
WRITTEN TEST 27/10/2022

First Name: Last Name:

This exam has 2 questions, with subparts. You have **2 hours** to complete the exam. There are a total of 30 points available (sufficiency at 18 points). You cannot consult any notes, books or aids of any kind except for the codes implemented during the lab sessions. Write answers legibly in the additional sheets provided, and show all of your work. Please **write your name** on the exam itself and on the extra sheets. Concerning the implementations using Eigen, upload only the `.cpp` main files. For the exercises requiring LIS, report the bash commands used to perform the computations in a unique `.txt` file. **Upload the files** following the received instructions.

Exercise 1

1. Consider the following problem: find $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}^n$ are given. State under which conditions the mathematical problem is well posed.
2. Describe the Gradient Method. Introduce the notation, the algorithm, the interpretation of the scheme as a minimization problem. Recall the main theoretical results.
3. Describe the Conjugate Gradient method. Introduce the notation, the algorithm, the interpretation of the scheme as a minimization problem. Recall the main theoretical results.
4. Comment on the main differences between the Gradient and Conjugate Gradient Methods.
5. Describe the preconditioned Gradient and the Conjugate Gradient Methods and state the main conditions the preconditioner P has to satisfy.
6. Download the matrix `diffreact.mtx` from the Exam folder in webeep and save it on the `/shared-folder/iter_sol++` directory. Load the matrix in a new file called `exer1.cpp` using the `unsupported/Eigen/SparseExtra` module and check if the matrix is symmetric. Report on the sheet $\|A\|$ and $\|A_{SS}\|$ where A_{SS} is the skew-symmetric part of A and $\|\cdot\|$ is the Euclidean norm.
7. Define an Eigen vector $\mathbf{b} = A\mathbf{x}^*$, where $\mathbf{x}^* = (1, 1, \dots, 1)^T$. Report on the sheet $\|\mathbf{b}\|$.
8. Solve the linear system $A\mathbf{x} = \mathbf{b}$ using both the Gradient Method and Conjugate Gradient method (implemented in the `grad.hpp` and `cg.hpp` templates, respectively). Fix a maximum number of iterations which is sufficient to reduce the (relative) residual below than 10^{-8} . Use the diagonal preconditioner provided by Eigen. Report on the sheet the iteration counts and the relative residual at the last iteration.
9. Solve the same linear system using the Conjugate Gradient method of the LIS library. Explore the Additive Schwarz method in order to define a proper preconditioner. Report on the sheet the iteration counts and the relative residual at the last iteration.
10. Compare and comment the results obtained at the two previous points.

Exercise 2

1. Consider the following eigenvalue problem: $A\mathbf{x} = \lambda\mathbf{x}$, where $A \in \mathbb{R}^{n \times n}$ is given. Describe the power method for the numerical approximation of the largest in modulus eigenvalue of A . Introduce the notation, the algorithm, and the applicability conditions.
2. State the main theoretical result.
3. Discuss how the power method can be suitably modified in order to approximate the smallest in modulus eigenvalue of A and comment on the computational costs.
4. Let A be a $n \times n$ symmetric, tridiagonal matrix defined such that

$$A = \begin{pmatrix} a_1 & 0.5 & 0 & 0 & \dots & 0 \\ 0.5 & a_2 & 0.5 & 0 & \dots & 0 \\ 0 & 0.5 & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0.5 \\ 0 & 0 & \dots & 0 & 0.5 & a_n \end{pmatrix} \quad \text{with } a_i = \left\lfloor \frac{n+1}{2} - i \right\rfloor + 1, \quad \text{for all } 1 \leq i \leq n;$$

Let $n = 99$. In a new file called `exer2.cpp`, define the matrix A in the sparse format. Report on the sheet a_1 and a_n .

5. Solve the eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$ using the proper solver provided by the `Eigen` library. Report on the sheet the smallest and largest computed eigenvalues of A .
6. Export matrix A in the matrix market format using the `unsupported/Eigen/SparseExtra` module (save it as `exer2.mtx`) and move it to the `lis-2.0.34/test` folder. Using the proper iterative solver available in the LIS library compute the largest eigenvalue of A up to a tolerance of 10^{-10} . Report the computed eigenvalue and motivate the choice made.
7. Can you use the Inverse method to perform the above operation? Why?
8. Compute the smallest eigenvalue of the `exer2.mtx` matrix. Explore different inner iterative methods and preconditioners (at least 2 alternative strategies). Report on the sheet the iteration counts and the residual at the last iteration. Comment the results.