Numerical Methods for Partial Differential Equations
A.Y. 2022/2023

Laboratory 07
Finite Element method for the heat equation

Exercise 1.

Let $\Omega = (0,1)^3$ be the unit cube and $T > 0$. Let us consider the following time-dependent problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} u) = f & \text{in } \Omega \times (0,T), & \text{(1a)} \\ \mu \boldsymbol{\nabla} u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), & \text{(1b)} \\ u = u_0 & \text{in } \Omega \times \{0\}, & \text{(1c)} \end{cases}$$

where $\mathbf{x} = (x,y,z)^T$, $\mu = 1$, $f(\mathbf{x},t) = 0$ and

$$u_0(\mathbf{x}) = x(x-1)\,y(y-1)\,z(z-1)\ .$$

1.1. Write the weak formulation to problem (1), and derive the semi-discrete formulation with the finite element method.

1.2. Write the fully discrete formulation of problem (1) using the theta method to dicsretize the time derivative.

1.3. Implement in `deal.II` a finite element solver for problem (1) using the theta method to approximate the time derivative.

1.4. Using the implicit Euler method (i.e. setting $\theta = 1$), compute the solution to the problem (1). Set $T = 1$, $\Delta t = 0.05$ and using linear polynomials (degree $r = 1$) on the mesh `mesh/mesh-cube-10.msh`.

Opening the solution in Paraview:

1. plot the solution along the line $y = z = \dfrac{1}{2}$;
2. compute the integral $\displaystyle\int_\Omega u\, d\mathbf{x}$ at times $t = 0$ and $t = T$; what can you observe?

1.5. Compute the solution to the problem using the explicit Euler method (i.e. setting $\theta = 0$), using the same discretization settings as in previous question.