Numerical Methods for Partial Differential Equations
A.Y. 2022/2023

Laboratory 06
Finite Element method for non linear equations and vectorial problems

Exercise 1. NON LINEAR EQUATIONS

Let $\Omega = (0,1)^3$, be the unit cube and let us consider the following non linear problem:

$$\begin{cases} -\nabla \cdot \left(\left(\mu_0 + \mu_1 u^2 \right) \nabla u \right) = f & \text{in } \Omega, \quad (1a) \\ u = 0 & \text{on } \partial\Omega, \quad (1b) \end{cases}$$

where $\mathbf{x} = (x, y, z)^T$, $\mu_0 = 1$, $\mu_1 = 10$ and $f(\mathbf{x}) = 1$.

1.1. Write the weak formulation of problem (1), expressing it in the residual form $R(u)(v) = 0$.

Solution. Let $V = H^1_0(\Omega)$ and $v \in V$. Following the usual procedure (multiply v to (1a), then integrate by parts), we obtain

$$\underbrace{\int_\Omega (\mu_0 + \mu_1 u^2) \nabla u \cdot \nabla v d\mathbf{x}}_{b(u)(v)} = \underbrace{\int_\Omega f v d\mathbf{x}}_{F(v)} .$$

By defining the *residual*

$$R(u)(v) = b(u)(v) - F(v) ,$$

we can write the weak formulation as

$$\text{find } u \in V \text{ such that } R(u)(v) = 0 \text{ for all } v \in V .$$

Notice that $b(u)(v)$, and thus $R(u)(v)$, are non linear in u.

1.2. Compute the Fréchet derivative $a(u)(\delta, v)$ of the residual $R(u)(v)$, then write Newton's method for the solution of problem (1).

Solution. We have, informally:

$$
\begin{aligned}
a(u)(\delta, v) = \frac{\mathrm{d}R}{\mathrm{d}u}(\delta, v) &= \frac{db}{du}(\delta, v) \\
&= \int_\Omega (2\mu_1 u \delta) \nabla u \cdot \nabla v d\mathbf{x} \int_\Omega (\mu_0 + \mu_1 u^2) \nabla \delta \cdot \nabla v d\mathbf{x} \,,
\end{aligned}
\tag{2}
$$

with $\delta \in V$ and $v \in V$. Notice that the bilinear form J is linear with respect to δ and v. The Newton method for this problem reads: given an initial guess $u^{(0)}$, iterate for $k = 0, 1, 2, \ldots$ and until convergence:

1. compute $\delta^{(k)}$ by solving the linear problem: $a(u^{(k)})(\delta^{(k)}, v) = -R(u^{(k)})(v)$ for all $v \in V$;
2. set $u^{(k+1)} = u^{(k)} + \delta^{(k)}$.

The problem at step 1 is a linear differential problem in weak form, and thus we can solve it using finite elements.

Upon finite element discretization, the bilinear form (2) gives rise to the following matrix:

$$
(A(u))_{ij} = a(u)(\varphi_j, \varphi_i) = \int_\Omega (2\mu_1 u \varphi_j) \nabla u \cdot \nabla \varphi_i d\mathbf{x} + \int_\Omega (\mu_0 + \mu_1 u^2) \nabla \varphi_j \cdot \nabla \varphi_i d\mathbf{x} \,,
$$

whereas the residual yields the vector

$$
(\mathbf{r}(u))_i = R(u)(\varphi_i) = \int_\Omega (\mu_0 + \mu_1 u^2) \nabla u \cdot \nabla \varphi_i d\mathbf{x} - \int_\Omega f \varphi_i d\mathbf{x} \,.
$$

1.3. Using Newton's method, implement a solver for problem (1). Then, solve the problem on the mesh `mesh/mesh-cube-20.msh`, with polynomial degree $r = 1$, and using a tolerance of 10^{-6} on the norm of the residual for the Newton's method.

Solution. See file `src/lab-06-exercise1.cpp` for the implementation. The solution is reported in Figure 1a.

Exercise 2. VECTORIAL PROBLEM

Let $\Omega = (0, 1)^3$ be the unit cube and let us consider the following linear elasticity problem: find a displacement field $\mathbf{u} : \Omega \to \mathbb{R}^3$ such that

$$
\begin{cases}
-\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} & \text{in } \Omega \,, \\
\mathbf{u} = \mathbf{g} & \text{on } \Gamma_0 \cup \Gamma_1 \,, \\
\sigma(\mathbf{u})\mathbf{n} = \mathbf{0} & \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \,,
\end{cases}
\tag{3}
$$

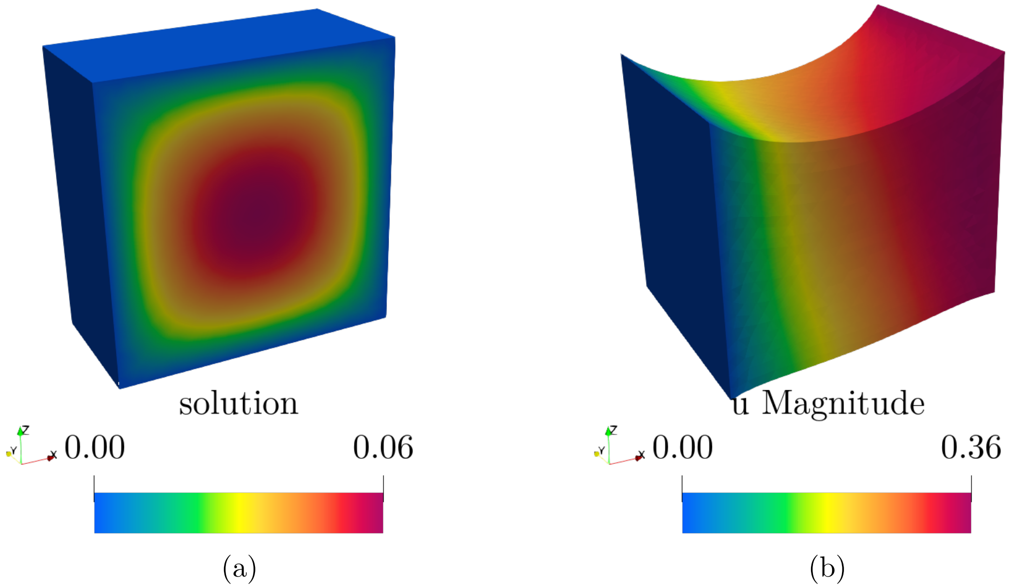

Figure 1: (a) Solution to exercise 1. The domain was clipped along the plane $y = 0.5$. (b) Solution to exercise 2. The domain was warped by the solution $\mathbf{u}$ (using the filter "Warp by vector").

where

$$\begin{aligned}
\sigma(\mathbf{u}) &= \mu \boldsymbol{\nabla} \mathbf{u} + \lambda (\boldsymbol{\nabla} \cdot \mathbf{u}) I \ , \\
\Gamma_0 &= \{x = 0, y \in (0,1), z \in (0,1)\} \ , \\
\Gamma_1 &= \{x = 1, y \in (0,1), z \in (0,1)\} \ , \\
\Gamma_2 &= \{x \in (0,1), y = 0, z \in (0,1)\} \ , \\
\Gamma_3 &= \{x \in (0,1), y = 1, z \in (0,1)\} \ , \\
\Gamma_4 &= \{x \in (0,1), y \in (0,1), z = 0\} \ , \\
\Gamma_5 &= \{x \in (0,1), y \in (0,1), z = 1\} \ ,
\end{aligned}$$

$\mu = 1$, $\lambda = 10$, $\mathbf{g}(\mathbf{x}) = (0.25x, 0.25x, 0)^T$ and $\mathbf{f}(\mathbf{x}) = (0, 0, -1)^T$.

2.1. Write the weak formulation of problem (3).

Solution. Let $V_0 = \left\{ \mathbf{v} \in \left[H^1(\Omega)\right]^3 \ : \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0 \cup \Gamma_1 \right\}$. We write $\mathbf{u} = \mathbf{u}_0 + \mathbf{R}(\mathbf{g})$, with $\mathbf{u}_0 \in V_0$ and $\mathbf{R}(\mathbf{g}) \in \left[H^1(\Omega)\right]^3$ such that $\mathbf{R}(\mathbf{g}) = \mathbf{g}$ on $\Gamma_0 \cup \Gamma_1$. Then, we proceed as usual for the weak formulation: let $\mathbf{v} \in V_0$, obtaining

$$\begin{aligned}\int_\Omega (\mu \nabla \mathbf{u}_0 : \nabla \mathbf{v} + \lambda(\nabla \cdot \mathbf{u}_0) I : \nabla \mathbf{v})\, d\mathbf{x} &= \int_\Omega \mathbf{f} \cdot \mathbf{v} d\mathbf{x} \\ &\quad - \int_\Omega (\mu \nabla \mathbf{R}(\mathbf{g}) : \nabla \mathbf{v} + \lambda(\nabla \cdot \mathbf{R}(\mathbf{g})) I : \nabla \mathbf{v})\, d\mathbf{x}\ , \\ \int_\Omega (\mu \nabla \mathbf{u}_0 : \nabla \mathbf{v} + \lambda(\nabla \cdot \mathbf{u}_0)(\nabla \cdot \mathbf{v}))\, d\mathbf{x} &= \int_\Omega \mathbf{f} \cdot \mathbf{v} d\mathbf{x} \\ &\quad - \int_\Omega (\mu \nabla \mathbf{R}(\mathbf{g}) : \nabla \mathbf{v} + \lambda(\nabla \cdot \mathbf{R}(\mathbf{g}))(\nabla \cdot \mathbf{v}))\, d\mathbf{x}\ .\end{aligned}$$

Introducing

$$\begin{aligned}a(\mathbf{u}_0, \mathbf{v}) &= \int_\Omega (\mu \nabla \mathbf{u}_0 : \nabla \mathbf{v} + \lambda(\nabla \cdot \mathbf{u}_0)(\nabla \cdot \mathbf{v}))\, d\mathbf{x}\ , \\ F(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} d\mathbf{x}\ ,\end{aligned}$$

the weak formulation reads:

$$\text{find } \mathbf{u}_0 \in V_0 \text{ such that } a(\mathbf{u}_0, \mathbf{v}) = F(\mathbf{v}) - a(\mathbf{R}(\mathbf{g}), \mathbf{v}) \text{ for all } \mathbf{v} \in V_0\ .$$

2.2. Implement in `deal.II` a finite element solver for problem (3).

Solution. See the file `src/lab-06-exercise2.cpp`. The solution is displayed in Figure 1b.

2.3. Consider now the domain Ω displayed in Figure 2. Solve the following problem:

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} & \text{in } \Omega\ , \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_\mathrm{D} \\ \sigma(\mathbf{u})\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \backslash \Gamma_\mathrm{D}\ , \end{cases}$$

with $\sigma(\mathbf{u}) = \mu \nabla \mathbf{u} + \lambda(\nabla \cdot \mathbf{u}) I$, $\mu = 10$, $\lambda = 1$ and $\mathbf{f}(\mathbf{x}) = (0, 0, -0.1)^T$. The domain is provided in the file `mesh/mesh-beam-10.msh`, and the boundary Γ_D has tag 0.

Solution. See the file `src/lab-06-exercise2.cpp`. The solution is shown in Figure 3.

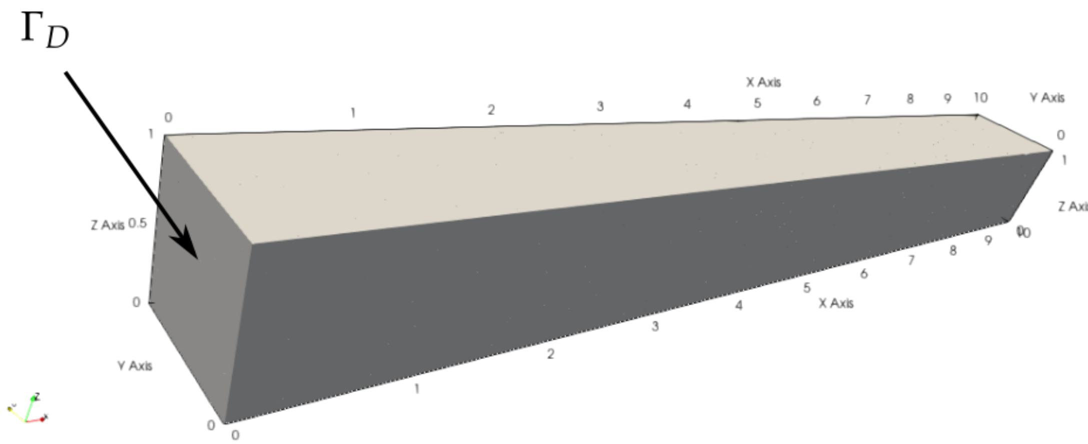

Figure 2: Computational domain for Exercise 2.3. The boundary Γ_D has tag 0 in the file `mesh/mesh-beam-10.msh`.

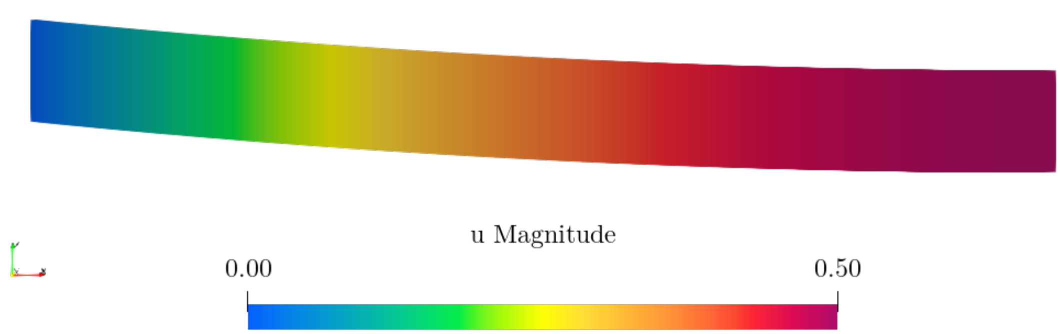

Figure 3: Lateral view of the solution to Exercise 2.3. The domain was warped by the solution $\mathbf{u}$ (using the filter "Warp by vector").