Probability density functions (PDFs)

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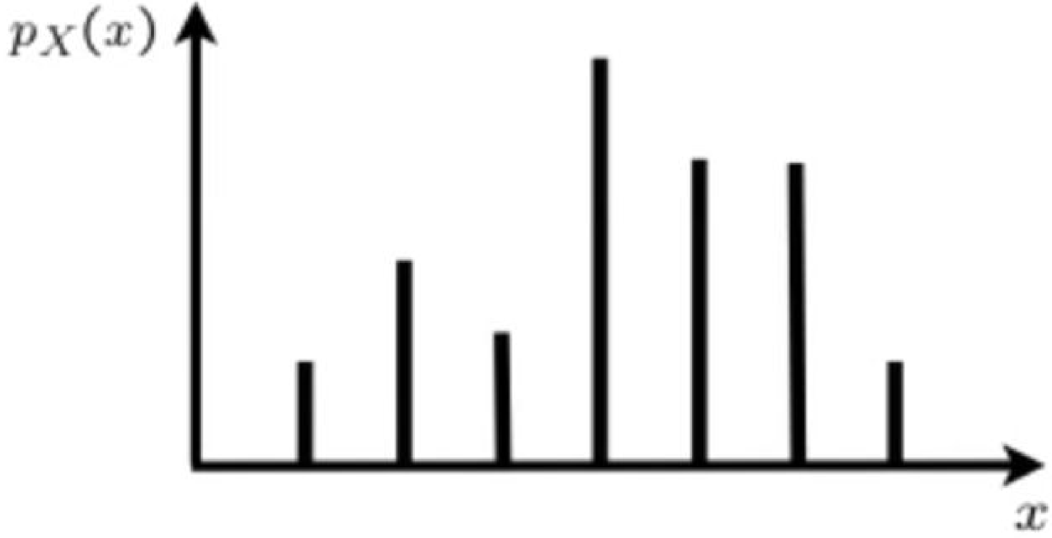
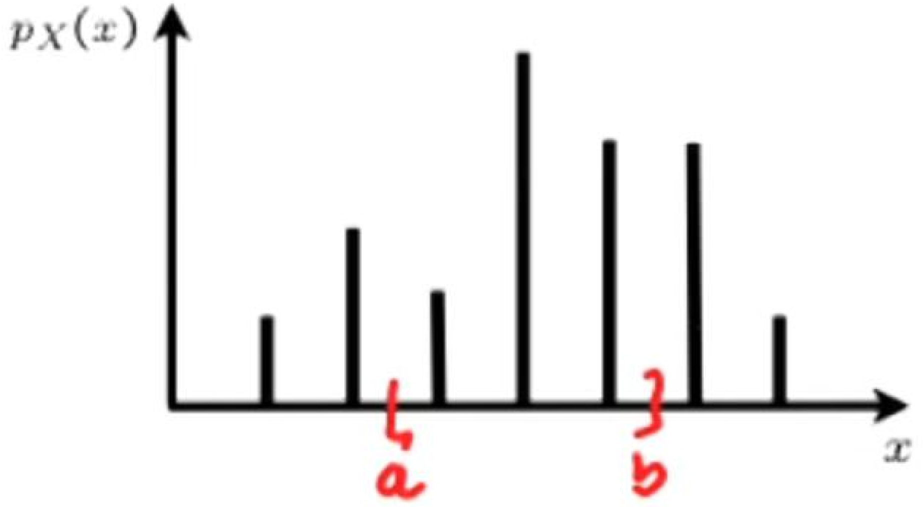
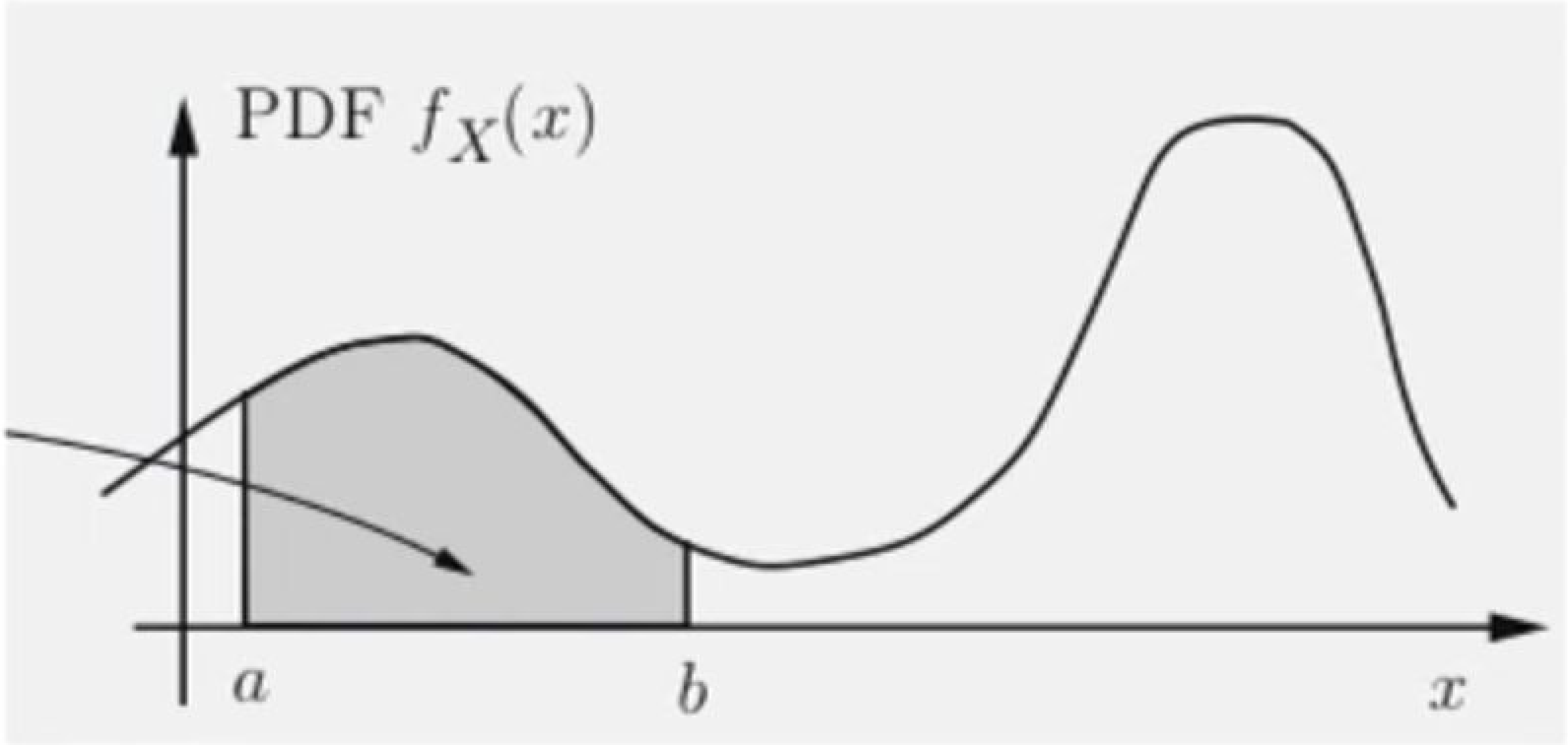


Probability density functions (PDFs)



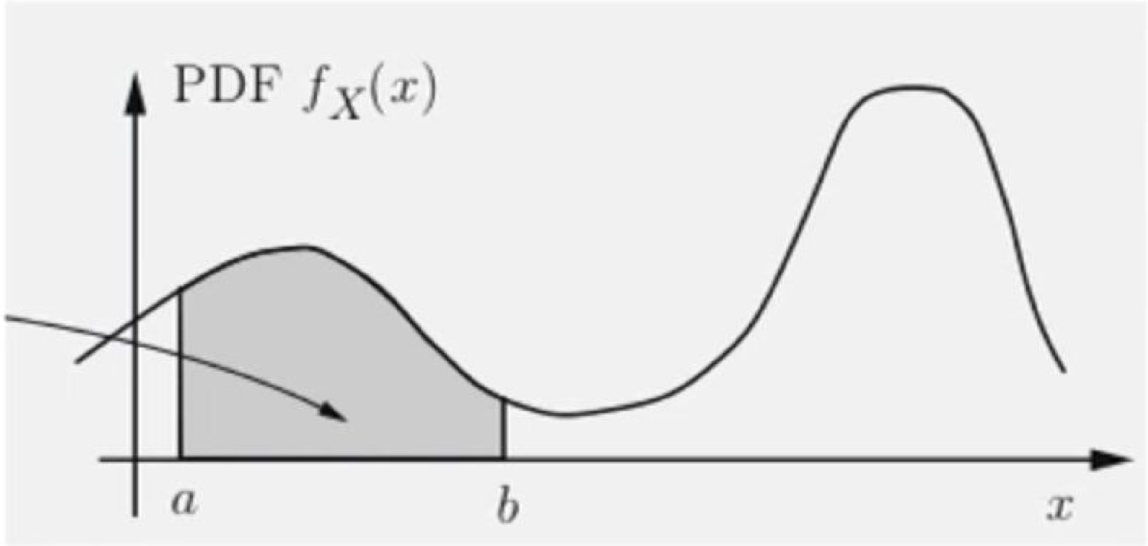
$$\mathbf{P}(a \leq X \leq b) = \sum_{x:\, a \leq x \leq b} p_X(x)$$

Probability density functions (PDFs)



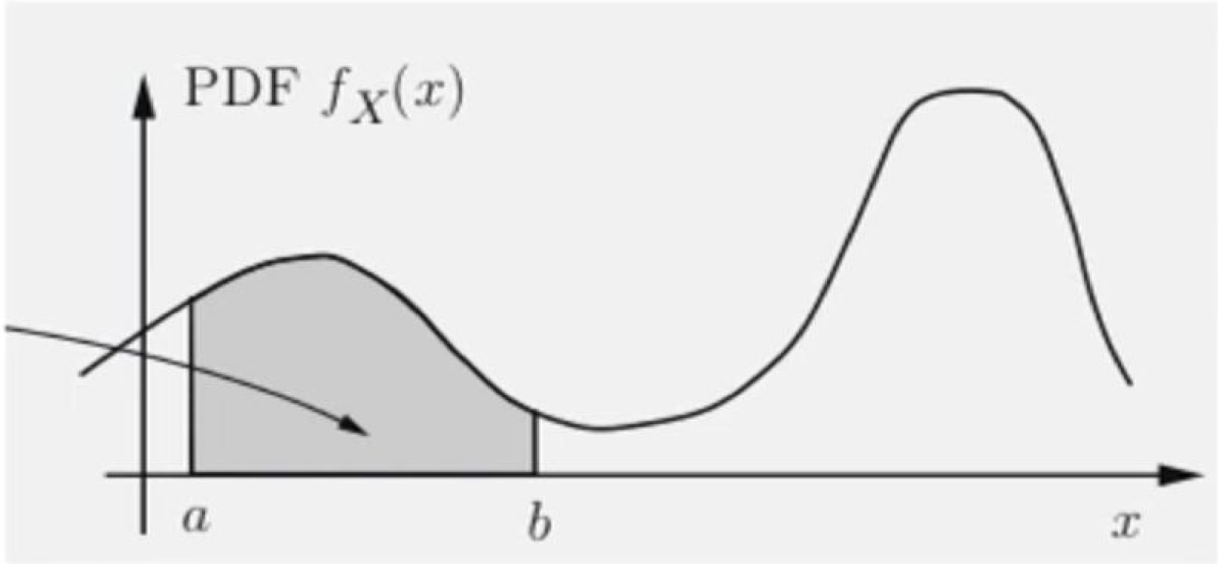
Probability density functions (PDFs)

$\mathbf{P}(a \leq X \leq b)$



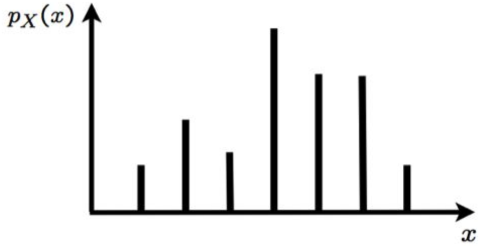
Probability density functions (PDFs)

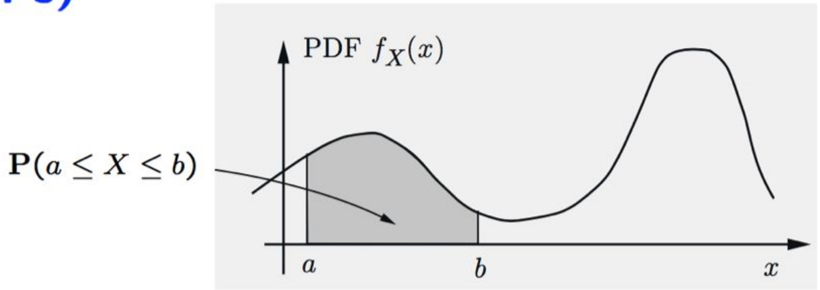
$\mathbf{P}(a \le X \le b)$



$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

Probability density functions (PDFs)





$$\mathbf{P}(a \le X \le b) = \sum_{x:\, a \le x \le b} p_X(x)$$

$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\, dx$$

$$p_X(x) \ge 0 \qquad \sum_x p_X(x) = 1$$

$$f_X(x) \ge 0 \qquad \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

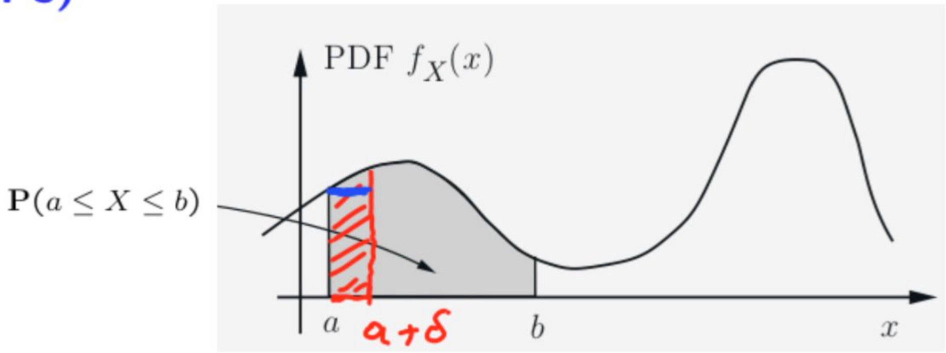
Definition: A random variable is **continuous if it can be described by a PDF**

Probability density functions (PDFs)

$\delta > 0$, small

$P(a \le X \le a+\delta)$

$\approx f_X(a) \cdot \delta$



$$\boxed{\mathbf{P}(a \le X \le a+\delta) \approx f_X(a) \cdot \delta}$$

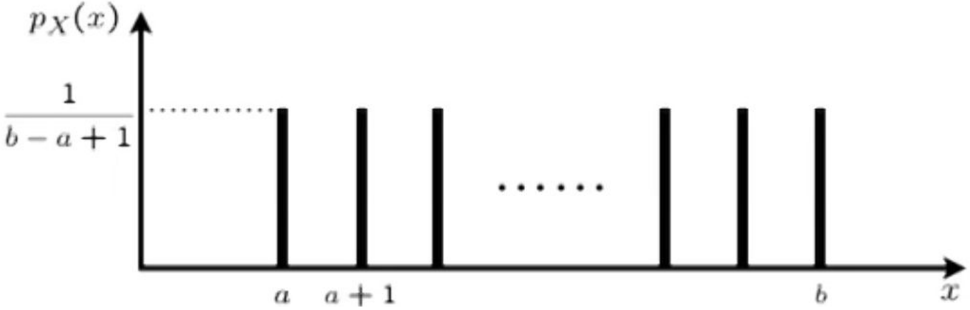
$$\boxed{\mathbf{P}(X = a) = 0}$$

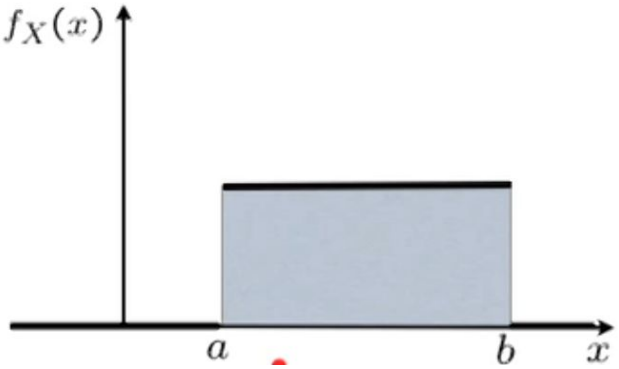
$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

$$f_X(x) \ge 0$$

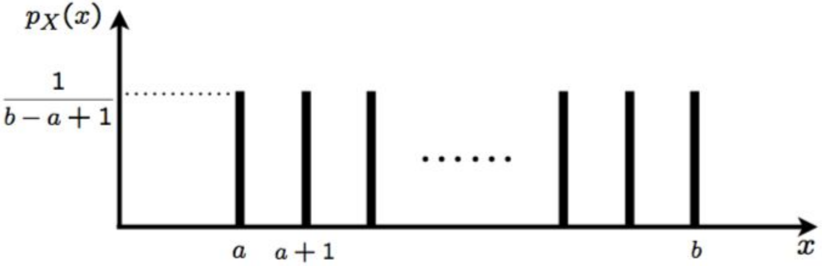
$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

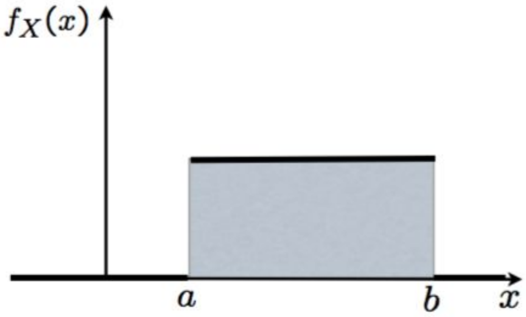
Example: continuous uniform PDF



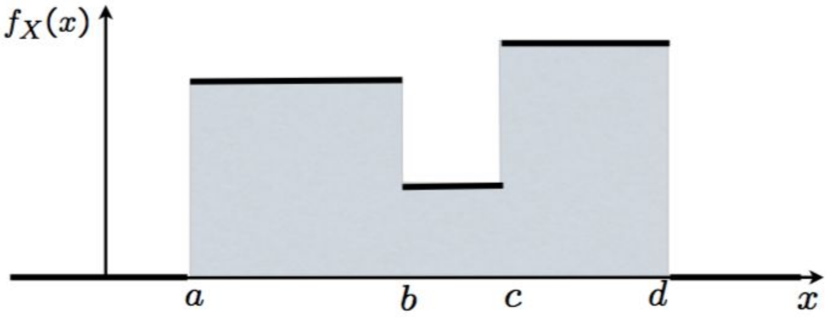


Example: continuous uniform PDF

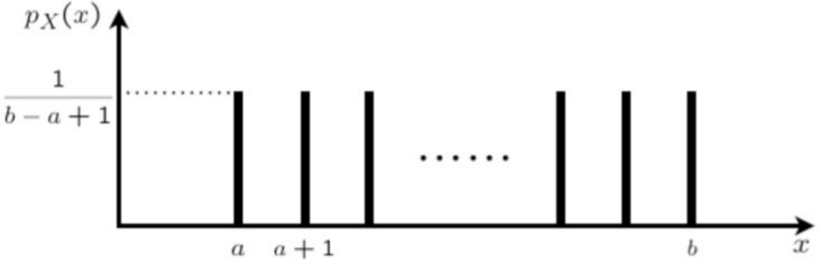


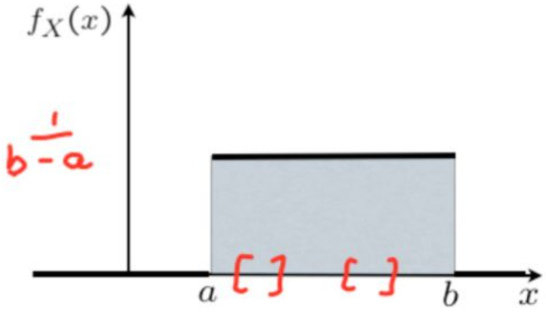


- Generalization: piecewise constant PDF

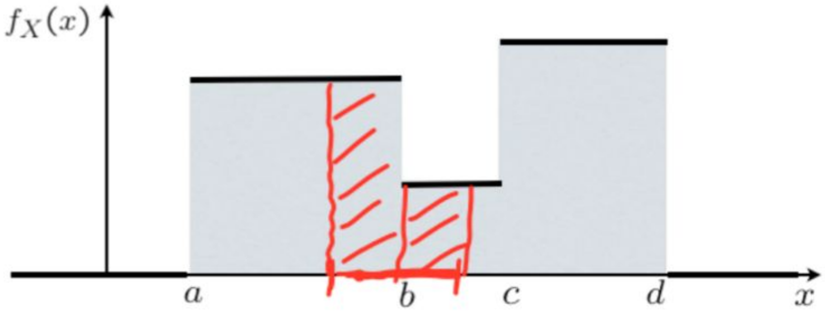


Example: continuous uniform PDF

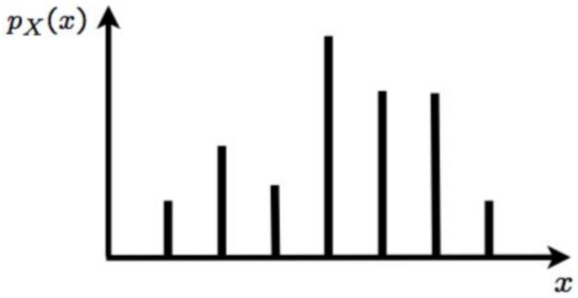


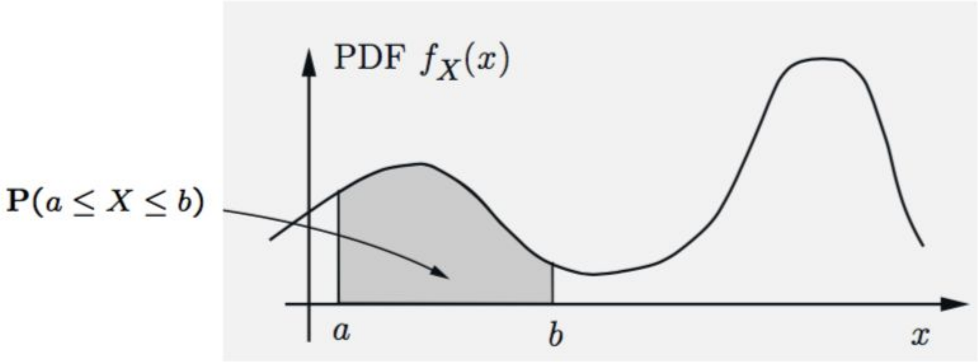


- Generalization: piecewise constant PDF



Expectation/mean of a continuous random variable





$$\mathbf{E}[X] = \sum_x x p_X(x)$$

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- **Interpretation:** Average in large number of independent repetitions of the experiment

Fine print:
Assume $\int_{-\infty}^{\infty} |x|\, f_X(x)\, dx < \infty$

Properties of expectations

- If $X \geq 0$, then $\mathbf{E}[X] \geq 0$
- If $a \leq X \leq b$, then $a \leq \mathbf{E}[X] \leq b$
- Expected value rule:

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

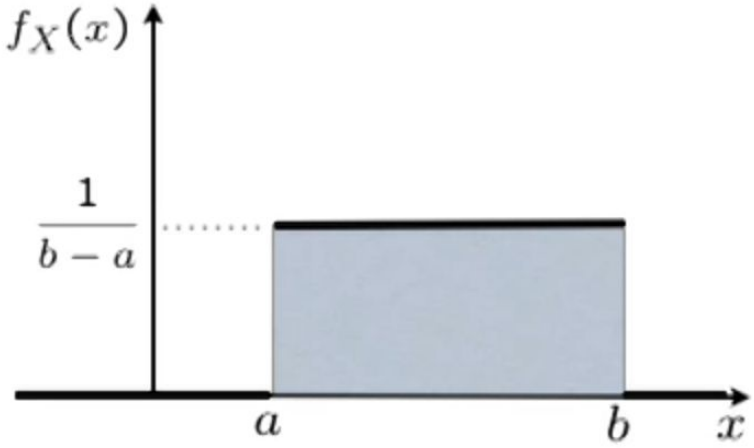
$$\boxed{\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

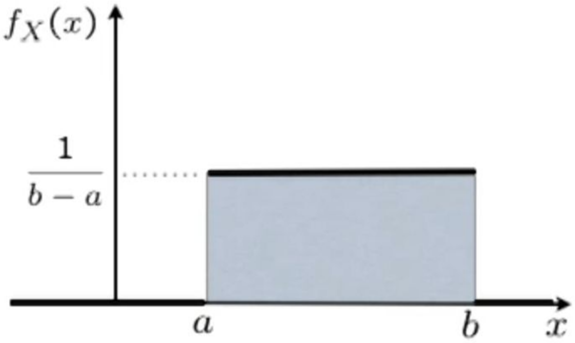
- Linearity

$$\boxed{\mathbf{E}[aX + b] = a\mathbf{E}[X] + b}$$

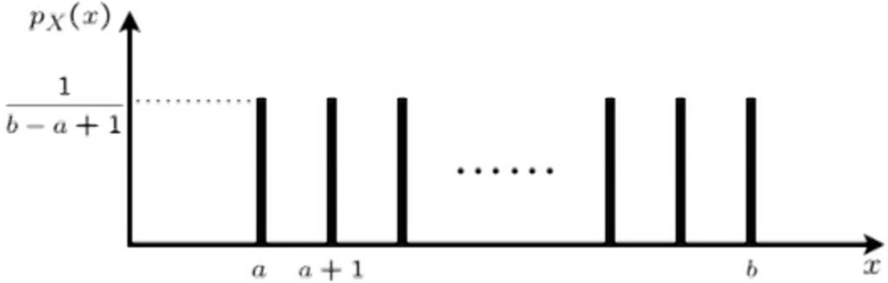
Continuous uniform random variable; parameters a, b



Continuous uniform random variable; parameters a, b



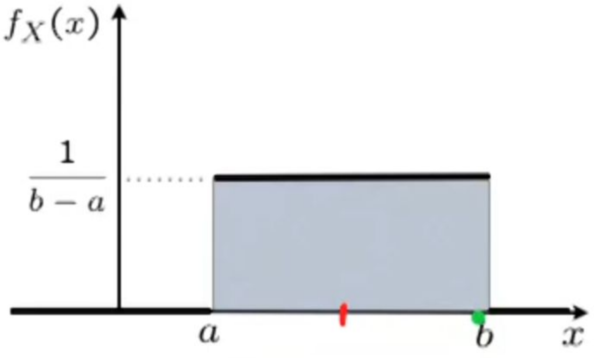
$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

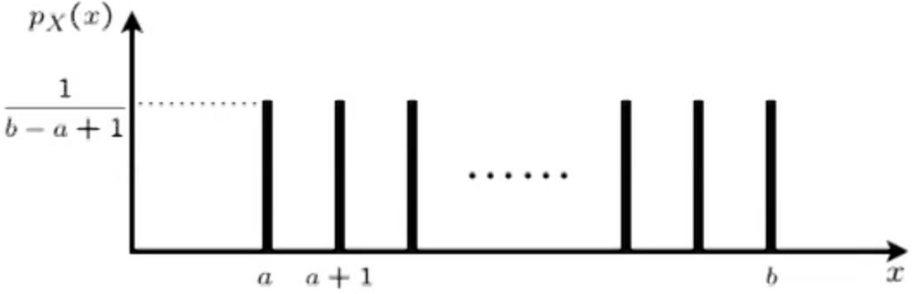


$$\mathbf{E}[X] = \frac{a+b}{2}$$

$$\text{var}(X) = \frac{1}{12}(b-a)(b-a+2)$$

Continuous uniform random variable; parameters a, b





$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{a+b}{2}$$

$$\mathbf{E}[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right)$$

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = \boxed{(b-a)^2/12}$$

$$\sigma = \frac{b-a}{\sqrt{12}}$$

$$\mathbf{E}[X] = \frac{a+b}{2}$$

$$\mathrm{var}(X) = \frac{1}{12}(b-a)(b-a+2)$$

Exponential random variable; parameter $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

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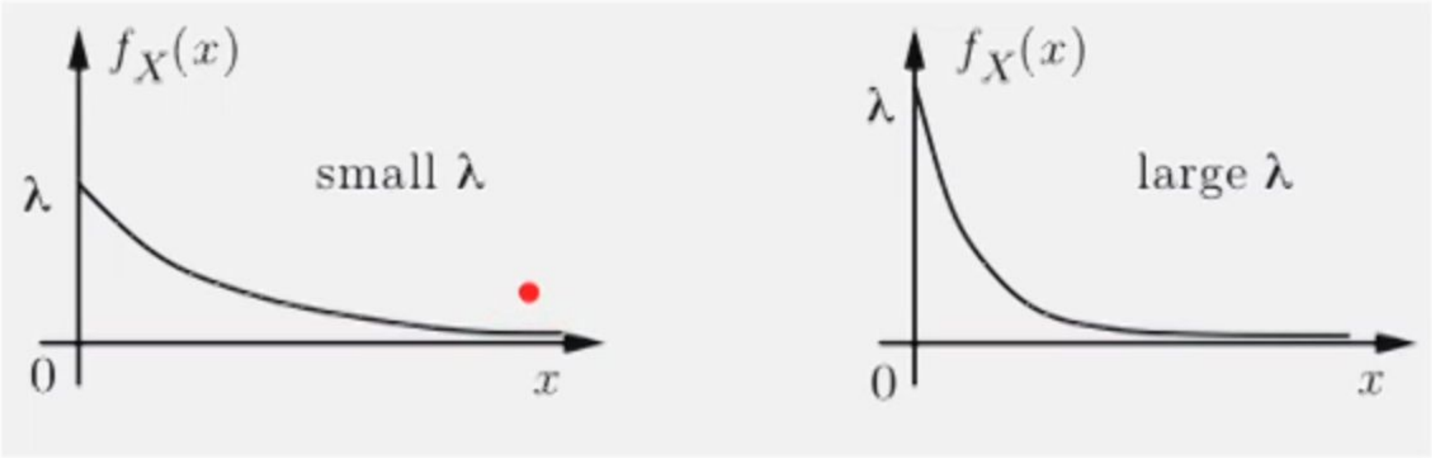

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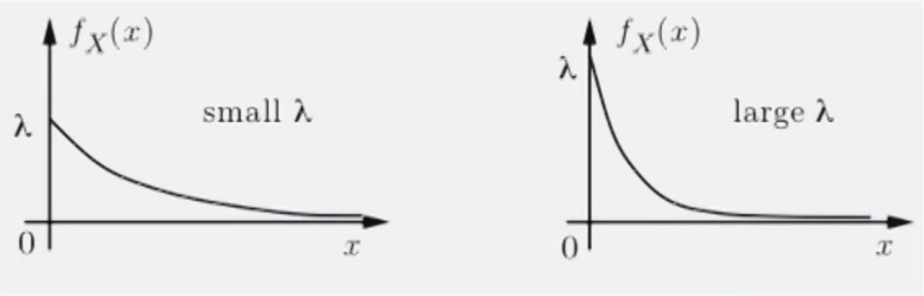

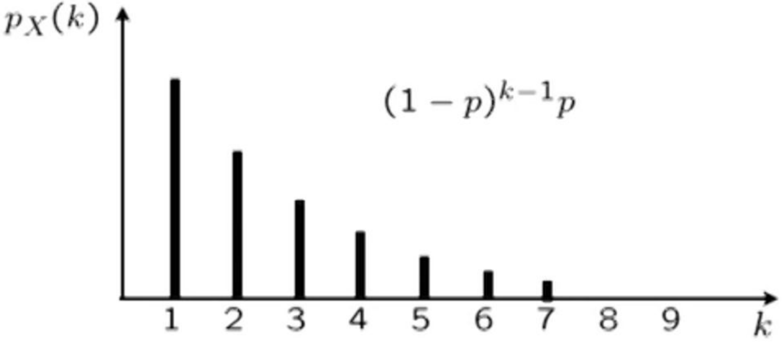

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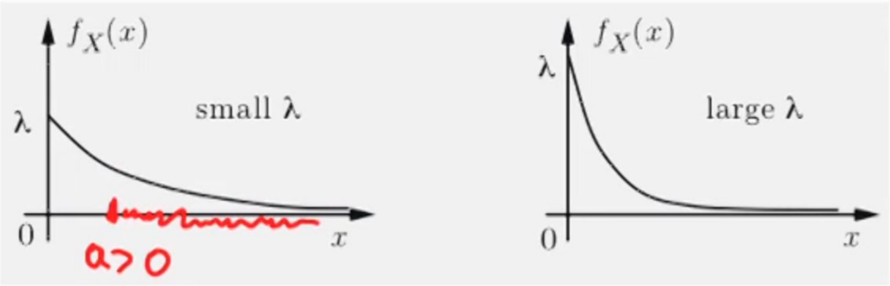
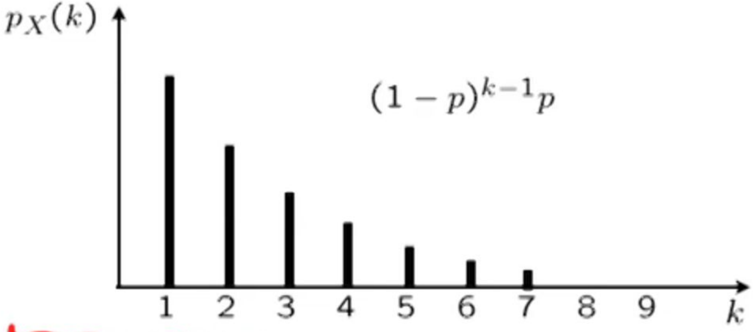




$$\boxed{P(X \ge a)} = \int_a^\infty \lambda e^{-\lambda x}\, dx$$

$$\left[\int e^{ax}\, dx = \frac{1}{a} e^{ax} \qquad a \leftrightarrow -\lambda \right]$$

$$= \lambda \cdot \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_a^\infty$$

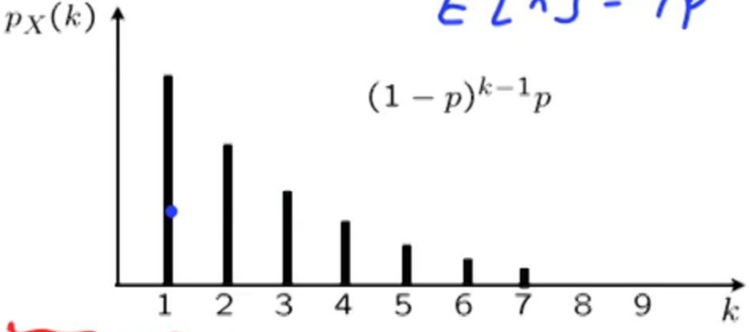
$$= -e^{-\lambda \cdot \infty} + e^{-\lambda a} = \boxed{e^{-\lambda a}}$$

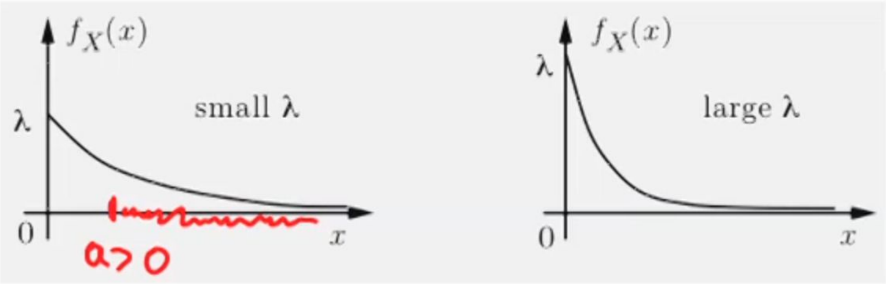
Exponential random variable; parameter $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$\int f_X(x)\, dx = 1$

$E[X] = 1/p$





$$\boxed{P(X \geq a)} = \int_a^\infty \lambda e^{-\lambda x}\, dx$$

$$\left[\int e^{ax}\, dx = \frac{1}{a} e^{ax} \quad a \leftrightarrow -\lambda \right]$$

$$= \lambda \cdot \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_a^\infty$$

$$= -e^{-\lambda \cdot \infty} + e^{-\lambda a} = \boxed{e^{-\lambda a}}$$

$$\mathbf{E}[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx = 1/\lambda$$

$$\mathbf{E}[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx = 2/\lambda^2$$

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = 1/\lambda^2$$

Cumulative distribution function (CDF)

CDF definition: $F_X(x) = \mathbf{P}(X \leq x)$

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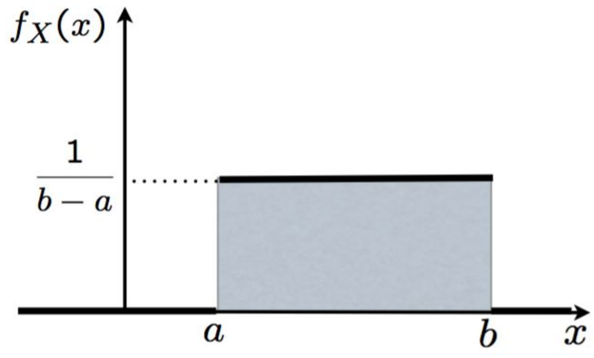
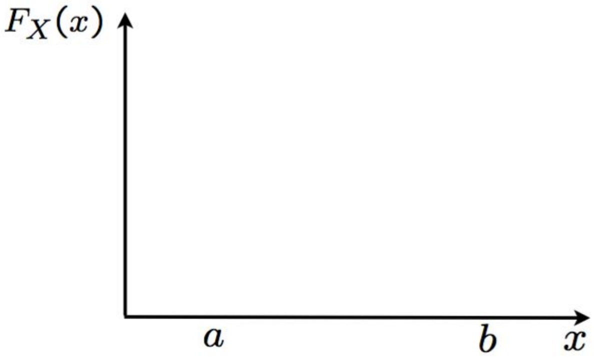


- Continuous random variables:

$$F_X(x) = \mathbf{P}(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$$



Cumulative distribution function (CDF)

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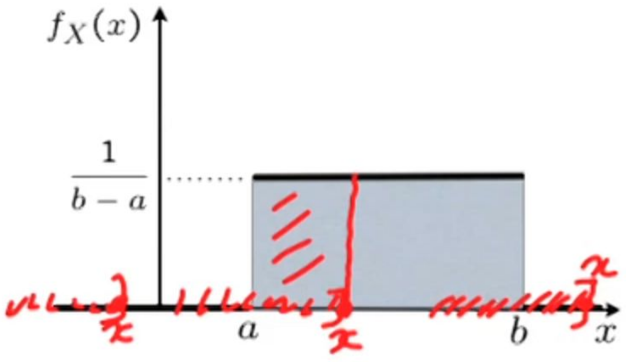
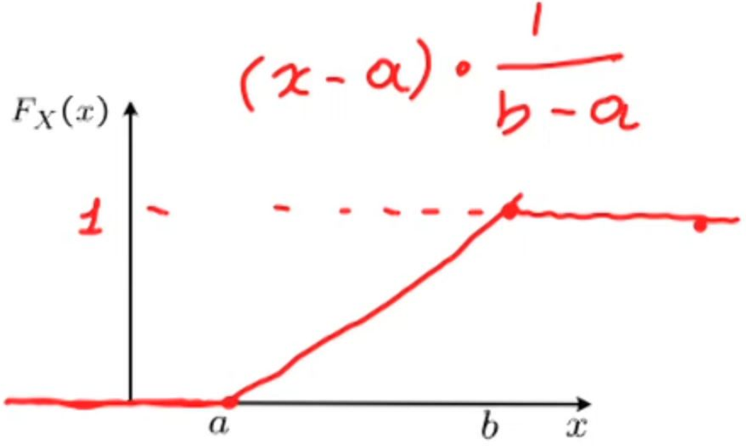


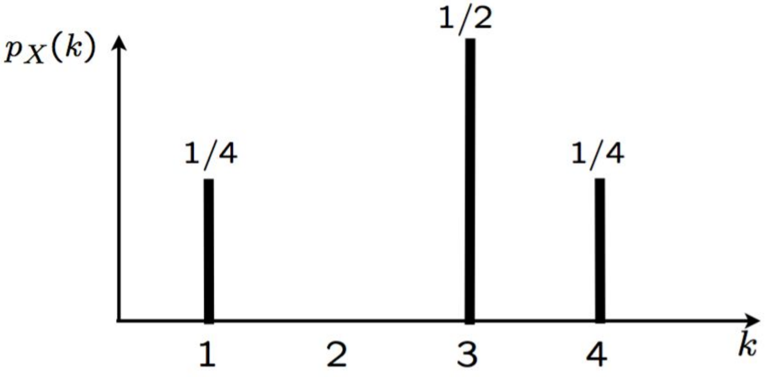


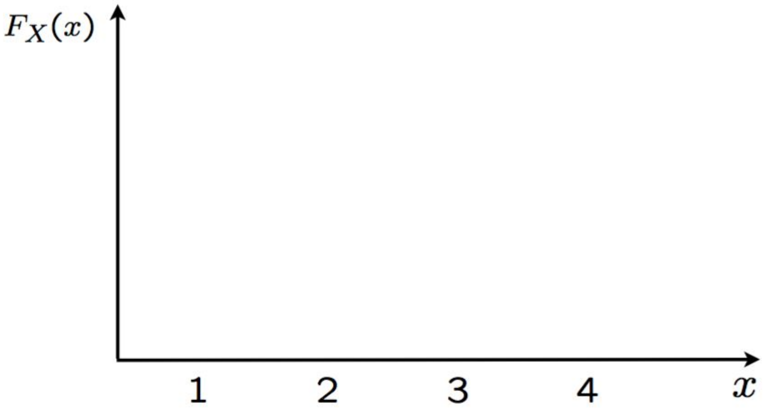
Cumulative distribution function (CDF)

CDF definition: $F_X(x) = \mathrm{P}(X \le x)$

- Discrete random variables:

$$F_X(x) = \mathrm{P}(X \le x) = \sum_{k \le x} p_X(k)$$





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CDF definition: $F_X(x) = \mathbf{P}(X \le x)$

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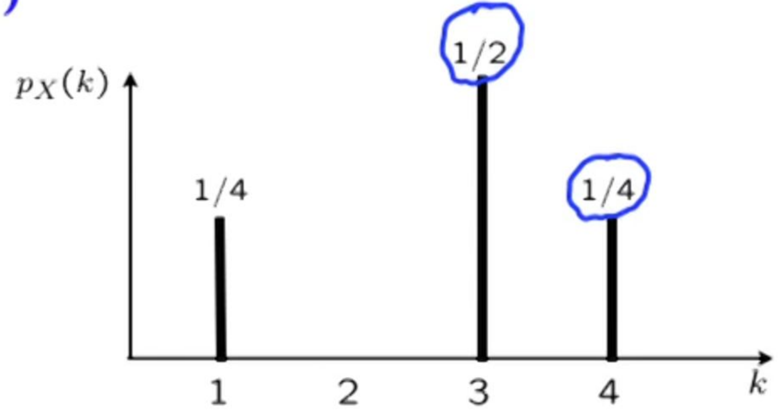
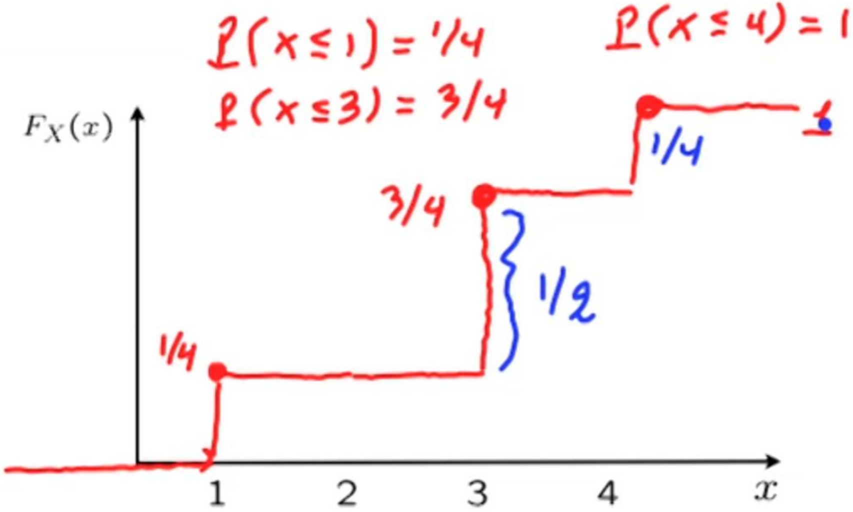




Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{3}{4} & \text{for } 0 \leq x \leq 1 \\ \frac{1}{4} & \text{for } 2 \leq x \leq 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Draw the graph of f.

Determine the distribution function F of X, and draw its graph.

Example

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait

(a) more than 10 minutes;

(b) between 10 and 20 minutes.

Example

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait

(a) more than 10 minutes;
(b) between 10 and 20 minutes.

Solution. Let X denote the length of the call made by the person in the booth. Then the desired probabilities are

(a)

$$\begin{aligned} P\{X > 10\} &= 1 - F(10) \\ &= e^{-1} \approx .368 \end{aligned}$$

(b)

$$\begin{aligned} P\{10 < X < 20\} &= F(20) - F(10) \\ &= e^{-1} - e^{-2} \approx .233 \end{aligned}$$

■

General CDF properties

$$F_X(x) = \mathbf{P}(X \leq x)$$

- Non-decreasing

- $F_X(x)$ tends to 1, as $x \to \infty$

- $F_X(x)$ tends to 0, as $x \to -\infty$

Normal (Gaussian) random variables

- Important in the theory of probability
  - Central limit theorem

- Prevalent in applications
  - Convenient analytical properties
  - Model of noise consisting of many, small independent noise terms

Standard normal (Gaussian) random variables

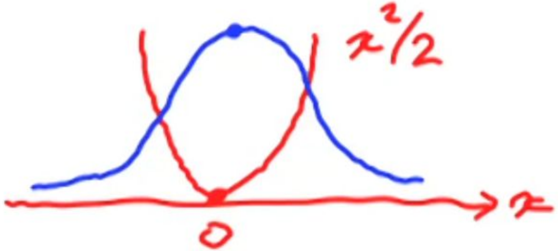
- Standard normal $N(0,1)$: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

- $\mathbf{E}[X] =$

- $\mathrm{var}(X) = 1$

Standard normal (Gaussian) random variables

- Standard normal $N(0,1)$: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



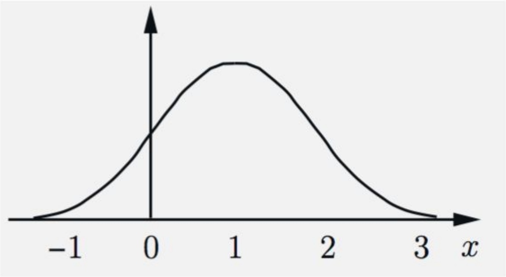
calculus:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

- $\mathbf{E}[X] = 0$

- $\mathrm{var}(X) = 1$ integrate by parts

General normal (Gaussian) random variables

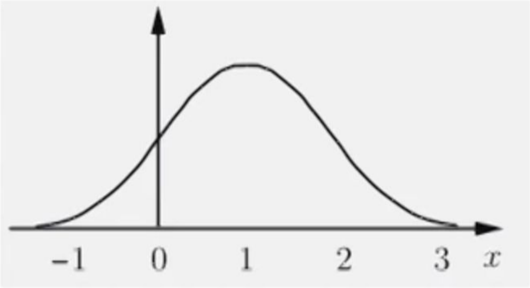


- General normal $N(\mu, \sigma^2)$: $f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

- $\mathbf{E}[X] =$

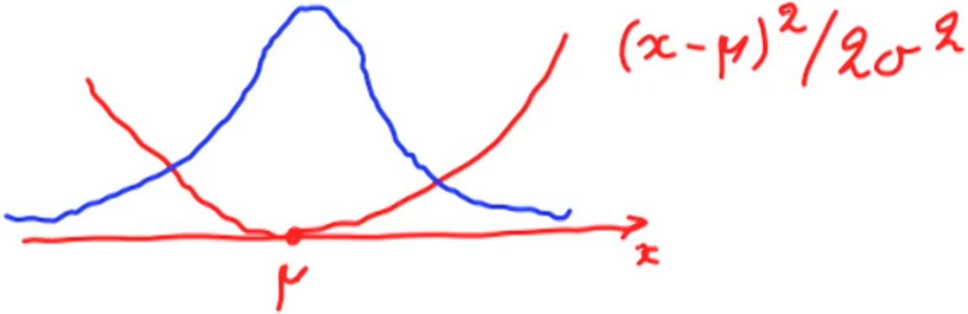
- $\text{var}(X) = \sigma^2$

General normal (Gaussian) random variables



- General normal $N(\mu, \sigma^2)$: $f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\underline{(x-\mu)^2/2\sigma^2}}$

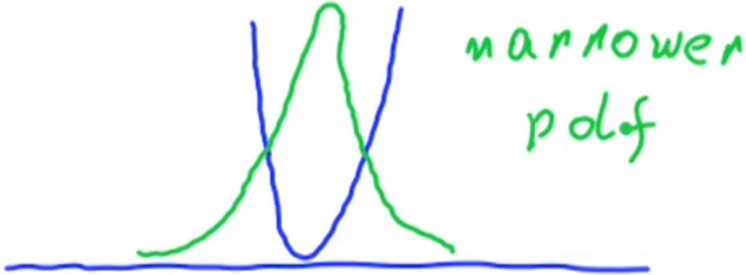
$\sigma > 0$



- $\mathbf{E}[X] = \mu$

- $\text{var}(X) = \sigma^2$

σ small



Linear functions of a normal random variable

- Let $Y = aX + b \quad X \sim N(\mu, \sigma^2)$

  $\mathbf{E}[Y] =$

  $\mathsf{Var}(Y) =$

- Fact (will prove later in this course):

$$\boxed{Y \sim N(a\mu + b, a^2\sigma^2)}$$

Linear functions of a normal random variable

- Let $Y = aX + b \quad X \sim N(\mu, \sigma^2)$

  $\mathbf{E}[Y] = a\mu + b$

  $\mathrm{Var}(Y) = a^2\sigma^2$

- Fact (will prove later in this course):

$$\boxed{Y \sim N(\underline{a\mu + b}, \underline{a^2\sigma^2})}$$

Standard normal tables

- No closed form available for CDF

but have tables, for the standard normal

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

Standard normal tables

- No closed form available for CDF

but have tables, for the standard normal

$Y \sim N(0,1)$

$\Phi(y) = F_Y(y) = P(Y \le y)$

$\Phi(0) = P(Y \le 0) = 0.5$

$\Phi(1.16) = 0.8770$ $\quad \Phi(2.9) = 0.9981$

$\Phi(-2)$

$= 1 - \Phi(2)$

$= 1 - .9772$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
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1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
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2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
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2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

Standardizing a random variable

- Let X have mean μ and variance $\sigma^2 > 0$

- Let $Y = \dfrac{X - \mu}{\sigma}$

Standardizing a random variable

- Let X have mean μ and variance $\sigma^2 > 0$

- Let $Y = \frac{X-\mu}{\sigma}$ $\quad E[Y] = 0 \quad \mathrm{Var}(Y) = \frac{1}{\sigma^2}\mathrm{Var}(X) = 1$

$$X = \mu + \sigma Y$$

- If also X is normal, then: $Y \sim N(0,1)$

Calculating normal probabilities

- Express an event of interest in terms of standard normal

$X \sim N(6,4) \qquad \sigma = 2$

$$\frac{2-6}{2} \le \frac{X-6}{2} \le \frac{8-6}{2}$$

st. normal

$$P(2 \le X \le 8) = P(-2 \le Y \le 1)$$

$$= P(Y \le 1) - P(Y \le -2)$$

$$= P(Y \le 1) - (1 - P(Y \le 2))$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
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0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
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1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
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2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

Example 3.8. Using the Normal Table. The annual snowfall at a particular geographic location is modeled as a normal random variable with a mean of $\mu = 60$ inches, and a standard deviation of $\sigma = 20$. What is the probability that this year's snowfall will be at least 80 inches?

Let X be the snow accumulation, viewed as a normal random variable, and let

$$Y = \frac{X - \mu}{\sigma} = \frac{X - 60}{20},$$

be the corresponding standard normal random variable. We want to find

$$\mathbf{P}(X \geq 80) = \mathbf{P}\left(\frac{X - 60}{20} \geq \frac{80 - 60}{20}\right) = \mathbf{P}\left(Y \geq \frac{80 - 60}{20}\right) = \mathbf{P}(Y \geq 1) = 1 - \Phi(1),$$

where Φ is the CDF of the standard normal. We read the value $\Phi(1)$ from the table:

$$\Phi(1) = 0.8413,$$

so that

$$\mathbf{P}(X \geq 80) = 1 - \Phi(1) = 0.1587.$$

Example 3.9. Signal Detection. A binary message is transmitted as a signal that is either -1 or $+1$. The communication channel corrupts the transmission with additive normal noise with mean $\mu = 0$ and variance σ^2. The receiver concludes that the signal -1 (or $+1$) was transmitted if the value received is < 0 (or ≥ 0, respectively); see Fig. 3.11. What is the probability of error?

An error occurs whenever -1 is transmitted and the noise N is at least 1 so that $N + S = N - 1 \geq 0$, or whenever $+1$ is transmitted and the noise N is smaller than -1 so that $N + S = N + 1 < 0$. In the former case, the probability of error is

$$\mathbf{P}(N \geq 1) = 1 - \mathbf{P}(N < 1) = 1 - \mathbf{P}\left(\frac{N - \mu}{\sigma} < \frac{1 - \mu}{\sigma}\right)$$
$$= 1 - \Phi\left(\frac{1 - \mu}{\sigma}\right) = 1 - \Phi\left(\frac{1}{\sigma}\right).$$

In the latter case, the probability of error is the same, by symmetry. The value of $\Phi(1/\sigma)$ can be obtained from the normal table. For $\sigma = 1$, we have $\Phi(1/\sigma) = \Phi(1) = 0.8413$, and the probability of the error is 0.1587.

See picture on the next slide

Picture to the previous slide