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Efficient numerical techniques for Burgers' equation

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ABSTRACT

This paper presents new efficient numerical techniques for solving one dimensional quasi-linear Burgers' equation. Burgers' equation is used as a model problem in the study of turbulence, boundary layer behavior, shock waves, convection dominated diffusion phenomena, gas dynamics, acoustic attenuation in fog and continuum traffic simulation. Using a non-linear Cole–Hopf transformation the Burgers' equation is reduced to one-dimensional diffusion equation. The linearized diffusion equation is semi discretized by using method of lines (MOL) which leads to a system of ordinary differential equations in time. Resulting system of ordinary differential equations is solved by backward differentiation formulas (BDF) of order one, two and three and the analysis of numerical errors are presented. Numerical results for modest values of kinematic viscosity are compared with the exact solution to demonstrate the efficiency of proposed numerical methods.

1. Introduction

The nonlinear parabolic partial differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

is known as the one-dimensional Burgers' equation, where $\nu > 0$ is the kinematic viscosity parameter. It is the simplest nonlinear partial differential equation combining both nonlinear propagation and diffusive effects. When $\nu \neq 0$ Eq. (1.1) is known as viscid Burgers' equation and if $\nu = 0$, it is a non-linear hyperbolic partial differential equation also known as inviscid Burgers' equation [1].

Historically, Burgers' equation was first introduced by Bateman [7] in 1915 who derived the steady state solution for one dimensional Burgers' equation. This equation was proposed as a model of turbulent fluid motion [8] by the Dutch scientist Johannes Martinus Burgers and due to his remarkable contribution the equation is named after him. J.M Burgers' investigated various aspects of turbulence and used in [8,9] to model turbulence. This equation may be used to test various numerical algorithms, due to the availability of an analytical expression for its solution for different sets of boundary and initial conditions. A great deal of efforts have been expended in last few years to compute efficiently the numerical solution of the Burgers' equation for small and large values of the kinematic viscosity.

This equation was solved analytically for restricted values of initial conditions [14]. In 1972, Benton and Platzman [10] surveyed the exact solution to the initial value problem for the one-dimensional Burgers' equation. Ozin [23], proposed finite element method for the solution of Burgers' equation. In [25], local discontinuous Galerkin finite element method is used for solving Burgers' equation. Tangent method is used in [20], for analytic study of generalized Burgers' Huxley equation.

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We proposed to use discretization method for solving one-dimensional Burgers equation. It is one of the well known technique due to its simple concept and easy practice. Darvishi and Javidi [12] studied a numerical solution by pseudospectral method and Darvishi's preconditioning. Kadalbajoo and Awasthi [19] have developed a numerical method based on Crank–Nicolson scheme for Burgers' equation. Zhang and Wang [22] solved the Burgers' equation by a predictor-corrector compact finite difference scheme. Haq et al. [15] introduced meshless method of lines for the numerical solution of nonlinear Burgers' type equation. Gao [14] introduced lattice Boltzmann model for Burgers' equation through selecting equilibrium distribution function properly. In 2013, [18] a numerical scheme based on weighted average differential quadrature method is proposed to solve time dependent Burgers' equation with appropriate initial and boundary conditions. Some of the efficient numerical tools for the solution of Burgers' equation are Automatic differentiation method [3], Implicit finite difference scheme [4], group explicit methods [13], explicit methods [5,21] and finite elements methods [2,16].

In our context we will consider the following one-dimensional Burgers' equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \quad a \le x \le b, \quad 0 \le t \le T \tag{1.2a}$$

with initial condition

$$u(x,0) = u_0(x), \quad a \le x \le b, \tag{1.2b}$$

and boundary conditions

$$u(a,t) = f_1(t), \quad 0 \le t \le T, \tag{1.2c}$$

$$u(b,t) = f_2(t), \quad 0 \le t \le T. \tag{1.2d}$$

where $\nu > 0$, is the kinematic viscosity parameter and $u_0(x), f_1$ and f_2 are given functions of the variables which are sufficiently smooth.

In this paper, Cole–Hopf transformation is used to map the solution of quasi-linear Burgers' equation to the linear diffusion equation. The linear diffusion equation is semi-discretized by method of lines (MOL) into a system of first order ordinary differential equations. The resulting system is solved by different backward differentiation formulas.

The method of lines is a powerful numerical technique for the solution of time-dependent partial differential equations which was introduced by Schiesser in 1991 [24]. In this method semi discretization is performed along the spatial direction and the spatial derivatives are approximated by finite differences. Thus the partial differential equation is reduced to a system of ordinary differential equations which can be integrated in time. In this paper, we have used backward differentiation formulas of order one, two and three known as BDF-1, BDF-2 and BDF-3 to solve the system of ordinary differential equations. Numerical error analysis shows that the fully discretized scheme along with BDF-1 is first order accurate in time and second order accurate in space, while for BDF-2 errors are quadratic over both space and time. BDF-3 has got accuracy of order three in time and two in space.

Backward differentiation formulas belong to a family of linear multi-step implicit methods used for numerical integration of ordinary differential equations. Backward differentiation formula give an approximation to the derivative of a variable at a time 't' in terms of its function values, at time 't' and earlier times. To demonstrate the accuracy of these methods we have compared different schemes for modest values of kinematic viscosity.

2. Cole–Hopf transformation

J.D. Cole (1951) [11] and [17] gave a non-linear transformation to reduce Burgers' equation into linear diffusion equation. Cole–Hopf transformation is a powerful analytic tool for the Burgers' equation to yield exact solution. Recently, it has been recognized that Cole–Hopf transformation can also be used to find numerical solution. It appeared first in a technical report by Langerstrom, Cole and Trilling. Outlines of the Cole–Hopf transformation may be given by the following Theorem.

Theorem 1. *In the context with initial and boundary conditions of the Eq. (1.2) let $\phi(x,t)$ be any solution to linear diffusion equation*

$$\frac{\partial \phi}{\partial t} = \nu\frac{\partial^2 \phi}{\partial x^2},$$

then the nonlinear transformation [Cole–Hopf]

$$u = -2\nu\frac{\phi_x}{\phi}, \tag{2.3}$$

is a solution to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}.$$

Let

$$u = \frac{\partial \psi}{\partial x}, \qquad \psi = \psi(x,t)$$

substituting in Eq. (1.2) and integrating *w.r.t.* x we obtain

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 = \nu \frac{\partial^2 \psi}{\partial x^2}.$$

Then introduce $\psi = -2\nu \log \phi$ to obtain

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2}$$

with initial condition

$$\phi(x,0) = \exp\left(-\frac{1}{2\nu}\int_0^x u_0(\xi)d\xi\right), \quad a \le x \le b,$$

and boundary conditions

$$\phi_x(0,t) = 0 = \phi_x(1,t), \quad t \ge 0.$$

Here

$$u = -2\nu \frac{\phi_x}{\phi} = \frac{\partial}{\partial x}(-2\nu \log \phi) = \frac{\partial \psi}{\partial x}, \qquad \psi = \psi(x,t)$$

substituting in Eq. (1.2)

$$\frac{\partial \psi_x}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \psi_x}{\partial x} = \nu \frac{\partial^2 \psi_x}{\partial x^2},$$

$$\frac{\partial \psi_t}{\partial x} + \frac{\partial}{\partial x}\left(\frac{1}{2}\psi_x^2\right) = \nu \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right)$$

Integrating *w.r.t.* x we obtain

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 = \nu \frac{\partial^2 \psi}{\partial x^2}. \tag{2.4}$$

We have $\psi = -2\nu \log \phi$

$$\frac{\partial \psi}{\partial t} = -2\nu \frac{\phi_t}{\phi}$$

$$\frac{\partial \psi}{\partial x} = -2\nu \frac{\phi_x}{\phi}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -2\nu \frac{\phi_{xx}}{\phi} + 2\nu \frac{\phi_x^2}{\phi^2}$$

substituting in Eq. (2.4) we get

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2} \tag{2.5a}$$

with initial condition

$$\phi(x,0) = \exp\left(-\frac{1}{2\nu}\int_0^x u_0(\xi)d\xi\right), \quad a \le x \le b, \tag{2.5b}$$

and boundary conditions

$$\phi_x(0,t) = 0 = \phi_x(1,t), \quad t \ge 0. \tag{2.5c}$$

3. Exact solution by Cole–Hopf transformation

The Fourier series solution to the linearized diffusion equation Eq. (2.5) is

$$\phi(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \exp\left(-n^2\pi^2\nu t\right)\cos(n\pi x),$$

with Fourier coefficients at $t = 0$ as

$$C_0 = \int_0^1 \exp\left[-\frac{1}{2\nu}\int_0^x u_0(\xi)d\xi\right]dx,$$

$$C_n = 2\int_0^1 \exp\left[-\frac{1}{2\nu}\int_0^x u_0(\xi)d\xi\right]\cos(n\pi x)dx,$$

Using the Cole–Hopf transformation, solution of the Burgers' equation Eq. (1.2) is

$$u(x,t) = 2\pi\nu \frac{\sum_{n=1}^{\infty} C_n \exp\left(-n^2\pi^2\nu t\right) n\sin(n\pi x)}{C_0 + \sum_{n=1}^{\infty} C_n \exp\left(-n^2\pi^2\nu t\right)\cos(n\pi x)} \tag{3.6}$$

4. Numerical scheme

The non-linear Burgers' equation is being reduced to one dimensional linear diffusion equation, with initial and boundary condition Eq. (2.5) by using Cole–Hopf transformation. The linear diffusion equation Eq. (2.5a) is discretized along 'x' direction by Method of lines (MOL) [26], while the variable 't' remains undiscretized.

4.1. Semi-discretization

Divide the interval $[a, b]$ into N equal subintervals, $a = x_0 \le x_1 \le \ldots \le x_N = b$ with constant spacing $h = (b-a)/N$ and $x_i = ih$ for $i = 1, 2, \ldots N$. On discretizing the 'x' variable, by method of lines, the second order spatial derivative ϕ_{xx} becomes

$$\frac{\partial^2\phi}{\partial x^2}(x_i, t) = \frac{1}{h^2}(\phi_{i+1}(t) - 2\phi_i(t) + \phi_{i-1}(t)), \quad i = 0, 1, 2, \ldots N.$$

Substituting in Eq. (2.5), we obtain a system of ordinary differential equations with initial condition

$$\frac{d\phi_i}{dt} = \frac{\nu}{h^2}(\phi_{i+1}(t) - 2\phi_i(t) + \phi_{i-1}(t)) \tag{4.7}$$

$$\phi_i(0) = \exp\left(-\frac{1}{2\nu}\int_0^{x_i} u_0(\xi)d\xi\right), \quad i = 0, 1, 2, \ldots N \tag{4.8}$$

where, $\phi_i(t) = \phi(x_i, t)$, taking into account the discretization of boundary condition $\phi_{-1}(t) = \phi_1(t)$ and $\phi_{N+1}(t) = \phi_{N-1}(t)$, we obtain system of ordinary differential equations which can be written in matrix form as

$$\mathbf{\Phi}'(t) = \frac{\nu}{h^2}A\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{\Phi_0} \tag{4.9}$$

where, $\mathbf{\Phi}(t) = [\phi_0(t), \ldots \phi_N(t)]^T$, $\mathbf{\Phi_0} = [\phi_0(0), \ldots, \phi_N(0)]^T$ is the initial condition and A is $(N+1)\times(N+1)$ tridiagonal matrix given by

$$A = \begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix}$$

4.2. Full discretization

Divide the time interval $[0, T]$ into M equal subintervals $0 = t_0 \le t_1 \le \ldots \le t_M = T$ with $\Delta t = T/M$, i.e $t_n = n\Delta t$, $n = 1, 2, .., M$. The system (4.9) of first order ordinary differential equations is solved numerically using different backward differentiation formulas. The general form of p^{th} order backward differentiation formula for solving the following IVP

$$\mathbf{\Phi}'(t) = \mathbf{f}(t, \mathbf{\Phi}), \quad \mathbf{\Phi}(0) = \mathbf{\Phi_0}$$

is given by

$$\mathbf{\Phi_{n+1}} = \sum_{j=0}^{p-1}\alpha_j\mathbf{\Phi_{n-j}} + \Delta t\beta\mathbf{f}(t_{n+1}, \mathbf{\Phi_{n+1}}) \tag{4.10}$$

where, $\mathbf{\Phi}_{n+1} = \mathbf{\Phi}(x_i, t_{n+1}) = [\phi_{0,n+1}, \ldots \phi_{N,n+1}]^T$, Δt is the time step, values of constants α_j for, $j = 0, 1, \ldots, p-1$ and β are given in [6].

4.2.1. Backward differentiation formula of order p = 1 (BDF-1)

In Eq. (4.10) when $\alpha_0 = 1$, $\beta = 1$ and $\mathbf{f}(t, \mathbf{\Phi}) = \frac{\nu}{h^2}A\mathbf{\Phi}$, we get

$$\mathbf{\Phi_{n+1}} = \mathbf{\Phi_n} + (\Delta t\nu/h^2)A\mathbf{\Phi_{n+1}}, \quad n = 0, 1, \ldots M-1 \tag{4.11}$$

$\mathbf{\Phi_0}$ is initial condition. It is also known as backward Euler Formula.

4.2.2. Backward differentiation formula of order p = 2 (BDF-2)

With $\alpha_0 = 4/3, \alpha_1 = -1/3, \beta = 2/3$ and $\mathbf{f}(t, \mathbf{\Phi}) = \frac{\nu}{h^2}A\mathbf{\Phi}$, Eq. (4.10) becomes

$$\mathbf{\Phi_{n+1}} = \frac{4}{3}\mathbf{\Phi_n} - \frac{1}{3}\mathbf{\Phi_{n-1}} + \frac{2}{3}(\Delta t\nu/h^2)A\mathbf{\Phi_{n+1}}, \quad n = 1, \ldots M-1 \tag{4.12}$$

the solution at first time level i.e. $\mathbf{\Phi_1}$ is obtained from BDF-1.

4.2.3. Backward differentiation formula of order p = 3 (BDF-3)

By putting $\alpha_0 = 18/11, \alpha_1 = -9/11, \alpha_2 = 2/11, \beta = 6/11$ and $\mathbf{f}(t, \mathbf{\Phi}) = \frac{\nu}{h^2}A\mathbf{\Phi}$ in Eq. (4.10), we have

$$\mathbf{\Phi_{n+1}} = \frac{18}{11}\mathbf{\Phi_n} - \frac{9}{11}\mathbf{\Phi_{n-1}} + \frac{2}{11}\mathbf{\Phi_{n-2}} + \frac{6}{11}(\Delta t\nu/h^2)A\mathbf{\Phi_{n+1}}, \quad n = 2, \ldots M-1 \tag{4.13}$$

the solution at first time level $\mathbf{\Phi_1}$ and second time level $\mathbf{\Phi_2}$ is obtained from BDF-1.

The numerical solution of the Burgers' Equation Eq. (1.2) in terms of the solution of fully discretized linear diffusion equation and Cole–Hopf transforamtion Eq. (2.3) is given by

$$\begin{aligned} u_{i,n} &= -(2\nu)\left\{\frac{\phi_{i+1,n} - \phi_{i-1,n}}{2h\phi_{i,n}}\right\} \\ &= -\left(\frac{\nu}{h}\right)\left\{\frac{\phi_{i+1,n} - \phi_{i-1,n}}{\phi_{i,n}}\right\}, \quad i = 1, .., N-1, \quad n = 1, .., M \end{aligned} \tag{4.14}$$

5. Error analysis

This section presents error analysis of the proposed numerical methods using backward differentiation formulas of order one, two and three. This analysis is based on Taylor series expansion of the fully discretized numerical scheme.

5.1. Backward differentiation formula of order one (BDF-1)

The fully discretized scheme along with backward differentiation formula of order one Eq. (4.11) can be expressed as

$$\phi(x_i, t_{n+1}) = \phi(x_i, t_n) + \frac{k\nu}{h^2}[\phi(x_{i-1}, t_{n+1}) - 2\phi(x_i, t_{n+1}) + \phi(x_{i+1}, t_{n+1})] \tag{5.15}$$

where $k = \Delta t, h = \Delta x$, $i = 1, 2, .., N-1$ and $n = 0, 1, ..., M-1$.

Local Truncation Error (LTE) is given by

$$LTE = \phi(x_i, t_{n+1}) - \phi(x_i, t_n) - \frac{k\nu}{h^2}[\phi(x_{i-1}, t_{n+1}) - 2\phi(x_i, t_{n+1}) + \phi(x_{i+1}, t_{n+1})] \tag{5.16}$$

Using the Taylor series expansion and after simplification we get

$$\begin{aligned} LTE &= k\frac{\partial\phi}{\partial t} + \frac{k^2}{2}\frac{\partial^2\phi}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3\phi}{\partial t^3} - \nu k\frac{\partial^2\phi}{\partial x^2} - \nu k^2\frac{\partial^3\phi}{\partial x^2\partial t} - \frac{\nu kh^2}{12}\frac{\partial^4\phi}{\partial x^4} - \frac{\nu k^3}{2}\frac{\partial^4\phi}{\partial x^2\partial t^2} + \cdots \\ &= \frac{k^2}{2}\frac{\partial^2\phi}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3\phi}{\partial t^3} - \nu k^2\frac{\partial^3\phi}{\partial x^2\partial t} - \frac{\nu kh^2}{12}\frac{\partial^4\phi}{\partial x^4} - \frac{\nu k^3}{2}\frac{\partial^4\phi}{\partial x^2\partial t^2} + \cdots \\ &= \frac{k^2}{2}\left(\frac{\partial^2\phi}{\partial t^2} - 2\nu\frac{\partial^3\phi}{\partial x^2\partial t}\right) - \frac{\nu h^2k}{12}\frac{\partial^4\phi}{\partial x^4} + \cdots \\ &= O(k^2 + h^2k) \end{aligned}$$

Truncation Error (TE) is given by

$$\begin{aligned} TE = k^{-1}(LTE) &= \frac{k}{2}\left(\frac{\partial^2\phi}{\partial t^2} - 2\nu\frac{\partial^3\phi}{\partial x^2\partial t}\right) - \frac{\nu h^2}{12}\frac{\partial^4\phi}{\partial x^4} + \cdots \\ &= O(k + h^2) \\ &= O(\Delta t + (\Delta x)^2) \end{aligned}$$

The numerical errors are proportional to the time step and the square of the space step. Hence the proposed fully discretized scheme along with backward differentiation formula of order one is second order accurate in space and first order accurate in time.

5.2. Backward differentiation formula of order two (BDF-2)

Local Truncation Error (LTE) for the fully discretized scheme along with backward differentiation formula of order two Eq. (4.12) is given by

$$LTE = \phi(x_i, t_{n+1}) - \frac{4}{3}\phi(x_i, t_n) - \frac{1}{3}\phi(x_i, t_{n-1}) - \frac{2k\nu}{3h^2}[\phi(x_{i-1}, t_{n+1}) - 2\phi(x_i, t_{n+1}) + \phi(x_{i+1}, t_{n+1})] \tag{5.17}$$

Using the Taylor series expansion and simplification we get

$$\begin{aligned} LTE &= \frac{2k}{3}\frac{\partial\phi}{\partial t} + \frac{2k^2}{3}\frac{\partial^2\phi}{\partial t^2} + \frac{k^3}{9}\frac{\partial^3\phi}{\partial t^3} - \frac{2\nu k}{3}\frac{\partial^2\phi}{\partial x^2} - \frac{2\nu k^2}{3}\frac{\partial^3\phi}{\partial x^2\partial t} - \frac{\nu kh^2}{18}\frac{\partial^4\phi}{\partial x^4} - \frac{\nu k^3}{3}\frac{\partial^4\phi}{\partial x^2\partial t^2} + \cdots \\ &= \frac{2k}{3}\left(\frac{\partial\phi}{\partial t} - \nu\frac{\partial^2\phi}{\partial x^2}\right) + \frac{2k^2}{3}\left(\frac{\partial^2\phi}{\partial t^2} - \nu\frac{\partial^3\phi}{\partial x^2\partial t}\right) + \frac{k^3}{3}\left(\frac{1}{3}\frac{\partial^3\phi}{\partial t^3} - \nu\frac{\partial^4\phi}{\partial x^2\partial t^2}\right) - \frac{\nu kh^2}{18}\frac{\partial^4\phi}{\partial x^4} + \cdots \\ &= \frac{k^3}{3}\left(\frac{1}{3}\frac{\partial^3\phi}{\partial t^3} - \nu\frac{\partial^4\phi}{\partial x^2\partial t^2}\right) - \frac{\nu kh^2}{18}\frac{\partial^4\phi}{\partial x^4} + \cdots \\ &= O(k^3 + kh^2) \end{aligned}$$

Truncation Error (TE) is given by

$$\begin{aligned} TE = k^{-1}(LTE) &= \frac{k^2}{3}\left(\frac{1}{3}\frac{\partial^3\phi}{\partial t^3} - \nu\frac{\partial^4\phi}{\partial x^2\partial t^2}\right) - \frac{\nu h^2}{18}\frac{\partial^4\phi}{\partial x^4} + \cdots \\ &= O(k^2 + h^2) \\ &= O((\Delta t)^2 + (\Delta x)^2) \end{aligned}$$

Hence the errors are quadratic over both time and space step.

5.3. Backward differentiation formula of order three (BDF-3)

In a similar manner we get Local Truncation Error (LTE) for the fully discretized scheme with backward differentiation formula of order three.

$$\begin{aligned} LTE &= \phi(x_i, t_{n+1}) - \tfrac{4}{3}\phi(x_i, t_n) - \tfrac{1}{3}\phi(x_i, t_{n-1}) - \tfrac{2k\nu}{3h^2}[\phi(x_{i-1}, t_{n+1}) - 2\phi(x_i, t_{n+1}) + \phi(x_{i+1}, t_{n+1})] \\ &= O(k^4 + h^2 k) \end{aligned}$$

Truncation Error (TE) is given by

$$\begin{aligned} TE = k^{-1}(LTE) &= O(k^3 + h^2) \\ &= O((\Delta t)^3 + (\Delta x)^2) \end{aligned}$$

The numerical errors are cubic over the time step and quadratic over the space step. Hence errors are least when solved by fully discretized scheme with BDF-3, which is also observed in the following test problems.

6. Numerical experiments and discussions

Burgers' equation is solved numerically by three different backward differentiation formulas and computed results generated by the proposed methods are compared with analytic results at different nodal points and at different final time T for modest values of kinematic viscosity ν.

6.1. Test problems

Several numerical experiments are conducted to compare proposed numerical method and exact solution. We have considered two test problems in which all the computations are carried out using MATLAB.

Example 1. Consider the Burgers' Eq. (1.2)with the initial condition

$$u(x, 0) = \sin(\pi x), \quad 0 \le x \le 1, \tag{6.18}$$

and the homogeneous boundary conditions

$$u(0, t) = u(1, t) = 0, \quad 0 \le t \le T. \tag{6.19}$$

By Cole–Hopf transformation

$$u(x, t) = -2\nu\frac{\phi_x}{\phi}, \tag{6.20}$$

Table 1
Comparison of the numerical solution (BDF-1) with the exact solution at different space points of example-1 at $T = 0.5$ for $\nu = 1$ and $\Delta t = 0.001$.

x	Computed solution				Exact solution
	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0.1	2.281E−03	2.271E−03	2.268E−03	2.268E−03	2.213E−03
0.2	4.339E−03	4.319E−03	4.315E−03	4.314E−03	4.210E−03
0.3	5.973E−03	5.947E−03	5.940E−03	5.939E−03	5.796E−03
0.4	7.024E−03	6.993E−03	6.985E−03	6.984E−03	6.816E−03
0.5	7.388E−03	7.356E−03	7.347E−03	7.347E−03	7.169E−03
0.6	7.029E−03	6.998E−03	6.990E−03	6.989E−03	6.821E−03
0.7	5.982E−03	5.955E−03	5.948E−03	5.948E−03	5.804E−03
0.8	4.347E−03	4.328E−03	4.323E−03	4.322E−03	4.218E−03
0.9	2.286E−03	2.276E−03	2.273E−03	2.273E−03	2.218E−03

Table 2
Comparison of the numerical solution (BDF-2) with the exact solution at different space points of example-1 at $T = 0.5$ for $\nu = 1$ and $\Delta t = 0.001$.

x	Computed solution				Exact solution
	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0.1	2.227E−03	2.217E−03	2.214E−03	2.214E−03	2.213E−03
0.2	4.236E−03	4.217E−03	4.212E−03	4.212E−03	4.210E−03
0.3	5.832E−03	5.806E−03	5.799E−03	5.798E−03	5.796E−03
0.4	6.858E−03	6.827E−03	6.820E−03	6.819E−03	6.816E−03
0.5	7.214E−03	7.181E−03	7.173E−03	7.172E−03	7.169E−03
0.6	6.863E−03	6.832E−03	6.824E−03	6.823E−03	6.821E−03
0.7	5.840E−03	5.814E−03	5.807E−03	5.806E−03	5.804E−03
0.8	4.244E−03	4.225E−03	4.220E−03	4.220E−03	4.218E−03
0.9	2.232E−03	2.221E−03	2.219E−03	2.219E−03	2.218E−03

the Burgers' equation is transformed in the following linear heat equation with initial and Neumann boundary conditions

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2}, \quad 0 \le x \le 1, \quad t > 0 \tag{6.21}$$

with the initial condition

$$\phi(x, 0) = \exp\left\{-\frac{1}{2\pi\nu}[1 - \cos(\pi x)]\right\}, \quad 0 \le x \le 1, \tag{6.22}$$

with the boundary conditions

$$\phi_x(0, t) = \phi_x(1, t) = 0, \quad 0 \le t \le T. \tag{6.23}$$

The exact solution of the problem is

$$u(x, t) = 2\pi\nu \frac{\sum_{n=1}^{\infty} C_n \exp\left(-n^2\pi^2\nu t\right) n \sin(n\pi x)}{C_0 + \sum_{n=1}^{\infty} C_n \exp\left(-n^2\pi^2\nu t\right) \cos(n\pi x)} \tag{6.24}$$

where

$$C_0 = \int_0^1 \exp\left\{-\frac{1}{2\pi\nu}[1 - \cos(\pi x)]\right\} dx, \tag{6.25}$$

$$C_n = 2\int_0^1 \exp\left\{-\frac{1}{2\pi\nu}[1 - \cos(\pi x)]\right\} \cos(n\pi x)\, dx, \tag{6.26}$$

The exact solution and numerical results obtained at different node points are tabulated for various values of kinematic viscosity ν.

Numerical solutions generated by backward differentiation formulas of order one, two and three have been tabulated and compared with exact solution at various nodal points. The comparison at various nodal points is carried out w.r.t kinematic viscosity ν, final time T and time increment Δt. Tables 1–3, have $\nu = 1$, $T = 0.5$ and $\Delta t = 0.001$; Tables 4–6, have $\nu = 10$, $T = 0.1$, and $\Delta t = 0.0001$, whereas Tables 7–9, have $\nu = 0.1$, $T = 2.3$ and $\Delta t = 0.01$. From the above results it is evident that as number of nodes increases numerical solution approaches to the exact solution.

A comparison of numerical results generated by different backward differentiation formulas with the exact solution has been presented in Tables 10–13. In Tables 10 and 11 the three BDF methods have been compared at fixed time but at different spatial

Table 3
Comparison of the numerical solution (BDF-3) with the exact solution at different space points of example-1 at $T = 0.5$ for $\nu = 1$ and $\Delta t = 0.001$.

x	Computed solution				Exact solution
	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0.1	2.226E−03	2.216E−03	2.213E−03	2.213E−03	2.213E−03
0.2	4.234E−03	4.215E−03	4.210E−03	4.209E−03	4.210E−03
0.3	5.829E−03	5.803E−03	5.796E−03	5.795E−03	5.796E−03
0.4	6.854E−03	6.824E−03	6.816E−03	6.815E−03	6.816E−03
0.5	7.210E−03	7.177E−03	7.169E−03	7.168E−03	7.169E−03
0.6	6.859E−03	6.828E−03	6.821E−03	6.820E−03	6.821E−03
0.7	5.837E−03	5.810E−03	5.804E−03	5.803E−03	5.804E−03
0.8	4.242E−03	4.223E−03	4.218E−03	4.217E−03	4.218E−03
0.9	2.230E−03	2.220E−03	2.218E−03	2.218E−03	2.218E−03

Table 4
Comparison of the numerical solution (BDF-1) with the exact solution at different space points of example-1 at $T = 0.1$ for $\nu = 10$ and $\Delta t = 0.0001$.

x	Computed solution				Exact solution
	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0.1	1.705E−05	1.684E−05	1.679E−05	1.679E−05	1.598E−05
0.2	3.242E−05	3.204E−05	3.194E−05	3.193E−05	3.040E−05
0.3	4.462E−05	4.409E−05	4.396E−05	4.395E−05	4.184E−05
0.4	5.246E−05	5.184E−05	5.168E−05	5.166E−05	4.919E−05
0.5	5.516E−05	5.450E−05	5.434E−05	5.432E−05	5.172E−05
0.6	5.246E−05	5.184E−05	5.168E−05	5.166E−05	4.919E−05
0.7	4.462E−05	4.409E−05	4.396E−05	4.395E−05	4.184E−05
0.8	3.242E−05	3.204E−05	3.194E−05	3.193E−05	3.040E−05
0.9	1.705E−05	1.684E−05	1.679E−05	1.679E−05	1.598E−05

Table 5
Comparison of the numerical solution (BDF-2) with the exact solution at different space points of example-1 at $T = 0.1$ for $\nu = 10$ and $\Delta t = 0.0001$.

x	Computed solution				Exact solution
	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0.1	1.625E−05	1.605E−05	1.600E−05	1.600E−05	1.598E−05
0.2	3.090E−05	3.053E−05	3.044E−05	3.043E−05	3.040E−05
0.3	4.254E−05	4.202E−05	4.190E−05	4.188E−05	4.184E−05
0.4	5.000E−05	4.940E−05	4.925E−05	4.923E−05	4.919E−05
0.5	5.258E−05	5.194E−05	5.179E−05	5.177E−05	5.172E−05
0.6	5.000E−05	4.940E−05	4.925E−05	4.923E−05	4.919E−05
0.7	4.254E−05	4.202E−05	4.190E−05	4.188E−05	4.184E−05
0.8	3.090E−05	3.053E−05	3.044E−05	3.043E−05	3.040E−05
0.9	1.625E−05	1.605E−05	1.600E−05	1.600E−05	1.598E−05

Table 6
Comparison of the numerical solution (BDF-3) with the exact solution at different space points of example-1 at $T = 0.1$ for $\nu = 10$ and $\Delta t = 0.0001$.

x	Computed solution				Exact solution
	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0.1	1.623E−05	1.603E−05	1.598E−05	1.598E−05	1.598E−05
0.2	3.087E−05	3.050E−05	3.041E−05	3.039E−05	3.040E−05
0.3	4.249E−05	4.198E−05	4.185E−05	4.183E−05	4.184E−05
0.4	4.995E−05	4.935E−05	4.920E−05	4.918E−05	4.919E−05
0.5	5.252E−05	5.189E−05	5.173E−05	5.171E−05	5.172E−05
0.6	4.995E−05	4.935E−05	4.920E−05	4.918E−05	4.919E−05
0.7	4.249E−05	4.198E−05	4.185E−05	4.183E−05	4.184E−05
0.8	3.087E−05	3.050E−05	3.041E−05	3.039E−05	3.040E−05
0.9	1.623E−05	1.603E−05	1.598E−05	1.598E−05	1.598E−05

Table 7
Comparison of the numerical solution (BDF-1) with the exact solution at different space points of example-1 at $T = 2.3$ for $\nu = 0.1$ and $\Delta t = 0.01$.

x	Computed solution				Exact solution
	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0.1	2.257E−02	2.256E−02	2.256E−02	2.256E−02	2.214E−02
0.2	4.363E−02	4.363E−02	4.363E−02	4.363E−02	4.280E−02
0.3	6.165E−02	6.164E−02	6.163E−02	6.163E−02	6.043E−02
0.4	7.497E−02	7.495E−02	7.495E−02	7.495E−02	7.344E−02
0.5	8.197E−02	8.194E−02	8.193E−02	8.193E−02	8.023E−02
0.6	8.120E−02	8.116E−02	8.115E−02	8.114E−02	7.940E−02
0.7	7.176E−02	7.171E−02	7.170E−02	7.170E−02	7.011E−02
0.8	5.380E−02	5.376E−02	5.375E−02	5.375E−02	5.252E−02
0.9	2.887E−02	2.885E−02	2.885E−02	2.885E−02	2.817E−02

Table 8
Comparison of the numerical solution (BDF-2) with the exact solution at different space points of example-1 at $T = 2.3$ for $\nu = 0.1$ and $\Delta t = 0.01$.

x	Computed solution				Exact solution
	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0.1	2.234E−02	2.234E−02	2.234E−02	2.234E−02	2.214E−02
0.2	4.319E−02	4.319E−02	4.319E−02	4.319E−02	4.280E−02
0.3	6.101E−02	6.100E−02	6.100E−02	6.100E−02	6.043E−02
0.4	7.418E−02	7.416E−02	7.415E−02	7.415E−02	7.344E−02
0.5	8.108E−02	8.105E−02	8.104E−02	8.104E−02	8.023E−02
0.6	8.028E−02	8.024E−02	8.023E−02	8.023E−02	7.940E−02
0.7	7.093E−02	7.088E−02	7.087E−02	7.087E−02	7.011E−02
0.8	5.316E−02	5.312E−02	5.311E−02	5.311E−02	5.252E−02
0.9	2.853E−02	2.850E−02	2.850E−02	2.850E−02	2.817E−02

Table 9
Comparison of the numerical solution (BDF-3) with the exact solution at different space points of example-1 at $T = 2.3$ for $\nu = 0.1$ and $\Delta t = 0.01$.

x	Computed solution				Exact solution
	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0.1	2.253E−02	2.253E−02	2.253E−02	2.253E−02	2.214E−02
0.2	4.357E−02	4.357E−02	4.357E−02	4.357E−02	4.280E−02
0.3	6.156E−02	6.155E−02	6.155E−02	6.155E−02	6.043E−02
0.4	7.487E−02	7.485E−02	7.485E−02	7.485E−02	7.344E−02
0.5	8.186E−02	8.183E−02	8.182E−02	8.182E−02	8.023E−02
0.6	8.109E−02	8.105E−02	8.104E−02	8.104E−02	7.940E−02
0.7	7.167E−02	7.162E−02	7.161E−02	7.161E−02	7.011E−02
0.8	5.373E−02	5.369E−02	5.368E−02	5.368E−02	5.252E−02
0.9	2.884E−02	2.882E−02	2.881E−02	2.881E−02	2.817E−02

Table 10
Comparison of computed solution generated by BDF-1, BDF-2, BDF-3 and the exact solution at different space points for example-1 at $T = 0.6$ for $\nu = 1$ and $\Delta t = 0.001$.

x	Computed solution			Exact solution
	BDF-1	BDF-2	BDF-3	
0.1	8.501E−04	8.259E−04	8.254E−04	8.254E−04
0.2	1.617E−03	1.571E−03	1.570E−03	1.570E−03
0.3	2.226E−03	2.163E−03	2.161E−03	2.161E−03
0.4	2.617E−03	2.543E−03	2.541E−03	2.541E−03
0.5	2.752E−03	2.674E−03	2.672E−03	2.672E−03
0.6	2.618E−03	2.543E−03	2.542E−03	2.542E−03
0.7	2.227E−03	2.164E−03	2.162E−03	2.162E−03
0.8	1.618E−03	1.572E−03	1.571E−03	1.571E−03
0.9	8.508E−04	8.266E−04	8.261E−04	8.260E−04

Table 11
Comparison of computed solution generated by BDF-1, BDF-2, BDF-3 and the exact solution at different space points for example-1 at $T = 2.5$ for $\nu = 0.1$ and $\Delta t = 0.01$.

x	Computed solution			Exact solution
	BDF-1	BDF-2	BDF- 3	
0.1	1.873E−02	1.852E−02	1.852E−02	1.852E−02
0.2	3.611E−02	3.571E−02	3.571E−02	3.571E−02
0.3	5.080E−02	5.022E−02	5.021E−02	5.021E−02
0.4	6.141E−02	6.070E−02	6.068E−02	6.069E−02
0.5	6.666E−02	6.587E−02	6.585E−02	6.586E−02
0.6	6.553E−02	6.473E−02	6.471E−02	6.471E−02
0.7	5.748E−02	5.676E−02	5.674E−02	5.675E−02
0.8	4.283E−02	4.227E−02	4.226E−02	4.226E−02
0.9	2.289E−02	2.259E−02	2.258E−02	2.258E−02

Table 12
Comparison of computed solution generated by BDF-1, BDF-2, BDF-3 and the exact solution at different times for example-1 at $\nu = 1$, $\Delta x = 0.0125$ and $\Delta t = 0.001$.

x	T	Computed solution			Exact solution
		BDF-1	BDF-2	BDF-3	
	0.8	2.730E−04	2.627E−04	2.625E−04	2.625E−04
0.25	1.0	3.830E−05	3.650E−05	3.646E−05	3.646E−05
	1.2	5.370E−06	5.070E−06	5.070E−06	5.060E−06
	0.8	3.861E−04	3.715E−04	3.712E−04	3.712E−04
0.5	1.0	5.417E−05	5.163E−05	5.157E−05	5.156E−05
	1.2	7.600E−06	7.170E−06	7.160E−06	7.160E−06
	0.8	2.730E−04	2.627E−04	2.625E−04	2.625E−04
0.75	1.0	3.830E−05	3.650E−05	3.646E−05	3.646E−05
	1.2	5.370E−06	5.070E−06	5.070E−06	5.060E−06

Table 13
Comparison of computed solution generated by BDF-1, BDF-2, BDF-3 and the exact solution at different times for example-1 at $\nu = 0.1$, $\Delta x = 0.0125$ and $\Delta t = 0.01$.

x	T	Computed solution			Exact solution
		BDF-1	BDF-2	BDF-3	
	2.4	4.807E−02	4.756E−02	4.755E−02	4.755E−02
0.25	2.6	4.003E−02	3.956E−02	3.955E−02	3.955E−02
	3.0	2.759E−02	2.721E−02	2.720E−02	2.720E−02
	2.4	7.354E−02	7.270E−02	7.268E−02	7.269E−02
0.5	2.6	6.043E−02	5.968E−02	5.966E−02	5.967E−02
	3.0	4.080E−02	4.021E−02	4.020E−02	4.020E−02
	2.4	5.664E−02	5.594E−02	5.593E−02	5.593E−02
0.75	2.6	4.582E−02	4.522E−02	4.520E−02	4.521E−02
	3.0	3.023E−02	2.978E−02	2.977E−02	2.977E−02

points while in Tables 12 and 13 numerical solution is compared with the exact one at different times. From the results it is seen that numerical scheme generated by backward differentiation formula of order three has better accuracy than the BDF-1 and BDF-2.

In Fig. 1, computed solution generated by backward differentiation formulas of order one, two and three and the exact solution are presented for different values of kinematic viscosity, ν. Accuracy of these BDF's are also tested by measuring error in L_2 and L_∞ norm.

$$\|u^a - u\|_2 = \left(h \sum_{i=1}^{N} (u_{i,M} - u(x_i, T)^2\right)^{\frac{1}{2}}$$
$$\|u^a - u\|_\infty = \max_{1<=i<=N} |u_{i,M} - u(x_i, T)^2|,$$

where u^a is the approximate solution, u is the exact solution, $u_{i,M} = u(x_i, t_M)$ is the approximate solution at $x = x_i$ and $t_M = T$, $u(x_i, T)$ is the exact solution at $x = x_i$ and final time T.

Tables 14 and 15 shows comparison of errors of BDF- 1, BDF-2 and BDF-3 for different values of T, ν and Δt. In Fig. 2, numerical method generated by backward differentiation formula of order three is compared with the exact solution at different times for modest values of kinematic viscosity, ν. It highlights the accuracy of BDF-3 which holds even for large values of T.

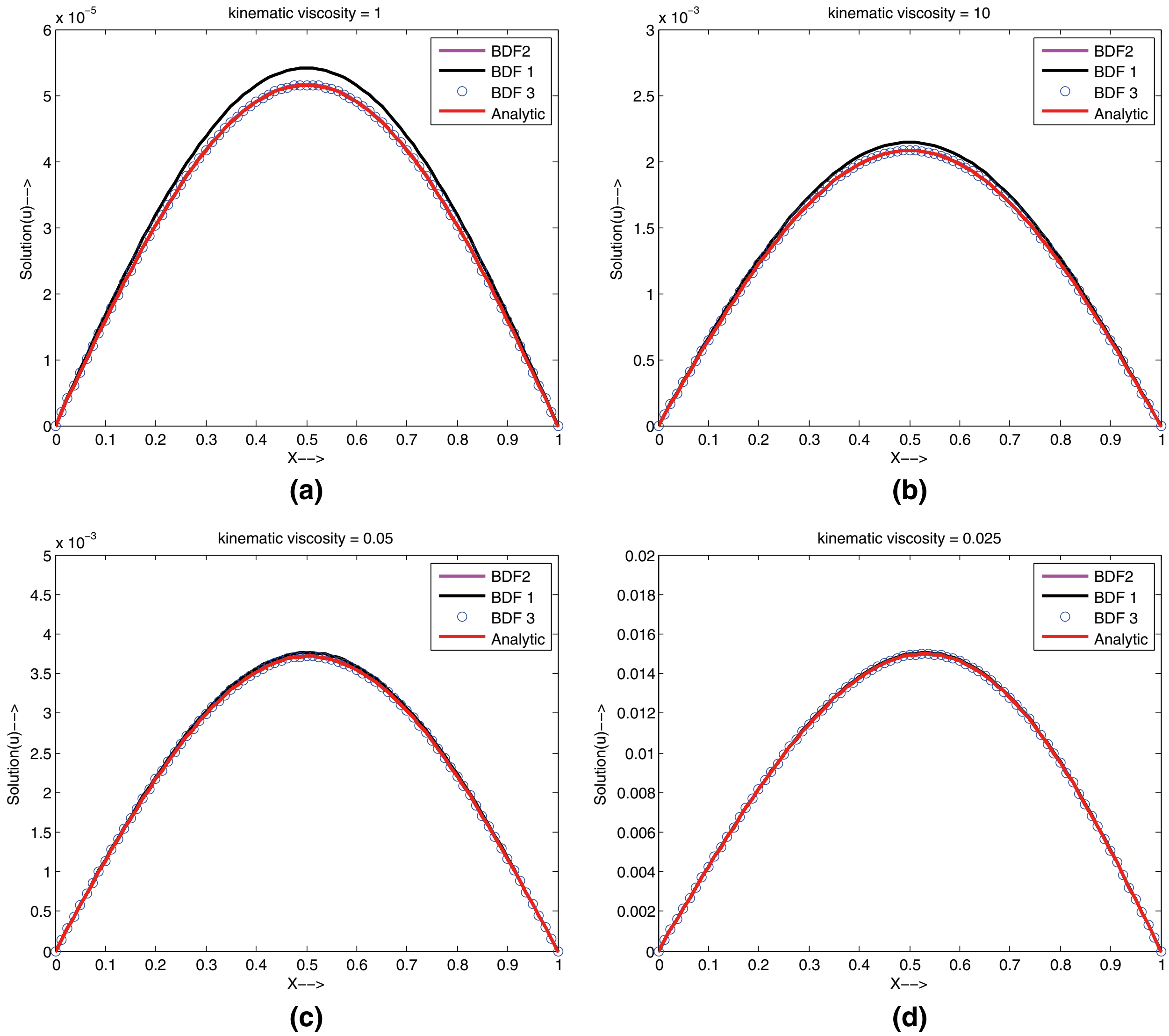

Fig. 1. Numerical results of Example 1 by backward differentiation formulas of order one, two and three for $\Delta x = 0.0125$ and different values of ν , Δt and T, (a) $\nu = 1$, $\Delta t = 0.001$, $T = 1$, (b) $\nu = 0.1$, $\Delta t = 0.01$, $T = 6$, (c) $\nu = 0.05$, $\Delta t = 0.01$, $T = 10$ and (d) $\nu = 0.25$, $\Delta t = 0.01$, $T = 12$.

Table 14
Comparison of errors in L_2 norm and L_∞ norm for different values of T, $N = 80$ and $\Delta t = 0.001$, corresponding to example-1.

$\nu = 1$	$T = 1$		$T = 2$	
	L_2	L_∞	L_2	L_∞
BDF-1	1.8457E−06	2.6102E−06	1.9600E−10	2.7713E−10
BDF-2	4.6043E−08	6.5114E−08	5.0898E−12	7.1885E−12
BDF-3	4.8173E−09	6.8127E−09	6.9285E−13	9.7552E−13

Table 15
Comparison of errors in L_2 norm and L_∞ norm for different values of T, $N = 80$ and $\Delta t = 0.01$, corresponding to example-1.

$\nu = 0.1$	$T = 3$		$T = 3.5$	
	L_2	L_∞	L_2	L_∞
BDF-1	4.2130E−04	5.9753E−04	3.0024E−04	4.2509E−04
BDF-2	7.1682E−06	1.0225E−05	5.6070E−06	7.9505E−06
BDF-3	1.5263E−06	2.1892E−06	7.0422E−07	1.0085E−06

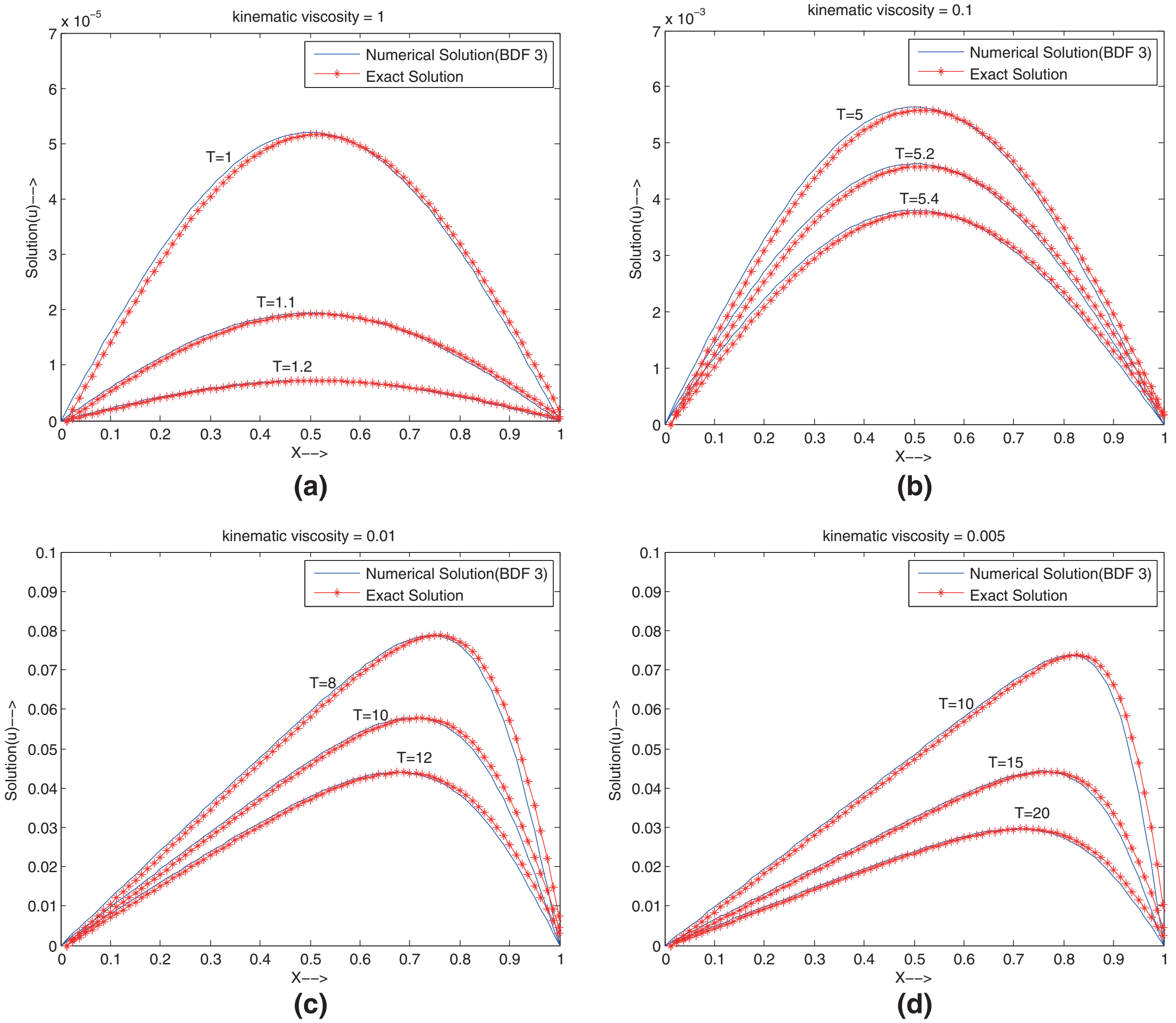

Fig. 2. Numerical solutions of Example 1 at different times for $\Delta x = 0.0125$ and different values of ν and Δt, (a) $\nu = 1$, $\Delta t = 0.0001$, (b) $\nu = 0.1$, $\Delta t = 0.01$, (c) $\nu = 0.01$, $\Delta t = 0.01$ and (d) $\nu = 0.005$, $\Delta t = 0.01$.

Example 2. Considering Burgers' Eq. (1.2) with the following initial condition

$$u(x, 0) = 4x(1 - x), \qquad 0 \le x \le 1, \tag{6.27}$$

and boundary conditions

$$u(0, t) = 0 = u(1, t), \qquad 0 \le t \le T. \tag{6.28}$$

The exact solution of this problem is obtained in similar way as in the example-1. Here we note that the Fourier coefficients C_0 and C_n are the following

$$C_0 = \int_0^1 \exp\left\{-\frac{1}{3\nu}[x^2(3 - 2x)]\right\} dx, \tag{6.29}$$

$$C_n = 2\int_0^1 \exp\left\{-\frac{1}{3\nu}[x^2(3 - 2x)]\right\} \cos(n\pi x)\, dx. \tag{6.30}$$

We summarize the numerical results obtained by three methods at different times and for different values of kinematic viscosity.

In Table 16, three different schemes BDF-1,BDF-2 and BDF-3 have been compared with exact solution at different times for $\nu = 1$, $\Delta t = 0.001$ and number of nodes is 81, while Table 17 shows comparison of numerical and exact solution at different times for $\nu = 0.1$, $\Delta t = 0.01$. In Tables 18 and 19, time is fixed while numerical solution is computed at different nodal points for

Table 16
Comparison of numerical solution with the exact solution at different times for example-2 at $\nu = 1$, $\Delta x = 0.0125$ and $\Delta t = 0.001$.

x	T	Computed solution			Exact solution
		BDF-1	BDF-2	BDF-3	
0.25	0.4	1.428E−02	1.401E−02	1.400E−02	1.400E−02
	0.6	2.007E−03	1.950E−03	1.949E−03	1.949E−03
	0.8	2.817E−04	2.711E−04	2.708E−04	2.708E−04
0.50	0.4	2.024E−02	1.986E−02	1.985E−02	1.985E−02
	0.6	2.840E−03	2.759E−03	2.757E−03	2.757E−03
	0.8	3.984E−04	3.834E−04	3.830E−04	3.830E−04
0.75	0.4	1.435E−02	1.407E−02	1.407E−02	1.407E−02
	0.6	2.009E−03	1.952E−03	1.950E−03	1.950E−03
	0.8	2.817E−04	2.711E−04	2.709E−04	2.708E−04

Table 17
Comparison of numerical solution with the exact solution at different times for example-2 at $\nu = 0.1$, $\Delta x = 0.0125$ and $\Delta t = 0.01$.

x	T	Computed solution			Exact solution
		BDF-1	BDF-2	BDF-3	
0.25	2.2	5.872E−02	5.814E−02	5.813E−02	5.814E−02
	2.4	4.902E−02	4.850E−02	4.849E−02	4.849E−02
	2.6	4.083E−02	4.035E−02	4.034E−02	4.035E−02
0.50	2.2	9.142E−02	9.046E−02	9.044E−02	9.045E−02
	2.4	7.512E−02	7.426E−02	7.424E−02	7.424E−02
	2.6	6.172E−02	6.096E−02	6.094E−02	6.094E−02
0.75	2.2	7.192E−02	7.110E−02	7.108E−02	7.109E−02
	2.4	5.796E−02	5.724E−02	5.723E−02	5.723E−02
	2.6	4.687E−02	4.625E−02	4.624E−02	4.624E−02

Table 18
Comparison of computed solution generated by BDF-1, BDF-2, BDF-3 and the exact solution at different space points for example-2 at $T = 2.5$, $\nu = 0.1$ and $\Delta t = 0.01$.

x	Computed solution			Exact solution
	BDF-1	BDF-2	BDF-3	
0.1	1.909E−02	1.888E−02	1.888E−02	1.888E−02
0.2	3.683E−02	3.642E−02	3.641E−02	3.641E−02
0.3	5.182E−02	5.123E−02	5.122E−02	5.122E−02
0.4	6.268E−02	6.195E−02	6.194E−02	6.194E−02
0.5	6.809E−02	6.728E−02	6.726E−02	6.727E−02
0.6	6.698E−02	6.616E−02	6.614E−02	6.614E−02
0.7	5.879E−02	5.805E−02	5.804E−02	5.804E−02
0.8	4.383E−02	4.326E−02	4.325E−02	4.325E−02
0.9	2.343E−02	2.313E−02	2.312E−02	2.312E−02

Table 19
Comparison of computed solution generated by BDF-1, BDF-2, BDF-3 and the exact solution at different space points for example-2 at $T = 1$, $\nu = 1$ and $\Delta t = 0.001$.

x	Computed solution			Exact solution
	BDF-1	BDF-2	BDF-3	
0.1	1.727E−05	1.646E−05	1.644E−05	1.644E−05
0.2	3.286E−05	3.131E−05	3.128E−05	3.127E−05
0.3	4.522E−05	4.310E−05	4.305E−05	4.304E−05
0.4	5.316E−05	5.066E−05	5.061E−05	5.060E−05
0.5	5.590E−05	5.327E−05	5.321E−05	5.320E−05
0.6	5.316E−05	5.066E−05	5.061E−05	5.060E−05
0.7	4.522E−05	4.310E−05	4.305E−05	4.304E−05
0.8	3.286E−05	3.131E−05	3.128E−05	3.127E−05
0.9	1.727E−05	1.646E−05	1.644E−05	1.644E−05

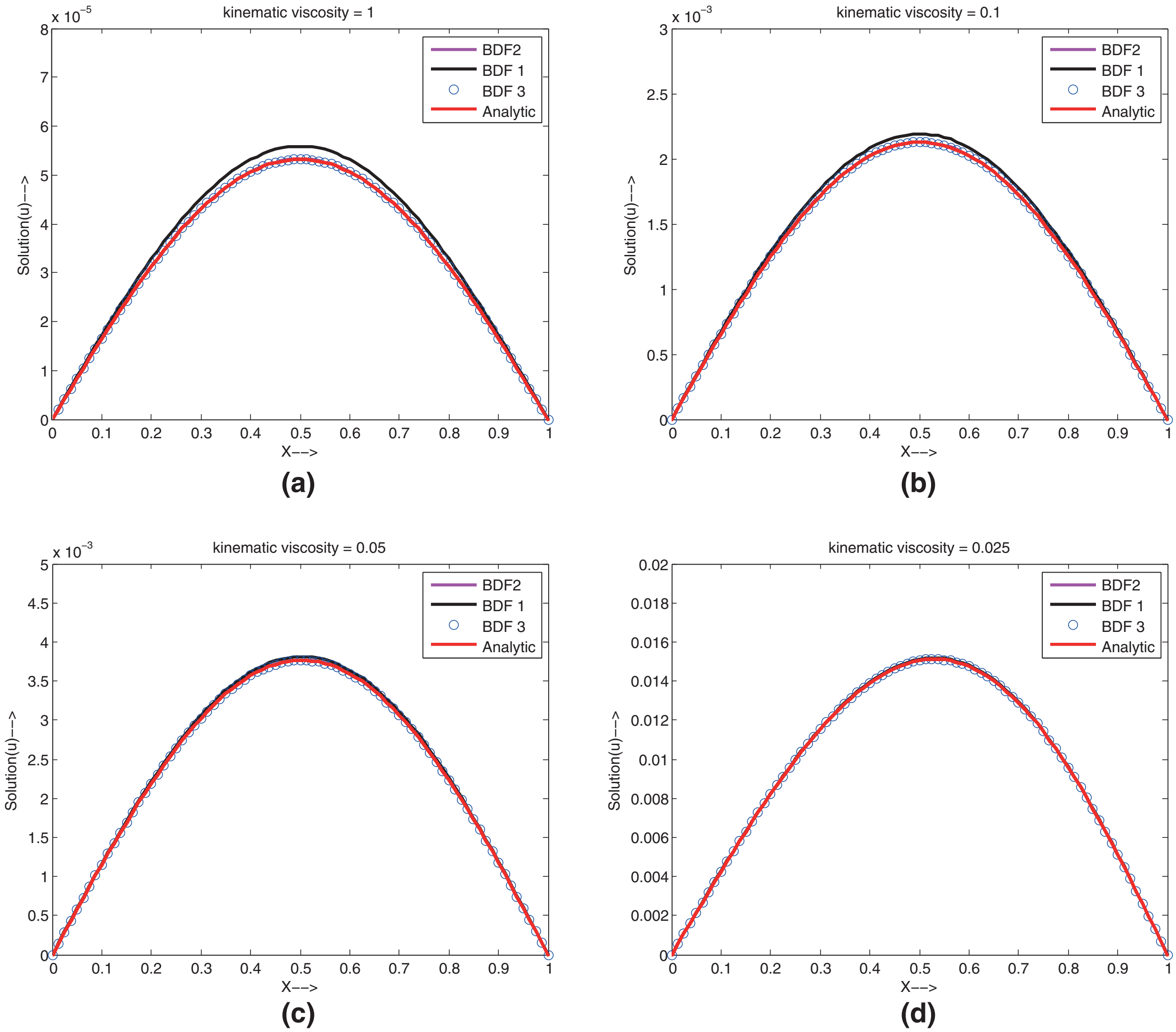

Fig. 3. Numerical solutions of Example 2 by backward differentiation methods for $\Delta x = 0.0125$ and different values of ν , Δt and T, (a) $\nu = 1$, $\Delta t = 0.001$, $T = 1$, (b) $\nu = 0.1$, $\Delta t = 0.01$, $T = 6$, (c)$\nu = 0.05$, $\Delta t = 0.01$, $T = 10$ and (d) $\nu = 0.25$, $\Delta t = 0.01$, $T = 12$.

Table 20
Comparison of the numerical results in L_2 norm and L_∞ norm for different values of T, $N = 80$ and $\Delta t = 0.01$, corresponding to example-2.

$\nu = 0.05$	$T = 5$		$T = 5.2$	
	L_2	L_∞	L_2	L_∞
BDF 1	1.8000E−03	2.8285E−04	1.7000E−03	2.6549E−04
BDF 2	5.3991E−05	8.9093E−06	5.2835E−05	8.6442E−06
BDF 3	4.4162E−05	6.9974E−06	3.9302E−05	6.2250E−06

some values of kinematic viscosity. Fig. 3 represent comparison of the exact and computed solution (BDF-1, BDF-2 and BDF-3) for different values of kinematic viscosity, ν.

The L_2 and L_∞ error norms are calculated at different values of T, $\nu = 0.05$, $\Delta t = 0.01$ and tabulated in Table 20. It shows that numerical results generated by BDF-3 is more accurate than other schemes as error is decreasing even for large values of T. In Fig. 4, we present plots of exact and computed solution generated by BDF-3 at different times. It is observed that the BDF-2 and BDF-3 give better accuracy than BDF-1.

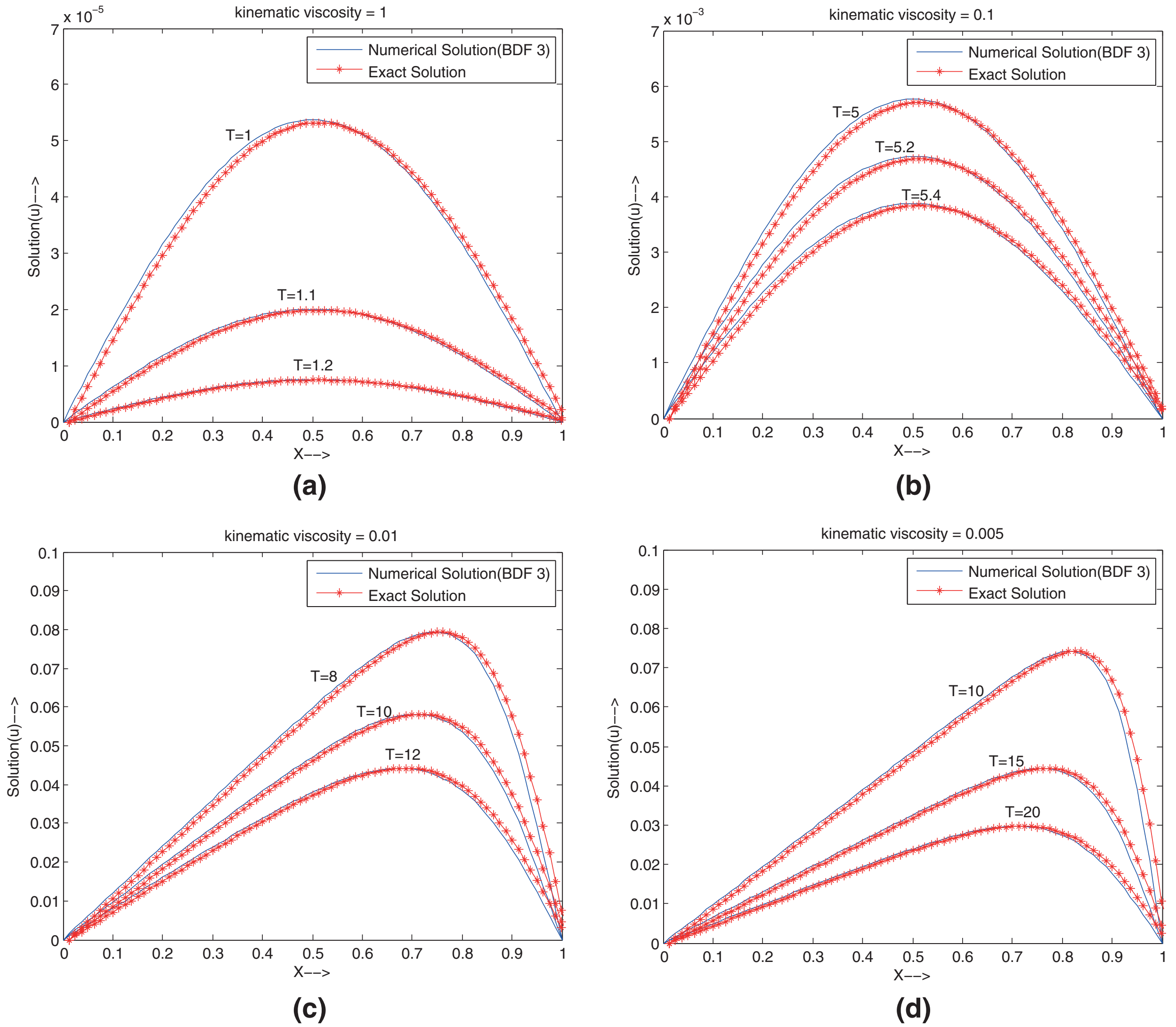

Fig. 4. Numerical solutions of Example 2 at different times for $\Delta x = 0.0125$ and different values of ν and Δt, (a) $\nu = 1$, $\Delta t = 0.0001$, (b) $\nu = 0.1$, $\Delta t = 0.01$, (c) $\nu = 0.01$, $\Delta t = 0.01$ and (d) $\nu = 0.005$, $\Delta t = 0.01$.

7. Conclusions

The quasi-linear Burgers' equation has been linearized by using the non-linear Cole–Hopf transformation into linear diffusion equation. The linear diffusion equation is semi-discretized in variable 'x' by method of lines which yields a set of first order ordinary differential equations. This system of first order ordinary differential equations is solved by backward differentiation formulas of order one, two and three. From the Tables and Figures it is evident that numerical results generated by BDF-2 and BDF-3 are more accurate than BDF-1. Error analysis shows that BDF-3 has accuracy of order $O((\Delta t)^3 + (\Delta x)^2)$, while that of BDF-2 is $O((\Delta t)^2 + (\Delta x)^2)$ and BDF-1 is accurate of order $O(\Delta t + (\Delta x)^2)$. The computed results are reasonably in good agreement with exact solution, moreover, the error decreases even for large values of T. Hence this scheme is an efficient method to solve the nonlinear time-dependent Burgers' equation.

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