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The Dirichlet Problem for p-Harmonic Functions on a Network

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We discuss the Dirichlet problem for p-harmonic functions on a network. We show that every continuous function on the p-Royden boundary is p-resolutive and that the set of regular boundary points coincides with the p-harmonic boundary. Also we prove that the p-Dirichlet solution is left continuous with respect to p.

KEYWORDS: discrete potential theory, nonlinear harmonic function, Dirichlet problem, Royden compactification, Royden boundary

1. Introduction

For $1 < p < \infty$ the p-Laplacian $\Delta_p u$ of a function $u \in C^1(M)$ on a Riemannian manifold M is defined as

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

A function u is called p-harmonic if $\Delta_p u = 0$ in M. The p-Dirichlet norm of u is

$$\sup_M |u| + \left(\int_M |\nabla u|^p \, dV\right)^{1/p},$$

where dV is the volume element in M. Let $\mathbf{BD}^p$ be the class of bounded functions with finite p-Dirichlet norms. The p-Royden compactification is a compactification of M to which every function in $\mathbf{BD}^p$ continuously extends and which is separated by functions in $\mathbf{BD}^p$. The p-Royden boundary is the set of points in the p-Royden compactification which are not in M. Let $\mathbf{BD}^p_0$ be the class of functions in $\mathbf{BD}^p$ which are represented as limits of functions in $C^1_0(M)$. The p-harmonic boundary is the set of boundary points at which every function in $\mathbf{BD}^p_0$ vanishes. Nakai [2] discussed the Dirichlet problem for p-harmonic functions. He showed that every continuous function on the p-Royden boundary is p-resolutive and that the set of regular boundary points coincides with the p-harmonic boundary.

In this paper we treat networks instead of Riemannian manifolds. It is possible to define p-harmonic functions on a network, see §2, and discuss the Dirichlet problem for p-harmonic functions on a network. We show some results on a network similar to Nakai's theorem mentioned above. Namely, after discussing the Royden compactification of a network, we prove Theorems 3.14 and 3.15, which show that every continuous function on the p-Royden boundary is p-resolutive and that the set of regular boundary points coincides with the p-harmonic boundary. Also we prove in §4 that, for each continuous function on the p-Royden boundary, the p-Dirichlet solution is left continuous with respect to p.

2. Preliminaries

Let (V, E) be an infinite graph, where V is the set of *vertices* and E is the set of *edges*. This means that V is a countable set and an element of E is an ordered pair (x, y) of vertices $x, y \in V$. If $(x, y) \in E$, then we write $x \sim y$. Let

$$\partial x = \{y \in V; x \sim y\} \quad \text{and} \quad Nx = \partial x \cup \{x\}.$$

Throughout this paper we always assume the following:

(i) $(x, y) \in E$ if $(y, x) \in E$;
(ii) there is no self-loop, i.e., $(x, x) \notin E$ for $x \in V$;
(iii) there is a *path* from x to y for every distinct pair $x, y \in V$, i.e., there is a sequence $\{x_i\}_{i=0}^{l}$ of distinct vertices such that $x = x_0 \sim x_1 \sim \cdots \sim x_l = y$;
(iv) (V, E) is locally finite, i.e., ∂x is a finite set for each $x \in V$.

Let r be a strictly positive function on E and call it a *resistance*. We assume that $r(y, x) = r(x, y)$ for each edge $(x, y) \in E$. A *network* is a triplet $\mathcal{N} = (V, E, r)$, where (V, E) is an infinite graph and r is a resistance.

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For a subset $D \subset V$ we let

$$\overline{D} = \bigcup_{x \in D} Nx \quad \text{and} \quad \partial D = \overline{D} \setminus D.$$

We denote by $L(D)$ the set of functions on D. Let $L_0(D)$ be the set of functions in $L(D)$ with finite supports. Also we let $L(E)$ be the set of functions on E such that $w(x,y) = -w(y,x)$ for $(x,y) \in E$. Let $L_0(E)$ be the set of functions in $L(E)$ with finite supports. For $u \in L(D)$ and $(x,y) \in E$ with $x, y \in D$, we define a *discrete derivative* ∇u at (x,y) by

$$\nabla u(x,y) = \frac{u(y) - u(x)}{r(x,y)}.$$

For $w_1, w_2 \in L(E)$ we let

$$\langle w_1, w_2 \rangle = \sum_{e \in E} r(e) w_1(e) w_2(e)$$

if it converges absolutely.

Let $1 < p < \infty$. We define the *p-Dirichlet sum* $D_p[u]$ of $u \in L(V)$ by

$$D_p[u] = \sum_{e \in E} r(e) |\nabla u(e)|^p.$$

We denote by $\mathbf{D}^p$ the class of functions with finite p-Dirichlet sums. We consider $\mathbf{D}^p$ as a Banach space with norm $(D_p[u] + |u(a_0)|^p)^{1/p}$ where a_0 is a fixed vertex. We denote by $\mathbf{D}_0^p$ the closure of $L_0(V)$ in $\mathbf{D}^p$. Also we denote by $\mathbf{BD}^p$ ($\mathbf{BD}_0^p$, resp.) the set of bounded functions in $\mathbf{D}^p$ ($\mathbf{D}_0^p$, resp.). Let

$$\varphi_p(t) = |t|^{p-1} \operatorname{sgn} t,$$

where sgn is the sign function, i.e., $\operatorname{sgn} t = 1$ if $t > 0$; $\operatorname{sgn} 0 = 0$; and $\operatorname{sgn} t = -1$ if $t < 0$. Note that φ_p is a strictly increasing function and that $\varphi_p(t) = |t|^{p-2} t$ for $t \neq 0$. We define the *p-Laplacian* $\Delta_p u$ of $u \in L(Nx)$ as

$$\Delta_p u(x) = \sum_{y \in \partial x} \varphi_p(\nabla u(x,y)).$$

For $x \in V$ a function $u \in L(Nx)$ is said to be *p-harmonic* (*p-subharmonic*, *p-superharmonic*, resp.) at x if u satisfies $\Delta_p u(x) = 0$ ($\Delta_p u(x) \geq 0$, $\Delta_p u(x) \leq 0$, resp.). Let $D \subset V$. A function $u \in L(\overline{D})$ is said to be *p-harmonic* (*p-subharmonic*, *p-superharmonic*, resp.) in D if u is p-harmonic (p-subharmonic, p-superharmonic, resp.) at each vertex in D. For details see Soardi and Yamasaki [3].

3. The Dirichlet Problem

Next lemma shows the Harnack inequality.

Lemma 3.1. *Let $x \in V$ and $c_x = \max_{y \in \partial x} r(x,y)(\sum_{z \in \partial x \setminus \{y\}} r(x,z)^{1-p})^{q-1}$, where q is the number with $1/p + 1/q = 1$.*
(i) *If u is p-subharmonic at x and $u \leq 0$ on Nx, then $u(y) \geq (1 + c_x) u(x)$ for $y \in \partial x$.*
(ii) *If u is p-superharmonic at x and $u \geq 0$ on Nx, then $u(y) \leq (1 + c_x) u(x)$ for $y \in \partial x$.*

Proof. We shall prove (ii) only. Since $\Delta_p u(x) \leq 0$ and $u(z) \geq 0$ for $z \in Nx$, we have

$$\varphi_p(\nabla u(x,y)) \leq -\sum_{z \in \partial x \setminus \{y\}} \varphi_p(\nabla u(x,z)) \leq \sum_{z \in \partial x \setminus \{y\}} \varphi_p\left(\frac{u(x)}{r(x,z)}\right) = u(x)^{p-1} \sum_{z \in \partial x \setminus \{y\}} r(x,z)^{1-p}$$

for $y \in \partial x$. The fact $\varphi_q(\varphi_p(t)) = t$ implies

$$\nabla u(x,y) \leq u(x) \left(\sum_{z \in \partial x \setminus \{y\}} r(x,z)^{1-p} \right)^{q-1},$$

and hence $u(y) - u(x) \leq c_x u(x)$, which shows the assertion. □

Lemma 3.2. *Let $u \in \mathbf{D}^p$ and $v \in L_0(V)$. Then*

$$\langle \varphi_p \circ \nabla u, \nabla v \rangle = -2 \sum_{x \in V} v(x) \Delta_p u(x).$$

Proof. A simple calculation shows that

$$\langle \varphi_p \circ \nabla u, \nabla v \rangle = \sum_{x \in V} \sum_{y \in \partial x} r(x,y) \varphi_p(\nabla u(x,y)) \nabla v(x,y) = \sum_{x \in V} \sum_{y \in \partial x} (v(y) - v(x)) \varphi_p(\nabla u(x,y))$$

$$= \sum_{x\in V} v(x)\left(-\sum_{y\in\partial x}\varphi_p(\nabla u(x,y)) + \sum_{y\in\partial x}\varphi_p(\nabla u(y,x))\right)$$
$$= -2\sum_{x\in V} v(x)\sum_{y\in\partial x}\varphi_p(\nabla u(x,y)) = -2\sum_{x\in V} v(x)\Delta_p u(x).$$

□

We consider that V is equipped with the discrete topology. A compactification of V is a compact Hausdorff space containing V as a dense open subset. There is a unique (up to a homeomorphism) compactification V^{*p} of V such that every function in $\mathbf{BD}^p$ is continuously extended to V^{*p} and that the class of extended functions separates points of V^{*p}. This compactification is called the *p-Royden compactification* of V and $\Gamma^p = V^{*p} \setminus V$ is called the *p-Royden boundary*. We denote by u^{*p} the extension of $u \in \mathbf{BD}^p$ to V^{*p}. See Yamasaki [6].

We let $s \vee t = \max(s,t)$ and $s \wedge t = \min(s,t)$ for real numbers s, t.

Lemma 3.3. *For $u \in \mathbf{D}^p$ there exists $\lim_{x\to\xi} u(x)$ for each $\xi \in \Gamma^p$.*

Proof. Let $\alpha > 0$ and let $v_\alpha = (u \wedge \alpha) \vee (-\alpha)$. Note that $v_\alpha \in \mathbf{BD}^p$ and $|v_\alpha^{*p}(\xi)| \le \alpha$. If there is $\alpha > 0$ such that $|v_\alpha^{*p}(\xi)| < \alpha$, then $v_\alpha = u$ in the intersection of V and some neighbourhood of ξ, so that $\lim_{x\to\xi} u(x) = v_\alpha^{*p}(\xi)$. Next we assume that $|v_\alpha^{*p}(\xi)| = \alpha$ for all $\alpha > 0$. Then either $v_\alpha^{*p}(\xi) = \alpha$ or $v_\alpha^{*p}(\xi) = -\alpha$ for each α. If there is $\alpha_0 > 0$ with $v_{\alpha_0}^{*p}(\xi) = \alpha_0$, then $v_{\alpha_0}^{*p} > \alpha_0 - \varepsilon$ in some neighbourhood of ξ, and $v_\alpha^{*p}(\xi) = \alpha$ for every $\alpha \ge \alpha_0$. This means $\lim_{x\to\xi} u(x) = \infty$. If there is α_0 such that $v_{\alpha_0}^{*p}(\xi) = -\alpha_0$, then similarly we have $\lim_{x\to\xi} u(x) = -\infty$. Therefore there exists $\lim_{x\to\xi} u(x)$ in any case. □

We let $u^{*p}(\xi) = \lim_{x\to\xi} u(x)$ for $u \in \mathbf{D}^p$ and for $\xi \in \Gamma^p$. We remark that $-\infty \le u^{*p} \le \infty$. The set

$$\Gamma_0^p = \{\xi \in \Gamma^p;\ u^{*p}(\xi) = 0 \text{ for every } u \in \mathbf{BD}_0^p\}$$

is called the *p-harmonic boundary* of V. A network is said to be of *p-hyperbolic type* if $\mathbf{D}^p \ne \mathbf{D}_0^p$; otherwise the network is said to be of *p-parabolic type*. Yamasaki [6, Theorems 4.3 and 5.2] showed the following.

Proposition 3.4 (Yamasaki [6, Theorem 4.3]). *A network is of p-hyperbolic type if and only if $\Gamma_0^p \ne \emptyset$.*

Lemma 3.5 (Yamasaki [6, Theorem 5.2]). *Let $u \in \mathbf{BD}^p$. Then $u \in \mathbf{BD}_0^p$ if and only if $u^{*p} = 0$ on Γ_0^p.*

In the sequel of this section we always assume that $\mathcal{N} = (V, E, r)$ is a network of p-hyperbolic type. For a function f on Γ^p we define the *upper class* $\mathcal{U}_f^p$ and the *lower class* $\mathcal{L}_f^p$ as

$$\mathcal{U}_f^p = \{u \in \mathbf{D}^p;\ u \text{ is } p\text{-superharmonic and bounded below in } V,\ u^{*p} \ge f \text{ on } \Gamma_0^p\},$$
$$\mathcal{L}_f^p = \{v \in \mathbf{D}^p;\ v \text{ is } p\text{-subharmonic and bounded above in } V,\ v^{*p} \le f \text{ on } \Gamma_0^p\}.$$

Also we define the *upper solution* $\overline{\mathcal{H}}_f^p$ and the *lower solution* $\underline{\mathcal{H}}_f^p$ as

$$\overline{\mathcal{H}}_f^p(x) = \inf\{u(x); u \in \mathcal{U}_f^p\},\quad \underline{\mathcal{H}}_f^p(x) = \sup\{v(x); v \in \mathcal{L}_f^p\}\quad \text{for } x \in V.$$

If $\mathcal{U}_f^p = \emptyset$, then we let $\overline{\mathcal{H}}_f^p \equiv \infty$. If $\mathcal{L}_f^p = \emptyset$, then we let $\underline{\mathcal{H}}_f^p \equiv -\infty$. From the definitions we easily have that

$$\overline{\mathcal{H}}_{f_1}^p \le \overline{\mathcal{H}}_{f_2}^p,\quad \underline{\mathcal{H}}_{f_1}^p \le \underline{\mathcal{H}}_{f_2}^p \quad \text{if } f_1 \le f_2;$$
$$\overline{\mathcal{H}}_{f+c}^p = \overline{\mathcal{H}}_f^p + c,\quad \underline{\mathcal{H}}_{f+c}^p = \underline{\mathcal{H}}_f^p + c \quad \text{for a constant } c.$$

Proposition 3.6. (i) *If $\overline{\mathcal{H}}_f^p \not\equiv \pm\infty$, then $\overline{\mathcal{H}}_f^p$ is p-harmonic in V.*
(ii) *If $\underline{\mathcal{H}}_f^p \not\equiv \pm\infty$, then $\underline{\mathcal{H}}_f^p$ is p-harmonic in V.*

Proof. We shall prove (i) only. Since $\overline{\mathcal{H}}_f^p \not\equiv \infty$, it follows that $\mathcal{U}_f^p \ne \emptyset$, and therefore $\overline{\mathcal{H}}_f^p(x) < \infty$ everywhere. Since $\overline{\mathcal{H}}_f^p \not\equiv -\infty$, we find $x \in V$ with $\overline{\mathcal{H}}_f^p(x) > -\infty$. Let $\{u_n\}_n \subset \mathcal{U}_f^p$ be such that $\lim_{n\to\infty} u_n(y) = \overline{\mathcal{H}}_f^p(y)$ for $y \in Nx$. We may assume that there is α such that $u_n \le \alpha$ in Nx for all n. Let $\tilde{u}_n$ be a function such that $\tilde{u}_n = u_n$ in $X \setminus \{x\}$ and that $\tilde{u}_n$ is p-harmonic at x. Since $\tilde{u}_n \le u_n$ and $\tilde{u}_n \in \mathcal{U}_f^p$, it follows that $\lim_{n\to\infty} \tilde{u}_n(y) = \overline{\mathcal{H}}_f^p(y)$ for $y \in Nx$. Since $\tilde{u}_n$ is p-harmonic at x, Lemma 3.1 (i) shows that $\tilde{u}_n(y) - \alpha \ge (1 + c_x)(\tilde{u}_n(x) - \alpha)$ for $y \in \partial x$, which implies $\overline{\mathcal{H}}_f^p(y) > -\infty$. Also we have that $\overline{\mathcal{H}}_f^p$ is p-harmonic at x. Repeating this argument we have that $\overline{\mathcal{H}}_f^p > -\infty$ and that $\overline{\mathcal{H}}_f^p$ is p-harmonic everywhere. □

Lemma 3.7. *It is true that $s + t$ and $\varphi_p(s) + \varphi_p(t)$ have the same sign for any $s, t \in \mathbb{R}$.*

Proof. Since φ_p is a strictly increasing function and $\varphi_p(-t) = -\varphi_p(t)$, it follows that $s = -t$ ($s > -t$, $s < -t$, resp.) implies $\varphi_p(s) = -\varphi_p(t)$ ($\varphi_p(s) > -\varphi_p(t)$, $\varphi_p(s) < -\varphi_p(t)$, resp.). This means that $s + t$ and $\varphi_p(s) + \varphi_p(t)$ have the same sign. □

Lemma 3.8. *Let $v_1, v_2 \in \mathbf{D}^p$ be two p-subharmonic functions which are bounded above. Suppose that $(v_1 + v_2)^{*p} \le 0$ on Γ_0^p. Then $v_1 + v_2 \le 0$ in V.*

Proof. Let $u = (v_1 + v_2) \vee 0$. Note that $u \in \mathbf{BD}^p$. By the hypothesis we have $u^{*p} = 0$ on Γ_0^p. Lemma 3.5 implies that $u \in \mathbf{BD}_0^p$, and that there is a sequence $\{u_m\}_m \subset L_0(V)$ such that $\lim_{m\to\infty} D_p[u_m - u] = 0$. We may assume $u_m \geq 0$. Then Lemma 3.2 shows that

$$\begin{aligned}\langle \varphi_p \circ \nabla v_1 + \varphi_p \circ \nabla v_2, \nabla u_m\rangle &= \langle \varphi_p \circ \nabla v_1, \nabla u_m\rangle + \langle \varphi_p \circ \nabla v_2, \nabla u_m\rangle \\ &= -2\sum_{x\in V} u_m(x)\Delta_p v_1(x) - 2\sum_{x\in V} u_m(x)\Delta_p v_2(x) \leq 0,\end{aligned}$$

so that

$$\langle \varphi_p \circ \nabla v_1 + \varphi_p \circ \nabla v_2, \nabla u\rangle = \lim_{m\to\infty} \langle \varphi_p \circ \nabla v_1 + \varphi_p \circ \nabla v_2, \nabla u_m\rangle \leq 0.$$

Let $(x, y) \in E$. Let $A = \{z \in X; u(z) > 0\}$. If $x, y \in A$, then $\nabla u(x, y) = \nabla v_1(x, y) + \nabla v_2(x, y)$. Lemma 3.7 implies

$$(\varphi_p(\nabla v_1(x, y)) + \varphi_p(\nabla v_2(x, y)))\nabla u(x, y) \geq 0,$$

and the equality holds only if $\nabla u(x, y) = 0$. If $x, y \notin A$, then $\nabla u(x, y) = 0$, so that

$$(\varphi_p(\nabla v_1(x, y)) + \varphi_p(\nabla v_2(x, y)))\nabla u(x, y) = 0.$$

If $x \notin A$ and $y \in A$, then $u(x) = 0 \geq v_1(x) + v_2(x)$ and $u(y) = v_1(y) + v_2(y) > 0$. We have $0 < \nabla u(x, y) \leq \nabla v_1(x, y) + \nabla v_2(x, y)$. Lemma 3.7 implies

$$(\varphi_p(\nabla v_1(x, y)) + \varphi_p(\nabla v_2(x, y)))\nabla u(x, y) > 0.$$

If $x \in A$ and $y \notin A$, then similarly we have

$$(\varphi_p(\nabla v_1(x, y)) + \varphi_p(\nabla v_2(x, y)))\nabla u(x, y) > 0.$$

We obtain

$$\langle \varphi_p \circ \nabla v_1 + \varphi_p \circ \nabla v_2, \nabla u\rangle = \sum_{(x,y)\in E} r(x, y)(\varphi_p(\nabla v_1(x, y)) + \varphi_p(\nabla v_2(x, y)))\nabla u(x, y) \geq 0.$$

Hence $(\varphi_p(\nabla v_1(x, y)) + \varphi_p(\nabla v_2(x, y)))\nabla u(x, y) = 0$ for each $(x, y) \in E$. We have that $\nabla u(x, y) = 0$ for each $(x, y) \in E$, and that u is a constant in V. Since $\Gamma_0^p \neq \emptyset$ by Proposition 3.4 and $u^{*p} = 0$ on Γ_0^p, it follows that $u \equiv 0$, which means $v_1 + v_2 \leq 0$ in V as required. □

Corollary 3.9. *Let $u \in \mathbf{D}^p$ be a p-subharmonic function which is bounded above. Then* $\sup_V u = \sup_{\Gamma_0^p} u^{*p}$.

Proof. It is obvious that $\sup_V u \geq \sup_{\Gamma_0^p} u^{*p}$. We shall show the reverse inequality. We may assume $\alpha := \sup_{\Gamma_0^p} u^{*p} < \infty$. Since $(u - \alpha)^{*p} \leq 0$ on Γ_0^p, Lemma 3.8 implies that $u - \alpha \leq 0$ in V. □

Lemma 3.10. $\overline{\mathcal{H}}_f^p \geq \underline{\mathcal{H}}_f^p$ *in* V.

Proof. If either $\overline{\mathcal{H}}_f^p \equiv \infty$ or $\underline{\mathcal{H}}_f^p \equiv -\infty$, then there is nothing to prove. We may assume that $\mathcal{U}_f^p \neq \emptyset$ and $\mathcal{L}_f^p \neq \emptyset$. Let $u \in \mathcal{U}_f^p$ and $v \in \mathcal{L}_f^p$. Let $\xi \in \Gamma_0^p$. If $-\infty < f(\xi) < \infty$, then $v^{*p}(\xi) - u^{*p}(\xi) \leq f(\xi) - f(\xi) = 0$; if $f(\xi) = \infty$, then $u^{*p}(\xi) = \infty$ and, since v is bounded above, $v^{*p}(\xi) < \infty$, so that $v^{*p}(\xi) - u^{*p}(\xi) = -\infty < 0$; if $f(\xi) = -\infty$, then similarly $v^{*p}(\xi) - u^{*p}(\xi) = -\infty < 0$. Therefore Lemma 3.8 shows that $v - u \leq 0$ in V, and we have the assertion. □

We introduce the *p-Royden decomposition* of a function in $\mathbf{BD}^p$.

Lemma 3.11. *Let $u \in \mathbf{BD}^p$. Then there exist a unique function $v_0 \in \mathbf{BD}_0^p$ and a unique p-harmonic function $h \in \mathbf{BD}^p$ such that $u = v_0 + h$. Moreover $D_p[h] \leq D_p[u]$.*

Proof. Yamasaki showed in the proof of [5, Theorem 2.1] that, for $u \in \mathbf{D}^p$, there exists a unique function $v_0 \in \mathbf{D}_0^p$ such that $D_p[u - v_0] = \inf\{D_p[u - v]; v \in \mathbf{D}_0^p\}$ and that $h := u - v_0$ is p-harmonic. Therefore $D_p[h] = D_p[u - v_0] \leq D_p[u - 0] = D_p[u]$. Yamasaki [6, Theorem 3.2] showed that, if u is bounded, then both v_0 and h are also bounded. □

A function f is said to be *p-resolutive* if $\overline{\mathcal{H}}_f^p = \underline{\mathcal{H}}_f^p \not\equiv \pm\infty$. In this case we simply write $\mathcal{H}_f^p := \overline{\mathcal{H}}_f^p = \underline{\mathcal{H}}_f^p$ and call it the *p-Dirichlet solution*. Let

$$\mathcal{F}^p = \{u^{*p}|_{\Gamma^p}; u \in \mathbf{BD}^p\}.$$

Theorem 3.12. *Every function $f \in \mathcal{F}^p$ is p-resolutive. Moreover $\mathcal{H}_f^p \in \mathbf{BD}^p$ and $(\mathcal{H}_f^p)^{*p} = f$ on Γ_0^p.*

Proof. We take $v \in \mathbf{BD}^p$ such that $v^{*p}|_{\Gamma^p} = f$. Lemma 3.11 shows that there exist $v_0 \in \mathbf{BD}_0^p$ and a p-harmonic function $h \in \mathbf{BD}^p$ such that $v = v_0 + h$. Lemma 3.5 implies that $v_0^{*p} = 0$ on Γ_0^p, and that $h^{*p} = v^{*p} = f$ on Γ_0^p. This means that $h \in \mathcal{U}_f^p \cap \mathcal{L}_f^p$, and that $\overline{\mathcal{H}}_f^p \leq h$ and $\underline{\mathcal{H}}_f^p \geq h$. By Lemma 3.10 we have $\overline{\mathcal{H}}_f^p = \underline{\mathcal{H}}_f^p = h \in \mathbf{BD}^p$. Therefore $(\mathcal{H}_f^p)^{*p} = h^{*p} = f$ on Γ_0^p. □

We denote by $\mathcal{C}^p$ the set of continuous functions on Γ^p.

Lemma 3.13. *$\mathcal{F}^p$ is dense in $\mathcal{C}^p$ with respect to the supremum norm.*

Proof. By means of the Stone-Weierstrass Theorem [1, Theorem 7.29] we have only to show that $f_1 \vee f_2 \in \mathcal{F}^p$ whenever $f_1, f_2 \in \mathcal{F}^p$. Let $u_i \in \mathbf{BD}^p$ be such that $f_i = u_i^{*p}|_{\Gamma^p}$ for $i = 1, 2$. Let $v = u_1 \vee u_2$. We shall show that $|\nabla v| \leq |\nabla u_1| \vee |\nabla u_2|$ for each edge. Let $(x, y) \in E$. We may assume that $v(y) \geq v(x)$ and that $v(y) = u_1(y)$. Since $v(x) \geq u_1(x)$, it follows that $0 \leq \nabla v(x, y) \leq \nabla u_1(x, y)$, so that $|\nabla v(x, y)| \leq |\nabla u_1(x, y)| \vee |\nabla u_2(x, y)|$, hence the claim. Therefore $|\nabla v|^p \leq |\nabla u_1|^p \vee |\nabla u_2|^p \leq |\nabla u_1|^p + |\nabla u_2|^p$, so that $v \in \mathbf{BD}^p$. This implies $f_1 \vee f_2 = (u_1 \vee u_2)^{*p}|_{\Gamma^p} = v^{*p}|_{\Gamma^p} \in \mathcal{F}^p$ as required. □

Theorem 3.14. *Every function $f \in \mathcal{C}^p$ is p-resolutive. Moreover $\lim_{x\to\xi} \mathcal{H}_f^p(x) = f(\xi)$ for $\xi \in \Gamma_0^p$.*

Proof. Let $\varepsilon > 0$. Lemma 3.13 shows that there exists $g \in \mathcal{F}^p$ such that $|f - g| < \varepsilon$. Then

$$\mathcal{H}_g^p - \varepsilon = \underline{\mathcal{H}}_{g-\varepsilon}^p \leq \underline{\mathcal{H}}_f^p \leq \overline{\mathcal{H}}_f^p \leq \overline{\mathcal{H}}_{g+\varepsilon}^p = \mathcal{H}_g^p + \varepsilon.$$

Since ε is arbitrary, we have $\overline{\mathcal{H}}_f^p = \underline{\mathcal{H}}_f^p \not\equiv \pm\infty$. Combining the inequality above and Theorem 3.12 we have

$$f(\xi) - 2\varepsilon \leq g(\xi) - \varepsilon \leq \liminf_{x\to\xi} \mathcal{H}_f^p(x) \leq \limsup_{x\to\xi} \mathcal{H}_f^p(x) \leq g(\xi) + \varepsilon \leq f(\xi) + 2\varepsilon$$

for $\xi \in \Gamma_0^p$, and therefore $\lim_{x\to\xi} \mathcal{H}_f^p(x) = f(\xi)$. □

Theorem 3.14 means that every point in Γ_0^p is regular for the Dirichlet problem. As a converse we have the following.

Theorem 3.15. *For every point $\xi \in \Gamma^p \setminus \Gamma_0^p$, there exists a continuous function f on Γ^p such that $\lim_{x\to\xi} \mathcal{H}_f^p(x) \neq f(\xi)$.*

Proof. The definition of Γ_0^p shows that there is $u \in \mathbf{BD}_0^p$ such that $u^{*p}(\xi) \neq 0$. Let $f = u^{*p}|_{\Gamma^p}$. Then $f \in \mathcal{F}^p \subset \mathcal{C}^p$ and $f(\xi) = u^{*p}(\xi) \neq 0$. On the other hand Lemma 3.5 implies $f = 0$ on Γ_0^p. This means that the constant function 0 is in $\mathcal{U}_f^p \cap \mathcal{L}_f^p$. Lemma 3.10 implies that $\mathcal{H}_f^p \equiv 0$. Hence $\lim_{x\to\xi} \mathcal{H}_f^p(x) = 0$. □

4. A Continuity of p-Dirichlet Solutions with Respect to p

In this section we suppose the condition that

$$\text{there exists a constant } r_0 > 0 \text{ such that } r \geq r_0 \text{ on } E \tag{A}$$

unless otherwise mentioned.

Lemma 4.1. *Let $1 < p_1 < p_2 < \infty$. Then*
(i) $\mathbf{D}^{p_1} \subset \mathbf{D}^{p_2}$;
(ii) $\mathbf{D}_0^{p_1} \subset \mathbf{D}_0^{p_2}$;
(iii) $\mathbf{BD}^{p_1} \subset \mathbf{BD}^{p_2}$;
(iv) $\mathbf{BD}_0^{p_1} \subset \mathbf{BD}_0^{p_2}$.

Proof. Let $u \in \mathbf{D}^{p_1}$. Since $r_0|\nabla u(e)|^{p_1} \leq r(e)|\nabla u(e)|^{p_1} \leq D_{p_1}[u]$ for every $e \in E$, it follows that there is a constant M such that $|\nabla u| \leq M$ on E. Then

$$D_{p_2}[u] = \sum_{e\in E} r(e)|\nabla u(e)|^{p_2} \leq M^{p_2-p_1} \sum_{e\in E} r(e)|\nabla u(e)|^{p_1} = M^{p_2-p_1} D_{p_1}[u].$$

This shows (i) and (iii).

Further assume that $u \in \mathbf{D}_0^{p_1}$. Let $\{v_m\}_m \subset L_0(V)$ be such that $\lim_{m\to\infty} D_{p_1}[u - v_m] = 0$. Since $r_0|\nabla u(e) - \nabla v_m(e)|^{p_1} \leq D_{p_1}[u - v_m]$, we find a constant M such that $|\nabla u - \nabla v_m| \leq M$ on E for all m. Then $D_{p_2}[u - v_m] \leq M^{p_2-p_1} D_{p_1}[u - v_m]$, so that (ii) and (iv) follow. □

Lemma 4.1 does not hold without Condition (A).

Example 4.2. Let $V = \{x_n\}_{n=1}^{\infty}$, $E = \{(x_{n-1}, x_n)\}_{n=2}^{\infty}$ and $r(x_{n-1}, x_n) = n^{-3}$. Let $u(x_n) = \sum_{k=1}^{n} k^{-2}$. Then $\nabla u(x_{n-1}, x_n) = n$, and $D_p[u] = \sum_{n=2}^{\infty} n^{p-3}$, which converges if and only if $p < 2$.

Lemma 4.3. *Let $1 < p_1 < p_2 < \infty$. Let $\xi \in \Gamma^{p_2}$. Let $\{x_n\}_n$ be a sequence in V which converges to ξ in V^{*p_2}. Then $\{x_n\}_n$ converges to a point $\eta \in \Gamma^{p_1}$ in V^{*p_1}. Moreover η is independent of the choice of $\{x_n\}_n$.*

Proof. The sequence $\{x_n\}_n$ has at least one accumulation point in V^{*p_1}. Let $\eta_1, \eta_2 \in \Gamma^{p_1}$ be two accumulation points of $\{x_n\}_n$ in V^{*p_1}. We take a subsequence $\{x_{j,n}\}_n \subset \{x_n\}_n$ which converges to η_j in V^{*p_1} for $j = 1, 2$. Let $u \in \mathbf{BD}^{p_1}$. Then $\lim_{n\to\infty} u(x_{j,n}) = u^{*p_1}(\eta_j)$. Since $u \in \mathbf{BD}^{p_2}$ by Lemma 4.1, we have $\lim_{n\to\infty} u(x_n) = u^{*p_2}(\xi)$, and so $u^{*p_1}(\eta_1) = u^{*p_2}(\xi) = u^{*p_1}(\eta_2)$. Since $\mathcal{F}^{p_1}$ separates Γ^{p_1}, it follows that $\eta_1 = \eta_2$. This means that $\{x_n\}_n$ has a unique accumulation point in V^{*p_1}, so that $\{x_n\}_n$ converges to that point.

Let $\{x_n\}_n$ and $\{y_n\}_n$ be two sequences in V which converge to ξ in V^{*p_2}. We put the sequence $\{z_n\}_n = \{x_1, y_1, x_2, y_2, \ldots\}$, which also converges to ξ in V^{*p_2}. Then $\{z_n\}_n$ converges to a point $\eta \in \Gamma^{p_1}$ in V^{*p_1}, and hence $\{x_n\}_n$ and $\{y_n\}_n$ also converge to η in V^{*p_1}. □

For $\xi \in \Gamma^{p_2}$ we denote by $\Psi_{p_1,p_2}(\xi)$ the point $\eta \in \Gamma^{p_1}$ found in Lemma 4.3. For simplicity we let $\Psi_{p,p}$ be the identity map on Γ^p. We note that

$$\Psi_{p_1,p_2} \circ \Psi_{p_2,p_3} = \Psi_{p_1,p_3} \quad \text{on } \Gamma^{p_3},$$

for $1 < p_1 < p_2 < p_3 < \infty$.

Lemma 4.4. *Let $1 < p_1 < p_2 < \infty$. For $u \in \mathbf{BD}^{p_1}$*

$$u^{*p_1} \circ \Psi_{p_1,p_2} = u^{*p_2} \quad \text{on } \Gamma^{p_2}.$$

Proof. Let $\xi \in \Gamma^{p_2}$ and let $\{x_n\}_n \subset V$ be a sequence converging to ξ in V^{*p_2}. Then $\{x_n\}_n$ converges to $\Psi_{p_1,p_2}(\xi)$ in V^{*p_1}. Therefore $u^{*p_1} \circ \Psi_{p_1,p_2}(\xi) = \lim_{n\to\infty} u(x_n) = u^{*p_2}(\xi)$. □

Lemma 4.5. *Let $1 < p_1 < p_2 < \infty$. Then*
(i) $\Gamma^{p_1} = \Psi_{p_1,p_2}(\Gamma^{p_2})$;
(ii) $\Gamma_0^{p_1} \supset \Psi_{p_1,p_2}(\Gamma_0^{p_2})$.

Proof. It is obvious that $\Gamma^{p_1} \supset \Psi_{p_1,p_2}(\Gamma^{p_2})$. We shall show the converse. Let $\eta \in \Gamma^{p_1}$. Let $\{x_n\}_n$ be a sequence in V which converges to η in V^{*p_1}. By taking a subsequence if necessary, we may assume that $\{x_n\}_n$ converges to a point $\xi \in \Gamma^{p_2}$ in V^{*p_2}. Then $\eta = \Psi_{p_1,p_2}(\xi)$, and (i) holds.

Let $\xi \in \Gamma_0^{p_2}$ and $u \in \mathbf{BD}_0^{p_1}$. Since $u \in \mathbf{BD}_0^{p_2}$ by Lemma 4.1, using Lemma 4.4 and the definition of $\Gamma_0^{p_2}$, we have $u^{*p_1} \circ \Psi_{p_1,p_2}(\xi) = u^{*p_2}(\xi) = 0$. Lemma 3.5 implies that $\Psi_{p_1,p_2}(\xi) \in \Gamma_0^{p_1}$, and that (ii) holds. □

Lemma 4.6. *Let $1 < p_1 < p_2 < \infty$. Then $\mathcal{F}^{p_1} \circ \Psi_{p_1,p_2} \subset \mathcal{F}^{p_2}$ and $\mathcal{C}^{p_1} \circ \Psi_{p_1,p_2} \subset \mathcal{C}^{p_2}$.*

Proof. First let $f \in \mathcal{F}^{p_1}$. Let $u \in \mathbf{BD}^{p_1}$ be such that $u^{*p_1}|_{\Gamma^{p_1}} = f$. Lemma 4.4 shows that $u^{*p_2} = u^{*p_1} \circ \Psi_{p_1,p_2} = f \circ \Psi_{p_1,p_2}$ on Γ^{p_2}. Since $u \in \mathbf{BD}^{p_2}$ by Lemma 4.1, we have $f \circ \Psi_{p_1,p_2} \in \mathcal{F}^{p_2}$.

Now let $f \in \mathcal{C}^{p_1}$. Let $\{f_n\}_n \subset \mathcal{F}^{p_1}$ be a sequence which converges uniformly to f on Γ^{p_1}. Then $f_n \circ \Psi_{p_1,p_2} \in \mathcal{F}^{p_2}$ and $\{f_n \circ \Psi_{p_1,p_2}\}_n$ converges uniformly to $f \circ \Psi_{p_1,p_2}$ on Γ^{p_2}. Therefore $f \circ \Psi_{p_1,p_2} \in \mathcal{C}^{p_2}$. □

Let

$$\operatorname{ind} \mathcal{N} = \sup\{p > 1;\ \mathcal{N} \text{ is of } p\text{-hyperbolic type}\},$$

which is called the *parabolic index* of $\mathcal{N}$. Note that $\mathcal{N}$ is of p-hyperbolic type for $1 < p < \operatorname{ind}\mathcal{N}$ and that $\mathcal{N}$ is of p-parabolic type for $\operatorname{ind}\mathcal{N} < p < \infty$. See Yamasaki [4].

Theorem 4.7. *Let $1 < p_0 < \operatorname{ind}\mathcal{N}$ and let $f \in \mathcal{C}^{p_0}$. Then $\mathcal{H}^p_{f\circ\Psi_{p_0,p}}(x)$ is a left continuous function of p in the interval $p_0 < p < \operatorname{ind}\mathcal{N}$ for each $x \in V$.*

Proof. Lemma 4.6 shows that $f \circ \Psi_{p_0,p} \in \mathcal{C}^p$, and that $\mathcal{H}^p_{f\circ\Psi_{p_0,p}}$ is well-defined.

Let $\{p_n\}_n$ be a sequence with $p_0 < p_n \le p$ and $p_n \to p$. First assume $f \in \mathcal{F}^{p_0}$. We let $h_n = \mathcal{H}^{p_n}_{f\circ\Psi_{p_0,p_n}}$ and $h = \mathcal{H}^p_{f\circ\Psi_{p_0,p}}$. Since f is bounded, it follows that the sequence $\{f \circ \Psi_{p_0,p_n}\}_n$ is uniformly bounded, and so is $\{h_n\}_n$. We take a subsequence of $\{h_n\}_n$ converging pointwise to a function u on V. Hence it suffices to show that $u = h$ whenever $\{h_n\}_n$ converges pointwise to u on V.

Let $\xi \in \Gamma_0^p$. Theorem 3.12 shows that $h_n \in \mathbf{BD}^{p_n}$ and that $h_n^{*p_n} = f \circ \Psi_{p_0,p_n}$ on $\Gamma_0^{p_n}$. Since $\Psi_{p_n,p}(\xi) \in \Gamma_0^{p_n}$ by Lemma 4.5 (ii), using Lemma 4.4 we have

$$h_n^{*p}(\xi) = h_n^{*p_n} \circ \Psi_{p_n,p}(\xi) = f \circ \Psi_{p_0,p_n} \circ \Psi_{p_n,p}(\xi) = f \circ \Psi_{p_0,p}(\xi).$$

Theorem 3.12 shows that $h \in \mathbf{BD}^p$ and $h^{*p}(\xi) = f \circ \Psi_{p_0,p}(\xi)$. Since $h_n \in \mathbf{BD}^{p_n} \subset \mathbf{BD}^p$ and $h_n^{*p} - h^{*p} \equiv 0$ on Γ_0^p, it follows that $h_n - h \in \mathbf{BD}_0^p$ by Lemma 3.5.

A similar argument shows that $h_0 := \mathcal{H}_f^{p_0}$ satisfies that $h_0^{*p_n} = f \circ \Psi_{p_0,p_n}$ on $\Gamma_0^{p_n}$ and that $h_0 - h_n \in \mathbf{BD}_0^{p_n}$. The p_n-harmonicity of h_n implies that $h_0 = (h_0 - h_n) + h_n$ is the p_n-Royden decomposition. Lemma 3.11 shows $D_{p_n}[h_n] \le D_{p_n}[h_0]$. Since $|h_n|$ is p_n-subharmonic, Corollary 3.9 and Lemma 4.5 (ii) show that

$$\begin{aligned}|\nabla h_n(x,y)| = \frac{|h_n(y) - h_n(x)|}{r(x,y)} &\le 2r_0^{-1}\sup_V |h_n| = 2r_0^{-1}\sup_{\Gamma_0^{p_n}} |h_n^{*p_n}| \\ &= 2r_0^{-1}\sup_{\Gamma_0^{p_n}} |f \circ \Psi_{p_0,p_n}| = 2r_0^{-1}\sup_{\Psi_{p_0,p_n}(\Gamma_0^{p_n})} |f| \le 2r_0^{-1}\sup_{\Gamma_0^{p_0}} |f|\end{aligned}$$

for $(x,y) \in E$. We find a constant M such that $|\nabla h_n| \le M$ on E for all n. Then

$$D_p[h_n] \le M^{p-p_n} D_{p_n}[h_n] \le M^{p-p_n} D_{p_n}[h_0] \le M^{p-p_n} M^{p_n-p_0} D_{p_0}[h_0] = M^{p-p_0} D_{p_0}[h_0].$$

This implies that $\{h_n - h\}_n$ is a sequence in $\mathbf{D}_0^p$ which converges pointwise to $u - h$ and satisfies that $\{D_p[h_n - h]\}_n$ is bounded. Hence, by Yamasaki [6, Theorem 4.1], we have $u - h \in \mathbf{D}_0^p$. It is easy to see that $\lim_{n\to\infty} \Delta_{p_n} h_n(x) = \Delta_p u(x)$, so that u is p-harmonic. Hence $u = h$.

Now let $f \in \mathcal{C}^{p_0}$. For $\varepsilon > 0$ we find $g \in \mathcal{F}^{p_0}$ such that $|f - g| \le \varepsilon$ on Γ^{p_0}. Since $|f \circ \Psi_{p_0,p_n} - g \circ \Psi_{p_0,p_n}| \le \varepsilon$ on Γ^{p_n}, we have $|\mathcal{H}^{p_n}_{f\circ\Psi_{p_0,p_n}} - \mathcal{H}^{p_n}_{g\circ\Psi_{p_0,p_n}}| \le \varepsilon$ in V. Similarly we have $|\mathcal{H}^{p}_{f\circ\Psi_{p_0,p}} - \mathcal{H}^{p}_{g\circ\Psi_{p_0,p}}| \le \varepsilon$ in V. Then

$$|\mathcal{H}^{p_n}_{f\circ\Psi_{p_0,p_n}}(x) - \mathcal{H}^{p}_{f\circ\Psi_{p_0,p}}(x)| \le 2\varepsilon + |\mathcal{H}^{p_n}_{g\circ\Psi_{p_0,p_n}}(x) - \mathcal{H}^{p}_{g\circ\Psi_{p_0,p}}(x)|.$$

Therefore the first part shows that $\mathcal{H}^{p_n}_{f\circ\Psi_{p_0,p_n}}(x) \to \mathcal{H}^{p}_{f\circ\Psi_{p_0,p}}(x)$ as $n \to \infty$. □

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