Homework 8

(Due Thursday, April 20)

Question 1. (*Probability bounds.*) Let $X_1, X_2, \ldots X_{10}$ be independent random variables, uniformly distributed over the unit interval $[0, 1]$.

(a) Find an upper bound on $P(X_1 + \ldots + X_{10} \geq 7)$ using the Markov inequality.

(b) Find an upper bound on $P(X_1 + \ldots + X_{10} \geq 7)$ using the Chebyshev inequality.

(c) Estimate $P(X_1 + \ldots + X_{10} \geq 7)$ using the central limit theorem.

Question 2. (*Using the CLT to estimate the probability of a wrong decision.*) Before starting to play the roulette in a casino, you want to look for biases that you can exploit. You therefore watch 100 rounds that result in a number between 1 and 36, and count the number of rounds for which the result is odd. If the count exceeds 55, you decide that the roulette is not fair. Assuming that the roulette is fair, find an approximation for the probability that that you will make the wrong decision.

Question 3. (*Using the CLT.*) A factory produces X_n gadgets on day n, where the X_n are independent and identically distributed random variables, with mean 5 and variance 9.

(a) Find an approximation to the probability that the total number of gadgets produced in 100 days is less than 440.

(b) Find (approximately) the largest value of n such that

$$P(X_1 + \ldots + X_n \geq 200 + 5n) \leq 0.05$$

(c) Let N be the first day on which the total number of gadgets produced exceeds 1000. Calculate an approximation to the probability that $N \geq 220$.

Question 4. (*Airline overbooking.*) For any given flight, an airline tries to sell as many tickets as possible. Suppose that on average, 10% of ticket holders fail to show up, all independent of one another. Knowing this, an airline will sell more tickets than there are seats available (i.e., overbook the flight) and hope that there is a sufficient number of ticket holders who do not show up to compensate for its overbooking. [1]

Using the Central Limit Theorem, determine n, the maximum number of tickets an airline should sell on a flight with 300 seats so that it can be approximately 99% confident that all ticket holders who do show up will be able to board the plane. Use the 1/2-correction we discussed in class in your calculations. [2]

Hint: You may have to solve numerically a quadratic equation.

[1] and, of course, you know what can happen when enough people actually do show up.

[2] which is also known as de Moivre-Laplace 1/2-correction.