Check Your Understanding of the Lecture Material

Finger Exercises with Solutions

Instructor: David Dobor

March 31, 2017

While the solutions to these questions are available, you are strongly encouraged to do your best to answer them on your own and to only peek into the solutions if you get stuck or to verify your answers.

Understanding this material is your key to success on your final examination.

Check Your Understanding: 1. Sum of normals

Let X and Y be independent normal random variables.

a) Is $2X - 4$ always normal?

True

False

b) Is $3X - 4Y$ always normal?

True

False

c) Is $X^2 + Y$ always normal?

True

False

Check Your Understanding: 2. Covariance calculation

Suppose that X, Y, and Z are independent random variables with unit variance. Furthermore, $\mathbf{E}[X] = 0$ and $\mathbf{E}[Y] = \mathbf{E}[Z] = 2$. Then,

$$\mathrm{cov}(XY, XZ) =$$

Check Your Understanding: 3. Covariance properties

a) Is it true that $\mathrm{cov}(X, Y) = \mathrm{cov}(Y, X)$?

True

False

b) Find the value of a in the relation $\mathrm{cov}(2X, -3Y + 2) = a \cdot \mathrm{cov}(X, Y)$.

$a =$

c) Suppose that X, Y, and Z are independent, with a common variance of 5. Then,

$\mathrm{cov}(2X + Y, 3X - 4Z) =$

Check Your Understanding: 4. The variance of a sum

The random variables $X_1, \ldots, X_8$ satisfy $\mathbf{E}[X_i] = 1$ and $\text{var}(X_i) = 4$ for $i = 1, 2, \ldots, 8$. Also, for $i \neq j$, $\mathbf{E}[X_i X_j] = 3$. Then,

$$\text{var}(X_1 + \cdots + X_8) =$$

Check Your Understanding: 5. Correlation coefficient

It is known that a for a standard normal random variable X, we have $\mathbf{E}[X^3] = 0$, $\mathbf{E}[X^4] = 3$, $\mathbf{E}[X^5] = 0$, $\mathbf{E}[X^6] = 15$. Find the correlation coefficient between X and X^3.

Check Your Understanding: 6. Correlation properties

As in the preceding (class) example, let Z, V, and W be independent random variables with mean 0 and variance 1, and let $X = Z + V$ and $Y = Z + W$. We have found that $\rho(X, Y) = 1/2$.

a) It follows that:

$\rho(X, -Y) =$

$\rho(-X, -Y) =$

b) Suppose that X and Y are measured in dollars. Let X' and Y' be the same random variables, but measured in cents, so that $X' = 100X$ and $Y' = 100Y$. Then,

$\rho(X', Y') =$

c) Suppose now that $\tilde{X} = 3Z + 3V + 3$ and $\tilde{Y} = -2Z - 2W$. Then

$\rho(\tilde{X}, \tilde{Y}) =$

d) Suppose now that the variance of Z is replaced by a very large number. Then

$\rho(X, Y)$ is close to

e) Alternatively, suppose that the variance of Z is close to zero. Then

$\rho(X, Y)$ is close to

Answers

QUESTION 1

a) This is a fact that we are already familiar with: a linear function of a normal random variable is normal.

b) Since X and Y are independent and normal, the random variables $3X$ and $-4Y$ are also independent and normal. Since the sum of independent normals is normal, it follows that $3X - 4Y$ is normal.

c) There is no reason for this to be the case. To see this, consider an extreme case where $Y = 0$ (a degenerate case of a normal). Then the random variable $X^2 + Y$ is nonnegative, which is incompatible with having a normal distribution.

QUESTION 2

Because of independence and the zero-mean assumption, it follows that $\mathbf{E}[XY] = \mathbf{E}[X] \cdot \mathbf{E}[Y] = 0$ and similarly $\mathbf{E}[XZ] = 0$. Thus,

$$\text{cov}(XY, XZ) = \mathbf{E}[XYXZ] = \mathbf{E}[X^2YZ] = \mathbf{E}[X^2] \cdot \mathbf{E}[Y] \cdot \mathbf{E}[Z] = \text{var}(X) \cdot \mathbf{E}[Y] \cdot \mathbf{E}[Z] = 4.$$

QUESTION 3

a) We have $\big(X - \mathbf{E}[X]\big)\big(Y - \mathbf{E}[Y]\big) = \big(Y - \mathbf{E}[Y]\big)\big(X - \mathbf{E}[X]\big)$, and after taking expectations we obtain $\text{cov}(X, Y) = \text{cov}(Y, X)$.

b) We have argued that $\text{cov}(aX + b, Y) = a \cdot \text{cov}(X, Y)$. Note that by symmetry, we also have $\text{cov}(X, aY + b) = a \cdot \text{cov}(X, Y)$. By using these relations,

$$\text{cov}(2X, -3Y + 2) = 2 \cdot \text{cov}(X, -3Y + 2) = 2 \cdot (-3) \cdot \text{cov}(X, Y) = -6\,\text{cov}(X, Y).$$

c) Using linearity,

$$\begin{aligned}\text{cov}(2X + Y, 3X - 4Z) &= \text{cov}(2X + Y, 3X) + \text{cov}(2X + Y, -4Z) \\ &= \text{cov}(2X, 3X) + \text{cov}(Y, 3X) + \text{cov}(2X, -4Z) + \text{cov}(Y, -4Z) \\ &= 6\,\text{var}(X) + 0 + 0 + 0 = 30,\end{aligned}$$

where the zeros are obtained because independent random variables have zero covariance.

QUESTION 4

For $i \neq j$, we have $\text{cov}(X_i, X_j) = \mathbf{E}[X_iX_j] - \mathbf{E}[X_i] \cdot \mathbf{E}[X_j] = 3 - 1 = 2$. Thus,

$$\text{var}(X_1 + \cdots + X_8) = 8 \cdot \text{var}(X_1) + 56 \cdot \text{cov}(X_1, X_2) = 32 + 112 = 144.$$

QUESTION 5

Since $\mathbf{E}[X] = \mathbf{E}[X^3] = 0$, we have $\text{cov}(X, X^3) = \mathbf{E}[X \cdot X^3] = \mathbf{E}[X^4] = 3$. Furthermore, since $\text{var}(X) = 1$ and $\text{var}(X^3) = \mathbf{E}[X^6] = 15$ we obtain

$$\rho(X, X^3) = \frac{3}{\sqrt{1} \cdot \sqrt{15}} = \sqrt{3/5}.$$

Interestingly, even though the random variables are strongly dependent (the value of one determines the value of the other), the value of the correlation coefficient is moderate.

QUESTION 6

We saw that a linear transformation $x \mapsto ax + b$ of a random variable does not change the value of the correlation coefficient, except for a possible sign change if the coefficient a is negative. Note that in the case of $\rho(-X, -Y)$, we have two sign changes, hence no sign change.

For the last two parts, if Z has a very large variance, then the terms V and W become insignificant, and $\rho(X, Y) \approx \rho(Z, Z) = 1$. And if Z has very small variance, then X and Y are approximately independent, so that $\rho(-X, -Y) \approx 0$. (These conclusions can also be justified by an exact calculation.)