


LAB 01

$$\Omega = (0, 1)$$

POISSON PROBLEM

$$\begin{cases} -(\mu(x)u'(x))' = f(x) & x \in \Omega \subset (0, 1) \\ u(0) = u(1) = 0 \end{cases}$$

$$\mu(x) = 1 \text{ FOR } x \in \Omega$$

$$f(x) = \begin{cases} 0 & \text{IF } x \leq \frac{1}{8} \text{ OR } x > \frac{1}{4} \\ -1 & \text{IF } \frac{1}{8} < x \leq \frac{1}{4} \end{cases}$$

① WEAK FORMULATION OF THE PROBLEM

LET'S SET

$$V = H_0^1(\Omega) = \{v \in H^1(\Omega) : v(0) = v(1) = 0\}$$

$$\text{RMK: } H^1(\Omega) = \{v \in L^2(\Omega) : v' \in L^2(\Omega)\}$$

$$-(\mu u')' = f$$

BECOMES

$$-\int_{\Omega} (\mu u')' v dx = \int_{\Omega} f v dx$$

$$-\int_{\Omega} (\mu u')' v dx = \int_{\Omega} f v dx$$

$$\bullet -\int_{\Omega} (\mu u')' v dx$$

$$(\mu u')' v = \mu' u' v + \mu u'' v$$

$$(\mu u' v)' = \mu' u' v + \mu u'' v + \mu u' v'$$

$$(\mu u' v)' = (\mu u')' v + \mu u' v'$$

$$\Rightarrow -(\mu u')' v = -(\mu u' v)' + \mu u' v'$$

$$\Rightarrow -\int_{\Omega} (\mu u')' v dx = -\int_{\Omega} (\mu u' v)' dx + \int_{\Omega} \mu u' v' dx$$

$$= - \left[\mu u' v \right]_0^\infty + \int_0^\infty \mu u' v' dx$$

$$\Rightarrow \int_{\Omega} \mu u' v' dx = \int_{\Omega} f v dx$$

FIND $u \in V = H_0^1(\Omega)$ s.t. $\delta(u, v) = F(v) \quad \forall v \in V$

$$\text{with } \delta(u, v) = \int_{\Omega} \mu u' v' dx$$

$$F(v) = \int_{\Omega} f v dx$$

(2) GALERKIN FORMULATION OF FEM PROBLEM

GIVEN $V_h \subset V$ s.t. $\dim\{V_h\} = N_h < +\infty$

(G.P.) is

FIND $u_h \in V_h$ s.t. $a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h$

$\Rightarrow V_h$ is to be chosen

(3) FINITE ELEMENT FORMULATION OF FEM PROBLEM, WITH PIECEWISE POLYNOMIALS OF DEGREE r

USE THE F.E. SPACE $X_h^r(\Omega)$

$$X_h^r(\Omega) := \{v_h \in C^0(\Omega) : v_h(x) \in P_r, \forall x \in k_i, \forall i=1, \dots, N+1\}$$

AND SET $V_h = V \cap X_h^r(\Omega)$

$$\text{s.t. } V_h = \{v_h \in X_h^r(\Omega) : v_h(0) = v_h(1) = 0\}$$

SINCE $V = H_0^1(\Omega)$ AND $X_h^r(\Omega) \subset H^1(\Omega)$

BASIS OF V_h : $\{\varphi_i\}_{i=1}^{N_h}$

THE SOLUTION u_h CAN BE WRITTEN AS A LINEAR COMBINATION OF THE BASIS FUNCTIONS

$$u_h(x) = \sum_{j=1}^{N_h} u_j \varphi_j(x)$$

AND USE THIS IN THE GALERKIN PROBLEM,
I CAN ALSO CHOOSE THE BASIS FUNCTIONS AS TEST FUNCTIONS, SINCE $\varphi_i \in V_h \quad \forall i=1, \dots, N_h$

$$\partial(u_h, v_h) = F(v_h)$$

$$3 \left(\sum_{j=1}^{N_h} u_j \varphi_j(x), v_h \right) = F(v_h)$$

USING BILINEARITY
AND CHOOSING φ_i
AS THE TEST FUNCTIONS
 $\forall i = 1, \dots, N_h$

$$\sum_{j=1}^{N_h} u_j \partial(\varphi_j, \varphi_i) = F(\varphi_i)$$

THIS IS A LINEAR SYSTEM OF EQUATIONS, WITH
 u_j UNKNOWN, $\partial(\varphi_i, \varphi_j)$ COEFFICIENTS OF THE
LINEAR SYSTEM, AND $F(\varphi_i)$ AS KNOWN TERMS
(RIGHT HAND SIDE)

$$\Rightarrow Au = f$$

$$(A)_{ij} = \partial(\varphi_j, \varphi_i) = \int_{\Omega} \mu \varphi_j \varphi_i' dx$$

$$= \mu \int_0^1 \varphi_j'(x) \varphi_i'(x) dx$$

$u = (u_1, u_2, \dots, u_{N_h})^T \in \mathbb{R}^{N_h}$ VECTOR OF
CONTROL VARIABLES

$$(f)_i = F(\varphi_i) = \int_{\Omega} f(x) \varphi_i(x) dx$$

$$= \int_0^1 f(x) \varphi_i(x) dx$$

1D POISSON PROBLEM

$$\begin{cases} -(\mu(x)u'(x)) = f(x) & \text{IN } \Omega = (0, 1) \\ u(0) = u(1) = 0 \end{cases}$$

$$\mu(x) = 1 \quad \text{IN } x \in \Omega$$

$$f(x) = \begin{cases} 0 & x \leq \frac{1}{8} \vee x > \frac{1}{4} \\ -1 & \frac{1}{8} < x \leq \frac{1}{4} \end{cases}$$



① WEAK FORMULATION OF THE PROBLEM

$$\int_0^1 -(\mu(x)u'(x))v(x)dx = \int_0^1 f(x)v(x)dx$$

\int 1. B.P.

o (bc's)

$$\int_0^1 \mu(x)u'(x)v'(x)dx - \left[\mu(x)u'(x)v(x) \right]_0^1 = \int_0^1 f(x)v(x)dx$$

$$\int_0^1 \mu(x)u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx$$

$$\Rightarrow u \in H_0^1(\Omega), v \in H_0^1(\Omega) = V$$

\Rightarrow WEAK FORMULATION

$$\text{FIND } u \in V \text{ st. } a(u, v) = F(v) \quad \forall v \in V$$

WITH $a(u, v) = \int_0^1 p(x) u'(x) v'(x) dx$

$$F(v) = \int_0^1 f(x) v(x) dx$$

② GALERKIN FORMULATION

$$V_h \subset V, \dim \{V_h\} < +\infty$$

$$\text{FIND } u_h \in V_h \text{ st. } a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

BUT we still don't know how V_h is

③ F.E. FORMULATION

\Rightarrow CREATE V_h



$$X_h^r = \left\{ v_h \in C^0(\bar{\Omega}), v_h|_{k_i} \in P_r \quad \forall i = 1, \dots, N+n \right\}$$

F.E.M. : $V_h = X_h^r \cap H_0^1(\Omega)$

↑
IMPOSE B.C.s

$\{\varphi_i\}_{i=1}^{N_h}$ BASIS OF V_h

$$u_h(x) = \sum_{j=1}^{N_h} v_j \varphi_j(x)$$

↳ USE THIS IN THE GALERKIN FORM

$$\partial_h \left(\sum_{j=1}^{N_h} v_j \varphi_j(x), v_h \right) = F(v_h)$$

INSTEAD OF TESTING
AGAINST EVERY v_h , TEST
AGAINST EVERY φ_i

$$\sum_{j=1}^{N_h} v_j \partial(\varphi_j, \varphi_i) = F(\varphi_i), \quad i = 1, 2, \dots, N_h$$

SET OF EQUATIONS

LINEAR SYSTEM

$$A \underline{v} = \underline{f}$$

$$\hookrightarrow (A)_{ij} = \partial(\varphi_j, \varphi_i) = \int_0^1 \mu(x) \varphi'_j(x) \varphi'_i(x) dx$$

$$\underline{v} = (v_1, v_2, \dots, v_{N_h})^\top$$

$$(F)_i = F(\varphi_i) = \int_0^1 f(x) \varphi_i(x) dx$$

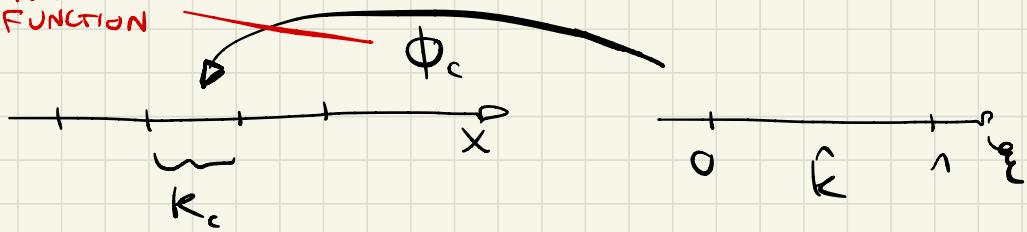
I CAN REWRITE THE INTEGRALS AS

$$A_{ij} = \sum_{c=1}^{N+1} \int_{K_c} \mu(x) \varphi_j^*(x) \varphi_i^*(x) dx$$

$$R_i = \sum_{c=1}^{N+1} \int_{K_c} f(x) \varphi_i(x) dx$$

AND, USING REFERENCE ELEMENTS, AS

TRANSFORMATION
FUNCTION



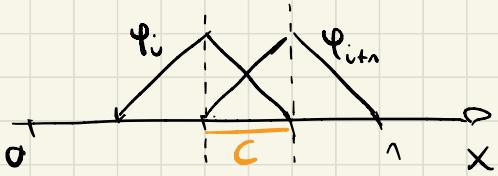
I DO A CHANGE OF VARIABLES IN INTEGRALS

$$A_{ij} = \sum_{c=1}^{N+1} \int_{\hat{K}} \mu(\phi_c(\xi)) \varphi_j^*(\phi_c(\xi)) \varphi_i^*(\phi_c(\xi)) J_c(\xi) d\xi$$

$$R_i = \sum_{c=1}^{N+1} \int_{\hat{K}} f(\phi_c(\xi)) \varphi_i(\phi_c(\xi)) \underbrace{J_c(\xi)}_{J_c(\xi) = \phi'_c(\xi)} d\xi$$

BUT I CAN SIMPLIFY THIS EVEN MORE BY LOOKING AT INTERSECTIONS BETWEEN BASIS FUNCTIONS TO SEE WHERE CONTRIBUTIONS ARE NON-NUL

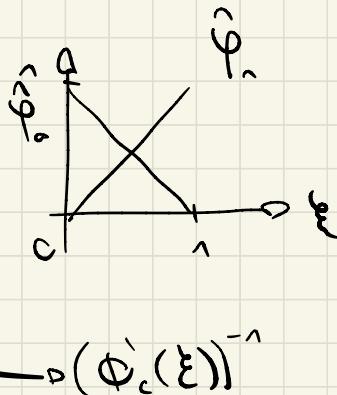
(CASE r = 1)



→ ALL OTHERS BASIS FUNCTIONS
ARE USELESS IN COMPUTING
INTEGRALS ON C

$$\varphi_i(\phi_c(\xi)) = \hat{\varphi}_i(\xi)$$

$$\varphi_i'(\phi_c(\xi)) = \hat{\varphi}'_{i,c}(\xi) J_c(\xi)$$



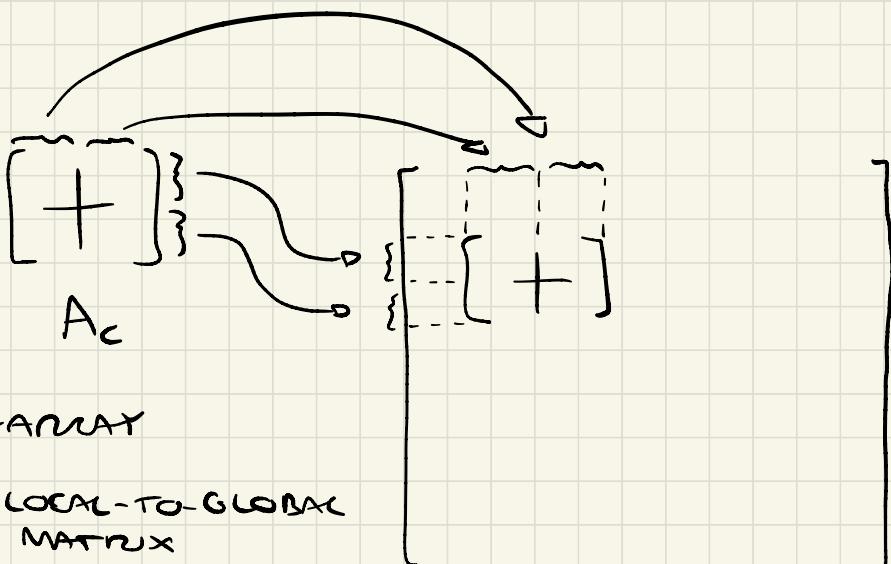
$$n_{\text{loc}} = 2r - 1$$

$$A_c \in \mathbb{R}^{n_{\text{loc}} \times n_{\text{loc}}}$$

$$(A_c)_{e,m} = \int_K \mu(\phi_c(\xi)) (\hat{\varphi}_m(\xi) J_c(\xi)^{-}) (\hat{\varphi}_e(\xi) J_c(\xi)^{-}) \cdot J_c(\xi) d\xi$$

$$(f_c)_e = \sum_K f(\phi_c(\xi)) \hat{\varphi}_e(\xi) J_c(\xi) d\xi$$

NOW I USE MATRICES ON SINGLE MESH ELEMENTS TO COMPOSE THE MATRIX OF THE WHOLE MESH



1D-ARRAY

↳ LOCAL-TO-GLOBAL
MATRIX

MULTIPLE MATRICES OVERLAPS (ELEMENTS ARE NEAR IN THE MESH COMMUNICATE WITH EACH OTHERS)

↳ OVERLAPPING VALUES ARE OVERLAPPED

INTEGRALS CAN'T BE COMPUTED INFECTICALLY, BUT
I CAN APPROXIMATE THEM WITH A COMPUTER

⇒ NUMERICAL QUADRATURE

$$\int_0^1 g(\xi) d\xi \approx \sum_{q=0}^{N_q} g(\xi_q) w_q$$

↑ ↓
 QUADRATURE QUADRATURE
 WEIGHTS NODES

⇒ GAUSS - LEGE ... QUADRATURE
OF ORDER $r+n$

→ DO THIS FOR INTEGRALS OF A AND f

$$(A)_{pq} = \sum_{q=0}^{n_q} \mu(\phi_c(\xi_q)) \underbrace{(\hat{\varphi}_m(\xi_q) J_c(\xi_q))}_{\text{II}} (\hat{\varphi}_p(\xi_q) J_c(\xi_q)) \cdot \underbrace{J_c(\xi_q) w_q}_{\text{III}}$$

$$(f_c)_p = \sum_{q=0}^{n_q} f(\phi_c(\xi_q)) \hat{\varphi}_c(\xi_q) J_c(\xi_q) w_q$$

F.F. SOFTWARE

4 STEPS

- ① PRE-PROCESSING
- ② ASSEMBLY PHASE (LINEAR SYSTEM IS BUILT)
- ③ LINEAR SOLVE (SOLVE THE L.S.)
- ④ POST-PROCESSING

$$A = \left[\begin{array}{c} \varphi_0 \\ \vdots \\ \varphi_{N_h} \end{array} \right] \in \mathbb{R}^{N_h \times N_h}$$

INTERIOR NODES

$$\hat{A} = \left[\begin{array}{c} \varphi_0 \\ \vdots \\ \varphi_{N_h+n} \end{array} \right] \in \mathbb{R}^{(N_h+2) \times (N_h+2)}$$

BOUNDARY NODES

$\varphi_0, \varphi_{N_h+n}$

$\psi_0 = 0, \psi_{N_h+n} = 0$

\Rightarrow BOUNDARY CONDITIONS

THIS IS THE MATRIX THAT DEALII ASSEMBLES

LAB 02

PROBLEM

$$\left\{ \begin{array}{l} -(\mu(x)u'(x))' = f(x) \quad x \in (0, 1) \\ u(0) = u(1) = 0 \end{array} \right.$$

$$\mu(x) = 1, \quad f(x) = 4\pi^2 \sin(2\pi x)$$

$u_{ex}(x) = \sin(2\pi x)$ IS THE EXACT SOLUTION

$$\left\{ \begin{array}{l} u_{ex}(0) = u_{ex}(1) = 0 \\ u_{ex}'(x) = 2\pi \cos(2\pi x) \\ -(\mu(x)u_{ex}')' = \dots = 4\pi^2 \sin(2\pi x) = f(x) \end{array} \right.$$

$$\|u_h - u_{ex}\|_2$$

↑
↑
known

COMPUTE WITH
SOFTWARE

$$\|u_h - u_{ex}\|_\infty$$

↑
↑
known

I WANT TO SEE HOW ERROR BEHAVES AS I CHANGE THE MESH

$$\|u_h - u_{ex}\|_2 = \sqrt{\int_0^1 |u_h(x) - u_{ex}(x)|^2 dx}$$

F.F. ANALYTICAL
FUNCTION EXPRESSION

rk USE DIFFERENT QUADRATURE FORMULAS TO COMPUTE THE ASSEMBLING OF MATRIX A AND THE ERROR

→ TO COMPUTE THE ERROR A MORE ACCURATE QUADRATURE FORMULA IS NECESSARY

rk $E_L = \|u_h - u_{ex}\| \leq \underbrace{C \|u_{ex}\|}_{D} h^{r+n}$ IF $u_{ex} \in H^{r+n}(a)$

$$E_L \approx D h^{r+n}$$

$$\log E_L \approx \log D + \log h^{r+n} = \log D + (r+n) \log h$$

↳ LINEAR FUNCTION IN $\log h$
 $(r+n)$ IS THE CONVERGENCE RATE

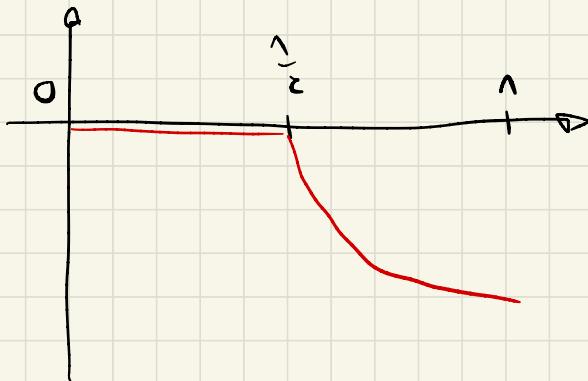
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1D PROBLEM

$$\left\{ \begin{array}{l} -(M(x)u'(x))' = f(x) \quad \text{IN } (0,1) \\ u(0) = u(1) = 0 \end{array} \right.$$

CHANGING FORCING TERM

$$f(x) = \begin{cases} 0 & x \leq \frac{1}{2} \\ -\sqrt{x - \frac{1}{2}} & x > \frac{1}{2} \end{cases}$$



$$u_{ex} = \begin{cases} Ax, & x \leq \frac{1}{2} \\ Ax + \frac{4}{15} \left(x - \frac{1}{2} \right)^{\frac{5}{2}}, & x > \frac{1}{2} \end{cases}$$

$$A = -\frac{4}{15} \left(\frac{1}{5} \right)^{\frac{5}{2}}$$

$$u_{ex} = \begin{cases} A, & x \leq \frac{1}{2} \\ A + \frac{4}{15} \cdot \frac{5}{2} \left(x - \frac{1}{2} \right)^{\frac{3}{2}} = A + \frac{2}{3} \left(x - \frac{1}{2} \right)^{\frac{3}{2}}, & x > \frac{1}{2} \end{cases}$$

$$f(x) \in L^2(\mathbb{R})$$

$$\sqrt{\int_0^\infty f(x)^2 dx} = \sqrt{\int_{-\frac{1}{2}}^{\frac{1}{2}} (x - \frac{1}{2})^2 dx} = \sqrt{\frac{x^2}{2} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \frac{1}{2}x \Big|_{-\frac{1}{2}}^{\frac{1}{2}}} \\ = \sqrt{\frac{1}{8} - \frac{1}{8} - \frac{1}{2} + \frac{1}{4}} = \sqrt{\frac{1}{8}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} < +\infty$$

$$f(x) = -u''(x) \Rightarrow u''(x) \in L^2(\mathbb{R})$$

THIS MEANS THAT $u_{ex}(x) \in H^2(\mathbb{R})$

$\Rightarrow u_{ex}(x) \in H^{r+1}(\mathbb{R})$, IF $r=1$

\rightarrow A PROOF ESTIMATE IS CONFIRMED

BUT IF $r=2$, I REQUEST THAT

$u_{ex}(x) \in H^3(\mathbb{R})$, WHICH MEANS THAT I
REQUEST THAT $f(x) \in H^1(\mathbb{R})$: IN OTHER

WORDS IN REQUIRING THAT

$$f'(x) \in L^2(\mathbb{R})$$

BUT THIS IS NOT TRUE. IN FACT

$$f'(x) = \begin{cases} 0 & \text{IF } x \leq \frac{1}{2} \\ -\frac{1}{2}(x - \frac{1}{2})^{-\frac{1}{2}} = -\frac{1}{2} \frac{1}{\sqrt{x-\frac{1}{2}}} & \text{IF } x > \frac{1}{2} \end{cases}$$

$$f'(x)^2 = \begin{cases} 0 & \text{IF } x \leq \frac{1}{2} \\ \frac{1}{4} \cdot \frac{1}{(x - \frac{1}{2})} & \text{IF } x > \frac{1}{2} \end{cases}$$

AND SINCE $\int_0^\infty \frac{1}{4} \cdot \frac{1}{(x - \frac{1}{2})} dx$ DOES NOT CONVERGE

$\Rightarrow f'(x) \notin L^2(\mathbb{R})$, WHICH MEANS THAT

$$f'(x) = -u''(x) \notin L^2(\Omega)$$

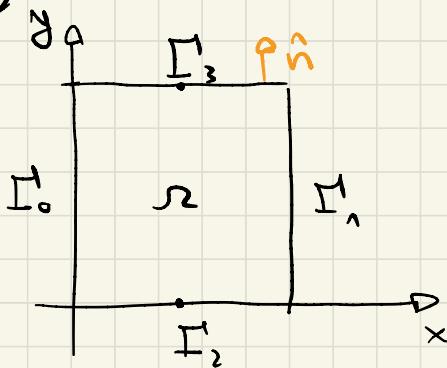
which in turn means that

$$u_{ex} \notin L^3(\Omega)$$

so the a priori error estimation tells us that for $r=2$, the method does not converge

LAB 03

(ZD)



$$\Omega = (0,1)^2$$

$$\begin{cases} -\nabla \cdot (\mu \nabla u) = f & x \in \Omega \\ u = g & x \in \Gamma_0 \cup \Gamma_n \\ \mu \nabla u \cdot n = h & x \in \Gamma_2 \cup \Gamma_3 \end{cases}$$

DIRICHLET
CONDITIONS

NEUMANN
CONDITIONS

$$\mu = 1, \quad f(x) = -5, \quad g(x) = x+y, \quad h(x) = y$$

WFI

$$\int_{\Omega} -\nabla \cdot (\mu \nabla u) v = \int_{\Omega} f v$$

I.b.p.

$$\int_{\Omega} \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_{\partial\Omega} \mu (\nabla u \cdot n) v \, d\sigma = \int_{\Omega} f v \, d\bar{x}$$

$v = 0$ on $\Gamma_0 \cup \Gamma_n$

$$\int_{\Omega} \mu \nabla u \cdot \nabla v \, d\bar{x} - \int_{\Gamma_2 \cup \Gamma_3} \mu (\nabla u \cdot n) v \, d\sigma = \int_{\Omega} f v \, d\bar{x}$$

$$\int_{\Omega} \mu \nabla u \cdot \nabla v dx = \int_{\Omega} fv + \int_{\Sigma_2 \cup \Sigma_3} hv d\sigma$$

$$V_0 = \{v \in H^1 : v = 0 \text{ on } \Gamma_0 \cup \Gamma_2\}$$

$$u = u_0 + Rg \quad , \quad Rg \in H^1, \quad Rg = g \text{ on } \Gamma_0 \cup \Gamma_1 \\ u_0 \in V_0$$

$$\underbrace{\int_{\Omega} \mu \nabla u_0 \cdot \nabla v dx}_{\mathcal{B}(u_0, v)} = \int_{\Omega} fv dx + \int_{\Sigma_2 \cup \Sigma_3} hv d\sigma - \int_{\Omega} \nabla Rg \cdot \nabla v dx$$

W.F. FIND $u_0 \in V_0$ s.t. $\forall v \in V_0$

$$\mathcal{B}(u_0, v) = F(v)$$

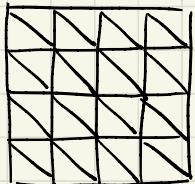
G.W.F. $V_{0,h} \subset V_0$, $\dim V_{0,h} < +\infty$

FIND $u_{0,h} \in V_{0,h}$ s.t. $\forall v_h \in V_{0,h}$

$$\mathcal{B}(u_{0,h}, v_h) = F(v_h)$$

F.E. : $V_{0,h} = X_h \cap V_0$

$$A\bar{u} = \bar{F}$$



LAB 04

DIFFUSION-REACTION EQUATION

[E1]

$$\begin{cases} -\nabla \cdot (\mu \nabla u) + \sigma u = f & \text{IN } \Omega = (0,1)^2 \\ u = 0 & \text{IN } \partial\Omega \text{ (HOMOGENEOUS)} \end{cases}$$

$$\mu(\bar{x}) = 1, \quad \sigma(\bar{x}) = 1 \quad \bar{x} = (x, y) \quad \text{Dirichlet B.C.s}$$

$$f(x) = (20\pi^2 + 1) \sin(2\pi x) \sin(4\pi y)$$

$$u_{ex}(\bar{x}) = \sin(2\pi x) \sin(4\pi y)$$

WEAK FORMULATION

$$V = H_0^1(\Omega) = \{ v \in H^1(\Omega) : v(\bar{x}) = 0, \bar{x} \in \partial\Omega \}$$

$$\int_{\Omega} \underbrace{-\nabla \cdot (\mu \nabla v) v}_{\text{i.b.p.}} d\bar{x} + \int_{\Omega} \sigma v v d\bar{x} = \int_{\Omega} f v d\bar{x}$$

$$\int_{\Omega} \mu \nabla u \cdot \nabla v d\bar{x} - \int_{\Omega} \mu (\nabla u \cdot \hat{n}) v d\bar{x} + \int_{\Omega} \sigma u v d\bar{x} = \int_{\Omega} f v d\bar{x}$$

o

$$a(u, v) = \int_{\Omega} \mu \nabla u \cdot \nabla v d\bar{x} + \int_{\Omega} \sigma u v d\bar{x}$$

$$F(v) = \int_{\Omega} f v d\bar{x}$$

WF: FIND $u \in V$ s.t. $a(u, v) = F(v) \quad \forall v \in V = H_0^1(\Omega)$

GALERKIN FORMULATION

$$V_h \subset V \text{ s.t. } \dim \{V_h\} = N_h < \infty$$

G.P. READS

$$\text{FIND } u_h \in V_h \text{ s.t. } a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h$$

FINITE ELEMENT FORMULATION

INTRODUCE THE FINITE ELEMENT SPACE X_h^r

$$X_h^r = \left\{ v_h \in C^0(\bar{\Omega}) : v_h|_{K_i} \in P_r, \quad \forall i = 0, 1, \dots, N+1 \right\}$$

AND CHOOSE $V_h = X_h^r \cap H_0^1(\Omega)$

$$\{\varphi_i\}_{i=1}^{N_h} \text{ BASIS OF } V_h$$

$$u_h = \sum_{j=1}^{N_h} u_j \varphi_j(\bar{x}) \quad u_j \text{ CONTROL VARIABLES} \quad j = 1, \dots, N_h$$

$$\begin{aligned} a(u_h, v_h) &= a\left(\sum_{i=1}^{N_h} u_i \varphi_i(\bar{x}), \varphi_i\right) \\ &= \sum_{i=1}^{N_h} u_i a(\varphi_i, \varphi_i) = F(\varphi_i) \quad \forall i = 1, \dots, N_h \end{aligned}$$

$$A \bar{u} = \bar{f} \quad \text{LINEAR SYSTEM}$$

A IS DIFFERENT W.R.T. THE POISSON PROBLEM

$$(A)_{ij} = \partial(\varphi_j, \varphi_i) = \int_R \mu \nabla \varphi_j \cdot \nabla \varphi_i d\bar{x} + \underbrace{\int_R \sigma \varphi_j \varphi_i d\bar{x}}_{\text{REACTION}}$$

$$u_{ex}(\bar{x}) = \sin(2\pi x) \sin(4\pi y)$$

$$\nabla u_{ex}(\bar{x}) = \begin{bmatrix} 2\pi \cos(2\pi x) \sin(4\pi y) \\ 4\pi \sin(2\pi x) \cos(4\pi y) \end{bmatrix}$$

LABOS

(E1)

$$\left\{ \begin{array}{l} -\nabla \cdot (\mu(\bar{x}) \nabla u) + \sigma u = f \\ \text{DIFFUSION TERM} \quad \text{REACTION TERM} \\ u = 0 \quad \text{ON} \quad \partial \Omega \end{array} \right.$$

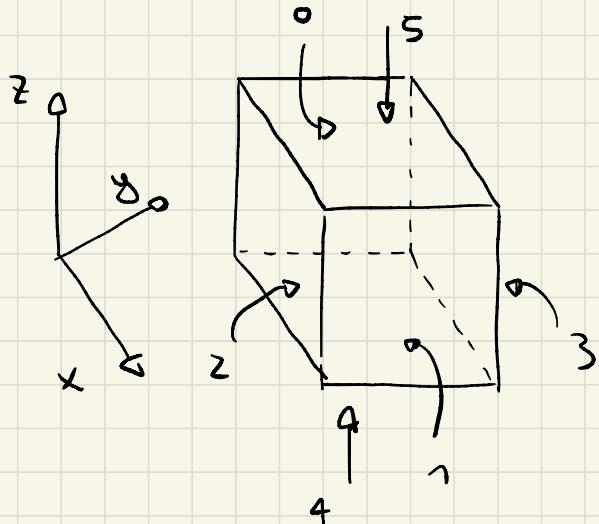
IN $\Omega = (0, 1)^3 \subset \mathbb{R}^3$

$$\mu(x) = \begin{cases} 100 & x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases} \quad \bar{x} = (x, y, z)^T$$

$$\sigma(\bar{x}) = 1$$

$$f(\bar{x}) = 1$$

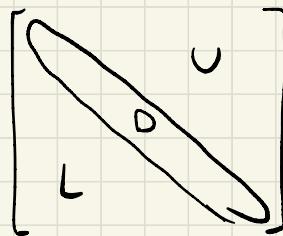
BOUNDARY CONVENTION IN CUBES



USE JACOBI, PRECONDITIONER

$$A = D + L + U$$

↑ ↑ ↑
DIAGONAL LOWER UPPER
OF A TRIANGULAR TRIANGULAR
(NO DIAG) (NO DIAG)



$$P_J = D$$

↓

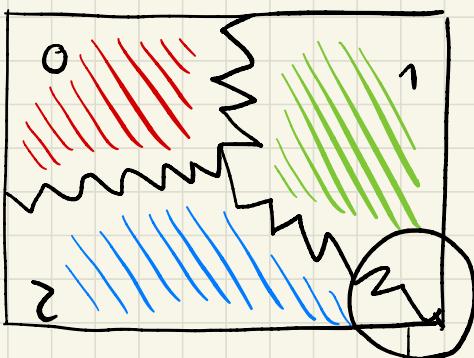
$$P_{GS} = D + L$$

$$P_{SOR} = \frac{1}{r} (D + r L)$$

$r > 0$

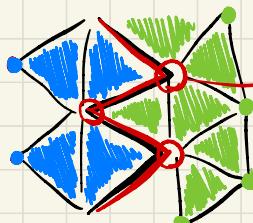
DETAILED IMPLEMENTATION

PARALLELIZATION



PARALLEL PARTITION
OVER MESH ELEMENTS

PARALLEL
PARTITION
OF MESH DOFs



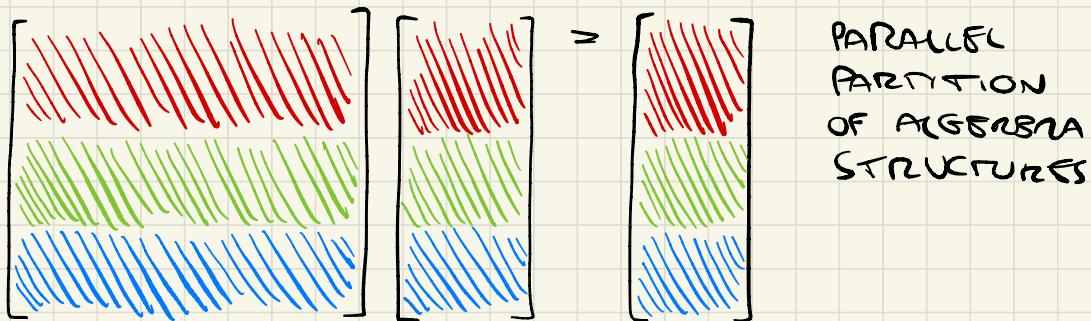
WHAT ABOUT THIS
POINTS? ONE OF THE
TWO PROCESSES
HANDLES THEM,

OWNED BY BLUE PROCESS,
 BUT ARE RELEVANT
 TO THE GREEN ONE
 ↪
 BUT THE OTHER ONE
 NEEDS INFORMATION
 ON THESE

↪ BLUE & GREEN PROCESSES NEEDS TO COOPERATE
 BY COMMUNICATING
 ↪ MPI: BY USING MESSAGES

THIS ALL ENDS UP IN A LINEAR SYSTEM

⇒ SINCE DoFs HAVE BEEN PARTITIONED, ALSO
 UNKNOWNS HAVE BEEN
 ⇒ UNKNOWN PARTITIONED BY ROWS



⇒ THIS IS THE PARTITIONING OF THE DATA
 BETWEEN THE PROCESSES
 ⇒ WHAT ABOUT THE WORK?

THE ASSEMBLY LOOP CAN BE PARALLELIZED:
 ⇒ EACH PROCESS ASSEMBLES ONLY OWNED ELEMENTS

BUT THERE ARE DoFs WHERE THE PARTITIONING OVERLAP → ON INTERFACES

↪ EACH PROCESS MAY WRITE ON ROWS OF A
 THAT HE DOESN'T OWN
 ⇒ COMMUNICATION AT THE END
 OF ASSEMBLY STEP

ALSO THE LINSAR SOLVER MUST BE PARALLELIZED
⇒ BUT THIS DOESN'T SCALE PERFECTLY
LINSARLY

LAB 08 NON LINEAR EQUATIONS & VECTORIAL PROBS

E1) NON LINEAR PDE NON LINEAR FUNCTION OF u

$$\begin{cases} -\nabla \cdot ((\mu_0 + \mu_1 \underline{u}^2) \nabla \underline{u}) = f & \text{IN } \Omega \\ u = 0 & \text{IN } \partial\Omega \end{cases}$$

$\Omega = (0,1)^3$ UNIT CUBE

$$x = (x, y, z)^T$$

$$\mu_0 = 1, \quad \mu_1 = 10,$$

1.1 WEAK FORMULATION

$V = H_0^1(\Omega)$ (HOMOGENEOUS DIRICHLET BCS)

$$v \in V$$

$$\underbrace{\int_{\Omega} (\mu_0 + \mu_1 u^2) \nabla u \cdot \nabla v \, d\bar{x}}_{b(u)(v)} = \underbrace{\int_{\Omega} f v \, d\bar{x}}_{F(v)}$$

IN RESIDUAL FORM

$$R(u)(v) = b(u)(v) - F(v) = 0$$

WF: FIND $u \in V$ s.t. $R(u)(v) = 0 \quad \forall v \in V$

\Rightarrow NEWTON METHOD

| NEED THE DERIVATIVE OF $R(u)(v)$ w.r.t. u

$$\begin{aligned}
a(u)(\delta, v) &= \lim_{h \rightarrow 0} \frac{1}{h} \left[R(u+h\delta)(v) - R(u)(v) \right] \\
&= \frac{1}{h} \left[b(u+h\delta)(v) - b(u)(v) \right] \\
&= \frac{1}{h} \left[\int_{\Omega} (\mu_0 + \mu_1(u^2 + 2h\delta u + h^2\delta^2)) \nabla(u+h\delta) \nabla v \, d\bar{x} \right. \\
&\quad \left. - \int_{\Omega} (\mu_0 + \mu_1 u^2) \nabla u \nabla v \, d\bar{x} \right] \\
&= \frac{1}{h} \left[\int_{\Omega} (\mu_0 + \mu_1(u^2 + 2h\delta u + h^2\delta^2)) \nabla u \cdot \nabla v \, d\bar{x} \right. \\
&\quad \left. + \int_{\Omega} (\mu_0 + \mu_1(u^2 + 2h\delta u + h^2\delta^2)) \nabla(h\delta) \nabla v \, d\bar{x} \right. \\
&\quad \left. - \int_{\Omega} (\mu_0 + \mu_1 u^2) \nabla u \nabla v \, d\bar{x} \right] = \\
&= \frac{1}{h} \left[\int_{\Omega} \mu_1(2h\delta u + h^2\delta^2) \nabla u \nabla v \, d\bar{x} \right. \\
&\quad \left. + \int_{\Omega} (\mu_0 + \mu_1(u^2 + 2h\delta u + h^2\delta^2)) h \nabla \delta \nabla v \, d\bar{x} \right] \\
&= \int_{\Omega} (2\mu_1 u \delta + \mu_1 h \delta^2) \nabla u \nabla v \, d\bar{x} \\
&\quad + \int_{\Omega} (\mu_0 + \mu_1(u^2 + 2h\delta u + h^2\delta^2)) \nabla \delta \nabla v \, d\bar{x}
\end{aligned}$$

$(\alpha + b(c+d))e - (\alpha + bc)e$
 $(\alpha + bc + bd)e - (\alpha + bc)e$
 ~~$(\alpha + bc)e + bde - (\alpha + bc)e$~~

FOR $h \rightarrow 0 \Rightarrow$

$$\begin{aligned}
&\int_{\Omega} (2\mu_1 u \delta) \nabla u \nabla v \, d\bar{x} \\
&+ \int_{\Omega} (\mu_0 + \mu_1 u^2) \nabla \delta \nabla v \, d\bar{x}
\end{aligned}$$

A.7) FRÉCHET DERIVATIVE

$$\partial(u)(\delta, v) = \frac{\partial R}{\partial u}(\delta, v) = \frac{\partial b}{\partial u}(\delta, v)$$

FRÉCHET
DERIVATIVE



$$= \int_{\Omega} (2\mu_n u \delta) \nabla u \cdot \nabla v \, d\bar{x} + \int_{\Omega} (\mu_0 + \mu_n u^2) \nabla \delta \cdot \nabla v \, d\bar{x}$$

Rmk: 3 BILINEAR IN SECOND & THIRD ELEMENTS

NEWTON METHOD:

GIVEN $u^{(0)} \in V$, ITERATE FOR $k=0, 1, 2, \dots$

$$(i) \quad \partial(u^{(k)})(\delta^{(k)}, v) = -R(u^{(k)})(v)$$

↳ LINEAR PROBLEM: FIND $\delta^{(k)}$ (TANGENT PROBLEM)

$$(ii) \quad u^{(k+1)} = u^{(k)} + \delta^{(k)}$$

↑
INFINITE DIMENSIONAL

UNTIL CONVERGENCE

↓
DISCRETIZE &

SOLVE

I NEED TO ASSEMBLE

A_{ij}

$$\begin{aligned} \partial(u^{(k)})(\varphi_j, \varphi_i) &= \int_{\Omega} (2\mu_n u^{(k)} \varphi_j) \nabla u^{(k)} \cdot \nabla \varphi_i \, d\bar{x} \\ &\quad + \int_{\Omega} (\mu_0 + \mu_n u^{(k)2}) \nabla \varphi_j \cdot \nabla \varphi_i \, d\bar{x} \end{aligned}$$

Rmk $u=0$ ON $\partial\Omega \Rightarrow \delta=0$ ON $\partial\Omega$

(E2) VECTORIAL PDEs

FIND $\bar{u} : \Omega \rightarrow \mathbb{R}^3$ s.t.

$$\begin{cases} -\nabla \cdot \underline{\sigma}(\bar{u}) = f & \text{in } \Omega = (0,1)^3 \\ g \end{cases}$$

TENSOR
VALUED
FUNCTION
(E.G.: A MATRIX)

$$\begin{cases} \bar{u} = \bar{g} & \text{on } \Gamma_0 \cup \Gamma_1 \\ \underline{\sigma}(\bar{u}) \bar{n} = \bar{\sigma} & \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \end{cases}$$

ELASTICITY PROBLEM:

f : LOAD , u : DISPLACEMENT

$\underline{\sigma}$: STRESS TENSOR, RELATES THE DISPLACEMENT u OF THE OBJECT TO THE STATE OF STRESS INSIDE THE OBJECT

$$\underline{\sigma}(\bar{u}) = \mu \nabla \bar{u} + \lambda (\nabla \cdot \bar{u}) \mathbf{I} \quad \leftarrow \text{LINEAR ELASTICITY}$$

↳ LINEAR FUNCTION OF \bar{u}

$$\mu = 1, \lambda = 10,$$

$$g(\bar{x}) = [0, 25x, 0, 25x, 0]^T$$

$$f(\bar{x}) = [0, 0, -1]^T$$

Z.1] WEAK FORMULATION

NON HOMOGENEOUS DIRICHLET BCs

\Rightarrow I NEED TO USE THE LIFTING FUNCTION

$$V_0 = \left\{ \bar{v} \in \underbrace{[H^1(\Omega)]^3}_{\text{EACH COMPONENT}} : v=0 \text{ ON } \Gamma_0 \cup \Gamma_n \right\}$$

OF v MUST BE IN $L^2(\Omega)$

$$\bar{u} = \bar{u}_0 + R(\bar{g}) \quad , \quad \bar{u}_0 \in V_0$$

LIFTING FUNCTION

$$\bar{R}(\bar{g}) \in [H^1(\Omega)]^3 \text{ s.t. }$$

$$\bar{R}(\bar{g}) = g \text{ ON } \Gamma_0 \cup \Gamma_n$$

$$\int_{\Omega} -\nabla \underline{\sigma} (\bar{u}_0 + \bar{R}(\bar{g})) \cdot \bar{v} \, d\bar{x} = \int_{\Omega} \bar{f} \cdot \bar{v} \, d\bar{x}$$

INTEGRATE BY PARTS

SCALAR PRODUCT
BETWEEN TENSORS

$$\int_{\Omega} \underline{\sigma} (\bar{u}_0 + \bar{R}(\bar{g})) : \nabla \bar{v} \, d\bar{x} - \int_{\partial\Omega} \underline{\sigma} (\bar{u}_0 + \bar{R}(\bar{g})) \bar{n} \cdot \bar{v} \, d\gamma = \int_{\Omega} \bar{f} \cdot \bar{v} \, d\bar{x}$$

○ \bar{n} NEUMANN
DIRICHLET

$$\begin{bmatrix} \partial_{11} & \partial_{12} \\ \partial_{21} & \partial_{22} \end{bmatrix} : \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \partial_{11}b_{11} + \partial_{12}b_{12} + \partial_{21}b_{21} + \partial_{22}b_{22}$$

$$\Rightarrow \int_{\Omega} \underline{\sigma} (\bar{u}_0) : \nabla \bar{v} \, d\bar{x} = \int_{\Omega} \bar{f} \cdot \bar{v} \, d\bar{x} - \int_{\Omega} \underline{\sigma} (\bar{R}(\bar{g})) : \nabla \bar{v} \, d\bar{x}$$

$\partial(u_0, v)$

$F(v)$

$$\int_{\Omega} (\mu \nabla u_0 : \nabla v + \lambda (\nabla \cdot u_0)(\nabla \cdot v)) \, d\bar{x} = \int_{\Omega} \bar{f} \cdot \bar{v} \, d\bar{x}$$

- $\int_{\Omega} (\mu \nabla \bar{R}(\bar{g}) : \nabla \bar{v} + \lambda (\nabla \cdot \bar{R}(\bar{g})(\nabla \cdot \bar{v})) \, d\bar{x}$

WF FIND $u_0 \in V_0$ s.t.

$$\mathcal{A}(\bar{u}_0, \bar{v}) = F(\bar{v}) - \mathcal{A}(\bar{R}(g), \bar{v}) \quad \forall v \in V_0$$

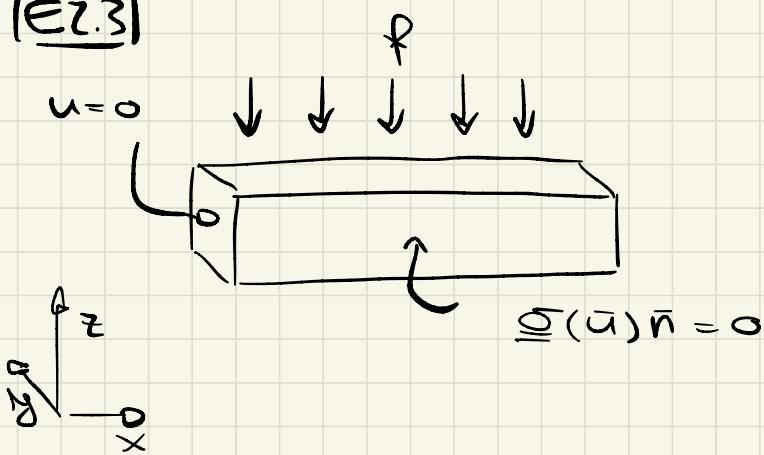
Rmk

$$\mathcal{A}(u, v) = \int_{\Omega} [\mu \nabla u : \nabla v + \lambda (\nabla \cdot u)(\nabla \cdot v)] dx$$

\downarrow F.E. DISCRETIZATION

$$A_{ij} = \mathcal{A}(\bar{\varphi}_j, \bar{\varphi}_i) = \int_{\Omega} [\mu \nabla \bar{\varphi}_j : \nabla \bar{\varphi}_i + \lambda (\nabla \cdot \bar{\varphi}_j)(\nabla \cdot \bar{\varphi}_i)] dx$$

E2.3



LAB OF PARABOLIC PROBLEMS: TIME DEPENDENT PDEs

HEAT EQUATION

$$u \in \mathcal{R} \times (0, T) \rightarrow \mathbb{R} , \quad \mathcal{R} = (0, 1)^3, T > 0$$

$u(\bar{x}, t)$ s.t.

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \nabla \cdot (\mu \nabla u) = f(\bar{x}, t) \quad \text{IN } \mathcal{R} \times (0, T) \\ \underbrace{\frac{\partial u}{\partial t}}_{\text{TIME DERIVATIVE}} - \underbrace{\nabla \cdot (\mu \nabla u)}_{\text{DIFFUSION TERM}} = f(\bar{x}, t) \\ \mu \nabla u \cdot \bar{n} = 0 \quad \text{ON } \partial \mathcal{R} \times (0, T) \\ \text{HEAT FLOW} \qquad \qquad \text{HOMOGENEOUS NEUMANN} \\ u = u_0 \quad \text{IN } \mathcal{R} \times \{0\} \\ \text{INITIAL CONDITION} \end{array} \right.$$

u TEMPERATURE IN A POINT

NEUMANN CONDITION: NO FLOW OF HEAT

ACROSS THE BOUNDARY

\Rightarrow ISOLATED OBJECT

u_0 INITIAL TEMPERATURE OF THE OBJECT

1.1) WEAK FORMULATION AND SEMI-DISCRETE FORMULATION WITH THE F.E. METHOD TO DISCRETIZE SPACE DERIVATIVES

$$t \in (0, T)$$

$$v \in V = H^1(\Omega)$$

MULTIPLY BY v AND INTEGRATE

$$\int_{\Omega} \frac{\partial u}{\partial t} v \, d\bar{x} + \underbrace{\int_{\Omega} M \nabla u \nabla v \, d\bar{x}}_{a(u, v)} = \underbrace{\int_{\Omega} f v \, d\bar{x}}_{F(v)}$$

WF: $\forall t \in (0, T)$, FIND $u(t) \in V$ s.t.

$$\begin{cases} \int_{\Omega} \frac{\partial u}{\partial t} v \, d\bar{x} + a(u, v) = F(v) & \forall v \in V \\ u(0) = u_0 \end{cases}$$

$$V_h = V \cap X_h^r(\Omega)$$

SEMI-DISCRETE WF: $\forall t \in (0, T)$, FIND $u_h(t) \in V_h$ s.t.

$$\int_{\Omega} \frac{\partial u}{\partial t} v_h \, d\bar{x} + a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

$$u_h(0) = u_{0,h} \rightarrow \text{F.E. APPROXIMATION OF } u_0$$

$$\left\{ \varphi_i \right\}_{i=1}^{N_h} \text{ BASIS FOR } V_h$$

$$\Rightarrow M \frac{d\bar{u}}{dt} + A\bar{u}(t) = \bar{f}(t) \quad \forall t \in (0, T)$$

\Rightarrow SYSTEM OF ODES

- $\bar{U}(t) = [U_1(t), U_2(t), \dots, U_{N_h}(t)]$ TIME DEPENDENT CONTROL VARIABLES

$$U_h(t) = \sum_{i=1}^{N_h} \varphi_i(t) U_i(t)$$

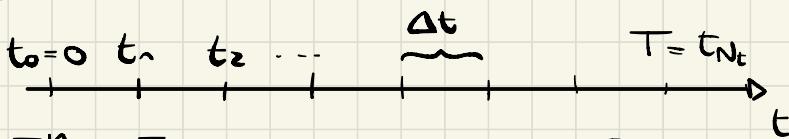
- $M_{i,j} = \int_R \varphi_i \varphi_j d\bar{x}$ MASS MATRIX
⇒ DISCRETIZATION OF TEMPORAL DERIVATIVES

- $A_{i,j} = \int_R M \nabla \varphi_i \cdot \nabla \varphi_j d\bar{x}$ STIFFNESS MATRIX
⇒ DISCRETIZATION OF DIFFUSION TERM

- $[\bar{F}(t)]_i = \int_R f(t) \varphi_i d\bar{x}$ FORCING TERM

1.2 FULLY-DISCRETE FORMULATION WITH THETA METHOD TO DISCRETIZE TIME DERIVATIVE

Θ-METHOD



$$\bar{U}_h^n = \bar{U}_h(t_n), \text{ GIVEN } \Theta \in [0, 1]$$

$$M \frac{\bar{U}^{n+1} - \bar{U}^n}{\Delta t} + \Theta A \bar{U}^{n+1} + (1-\Theta) A \bar{U}^n = \Theta \bar{F}^{n+1} + (1-\Theta) \bar{F}^n$$

$$\Rightarrow \begin{cases} \left(\frac{1}{\Delta t} M + \Theta A \right) \bar{U}^{n+1} = \left(\frac{1}{\Delta t} M - (1-\Theta) A \right) \bar{U}^n + \Theta \bar{F}^{n+1} + (1-\Theta) \bar{F}^n \\ \bar{U}_0 = \bar{U}_h^0 \end{cases}$$

Blus

Brus

Blus AND Brus

ARE NOT TIME DEPENDANT

⇒ ALWAYS

THE SAME

$\Theta = 0$ EXPLICIT EULER (FORWARD)

$\Theta = 1$ IMPLICIT EULER (BACKWARD)

$\Theta = \frac{1}{2}$ CRANCK-NICOLSON (IMPLICIT)

ASSEMBLE B_{LHS}, B_{RHS}

FOR $n = 0, 1, \dots, N_t - 1$

• SOLVE \star :

- ASSEMBLE f^{n+1}, f^n
- SOLVE WITH CG \star

$$\frac{\partial u}{\partial t} - \nabla \cdot (\mu \nabla u) = 0$$

$$\int_R \frac{\partial u}{\partial t} dx - \int_R \nabla \cdot (\mu \nabla u) d\bar{x} = 0$$

$$\frac{d}{dt} \int_R u d\bar{x} = \int_R \nabla \cdot (\mu \nabla u) d\bar{x} = \underbrace{\int_{\partial R} \mu \nabla u \cdot \bar{n} d\sigma}_{\text{HEAT FLUX}} = 0$$

Now it is
SET TO ZERO 

WHY EXPLICIT EULER DOESN'T WORK

EXPLICIT EULER

$$M \frac{u^{n+1} - u^n}{\Delta t} + Au^n = f^n$$

$$\underbrace{\frac{M}{\Delta t} u^{n+1}}_{\text{LHS}} = \underbrace{\left(\frac{M}{\Delta t} - A \right)}_{\text{BRHS}} u^n + f^n$$

LHS

BRHS

STABILITY

CONDITIONALLY STABLE

$u \rightarrow \text{CONSTANT AS } t \rightarrow +\infty$

NOT STABLE METHOD:

\Rightarrow NO GUARANTEE THAT

$u_h^n \rightarrow \text{CONSTANT AS } t \rightarrow +\infty$

E.E. IS CONDITIONALLY STABLE

\hookrightarrow STABILITY IF $\Delta t \leq c \cdot h^2$

$\underbrace{c}_{\substack{\text{CONSTANT DEPENDING} \\ \text{ON } \sigma, \mu}}$ $\underbrace{h^2}_{\substack{\text{MESH SIZE}}}$

$\Rightarrow \Delta t$ IS NOT SMALL ENOUGH IN THIS CASE

LAB 08 CONVERGENCE ANALYSIS

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (\mu \nabla u) = f & \text{IN } \Omega \times (0, T) \\ u = g & \text{IN } \partial\Omega \times (0, T) \\ u = u_0 & \text{IN } \Omega \times \{0\} \end{cases}$$

$$\Omega = (0, 1)^3$$

$$T = 1$$

$$\mu = 1$$

1.1 $u_{ex}(x, t) = \sin(5\pi t) \sin(2\pi x) \sin(3\pi y) \sin(4\pi z)$

COMPUTE THE FORCING TERM, THE BOUNDARY DATUM g , AND THE INITIAL CONDITION u_0

$$\frac{\partial u}{\partial t} = 5\pi \cos(5\pi t) \sin(2\pi x) \sin(3\pi y) \sin(4\pi z)$$

$$\frac{\partial^2 u}{\partial x^2} = -4\pi^2 \sin(5\pi t) \sin(2\pi x) \sin(3\pi y) \sin(4\pi z)$$

$$\frac{\partial^2 u}{\partial y^2} = -9\pi^2 \sin(5\pi t) \sin(2\pi x) \sin(3\pi y) \sin(4\pi z)$$

$$\frac{\partial^2 u}{\partial z^2} = -16\pi^2 \sin(5\pi t) \sin(2\pi x) \sin(3\pi y) \sin(4\pi z)$$

- $f = \frac{\partial u}{\partial t} - \nabla \cdot (\mu \nabla u) = \frac{\partial u}{\partial t} - \mu \Delta u =$

$$= [5\pi \cos(5\pi t) + 29\pi^2 \sin(5\pi t)] \sin(2\pi x) \sin(3\pi y) \sin(4\pi z)$$

• $g = u_{ex}|_{\partial\Omega} = 0$ (AT LEAST ONE BETWEEN x, y, z IS ZERO OR ONE ON $\partial\Omega$)
 $\Rightarrow \sin(0) = \sin(2\pi) = \sin(3\pi) = \sin(4\pi) = 0$

• $u_0 = u_{ex}|_{t=0} = 0$

\Rightarrow "METHOD OF MANUFACTURED SOLUTIONS"
 \Rightarrow TEST THE METHOD TO CHECK IF IT WORKS

CONVERGENCE TABLE (W.R.T. TIME DISCRETIZATION PARAMETER)

$$e_{L^2} = \|u_h^{N_T} - u_{ex}(T)\|_{L^2(\Omega)} \leq C_h h^{\frac{p}{q}} + C_{\Delta t} \Delta t^{\frac{q}{p}}$$

CONVERGENCE RATE W.R.T. h CONVERGENCE RATE W.R.T. Δt

ASSUMPTIONS:

- (i) $C_h h^{\frac{p}{q}} \approx 0$
- (ii) $e_{L^2} \approx C_{\Delta t} \Delta t^{\frac{q}{p}}$
- (iii) $e_{L^2}(\Delta t_1) = e_1 \approx C_{\Delta t} \Delta t_1^{\frac{q}{p}}$
 $e_{L^2}(\Delta t_2) = e_2 \approx C_{\Delta t} \Delta t_2^{\frac{q}{p}}$

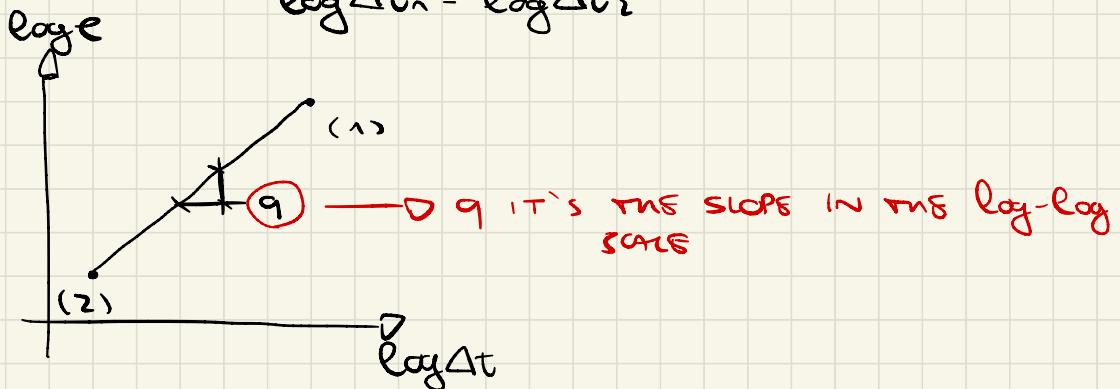
$$\frac{e_1}{e_2} \approx \frac{C_{\Delta t} \Delta t_1^{\frac{q}{p}}}{C_{\Delta t} \Delta t_2^{\frac{q}{p}}} = \left(\frac{\Delta t_1}{\Delta t_2}\right)^{\frac{q}{p}}$$

I WANT TO FIND q

$$\log \frac{e_1}{e_2} \approx q \log \frac{\Delta t_1}{\Delta t_2} \Rightarrow q \approx \frac{\log \frac{e_1}{e_2}}{\log \frac{\Delta t_1}{\Delta t_2}}$$

DEAL.W CONVERGENCE TABLE ASSUMES $\frac{\Delta t_1}{\Delta t_2} = 2$
(THAT'S WHY I NEED TIME STEPS & MESH SIZES THAT ALWAYS HALVE)

$$\Rightarrow q = \frac{\log e_1 - \log e_2}{\log \Delta t_1 - \log \Delta t_2}$$



THE SMALLER Δt , THE FASTER THE LINEAR SYSTEM IS SOLVED

LINEAR SYSTEM MATRIX IS

$$\frac{M}{\Delta t} + \alpha A \underset{\Delta t \rightarrow 0}{\sim} \frac{M}{\Delta t} \quad \text{→ WELL CONDITIONED MATRIX W.R.T. THE STIFFNESS MATRIX}$$

ALSO, WE'RE USING AN ITERATIVE SOLVER

\Rightarrow WE HAVE TO PROVIDE AN INITIAL GUESS, AND THE NUMBER OF ITERATIONS ALSO DEPENDS ON IT
IN THIS CASE THE I.G. AT A SPECIFIC TIME STEP IS THE SOLUTION FOUND AT THE PREVIOUS TIME STEP
 \Rightarrow I'M CLOSE TO THE SOLUTION

E2 NON LINEAR, TIME DEPENDENT PROBLEM

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot ((\mu_0 + \mu_1 u^2) \nabla u) = f & \text{IN } \Omega \times (0, T) \\ u = g & \text{ON } \partial\Omega \times (0, T) \\ u = u_0 & \text{IN } \Omega \times \{0\} \end{cases}$$

$$\mu_0 = 0, 1, \quad \mu_1 = 1, \quad g(\bar{x}, t) = 0, \quad u_0 = 0$$

$$\Omega = (0, 1)^3, \quad T = 1 \quad f(\bar{x}, t) = \begin{cases} 2 & \text{if } t < 0, 75 \\ 0 & \text{if } t \geq 0, 75 \end{cases}$$

HEAT NON LINEAR

- ↳ THE OBJECT HAS A CONDUCTIVITY THAT INCREASES WITH THE TEMPERATURE
- ↳ f IS A SOURCE TERM \Rightarrow HEATER THAT SHUTS DOWN AFTER $t_h = 0, 75$

(2.1) WEAK FORMULATION

$$V = H_0^1(\Omega)$$

$\forall t \in (0, T)$, FIND $u(t) \in V$ s.t.

$$(a) u(0) = u_0$$

$$(b) \int_{\Omega} \frac{\partial u}{\partial t} v d\bar{x} + \underbrace{\int_{\Omega} (\mu_0 + \mu_1 u^2) \nabla u \cdot \nabla v d\bar{x}}_{b(u)(v)} = \underbrace{\int_{\Omega} f v d\bar{x}}_{F(v)}$$

$\forall v \in V$

IN RESIDUAL FORM

$$R(u)(v) = \int_{\Omega} \frac{\partial u}{\partial t} v dx - b(u)(v) - F(v) = 0 \quad \forall v \in V$$

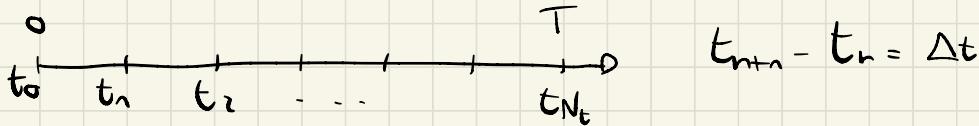
INTRODUCE $V_h = V \cap X_h^r(\Omega)$ DISCRETE SPACE

SEMI-DISCRETE LWF: $\forall t \in (0, T)$, FIND $u_h \in V_h$ st

$$(a) \quad u_h(0) = u_0$$

$$(b) \quad R(u_h)(v_h) = 0 \quad \forall v_h \in V_h$$

INTRODUCE TIME DISCRETIZATION (IMPLICIT EULER)



FOR $n = 0, 1, 2, \dots, N_t - 1$:

$$\underbrace{\int_{\Omega} \frac{u_h^{n+1} - u_h^n}{\Delta t} v_h dx + b(u_h^{n+1})(v_h) - F^{n+1}(v_h)}_{R^{n+1}(u_h^{n+1})(v_h)} = 0 \quad \forall v_h \in V_h$$

NON LINEAR PROBLEM

\Rightarrow USE NEWTON METHOD

I NEED THE DERIVATIVE OF THE RESIDUAL

$$\partial(u_h^{n+1})(\delta_h, v_h)$$

$$\partial(u_h^{n+1})(\delta_h, v_h) = \int_{\Omega} \frac{\delta_h}{\Delta t} v_h dt + \frac{db}{du}(\delta_h, v_h) - \cancel{\frac{\partial F}{\partial u}}(v_h)$$

$$= \int_{\Omega} \frac{\delta_h}{\Delta t} v_h dx + \int_{\Omega} (\gamma \mu_1 u_h^{n+1} \delta_h) \nabla u_h^{n+1} \nabla v_h dx + \int_{\Omega} (\mu_0 + \mu_1 (u_h^{n+1})^2) \nabla \delta_h \nabla v_h dx$$

AT TIME t_n : SET $u_h^{n+1,(0)} = u_h^n$, THEN ITERATE

FOR $k=0, 1, \dots$ UNTIL CONVERGENCE

$$(i) \quad \partial(u_h^{n+1,(k)}) (\delta_h^{(k)}, v_h) = -R(u_h^{n+1,(k)}) (v_h) \quad \forall v_h \in V_h$$

$$\rightarrow \delta_h^{(k)}$$

$$(ii) \quad u_h^{n+1,(k+1)} = u_h^{n+1,(k)} + \delta_h^{(k)}$$

I HAVE TWO LOOPS IN THE SOLVE STEP

FOR $n=0, 1, \dots, N_T - 1$ (TIME DISCRETIZATION)

FOR $k=0, 1, \dots$, UNTIL CONVERGENCE

[NEWTON ITERATIONS]

END

END

$$A(\bar{U}^{n+\alpha(k)}) \bar{\delta} = -\bar{v}$$

$$\bar{\delta} = U^{n+\alpha, (k+\alpha)} - U^{n+\alpha, (k)}$$

$$\begin{aligned} U_h^{n+\alpha, (k+\alpha)}|_{\partial\Omega} &= g \\ U_h^{n+\alpha, (k)}|_{\partial\Omega} &= g \end{aligned} \Rightarrow \delta_h^{(k)}|_{\partial\Omega} = 0$$

$$\left. \begin{aligned} U_h^{n+\alpha, (0)}|_{\partial\Omega} &= g \\ \delta_h^{(k)}|_{\partial\Omega} &= 0 \end{aligned} \right\} \Rightarrow U_h^{n+\alpha, (k)}|_{\partial\Omega} = g$$

IF INSTEAD I HAD USED EXPLICIT EULER

$$\int_{\Omega} \frac{(U_h^{n+\alpha} - U_h^n) v_h}{\Delta t} dx + b(U_h^n)(v_h) - F^n(v_h) = 0$$

ONLY UNKNOWN

\Rightarrow LINEAR PROBLEM w.r.t. $U_h^{n+\alpha}$

NO NEED OF NEWTON AT EACH TIME STEP

BUT I MAY HAVE ISSUES OF ABSOLUTE STABILITY

$$\frac{1}{2} (\nabla \bar{u} + (\nabla \bar{u})^T)$$

$$\nabla \bar{u} = \begin{bmatrix} \nabla^T u_1 \\ \nabla^T u_2 \\ \nabla^T u_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

LAB09 STOKES PROBLEM

$$\Omega \subset \mathbb{R}^3$$

$$\bar{U}: \Omega \rightarrow \mathbb{R}^3$$

FLUID VELOCITY

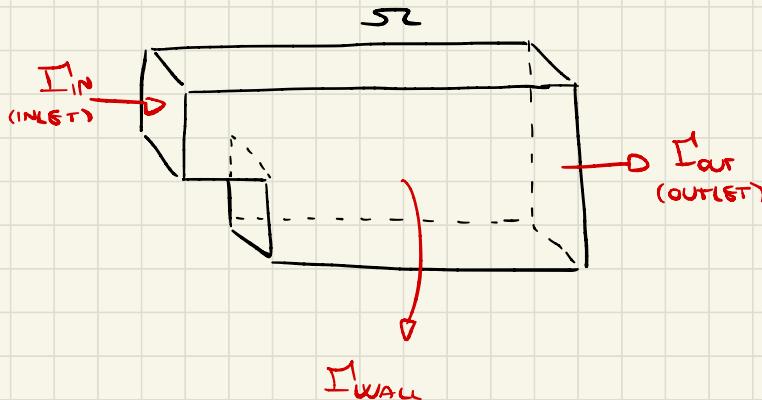
$$P: \Omega \rightarrow \mathbb{R}$$

FLUID PRESSURE

VISCOSITY



$$\left\{ \begin{array}{ll} -\nu \Delta \bar{U} + \nabla P = \bar{f} & \text{IN } \Omega \\ \nabla \cdot \bar{U} = 0 & \text{IN } \Omega \\ \bar{U} = \bar{U}_{in} & \text{ON } \Gamma_{in} \\ \bar{U} = \bar{0} & \text{ON } \Gamma_{wall} \quad \text{NO-SLIP CONDITION} \\ \nu (\nabla \bar{U}) \hat{n} - P \hat{n} = -P_{out} \hat{n} & \text{ON } \Gamma_{out} \end{array} \right.$$



1.1 WEAK FORMULATION

$$V = \left\{ \bar{v} \in [H^1(\Omega)]^3 : \bar{v} = \bar{u}_{in} \text{ ON } \Gamma_{in}, \bar{v} = \bar{0} \text{ ON } \Gamma_{wall} \right\}$$

$$V_0 = \left\{ \bar{v} \in [H^1(\Omega)]^3 : \bar{v} = \bar{0} \text{ ON } \Gamma_{in} \cup \Gamma_{wall} \right\}$$

LIFTING
FUNCTION
TRICK

$$Q = L^2(\Omega)$$

WVF MOMENTUM EQUATION $\bar{v} \in V_0$

$$\int_{\Omega} - \nabla \cdot \Delta \bar{u} \cdot \bar{v} \, d\bar{x} + \int_{\Omega} \nabla p \cdot \bar{v} \, d\bar{x} = \int_{\Omega} \bar{f} \cdot \bar{v} \, d\bar{x}$$

$$\begin{aligned} \int_{\Omega} \nabla \cdot \nabla \bar{u} : \nabla \bar{v} \, d\bar{x} - \int_{\partial\Omega} (\nabla \bar{u}) \hat{n} \cdot \bar{v} \, d\sigma - \int_{\Omega} p \nabla \cdot \bar{v} \, d\bar{x} + \\ + \int_{\partial\Omega} p \hat{n} \cdot \bar{v} \, d\sigma = \int_{\Omega} \bar{f} \cdot \bar{v} \, d\bar{x} \end{aligned}$$

$$\begin{aligned} \int_{\Omega} \nabla \cdot \nabla \bar{u} : \nabla \bar{v} \, d\bar{x} - \underbrace{\int_{\partial\Omega} (\nabla \bar{u}) \hat{n} \cdot \bar{v} \, d\sigma}_{BCS \text{ ON } \Gamma_{out}} - \int_{\Omega} p \nabla \cdot \bar{v} \, d\bar{x} + \\ + \underbrace{\int_{\partial\Omega} p \hat{n} \cdot \bar{v} \, d\sigma}_{BCS \text{ ON } \Gamma_{out}} = \int_{\Omega} \bar{f} \cdot \bar{v} \, d\bar{x} \end{aligned}$$

1 OBTAIN

$$\underbrace{\int_{\Omega} \nabla \cdot \nabla \bar{u} : \nabla \bar{v} \, d\bar{x}}_{a(u,v)} - \underbrace{\int_{\Omega} p \nabla \cdot \bar{v} \, d\bar{x}}_{b(v,p)} = \underbrace{\int_{\Omega} \bar{f} \cdot \bar{v} \, d\bar{x}}_{F(v)} + \underbrace{\int_{\partial\Omega} (-p_{out} \hat{n}) \bar{v} \, d\sigma}_{BCS \text{ ON } \Gamma_{out}}$$

LIFTING FUNCTION:

$$\bar{u} = \bar{u}_0 + \bar{R}(\bar{u}_m) \quad \bar{u}_0 \in V_0$$

$$\bar{R}(\bar{u}_m) \in V : \bar{R}(\bar{u}_m) = \bar{u}_m \text{ in } I_m \\ \nabla \cdot \bar{R}(\bar{u}_m) = 0 \text{ in } \Omega$$

$$a(\bar{u}_0, \bar{v}) + b(\bar{v}, p) = F(\bar{v}) - a(\bar{R}(\bar{u}_m), \bar{v}) \quad \forall \bar{v} \in V_0$$

WF CONTINUITY EQUATION $q \in Q$

$$\int_{\Omega} (\nabla \cdot \bar{u}) q \, d\bar{x} = 0 \Rightarrow b(\bar{u}, q) = 0 \\ b(\bar{u}_0, q) = 0$$

WF STOKES:

FIND $\bar{u}_0 \in V_0, p \in Q$, s.t. $\forall v \in V_0, \forall q \in Q$

$$\begin{cases} a(\bar{u}_0, \bar{v}) + b(\bar{v}, p) = F(\bar{v}) - a(\bar{R}(\bar{u}_m), \bar{v}) \\ b(\bar{u}, q) = 0 \end{cases}$$

DISCRETIZATION:

$$V_h = V_0 \cap [X_h^{r_u}(\Omega)]^3, \quad Q_h = Q \cap X_h^{r_p}(\Omega)$$

FIND $\bar{u}_{oh} \in V_h, p_h \in Q_h$ s.t. $\forall \bar{v}_h \in V_h, \forall q_h \in Q_h$

$$\begin{cases} a(\bar{u}_{oh}, \bar{v}_h) + b(\bar{v}_h, p_h) = F(\bar{v}_h) - a(\bar{R}_h(\bar{u}_m), \bar{v}_h) \\ b(\bar{u}_h, q_h) = 0 \end{cases}$$

BASIS FUNCTIONS

$$\{\bar{\varphi}_i\}_{i=1}^{N_h^u}$$
 BASIS OF V_h

$$\{\bar{\psi}_i\}_{i=1}^{N_h^p}$$
 BASIS OF Q_h

↓
F.E. DISCRETIZATION

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \bar{U} \\ \bar{P} \end{bmatrix} = \begin{bmatrix} \bar{F} \\ \bar{0} \end{bmatrix}$$

$$A_{ij} = \int_{\Omega} \nabla \bar{\varphi}_j : \nabla \bar{\varphi}_i \, d\bar{x}$$

$$B_{ij} = - \int_{\Omega} \bar{\psi}_j \cdot \nabla \cdot \bar{\varphi}_i \, d\bar{x}$$

$$\bar{F}_i = \int_{\Omega} \bar{f} \cdot \bar{\varphi}_i \, d\bar{x} + \sum_{\Sigma_{out}} -p_{out} \hat{n} \cdot \bar{\varphi}_i \, d\bar{x}$$

$$S = \begin{bmatrix} A & B \\ B & 0 \end{bmatrix}$$

"SADDLE-POINT" PROBLEM MATRIX

\Rightarrow HARD TO SOLVE

BLOCK BLOCK PRECONDITIONERS FAIL
TO REDUCE THE CONDITION NUMBER
OF THIS BLOCK MATRIX

\Rightarrow I NEED A SPECIFIC PRECONDITIONER

$$P_n = \begin{bmatrix} A & 0 \\ 0 & \frac{1}{\gamma} M_p \end{bmatrix} \quad (M_p)_{ij} = \int_{\Omega} \Psi_i \Psi_j dx$$

\hookrightarrow PRESSURE MASS MATRIX

$$P_n^{-1} S \bar{x} = P_n^{-1} \bar{y}$$

$$P_n^{-1} \bar{z}$$

$$\bar{z} = \begin{bmatrix} \bar{z}_u \\ \bar{z}_p \end{bmatrix}$$

$$P_n^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & \gamma M_p^{-1} \end{bmatrix}$$

$$P_n^{-1} \bar{z} = \begin{bmatrix} A^{-1} \bar{z}_u \\ \gamma M_p^{-1} \bar{z}_p \end{bmatrix}$$

$$\frac{1}{\gamma} M_p^{-1} \bar{z} = \bar{z}_p$$

SAME AS SOLVING

$$A \bar{w} = \bar{z}_u$$

ANOTHER POSSIBLE PRECONDITIONER IS

$$P_2 = \begin{bmatrix} A & 0 \\ B & \frac{1}{\gamma} M_p \end{bmatrix}$$

DATA: $\gamma = 1 \frac{m^2}{s}$ $\bar{P} = \bar{\Theta} \frac{m}{s^2}$

$$\bar{U}_{in} = \begin{bmatrix} -\alpha \gamma (z-y)(z-z)(z-z) \\ 0 \frac{m}{s} \\ 0 \frac{m}{s} \end{bmatrix} \quad \alpha = 1 \text{ m}^{-1}\text{s}^{-1}$$

$$P_{air} = 10 \text{ Pa}$$

$$P_1 = \begin{bmatrix} A & 0 \\ 0 & \frac{1}{2} M_p \end{bmatrix}$$

$$P_2 = \begin{bmatrix} A & 0 \\ B & \frac{1}{2} M_p \end{bmatrix}$$


THIS MAKES THE PRECONDITIONER
MORE SIMILAR TO THE
SYSTEM MATRIX
==> BETTER PERFORMANCE