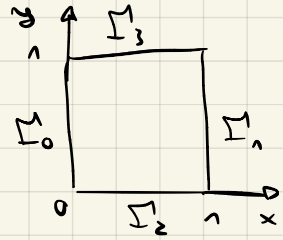
$\boxed{E1}$ DIFFUSION - REACTION PROBLEM



$$\begin{cases} -\mu\Delta u + \sigma u = f & \text{IN } \Omega \\ u = 0 & \text{IN } \Gamma_D = \Gamma_0 \cup \Gamma_2 \\ \mu \nabla u \cdot \hat{n} = q & \text{IN } \Gamma_N = \Gamma_1 \cup \Gamma_3 \end{cases}$$

$\boxed{1.1}$ WEAK FORM

$$\int_\Omega (-\mu\Delta u v + \sigma u v)\, d\bar{x} = \int_\Omega f \cdot v \, d\bar{x}$$

$$\nabla \circ (\nabla u \cdot v) = \Delta u \cdot v + \nabla u \cdot \nabla v$$

$$-\Delta u \cdot v = \nabla u \cdot \nabla v - \nabla \circ (\nabla u \cdot v)$$

$$\int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_\Omega \mu \nabla \circ (\nabla u \cdot v)\, d\bar{x} + \int_\Omega \sigma u v \, d\bar{x} = \int_\Omega f \cdot v \, d\bar{x}$$

DIVERGENCE THEOREM

$$\int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma + \int_\Omega \sigma u v \, d\bar{x} = \int_\Omega f \cdot v \, d\bar{x}$$

I CAN SPLIT THE INTEGRAL OVER THE BORDER IN TWO

$$\int_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma = \int_{\Gamma_D} \mu (\nabla u \cdot \hat{n}) v \, d\sigma + \int_{\Gamma_N} \mu (\nabla u \cdot \hat{n}) v \, d\sigma$$

WHICH BY TAKING $V = H^1_0(\Omega) = \{ w \in H^1(\Omega) : w = 0 \text{ ON } \Gamma_D \}$
AND BY REPLACING BOUNDARY CONDITIONS BECOMES
($v = 0$ ON Γ_D, $\mu \nabla u \cdot \hat{n} = q$ ON Γ_N)

$$\int_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma = \int_{\Gamma_N} \mu q \cdot v \, d\sigma$$

SO I OBTAINED

$$\int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} + \int_\Omega \sigma u \cdot v \, d\bar{x} = \int_\Omega f \cdot v \, d\bar{x} + \int_{\Gamma_N} q \cdot v \, d\sigma$$

BY INTRODUCING THE BILINEAR FORM

$$a(u,v)=\int_{\Omega}\mu\nabla u\cdot\nabla v\,d\bar{x}+\int_{\Omega}\sigma u\cdot v\,d\bar{x}$$

AND THE LINEAR FUNCTIONAL

$$F(v)=\int_{\Omega}f\cdot v\,d\bar{x}+\int_{\Gamma_N}g\cdot v\,d\sigma$$

I OBTAINED THE WEAK FORMULATION AS

FIND $u\in V=H^1_0(\Omega)$ S.t.

$$a(u,v)=F(v)\qquad\forall v\in V$$

$$u_{ex}(x,y) = (e^x - 1)(e^y - 1)$$

$$\frac{\partial u_{ex}}{\partial x} = \frac{\partial}{\partial x}(e^x - 1)\cdot(e^y - 1) + \cancel{\frac{\partial}{\partial x}(e^y - 1)} \to 0 \cdot (e^x - 1) = e^x(e^y - 1)$$

$$\frac{\partial u_{ex}}{\partial y} = e^y(e^x - 1)$$

$$\nabla u_{ex} = \begin{bmatrix} e^x(e^y - 1) \\ e^y(e^x - 1) \end{bmatrix}$$

$$\Gamma_1 = \{x = 1,\ y \in (0,1)\} \quad , \quad \Gamma_3 = \{y = 1,\ x \in (0,1)\}$$

$$\nabla u_{ex}\big|_{\Gamma_1} = \begin{bmatrix} e(e^y - 1) \\ e^y(e - 1) \end{bmatrix} \qquad \hat{n}\big|_{\Gamma_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(\nabla u_{ex} \cdot \hat{n})\big|_{\Gamma_1} = e(e^y - 1)$$

$$\nabla u_{ex}\big|_{\Gamma_3} = \begin{bmatrix} e^x(e - 1) \\ e(e^x - 1) \end{bmatrix} \qquad \hat{n}\big|_{\Gamma_3} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(\nabla u_{ex} \cdot \hat{n})\big|_{\Gamma_3} = e(e^x - 1)$$

$$\Longrightarrow \quad g = \begin{cases} e(e^y - 1) & \text{ON } \Gamma_1 \\ e(e^x - 1) & \text{ON } \Gamma_3 \end{cases}$$

$$\frac{\partial^2 u_{ex}}{\partial x^2} = \frac{\partial}{\partial x} e^x \cdot (e^y - 1) + \cancel{\frac{\partial}{\partial x}(e^y - 1)} \to 0 \; e^x = e^x(e^y - 1)$$

$$\frac{\partial^2 u_{ex}}{\partial y^2} = e^y(e^x - 1)$$

$$\Delta u_{ex} = \frac{\partial^2 u_{ex}}{\partial x^2} + \frac{\partial^2 u_{ex}}{\partial y^2} = e^x(e^y - 1) + e^y(e^x - 1)$$

$$= 2e^{x+y} - e^x - e^y$$

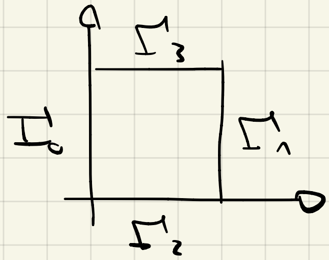
$$f = -\mu \Delta u_{ex} + \sigma u_{ex} = -2e^{x+y} + e^x + e^y + (e^x - 1)(e^y - 1)$$

$$= -2e^{x+y} \cancel{+e^x} \cancel{+e^y} + e^{x+y} \cancel{-e^x} \cancel{-e^y} + 1$$

$$= 1 - e^{x+y}$$

E2 POISSON TIME DEPENDENT

$\Omega : (0,1)^2$



FIND $u : \Omega \times (0,T] \to \mathbb{R}$ s.t.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu \Delta u = f & \text{IN } \Omega \times (0,T] \\ u = 0 & \text{ON } \Gamma_D \times (0,T] \\ \mu \Delta u \cdot \hat{n} = 0 & \text{ON } \Gamma_N \times (0,T] \\ u = u_0 & \text{IN } \Omega \times \{0\} \end{cases}$$

$u = u(\bar{x}, t)$ $\qquad f : \Omega \times (0,T) \to \mathbb{R}$

$\mu \in \mathbb{R}, \mu > 0$ $\qquad u_0 : \Omega \to \mathbb{R}$

2.1 WEAK FORMULATION OF THE PROBLEM

TAKE $v \in V$ FROM A SUITABLE FUNCTION SPACE V, TO BE DEFINED

MULTIPLY v TO 2a AND INTEGRATE OVER THE DOMAIN Ω

$$\int_\Omega \left(\frac{\partial u}{\partial t} - \mu \Delta u \right) v \, d\bar{x} = \int_\Omega f \cdot v \, d\bar{x}$$

$$\int_\Omega \frac{\partial u}{\partial t} v \, d\bar{x} - \int_\Omega \mu \Delta u \, v \, d\bar{x} = \int_\Omega f \cdot v \, d\bar{x}$$

I CAN EXPLOIT PARTIAL INTEGRATION TO OBTAIN

$$\nabla \cdot (\mu \nabla u \cdot v) = \mu \Delta u \cdot v + \mu \nabla u \cdot \nabla v$$

$$-\mu \Delta u \cdot v = \mu \nabla u \cdot \nabla v - \nabla \cdot (\mu \nabla u \cdot v)$$

$$\int_\Omega -\mu \Delta u \cdot v \, d\bar{x} = \int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_\Omega \nabla \cdot (\mu \nabla u \cdot v) \, d\bar{x}$$

THANKS TO THE DIVERGENCE THEOREM

$$\int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_{\partial \Omega} \mu \nabla u \cdot \hat{n} \cdot v \, d\sigma$$

I OBTAINED

$$\int_\Omega \frac{\partial u}{\partial t}\, d\bar{x} + \int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_{\partial\Omega} \mu \nabla u \cdot \hat{n}\, v \, d\sigma = \int_\Omega f \cdot v \, d\bar{x}$$

NOW INTRODUCE THE FUNCTION SPACE $H_0^1(\Omega)$ s.t.

$$H_0^1(\Omega) = \{ w \in H^1(\Omega) : w = 0 \text{ ON } \Gamma_D \}$$

AND CHOOSE $V = H_0^1(\Omega)$

SINCE $v \in V$ AND

NEUMANN BCS

$$\int_{\partial\Omega} \mu \nabla u \cdot \hat{n} \cdot v \, d\sigma = \cancel{\int_{\Gamma_D} \mu \nabla u \cdot \hat{n} \cdot v \, d\sigma} + \cancel{\int_{\Gamma_N} \mu \nabla u \cdot \hat{n} \cdot v \, d\sigma} = 0$$

I OBTAINED THE WEAK FORMULATION AS

$$\int_\Omega \frac{\partial u}{\partial t}\, d\bar{x} + \int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} = \int_\Omega f \cdot v \, d\bar{x}$$

INTRODUCE THE BILINEAR FORM $a(u,v)$ AND THE LINEAR FUNCTIONAL $F(v)$ s.t.

$$a(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x}$$

$$F(v) = \int_\Omega f \cdot v \, d\bar{x}$$

I CAN WRITE THE WEAK FORMULATION AS

FOR ALL $t \in (0,T)$, FIND $u \in V$ s.t.

$$\int_\Omega \frac{\partial u}{\partial t}\, d\bar{x} + a(u,v) = F(v) \qquad \forall v \in V$$

I CAN NOW DISCRETIZE IN SPACE TO OBTAIN THE SEMI DISCRETE FORMULATION OF THE PROBLEM

PARTITION THE DOMAIN IN TRIANGLES k TO OBTAIN A MESH $\mathcal{T}_h$

INTRODUCE THE F.E. SPACE $X_h^r(\Omega)$ OF PIECEWISE POLYNOMIALS OF MAXIMUM DEGREE r

$$X_h^r(\Omega) = \{ w_h \in C^0(\bar{\Omega}) : w_h(\bar{x}) \in \mathbb{P}_r, \ \forall \bar{x} \in k, \forall k \in \mathcal{T}_h \}$$

$\dim\{X_h^r\} = \hat{N}_h < +\infty$ $\quad \{\varphi_i\}_{i=1}^{\hat{N}_h}$ LAGRANGIAN BASIS FUNCTIONS OF X_h^r, $\varphi_i(\bar{x}_j) = \delta_{ij}$

I NOW CONSIDER THE FUNCTION SPACE $V_h = V \cap X_h^r(\Omega)$

I KNOW $\dim\{V_h\} = N_h \leq \hat{N}_h < +\infty$

BY DOING THIS, THE SEMI DISCRETE FORMULATION READ AS

$\forall t \in (0,T)$, FIND $u_h(t)$ s.t.

$$\int_\Omega \frac{\partial u_h}{\partial t}(t) v \, d\bar{x} + a(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h$$

BY WRITING $u_h(t) = \sum_{i=1}^{N_h} u_j(t) \varphi_j$

AND BY TAKING $\{\varphi_i\}_{i=1}^{N_h}$ AS THE TEST FUNCTIONS

I CAN REWRITE THE SEMI DISCRETE FORMULATION AS

- $$\int_\Omega \frac{\partial u_h}{\partial t}(t) v \, d\bar{x} = \int_\Omega \sum_{i=1}^{N_h} \frac{\partial u_j}{\partial t}(t) \cdot \varphi_j \varphi_i \, d\bar{x}$$
$$= \sum_{i=1}^{N_h} \frac{d u_j}{d t}(t) \int_\Omega \varphi_j \varphi_i \, d\bar{x}$$

$$\Longrightarrow M \frac{d\bar{u}}{dt}(t) \quad \text{WITH} \quad (M)_{ij} = \int_\Omega \varphi_j \varphi_i \, d\bar{x}$$

- $$a(u_h, v_h) = a\left(\sum u_j(t) \varphi_j, \varphi_i\right) = \sum u_j(t) \, a(\varphi_j, \varphi_i)$$

$$\Longrightarrow A\bar{u}(t) \quad \text{WITH} \quad (A)_{ij} = a(\varphi_j, \varphi_i) = \int_\Omega \mu \nabla\varphi_j \nabla\varphi_i \, d\bar{x}$$

- $$F(v_h) = F(\varphi_i) = \int_\Omega f \varphi_i \, d\bar{x}$$

$$\Longrightarrow \bar{f} \quad \text{WITH} \quad (\bar{f})_i = F(\varphi_i) = \int_\Omega f \varphi_i \, d\bar{x}$$

$$\begin{cases} M \dfrac{d\bar{u}}{dt}(t) + A\bar{u}(t) = \bar{f}(t) \\ \bar{u}(0) = u_0 \end{cases}$$

I CAN NOW PARTITION THE TIME INTERVAL INTO N_T SUB INTERVALS (t^n, t^{n+1}) OF SIZE Δt, $n = 0, 1, \dots, N_T - 1$

LET ALSO $\theta \in [0,1]$ BE THE PARAMETER FOR THE THETA METHOD

THE FULLY DISCRETE FORMULATION READ AS

$$\begin{cases} M \dfrac{u^{n+1} - u^n}{\Delta t} + (1-\theta) A u^n + \theta A u^{n+1} = (1-\theta) f^n + \theta f^{n+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall n = 0, 1, \dots, N_T - 1 \\ u^0 = u_0 \end{cases}$$

WHICH CAN BE REWRITTEN AS

$$\begin{cases} \left(\dfrac{M}{\Delta t} + \theta A\right) u^{n+1} = \left(\dfrac{M}{\Delta t} - (1-\theta) A\right) u^n + (1-\theta) f^n + \theta f^{n+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall n = 0, 1, \dots, N_T - 1 \\ u^0 = u_0 \end{cases}$$

THIS IS THE FULLY DISCRETE FORM OF THE WEAK FORMULATION

$\boxed{2.4}$ $f = 0$

θ METHOD IS UNCONDITIONALLY STABLE FOR $\theta \in [\frac{1}{2}, 1]$, WHILE IT'S CONDITIONALLY STABLE FOR $\theta \in [0, \frac{1}{2})$

THIS MEANS THAT WITH $\theta = 0$ (EXPLICIT/FORWARD EULER), THE METHOD IS STABLE ONLY FOR A SMALL ENOUGH TIMESTEP, IN PARTICULAR FOR A Δt S.T.

$$\Delta t < c(\theta) h^2$$

WITH $h = 0{,}1$, $\Delta t = 0{,}25$ IT'S NOT SMALL ENOUGH TO ASSURE THE ABSOLUTE STABILITY OF THE METHOD, AND IN FACT BY TRYING TO EXECUTE IT WE CAN SEE HOW IT DIVERGES.

BY TRYING TO REDUCE Δt I WASN'T ABLE TO FIND A SUITABLE Δt FOR WHICH THE METHOD CONVERGES