Check Your Understanding of the Lecture Material

Finger Exercises with Solutions

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While the solutions to these questions are available, you are strongly encouraged to do your best to answer them on your own and to only peek into the solutions if you get stuck or to verify your answers.

Understanding this material is your key to success on your final examination.

PDFs

Check Your Understanding: 1. PDFs

Let X be a continuous random variable with a PDF of the form

$$f_X(x) = \begin{cases} c(1-x), & \text{if } x \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Find the following values.

1. $c =$
2. $\mathbf{P}(X = 1/2) =$
3. $\mathbf{P}\big(X \in \{1/k : k \text{ integer}, k \geq 2\}\big) =$
4. $\mathbf{P}(X \leq 1/2) =$

Check Your Understanding: 2. Piecewise constant PDF

Consider a piecewise constant PDF of the form

$$f_X(x) = \begin{cases} 2c, & \text{if } 0 \leq x \leq 1, \\ c, & \text{if } 1 < x \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find the following values.

a) $c =$

b) $\mathbf{P}(1/2 \leq X \leq 3/2) =$

Uniform PDF

Check Your Understanding: 3. Uniform PDF

Let X be uniform on the interval $[1,3]$. Suppose that $1 < a < b < 3$. Then,

(a) $\mathbf{P}(a \le X \le b) =$

(Your answer to part (a) should be an algebraic expression involving a and b.)

(b) $\mathbf{E}[X] =$

(c) $\mathbf{E}[X^3] =$

Exponential PDF

Check Your Understanding: 4. Exponential PDF

Let X be an exponential random variable with parameter $\lambda = 2$. Find the values of the following.

a) $\mathbf{E}[(3X+1)^2] =$

b) $\mathbf{P}(1 \le X \le 2) =$

Cumulative distribution functions

Check Your Understanding: 5. Exponential CDF

Let X be an exponential random variable with parameter 2.

Find the CDF of X. Express your answer in terms of x.

a) For $x \leq 0$, $F_X(x) =$

b) For $x > 0$, $F_X(x) =$

Normal random variables

Check Your Understanding: 6. Normal random variables

Choose the correct answer below.

According to our conventions, a normal random variable

$$X \sim N(\mu, \sigma^2)$$

is a continuous random variable

a) always.

b) if and only if $\sigma \neq 0$.

c) if and only if $\mu \neq 0$ and $\sigma \neq 0$.

Check Your Understanding: 7. Using the normal tables

Let X be a normal random variable with mean 4 and variance 9. Use the normal table to find the following probabilities, to an accuracy of 4 decimal places.

a) $\mathbf{P}(X \leq 5.2) =$

b) $\mathbf{P}(X \geq 2.8) =$

c) $\mathbf{P}(X \leq 2.2) =$

Answers

QUESTION 1

1. We have $1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 c(1-x) = c(x - x^2/2)\Big|_0^1 = c/2$, and therefore $c = 2$.

2. Individual points have zero probability.

3. Using countable additivity and the fact that single points have zero probability, we have

$$\mathbf{P}\big(X \in \{1/2, 1/3, 1/4, 1/5, \ldots\}\big) = \sum_{n=2}^{\infty} \mathbf{P}(X = 1/n) = \sum_{n=2}^{\infty} 0 = 0.$$

4.

$$\mathbf{P}(X \leq 1/2) = \int_{-\infty}^{1/2} f_X(x)\,dx = \int_0^{1/2} 2(1-x)\,dx = 2(x - x^2/2)\Big|_0^{1/2} = \frac{3}{4}.$$

QUESTION 2

a) The total area under the PDF is the sum of the areas of two rectangles and is equal to $(2c) \cdot 1 + c \cdot 2 = 4c$. Therefore, $c = 1/4$.

b) The total area under the PDF over the interval of interest is the sum of the areas of two smaller rectangles and is equal to$(2c) \cdot (1/2) + c \cdot (1/2) = c \cdot (3/2) = 3/8$.

QUESTION 3

(a) The value of the PDF on the interval $[1, 3]$ must be equal to 1/2, so that it integrates to 1. Thus,

$$\mathbf{P}(a \leq X \leq b) = \int_a^b \frac{1}{2}\,dx = \frac{b-a}{2}.$$

(b) The expected value of a uniform is the midpoint of its range: $\mathbf{E}[X] = (1+3)/2 = 2$

(c) Using the expected value rule,

$$\mathbf{E}[X^3] = \int_1^3 x^3 \cdot \frac{1}{2}\,dx = \frac{1}{2} \cdot \frac{1}{4}x^4\Big|_1^3 = \frac{1}{2} \cdot \frac{1}{4} \cdot (81 - 1) = 10.$$

QUESTION 4

a) By expanding the quadratic, using linearity of expectations, and the facts that $\mathbf{E}[X] = 1/\lambda$ and $\mathbf{E}[X^2] = 2/\lambda^2$, we have

$$\mathbf{E}[(3X+1)^2] = 9\mathbf{E}[X^2] + 6\mathbf{E}[X] + 1 = 9 \cdot \frac{2}{2^2} + 6 \cdot \frac{1}{2} + 1 = \frac{17}{2}.$$

b) We have seen that for $a > 0$, we have $\mathbf{P}(X \geq a) = e^{-\lambda a}$, so that $\mathbf{P}(X \leq a) = 1 - e^{-\lambda a}$. Therefore,

$$\mathbf{P}(1 \leq X \leq 2) = \mathbf{P}(X \leq 2) - \mathbf{P}(X \leq 1) = (1 - e^{-4}) - (1 - e^{-2}) = e^{-2} - e^{-4}.$$

QUESTION 5

a) Since X is a nonnegative random variable, $F_X(x) = \mathbf{P}(X \leq x) = 0$ for $x \leq 0$.

b) We have seen that for an exponential random variable with parameter λ and for any $a > 0$, we have $\mathbf{P}(X \geq a) = e^{-\lambda a}$. Therefore, $F_X(x) = \mathbf{P}(X \leq x) = 1 - \mathbf{P}(X \geq x) = 1 - e^{-\lambda x} = 1 - e^{-2x}$.

QUESTION 6

When $\sigma \neq 0$, the distribution of X is described by a PDF, and so X is a continuous random variable. But when $\sigma = 0$, then X has all of its probability assigned to a single point, and therefore it is not a continuous random variable. (For continuous random variables, any single point must have zero probability.)

QUESTION 7

a) Note that the standard deviation is 3. Subtracting the mean and dividing by the standard deviation, we obtain

$$\mathbf{P}(X \leq 5.2) = \mathbf{P}\left(\frac{X-4}{3} \leq \frac{5.2-4}{3}\right) = \Phi(0.4) = 0.6554.$$

b) Because of the symmetry around the mean,

$$\mathbf{P}(X \geq 2.8) = \mathbf{P}(X \leq 5.2) = 0.6554$$

.

c)

$$\mathbf{P}(X \leq 2.2) = \mathbf{P}\left(\frac{X-4}{3} \leq \frac{2.2-4}{3}\right) = \Phi(-0.6) = 1 - \Phi(0.6) = 1 - 0.7257 = 0.2743.$$