Homework 6

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The state of a body falling vertically through the atmosphere is $\underline{\mathrm{x}} = [x\ \dot{x}\ \beta]^T$, where x is its height above the Earth's surface and β its ballistic coefficient. The ballistic coefficient is included as a state because it is not well known, so it must be estimated. Use EKF for continuous time models and discrete observations.

The state equations are

$$\frac{dx}{dt} = \dot{x} \tag{1}$$

$$\frac{d\dot{x}}{dt} = d - g \tag{2}$$

$$\frac{d\beta}{dt} = \xi(t) \tag{3}$$

where $g = 9.8$ $[m/s^2]$ is acceleration due to gravity, $\xi(t)$ is the zero-mean white noise of intensity 1000 $[g^2/(m^2 s^6)]$, and the drag is given by

$$d = \frac{\rho \dot{x}^2}{2\beta} \tag{4}$$

Units are provided in the rectangular brackets, however the problem is defined so that you do not need to worry about them.

Atmospheric density is

$$\rho = \rho_0 e^{-x/c} \tag{5}$$

with $\rho_0 = 1220$ $[g/m^3]$ being the density at sea level and $c = 10263$ $[m]$ a decay constant.

Suppose range measurements are taken every second (T=1 [s]) as in the figure (see the next page). Thus,

$$z_k = \sqrt{r_1^2 + (x_k - r_2)^2} + v_k \tag{6}$$

with $r_1 = 1000$ $[m]$, $r_2 = 500$ $[m]$, $x_k = x(kT)$ and $v_k \sim N(0, \sigma_r^2)$, $\sigma_r^2 = 5$ $[m^2]$.

(a) Write down the prediction step of EKF; **(b)** Write down the update step of EKF; **(c)** Assume that at $t = 0$, $E\{x(0)\} = 10000[m]$, $var\{x(0)\} = 50[m^2]$, $E\{\dot{x}(0)\} = -500[m/s]$, $var\{\dot{x}(0)\} = 200[m^2/s^2]$, $E\{\beta(0)\} = 6 \times 10^7[g/ms^2]$, $var(\beta(0)) = 2 \times 10^{12}[g^2/m^2 s^4]$ and that the measured data are taken every second beginning at $t_1 = 1[s]$, and are given in meters by

$$z_1, z_2, \ldots.z_{10} = 9055, 8560, 7963, 7467, 7000, 6378, 5885, 5400, 4928, 4503 \tag{7}$$

Plot the $x_m(k) = \sqrt{z_k^2 - r_1^2} + r_2$ and the EKF estimates for $x(t)$ on the same diagram. On separate diagrams, plot the velocity ($\dot{x}$) and the ballistic coefficient (β) estimates.

ANS:
For equation 1, I can rewrite as:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ \beta \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{\rho_0 e^{-x/c}\dot{x}^2}{2\beta} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \xi(t) \end{bmatrix} \tag{8}$$

In Extend Kalman Filter, equation 1 could refer to:

$$dx = f(x,t)dt + b(t)dw + u \tag{9}$$

then, for computing, I could set:

$$f(x,t) = \begin{bmatrix} \dot{x} \\ \frac{\rho_0 e^{-x/c}\dot{x}^2}{2\beta} \\ 0 \end{bmatrix}, \mathrm{u} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} \tag{10}$$

Because $\xi(t)$ is the zero-mean white noise of intensity $1000\ [g^2/(m^2 s^6)]$, matrix $b(t)$ could set as:

$$b = \begin{bmatrix} 0 \\ 0 \\ \sqrt{1000} \end{bmatrix} \tag{11}$$

And, equation 2 can refer to the equation in discrete observation step:

$$\underline{y}(t_k) = \underline{h_k}(\underline{x}(t_k)) + \underline{v}(t_k) \tag{12}$$

a. In the prediction step of EKF:
I use following equations and compute with MATLAB ode45 function for continuous time model:

$$\dot{\hat{\underline{x}}}(t) = f(\hat{\underline{x}}, t) + ut \in [t_k, t_{k+1}] \tag{13}$$

$$\underline{\dot{A}}(t) = \frac{\partial f(\hat{\underline{x}}, t)}{\partial \hat{\underline{x}}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{\rho_0 e^{-x/c}\dot{x}^2}{2\beta c} & \frac{\rho_0 e^{-x/c}\dot{x}}{\beta} & -\frac{\rho_0 e^{-x/c}\dot{x}^2}{2\beta^2} \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

$$\underline{\dot{P}}(t) = \underline{A}(t)P(t) + P(t)\underline{A}^T(t) + bb^T(t), t \in [t_k, t_{k+1}] \tag{15}$$

For initial guess of $\hat{\underline{x}}_0$, I use expected value provided in (c) part:

$$\hat{\underline{x}}0 = \begin{bmatrix} E\{x(0)\} \\ E\{\dot{x}(0)\} \\ E\{\beta(0)\} \end{bmatrix} = \begin{bmatrix} 10000 \\ -500 \\ 6 \times 10^7 \end{bmatrix} \tag{16}$$

For initial guess of $\hat{\underline{P}}_0$, I use variance value provided in (c) part:

$$\hat{\underline{P}}_0 = \begin{bmatrix} \mathrm{var}\{x(0)\} & P_{12} & P_{13} \\ P_{21} & \mathrm{var}\{\dot{x}(0)\} & P_{23} \\ P_{31} & P_{32} & \mathrm{var}\{\beta(0)\} \end{bmatrix} = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 2 \times 10^{12} \end{bmatrix} \tag{17}$$

In my computing, the covriance element of $\underline{\hat{P}}_0$ are set to zero for simplified. After doing ode45 function, ode45 function will return two sequence of the result (time and array of $\underline{\hat{x}}(t_{k+1}), \underline{P}(t_{k+1})$), and I will assign the last array in the sequence as the prediction of EKF:

$$\begin{aligned}\underline{\hat{x}}_{k+1}(-) = \underline{\hat{x}}(t_{k+1})\\ \underline{\hat{P}}_{k+1}(-) = \underline{\hat{P}}(t_{k+1})\end{aligned} \tag{18}$$

b. In the prediction step of EKF:
First, I use $\underline{\hat{x}}(t_{k+1})$ to update $\underline{H}_{k+1}$:

$$\underline{H}_{k+1} = \frac{\partial h_{k+1}(x_{k+1}(-))}{\partial x_{k+1}} = \frac{(x_k - r_2)}{\sqrt{r_1^2 + (x_k - r_2)}} \tag{19}$$

Then, I use the new $\underline{H}_{k+1}$ to compute the new kalman filter gain:

$$\underline{H}_{k+1} = \underline{P}_{K+1}(-)\underline{H}_{k+1}^T(\underline{H}_{k+1}\underline{P}_{K+1}(-)\underline{H}_{k+1}^T + \underline{R}_{k+1})^{-1} \tag{20}$$

with new $\underline{H}_{k+1}$ and $\underline{K}_k + 1$, I can update $\underline{\hat{x}}_{k+1}$ and $\underline{P}_{k+1}$:

$$\begin{aligned}\underline{\hat{x}}_{k+1} = \underline{\hat{x}}_{k+1}(-) + \underline{K}_{k+1}(\underline{y}_{k+1} - \underline{h}_{k+1}(\underline{\hat{x}}_{k+1}(-)))\\ \underline{P}_{k+1} = (I - \underline{K}_{k+1}\underline{H}_{k+1})\underline{P}_{k+1}(-)\end{aligned} \tag{21}$$

After updating, do another extend kalman filter with next measurement z_k.
c.
First, I use $x_m(k) = \sqrt{z_k^2 - r_1^2} + r_2$ with z_k to get the sequence of x_m. In all figures, I also add cumulating results of $x(t)$ from ode45 (as green line). The following is the diagram of x_m and $Ex(t_k)$:

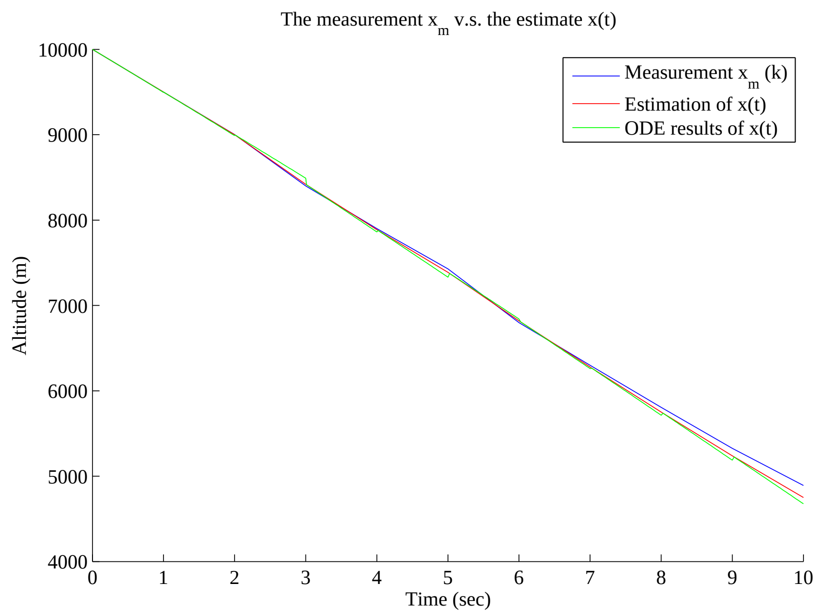

Figure 1: The measurement x_m and the EKF estimate x(t).

The following is the diagram for the velocity $\dot{x}(t)$:

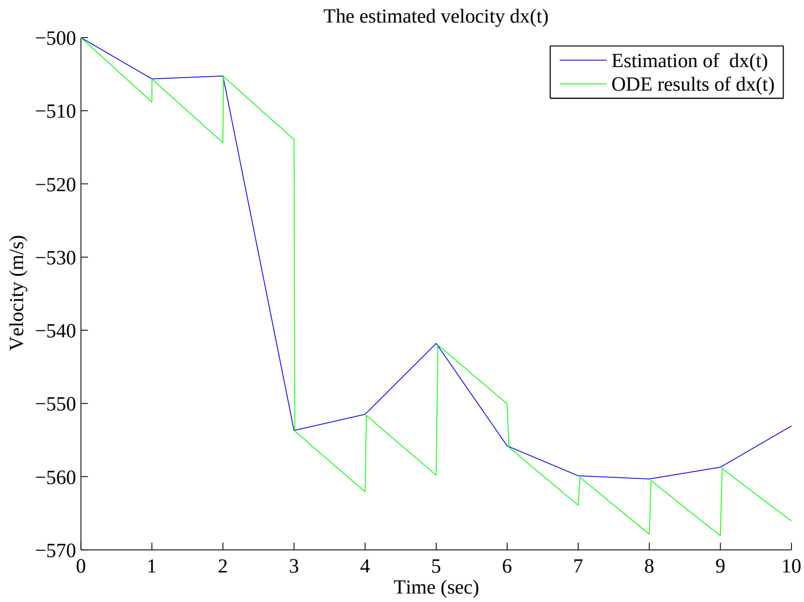

Figure 2: The EKF estimate $\dot{x}(t)$.

The following is the diagram for the ballistic coefficient $\beta(t)$:

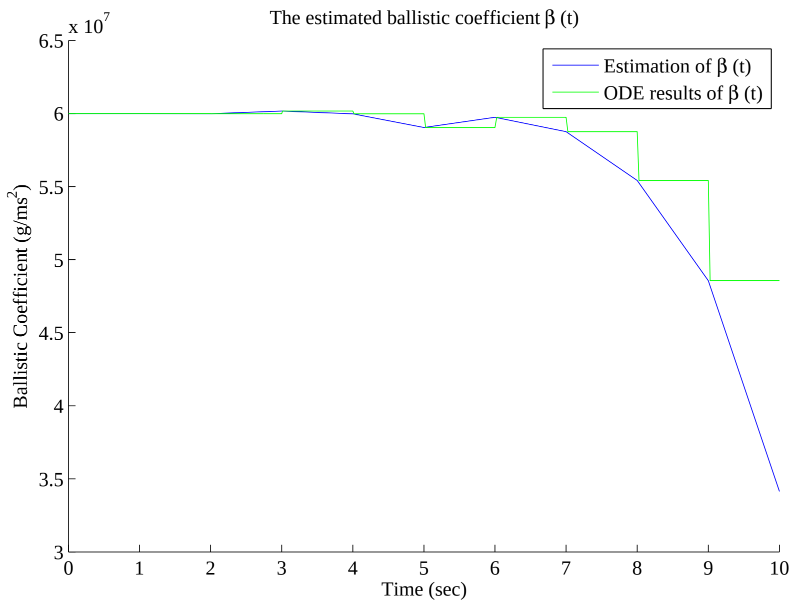

Figure 3: The EKF estimate $\beta(t)$.