Extra Practice Problems

Question 1. Alice and Bob each choose at random a real number between zero and one. We assume that the pair of numbers is chosen according to the uniform probability law on the unit square, so that the probability of an event is equal to its area.

We define the following events:

$A = \{$The magnitude of the difference of the two numbers is greater than $1/3\}$
$B = \{$At least one of the numbers is greater than $1/4\}$
$C = \{$The sum of the two numbers is $1\}$
$D = \{$Alice's number is greater than $1/4\}$

Find the following probabilities:

(a) $P(A)$.

(b) $P(B)$.

(c) $P(A \cap B)$.

(d) $P(C)$.

(e) $P(D)$

(f) $P(A \cap D)$.

Answers:

(a) $P(A) = 4/9$.

(b) $P(B) = 15/16$.

(c) $P(A \cap B) = 4/9$.

(d) $P(C) = 0$.

(e) $P(D) = 3/4$

(f) $P(A \cap D) = 89/288$.

Question 2. The Parking Lot Problem. Sanjeev and Paula park their corvettes in an empty parking lot with $n \geq 2$ consecutive parking spaces (i.e, n spaces in a row, where only one car fits in each space). Sanjeev and Paula pick parking spaces at random. (All pairs of parking spaces are equally likely.) What is the probability that there is at most one empty parking space between them?

Answers: $\frac{4n-6}{n(n-1)}$

Question 3. You flip a fair coin (i.e., the probability of obtaining Heads is 1/2) three times. Assume that all sequences of coin flip results, of length 3, are equally likely. Determine the probability of each of the following events.

(a) $\{HHH\}$.

(b) $\{HTH\}$.

(c) Any sequence with 2 Heads and 1 Tails (in any order).

(d) Any sequence in which the number of Heads is greater than or equal to the number of Tails.

Answers:

(a) 1/8

(b) 1/8

(c) 3/8

(d) 1/2

Question 4. Find the value of $P(A \cup (B^c \cup C^c)^c)$ for each of the following cases:

(a) The events A, B, C are disjoint events and $P(A) = 2/5$.

$$P(A \cup (B^c \cup C^c)^c) = \dots$$

(b) The events A and C are disjoint, and $P(A) = 1/2$ and $P(B \cap C) = 1/4$.

$$P(A \cup (B^c \cup C^c)^c) = \dots$$

(c) $P(A^c \cup (B^c \cup C^c)) = 0.7$.

$$P(A \cup (B^c \cup C^c)^c) = \dots$$

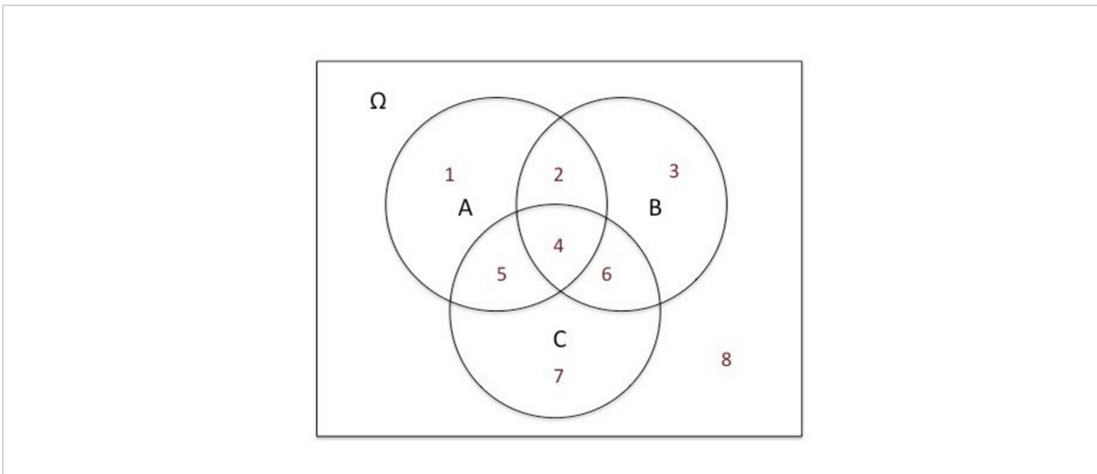
Answers:

(a) 2/5

(b) 3/4

(c) 0.3

Question 5. In this problem, you are given descriptions in words of certain events (e.g., "at least one of the events A, B, C occurs"). For each one of these descriptions, identify the correct symbolic description in terms of A, B, C from Events $E1 - E7$ below. Also identify the correct description in terms of regions (i.e., subsets of the sample space Ω) as depicted in the Venn diagram below. (For example, Region 1 is the part of A outside of B and C.)



Symbolic descriptions:

- Event E1: $A \cap B \cap C$
- Event E2: $(A \cap B \cap C)^c$
- Event E3: $A \cap B \cap C^c$
- Event E4: $B \cup (B^c \cap C^c)$
- Event E5: $A^c \cap B^c \cap C^c$
- Event E6: $(A \cap B) \cup (A \cap C) \cup (B \cap C)$
- Event E7: $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

1. At least two of the events A, B, C occur.

2. At most two of the events A, B, C occur.

3. None of the events A, B, C occurs.

4. All three events A, B, C occur.

5. Exactly one of the events A, B, C occurs.

6. Events A and B occur, but C does not occur.

7. Either event B occurs or, if not, then C also does not occur.

Solution to Question 5

1. At least two of the events A, B, C occur.

   Event E6 ✔

   Regions: 2 4 5 6 ✔

2. At most two of the events A, B, C occur.

   Event E2 ✔

   Regions: 1 2 3 5 6 7 8 ✔

3. None of the events A, B, C occurs.

   Event E5 ✔

   Region: 8 ✔

4. All three events A, B, C occur.

   Event E1 ✔

   Region: 4 ✔

5. Exactly one of the events A, B, C occurs.

   Event E7 ✔

   Regions: 1 3 7 ✔

6. Events A and B occur, but C does not occur.

   Event E3 ✔

   Region: 2 ✔

7. Either event B occurs or, if not, then C also does not occur.

   Event E4 ✔

   Regions: 1 2 3 4 6 8 ✔

Question 6. You roll two five-sided dice. The sides of each die are numbered from 1 to 5. The dice are ″fair″ (all sides are equally likely), and the two die rolls are independent.

(a) Event A is ″the total is 10″ (i.e., the sum of the results of the two die rolls is 10).

1. Is event A independent of the event ″at least one of the dice resulted in a 5″?
2. Is event A independent of the event ″at least one of the dice resulted in a 1″?

(b) Event B is ″the total is 8.″

1. Is event B independent of getting ″doubles″ (i.e., both dice resulting in the same number)?
2. Given that the total was 8, what is the probability that at least one of the dice resulted in a 3?

Answers:

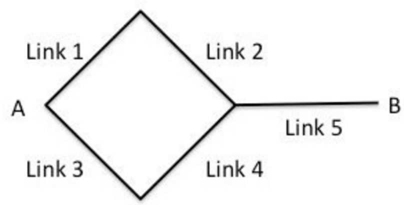
(a) no

(b) no

(c) no

(d) 2/3

Question 7. A reliability problem Consider the communication network shown in the figure below and suppose that each link can fail with probability p. Assume that failures of different links are independent.



(a) Assume that $p = 1/3$. Find the probability that there exists a path from A to B along which no link has failed. (Give a numerical answer.)

(b) Given that exactly one link in the network has failed, find the probability that there exists a path from A to B along which no link has failed. (Give a numerical answer.)

Answers:

(a) $112/243 = [1 - [1 - (1-p)^2]^2] \cdot (1-p)$

(b)

Question 8. Oscar's lost dog in the forest

Oscar has lost his dog in either forest A (with probability 0.4) or in forest B (with probability 0.6).

If the dog is in forest A and Oscar spends a day searching for it in forest A, the conditional probability that he will find the dog that day is 0.25. Similarly, if the dog is in forest B and Oscar spends a day looking for it there, he will find the dog that day with probability 0.15.

The dog cannot go from one forest to the other. Oscar can search only in the daytime, and he can travel from one forest to the other only overnight.

The dog is alive during day 0, when Oscar loses it, and during day 1, when Oscar starts searching. It is alive during day 2 with probability $2/3$. In general, for $n \geq 1$, if the dog is alive during day $n-1$, then the probability it is alive during day n is $2/(n+1)$. The dog can only die overnight. Oscar stops searching as soon as he finds his dog, either alive or dead.

(a) In which forest should Oscar look on the first day of the search to maximize the probability he finds his dog that day?

(b) Oscar looked in forest A on the first day but didn't find his dog. What is the probability that the dog is in forest A?

(c) Oscar flips a fair coin to determine where to look on the first day and finds the dog on the first day. What is the probability that he looked in forest A?

(d) Oscar decides to look in forest A for the first two days. What is the probability that he finds his dog alive for the first time on the second day?

(e) Oscar decides to look in forest A for the first two days. Given that he did not find his dog on the first day, find the probability that he does not find his dog dead on the second day.

(f) Oscar finally finds his dog on the fourth day of the search. He looked in forest A for the first 3 days and in forest B on the fourth day. Given this information, what is the probability that he found his dog alive?

Question 9. Victor and his umbrella

Before leaving for work, Victor checks the weather report in order to decide whether to carry an umbrella. On any given day, with probability 0.2 the forecast is "rain" and with probability 0.8 the forecast is "no rain". If the forecast is "rain", the probability of actually having rain on that day is 0.8. On the other hand, if the forecast is "no rain", the probability of actually raining is 0.1.

1. One day, Victor missed the forecast and it rained. What is the probability that the forecast was "rain"?

2. Victor misses the morning forecast with probability 0.2 on any day in the year. If he misses the forecast, Victor will flip a fair coin to decide whether to carry an umbrella. (We assume that the result of the coin flip is independent from the forecast and the weather.) On any day he sees the forecast, if it says "rain" he will always carry an umbrella, and if it says "no rain" he will not carry an umbrella. Let U be the event that "Victor is carrying an umbrella", and let N be the event that the forecast is "no rain". Are events U and N independent?

3. Victor is carrying an umbrella and it is not raining. What is the probability that he saw the forecast?

Question 10. A student takes a multiple-choice exam. Suppose for each question she either knows the answer or gambles and chooses an option at random. Further suppose that if she knows the answer, the probability of a correct answer is 1, and if she gambles this probability is 1/4. To pass, students need to answer at least 60% of the questions correctly. The student has ”studied for a minimal pass”, i.e., with probability 0.6 she knows the answer to a question. Given that she answers a question correctly, what is the probability that she actually knows the answer?

Question 11.

(a) You go to see the doctor about an ingrowing toenail. The doctor selects you at random to have a blood test for swine flu, which for the purposes of this exercise we will say currently affects 1 in 10,000 people in the US. The test is 99% accurate, in the sense that the probability of a false positive is 1% (that is, if you don't have the disease, the test says that you do have it 1% of the time). The probability of a false negative is zero (that is, if you do have the disease, the test always recognizes that you do). You test positive. What is the new probability that you have swine flu?

(b) Now imagine that you went to a friend's wedding in the country of Agriphinia recently, and (for the purposes of this exercise) it is known that 1 in 200 people who visited Agriphinia recently come back with swine flu. Given the same test result as above, what should your revised estimate be for the probability that you have the disease?

Question 12. A lollipop is drawn at random from a box containing one red and one white lollipop. If the white lollipop is drawn, it is put back into the box. If the red lollipop is drawn, it is returned to the box together with two more red lollipops. Then a second draw is made. What is the probability a red lollipop was drawn on both the first and the second draws?

Question 13. A Temple University study indicates that typically 8% of all students forget to turn off their cell phones when attending classes. (Your instructor claims that this is a bogus study whose claims have been greatly exaggerated, but you decide to take the study's results for granted and continue to practice for your midterm). Assume that 20 people attend your next CIS 2033 class (again, suspend your disbelief).

(a) What is the likelihood that 5 cell phones will go off during the class? (assume that if somebody's cell phone rings, nobody else bothers to check if theirs' is still on)

(b) What is the likelihood that everybody's cell phone rings?

(c) What is the likelihood that nobody's cell phone rings?

(d) What is the likelihood that at least 2 people's cell phones go off?

Just write down the correct formulas and plug in the numbers. You may skip the actual number-crunching, for now. What's your random variable here? How is it distributed?