Homework 4

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Problem 1. A linear system describing the dynamics of $x_1(k)$ and $x_2(k)$ can be written as

$$\begin{aligned} x_1(k+1) &= x_1(k) - 0.8x_1(k) + 0.4x_2(k) + w_1(k) & (1) \\ x_2(k+1) &= x_2(k) - 0.4x_1(k) + 1 + w_2(k) & (2) \end{aligned}$$

where $w_1(k)$, $w_2(k)$ are zero-mean value independent Gaussian random variables with variance 1. The initial expected values are $\overline{x}_1(0) = 10$, $\overline{x}_2(0) = 20$ and the covariance matrix of the vector $x(0) = [x_1(0)\ x_2(0)]^T$ is $P(0) = diag(40, 40)$. Plot the evolution of the expected values for $k = 0..14$. In a separate figure, plot the evolution of the variance corresponding to $x_1(k)$ and the one corresponding to $x_2(k)$, $k = 0..14$.

ANS:
Equation 1 and 2 can rewrite as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (3)$$

And, the expected values of $x(0) = [x_1(k)\ x_2(k)]^T$ is:

$$E\left\{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}\right\} = \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & 1 \end{bmatrix} E\left\{\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}\right\} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} \bar{x}_1(k+1) \\ \bar{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (5)$$

Using equation 5 with the initial values $\overline{x}_1(0) = 10$, $\overline{x}_2(0) = 20$ to plot the evolutions of expected value for (time step) $k = 0..14$ in MATLAB:

For $\overline{x}(k)$:

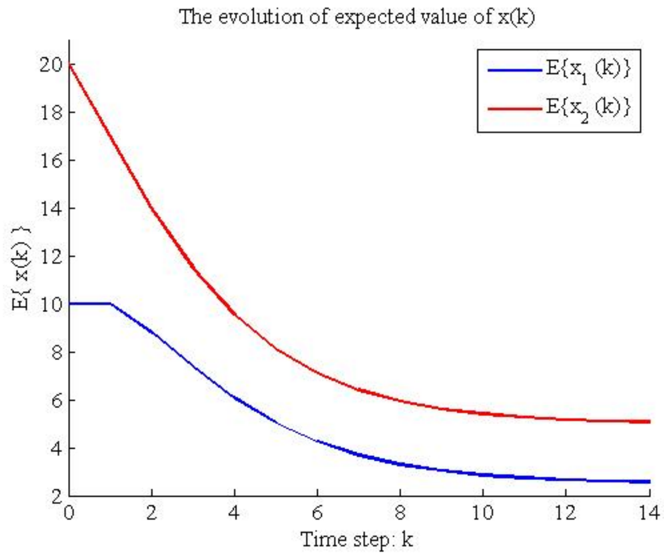

Figure 1: The evolution of expected values of $x(k)$

Next, for the variance of $x_1(k)$ and $x_2(k)$, I use equation:

$$P_{k+1} = AP_kA^T + \Gamma Q\Gamma^T \tag{6}$$

with initial value:

$$P_0 = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix} \tag{7}$$

and covariance matrix:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8}$$

to plot the evolution of variance of $x_1(k)$ and $x_2(k)$ for (time step) $k = 0..14$ in MATLAB:

For the covariance of $x(k)$:

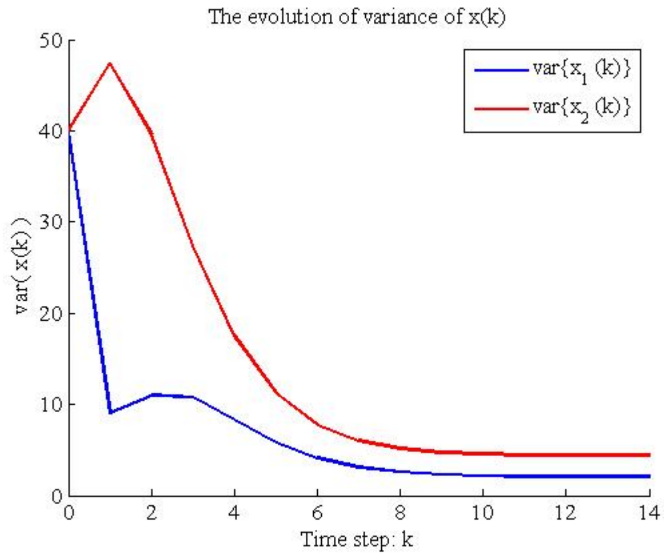

Figure 2: The evolution of variance of $x(k)$

Problem 2. While moving, the position and velocity of a university campus bus along its route are described by the following stochastic differential equation

$$\begin{aligned} ds &= vdt \quad (9) \\ dv &= -2vdt + 22dt + 20dw \quad (10) \end{aligned}$$

where $s(t)$ is the distance traveled along the route and $v(t)$ is the bus velocity. The initial position of the bus at $t = 0$ is known with the variance $var(s(0)) = 100$ and the velocity variance corresponds to its value at the steady state, i.e., $var(v(0)) = \lim_{t\to\infty} var(v(t))$. The variables $s(0)$ and $v(0)$ are independent at the time point $t = 0$. At what point of time t^*, the variance $var(s(t^*)) = 2var(s(0))$? Provide differential equations describing dynamics of the variances, derive the steady state of $var(v(t))$ and explain your conclusions. Numerical solutions are accepted.
ANS:
First, I derive the steady state of the variance of $v(t)$. For linear stochastic differential equation (such as equation 10) could express as:

$$dx(t) = Ax(t)dt + udt + bdw \quad (11)$$

Then, A is -2, b is 20, and u is 22 in equation 10.
Next, I could use equation:

$$\dot{P} = PA^T + AP + bb^T \quad (12)$$

to compute the derivative of the variance of $v(t)$:

$$\dot{\sigma}^2_{v(t)} = -2 \times \sigma^2_{v(t)} + \sigma^2_{v(t)} \times -2 + 20 \times 20 \tag{13}$$

$$\dot{\sigma}^2_{v(t)} = -4\sigma^2_{v(t)} + 400 \tag{14}$$

At steady state, when $t \to \infty$, I could assume that $\dot{\sigma}^2_{v(\infty)} \to 0$ and $\sigma^2_{v(\infty)} = \sigma^2_{v(0)}$. Then from equation 14, I could get:

$$\underbrace{\dot{\sigma}^2_{v(\infty)}}_{0} = -4\underbrace{\sigma^2_{v(\infty)}}_{\sigma^2_{v(0)}} + 400 \tag{15}$$

$$\sigma^2_{v(0)} = 100 \tag{16}$$

Next, I could also rewrite equation 9 and 10 as equation 11:

$$d\begin{bmatrix} s(t) \\ v(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}}_{A} \begin{bmatrix} s(t) \\ v(t) \end{bmatrix} dt + \underbrace{\begin{bmatrix} 0 \\ 22 \end{bmatrix}}_{u} dt + \underbrace{\begin{bmatrix} 0 \\ 20 \end{bmatrix}}_{b} dw \tag{17}$$

Then, I could use equation 12 to express differential equations describing dynamics of the variance of $s(t)$ and $v(t)$ as:

$$d\underbrace{\begin{bmatrix} \mathrm{var}(s(t)) & \mathrm{cov}(s,v) \\ \mathrm{cov}(s,v) & \mathrm{var}(v(t)) \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}}_{A} p + P \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}^T + \underbrace{\begin{bmatrix} 0 \\ 20 \end{bmatrix}}_{b} \begin{bmatrix} 0 \\ 20 \end{bmatrix}^T \tag{18}$$

Finally, I use equation 18 to compute the evolution of the variance of $s(t)$ and $v(t)$ in MATLAB with function ode45.
For the variance of $s(t)$:

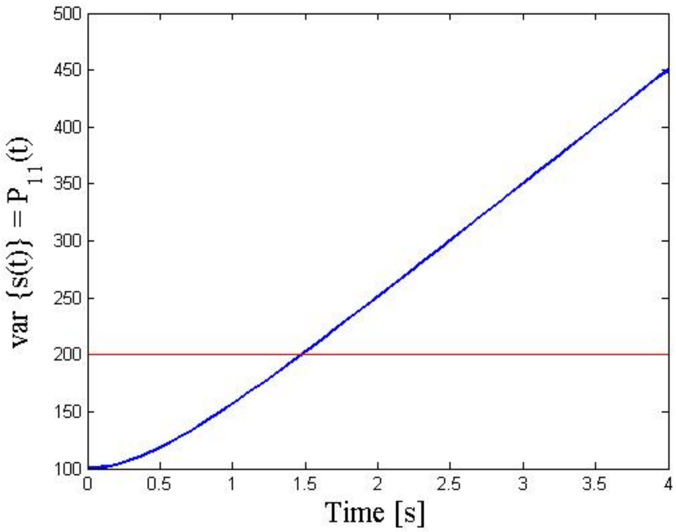

Figure 3: The evolution of variance of $s(t)$

The horizontal red line in the figure indicate the value of $2 * var(s(0))$, so I find out that $var(s(t)) = var(s(0))$ at time $= 1.42$ to 1.52.
And, from the sub-equation of equation 18, the differential of the variance of $s(t)$:

$$\dot{P}_{11} = P_{21} + P_{12} = 2P_{12} \tag{19}$$

It shows that, at steady state, $var(s(t))$ will continuously increase (because there is no negative term to limit $var(s(t))$), and its increasing rate is 2 times of the value of the covariance of $s(t)$ and $v(t)$.

For the variance of $v(t)$:

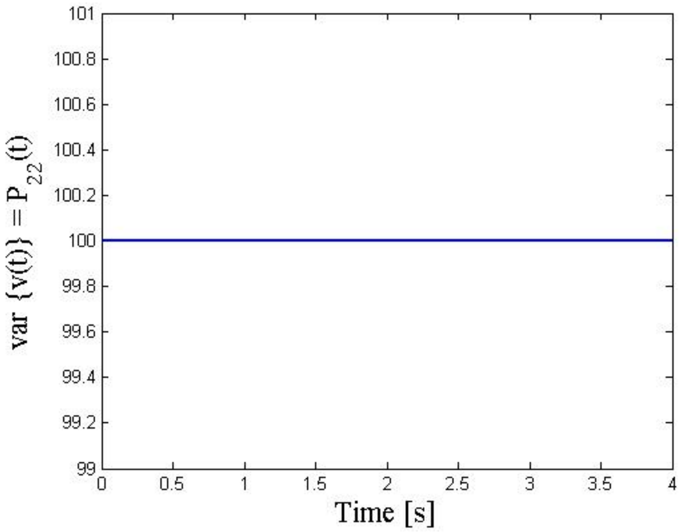

Figure 4: The evolution of variance of $v(t)$

From the sub-equation of equation 18, the differential of the variance of $v(t)$:

$$\dot{P}_{22} = -4P_{22} + 400 \tag{20}$$

and, at steady state, when $\dot{P_{22}} = 0$, I get:

$$P_{22} = 100 \tag{21}$$

Because our initial condition set $P_{22}(0) = P_{22}(infty)$, the value of P_{22} will always be 100 as figure 4.

For the covariance of $s(t)$ and $v(t)$:

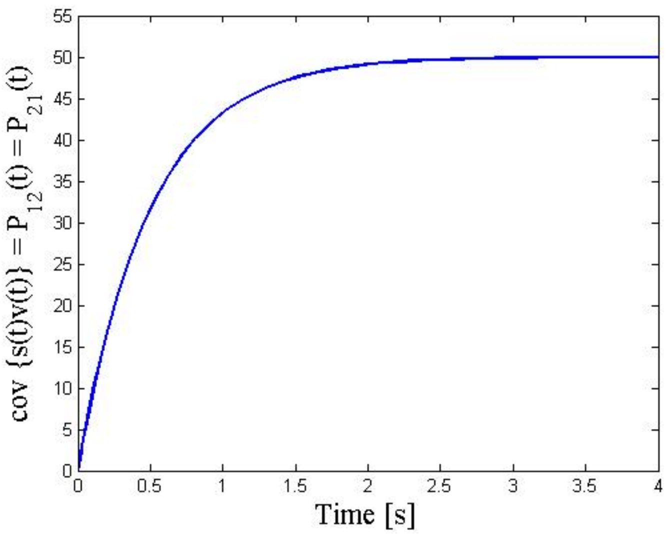

Figure 5: The evolution of covariance of $s(t), v(t)$

From the sub-equation of 18, the differential of the covariance of $s(t)$ and $v(t)$:

$$\dot{P}_{12} = -2P_{12} + P_{22} \tag{22}$$

and, at steady state, when $\dot{P_{12}} = 0$, I get:

$$P_{12} = \frac{1}{2}P_{22} = 50 \tag{23}$$

It shows that, at steady state, the covariance of $s(t)$ and $v(t)$ will converge to 50 as figure 5.