Homework 3

(Due at 11 AM on February 16.)

Question 1. How many different letter arrangements can be formed from the letters PEPPER?

For example, Here is one way of arranging the letters: PPREPE, and here's another: PPEREP – distinct from the first. Note that some permutations of the letters will result in the same letter arrangement (i.e. will "read the same"). We are interested in the number of distinct arrangements.

Question 2.

(a) From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?

(b) What if 2 of the men are feuding and refuse to serve on the committee together?

(*Hint:* It may be easier to count, first, out of all possible groups of the 3 men - of which there should be $\binom{7}{3}$ - the number of groups that actually *do* contain the 2 feuding men.)

Question 3.

(a) An urn contains n balls, one of which is special. If k of these balls are withdrawn one at a time, with each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

(b) An urn contains n balls, out of which exactly m are red. We select k of the balls at random, without replacement (i.e., selected balls are not put back into the urn before the next selection). What is the probability that i of the selected balls are red?

Question 4. A poker hand consists of 5 cards. If the cards have distinct consecutive values and are not all of the same suit, we say that the hand is a straight. For instance, a hand consisting of the five of spades, six of spades, seven of spades, eight of spades, and nine of hearts is a straight. What is the probability that one is dealt a straight?

Question 5. It is known that screws produced by a certain company will be defective with probability .01, independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Question 6. Consider the (completely fictitious, of course) situation that you attend, completely unprepared, a multiple-choice exam. It consists of 10 questions, and each question has four alternatives (of which only one is correct). You will pass the exam if you answer six or more questions correctly. You decide to answer each of the questions in a random way, in such a way that the answer of one question is not affected by the answers of the others. What is the probability that you will pass?

Question 7. A three-sided die.
The newest invention of one of your CIS-2033 classmates is a three-sided die. On any roll of this die, the result is 1 with probability 1/2, 2 with probability 1/4, and 3 with probability 1/4.

Consider a sequence of six independent rolls of this die.

1. Find the probability that exactly two of the rolls result in a 3.

   (a) $\binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4$

   (b) $\binom{6}{2}\left(\frac{1}{4}\right)^2$

   (c) $\binom{6}{2}\left(\frac{1}{4}\right)^2\binom{6}{4}\left(\frac{3}{4}\right)^4$

   (d) $\binom{6}{2}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^2$

2. Given that exactly two of the six rolls resulted in a 1, find the probability that the first roll resulted in a 1.

3. We are told that exactly three of the rolls resulted in a 1 and exactly three rolls resulted in a 2. Given this information, find the probability that the six rolls resulted in the sequence (1, 2, 1, 2, 1, 2).

Question 8. *Extra Credit of 2 points. To receive this credit, explain your solution to your instructor.*

Eight rooks are placed in distinct squares of an 8-by-8 chessboard, with all possible placements being equally likely. Find the probability that all the rooks are safe from one another, i.e., that there is no row or column with more than one rook.

Appendix

Solved Problems

Question 9. Letter Arrangements.

(a) How many different letter arrangements can be formed from the letters HIT?

(b) How many different letter arrangements can be formed from the letters $MISS$?

(a) There are $3! = 6$ permutations of the letters HIT. Let's list all of them:

$$HIT \quad HTI \quad TIH \quad THI \quad IHT \quad ITH$$

(b) We first note that there are $4! = 24$ permutations of the letters MIS_1S_2 when the 2 S's are distinguished from each other. However, consider any one of these permutations, for example S_1IMS_2. If we now permute the S's amongst themselves, then the resulting arrangement would still be $SIMS = S_2IMS_1$. Hence there are a total of $4!/2! = 12$ possible letter arrangements of the letters $MISS$. Let's list them all:

$$\begin{array}{cccccc} MISS & MSSI & SSMI & IMSS & ISSM & SSIM \\ SMIS & SIMS & SMSI & SISM & MSIS & ISMS \end{array}$$

Question 10. If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two black?

One way of appoaching this problem: If we regard the order in which the balls are selected as being relevant, then the sample space consists of $11 \cdot 10 \cdot 9 = 990$ outcomes. Furthermore, there are $6 \cdot 5 \cdot 4 = 120$ outcomes in which the first ball selected is white and the other two are black; $5 \cdot 6 \cdot 4 = 120$ outcomes in which the first is black, the second is white, and the third is black; and $5 \cdot 4 \cdot 6 = 120$ in which the first two are black and the third is white. Hence, assuming that "randomly drawn? means that each outcome in the sample space is equally likely to occur, we see that the desired probability is

$$\frac{120 + 120 + 120}{990} = \frac{4}{11}$$

Another way of appoaching this problem: This problem could also have been solved by regarding the outcome of the experiment as the unordered set of drawn balls. From this point of view, there are $\binom{11}{3} = 165$ outcomes in the sample space. Now, each set of 3 balls corresponds to 3! outcomes when the order of selection is noted. As a result, if all outcomes are assumed equally likely when the order of selection is noted, then it follows that they remain equally likely when the outcome is taken to be the unordered set of selected balls. Hence, using the latter representation of the experiment, we see that the desired probability is

$$\frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{4}{11}$$

which, of course, agrees with the answer obtained previously.

Question 11. A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Because each of the $\binom{15}{5}$ possible committees is equally likely to be selected, the desired probability is

$$\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}$$