$$f(\underline{x}) = \frac{1}{2}(x_1^2 + \eta x_2^2)$$

$$\frac{\partial f(\underline{x})}{\partial x_1} = x_1 \qquad \Longrightarrow \nabla \underline{f} = \begin{bmatrix} x_1 \\ \eta x_2 \end{bmatrix}$$

$$\frac{\partial f(\underline{x})}{\partial x_2} = \eta x_2$$

$$\frac{\partial^2 f(\underline{x})}{\partial x_1^2} = 1 \qquad \Longrightarrow Hf = \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix}$$

$$\frac{\partial^2 f(\underline{x})}{\partial x_1 \partial x_2} = 0$$

$$\lambda_{max}(Hf) = max(1, \eta)$$

$$\frac{\partial^2 f(\underline{x})}{\partial x_2^2} = \eta$$

$$\frac{\partial^2 f(\underline{x})}{\partial x_2 \partial x_1} = 0$$

$$\Longrightarrow \quad \tau_{max} = \frac{2}{\lambda_{max}(1, \eta)}$$

IF $\eta = 4$, WE EXPECT $\tau_{max} = \frac{2}{4} = \frac{1}{2}$

$$\tau_{opt} = \underset{s}{argmin}\; f(x - s\nabla f(x))$$

$$= \underset{s}{argmin}\; f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - s\begin{bmatrix} x_1 \\ \eta x_2 \end{bmatrix}\right)$$

$$= \underset{s}{argmin}\; f\left(\begin{bmatrix} (1-s)x_1 \\ (1-s\eta)x_2 \end{bmatrix}\right)$$

$$= \underset{s}{argmin}\; \frac{1}{2}\left[(1-s)^2 x_1^2 + (1-s\eta)^2 x_2^2\right]$$

$$\frac{\partial}{\partial s}\left[\frac{1}{2}(1 - 2s + s^2)x_1^2 + \frac{1}{2}(1 - 2s\eta + s^2\eta^2)x_2^2\right]$$

$$\frac{\partial}{\partial s}\left[\cancel{\frac{1}{2}(x_1^2 + x_2^2)} + \frac{1}{2}(-2s + s^2)x_1^2 + \frac{1}{2}(-2s\eta + s^2\eta^2)x_2^2\right]$$

$$= \frac{1}{\cancel{2}}(-\cancel{2} + \cancel{2}s)x_1^2 + \frac{1}{\cancel{2}}(-\cancel{2}\eta + \cancel{2}s\eta^2)x_2^2$$

$$= (-1 + s)x_1^2 + (-\eta + s\eta^2)x_2^2$$

$$(-1+s)x_1^2 + \eta(-1+s\eta)x_2^2 = 0$$

$$-x_1^2 + sx_1^2 + -\eta x_2^2 + s\eta^2 x_2^2 = 0$$

$$s(x_1^2 + \eta^2 x_2^2) = x_1^2 + \eta x_2^2$$

$$s = \frac{x_1^2 + \eta x_2^2}{x_1^2 + \eta^2 x_2^2}$$