Homework 7

Question 1. The joint distribution of discrete random variables X and Y is given in the following table.

Joint Probabilities $P(X = a, Y = b)$

b	a 0	1	2
0	0	1/4	0
1	1/4	0	1/4
2	0	1/4	0

(a) Compute $\mathbf{E}[X + Y]$ and $\mathbf{E}[XY]$.

(b) Compute the marginal distributions of X and Y.

(c) Are X and Y independent? Why?

(d) For the random variables given in this problem, it true that

$$\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]?$$

(e) Is it always true that $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$?

(f) Compute the covariance between X and Y, $\mathrm{cov}(X, Y)$.

(g) Compute the correlation coefficient between X and Y, $\rho(X, Y)$.

Question 2. To investigate the relation between hair color and eye color, the hair color and eye color of 5383 persons was recorded. The data are given in the following table:

Eye color	Hair color		
	Fair/red	Medium	Dark/black
Light	1168	825	305
Dark	573	1312	1200

Eye color is encoded by the values 1 (Light) and 2 (Dark), and hair color by 1 (Fair/red), 2 (Medium), and 3 (Dark/black). By dividing the numbers in the table by 5383, the table is turned into a joint probability distribution for random variables X (hair color) taking values 1 to 3 and Y (eye color) taking values 1 and 2.

As a result, the joint probability distribution of X and Y, as well as the marinal distributions of X and Y, are given by

b	a			$P(Y = b)$
	1	2	3	
1	0.217	0.153	0.057	0.427
2	0.106	0.244	0.223	0.573
$P(X = a)$	0.323	0.397	0.280	1

(a) Find out whether X and Y are dependent or independent.

(b) Compute $\mathrm{cov}(X, Y)$. Are X and Y positively correlated, negatively correlated, or uncorrelated?

(c) Compute the correlation coefficient between X and Y.

Question 3. The joint probability distribution of two discrete random variables X and Y is partly given in the following table.

b	0	a 1	2	$P(Y = b)$
-1	...	...	...	$1/2$
1	...	$1/2$	...	$1/2$
$P(X = a)$	$1/6$	$2/3$	$1/6$	1

(a) Complete the table.

(b) Are X and Y dependent or independent? Why?

(c) Determine $\mathbf{E}[XY]$.

(d) Are X and Y correlated or uncorrelated? Why?

(e) Determine $\mathrm{var}(X + Y)$.

(f) Determine $\mathrm{var}(X - Y)$.

Question 4. The joint probability distribution of two discrete random variables X and Y is given in the following table. Take ϵ to be fixed between $-1/4$ and $1/4$:

a	b 0	1	$p_X(a)$
0	$1/4 - \varepsilon$	$1/4 + \varepsilon$	$1/2$
1	$1/4 + \varepsilon$	$1/4 - \varepsilon$	$1/2$
$p_Y(b)$	$1/2$	$1/2$	1

(a) Take $\epsilon = 1/8$ and compute $\operatorname{cov}(X, Y)$.

(b) Take $\epsilon = 1/8$ and compute $\rho(X, Y)$.

(c) For which values of ϵ is $\rho(X, Y)$ equal to -1, 0, 1?