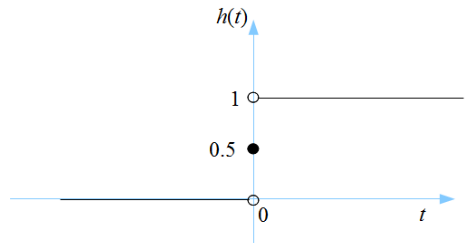
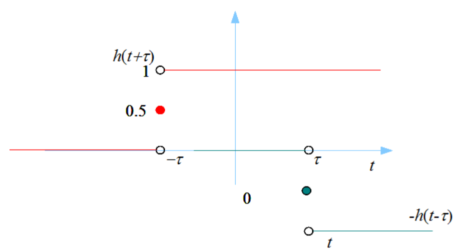
- What is the Heaviside (step) function, or unit step function ?

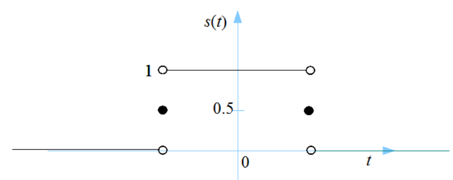
$$h(t) = \begin{cases} 0, & t < 0 \\ 0.5, & t = 0 \\ 1, & t > 0 \end{cases}$$



- The symmetric square pulse

$$s(t) = h(t+\tau) - h(t-\tau)$$





- The Dirac pulse

The surface under the square pulse $s(t)$ is $S = 2\tau$. Let us define the signal which is

$$\delta^{\tau}(t) = \frac{1}{2\tau}[h(t+\tau) - h(t-\tau)]$$

For any finite value τ, the surface of the function δ^{τ} is 1, i.e., $\int_{-\infty}^{\infty} \delta^{\tau}(t)dt = 1$

Consider now the Dirac pulse function $\delta(t) = \lim_{t\to 0} \delta^{\tau}(t)$

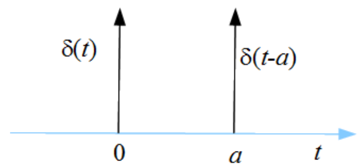
$$\delta(t) = \lim_{t\to 0} \delta^{\tau}(t) = \lim_{t\to 0} \frac{1}{2\tau}[h(t+\tau) - h(t-\tau)] \Rightarrow \delta(t) = \frac{dh(t)}{dt}$$

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

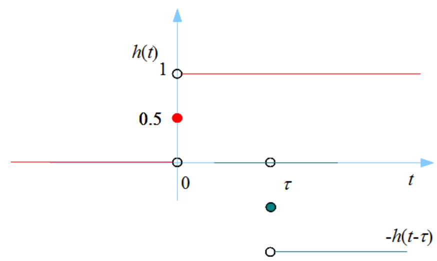


For causal systems, it is more convenient to define the Dirac pulse based on the asymmetric square pulse.

- The asymmetric square pulse

$$s(t) = h(t) - h(t - \tau)$$

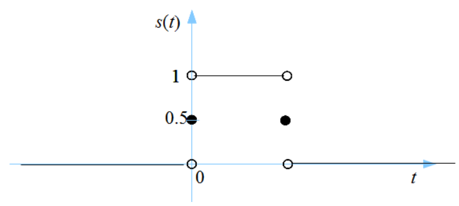
The surface under the pulse is $S = \tau$



- The Dirac pulse

$$\delta(t) = \lim_{t \to 0} \frac{1}{\tau}[h(t) - h(t - \tau)]$$

and under this definition of the derivative, we can conclude again that the Dirac pulse is the derivative of the step function

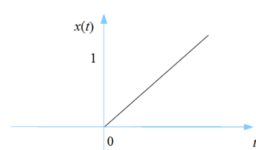


$$\delta(t) = \frac{dh(t)}{dt}$$

- Examples

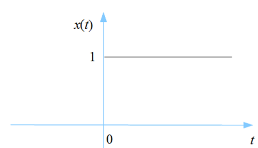
$$\dot{x} = h(t), \text{ for } x(0) = x_0 = 0$$

$$\text{Solution}: \quad x(t) = x_0 + t = t, t > 0$$



$$\dot{x} = \delta(t), \text{ for } x(0) = x_0 = 0$$

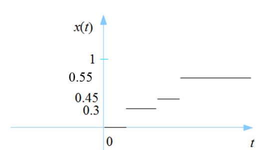
$$\text{Solution}: \quad x(t) = \int_0^t \delta(t)dt = h(t)$$



$$\dot{x} = \sum_{k=1}^{3} 0.3/k \ \delta(t - t_k), \text{ for } x(0) = x_0 = 0$$

$$t_1 = 0.5, \ t_2 = 1, \ t_3 = 1.2$$

$$x(t) = 0.3h(t - t_1) + 0.15(t - t_1) + 0.1h(t - t_3)$$



Homework hints:

$$\dot{x}_1 = x_2 \tag{1}$$
$$\dot{x}_2 = -0.1 \sin x_1 + \sum_k f(t_k)\delta(t - t_k) \tag{2}$$

where $f(t_k) \sim \mathcal{N}(0, \sigma = 0.2)$ and $t_k - t_{k-1}$ are random increments from the exponential distribution with parameter λ.

1. clear all

2. Define the interval on which you want to find the solution, say $[0, T]$

3. $t_1 = 0$, $k = 1$
   while $(t_k < T)$,
      $k = k + 1$
      $t_k = t_{k-1} + exp\_rand\_number$
   end
   $numOfEvents = k$
   $t_k = T$

4. $x_0 = [0, 0]$, $t = [\ ]$, $x = [\ ]$
   for $k = 1 : (numOfEvents - 1)$,
      $[t_s, x_s]$ =solve$(function, [t_k, t_{k+1}], x_0)$
      t =concatenate(t, t_s)
      x =concatenate(x, x_s)
      clear t_s
      clear x_s
      $f = normal\_rand\_number$
      x_0 =lastOf$(x) + [0, f]$
   end

5. Plot the solution.