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THE LIOUVILLE THEOREM FOR HOMOGENEOUS ELLIPTIC DIFFERENTIAL INEQUALITIES

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Dedicated to Nicolai Krylov with regards of the author

We give a new proof method for various similar Liouville type theorems. This allows us to extend in some cases the results of E. Mitidieri and S. I. Pohozaev and of R. Filippucci. Bibliography: 8 *titles.*

We consider Liouville type theorems for entire solutions of homogeneous elliptic differential inequalities of the form

$$\operatorname{div}\{\mathscr{A}(|Du|)Du\} \leqslant 0 \quad \text{in } \mathbb{R}^n. \tag{1}$$

For inequalities of this type, a number of results are presently known, for example, a nonnegative entire solution $u = u(x)$ of the harmonic inequality

$$\Delta u \leqslant 0 \quad \text{in } \mathbb{R}^2$$

must be constant [1, p. 115]. A generalization to the p-Laplacian operator also holds, i.e., a nonnegative entire solution of

$$\Delta_p u \equiv \operatorname{div}\{|Du|^{p-2}Du\} \leqslant 0 \quad \text{in } \mathbb{R}^n$$

again must be constant, provided now $p \geqslant n$. For this result, see Serrin and Zou [2] and, independently (cf. received dates), Mitidieri and Pohozaev [3]. These results were generalized by Mitidieri and Pohozaev [4, 3]; they proved two main results. For the first, if

$$0 < \mathscr{A}(t) < C \tag{2}$$

for some positive constant C and for all $t \geqslant 0$, then any entire solution of (1) which is bounded from below must be constant provided that $n = 2$. (Also for $n = 1$, but this is trivial.) The most important example for this result is the mean curvature inequality

$$\operatorname{div}\frac{Du}{\sqrt{1+|Du|^2}} \leqslant 0 \quad \text{in } \mathbb{R}^2.$$

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For the second result, they allow a more general version of (1), in which $\mathscr{A}$ is now allowed to depend on the complete set of variables x, u, Du. Their main result is that if there exist positive constants c_1 and c_2 such that

$$c_1 s^{p-2} \leqslant \mathscr{A}(x, u, s) \leqslant c_2 s^{p-2}, \quad p > 1, \tag{3}$$

then any entire solution of the inequality

$$\operatorname{div}\{\mathscr{A}(x, u, Du)Du\} \leqslant 0 \quad \text{in } \mathbb{R}^n$$

which is bounded from below must be constant provided that $n \leqslant p$. This result obviously includes the earlier result for the p-Laplace inequality.

Moreover, in all the cases above, the restriction on the dimension n is necessary (cf. Section 5), as is also the direction of the inequality in (1).

More recently, Filippucci [5, Theorem 3.3] extended the results of Mitidieri and Pohozaev, by considering the inequality

$$\operatorname{div}\{h(x)\ g(u)\ \mathscr{A}(|Du|)Du\} \leqslant 0 \quad \text{in } \mathbb{R}^n$$

under appropriate conditions on the functions g and h, and assuming also for $t > 0$

$$c_1 t^{p-1} \leqslant t\mathscr{A}(t) \leqslant c_2 t^{p-1}, \quad p > 1.$$

Another result of interest, in which the principal operator is a sum of p-Laplacian operators of different degree, has been obtained by D'Ambrosio [6].

The purpose of this paper is to give a new proof method for various similar Liouville type theorems. This allows us to extend in some cases the results of Mitidieri and Pohozaev, and of Filippucci (cf. Theorems 1 and 2 below). For example, even for the harmonic inequality $\Delta u \leqslant 0$ we obtain the following new result.

For the harmonic inequality $\Delta u \leqslant 0$ the domain of u in the statement of Theorem 1 can omit any finite set of isolated points, without affecting the conclusion, provided that $n = 2$.

In the sequel, we shall specifically consider the inequality (1) with the main operator $\mathscr{A}$ having the special form $\mathscr{A}(t) = t^{p-2}A(t)$, $t > 0$. We require only that

(i) $p > 1$

(ii) $A \in C([0, \infty))$ with A(0) > 0

(iii) $t^{p-1}A(t)$ is strictly increasing for $t > 0$.

The Laplace case $A(t) \equiv 1$, $p = 2$; the p-Laplacian, where $p > 1$, $A(t) \equiv 1$; and the mean curvature case $p = 2$, $A(t) = 1/\sqrt{1 + |t|^2}$, obviously satisfy these conditions, where for the mean curvature case condition (iii) follows from

$$\frac{d}{dt}\frac{t}{\sqrt{1+t^2}} = \frac{1}{(1+t^2)^{3/2}} > 0.$$

Note also that if $A \in C^1(\mathbb{R}_+$ then (iii) is equivalent to the condition $p - 1 + tA_t/A > 0$ for $t > 0$. We prove the following result.

Theorem 1. *Let $u = u(x) \in C^1(\mathbb{R}^n)$ be an entire solution of the differential inequality* (1), *which is uniformly bounded from below. If the function $\mathscr{A}(t)$ $(= t^{p-1}A(t))$ satisfies conditions* (i)–(iii) *and $p \geqslant n$, then u is constant.*

When the function $A(t)$ is continuously differentiable, condition (iii) can be dropped. More precisely, the following result holds.

Theorem 2. *Let $A(t) \in C^1([0, \infty])$ with $A(t) > 0$ for $t \geqslant 0$. If u is an entire solution of the inequality* (1) *which is bounded from below, then $u \equiv$ constant provided that $p \geqslant n$.*

It is obvious that the p-Laplace operator and the mean curvature operator above are covered by Theorem 2, as well as by Theorem 1.

Finally, a surprising removable singularity type conclusion also holds, for example:

The domain of u in Theorem 1 *can omit any finite set of isolated points, without affecting the conclusion, provided that the function $A(t)$ satisfies the additional condition*

(iv) $A(t)$ is bounded from above and also bounded below for all $t \geqslant 0$ respectively by positive constants $K > 0$ and $\delta > 0$.

See Theorem 3 in Section 4. Note that condition (iv) is obviously satisfied by the Laplace operator and the p-Laplace operator (with $A(t) \equiv 1$). We prove Theorem 1 in Section 2, Theorem 2 in Section 3, and Theorem 3 in Section 4. We give some counterexamples in Section 5. The relative simplicity of the proofs seems remarkable.

Remark. The case $p = 2$ of Theorem 1 extends the first result of Mitidieri and Pohozaev in that condition (2) is replaced by conditions (ii) and (iii). An explicit example where (2) fails, but (ii) and (iii) hold is $\mathscr{A}(t) = A(t) = \ln(2 + t)$, $t \geqslant 0$. Theorem 1 also extends their second result by replacing condition (3) with conditions (ii) and (iii). Again, an example where (3) fails, but (ii) and (iii) hold is $A(t) = \ln(2 + t)$. It almost goes without saying that Theorem 2 provides similar extensions of the results of Mitidieri and Pohozaev.

1. Preliminaries.

The proofs of Theorem 1 and 2 depend on the construction of a special logarithmic type radial solution $v = v(r)$ of the reverse differential inequality

$$\operatorname{div}\{\mathscr{A}(|Dv|)Dv\} \geqslant 0. \tag{1.1}$$

We carry out this construction in the present section. First, we modify $A(t)$ by replacing it with the function

$$A_1(t) = \begin{cases} A(t), & 0 \leqslant t \leqslant 1, \\ A(1), & t \geqslant 1, \end{cases}$$

and, to begin with, consider the inequality

$$\operatorname{div}\{\mathscr{A}_1(|Dv|)Dv\} \geqslant 0, \tag{1.2}$$

where $\mathscr{A}_1(t) = t^{p-2}A_1(t)$. Choosing an arbitrary origin 0 and writing $v = v(r)$, with $r = |x|$ the radial distance from a point x to 0, an easy calculation shows that v satisfies the ordinary differential inequality

$$\frac{d}{dr}\{\mathscr{A}_1(|v'|)v'\} + \frac{n-1}{r}\{\mathscr{A}_1(|v'|)v'\} \geqslant 0. \tag{1.3}$$

Let $0 < R < S < \infty$, and let $m > 0$. We seek specifically a solution of (1.3) for $R \leqslant r \leqslant S$ under the end conditions

$$v(R) = m, \quad v(S) = 0 \quad (v'(r) < 0). \tag{1.4}$$

In fact, since $p \geqslant n$, it is clearly enough that v satisfies the ordinary differential equation

$$\frac{d}{dr}\{\mathscr{A}_1(|v'|v')\} + \frac{p-1}{r}\mathscr{A}_1(|v'|)v' = 0. \tag{1.5}$$

To show the existence of a (unique) solution of (1.4), (1. 5), note first that (1.5) can be written equivalently as

$$\frac{d}{dr}\log\{\mathscr{A}_1(|v'|)|v'|\} + \frac{p-1}{r} = 0, \quad v' < 0. \tag{1.6}$$

Integrating and forming the exponential, we get

$$\mathscr{A}_1(|v'|)|v'| = \frac{a}{r^{p-1}}, \tag{1.7}$$

where $a > 0$ is a constant of integration. By assumption (iii), this can be written as

$$\varphi(|v'|) = \frac{a}{r^{p-1}},$$

where $\varphi(t) = t^{p-1}A_1(t)$ is a strictly increasing function for $t > 0$, approaching ∞ as $t \to \infty$. Letting $\varphi^{-1}(t)$, $0 \leqslant t < \infty$, be the inverse of $\varphi(t)$ gives

$$|v'| = \varphi^{-1}\left(\frac{a}{r^{p-1}}\right). \tag{1.8}$$

Thus, by the second condition of (1.4),

$$v(r) = \int\limits_r^S \varphi^{-1}\left(\frac{a}{\tau^{p-1}}\right) d\tau \qquad \text{for} \quad R \leqslant r \leqslant S. \tag{1.9}$$

Putting $r = R$ in (1.9) and using (1.4), we obtain the following main condition on the integration constant a:

$$m = \int\limits_R^S \varphi^{-1}\left(\frac{a}{\tau^{p-1}}\right) d\tau.$$

We assert that this (uniquely) determines $a = a(m, S)$ for R fixed. Indeed, the right-hand side vanishes when $a = 0$, is increasing in a, and, since $\varphi^{-1}(\infty)$ is necessarily ∞, becomes infinite as a approaches infinity, which proves the assertion. In summary, the function $v(r) = v(r; m, S)$ given by (1.9) with $a = a(m, S)$, is a solution of (1.4), (1.5). We now discuss further properties of the solution (1.5). Since $A_1(0) > 0$ and $A_1(t) \equiv A(1) > 0$ for $t \geqslant 1$, it follows that A_1 has a positive upper bound K for all $t \geqslant 0$. Hence, by (1.7), after forming the $1/(p-1)$ root, we find that

$$|v'| \geqslant \left(\frac{a}{K}\right)^{1/(p-1)} \frac{1}{r}, \qquad a = a(m, S). \tag{1.10}$$

Integration from R to S leads to

$$m \geqslant \left(\frac{a}{K}\right)^{1/(p-1)} \log\frac{S}{R}.$$

Hence

$$a \leqslant m^{p-1} K / (\log \left(\frac{S}{R}\right)^{p-1}, \tag{1.11}$$

providing an upper bound for $a = a(m, S)$. In turn, by (1.8),

$$|v'(r)| \leqslant \varphi^{-1}\left(\frac{m^{p-1} K}{(R \log S/R)^{p-1}}\right), \qquad R \leqslant r \leqslant S. \tag{1.12}$$

Then if S *is suitably large*, say $S \geqslant Re^{mK^{1/(p-1}/RA(1)^{1/(p-1)}} = \sigma(m)$, it follows that $|v'(r)| \leqslant \varphi^{-1}(A_1(1)) = 1$. Thus, $\mathscr{A}_1(|v'|) = \mathscr{A}(|v'|)$ and *the function* v *given by* (1.9) *satisfies the* (*reversed*) *inequality* (1.1). Next, by the integration of (1.10) from r to S and use of (1.4),

$$v(r) \geqslant \left(\frac{a}{K}\right)^{1/(p-1)} \log \frac{S}{r}, \quad a = a(s, m). \tag{1.13}$$

On the other hand, corresponding to (1.10), we also have

$$|v'| \leqslant \left(\frac{a}{\delta}\right)^{1/(p-1)} \frac{1}{r},$$

where δ is a positive lower bound for $A_1(t)$, $t > 0$. Thus,

$$v(r) \leqslant \left(\frac{a}{\delta}\right)^{1/(p-1)} \log \frac{S}{r}.$$

Putting $r = R$, we get

$$m \leqslant \left(\frac{a}{\delta}\right)^{1/(p-1)} \log \frac{S}{R}. \tag{1.14}$$

Then eliminating a between (1.13) and (1.14) gives

$$v(r) \geqslant m \left(\frac{\delta}{K}\right)^{1/(p-1)} \frac{\log S/r}{\log S/R}, \quad R \leqslant r \leqslant S. \tag{1.15}$$

2. Proof of Theorem 1. Since the given solution $u = u(x)$ of (1) is bounded from below, we can let $M = \inf_{\mathbb{R}^n} u(x) > -\infty$. Thus, $\overline{u}(x) = u(x) - M$ has infimum 0 and obviously is also a $C^1(\mathbb{R}^n)$ solution of (1) with greatest lower bound 0. We suppose to begin with that $\overline{u} > 0$ in $\mathbb{R}^n$, for *otherwise* we should have $\overline{u}(x_0) = 0$ at some x_0 in $\mathbb{R}^n$. But then, by the strong maximum principle (cf. [7, Theorem 1.1]), we get

$$\overline{u}(x) \equiv 0 \quad \text{in } \mathbb{R}^n,$$

the required conclusion, i.e., $u(x) \equiv M$.

With $\overline{u}(x) > 0$, now let R be fixed, and choose

$$m = m(R) = \inf_{|x|=R} u(x) > 0 \tag{2.1}$$

The function $v(r)$, $R \leqslant r \leqslant S$, then satisfies the inequality (1.1) when $S \geqslant \sigma(m)$, while $\overline{u}$ satisfies (1). We claim now that

$$\overline{u}(x) \geqslant v(|x|) \qquad \text{for } R \leqslant |x| \leqslant S. \tag{2.2}$$

Assuming for the moment that this claim is valid, we can complete the proof. Indeed, taking $|x| = r = \sqrt{RS}$ in (1.15), we obtain the remarkable inequality

$$\overline{u}(x) \geqslant m\left(\frac{\delta}{K}\right)^{1/(p-1)} \frac{\log S/\sqrt{RS}}{\log S/R} = \frac{1}{2}m\left(\frac{\delta}{K}\right)^{1/(p-1)}. \tag{2.3}$$

Since (2.3) is valid for all $S \geqslant \sigma(m)$, it now follows (!) that

$$\inf_{|x|\to\infty} \overline{u}(x) \geqslant \frac{1}{2}m\left(\frac{\delta}{K}\right)^{1/(p-1)}. \tag{2.4}$$

But $\overline{u} > 0$ in $\mathbb{R}^n$ and $\inf_{\mathbb{R}^n} \overline{u} = 0$, i.e., necessarily $\inf_{|x|\to\infty} \overline{u}(x) = 0$. This contradicts (2.4) and completes the proof of Theorem 1.

Appendix to Section 3. Proof of the claim above. Suppose for contradiction that the claim is false. Then there exists a point x_0 with $R < |x_0| < S$ such that $\overline{u}(x_0) < v(|x_0|)$, while also $\overline{u} \geqslant v$ when $|x| = R$ and $|x| = S$. By adding an appropriate constant $c > 0$ to $\overline{u}$, say $\overline{\overline{u}} = \overline{u} + c$, it can then be arranged that

$$\overline{\overline{u}}(x) \geqslant v(|x|)$$

for all $R \leqslant |x| \leqslant S$, and there is a point x_1 with $R < |x_1| < S$ such that $\overline{\overline{u}}(x_1) = v(|x_1|)$.

Now, let $\varepsilon \in (0, c)$ and define

$$\Gamma = \{x \in \mathbb{R}^n : \ R \leqslant |x| \leqslant S, \overline{\overline{u}}(x) - \varepsilon < v(|x|)\}. \tag{2.5}$$

It is clear that $x_1 \in \Gamma$. Moreover, by (2.1) and (1.4),

$$\overline{\overline{u}} - \varepsilon > v \quad \text{when} \quad |x| = R \text{ and when } |x| = S.$$

Thus, Γ is an open nonempty precompact subset of $R < |x| < S$. Consider the test function

$$\phi = (v - \overline{\overline{u}} + \varepsilon)^+. \tag{2.6}$$

It is clear that $\phi \geqslant 0$, while ϕ vanishes outside Γ. Moreover, by [8, Lemma 7.6], $D\phi = Dv - Du$ in Γ and $D\phi = 0$ a.e. outside Γ.

Next, by (1) and (1.1), respectively,

$$\int \langle \mathscr{A}(|Du|)\ Du,\ D\phi\rangle \geqslant 0,$$

$$\int \langle \mathscr{A}(|Dv|)\ Dv,\ D\phi\rangle \leqslant 0.$$

By subtraction,

$$\int \langle \mathscr{A}(|Dv|)\ Dv - \mathscr{A}(|Du|)\ Du,\ D\phi\rangle \leqslant 0.$$

Therefore, in particular,

$$\int_\Gamma \langle \mathscr{A}(|Dv|)\ Dv - \mathscr{A}(|Du|)\ Du,\ D\phi\rangle \leqslant 0. \tag{2.7}$$

By [7, Theorem 2.4.1] together with the assumption (iii), it follows that the integrand in (2.7) is nonnegative. The integrand must therefore vanish a.e. in Γ. Therefore, by Theorem 2.4.1 again, we get

$$Dv = Du \quad \text{in } \Gamma.$$

But then, equivalently, $D\phi = 0$ in Γ, and in turn $\phi \equiv$ constant in Γ. In fact, the constant must be zero since $\phi = 0$ on $\partial\Gamma$, i.e., $\phi \equiv 0$ in Γ. Stated in other words,

$$v - \overline{\overline{u}} + \varepsilon \equiv 0 \quad \text{in } \Gamma$$

(cf. (2.6)). This contradicts the definition (2.5) of the (nonempty) set Γ and completes the proof of the claim. □

3. Proof of Theorem 2. Let $A(t) \leqslant C$ for $t \leqslant 1$, and let $c > 0$ be a constant such that $tA_t(t)/A < p-1$ (such a constant exists since $A(t) > 0$ for $t \geqslant 0$ and A is continuous.) It follows that $t^{p-1}A(t)$ is strictly increasing for $0 < t < c$. Now, let

$$A_c(t) = \begin{cases} A(c), & t \geqslant c, \\ A(t), & t \leqslant c. \end{cases}$$

It is clear that $t^{p-1}A_c(t)$ is then strictly increasing for all $t > 0$; moreover, there are constants $K > 0$ and $\delta > 0$ such that

$$\delta \leqslant A_c(t) \leqslant K, \quad t \geqslant 0.$$

We can now proceed exactly as in the proof of Theorem 1 to obtain Theorem 2 provided only that S is chosen so large that

$$|v'(r)| \leqslant c \quad \text{for } R \leqslant r \leqslant S.$$

Again this choice of S is possible by virtue of the gradient condition (1.12). This completes the proof. □

4. Proof of Theorems 3. We turn now to the removable singularity generalization of Theorem 1 noted in the introduction, which can be stated explicitly as the following assertion.

Theorem 3. *Let P be a finite set of isolated points in $\mathbb{R}^n$, and let $u \in C1(\mathbb{R}^n \backslash P)$ be a solution of the differential inequality* (1.1) *in $\mathbb{R}^n \backslash P$, which is uniformly bounded from below. If the function $\mathscr{A}(t) = t^{p-1}A(t)$ satisfies conditions* (i)–(iii) *in the introduction, condition* (iv) *also holds, and $p \geqslant n$, then $u \equiv 0$ in $\mathbb{R}^n \backslash P$.*

Proof. By condition (iv), there exist constants $K > 0$ and $\delta 0$ such that $\delta \leqslant A(t) \leqslant K$ for all $t \geqslant 0$. In consequence, there is no need at the beginning of the proof to modify $A(t)$ to the form $A_1(t)$, and so also no need later to make $|v'(r)| \leqslant 1$, as in the proof of Theorem 1. We now proceed as in the proof of Theorem 1. It is obvious that for each point $x_i \in P$, $i = 1, \cdots, k$, there is a positive constant m_i and a constant $R_i > 0$ such that

$$\overline{u}(x) \geqslant m_i$$

when $|x| = R_i$, where R_i is now so small that u is a solution of (1) in the set $B_{2R_i}\backslash\{x_i\}$. We can then assert that $\overline{u} = u - M$ obeys

$$\overline{u}(x) \geqslant m_i \left(\frac{\delta}{K}\right)^{1/(p-1)} \frac{\log |x|/S}{\log R/S}, \qquad S \leqslant |x| \leqslant R_i. \tag{4.1}$$

Indeed, the construction of Section 1 supplies a positive solution $v_i(r)$ in the set $0 < S \leqslant r \leqslant R_i$; (essentially, one replaces (1.4) by

$$v_i(R_i) = m_i,\ v_i(S) = 0, \qquad (v' > 0)$$

and otherwise proceeds almost exactly as before.) Here (4.1) corresponds to (1.15).

Now, let $|x| = r = \sqrt{R_i S}$ in (2.4) to obtain

$$\overline{u}(x) \geqslant \frac{1}{2} m_i \left(\frac{\delta}{K}\right)^{1/(p-1)}, \qquad |x| < R^i \tag{4.2}$$

as in (2.2). In turn

$$\inf_{x\to x_i} \overline{u}(x) \geqslant \frac{1}{2} m_i \left(\frac{\delta}{K}\right)^{1/(p-1)} > 0. \tag{4.3}$$

By agreement $\overline{u} > 0$ in $\mathbb{R}^n\backslash P$ and $\inf_{\mathbb{R}^n\backslash P} \overline{u}(x) = 0$. But P is a *finite* set. It follows that there is some point $x_1 \in P$ where $\inf_{x\to x_i} \overline{u}(x) = 0$. This contradicts (4.3) and completes the proof.

5. Counterexamples. We treat three different counterexamples: (1) that Theorem 1 fails when $p < n$; (2) that Theorem 2 fails when $p = n$ and the solution domain is $B\backslash P$, where B is a finite ball; (3) that Theorem 1 fails when the inequality sign in equation (1) is reversed.

(1) Consider the function $w = \varepsilon(1+|x|^2)^{k/2}$, where $\varepsilon > 0,\ k < 0$. We evaluate the expression (1), here denoted by I, yielding after a lengthy but direct calculation, using the relation $\mathscr{A}(t) = t^{p-2}A(t)$ at the final step,

$$\begin{aligned} I &= \operatorname{div}\{\mathscr{A}(|Dw|)Dw\} \\ &= \frac{\partial}{\partial x_i}\{\mathscr{A}(|w'|)w'\frac{x_i}{r}\} \\ &= -\frac{t\mathscr{A}(t)}{r}\left[\left(1 + \frac{t\mathscr{A}'(t)}{\mathscr{A}'(t)}\mathscr{A}\right)\frac{rw''}{w'} + (n-1)\right], \qquad t = |w'| \\ &= -\frac{t\mathscr{A}(t)}{r}\left[(p-1+E)\frac{rw''}{w'} + (n-1)\right], \qquad E = E(t) = \frac{tA'(t)}{A(t)}, \end{aligned}$$

it being assumed for the sake of the counterexamples that $A \in C^1(\mathbb{R}_+)$. Calculating w' and w'', we have

$$\begin{aligned} &w' = \varepsilon k(1+r^2)^{(k-2)/2} r \leqslant 0, \\ &w'' = \varepsilon k(k-2)(1+r2)^{(k-4)/2} r^2 + \varepsilon k(1+r^2)^{(k-2)/2}, \\ &\frac{rw''}{w'} = \frac{1+(k-1)r^2}{1+r^2}. \end{aligned}$$

Hence up to a positive factor

$$I = -(p - 1 + E + (n - 1)) - [(p - 1)k + n - p + (k - 1)E]\, r^2$$

As observed in the introduction, $p - 1 + E > 0$. Thus, $I \leqslant 0$ whenever

$$(p - 1)k + (n - p) + (k - 1)E \geqslant 0. \tag{5.1}$$

To begin with, when $p < n$ we shall identify a subset $\mathscr{S}$ of those operators $\mathscr{A}$ which satisfy (i) - (iii) and for which an appropriate (bounded) choice of k makes $I \leqslant 0$; that is, this produces a counterexample to Theorem 1 for the set $\mathscr{S}$ and for $p < n$. Specifically, we take $\mathscr{S}$ to be the subset of operators obeying (i) - (iii) such that

$$\overline{\lim}_{t\to 0} E(t) < n - p \qquad (p < n). \tag{5.2}$$

In this case, if ε is suitably small, say$\varepsilon \leqslant \varepsilon_0$), it is easy to see that for some constant $\mu \in (0, 1]$ we have

$$E = E(t) \leqslant (1 - \mu)(n - p) \tag{5.3}$$

(note here that $|w'| = \varepsilon k|r(1 + r^2)^{(k-2)/2} \leqslant \varepsilon|k|)$. Using (5.3), condition (5.1) is now guaranteed by

$$[(p - 1) + (1 - \mu)(n - p)]k + \mu(n - p) \geqslant 0.$$

This is satisfied for $p < n$ and for all parameters k such that

$$0 > k > \frac{\mu(p - n)}{(p - 1) + (1 - \mu)(n - p)}.$$

Since the expression on the right is bounded in absolute value by $(n - p)/(p - 1)$, this completes the counterexample to Theorem 1 for operators in the set $\mathscr{S}$ and with $p < n$.. This being shown, we now observe that the set $/\mathscr{S}$ is empty. Indeed $A(0) > 0$ and thus, by contradiction of (5.2), there must be an interval $0 < t < t_0$ in which

$$tA_t(t) >\geqslant c > 0.$$

Integration yields $A(t) \to -\infty$ as $t \to 0$, which is impossible. Thus $/\mathscr{S}$ is empty. This completes the counterexample for Theorem 1 when $p < n$.

Example. When $A \equiv 1$, as is the case for the p-Laplace operator, we have $E = 0$, and so $I \leqslant 0$ when $k = (p - n)/(p - 1)$. Similarly, $E \leqslant 0$ for the mean curvature operator, and the same value of k suffices.

(2) Without loss of generality we can suppose that $P = \{0\}$ and $B = \{|x| < 1\}$. We seek a function w on $B\backslash\{0\}$ such that (1) holds, but w is not a constant (or even continuous). Consider $w = \log(1/r)$, $r < 1$. Here $|w'| = 1/r$ is arbitrarily large when r is small, so that it is reasonable to consider functions A such that $E \leqslant 0$ when t is large. An example of this is the p-Laplacian where $A \equiv 1$ and $E = 0$. We find, with $p = n$, that up to a positive factor,

$$I = p - n + E = E \leqslant 0,$$

as required.

(3) Consider the function $w = |x|^2$. Then $rw''/w' = 1$ and

$$I = (p - 1 + E) + n - 1 > 0,$$

where $p - 1 + E > 0$ as noted in the introduction, I.e., the nonconstant positive function $|x|^2$ satisfies the inequality (1) with the inequality sign reversed.

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[1] See especially Corollary II (a), p 84.
[2] See especially pp. 64-70.