EXAM SIMULATION

$\boxed{E1}$ POISSON PROBLEM WITH ROBIN BCS

$$\Omega = (0,1)^2, \quad \partial\Omega = \Gamma_D \cup \Gamma_R = \cup_{i=0}^{3} \Gamma_i$$

$$\Gamma_D = \Gamma_2, \quad \Gamma_R = \Gamma_0 \cup \Gamma_1 \cup \Gamma_3$$

STRONG PROBLEM:

FIND $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} -\mu \Delta u = f & \text{IN } \Omega \quad (1a) \\ u = 0 & \text{IN } \Gamma_D \\ -\mu \nabla u \cdot \hat{n} + \gamma (u - u_R) = 0 & \text{IN } \Gamma_R \end{cases}$$

$\boxed{1.1}$ WRITE THE WEAK FORMULATION OF THE PROBLEM BY INCLUDING DEFINITIONS, CHOICE OF FUNCTION SPACES AND THE DERIVATION OF THE FORMULATION

LET $v \in V$ BE A TEST FUNCTION IN A SUITABLE FUNCTION SPACE V (T.B.D.)

MULTIPLY v TO 1a AND INTEGRATE THE RESULT OVER Ω

$$\int_\Omega -\mu \Delta u \cdot v \, d\bar{x} = \int_\Omega f \cdot v \, d\bar{x}$$

I CAN PARTIALLY INTEGRATE THIS TO OBTAIN

$$\nabla \cdot (\nabla u \cdot v) = \Delta u \cdot v + \nabla u \cdot \nabla v$$

$$\Longrightarrow -\Delta u \cdot v = \nabla u \cdot \nabla v - \nabla \cdot (\nabla u \cdot v)$$

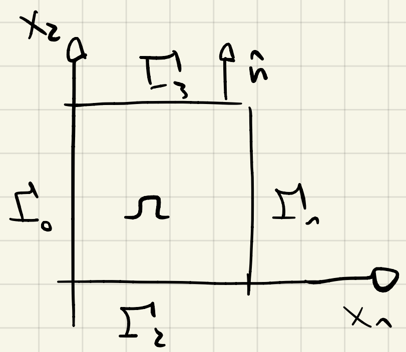
BY REPLACING THE OBTAINED RESULT IN THE INTEGRAL WE HAVE

$\mu \in \mathbb{R}, \mu > 0$

$\gamma \in \mathbb{R}, \gamma \geq 0$

$u_R : \Gamma_R \to \mathbb{R}$

$f : \Omega \to \mathbb{R}$



$$\int_\Omega -\mu \Delta u \cdot v \, d\bar{x} = \int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_\Omega \mu \nabla \cdot (\nabla u \cdot v) \, d\bar{x}$$

AND BY APPLYING THE DIVERGENCE THEOREM

$$\int_\Omega \mu \Delta u \cdot v \, d\bar{x} = \int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \oint_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma$$

I CAN SPLIT THE INTEGRAL OVER THE BOUNDARY AS THE INTEGRAL OVER THE SINGLE INTERFACES Γ_i, AND OBTAIN

$$\oint_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma = \int_{\Gamma_D} \mu (\nabla u \cdot \hat{n}) \underbrace{\cancel{v}}_{=0} \, d\sigma + \int_{\Gamma_R} \mu (\nabla u \cdot \hat{n}) v \, d\sigma$$

DIRICHLET BCS

$$\oint_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma = \int_{\Gamma_R} \mu (\nabla u \cdot \hat{n}) v \, d\sigma$$

AND BY REPLACING IT WITH THE ROBIN BOUNDARY CONDITIONS I OBTAIN

$$\oint_{\partial\Omega} \mu (\nabla u \cdot \hat{n}) v \, d\sigma = \int_{\Gamma_R} \gamma (u - u_R) v \, d\sigma$$

$$= \int_{\Gamma_R} \gamma u v \, d\sigma - \int_{\Gamma_R} \gamma u_R v \, d\sigma$$

PUTTING ALL TOGETHER

$$\int_\Omega \mu \nabla u \, \nabla v \, d\bar{x} - \int_{\Gamma_R} \gamma u v \, d\sigma = \int_\Omega f \cdot v \, d\bar{x} - \int_{\Gamma_R} \gamma u_R v \, d\sigma$$

FOR THE WEAK FORMULATION TO BE WELL DEFINED I NEED

$$u, v \in H^1(\Omega) = \{ w \in L^2(\Omega) : \nabla w \in [L^2(\Omega)]^2 \}$$

SO I CHOOSE THE FUNCTION SPACE $V = H^1_D(\Omega)$, WITH

$$H^1_D(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ ON } \Gamma_D \}$$

BY INTRODUCING THE NOTATION

$$a(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_{\Gamma_R} \gamma u v \, d\sigma$$

$$F(v) = \int_\Omega f \cdot v \, d\bar{x} - \int_{\Gamma_R} \gamma u_R v \, d\sigma$$

I CAN WRITE THE WEAK FORMULATION AS:

FIND $u \in V = H^1(\Omega)$ s.t.

$$a(u,v) = F(v) \quad \forall v \in V$$

1.2 TO WRITE THE GALERKIN - F.E. APPROXIMATION OF THE PROBLEM, I NEED TO DISCRETIZE THE FUNCTION SPACE V, IN ORDER TO BE ABLE TO SOLVE THE PROBLEM ON A CALCULATOR, SINCE $\dim\{V\} = +\infty$

I WANT TO RESTRICT V_h s.t. $\dim\{V_h\} = N_h < +\infty$

BY DOING THIS, THE WEAK FORMULATION BECOMES THE GALERKIN PROBLEM

FIND $u_h \in V_h$ s.t. $a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

LET'S NOW INTRODUCE A PARTITION OF THE DOMAIN (THE MESH $\mathcal{T}_h$) INTO TRIANGULAR SUBDOMAINS (ELEMENTS K), AND THE SPACE X_h^r OF PIECEWISE POLYNOMIALS OF DEGREE r

$$X_h^r(\Omega) = \{ w_h \in C^0(\bar{\Omega}) : w_h \in \mathbb{P}_r \ \forall \bar{x} \in K, \forall K \in \mathcal{T}_h \}$$

$\dim\{X_h^r(\Omega)\} = \hat{N}_h < +\infty$, AND DEPENDS ON THE NUMBER OF ELEMENTS IN THE MESH, THE DEGREE r, AND HOW TRIANGLES ARE ORGANIZED WITHIN THE MESH

THIS MEANS THAT I CAN CONSIDER A SET OF LAGRANGIAN BASIS FUNCTIONS $\varphi_i(\bar{x})$ THAT ARE PIECEWISE POLYNOMIALS OF DEGREE r $(\varphi_i(\bar{x}_j) = \delta_{ij}, \forall i,j = 1, \dots, \hat{N}_h)$

↳ NODES OF THE MESH $\mathcal{T}_h$

NOW I SET $V_h = V \cap X_h^r(\Omega)$

SINCE $\dim\{V_h\} = N_h \leq \hat{N}_h < +\infty$, I CAN CONSIDER N_h OVER THE $\hat{N}_h$ LAGRANGIAN BASIS FUNCTIONS OF THE F.E. SPACE X_h^r AS BASIS FUNCTIONS ALSO FOR THE SPACE V_h

$$\{\varphi_i\}_{i=1}^{N_h}$$

THIS WAY, THE APPROXIMATE SOLUTION u_h CAN BE WRITTEN AS

$u_h = \sum_{j=1}^{N_h} u_j \varphi_j$ WHERE $u_j \in \mathbb{R}$ ARE CALLED CONTROL VARIABLES (DEGREES OF FREEDOM)

THIS MEANS THAT I CAN WRITE

$$a(u_h, v_h) = a\left(\sum_{j=1}^{N_h} u_j \varphi_j, v_h\right)$$

AND SINCE $\varphi_i \in V_h \ \forall i = 1, \dots, N_h$, I CAN USE THESE AS TEST FUNCTIONS, INSTEAD OF CHECKING WITH ALL $v_h \in V_h$

THIS MEANS THAT THE GALERKIN PROBLEM BECOMES, BY ALSO EXPLOITING BILINEARITY OF $a(\cdot,\cdot)$

$$a(u_h, v_h) = a\left(\sum_{j=1}^{N_h} u_j \varphi_j, \varphi_i\right) = \sum_{j=1}^{N_h} u_j \, a(\varphi_j, \varphi_i) = F(\varphi_i)$$

$$\forall i = 1, 2, \dots, N_h$$

THE RESULT IS A LINEAR SYSTEM $A\bar{u} = \bar{f}$ WITH

- $\bar{u} = (u_1, u_2, \dots, u_{N_h})^T \in \mathbb{R}^{N_h}$
- $A \in \mathbb{R}^{N_h \times N_h}$ s.t.

$$(A)_{ij} = a(\varphi_j, \varphi_i) = \int_\Omega \mu \nabla \varphi_j \cdot \nabla \varphi_i \, d\bar{x} - \int_{\Gamma_R} \gamma \varphi_j \varphi_i \, d\sigma$$

- $\bar{f} \in \mathbb{R}^{N_h}$ s.t.

$$(f)_i = \int_\Omega f \cdot \varphi_i \, d\bar{x} - \int_{\Gamma_R} \gamma u_R \varphi_i \, d\sigma$$

TO CONSTRUCT THE MATRIX A AND THE VECTOR f, THE REFERENCE ELEMENT TRICK IS USED;

A REFERENCE ELEMENT $\hat{K}$ IS USED, THAT HAS IT'S REFERENCE BASIS FUNCTIONS $\hat{\varphi}_e$, $e = 1, \dots, N_r$ WITH N_r NUMBER OF BASIS FUNCTIONS IN EACH ELEMENT $\left(N_r = \binom{r+d}{r}\right)$.

INTEGRALS ARE SOLVED ON THE REFERENCE ELEMENT THROUG QUADRATURE FORMULAS (FOR EXAMPLE GAUSS-LEGENDRE) AND THEN ARE BROUGHT BACK TO THE ORIGINAL ELEMENTS K THROUGH THE AFFINE LINEAR MAPPING

$$\phi_K : \hat{K} \to K \in \mathcal{T}_h \quad , \quad x = \phi_K(\xi) = B_K \xi + g_K$$

$$\Rightarrow \varphi_i(x) = (\hat{\varphi}_e \circ \phi_K)^{-1}(x) \quad , x \in K$$

1.3 $\mu = 1, \ \gamma = 1$

$$f(\bar{x}) = \pi^2 (x_1^2 + x_2^2) \sin(\pi x_1 x_2)$$

$$u_R(\bar{x}) = \begin{cases} \pi x_2 & x_1 = 0, \ x_2 \in (0,1) \quad (\text{ON } \Gamma_0) \\ \sin(\pi x_2) - \pi x_2 \cos(\pi x_2) & x_1 = 1, \ x_2 \in (0,1) \quad (\text{ON } \Gamma_1) \\ \sin(\pi x_1) - \pi x_1 \cos(\pi x_1) & x_1 \in (0,1), \ x_2 = 1 \quad (\text{ON } \Gamma_3) \end{cases}$$

1.5 $u_{ex} = \sin(\pi x_1 x_2)$

$$\nabla u_{ex} = \begin{cases} \pi x_2 \cos(\pi x_1 x_2) \\ \pi x_1 \cos(\pi x_1 x_2) \end{cases}$$

1.6 THE ERROR COMPUTED BEHAVES AS THE THEORY EXPECTED.

IN FACT, WE HAVE THAT IF $u_{ex} \in H^{r+1}(\Omega)$, THEN

$$e_{L^2} = \| u - u_{ex} \|_{L^2} \leq C_{L^2} h^{r+1} |u_{ex}|_{H^{r+1}(\Omega)}$$

$$e_{H^1} = \| u - u_{ex} \|_{H^1} \leq C_{H^1} h^{r} |u_{ex}|_{H^{r+1}(\Omega)}$$

IN OUR CASE, $u_{ex} \in H^p(\Omega) \ \forall p = 1, 2, \dots$
SO THE METHOD CONVERGES WITH ORDER $r+1$ IN NORM L^2 WITH REFERENCE TO THE MESH SIZE h, AND WITH ORDER r IN NORM H^1, WHERE r IS THE DEGREE OF THE PIECEWISE POLINOMIALS IN THE F.E. SPACE $X_h^r(\Omega)$

IN FACT, AS CAN BE SEEN IN THE CONVERGENCE TABLE, WITH $r = 2$, THE ERROR BEHAVES PROPORTIONALLY TO h^3 IN L^2 NORM AND TO h^2 IN H^1 NORM, AS EXPECTED

ROBIN BCs

$$\nabla u \cdot \hat{n} = u - u_R$$

$$\Gamma_0 = \{ x_1 = 0, x_2 \in (0,1) \} \qquad \hat{n}|_{\Gamma_0} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\nabla u = \begin{bmatrix} \pi x_2 \cos(\pi x_1 x_2) \\ \pi x_1 \cos(\pi x_1 x_2) \end{bmatrix}$$

$$(\nabla u \cdot \hat{n})_{\Gamma_0} = -\pi x_2 \cancelto{1}{\cos(\pi x_1 x_2)} = -\pi x_2$$

$$(u - u_R)|_{\Gamma_0} = \cancelto{0}{\sin(\pi x_1 x_2)} - \pi x_2$$

$$\Rightarrow (\nabla u \cdot \hat{n})|_{\Gamma_0} = (u - u_R)|_{\Gamma_0}$$

$$\Gamma_1 : \{ x_1 = 1, x_2 \in (0,1) \} \qquad \hat{n}|_{\Gamma_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(\nabla u \cdot \hat{n})|_{\Gamma_1} = \pi x_2 \cdot \cos(\pi x_2)$$

$$u - u_R|_{\Gamma_1} = \cancel{\sin(\pi x_2)} - \cancel{\sin(\pi x_2)} + \pi x_2 \cos(\pi x_2)$$

$$\Rightarrow (\nabla u \cdot \hat{n})|_{\Gamma_1} = (u - u_R)_{\Gamma_1}$$

$$\Gamma_3 : \{ x_1 \in (0,1), x_2 = 1 \} \qquad \hat{n}|_{\Gamma_3} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(\nabla u \cdot \hat{n})|_{\Gamma_3} = \pi x_1 \cos(\pi x_1)$$

$$(u - u_R)|_{\Gamma_3} = \sin(\pi x_1) - \sin(\pi x_1) + \pi x_1 \cos(\pi x_1)$$

$$\Rightarrow (\nabla u \cdot \hat{n})|_{\Gamma_3} = (u - u_R)|_{\Gamma_3}$$

E2 STOKES PROBLEM

$\Omega = (0,1)^2$

FIND $\bar{u} : \Omega \to \mathbb{R}^2$ AND $p : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} -\mu \Delta \bar{u} + \nabla p = \bar{0} & \text{IN } \Omega \quad (2a) \\ \nabla \cdot \bar{u} = 0 & \text{IN } \Omega \quad (2b) \\ \bar{u} = \bar{g} & \text{ON } \partial\Omega = \cup_{i=0}^{3} \Gamma_i \end{cases}$$

$\mu \in \mathbb{R}, \mu > 0 \quad , \quad g : \Omega \to \mathbb{R}^2$

2.1 WEAK FORMULATION OF THE PROBLEM

$\bar{v} \in V$ AND $q \in Q$ TEST FUNCTIONS IN SUITABLE FUNCTION SPACES V AND Q (TO BE DEFINED)

2a MULTIPLY $\bar{v}$ TO 2a AND INTEGRATE OVER Ω

$$\int_\Omega (-\mu \Delta \bar{u} \bar{v} + \nabla p \cdot \bar{v})\, d\bar{x} = 0$$

IN CAN EXPLOIT PARTIAL INTEGRATION ON BOTH TERMS

- $\nabla \cdot (\nabla \bar{u} \cdot \bar{v}) = \Delta \bar{u} \cdot \bar{v} + \nabla \bar{u} : \nabla \bar{v}$

$\Rightarrow -\Delta \bar{u} \cdot \bar{v} = \nabla \bar{u} : \nabla \bar{v} - \nabla \cdot (\nabla \bar{u} \cdot \bar{v})$

SO $\int_\Omega (-\mu \Delta \bar{u} \cdot \bar{v})\, d\bar{x} = \int_\Omega \mu \nabla \bar{u} : \nabla \bar{v}\, d\bar{x} - \int_\Omega \mu \nabla \cdot (\nabla \bar{u} \cdot \bar{v})\, d\bar{x}$

AND BY THE DIVERGENCE THEOREM

$$\Rightarrow \int_\Omega \mu \nabla \bar{u} : \nabla \bar{v}\, d\bar{x} - \int_{\partial\Omega} \mu (\nabla \bar{u} \cdot \hat{n})\, \bar{v}\, d\sigma$$

AND BY REPLACING DIRICHLET BCS

$$\int_\Omega \mu \nabla \bar{u} : \nabla \bar{v}\, d\bar{x} - \int_{\partial\Omega} \mu (\nabla \bar{g} \cdot \hat{n}) \cdot \bar{v}\, d\sigma$$

- $\int_\Omega \nabla p \cdot \bar{v}\, d\bar{x}$

I CAN EXPLOIT PARTIAL INTEGRATION ALSO HERE

$\nabla \cdot (p \cdot \bar{v}) = \nabla p \cdot \bar{v} + p \cdot \nabla \cdot \bar{v}$

$\Rightarrow \int_\Omega \nabla p \cdot \bar{v}\, d\bar{x} = \int_\Omega \nabla \cdot (p \cdot \bar{v})\, d\bar{x} - \int_\Omega p \cdot \nabla \cdot \bar{v}\, d\bar{x}$

BY USING THE DIVERGENCE THEOREM

$$\int_{\partial\Omega} p \cdot \bar{v} \cdot \hat{n}\, d\sigma - \int_\Omega p \cdot \nabla \cdot \bar{v}\, d\bar{x}$$

PUTTING ALL TOGETHER

$$\int_\Omega \mu \nabla \bar{u} : \nabla \bar{v}\, d\bar{x} - \int_{\partial\Omega} \mu (\nabla \bar{g} \cdot \hat{n}) \bar{v}\, d\sigma + \int_{\partial\Omega} p \cdot \hat{n} \cdot \bar{v}\, d\sigma - \int_\Omega p \cdot \nabla \cdot \bar{v}\, d\bar{x} = \bar{0}$$

WHICH CAN BE REWRITTEN AS

$$\int_\Omega \mu \nabla \bar{u} : \nabla \bar{v}\, d\bar{x} \underbrace{- \int_\Omega p \cdot \nabla \cdot \bar{v}\, d\bar{x}}^{b(\bar{v},p)} + \int_{\partial\Omega} p \cdot \hat{n} \cdot \bar{v}\, d\sigma = \int_{\partial\Omega} \mu (\nabla \bar{g} \cdot \hat{n}) \bar{v}\, d\sigma$$

$\to 0???$ $\quad \to 0$

$a(\bar{u}, \bar{v}) = \int_\Omega \mu \nabla \bar{u} : \nabla \bar{v}\, d\bar{x}$

$b(\bar{v}, p) = -\int_\Omega p \cdot \nabla \cdot \bar{v}\, d\bar{x}$

2b MULTIPLY $q \in Q$ TO 2a AND INTEGRATE OVER Ω

$$\int_\Omega \nabla \cdot \bar{u} \cdot q\, d\bar{x} = \int_\Omega q \cdot \nabla \cdot \bar{u}\, d\bar{x} = b(\bar{u}, q)$$

$V = \{ \bar{v} \in [H^1(\Omega)]^2 : \bar{v} = \bar{g} \text{ ON } \partial\Omega \}$

$V_0 = \{ \bar{v} \in [H^1(\Omega)]^2 : \bar{v} = \bar{0} \text{ ON } \partial\Omega \}$

ONLY DIRICHLET BCS:

$$Q = L^2_0(\Omega) = \{ q \in L^2(\Omega) : \int_\Omega q\, d\bar{x} = 0 \}$$

INTRODUCE THE LIFTING FUNCTION

$\bar{u} = \bar{u}_0 + \bar{R}_g \qquad \bar{u}_0 \in V_0$

$\bar{R}_g \in V \quad : \quad \bar{R}_g = \bar{g}$ ON $\partial\Omega$, $\quad \nabla \cdot \bar{R}_g = 0$ IN Ω

$$\begin{cases} a(\bar{u}_0, \bar{v}) + b(\bar{v}, p) = -a(\bar{R}_g, \bar{v}) \\ b(\bar{u}, q) = 0 \end{cases}$$

$\boxed{2.2}$ G - FE

X_h^r , X_h^s FINITE ELEMENT SPACES FOR VELOCITY AND PRESSURE

$$V_{o,h} = V_o \cap [X_h^r(\Omega)]^2 \quad , \quad Q_h = Q \cap X_h^s(\Omega)$$

FIND $\bar{u}_{o,h} \in V_{o,h}$, $p_h \in Q_h$ s.t.

$$\begin{cases} a(\bar{u}_{oh}, \bar{v}_h) + b(\bar{v}_h, p_h) = -a(\bar{R}_{g_h}, \bar{v}_h) \\ b(\bar{u}_h, q_h) = 0 \end{cases}$$

$\forall \bar{v}_h \in V_{oh}$, $\forall q_h \in Q_h$

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} \bar{F} \\ \bar{0} \end{bmatrix} \qquad \begin{aligned} (A)_{ij} &= a(\bar{\varphi}_j, \bar{\varphi}_i) \\ (B)_{ij} &= b(\bar{\varphi}_j, \psi_i) \\ F_i &= a(\bar{R}_g, \bar{\varphi}_i) \end{aligned}$$

WITH $\{\bar{\varphi}_i\}_{i=1}^{N_h^r}$ BASIS FUNCTIONS OF V_{oh}

$\{\psi_i\}_{i=1}^{N_h^s}$ BASIS FUNCTIONS OF Q_h

$\boxed{2.5}$ AS WE KNOW FROM THE THEORY, FINITE ELEMENT PAIRS $\mathbb{P}_r - \mathbb{P}_r$ ARE UNSTABLE: IN FACT, THE PROGRAM DIVERGES WITH THE PAIR $\mathbb{P}_1 - \mathbb{P}_1$

THE METHOD IS STABLE FOR F.E. PAIRS $\mathbb{P}_{r+1} - \mathbb{P}_r$, FOR $r \geq 1$