

### Counting and Probability Theory:

Counting is fundamental in probability theory because it allows us to determine the number of possible outcomes in a given scenario. Probability theory deals with quantifying uncertainty and likelihood of events occurring. By understanding counting principles, we can compute probabilities more accurately.

#### 1. Counting:

- Counting is a fundamental concept in mathematics that involves determining the number of elements in a finite set.
- In the context of probability theory, counting is often used to enumerate the number of favorable outcomes in a sample space or event space.
- Techniques of counting, such as permutations, combinations, and the binomial coefficient, are essential tools in calculating probabilities, particularly in situations involving discrete outcomes or arrangements.

#### 2. Probability Theory:

- Probability theory is a branch of mathematics concerned with quantifying uncertainty and analyzing random phenomena.
- It provides a framework for assigning numerical measures to the likelihood of various outcomes in a given experiment or situation.
- Key concepts in probability theory include sample spaces, events, probability distributions, random variables, and stochastic processes.
- Probability theory enables the formulation of rules and principles for reasoning about uncertain events and making informed decisions in the face of randomness.
- Applications of probability theory are vast and diverse, ranging from gambling and games of chance to risk assessment, statistics, physics, finance, and many other fields.

Counting techniques are often employed in probability theory to determine the total number of possible outcomes and calculate probabilities. Together, counting and probability theory provide powerful tools for analyzing and understanding uncertain situations and making informed decisions based on probabilistic reasoning.

### Basics of Sample and Event Space:

In probability theory, the concepts of sample space and event space are fundamental.

#### 1. Sample Space (S): The set of all possible outcomes of a random experiment.

- The sample space, denoted by  $S$ , is the set of all possible outcomes of a random experiment.
- It encompasses every conceivable outcome that could result from an experiment or observation.
- For example, when rolling a fair six-sided die, the sample space is  $S=\{1,2,3,4,5,6\}$ , as these are all the possible outcomes.

#### 2. Event Space (E): A subset of the sample space, consisting of outcomes that satisfy a particular condition or criteria.

- The event space, denoted by  $E$ , is the set of all possible subsets of the sample space.
- It represents any collection of outcomes from the sample space that is of interest.
- Events can be simple, consisting of just one outcome, or compound, involving multiple outcomes.
- For example, if we define the event "rolling an even number" when rolling a fair six-sided die, the event space would be  $E=\{2,4,6\}$ .

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In summary, the sample space encompasses all possible outcomes of an experiment, while the event space represents specific sets of outcomes within the sample space that we are interested in analyzing or considering. These concepts are fundamental in understanding probability theory and conducting probabilistic analyses.

### Axioms of Probability:

The axioms of probability are fundamental principles that form the basis of probability theory. They were first formulated by the mathematician Andrey Kolmogorov and provide a rigorous framework for assigning probabilities to events. There are three main axioms:

**1. Non-negativity:**

- Probability values are always non-negative.
- Mathematically, for any event A, the probability of A is greater than or equal to zero:  $P(A) \geq 0$

**2. Normalization:**

- The probability of the entire sample space is 1.
- Mathematically, the probability of the sample space S (the set of all possible outcomes) is equal to 1:  $P(S)=1$

**3. Additivity (Countable):** The probability of the union of disjoint events is the sum of their individual probabilities.

- For mutually exclusive (disjoint) events, the probability of the union of these events is equal to the sum of their individual probabilities.
- Mathematically, if  $A_1, A_2, A_3, \dots$  are pairwise disjoint events (i.e., no two events have any outcomes in common), then the probability of the union of these events is equal to the sum of their individual probabilities:  $P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$

These axioms provide a solid foundation for defining probabilities and ensure that probability measures behave consistently and logically. From these axioms, many other properties and theorems of probability theory can be derived.

### Total Probability Theorem and Bayes Theorem:

1. **Total Probability Theorem:** It states that the probability of an event can be found by summing the probabilities of that event across all possible conditions or scenarios.
2. **Bayes Theorem:** A fundamental theorem in probability that describes the probability of an event, based on prior knowledge of conditions that might be related to the event.

### Random Variables, PMF, and CDF:

1. **Random Variable:** A variable whose possible values are numerical outcomes of a random phenomenon.
2. **Probability Mass Function (PMF):** Gives the probability that a discrete random variable is exactly equal to some value.
3. **Cumulative Distribution Function (CDF):** Gives the probability that a random variable is less than or equal to a certain value.

### Discrete Distributions:

- **Bernoulli Distribution:** Models the outcome of a single experiment with two possible outcomes (e.g., success or failure).
- **Binomial Distribution:** Models the number of successes in a fixed number of independent Bernoulli trials.
- **Geometric Distribution:** Models the number of trials needed to achieve the first success in a sequence of Bernoulli trials.

### Expectation and its Properties:

Expectation (or Expected Value): Represents the average value of a random variable. For a discrete random variable  $X$  with probability mass function (PMF)  $P(X)$ , the expectation is given by  $E(X) = \sum x * P(x)$ , where the sum is taken over all possible values of  $X$ .

Properties:

- Linearity:  $E(aX + bY) = aE(X) + bE(Y)$ , where  $a$  and  $b$  are constants.
- Independence: If  $X$  and  $Y$  are independent random variables,  $E(XY) = E(X)E(Y)$ .

### Variance and its Properties:

Variance: Measures the spread of a random variable around its mean. For a random variable  $X$  with mean  $\mu$ , the variance is given by  $\text{Var}(X) = E((X - \mu)^2)$ .

Properties:

- $\text{Var}(aX + b) = a^2 * \text{Var}(X)$ , where  $a$  and  $b$  are constants.
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$ , where  $\text{Cov}(X, Y)$  is the covariance of  $X$  and  $Y$ .

### Continuous Distributions:

1. Uniform Distribution: All values within a given range are equally likely. It has a constant probability density function (PDF) over the interval  $[a, b]$ .
2. Exponential Distribution: Models the time until an event occurs in a continuous-time process. It has a PDF given by  $f(x) = \lambda * \exp(-\lambda x)$  for  $x \geq 0$ , where  $\lambda$  is the rate parameter.
3. Normal Distribution: Also known as the Gaussian distribution, it's one of the most widely used distributions in statistics. It's characterized by its bell-shaped curve and is completely determined by its mean ( $\mu$ ) and standard deviation ( $\sigma$ ).

### Sampling from Continuous Distributions:

- `numpy.random.uniform`: Generates samples from a uniform distribution.
- `numpy.random.exponential`: Generates samples from an exponential distribution.
- `numpy.random.normal`: Generates samples from a normal distribution.

Example of sampling from a normal distribution using NumPy:

```
import numpy as np
```

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```
# Parameters for the normal distribution
mean = 0
std_dev = 1
sample_size = 1000

# Generate samples from a normal distribution
samples = np.random.normal(mean, std_dev, sample_size)

print("Generated samples:", samples)
```

### **Simulation Techniques using NumPy:**

NumPy provides a powerful platform for simulating various scenarios and processes.

- **Random Sampling:** Simulating random events, such as coin flips, dice rolls, etc.
- **Monte Carlo Simulation:** Using random sampling to estimate numerical results, such as integrals or probabilities.
- **Markov Chain Monte Carlo (MCMC):** A method for estimating complex probability distributions or solving optimization problems.

Example of simulating random sampling in NumPy:

```
import numpy as np

# Simulate 10 random numbers from a uniform distribution between 0 and 1
uniform_samples = np.random.uniform(0, 1, 10)
print("Uniform samples:", uniform_samples)

# Simulate 10 random numbers from an exponential distribution with lambda = 0.5
exponential_samples = np.random.exponential(scale=1/0.5, size=10)
print("Exponential samples:", exponential_samples)

# Simulate 10 random numbers from a normal distribution with mean 0 and standard deviation 1
normal_samples = np.random.normal(0, 1, 10)
print("Normal samples:", normal_samples)
```

### **Inferential statistics - sample vs population**

Inferential statistics is a branch of statistics concerned with making inferences or conclusions about a population based on data collected from a sample of that population. It involves using sample data to make predictions, estimates, or decisions about a larger population.

- **Population:** The complete set of individuals, items, or data under study.
- **Sample:** A subset of the population used to make inferences or draw conclusions about the entire population.

### **Central Limit Theorem (CLT) and its Proof:**

**Central Limit Theorem (CLT):** States that the sampling distribution of the sample mean approaches a normal distribution as the sample size increases, regardless of the shape of the population distribution.

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### Proof Outline:

1. **Independence:** Samples must be independent.
2. **Sample Size:** The sample size should be "sufficiently large" (typically  $>30$ ).
3. **Sample Mean:** The distribution of the sample means approaches normality regardless of the shape of the population distribution.
4. **Sample Variance:** The variance of the sample means is the population variance divided by the sample size.

### Chi-Squared Distribution and its Properties:

**Chi-Squared Distribution:** A continuous probability distribution that arises in the context of hypothesis testing, particularly in tests concerning the variance or the standard deviation of a normal distribution.

#### Properties:

- Non-negative values only.
- Shape depends on the degrees of freedom (df).
- Mean:  $\mu = df$ .
- Variance:  $\sigma^2 = 2df$ .

### Point and Interval Estimators:

1. **Point Estimator:** A statistic that provides a single, plausible value for an unknown parameter of a population.
2. **Interval Estimator:** An estimate of an unknown parameter in the form of an interval or range, along with a confidence level.

### Estimation Technique - Maximum Likelihood Estimation (MLE):

Maximum Likelihood Estimation (MLE): A method used to estimate the parameters of a statistical model. It seeks parameter values that maximize the likelihood function, which measures the probability of observing the given sample.

### Interval Estimator of $\mu$ with Unknown $\sigma$ :

When the population standard deviation  $\sigma$  is unknown, the sample standard deviation  $s$  is used instead. The confidence interval for the population mean  $\mu$  can be calculated using the t-distribution.

The formula for the confidence interval for  $\mu$  with unknown  $\sigma$  is:

$$\bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

Where:

- $\bar{x}$  is the sample mean.
- $t_{\alpha/2}$  is the critical value from the t-distribution with  $n - 1$  degrees of freedom and confidence level  $1 - \alpha$ .
- $s$  is the sample standard deviation.
- $n$  is the sample size.

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This interval estimator provides an interval estimate for the population mean  $\mu$  with a specified level of confidence.

### Examples of Estimators:

An estimator is a statistic (function of the data) used to estimate an unknown parameter in a statistical model. Here are some common examples:

1. **Sample Mean ( $\bar{X}$ ):** Estimator for the population mean ( $\mu$ ).
2. **Sample Variance ( $S^2$ ):** Estimator for the population variance ( $\sigma^2$ ).
3. **Sample Proportion ( $\hat{p}$ ):** Estimator for the population proportion ( $p$ ).
4. **Maximum Likelihood Estimator (MLE):** An estimator that maximizes the likelihood function of the data.
5. **Method of Moments Estimator:** An estimator that equates sample moments to population moments.

### Hypothesis Testing:

Hypothesis testing is a statistical method used to make inferences about population parameters based on sample data. It involves defining null and alternative hypotheses, calculating a test statistic, and making a decision based on the test statistic and a pre-defined significance level ( $\alpha$ ). Hypothesis testing can be categorized into three types:

#### **Type I Error (False Positive):**

- Rejecting the null hypothesis when it is actually true.
- Probability of Type I error is denoted by  $\alpha$ , the significance level.

#### **Type II Error (False Negative):**

- Failing to reject the null hypothesis when it is actually false.
- Probability of Type II error is denoted by  $\beta$ .

#### **Type III Error (Non-Standard Error):**

- Incorrect specification of null and alternative hypotheses or an incorrect interpretation of the test results.
- It's not a conventional error in hypothesis testing but rather an error in the formulation or interpretation of the hypotheses.

Examples of hypothesis testing scenarios include testing the effectiveness of a new drug (comparing its effect to a placebo), assessing the impact of a marketing campaign, or determining whether there is a significant difference between the means of two populations.

Here's a basic example of hypothesis testing:

1. **Scenario:** A company claims that their new product increases customer satisfaction by at least 10%. You want to test this claim.
2. **Null Hypothesis ( $H_0$ ):** The new product does not increase customer satisfaction by at least 10% ( $p \leq 0.10$ ).
3. **Alternative Hypothesis ( $H_a$ ):** The new product increases customer satisfaction by more than 10% ( $p > 0.10$ ).

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4. **Test Statistic:** You can use a z-test or t-test depending on the sample size and distribution of data.
5. **Decision Rule:** If the p-value (probability of observing the data, given that the null hypothesis is true) is less than the significance level ( $\alpha$ ), typically 0.05, you reject the null hypothesis.