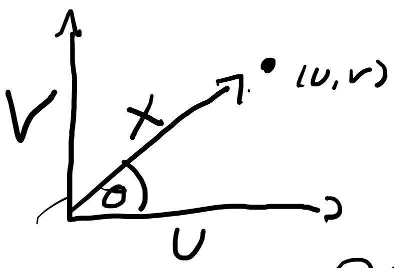
P1. Se quiere estudiar el comportamiento de un vector bidimensional que tiene sus dos componentes ortogonales, independientes y que siguen una distribución normal. Al realizar las mediciones respectivas de cada componente, se obtiene una MAS $U = (U_1, ..., U_n)$ de n observaciones con $U_n \sim \mathcal{N}(0, \sigma^2)$ y una MAS $V = (V_1, ..., V_n)$ de n observaciones con $V_n \sim \mathcal{N}(0, \sigma^2)$. En específico, se busca estudiar el comportamiento de los módulos de los vectores obtenidos. Se obtiene una nueva MAS $X = (X_1, ..., X_n)$

i. Encuentre la función de densidad de X_1



$$\underline{X_i = \sqrt{U_i^2 + V_i^2}}$$

$$X = \sqrt{U^2 + W^2}$$

$$\mathbb{P}(X \leq x) = \mathbb{P}(\underbrace{\sqrt{U^2 + W^2}} \leq x)$$

$$D_x = \{(u, w) \mid \sqrt{u^2 + w^2} \leq x\}$$

Entonces

$$\mathbb{P}(X \leq x) = F_X(x) = \iint_{D_x} f_{U,W}(u, w)\, du\, dw$$

$U \perp\!\!\!\perp W$

$$= \iint_{D_x} f_U(u) f_W(w)\, du\, dw$$

$$U \sim \mathcal{N}(0, \sigma^2), \quad W \sim \mathcal{N}(0, \sigma^2)$$

$$F_X(x) = \iint_{D_x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\frac{w^2}{\sigma^2}}\, du\, dw$$

$$= \frac{1}{2\pi\sigma^2} \iint_{D_x} e^{-\frac{1}{2\sigma^2}(u^2 + w^2)}\, du\, dw$$

$$r^2 = u^2 + w^2, \quad \theta \in [0, 2\pi]$$

$$= \frac{1}{2\pi\sigma^2} \int_0^{2\pi} \int_0^{x} e^{-\frac{1}{2\sigma^2} r^2}\, r\, dr\, d\theta$$

$$= \frac{1}{\sigma^2} \int_0^{x} e^{-\frac{1}{2\sigma^2} r^2}\, r\, dr$$

$$\frac{\partial F_X(x)}{\partial x} = f_X(x)$$

$$\Rightarrow \boxed{f_X(x) = \frac{x}{\sigma^2} e^{-\frac{1}{2\sigma^2}x^2}} \quad X \sim Rayleigh(\sigma)$$

P2: Consideremos el estimador de la varianza

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\overline{X_n} - X_i)^2 \quad // \quad \overline{X_n}$$

i) $\mathbb{E}(S^2) = \sigma^2$.

$$\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n} \overline{X_n}^2 + X_i^2 - 2X_i\overline{X_n}$$

$$= \frac{1}{n-1}\left[n\overline{X_n}^2 - 2\overline{X_n}\underbrace{\sum X_i} + \sum X_i^2\right]$$

$$= \frac{1}{n-1}\left[n\overline{X_n}^2 - 2n\overline{X_n}\,\overline{X_n} + \sum X_i^2\right] = \frac{1}{n-1}\left[\sum X_i^2 - n\overline{X_n}^2\right]$$

$$\mathbb{E}(S^2) = \mathbb{E}\left[\frac{1}{n-1}\left(\sum X_i^2 - n\overline{X_n}^2\right)\right]$$

$$= \frac{1}{n-1}\left(\underbrace{\mathbb{E}\left(\sum X_i^2\right)}_{1} - n\underbrace{\mathbb{E}(\overline{X_n}^2)}_{2}\right) = (\star)$$

Antes de calcular 1, veamos lo siguiente

$$\mathbb{V}ar(X) = \mathbb{E}\left[(X - \mathbb{E}(X))^2\right]$$

$$= \mathbb{E}\left[X^2 - 2\mathbb{E}(X)X + \mathbb{E}(X)^2\right]$$

$$= \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + \mathbb{E}(X)^2$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \qquad \mathbb{V}ar(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$1 = \mathbb{E}\left(\sum X_i^2\right) = \sum \mathbb{E}(X_i^2)$$

$$= \sum \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2 + \mathbb{E}(X_i)^2$$

$$= \sum \mathbb{V}ar(X_i) + \mathbb{E}(X_i)^2$$

$X_i \sim N(\mu, \sigma^2)$

$$= \sum \sigma^2 + \mu^2$$

$$= n(\sigma^2 + \mu^2) = 1$$

$$2 = \mathbb{E}(\overline{X_n}^2) = \mathbb{E}(\overline{X}_n^2) - \mathbb{E}(\overline{X_n})^2 + \mathbb{E}(\overline{X_n})^2$$

$(2) = \mathbb{E}(\overline{X_n}^2) = \mathbb{E}(\overline{X_n}^2) - \mathbb{E}(\overline{X_n})^2 + \mathbb{E}(\overline{X_n})^2$

$(\star) = \mathbb{V}ar(\overline{X_n}) + \mathbb{E}(\overline{X_n})^2$

- $X_i \sim N(\mu, \sigma^2)$
- $\sum X_i \sim N(n\mu, n\sigma^2)$
- $\frac{1}{n}\sum X_i \sim \frac{1}{n} N(n\mu, n\sigma^2) = N\left(\frac{n\mu}{n}, \frac{n\sigma^2}{n^2}\right)$

$= N\left(\mu, \frac{\sigma^2}{n}\right)$

$$\overline{X_n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$(\star) = \boxed{\frac{\sigma^2}{n} + \mu^2 = (2)}$$

$$(\star\star) = \frac{1}{n-1}\left(n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right)$$

$= \sigma^2$

Se tiene que $\mathbb{E}(S^2) = \sigma^2$.
Esto es que el estimador sea insesgado

ii) Calcular la varianza de S^2.

- $\overline{X_n} \perp\!\!\!\perp S^2$. (Se verá despues) $\to$ Necesario para el desarrollo "a mano" y el que veremos ahora.
- $W = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$

$= \sum_{i=1}^{n} \left(\frac{(X_i - \overline{X_n}) + (\overline{X_n} - \mu)}{\sigma}\right)^2$

$= \sum_{i=1}^{n} \left(\frac{X_i - \overline{X_n}}{\sigma}\right)^2 + \sum_{i=1}^{n}\left(\frac{\overline{X_n} - \mu}{\sigma}\right)^2 + 2\left(\frac{\overline{X_n}-\mu}{\sigma}\right)\cancel{\sum_{i=1}^{n}\left(\frac{X_i - \overline{X_n}}{\sigma}\right)}^{0}$

$\sum X_i - \overline{X_n}$
$= n\overline{X_n} - n\overline{X_n} = 0$

$= \underbrace{\frac{1}{\sigma^2}\sum (X_i - \overline{X_n})^2}_{} + n\left(\frac{\overline{X_n} - \mu}{\sigma}\right)^2$

$W = \frac{(n-1)S^2}{\sigma^2} + n\left(\frac{\overline{X_n} - \mu}{\sigma}\right)^2$

$$\underbrace{W}_{\cdot} = \underbrace{\frac{(n-1)\,S^2}{\sigma^2}}_{¿?} + \underbrace{n\left(\frac{\overline{X}_n - \mu}{\sigma}\right)^2}_{Z^2}$$

Z^2 distribuye como sigue:

$$\frac{\overline{X}_n - \mu}{\sigma} \sim N\left(0, \cancel{\frac{\sigma^2}{n}} \cdot \frac{1}{\cancel{\sigma^2}}\right)$$

$$\left(\frac{\overline{X}_n - \mu}{\sigma}\right) \sim N\left(0, \frac{1}{n}\right)$$

$$Z = \sqrt{n}\left(\frac{\overline{X}_n - \mu}{\sigma}\right) \sim N(0,1)$$

$$Z^2 \sim \chi^2_1$$

$$\cdot \sum \underbrace{\left(\frac{X_i - \mu}{\sigma}\right)^2}_{N(0,1)} \qquad X_i \sim N(\mu, \sigma^2)$$

$$X_i - \mu \sim N(0, \sigma^2)$$

$$\frac{X_i - \mu}{\sigma} \sim N(0,1)$$

$$W \sim \chi^2_n$$

$$\rightarrow M_W(t) = \mathbb{E}\left[e^{tW}\right] = \mathbb{E}\left[e^{t\left(\frac{(n-1)S^2}{\sigma^2} + Z^2\right)}\right]$$

$$S^2 \perp\!\!\!\perp \overline{X}_n \quad = \mathbb{E}\left[\underbrace{e^{t\left(\frac{(n-1)}{\sigma^2}S^2\right)}}\cdot\underbrace{e^{tZ^2}}\right]$$

$$\hookrightarrow \quad = M_{\frac{(n-1)S^2}{\sigma^2}}(t) \cdot M_{Z^2}(t)$$

$$\cdot \quad M_W(t) = M_{\frac{(n-1)S^2}{\sigma^2}}(t) \cdot M_{Z^2}(t)$$

$$W \sim \chi^2_n, \quad Z^2 \sim \chi^2_1$$

$$(1-2t)^{-\frac{n}{2}} = M_{\frac{(n-1)S^2}{\sigma^2}}(t) \cdot (1-2t)^{-\frac{1}{2}}$$

$$(1-2t)^{-\frac{(n-1)}{2}} = \underbrace{M_{\frac{(n-1)}{\sigma^2}S^2}}(t)$$

Se concluye que

$$\frac{(n-1)}{\sigma^2}S^2 \sim \chi^2_{n-1}$$

Entonces, ahora podemos calcular $\mathbb{V}ar(S^2)$

$$\mathbb{V}ar(S^2) = \mathbb{V}ar\left(\frac{1}{n-1}\sum (X_i - \overline{X}_n)^2\right)$$

$$= \mathbb{V}ar\left(\frac{1}{n-1}\frac{\sigma^2}{\sigma^2}\sum (X_i - \overline{X}_n)^2\right)$$

$$= \left(\frac{\sigma^2}{n-1}\right)^2 \mathbb{V}ar\Big(\underbrace{\frac{n-1}{\sigma^2} S^2}_{\chi^2_{n-1}}\Big)$$

$$= \frac{\sigma^4}{(n-1)^{\cancel{2}}}\, 2\cancel{(n-1)}$$

$$= \boxed{\frac{2\sigma^4}{n-1} = \mathbb{V}ar(S^2)} \;//$$