# Extra Practice Problems

**Question 1.** Alice and Bob each choose at random a real number between zero and one. We assume that the pair of numbers is chosen according to the uniform probability law on the unit square, so that the probability of an event is equal to its area.

We define the following events:

$A = \{$The magnitude of the difference of the two numbers is greater than $1/3\}$
$B = \{$At least one of the numbers is greater than $1/4\}$
$C = \{$The sum of the two numbers is $1\}$
$D = \{$Alice's number is greater than $1/4\}$

Find the following probabilities:

(a) $P(A)$.

(b) $P(B)$.

(c) $P(A \cap B)$.

(d) $P(C)$.

(e) $P(D)$

(f) $P(A \cap D)$.

Answers:

(a) $P(A) = 4/9$.

(b) $P(B) = 15/16$.

(c) $P(A \cap B) = 4/9$.

(d) $P(C) = 0$.

(e) $P(D) = 3/4$

(f) $P(A \cap D) = 89/288$.

**Question 2.** You flip a fair coin (i.e., the probability of obtaining Heads is 1/2) three times. Assume that all sequences of coin flip results, of length 3, are equally likely. Determine the probability of each of the following events.

(a) $\{HHH\}$.

(b) $\{HTH\}$.

(c) Any sequence with 2 Heads and 1 Tails (in any order).

(d) Any sequence in which the number of Heads is greater than or equal to the number of Tails.

Answers:

(a) 1/8

(b) 1/8

(c) 3/8

(d) 1/2

**Question 3.** Find the value of $P(A \cup (B^c \cup C^c)^c)$ for each of the following cases:

(a) The events $A, B, C$ are disjoint events and $P(A) = 2/5$.

$$P(A \cup (B^c \cup C^c)^c) = \dots$$

(b) The events $A$ and $C$ are disjoint, and $P(A) = 1/2$ and $P(B \cap C) = 1/4$.

$$P(A \cup (B^c \cup C^c)^c) = \dots$$

(c) $P(A^c \cup (B^c \cup C^c)) = 0.7$.

$$P(A \cup (B^c \cup C^c)^c) = \dots$$

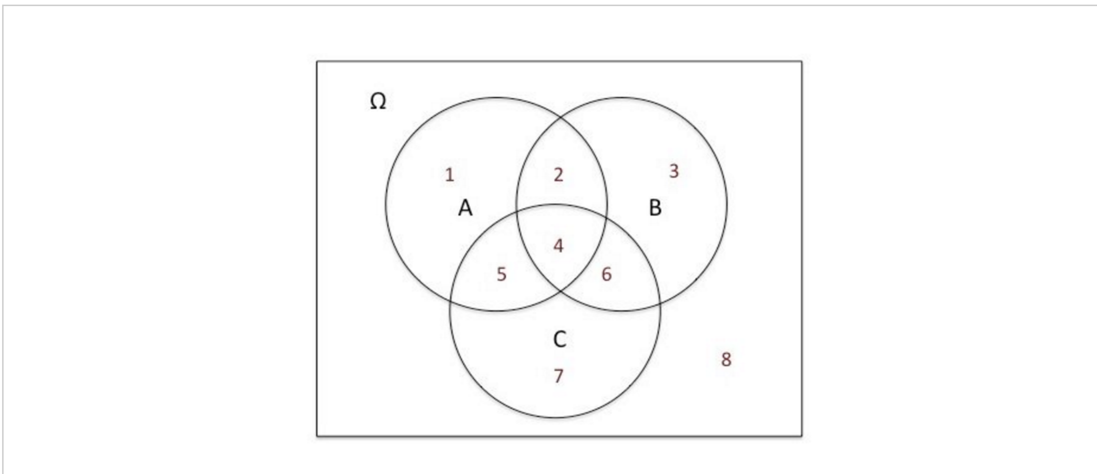
Answers:

(a) 2/5

(b) 3/4

(c) 0.3

**Question 4.** In this problem, you are given descriptions in words of certain events (e.g., "at least one of the events $A, B, C$ occurs"). For each one of these descriptions, identify the correct symbolic description in terms of $A, B, C$ from Events $E1 - E7$ below. Also identify the correct description in terms of regions (i.e., subsets of the sample space $\Omega$) as depicted in the Venn diagram below. (For example, Region 1 is the part of $A$ outside of $B$ and $C$.)




Symbolic descriptions:

- Event E1: $A \cap B \cap C$
- Event E2: $(A \cap B \cap C)^c$
- Event E3: $A \cap B \cap C^c$
- Event E4: $B \cup (B^c \cap C^c)$
- Event E5: $A^c \cap B^c \cap C^c$
- Event E6: $(A \cap B) \cup (A \cap C) \cup (B \cap C)$
- Event E7: $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

1. At least two of the events $A$, $B$, $C$ occur.

2. At most two of the events $A$, $B$, $C$ occur.

3. None of the events $A$, $B$, $C$ occurs.

4. All three events $A$, $B$, $C$ occur.

5. Exactly one of the events $A$, $B$, $C$ occurs.

6. Events $A$ and $B$ occur, but $C$ does not occur.

7. Either event $B$ occurs or, if not, then $C$ also does not occur.

# Solution to Question 5

1. At least two of the events $A$, $B$, $C$ occur.

   Event E6 ✓

   Regions: 2 4 5 6 ✓

2. At most two of the events $A$, $B$, $C$ occur.

   Event E2 ✓

   Regions: 1 2 3 5 6 7 8 ✓

3. None of the events $A$, $B$, $C$ occurs.

   Event E5 ✓

   Region: 8 ✓

4. All three events $A$, $B$, $C$ occur.

   Event E1 ✓

   Region: 4 ✓

5. Exactly one of the events $A$, $B$, $C$ occurs.

   Event E7 ✓

   Regions: 1 3 7 ✓

6. Events $A$ and $B$ occur, but $C$ does not occur.

   Event E3 ✓

   Region: 2 ✓

7. Either event $B$ occurs or, if not, then $C$ also does not occur.

   Event E4 ✓

   Regions: 1 2 3 4 6 8 ✓

**Question 5.** You roll two five-sided dice. The sides of each die are numbered from 1 to 5. The dice are "fair" (all sides are equally likely), and the two die rolls are independent.

(a) Event $A$ is "the total is 10" (i.e., the sum of the results of the two die rolls is 10).

1. Is event $A$ independent of the event "at least one of the dice resulted in a 5"?
2. Is event $A$ independent of the event "at least one of the dice resulted in a 1"?

(b) Event $B$ is "the total is 8."

1. Is event $B$ independent of getting "doubles" (i.e., both dice resulting in the same number)?
2. Given that the total was 8, what is the probability that at least one of the dice resulted in a 3?

Answers:

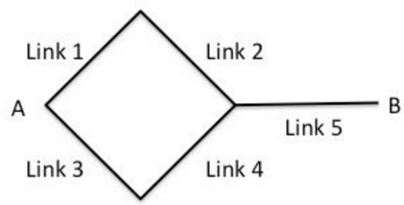
(a) no

(b) no

(c) no

(d) 2/3

**Question 6.** A reliability problem Consider the communication network shown in the figure below and suppose that each link can fail with probability $p$. Assume that failures of different links are independent.




(a) Assume that $p = 1/3$. Find the probability that there exists a path from A to B along which no link has failed. (Give a numerical answer.)

(b) Given that exactly one link in the network has failed, find the probability that there exists a path from A to B along which no link has failed. (Give a numerical answer.)

Answers:

(a) $112/243 = [1 - [1 - (1-p)^2]^2] \cdot (1-p)$

(b) $4/5$

**Question 7.** A binary communication system is used to send one of two messages:

- message A is sent with probability 2/3, and consists of an infinite sequence of zeroes,
- message B is sent with probability 1/3, and consists of an infinite sequence of ones.

The $i$th received bit is "correct" (i.e., the same as the transmitted bit) with probability 3/4, and is ?incorrect" (i.e., a transmitted 0 is received as a 1, and vice versa), with probability 1/4. We assume that **conditioned on any specific message sent**, the received bits, denoted by $Y_1, Y_2, \ldots$ are independent.

1. Find $\mathbf{P}(Y_1 = 0)$, the probability that the first bit received is 0.
2. Given that message A was transmitted, what is the probability that exactly 6 of the first 10 received bits are ones? (Answer with at least 3 decimal digits.)
3. Find the probability that the first and second received bits are the same.
4. Given that $Y_1, \ldots, Y_5$ were all equal to 0, what is the probability that $Y_6$ is also zero?
5. Find the mean of $K$, where $K = \min\{i : Y_i = 1\}$ is the index of the first bit that is 1.
6. Is $Y_2 + Y_3$ independent of $Y_1$?
7. Is $Y_2 - Y_3$ independent of $Y_1$?

Answers:

1. 7/12
2. 0.0162220
3. 10/16
4. 0.7489733
5. 28/9
6. No
7. Yes

**Question 8.** A three-sided die.
The newest invention of one of your CIS-2033 classmates is a three-sided die. On any roll of this die, the result is 1 with probability 1/2, 2 with probability 1/4, and 3 with probability 1/4.

Consider a sequence of six independent rolls of this die.

1. Find the probability that exactly two of the rolls result in a 3.

   (a) $\binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4$

   (b) $\binom{6}{2}\left(\frac{1}{4}\right)^2$

   (c) $\binom{6}{2}\left(\frac{1}{4}\right)^2\binom{6}{4}\left(\frac{3}{4}\right)^4$

   (d) $\binom{6}{2}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^2$

2. Given that exactly two of the six rolls resulted in a 1, find the probability that the first roll resulted in a 1.

3. We are told that exactly three of the rolls resulted in a 1 and exactly three rolls resulted in a 2. Given this information, find the probability that the six rolls resulted in the sequence (1, 2, 1, 2, 1, 2).

4. The conditional probability that exactly k rolls resulted in a 3, given that at least one roll resulted in a 3, is of the form:

$$\frac{1}{1-(c_1/c_2)^{c_3}}\binom{c_3}{k}\left(\frac{1}{c_2}\right)^k\left(\frac{c_1}{c_2}\right)^{c_3-k}, \quad \text{for } k = 1, 2, \ldots, 6.$$

   Find the values of constants $c_1, c_2$, and $c_3$.

1. (a)
2. 1/3
3. 1/20
4. $c_1 = 3, c_2 = 4, c_3 = 6$

**Question 9.** Let random variables $X$ and $Y$ be described a joint PMF shown in this table:

| y \ x | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 4 | 1/20 | 2/20 | 2/20 | |
| 3 | 2/20 | 4/20 | 1/20 | 2/20 |
| 2 | | 1/20 | 3/20 | 1/20 |
| 1 | | 1/20 | | |

1. Compute $p_X(1)$.
2. Compute $p_{X|Y}(1 \mid 1)$.
3. Are $X$ and $Y$ independent?
4. Consider the event $A = \{X \leq 2\} \cap \{Y \geq 3\}$. Compute the joint PMF of $X$ and $Y$, given $A$ (show the 2-by-2 joint PMF table).
5. Compute the marginal probabilities in the conditional model (i.e. compute $p_{X|A}(1), p_{X|A}(2), p_{Y|A}(3), p_{Y|A}(4)$).
6. Are $X$ and $Y$ independent, given $A$?

1. 3/20
2. 0
3. No
4. $p_{XY|A}(1,3) = 2/9, p_{XY|A}(1,4) = 1/9, p_{XY|A}(2,3) = 4/9, p_{XY|A}(2,4) = 2/9$.
5. $p_{X|A}(1) = 1/3, p_{X|A}(2) = 2/3, p_{Y|A}(3) = 2/3, p_{Y|A}(4) = 1/3$.
6. Yes

## The Bayes Rule, problems 10 -12

**Question 10.** A student takes a multiple-choice exam. Suppose for each question she either knows the answer or gambles and chooses an option at random. Further suppose that if she knows the answer, the probability of a correct answer is 1, and if she gambles this probability is 1/4. To pass, students need to answer at least 60% of the questions correctly. The student has "studied for a minimal pass", i.e., with probability 0.6 she knows the answer to a question. Given that she answers a question correctly, what is the probability that she actually knows the answer?

**Question 11.**

(a) You go to see the doctor about an ingrowing toenail. The doctor selects you at random to have a blood test for swine flu, which for the purposes of this exercise we will say currently affects 1 in 10,000 people in the US. The test is 99% accurate, in the sense that the probability of a false positive is 1% (that is, if you don't have the disease, the test says that you do have it 1% of the time). The probability of a false negative is zero (that is, if you do have the disease, the test always recognizes that you do). You test positive. What is the new probability that you have swine flu?

(b) Now imagine that you went to a friend's wedding in the country of Agriphinia recently, and (for the purposes of this exercise) it is known that 1 in 200 people who visited Agriphinia recently come back with swine flu. Given the same test result as above, what should your revised estimate be for the probability that you have the disease?

**Question 12.** A Temple University study indicates that typically 8% of all students forget to turn off their cell phones when attending classes. (Your instructor claims that this is a bogus study whose claims have been greatly exaggerated, but you decide to take the study's results for granted and continue to practice for your midterm). Assume that 20 people attend your next CIS 2033 class (again, suspend your disbelief).

(a) What is the likelihood that 5 cell phones will go off during the class? (assume that if somebody's cell phone rings, nobody else bothers to check if theirs' is still on)

(b) What is the likelihood that everybody's cell phone rings?

(c) What is the likelihood that nobody's cell phone rings?

(d) What is the likelihood that at least 2 people's cell phones go off?

**Question 13.** A lollipop is drawn at random from a box containing one red and one white lollipop. If the white lollipop is drawn, it is put back into the box. If the red lollipop is drawn, it is returned to the box together with two more red lollipops. Then a second draw is made. What is the probability a red lollipop was drawn on both the first and the second draws?