) = 0 \quad \text{in } \Omega,$$
$$u_i = 1 \quad \text{on } S_R,$$
$$u_i \asymp v_i \quad \text{near } \zeta.$$

Due to the existence of the ratio limit near ζ, we have $u_i \sim v_i$ near ζ. Moreover, using an elementary comparison argument we get that either $u_1 \geqslant u_2$ or $u_2 \geqslant u_1$. So, without loss of generality, we assume that $u_1 \geqslant u_2$. We claim that $u_1 = u_2$ and this implies the lemma.

We use Lindqvist technique [13]. For completeness, we give a self-contained proof. Nevertheless, the reader is advised to consult the above paper for details.

By our assumption $u_i(x) \leqslant C \min\{1, |x|^{\alpha^*}\}$, and therefore,

$$\int_\Omega u_i^p \frac{1}{|x|^p}\, \mathrm{d}x < \infty. \tag{4.1}$$

On the other hand, Lemma 2.6 implies that $|\nabla u_i(x)| \leqslant C_2 u_i(x)|x|^{-1}$, and hence $\nabla u_i \in L^p(\Omega)$. Consequently, for $0 < \varepsilon < 1$, the $C^{1,\alpha}$-function

$$\mu_\varepsilon := \frac{(u_1+\varepsilon)^p - (u_2+\varepsilon)^p}{(u_1+\varepsilon)^{p-1}}$$

that vanishes on S_R is a valid test function for the equation $Q'(u)=0$. In particular,

$$\int_{\Omega} |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla \mu_\varepsilon \,\mathrm{d}x + \int_{\Omega} V u_1^{p-1}\mu_\varepsilon \,\mathrm{d}x = 0.$$

Adding together this equation and the one with u_1, u_2 interchanged, we obtain

$$\begin{aligned}
-\int_{\Omega} V(x)\left[\frac{u_1^{p-1}}{(u_1+\varepsilon)^{p-1}} - \frac{u_2^{p-1}}{(u_2+\varepsilon)^{p-1}}\right]\bigl((u_1+\varepsilon)^p - (u_2+\varepsilon)^p\bigr)\,\mathrm{d}x \\
= \int_{\Omega} (u_1+\varepsilon)^p L\bigl(\nabla \log(u_1+\varepsilon), \nabla \log(u_2+\varepsilon)\bigr)\,\mathrm{d}x \\
+ \int_{\Omega} (u_2+\varepsilon)^p L\bigl(\nabla \log(u_2+\varepsilon), \nabla \log(u_1+\varepsilon)\bigr)\,\mathrm{d}x,
\end{aligned} \tag{4.2}$$

where (see [13] or [20])

$$L(a,b) := |a|^p - |b|^p - p|b|^{p-2}b\cdot(a-b) \geqslant 0 \quad \forall a,b \in \mathbb{R}^d \tag{4.3}$$

and equality occurs in (4.3) if and only if $a=b$. In particular, (recalling that $u_1=u_2=1$ on S_R) the right-hand side of (4.2) is zero if and only if $u_1=u_2$.

On the other hand, the integrand of the left-hand side of (4.2) tends to zero as $\varepsilon \to 0$. Moreover, due to Lagrange theorem (recall that $u_1 \geqslant u_2$) we have for $0<\varepsilon<1$

$$\begin{aligned}
&\bigl|V(x)\bigr|\left|\left[\frac{u_1^{p-1}}{(u_1+\varepsilon)^{p-1}} - \frac{u_2^{p-1}}{(u_2+\varepsilon)^{p-1}}\right]\bigl((u_1+\varepsilon)^p - (u_2+\varepsilon)^p\bigr)\right| \\
&\leqslant \bigl|V(x)\bigr|\frac{u_1^{p-1}}{(u_1+\varepsilon)^{p-1}} p(u_1-u_2)(u_1+\varepsilon)^{p-1} \\
&\leqslant p\bigl|V(x)\bigr|u_1^p \in L^1(\Omega).
\end{aligned}$$

Therefore, by letting $\varepsilon \to 0$ in (4.2) and using Lebesgue's dominated convergence theorem and Fatou's lemma, we arrive at the desired conclusion that $u_1=u_2$. $\square$

The uniqueness of the Dirichlet problem for small solutions is an immediate corollary.

Corollary 4.2. *Suppose that V has a weak Fuchsian type singularity at ζ, and let Ω, Ω' be punctured neighborhoods of ζ such that $\partial\Omega \setminus \{\zeta\}$ is smooth, $\overline{\Omega} \setminus \{\zeta\} \subset \Omega'$, and $Q \geqslant 0$ on $C_0^\infty(\Omega')$. Then for any positive $\phi \in C(\partial\Omega \setminus \{\zeta\})$ the following Dirichlet problem*

$$\begin{gathered}
Q'(u) = 0 \quad \text{in } \Omega, \\
u = \phi \quad \text{on } \partial\Omega \setminus \{\zeta\}, \\
u(x) \leqslant C u_{\mathrm{small}}^{(\zeta)}(x) \quad \text{near } \zeta
\end{gathered}$$

has at most one solution.

5. Three-spheres theorem

In this section we prove a three-spheres theorem for certain positive super- and subsolutions u of the equation $Q'_V(v)=0$ near an isolated singular point ζ. Recall that for a fixed function u and $r>0$, we denote

$$m(r) := \inf_{x\in S_r} u(x), \qquad M(r) := \sup_{x\in S_r} u(x).$$

The classical Hadamard three-spheres theorem states that for any positive subharmonic function u in an open ball $B_R \subset \mathbb{R}^d$, $d>2$ (respectively, $d=2$), we have that $M(r)$ is a convex function of r^{2-d} (respectively, $\log r$) (see for example [22]).

Suppose that V satisfies condition (C1) near ζ, and let $\tilde{W}_G = W_G^\zeta$ be the Wolff potential around ζ. We show that if u is a certain positive sub/supersolution of the equation $Q'(v)=0$ near ζ, then the function $m_u(r)$ (respectively, $M_u(r)$) is a concave (respectively, convex) function of $\tilde{W}_G$ *near* the isolated singular point ζ. Let

$$f\big(|x|\big) := C_1 + C_2\tilde{W}_G\big(|x|\big), \tag{5.1}$$

where C_1, C_2 depend on u. The proof is based on Lemma 3.1 that claims that for certain values of C_1, C_2, the function $f(|x|)$ is either a super- or subsolution near the singular point ζ.

Theorem 5.1. *Assume that V satisfies condition* (C1)*, and let $W := W_G^\zeta$ be the corresponding Wolff potential around ζ.*

Suppose that u is a positive subsolution of the equation $Q'(v)=0$ near a punctured neighborhood of ζ, satisfying one of the following conditions:

1.1	If $\zeta=0$ *and* $p>d$,	*then* $\lim_{x\to 0} u(x)=0$
1.2	If $\zeta=0$ *and* $p\leqslant d$,	*then* $\lim_{x\to 0} u(x)=\infty$
1.3	If $\zeta=\infty$ *and* $p\geqslant d$,	*then* $\lim_{x\to\infty} u(x)=\infty$
1.4	If $\zeta=\infty$ *and* $p<d$,	*then* $\lim_{x\to\infty} u(x)=0$

Then for the case $\zeta=0$ (respectively, $\zeta=\infty$) there is $0<R_2$ such that for every $r_2<R_2$ (respectively, $R_2<r_2$) there exists $0<R_1(r_2)<r_2$ (respectively, $0<r_2<R_1(r_2)$), such that for every $0<r_1<R_1(r_2)<r_2<R_2$ (respectively, $0<R_2<r_2<R_1(r_2)<r_1$) the following convex three-spheres inequality

$$M(r_3) \leqslant M(r_1)\frac{\tilde{W}(r_2)-\tilde{W}(r_3)}{\tilde{W}(r_2)-\tilde{W}(r_1)} + M(r_2)\frac{\tilde{W}(r_3)-\tilde{W}(r_1)}{\tilde{W}(r_2)-\tilde{W}(r_1)} \tag{5.2}$$

holds true for all $r_3\in(r_1,r_2)$ (respectively, $r_3\in(r_2,r_1)$). Moreover, M is monotone near ζ.

Similarly, suppose that u is a positive supersolution of the equation $Q'(v)=0$ near a punctured neighborhood of ζ, satisfying one of the following conditions:

2.1	If $\zeta=0$ *and* $p>d$,	*then* $\lim_{x\to 0} u(x)=\infty$
2.2	If $\zeta=0$ *and* $p\leqslant d$,	*then* $\lim_{x\to 0} u(x)=0$
2.3	If $\zeta=\infty$ *and* $p\geqslant d$,	*then* $\lim_{x\to\infty} u(x)=0$
2.4	If $\zeta=\infty$ *and* $p<d$,	*then* $\lim_{x\to\infty} u(x)=\infty$

Then for the case $\zeta=0$ (respectively, $\zeta=\infty$) there is $0<R_2$ such that for every $r_2<R_2$ (respectively, $R_2<r_2$) there exists $0<R_1(r_2)<r_2$ (respectively, $0<r_2<R_1(r_2)$), such that for every $0<r_1<R_1(r_2)<r_2<R_2$ (respectively, $0<R_2<r_2<R_1(r_2)<r_1$) the following concave three-spheres inequality

$$m(r_1)\frac{\tilde{W}(r_2)-\tilde{W}(r_3)}{\tilde{W}(r_2)-\tilde{W}(r_1)}+m(r_2)\frac{\tilde{W}(r_3)-\tilde{W}(r_1)}{\tilde{W}(r_2)-\tilde{W}(r_1)}\leqslant m(r_3) \tag{5.3}$$

holds true for all $r_3 \in (r_1, r_2)$ (respectively, $r_3 \in (r_2, r_1)$). Moreover, m is monotone near ζ.

Proof. We will only prove the case 2.1 of the theorem, namely, the concave three-spheres inequality (5.3) for $m(r)$ under the assumption $p > d$ and $u \to \infty$ as $x \to 0$ (see Remark 5.2). The proofs for the other cases are similar.

Since $\lim_{x\to 0} u(x) = \infty$, it follows that for every $0 < r_2 < R$ (to be determined later) there exists $R_1 := R_1(r_2) < r_2$ such that

$$m(r_1) > 2m(r_2) \quad \forall 0 < r_1 < R_1. \tag{5.4}$$

For $0 < r_2 < R$ and $0 < r_1 < R_1(r_2)$, define an auxiliary function

$$\begin{aligned} &f : \{x \mid r_1 < |x| < r_2\} \to \mathbb{R}_+, \\ f(x) := \tilde{f}(|x|) &:= m(r_1)\frac{\tilde{W}(r_2)-\tilde{W}(|x|)}{\tilde{W}(r_2)-\tilde{W}(r_1)}+m(r_2)\frac{\tilde{W}(|x|)-\tilde{W}(r_1)}{\tilde{W}(r_2)-\tilde{W}(r_1)} \\ &= \frac{m(r_1)\tilde{W}(r_2)-m(r_2)\tilde{W}(r_1)}{\tilde{W}(r_2)-\tilde{W}(r_1)}+\frac{m(r_2)-m(r_1)}{\tilde{W}(r_2)-\tilde{W}(r_1)}\tilde{W}(|x|) \\ &= C_1 + C_2\tilde{W}(|x|). \end{aligned} \tag{5.5}$$

Note that assumption 2.1 and $\tilde{W}(r_2) > \tilde{W}(r_1)$ (Lemma 2.1) imply that $C_2 < 0$. Therefore, f is a positive monotone decreasing function of $|x|$ satisfying $f(r_i) = m(r_i)$, $i = 1, 2$. In particular,

$$f(x) \leqslant m(r_1) \quad \forall r_1 \leqslant |x| \leqslant r_2. \tag{5.6}$$

Note that f is of the form of u_- of Lemma 3.1, therefore, for the case 2.1 f should be a positive subsolution near $\zeta = 0$. Indeed, using Lemma 2.1 we obtain

$$\begin{aligned} Q'_V(f) &= -\Delta_p f + V(x)\big(f(|x|)\big)^{p-1} \\ &\leqslant -\left(\frac{m(r_1)-m(r_2)}{\tilde{W}(r_2)-\tilde{W}(r_1)}\right)^{p-1} G(|x|) + G(|x|)f(|x|)^{p-1} \\ &\leqslant -\left(\frac{m(r_1)}{2\tilde{W}(R)}\right)^{p-1} G(|x|) + \big(m(r_1)\big)^{p-1}G(|x|). \end{aligned} \tag{5.7}$$

It follows (recall that $\tilde{W}(r) \to 0$ as $r \to 0$) that there exists $0 < R =: R_2$ such that for every $r_2 < R_2$ there exists $0 < R_1(r_2) < r_2$, such that for every $0 < r_1 < R_1(r_2) < r_2 < R_2$ the function f is a subsolution in $\{x \mid r_1 < |x| < r_2\}$.

Consequently, for such r_1 and r_2, the weak comparison principle implies that

$$m(r_1)\frac{\tilde{W}(r_2)-\tilde{W}(|x|)}{\tilde{W}(r_2)-\tilde{W}(r_1)}+m(r_2)\frac{\tilde{W}(|x|)-\tilde{W}(r_1)}{\tilde{W}(r_2)-\tilde{W}(r_1)} = f(|x|) \leqslant u(x) \quad \text{in } \{x \mid r_1 < |x| < r_2\}. \tag{5.8}$$

Finally, for $r_3 \in (r_1, r_2)$ we take in (5.8) the infimum over the spheres S_{r_3}, and we arrive at the desired three-spheres inequality (5.3).

It remains to prove the monotonicity of m as $r \to 0^+$. Indeed, using (5.3), it follows that for $r_2 < R_2$ and $0 < r_1 < R_1(r_2)$ we have

$$\min\bigl(m(r_1), m(r_2)\bigr) \leqslant m(r_3) \quad \forall r_3 \in (r_1, r_2). \tag{5.9}$$

Since by our assumption $\lim_{r\to 0^+} m(r) = \infty$, (5.9) clearly implies that m is a decreasing function of r near 0. □

Remarks 5.2. 1. According to Theorems 1.11 and 1.14, positive solutions of the equation $Q'(u) = 0$ in a punctured neighborhood of ζ never satisfy one of the conditions 2.1–2.4 of Theorem 5.1. Nevertheless, the validity of the concave three-spheres inequality (5.3) in these cases is an essential part of the proof of Theorem 1.14. Therefore, we chose to prove one of these cases in detail.

2. Theorem 5.1 can be strengthened by replacing the assumptions $\lim_{x\to\zeta} u(x) = 0, \infty$, by the weaker assumptions $\liminf_{x\to\zeta} u(x) = 0$ and $\limsup_{x\to\zeta} u(x) = \infty$, respectively. We shall not elaborate this point in the present paper.

6. Proof of Theorem 1.14

We first prove the following lemma.

Lemma 6.1. *Suppose that V satisfies condition* (C1) *near ζ. Let u be a positive solution of the equation $Q'(u) = 0$ in a punctured neighborhood of ζ. Then*

$$\lim_{x\to\zeta} u(x) \quad \textit{exists}; \tag{6.1}$$

the limit might be infinite.

Proof. By Lemma 3.3, the equation $Q'(w) = 0$ admits a positive solution v in a punctured neighborhood of ζ satisfying

$$\lim_{x\to\zeta} v(x) = 1. \tag{6.2}$$

Let u be a positive solution of the equation $Q'(w) = 0$ in such a punctured neighborhood. By Proposition 2.8,

$$\lim_{x\to\zeta} \frac{u(x)}{v(x)} \quad \text{exists}.$$

In view of (6.2), it follows that $\lim_{x\to\zeta} u(x)$ exists. □

Proof of Theorem 1.14. The first part of the theorem follows from Lemma 6.1.

Consider now the classical cases with $p \neq d$. It follows from Theorem 4.1 and from the existence of a positive solution v satisfying $v(x) \sim 1$ near ζ (see Lemma 3.3) that $\lim_{x\to\zeta} u(x) \neq 0$.

Let us turn to the nonclassical cases. Suppose that $\zeta = 0$ and $p > d$ (respectively, $\zeta = \infty$ and $p < d$). By Lemma 6.1 $\lim_{x\to\zeta} u(x)$ exists. We need to prove that the limit in (6.1) is finite. Assume to the contrary that

$$\lim_{x\to\zeta} u(x) = \infty.$$

So, $\lim_{r\to\zeta} m(r) = \infty$, and we are in the situation to use the three-spheres inequality (5.3). Hence, for appropriate fixed $r_3 < r_2$ (respectively, $r_2 < r_3$) and any small enough r_1 satisfying $0 < r_1 < r_3$ (respectively, large enough r_1 satisfying $r_3 < r_1$) we have

$$m(r_1)\frac{\tilde{W}(r_2)-\tilde{W}(r_3)}{\tilde{W}(r_2)-\tilde{W}(r_1)}+m(r_2)\frac{\tilde{W}(r_3)-\tilde{W}(r_1)}{\tilde{W}(r_2)-\tilde{W}(r_1)}\leqslant m(r_3) \tag{6.3}$$

for all $r_3 \in (r_1, r_2)$ (respectively, $r_3 \in (r_2, r_1)$).

After some simple algebraic manipulations we arrive at

$$m(r_1) \leqslant \frac{\tilde{W}(r_2)}{\tilde{W}(r_2)-\tilde{W}(r_3)} m(r_3) \quad \forall 0 < r_1 < R_1 \quad (\text{respectively}, \forall r_1 > R_2).$$

Thus, $m(r_1)$ is bounded near ζ, a contradiction. Therefore, $\lim_{x\to\zeta} u(x)$ exists and is finite.

Assume now that V satisfies also condition (C2). By Lemma 3.3, there exist two positive solutions $u^{(\zeta)}_{\mathrm{small}}$ and $u^{(\zeta)}_{\mathrm{large}}$ of the equation $Q'(u) = 0$ defined in a punctured neighborhood of ζ satisfying

$$u^{(\zeta)}_{\mathrm{small}}(x) \underset{x\to\zeta}{\sim} \min\{1, v_{\alpha^*}(x)\}$$

and

$$u^{(\zeta)}_{\mathrm{large}}(x) \underset{x\to\zeta}{\sim} \max\{1, v_{\alpha^*}(x)\}.$$

These two solutions clearly satisfy the requirements of the last claim of the theorem. □

Corollary 6.2. *Let $p > d$ and $\zeta = 0$ (respectively, $p < d$ and $\zeta = \infty$). Assume that V satisfies condition (C1) near ζ. Then any solution u of the equation $Q'(u) = 0$ in $B_R \setminus \{0\}$ (respectively, B^*_R) which is unbounded near ζ changes its sign in any punctured neighborhood of ζ.*

Moreover, if in addition, $V = 0$, then

$$\lim_{r\to\zeta} M_u(r) = -\lim_{r\to\zeta} m_u(r) = \infty.$$

Similarly, we have

Corollary 6.3. *Let $p < d$ and $\zeta = 0$ (respectively, $p > d$ and $\zeta = \infty$). Assume that V satisfies condition (C1) near ζ. Then any solution u of the equation $Q'(u) = 0$ in $B_R \setminus \{0\}$ (respectively, B^*_R) satisfying*

$$\liminf_{z\to\zeta} |u(x)| = 0$$

changes its sign in any punctured neighborhood of ζ.

7. The asymptotics of positive solutions

In this section we discuss the asymptotic behavior of positive solutions near a singular point ζ. First, we find the asymptotic behavior near an interior singularity for the classical case $p \leqslant d$ and $\zeta = 0$.

Proof of Theorem 1.11. It follows from Remarks 1.8 and Lemma 3.3 that there exists a solution w, such that $w \sim v_{\alpha^*}$ near the origin. On the other hand, by Serrin's removable singularity result (Theorem 1.10), the solution u has either a removable singularity or $u \asymp v_{\alpha^*}$. In the latter case, the ratio limit theorem (Proposition 2.8) implies that $u \sim w \sim v_{\alpha^*}$ near the origin. □

We turn now to the proof of the asymptotic behavior of positive solutions in the nonclassical cases under the integrability assumption.

Proof of Theorem 1.17. Let u be a positive solution of $Q'(u)=0$ near ζ. By Theorem 1.14 we have $\lim_{x\to\zeta} u(x)<\infty$. So we may assume that $\lim_{x\to\zeta} u(x)=0$ and consequently, Theorem 5.1 applies.

In particular, by fixing r_2 and letting $r_1\to 0$ (respectively, $r_1\to\infty$) in (5.2) we obtain

$$M(r_3)\leqslant \frac{M(r_2)}{\tilde{W}(r_2)}\tilde{W}(r_3),\quad 0<r_3<r_2\quad (\text{respectively}, r_2<r_3<\infty). \tag{7.1}$$

Recall that for V integrable near ζ we have $\tilde{W}\sim v_{\alpha^*}$. Consequently, (7.1) implies $u\leqslant Cv_{\alpha^*}$ near ζ.

On the other hand, by Lemma 3.3 there exists a positive solution w near ζ such that $w\sim v_{\alpha^*}$. Therefore, Theorem 4.1 implies that $u\sim w\sim v_{\alpha^*}$ near ζ. $\square$

8. Applications and conjectures

In this section we present some applications of Theorem 1.14. First we recall the notion of positive solutions of minimal growth [2,21].

Definitions 8.1. 1. Let $K\Subset\Omega$, and let u be a positive solution of the equation $Q'(w)=0$ in $\Omega\setminus K$. We say that u is *a positive solution of minimal growth in a neighborhood of infinity in* Ω if for any $K\Subset K'\Subset\Omega$ with smooth boundary and any positive supersolution $v\in C((\Omega\setminus K')\cup\partial K')$ of the equation $Q'(w)=0$ in $\Omega\setminus K'$ satisfying $u\leqslant v$ on $\partial K'$, we have $u\leqslant v$ in $\Omega\setminus K'$.

2. Let u be a positive solution of the equation $Q'(w)=0$ in a punctured neighborhood of ζ. We say that u is *a positive solution of minimal growth at* ζ if for any smaller punctured neighborhood K of ζ such that $\partial K\setminus\{\zeta\}$ is smooth, and any positive supersolution $v\in C(\overline{K}\setminus\{\zeta\})$ of the equation $Q'(w)=0$ in $K\setminus\{\zeta\}$ satisfying $u\leqslant v$ on $\partial K\setminus\{\zeta\}$, we have $u\leqslant v$ in $K\setminus\{\zeta\}$.

Assume that $V\in L^\infty_{\mathrm{loc}}(\Omega)$, and that the equation $Q'(u)=0$ admits a positive solution in Ω. Then for any $\zeta\in\Omega$ the equation $Q'(u)=0$ admits a positive solution u_ζ of the equation $Q'(u)=0$ in $\Omega\setminus\{\zeta\}$ of minimal growth in a neighborhood of infinity in Ω [21]. This solution is known to be unique (up to a multiplicative constant); see [21] for the case $1<p\leqslant d$, and [6] for $p>d$. The functional Q is said to be *critical* in Ω if u_ζ is in fact a positive solution of the equation $Q'(u)=0$ in Ω, and *subcritical* otherwise.

We note that for $\Omega=\mathbb{R}^d$, the solutions of minimal growth in the neighborhood of infinity of Ω are solutions with minimal growth at $\zeta=\infty$.

For solutions of minimal growth at ζ we have the following asymptotic behavior.

Theorem 8.2. *Assume that V satisfies conditions* (C1) *and* (C2) *near ζ and $p\neq d$. Let u be solution of the equation $Q'(w)=0$ of minimal growth at ζ. Then*

$$u(x)\underset{x\to\zeta}{\sim}\min\{1,|x|^{\alpha^*}\}.$$

Proof. In the classical cases the theorem is just a reformulation of Theorem 1.14. In the nonclassical cases the existence of a solution with the required asymptotic behavior near ζ is guaranteed by Lemma 3.3, and therefore the asymptotic behavior of u near ζ follows from Theorem 4.1. $\square$

In a recent paper [10], Jaye and Verbitsky study the behavior of the unique positive solution u of the equation $Q'(u)=0$ in $\mathbb{R}^d$ of minimal growth in a neighborhood of infinity in $\mathbb{R}^d$ with an isolated singularity at the origin, where $p<d$. It is proved that if V is a *nonpositive* potential which belongs to a certain class of measures, then $u(x)\asymp|x|^{\alpha^*}$ in $\mathbb{R}^d$. Our results here allow for the following generalization.

Corollary 8.3. *Let $\Omega=\mathbb{R}^d$ and $p<d$. Assume that V belongs to $L^\alpha(B_R)$ with $\alpha>d/p$, and V satisfies conditions* (C1) *and* (C2) *at infinity. Suppose that Q is subcritical in $\mathbb{R}^d$. Then the unique positive solution u of the equation $Q'(u)=0$ in $\Omega\setminus\{0\}$ of minimal growth in a neighborhood of infinity in Ω satisfies $u\asymp|x|^{\alpha^*}$.*

Proof. According to Theorem 1.10, $u \asymp |x|^{\alpha^*}$ near zero, and according to Theorem 8.2, $u \sim |x|^{\alpha^*}$ near infinity. Hence the claim follows by a compactness argument. $\square$

The next result answers a question posed in [21, Section 5], by showing that for a general domain Ω, and $p > d$, positive solutions of the equation $Q'(u) = 0$ in $\Omega \setminus \{\zeta\}$ of minimal growth in a neighborhood of infinity in Ω are comparable to 1 near the isolated singular point ζ.

Theorem 8.4. *Let Ω be a domain in $\mathbb{R}^d$. Fix $\zeta \in \Omega$; without loss of generality assume $\zeta = 0$. Suppose that V satisfies condition* (C1) *with respect to $\zeta = 0$. Assume further that the equation $Q'(u) = 0$ admits a positive solution in Ω, and let u_0 be a positive solution of $Q'(u) = 0$ in $\Omega \setminus \{0\}$ of minimal growth in a neighborhood of infinity in Ω. If $p > d$, then there exists a positive constant C such that*

$$\lim_{x \to 0} u_0(x) = C > 0. \tag{8.1}$$

If $p < d$, then (8.1) *holds if and only if Q is critical in Ω.*

Proof. Suppose that $p > d$. By Theorem 1.14, u_0 is in fact continuous at 0. It remains to prove that $u_0(0) > 0$. Assume to the contrary that $u_0(0) = 0$, and let v be a positive solution of $Q'(u) = 0$ in Ω satisfying $v(0) = 1$. Then for any $\varepsilon > 0$ there exists $r = r(\varepsilon) > 0$ such that

$$u_0(x) \leqslant \varepsilon v(x) \quad \forall x \in S_r,$$

and by Definition 8.1,

$$u_0(x) \leqslant \varepsilon v(x) \quad \forall x \in \Omega \setminus B_r.$$

Clearly, $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. Consequently, $u_0 = 0$, a contradiction (cf. [21, Section 5]).

The last statement of the theorem follows from Corollary 2.7. $\square$

The following example illustrates Theorem 8.4.

Example 8.5. Let $\Omega = B_R$, $Q'(u) = -\Delta_p(u)$, $p > d$, and $\zeta = 0$. The corresponding positive solution u_0 of minimal growth in a neighborhood of infinity in Ω is given by

$$u_0(x) := R^{\alpha^*} - |x|^{\alpha^*}, \quad x \in B_R \setminus \{0\}.$$

Note that $\lim_{x \to 0} u_0(x) = R^{\alpha^*} > 0$.

Next, we present a positive Liouville theorem in $\mathbb{R}^d$, where $p < d$. This result is well known for $p = 2$ under weaker assumptions (see for example [18]).

Theorem 8.6. *Let $p < d$, and suppose that $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ satisfies conditions* (C1) *and* (C2) *near infinity. Assume further that*

$$Q(u) := \int_{\mathbb{R}^d} \left(|\nabla u|^p + V|u|^p\right) \mathrm{d}x \geqslant 0 \quad \forall u \in C_0^\infty\left(\mathbb{R}^d\right). \tag{8.2}$$

Then the equation $Q'(u) = 0$ admits a unique (*up to a multiplicative constant*) *positive solution u in $\mathbb{R}^d$. Moreover, there exists $C \geqslant 0$ such that*

$$\lim_{x \to \infty} u(x) = C,$$

and $C = 0$ if Q is critical in $\mathbb{R}^d$. Furthermore, if V is integrable near infinity, then $C = 0$ if and only if Q is critical in $\mathbb{R}^d$.

Proof. Assumption (8.2) implies that the equation $Q'(u) = 0$ admits an entire positive solution u in $\mathbb{R}^d$ [20]. The *uniqueness* follows from [6, Theorem 2.6], where a positive Liouville theorem is proved under weaker assumptions. The existence of $\lim_{x\to\infty} u(x) = C < \infty$ follows from Theorem 1.14.

By Theorem 8.2, a positive solution u of the equation $Q'(u) = 0$ of minimal growth in a neighborhood of infinity in $\mathbb{R}^d$ satisfies $u \sim v_{\alpha^*}$, and in particular u tends to zero as $x \to \infty$. Thus, if Q is critical in $\mathbb{R}^d$, then $C = 0$.

Moreover, if V is integrable, then by Theorem 1.17 we have $\lim_{x\to\infty} u(x) = 0$ if and only if

$$u(x) \underset{x\to\infty}{\sim} |x|^{\alpha^*}, \tag{8.3}$$

which takes place if and only if u is a positive solution of minimal growth in a neighborhood of infinity in $\mathbb{R}^d$. □

Remark 8.7. Under the assumptions of Theorem 8.6, the uniqueness of a positive entire solution (up to a multiplicative constant) can also be obtained using the existence of the limit $\lim_{x\to\infty} u(x) = C$.

We conclude our paper with a conjecture. Let $\zeta \in \{0, \infty\}$ be an isolated singular point of the equation $Q'(w) = 0$ in a domain Ω. Denote by $\mathcal{G}_\zeta$ the germ of all positive solutions u of the equation $Q'(w) = 0$ in some punctured neighborhood $\Omega' \subset \Omega$ of ζ (the neighborhood Ω' might depend on u). Let $u, v \in \mathcal{G}_\zeta$. We use the following notations.

- We denote $u \underset{x\to\zeta}{\sim} v$ if $\lim_{\substack{x\to\zeta\\ x\in\Omega}} \frac{u(x)}{v(x)} = C$ for some positive constant C.
- By $u \underset{x\to\zeta}{\prec} v$ we mean that $\lim_{\substack{x\to\zeta\\ x\in\Omega}} \frac{u(x)}{v(x)} = 0$.
- By $u \underset{x\to\zeta}{\precsim} v$ we mean that either $u \underset{x\to\zeta}{\sim} v$ or $u \underset{x\to\zeta}{\prec} v$.
- We denote $u \underset{x\to\zeta}{\succ} u$ if $v \underset{x\to\zeta}{\prec} u$. Similarly, $u \underset{x\to\zeta}{\succsim} v$ if $v \underset{x\to\zeta}{\precsim} u$.

Clearly, $u \underset{x\to\zeta}{\sim} v$ defines an equivalence relation and equivalence classes on $\mathcal{G}_\zeta$.

Definition 8.8. We say that ζ is a *regular point of the equation $Q'(w) = 0$ in Ω* if for any two positive solutions $u, v \in \mathcal{G}_\zeta$ we have either $u \underset{x\to\zeta}{\precsim} v$ or $u \underset{x\to\zeta}{\succsim} v$.

Conjecture 8.9. *Suppose that* (1.1) *admits a* (*global*) *positive solution and V has a Fuchsian type singularity at the isolated singular point ζ. Then*:

(i) *ζ is a regular point of Eq.* (1.1).
(ii) *Eq.* (1.1) *admits a unique* (*global*) *positive solution of minimal growth in a neighborhood of infinity in $\Omega \setminus \{\zeta\}$.*
(iii) *$\mathcal{G}_\zeta$ admits exactly two equivalence classes with respect to $\sim$.*

We note that the results discussed in the present paper and in [6,19] give partial answers to Conjecture 8.9. In particular, in [6] the authors proved that (i) implies (ii), and proved the regularity for potentials with a weak Fuchsian isolated singularity and for spherically symmetric potentials with a Fuchsian isolated singularity.

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