CMPE 245 Homework 2

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Problem 1. A stochastic process is defined as

$$x_k = x_{k-1} + m_k, \quad x_0 = 0 \tag{1}$$

where the random variable m_k is defined by the probability density function $p(m_k)$

$$p(m_k) \sim \begin{cases} 0, \quad m_k < -1/2 \\ 4(m_k + 1/2), \quad -1/2 \leq m_k < 0 \\ 2 - 4m_k, \quad 0 \leq m_k \leq 1/2 \\ 0, \quad m_k > 1/2 \end{cases} \tag{2}$$

Find the expected value $E\{x_k\}$ and the variance $E\{(x_k - \overline{x}_k)^2\}$ of this process.

ANS:
equation 1. can rewrite as:

$$\begin{array}{l} x_k = x_{k-1} + m_k = x_{k-2} + m_k + m_k = x_{k-2} + 2(m_k) \\ x_k = x_{k-3} + m_k + 2(m_k) = x_{k-3} + 3(m_k) \\ x_k = ... = x_0 + k(m_k), \quad where\ x_0 = 0 \\ x_k = k(m_k) \end{array} \tag{3}$$

Therefore, the expected value can compute as:

$$\begin{array}{l} E\{x_k\} = E\{x_{k-1} + m_k\} = E\{k(m_k)\} = k * E\{m_k\} = k * \int_{-\infty}^{\infty} m_k\ p(m_k)\ dm_k \\ E\{x_k\} = k\ \int_{-1/2}^{1/2} m_k\ p(m_k)\ dm_k \\ E\{x_k\} = k\ \int_{-1/2}^{0} m_k\ 4\ (m_k + 1/2)\ dm_k + \int_{0}^{1/2} m_k\ (2 - 4m_k)\ dm_k \\ E\{x_k\} = k\ (\frac{4}{3}m_k^3 + m_k^2 \Big|_{-1/2}^{0} + \frac{-4}{3}m_k^3 + m_k^2 \Big|_{0}^{1/2}) = k\ \frac{1}{6} \end{array} \tag{4}$$

And, the variance can compute as:

$$\begin{array}{l} E\{(x_k - \overline{x}_k)^2\} = \int_{-\infty}^{\infty} (x_k - \overline{x}_k)^2\ p(x)\ dx, \quad where\ x_k = k\ m_k\ and\ \overline{x}_k = E\{x_k\} = k\ \frac{1}{6} \\ E\{(x_k - \overline{x}_k)^2\} = \int_{-\infty}^{\infty} k\ (m_k - \frac{1}{6})^2\ p(m_k)\ dm_k \\ E\{(x_k - \overline{x}_k)^2\} = k(\ \int_{-1/2}^{0} (m_k - \frac{1}{6})^2\ 4\ (m_k + 1/2)\ dm_k + \int_{0}^{1/2} (m_k - \frac{1}{6})^2\ (2 - 4m_k)\ dm_k) \\ E\{(x_k - \overline{x}_k)^2\} = k\ (m_k^4 + \frac{2}{9}m_k^3 - \frac{5}{18}m_k^2 + 18m_k \Big|_{-1/2}^{0} - m_k^4 + \frac{10}{9}m_k^3 - \frac{7}{18}m_k^2 + 18m_k \Big|_{0}^{1/2}) \\ E\{(x_k - \overline{x}_k)^2\} = k\ \frac{1}{18} \end{array} \tag{5}$$

Problem 2 (only for graduate students) It is known that if x is a random variable with a pdf $p_x(x)$, i.e., $x \sim p_x(x)$, and y is a random variable $y \sim p_y(y)$, then $z = x + y$ is a random variable $z \sim p_z(z)$, where $p_z(z) = p_x(x) * p_y(y)$. The symbol $*$ denotes the convolution, i.e.,

$$p_z(z) = p_z(x + y) = \int_{-\infty}^{\infty} p_y(z - x) p_x(x) dx \tag{6}$$

(a) If $m_k \sim p(m_k)$, where $p(m_k)$ is depicted in the figure, and $m_k = n + h$, figure out $p_n(n)$ and $p_h(h)$.

ANS:
Because $m_k = n + h$ and $p_z(z) = p_x(x) * p_y(y)$, I could derive that $p(m_k) = p(n + h) = p_n(n) * p_h(h)$. $p(m_k)$ is a symmetrical triangular-shaped distribution, and, it is the result of convolution of two identical uniform distribution. Therefore, $p_n(n)$ and $p_h(h)$ can be two identical uniform distribution as:

$$p_n(n) = \begin{cases} 0, & n < a \\ 1/(b-a), & a \leq n \leq b \qquad where\ a = -1/4,\ b = 1/4 \\ 0, & n > b \end{cases} \tag{7}$$

(b) Use your conclusion from (a) to write a MATLAB code that generates the sequence from Problem 1. Generate the sequence 100 (or more) times and based on these sequences, verify the results obtained in Problem 1.

ANS:
First, I use MATLAB to create uniform-shaped function $p_n(n)$ (as figure 1). Then, I use MATLAB function (conv) to do the convolution of $p_n(n)$ and itself (as figure 2).
Finally, I generate the sequence of 100 times $x_k = x_{k-1} + m_k$, and I got x_k about -5 to 5. While, according to my result of Problem 1 that $E\{x_k\} = k\,\frac{1}{6} = 100 * \frac{1}{6} = 16.667$.

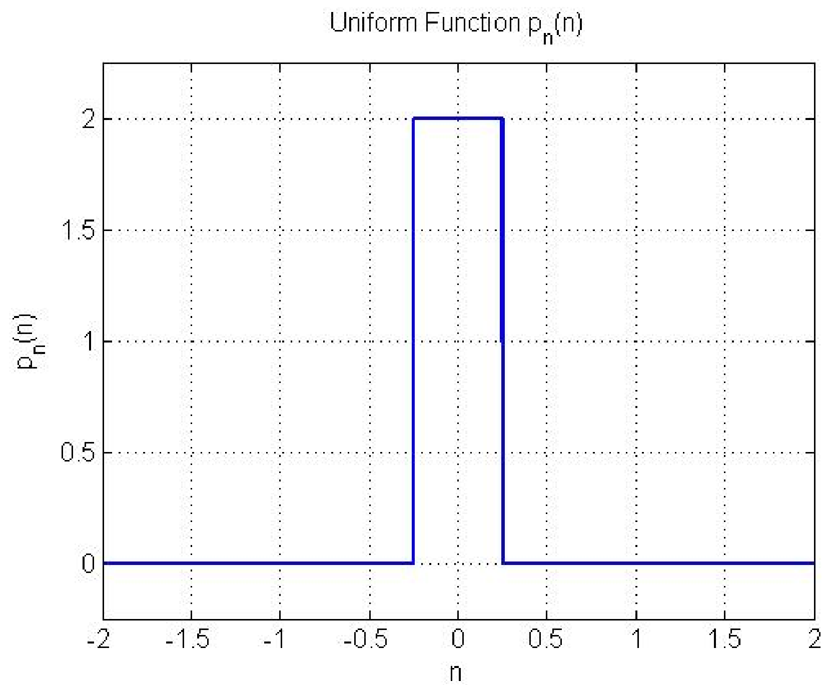

Figure 1: Uniform Function $p_n(n)$

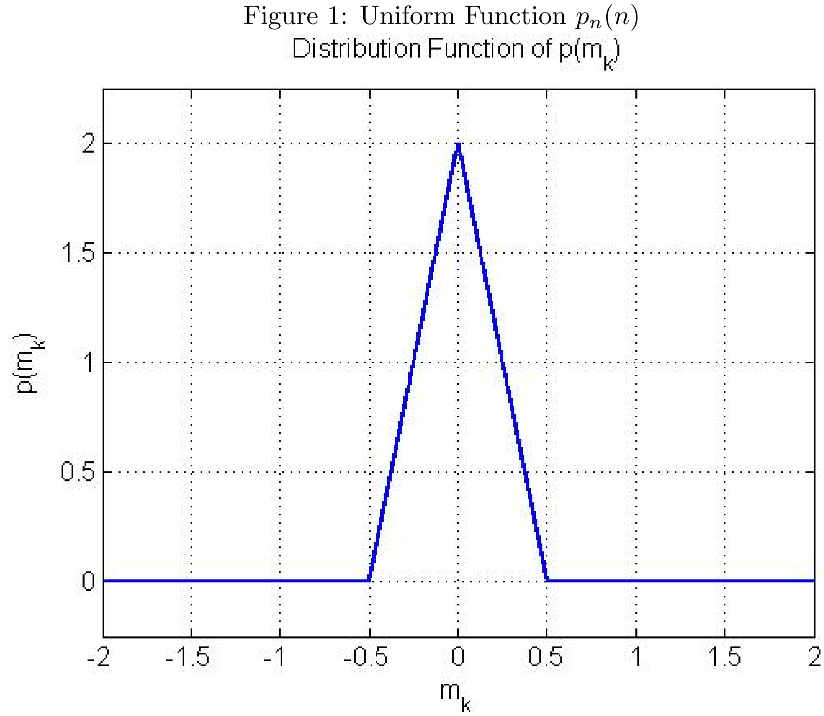

Figure 2: Distribution Function $p(m_k)$