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p -HARMONIC FUNCTIONS IN THE PLANE

JUAN J. MANFREDI

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ABSTRACT. Given $p > 1$, let u be a solution to $\operatorname{div}(\nabla u|^{p-2}\nabla u) = 0$, on a domain Ω of the plane. Using the theory of quasiregular mappings we prove that the zeros of ∇u are isolated in Ω , obtain bounds for the Hölder exponent of ∇u and prove a strong form of the comparison principle.

1. Introduction and statements of results. Let Ω be a domain in \mathbf{R}^2 , $p > 1$ and $u: \Omega \rightarrow \mathbf{R}$ be in the Sobolev space $W_{\text{loc}}^{1,p}(\Omega)$; i.e. functions in $L_{\text{loc}}^p(\Omega)$ whose first distributional derivatives are also functions in $L_{\text{loc}}^p(\Omega)$. $W^{1,p}(\Omega)$ is the Sobolev space consisting of functions in $W_{\text{loc}}^{1,p}(\Omega)$ for which

$$\|u\|_{1,p} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} < \infty,$$

and $\dot{W}^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in this norm.

A function $u \in W_{\text{loc}}^{1,p}(\Omega)$ is p -harmonic if it is a weak solution of the p -harmonic equation

$$(1) \quad \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0;$$

that is, for all functions φ in $\dot{W}^{1,p}(\Omega)$ with compact support we have

$$\int_{\Omega} |\nabla u|^{p-2} [\nabla u, \nabla \varphi] dx = 0.$$

Here $[\cdot, \cdot]$ is the standard inner product in \mathbf{R}^2 . Two-harmonic functions are the classical harmonic functions. When $p \neq 2$ note that equation (1) is nonlinear and degenerate, at points at which $\nabla u = 0$.

As shown by Evans [E] and Ural'tseva [U] in the case $p > 2$, and by Lewis [L2] for the full range $1 < p < \infty$, p -harmonic functions are in the class $C_{\text{loc}}^{1,\alpha}$, where $\alpha = \alpha(p) \in (0, 1]$. Indeed these results hold also for p -harmonic functions of n -variables, $n \geq 2$, where now $\alpha = \alpha(p, n)$. Examples in [L1] show that there are p -harmonic functions which are not in the class $C_{\text{loc}}^{1,1}$ when $p > 2$ and $n \geq 2$.

Bojarskii and Iwaniec [BI] proved that the complex gradient $(u_x, -u_y)$ of a p -harmonic function u is a quasiregular mapping for $p > 2$. In §2 we recall the definition and some properties of quasiregular mappings and prove the following.

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THEOREM 1. *Let $\Omega \subset \mathbf{R}^2$ be a domain and $u \in W_{\text{loc}}^{1,p}(\Omega)$ a p -harmonic function, $1 < p < \infty$. Then*

(i) *there exists $\eta = \eta(p) > 0$ such that $u \in W_{\text{loc}}^{2,2+\eta}(\Omega)$; that is the distributional derivatives of ∇u are functions in $L_{\text{loc}}^{2+\eta}(\Omega)$,*

(ii) *$u \in C_{\text{loc}}^{1,\alpha}(\Omega)$, where*

$$\alpha = \alpha(p) = \begin{cases} p-1 & \text{for } 1 < p \leq 2, \\ 1/(p-1) & \text{for } p > 2, \end{cases}$$

(iii) *$F = (u_x, -u_y)$ is K -quasiregular in Ω , where*

$$K = K(p) = \frac{1}{2}[(p-1) + 1/(p-1)].$$

Let $S(u) = \{x \in \Omega : \nabla u(x) = 0\}$ be the singular set of u . From Theorem 1 and the fact that nonconstant quasiregular mappings are discrete, we immediately deduce:

COROLLARY 1. *If u is a nonconstant p -harmonic function, $S(u)$ is a discrete set; that is the zeros of ∇u are isolated in Ω .*

In addition, $u \in C^\infty(\Omega \setminus S(u))$, even real analytic in $\Omega \setminus S(u)$ by standard results for elliptic equations. See [GT].

Let F be a K -quasiregular mapping in \mathbf{R}^2 whose components we denote by F^1 and F^2 . Then F^1 , F^2 and $\log |F|$, when $|F| \neq 0$, are solutions of a linear equation of divergence type

$$(2) \quad \operatorname{div}(\sigma(x)\nabla v(x)) = 0$$

(see for example [GLM]) whose ellipticity constants depend only on K . On applying these results to the complex gradient of a p -harmonic function we obtain

COROLLARY 2. *If u is a nonconstant p -harmonic function then u_x, u_y and $\log |\nabla u|$, when $|\nabla u| \neq 0$ satisfy an equation of type (2), where the ellipticity constants depend exclusively on p .*

We remark here that Alessandrini [A] has given a different proof of the fact that $\log |\nabla u|$, when $|\nabla u| \neq 0$ satisfies a linear equation like (2) and he uses it to prove that $S(u)$ must be a discrete set.

From Corollary 2 we see that u_x, u_y and $\log |\nabla u|$ satisfy the strong maximum principle in Ω .

In §3 we shall prove the following strong comparison principle:

THEOREM 2. *Let u and v be p -harmonic functions in a domain $\Omega \subset \mathbf{R}^2$, such that $u \leq v$ in Ω . Then either $u \equiv v$ in Ω or $u < v$ in Ω .*

When one of the functions u or v has nonvanishing gradient the strong comparison principle can be reduced to the linear case and thus holds even in higher dimensions. See [T] and [FV]. When the singular sets $S(u)$ and $S(v)$ are not disjoint the strong comparison principle remains open in dimension three or larger.

Finally in §4 we discuss the relation between quasiregular mappings and elliptic equations in general form in two dimensions. It is known [GT, Chapter 12] that if u is a solution of the equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$$

in a domain Ω , where $A, B, C \in L^\infty(\Omega)$ and

$$\lambda|\zeta|^2 \leq A\zeta_1^2 + 2B\zeta_1\zeta_2 + C\zeta_2^2 \leq \Lambda|\zeta|^2$$

for all $\zeta \in \mathbf{R}^2$ and a.e. $x \in \Omega$, $0 < \lambda \leq \Lambda < \infty$, then $F = (u_x, -u_y)$ is a K -quasiregular mapping in Ω with $1 \leq K \leq C(\Lambda/\lambda)$. In §4 we prove Theorem 3 which is a converse of this result and may be thought as a generalization of the fact u harmonic $\Leftrightarrow F$ is analytic.

THEOREM 3. *Suppose $F = (u_x, -u_y)$ is a K -quasiregular mapping in a domain $\Omega \subset \mathbf{R}^2$. Then there are functions $A, B, C \in L^\infty(\Omega)$ such that $Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$ and the matrix*

$$M = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

is positive definite. Moreover if Λ and λ denote respectively the largest and the smallest eigenvalue of M , we have $1 \leq \Lambda/\lambda \leq c(K)$. In addition $c(K) \rightarrow 1$ as $K \rightarrow 1$.

As the proof will show the functions A, B, C and the constant $c(K)$ have simple expressions in terms of the “complex dilatation” of F , to be defined in §4.

2. Quasiregular mappings and the proof of Theorem 1. Let Ω be a domain in \mathbf{R}^2 and $F: \Omega \rightarrow \mathbf{R}^2$. Write $F = (F^1, F^2)$. We say that F is K -quasiregular, $K \geq 1$, if $F \in W_{\text{loc}}^{1,2}(\Omega)$ and for a.e. $z \in \Omega$

$$(3) \quad \|DF(z)\|^2 \leq 2KJF(z)$$

where

$$\|DF(z)\|^2 = (F_x^1(z))^2 + (F_y^1(z))^2 + (F_x^2(z))^2 + (F_y^2(z))^2$$

and

$$JF(z) = F_x^1(z) \cdot F_y^2(z) - F_y^1(z) \cdot F_x^2(z).$$

One-quasiregular mappings are precisely analytic functions, since inequality (3) becomes an equality equivalent to the Cauchy-Riemann equations for F^1 and F^2 .

Quasiregular mappings share many common properties with analytic functions. We state now those that we use in the proof of Theorem 1. All of them can be bound in [LV].

—A nonconstant quasiregular mapping is open, discrete and continuous.

—If F is K -quasiregular in Ω and Ω' is a domain whose closure is contained in Ω , then F is Hölder continuous in Ω' with exponent $K - \sqrt{K^2 - 1}$ and constant depending only on K and the norm of F in $W^{1,2}(\Omega')$.

—The uniform limit on compact subsets of K -quasiregular mappings is again K -quasiregular.

—There exists $\delta(K) > 0$ such that if F is K -quasiregular in Ω , the first derivatives of F are in $L_{\text{loc}}^{2+\delta(K)}(\Omega)$. Moreover, if Ω' is a domain whose closure is contained in Ω , $\|DF\|_{L^{2+\delta(K)}(\Omega')}$ is bounded by a quantity that depends only on K and the norm of F in $W^{1,2}(\Omega')$.

—If F is K -quasiregular and bounded in Ω and Ω' is a domain whose closure is contained in Ω the norm $\|F\|_{W^{1,2}(\Omega')}$ is bounded by a quantity depending only on $d(\Omega', \partial\Omega)$ and $\|F\|_{L^\infty(\Omega)}$.

PROOF OF THEOREM 1. Let $\Omega' \subset \Omega$ a domain whose closure is also contained in Ω . Let $\varepsilon \in \mathbf{R}$ so that $0 < \varepsilon < 1$. Define

$$(4) \quad L_\varepsilon v = \operatorname{div}((\varepsilon + |\nabla v|^2)^{p/2-1} \nabla v).$$

Let v^ε be the solution of the Dirichlet problem

$$L_\varepsilon v^\varepsilon = 0 \quad \text{in } \Omega', \quad v^\varepsilon - u \in \dot{W}^{1,p}(\Omega').$$

Since L_ε is no longer degenerate, it follows that $v^\varepsilon \in C^\infty(\Omega')$. Moreover, $v^\varepsilon \in W^{1,p}(\Omega')$ and it is easy to see that $v^\varepsilon \rightarrow u$ in $W^{1,p}(\Omega')$ as $\varepsilon \rightarrow 0$.

Set $F^\varepsilon = (v_x^\varepsilon, -v_y^\varepsilon)$. Differentiating (4), we obtain

$$L_\varepsilon v^\varepsilon = (\varepsilon + |\nabla v^\varepsilon|^2)^{p/2-2} [(\varepsilon + |\nabla v^\varepsilon|^2 + (p-2)(v_x^\varepsilon)^2)v_{xx}^\varepsilon + 2(p-2)v_x^\varepsilon v_y^\varepsilon v_{xy}^\varepsilon + (\varepsilon + |\nabla v^\varepsilon|^2 + (p-2)(v_y^\varepsilon)^2)v_{yy}^\varepsilon]$$

Write $L_\varepsilon v^\varepsilon = 0$ as

$$av_{xx}^\varepsilon + 2bv_{xy}^\varepsilon + cv_{yy}^\varepsilon = 0$$

where

$$a = (\varepsilon + |\nabla v^\varepsilon|^2) + (p-2)(v_x^\varepsilon)^2,$$

$$b = (p-2)v_x^\varepsilon v_y^\varepsilon,$$

$$c = (\varepsilon + |\nabla v^\varepsilon|^2) + (p-2)(v_y^\varepsilon)^2.$$

Momentarily set $P = v_x^\varepsilon$, $Q = v_y^\varepsilon$. Then $F^\varepsilon = (P, -Q)$. We have

$$(5) \quad \begin{aligned} DF^\varepsilon &= \begin{bmatrix} P_x & -Q_x \\ P_y & Q_y \end{bmatrix}, & P_y &= -Q_x, \\ JF^\varepsilon &= -P_x Q_y + P_y^2, & aP_x + 2bP_y + cQ_y &= 0, \\ \|DF^\varepsilon\|^2 &= P_x^2 + 2P_y^2 + Q_y^2. \end{aligned}$$

Thus

$$\frac{\|DF^\varepsilon\|^2}{JF^\varepsilon} = \frac{P_x^2 + 2P_y^2 + Q_y^2}{P_y^2 - P_x Q_y}.$$

From Exercise 12.3, p. 317 in [GT] we deduce

$$\begin{aligned} \frac{\|DF^\varepsilon\|^2}{JF^\varepsilon} &\leq \frac{a^2 + 2b^2 + c}{ac - b^2} \\ &= \frac{(p^2 - 2p + 2)|\nabla v^\varepsilon|^4 + 2\varepsilon(p|\nabla v^\varepsilon|^2 + \varepsilon)}{(p-1)|\nabla v^\varepsilon|^4 + \varepsilon(p|\nabla v^\varepsilon|^2 + \varepsilon)} \\ &\leq \frac{(p^2 - 2p + 2)|\nabla v^\varepsilon|^4}{(p-1)|\nabla v^\varepsilon|^4} = (p-1) + 1/(p-1) = 2K(p). \end{aligned}$$

Therefore F^ε is a $K(p)$ -quasiregular mapping. Thus we will have proved (i), (ii) and (iii) of Theorem 1 with constants independent of ε if we can show that $\|F^\varepsilon\|_{L^\infty(\Omega')}$ is bounded independent of ε , where Ω'' is a domain whose closure is contained in Ω' . This follows from the uniform bound of the v^ε in $W^{1,p}(\Omega')$ and Theorem 1 in [L2].

Next we take limits as $\varepsilon \rightarrow 0$. Since $F^\varepsilon \rightarrow F$ in $L^p(\Omega')$ for a certain subsequence $\varepsilon_i \rightarrow 0$, $F \xrightarrow{\varepsilon_i} F$ a.e. in Ω' . The sequence $(F^{\varepsilon_i})_{i=1}^\infty$ is equicontinuous since we

have a uniform Hölder estimate independent of ε . Extracting a subsequence, again denoted $(F^{\varepsilon_i})_{i=1}^\infty$, we conclude that $F^{\varepsilon_i} \rightarrow F$ uniformly on compact subsets of Ω' , possibly after modifying F in a set of measure zero. Thus F is $K(p)$ -quasiregular, which proves (iii). Now (i) and (ii) follows from (iii) and properties of quasiregular mappings.

It remains to consider the sharpness of $\eta(p)$ and $\alpha(p)$. A well-known conjecture in the theory of quasiregular mappings, see [GR], states that the exponent of integrability of the first derivatives of a K -quasiregular mapping is $2 + \delta(K)$, where

$$\delta(K) = \left(\frac{K+1}{K-1} \right)^{1/2} - 1.$$

This conjecture would imply that $\eta(p) \geq 2/(p-2)$ for $p > 2$ and $\eta(p) \geq 2(p-1)/(2-p)$ for $p < 2$, which may make the problem of finding the best $\eta(p)$ rather interesting.

All p -harmonic functions which are pseudoradial, that is of the form $|x|^\gamma \omega(x/|x|)$, $\gamma \in \mathbf{R}$, ω periodic, have been found by Krol and Mazja [KM]. See also Theorem 3.2 in [KV] and [Ar]. It is not hard to see, [L1], that the Hölder exponent of the first derivatives of these solutions, which we denote $\alpha_r(p)$, satisfies: $\alpha_r(2) = 1$, $\alpha_r(p)$ decreasing in p and $\lim_{p \rightarrow \infty} \alpha_r(p) = 4/3$. Thus $\alpha(p) = 1/(p-1)$ is not sharp within the class of pseudo-radial p -harmonic functions. However it is unknown in general whether $1/(p-1)$ is sharp or not.

3. Proof of the strong comparison principle. Let u and v be two p -harmonic functions satisfying the hypothesis of Theorem 2. Set $w = v - u$ and for $0 \leq t \leq 1$, $u_t = tv + (1-t)u$. Write $\mathbf{A}(h) = |h|^{p-2}h$ for $h \in \mathbf{R}^2$. For $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_{\Omega} \langle \mathbf{A}(\nabla v) - \mathbf{A}(\nabla u), \nabla \varphi \rangle dx = 0.$$

Writing

$$a_{ij}(x) = \int_0^1 \frac{\partial}{\partial h_j} \mathbf{A}^i(\nabla u_t) dt$$

and

$$\mathbf{A}(\nabla v) - \mathbf{A}(\nabla u) = \int_0^1 \frac{d}{dt} \mathbf{A}(\nabla u_t) dt$$

we obtain

$$(5) \quad \int_{\Omega} \sum_{i,j=1}^2 a_{ij}(x) \frac{\partial w}{\partial x_j} \cdot \frac{\partial \varphi}{\partial x_i} dx = 0.$$

Denote by $\tau(x)$ the matrix whose entries are $a_{ij}(x)$, $1 \leq i, j \leq 2$. From (5) we deduce

$$(6) \quad \operatorname{div}(\tau(x) \nabla w) = 0.$$

$\tau(x)$ is a symmetric matrix. Let $\lambda(x)$ and $\Lambda(x)$ denote the minimum and maximum eigenvalues of $\tau(x)$. A computation shows that

$$\begin{aligned} \Lambda(x) &\leq \max(1, p-1) \int_0^1 |\nabla u_t|^{p-2} dt, \\ \lambda(x) &\geq \min(1, p-1) \int_0^1 |\nabla u_t|^{p-2} dt. \end{aligned}$$

In particular (6) is uniformly elliptic in the following sense

$$1 \leq \frac{\Lambda(x)}{\lambda(x)} \leq \begin{cases} p-1 & \text{if } p > 2, \\ 1/(p-1) & \text{if } 1 < p < 2. \end{cases}$$

But, in general, (6) is a degenerate equation. The eigenvalues vanish in the intersection of the singular sets $S = S(u) \cap S(v)$. We now take advantage of the fact that S is discrete.

Notice that the bounds for $\Lambda(x)$ and $\lambda(x)$ are continuous in $\Omega \setminus S$. Set

$$\Omega' = \{x \in \Omega \setminus S \text{ such that } w(x) = 0\}.$$

Since w is continuous Ω' is closed relative to $\Omega \setminus S$. Pick now $x_0 \in \Omega'$. We can find $\varepsilon > 0$ and $\delta > 0$ such that $\lambda(x) \geq \delta$ on $|x - x_0| \leq \varepsilon$. By the strong maximum principle for nondegenerate elliptic equations $w \equiv 0$ on $|x - x_0| \leq \varepsilon$. Thus Ω' is open. Since $\Omega \setminus S$ is a domain, there are two possibilities.

(a) $\Omega' = \emptyset$. In this case $w(x) > 0$ for $x \in \Omega \setminus S$. Pick now $x_0 \in S$. Find $\rho > 0$ and $\varepsilon > 0$ such that $w(x) > \rho$ for x satisfying $|x - x_0| = \varepsilon$. By the standard maximum principle $w \geq \rho$ on $|x - x_0| \leq \varepsilon$. Therefore $w(x) > 0$ for all $x \in \Omega$.

(b) $\Omega' = \Omega \setminus S$. Clearly we have $w \equiv 0$ in Ω .

4. Quasiregular mappings and elliptic equations in two dimensions. In this section we will use complex notation. That is, we rewrite $z = x + iy$, $F = u_x - iu_y$, $F_z = \frac{1}{2}(F_x - iF_y)$ and $F_{\bar{z}} = \frac{1}{2}(F_x + iF_y)$. Thus

$$F_z = \frac{u_{xx} - u_{yy}}{2} - iu_{xy} \quad \text{and} \quad F_{\bar{z}} = \frac{u_{xx} + u_{yy}}{2}.$$

The complex dilatation of F , $\mu(z)$ is defined via the Beltrami equation

$$(7) \quad F_{\bar{z}} = \mu F_z.$$

A mapping F is K -quasiregular if and only if, [LV], $\|\mu\|_\infty \leq \kappa(K) < 1$. With our definition of K in §2 we may take $\kappa(K) = (K^2 - 1)^{1/2}/(K + 1)$. Write $\mu = \alpha + i\beta$, where $\alpha = \text{Re}(\mu)$ and $\beta = \text{Im}(\mu)$ are real functions bounded by $\kappa(K)$. The Beltrami equation (7) becomes

$$\frac{u_{xx} + u_{yy}}{2} = (\alpha + i\beta) \left(\frac{u_{xx} - u_{yy}}{2} - iu_{xy} \right).$$

Separating real and imaginary parts

$$\begin{aligned} (\alpha - |\mu|^2)u_{xx} + (\alpha + |\mu|^2)u_{yy} &= 0, \\ \beta u_{xx} - (\alpha + |\mu|^2)u_{xy} &= 0. \end{aligned}$$

We are looking for functions A , B and C such that

$$u_{xx}(A(|\mu|^2 + \alpha) - 2B\beta + C(|\mu|^2 - \alpha)) = 0.$$

Thus, we need to solve the following calculus problem:

$$\text{Find min } \left\{ \Lambda/\lambda \text{ where } \Lambda \text{ and } \lambda \text{ are the eigenvalues of } \begin{pmatrix} A & B \\ B & C \end{pmatrix}, \lambda \leq \Lambda \right\},$$

where A , B and C satisfy

$$A(|\mu|^2 + \alpha) - 2B\beta + C(|\mu|^2 - \alpha) = 0.$$

Using the Lagrange's multiplier method we obtain that the minimum is attained at $A = 1 - \alpha$, $B = \beta$, $C = 1 + \alpha$ and it has the value $\Lambda/\lambda = (1 + |\mu|)/(1 - |\mu|)$. Thus we may take

$$c(K) = \frac{1 + \kappa(K)}{1 - \kappa(K)}.$$

Let us remark that our proof shows that $c(K)$ is sharp.

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