Check Your Understanding of the Lecture Material

Finger Exercises with Solutions

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While the solutions to these questions are available, you are strongly encouraged to do your best to answer them on your own and to only peek into the solutions if you get stuck or to verify your answers.

Understanding this material is your key to success on your final examination.

A Conditional PDF

Check Your Understanding: 1. A Conditional PDF

Suppose that X has a PDF of the form

$$f_X(x) = \begin{cases} 1/x^2, & \text{if } x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For any $x > 2$, the conditional PDF of X, given the event $X > 2$ is

$$\ldots$$

(Your answer should be an algebraic function of x.)

Memorylessness of the exponential PDF

Check Your Understanding: 2. Memorylessness of the exponential PDF

Let X be an exponential random variable with parameter λ.

1) The probability that $X > 5$ is

   (a) $\lambda e^{-5\lambda}$

   (b) $e^{-5\lambda}$

   (c) none of the above

2) The probabilty that $X > 5$ given that $X > 2$ is

   (a) $\lambda e^{-5\lambda}$

(b) $e^{-5\lambda}$

(c) $\lambda e^{-3\lambda}$

(d) $e^{-3\lambda}$

(e) none of the above

3) Given that $X > 2$, and for a small $\delta > 0$, the probability that $4 \leq X \leq 4 + 2\delta$ is approximately

(a) $\lambda\delta$

(b) $2\lambda\delta$

(c) $\delta e^{-4\lambda}$

(d) $\lambda\delta e^{-4\lambda}$

(e) $\lambda\delta e^{-2\lambda}$

(f) $2\lambda\delta e^{-2\lambda}$

(g) none of the above

The total probability theorem

Check Your Understanding: 3. The total probability theorem

On any given day, mail gets delivered by either Alice or Bob. If Alice delivers it, which happens with probability 1/4, she does so at a time that is uniformly distributed between 9 and 11. If Bob delivers it, which happens with probability 3/4, he does so at a time that is uniformly distributed between 10 and 12. The PDF of the time X that mail gets delivered satisfies

a) $f_X(9.5) =$

b) $f_X(10.5) =$

A mixed random variable

Check Your Understanding: 4. A mixed random variable

A lightbulb is installed. With probability 1/3, it burns out immediately when it is first installed. With probability 2/3, it burns out after an amount of time that is uniformly distributed on $[0, 3]$.

The expected value of the time until the lightbulb burns out is ...

Joint PDFs

Check Your Understanding: 5. Jointly continuous r.v.'s

(a) The random variables X and Y are continuous. Is this enough information to determine the value of $\mathbf{P}\big(X^2 = e^{3Y}\big)$?

(b) The random variables X and Y are jointly continuous. Is this enough information to determine the value of $\mathbf{P}\big(X^2 = e^{3Y}\big)$?

Check Your Understanding: 6. From joint PDFs to probabilities

(a) The probability of the event that $0 \le Y \le X \le 1$ is of the form $\displaystyle\int_a^b \left(\int_c^d f_{X,Y}(x,y)\,dx \right)\,dy.$

Find the values of a, b, c, d. Each one of your answers should be one of the following: $0, x, y,$ or 1.

$a =$

$b =$

$c =$

$d =$

(b) The probability of the event that $0 \le Y \le X \le 1$ is also of the form $\displaystyle\int_a^b \left(\int_c^d f_{X,Y}(x,y)\,dy \right)\,dx$. Note the different order of integration as compared to part (a).

Find the values of a, b, c, d. Each one of your answers should be one of the following: $0, x, y,$ or 1.

$a =$

$b =$

$c =$

$d =$

Finding a marginal PDF

Check Your Understanding: 7. Finding a marginal PDF

The random variables X and Y are described by a uniform joint PDF of the form $f_{X,Y}(x,y) = 3$ on the set $\{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ y \leq x^2\}$.

Then, $f_X(0.5) =$

Check Your Understanding: 8. From joint PDFs to the marginals

For each one of the following formulas, identify those that are always true. All integrals are meant to be from $-\infty$ to ∞.

(a) $f_{X,Z}(a,b) = \int f_{X,Y,Z}(a',b,c)\,da'$

- Yes
- No

(b) $f_{X,Z}(a,c) = \int f_{X,Y,Z}(a,b,c)\,db$

- Yes
- No

(c) $f_{X,Z}(a,b) = \int f_{X,Y,Z}(a,b,c)\,dc$

- Yes
- No

(d) $f_Y(a) = \int\int\int f_{U,V,X,Y}(a,b,c,s)\,db\,dc\,ds$

- Yes
- No

(e) $f_Y(a) = \int\int\int f_{U,V,X,Y}(s,c,b,a)\,db\,dc\,ds$

- Yes
- No

Joint CDFs

Check Your Understanding: 9. Joint CDFs

a) Is it always true that if $x < x'$, then $F_{X,Y}(x, y) \leq F_{X,Y}(x', y)$?

- Yes
- No

b) Suppose that the random variables X and Y are jointly continuous and take values on the set where $0 \leq x, y \leq 1$. Is $F_{X,Y}(x, y) = (x + 2y)^2/9$ a legitimate joint CDF? *Hint:* Consider $F_{X,Y}(0, 1)$.

- Yes
- No

c) Suppose that the random variables X and Y are jointly continuous and take values on the unit square, i.e., $0 \leq x \leq 1$ and $0 \leq y \leq 1$. The joint CDF on that set is of the form $xy(x + y)/2$. Find an expression for the joint PDF which is valid for (x, y) in the unit square.

Answers

QUESTION 1

The conditional PDF will be a scaled version of the unconditional, of the form $\frac{f_X(x)}{\mathbf{P}(X>2)}$. Now, $\mathbf{P}(X > 2) = \int_2^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_2^\infty = 1/2$, and so the answer is $2/x^2$.

QUESTION 2

1) We have seen in the past that for an exponential random variable with parameter λ, $\mathbf{P}(X > a) = e^{-\lambda a}$, and so $\mathbf{P}(X > 5) = e^{-5\lambda}$.

2) Because of the memorylessness property, given that $X > 2$, the remaining time $X?2$ is again exponential with the same parameter. Thus, $\mathbf{P}(X > 5 \,|\, X > 2) = \mathbf{P}(X - 2 > 3 \,|\, X > 2) = \mathbf{P}(X > 3) = e^{-3\lambda}$.

3) By memorylessness, this is the same as the unconditional probability that an exponential takes values in the interval $[2, 2 + 2\delta]$, which is approximately the length, 2δ, of the small interval times the density evaluated at 2, yielding $2\lambda\delta e^{-2\lambda}$.

QUESTION 3 The PDF is $1/4$ times a uniform on $[9, 11]$ (of height $1/2$) plus $3/4$ times a uniform on $[10, 12]$ (again of height $1/2$).

a) At time 9.5, only the first uniform is nonzero, yielding $f_X(9.5) = (1/4) \cdot (1/2) = 1/8$

b) At time 10.5 both uniforms are nonzero, yielding $f_X(10.5) = (1/4) \cdot (1/2) + (3/4) \cdot (1/2) = 1/2$

QUESTION 4

The expected value of a uniform on $[0, 3]$ is $3/2$. Using the definition of expectation of mixed random variables, the expected value is $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot \frac{3}{2} = 1$.

QUESTION 5

(a) There is no information on the relation between the two random variables. If, for example, $X = \sqrt{e^{3Y}}$, the probability is 1, whereas if $X = \sqrt{e^{3Y}} + 1$, then the probability is zero.

(b) The set of points on the $x - y$ plane that correspond to the event $X^2 = e^{3Y}$ is a one-dimensional curve, which has zero area, and therefore zero probability.

QUESTION 6

(a) For any given $y \in [0,1]$, x ranges from y to 1, yielding $\int_0^1 \int_y^1 f_{X,Y}(x,y)\,dx\,dy$.

(b) For any given $x \in [0,1]$, y ranges from 0 to x, yielding $\int_0^1 \int_0^x f_{X,Y}(x,y)\,dy\,dx$.

QUESTION 7

For any $x \in [0,1]$, and using also the fact that the PDF is zero outside the specified set of $x-y$ pairs, we have $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{x^2} 3\,dy = 3x^2$. Threfore, $f_X(0.5) = 3/4$

QUESTION 8

In each case, we need to "integrate out" the arguments associated with random variables that do not appear on the left-hand side. Thus, the correct formulas are:

$$f_{X,Z}(a,c) = \int f_{X,Y,Z}(a,b,c)\,db$$

and

$$f_Y(a) = \int\int\int f_{U,V,X,Y}(s,c,b,a)\,db\,dc\,ds.$$

QUESTION 9

a) Since $x < x'$, the event $\{X \leq x, Y \leq y\}$ is a subset of the event $\{X \leq x', Y \leq y\}$ and therefore $F_{X,Y}(x,y) = \mathbf{P}(X \leq x, Y \leq y) \leq \mathbf{P}(X \leq x', Y \leq y) = F_{X,Y}(x',y)$.

b) Since the random variables are nonnegative, we have $F_{X,Y}(0,1) = \mathbf{P}(X \leq 0 \text{ and } Y \leq 1) = \mathbf{P}(X = 0 \text{ and } Y \leq 1) \leq \mathbf{P}(X = 0) = 0$, where the last equality holds because X is a continuous random variable. But zero is different from $(0 + 2 \cdot 1)^2/9$. Therefore, we do not have a legitimate joint CDF.

c) The joint CDF is of the form $x^2y/2 + y^2x/2$. The partial derivative with respect to x is $xy + y^2/2$. Taking now the partial derivative with respect to y, we obtain $x + y$.