Lenguajes de Programación

2020 — 2º Semestre

Clase 4:

- Inductive datatypes and recursion

Federico Olmedo

Induction principle:

To prove a property P over the set of natural numbers we must:

- Prove that it holds for 0
- Prove that it holds for n+1 assuming it holds for n

$$\frac{P(0) \qquad \forall n.\ P(n) \implies P(n+1)}{\forall n.\ P(n)}$$

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Recursion scheme:

To define a function f over the set of natural numbers we must:

- Define it for 0
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$$\frac{\begin{array}{lcl} f(0) & = & \ldots\ldots\ldots \\ f(n+1) & = & \ldots f(n) \ldots \end{array}}{\forall n.\ \text{"} f(n) \text{ is univocously defined"}}$$

The natural numbers as an inductive set

$\mathbb{N}$ is the least set satisfying the following rules:

$$\frac{}{0 \in \mathbb{N}} \qquad \frac{n \in \mathbb{N}}{n+1 \in \mathbb{N}}$$

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$$\mathit{length}(\text{empty}) = 0$$

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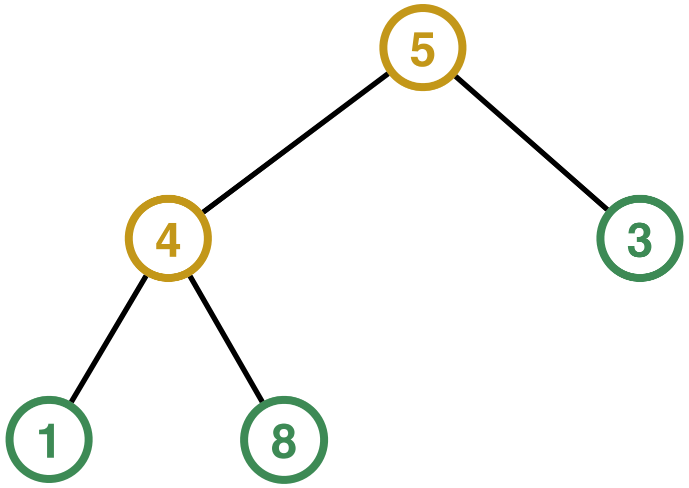
Inductive sets



Binary trees:

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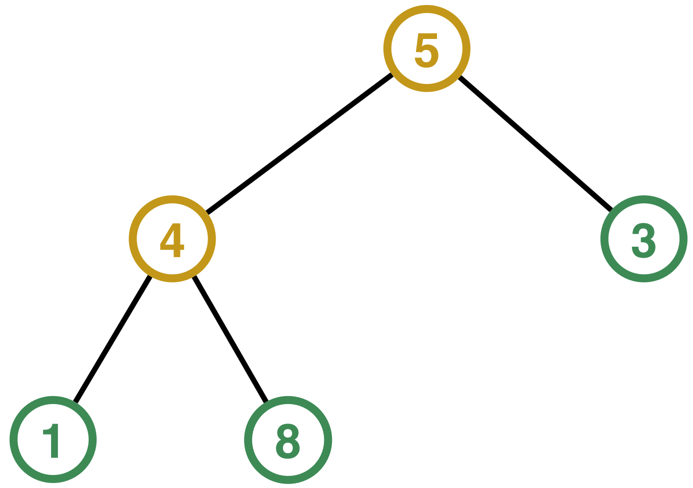

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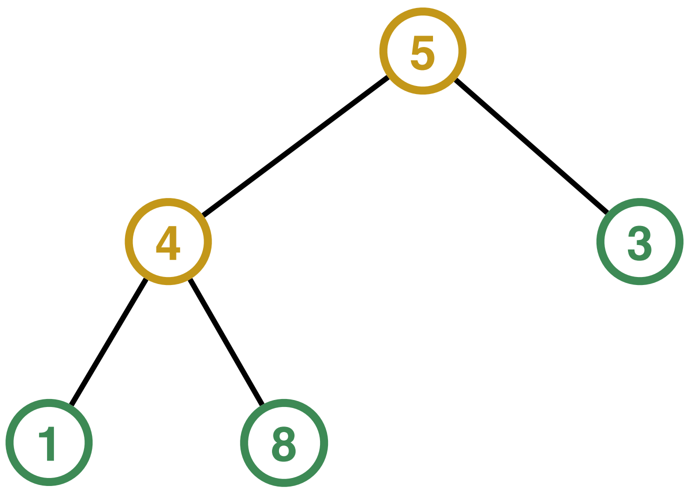

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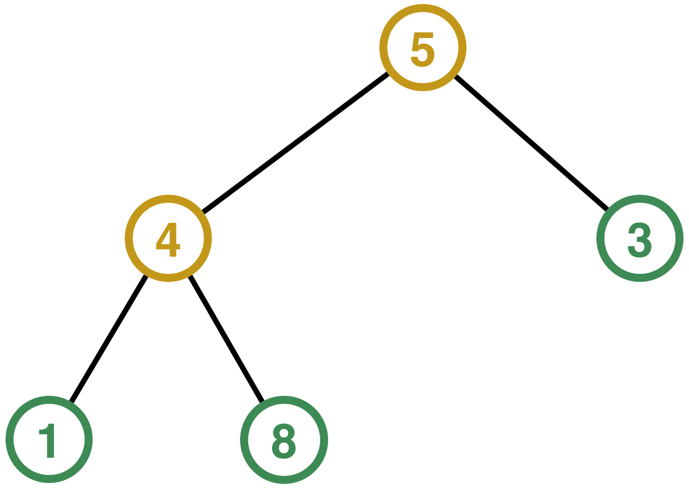

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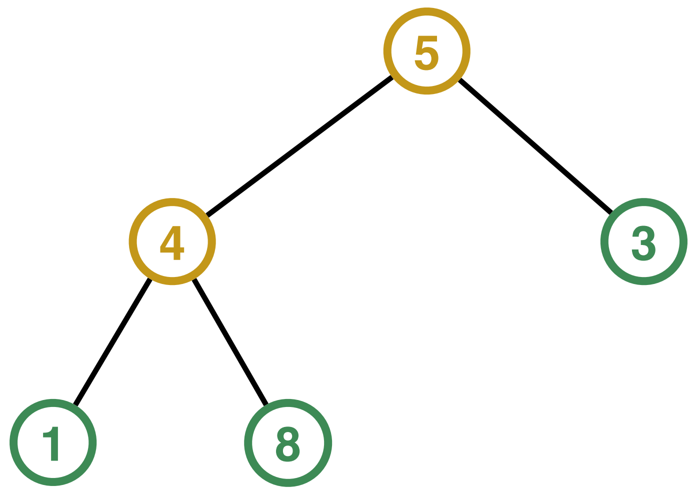

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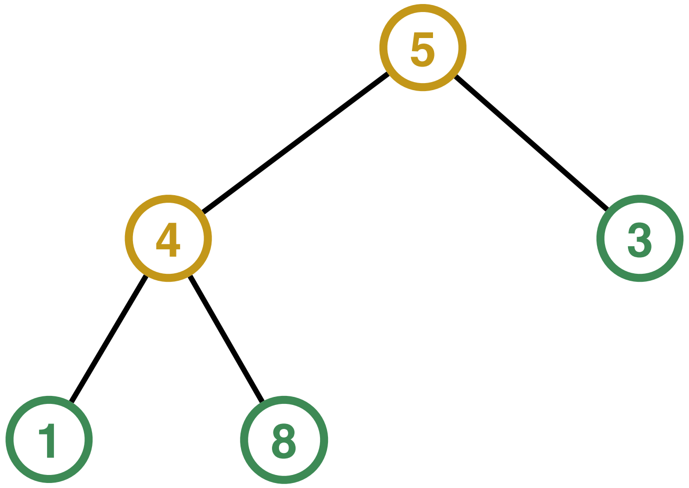

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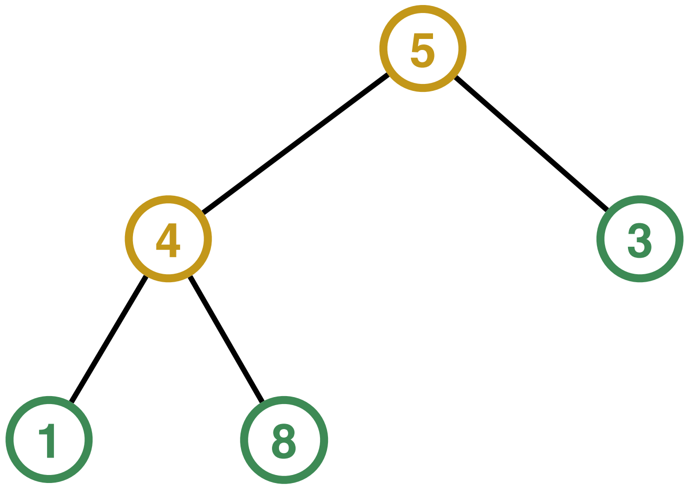

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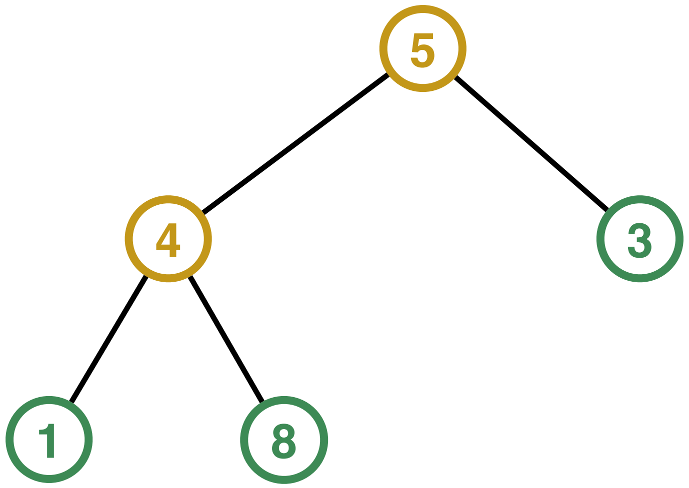

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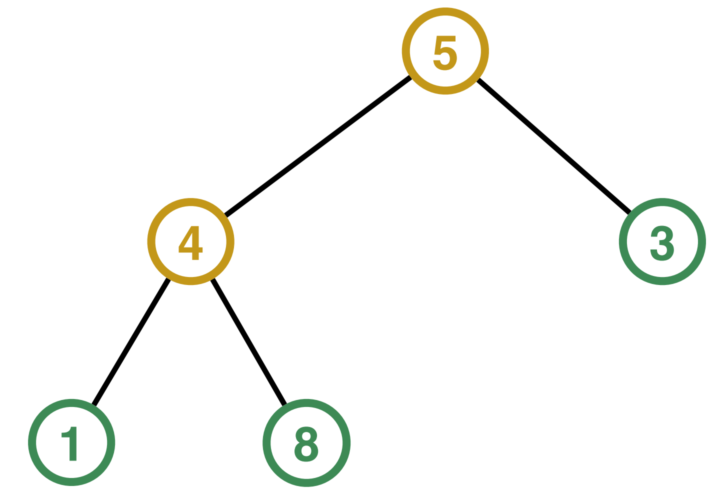

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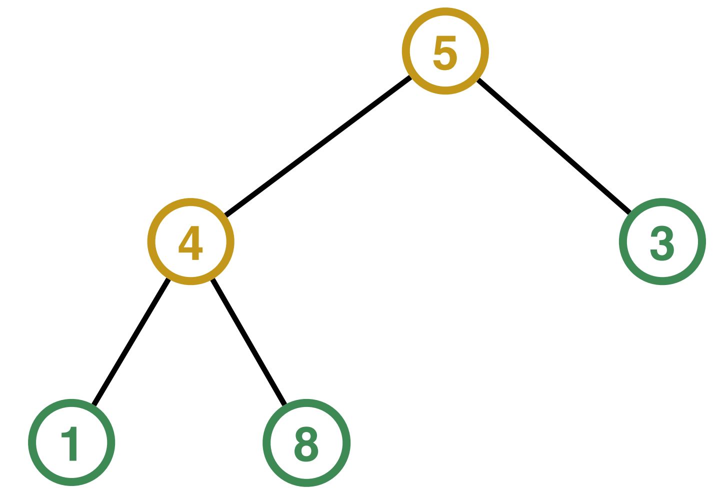

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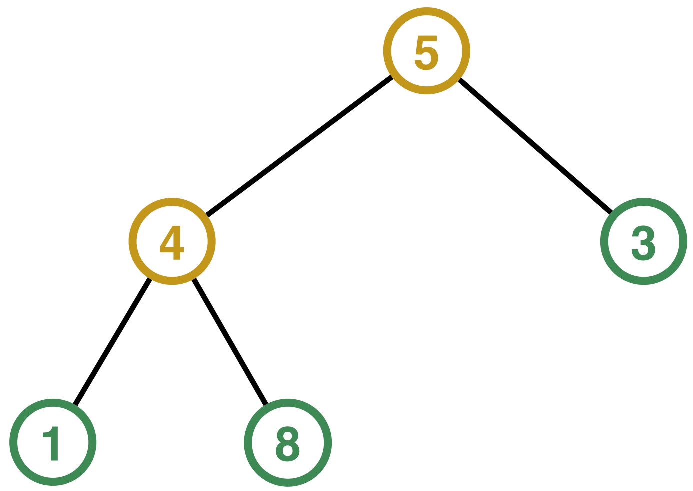

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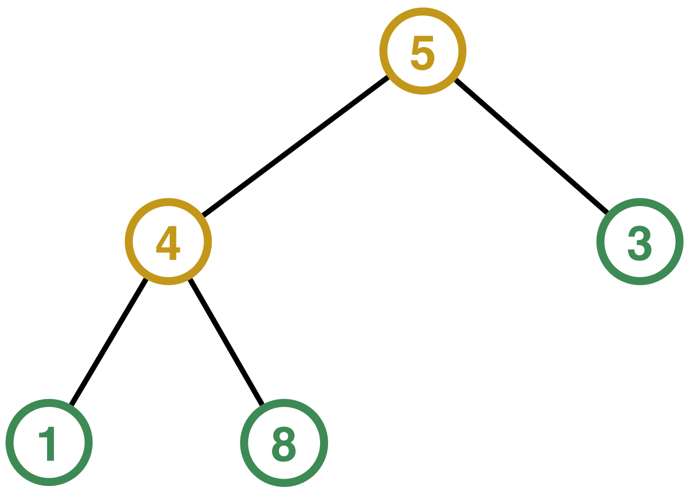

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5

Inductive types and recursion in Racket

Inductive types and recursion in Racket

```
#|
<BinTree> ::=  (leaf <value>)
            |  (in-node <value> <BinTree> <BinTree>)
|#
;; Inductive type for representing binary trees
(deftype BinTree
  (leaf value)
  (in-node value left right))


;; height :: BinTree -> Integer
;; Devuelve la altura del arbol binario
(define (height bt)
  (match bt
    [(leaf _) 0]
    [(in-node _ l r) (+ 1 (max (height l) (height r)))]))
```

Inductive types and recursion in Racket

All recursive function over binary trees will have the following template:

(which is syntactically derived from the grammar of binary trees)

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(define (func-to-define bt)
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1. Define function `(sum-bintree bt)` that returns the sum of the elements in the nodes of (binary tree) `bt`.
2. Define function `(max-bintree bt)` that returns the maximum of the elements in the nodes of (binary tree) `bt`.

Capturing the recursive scheme

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```
;; fold-bintree :: (Number -> A) (Number A A -> A) -> (Bintree -> A)
;; fold over numeric binary trees
(define (fold-bintree f g)
  (λ (bt)
    (match bt
    [(leaf v) (f v)]
    [(in-node v l r) (g v
                        ((fold-bintree f g) l)
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;; max-bintree :: BinTree -> Number
;; Returns the maximum element of a (numeric) binary tree
(define max-bintree
  (fold-bintree identity max))

;; sum-bintree :: Bintree -> Number
;; Returns the sum of the elements of a numeric binary tree
(define sum-bintree
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Define function `(contains-bintree? bt v)` using fold-bintree.

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- Values built from different constructors are always different. For instance `(equal? (leaf a) (in-node b (…) (…)))` reduces to `#f` (for all a and b)

- The only way of building values of the data structure is via the provided constructors.

9 Takeaways

Follow the Grammar!

When defining a function that operates on an inductive datatype, the definition should be patterned after the grammar of the datatype.

Lecture material

Bibliography

- PrePLAI: Introduction to functional programming in Racket [Sections 4-5]

- Essentials of Programming Languages (3rd Edition)
  Daniel P. Friedman
  [Chapter 1]

- Source code from the lecture [Download]