LAB 01

$\Omega = (0, 1)$

POISSON PROBLEM

$$\begin{cases} -(\mu(x) u'(x))' = f(x) & x \in \Omega(0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$\mu(x) = 1$ FOR $x \in \Omega$

$$f(x) = \begin{cases} 0 & \text{IF } x \le \frac{1}{8} \text{ OR } x > \frac{1}{4} \\ -1 & \text{IF } \frac{1}{8} < x \le \frac{1}{4} \end{cases}$$

1 WEAK FORMULATION OF THE PROBLEM

LET'S SET

$$V = H_0^1(\Omega) = \{ v \in H^1(\Omega) : v(0) = v(1) = 0 \}$$

RMK: $H^1(\Omega) = \{ v \in L^2(\Omega) : v' \in L^2(\Omega) \}$

$$-(\mu u')' = f$$

BECOMES

$$-\int_\Omega (\mu u')' v \, dx = \int_\Omega f v \, dx$$

$$-\int_\Omega (\mu u')' v\,dx = \int_\Omega f v\,dx$$

- $-\int_\Omega (\mu u')' v\,dx$

$$(\mu u')' v = \mu' u' v + \mu u'' v$$
$$(\mu u' v)' = \mu' u' v + \mu u'' v + \mu u' v'$$
$$(\mu u' v)' = (\mu u')' v + \mu u' v'$$
$$\Rightarrow -(\mu u')' v = -(\mu u' v)' + \mu u' v'$$

$$\Rightarrow -\int_\Omega (\mu u')' v\,dx = -\int_\Omega (\mu u' v)'\,dx + \int_\Omega \mu u' v'\,dx$$
$$= -\underbrace{\cancel{[\mu u' v]_0^1}}_{0} + \int_0^1 \mu u' v'\,dx$$

$$\Rightarrow \int_\Omega \mu u' v'\,dx = \int_\Omega f v\,dx$$

FIND $u \in V = H_0^1(\Omega)$ s.t. $a(u,v) = F(v)$ $\forall v \in V$

WITH $a(u,v) = \int_\Omega \mu u' v'\,dx$

$F(v) = \int_\Omega f v\,dx$

② GALERKIN FORMULATION OF THE PROBLEM

GIVEN $V_h \subset V$ s.t. $\dim\{V_h\} = N_h < +\infty$

G.P. IS

FIND $u_h \in V_h$ s.t. $a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

$\Longrightarrow$ V_h IS TO BE CHOSEN

③ FINITE ELEMENT FORMULATION OF THE PROBLEM, WITH PIECEWISE POLYNOMIALS OF DEGREE r

USE THE F.E. SPACE $X_h^r(\Omega)$

$$X_h^r(\Omega) := \{ v_h \in C^0(\Omega) : v_h(x) \in \mathbb{P}_r, \ \forall x \in K_i, \ \forall i = 1, \dots, N_{h+1} \}$$

AND SET $V_h = V \cap X_h^r(\Omega)$

s.t. $V_h = \{ v_h \in X_h^r(\Omega) : v_h(0) = v_h(1) = 0 \}$

SINCE $V = H_0^1(\Omega)$ AND $X_h^r(\Omega) \subset H^1(\Omega)$

BASIS OF V_h : $\{\varphi_i\}_{i=1}^{N_h}$

THE SOLUTION u_h CAN BE WRITTEN AS A LINEAR COMBINATION OF THE BASIS FUNCTIONS

$$u_h(x) = \sum_{j=1}^{N_h} u_j \varphi_j(x)$$

AND USE THIS IN THE GALERKIN PROBLEM, I CAN ALSO CHOOSE THE BASIS FUNCTIONS AS TEST FUNCTIONS, SINCE $\varphi_i \in V_h \ \forall i = 1, \dots, N_h$

$$a(u_h, v_h) = F(v_h)$$

$$a\left(\sum_{j=1}^{N_h} u_j \varphi_j(x), v_h\right) = F(v_h)$$ USING BILINEARITY AND CHOOSING φ_i AS THE TEST FUNCTIONS

$$\sum_{j=1}^{N_h} u_j \, a(\varphi_j, \varphi_i) = F(\varphi_i) \quad \forall i = 1, \dots, N_h$$

THIS IS A LINEAR SYSTEM OF EQUATIONS, WITH u_j UNKNOWNS, $a(\varphi_i, \varphi_j)$ COEFFICIENTS OF THE LINEAR SYSTEM, AND $F(\varphi_i)$ AS KNOWN TERMS (RIGHT HAND SIDE)

$$\Rightarrow \quad Au = f$$

$$(A)_{ij} = a(\varphi_j, \varphi_i) = \int_\Omega \mu \varphi_j' \varphi_i' \, dx$$

$$= \mu \int_0^1 \varphi_j'(x) \varphi_i'(x) \, dx$$

$u = (u_1, u_2, \dots, u_{N_h})^T \in \mathbb{R}^{N_h}$ VECTOR OF CONTROL VARIABLES

$$(f)_i = F(\varphi_i) = \int_\Omega f(x) \varphi_i(x) \, dx$$

$$= \int_0^1 f(x) \varphi_i(x) \, dx$$

1D POISSON PROBLEM

$$\begin{cases} -\left(\mu(x)u'(x)\right) = f(x) & \text{IN } \Omega = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$\mu(x) = 1$ IN $x \in \Omega$

$$f(x) = \begin{cases} 0 & x \leq \frac{1}{8} \vee x > \frac{1}{4} \\ -1 & \frac{1}{8} < x \leq \frac{1}{4} \end{cases}$$

1 WEAK FORMULATION OF THE PROBLEM

$$\int_0^1 -\left(\mu(x)u'(x)\right)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

$\Big\downarrow$ I.B.P.

$$\int_0^1 \mu(x)u'(x)v'(x)\,dx - \overbrace{\Big[\mu(x)u'(x)v(x)\Big]_0^1}^{0 \text{ (bcs)}} = \int_0^1 f(x)v(x)\,dx$$

$$\int_0^1 \mu(x)u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

$\Rightarrow u \in H_0^1(\Omega), \; v \in H_0^1(\Omega) = V$

$\Rightarrow$ WEAK FORMULATION

FIND $u \in V$ st $a(u,v) = F(v) \quad \forall v \in V$

WITH $a(u,v) = \int_0^1 \mu(x) u'(x) v'(x) dx$

$F(v) = \int_0^1 f(x) v(x) dx$

(2) GALERKIN FORMULATION

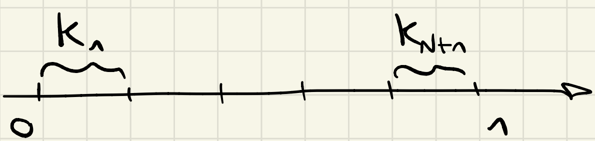
$V_h \subset V \quad , \quad \dim\{V_h\} < +\infty$

FIND $u_h \in V_h$ st. $a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

BUT WE STILL DON'T KNOW HOW V_h IS

(3) F.E. FORMULATION

$\Rightarrow$ CREATE V_h



$X_h^r = \{ v_h \in C^0(\bar{\Omega}), \; v_h|_{K_i} \in \mathbb{P}_r \quad \forall i = 1, \dots, N+1 \}$

F.E.M. : $V_h = X_h^r \cap H_0^1(\Omega)$

IMPOSE BCs

$\{\varphi_i\}_{i=1}^{N_h}$ BASIS OF V_h

$$u_h(x) = \sum_{j=1}^{N_h} U_j \varphi_j(x)$$

↳ USE THIS IN THE GALERKIN FORM

$$a_h\Big(\sum_{j=1}^{N_h} U_j \varphi_j(x),\, v_h\Big) = F(v_h)$$

INSTEAD OF TESTING AGAINST EVERY v_h, TEST AGAINST EVERY φ_i

$$\underbrace{\underbrace{\sum_{j=1}^{N_h} U_j\, a(\varphi_j, \varphi_i)}_{\text{SET OF EQ.S}} = F(\varphi_i) \quad , \quad i = 1, 2, \dots, N_h}_{\text{LINEAR SYSTEM}}$$

$$A\underline{u} = \underline{f}$$

$$\hookrightarrow (A)_{ij} = a(\varphi_j, \varphi_i) = \int_0^1 \mu(x)\varphi_j'(x)\varphi_i'(x)\,dx$$

$$\underline{u} = (U_1, U_2, \dots, U_{N_h})^T$$

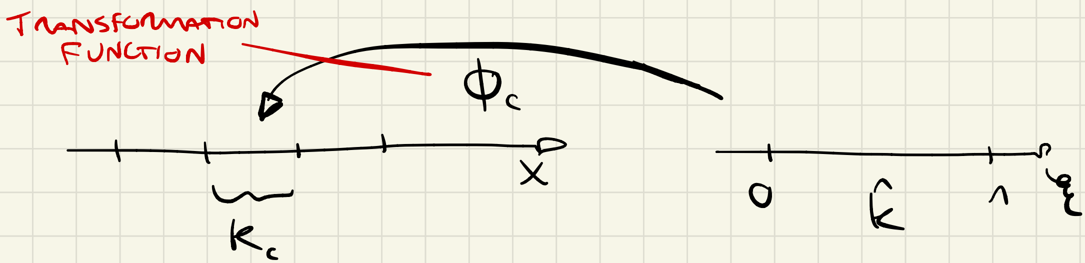
$$(\underline{f})_i = F(\varphi_i) = \int_0^1 f(x)\varphi_i(x)\,dx$$

I CAN REWRITE THE INTEGRALS AS

$$A_{ij} = \sum_{c=1}^{N+1} \int_{K_c} \mu(x)\varphi_j'(x)\varphi_i'(x)\,dx$$

$$f_i = \sum_{c=1}^{N+1} \int_{K_c} f(x)\varphi_i(x)\,dx$$

AND, USING REFERENCE ELEMENTS, AS



I DO A CHANGE OF VARIABLES IN INTEGRALS

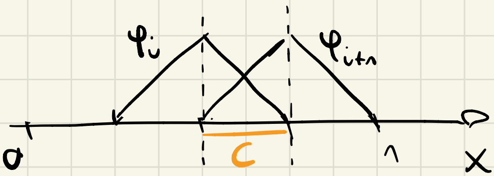
$$A_{ij} = \sum_{c=1}^{N+1} \int_{\hat{K}} \mu(\phi_c(\xi))\,\varphi_j'(\phi_c(\xi))\,\varphi_i'(\phi_c(\xi))\,\underbrace{J_c(\xi)}\,d\xi$$

$$J_c(\xi) = \phi_c'(\xi)$$

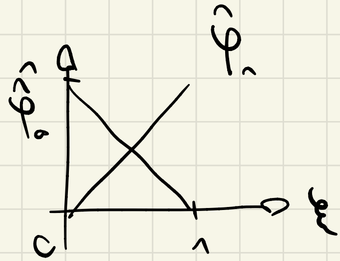
$$f_i = \sum_{c=1}^{N+1} \int_{\hat{K}} f(\phi_c(\xi))\,\varphi_i(\phi_c(\xi))\,\overbrace{J_c(\xi)}\,d\xi$$

BUT I CAN SIMPLIFY THIS EVEN MORE BY LOOKING AT INTERSECTIONS BETWEEN BASIS FUNCTIONS TO SEE WERE CONTRIBUTIONS ARE NON-NULL

(CASE $r = 1$)



$\to$ ALL OTHERS BASIS FUNCTIONS ARE USELESS IN COMPUTING INTEGRALS ON C



$$\varphi_i(\phi_c(\xi)) = \hat\varphi_{\hat i_c}(\xi)$$

$$\varphi_i'(\phi_c(\xi)) = \hat\varphi'_{\hat i_c}(\xi)\, J_c^{-1}(\xi)$$

$$J_c^{-1}(\xi) \to (\phi_c'(\xi))^{-1}$$

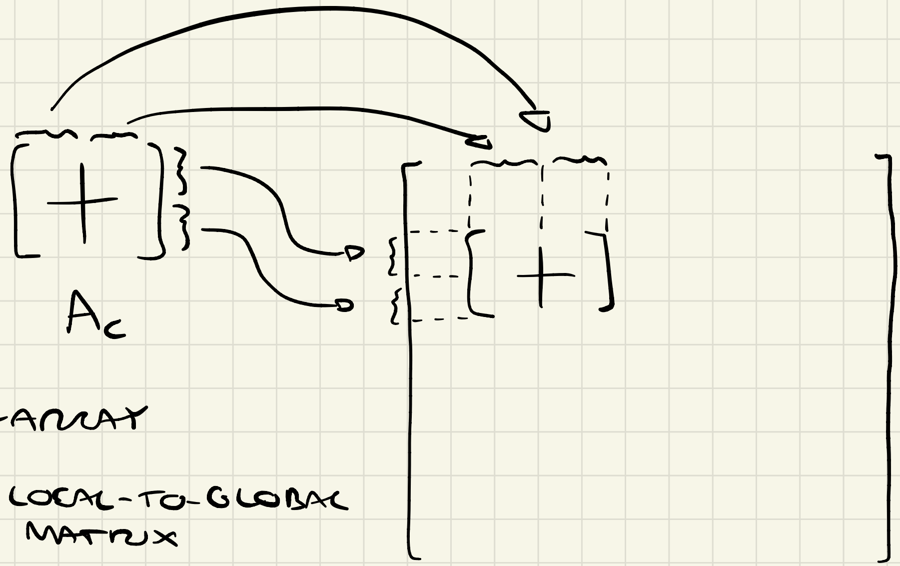
$$n_{loc} = 2r - 1$$

$$A_c \in \mathbb{R}^{n_{loc} \times n_{loc}}$$

$$(A_c)_{e,m} = \int_{\hat K} \mu(\phi_c(\xi)) \left(\hat\varphi_m(\xi) J_c(\xi)^{-1}\right)\left(\hat\varphi_e(\xi) J_c(\xi)^{-1}\right) \cdot$$
$$\cdot\, J_c(\xi)\, d\xi$$

$$(f_c)_e = \int_{\hat K} f(\phi_c(\xi))\, \hat\varphi_e(\xi)\, J_c(\xi)\, d\xi$$

NOW I USE MATRICES ON SINGLE MESH ELEMENTS TO COMPOSE THE MATRIX OF THE WHOLE MESH



ID-ARRAY

↳ LOCAL-TO-GLOBAL MATRIX

MULTIPLE MATRICES OVERLAPS (ELEMENTS THAT ARE NEAR IN THE MESH COMMUNICATE WITH EACH OTHERS)

↳ OVERLAPPING VALUES ARE OVERLAPPED

INTEGRALS CAN'T BE COMPUTED THEORETICALLY, BUT I CAN APPROXIMATE THEM WITH A COMPUTER

⟹ NUMERICAL QUADRATURE

$$\int_0^{\wedge} g(\xi)\,d\xi \simeq \sum_{q=0}^{N_q} g(\xi_q)\,w_q$$

ξ_q: QUADRATURE NODES

w_q: QUADRATURE WEIGHTS

⟹ GAUSS-LEGE... QUADRATURE OF ORDER $r+1$

→ DO THIS FOR INTEGRALS OF A AND f

$$(A)_{\ell m} = \sum_{q=0}^{n_q} \mu(\Phi_c(\xi_q)) \underbrace{\left(\hat{\varphi}_m(\xi_q) J_c(\xi_q)^{-1}\right)}_{\mathrm{I}} \underbrace{\left(\hat{\varphi}_\ell(\xi_q) J_c(\xi_q)^{-1}\right)}_{\mathrm{II}} \cdot \underbrace{J_c(\xi_q)\, w_q}_{\mathrm{III}}$$

$$(f_c)_\ell = \sum_{q=0}^{n_q} f(\Phi_c(\xi_q)) \hat{\varphi}_i(\xi_q) J_c(\xi_q) w_q$$

F.E. SOFTWARE

4 STEPS

1. PRE-PROCESSING
2. ASSEMBLY PHASE (LINEAR SYSTEM IS BUILT)
3. LINEAR SOLVE (SOLVE THE L.S)
4. POST-PROCESSING

$$
\begin{array}{c}
\varphi_1 \qquad\qquad\qquad\qquad \varphi_{N_h} \\
A = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \in \mathbb{R}^{N_h \times N_h}
\end{array}
$$

INTERIOR NODES

BOUNDARY NODES

$$
\begin{array}{c}
\varphi_0 \qquad\qquad\qquad\qquad \varphi_{N_h+1} \\
\hat{A} = \begin{bmatrix} - & - & - & - & - \\ & & & & \\ & & & & \\ - & - & - & - & - \end{bmatrix} \in \mathbb{R}^{(N_h+2)(N_h+2)}
\end{array}
$$

$U_0, \quad U_{N_h+1}$

$U_0 = 0, \quad U_{N_h+1} = 0$

$\Longrightarrow$ BOUNDARY CONDITIONS

THIS IS THE MATRIX THAT DEALII ASSEMBLES

LAB 02

PROBLEM

$$\begin{cases} -(\mu(x)u'(x))' = f(x) & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$$\mu(x) = 1 \quad , \quad f(x) = 4\pi^2 \sin(2\pi x)$$

$u_{ex}(x) = \sin(2\pi x)$ IS THE EXACT SOLUTION

- $u_{ex}(0) = u_{ex}(1) = 0$
- $u_{ex}'(x) = 2\pi\cos(2\pi x)$
- $-(\mu(x)u_{ex}')' = \dots = 4\pi^2 \sin(2\pi x) = f(x)$

$\|u_h - u_{ex}\|_{L^2}$ $\qquad$ $\|u_h - u_{ex}\|_{H^1}$

KNOWN $\qquad$ KNOWN

COMPUTE WITH SOFTWARE

I WANT TO SEE HOW ERROR BEHAVES AS I CHANGE THE MESH

$$\|u_h - u_{ex}\|_{L^2} = \sqrt{\int_0^1 |\underbrace{u_h(x)}_{\text{F.E. FUNCTION}} - \underbrace{u_{ex}(x)}_{\text{ANALYTICAL EXPRESSION}}|^2 dx}$$

RMK USE DIFFERENT QUADRATURE FORMULAS TO COMPUTE THE ASSEMBLING OF MATRIX A AND THE ERROR
$\Longrightarrow$ TO COMPUTE THE ERROR A MORE ACCURATE QUADRATURE FORMULA IS NECESSARY

RMK $e_{L^2} = \|u_h - u_{ex}\| \leq \underbrace{C\,|u_{ex}|}_{D}{}_{H^{r+1}}\, h^{r+1}$ IF $u_{ex} \in H^{r+1}(\Omega)$

$$e_{L^2} \approx D h^{r+1}$$

$$\log e_{L^2} \approx \log D + \log h^{r+1} = \log D + (r+1)\log h$$

$\hookrightarrow$ LINEAR FUNCTION IN $\log h$
$(r+1)$ IS THE CONVERGENCE RATE

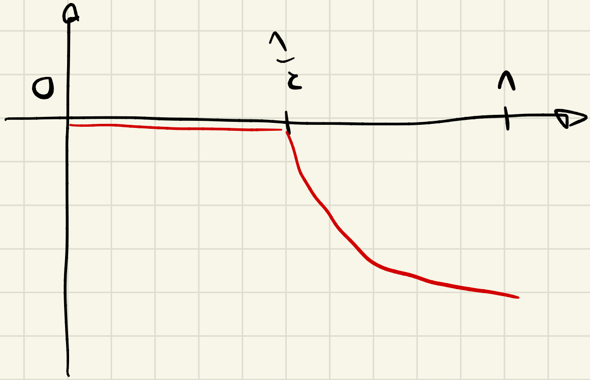
$\boxed{E\ 1.5}$

1D PROBLEM

$$\begin{cases} -(\mu(x)\,u'(x))' = f(x) & \text{IN } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

CHANGING FORCING TERM

$$f(x) = \begin{cases} 0 & x \leq \frac{1}{2} \\ -\sqrt{x - \frac{1}{2}} & x > \frac{1}{2} \end{cases}$$



$$u_{ex} = \begin{cases} Ax, & x \leq \frac{1}{2} \\ Ax + \frac{4}{15}\left(x - \frac{1}{2}\right)^{\frac{5}{2}}, & x > \frac{1}{2} \end{cases}$$

$$A = -\frac{4}{15}\left(\frac{1}{5}\right)^{\frac{5}{2}}$$

$$u'_{ex} = \begin{cases} A, & x \leq \frac{1}{2} \\ A + \frac{\cancel{4}^{\,2}}{{}_{3}\cancel{15}} \cdot \frac{\cancel{5}}{\cancel{2}}\left(x - \frac{1}{2}\right)^{\frac{3}{2}} = A + \frac{2}{3}\left(x - \frac{1}{2}\right)^{\frac{3}{2}}, & x > \frac{1}{2} \end{cases}$$

$f(x) \in L^2(\Omega)$

$$\sqrt{\int_0^1 f(x)^2 dx} = \sqrt{\int_{\frac{1}{2}}^{1} \left(x - \frac{1}{2}\right) dx} = \sqrt{\frac{x^2}{2}\Big|_{\frac{1}{2}}^{1} - \frac{1}{2}x\Big|_{\frac{1}{2}}^{1}}$$

$$= \sqrt{\cancel{\frac{1}{2}} - \frac{1}{8} - \cancel{\frac{1}{2}} + \frac{1}{4}} = \sqrt{\frac{1}{8}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} < +\infty$$

$$f(x) = -u''(x) \Rightarrow u''(x) \in L^2(\Omega)$$

THIS MEANS THAT $u_{ex}(x) \in H^2(\Omega)$

$\Rightarrow u_{ex}(x) \in H^{r+1}(\Omega)$, IF $r=1$

$\Rightarrow$ A PRIORI ESTIMATE IS CONFIRMED

BUT IF $r=2$, I REQUEST THAT $u_{ex}(x) \in H^3(\Omega)$, WHICH MEANS THAT I REQUEST THAT $f(x) \in H^1(\Omega)$: IN OTHER WORDS I'M REQUIRING THAT

$$f'(x) \in L^2(\Omega)$$

BUT THIS IS NOT TRUE. IN FACT

$$f'(x) = \begin{cases} 0 & \text{IF } x \leq \frac{1}{2} \\ -\frac{1}{2}\left(x-\frac{1}{2}\right)^{-\frac{1}{2}} = -\frac{1}{2}\frac{1}{\sqrt{x-\frac{1}{2}}} & \text{IF } x > \frac{1}{2} \end{cases}$$

$$f'(x)^2 = \begin{cases} 0 & \text{IF } x \leq \frac{1}{2} \\ \frac{1}{4}\cdot\frac{1}{\left(x-\frac{1}{2}\right)} & \text{IF } x > \frac{1}{2} \end{cases}$$

AND SINCE $\int_0^1 \frac{1}{4}\cdot\frac{1}{\left(x-\frac{1}{2}\right)}dx$ DOES NOT CONVERGE

$\Rightarrow f'(x) \notin L^2(\Omega)$, WHICH MEANS THAT

$$f'(x) = -u'''(x) \notin L^2(\Omega)$$

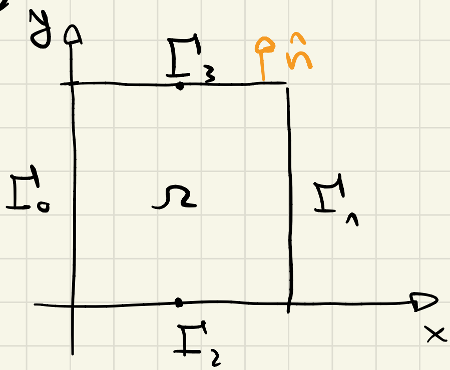
WHICH IN TURN MEANS THAT

$$u_{ex} \notin H^3(\Omega)$$

SO THE A PRIORI ERROR ESTIMATION TELLS US THAT FOR $r = 2$, THE METHOD DOES NOT CONVERGE

LAB 03

2D



$\Omega = (0,1)^2$

$$\begin{cases} -\nabla\cdot(\mu \nabla u) = f & x \in \Omega \\ u = g & x \in \Gamma_0 \cup \Gamma_1 \\ \mu \nabla u \cdot n = h & x \in \Gamma_2 \cup \Gamma_3 \end{cases}$$

DIRICHLET CONDITIONS

NEUMANN CONDITIONS

$\mu = 1, \quad f(x) = -5, \quad g(x) = x+y, \quad h(x) = y$

WF

$$\underbrace{\int_\Omega -\nabla\cdot(\mu\nabla u)v}_{\text{I.b.p.}} = \int_\Omega f v$$

$$\int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_{\partial\Omega} \mu(\nabla u \cdot n) v \, d\sigma = \int_\Omega f v \, d\bar{x}$$

$v = 0$ ON $\Gamma_0 \cup \Gamma_1$

$$\int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_{\Gamma_2 \cup \Gamma_3} \mu(\nabla u \cdot n) v \, d\sigma = \int_\Omega f v \, dx$$

$$\int_\Omega \mu \nabla u \cdot \nabla v \, dx = \int_\Omega f v + \int_{\Gamma_2 \cup \Gamma_3} h v \, d\sigma$$

$$V_0 = \{ v \in H^1 : v = 0 \text{ ON } \Gamma_0 \cup \Gamma_1 \}$$

$$u = u_0 + R_g \quad , \quad R_g \in H^1 , \; R_g = g \text{ ON } \Gamma_0 \cup \Gamma_1$$

$$u_0 \in V_0$$

$$\underbrace{\int_\Omega \mu \nabla u_0 \cdot \nabla v \, dx}_{a(u_0, v)} = \underbrace{\int_\Omega f v \, dx + \int_{\Gamma_2 \cup \Gamma_3} h v \, d\sigma - \int_\Omega \mu \nabla R_g \cdot \nabla v \, dx}_{F(v)}$$

$\boxed{\text{WF}}$ FIND $u_0 \in V_0$ s.t. $\forall v \in V_0$

$$a(u_0, v) = F(v)$$

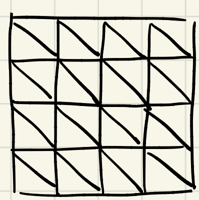
$\boxed{\text{GWF}}$ $V_{0,h} \subset V_0$, $\dim V_{0,h} < +\infty$

FIND $u_{0,h} \in V_{0,h}$ s.t. $\forall v_h \in V_{0,h}$

$$a(u_{0,h}, v_h) = F(v_h)$$

$\boxed{\text{F.E.}}$: $V_{0,h} = X_h^r \cap V_0$

$$A \bar{u} = \bar{f}$$



LAB 04 DIFFUSION-REACTION EQUATION

$\boxed{E1}$

$$\begin{cases} -\nabla \cdot (\mu \nabla u) + \sigma u = f & \text{IN } \Omega = (0,1)^2 \\ u = 0 & \text{IN } \partial\Omega \end{cases}$$

(HOMOGENEOUS DIRICHLET BCS)

$\mu(\bar{x}) = 1$, $\sigma(\bar{x}) = 1$ $\bar{x} = (x, y)$

$f(x) = (20\pi^2 + 1) \sin(2\pi x) \sin(4\pi y)$

$u_{ex}(\bar{x}) = \sin(2\pi x) \sin(4\pi y)$

WEAK FORMULATION

$$V = H_0^1(\Omega) = \{ v \in H^1(\Omega) : v(\bar{x}) = 0, \ \bar{x} \in \partial\Omega \}$$

$$\int_\Omega \underbrace{-\nabla \cdot (\mu \nabla u) v}_{\text{i.b.p.}} \, d\bar{x} + \int_\Omega \sigma u v \, d\bar{x} = \int_{dx} f v \, d\bar{x}$$

$$\int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \underbrace{\int_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, dx}_{0} + \int_\Omega \sigma u v \, d\bar{x} = \int_\Omega f v \, d\bar{x}$$

$$a(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} + \int_\Omega \sigma u v \, d\bar{x}$$

$$F(v) = \int_\Omega f v \, d\bar{x}$$

WF: FIND $u \in V$ s.t. $a(u,v) = F(v) \quad \forall v \in V = H_0^1(\Omega)$

GALERKIN FORMULATION

$V_h \subset V$ s.t. $\dim\{V_h\} = N_h < +\infty$

G.P. READS

FIND $u_h \in V_h$ s.t. $a(u_h, v_h) = F(v_h) \ \forall v_h \in V_h$

FINITE ELEMENT FORMULATION

INTRODUCE THE FINITE ELEMENT SPACE X_h^r

$$X_h^r = \left\{ v_h \in C^0(\bar{\Omega}) : v_h\big|_{K_i} \in \mathbb{P}_r \ , \ \forall i = 0, 1, \dots, N+1 \right\}$$

AND CHOOSE $V_h = X_h^r \cap H_0^1(\Omega)$

$\{\varphi_i\}_{i=1}^{N_h}$ BASIS OF V_h

$u_h = \sum_{j=1}^{N_h} u_j \varphi_j(\bar{x})$ $\quad$ u_j CONTROL VARIABLES $j = 1, \dots, N_h$

$$a(u_h, v_h) = a\left(\sum_{i=1}^{N_h} u_j \varphi_j(\bar{x}), \varphi_i\right)$$

$$= \sum_{i=1}^{N_h} u_j \, a(\varphi_j, \varphi_i) = F(\varphi_i) \quad \forall i = 1, \dots, N_h$$

$A\bar{u} = \bar{f}$ $\quad$ LINEAR SYSTEM

A IS DIFFERENT W.R.T. THE POISSON PROBLEM

$$(A)_{ij} = a(\varphi_j, \varphi_i) = \int_\Omega \mu \nabla \varphi_j \nabla \varphi_i \, d\bar{x} + \underbrace{\int_\Omega \sigma \varphi_j \varphi_i \, d\bar{x}}_{\text{REACTION}}$$

$$u_{ex}(\bar{x}) = \sin(2\pi x)\sin(4\pi y)$$

$$\nabla u_{ex}(\bar{x}) = \begin{bmatrix} 2\pi \cos(2\pi x)\sin(4\pi y) \\ 4\pi \sin(2\pi x)\cos(4\pi y) \end{bmatrix}$$

LAB05

E1

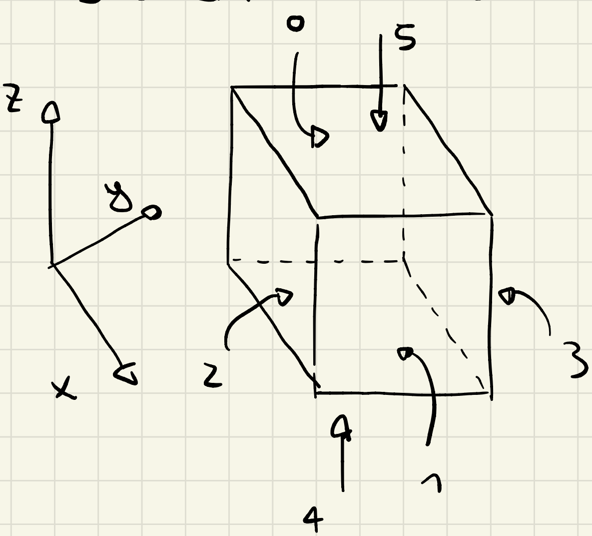
$$\begin{cases} -\underbrace{\nabla\cdot(\mu(\bar{x})\nabla u)}_{\text{DIFFUSION TERM}} + \underbrace{\sigma u}_{\text{REACTION TERM}} = f & \text{IN } \Omega = (0,1)^3 \subset \mathbb{R}^3 \\ u = 0 & \text{ON } \partial\Omega \end{cases}$$

$$\mu(x) = \begin{cases} 100 & x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases} \qquad \bar{x} = (x, y, z)^T$$

$$\sigma(\bar{x}) = 1$$

$$f(\bar{x}) = 1$$

BOUNDARY CONVENTION IN CUBES



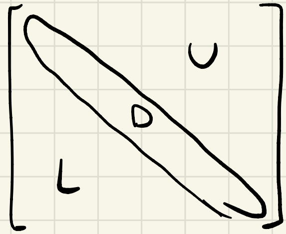
USE JACOBI PRECONDITIONER

$$A = D + L + U$$

D: DIAGONAL OF A

L: LOWER TRIANGULAR (NO DIAG)

U: UPPER TRIANGULAR (NO DIAG)

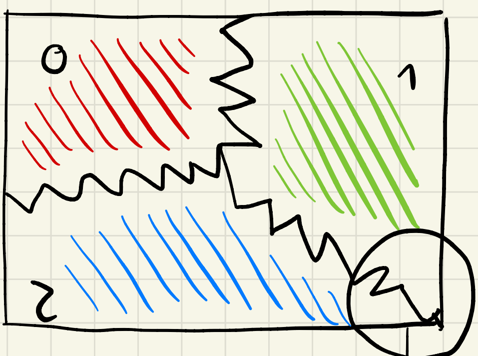


$$P_J = D \qquad P_{GS} = D + L \qquad P_{SOR} = \frac{1}{r}(D + rL)$$

DEALII IMPLEMENTATION

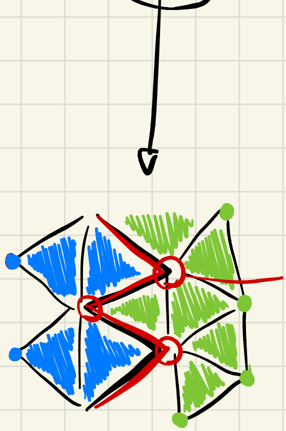
$r > 0$

PARALLELIZATION



PARALLEL PARTITION OVER THE ELEMENTS

PARALLEL PARTITION OF THE DOFs



WHAT ABOUT THIS POINTS? ONE OF THE TWO PROCESSES OWNS THEM,

BUT THE OTHER ONE NEEDS INFORMATIONS ON THESE

OWNED BY BLUE PROCESS, BUT ARE RELEVANT TO THE GREEN ONE ⟵

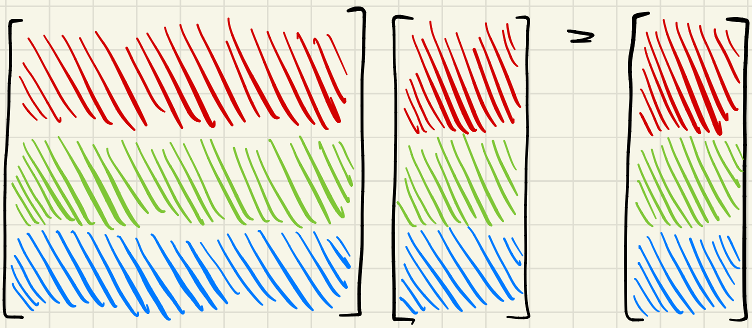
↳ BLUE & GREEN PROCESSES NEEDS TO COOPERATE BY COMMUNICATING

↳ MPI: BY USING MESSAGES

THIS ALL ENDS UP IN A LINEAR SYSTEM

⟹ SINCE DoFs HAVE BEEN PARTITIONED, ALSO UNKNOWNS HAVE BEEN

⟹ UNKNOWN PARTITIONED BY ROWS



PARALLEL PARTITION OF ALGEBRA STRUCTURES

⟹ THIS IS THE PARTITIONING OF THE DATA BETWEEN THE PROCESSES

⟹ WHAT ABOUT THE WORK?

THE ASSEMBLY LOOP CAN BE PARALLELIZED:

⟹ EACH PROCESS ASSEMBLES ONLY OWNED ELEMENTS

BUT THERE ARE DoFs WHERE THE PARTITIONING OVERLAP ⟶ ON INTERFACES

↳ EACH PROCESS MAY WRITE ON ROWS OF A THAT HE DOESN'T OWN

⟹ COMMUNICATION AT THE END OF ASSEMBLY STEP

ALSO THE LINEAR SOLVER MUST BE PARALLELIZED

$\Rightarrow$ BUT THIS DOESN'T SCALE PERFECTLY LINEARLY

LAB06 NON LINEAR EQUATIONS & VECTORIAL PROBs

$\boxed{E1}$ NON LINEAR PDE

NON LINEAR FUNCTION OF u

$$\begin{cases} -\nabla\cdot((\mu_0+\mu_1 \underline{\underline{u}}^2)\nabla \underline{\underline{u}}) = f & \text{IN } \Omega \\ u = 0 & \text{IN } \partial\Omega \end{cases}$$

$\Omega = (0,1)^3$ UNIT CUBE

$x = (x,y,z)^T$

$\mu_0 = 1, \quad \mu_1 = 10,$

$\boxed{1.1}$ WEAK FORMULATION

$V = H_0^1(\Omega)$ (HOMOGENEOUS DIRICHLET BCs)

$v \in V$

$$\underbrace{\int_\Omega (\mu_0+\mu_1 u^2)\nabla u\cdot\nabla v \, d\bar{x}}_{b(u)(v)} = \underbrace{\int_\Omega f v \, d\bar{x}}_{F(v)}$$

IN RESIDUAL FORM

$$R(u)(v) = b(u)(v) - F(v) = 0$$

WF: FIND $u \in V$ s.t. $R(u)(v) = 0 \quad \forall v \in V$

$\Rightarrow$ NEWTON METHOD

I NEED THE DERIVATIVE OF $R(u)(v)$ w.r.t. u

$$a(u)(\delta, v) = \lim_{h \to 0} \frac{1}{h}\Big[R(u+h\delta)(v) - R(u)(v)\Big]$$

$$= \frac{1}{h}\Big[b(u+h\delta)(v) - b(u)(v)\Big]$$

$$= \frac{1}{h}\Big[\int_\Omega (\mu_0 + \mu_1(u^2 + 2h\delta u + h^2\delta^2))\nabla(u+h\delta)\nabla v \, d\bar{x} - \int_\Omega (\mu_0 + \mu_1 u^2)\nabla u \nabla v \, d\bar{x}\Big]$$

$$= \frac{1}{h}\Big[\int_\Omega (\underset{(a}{\mu_0} + \underset{b}{\mu_1}(\underset{(c}{u^2} + \overbrace{2h\delta u + h^2\delta^2}^{d}))\underset{e}{\nabla u \cdot \nabla v} \, d\bar{x}$$

$$+ \int_\Omega (\mu_0 + \mu_1(u^2 + 2h\delta u + h^2\delta^2))\nabla(h\delta)\nabla v \, d\bar{x}$$

$$- \int_\Omega (\underset{(a}{\mu_0} + \underset{b}{\mu_1} \cdot \underset{c)}{u^2}) \cdot \underset{e}{\nabla u \nabla v} \, d\bar{x}\Big] =$$

$$= \frac{1}{h}\Big[\int_\Omega \mu_1(2h\delta u + h^2\delta^2)\nabla u \nabla v \, d\bar{x} + \int_\Omega (\mu_0 + \mu_1(u^2 + 2h\delta u + h^2\delta^2))h\nabla\delta\nabla v \, d\bar{x}\Big]$$

$$= \int_\Omega (2\mu_1 u\delta + \mu h\delta^2)\nabla u \nabla v \, d\bar{x} + \int_\Omega (\mu_0 + \mu_1(u^2 + 2hu\delta + h^2\delta^2))\nabla\delta\nabla v \, d\bar{x}$$

$(a+b(c+d))e - (a+bc)e$

$(a+bc+bd)e - (a+bc)e$

$\cancel{(a+bc)e} + bde - \cancel{(a+bc)e}$

FOR $h \to 0 \Rightarrow \int_\Omega (2\mu_1 u\delta)\nabla u \nabla v \, d\bar{x} + \int_\Omega (\mu_0 + \mu_1 u^2)\nabla\delta\nabla v \, d\bar{x}$

1.2) FRÉCHET DERIVATIVE

$$a(u)(\delta, v) = \frac{dR}{du}(\delta, v) = \frac{db}{du}(\delta, v)$$

FRÉCHET DERIVATIVE

$$= \int_\Omega (2\mu_1 u \delta) \nabla u \cdot \nabla v \, d\bar{x} + \int (\mu_0 + \mu_1 u^2) \nabla \delta \cdot \nabla v \, d\bar{x}$$

RMK: a BILINEAR IN SECOND & THIRD ELEMENTS

NEWTON METHOD:

GIVEN $u^{(0)} \in V$, ITERATE FOR $k = 0, 1, 2, \dots$

(i) $a(u^{(k)})(\delta^{(k)}, v) = -R(u^{(k)})(v)$

↳ LINEAR PROBLEM: FIND $\delta^{(k)}$ (TANGENT PROBLEM)

INFINITE DIMENSIONAL

⇓

DISCRETIZE & SOLVE

(ii) $u^{(k+1)} = u^{(k)} + \delta^{(k)}$

UNTIL CONVERGENCE

I NEED TO ASSEMBLE

$$\underbrace{a(u^{(k)})(\varphi_j, \varphi_i)}_{A_{ij}} = \int_\Omega (2\mu_1 u^{(k)} \varphi_j) \nabla u^{(k)} \nabla \varphi_i \, d\bar{x} + \int_\Omega (\mu_0 + \mu_1 u^{(k)2}) \nabla \varphi_j \nabla \varphi_i \, d\bar{x}$$

RMK $u = 0$ ON $\partial\Omega \Rightarrow \delta = 0$ ON $\partial\Omega$

E2 VECTORIAL PDEs

FIND $\bar{u} : \Omega \to \mathbb{R}^3$ s.t.

$$\begin{cases} -\nabla \cdot \underline{\underline{\sigma}}(\bar{u}) = f & \text{IN } \Omega = (0,1)^3 \\ \bar{u} = \bar{g} & \text{ON } \Gamma_0 \cup \Gamma_1 \\ \underline{\underline{\sigma}}(\bar{u})\bar{n} = \bar{0} & \text{ON } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \end{cases}$$

TENSOR VALUED FUNCTION (E.G.: A MATRIX)

ELASTICITY PROBLEM:

f: LOAD , u: DISPLACEMENT

$\underline{\underline{\sigma}}$: STRESS TENSOR, RELATES THE DISPLACEMENT u OF THE OBJECT TO THE STATE OF STRESS INSIDE THE OBJECT

$$\underline{\underline{\sigma}}(\bar{u}) = \mu \nabla \bar{u} + \lambda (\nabla \cdot \bar{u}) I$$ ← LINEAR ELASTICITY

→ LINEAR FUNCTION OF $\bar{u}$

$\mu = 1$, $\lambda = 10$,

$g(\bar{x}) = [0{,}25x, \; 0{,}25x, \; 0]^T$

$f(\bar{x}) = [0, \; 0, \; -1]^T$

2.1 WEAK FORMULATION

NON HOMOGENEOUS DIRICHLET BCs

$\Longrightarrow$ I NEED TO USE THE LIFTING FUNCTION

$$V_0 = \{ \underbrace{\bar{v} \in [H^1(\Omega)]^3}_{} : v = 0 \text{ on } \Gamma_0 \cup \Gamma_1 \}$$

EACH COMPONENT OF v MUST BE IN $H^1(\Omega)$

$$\bar{u} = \bar{u}_0 + R(\bar{g}) \quad , \quad \bar{u}_0 \in V_0$$

LIFTING FUNCTION $\bar{R}(\bar{g}) \in [H^1(\Omega)]^3$ s.t.

$$\bar{R}(\bar{g}) = g \text{ on } \Gamma_0 \cup \Gamma_1$$

$$\int_\Omega -\nabla \underline{\underline{\sigma}}(\bar{u}_0 + \bar{R}(\bar{g})) \cdot \bar{v}\, d\bar{x} = \int_\Omega \bar{f} \cdot \bar{v}\, d\bar{x}$$

INTEGRATE BY PARTS

SCALAR PRODUCT BETWEEN TENSORS

0

NEUMANN

$$\int_\Omega \underline{\underline{\sigma}}(\bar{u}_0 + \bar{R}(\bar{g})) : \nabla \bar{v}\, d\bar{x} - \cancel{\int_{\cancel{\partial\Omega}\ \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5} \underline{\underline{\sigma}}(\bar{u}_0 + \bar{R}(\bar{g}))\bar{n} \cdot \bar{v}\, d\gamma} = \int_\Omega \bar{f} \cdot \bar{v}\, d\bar{x}$$

DIRICHLET

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{12} + a_{21}+b_{21} + a_{22}b_{22}$$

$$\Longrightarrow \underbrace{\int_\Omega \underline{\underline{\sigma}}(\bar{u}_0) : \nabla \bar{v}\, d\bar{x}}_{a(u_0, v)} = \underbrace{\int_\Omega \bar{f} \cdot \bar{v}\, d\bar{x} - \int_\Omega \underline{\underline{\sigma}}(\bar{R}(\bar{g})) : \nabla \bar{v}\, d\bar{x}}_{F(v)}$$

$$\int_\Omega (\mu \nabla u_0 : \nabla v + \lambda (\nabla \cdot u_0)(\nabla \cdot v))\, d\bar{x} = \int_\Omega f \cdot \bar{v}\, d\bar{x} - \int_\Omega (\mu \nabla \bar{R}(\bar{g}) : \nabla \bar{v} + \lambda (\nabla \cdot \bar{R}(\bar{g}))(\nabla \cdot v))\, d\bar{x}$$

WF FIND $u_0 \in V_0$ s.t.

$$a(\bar{u}_0, \bar{v}) = F(\bar{v}) - a(\bar{R}(\bar{g}), \bar{v}) \quad \forall v \in V_0$$

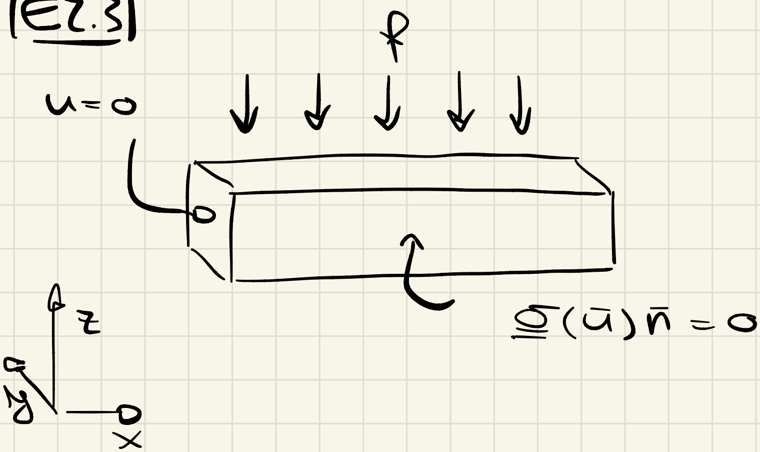
Rmk

$$a(u,v) = \int_\Omega [\mu \nabla \bar{u} : \nabla \bar{v} + \lambda (\nabla \cdot \bar{u})(\nabla \cdot \bar{v})] d\bar{x}$$

$\downarrow$ F.E. DISCRETIZATION

$$A_{ij} = a(\bar{\varphi}_j, \bar{\varphi}_i) = \int_\Omega [\mu \nabla \bar{\varphi}_j : \nabla \bar{\varphi}_i + \lambda (\nabla \cdot \bar{\varphi}_j)(\nabla \cdot \bar{\varphi}_i)] d\bar{x}$$

$\boxed{\text{EZ.3}}$



LAB 07 PARABOLIC PROBLEMS: TIME DEPENDENT PDEs

HEAT EQUATION

$u \in \Omega \times (0,T) \to \mathbb{R}$, $\Omega = (0,1)^3$, $T > 0$

$u(\bar{x}, t)$ s.t.

$$
\begin{cases}
\underbrace{\dfrac{\partial u}{\partial t}}_{\text{TIME DERIVATIVE}} \underbrace{- \nabla \cdot (\mu \nabla u)}_{\text{DIFFUSION TERM}} = f(\bar{x}, t) & \text{IN } \Omega \times (0,T) \\
\underbrace{\mu \nabla u \cdot \bar{n}}_{\text{HEAT FLOW}} = 0 \quad \text{HOMOGENEOUS NEUMANN} & \text{ON } \partial\Omega \times (0,T) \\
\underbrace{u = u_0}_{\text{INITIAL CONDITION}} & \text{IN } \Omega \times \{0\}
\end{cases}
$$

u TEMPERATURE IN A POINT

NEUMANN CONDITION: NO FLOW OF HEAT ACROSS THE BOUNDARY

$\Rightarrow$ ISOLATED OBJECT

u_0 INITIAL TEMPERATURE OF THE OBJECT

1.1 WEAK FORMULATION AND SEMI-DISCRETE FORMULATION WITH THE F.E. METHOD TO DISCRETIZE SPACE DERIVATIVE

$t \in (0,T)$

$v \in V = H^1(\Omega)$

MULTIPLY BY v AND INTEGRATE

$$\int_\Omega \frac{\partial u}{\partial t} v \, d\bar{x} + \underbrace{\int_\Omega \mu \nabla u \nabla v \, d\bar{x}}_{a(u,v)} = \underbrace{\int_\Omega f v \, d\bar{x}}_{F(v)}$$

<u>WF</u>: $\forall t \in (0,T)$, FIND $u(t) \in V$ s.t.

$$\begin{cases} \int_\Omega \frac{\partial u}{\partial t} v \, dx + a(u,v) = F(v) & \forall v \in V \\ u(0) = u_0 \end{cases}$$

$V_h = V \cap X_h^r(\Omega)$

<u>SEMI DISCRETE WF</u>: $\forall t \in (0,T)$, FIND $u_h(t) \in V_h$ s.t.

$$\int_\Omega \frac{\partial u}{\partial t} v_h \, d\bar{x} + a(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h$$

$u_h(0) = u_{0,h}$ $\Rightarrow$ F.E. APPROXIMATION OF u_0

$\{\varphi_i\}_{i=1}^{N_h}$ BASIS FOR V_h

$$\Rightarrow M \frac{d\bar{u}}{dt} + A\bar{u}(t) = \bar{f}(t) \qquad \forall t \in (0,T)$$

$\Rightarrow$ SYSTEM OF ODEs

- $\bar{U}(t) = [U_1(t), \; U_2(t), \; \dots, U_{N_h}(t)]$ TIME DEPENDENT CONTROL VARIABLES

$$u_h(t) = \sum_{i=1}^{N_h} \varphi_i U_i(t)$$

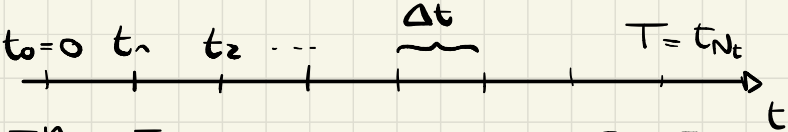
- $M_{ij} = \int_\Omega \varphi_i \varphi_j \, d\bar{x}$ MASS MATRIX $\Rightarrow$ DISCRETIZATION OF TEMPORAL DERIVATIVE

- $A_{ij} = \int_\Omega \mu \nabla \varphi_i \nabla \varphi_j \, d\bar{x}$ STIFFNESS MATRIX $\Rightarrow$ DISCRETIZATION OF DIFFUSION TERM

- $[\bar{f}(t)]_i = \int_\Omega f(t) v \, d\bar{x}$ FORCING TERM

$\boxed{1.2}$ FULLY-DISCRETE FORMULATION WITH THETA METHOD TO DISCRETIZE TIME DERIVATIVE

θ-METHOD



$\bar{u}_h^n = \bar{u}_h(t_n)$, GIVEN $\theta \in [0,1]$

$$M \frac{\bar{u}^{n+1} - \bar{u}^n}{\Delta t} + \theta A \bar{u}^{n+1} + (1-\theta) A \bar{u}^n = \theta \bar{f}^{n+1} + (1-\theta) \bar{f}^n$$

$$\Rightarrow \textcircled{*} \begin{cases} \underbrace{\left(\frac{1}{\Delta t} M + \theta A\right)}_{B_{LHS}} \bar{u}^{n+1} = \underbrace{\left(\frac{1}{\Delta t} M - (1-\theta) A\right)}_{B_{RHS}} \bar{u}^n + \theta \bar{f}^{n+1} + (1-\theta) \bar{f}^n \\ \bar{u}_0 = \bar{u}_h^0 \end{cases}$$

RMK B_{LHS} AND B_{RHS} ARE NOT TIME DEPENDANT $\Rightarrow$ ALWAYS THE SAME

$\theta = 0$ EXPLICIT EULER (FORWARD)

$\theta = 1$ IMPLICIT EULER (BACKWARD)

$\theta = \frac{1}{2}$ CRANCK-NICOLSON (IMPLICIT)

ASSEMBLE B_{LHS}, B_{RHS},

FOR $n = 0, 1, \ldots, N_t - 1$

- SOLVE ⊛:

  - ASSEMBLE f^{n+1}, f^n
  - SOLVE WITH CG ⊛

$$\frac{\partial u}{\partial t} - \nabla(\mu \nabla \circ u) = 0$$

$$\int_\Omega \frac{\partial u}{\partial t} dx - \int_\Omega \nabla(\mu \nabla \circ u) d\bar{x} = 0$$

NOW IT IS SET TO ZERO

$$\frac{d}{dt}\int_\Omega u \, d\bar{x} = \int_\Omega \nabla \cdot (\mu \nabla \circ u) \, d\bar{x} = \underbrace{\int_{\partial\Omega} \mu \nabla u \cdot \bar{n} \, d\sigma}_{\text{HEAT FLUX}} = 0$$

WHY EXPLICIT EULER DOESN'T WORK

EXPLICIT EULER

$$M \frac{u^{n+1} - u^n}{\Delta t} + A u^n = f^n$$

$$\underbrace{\frac{M}{\Delta t}}_{B_{LHS}} u^{n+1} = \underbrace{\left(\frac{M}{\Delta t} - A\right)}_{B_{RHS}} u^n + f^n$$

STABILITY

CONDITIONALLY STABLE

$u \to$ CONSTANT AS $t \to +\infty$

NOT STABLE METHOD:

$\Rightarrow$ NO GUARANTEE THAT

$u_h^n \to$ CONSTANT AS $t \to +\infty$

E.E. IS CONDITIONALLY STABLE

$\hookrightarrow$ STABILITY IF $\Delta t \leq \underbrace{c}_{\text{CONSTANT DEPENDING ON } \Omega, \mu} \cdot \underbrace{h}_{\text{MESH SIZE}}^2$

$\Rightarrow$ Δt IS NOT SMALL ENOUGH IN THIS CASE

LAB 08 CONVERGENCE ANALYSIS

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot (\mu \nabla u) = f & \text{IN } \Omega \times (0,T) \\ u = g & \text{IN } \partial\Omega \times (0,T) \\ u = u_0 & \text{IN } \Omega \times \{0\} \end{cases}$$

$$\Omega = (0,1)^3$$
$$T = 1$$
$$\mu = 1$$

1.1 $u_{ex}(\bar{x}, t) = \sin(5\pi t)\sin(2\pi x)\sin(3\pi y)\sin(4\pi z)$

COMPUTE THE FORCING TERM, THE BOUNDARY DATUM g, AND THE INITIAL CONDITION u_0

$$\frac{\partial u}{\partial t} = 5\pi\cos(5\pi t)\sin(2\pi x)\sin(3\pi y)\sin(4\pi z)$$

$$\frac{\partial^2 u}{\partial x^2} = -4\pi^2\sin(5\pi t)\sin(2\pi x)\sin(3\pi y)\sin(4\pi z)$$

$$\frac{\partial u^2}{\partial y^2} = -9\pi^2\sin(5\pi t)\sin(2\pi x)\sin(3\pi y)\sin(4\pi z)$$

$$\frac{\partial u^2}{\partial z^2} = -16\pi^2\sin(5\pi t)\sin(2\pi x)\sin(3\pi y)\sin(4\pi z)$$

- $f = \dfrac{\partial u}{\partial t} - \nabla \cdot (\mu \nabla u) = \dfrac{\partial u}{\partial t} - \mu \Delta u =$
  $= \left[5\pi\cos(5\pi t) + 29\pi^2\sin(5\pi t)\right]\sin(2\pi x)\sin(3\pi y)\sin(4\pi z)$

- $g = u_{ex}\big|_{\partial\Omega} = 0$ (AT LEAST ONE BETWEEN x, y, z IS ZERO OR ONE ON $\partial\Omega$
  $\Rightarrow \sin(0) = \sin(2\pi) = \sin(3\pi) = \sin(4\pi) = 0$)

- $u_0 = u_{ex}\big|_{t=0} = 0$

$\Rightarrow$ "METHOD OF MANUFACTURED SOLUTIONS"
$\Rightarrow$ TEST THE METHOD TO CHECK IF IT WORKS

CONVERGENCE TABLE (W.R.T. TIME DISCRETIZATION PARAMETER)

$$e_{L^2} = \| u_h^{N_T} - u_{ex}(T) \|_{L^2(\Omega)} \leq C_h h^{p} + C_{\Delta t} \Delta t^{q}$$

CONVERGENCE RATE W.R.T. h

CONVERGENCE RATE W.R.T. Δt

ASSUMPTIONS:

(i) $C_h h^p \simeq 0$

(ii) $e_{L^2} \simeq C_{\Delta t} \Delta t^q$

(iii) $e_{L^2}(\Delta t_1) = e_1 \simeq C_{\Delta t} \Delta t_1^q$

$e_{L^2}(\Delta t_2) = e_2 \simeq C_{\Delta t} \Delta t_2^q$

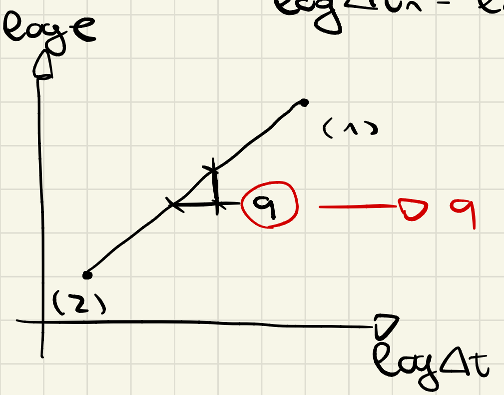
$$\frac{e_1}{e_2} \simeq \frac{\cancel{C_{\Delta t}} \Delta t_1^q}{\cancel{C_{\Delta t}} \Delta t_2^q} = \left(\frac{\Delta t_1}{\Delta t_2} \right)^q$$

I WANT TO FIND q

$$\log \frac{e_1}{e_2} \simeq q \log \frac{\Delta t_1}{\Delta t_2} \Rightarrow q \simeq \frac{\log e_1/e_2}{\log \frac{\Delta t_1}{\Delta t_2}}$$

DEAL.II CONVERGENCE TABLE ASSUMES $\frac{\Delta t_1}{\Delta t_2} = 2$
(THAT'S WHY I NEED TIME STEPS & MESH SIZES THAT ALWAYS HALVES)

$$\Rightarrow q = \frac{\log e_1 - \log e_2}{\log \Delta t_1 - \log \Delta t_2}$$



→ q IT'S THE SLOPE IN THE log-log SCALE

THE SMALLER Δt, THE FASTER THE LINEAR SYSTEM IS SOLVED

LINEAR SYSTEM MATRIX IS

$$\frac{M}{\Delta t} + \theta A \underset{\sim}{\Delta t \to 0} \frac{M}{\Delta t}$$ → WELL CONDITIONED MATRIX W.R.T. THE STIFFNESS MATRIX

ALSO, WE'RE USING AN ITERATIVE SOLVER

⇒ WE HAVE TO PROVIDE AN INITIAL GUESS, AND THE NUMBER OF ITERATIONS ALSO DEPENDS ON IT

IN THIS CASE THE I.G. AT A SPECIFIC TIME STEP IS THE SOLUTION FOUND AT THE PREVIOUS TIME STEP

⇒ I'M CLOSE TO THE SOLUTION

E2 NON LINEAR, TIME DEPENDENT PROBLEM

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot ((\mu_0 + \mu_1 u^2)\nabla u) = f & \text{IN } \Omega \times (0,T) \\ u = g & \text{ON } \partial\Omega \times (0,T) \\ u = u_0 & \text{IN } \Omega \times \{0\} \end{cases}$$

$\mu_0 = 0{,}1$, $\mu_1 = 1$, $g(\bar{x}, t) = 0$, $u_0 = 0$

$\Omega = (0,1)^3$, $T = 1$ $\quad f(\bar{x}, t) = \begin{cases} 2 & \text{IF } t < 0{,}25 \\ 0 & \text{IF } t \geq 0{,}25 \end{cases}$

HEAT NON LINEAR

- → THE OBJECT HAS A CONDUCTIVITY THAT INCREASES WITH THE TEMPERATURE
- → f IS A SOURCE TERM ⟹ HEATER THAT SHUTS DOWN AFTER $t_H = 0{,}25$

2.1 WEAK FORMULATION

$V = H_0^1(\Omega)$

$\forall t \in (0,T)$, FIND $u(t) \in V$ s.t.

(a) $u(0) = u_0$

(b) $\displaystyle\int_\Omega \frac{\partial u}{\partial t} v \, d\bar{x} + \underbrace{\int_\Omega (\mu_0 + \mu_1 u^2) \nabla u \cdot \nabla v \, d\bar{x}}_{b(u)(v)} = \underbrace{\int_\Omega f v \, d\bar{x}}_{F(v)}$

$\forall v \in V$

IN RESIDUAL FORM

$$R(u)(v) = \int_\Omega \frac{\partial u}{\partial t} v \, d\bar{x} - b(u)(v) - F(v) = 0 \quad \forall v \in V$$

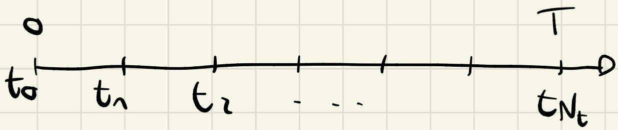
INTRODUCE $V_h = V \cap X_h^r(\Omega)$ DISCRETE SPACE

SEMI-DISCRETE WF: $\forall t \in (0,T)$, FIND $u_h \in V_h$ ST

(a) $u_h(0) = u_0$

(b) $R(u_h)(v_h) = 0 \quad \forall v_h \in V_h$

INTRODUCE TIME DISCRETIZATION (IMPLICIT EULER)



$t_{n+1} - t_n = \Delta t$

FOR $n = 0, 1, 2, \dots, N_t - 1$:

$$\underbrace{\int_\Omega \frac{u_h^{n+1} - u_h^n}{\Delta t} v_h \, dx + b(u_h^{n+1})(v_h) - F^{n+1}(v_h)}_{R^{n+1}(u_h^{n+1})(v_h)} = 0 \quad \forall v_h \in V_h$$

NON LINEAR PROBLEM

$\Rightarrow$ USE NEWTON METHOD

I NEED THE DERIVATIVE OF THE RESIDUAL

$$a(u_h^{n+1})(\delta_h, v_h)$$

$$a(u_h^{n+1})(\delta_h, v_h) = \int_\Omega \frac{\delta_n}{\Delta t} v_h \, dt + \frac{db}{du}(\delta_h, v_h) - \cancel{\frac{dF}{du}}_{0}$$

$$= \int_\Omega \frac{\delta_h}{\Delta t} v_h \, dx + \int_\Omega (2\mu_1 u_h^{n+1} \delta_h) \nabla u_h^{n+1} \nabla v_h \, dx +$$

$$+ \int_\Omega (\mu_0 + \mu_1 (u_h^{n+1})^2) \nabla \delta_h \nabla v_h \, dx$$

AT TIME t_n: SET $u_h^{n+1,(0)} = u_h^n$, THEN ITERATE

FOR $k = 0, 1, \ldots$ UNTIL CONVERGENCE

(i) $a(u_h^{n+1,(k)})(\delta_h^{(k)}, v_h) = -R(u_h^{n+1,(k)})(v_h) \quad \forall v_h \in V_h$
$\longrightarrow \delta_h^{(k)}$

(ii) $u_h^{n+1(k+1)} = u_h^{n+1(k)} + \delta_h^{(k)}$

I HAVE TWO LOOPS IN THE SOLVE STEP

FOR $n = 0, 1, \ldots, N_T - 1$ (TIME DISCRETIZATION

FOR $k = 0, 1, \ldots,$ UNTIL CONVERGENCE

[NEWTON ITERATIONS]

END

END

$$A(\bar{u}^{n+1,(k)})\,\bar{\delta} = -\bar{V}$$

$$\bar{\delta} = u^{n+1,(k+1)} - u^{n+1,(k)}$$

$$\left.\begin{array}{l} u_h^{n+1,(k+1)}\big|_{\partial\Omega} = g \\ u_h^{n+1,(k)}\big|_{\partial\Omega} = g \end{array}\right. \Rightarrow \delta_h^{(k)}\big|_{\partial\Omega} = \bar{0}$$

$$\left.\begin{array}{l} u_h^{n+1,(0)}\big|_{\partial\Omega} = g \\ \delta_h^{(k)}\big|_{\partial\Omega} = 0 \end{array}\right\} \Rightarrow u_h^{n+1,(k)}\big|_{\partial\Omega} = g$$

IF INSTEAD I HAD USED EXPLICIT EULER

$$\int_\Omega \frac{u_h^{n+1} - u_h^n}{\Delta t} v_h \, dx + b(u_h^n)(v_h) - F^n(v_h) = 0$$

ONLY UNKNOWN

$\Rightarrow$ LINEAR PROBLEM W.R.T. u_h^{n+1}
NO NEED OF NEWTON AT EACH TIMESTEP
BUT I MAY HAVE ISSUES OF ABSOLUTE STABILITY

$$\frac{1}{2}\left(\nabla\bar{u} + (\nabla\bar{u})^T\right)$$

$$\nabla\bar{u} = \begin{bmatrix} \nabla^T u_1 \\ \nabla^T u_2 \\ \nabla^T u_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

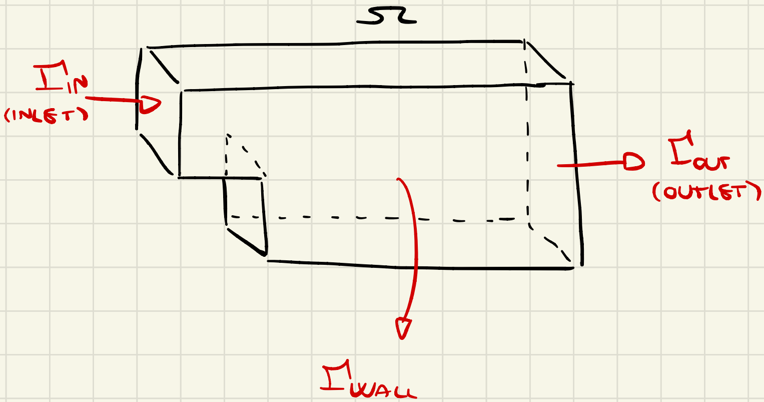
LAB09 STOKES PROBLEM

$\Omega \subset \mathbb{R}^3$ $\quad$ $\bar{u} : \Omega \to \mathbb{R}^3$ FLUID VELOCITY

$p : \Omega \to \mathbb{R}$ FLUID PRESSURE

VISCOSITY

$$\begin{cases} -\nu \Delta \bar{u} + \nabla p = \bar{f} & \text{IN } \Omega \quad \text{MOMENTUM EQUATION} \\ \nabla \cdot \bar{u} = 0 & \text{IN } \Omega \quad \text{CONTINUITY EQUATION} \\ \bar{u} = \bar{u}_{IN} & \text{ON } \Gamma_{IN} \\ \bar{u} = \bar{0} & \text{ON } \Gamma_{WALL} \quad \text{NO-SLIP CONDITION} \\ \nu (\nabla \bar{u}) \hat{n} - p \hat{n} = -p_{out} \hat{n} & \text{ON } \Gamma_{OUT} \end{cases}$$



1.1 WEAK FORMULATION

$$V = \{\bar{v} \in [H^1(\Omega)]^3 : \bar{v} = \bar{u}_{IN} \text{ ON } \Gamma_{IN},\ \bar{v} = \bar{0} \text{ ON } \Gamma_{WALL}\}$$

$$V_0 = \{\bar{v} \in [H^1(\Omega)]^3 : \bar{v} = \bar{0} \text{ ON } \Gamma_{IN} \cup \Gamma_{WALL}\}$$ LIFTING FUNCTION TRICK

$$Q = L^2(\Omega)$$

<u>WF</u> MOMENTUM EQUATION $\quad \bar{v} \in V_0$

$$\int_\Omega -\nu \Delta \bar{u} \cdot \bar{v}\, d\bar{x} + \int_\Omega \nabla p \cdot \bar{v}\, d\bar{x} = \int_\Omega \bar{f} \cdot \bar{v}\, d\bar{x}$$

$$\int_\Omega \nu \nabla \bar{u} : \nabla \bar{v}\, d\bar{x} - \int_{\partial\Omega} \nu (\nabla \bar{u})\hat{n} \cdot \bar{v}\, d\sigma - \int_\Omega p \nabla \cdot v\, d\bar{x} + $$
$$+ \int_{\partial\Omega} p \hat{n} \cdot \bar{v}\, d\sigma = \int_\Omega \bar{f} \cdot \bar{v}\, d\bar{x}$$

$$\int_\Omega \nu \nabla \bar{u} : \nabla \bar{v}\, d\bar{x} - \int_{\Gamma_{OUT}} \nu (\nabla \bar{u})\hat{n} \cdot \bar{v}\, d\sigma - \int_\Omega p \nabla \cdot \bar{v}\, d\bar{x} + $$
$$+ \int_{\Gamma_{OUT}} p \hat{n} \cdot \bar{v}\, d\sigma = \int_\Omega \bar{f} \cdot \bar{v}\, d\bar{x}$$

BCS ON Γ_{OUT}

I OBTAIN

$$\underbrace{\int_\Omega \nu \nabla \bar{u} : \nabla \bar{v}\, d\bar{x}}_{a(u,v)} - \underbrace{\int_\Omega p \nabla \cdot v\, d\bar{x}}_{b(v,p)} = \underbrace{\int_\Omega \bar{f} \cdot \bar{v}\, d\bar{x} + \int_{\Gamma_{OUT}} (-p_{OUT}\hat{n})\bar{v}\, d\sigma}_{F(v)}$$

LIFTING FUNCTION:

$$\bar{u} = \bar{u}_0 + \bar{R}(\bar{u}_{in}) \qquad \bar{u}_0 \in V_0$$

$$\bar{R}(\bar{u}_{in}) \in V : \bar{R}(\bar{u}_{in}) = \bar{u}_{in} \text{ IN } \Gamma_{IN}$$
$$\nabla \cdot \bar{R}(\bar{u}_{in}) = 0 \text{ IN } \Omega$$

$$a(\bar{u}_0, \bar{v}) + b(\bar{v}, p) = F(\bar{v}) - a(\bar{R}(\bar{u}_{in}), \bar{v}) \quad \forall \bar{v} \in V_0$$

<u>WF</u> CONTINUITY EQUATION $\quad q \in Q$

$$\int_\Omega (\nabla \cdot \bar{u}) q \, d\bar{x} = 0 \implies b(\bar{u}, q) = 0$$
$$b(\bar{u}_0, q) = 0$$

<u>WF</u> STOKES:

FIND $\bar{u}_0 \in V_0$, $p \in Q$, s.t. $\forall v \in V_0, \forall q \in Q$

$$\begin{cases} a(\bar{u}_0, \bar{v}) + b(\bar{v}, p) = F(\bar{v}) - a(\bar{R}(\bar{u}_{in}), \bar{v}) \\ b(\bar{u}, q) = 0 \end{cases}$$

DISCRETIZATION:

$$V_{0h} = V_0 \cap [X_h^{r_u}(\Omega)]^3 \quad , \quad Q_h = Q \cap X_h^{r_p}(\Omega)$$

FIND $\bar{u}_{0h} \in V_{0h}$, $p_h \in Q_h$ s.t. $\forall \bar{v}_h \in V_{0h}$, $\forall q_h \in Q_h$

$$\begin{cases} a(\bar{u}_{0h}, \bar{v}_h) + b(\bar{v}_h, p_h) = F(\bar{v}_h) - a(\bar{R}_h(\bar{u}_{in}), \bar{v}_h) \\ b(\bar{u}_h, q_h) = 0 \end{cases}$$

BASIS FUNCTIONS

$\{\bar{\varphi}_i\}_{i=1}^{N_h^u}$ BASIS OF V_{0h}

$\{\psi_i\}_{i=1}^{N_h^p}$ BASIS OF Q_h

$\Big\downarrow$ F.E. DISCRETIZATION

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \bar{U} \\ \bar{P} \end{bmatrix} = \begin{bmatrix} \bar{F} \\ \bar{0} \end{bmatrix}$$

$$A_{ij} = \int_\Omega \nu \nabla \bar{\varphi}_j : \nabla \bar{\varphi}_i \, d\bar{x}$$

$$B_{ij} = -\int_\Omega \psi_j \nabla \cdot \bar{\varphi}_i \, d\bar{x}$$

$$\bar{F}_i = \int_\Omega \bar{f} \cdot \bar{\varphi}_i \, d\bar{x} + \int_{\Gamma_{out}} -p_{out} \hat{n} \cdot \bar{\varphi}_i \, d\bar{x}$$

$$S = \begin{bmatrix} A & B \\ B & 0 \end{bmatrix}$$

"SADDLE-POINT" PROBLEM MATRIX

$\Rightarrow$ HARD TO SOLVE

BLACK BLOCK PRECONDITIONERS FAIL TO REDUCE THE CONDITION NUMBER OF THIS BLOCK MATRIX

$\Rightarrow$ I NEED A SPECIFIC PRECONDITIONER

$$P_1 = \begin{bmatrix} A & 0 \\ 0 & \frac{1}{\gamma} M_p \end{bmatrix} \qquad (M_p)_{ij} = \int_\Omega \psi_i \psi_j \, d\bar{x}$$

$\hookrightarrow$ PRESSURE MASS MATRIX

$$P_1^{-1} S \bar{x} = P_1^{-1} \bar{y} \qquad P_1^{-1} \bar{z} \qquad \bar{z} = \begin{bmatrix} \bar{z}_u \\ \bar{z}_p \end{bmatrix}$$

$$P_1^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & \gamma M_p^{-1} \end{bmatrix} \qquad P_1^{-1} \bar{z} = \begin{bmatrix} A^{-1} \bar{z}_u \\ \gamma M_p^{-1} \bar{z}_p \end{bmatrix}$$

$$\frac{1}{\gamma} M_p \bar{w} = \bar{z}_p$$

SAME AS SOLVING

$$A \bar{w} = \bar{z}_u$$

ANOTHER POSSIBLE PRECONDITIONER IS

$$P_2 = \begin{bmatrix} A & 0 \\ B & \frac{1}{\gamma} M_p \end{bmatrix}$$

DATA: $\gamma = 1 \frac{m^2}{s}$ $\bar{f} = \bar{0} \frac{m}{s^2}$

$$\bar{u}_{in} = \begin{bmatrix} -\alpha\, y(2-y)(1-z)(2-z) \\ 0 \frac{m}{s} \\ 0 \frac{m}{s} \end{bmatrix}$$

$\alpha = 1\ m^{-1} s^{-1}$

$p_{out} = 10\,Pa$

$$P_1 = \begin{bmatrix} A & 0 \\ 0 & \frac{1}{\gamma} M_p \end{bmatrix} \qquad P_2 = \begin{bmatrix} A & 0 \\ B & \frac{1}{\gamma} M_p \end{bmatrix}$$

THIS MAKES THE PRECONDITIONER MORE SIMILAR TO THE SYSTEM MATRIX

$\Rightarrow$ BETTER PERFORMANCES