Homework 4

(Due at 11 AM on Thursday, February 23
along with this week's reading assignment.)

Question 1. Consider 10 independent tosses of a biased coin with the probability of Heads at each toss equal to p, where $0 < p < 1$.

1. Let A be the event that there are 6 Heads in the first 8 tosses. Let B be the event that the 9th toss results in Heads.

   Find $P(B \mid A)$ and express it in terms of p.

2. Find the probability that there are 3 Heads in the first 4 tosses and 2 Heads in the last 3 tosses.

3. Given that there were 4 Heads in the first 7 tosses, find the probability that the 2nd Heads occurred at the 4th toss. Give a numerical answer.

4. We are interested in calculating the probability that there are 5 Heads in the first 8 tosses and 3 Heads in the last 5 tosses. Give the numerical values of $a, b, c, d, e,$ and f that would match the answer

$$ap^7(1-p)^3 + bp^c(1-p)^d + ep^f(1-p)^f$$

   (a) $a =$

   (b) $b =$

   (c) $c =$

   (d) $d =$

   (e) $e =$

   (f) $f =$

Question 2. We have two fair three-sided dice, indexed by $i = 1, 2$. Each die has sides labeled $1, 2,$ and 3. We roll the two dice independently, one roll for each die. For $i = 1, 2$, let the random variable X_i represent the result of the i-th die, so that X_i is uniformly distributed over the set $\{1, 2, 3\}$. Define $X = X_2 - X_1$.

1. Calculate the numerical values of following probabilities:
   (a) $P(X = 0) =$
   (b) $P(X = 1) =$
   (c) $P(X = -2) =$
   (d) $P(X = 3) =$
2. Let $Y = X^2$. Calculate the following probabilities:
   (a) $P(Y = 0) =$
   (b) $P(Y = 1) =$
   (c) $P(Y = 2) =$

Question 3. *You've already solved this problem in the last homework. Here you are simply asked to rewrite your solution in a different notation.*

Consider the (completely fictitious, of course) situation that you attend, completely unprepared, a multiple-choice exam. It consists of 10 questions, and each question has four alternatives (of which only one is correct). You will pass the exam if you answer six or more questions correctly. You decide to answer each of the questions in a random way, in such a way that the answer of one question is not affected by the answers of the others.

Let the random variable X denote the total number of correctly answered questions.

For $i = \{1, 2, \ldots, 10\}$ let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th answer is correct} \\ 0, & \text{if the } i\text{th answer is incorrect} \end{cases}$$

Thus X is given by

$$X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 + X_9 + X_{10}$$

1. What is the distribution of the random variable X_7?
2. What is the range of values that the random variable X takes?
3. What is the distribution of the random variable X?
4. Calculate the probability that you answered the first question correctly and the second one incorrectly.
5. Calculate $P(X = 0)$.
6. Calculate $P(X = 1)$.
7. Calculate $P(X = 7)$.
8. Write down the expression for $P(X = k)$ for $k = \{1, 2, \ldots, 10\}$
9. What is the probability that you will pass. In other words, what is $P(X \geq 6)$?