Check Your Understanding of the Lecture Material

Finger Exercises with Solutions

Instructor: David Dobor

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While the solutions to these questions are available, you are strongly encouraged to do your best to answer them on your own, and to only peek into the solutions if you get stuck or to verify your answers.

Understanding this material is your key to success on your first midterm examination.

Bernoulli and indicator random variables

Check Your Understanding: 1. Indicator variables

Let A and B be two events (subsets of the same sample space Ω), with nonempty intersection. Let I_A and I_B be the associated indicator random variables.

For each of the two cases below, select one statement that is true.

(A) $I_A + I_B$:

- (a) is an indicator variable of $A \cup B$.
- (b) is an indicator variable of $A \cap B$.
- (c) is not an indicator variable of any event.

(B) $I_A \cdot I_B$:

- (a) is an indicator variable of $A \cup B$.
- (b) is an indicator variable of $A \cap B$.
- (c) is not an indicator variable of any event.

Binomial random variables

Check Your Understanding: 2. The binomial PMF

You roll a fair six-sided die (all 6 of the possible results of a die roll are equally likely) 5 times, independently. Let X be the number of times that the roll results in 2 or 3. Find the numerical values of the following.

(a) $p_X(2.5)$

(b) $p_X(1)$

Geometric random variables

Check Your Understanding: 3. Geometric random variables

Let X be a geometric random variable with parameter p. Find the probability that $X \geq 10$.

Expectation

Check Your Understanding: 4. Expectation calculation

The PMF of the random variable Y satisfies $p_Y(-1) = 1/6, p_Y(2) = 2/6, p_Y(5) = 3/6$, and $p_Y(y) = 0$ for all other values y. The expected value of Y is:

$$E[Y] = \dots$$

Elementary properties of expectation

Check Your Understanding: 5. Random variables with bounded range

Suppose a random variable X can take any value in the interval $[-1, 2]$ and a random variable Y can take any value in the interval $[-2, 3]$.

(A) The random variable $X - Y$ can take any value in an interval $[a, b]$. Find the values of a and b:

  (a) $a = \dots$

  (b) $b = \dots$

(B) Can the expected value of $X + Y$ be equal to 6?

  (a) Yes, why not?

  (b) No, no way!

The expected value rule

Check Your Understanding: 6. The expected value rule

Let X be a uniform random variable on the range $\{-1, 0, 1, 2\}$. Let $Y = X^4$. Use the expected value rule to calculate $E[Y]$.

$$E[Y] = \ldots$$

Linearity of expectations

Check Your Understanding: 7. Linearity of expectations

The random variable X is known to satisfy $E[X] = 2$ and $E[X^2] = 7$. Find the expected value of $8 - X$ and of $(X-3)(X+3)$.

(a) $E[8 - X] = \ldots$

(b) $E[(X-3)(X+3)] = \ldots$

Variance

Check Your Understanding: 8. Variance calculation

Suppose that $var(X) = 2$. The variance of $2 - 3X$ is:

$$var(2 - 3X) = \ldots$$

Variance Properties

Check Your Understanding: 9. Variance properties

Is it always true that $E[X^2] \geq (E[X])^2$?

(a) Yes, of course! (Why?)

(b) No, that can't always be true! (Why not?)

Variance of the uniform

Check Your Understanding: 10. Variance of the uniform

Suppose that the random variable X takes values in the set $\{0, 2, 4, 6, \ldots, 2n\}$ (the even integers between 0 and $2n$, inclusive), with each value having the same probability. What is the variance of X?

Hint: Consider the random variable $Y = X/2$ and recall that the variance of a uniform random variable on the set $\{0, 1, \ldots, n\}$ is equal to $n(n+2)/12$.

$$Var(X) = \ldots$$

Conditional PMFs and expectations given an event

Check Your Understanding: 11. Conditional variance

In class, we saw that the conditional distribution of X, which was a uniform over a smaller range (and in some sense, less uncertain), had a smaller variance, i.e., $var(X|A) \leq var(X)$. Here is an example where this is not true. Let Y be uniform on $\{0, 1, 2\}$ and let B be the event that Y belongs to $\{0, 2\}$.

(a) What is the variance of Y?

(b) What is the conditional variance $var(Y|B)$?

Total expectation theorem

Check Your Understanding: 12. Total expectation calculation

We have two coins, A and B. For each toss of coin A, we obtain Heads with probability 1/2; for each toss of coin B, we obtain Heads with probability 1/3. All tosses of the same coin are independent. We select a coin at random, where the probabilty of selecting coin A is 1/4, and then toss it until Heads is obtained for the first time.

The expected number of tosses until the first Heads is: ...

Check Your Understanding: 13. Memorylessness of the geometric

Let X be a geometric random variable, and assume that $var(X) = 5$.

(A) What is the conditional variance $var(X-4|X>4)$?

(B) What is the conditional variance $var(X-8|X>4)$?

Joint PMF calculation

Check Your Understanding: 14. Joint PMF calculation

The random variable V takes values in the set $\{0,1\}$ and the random variable W takes values in the set $\{0,1,2\}$. Their joint PMF is of the form

$$p_{V,W}(v,w) = c \cdot (v+w),$$

where c is some constant, for v and w in their respective ranges, and is zero everywhere else.

(a) Find the value of c.

(b) Find $p_V(1)$.

Properties of Expectation

Check Your Understanding: 15. Expected value rule

Let X and Y be discrete random variables. For each one of the formulas below, state whether it is true or false.

(a) $E[X^2] = \sum_x x p_X(x^2)$

(b) $E[X^2] = \sum_x x^2 p_X(x)$

(c) $E[X^2] = \sum_x x^2 p_{X,Y}(x)$

(d) $E[X^2] = \sum_x x^2 p_{X,Y}(x,y)$

(e) $E[X^2] = \sum_x \sum_y x^2 p_{X,Y}(x,y)$

(f) $E[X^2] = \sum_z z p_X^2(z)$

Check Your Understanding: 16. Linearity of expectations drill

Suppose that $E[X_i] = i$ for every i. Then,

$$E[X_1 + 2X_2 - 3X_3] = \dots$$

Check Your Understanding: 17. Using linearity of expectations

We have two coins, A and B. For each toss of coin A, we obtain Heads with probability $1/2$; for each toss of coin B, we obtain Heads with probability $1/3$. All tosses of the same coin are independent.

We toss coin A until Heads is obtained for the first time. We then toss coin B until Heads is obtained for the first time with coin B.

The expected value of the total number of tosses is: ...

Answers

QUESTION 1

(A) (c)

(B) (b)

QUESTION 2

(a) 0

(b) 0.3292

QUESTION 3

$$(1-p)^9$$

QUESTION 4

$$E[Y] = (-1)\cdot\frac{1}{6} + 2\cdot\frac{2}{6} + 5\cdot\frac{3}{6} = \frac{18}{6} = 3.$$

QUESTION 5

(A) The smallest possible value of $X - Y$ is obtained if X takes its smallest value, -1, and Y takes its largest value, 3, resulting in $X - Y = -1 - 3 = -4$. Similarly, the largest possible value of $X - Y$ is obtained if X takes its largest value, 2, and Y takes its smallest value, -2, resulting in $X - Y = 2 - (-2) = 4$.

(B) No, no way! No matter what the outcome of the experiment is, the value of $X + Y$ will be at most 5, and so the expected value can be at most 5.

QUESTION 6

We are dealing with $Y = g(X)$, where g is the function defined by $g(x) = x^4$. Thus,

$$\begin{aligned}
E[Y] = E[X^4] = \sum_x x^4 p_X(x) &= (-1)^4\cdot\frac{1}{4} + 0^4\cdot\frac{1}{4} + 1^4\cdot\frac{1}{4} + 2^4\cdot\frac{1}{4} \\
&= \frac{1}{4} + \frac{1}{4} + \frac{16}{4} \\
&= 4.5
\end{aligned}$$

QUESTION 7

(a) The random variable $8 - X$ is of the form $aX + b$, with $a = -1$ and $b = 8$. By linearity, $E[8 - X] = -E[X] + 8 = -2 + 8 = 6$.

(b) The random variable $(X-3)(X+3)$ is equal to $X^2 - 9$ and therefore its expected value is $E[X^2] - 9 = 7 - 9 = -2$.

QUESTION 8

The random variable $2 - 3X$ is of the form $aX + b$, with $a = -3$ and $b = 2$. Thus, $var(2-3X) = (-3)^2 var(X) = 9 \cdot 2 = 18$.

QUESTION 9

We know that variances are always nonnegative and that $var(X) = E[X^2] - (E[X])^2$. Therefore, $0 \leq var(X) = E[X^2] - (E[X])^2$, or, equivalently, $E[X^2] \geq (E[X])^2$.

QUESTION 10

Following the hint, let $Y = X/2$. The random variable Y takes values in the set $\{0, 1, 2, \ldots, n\}$, each value having the same probability. Therefore, Y is uniform and has a variance of $n(n+2)/12$. Since $X = 2Y$,

$$var(X) = var(2 \cdot Y) = 4 \cdot var(Y) = \frac{4}{12} n(n+2).$$

QUESTION 11

(a) The calculation of the variance of Y is exactly the same as the calculation of $var(X|A)$ in the preceding example, yielding $2/3$.

(b) In the conditional model, the conditional mean is $E[Y|B] = 1$. Since Y is either 0 or 2 in the conditional model, the difference between Y and the conditional mean is either 1 or -1, so that $(Y - E[Y|B])^2$ is always equal to 1. It follows that the conditional variance is equal to 1.

Note that in this example, $var(Y|B) > var(Y)$.

QUESTION 12

Let T be the number of tosses until the first Heads. Once a coin is selected, the conditional distribution of T is geometric, with a mean of $1/p$, where p is the probability of Heads for the selected coin. Let C_A and C_B denote the events that coin A or B, respectively, is selected.

$$E[T] = P(C_A)E[T \mid C_A] + P(C_B)E[T \mid C_B] = \frac{1}{4} \cdot 2 + \frac{3}{4} \cdot 3 = \frac{11}{4}.$$

QUESTION 13

(a) The conditional distribution of $X-4$ given $X > 4$ is the same geometric PMF that describes the distribution of X. Hence $var(X-4|X>4) = var(X) = 5$.

(b) In the conditional model (i.e., given that $X > 4$), the random variables $X-4$ and $X-8$ differ by a constant. Hence they have the same variance and the answer is again 5.

QUESTION 14

(a) The sum of the entries of the PMF is $c \cdot (0+0) + c \cdot (0+1) + c \cdot (0+2) + c \cdot (1+0) + \ldots = 9c$. Since this sum must be equal to 1, we have $c = 1/9$.

(b)

$$p_V(1) = \sum_{w=0}^{2} p_{V,W}(1,w) = p_{V,W}(1,0) + p_{V,W}(1,1) + p_{V,W}(1,2) = \frac{1}{9}(1+2+3) = \frac{6}{9}.$$

QUESTION 15

(a) False. This does not follow from any of our formulas.

(b) True. This is the expected value rule for a function of a single random variable.

(c) False. This is syntactically wrong since the function $p_{X,Y}$ needs two arguments.

(d) False. The left-hand side is a number whereas the right-hand side is actually a function of y.

(e) True. This is the expected value rule

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y).$$

for the function $g(x,y) = x^2$.

(f) True. This is just the definition of the expectation $E[Z] = \sum_z p_Z(z)$, where Z is the random variable X^2.

QUESTION 16

Using linearity,

$$
\begin{aligned}
E[X_1 + 2X_2 - 3X_3] &= E[X_1] + E[2X_2] + E[3X_3] \\
&= E[X_1] + 2E[X_2] + 3E[X_3] \\
&= 1 + 2 \cdot 2 - 3 \cdot 3 \\
&= -4
\end{aligned}
$$

QUESTION 17

Let T_A and T_B be the number of tosses of coins A and B, respectively. We know that T_A is geometric with parameter $p = 1/2$, so that $E[T_A] = 1/p = 1/(1/2) = 2$. Similarly, $E[T_B] = 3$. The total number of coin tosses is $T_A + T_B$. Using linearity,

$$E[T_A + T_B] = E[T_A] + E[T_B] = 2 + 3 = 5.$$