CMPE 245 Homework 3

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Problem 1. A dynamical system is governed by the stochastic difference equation

$$\underline{x}(k+1)=\begin{bmatrix}1.5 & 1\\ -0.7 & 0\end{bmatrix}\underline{x}(k)+\begin{bmatrix}1.0\\ 0.5\end{bmatrix}w(k) \qquad (1)$$

where w(k) is a sequence of independent Normal(mean=0,std=1) (Gaussian) random variables.

a) Derive the expression for the state covariance and determine the state covariance in the steady state P_∞.

ANS:

To simplify derivation, in equation 1, I set two real number matrix as

$\Phi=\begin{bmatrix}1.5 & 1\\ -0.7 & 0\end{bmatrix}$, and $\Gamma=\begin{bmatrix}1.0\\ 0.5\end{bmatrix}$, and equation 1 can rewrite to

$$\underline{x}(k+1)=\Phi\underline{x}(k)+\Gamma w(k) \qquad (2)$$

Next, follow course material (Slide 4).

First, I derive the state vector expectation:

$$\begin{aligned}
E\left\{\underline{x}(k+1)\right\}&=E\left\{\underline{\Phi}\underline{x}(k)+\underline{\Gamma}w(k)\right\}\\
E\left\{\underline{x}(k+1)\right\}&=E\left\{\underline{\Phi}\underline{x}(k)\right\}+E\left\{\underline{\Gamma}w(k)\right\}\\
E\left\{\underline{x}(k+1)\right\}&=\underline{\Phi}E\left\{\underline{x}(k)\right\}+\underline{\Gamma}E\left\{w(k)\right\},\quad \text{where } E\left\{w(k)\right\}=0\\
E\left\{\underline{x}(k+1)\right\}&=\underline{\Phi}E\left\{\underline{x}(k)\right\}+0\\
\underline{\overline{x}}(k+1)&=\underline{\Phi}\underline{\overline{x}}(k) \qquad (3)
\end{aligned}$$

Next, I derive the state covariance:

$$\underline{x}(k+1)-\underline{\overline{x}}(k+1)=\underline{\Phi}\underline{x}(k)+\underline{\Gamma}w(k)-\underline{\Phi}\underline{\overline{x}}(k)$$
$$\underline{x}(k+1)-\underline{\overline{x}}(k+1)=\underline{\Phi}(\underline{x}(k)-\underline{\overline{x}}(k))+\underline{\Gamma}w(k)$$
$$\Rightarrow$$
$$E\left\{(\underline{x}(k+1)-\underline{\overline{x}}(k+1))(\underline{x}(k+1)-\underline{\overline{x}}(k+1))^T\right\}=E\left\{(\underline{\Phi}(\underline{x}(k)-\underline{\overline{x}}(k))+\underline{\Gamma}w(k))(\underline{\Phi}(\underline{x}(k)-\underline{\overline{x}}(k))+\underline{\Gamma}w(k))^T\right\}$$
$$\underline{P}_{k+1}=E\left\{(\underline{\Phi}(\underline{x}(k)-\underline{\overline{x}}(k))+\underline{\Gamma}w(k))(\underline{\Phi}(\underline{x}(k)-\underline{\overline{x}}(k))+\underline{\Gamma}w(k))^T\right\}$$
$$\underline{P}_{k+1}=\underline{\Phi}E\underbrace{\left\{(\underline{x}(k)-\underline{\overline{x}}(k))(\underline{x}(k)-\underline{\overline{x}}(k))^T\right\}}_{\underline{P}_k}\underline{\Phi}^T+\underline{\Phi}E\underbrace{\left\{(\underline{x}(k)-\underline{\overline{x}}(k))w(k)^T\right\}}_{0}\underline{\Gamma}^T$$
$$+\underline{\Gamma}E\underbrace{\left\{w(k)(\underline{x}(k)-\underline{\overline{x}}(k))^T\right\}}_{0}\underline{\Phi}^T+\underline{\Gamma}E\underbrace{\left\{w(k)w(k)^T\right\}}_{\underline{Q}_k}\underline{\Gamma}^T$$

$$\underline{P}_{k+1}=\underline{\Phi}\underline{P}_k\underline{\Phi}^T+\underline{\Gamma}\underline{Q}_k\underline{\Gamma}^T \qquad (4)$$

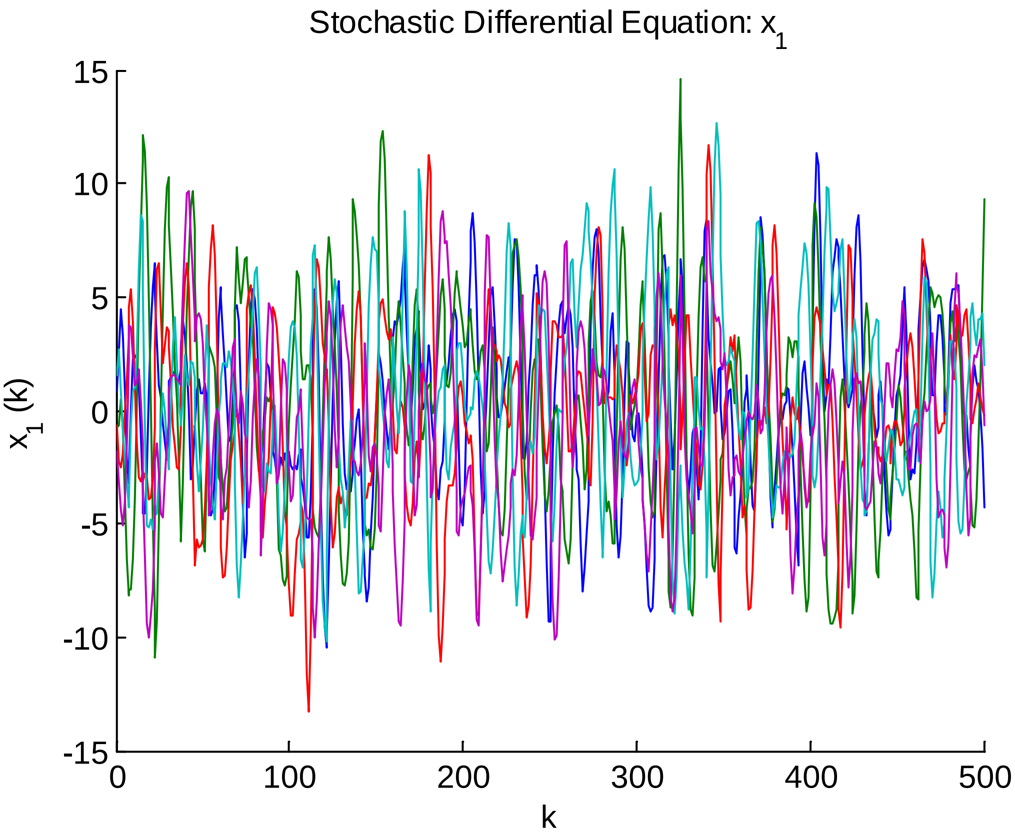
Because Φ and Γ are time invariant vector, and $\max\left(\mathrm{eig}(\Phi)\right)<1$,
for $k\to\infty$:
If $\Gamma w_k=0$, then

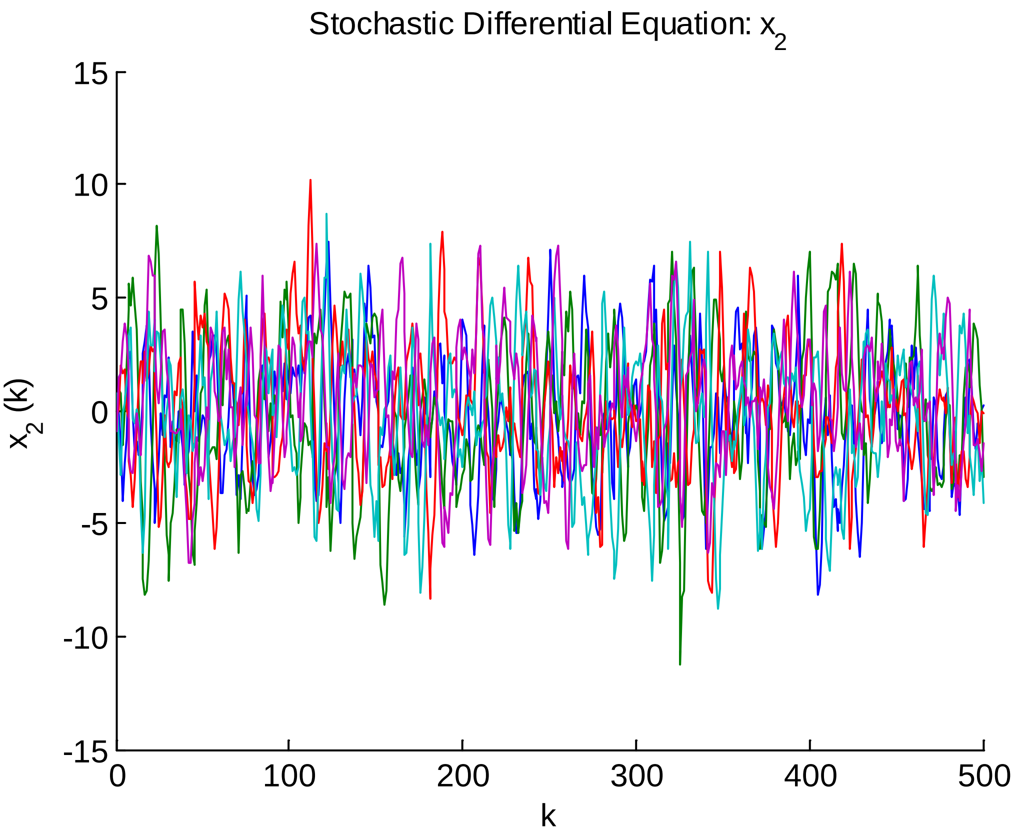
$$\underline{P}_{k+1}=\underline{\Phi}\underline{P}_k\underline{\Phi}^T=\underline{\Phi}\underline{\Phi}\underline{P}_{k-1}\underline{\Phi}^T\underline{\Phi}^T=(\underline{\Phi})^k\underline{P}_0(\underline{\Phi}^T)^k=0 \qquad (5)$$

If $\Gamma w_k\neq0$, and $\underline{\Gamma}\underline{Q}_k\underline{\Gamma}^T=const=\begin{bmatrix}1.0 & 0.5\\0.5 & 0.25\end{bmatrix}$ and $P_0=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ (since $x_0=\begin{bmatrix}0\\0\end{bmatrix}$), then

$$\underline{P}_\infty=\underline{\Phi}\underline{P}_\infty\underline{\Phi}^T+\underline{\Gamma}\underline{Q}\underline{\Gamma}^T \qquad (6)$$

b) Write a MATLAB code that generates x(k) for k = 1..500 and x(0) = $[0\ 0]^T$. Generate five sequences of x(k) and plot them on two diagrams, one for x_1(k) and one for x_2(k).
ANS:
I use MATLAB function (random) to create normal distribution w(k), and I run equation 1 for 500 to create one sequence. Then, I run the same steps 5 time to create 5 sequences.
The following are two diagram for x_1(k) and x_2(k). In each diagram, I use different colors to show different sequences.


Stochastic Differential Equation: x_1
15
10
5
0
-5
-10
-15
0
100
200
300
400
500
k
x_1 (k)


Stochastic Differential Equation: x_2
15
10
5
0
-5
-10
-15
0
100
200
300
400
500
k
x_2 (k)

c) Generate 50 trajectories for x(k) and from them compute the standard deviation for $x_1(k)$ and $x_2(k)$.

ANS:

First, I do similar step as problem (b), but in (c) I run 50 times to generate 50 trajectory for x(k). Next, I concatenate these 50 trajectories, and use MATLAB function (std) to compute standard deviation for $x_1(k)$ and $x_2(k)$

Standard deviation for $x_1(k)$ = 4.5724

Standard deviation for $x_1(k)$ = 3.2759

And, covariance of x1(k) and x2(k) is

$$\text{numerical } P_k = \begin{Bmatrix} 20.9070 & -12.9112 \\ -12.9112 & 10.7316 \end{Bmatrix}$$

d) Use the analytical results to compute the state x(k) covariance matrix P(k) and corresponding standard deviations for $x_1(k)$ and $x_2(k)$. Compare them with the computed values in point c.

ANS

Fist, I use equation 7 to run 500 times to get the analytical result of P(k)

$$\begin{aligned}
&\underline{P}_1 = \underline{\Phi}\underline{P}_0\underline{\Phi}^T + \underline{\Gamma}\underline{Q}\underline{\Gamma}^T, \quad \text{where } \underline{P}_0 \text{ is zero matrix} \\
&\underline{P}_k = \underline{\Phi}\underline{P}_{k-1}\underline{\Phi}^T + \underline{\Gamma}\underline{Q}\underline{\Gamma}^T \\
&\ldots \\
&\underline{P}_\infty = \underline{\Phi}\underline{P}_\infty\underline{\Phi}^T + \underline{\Gamma}\underline{Q}\underline{\Gamma}^T
\end{aligned} \tag{7}$$

After computation, I get the analytic result of the cocariance matrix P(k)

$$P_k = \begin{Bmatrix} 18.8802 & -11.3672 \\ -11.3672 & 9.5013 \end{Bmatrix}$$

From P(k), we can get

STD. of x1 is $\sqrt{\underline{P}_k^{11}} = 4.3451$, and

STD. of x2 is $\sqrt{\underline{P}_k^{22}} = 3.0824$

The analytical value of covariance, standard deviation is close to what I get in part(c), but they will not be the same (because of random process).

Problem 2. (only for graduate students) A 2D random walk is described by the following system of stochastic differential equations

$$dx = dw_1 \tag{8}$$

$$dy = dw_2 \tag{9}$$

where dw_1 and dw_2 are increments of two independent unit intensity Wiener processes.

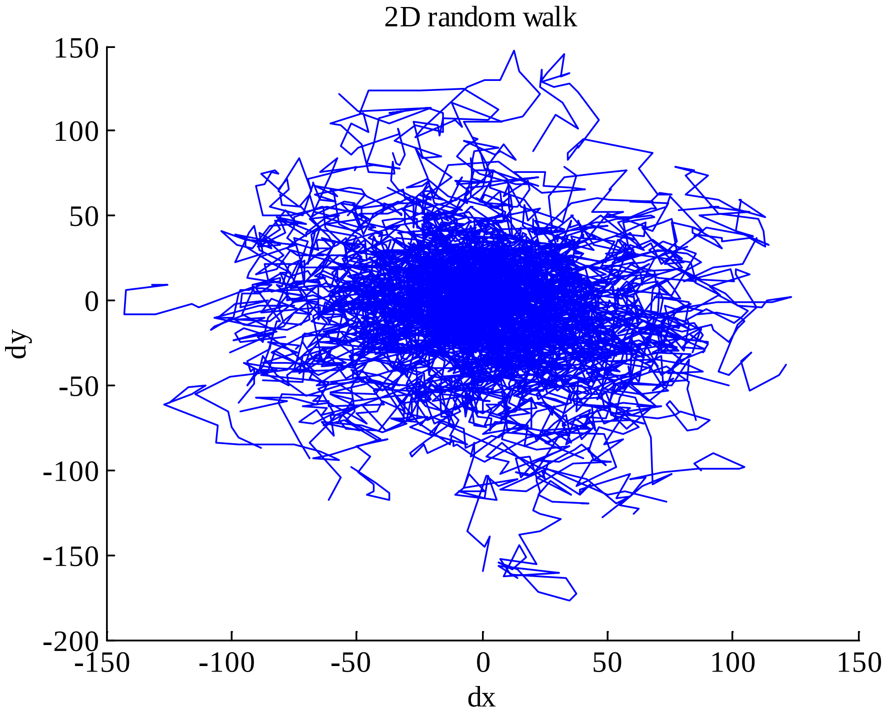
(a) Use the initial condition x(0) = y(0) = 0 to compute a trajectory x(kΔT), y(kΔT) for ΔT = 0.1s and the final time point T_f = 10s. Repeat the computations 100 times and plot all trajectories in the x-y coordinate system.

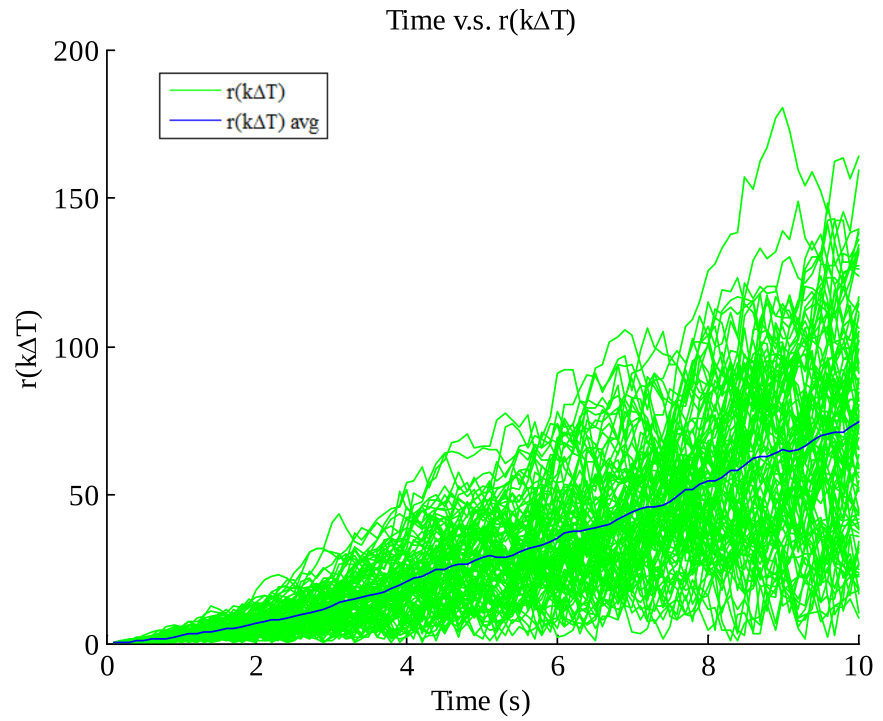
(b) For each trajectory, compute $r(k\Delta T) = \sqrt{x(k\Delta T)^2 + y(k\Delta T)^2}$ and their average for each k, $\overline{r}(k\Delta T)$. Plot on the same diagram $r(k\Delta T)$ versus the time in green, as well as $\overline{r}(k\Delta T)$ in blue.

ANS:

For question (a) and (b), I use MATLAB function (random) to create normal distribution $w(k)$ and its standard deviation $\delta = k\Delta T$, and I run equation

$x_k = x_{k-1} + dw_k = x_0 + dw_1 + \cdots + dw_k$ for 100 time to create one sequent. Then, I run the same steps 100 times to create 100 trajectories. The following are diagram of the x-y coordinate system and diagram of $r(k\Delta T)$ and $\overline{r}(k\Delta T)$.





(c) What polynomial is the best fit to $\bar{r}(k\Delta T)$?

ANS:

A second order polynomial relate to time , such as

$\bar{r}(k\Delta T) = C(k\Delta T)^2$, where C is constant about 0.9 to 0.7