


EXAM SIMULATION

E1 POISSON PROBLEM WITH ROBIN BCS

$$\Omega = (0,1)^2, \partial\Omega = \Gamma_0 \cup \Gamma_R = \bigcup_{i=0}^3 \Gamma_i$$

$$\Gamma_0 = \Gamma_2, \Gamma_R = \Gamma_0 \cup \Gamma_1 \cup \Gamma_3$$

STRONG PROBLEM:

FIND $u: \Omega \rightarrow \mathbb{R}$ s.t.

$$\begin{cases} -\mu \Delta u = f & \text{in } \Omega \\ u = 0 & \text{in } \Gamma_0 \\ -\mu \nabla u \cdot \hat{n} + \gamma(u - u_R) = 0 & \text{in } \Gamma_R \end{cases}$$

A.1 WRITE THE WEAK FORMULATION OF THE PROBLEM BY INCLUDING DEFINITIONS, CHOICE OF FUNCTION SPACES AND THE DERIVATION OF THE FORMULATION

LET $v \in V$ BE A TEST FUNCTION IN A SUITABLE FUNCTION SPACE V (T.B.D.)

MULTIPLY v TO Γ_R AND INTEGRATE THE RESULT OVER Ω

$$\int_{\Omega} -\mu \Delta u \cdot v \, d\bar{x} = \int_{\Omega} f \cdot v \, d\bar{x}$$

I CAN PARTIALLY INTEGRATE THIS TO OBTAIN

$$\begin{aligned} \nabla \cdot (\nabla u \cdot v) &= \Delta u \cdot v + \nabla u \cdot \nabla v \\ \Rightarrow -\Delta u \cdot v &= \nabla u \cdot \nabla v - \nabla \cdot (\nabla u \cdot v) \end{aligned}$$

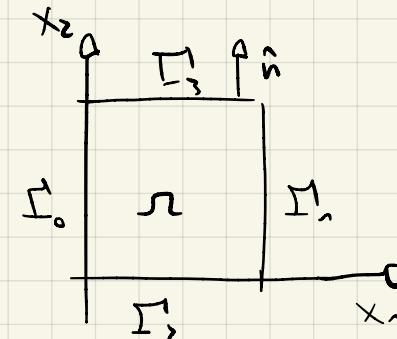
BY REPLACING THE OBTAINED RESULT IN THE INTEGRAL WE HAVE

$$\mu > 0$$

$$\gamma \geq 0$$

$$u_R: \Gamma_R \rightarrow \mathbb{R}$$

$$f: \Omega \rightarrow \mathbb{R}$$



$$\int_{\Omega} -\mu \Delta u \cdot v \, d\bar{x} = \int_{\Omega} \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_{\Omega} \mu \nabla \cdot (\nabla u \cdot v) \, d\bar{x}$$

AND BY APPLYING THE DIVERGENCE THEOREM

$$\int_{\Omega} \mu \Delta u \cdot v \, d\bar{x} = \int_{\Omega} \mu \nabla u \cdot \nabla v \, d\bar{x} - \oint_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma$$

I CAN SPLIT THE INTEGRAL OVER THE BOUNDARY AS THE INTEGRAL OVER THE SINGLE INTERFACES Γ_i , AND OBTAIN

$$\oint_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma = \int_{\Gamma_0} \mu (\nabla u \cdot \hat{n}) v \, d\sigma + \int_{\Gamma_R} \mu (\nabla u \cdot \hat{n}) v \, d\sigma$$

DIRICHLET BCS

$$\oint_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma = \int_{\Gamma_R} \mu (\nabla u \cdot \hat{n}) v \, d\sigma$$

AND BY REPLACING IT WITH THE ROBIN BOUNDARY CONDITIONS I OBTAIN

$$\begin{aligned} \oint_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma &= \int_{\Gamma_R} \gamma(u - u_R) v \, d\sigma \\ &= \int_{\Gamma_R} \gamma u v \, d\sigma - \int_{\Gamma_R} \gamma u_R v \, d\sigma \end{aligned}$$

PUTTING ALL TOGETHER

$$\int_{\Omega} \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_{\Gamma_R} \gamma u v \, d\sigma = \int_{\Omega} f \cdot v \, d\bar{x} - \int_{\Gamma_R} \gamma u_R v \, d\sigma$$

FOR THE WEAK FORMULATION TO BE WELL DEFINED I NEED

$$u, v \in H^1(\Omega) = \left\{ w \in L^2(\Omega) : \nabla w \in [L^2(\Omega)]^2 \right\}$$

SO I CHOOSE THE FUNCTION SPACE $V = H_0^1(\Omega)$, WITH

$$H_0^1(\Omega) = \left\{ v \in H^1(\Omega) : v = 0 \text{ ON } \Gamma_0 \right\}$$

BY INTRODUCING THE NOTATION

$$\mathcal{A}(u, v) = \int_{\Omega} \mu \nabla u \cdot \nabla v \, dx - \int_{\Gamma} g_u v \, ds$$

$$F(v) = \int_{\Omega} f \cdot v \, dx - \int_{\Gamma} g_f v \, ds$$

I CAN WRITE THE WEAK FORMULATION AS:

FIND $u \in V = H^1(\Omega)$ s.t.

$$\mathcal{A}(u, v) = F(v) \quad \forall v \in V$$

1.2 TO WRITE THE GALERKIN-F.E. APPROXIMATION OF THE PROBLEM, I NEED TO DISCRETIZE THE FUNCTION SPACE V , IN ORDER TO BE ABLE TO SOLVE THE PROBLEM ON A CALCULATOR, SINCE $\dim\{V\} = +\infty$

I WANT TO RESTRICT V_h s.t. $\dim\{V_h\} = N_h < +\infty$

BY DOING THIS, THE WEAK FORMULATION BECOMES THE GALERKIN PROBLEM

FIND $u_h \in V_h$ s.t. $\mathcal{A}(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

LET'S NOW INTRODUCE A PARTITION OF THE DOMAIN (THE MESH Z_h) INTO TRIANGULAR SUBDOMAINS (ELEMENTS K), AND THE SPACE X_h OF PIECEWISE POLYNOMIALS OF DEGREE r

$$X_h(\Omega) = \{u_h \in C^0(\bar{\Omega}) : u_h \in P_r \quad \forall x \in K, \forall k \in Z_h\}$$

$\dim\{X_h(\Omega)\} = N_h < +\infty$, AND DEPENDS ON THE NUMBER OF ELEMENTS IN THE MESH, THE DEGREE r , AND HOW TRIANGLES ARE ORGANIZED WITHIN THE MESH
THIS MEANS THAT I CAN CONSIDER A SET OF LAGRANGIAN BASIS FUNCTIONS $\varphi_i(x)$ THAT ARE PIECEWISE POLYNOMIALS OF DEGREE r ($\varphi_i(x_j) = \delta_{ij}, \forall i,j = 1, \dots, N_h$)

LO NODES OF THE MESH Z_h

NOW I SET $V_h = V \cap X_h(\Omega)$

SINCE $\dim\{V_h\} = N_h \leq N_h < +\infty$, I CAN CONSIDER N_h OVER THE N_h LAGRANGIAN BASIS FUNCTIONS OF THE F.E. SPACE X_h AS BASIS FUNCTIONS ALSO FOR THE SPACE V_h

$$\{\varphi_i\}_{i=1}^{N_h}$$

APPROXIMATE THIS WAY, THE SOLUTION u_h CAN BE WRITTEN AS

$$u_h = \sum_{j=1}^{N_h} u_j \varphi_j \quad \text{WHERE } u_j \in \mathbb{R} \text{ ARE CALLED CONTROL VARIABLES (DEGREES OF FREEDOM)}$$

THIS MEANS THAT I CAN WRITE

$$\mathcal{A}(u_h, v_h) = \mathcal{A}\left(\sum_{i=1}^{N_h} u_i \varphi_i, v_h\right)$$

AND SINCE $\varphi_i \in V_h \quad \forall i = 1, \dots, N_h$, I CAN USE THESE AS TEST FUNCTIONS, INSTEAD OF CHECKING WITH ALL $v_h \in V_h$
THIS MEANS THAT THE GALERKIN PROBLEM BECOMES, BY ALSO EXPLOITING BILINEARITY OF $\mathcal{A}(\cdot, \cdot)$

$$\mathcal{A}(u_h, v_h) = \mathcal{A}\left(\sum_{i=1}^{N_h} u_i \varphi_i, \varphi_j\right) = \sum_{i=1}^{N_h} u_i \mathcal{A}(\varphi_i, \varphi_j) = F(\varphi_j)$$

$$i = 1, 2, \dots, N_h$$

THE RESULT IS A LINEAR SYSTEM $A \bar{u} = \bar{f}$ WITH

- $\bar{u} = (u_1, u_2, \dots, u_{N_h})^\top \in \mathbb{R}^{N_h}$
- $A \in \mathbb{R}^{N_h \times N_h}$ s.t.

$$(A)_{ij} = \mathcal{A}(\varphi_i, \varphi_j) = \int_{\Omega} \mu \nabla \varphi_i \cdot \nabla \varphi_j \, dx - \int_{\Gamma} g \varphi_i \varphi_j \, ds$$

- $\bar{f} \in \mathbb{R}^{N_h}$ s.t.

$$(f)_i = \int_{\Omega} f \cdot \varphi_i \, dx - \int_{\Gamma} g_f \varphi_i \, ds$$

TO CONSTRUCT THE MATRIX A AND THE VECTOR f , THE REFERENCE ELEMENT TRICK IS USED;

A REFERENCE ELEMENT \hat{K} IS USED, THAT HAS IT'S REFERENCE BASIS FUNCTIONS $\hat{\varphi}_e$, $e = 1, \dots, N_r$ WITH N_r NUMBER OF BASIS FUNCTIONS IN EACH ELEMENT ($N_r = \binom{r+1}{2}$).

INTEGRALS ARE SOLVED ON THE REFERENCE ELEMENT THROUGH QUADRATURE FORMULAS (FOR EXAMPLE GAUSS-LEGENDRE) AND THEN ARE BROUGHT BACK TO THE ORIGINAL ELEMENTS K THROUGH THE AFFINE LINEAR MAPPING

$$\begin{aligned} \Phi_k: \hat{K} &\rightarrow K \in Z_h, \quad x = \Phi_k(\xi) = B_k \xi + g_k \\ \Rightarrow \varphi_i(x) &= (\hat{\varphi}_e \circ \Phi_k)^{-1}(x), \quad x \in K \end{aligned}$$

1.3) $\mu = 1, \delta = 1$

$$f(\bar{x}) = \pi^2 (x_1^2 + x_2^2) \sin(\pi x_1 x_2)$$

$$u_R(\bar{x}) = \begin{cases} \pi x_2 & x_1 = 0, x_2 \in (0, 1) \text{ (ON } \Gamma_0) \\ \sin(\pi x_2) - \pi x_2 \cos(\pi x_2) & x_1 = 1, x_2 \in (0, 1) \text{ (ON } \Gamma_1) \\ \sin(\pi x_1) - \pi x_1 \cos(\pi x_1) & x_1 \in (0, 1), x_2 = 1 \text{ (ON } \Gamma_3) \end{cases}$$

1.5) $u_{ex} = \sin(\pi x_1 x_2)$

$$\nabla u_{ex} = \begin{cases} \pi x_2 \cos(\pi x_1 x_2) \\ \pi x_1 \cos(\pi x_1 x_2) \end{cases}$$

1.6) THE ERROR COMPUTED BEHAVES AS THE THEORY EXPECTED.

IN FACT, WE HAVE THAT IF $u_{ex} \in H^{r+1}(\Omega)$, THEN

$$e_{L^2} = \|u - u_{ex}\|_{L^2} \leq C_{L^2} h^{r+1} \|u_{ex}\|_{H^{r+1}(\Omega)}$$

$$e_{H^1} = \|u - u_{ex}\|_{H^1} \leq C_{H^1} h^r \|u_{ex}\|_{H^{r+1}(\Omega)}$$

IN OUR CASE, $u_{ex} \in H^p(\Omega)$ $\forall p = 1, 2, \dots$

SO THE METHOD CONVERGES WITH ORDER $r+1$ IN NORM L^2 WITH REFERENCE TO THE MESH SIZE h , AND WITH ORDER r IN NORM H^1 , WHERE r IS THE DEGREE OF THE PIECEWISE POLYNOMIALS IN THE F.E. SPACE $X_h^r(\Omega)$

IN FACT, AS CAN BE SEEN IN THE CONVERGENCE TABLE, WITH $r=2$, THE ERROR BEHAVES PROPORTIONALLY TO h^3 IN L^2 NORM AND TO h^2 IN H^1 NORM, AS EXPECTED

ROBIN B.Cs

$$\nabla u \cdot \hat{n} = u - u_n$$

$$\Gamma_0 = \{x_1 = 0, x_2 \in (0, 1)\}$$

$$\hat{n}|_{\Gamma_0} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\nabla u = \begin{bmatrix} \pi x_2 \cos(\pi x_1 x_2) \\ \pi x_1 \cos(\pi x_1 x_2) \end{bmatrix}$$

$$(\nabla u \cdot \hat{n})|_{\Gamma_0} = -\pi x_2 \cos(\pi x_1 x_2) = -\pi x_2$$

$$(u - u_R)|_{\Gamma_0} = \sin(\pi x_1 x_2) - \pi x_2$$

$$\Rightarrow (\nabla u \cdot \hat{n})|_{\Gamma_0} = (u - u_R)|_{\Gamma_0}$$

$$\Gamma_1 : \{x_1 = 1, x_2 \in (0, 1)\} \quad \hat{n}|_{\Gamma_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(\nabla u \cdot \hat{n})|_{\Gamma_1} = \pi x_2 \cos(\pi x_2)$$

$$u - u_R|_{\Gamma_1} = \sin(\pi x_2) - \sin(\pi x_2) + \pi x_2 \cos(\pi x_2)$$

$$\Rightarrow (\nabla u \cdot \hat{n})|_{\Gamma_1} = (u - u_R)|_{\Gamma_1}$$

$$\Gamma_3 : \{x_1 \in (0, 1), x_2 = 1\} \quad \hat{n}|_{\Gamma_3} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(\nabla u \cdot \hat{n})|_{\Gamma_3} = \pi x_1 \cos(\pi x_1)$$

$$(u - u_R)|_{\Gamma_3} = \sin(\pi x_1) - \sin(\pi x_1) + \pi x_1 \cos(\pi x_1)$$

$$\Rightarrow (\nabla u \cdot \hat{n})|_{\Gamma_3} = (u - u_R)|_{\Gamma_3}$$

(E2) STOKES PROBLEM

$$\Omega = (0, 1)^2$$

FIND $U: \Omega \rightarrow \mathbb{R}^2$ AND $p: \Omega \rightarrow \mathbb{R}$ s.t.

$$\begin{cases} -\mu \Delta u + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega = \cup_{i=0}^3 \Gamma_i \end{cases}$$

$$\mu \in \mathbb{R}, \mu > 0, g: \Omega \rightarrow \mathbb{R}^2$$

(2.1) WEAK FORMULATION OF THE PROBLEM