Homework 2

Solutions to Selected Questions.

(Homework due in class on Thursday, September 15)

Question 1. We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

(a) Find the probability that doubles were rolled.

(b) Given that the roll resulted in a sum of 4 or less, find the conditional probability that doubles were rolled.

(c) Find the probability that at least one die is a 6.

(d) Given that the two dice land on different numbers, find the conditional probability that at least one die is a 6.

Solution to Problem 1.14. (a) Each possible outcome has probability $1/36$. There are 6 possible outcomes that are doubles, so the probability of doubles is $6/36 = 1/6$.

(b) The conditioning event (sum is 4 or less) consists of the 6 outcomes

$$\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\},$$

2 of which are doubles, so the conditional probability of doubles is $2/6 = 1/3$.

(c) There are 11 possible outcomes with at least one 6, namely, $(6,6)$, $(6,i)$, and $(i,6)$, for $i = 1, 2, \ldots, 5$. Thus, the probability that at least one die is a 6 is $11/36$.

(d) There are 30 possible outcomes where the dice land on different numbers. Out of these, there are 10 outcomes in which at least one of the rolls is a 6. Thus, the desired conditional probability is $10/30 = 1/3$.

Question 2. A coin is tossed twice. Alice claims that the event of two heads is at least as likely if we know that the first toss is a head than if we know that at least one of the tosses is a head.

(a) Is she right? (try to answer this without peeking into parts (b), (c), and (d) of this question.)

(b) Assume the coin is fair, i.e. all outcomes $\{HH, HT, TH, TT\}$ are equally likely. Let A be the event that the first toss is a head, and B be the event that the second toss is a head. What are - in words - the events $A \cap B$ and "$A \cap B$ given A"? Compute $P(A \cap B \mid A)$.

(c) With the same assumptions and notation as in the previous part, what is the event "$A \cap B$ given $A \cup B$"? Compute $P(A \cap B \mid A \cup B)$.

(d) Compare the results you got in parts (b) and (c).

(e) Was Alice right?

(f) Does it make a difference if the coin is fair or unfair?

Solution to Problem 1.15. Let A be the event that the first toss is a head and let B be the event that the second toss is a head. We must compare the conditional probabilities $\mathbf{P}(A \cap B \mid A)$ and $\mathbf{P}(A \cap B \mid A \cup B)$. We have

$$\mathbf{P}(A \cap B \mid A) = \frac{\mathbf{P}\big((A \cap B) \cap A\big)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)},$$

and

$$\mathbf{P}(A \cap B \mid A \cup B) = \frac{\mathbf{P}\big((A \cap B) \cap (A \cup B)\big)}{\mathbf{P}(A \cup B)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A \cup B)}.$$

Since $\mathbf{P}(A \cup B) \geq \mathbf{P}(A)$, the first conditional probability above is at least as large, so Alice is right, regardless of whether the coin is fair or not. In the case where the coin is fair, that is, if all four outcomes HH, HT, TH, TT are equally likely, we have

$$\frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{1/4}{1/2} = \frac{1}{2}, \qquad \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A \cup B)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Question 3. At a certain stage of a criminal investigation, the inspector in charge is 60 percent convinced of the guilt of a certain suspect. Suppose, however, that a *new* piece of evidence which shows that the criminal has a certain characteristic (such as left-handedness, baldness, or brown hair) is uncovered. If 20 percent of the population possesses this characteristic, how certain of the guilt of the suspect should the inspector now be if it turns out that the suspect has the characteristic?

Answer (example 3f, Ross): Letting G denote the event that the suspect is guilty and C the event that he possesses the characteristic of the criminal, we have

$$
\begin{aligned}
P(G \mid C) &= \frac{P(G \cap C)}{P(C)} \\
&= \frac{P(C \mid G)P(G)}{P(C \mid G)P(G) + P(C \mid G^c)P(G^c)} \\
&= \frac{1 \times 0.6}{1 \times 0.6 + 0.2 \times 0.4} \\
&\approx .882
\end{aligned}
$$

Question 4. A ball is drawn at random from an urn containing one red and one white ball. If the white ball is drawn, it is put back into the urn. If the red ball is drawn, it is returned to the urn together with two more red balls. Then a second draw is made. What is the probability a red ball was drawn on both the first and the second draws.