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A systematic literature review of Burgers' equation with recent advances

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Abstract. Even if numerical simulation of the Burgers' equation is well documented in the literature, a detailed literature survey indicates that gaps still exist for comparative discussion regarding the physical and mathematical significance of the Burgers' equation. Recently, an increasing interest has been developed within the scientific community, for studying non-linear convective–diffusive partial differential equations partly due to the tremendous improvement in computational capacity. Burgers' equation whose exact solution is well known, is one of the famous non-linear partial differential equations which is suitable for the analysis of various important areas. A brief historical review of not only the mathematical, but also the physical significance of the solution of Burgers' equation is presented, emphasising current research strategies, and the challenges that remain regarding the accuracy, stability and convergence of various schemes are discussed. One of the objectives of this paper is to discuss the recent developments in mathematical modelling of Burgers' equation and thus open doors for improvement. No claim is made that the content of the paper is new. However, it is a sincere effort to outline the physical and mathematical importance of Burgers' equation in the most simplified ways. We throw some light on the plethora of challenges which need to be overcome in the research areas and give motivation for the next breakthrough to take place in a numerical simulation of ordinary/partial differential equations.

Keywords. Burgers' equation; non-linear convection–diffusion equation; Hopf–Cole transformation; numerical solutions.

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1. Introduction

The numerical simulation of non-linear convective–diffusive partial differential equations encountered in computational fluid dynamics (CFD) has been a significant research topic for many decades, both in heat transfer and in fluid mechanics. CFD has been extensively used for various engineering analysis. Since the time of Newton, people have been aware of numerical methods. During this period, various solution methods of ordinary differential equations (ODEs) and partial differential equations (PDEs) were conceptually conceived on paper by the scientific community. Owing to extraordinary progress in computational capacity, computer speed has improved much more rapidly compared to its cost. A CFD is a powerful tool for solving a wide variety of modern engineering problems. CFD provides quantitative as well as qualitative analysis of the system. It helps in the simulation of solid–fluid interaction. More and more complex PDEs are developed to consider various real life parameters affecting the engineering systems. It is necessary to have a deep understanding of the mechanism of fluid flow through a pipe, as it is extensively used in advanced engineering systems, industry and our day-to-day life. In human body, the blood is flowing continuously through the arteries and veins. All the modern biomedical instruments like artificial hearts and dialysis systems work on the basis of fluid flow through pipes. In refrigeration and air conditioning application, refrigerants are flowing through pipes. On a broader scale, PDEs

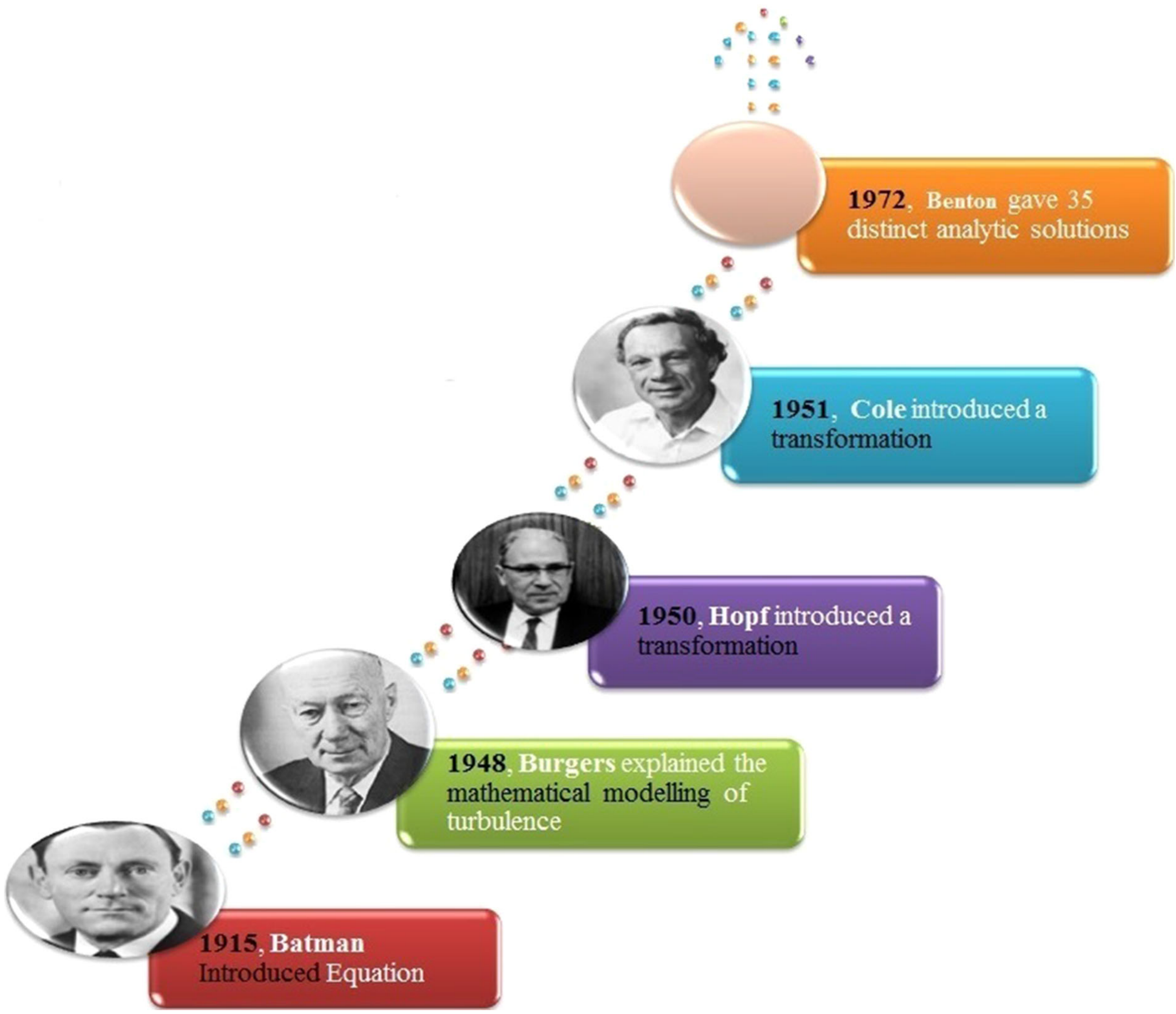

Figure 1. Historical milestones in the development of Burgers' equation.

explaining all these physical processes play important roles in the design and analysis of modern engineering systems.

In 1915, Harry Bateman (1882–1946) [1], an English mathematician, introduced a partial differential equation in his paper along with its corresponding initial and boundary conditions given by eqs (1)–(3)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad 0 < t < \tau, \tag{1}$$

$$u(x,0) = \psi(x), \quad 0 < x < L, \tag{2}$$

$$u(0,t) = \zeta_1(t), \quad u(L,t) = \zeta_2(t), \quad 0 < t < \tau, \tag{3}$$

where u, x, t and ν are the velocity, spatial coordinate, time and kinematic viscosity, respectively. ψ, ζ_1 and ζ_2 are prescribed functions of variables depending upon the specific conditions for the problem to be solved. Later in 1948, Johannes Martinus Burgers (1895–1981) [2–4], a Dutch physicist, explained the mathematical modelling of turbulence with the help of eq. (1). Burgers went on to become one of the leading figures in the field of fluid mechanics. To honour the contributions of Burgers, this equation is well known as the Burgers' equation. Eberhard Hopf (1902–1983) [5] and Julian David Cole (1925–1999) [6] independently introduced a transformation to convert Burgers' equation into a linear heat equation and solved exactly for an arbitrary initial condition. Hence, the transformation is famously known as the Hopf–Cole transformation given by eq. (4).

$$u(x,t) = -2\nu\frac{\theta_x}{\theta}, \tag{4}$$

where θ satisfies the famous heat equation

$$\frac{\partial \theta}{\partial t} = \nu\frac{\partial^2 \theta}{\partial x^2}. \tag{5}$$

The historical milestones in the development of Burgers' equation is given in figure 1.

2. Physical significance

During the last few decades, significant efforts have been oriented towards the development of robust computational schemes to handle non-linear PDEs found in fluid mechanics and heat transfer. One of the most celebrated equation involving both non-linear propagation effects and diffusive effects is the Burgers' equation. Burgers' equation, being a non-linear PDE, represents various physical problems arising in engineering, which are inherently difficult to solve. The simultaneous presence of non-linear convective term ($u(\partial u/\partial x)$) and diffusive term ($\nu(\partial^2 u/\partial x^2)$) add an additional feature to the Burgers' equation. When ν approaches zero, eq. (1) become inviscid Burgers' equation, which is a model for non-linear wave propagation. The unsteady heat equation without internal heat generation arises in the mathematical modelling of many physical phenomena occurring in nature. When u approaches zero, eq. (1) becomes the heat equation. When temperature varies with respect to time in a solid material, there will be a corresponding variation of the rate of heat transfer within the solid as well as at its boundary. In many practical engineering problems, the determination of temperature distribution is required to calculate the local heat transfer rate, thermal expansion and thermal stress at some crucial location. Burgers' equation, which is a fundamental partial differential equation from fluid mechanics, proves to be a very good example because of the following reasons:

1. The exact solution of the partial differential equation is well known.
2. It can be thought of as a hyperbolic problem with artificial diffusion for small kinematic viscosity ν and heat equation for very small u.
3. It can be used in boundary layer calculation for the flow of viscous fluid.
4. It forms a standard test problem for the PDE solvers.
5. It is suitable for analysis in various fields like sedimentation of polydispersive suspensions and colloids, aspect of turbulence, non-linear wave propagation, growth of molecular interfaces, longitudinal elastic waves in isotropic solids, traffic flow, cosmology, gas dynamics and shock wave theory.

Hence, a multidisciplinary approach is required to study Burgers' equation as shown in figure 2.

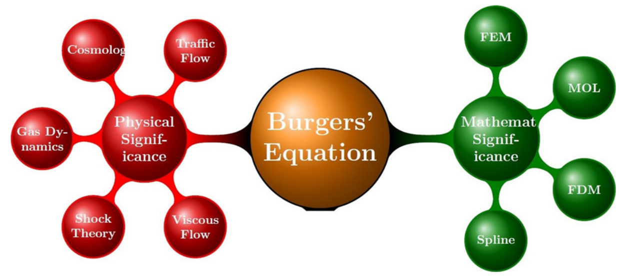

Figure 2. A multidisciplinary approach.

2.1 *Viscous flow and turbulence*

Several important developments have been made in the field of viscous flow and turbulence describing aerodynamic flow at standard temperatures and pressures. Burgers' equation, having much in common with the Navier–Stokes equation, plays a vital role in analysing fluid turbulence. Murray [7] found that there is an ultimate steady turbulent state. The author concluded that small disturbances ultimately grow into a single large domain of relatively smooth flow, accompanied by a vortex sheet in which strong vorticity is concentrated. The interface of the control theory and fluid dynamics has a large number of practical applications including feedback control of turbulence for drag reduction. A non-linear Galerkin's method was used by Baker *et al* [8] to propose a methodology for the synthesis of non-linear finite-dimensional feedback controllers. They used one-dimensional Burgers' equation (6) with distributed control

$$\frac{\partial U}{\partial t} = \frac{1}{\mathrm{Re}}\frac{\partial^2 U}{\partial z^2} - U\frac{\partial U}{\partial z} + b(z)u(t), \tag{6}$$

where $b(z)$ is the actuator distribution function. A spatially periodic inviscid random forced Burgers' equation in arbitrary dimension and the randomly time-dependent Lagrangian system related to it was considered by Iturriaga and Khanin [9]. Bogaevsky [10] showed that matter accumulates in the shock discontinuities. Bec and Kahanin [11] explained Burgers' turbulence as the study of solutions to the one- or multidimensional Burgers' equation with random initial conditions or random forcing. The study of random Lagrangian systems, of stochastic partial differential equations and their invariant measures, the theory of dynamical systems, the applications of field theory to the understanding of dissipative anomalies and of multiscaling in hydrodynamic turbulence have benefited significantly from progress in Burgers' turbulence. Recently, it become a point of attraction to many researchers due to the new emerging application of a Burgers' model to statistical physics, cosmology and fluid dynamics. Navier–Stokes turbulence has one important property: sensitivity to small perturbations in the initial data and thus the spontaneous rise of randomness by chaotic dynamics.

The results obtained by Chekhlov and Yakhot [12] in a simple one-dimensional system, were similar to the outcome of experimental investigations of the real-life three-dimensional turbulence. They considered the dynamics of velocity fluctuations driven by a white-in-time random force, in the one-dimensional Burgers' equation, which showed a biscaling behaviour [13]. WKB calculations are very useful for understanding the behaviour of the randomly driven Burgers' equation. The instanton solution, which describes the exponential tail of the probability distribution function of velocity differences for large positive velocity differences, was found by Gurarie and Migdal [14]. The properties of the probability density function of velocity differences were investigated for the three different cases by Yakhot and Chekhlov [15]. In turbulence, the deviation of the probability density function tails from Gaussian was regarded as a manifestation of intermittency [16]. Particular interest, of solutions, of the randomly forced Burgers' equation, were the asymptotic properties of probability distribution functions associated with velocity gradients and velocity increments. Weinan *et al* [17] studied the asymptotic behaviour of the probability distribution of velocity gradients and velocity increments using a new and direct approach for analysing the scaling properties of the various distribution functions for the randomly forced Burgers' equation.

The randomly forced Burgers' equation, which is periodic in x with period 1, and with white noise in t, is a prototype for a very wide range of problems in non-equilibrium statistical physics, where strong non-linear effects are present [18]. Bec *et al* [19] presented the fast Legendre transform numerical scheme for space-periodic kicked Burgers' turbulence with spatially smooth forcing. Bec *et al* [20] had shown theoretically and numerically that the shocks have a universal global structure which was determined by the topology of the configuration space. They studied the dynamics of the multidimensional randomly forced Burgers' equation in the limit of vanishing viscosity. Bec and Khanin [21] studied the inviscid randomly forced Burgers' equation with non-periodic forcing on the whole real line started at $t = -\infty$. They introduced variational approach to Burgers' turbulence and presented results on the existence and uniqueness of the main shock and the global minimiser. Gomes *et al* [22] presented a very simple and soft approach, which allows to construct stationary distributions for randomly forced viscous Hamilton–Jacobi and Burgers' equations and proved convergence of this distributions in the limit of vanishing viscosity. The global solutions satisfy uniform Lipschitz and semiconvexity properties. The nature of the fluctuational mechanism is very general, but it guarantees only a very slow rate of convergence. It was used for the uniqueness of global solutions and convergence of the solutions to the Cauchy problems.

2.2 *Shock theory*

Burgers' equation is similar to the Navier–Stokes equation without the pressure term. For low kinematic viscosity, there can be velocity discontinuities, i.e. shocks. From a physical point of view, viscous model better explains what happens in actual than in inviscid equation. Kreiss and Kreiss [23] considered the viscous Burgers' equation with the initial and non-homogeneous Dirichlet boundary conditions to study the effect of shock on the convergence of steady-state solution.

$$u_t + \frac{1}{2}(u^2)_x = \epsilon u_{xx} + f(x),$$
$$t \geq 0, 0 \leq x \leq 1, \epsilon > 0, \tag{7}$$
$$u(x, 0) = g(x), \tag{8}$$
$$u(0, t) = a, \quad u(1, t) = b \tag{9}$$

and the corresponding steady-state problem

$$\frac{1}{2}(y^2)_x = \epsilon y_{xx} + f(x), \quad 0 \leq x \leq 1, \epsilon > 0, \tag{10}$$
$$y(0) = a, \tag{11}$$
$$y(1) = b. \tag{12}$$

They investigated uniqueness, the existence and properties of the steady-state solution. It was proved that the steady-state problem has a unique solution and for sufficiently large ϵ, the steady-state equation has a solution. Moreover, steady-state equation has a unique solution for all $\epsilon > 0$. To speed up the convergence to steady state, the shock should be located at the boundary. On the other hand, location of shock in the interior leads to very slow convergence. The speed of convergence was studied by analysing the corresponding eigenvalue problem. The non-linear Burgers' equation was solved by using shock-capturing schemes for problems involving formation and propagation of shocks, shock collisions and expansion of discontinuities [24].

Reyna and Ward [25] were motivated by the work of Kreiss and Kreiss on the initial boundary value problem for Burgers' equation. They investigated the shock layer behaviour associated with the following viscous shock problem in the limit $\epsilon \to 0$:

$$u_t + [f(u)]_x = \epsilon u_{xx}, \quad 0 < x < 1, t > 0, \tag{13}$$
$$u(x, 0) = u_0(x), \quad u(0, t) = \alpha, \quad u(1, t) = -\alpha. \tag{14}$$

Here α is a positive constant and the convex non-linearity $f(u)$ satisfies $f(0) = f'(0) = 0, uf'(0) > 0$ for $u \neq 0$ and $f(\alpha) = f(-\alpha)$. For Burgers' equation $f(u) = u^2/2$. Their main goal was to analyse the slow shock layer motion for eq. (13) analytically in the limit $\epsilon \to 0$. Analytical explanation for the exponentially slow phase where the shock layer drifts to its equilibrium point was given. The method of matched asymptotic expansions (MMAE) was used to construct equilibrium solution to Burgers' equation. From symmetry, the equilibrium shock layer solution for eq. (13) is at $x = 1/2$. Using MMAE, they obtained one-parameter family of approximate solutions parametrised by x_0.

$$u = -\alpha \tanh[\alpha\epsilon^{-1}(x - x_0)/2]x_0. \tag{15}$$

In the case of equilibrium problem, to eliminate the indeterminacy in the MMAE solution for the location of the shock layer, projection method was used. Burgers' equation is applicable for time-dependent problems. Hence, they used projection method to derive an equation of motion for the trajectory of the slow moving shock layer. The result shows that the solution to eq. (13) is given asymptotically by $u \sim -\alpha \tanh[\alpha\epsilon^{-1}(x-x_0(t))/2]$, during the slow evolutionary phase, where $x_0(t)$ satisfies the ODE

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = \alpha[\mathrm{e}^{-\alpha\epsilon^{-1}x_0} - \mathrm{e}^{-\alpha\epsilon^{-1}(1-x_0)}]. \tag{16}$$

Their main contribution is to show that even if the boundary operator is changed by small value it may destabilise the equilibrium solution. It results in hitting the shock layer to one of the physical boundaries instead of drifting to its equilibrium position. Numerical method based on preliminary WKB-type transformation of eq. (13) was formulated to study the shock layer behaviour for $\epsilon \ll 1$. It is advantageous because it magnifies the exponentially weak interactions and exponentially long time-scale associated with the shock layer motion becomes transparent. Simple finite-difference method can be used on the well condition equilibrium transformed problem to study the shock layer behaviour numerically for rather small ϵ.

2.3 *Gas dynamics*

The Burgers' equation shows the interplay of convection and diffusion present in viscous fluid flow engineering problem [26]. Also, Burgers' equation with source terms appeared in the aerodynamics theory [27].

$$u_t + uu_x = \nu u_x x - \lambda u. \tag{17}$$

The contribution was the manipulation of a mathematical problem from the gas dynamics theory, which was related to the heat exchange in the boundary layer. Heat can be released in physical application of gas dynamics, which can be presented in chemically reacting boundary layer or in the propagation of the detonation wave. Near-equilibrium boundary layers of a gas by presenting heat source terms can be considered using Burgers' equation, i.e.,

$$u_t + uu_x = \nu u_x x - \lambda u; \quad x \geq 0, t \geq 0, \lambda \geq 0, \tag{18}$$

$$u(0, x) = \phi(x), \quad u(t, 0) = f(t). \tag{19}$$

The term λu in eq. (18) represents the heat released in the boundary layer. Analytical solution in parametric form of the Burgers' equation with source term was constructed. Lagrange subsidiary equations and splitting of the second-order non-linear PDE were used for constructing solutions. As a result, Monge's equation was generated.

2.4 *Cosmology*

The expanding Universe is irregular and clumpy compared to its uniform initial state [28]. Three-dimensional Burgers' equation describes the characteristics of a random potential vector field, which was investigated by Gurbatov and Saichev [29]. In this paper, they discussed the possible relationship between the large-scale structure of the Universe and the cellular structure. The three-dimensional Burgers' equation

$$V_t + (V\triangledown)V = \nu\Delta V \tag{20}$$

$$V(r, t = 0) = V_0(r) \tag{21}$$

may also be an acceptable model for the turbulence theory. The way the field $V(r, t)$ behaves described by eq. (20) is similar to the growth of the Universe lies in the fact that the field $V(r, t)$ acquires in the time a universal cellular structure, the presence and reason of formation. As per the author, one of the mechanisms of formation of cellular structure of the Universe, the inertial instability, can be described by analysing the dynamics and statistics of the fields described by eq. (20). They investigated the dynamics and statistics of random potential fields, described by eq. (20). The possible connection of their characteristics with the large-scale structure of the Universe was discussed. The adhesion model of large-scale distribution of matter in the Universe is formulated by Burgers' turbulence [30–32].

Peebles [33] explained that right after the decoupling between baryons and photons, the promotive Universe is a rarefied medium without pressure composed mainly of non-collisional dust interacting through Newtonian gravity. The initial density of this dark matter fluctuation is responsible for the formulation of the large-scale structure in which both the dark non-baryonic matter and the luminous baryonic matter concentrate. A hydrodynamical formulation of the cosmological problem leads to a description where matter evolves with a velocity $\bar{V}$. Molchanov *et al* [34] studied the dynamics of the structure of shock fronts in the inviscid non-homogeneous Burgers' equation in R^d in the presence of random forcing due to a degenerate potential. The influence of the clusters through their gravitational field was considered by Burgers' equation with external force field $\vec{F}$ as given by eq. (22). The development of cluster is a relative slow process.

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\bigtriangledown)\vec{v} = \frac{1}{2}\mu \bigtriangledown^2 \vec{v} - \vec{F}, \quad \vec{v} \in R^3. \tag{22}$$

2.5 *Traffic flow*

Burgers' equation shows a strong connection between fluid dynamics and traffic flow models. In the theory of traffic flow, use of Greenshields [35] model leads to the formation of Burgers' equation. Traffic current on expressway is defined as the number of cars passing at a reference point per unit time [36]. If vehicle concentration is linearly related to the drift speed in the one-dimensional case, then vehicle concentration on the road can be explained by Burgers' equation in a moving frame of reference [37]. From the sophisticated level of fluid-dynamical approach, no traffic jam forms spontaneously from a state of uniform density. Leibig [38] has studied how a random initial distribution of steps in the density profile evolves with time. If we treat traffic as an effectively one-dimensional compressible fluid, then in analogy with the hydrodynamic theory of fluids, a 'macroscopic' theory of traffic can be developed [39]. Traffic flow can be treated as an effectively one-dimensional compressible fluid (a continuum) when viewed from a long distance [40]. Second-order model of very light traffic flow is given by Aw and Rascle [41]. The density wave in traffic flow, which gradually changes from non-uniform to uniform distribution, is described by Burgers' equation [42]. Equation (23) is considered as one-dimensional non-linear non-homogeneous Burgers' equation, which is applicable to other physical phenomena such as design of feedback control [43], electrohydrodynamic field in plasma physics [44] and wind forcing the build-up of water waves [45].

$$u_t + uu_x = \nu u_{xx} + F(x, t). \tag{23}$$

2.6 *Quantum field*

The Burgers' equation is the simplest example where a quantum computer was demonstrably more efficient at numerically predicting the time-dependent solutions [46]. The Burgers' equation is derived as an effective field theory governing the behaviour of the quantum computer at its macroscopic scale, where both the lattice cell size and the time step interval become infinitesimal. A microscopic scale algorithm for a type-II quantum computer was presented for modelling the time evolution of a continuous field governed by the non-linear Burgers' equation in one spatial dimension in [47]. The quantum model is a system of qubits, where there exists a minimum time interval in the time-dependent dynamics. The measurement steps are dispersed periodically in time and across all the elements of the quantum system. Yepez [48] presented an analysis of an open quantum model of the time-dependent evolution of a flow field governed by the non-linear Burgers' equation in one spatial dimension. This allows to use the quantum algorithm for modelling highly non-linear shock formation, even severely under-resolved shock fronts, without the model breaking down. The flow field evolving under the Burgers' equation develops sharp features over time. Therefore, it is a better test of liquid-state nuclear magnetic resonance (NMR) implementations of type-II quantum computers than examples using the diffusion equation. The practicality of implementing the quantum lattice gas (QLG) algorithm using a spatial NMR technique was proved by the numerical data and the exact analytical results for the non-linear Burgers' equation [49].

3. Further significance

The Burger's equation has been used as a benchmark problem in parallel and distributed numerical computation, i.e., algorithms can be tested with the help of known analytical solutions of Burger's equation. An OpenMPI-based hybrid space–time parallel algorithm is discussed in [50]. GPU-based parallel computing algorithm for the numerical solution of one-dimensional Burger's equation is presented in [51]. In [52], CUDA Fortran was used for the problem presented in [51]. A parallel computing technique for two-dimensional Burger's equation is discussed in [53] with OpenMPI and GPU implementation. To find solutions of non-linear partial differential equations, Puffer *et al* [54] have used the cellular neural network and Hayati and Karami [55] have used feedforward neural network learning algorithms. It was tested and validated for Burgers' equation.

Zueco [56] has introduced a new idea for the simulation of Burger's equation through network simulation method based on an electrical motion analogy. It was described that the discretised equation was analogous to an electric network model, and hence, solved using an electrical circuit simulation software called Pspice. In a study by Hetmanczyk and Ochs [57], inviscid as well as the viscous Burger's equations were simulated using wave digital simulation technique. First, they derived an electrical circuit corresponding to the equations and then found wave-digital realisation by using a circuit equivalent to a Gears 2-step backward differentiation formula. The results were compared with exact solution and solutions from the trapezoidal rule. They found that Gears method handles discontinuities efficiently.

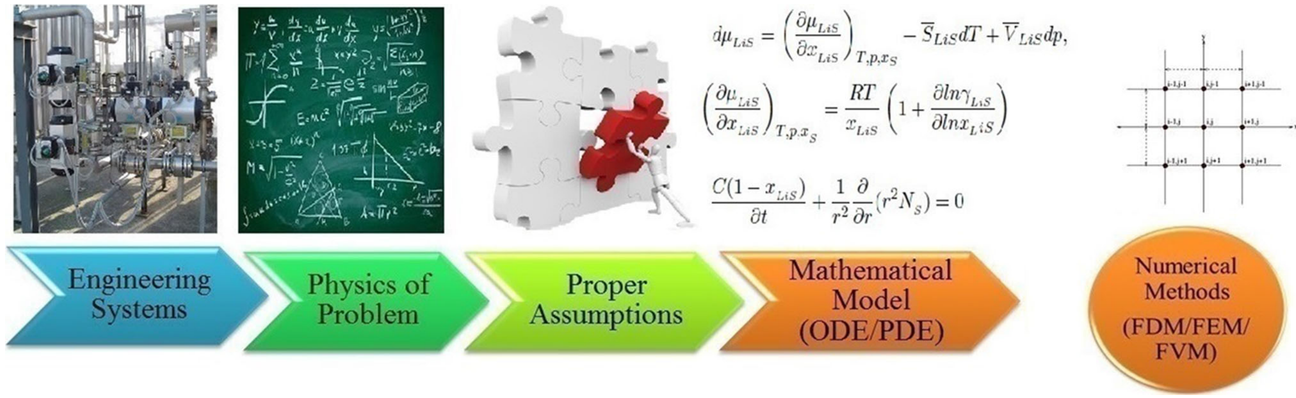

Figure 3. Process map for numerical simulation.

In 1995, Esipov [58] derived and studied the physics of coupled Burgers' equation and considered it as a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids under the effect of gravity. The study of the fluctuations of concentration during sedimentation may lead to an experimental realisation of Burgers' turbulence. Tsai *et al* [59] employed inviscid Burgers' equation for the validation of a parallel domain-decomposed Chebyshev collection method developed for atmospheric model simulations. In [60], Kalman filter of Burgers' equation is used to examine the problems associated with atmospheric data assimilation and numerical weather prediction. The forced Burgers' equation is considered in [61] for studying various errors encountered in the simulations of atmospheric flow. Propagation of high-intensity noise in the atmosphere can be simulated using Burgers' equation [62], and this is the key idea used for the prediction of propagation of noise from jet aircraft and helicopters [63–65]. A recent study [66] describes Burgers' equation as a mathematical model for one-dimensional groundwater recharge by spreading.

Control problems have broad applications in the real life, but introducing a control variable in the governing equation such as non-linear partial differential equation like Burgers' is not easy. Plenty of articles are available for the control problems in Burgers' equation, boundary control techniques, and its numerical methods are discussed in [67–69] and various distributed control methods can be found in [70–72]. Smaoui [73] studied boundary and distributed control of Burgers' equation and conducted numerical and analytical stability analysis. Glass and Guerrero [74] have proposed a boundary control for the viscous Burgers' equation for small kinematic viscosity, and proved the exact controllability property to a non-zero steady-state situation.

4. Mathematical significance

Consequently, numerical solution of the PDEs has been a significant research topic for many decades, in both heat transfer and fluid mechanics. Process map for numerical simulation is shown in figure 3. First step in this process is to understand the physics of the problem, which lead to the formation of a mathematical model with the help of equations. In most of the cases, these equations are either ODEs or PDEs. Some assumptions have to be made, because the real life problems in engineering are a bit complex to analyse. After this, these equations are solved by numerical methods like finite-difference method, finite-element method and finite-volume method.

Rodin [75] studied some approximate and exact solutions of boundary value problem for Burgers' equation with the help of Hopf–Cole transformation. Benton and Platzman [76] have given 35 distinct analytical solutions of Burgers' equation with different initial conditions. Wolf *et al* [77] discovered a procedure to extend the analytical solution of Burgers' equation to the n-dimensional problems by employing group actions on coset bundles. Nerney *et al* [78] extended the solutions to the curvilinear coordinate systems. Kudryavtsev and Sapozhnikov [79] proposed a method to find exact solution of inhomogeneous Burgers' equation using Hopf–Cole and Darboux transformations. In [80], analytical solutions of $(1+n)$-dimensional Burgers' equation have been computed using various semianalytical methods like homotopy perturbation method, Adomian decomposition method and differential transform method. The closed form of the solution has been successfully computed for $(1+n)$, $(1+3)$ and $(1+2)$ dimensions with an initial condition equal to the sum of its spatial coordinates, i.e., $u(x_1, x_2, \ldots, x_n, t=0) = x_1 + x_2 + \cdots + x_n$ and for $(1+1)$ dimension with an initial condition $u(x, t=0) = 2x$. In [81], Elzaki transformation and homotopy perturbation method have been combined to get a semianalytic solution of Burgers' equation. Various numerical schemes and a few of the commonly used analytical solutions are listed here.

4.1 *Solution for a smooth initial condition*

$$u(x,t) = \frac{2\pi\nu}{L} \frac{\sum_{n=1}^{\infty} C_n \exp(-n^2\pi^2\nu t/L^2) n \sin(n\pi x/L)}{C_0 + \sum_{n=1}^{\infty} C_n \exp(-n^2\pi^2\nu t/L^2)\cos(n\pi x/L)}. \tag{24}$$

Cole [6] has introduced this analytical solution (eq. (24)) of Burgers' equation (eq. (1)) for a periodic initial disturbance $\psi(x)$ and boundary conditions $\zeta_1(t) = \zeta_2(t) = 0$. C_0 and C_n are the Fourier coefficients and Cole [6] has also provided a formula to compute coefficients for an arbitrary periodic initial condition. For a sinusoidal initial condition (25), these coefficients can be determined using eqs (26) and (27).

$$\psi(x) = u_0 \sin\left(\frac{\pi x}{L}\right), \tag{25}$$

$$C_0 = \frac{1}{L}\int_0^L \exp\left[-\frac{u_0 L}{2\pi\nu}\left(1-\cos\frac{\pi x}{L}\right)\right] \mathrm{d}x, \tag{26}$$

$$C_n = \frac{2}{L}\int_0^L \exp\left[-\frac{u_0 L}{2\pi\nu}\left(1-\cos\frac{\pi x}{L}\right)\right] \times \cos\frac{n\pi x}{L} \mathrm{d}x. \tag{27}$$

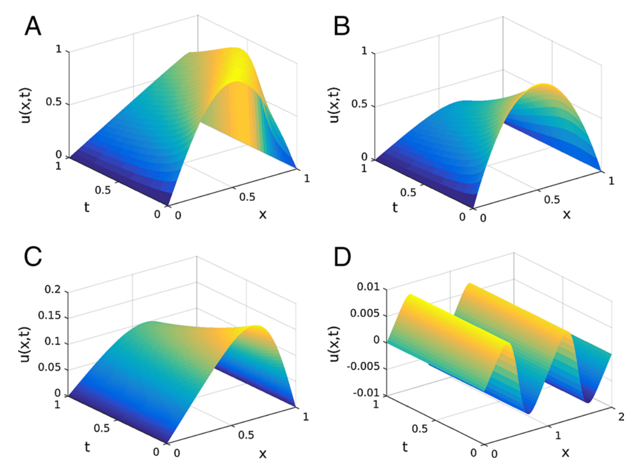

Figure 4. Physical behaviour of Burgers' equation for smooth initial conditions.

Wood [82] has introduced a closed form of exact solution for the Burgers' equation over the domain $0 \le x \le 1$ and $t \ge 0$, in the form

$$u(x,t) = 2\pi\nu \frac{\sin(\pi x)\exp(-\pi^2\nu t)}{\alpha + \cos(\pi x)\exp(-\pi^2\nu t)}. \tag{31}$$

Initial and boundary conditions can be derived from the exact solution (eq. (31)). Figure 4C shows the surface plot of $u(x,t)$ for $\alpha = 2$ and $\nu = 0.05$. Cecchi *et al* [83] have considered an analytical solution analogous to the sinusoidal wave propagation in a viscous medium as given in eq. (32) for a homogeneous boundary condition and periodic initial condition $\psi(x) = u(x, t = 0)$ over $0 \le x \le 2$.

$$u(x,t) = 2\pi\nu \frac{\sin(\pi x)\exp(-\pi^2\nu^2 t) + 4\sin(2\pi x)\exp(-4\pi^2\nu^2 t)}{4 + \cos(\pi x)\exp(-\pi^2\nu^2 t) + 2\cos(2\pi x)\exp(-4\pi^2\nu^2 t)}. \tag{32}$$

The solution corresponding to this initial condition is plotted in figure 4A for $u_0 = 1$ and $\nu = 0.01$ over a space domain $0 \le x \le 1$.

$$\psi(x) = u_0 x\,(L - x). \tag{28}$$

For an initial condition like eq. (28), the coefficients C_0 and C_n can be determined as

$$C_0 = \frac{1}{L}\int_0^L \exp\left(-\frac{u_0 x^2(3L-2x)}{12\nu}\right) \mathrm{d}x, \tag{29}$$

$$C_n = \frac{2}{L}\int_0^L \exp\left(-\frac{u_0 x^2(3L-2x)}{12\nu}\right) \times \cos\frac{n\pi x}{L} \mathrm{d}x. \tag{30}$$

Physical behaviour of the solution corresponding to this initial condition is plotted in figure 4B for $u_0 = 4$ and $\nu = 0.1$ over a space domain $0 \le x \le 1$.

For $\mu = 0.01$, the behaviour of $u(x,t)$ is plotted in figure 4D.

4.2 *Shock and travelling wave solutions*

$$u(x,t) = U - u_1 \tanh\frac{u_1\,(x - x_1 - Ut)}{2\nu}. \tag{33}$$

Equation (33) represents a shock wave with a velocity U. u_1 and x_1 are constants, such that $u_1 = u(-\infty, 0)$, $u_2 = u(+\infty, 0)$ and $u_1 > u_2$, $x_1 = x$ at which discontinuity occurs. This problem is presented in Cole's study [6] and he mentioned that Bateman [1] has used eq. (33) with $U = 0$ as a steady-state solution. Many researchers have been using a simplified version of eq. (33) by making $U = u_1 = \lambda/2$ and $x_1 = 0$ as given in eq. (34). Equation (34) is plotted in figure 5A for $\lambda = 1$ and $\nu = 0.01$.

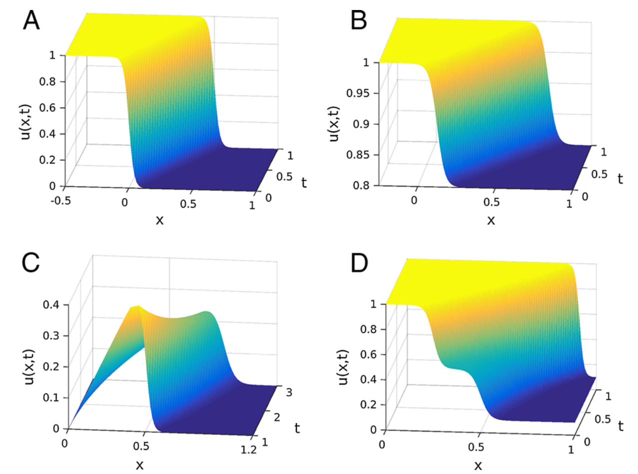

Figure 5. Examples of travelling wave and shock wave solutions of viscous Burgers' equation.

$$u(x,t) = \frac{\lambda}{2}\left(1 + \tanh\left(\frac{\lambda}{8\nu}(\lambda t - 2x)\right)\right). \tag{34}$$

Christie *et al* [84] constructed a travelling wave analytical solution for Burgers' equation in the form

$$u(x,t) = \frac{\mu + \alpha + (\mu - \alpha)\exp(\eta)}{1 + \exp(\eta)}, \tag{35}$$

$$\eta = \frac{\alpha(x - \mu t - \beta)}{\nu}, \tag{36}$$

where α, β and μ are constants, μ is the speed of the wave and β represents the point in x at which initial discontinuity occurs. In most of the articles, values of these constants were chosen as $\alpha = 0.4$, $\beta = 0.125$ and $\mu = 0.6$. The behaviour of the solution (eq. (35)) corresponding to these constants and $\nu = 0.01$ is given in figure 5B. An example of shock wave (eq. (37)) can be found in [85] and its surface plot is given in figure 5C for $\nu = 0.005$.

$$u(x,t) = \frac{x/t}{1 + (t/t_0)^{1/2}\exp(x^2/4\nu t)}, \quad t \geq 1, \tag{37}$$

$$t_0 = \exp\left(\frac{1}{8\nu}\right). \tag{38}$$

Another interesting shock wave solution of Burgers' equation is given in eq. (39) and in figure 5D for $\nu = 0.5$.

$$u(x,t) = \frac{0.1\mathrm{e}^{-A} + 0.5\mathrm{e}^{-B} + \mathrm{e}^{-C}}{\mathrm{e}^{-A} + \mathrm{e}^{-B} + \mathrm{e}^{-C}}, \tag{39}$$

$$A = \left(\frac{0.05}{\nu}\right)(x - 0.5 + 4.95t), \tag{40}$$

$$B = \left(\frac{0.25}{\nu}\right)(x - 0.5 + 0.75t), \tag{41}$$

$$C = \left(\frac{0.5}{\nu}\right)(x - 0.375). \tag{42}$$

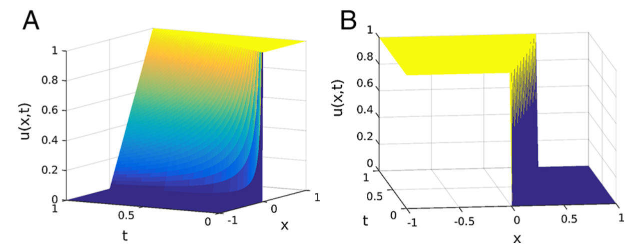

Figure 6. Examples of shock wave and rarefaction wave solutions of inviscid Burgers' equation.

The initial and boundary conditions related to the exact solutions considered in this section can be deduced from the exact solution itself, by proper substitution of x and t values. For more travelling wave solutions, refer to Salas [86].

4.3 *Shock and rarefaction waves in inviscid Burgers' equation*

Consider Riemann-type initial conditions as given in eq. (43), such that $u_1 \neq u_2$ at $x = 0$. Inviscid Burgers' equation possesses a shock or rarefaction wave solution depending on whether $u_1 > u_2$ or $u_1 < u_2$ in eq. (43). For $u_1 = 1$ and $u_2 = 0$, the solution is given in eq. (44) and $S = 1/2$ represents the shock speed. The shock propagation is captured in figure 6A.

$$\psi(x) = \begin{cases} u_1, & x \leq 0, \\ u_2, & x > 0, \end{cases} \tag{43}$$

$$u(x,t) = \begin{cases} 1, & x - St < 0, \\ 0, & x - St > 0. \end{cases} \tag{44}$$

When $u_1 = 0$ and $u_2 = 1$, inviscid Burgers' equation has a rarefaction wave solution given in eq. (45) and its propagation is captured in figure 6B. Simulation of the inviscid Burgers' equation is a challenge for many numerical schemes because of the presence of discontinuity in the solution.

$$u(x,t) = \begin{cases} 0, & (x/t) < 0 \\ x/t, & 0 < (x/t) < 1. \\ 1, & (x/t) > 1 \end{cases} \tag{45}$$

4.4 *Finite-difference method (FDM)*

Various methods proposed for mathematical modelling have their own advantages and disadvantages. Out of these, FDM, the discretisation method, is the most simple and oldest method to solve ODEs/PDEs. In 1983, Fletcher [87] gave exact solutions of some specified two-dimensional Burgers' equation based on two-dimensional Hopf–Cole transformation. Unlike the one-dimensional Burgers' equation, two-dimensional Hopf–Cole transformation cannot be used to convert two-dimensional Burgers' equation into a linear heat equation. For using two-dimensional Hopf–Cole transformation, the condition of potential symmetry must be satisfied by the two-dimensional Burgers' equation. The characteristics of Burgers' equation was studied by Aref and Daripa [88] using phase plane analysis. They have applied the finite-difference method for the semidiscretisation of space variable over a few grid points resulting in a system of coupled ODEs. Phase plane analysis of these systems of ODEs has been conducted to study the characteristics of Burgers' equation. Kutluay *et al* [89] used Hopf–Cole transformation to convert Burgers' equation to heat equation. The transformed heat equation with the insulated boundary conditions was solved by explicit and exact-explicit finite-difference method. In 2003, Bahadir [90] proposed a fully implicit finite-difference scheme, while the non-linear system is solved by Newton's method. Hassanien *et al* [91] developed a two-level three-point finite-difference scheme, which was fourth-order accurate in space and second-order accurate in the time. Stability using von-Neumann stability analysis showed that the method is unconditionally stable. Kadalbajoo *et al* [92] developed an implicit scheme for solving the Burgers' equation. They used a standard backward Euler scheme with constant time step to discretise in the temporal direction and a standard upwind finite-difference scheme to discretise in spatial direction on piecewise uniform mesh. The quasilinearisation process was used to tackle non-linearity. The sequence of solutions of the linear equations obtained after applying quasilinearisation was shown to converge quadratically to the solution of the original non-linear problem. A numerical method based on Crank–Nicolson scheme was put forward by Kadalbajoo and Awasthi [93], where they first reduced the Burgers' equation to a linear heat equation using Hopf–Cole transformation and then decsretised using Crank–Nicolson scheme. The mesh size could be chosen without any restriction. It was shown that the scheme was second-order accurate in both space and time and also unconditionally stable. Liao [94] proposed a method which transforms the original non-linear Burgers' equation into a linear heat equation using Hopf–Cole transformation, and transforms the Dirichlet boundary condition into the Robin boundary condition. The linear heat equation is then solved by an implicit fourth-order compact finite-difference scheme. Descretisation in temporal direction was performed using Crank–Nicolson scheme and the accuracy was improved by Richardson extrapolation. Comparison proved the superiority of the method over the existing schemes. A second-order accurate difference scheme is discussed [95] in which the non-linear system is solved by both Newton's method and predictor–corrector method. The authors have also presented the uniqueness of the difference solution, the stability and L_2 convergence of the difference scheme by the energy method. Pandey *et al* [96] tried another method by reducing Burgers' equation to the heat equation and applying Douglas finite-difference scheme on the reduced equation. The method was shown to be unconditionally stable, fourth-order accurate in space and second-order accurate in time. The modified local Crank–Nicolson (MLCN) method for one- and two-dimensional Burgers' equations was presented in [97]. The MLCN is an explicit difference scheme with simple computation and is unconditionally stable. Liao [98] extended the method in [94] to solve two-dimensional Burgers' equation using an unconditionally stable, compact fourth-order finite-difference scheme. The author has assumed potential symmetry condition and hence used two-dimensional Hopf–Cole transformation to convert non-linear Burgers' equation into two-dimensional linear heat equation. The linear heat equation was then solved by an implicit fourth-order compact finite-difference scheme. The author has also developed a compact fourth-order formula to approximate the boundary conditions of the heat equation, while the initial condition for the heat equation was approximated using Simpson's rule. A semi-implicit finite-difference method was used to find the numerical solution of two-dimensional coupled Burgers' equation in [99]. In 2012, Mousa *et al* [100] proposed combined compact finite-difference scheme for the treatment of one-dimensional Burgers' equation. They have used Hopf–Cole transformation to convert non-linear Burgers' equation to heat equation. They have implemented a compact finite-difference scheme to approximate the space derivatives and a low storage Runge–Kutta scheme to approximate the time integration. In [101], Kweyu *et al* have generated three sets of varied initial and boundary conditions from general analytic solution obtained by using Hopf–Cole transformation and method of separation of variables. Numerical solution was obtained by using Crank–Nicolson scheme and explicit scheme. The accuracy in terms of consistency, convergence and stability was determined by means of L_1 error. Another scheme using Crank–Nicolson method was discussed by Wani and Thakar [102], where they approximated u_t by forward difference and uu_x by central difference at $t = t_n$ and $t = t_{n+1}$. The scheme remained linear in u at $t = t_{n+1}$ and νu_{xx} was approximated by usual Crank–Nicolson expression. The results came out to be better than those by Kadalbajoo and Awasthi [93]. Srivastava *et al* [103] proposed an implicit logarithmic finite-difference method, for the numerical solution of two-dimensional time-dependent coupled viscous Burgers' equation on the uniform grid points. Also in [104], an implicit exponential finite-difference scheme has been proposed for solving two-dimensional non-linear coupled viscous Burgers' equations with appropriate initial and boundary conditions.

4.5 *Method of lines (MOL)*

In 1930, Rothe [105], who was from former Soviet Union, introduced method of lines (MOL) in his paper. MOL does not involve the discretisation of all variables. MOL is used

- to convert the system of PDE into ODE initial value problem,
- to discretise the spatial derivatives together with the boundary conditions,
- to integrate the resulting ODEs using a sophisticated ODE solver, which takes the burden of time discretisation by choosing the time steps to maintain the accuracy and stability of the evolving solution.

The MOL semidiscretisation approach was used to transfer

$$u_t + uu_x = \nu u_{xx} \tag{46}$$

into a system of first-order linear ODE. In the MOL approach, the ODEs are integrated directly with a standard code for the task. The stability analysis of the MOL is the most important and critical factor in their solution. Stability analysis includes the study of numerical solution as it gives the means by which the step size and the numerical integration scheme could be selected. The stability analysis of MOL for discretisation that produce initial value type in ODE is easy compared to the analysis of MOL for discretisation that produce boundary-valued problems.

The relative merits and demerits of MOL with an ordinary differential equation solver to classical explicit and implicit finite-difference techniques were compared by Kurtz *et al* [106]. Shampine [107] investigated factors influencing the choice of ODE solver for the numerical solution of PDEs by MOL. The numerical solution of the advection–diffusion equation was studied. The system arising from the solution of the advection–diffusion equation by MOL has a Jacobian with a very simple form. A novel approach to the development of the code was given by Oymak and Selcuk [108], which involves coupling between MOL and a parabolic algorithm. This code removes the necessity of iterative solution on pressure and solution of a Poisson-type equation for the pressure. The main contribution of this paper is the proposal of a time-accurate Navier–Stokes code based on the MOL approach with a non-iterative algorithm for the pressure. The spatial derivatives of dependent variables were approximated by using a 5-point Lagrange interpolation polynomial. Convective terms were discretised using upwinds and diffusion terms were discretised centrally. 2D Navier–Stokes equation was used by Selcuk *et al* [109] to test the performance of MOL and FDM for checking solution accuracy and CPU time. The results were compared with the previously reported results in literature. A parabolic algorithm was used, which removed the necessity of iterative solution and did not require the solution of a Poisson-type equation for the pressure. The method was used to calculate

- axial velocity and pressure distribution in pipe flow,
- steady-state reattachment lengths in sudden expansion flow on uniform grid distribution.

Javidi [110] presented a new method for solving the Burger's equation by combination of method of lines (MOL) and matrix-free modified extended backward differential formula (MF-MEBDF), where a difference scheme of $O(h^4)$ was used to approximate $u(x,t)$ and $u_{xx}(x,t)$. The resulting set of ordinary differential equation in t was solved by MF-MEBDF, where an exact Newton method and then the IOM algorithm is used to solve the resulting system of differential equations. The advantage of this method lies in the fact that there would be no need to find the Jacobian matrix and its related decomposed matrix, thus saving our computational cost and running time. A non-central 7-point formula was used by Bakodah [111] to develop numerical scheme, which gives approximate solution to Burgers' equation in three different cases. The solutions were compared using the results of numerical experiments with 3- and 5-point formulae. The 7-point formula in MOL was used for solving Burgers' equation for arbitrary initial conditions.

The purpose of using higher-order discretisation scheme was 'the convective term'. MOL was used to yield non-oscillatory solutions for recirculating flow. MOL was found to be superior to FDM with respect to CPU and set-up time. MOL has the simplicity of the explicit method and stability advantage of the implicit method. It is possible to achieve higher-order approximation in the discretisation of spatial derivative. Due to MOL, comparable orders of accuracy can be achieved without using extremely small time steps. Particularly, for more complex and higher Reynold number problem, MOL is superior to FDM with respect to CPU time. To decrease the computation time considerably

- we have to increase the order of the spatial discretisation method (high accuracy with less grid points),
- highly accurate and stable numerical algorithm for the time integration is needed.

4.6 *Finite-element method (FEM) and splines*

Finite-element method (FEM) represents a powerful and general class of techniques for the approximate solution of PDEs. It gives more accurate solutions compared to finite-difference methods and can be used in problems having complicated domain geometry. Another important method to solve Burgers' equation is by using splines. Spline functions have some attractive properties. Being piecewise polynomial, they can be integrated and differentiated easily. As they have compact support, numerical methods in which spline functions are used as a basis function lead to matrix systems including band matrices. Such systems have solution algorithms with low computational cost. Therefore, spline solutions of the Burgers' equation are suggested in many studies. Splines were used in a mathematical context for the first time by Schoenberg [112] in connection with piecewise polynomial approximation. Since then, it has come a long way in solving the Burgers' equation.

In 1978, Jain and Holla [113] used cubic spline functions for solving coupled Burgers' equation in two space variables. They have analysed the algorithms for their stability and convergence. A finite-element method based on rectangular elements was developed in [114]. As exact solutions for a two-dimensional Burgers' equation were not available, the accuracy of these methods was checked via grid refinement. In 1980, Varoglu and Finn [115] applied a finite-element method to solve the Burgers' equation, which was based on the combination of the space–time elements and the characteristics. Caldwell *et al* [116] attempted a piecewise approximation method (finite-element method) using two elements with the aim of 'chasing the peak' by altering the size of the elements at each stage by using the information at the previous step. Caldwell and Smith [117] extended this method to the general case of n elements. They observed that for large R, finite-element results were much superior to the finite-difference results. Many researchers used moving node finite-element method to solve the Burgers' equation. Gelinas *et al* [118] presented a node moving finite-element method, which can be applied to large gradients or shocks with high resolution and accuracy. In their system, the nodes move systematically and continuously to those regions where they are required the most. Caldwell *et al* [119] further developed a moving node finite-element method by using an algorithm, which was a generalisation of the one considered by Caldwell *et al* [116]. Ali and Gardner [120] developed a collocation solution of Burgers' equation using cubic B-spline finite-element method. For solving the two-dimensional Burgers' equations in inhomogeneous form, a stable scheme based on the operator-splitting technique with cubic spline functions was derived by Shankar *et al* [121]. The unconditionally stable scheme was of first-order accuracy in time and second-order accuracy in space directions. Öziş *et al* [122] reduced the Burgers' equation to heat equation by Hopf–Cole transformation, which was then solved by FEM. They used linear functions as test functions over the intervals. But when Hopf–Cole transformation is used, due to the series solution involved, it would be required to consider innumerous terms of the series to ensure fast convergence and minimise error. Dogan [123] came up with a Galerkin FEM solution of Burgers' equation using linear elements. The system of ordinary differential equation obtained on applying FEM was solved by Crank–Nicolson scheme. Many works on finite-element method have used splines as weight functions. Kutluay *et al* [124] developed a least squares quadratic B-spline FEM. They reduced the Burgers' equation to a pentadiagonal system by applying classic weighted residual method over the finite elements, which was then solved by a variant of Thomas algorithm together with an iteration process at each time step. Azkan and Ozdes [125] developed a variational method which was constructed on descretisation in time. The descretisation was done by replacing the time derivative with its difference coefficient, thus converting it into an ordinary differential equation which was then solved by Galerkin method. Özis *et al* [126] developed Galerkin quadratic B-spline finite-element method where the test function used for FEM was quadratic spline. The initial interval [0,1] was divided into 80 finite elements of equal length. The resulting system was first-order ordinary differential equation with pentadiagoanal matrices as coefficients which was then solved by Thomas algorithm. Aksan [127] brought forward another method in which Burgers' equation was converted to a set of non-linear ordinary differential equations by the method of discretisation in time and then each of them was solved by applying the quadratic B-spline finite-element method. Two test examples established the efficiency of the presented method. A numerical solution based on collocation using septic splines was developed by Ramadan *et al* [128]. Here, the time descretisation was done using Crank–Nicolson scheme. The method was proved to be unconditionally stable. A cubic B-spline FEM solution of time-splitted Burgers' equation and quadratic B-spline FEM solution of space-splitted Burgers' equation was studied by Dağ *et al* [129]. Space splitting was done by setting $V = -u_x$. The first-order system thus obtained was solved by quadratic B-spline. The time-splitted Burgers' equation was

$$u_t + 2uu_x = 0; \quad u_t - 2\nu u_{xx} = 0 \tag{47}$$

which was then solved using the cubic B-spline. They found that splitted solution of Burgers' equation using splines gave better results compared to non-splitted solution. Dağ *et al* [130] further came up with a cubic B-spline collocation solution of Burgers' equation, where the non-linear term uu_x was approximated by using a form of quasilinearisation as follows:

$$\begin{aligned}(uu_x)_m^{n+1} = {} & u_m^{n+1}(u_x)_m^n + u_m^n(u_x)_m^{n+1} \\ & + u_m^n(u_x)_m^n.\end{aligned} \tag{48}$$

Kumar and Mehra [131] proposed a wavelet-Taylor Galerkin method. In deriving the computational scheme, Taylor-generalised Euler time discretisation is performed prior to wavelet-based Galerkin spatial approximation. The linear system of equations obtained in the process is solved by approximate-factorisation-based simple explicit schemes. Burgers' equation is also solved by splitting-up method using a wavelet-Taylor Galerkin approach. Here, the advection and diffusion terms in the Burgers' equation are separated, and the solution is computed in two phases by appropriate wavelet-Taylor Galerkin schemes. Ramadan *et al* [132] proposed a non-polynomial spline solution of Burgers' equation. The non-polynomial spline function in this work has a trigonometric part and a polynomial part of the first degree. The C^∞ differentiability of the trigonometric part of non-polynomial spline compensates for the loss of smoothness inherited by polynomial splines. Moving boundary conditions were used to improve the accuracy. In another study of Burgers' equation by Saka and Dağ [133], time and space splitting techniques were applied to the Burgers' equation and the modified Burgers' equation and then collocation procedure using quintic B-spline was employed to approximate the resulting systems. The method was proven to be better than quadratic B-spline FEM and quartic B-spline collocation method. Another spline interpolation method was devised by Jiang and Wang [134] by using the derivative of the quasi-interpolation to approximate the spatial derivative and a second-order compact finite-difference scheme to approximate the time derivative. Two-dimensional Burgers' equation is solved by local discontinuous Galerkin (LDG) finite-element method [135]. The authors have transformed the system of Burgers' equations to a linear heat equation by means of Hopf–Cole transformation. Then the LDG method is used to discretise the heat equation in space. A forward Euler and a third-order Runge–Kutta method is used to discretise the corresponding ordinary differential equations. Finally, the numerical solution for the heat equation is used to obtain numerical solutions of the system of Burgers' equations directly. Mittal and Jain [136] developed a method for solving Burgers' equation based on collocation of the modified cubic B-splines over finite elements. A limiter-free high-order spectral volume formulation was developed to solve the Burgers' equation in [137]. They have used Hopf–Cole transformation to convert non-linear Burgers' equation to a linear diffusion equation. The local discontinuous Galerkin (LDG) and the LDG2 viscous flux discretisation methods were used to solve this heat conduction equation. A three-stage SSP Runge–Kutta scheme was used for time advancement. Numerical solutions are presented for 1D and 2D Burgers' equations. Numerical scheme based on weighted average differential quadrature method was proposed by Jiwari *et al* [138]. They have used forward difference method for discretisation of time variable followed by quasilinearisation technique and polynomial quadrature method for spatial discretisation. The resulting linear equations were solved by Gauss elimination method. Stability and convergence analysis of the proposed scheme was presented. Yang [139] presented a finite-volume element method for approximating the solution of two-dimensional Burgers' equation. In this paper, Yang used the upwind technique to handle the non-linear convection term. Also, the semidiscrete scheme and the fully discrete scheme are presented. Arora and Singh [140] used the modified cubic B-splines in differential quadrature method to give a system of ordinary differential equation, which was solved by strong stability-preserving time-stepping Runge–Kutta (SSP-RK43) scheme.

4.7 *Other methods*

Apart from the methods discussed already, Burgers' equation has attracted many researchers worldwide to develop innumerous other methods. Computing the pressure is the most difficult and CPU consuming part which requires the solution of a Poisson-type equation introducing an elliptic nature. Pressure calculation is difficult and time-consuming leading to the use of pressure correction methods. The methods are

- projection (fractional step) method following Chorin [141] algorithm,
- artificial compressibility method proposed by Chorin [142],
- pressure-based finite volume method (the most well-known algorithm is the SIMPLE method of Patankar [143]. Other algorithms are SIMPLER and SIMPLEC),
- pressure implicit by splitting of operators (PISO) algorithm by Issa [144].

Mittal and Singhal [145] gave a spectral method to solve Burgers' equation with periodic boundary condition. They used the fact that the non-linear term uu_x is finitely reproducing with respect to basis functions $\sum_{j=-\infty}^{\infty}((1/\sqrt{2\pi})\mathrm{e}^{ijx})$. This method gives a system of ordinary differential equation which is solved by the s-stage Runge–Kutta–Chebyshev method. Elton [146] considered three-lattice Boltzmann methods and an analogous finite-difference method for solving two-dimensional viscous Burgers' equation with periodic boundary conditions. He proved that in the l_1 norm, the lattice Boltzmann methods converge first-order temporally and second-order spatially. Distributed approximating functional (DAF) method to Burgers' equation for large Reynolds number was developed by Zhang *et al* [147]. The method required small number of grid points and permitted the use of large time steps. Zhang *et al* used just about 25–35 grid points to produce highly accurate solutions when Re $\leq 10^5$. The method was further extended to the case of Re $> 10^5$ by using 200 grid points. They used a large mapping parameter to shift most of the 200 grid points to the boundary region to obtain oscillation-free solutions. The solution proved to be much better than those existing in literature. Park *et al* [68] devised a method based on the Karhunen–Loeve decomposition which is a technique for obtaining empirical eigenfunctions from the experimental or numerical data of a system. Employing these empirical eigenfunctions as basis functions of a Galerkin procedure, one can *a priori* limit the function space considered to the smallest linear subspace that is sufficient to describe the observed phenomena and consequently reduce the Burgers' equation to a set of ordinary differential equations with a minimum degree of freedom. Burns *et al* [148] have considered Burgers' equation with zero-Neumann boundary conditions to show that for moderate values of viscosity, numerical solution approaches non-constant shock-type stationary solution. Based on Hopf–Cole linearisation, Brander and Hedenfalk [149] solved Burgers' equation in one space dimension for an arbitrary incident pulse of finite length.

To get accurate results, creation of mesh plays a crucial role in the methods which use discretisation of PDEs into meshes. But, for discontinuous and high gradient problems mesh generation is a time-consuming process. Mesh-free or meshless method is a very good alternative to avoid this trouble. In these methods, the scattered nodes are only used instead of meshing the domain of the problem. Hon and Mao [150] applied a mesh-free method called multiquadratic (MQ) which is a special type of radial basis function. This method was originally developed to approximate two-dimensional geographic surface. Hon and Mao [150] further developed an adaptive algorithm to adjust MQ interpolation points to the peak of the shock wave so as to get better results. Malek and Mansi [151] found a group theoretic method to solve Burgers' equation. They applied one-parameter group transformation to Burgers' equation along with the initial and boundary values. This reduced the number of independent variables from two to one, giving rise to an ordinary differential equation. The ordinary differential equation is then solved analytically to get the solution in its closed form. Abbasbandy and Darvishi [152] studied the Adomian decomposition method, which decompose the solution $\theta(x,t)$ by an infinite series of components $\sum_{n=0}^{\infty}\theta_n(x,t)$, where θ_n will be obtained recursively. This gives the solution as an infinite series usually converging to an accurate solution. The method has the advantage that it can be applied directly and does not require any linearisation. It also does not require descretisation of the variables and, therefore, it is not affected by errors associated with discretisation. Abbasbandy and Darvishi [153] applied modified Adomian's method (constucted on method of descretisation in time) to Burgers' equation. They used Fourier cosine series approximated to n terms for $\theta(x,t_0) \approx a_0 + \sum_{n=1}^{N} a_n\cos(n\pi x)$. The method was shown to give results better than the FEM results by Öziş *et al* [122]. A variational iteration method for solving Burgers' and coupled Burgers' equations was developed by Abdou and Soliman [154], where they constructed a functional to approximate $u_{n+1}(x)$. The accuracy of the method was established by comparing the results with those obtained by Adomian decomposition method. In [155], higher-order accurate two-point compact alternating direction implicit algorithm was developed to solve the two-dimensional unsteady Burgers' equation. The method is the extension of A-stable fourth-order accurate second-diagonal Pade approximation to solve multidimensional flow problems. A comparison with fourth-order Du Fort Frankel scheme is also presented which is a conditionally stable explicit scheme. Sakai and Kimura [156] applied two-dimensional Hopf–Cole transformation to convert Burgers' equation into linear heat equation and the resulting equation is solved by spectral method. A pseudospectral method to solve Burgers' equation was explained by Darvishi and Javidi [157]. This method involves the use of spectral differentiation matrices to find the derivative of $u(x)$ at the collocation point x_j. Runge–Kutta method of fourth order was used to advance in time. Restrictive Pade approximation classical implicit finite-difference method was implemented by Gulsu [158] whose accuracy was demonstrated by the two test problems. Alice Gorguis [159] made a comparative study of decomposition method and Hopf–Cole transformation and established the advantages such as simplicity and reliability of the former over the latter. However, this method demands the use of truncated series. If the solution series has small convergence radius, then the truncated series may be inaccurate in many regions. To enlarge the convergence domain of the truncated series, Pade approximants (PAs) to the Adomian's series solution have been tested and applied to partial and ordinary differential equations with good results. Basto *et al* [160] applied PAs both in x and t directions to the truncated series solution given by Adomian's decomposition technique for Burgers' equation. This enlarged the domain of convergence of the solution and improved the accuracy. In 2008, Wu and Zhang [161] introduced artificial boundary method to solve two-dimensional Burgers' equation in unbounded domain by means of Hopf–Cole transformation. Artificial boundaries were introduced to make the computational domain finite and boundary conditions on the artificial boundaries were found which reduced the original problem to an equivalent problem on a bounded domain. They have also presented the stability of the reduced problem. A mesh-free method named element-free characteristic Galerkin method (EFCGM) was proposed by Zhang *et al* [162] for solving Burgers' equation with various values of viscosity. Based on the characteristic method, the convection terms of Burgers' equation disappear and this process makes Burgers' equation self-adjoint, which ensures that the spatial discretisation by the Galerkin method can be optimal. The results were tested for both one-dimensional and two-dimensional Burgers' equations. A variational iteration method was proposed in [163] for solving non-linear Burgers' equation in one and two dimensions. Zhang *et al* [164] developed another meshless method based on the coupling between variational multiscale method and mesh-free methods, for 2D Burgers' equation. Zhu *et al* [165] proposed discrete Adomian decomposition method (ADM) to numerically solve the two-dimensional Burgers' non-linear difference equations.

Asaithambi [166] presented a simple numerical method based on automatic differentiation or algorithmic differentiation that used second-order finite differences for the spatial derivatives and marched the solution in time using a Taylor series expansion. The advantage of automatic differentiation is that it is not necessary to obtain lengthy algebraic expressions for calculating higher-order derivatives – they were computed using recursive formulae obtained from the spatially discretised form of the differential equation itself. Liu and Shi [167] solved numerically the two-dimensional Burgers' equations with two variables by the lattice Boltzmann method. In 2011, Allery *et al* [168] introduced *a priori* reduction method for solving two-dimensional Burgers' equation. This method is based on an iterative procedure which consists of building a basis for the solution where at each iteration the basis is improved. They have shown that the proposed method takes less computational time. Another meshless method, the Petrov–Galerkin method, was presented by Sarboland *et al* [169]. In this approach, the trial space was generated by the multiquadratic radial basis function (MQRBF) and the test space was generated by the compactly supported RBF. In 2013, Aminikhah [170] combined Laplace transform and new homotopy perturbation methods (LTNHPM) to obtain closed form solutions of the coupled Burgers' equation. Aminikhah claimed that the proposed method can be applied to many complicated linear and non-linear partial differential equations without doing linearisation or discretisation. Xie and Li [171] used radial basis functions to approximate the solution of non-linear Burgers' equation. They have used multiquadric radial basis function for spatial discretisation and a second-order compact finite-difference scheme for temporal approximation. A homotopy analysis method (HAM) based multiscale meshless method was proposed by Mei [172]. Mei has constructed a multiscale interpolation operator with radial basis function, while HAM was used to solve the resulting ODE system. In [173], it was shown that combining the interval interpolation wavelet collocation method, HAM-based adaptive precise integration method can be employed to solve non-linear Burgers' equation. A lattice Boltzmann method was proposed in [174], based on BGK model and Chapman–Enskoy expansion for computing solutions of Burgers' equation.

The Lie group method, also called symmetry analysis, is a powerful and direct approach to construct exact solutions of non-linear differential equations [175]. The Lie symmetry analysis is performed for the general Burgers' equation, which possesses rich solutions and similarity reductions [176]. Such exact explicit solutions and similarity reductions are important in both applications and the theory of non-linear science. Entropic lattice Boltzmann models are discrete velocity models of hydrodynamics that possess a Lyapunov function, which makes them useful as non-linearly stable numerical methods for integrating hydrodynamic equations. An entropic lattice Boltzmann model for Burgers' equation was derived, and used to perform a fully explicit, unconditionally stable numerical integration of these equations [177].

4.8 *Recent developments*

As the Burgers' equation is non-linear, the numerical schemes lead to a system of non-linear equations. Implicit exponential finite-difference method and fully implicit exponential finite-difference method for solving Burgers' equation was given in [178]. In 2014, Biazar *et al* [179] used method of lines to approximate u_x and u_{xx} of one-dimensional quasilinear Burgers' equation. They discretised spatial derivatives u_x and u_{xx} using first- and second-order central differences. The resulting system of ordinary differential equation was solved to arrive at the solution. Another work on Burgers' equation in 2014 was by Diyer *et al* [180], where they gave a numerical scheme based on cubic B-spline quasi-interpolants and some techniques of matrix arguments. They applied the derivative of the cubic B-spline quasi-interpolant to approximate the spatial derivative of the differential equations and employed a first-order accurate forward difference for approaching the temporal derivative. So they did not have a system where they had to invert a matrix but an iterative relationship easy to implement. Sarboland *et al* [181], who had developed mesh-free method in [169] came up with two more mesh-free methods based on the multiquadric (MQ) quasi-interpolation operator $\mathscr{L}_{\mathscr{W}_2}$ and direct and indirect radial basis function network (RBFNs) schemes. Ganaie and Kukreja [182] in 2014 discussed cubic Hermite collocation method (CHCM) for Burgers' equation. They handled the non-linear term using quasilinearisation. Time discretisation was done using Crank–Nicolson scheme. Hermite functions are a class of piecewise polynomials having continuity properties and play an important role in setting approximate functions. They have compact support and can be easily differentiated. Ganaie and Kukreja performed a linear stability analysis and proved that the method is unconditionally stable. Talwar and Mohanty [183] proposed a new modified alternating group explicit (MAGE) iterative method in 2014 for solving one-space-dimensional linear and non-linear singular parabolic equations. These methods are explicit in nature and if coupled compactly, they are suitable for use in parallel computers. Wavelet solutions of Burgers' equation with high Reynolds numbers were presented by Liu *et al* [184]. Following this method, Burgers' equation was first transformed into a system of ordinary differential equations using the modified wavelet Galerkin method. Then, the classical fourth-order explicit Runge–Kutta method was employed to solve the resulting system of ordinary differential equations. The wavelet algorithm had a much better accuracy and a much faster convergence rate than many other numerical methods present in literature such as finite-difference method, classical weighted residual method, etc. Gao and Chi [185] proposed a numerical scheme based on high accuracy multiquadric quasi-interpolation operator L_w. They have used a multiquadric quasi-interpolant to approximate derivatives of the solution in spatial domain. Finite difference was used to approximate the derivatives of solution in temporal domain. An all-at-once approach for the optimal control of the unsteady Burgers' equation is given by Yilmaz and Karasožen [186]. They have discretised the non-linear Burgers' equation in time using the semi-implicit discretisation and the resulting first-order optimality conditions are solved iteratively by Newton's method. An explicit solution of Burgers' equation with stationary point source is given in [187]. This paper also explains the role of diffusion and convection when a non-autonomous reaction term produces heat constantly. In [188], a local RBF-based method of approximate particular solutions for two-dimensional unsteady Burgers' equations is developed. A comparative study between the lattice Boltzmann method (LBM) and the alternating direction implicit (ADI) method is presented in [189] using the 2D steady Burgers' equation. The comparative study showed that the LBM performs comparatively poor on high-resolution meshes due to smaller time step sizes, while on coarser meshes where the time step size is similar for both methods, the cache-optimized LBM performance is superior. In [190], an implicit logarithmic finite-difference method is introduced for the numerical solution of one-dimensional coupled non-linear Burgers' equation. The numerical scheme provides a system of non-linear difference equations which are linearised using Newton's method. The obtained linear system is solved by Gauss elimination with partial pivoting algorithm. Goyal and Mehra [191] developed a fast adaptive diffusion wavelet method for solving 1D and 2D coupled Burgers' equation with Dirichlet and periodic boundary conditions. They constructed diffusion wavelet from the diffusion operator obtained by discretising the Burgers' equation. They have used diffusion wavelet for the construction of an adaptive grid as well as for the computations involved in the numerical solution of Burgers' equation. Moreover, by comparing the CPU time, they claimed that the proposed method is faster. Laplace decomposition method (LDM) is proposed to solve the two-dimensional non-linear Burgers' equations in [192]. Convergence and accuracy of the proposed scheme are shown through test problems. Kumar and Pandit [193] introduced a numerical scheme based on finite difference and Haar wavelets to solve coupled Burgers' equation. They have discretised the time derivative by forward difference, followed by quasilinearisation technique to linearise the coupled Burgers' equation. Space derivatives are discretised with Haar wavelets resulting in a system of linear equations which is solved using Matlab 7.0.

After Hopf and Cole introduced the transformation, several attempts have been made to generalise Cole–Hopf transformation, which were nicely documented in [194]. For solving the two-dimensional non-linear

Burgers' equation, a numerical scheme based on high accuracy MQ quasi-interpolation scheme was presented by Sarboland *et al* [195] in 2015. They used equidistant data in numerical experiments. Yet again in 2015, Mukundan and Awasthi [196] presented new and efficient numerical techniques for solving Burgers' equation. They used Hopf–Cole transformation to get one-dimensional diffusion equation which was semidiscretised by using method of lines (MOL). Resulting system of ODEs was solved by backward differentiation formulae (BDF) of order one, two and three. Amit Prakash *et al* [197] used the fractional variational iteration method (FVIM) to solve a time- and space-fractional coupled Burgers' equations. The results obtained by FVIM were more accurate than FVIM, ADM, GDTM and HPM methods. In 2015, Saleeby [198] characterised the meromorphic solution of generalised Burgers' equation given by $u_x + u^m u_y = 0$, where $m \geqq 0$ is an integer. In the same year, Jiwari [199] developed a hybrid numerical scheme based on forward finite difference, quasilinearisation and uniform Haar wavelets. In this paper, the non-linear Burgers' equation was discretised along temporal direction by means of Euler implicit method. It was followed by the quasilinearisation technique in order to linearise the stationary Burgers' equation. Finally, uniform Haar wavelets were used for spatial discretisation. Jiwari showed that the proposed scheme provides better accuracy than other existing numerical techniques. Moreover, the proposed scheme can capture the behaviour of numerical solution for small values of kinematic viscosity, ν, and hence overcome the drawback of their previous paper [200]. Higher-order accurate numerical solution for one version of two-dimensional unsteady Burgers' equation was proposed by Zhanlav *et al* [201] in 2015. They used the linear transformation $z = x + y$, $s = x - y$ and $\bar{t} = 2t$ to reduce 2D Burgers' equation

$$u_t + u(u_x + u_y) \\ = \nu(u_{xx} + u_{yy}), \quad (x, y) \in [a, b]X[c, d]$$

into

$$u_t + uu_z = \nu(u_{zz} + u_{ss}), \quad a + c \leq z \leq b + d. \tag{49}$$

If solution $u(z, s, \bar{t})$ depends only on s and $\bar{t}$ variables, eq. (49) reduces to heat equation. On the other hand, if solution $u(z, s, \bar{t})$ depends only on z and $\bar{t}$ variables, eq. (49) reduces to one-dimensional Burgers' equation. They have used Cole–Hopf transformation to reduce 1D Burgers' equation into heat equation. Finally, heat equation with Robin boundary conditions is solved by a three-level explicit finite-difference scheme. This scheme is sixth-order accurate in space and third-order accurate in the time variable. Numerical solution of heat equation was used to find numerical solution of 1D Burgers' equation which in turn was used to find numerical solution of 2D unsteady Burgers' equation. Zhanlav *et al* [201] claimed that adoption of this method reduces the computational costs compared to other direct methods for solving the 2D unsteady Burgers' equation. Numerical scheme for the coupled Burgers' equation based on collocation of the modified bi-cubic B-spline functions was proposed by Mittal and Tripathi [202]. They have used these functions for space variables and for their derivatives. The resulting system of first-order ordinary differential equations was solved by strong stability preserving Runge–Kutta method (SSP-RK54). They have compared the obtained numerical results with those suggested in earlier studies. Mittal *et al* [203] proposed a numerical method based on the properties of uniform Haar wavelets together with a collocation method and semidiscretisation along the space direction for solving a coupled viscous Burgers' equation. The semidiscretisation scheme forms a system of non-linear ordinary differential equations which is solved by the fourth-order Runge–Kutta method. Mohanty *et al* [204] presented a new two-level implicit compact operator method for the numerical simulation of coupled viscous Burgers' equation in one spatial dimension. This scheme has accuracy of order two in time and four in space. Mohanty *et al* have used three spatial grid points and the obtained non-linear system was solved by Newton's iterative method. Lie group method was used for the analysis of the generalised system of 2D Burgers' equations with infinite Reynolds number [205]. Abdulwahhab [205] has derived optimal system of one-dimensional subalgebras which was further used to obtain generalised distinct exact solutions of the velocity components.

Recently in 2016, two new modified fourth-order exponential time differencing Runge–Kutta (ETDRK) schemes in combination with a global fourth-order compact finite-difference scheme (in space) for direct integration of non-linear coupled viscous Burgers' equations is presented in [206]. One of the scheme is a modification of the Cox and Matthews ETDRK4 scheme based on (1,3)-Padé approximation and the other is a modification of Krogstads ETDRK4-B scheme based on (2,2)-Padé approximation. Bhatt and Khaliq [206] have presented and compared the accuracy and efficiency of the proposed modified schemes. Bonkile *et al* [207] sketched a new implicit scheme with second-order accuracy in space and time, which was proposed to solve Burgers' equation without using Hopf–Cole transformation. Seydaoğlu *et al* [208] proposed higher-order splitting methods with complex coefficients and extrapolation methods for treating one-dimensional Burgers' equation. As splitting methods with real coefficients of order higher than two involve negative time steps, it is not suitable for time-irreversible systems such as Burgers' equation. Hence, the splitting technique with complex coefficients is used for solving Burgers' equation with periodic, Dirichlet, Neumann and Robin boundary conditions. A high-order finite-volume compact scheme is introduced by Guo *et al* [209] to solve one-dimensional Burgers' equation. They have computed the non-linear advective terms by the fifth-order finite-volume weighted upwind compact scheme. The diffusive terms are discretised by using the finite-volume six-order Padé scheme and the strong stability preserving third-order Runge–Kutta time discretisation is used in this work.

Two-dimensional Burgers' equation is given by

$$u_t + u(u_x + u_y) = \nu(u_{xx} + u_{yy}), \tag{50}$$

$$v_t + u(u_x + u_y) = \nu(v_{xx} + v_{yy}) \tag{51}$$

subject to initial condition

$$u(x, y, 0) = f(x, y), \quad (x, y) \in D, \tag{52}$$

$$v(x, y, 0) = g(x, y), \quad (x, y) \in D \tag{53}$$

and boundary conditions

$$u(x, y, t) = f_1(x, y, t), \quad (x, y) \in \partial D, t > 0, \tag{54}$$

$$v(x, y, t) = g_1(x, y, t), \quad (x, y) \in \partial D, t > 0, \tag{55}$$

where D is the domain, ∂D is its boundary, $u(x, y, t)$ and $v(x, y, t)$ are the velocity components to be determined, f, g, f_1 and g_1 are known functions and ν is the kinematic viscosity parameter. Sinuvasan *et al* [210] introduced an inhomogeneous term, the function $f(t, x)$ in Burgers' equation, such that there exists at least one symmetry for the whole equation. For the special cases, the given equation was reduced to the equation for a linear oscillator with non-constant coefficient.

5. Conclusions

Advances in computational capacities and new robust schemes will encourage multidisciplinary researchers to consider Burgers' equation more enthusiastically. Meanwhile, availability of exact solution gives an extra advantage to Burgers' equation. The main objective of this paper is to review a brief history from physical and mathematical point of view and recent developments in numerical simulation of Burgers' equation from simple schemes to the most efficient one. Additionally, a systematic categorisation of large scientific data available in the literature due to the tremendous achievements of various research groups is done. Researchers are in constant search of more accurate, stable, robust and efficient scheme, which will lead to the development of significantly improved schemes. In this detailed discussion, we sincerely emphasised the importance of Burgers' equation in modern engineering scenario. We would like to predict a bright future ahead.

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