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A New Proof of Local $C^{1,\alpha}$ Regularity for Solutions of Certain Degenerate Elliptic P.D.E.

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We prove $C^{1,\alpha}_{\text{loc}}$ estimates for solutions $u \in W^{1,p+2}$ of the degenerate elliptic p.d.e.

$$\operatorname{div}(|Du|^p Du) = 0 \qquad (p > 0).$$

1. INTRODUCTION

Denote by Ω a bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$. For $p > 0$ the degenerate elliptic equation

$$\operatorname{div}(|Du|^p Du) = 0 \qquad \text{in } \Omega \tag{1.1}$$

is the Euler–Lagrange equation for the problem of minimizing the functional

$$\Phi(v) \equiv \int_\Omega |Dv|^{p+2}\, dx$$

over all $v \in W^{1,p+2}(\Omega)$ satisfying the boundary condition

$$v - h \in W_0^{1,p+2}(\Omega)$$

for some given function $h: \mathbb{R}^n \to \mathbb{R}$. It is well known that should h be sufficiently regular, then Φ in fact attains its minimum at a unique function u, which in turn is a weak solution of (1.1):

$$\int_\Omega |Du|^p Du \cdot D\eta\, dx = 0 \qquad \text{for all} \quad \eta \in W_0^{1,p+2}(\Omega). \tag{1.2}$$

See, for example, Lewis [4] for proofs and applications of these ideas in studying p-capacitary extremals (this is the case $h \equiv 1$ in Ω^c).

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There is interest in discovering what additional regularity properties u may possess; and for this the best results to date are due to Ural'ceva [7] and Uhlenbeck [6], who independently proved $u \in C^{1;\alpha}_{\mathrm{loc}}$ for some (small) Hölder exponent $\alpha > 0$. Ural'ceva's method exploits some ideas originating with DeGiorgi (cf. [3]), whereas Uhlenbeck makes use of a perturbation technique due to Almgren [1]. And both authors actually obtain regularity for a wider class of nonliner problems.

This paper describes a new, simpler, and relatively short proof of $C^{1;\alpha}_{\mathrm{loc}}$ estimates for a weak solution of (1.1), and very briefly discusses an extension of the ideas to cover more general nonlinear problems as in [6, 7]. The method here is a combination of the DeGiorgi and the Moser [5] techniques for investigating uniformly elliptic, divergence structure p.d.e.; and the idea is to "truncate" the various test functions in those regions where $Du|$ is small (i.e., where (1.1) becomes degenerate). This done, we can estimate and then cancel out the effects of term $|Du|^p$, obtaining thereby "uniform" estimates even in regions where $|Du|$ is small.

Our main result is this:

THEOREM 1. *Suppose that $u \in W^{1,p+2}(\Omega)$ is a weak solution of* (1.1). *There then exist a constant $\alpha = \alpha(p, n) > 0$ and, for each $\Omega' \Subset \Omega$, a constant $C(\Omega') = C(\Omega', p, n, \|u\|_{W^{1,p+2}})$ such that*

$$\max_{\Omega'} |Du| \leqslant C(\Omega') \tag{1.3}$$

and

$$[Du]_{C^{\alpha}(\Omega')} \leqslant C(\Omega'). \tag{1.4}$$

In Section 2 we first present a formal derivation of an estimate on the Hölder continuity of Du near a point of degeneracy (where $Du = 0$), and then indicate how to modify the estimates to cover a related, approximate problem. Section 3 comprises an a priori Hölder estimate of Du in the interior of some ball, this for the solutions of both (1.1) and a related problem. In Section 4 we prove Theorem 1 by first constructing a sequence of approximate problems and obtaining uniform interior sup-norm estimates on the gradients of the solutions. These bounds and the interior gradient Hölder estimate from Section 3 then allow passage to limits in the approximations to obtain (1.3) and (1.4).

Finally it may be worth noting here that the techniques to follow in fact extend without much difficulty to certain related problems with a nonlinear term of the form $\phi(|Du|)$ in place of $|Du|^p$. We do not however wish to lengthen this paper by noting the appropriate hypothesis on ϕ and providing full proofs, and instead refer the interested reader to [6] or [7], where problems involving these more general nonlinearities are described.

Notation.

$$[f]_{C^\alpha(K)} \equiv \max_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \qquad (0<\alpha<1),$$

$$\|u\|_{W^{1,q}(\Omega)} \equiv \|u\|_{L^q(\Omega)} + \|Du\|_{L^q(\Omega)} \qquad (1\leqslant q\leqslant\infty),$$

$$⨍_K f(x)\,dx = \text{average of } f \text{ over } K = \frac{1}{\text{meas}\, K}\int_K f(x)\,dx.$$

The summation convention is used throughout; the letter C denotes various constants depending only on known quantities.

2. An a Priori Estimate on the Oscillation of $|Du|$ Near a Point of Degeneracy

Let us suppose for the time being u is a *smooth* solution of

$$\operatorname{div}(|Du|^p\, Du) = 0 \tag{2.1}$$

in some ball $B(R_0)$, that

$$Du(0) = 0, \tag{2.2}$$

and

$$\max_{B(R_0)} |Du| \leqslant K. \tag{2.3}$$

Define

$$M(R) \equiv \max_{B(R)} |Du| \qquad (0<R<R_0). \tag{2.4}$$

The principal assertion of this section is that (2.2) forces $M(R)$ to grow no faster than some fractional power of R:

PROPOSITION 2.1. *There exist constants $C_1 = C_1(p,n)$ and $\beta = \beta(p,n) > 0$ such that*

$$M(R) \leqslant C_1 K\left(\frac{R}{R_0}\right)^\beta \qquad (0<R<R_0). \tag{2.5}$$

The proof results from the sequence of lemmas below and a few preliminary remarks. First of all fix some $0<R<R_0$ and define

$$M_k^\pm(R) \equiv \max_{B(R)} \pm u_{x_k} \qquad (k=1,2,\dots,n). \tag{2.6}$$

Since $|Du| = (u_{x_1}^2 + \cdots + u_{x_n}^2)^{1/2}$, there must exist some index i such that either

$$M_i^+(R) \geqslant \frac{1}{\sqrt{n}} M(R)$$

or

$$M_i^-(R) \geqslant \frac{1}{\sqrt{n}} M(R).$$

Let us therefore assume, upon relabelling the coordinate axes if necessary,

$$M_1^+(R) \geqslant \frac{1}{\sqrt{n}} M(R) > 0. \tag{2.7}$$

The first lemma uses a modification of the method of DeGiorgi to prove that if u_{x_1} is on the average very close to its positive maximum $M_+^1(R)$ on $B(R)$, then u_{x_1} is strictly positive on $B(R/2)$. (Here we temporarily drop assumption (2.2).)

LEMMA 2.1. *There exists a constant $\varepsilon_0 = \varepsilon_0(p, n) > 0$ such that*

$$⨍_{B(R)} (M_1^+(R) - u_{x_1})^{+2}\, dx \leqslant \varepsilon_0 M_+^1(R)^2 \tag{2.8}$$

implies

$$\min_{B(R/2)} u_{x_1} \geqslant \frac{M_1^+(R)}{2}.$$

Proof. Let us write $M_1 \equiv M_1^+(R)$, $v \equiv M_1 - u_{x_1}$. We differentiate (2.1) with respect to x_1 to discover v solves the p.d.e.

$$-(a_{ij}|Du|^p v_{x_i})_{x_j} = 0 \qquad \text{in } B(R_0), \tag{2.9}$$

where

$$\begin{aligned} a_{ij} &\equiv \delta_{ij} + p\frac{u_{x_i}u_{x_j}}{|Du|^2} && \text{if} \quad Du \neq 0 \\ &\equiv \delta_{ij} && \text{if} \quad Du = 0. \end{aligned} \tag{2.10}$$

Now multiply (2.9) by $\zeta^2(v-k)^+$, where ζ is a smooth cutoff function vanishing outside $B(R)$ and

$$0 \leqslant k \leqslant \frac{M_1}{2} \tag{2.11}$$

is a constant to be selected below. After routine calculations we obtain the inequality

$$\int_{B(R)\cap\{k<v\}} |Dv|^2 |Du|^p \zeta^2 \, dx \leqslant C \int_{B(R)} (v-k)^{+2} |Du|^p |D\zeta|^2 \, dx$$

$$\leqslant CM(R)^p \int_{B(R)} (v-k)^{+2} |D\zeta|^2 \, dx. \quad (2.12)$$

The term on the left is greater than or equal to

$$\int_{B(R)\cap\{k<v<k+M_1/4\}} |Dv|^2 |Du|^p \zeta^2 \, dx.$$

Now if $v = M_1 - u_{x_1} < k + M_1/4$, we have

$$u_{x_1} \geqslant \frac{3}{4} M_1 - k \geqslant \frac{1}{4} M_1 \qquad \text{by (2.11)}$$

$$\geqslant \frac{1}{4\sqrt{n}} M(R), \qquad \text{by (2.7);}$$

hence

$$|Du|^p \geqslant CM(R)^p \qquad (C > 0)$$

on $\{k < v < k + M_1/4\}$. We use this estimate in (2.12) to obtain

$$M(R)^p \int_{B(R)\cap\{k<v<k+M_1/4\}} |Dv|^2 \zeta^2 \, dx \leqslant CM(R)^p \int_{B(R)} (v-k)^{+2} |D\zeta|^2 \, dx,$$

and therefore—after cancellation—

$$\int_{B(R)} |D\phi_k(v)|^2 \zeta^2 \, dx \leqslant C \int_{B(R)} (v-k)^{+2} |D\zeta|^2 \, dx,$$

for

$$\begin{aligned} \phi_k(x) &\equiv 0, & x &< k \\ &\equiv x - k, & k &\leqslant x \leqslant k + M_1/4 \\ &\equiv M_1/4, & k &+ M_1/4 < x. \end{aligned}$$

Sobolev's inequality therefore implies

$$\left(\int_{B(R)} [\zeta\phi_k(v)]^{2n/(n-2)} \, dx\right)^{(n-2)/n} \leqslant C \max_{B(R)} |D\zeta|^2 \int_{B(R)} (v-k)^{+2} \, dx. \quad (2.13)$$

Next we define for $m = 0, 1, 2, \ldots$

$$k_m \equiv \frac{M_1}{2}\left(1 - \frac{1}{2^m}\right) < \frac{M_1}{2} \qquad \text{(cf. (2.11))},$$

$$R_m \equiv \frac{R}{2}\left(1 + \frac{1}{2^m}\right),$$

and choose smooth cutoff functions ζ_m such that

$$0 \leqslant \zeta_m \leqslant 1, \qquad \zeta_m \equiv 1 \qquad \text{on } B(R_{m+1}),$$

$$\zeta_m \equiv 0 \qquad \text{outside } B(R_m),\ |D\zeta_m| \leqslant \frac{C2^m}{R}.$$

Set $R = R_m$, $\zeta = \zeta_m$, $k = k_m$ in (2.13):

$$\left(\int_{B(R_{m+1})} \phi_{k_m}(v)^{2n/(n-2)}\, dx\right)^{(n-2)/n} \leqslant \frac{C4^m}{R^2} \int_{B(R_m)} (v - k_m)^{+2}\, dx. \quad (2.14)$$

Define

$$J_m \equiv \int_{B(R_m)} \phi_{k_m}(v)^2\, dx \qquad (m = 0, 1, \ldots).$$

Next notice

$$J_m \leqslant \int_{B(R_m)} (v - k_m)^{+2}\, dx \leqslant CJ_m, \quad (2.15)$$

since $(v - k_m)^+ \neq \phi_{k_m}(v)$ only if $v > M_1/4 + k_m$, and on the set $\{v > M_1/4 + k_m\}$

$$(v - k_m)^{+2} \leqslant CM(R)^2 \leqslant CM_1^2 \leqslant C\phi_{k_m}^2.$$

Furthermore

$$\begin{aligned} &\text{meas}\{x \in B(R_{m+1}) \mid \phi_{k_{m+1}}(v) > 0\} \\ &\quad = \text{meas}\{x \in B(R_{m+1}) \mid v > k_{m+1}\} \\ &\quad \leqslant \frac{1}{(k_{m+1} - k_m)^2} \int_{B(R_m)} (v - k_m)^{+2}\, dx \leqslant \frac{C4^m}{M_1^2} J_m \qquad \text{by (2.15).} \end{aligned}$$

Consequently

$$\begin{aligned} J_{m+1} &= \int_{B(R_{m+1})} \phi_{k_{m+1}}(v)^2\,dx \\ &\leqslant \left(\int_{B(R_{m+1})} \phi_{k_{m+1}}(v)^{2n/(n-2)}\,dx\right)^{(n-2)/n} \\ &\quad \times (\operatorname{meas}\{x \in B(R_{m+1}) \mid \phi_{k_{m+1}}(v) > 0\})^{2/n} \\ &\leqslant \frac{C_2 C_3^m}{R^2 M_1^{4/n}} J_m^{1+2/n} \qquad (m = 0, 1, 2,\dots). \end{aligned}$$

According to [3, Lemma II.4.7, p. 55] therefore

$$J_m \to 0,$$

provided

$$J_0 \leqslant \int_{B(R)} v^{+2}\,dx = \int_{B(R)} (M_1 - u_{x_1})^{+2}\,dx \leqslant \varepsilon_0 M_1^2 \operatorname{meas} B(R).$$

In this case

$$\max_{B(R/2)} v \leqslant \frac{M_1}{2},$$

and the lemma follows at once. ▮

LEMMA 2.2. *Under assumptions* (2.2) *and* (2.7) *there exist constants* $0 < \lambda, \mu < 1$ *such that*

$$\operatorname{meas}\{x \in B(R) \mid u_{x_1}(x) \leqslant \lambda M_1^+(R)\} \geqslant \mu \operatorname{meas} B(R); \tag{2.16}$$

λ and μ depend only on p and n.

Proof. Again we set $M_1 = M_1^+(R)$.

Suppose (2.16) fails (for λ, μ as selected below). Then

$$\begin{aligned} \fint_{B(R)} (M_1 - u_{x_1})^{+2}\,dx &= \fint_{B(R)\cap\{u_{x_1} < \lambda M_1\}} (M_1 - u_{x_1})^{+2}\,dx \\ &\quad + \fint_{B(R)\cap\{\lambda M_1 \leqslant u_{x_1} \leqslant M_1\}} (M_1 - u_{x_1})^{+2}\,dx \\ &\leqslant CM_1^2 \frac{\operatorname{meas}\{u_{x_1} < \lambda M_1\}}{R^n} \\ &\quad + C(1-\lambda)^2 M_1^2 \\ &\leqslant C((1-\lambda)^2 + \mu) M_1^2 \\ &\leqslant \varepsilon_0 M_1^2, \end{aligned}$$

if $0 < \mu$ is small enough and $\lambda < 1$ is close enough to 1. The hypotheses of Lemma 2.1 are therefore verified; whence

$$\min_{B(R/2)} u_{x_1} \geqslant \frac{M_1^+(R)}{2} > 0,$$

a contradiction to (2.2). ∎

In view of this lemma, u_{x_1} must be strictly less than its maximum $M_1^+(R)$ on an "appreciable" subset of $B(R)$; and the next lemma asserts that therefore $M_1^+(R/2)$ is strictly less than $M_1^+(R)$.

LEMMA 2.3. *There exists a positive constant $\gamma = \gamma(p, n) < 1$ such that*

$$M_1^+\left(\frac{R}{2}\right) \leqslant \gamma M_1^+(R).$$

Proof. We modify a method due to Moser [5].

Once more let us write M_1 for $M_1^+(R)$. Define for $\delta > 0$

$$\phi(x) = \phi_\delta(x) \equiv \left(-\log\left(\frac{M_1 - x + \delta}{M_1(1-\lambda)}\right)\right)^+ \qquad \text{for} \quad x \leqslant M_1;$$

it is easy to check

$$\begin{aligned} &\phi \text{ is nondecreasing and convex,} \\ &(\phi')^2 = \phi'', \qquad x \neq M_1\lambda + \delta, \\ &\phi = 0, \qquad x < M_1\lambda + \delta. \end{aligned} \tag{2.17}$$

Now set

$$w \equiv \phi(u_{x_1}). \tag{2.18}$$

Then Lemma 2.2 and (2.17) imply

$$\operatorname{meas}\{x \in B(R) \mid w = 0\} \geqslant \mu \operatorname{meas} B(R),$$

and so

$$\operatorname{meas}\{x \in B(\theta R) \mid w = 0\} \geqslant \frac{\mu}{2} \operatorname{meas} B(R)$$

for some $\theta = \theta(\mu, n)$, $\frac{3}{4} < \theta < 1$. Accordingly

$$⨍_{B(\theta R)} w^2\, dx \leqslant CR^2 ⨍_{B(\theta R)} |Dw|^2\, dx, \qquad C = C(\mu, \theta, n) \tag{2.19}$$

(cf. [5, Lemma 2]).

Furthermore since ϕ satisfies (2.17) and $v = u_{x_1}$ solves (2.9), w is a nonnegative (weak) subsolution of the same equation:

$$-(a_{ij}\,|Du|^p\, w_{x_i})_{x_j} = -a_{ij}\,|Du|^p\, w_{x_i} w_{x_j} \leqslant 0. \tag{2.20}$$

In addition *on the set where* $w > 0$, we have $u_{x_1} \geqslant M_1\lambda + \delta \geqslant M_1\lambda$ and so

$$M(R)^p \leqslant CM_1^p \leqslant C\,|Du|^p \leqslant CM(R)^p \tag{2.21}$$

according to (2.7), the constants C depending only on n and p. As a consequence we can apply the Moser iteration method [5] to (2.20), invoke (2.21) to estimate the $|Du|^p$ terms in the integrals, cancel the resulting expressions $M(R)^p$, and arrive therefore at the estimate

$$\max_{B(R/2)} w^2 \leqslant C \fint_{B(\theta R)} w^2\, dx, \qquad C = C(p, n, \theta). \tag{2.22}$$

Finally choose a smooth cutoff function

$$\zeta \equiv 1 \qquad \text{on } B(\theta R), \qquad \zeta \equiv 0 \qquad \text{near } \partial B(R),$$

$$0 \leqslant \zeta \leqslant 1, \qquad |D\zeta| \leqslant \frac{C}{(1-\theta)R}.$$

Multiply (2.20) by ζ^2, and perform some routine calculations—again using (2.21) to estimate and then cancel the terms involving $|Du|^p$—to obtain

$$\fint_{B(\theta R)} |Dw|^2\, dx \leqslant \frac{C}{R^2}, \qquad C = C(p, n, \theta). \tag{2.23}$$

Now combine estimates (2.22), (2.19), (2.23):

$$\max_{B(R/2)} w \leqslant C_4, \qquad C_4 = C(p, n, \theta, \mu). \tag{2.24}$$

Therefore if $x \in B(R/2)$,

$$\begin{aligned} u_{x_1}(x) &\leqslant M_1(1 - (1-\lambda)\, e^{-w(x)}) + \delta \\ &\leqslant \gamma M_1 + \delta \end{aligned}$$

for $\gamma \equiv 1 - (1-\lambda)\, e^{-C_4} < 1$, according to (2.24). Now send $\delta \to 0$ to conclude

$$M_1^+\left(\frac{R}{2}\right) \leqslant \gamma M_1 = \gamma M_1^+(R). \quad \blacksquare$$

The final assertion we require is [2, Lemma 12.5, p. 273]:

LEMMA 2.4. *Let* $\omega_1,\ldots,\omega_M$ *and* $\bar{\omega}_1,\ldots,\bar{\omega}_N$ *be nonnegative nondecreasing functions on an interval* $(0, R_0)$. *Suppose there exist constants* $\delta_0 > 0$, $0 < \sigma < 1$, $0 < \eta < 1$, *such that for each* $0 < R < R_0$,

(a) $\delta_0 \max_{1 \leqslant i \leqslant M} \omega_i(R) \leqslant \bar{\omega}_i(R)$ *for some* $i \in \{1,\ldots,N\}$ *and*

(b) $\bar{\omega}_i(\eta R) \leqslant \sigma \bar{\omega}_i(R)$.

Then there exist constants $C = C(N, M, \delta_0, \sigma, \eta)$ *and* $\beta = \beta(N, M, \delta_0, \sigma, \eta) > 0$ *such that for each* $i = 1, 2,\ldots, M$

$$\omega_i(R) \leqslant C \left(\frac{R}{R_0}\right)^\beta \max_{1 \leqslant i \leqslant N} \bar{\omega}_i(R_0) \qquad (0 < R < R_0).$$

Proof of Proposition 2.1. We apply Lemma 2.4 with

$$\omega_1(R) = M(R), \qquad \bar{\omega}_i(R) = M_i^+(R) \qquad (i = 1,\ldots, n),$$

$$\bar{\omega}_i(R) = M_i^-(R) \qquad (i = n+1,\ldots, 2n),\ M = 1,\ N = 2n,\ \eta = \tfrac{1}{2},$$

$$\delta_0 = \frac{1}{\sqrt{n}}, \qquad \sigma = \gamma \qquad \text{(from Lemma 2.3).} \quad \blacksquare$$

Since we do not know a priori (or a posteriori) that the (weak) solution of (1.1) is smooth, it will be necessary later to study a sequence of approximate problems of the form

$$\operatorname{div}(|Du|^p Du) + \varepsilon \Delta u = 0 \qquad (\varepsilon > 0) \tag{2.25$_\varepsilon$}$$

in some ball $B(R_0)$. Let us therefore suppose now u is a smooth solution of $(2.25)_\varepsilon$, and that (2.2), (2.3) hold. Then the Hölder estimate near 0 is still valid:

PROPOSITION 2.2. *There exist constants* $C_1 = C_1(p, n)$ *and* $\beta = \beta(p, n) > 0$ *such that*

$$M(R) \leqslant C_1 K \left(\frac{R}{R_0}\right)^\beta \qquad (0 < R < R_0); \tag{2.26}$$

C_1 *and* β *do not depend on* ε.

Proof. The calculations preceding modify without difficulty to the case at hand, and the only noteworthy change is that the term $M(R)^p + \varepsilon$, instead of $M(R)^p$ as before, cancels in the course of the proofs of Lemmas 2.1 and 2.3. $\blacksquare$

3. An a Priori Hölder Estimate for Du

Proposition 2.1 provides a Hölder estimate on the oscillation of Du near a point of degeneracy, where $Du = 0$. In this section we combine that result and the DeGiorgi–Moser estimates for nondegenerate equations to obtain an a priori Hölder estimate for Du at all interior points.

So let us once more suppose u to be a smooth solution of (2.1) in some ball $B(R_0)$, with estimate (2.3) holding. (We drop the assumption (2.2).)

PROPOSITION 3.1. *There exist constants $C_5 = C_5(R_0, p, n, K)$ and $\alpha = \alpha(p, n) > 0$ such that*

$$|Du|_{C^\alpha(B(R_0/2))} \leqslant C_5. \tag{3.1}$$

Proof. According to Proposition 2.1 Du is Hölder continuous with exponent β at any point $x_0 \in B(R_0/2)$ at which $Du = 0$. Suppose now instead

$$|Du(x_0)| > 0. \tag{3.2}$$

Define for $k = 1, 2,..., n$, $0 < R < R_0/2$:

$$M(R) \equiv \max_{B(x_0,R)} |Du|,$$

$$M_k^\pm(R) \equiv \max_{B(x_0,R)} \pm u_{x_k},$$

$$\operatorname*{osc}_{B(x_0,R)} u_{x_k} \equiv \max_{B(x_0,R)} u_{x_k} - \min_{B(x_0,R)} u_{x_k} = M_k^+(R) + M_k^-(R).$$

Let $\gamma < 1$ be the constant from Lemma 2.3.

Define R_1 to be the supremum of the set of numbers $0 < R \leqslant R_0/2$ for which

$$M_k^\varepsilon\left(\frac{R}{2}\right) \leqslant \gamma M_k^\varepsilon(R) \tag{3.3}$$

fails for some choice of $k \in \{1, 2,..., n\}$, $\varepsilon \in \{+, -\}$ such that

$$M_k^\varepsilon(R) \geqslant \frac{1}{\sqrt{n}} M(R) > 0 \qquad \text{(cf. (2.7))}. \tag{3.4}$$

Then $R_1 > 0$, since otherwise we could conclude as in Section 2

$$M(R) \leqslant C\left(\frac{R}{R_0}\right)^\beta \qquad \left(0 < R \leqslant \frac{R_0}{2}\right),$$

a contradiction to (3.2). Therefore there exists $R_1/2 < R_2 \leqslant R_1$ such that, say,

$$M_1^+(R_2) \geqslant \frac{1}{\sqrt{n}} M(R_2) > 0,$$

but (3.3) fails for $R = R_2$, $k = 1$, $\varepsilon = +$. It must therefore follow that (2.8) holds since otherwise the argument in Lemmas 2.2 and 2.3 would imply (3.3) with $R = R_2$, $k = 1$, $\varepsilon = +$ (here $B(x_0, R_0)$ replaces $B(R)$). Hence Lemma 2.1 yields

$$\min_{B(x_0, R_2/2)} u_{x_1} \geqslant \frac{1}{2} \max_{B(x_0, R_2)} u_{x_1} \geqslant \frac{1}{2\sqrt{n}} M(R_2) > 0. \tag{3.5}$$

Accordingly $v = u_{x_k}$ $(k = 1, 2, \ldots, n)$ satisfies the nondegenerate equation (2.9) in $B(x_0, R_2/2)$:

$$-(a_{ij} |Du|^p v_{x_i})_{x_j} = 0 \qquad \text{in } B\left(x_0, \frac{R_2}{2}\right), \tag{3.6}$$

the a_{ij} defined by (2.10). Indeed, since (3.5) implies

$$M(R_2)^p \geqslant |Du|^p \geqslant \left(\frac{1}{2\sqrt{n}} M(R_2)\right)^p \qquad \text{in } B\left(x_0, \frac{R_2}{2}\right),$$

we have

$$\lambda |\xi|^2 \leqslant a_{ij} |Du|^p \xi_i \xi_j \leqslant \mu |\xi|^2 \qquad (\xi \in \mathbb{R}^n)$$

for

$$\lambda \equiv \left(\frac{1}{2\sqrt{n}} M(R_2)\right)^p$$

and

$$\mu \equiv (1 + p) M(R_2)^p.$$

As

$$1 > \frac{\lambda}{\mu} = (2^p n^{p/2} (p + 1))^{-1} > 0,$$

no matter how small $M(R_2)$ is, we may invoke the DeGiorgi–Moser estimates (cf. [5, p. 465]) to prove the existence of a constant $\delta = \delta(p, n) < 1$ such that

$$\operatorname*{osc}_{B(x_0, R/4)} u_{x_k} \leqslant \delta \operatorname*{osc}_{B(x_0, R)} u_{x_k} \qquad \left(k = 1, 2, \ldots, n;\ 0 < R < \frac{R_2}{2}\right). \tag{3.7}$$

Next we employ Lemma 2.4 with

$$\omega_i(R) = \bar{\omega}_i(R) = \operatorname*{osc}_{B(x_0,R)} u_{x_i}, \qquad (i = 1, 2,..., n),$$

$$\bar{\omega}_i(R) = M_i^+(R) \qquad (i = n+1,..., 2n),$$

$$\bar{\omega}_i(R) = M_i^-(R) \qquad (i = 2n+1,..., 3n),$$

$$M = n, \qquad N = 3n, \qquad \eta = \frac{1}{4}, \qquad \delta_0 = \frac{1}{2\sqrt{n}}, \qquad \sigma = \max(\delta, \gamma).$$

The hypotheses of this lemma are valid owing to (3.3), (3.4) and (3.7), *independently* of the values of R_1, R_2. Hence if $0 < R \leqslant R_0/2$,

$$\operatorname*{osc}_{B(x_0,R)} u_{x_k} \leqslant C \left(\frac{R}{R_0}\right)^{\alpha} \qquad (k = 1, 2,..., n) \tag{3.8}$$

for certain positive constants C, α, depending only on known quantities. Since estimate (3.8) holds for any $x_0 \in B(R_0/2)$ where $Du \neq 0$, this and Proposition 2.1 complete the proof. ∎

As in Section 2 we will actually need the gradient Hölder estimate for a solution of the approximate problem $(2.25)_\varepsilon$:

PROPOSITION 3.2. *If u is a smooth solution of* $(2.25)_\varepsilon$ *in* $B(R_0)$ *and estimate* (2.3) *holds, then*

$$[Du]_{C^\alpha(B(R_0/2))} \leqslant C_5$$

for certain constants $C_5 = C_5(R_0, p, n, K)$, $\alpha = \alpha(p, n) > 0$, *independent of* $\varepsilon > 0$.

The proof is a straightforward modification of the calculations above.

4. Proof of Theorem 1

As the weak solution of (1.1) constructed by the variational principle is known a priori only to lie in the space $W^{1,p+2}(\Omega)$, the formal calculations from Sections 2 and 3 are not directly applicable. We therefore construct a sequence of approximate problems the solutions of which are smooth, and to which the estimates of Sections 2 and 3 apply with relatively minor modifications.

Accordingly assume the ball $B(R_0)$ lies in Ω, and define

$$g = g_\delta \equiv \rho_\delta * u, \tag{4.1}$$

where ρ_δ is a standard mollification kernel, $\delta > 0$. Fix $\varepsilon > 0$ and consider the p.d.e.:

$$\begin{cases} \operatorname{div}(|Du^\varepsilon|^p Du^\varepsilon) + \varepsilon \Delta u^\varepsilon = 0 & \text{in } B(R_0), \\ u^\varepsilon = g & \text{on } \partial B(R_0). \end{cases} \tag{4.2$_\varepsilon$}$$

LEMMA 4.1. (a) *There exists a constant* $C = C(R_0, \delta)$ *such that*

$$\max_{B(R_0)} |Du^\varepsilon| \leqslant C, \tag{4.3}$$

if u^ε *is a smooth solution of* $(4.2)_\varepsilon$; C *does not depend on* ε.

(b) *Problem* $(4.2)_\varepsilon$ *has a unique smooth solution.*

Proof. (a) To simplify notation drop the superscript "ε." First we claim

$$\max_{\partial B(R_0)} |Du| \leqslant C_6 \tag{4.4}$$

for $C_6 = C_6(R_0, \delta)$. To see this choose any point x^* belonging to $\partial B(R_0)$; we may assume $x^* = (0, 0, \ldots, -R_0)$. Define

$$w(x) \equiv g(x^*) + \sum_{i=1}^{n-1} g_{x_i}(x^*)\, x_i + \max |D^2 g| \sum_{i=1}^{n-1} x_i^2 + \mu(x_n + R_0) - \lambda(x_n + R_0)^2$$

for μ, $\lambda > 0$ selected as follows. Assuming for the moment

$$|w_{x_n}| \geqslant 1, \tag{4.5}$$

we have

$$\begin{aligned} -\operatorname{div}(|Dw|^p Dw) - \varepsilon \Delta w &= -(|Dw|^p + \varepsilon)\, \Delta w - p\, |Dw|^{p-2}\, w_{x_i} w_{x_j} w_{x_i x_j} \\ &= -(|Dw|^p + \varepsilon)(2(n-1) \max |D^2 g| - 2\lambda) \\ &\quad - p\, |Dw|^{p-2} \left(\sum_{i=1}^{n-1} 2(w_{x_i})^2 \max |D^2 g| - 2(w_{x_n})^2 \lambda \right) \\ &\geqslant 0, \end{aligned}$$

provided $\lambda > 0$ is large enough. With λ now fixed we select $\mu > 0$ so large that (4.5) is valid, and also

$$w \geqslant g \qquad \text{on } \partial B(R_0).$$

The maximum principle implies $w \geqslant u$ in $B(R_0)$; and since $w(x^*) = g(x^*) = u(x^*)$,

$$\frac{\partial u}{\partial n}(x) \geqslant \frac{\partial w}{\partial n}(x^*) = -\mu.$$

A bound from above is derived similarly. As x^* stood for an arbitrary point on $\partial B(R_0)$, and since the tangential derivatives of u and g agree, we have proved (4.4).

To obtain a global gradient bound we fix $k \in \{1, 2,..., n\}$ and differentiate $(4.2)_\varepsilon$:

$$-(a_{ij}\,|Du|^p\, u_{x_k x_i})_{x_j} + \varepsilon\, \Delta u_{x_k} = 0, \tag{4.6}$$

the a_{ij} defined by (2.10). Multiply this equation by $(\pm u_{x_k} - C_6)^+$ and integrate by parts to find

$$0 = \int_{B(R_0) \cap \{\pm u_{x_k} > C_5\}} a_{ij}\,|Du|^p\, u_{x_k x_i} u_{x_k x_j} + \varepsilon\,|Du_{x_k}|^2\, dx;$$

and so

$$\pm u_{x_k} \leqslant C_6 \qquad (k = 1, 2,..., n)$$

in $B(R_0)$.

(b) Owing to the a priori estimate (4.3) and the uniform ellipticity of $(4.2)_\varepsilon$ standard quasilinear elliptic theory implies the existence of a unique solution: see, for example,. [2]. ∎

Next we require estimates independent of g and therefore δ (cf. (4.1)):

LEMMA 4.2. (a)

$$\int_{B(R_0)} |Du^\varepsilon|^{p+2}\, dx \leqslant C \left(\int_\Omega |Du|^{p+2}\, dx + 1 \right) \tag{4.7}$$

for some constant $C = C(R_0)$.

(b) *There exists* $C_7 = C_7(R_0)$ *such that*

$$\max_{B(R_0/2)} |Du^\varepsilon| \leqslant C_7. \tag{4.8}$$

The constants here do not depend on ε *or* δ.

Proof. (a) Multiply $(4.2)_\varepsilon$ by $u^\varepsilon - g$ and integrate by parts

$$\int_{B(R_0)} (|Du^\varepsilon|^p + \varepsilon)\,|Du^\varepsilon|^2\,dx \leqslant \int_{B(R_0)} (|Du^\varepsilon|^p + \varepsilon)\,|Dg|^2\,dx.$$

Hence

$$\begin{aligned}\int_{B(R_0)} |Du^\varepsilon|^{p+2} + \varepsilon\,|Du^\varepsilon|^2\,dx &\leqslant C\int_{B(R_0)} |Dg|^{p+2} + \varepsilon\,|Dg|^2\,dx\\ &\leqslant C\left(\int_{B(R_0)} |Dg|^{p+2}\,dx + 1\right)\\ &\leqslant C\left(\int_{\Omega} |Du|^{p+2}\,dx + 1\right),\end{aligned}$$

by (4.1).

(b) Equation (4.6) implies

$$-(b_{ij}\,|Du^\varepsilon|^p\,u_{x_k x_i})_{x_j} = 0 \tag{4.9}$$

on

$$W = W_\varepsilon \equiv \{x \in B(R_0) \mid |Du^\varepsilon(x)| > 1\},$$

where

$$b_{ij} \equiv a_{ij} + \frac{\varepsilon\delta_{ij}}{|Du^\varepsilon|^p} \qquad (1 \leqslant i, j \leqslant n).$$

Note that on W

$$|\xi|^2 \leqslant b_{ij}\xi_i\xi_j \leqslant (1 + p + \varepsilon)\,|\xi|^2 \qquad (\xi \in \mathbb{R}^n). \tag{4.10}$$

Now set

$$w = w_\varepsilon \equiv |Du^\varepsilon|^{p+2}.$$

We have

$$w_{x_i} = (p+2)\,|Du^\varepsilon|^p\,u^\varepsilon_{x_k x_i} u^\varepsilon_{x_k},$$

and so

$$\begin{aligned}-(b_{ij}w_{x_i})_{x_j} &= -(p+2)(b_{ij}\,|Du^\varepsilon|^p\,u^\varepsilon_{x_k x_l} u^\varepsilon_{x_k})_{x_j}\\ &= -(p+2)\,b_{ij}\,|Du^\varepsilon|^p\,u^\varepsilon_{x_k x_i} u^\varepsilon_{x_k x_j} \qquad \text{by (4.9)} \qquad (4.11)\\ &\leqslant 0 \qquad \text{on } W.\end{aligned}$$

Define

$$v \equiv (w-1)^+.$$

Then

$$-(b_{ij} v_{x_i})_{x_j} \leqslant 0 \qquad \text{in } B(R_0) \tag{4.12}$$

(in the weak sense) and the b_{ij} satisfy (4.10) on the set W, where $v \neq 0$.

Accordingly a standard elliptic estimate (cf. [2, p. 184]) implies

$$\max_{B(R_0/2)} v \leqslant C(q) \left(⨍_{B(3R_0/4)} v^q \, dx \right)^{1/q} \qquad \text{for any} \quad q > 1, \tag{4.13}$$

say

$$q = 1^* = \frac{n}{n-1}.$$

Using a Sobolev-type inequality we obtain

$$\begin{aligned} \left(⨍_{B(3R_0/4)} v^{1^*} \, dx \right)^{1/1^*} &\leqslant C ⨍_{B(3R_0/4)} |Dv| + v \, dx \qquad (C = C(n)) \\ &\leqslant C \left(⨍_{B(3R_0/4)} |Du^\varepsilon|^{p+2} \, |D^2 u^\varepsilon| + |Du^\varepsilon|^{p+2} \, dx \right) \\ &\leqslant C \left(⨍_{B(3R_0/4)} |Du^\varepsilon|^p \, |D^2 u^\varepsilon|^2 + |Du^\varepsilon|^{p+2} \, dx \right). \end{aligned} \tag{4.14}$$

Next multiply (4.11) by ζ^2 and integrate by parts, where ζ is a smooth cutoff function, $\zeta \equiv 1$ on $B(3R_0/4)$, $\zeta \equiv 0$ near $\partial B(R_0)$. After some simple calculations we obtain

$$⨍_{B(3R_0/4)} |Du^\varepsilon|^p \, |D^2 u^\varepsilon|^2 \, dx \leqslant \frac{C}{R_0^2} ⨍_{B(R_0)} |Du^\varepsilon|^{p+2} \, dx.$$

This estimate, (4.13), (4.14), the definitions of v and w, and (4.7) complete the proof. ∎

Remark. Part of our proof of (b) is based upon [6, p. 228].

Proof of Theorem 1. In view of estimate (4.8) and Proposition 3.2,

$$[Du^\varepsilon]_{C^\alpha(B(R_0/4))} \leqslant C \tag{4.15}$$

for some $\alpha > 0$, the constants independent of ε and δ. Furthermore a small variant of the proofs so far show in fact

$$\max_{\Omega''} |Du^{\varepsilon}| + [Du^{\varepsilon}]_{C^{\alpha}(\Omega'')} \leqslant C(\Omega'')$$

for any $\Omega'' \Subset B(R_0)$. These facts and estimate (4.7) imply there exists a subsequence (denoted "u^{ε}") converging as $\varepsilon, \delta \searrow 0$ to a function v,

$$\begin{aligned} u^{\varepsilon} &\rightharpoonup v && \text{weakly in } W^{1,p+2}(B(R_0)), \\ u^{\varepsilon} &\to v && \text{uniformly on each } \Omega'' \Subset B(R_0), \\ Du^{\varepsilon} &\to Dv && \text{uniformly on each } \Omega'' \Subset B(R_0). \end{aligned}$$

It is easy to check v solves (1.1) in $B(R_0)$, $u = v$ (in the trace sense) on $\partial B(R_0)$; so that by uniqueness $u = v$.

Then estimates (4.8) and (4.15) imply

$$\max_{B(R_0/2)} |Du| \leqslant C, \qquad [Du]_{C^{\alpha}(B(R_0/4))} \leqslant C.$$

Finally, given any subdomain $\Omega' \Subset \Omega$, we may cover it with finitely many balls, themselves contained in Ω and to which the preceding estimates apply. This proves the theorem. ∎

Note added in proof. J. Lewis and E. DiBenedetto have recently and independently proved $C^{1,\alpha}$ estimates for the case $-1 < p < 0$.

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