About Topelitz operator

1.1. Motivation and mains ideas

Roughly speaking a **Toeplitz operator** is a speciall kind of operator in a Hilbert space $\mathscr{H}$, wich is the composition of a multiplication operator and a orthogonal projection. The theory of Toeplitz operator is important because provide us with a bunch of concrete examples of operator on Hilbert spaces, and can help us to descrive the Banach algebra $B(\mathscr{H})$ of bounded linear operator on the Hilbert space $\mathscr{H}$.

1.2. Biholomorphism of the upper half-plane and group actions

Let us consider the upper half-plane, i.e

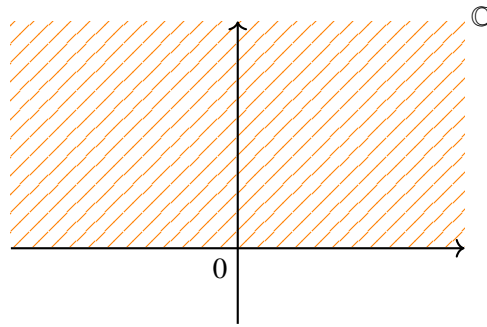$$\mathbf{H} := \{z \in \mathbb{C} : \mathfrak{I}(Z) > 0\}$$

Figura 1.1: Cartesian plane with shaded upper area and diagonal lines.

For every $A \in SL(2; \mathbb{R})$, a maatrix of the form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we asigned a Moebious transform

$$\varphi_A : \mathbf{H} \to \mathbf{H}$$
$$z \mapsto \frac{az + b}{cz + d}$$

the collection

$$M = \{\varphi_A : a \in SL(2, \mathbb{R})\}$$

is the set of biholomorphisms of the upper half plane, then we can define a function

$$\Phi : SL(2, \mathbb{R}) \to M$$
$$A \mapsto \varphi_A$$

defines a group homomorfism (a group action) for wich ker(Φ) = {$\pm Id$} so by the first isomorphism theorem for groups we have

$$M = SL(2, \mathbb{R})/\{\pm Id\} := PSL(2, \mathbb{R})$$

With the quotient topology this is a topological group. This define an acction of the Lie group $SL(2; \mathbb{R})$ in the upper half plane given by

$$\Psi : SL(2; \mathbb{R}) \times \mathbf{H} \to \mathbf{H}$$
$$(A, z) \mapsto \varphi_A(z) \tag{1.1}$$

Proposition 1.2.1. *The action defined below is transitive*

Proof. s

Definition 1. *The isotropy group of the element i, is the set*

$$SL(2, \mathbb{R})_i := \{A \in SL(2, \mathbb{R}) : \varphi_A(i) = i\}.$$

An easy computation shows us that

$$SL(2, \mathbb{R})_i = \left\{ \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} : \theta \in (-\pi, \pi) \right\}$$

as the action of the group $SL(2, \mathbb{R})$ in $\mathbf{H}$ is transitive, the theory of homogeneous spaces tells us that

$$\mathbf{H} = SL(2, \mathbb{R})/SL(2, \mathbb{R})_i$$

1.2.1.   Subgroups and their orbits

For the theory of Toeplitz operator there are three important subgroups of $SL(2, \mathbb{R})$ and their respective orbits.

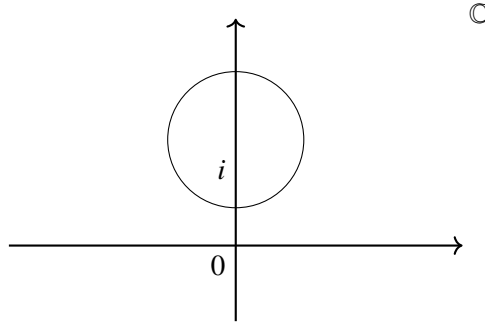**Isotropy group**: Their irbits are hiperbolic circunférences with center in i

Figura 1.2: Orbit of the isotropy group

Abelian group: The group

$$G_a = \left\{ A = \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/r \end{pmatrix} : r \in \mathbb{R}_+ \cong (\mathbb{R}_+, \cdot) \right\}$$

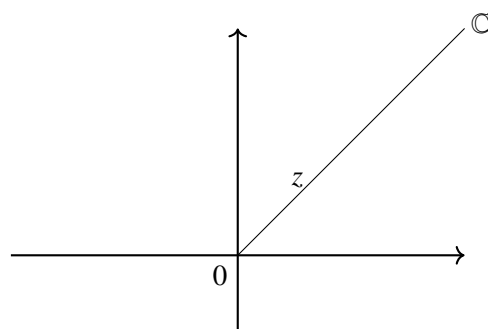for $A \in G_a$, $\varphi_A(z) = rz$ its a homothecy and their orbits are rays passing through z

Figura 1.3: Orbit of the isotropy group