CIS 2033 Lectures 3 and 4, Spring 2017[1]

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[1] Supplemental reading from Dekking's textbook: Chapter 3.

In the first two lectures, we introduced probabilities as a way of describing our beliefs about the likelihood that a given event will occur. But our beliefs will in general depend on the information that we have. Taking into account new information leads us to consider the so-called conditional probabilities – revised probabilities that take into account this new information.

Conditional probabilities are very useful whenever we want to break up a model into simpler pieces using a divide and conquer strategy. This is done using tools that we develop in this lecture and which we will keep applying throughout this course in different guises. They are also the foundation of the field of inference, and we will see how they arise in that context as well.

Next, we will consider a special case where one event does not convey useful information about another, a situation that we call *independence*. Independence usually describes a situation where the occurrence or non-occurrence of different events is determined by factors that are completely unrelated. Independence is what allows us to build complex models out of simple ones. This is because it is often the case that a complex system is made up of several components that are affected by unrelated, that is, independent sources of randomness.

And so with the tools to be developed over the next couple of lectures, we will be ready to calculate probabilities in fairly complex probabilistic models.

Introduction

Suppose I look at the registry of residents of my town and pick a person at random. What is the probability that this person is under 18 years of age? Let's say I find that the answer is about 25%.

Suppose now that I tell you that this person is married. Will you give the same answer? Of course not. The probability of being less than 18 years old is now much smaller.

What happened here? We started with some initial probabilities that reflect what we know or believe about the world. But we then acquired some additional knowledge, some new evidence– for example, about this person's family situation. This new knowledge should cause our beliefs to change, and the original probabilities must be replaced with new probabilities that take into account the new information. These revised probabilities are what we call conditional probabilities. And this is the subject of this lecture.

We will start with a formal definition of conditional probabilities together with the motivation behind this particular definition. We will then proceed to develop three tools that rely on conditional probabilities, including the Bayes rule, which provides a systematic way for incorporating new evidence into a probability model. The three tools that we introduce in this lecture involve very simple and elementary mathematical formulas, yet they encapsulate some very powerful ideas.

It is not an exaggeration to say that much of this class will revolve around the repeated application of variations of these three tools to increasingly complicated situations. In particular, the Bayes rule is the foundation for the field of inference. It is a guide on how to process data and make inferences about unobserved quantities or phenomena. As such, it is a tool that is used all the time, all over science and engineering.

Conditional Probabilities

Conditional probabilities are probabilities associated with a revised model that takes into account some additional information about the outcome of a probabilistic experiment. The question is how to carry out this revision of our model. We will give a mathematical definition of conditional probabilities, but first let us motivate this definition by examining a simple concrete example.

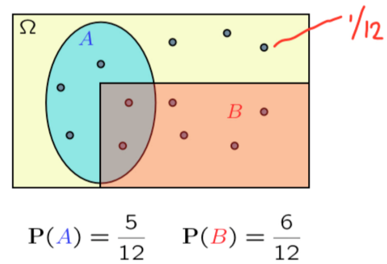


Figure 1: **Equally Likely Outcomes.** In this model with equally likely outcomes, the probabilities associated with sets A and B are $5/12$ and $1/2$, respectively.

EXAMPLE 1. Consider a probability model with 12 equally likely possible outcomes – each outcome has probability equal to $1/12$. We will focus on two particular events, event A and B, two subsets of the sample space. Event A has five elements, so its probability is $5/12$, and event B has six elements, so it has probability $6/12$.

Suppose now that someone tells you that event B has occurred, but tells you nothing more about the outcome. *How should the model change?* First, those outcomes that are outside event B are no longer possible. So we can either eliminate them, as was done in Figure 2, or we might keep them in the picture but assign them 0 probability, so that they cannot occur.

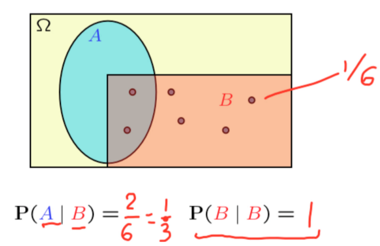


Figure 2: **Event** B **has occurred.** If is known that B has occurred, we now update our model by assigning 0 probability to the events that are outside of B.

How about the outcomes inside the event B? We're told that one of these 6 outomes inside the event B has occurred. Now these 6 outcomes were equally likely in the original model, and there is no reason to change their relative probabilities. So they should remain equally likely in revised model as well, so each one of them should have now probability $1/6$ since there's 6 of them. And this is our revised model, the conditional probability law:

0 probability to outcomes outside B, and probability 1/6 to each one of the outcomes that is inside the event B.

Let us write now this down mathematically. We will use the notation $P(A \mid B)$ to describe the conditional probability of an event A given that some other event B is known to have occurred. We read this expression as probability of A given B.

$\mathbf{P}(A \mid B) =$ "probability of A, given that B occurred"

So what are these conditional probabilities in our example? In the new model, where these outcomes are equally likely, we know that event A can occur in two different ways. Each one of them has probability $1/6$. So the probability of event A is $2/6$ (i.e. $1/3$).

How about event B. Well, B consists of 6 possible outcomes each with probability $1/6$. So event B in this revised model should have probability equal to 1. Of course, this is just saying the obvious. Given that we already know that B has occurred, the probability that B occurs in this new model should be equal to 1.

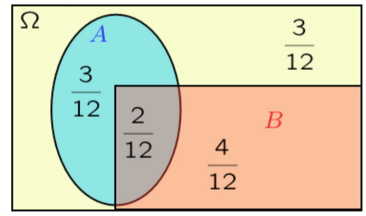


Figure 3: **Sample Space for Example 2.** The probability distribution here is such that each individual outcome is not necessarily equally likely.

EXAMPLE 2. In this example, the sample space does not consist of equally likely outcomes, but instead we're given the probabilities of different pieces of the sample space. Notice here that the probabilities are consistent with what was used in the original example. So this part of A that lies outside B has probability $3/12$, but in this case I'm not telling you how that probability is made up. I'm not telling you that it consists of 3 equally likely outcomes. So all I'm telling you is that the collective probability in the blue region in Figure 3 is $3/12$.

The total probability of A is, again, $5/12$ as before. The total probability of B is $2/12 + 4/12 = 6/12$, exactly as before. So it's a sort of similar situation as before. How should we revise our probabilities and create – construct – conditional probabilities once we are told that event B has occurred?

First, the relation $P(B \mid B) = 1$ should remain true: once we are told that B has occurred, then B is certain to occur, so it should have conditional probability equal to 1.

How about the conditional probability $P(A \mid B)$? Well, we can reason as follows. In the original model, and if we just look inside event B, those outcomes that make event A happen had a collective probability which was $1/3$ of the total probability assigned to B (because $2/12$ is one-third of $2/12 + 4/12$). So out of the overall probability assigned to B, $1/3$ of that probability corresponds to outcomes in which event A is happening. Therefore, if I tell you that B has occurred, I should assign probability equal to $1/3$ that event A is also going to happen. So that $P(A \mid B)$ should be equal to $1/3$.

Check Your Understanding: Are the following statements true of false?

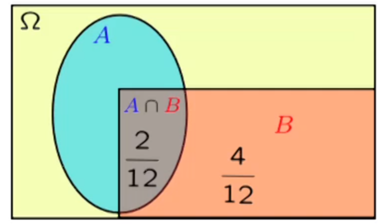
1. If Ω is finite and we have a discrete uniform probability law, and if $B \neq \emptyset$, then the conditional probability law on B, given that B occurred, is also discrete uniform.

2. If Ω is finite and we have a discrete uniform probability law, and if $B \neq \emptyset$, then the conditional probability law on Ω, given that B occurred, is also discrete uniform.

(1) True, because the outcomes inside B maintain the same relative proportions as in the original probability law.

(2) False. Outcomes in Ω that are outside B have zero conditional probability, so it cannot be the case that all outcomes in Ω have the same conditional probability.

Definition of Conditional Probability



By now, we should be satisfied that this approach is a reasonable way of constructing conditional probabilities. But now let us translate our reasoning into a formula. So we wish to come up with a formula that gives us the conditional probability of an event given another event.

The particular formula that captures our way of thinking, as motivated before, is the following. Out of the total probability assigned to B, we ask the question, which fraction of that probability is assigned to outcomes under which event A also happens? So we are living inside event B, but within that event, we look at those outcomes for which event A also happens – which is the intersection of A and B – and we ask: out of the total probability of B, what fraction of that probability is allocated to that intersection of A with B?

So the following formula captures our intuition of what we did before to construct conditional probabilities in examples 1 and 2.

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}$$

defined only when $\mathbf{P}(B) > 0$

Let us check that the definition indeed does what it's supposed to do. In this example, the probability of the intersection was 2/12 and the total probability of B was 6/12, which gives us 1/3, which is the answer that we had gotten intuitively a little earlier.

As a side point, let me also make a comment that this definition of conditional probabilities makes sense only if we do not attempt to divide by zero, i.e. only if the event B on which we're conditioning has positive probability. If event B has 0 probability, then conditional probabilities given B will be left undefined.

And one final comment. This is a definition. It's not a theorem. What does that mean? It means that there is no question whether this equality is correct or not. It's just a definition. There's no issue of correctness. The earlier argument that we gave was just a motivation of the definition. We tried to figure out what the definition should be if we want to have a certain intuitive and meaningful interpretation of the conditional probabilities. Let us now continue with a simple example.

Example 3. This is a example where we want to just apply the formula for conditional probabilities and see what we get. The example involves a four-sided die, which we roll twice, and we record the first roll, and the second roll. So there are 16 possible outcomes. We assume, to keep things simple, that each one of those 16 possible outcomes has the same probability, so each outcome has the probability 1/16.

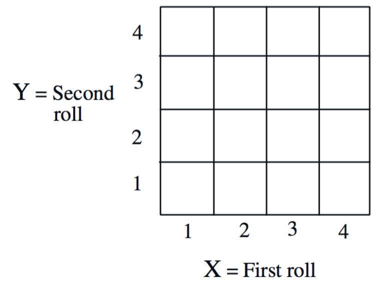


Figure 4: **Tetrahedral die, again.**

Let us consider now a particular event B on which we're going to condition. This is the event under which the smaller of the two die rolls is equal to 2, which means that one of the dice must have resulted in two, and the other die has resulted in something which is 2 or larger.

Let B be the event: $\min(X, Y) = 2$

Event B can happen in multiple ways: at 2, 2, or 2, 3, or 2, 4; then a 3, 2 and a 4, 2. All of these are outcomes in which one of the dice has a value equal to 2, and the other die is at least as large.

So we condition on event B. This results in a conditional model where each one of those five outcomes (pink squares in Figure 5) are equally likely since they used to be equally likely in the original model.

Now let's look at another quantity, let's call it M, the maximum of the two die rolls – that is, the largest of the results.

Let $M = \max(X, Y)$

We now would like to calculate the conditional probability that the maximum is equal to 1 given that the minimum is equal to 2:

$$P(M = 1 \mid B = 2)$$

Well, such outcome cannot happen! If I tell you that the smaller number is 2, then the larger number cannot be equal to 1, so this outcome is impossible, and therefore this conditional probability is equal to 0.

$$P(M = 1 \mid B = 2) = 0$$

Let's do something a little more interesting.

Let us now look at the conditional probability that the maximum is equal to 3, given the information that event B has occurred. It's best to draw a picture and see what that event corresponds to. M is equal to 3 – the maximum is equal to 3 – if one of the dice resulted in a 3, and the other die resulted in something that's 3 or less. So this event here corresponds to the blue region in this diagram.

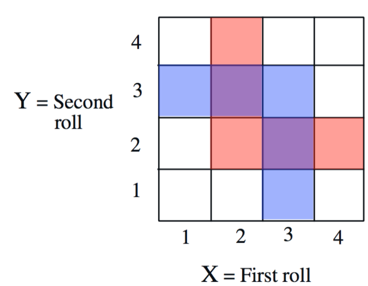


Figure 5: **Tetrahedral die.** Pink squares correspond to event B. Blue squares – to event A.

Now let us try to calculate the conditional probability by just following the definition. The conditional probability of one event given another is the probability that both of the two events occur, divided by the probability of the conditioning event. That is, out of the total probability in the conditioning event, we ask, what fraction of that probability is assigned to outcomes in which the event of interest is also happening?

So what is this event? The maximum is equal to 3, which is the blue event. And simultaneously, the pink event is happening. These two events intersect only in two places. Figure 5 shows the intersection of the two events - these are the squares that are shaded both blue and pink. And the probability of that intersection is 2 out of 16, since there are 16 outcomes and that event happens only with two particular outcomes. So this gives us 2/16 in the numerator.

How about the denominator? Event B consists of a total of five possible outcomes. Each one has probability 1/16, so this is 5/16. Thus the final answer is 2/16 divided by 5/16, or 2/5.

We could have gotten that same answer in a simple and perhaps more intuitive way. In the original model, all outcomes were equally likely. Therefore, in the conditional model, the five outcomes that belong to B should also be equally likely. Out of those five, there are two of them that make the event of interest to occur. So given that we live in B, there's two ways out of five that the event of interest will materialize. So the event of interest has conditional probability equal to 2/5.

Check Your Understanding: Let the sample space be the unit square, $\Omega = [0,1]^2$, and and let the probability of a set be the area of the set. Let A be the set of points $(x, y) \in [0,1]^2$ for which $y \leq x$. Let B be the set of points for which $x \leq 1/2$. Find $P(A \mid B)$.

theanswerisonefourth

Conditional Probabilities Obey the Same Axioms

(Skim this section on first reading.)

We want to emphasize an important point. Conditional probabilities are just the same as ordinary probabilities applied to a different situation. They do not taste or smell or behave any differently than ordinary probabilities. What do I mean by that?

$\mathbf{P}(A \mid B) \geq 0 \qquad \text{assuming } \mathbf{P}(B) > 0$

Figure 6: Conditional probabilities also satisfy the familiar axioms. This one was axiom 1.

Axiom 1. I mean that they satisfy the usual probability axioms. For example, ordinary probabilities must be non-negative. Is this true for conditional probabilities? Of course it is true, because conditional probabilities are defined as a ratio of two probabilities. Probabilities are non-negative. So the ratio will also be non-negative, of course as long as it is well-defined. And here we need to remember that we only talk about conditional probabilities when we condition on an event that itself has positive probability.

Axiom 2. Let's check it out the second axiom - what is the probability of the entire sample space, given the event B? By definition, the conditional probability is the probability of the intersection of the two events involved divided by the probability of the conditioning event. Now, what is the intersection of Ω with B? B is a subset of Ω. So when we intersect the two sets, we're left just with B itself.

So the numerator becomes the probability of B. We're dividing by the probability of B, and so the answer is equal to 1. So indeed, the sample space has unit probability, even under the conditional model.

Now, remember that when we condition on an event B, we could still work with the original sample space. However, outcomes that do not belong to B are considered impossible, so we might as well think of B itself as being our sample space.

If we proceed like that and think now of B as being our new sample space, what is the probability of this new sample space in the conditional model? Let's apply the definition once more. It's the probability of the intersection of the two events involved, B intersection B, divided by the probability of the conditioning event.

What is the numerator? The intersection of B with itself is just B, so the numerator is the probability of B. We're dividing by the probability of B. So the answer is, again, 1.

$$\mathbf{P}(\Omega \mid B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

$$\mathbf{P}(B \mid B) = \frac{P(B \cap B)}{P(B)} = 1$$

Figure 7: And this one was axiom 2.

Axiom 3. Finally, we need to check the additivity axiom. Recall what the additivity axiom says. If we have two events, two subsets of the sample space that are disjoint, then the probability of their union is equal to the sum of their individual probabilities. Is this going to be the case if we now condition on a certain event?

What we want to prove is the following statement. If we take two events that are disjoint (they have empty intersection) then the probability of the union is the sum of their individual probabilities, but where now the probabilities that we're employing are the conditional probabilities, given the event B. That is, we want to show that

$$\text{If } A \cap C = \varnothing \text{ then } P(A \cup C \mid B) = P(A \mid B) + P(C \mid B)$$

So let us verify whether this relation is correct or not.

By definition,

$$P(A \cup C \mid B) = \frac{P((A \cup C) \cap B)}{P(B)} \tag{1}$$

Now, let's look at this quantity, what is it?

We take the union $A \cup C$, and then intersect it with B. The result consists of the two gray pieces in Figure 8 .

So we rewrite expression (1) as follows:

$$P(A \cup C \mid B) = \frac{P\left((A \cup C) \cap B\right)}{P(B)}$$
$$= \frac{P\left((A \cap B) \cup (C \cap B)\right)}{P(B)}$$

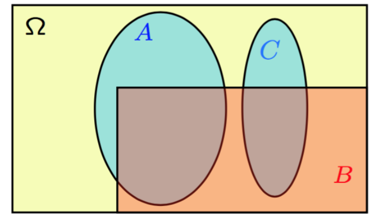


Figure 8: And this picture illustrates axiom 3 for conditional probabilities.

OK, now here's an observation. We assumed that the events A and C were disjoint. Then the piece of A that also belongs in B, therefore, is disjoint from the piece of C that also belongs to B.

Since $A \cap B$ and $C \cap B$ are disjoint, the probability of their union has to be equal to the sum of their individual probabilities. So here we're using the additivity axiom on the original probabilities to break this probability up into two pieces.

Thus we continue the previous expression as follows:

$$\frac{P\left((A \cap B) \cup (C \cap B)\right)}{P(B)} = \frac{P(A \cap B) + P(C \cap B)}{P(B)}$$
$$= \frac{P(A \cap B)}{P(B)} + \frac{P(C \cap B)}{P(B)}$$

And now we observe that here we have ratios of probabilities of intersections by the probability of B. So we see that the first term is just the conditional probability $P(A \mid B)$, using the definition of conditional probabilities. Similarly, the second part is just $P(C \mid B)$.

So we have indeed checked that this additivity property is true for the case of conditional probabilities when we consider two disjoint events.

Now, we could repeat the same derivation and verify that it is also true for the case of a disjoint union of finitely many events, or even for countably many disjoint events. We're not proving it, but the argument is exactly the same as for the case of two events.

To conclude, conditional probabilities do satisfy all of the standard axioms of probability theory. So conditional probabilities are just like ordinary probabilities.

This actually has a very important implication. Since conditional probabilities satisfy all of the probability axioms, any formula or theorem that we ever derive for ordinary probabilities will remain true for conditional probabilities as well.

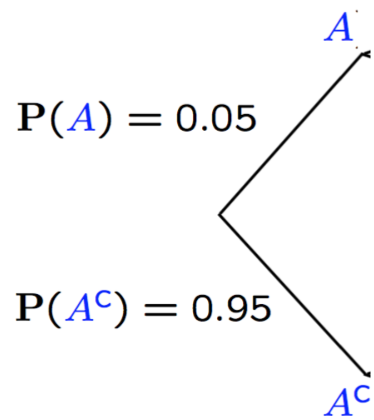
Models Based on Conditional Probabilities

Let us now examine what conditional probabilities are good for. We have already discussed that they are used to revise a model when we get new information, but there is another way in which they arise. We can use conditional probabilities to build a multi-stage model of a probabilistic experiment. We will illustrate this through an example involving the detection of an object up in the sky by a radar. We will keep our example very simple. On the other hand, it turns out to have all the basic elements of a real-world model.

So, we are looking up in the sky, and either there's an airplane flying up there or not. Let us call Event A the event that an airplane is indeed flying up there:

$$\text{Event } A: \text{ Airplane is flying above,}$$

and we have two possibilities: either event A occurs, or the complement of A occurs, in which case nothing is flying up there. At this point, we can also assign some probabilities to these two possibilities. Let us say that through prior experience, perhaps, or some other knowledge, we know that the probability that something is indeed flying up there is 5% and with probability 95% nothing is flying.



Now, we also have a radar that looks up into the sky. Let event B be that the radar detects something:

$$\text{Event } B: \text{ Something registers on the radar screen}$$

There are two things that can happen: either something registers on the radar screen or nothing registers. Of course, if it's a good radar, probably event B will tend to go together with event A. But it's also possible that the radar will make some mistakes.

And so we have various possibilities. If there's a plane up there, it's possible that the radar will detect it, in which case event B will also happen. But it's also conceivable that the radar will not detect it, in which case we have a so-called miss. So "miss" happens if a plane is flying up there, but the radar does not detect it.

Another possibility is that nothing is flying up there, but the radar does detect something, and this is a situation that's called a *false alarm*. Finally, there's the possibility that nothing is flying up there, and the radar did not see anything either.

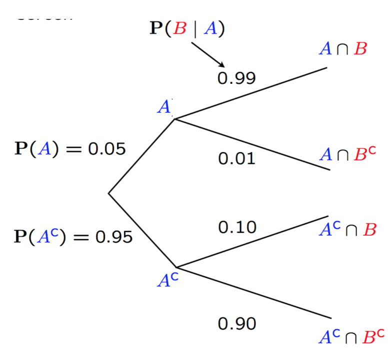


Figure 9: Event $A \cap B^c$ is called a "*miss*". Event $A^c \cap B$ is called a "*false alarm*". Pause here a second and make sure this terminology makes sense to you.

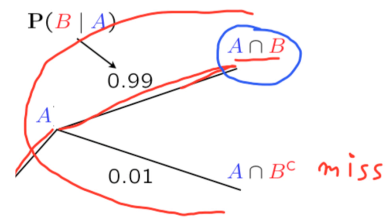


Figure 10: Part of the tree where Event A has occurred. This is the upper part of Figure 9 on the previous page.

Now, let us focus on one particular situation. Suppose that event A has occurred. In this universe where event A has occurred, there are two possibilities, and we can assign probabilities to these two possibilities. Let's say that if something is flying up there, our radar will find it with probability 99%, but will also miss it with probability 1%.

What's the meaning of this number, 99%? Well, this is a probability that applies to a situation where an airplane is up there. So it is really a conditional probability: It's the conditional probability that the radar will detect the plane, given that the plane is already flying up there. And similarly, this 1% can be thought of as the conditional probability that the complement of B occurs, so the radar doesn't see anything, given that there is a plane up in the sky.

We can assign similar probabilities under the other scenario. If there is no plane, there is a probability that there will be a false alarm, and there is a probability that the radar will not see anything.

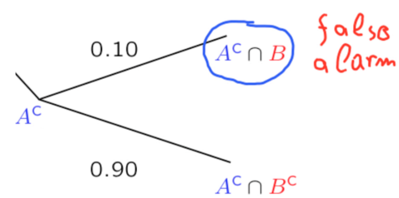


Figure 11: Part of the tree where event A^c has occurred. This is the lower part of Figure 9 on the previous page.

These four numbers – $P(B \mid A), P(B^c \mid A), P(B \mid A^c), P(B^c \mid A^c)$ – are, in essence, the specs of our radar. They describe how the radar behaves in a world in which an airplane has been placed in the sky, and some other numbers that describe how the radar behaves in a world where nothing is flying up in the sky.

So we have described various probabilistic properties of our model, but is it a complete model? Can we calculate anything that we might wish to calculate?

For example, can we calculate the probability that both A and B occur? This corresponds to reaching the upper right leaf of the tree in Figure 9 or Figure 10, the one labeled $A \cap B$. How can we calculate it?

Well, let us remember the definition of conditional probabilities. The conditional probability of an event given another event is the probability of their intersection divided by the probability of the conditioning event:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

Or, rearranging,

$$P(A \cap B) = P(B \cap A) = P(B \mid A)P(A)$$

So, in words, the probability that A and B occur is equal to the probability that A occurs times the conditional probability that B occurs given that A occurred. And in our example, this is

$$P(B \mid A)P(A) = 0.99 \times 0.05$$

Before we go on, let me take a moment to rewrite this for nicer looks. (Heads up: It helps on quizzes to know these nside out.) We have:

$$\boxed{\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} \qquad \mathbf{P}(B \mid A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)}}$$

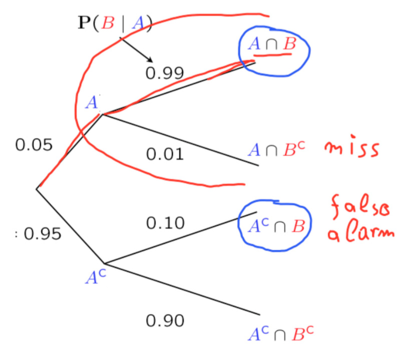


Figure 12: The branches leading to event $A \cap B$ are traced in red.

So we can calculate the probability of event $A \cap B$ by multiplying probabilities and conditional probabilities along the path in the tree diagram that leads us to the top right leaf.

And we can do the same for any other leaf in this diagram. So for example, the probability that $A^c \cap B^c$ happens is going to be the probability event A^c times the conditional probability $P(B^c \mid A^c)$.

So you could trace these two branches too to get the probability $0.95 \times 0.90 = 0.855$ of event $A^c \cap B^c$ occurring.

How about a different question? What is the probability, the total probability, that the radar sees something? Let us try to identify this event. The radar can see something under two scenarios:

1. A plane is up in the sky and the radar sees it.
2. Nothing is up in the sky, but the radar thinks that it sees something.

These two possibilities together make up the event B.

And so to calculate the probability of B, we need to add the probabilities of these two events.

For the first event, we already calculated its probability. It's 0.05×0.99. For the second possibility, we need to do a similar calculation. The probability that this second event occurs is equal to 0.95 times the conditional probability of B occurring under the scenario where A^c has occurred, and this is 0.10. So we have $P(A^c \cap B) = 0.95 \times 0.10$.

We add those two numbers together, the answer turns out to be:

$$\begin{aligned} P(B) &= P(A)P(B \mid A) + P(A^c)P(B \mid A^c) \\ &= 0.05 \times 0.99 + 0.95 \times 0.10 \\ &= 0.1445 \end{aligned}$$

Finally, the last question, which is perhaps the most interesting one. Suppose that the radar registered something. What is the probability that there is an airplane up there? How do we do this calculation? Well, we can start from the definition of the conditional probability of A given B, and note that we already have in our hands both the numerator and the denominator.

$$\begin{aligned} P(A \mid B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.05 \times 0.99}{0.05 \times 0.99 + 0.95 \times 0.10} \\ &= \frac{0.0495}{0.1445} \\ &= 0.3426 \end{aligned}$$

So there is a 34% probability that an airplane is there given that the radar has seen or thinks that it sees something.

Now, the numerical value of this answer is somewhat interesting because it's pretty small. Even though we have a very good radar that tells us the right thing 99% of the time under one scenario and 90% under the other scenario. Despite that, given that the radar has seen something, this is not really convincing or compelling evidence that there is an airplane up there. The probability that there's an airplane up there is only 34% in a situation where the radar thinks that it has seen something.

In the next few segments, we are going to revisit these three calculations and see how they can generalize. In fact, a large part of what is to happen in the remainder of this class will be elaboration on these three ideas. They are three types of calculations that will show up over and over, of course, in more complicated forms, but the basic ideas are essentially captured in this simple example.

Multiplication Rule

As promised, we will now start developing generalizations of the different calculations that we carried out in the context of the radar example. The first kind of calculation that we carried out goes under the name of the multiplication rule. And it goes as follows. Our starting point is the definition of conditional probabilities.

The conditional probability of A given another event, B, is the probability that both events have occurred divided by the probability of the conditioning event.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

We now take the denominator term and send it to the other side of this equality to obtain this relation

$$P(A \cap B) = P(A \mid B)P(B)$$

which we can interpret as follows. The probability that two events occur is equal to the probability that a first event occurs, event B in this case, times the conditional probability that the second event, event A, occurs, given that event B has occurred.

Now, out of the two events, A and B, we're of course free to choose which one we call the first event and which one we call the second event. So the probability of the two events happening is also equal to an expression of this form

$$P(A \cap B) = P(B \mid A)P(A)$$

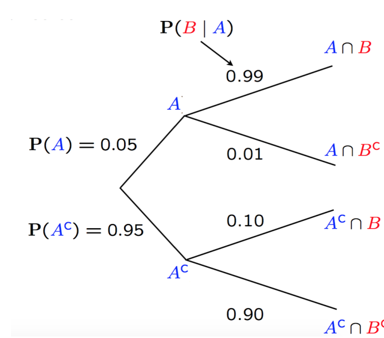


Figure 13: Same tree diagram as before.

We used this formula in the context of a tree diagram, and we used it to calculate the probability of the $A \cap B$ leaf of the tree shown in Figure 13 by multiplying the probability of taking the branch labeled $P(A)$ and multiplying that probability by the the conditional probability of taking the branch going from A to $A \cap B$, which is the probability that event B also occurs given that event A has occurred, i.e. $P(B \mid A)$.

How do we generalize this calculation? Consider a situation in which the experiment has an additional third stage that has to do with another event, C, that may or may not occur.

For example, if we have arrived at the node labeled $A \cap B$, A and B have both occurred. Next, if C also occurs, then we reach the leaf of the tree that is labeled $A \cap B \cap C$. Or there could be other scenarios. For example, it could be the case that A did not occur. Then event B occurred, and finally, event C did not occur, in which case we end up at the leaf labeled $A^c \cap B \cap C^c$.

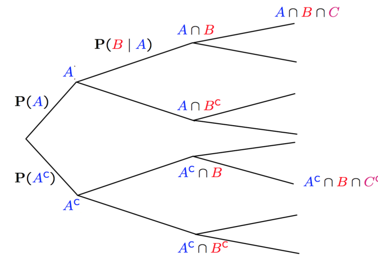


Figure 14: Here our experiment has the third stage that has to do with another event C.

What is the probability of this scenario happening? Let us try to do a calculation similar to the one that we used for the case of two events. However, we need to deal here with three events. What should we do? Well, we look at the intersection of these three events and think of it as the intersection of a composite event, $A^c \cap B$, then intersected with the event C^c.

$$P(A^c \cap B \cap C^c) = P\left((A^c \cap B) \cap C^c\right)$$

Clearly, you can form the intersection of three events by first taking the intersection of two of them and then intersecting with a third. After we group things this way, we're dealing with the probability of two events happening, the composite event$(A^c \cap B)$ and the ordinary event C^c. And the probability of two events happening is equal to the probability that the first event happens, and then the probability that the second event happens, given that the first one has happened.

$$P\left((A^c \cap B) \cap C^c\right) = P(A^c \cap B)\ P(C^c \mid A^c \cap B)$$

Can we simplify this even further? Yes. The first term is the probability of two events happening. So it can be simplified further as the probability that A^c occurs times the conditional probability that B occurs, given that A^c has occurred. And then we carry over the last term exactly the way it is.

$$\begin{aligned} P(A^c \cap B \cap C^c) &= P(A^c \cap B)\ P(C^c \mid A^c \cap B) \\ &= P(A^c)\ P(B \mid A)\ P(C^c \mid A^c \cap B) \end{aligned}$$

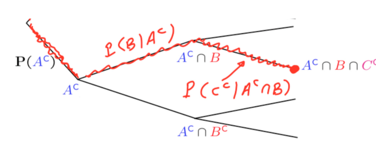


Figure 15: Lower part of the same tree as in Figure 14. We multiply the probabilities associated with the branches traced in red to get the probability of the event $A^c \cap B \cap C^c$.

The conclusion is that we can calculate the probability of the leaf labeled $A^c \cap B \cap C^c$ by multiplying the probabilities of the red branches shown in figure 15.

At this point, you can probably see that such a formula should also be valid for the case of more than three events. The probability that a bunch of events all occur should be the probability of the first event times a number of factors, each corresponding to a branch in a tree of this kind.

In particular, the probability that events $A_1, A_2, \ldots A_n$ all occur is going to be

$$P(A_1 \cap A_2 \cap \ldots A_n) = P(A_1) \prod_{i=2}^{n} P(A_i \mid A_1 \cap \ldots \cap A_{i-1}) \qquad (2)$$

This is the most general version of the multiplication rule and allows you to calculate the probability of several events happening by multiplying probabilities and conditional probabilities.

Check Your Understanding: Are the following statements true or false? (Assume that all conditioning events have positive probability.)

1. $P(A \cap B \cap C^c) = P(A \cap B)P(C^c \mid A \cap B)$
2. $P(A \cap B \cap C^c) = P(A)P(C^c \mid A)P(B \mid A \cap C^c)$
3. $P(A \cap B \cap C^c) = P(A)P(C^c \cap A \mid A)P(B \mid A \cap C^c)$
4. $P(A \cap B \mid C) = P(A \mid C)P(B \mid A \cap C)$

1. This is the usual multiplication rule applied to the two events $A \cap B$ and C^c.True.
2. This is the usual multiplication rule. In equation (2) above, consider the sequence of events A, C^c, and B.True.
3. This is because
$$P(C^c \cap A \mid A) = \frac{P(C^c \cap A \cap A)}{P(A)} = \frac{P(C^c \cap A)}{P(A)} = P(C^c \mid A).$$
So, this statement is equivalent to the one in part 2.True.
4. This is the usual multiplication rule $P(A \cap B) = P(A)P(B|A)$, applied to a model/universe in which event C is known to have occurred.True.

The Total Probability Theorem

Let us now revisit the second calculation that we carried out in the context of our earlier example. In that example, we calculated the total probability of an event that can occur under different scenarios. And it involves the powerful idea of divide and conquer where we break up complex situations into simpler pieces.

Here is what is involved. We have our sample space which is partitioned into a number of subsets (we can call these subsets events, as usual). In Figure 16 we take that number to be 3, so we'll have it partitioned into three possible scenarios. By "partitioned" we mean that 1) these three events cover the entire sample space and 2) that they're disjoint from each other. Moreover, for each one of the scenarios we're given their probabilities: $P(A_1), P(A_2), P(A_3)$ are known.

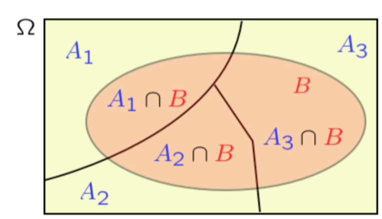


Figure 16: Decomposing Ω into non-overlapping events to compute the total probability.

If you prefer, you can also draw this situation in terms of a tree. There are three different scenarios that can happen. We're interested in a particular event, B. That event B can happen in three different ways, And these three ways correspond to these particular sub-events $A_1 \cap B$, $A_2 \cap B$ and $A_3 \cap B$ and those are also the labeled leaves in Figure 17.

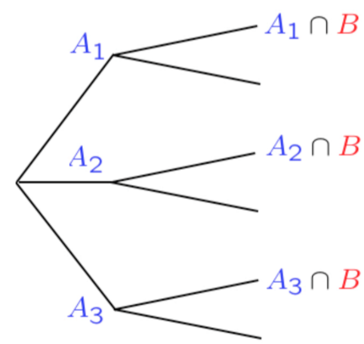


Figure 17: Partitioning of Ω can also be represented as a tree.

Finally, we are given conditional probabilities that event B will materialize under each one of the different possible scenarios. So we know $P(B \mid A_i)$ for every i.

Under those circumstances, can we calculate the probability of event B? Of course we can. And here's how we do it.

First we realize that event B consists of a number of disjoint pieces. One piece is when event B occurs together with event A_1. Another piece is when event B occurs together with A_2. Another piece is when event B occurs together with A_3. These three sets are disjoint from each other, as we see in Figure 16. And together they form the event B. Therefore, the probability of B is going to be, by the additivity axiom of probabilities, equal to the sum of the probabilities of these sub-events:

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$

Furthermore, for each one of these sub-events we can use the multiplication rule and write their probabilities as follows:

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \\ &= P(A_1)P(A_1 \mid B) + P(A_2)P(A_2 \mid B) + P(A_3)P(A_3 \mid B) \end{aligned}$$

So putting everything together, we have arrived at a formula of the following form. The total probability of event B is the sum of the probabilities of the different ways that B may occur, that is, B occurring under the different scenarios. And those particular probabilities are the product of the probability of the scenario times the conditional probability of B given that scenario:

$$\boxed{\mathbf{P}(B) = \sum_i \mathbf{P}(A_i)\,\mathbf{P}(B \mid A_i)}$$

Now, note that the sum of the probabilities of the different scenarios is of course equal to 1. And this is because the scenarios form a partition of our sample space. So if we look at this last formula, we realize that it is a weighted average of the conditional probabilities of event B - weighted average of the conditional probabilities where the probabilities of the individual scenarios are the weights. In words, the probability that an event occurs is a weighted average of the probability that it has under each possible scenario, where the weights are the probabilities of the different scenarios.

One final comment– our derivation was for the case of three events. But you can certainly see that the same derivation would go through if we had any finite number of events. But even more, if we had a partition of our sample space into an infinite sequence of events, the same derivation would still go through, except that in this place in the derivation, instead of using the ordinary additivity axiom we would have to use the countable additivity axiom. But other than that, all the steps would be the same. And we would end up with the same formula, except that now we would have an infinite sum over the infinite set of scenarios.

Check Your Understanding: We have an infinite collection of biased coins, indexed by the positive integers. Coin i has probability 2^{-i} of being selected. A flip of coin i results in Heads with probability 3^{-i}. We select a coin and flip it. What is the probability that the result is Heads?

The geometric sum formula may be useful here: $\sum_{i=1}^{\infty} \alpha^i = \frac{\alpha}{1-\alpha}$, when $|\alpha| < 1$.

theanswerisonefifth.
We think of the selection of coin i as scenario/event A_i. By the total probability theorem, for the case of infinitely many scenarios,

$$P(\text{Heads}) = \sum_{i=1}^{\infty} P(A_i)P(\text{Heads} \mid A_i) = \sum_{i=1}^{\infty} 2^{-i}3^{-i} = \sum_{i=1}^{\infty} (1/6)^i = \frac{1/6}{1-(1/6)}.$$

The Bayes Rule

We now come to the third and final kind of calculation out of the calculations that we carried out in our earlier example. The setting is exactly the same as in our discussion of the total probability theorem. We have a sample space which is partitioned into a number of disjoint subsets, or events, which we think of as scenarios. We're given the probability of each scenario.

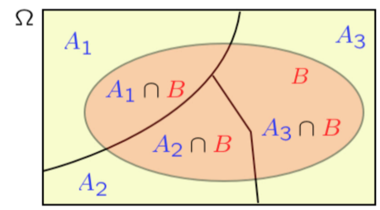


Figure 18: The setting is exactly the same as in our calculations of total probability.

And we think of these probabilities A_i as being some kind of initial beliefs. They capture how likely we believe each scenario to be. Now, under each scenario, we also have the probability that an event of interest, event B, will occur. Then the probabilistic experiment is carried out, and we observe that event B did indeed occur.

Once that happens, maybe this should cause us to revise our beliefs about the likelihood of the different scenarios. Having observed that B occurred, perhaps certain scenarios are more likely than others. How do we revise our beliefs? By calculating conditional probabilities.

And how do we calculate conditional probabilities? We start from the definition of conditional probabilities. The probability of one event given another is the probability that both events occur divided by the probability of the conditioning event:

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

How do we continue? We simply realize that the numerator is what we can calculate using the multiplication rule. And the denominator is exactly what we calculate using the total probability theorem. So we have everything we need to calculate those revised beliefs, i.e. conditional probabilities.

$$\boxed{\mathbf{P}(A_i \mid B) \; = \; \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\sum_j \mathbf{P}(A_j)\mathbf{P}(B \mid A_j)}}$$

And this is all there is in the Bayes rule. It is actually a very simple calculation. However, it is a quite important one.

Its history goes way back. In the middle of the 18th century, a Presbyterian minister, Thomas Bayes, worked it out. It was published a few years after his death and was quickly reorganized for its significance.

Bayes rule is a systematic way for incorporating new evidence, for learning from experience. And it forms the foundation of a major branch of mathematics, so-called Bayesian inference, which we will study in some detail later in this course.

The general idea of Bayesian inference is that we start with a probabilistic model which involves a number of possible scenarios, A_i, and we have some initial beliefs, $P(A_i)$, on the likelihood of each possible scenario.

There's also some particular event B that may occur under each scenario. And we know how likely it is to occur under each scenario.

We then actually observe that B occurred, and then we use that information to draw conclusions about the possible causes of B, or conclusions about the more likely or less likely scenarios that may have caused this events to occur. That is, having observed B, we make inferences as to how likely a particular scenario, say A_i, is going to be. That likelihood is captured by the conditional probabilities of A_i, given the event B, that is, by $P(A_i \mid B)$.

That's exactly what inference is all about, as we're going to see later in this class.

Check Your Understanding: A test for a certain rare disease is assumed to be correct 95% of the time: if a person has the disease, the test result is positive with probability 0.95, and if the person does not have the disease, the test result is negative with probability 0.95. A person drawn at random from a certain population has probability 0.001 of having the disease.

1. Find the probability that a random person tests positive.

2. Given that the person just tested positive, what is the probability he actually has the disease?

Let A be the event that the person has the disease, and B the event that the test result is positive.

1. The desired probability is

$$P(B) = P(A)P(B \mid A) + P(A^c)P(B \mid A^c) = 0.001 \times 0.95 + 0.999 \times 0.05 = 0.0509$$

2. The desired probability is

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)} = \frac{0.001 \times 0.95}{0.0509} \approx 0.01866$$

Note that even though the test was assumed to be fairly accurate, a person who has tested positive is still very unlikely (probability less than 2%) to have the disease. The explanation is that when testing 1000 people, we expect about 1 person to have the disease (and most likely test positive), but also expect about $1000 \times 0.999 \times 0.05 \approx 50$ people to test positive without having the disease. Hence, when we see a positive test, it is about 50 times more likely to correspond to one of the 50 false positives.