Numerical Methods for Partial Differential Equations
A.Y. 2022/2023

Laboratory 02
Finite Element method for the Poisson equation in 1D: convergence analysis

Exercise 1.

Let $\Omega = (0, 1)$. Let us consider the Poisson problem

$$\begin{cases} -\left(\mu(x)\, u'(x)\right)' = f(x) & x \in \Omega = (0,1) \qquad \text{(1a)} \\ u(0) = u(1) = 0 & \qquad \text{(1b)} \end{cases}$$

$\mu(x) = 1$ and $f(x) = 4\,\pi^2 \sin(2\pi x)$ for $x \in \Omega$.

1.1. Show that $u_{\mathrm{ex}}(x) = \sin(2\pi x)$ is the exact solution to (1).

1.2. Starting from the solution of the first laboratory, implement a method `double Poisson1D::compute_error(const VectorTools::NormType &norm_type) const` that computes the $L^2(\Omega)$ or $H^1(\Omega)$ norm (depending on the input argument) of the error between the computed solution and the exact solution:

$$e_{L^2} = \|u_h - u_{\mathrm{ex}}\|_{L^2} = \sqrt{\int_0^1 |u_h - u_{\mathrm{ex}}|^2\, dx}\ ,$$

$$e_{H^1} = \|u_h - u_{\mathrm{ex}}\|_{H^1} = \sqrt{\int_0^1 |u_h - u_{\mathrm{ex}}|^2\, dx + \int_0^1 |\boldsymbol{\nabla} u_h - \boldsymbol{\nabla} u_{\mathrm{ex}}|^2\, dx}\ .$$

1.3. With polynomial degree $r = 1$, solve the problem (1) with finite elements, setting $N + 1 = 10, 20, 40, 80, 160$. Compute the error in both the $L^2(\Omega)$ and $H^1(\Omega)$ norms as a function of the mesh size h, and compare the results with the theory.

1.4. Repeat the previous point setting $r = 2$.

1.5. Let us now redefine the forcing term as

$$f(x) = \begin{cases} 0 & \text{if } x \le \dfrac{1}{2}\ , \\ -\sqrt{x - \dfrac{1}{2}} & \text{if } x > \dfrac{1}{2}\ . \end{cases}$$

The exact solution in this case is

$$
\begin{aligned}
u_{\mathrm{ex}}(x) &= \begin{cases} Ax & \text{if } x \leq \dfrac{1}{2}\,, \\[2ex] Ax + \dfrac{4}{15}\left(x - \dfrac{1}{2}\right)^{\frac{5}{2}} & \text{if } x > \dfrac{1}{2}\,, \end{cases} \\
A &= -\frac{4}{15}\left(\frac{1}{2}\right)^{\frac{5}{2}}\,.
\end{aligned}
$$

Check the convergence order of the finite element method in this case, with polynomial degrees $r = 1$ and $r = 2$. What can you observe?