ENGINEERING PROJECT REPORT

CENTRALE NANTES — APPLIED MATHEMATICS DEPARTMENT

THE BLACK-SCHOLES-MERTON FRAMEWORK: STOCHASTIC CALCULUS AND DYNAMIC HEDGING

ROBIN GUICHON

30/01/2024 - 30/05/2024

Abstract : *This project presents a comprehensive reconstruction of the Black-Scholes-Merton option pricing model, moving from discrete probabilistic foundations to continuous-time hedging strategies. Starting with the Bernoulli random walk, we rigorously derive the Wiener process, establishing the necessity of Itô Calculus through quadratic variation analysis. We then construct the no-arbitrage Partial Differential Equation (PDE) via dynamic replication and solve it using the heat equation transformation. Finally, we provide a detailed geometric and analytical examination of the risk sensitivities (the Greeks), illustrating the mechanics of delta-hedging and volatility exposure using 3D visualization.*

Keywords : *Stochastic Calculus, Black-Scholes-Merton Model, Geometric Brownian Motion, Itô's Lemma, Dynamic Hedging, Option Pricing, Quadratic Variation, Risk-Neutral Valuation, The Greeks, Volatility*

Table of Contents

Introduction

Financial markets are systems of organized chaos. At any given moment, the price of an asset reflects a consensus of millions of disparate interactions, incorporating news, risk preferences, and capital flows. To the naked eye, these price movements appear erratic and unpredictable. However, beneath this apparent randomness lies a rigorous mathematical structure. The objective of quantitative finance is not to predict the direction of these movements, but to model their probability distribution and, crucially, to price the derivatives whose value depends on them.

This paper aims to demystify the Black-Scholes-Merton (BSM) framework, arguably the most influential achievement in modern financial economics. Rather than treating the BSM formula as a "black box," we reconstruct it from first principles. We begin with the atomic unit of randomness—the coin toss—and trace the mathematical limit as time accelerates, leading us to Brownian motion, stochastic calculus, and finally, the dynamic replication of risk.

Historical Context: From Bachelier to Black-Scholes

The intellectual history of option pricing is a journey from arithmetic intuition to stochastic rigor. It began in 1900 with Louis Bachelier, whose doctoral thesis, Théorie de la Spéculation, anticipated the mathematical description of Brownian motion five years before Albert Einstein. Bachelier modeled stock prices as an arithmetic random walk. While revolutionary, his model had a fatal flaw: it allowed share prices to become negative, a financial impossibility for limited liability assets.

It was not until the mid-20th century that Paul Samuelson (1965) corrected this by proposing the Geometric Brownian Motion (GBM), ensuring positive prices by modeling returns rather than absolute price changes. However, the question of valuation remained unsolved. Investors demanded a "risk premium" for holding volatile assets, meaning that pricing an option seemed to require knowing the market's subjective risk appetite.

The breakthrough came in 1973 with Fischer Black, Myron Scholes, and Robert C. Merton. They demonstrated that by continuously trading the underlying asset and the risk-free bond, one could construct a portfolio that perfectly replicates the payoff of an option. In the absence of arbitrage opportunities, this portfolio must earn the risk-free rate. This insight eliminated the need to estimate the market's expected return (), making the pricing formula dependent only on observable variables: the stock price, time, strike price, risk-free rate, and volatility.

The Modeling Challenge: Deterministic versus Stochastic Calculus

The necessity of Stochastic Calculus arises from the fundamental limitations of Newtonian calculus when applied to financial time series. In classical analysis, functions are assumed to be differentiable, implying that local variations are linear with respect to time. Consequently, in a Taylor series expansion, second-order terms scale with $(dt)^2$ and vanish as the time step dt approaches zero.

Financial asset prices violate this smoothness assumption as they exhibit infinite variation and non-differentiability at any scale. The stochastic increment, denoted as dW_t, scales with the square root of time $\sqrt{dt}$ rather than time itself. This fractal property leads to a profound result that differentiates stochastic calculus from its deterministic counterpart:

$$(dW_t)^2 = dt \tag{1.1}$$

Since the squared differential is of the order dt, it does not vanish in the limit. This persistent term represents the convexity of the price path. The omission of this second-order effect would result in a systematic mispricing of derivative contracts. Itô Calculus addresses this irregularity by explicitly retaining the quadratic variation, thereby introducing the necessary convexity correction term $\frac{1}{2}\sigma^2$ into the drift of the asset.

The Probabilistic Foundations

II.1 The Microscopic Genesis: Discrete Random Walks

The construction of continuous-time market models begins with the simplest description of uncertainty: the discrete random walk. This mathematical idealization of a fair coin toss serves as the atomic unit from which complex stochastic processes are built.

II.1.1 The Bernoulli Process

Consider a discrete time scale $t = 0, 1, 2, \ldots, n$ where the asset price moves in fixed increments. Let ξ_i be a sequence of independent and identically distributed (i.i.d.) random variables representing the price change at step i. In the symmetric Bernoulli model, ξ_i takes values in the set $\{+1, -1\}$ with equal probability:

$$P(\xi_i = +1) = P(\xi_i = -1) = \frac{1}{2} \tag{1.1}$$

The position of the random walk after n steps, denoted by S_n, is the cumulative sum of these increments:

$$S_n = S_0 + \sum_{i=1}^{n} \xi_i \tag{1.2}$$

This process generates a jagged trajectory that fluctuates around its origin without a preferred direction. Figure II.1.1 illustrates a single realization of this process over ten steps. The path exhibits local unpredictability while maintaining discrete continuity.

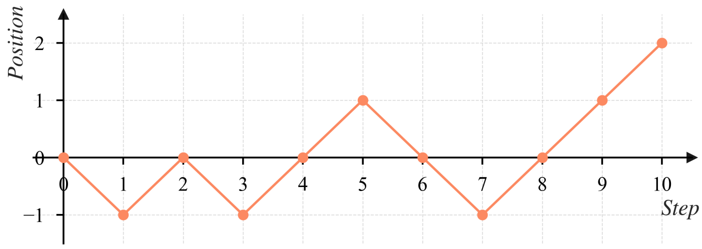

Figure II.1.1 – **Discrete Bernoulli Random Walk.** A single realization over 10 steps ($S_0 = 0$). The position evolves through discrete jumps of unit size $\Delta x = 1$, illustrating the fundamental unpredictability of the path at the microscopic scale.

II.1.2 Statistical Properties: Expectation and Variance

The behavior of the random walk is characterized by its moments. Since the elementary steps are symmetric, the expected value of each increment is zero:

$$\mathbb{E}[\xi_i] = (+1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0 \tag{1.3}$$

Linearity of expectation implies that the expected position of the walk remains constant at its initial value S_0. This property defines the symmetric random walk as a *martingale* in discrete time. While the mean trajectory is flat, the uncertainty regarding the position increases with every step. We quantify this uncertainty via the variance. The variance of a single step is:

$$\text{Var}(\xi_i) = \mathbb{E}[\xi_i^2] - (\mathbb{E}[\xi_i])^2 = 1 - 0 = 1 \tag{1.4}$$

Due to the independence of the increments, the variance of the sum is equal to the sum of the variances. Thus, after n steps, the total variance is:

$$\text{Var}(S_n) = \sum_{i=1}^{n} \text{Var}(\xi_i) = n \tag{1.5}$$

II.1.3 Scaling Laws and the Square Root of Time

The relationship between variance and time steps reveals the fundamental scaling law of stochastic processes. While the variance grows linearly with n, the standard deviation—which measures the typical deviation from the mean—scales with the square root of n:

$$\sigma_{S_n} = \sqrt{\text{Var}(S_n)} = \sqrt{n} \tag{1.6}$$

This square-root scaling implies that diffusion is rapid initially but decelerates over time. Figure II.1.2 displays multiple simulated paths of the random walk. The enveloping curves represent the standard deviation bounds $\pm 1\sqrt{n}$, $\pm 2\sqrt{n}$, and $\pm 3\sqrt{n}$.

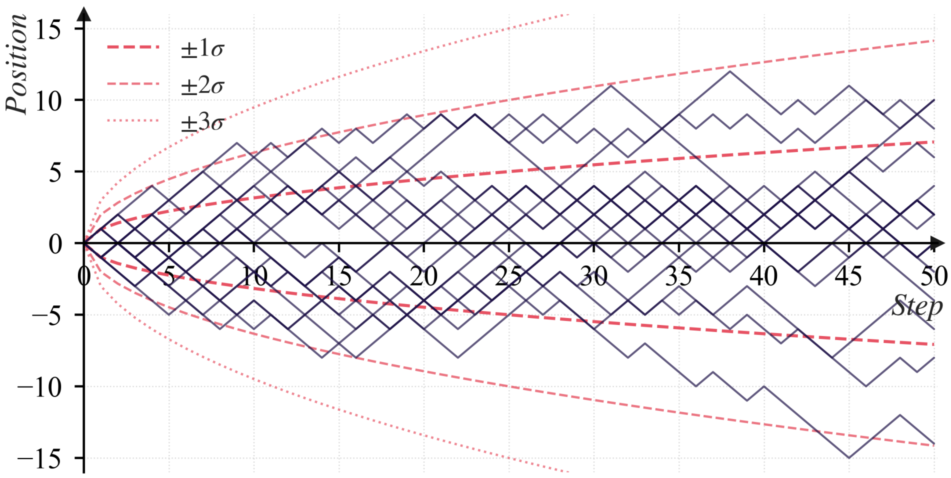

Figure II.1.2 – **Dispersion and Scaling Laws.** Multiple realizations of the random walk over 50 steps. The trajectories spread out over time, strictly contained probabilistically within the parabolic envelopes defined by $\sigma \propto \sqrt{n}$. This visualization confirms that uncertainty grows non-linearly with time.

The containment of the majority of paths within these bounds anticipates the emergence of the Gaussian distribution, as dictated by the Central Limit Theorem. This scaling property $\sigma \propto \sqrt{t}$ is the defining characteristic that necessitates the use of Itô calculus in the continuous limit, as the path variation becomes infinite.

II.2 The Passage to the Limit: Brownian Motion

The discrete random walk provides an intuitive model for price movements, but financial markets operate in continuous time. To capture this reality, we must take the limit of the random walk as the time step Δt approaches zero and the number of steps n approaches infinity. This limiting process yields the Wiener process, the fundamental building block of continuous-time finance.

II.2.1 The Central Limit Theorem (CLT)

Let us consider the scaled random walk over a fixed time horizon T. We divide this interval into n steps of length $\Delta t = T/n$. To ensure the variance of the process remains finite and non-zero as $n \to \infty$, the spatial step size Δx must scale according to the square root of time: $\Delta x = \sigma\sqrt{\Delta t}$.

The final position $W_T^{(n)}$ is the sum of n independent Bernoulli increments. According to the Central Limit Theorem (Lindeberg-Lévy), the distribution of this normalized sum converges in distribution to a Gaussian random variable as $n \to \infty$:

$$W_T^{(n)} \xrightarrow{d} \mathcal{N}(0, T) \tag{2.1}$$

Figure II.2.1 illustrates this convergence. The histogram represents the distribution of final positions for a simulated random walk with a large number of steps. The superimposed curve is the theoretical probability density function of the Normal distribution $\mathcal{N}(0, n\Delta x^2)$. The precise alignment demonstrates that macroscopic Gaussian behavior emerges from microscopic binary randomness.

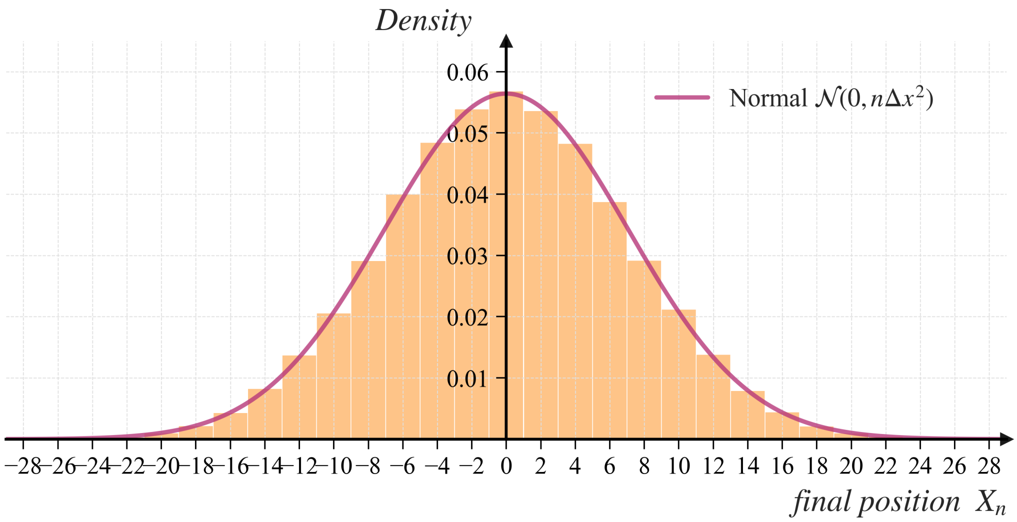

Figure II.2.1 – **Convergence to Normality.** Histogram of final positions X_n after 50 steps across 10,000 simulated walks. The red curve represents the theoretical Normal density $\mathcal{N}(0, n\Delta x^2)$. This visual evidence confirms the Central Limit Theorem, justifying the use of Gaussian processes in continuous-time modeling.

II.2.2 The Limiting Process: From Discrete to Continuous

The convergence illustrated by the Central Limit Theorem motivates the transition to continuous time. However, financial markets do not move in fixed jumps at fixed intervals. To model reality, we must imagine making the time steps smaller and smaller ($\Delta t \to 0$) while increasing their frequency to infinity ($n \to \infty$).

Mathematically, the continuous process W_t is defined as the limit of the discrete partial sums:

$$W_t = \lim_{\Delta t \to 0} \sum_{i=1}^{t/\Delta t} \xi_i \tag{2.2}$$

Crucially, as we refine the time grid, we must also adjust the size of the price jumps. If the step size Δx remained constant while the number of steps exploded, the variance would diverge to infinity. To preserve a finite, non-zero variance—preserving the "texture" of randomness—the step size must scale according to the square root of the time increment:

$$\Delta x = \sigma\sqrt{\Delta t} \tag{2.3}$$

This diffusive scaling rule is fundamental. It ensures that uncertainty accumulates linearly with time ($\mathrm{Var} \propto t$), preventing the process from either flattening out or exploding.

Figure II.2.2 illustrates this transition. It displays three random walks with decreasing time steps. As the grid becomes finer, the jagged, discrete path converges visually towards a continuous curve. This limit is the Brownian motion.

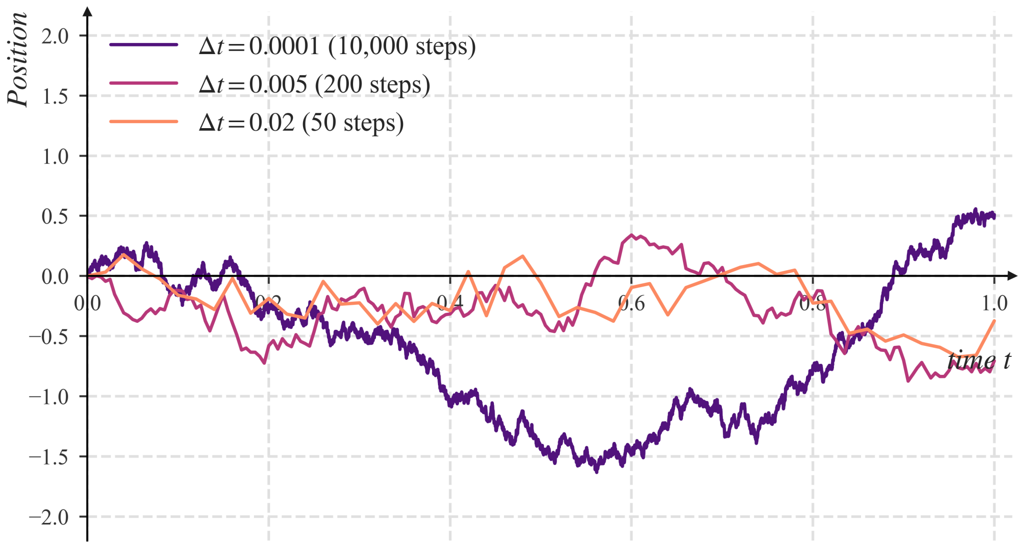

Figure II.2.2 – **From Discrete to Continuous.** Three realizations of random walks with decreasing time steps ($\Delta t = 0.02, 0.005$, and 0.0001). As the time grid becomes finer, the discrete zig-zag path converges to the continuous Brownian motion W_t.

II.2.3 Formal Definition of the Wiener Process (W_t)

The object that emerges from this limit is known as the Wiener process (or Standard Brownian Motion), denoted by W_t. It is the fundamental building block of continuous-time finance and is defined by four axiomatic properties:

1. **Initialization:** The process starts at zero with probability 1: $W_0 = 0 \quad \text{a.s.}$ This provides a fixed reference point for the accumulation of uncertainty.
2. **Independent Increments:** For any partition of time $0 \le t_1 < t_2 < \cdots < t_k$, the increments $W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}}$ are statistically independent. This implies that the process is memoryless; future movements depend only on the current state, not on the path taken to reach it.
3. **Gaussian Increments:** The increment over any time interval follows a Normal distribution with mean zero and variance equal to the elapsed time: $W_t - W_s \sim \mathcal{N}(0, t-s) \quad \text{for } 0 \le s < t$
4. **Path Continuity:** The sample paths $t \mapsto W_t$ are continuous almost surely. The process does not exhibit jumps, meaning prices evolve through an unbroken series of infinitesimal movements, even though they are nowhere differentiable.

These definitions establish W_t as a martingale with quadratic variation $[W, W]_t = t$, a property that will necessitate the development of stochastic calculus in the following sections.

II.3 The Stochastic Process with Drift and Diffusion

Up to now, we have seen that the standard Brownian motion W_t represents a martingale with zero mean and variance increasing linearly over time. It captures randomness in its purest form. However, real-world quantities are rarely so indifferent: they exhibit varying intensities of fluctuation and often follow a systematic trend.

To model this, we decompose the infinitesimal change in X_t into two parts:

$$\Delta X_t = \text{deterministic part} + \text{random part} \tag{3.1}$$

II.3.1 Scaling Brownian Motion: Introducing the Random Component (σ)

If we multiply a standard Brownian motion by a constant σ, we obtain a scaled Brownian motion σW_t. This new process has variance $\text{Var}(\sigma W_t) = \sigma^2 t$. The parameter σ acts as the **volatility coefficient** (or diffusion coefficient).

It controls the amplitude of the random fluctuations: a small σ gives gentle fluctuations, while a large σ makes the motion wilder. This scaled version is what drives the stochastic term in financial models.

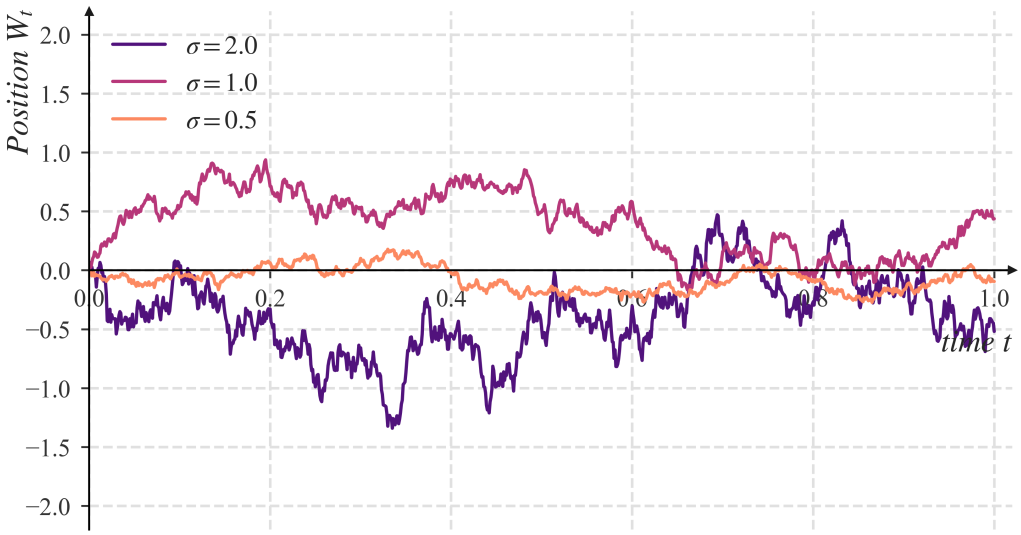

Figure II.3.1 – **Scaled Brownian Motions with Different Volatilities σ.** Three realizations of scaled Brownian motions with different volatilities ($\sigma = 0.5, 1.0, 2.0$). The paths demonstrate how the amplitude of fluctuations increases directly with σ.

II.3.2 Introducing Drift: The Deterministic Component (μ)

While the diffusion term σW_t introduces randomness, most real-world processes also exhibit a predictable, directional trend. This component is called the **drift** and is represented by a constant rate μ.

Mathematically, a pure drift process evolves as $X_t = X_0 + \mu t$. Unlike Brownian motion, this process increases or decreases linearly and deterministically. A positive drift indicates an upward tendency, while a negative drift indicates a decline.

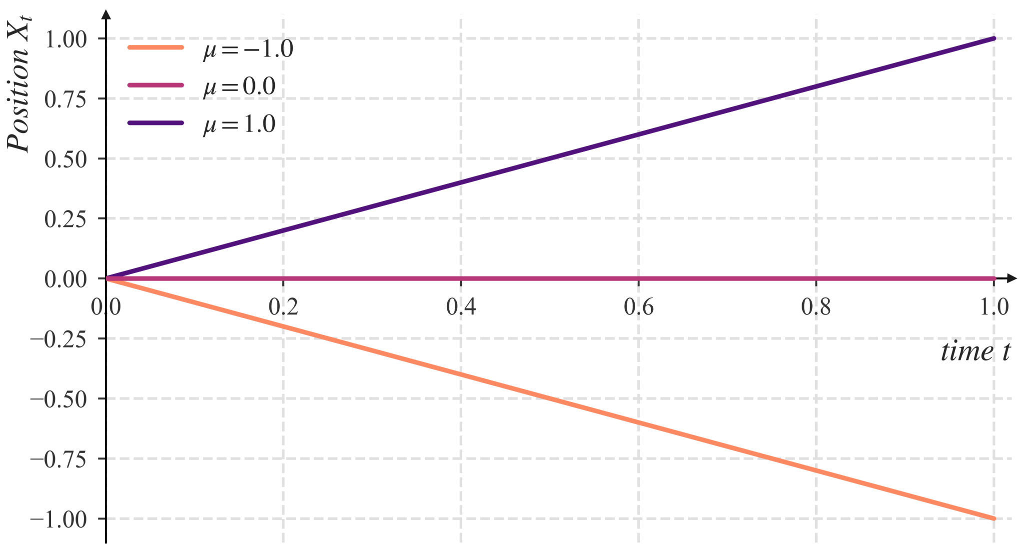

Figure II.3.2 – **Pure Drift Processes with Different Drift Parameters μ.** Three realizations of deterministic drift processes with $\mu = -1.0, 0.0, +1.0$. The paths illustrate how the drift parameter controls the direction and slope of the mean trajectory. Unlike Brownian motion, there is no randomness here.

II.3.3 Combining Drift and Diffusion: The Stochastic Differential Equation

While the diffusion term determines the spread around the path, the drift determines where the center of that spread moves through time. Together, they produce the characteristic shape of a stochastic process.

Mathematically, we express this combination using a **Stochastic Differential Equation (SDE)**:

$$dX_t = \mu dt + \sigma dW_t \tag{3.2}$$

where:
— dX_t is the infinitesimal change in the process,
— μdt is the deterministic drift component,
— σdW_t is the stochastic diffusion component.

II.3.4 From Finite to Infinitesimal Increments

To rigorously derive the Stochastic Differential Equation (SDE), we examine the behavior of the asset price change ΔX_t over a discrete time step Δt. In the discrete model, the evolution is given by:

$$\Delta X_t = \mu \Delta t + \sigma \epsilon_t \sqrt{\Delta t} \tag{3.3}$$

where $\epsilon_t \sim \mathcal{N}(0, 1)$ represents a standard normal shock.

As the time step Δt approaches zero, the scaled random term $\epsilon_t \sqrt{\Delta t}$ does not vanish but converges to the differential of the Brownian motion, denoted dW_t. This infinitesimal increment is characterized by its statistical moments :

$$\mathbb{E}[dW_t] = 0, \quad \mathrm{Var}(dW_t) = dt \tag{3.4}$$

Consequently, taking the limit $\Delta t \to 0$ transforms the discrete difference equation into the continuous SDE:

$$dX_t = \mu dt + \sigma dW_t \tag{3.5}$$

This equation encapsulates the fundamental dynamics of the asset: a deterministic drift μdt superimposed with a stochastic diffusion σdW_t. This formulation necessitates the use of stochastic calculus, as the term dW_t is of order $\sqrt{dt}$, creating non-trivial second-order effects.

The Mathematical Engine: Stochastic Calculu

III.1 From Ordinary to Stochastic Calculus

Before we can derive the geometric Brownian motion and ultimately the Black-Scholes model, we must first understand how to handle calculus when randomness is involved. In the ordinary world of smooth curves and predictable functions, the rules of differentiation and Taylor expansion work perfectly. But once we introduce Brownian motion, the mathematical landscape changes: paths become rough, and small time steps carry unpredictable "jumps" that cannot be ignored.

This is why we first need stochastic calculus: a version of calculus adapted to random motion. Its cornerstone is Itô's Lemma, which will soon let us correctly differentiate functions of stochastic processes like asset prices.

III.1.1 Why Ordinary Calculus Fails

In ordinary calculus, we study smooth functions—curves that can be approximated locally by straight lines. When a function $f(t, x)$ depends on a smooth variable $x(t)$, we can expand it using the Taylor series:

$$f(t + \Delta t, x + \Delta x) = f(t, x) + f_t \Delta t + f_x \Delta x + \frac{1}{2} f_{xx} (\Delta x)^2 + \dots \tag{1.1}$$

Now, if $x(t)$ is smooth, its change over a small time step behaves roughly like $(\Delta x \approx x'(t)\Delta t)$. That means the squared term behaves like:

$$(\Delta x)^2 \approx (x'(t))^2 (\Delta t)^2 \tag{1.2}$$

When we take the limit $\Delta t \to 0$, the term $(\Delta t)^2$ becomes negligible compared to Δt. Therefore, in ordinary calculus, we can safely ignore all higher-order terms, keeping only:

$$df = f_t dt + f_x dx \tag{1.3}$$

This simplification is what makes ordinary calculus elegant—smooth curves are "locally linear", so their tiny variations behave predictably.

In stochastic calculus, however, that trick breaks. Brownian motion W_t isn't smooth. Its increments don't shrink linearly with time; they shrink only like the square root of time:

$$\Delta W_t \sim \mathcal{N}(0, \Delta t) \tag{1.4}$$

This means that while Δt might be tiny, ΔW_t is roughly of size $\sqrt{\Delta t}$.

Now square the increment dW_t:

$$(dW_t)^2 \approx dt \tag{1.5}$$

Unlike the deterministic case, this term no longer vanishes faster than Δt. It is of the same order. So in the stochastic world, the "higher-order" term isn't negligible—it's essential. That's why the standard calculus formula must be modified, and why Itô's Lemma keeps the extra $\frac{1}{2}\sigma^2 f_{xx}$ term.

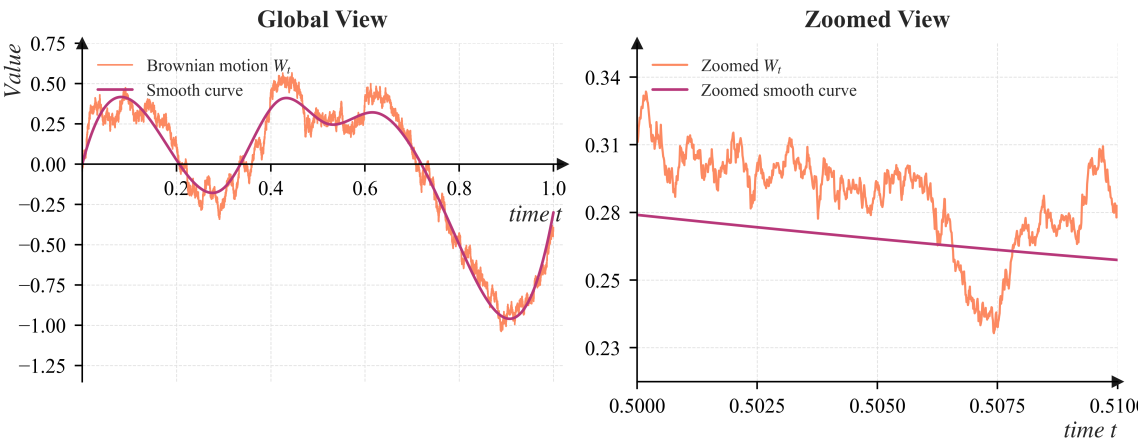

Figure III.1.1 – **Smooth Curve vs. Brownian Motion at Different Scales.** *Left:* Global View. At a broad scale, both the smooth curve and the Brownian motion appear similarly irregular. *Right:* Zoomed View. When we zoom in, the smooth curve becomes linear (differentiable), while the Brownian path remains jagged at every scale.

III.1.2 Drift and Diffusion: Two Kinds of Motion

Let's now recall our general stochastic process:

$$dX_t = \mu dt + \sigma dW_t \tag{1.6}$$

It contains two parts:

— The **drift** (μdt): the smooth, deterministic trend (expected change over time).

— The **diffusion** (σdW_t): the random, unpredictable fluctuation driven by Brownian motion.

The diffusion term introduces "second-order" randomness. It ensures that the squared increment $(dX_t)^2$ is proportional to dt. That non-zero second-order term will be what gives rise to Itô's Lemma.

III.1.3 The Taylor Expansion with Randomness

In ordinary calculus, the Taylor expansion of a function $f(X_t)$ allows us to approximate its value near a point. For a stochastic process, we must expand $f(X_t)$ up to the second order to capture the effects of curvature:

$$df = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 + \dots \tag{1.7}$$

In the deterministic world, the higher powers of dX_t vanish. However, we must substitute our stochastic process $dX_t = \mu dt + \sigma dW_t$ into this expansion. Squaring the increment yields:

$$(dX_t)^2 = (\mu dt + \sigma dW_t)^2 = \mu^2(dt)^2 + 2\mu\sigma(dt)(dW_t) + \sigma^2(dW_t)^2 \tag{1.8}$$

As we shrink the time step dt toward zero, we analyze the magnitude of each term:

1. The term $\mu^2(dt)^2$ becomes negligible compared to dt (it is of order dt^2).
2. The term $2\mu\sigma(dt)(dW_t)$ also vanishes, as it scales with $dt^{3/2}$.
3. However, the term $\sigma^2(dW_t)^2$ behaves like $\sigma^2 dt$, because the variance of dW_t is exactly dt.

Thus, unlike the deterministic case, the second-order term does not vanish. It converges to a finite value:

$$(dX_t)^2 = \sigma^2 dt \tag{1.9}$$

This result confirms that the quadratic variation is non-zero. Plugging this back into the Taylor expansion gives the fundamental approximation for stochastic differentials:

$$df = f'(X_t)(\mu dt + \sigma dW_t) + \frac{1}{2}f''(X_t)\sigma^2 dt \tag{1.10}$$

Note that we only keep terms up to second-order, because higher powers of dW_t shrink much faster than dt. Recall that dW_t behaves like $\sqrt{dt}$. That means:

$$(dW_t)^2 \sim dt, \quad (dW_t)^3 \sim (dt)^{3/2}, \quad (dW_t)^4 \sim (dt)^2, \quad \text{and so on.} \tag{1.11}$$

When $dt \to 0$, all terms involving powers higher than $(dW_t)^2$ vanish much faster than dt itself – they have no lasting contribution in the limit. Therefore, the second-order term is the highest one that still matters, because it captures the nonzero "quadratic variation" of Brownian motion. All higher-order terms disappear, leaving only up to the second order in the Itô expansion.

III.1.4 Arriving at Itô's Lemma

At this point, we can summarize what we have derived: when randomness enters the picture through Brownian motion, the usual rules of calculus must adapt. The key difference lies in the nonvanishing second-order term—it carries real information about the process's variability.

We now want to know how a smooth function of a stochastic process, say $f(t, X_t)$, evolves over time. In other words, if X_t itself follows a stochastic path, how does this affect the motion of $f(t, X_t)$?

To find out, we extend our expansion to include the time-dependence of f:

$$df = f_t dt + f_x dX_t + \frac{1}{2} f_{xx} (dX_t)^2 \tag{1.12}$$

Here, f_t and f_x denote the partial derivatives of f with respect to time t and the process X_t, respectively. We now substitute our stochastic process:

$$dX_t = \mu dt + \sigma dW_t \tag{1.13}$$

Expanding step by step gives:

$$\begin{aligned} df &= f_t dt + f_x(\mu dt + \sigma dW_t) + \frac{1}{2} f_{xx}(\mu dt + \sigma dW_t)^2 \\ &= f_t dt + f_x(\mu dt + \sigma dW_t) + \frac{1}{2} f_{xx}\sigma^2 (dW_t)^2 \\ &= f_t dt + f_x \mu dt + f_x \sigma dW_t + \frac{1}{2} f_{xx}\sigma^2 dt \end{aligned} \tag{1.14}$$

since $(dW_t)^2 = dt$ and all higher-order terms vanish. Collecting terms by powers of dt and dW_t yields:

$$df = \underbrace{\left(f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} \right) dt}_{\text{drift (deterministic) term}} + \underbrace{\sigma f_x}_{\text{diffusion (random) term}} dW_t \tag{1.15}$$

This is **Itô's Lemma** — the stochastic equivalent of the chain rule.

Alternatively, Itô's Lemma is often written in fully expanded partial-derivative notation, which highlights its structure in applications to finance:

$$dF(t, X_t) = \frac{\partial F}{\partial t} dt + \mu \frac{\partial F}{\partial X} dt + \sigma \frac{\partial F}{\partial X} dW + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} \sigma^2 dt \tag{1.16}$$

The four terms have a clear interpretation:

— $\frac{\partial F}{\partial t} dt$: explicit time dependence,
— $\mu \frac{\partial F}{\partial X} dt$: deterministic drift,
— $\sigma \frac{\partial F}{\partial X} dW$: random diffusion,
— $\frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial X^2} dt$: curvature correction from randomness.

This lemma will serve as the foundation for everything that follows, especially for deriving the dynamics of the geometric Brownian motion and the Black-Scholes equation.

III.2 Geometric Brownian Motion

In financial modeling, a fundamental requirement is that the price of limited-liability assets (such as stocks) cannot be negative. Furthermore, asset prices tend to evolve continuously while exhibiting random fluctuations. While we have established the mathematical rules of stochastic calculus, we must now select the appropriate stochastic process to model these prices. The industry standard, which forms the backbone of the Black-Scholes framework, is the **Geometric Brownian Motion (GBM)**.

III.2.1 From Arithmetic to Geometric Brownian Motion

To understand the necessity of the GBM, let us first revisit the Arithmetic Brownian Motion (ABM) introduced earlier:

$$dX_t = \mu dt + \sigma dW_t \tag{2.1}$$

In this model, changes in value are additive. A random shock adds a fixed amount of currency to the price, regardless of the current price level. This leads to two major issues:

1. **Negative Prices:** Since Brownian motion ranges from $-\infty$ to $+\infty$, an additive shock can easily push the asset price X_t below zero, which is financially impossible for equities.
2. **Scale Invariance:** Investors typically think in terms of returns (percentages), not absolute changes. A €5 move matters much more if the stock is at €10 than if it is at €1000.

To resolve this, we model the *rate of return* rather than the absolute change. We postulate that the percentage change follows a Brownian process:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{2.2}$$

Multiplying both sides by S_t yields the standard Stochastic Differential Equation (SDE) for the Geometric Brownian Motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.3}$$

Here, the drift μ represents the expected instantaneous rate of return, and σ represents the volatility of that return. Crucially, because the noise is multiplied by the current price S_t, the magnitude of fluctuations scales with the price, ensuring that if $S_0 > 0$, the price S_t remains positive forever.

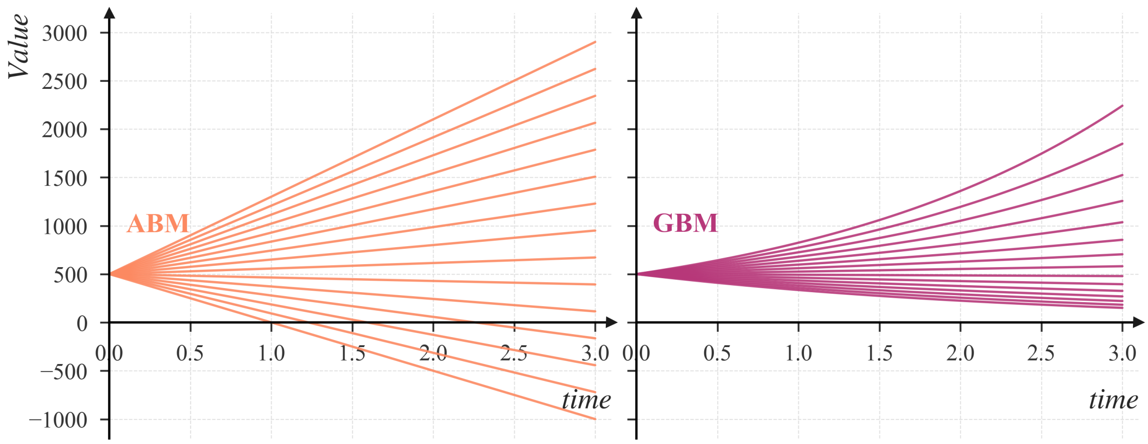

Figure III.2.1 – **Price Paths under Arithmetic vs. Geometric Brownian Motion.** *Left:* The ABM (red) adds noise linearly, allowing paths to cross into negative territory. *Right:* The GBM (green) applies multiplicative noise. As the price increases, the variance increases, creating an exponential growth pattern that stays strictly positive.

III.2.2 Logarithmic Transformation and the Role of the Exponential Function

The term "geometric" within the Geometric Brownian Motion (GBM) signifies that the process follows a trajectory of exponential growth. To gain a clearer analytical perspective, it is instructive to apply a logarithmic transformation. Let us define:

$$X_t = \ln S_t \tag{2.4}$$

Given that $S_t = e^{X_t}$, the evolution of S_t can be mapped directly to the dynamics of X_t, often yielding a form that is mathematically more tractable. Crucially, the logarithm converts multiplicative growth into additive increments, allowing for the application of standard linear analysis tools. In the context of finance, this is equivalent to working with log returns, which possess the convenient property of aggregating additively over time.

The next step is to determine the evolution of $X_t = \ln S_t$ assuming S_t follows a GBM. This transformation cannot be performed using ordinary calculus, since S_t is not a smooth function of time—it contains stochastic noise. Instead, we must apply Itô's Lemma, a fundamental result in stochastic calculus that generalizes the chain rule to random processes.

III.2.3 Applying Itô's Lemma

Given $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $f(S_t) = \ln S_t$, Itô's Lemma states:

$$df = f'(S_t)dS_t + \frac{1}{2}f''(S_t)(dS_t)^2 \tag{2.5}$$

The derivatives of f are:

$$f'(S_t) = \frac{1}{S_t}, \quad f''(S_t) = -\frac{1}{S_t^2} \tag{2.6}$$

We also use the stochastic property $(dW_t)^2 = dt$, implying $(dS_t)^2 = \sigma^2 S_t^2 dt$. Substituting, we obtain:

$$\begin{aligned} d(\ln S_t) &= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2}(\sigma^2 S_t^2 dt) \\ &= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \end{aligned} \tag{2.7}$$

Simplifying terms gives the differential equation for the logarithm of the price:

$$d(\ln S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \tag{2.8}$$

This is a stochastic process with a deterministic drift $(\mu - \frac{1}{2}\sigma^2)$ and volatility σ.

III.2.4 Integration and the Closed-Form Solution

Integrating both sides of the differential equation from the initial time 0 to any future time t yields the following expression:

$$\int_0^t d(\ln S_u) = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^t \sigma dW_u \tag{2.9}$$

The integral on the left-hand side computes directly to:

$$\ln S_t - \ln S_0 \tag{2.10}$$

On the right-hand side, the first integral is purely deterministic and evaluates linearly:

$$\int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du = \left(\mu - \frac{1}{2}\sigma^2\right) t \tag{2.11}$$

The second integral represents a stochastic component. Since W_t is a standard Brownian motion, this integral follows a normal distribution with a mean of 0 and a variance of $\sigma^2 t$. Specifically:

$$\int_0^t \sigma dW_u = \sigma W_t \tag{2.12}$$

This relies on the property that the increment $W_t - W_0 = W_t$ has variance t. Combining these parts results in the integrated form:

$$\ln S_t - \ln S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \tag{2.13}$$

To express the process in terms of the asset price S_t rather than its logarithm, we exponentiate both sides:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{2.14}$$

This equation provides the closed-form solution for the Geometric Brownian Motion. It explicitly demonstrates that S_t is lognormally distributed, as $\ln S_t$ follows a normal distribution.

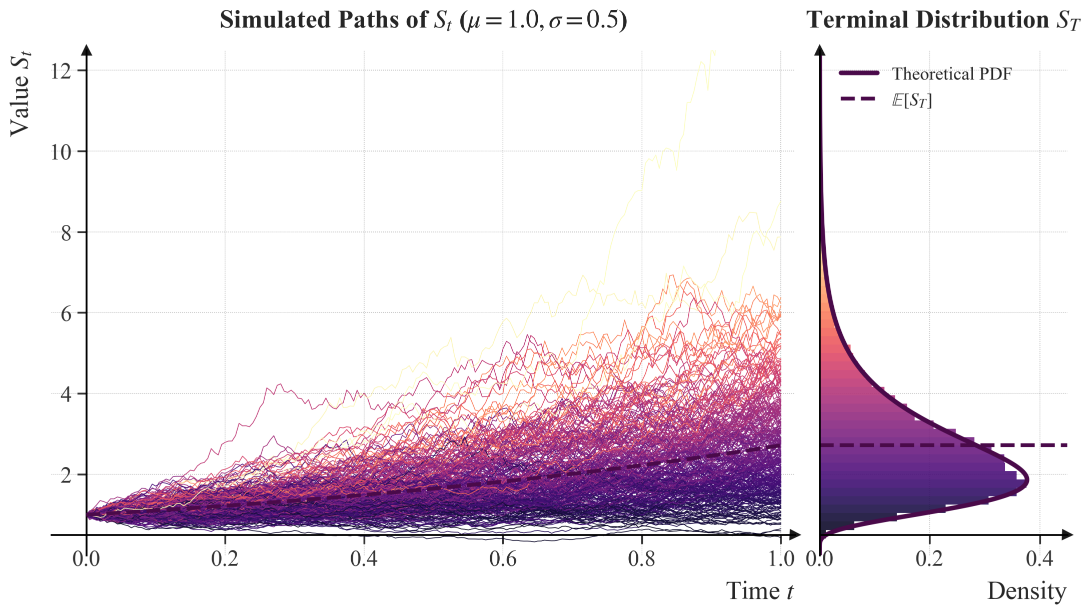

Figure III.2.2 – **Figure III.2.2: Geometric Brownian Motion Paths Generated by the Closed-Form Solution.** *The left panel displays simulated GBM trajectories ($\mu = 1.0, \sigma = 0.5$), colored by terminal value to visually connect with the right panel. The latter shows the resulting lognormal distribution (histogram and theoretical density) along with the expected value $\mathbb{E}[S_T]$, demonstrating the right-skewed outcomes of multiplicative stochastic growth.*

III.2.5 Interpretation: Expected Return and Volatility

The Geometric Brownian Motion model elegantly decomposes price evolution into two distinct components: a deterministic exponential trend and a stochastic random fluctuation. The expected rate of return is governed by μ, while σ captures the amplitude of uncertainty.

The term $-\frac{1}{2}\sigma^2$ originates from the curvature correction in Itô's Lemma and carries a significant economic interpretation. It indicates that higher volatility effectively reduces the expected growth rate of log prices. This phenomenon, often termed "volatility drag," reflects the fact that uncertainty lowers the geometric (compound) average return, even if the arithmetic mean return μ remains constant.

Thus, the GBM captures two fundamental aspects of asset prices:

- Expected return (μ): the central tendency or drift of the price process.
- Volatility (σ): the magnitude of randomness that drives price fluctuations.

The interplay between these parameters defines the complete probabilistic structure of future prices—a relationship that will be pivotal in deriving the Black-Scholes equation in the subsequent section.

The Black-Scholes-Merton Framework

IV.1 Theoretical Framework: Fundamental Pricing Principles

Up to this point, we have modeled the behaviour of an asset's price using the Geometric Brownian Motion (GBM):

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1.1}$$

This equation describes how a stock evolves under the influence of a deterministic drift and random fluctuations. However, the fundamental question in valuation is different: given this uncertain evolution, how do we determine the fair value of a derivative whose payoff depends on S_t? To answer this, we introduce three central ideas in modern financial economics: hedging, no-arbitrage, and risk-neutral valuation.

IV.1.1 Hedging: Removing Risk through Combination

So far, we have described how the underlying asset, such as a stock, evolves randomly through time. But in modern financial markets, investors often trade not only the asset itself but also contracts whose value depends on it—these are called derivatives.

A derivative is any financial instrument whose price is derived from another underlying variable, typically a stock price, an interest rate, or an index. Among many types of derivatives (futures, forwards, swaps, etc.), the most fundamental for our purpose is the option.

An option gives its holder the right, but not the obligation, to buy or sell the underlying asset at a predetermined price (the strike price) on or before a certain date (the maturity). For example, a European call option allows its holder to buy the stock at the strike price on the expiration date T, while a put option allows selling it at T.

Thus, the option's value $V(S_t, t)$ depends on both the current stock price S_t, which evolves stochastically, and the remaining time to maturity $T - t$.

Because S_t is random, the option's value is random as well. However, as Black and Scholes famously realized, it is possible to construct a combination of the option and the underlying stock that eliminates this randomness over an infinitesimal time interval. That's the idea of hedging.

Imagine forming a portfolio Π_t consisting of:

$$\Pi_t = V(S_t, t) - \Delta S_t \tag{1.2}$$

where Δ is the number of shares held short (if $\Delta > 0$) or long (if $\Delta < 0$). The goal is to choose Δ such that the random component—the term involving dW_t—disappears from $d\Pi_t$. If this can be achieved, the portfolio becomes locally risk-free over a small time step dt.

This idea—that a combination of risky assets can eliminate risk—is the seed of replicating portfolios and the key to the Black-Scholes construction.

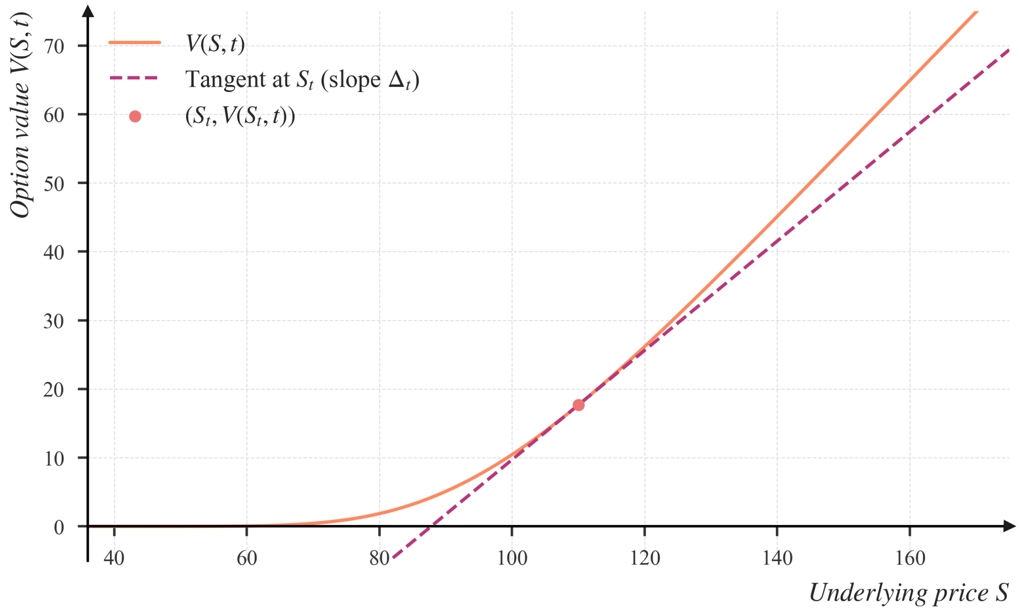

Figure IV.1.1 – **Local Delta-Hedge: Option Value and Tangent at S_t.** This figure illustrates how the option value $V(S_t, t)$ can be locally approximated by a straight line with slope $\Delta_t = \frac{\partial V(S_t, t)}{\partial S_t}$. The curved line represents the non-linear relationship between the option price and the underlying asset price, while the dashed tangent shows how holding Δ_t units of the underlying replicates the option's instantaneous sensitivity to small changes in S_t. By constructing this local linear combination, the random component of the portfolio's value can be neutralized—forming the core idea of delta-hedging in the Black-Scholes framework.

IV.1.2 No-Arbitrage: Equal Risk, Equal Return

Once the portfolio is made risk-free, logic demands that it must earn the same rate of return as any other risk-free investment. If it didn't—say, if it earned more than the risk-free rate r—investors could borrow money at r, form the same portfolio, and make a guaranteed profit. Such "free money" opportunities are called arbitrage, and their absence is a cornerstone assumption of rational markets.

IV.1.3 Risk-Neutral Valuation: Changing the Measure

So far, everything we've done—the stochastic model, the hedging argument, and the no-arbitrage condition—has been expressed in the real world, the one investors actually live in. In this world, the stock's expected growth rate μ exceeds the risk-free rate r because investors require a risk premium for bearing uncertainty. The GBM we derived earlier reflects that reality (denoted by the physical measure $\mathbb{P}$):

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}} \tag{1.3}$$

Now imagine a different world—a fictional but very convenient one—where investors are completely indifferent to risk. In this risk-neutral world, nobody demands extra return for uncertainty, because risk can be perfectly hedged away (as we just saw with the replicating portfolio). If risk can be eliminated, then it should no longer be rewarded.

That single idea changes everything: in the risk-neutral world, every asset, whether risky or not, grows on average at the same rate as the risk-free asset. Mathematically, the stock's drift μ is replaced by r:

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}} \tag{1.4}$$

This is not a new stochastic process—it's the same random evolution viewed through a different probabilistic lens, called the risk-neutral measure $\mathbb{Q}$. Under this measure, prices evolve as if all investors were risk-neutral.

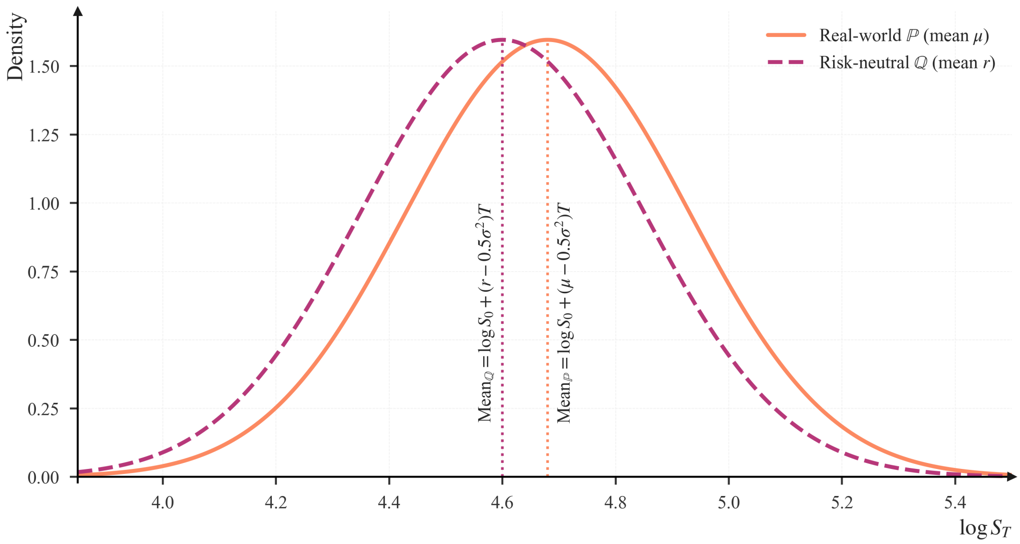

Figure IV.1.2 – **Real-World vs. Risk-Neutral Stock Price Dynamics.** This figure illustrates the probability density of the logarithmic terminal price, $\log S_T$, under the real-world measure $\mathbb{P}$ (drift μ) and the risk-neutral measure $\mathbb{Q}$ (drift r). Both distributions share the same variance, $\sigma^2 T$, but differ in their means. This horizontal shift visually represents the transition from the real world—where investors demand a risk premium—to the risk-neutral world, where all assets are expected to grow at the risk-free rate (r).

IV.2 The Black-Scholes Model

In the preceding sections, we established the stochastic foundation of asset price dynamics using the Geometric Brownian Motion (GBM) and analyzed how randomness propagates through functions using Itô's Lemma. Our objective now shifts from merely describing how prices behave to determining what the fair value of a derivative on that asset should be.

The conceptual link between stochastic dynamics and valuation relies on three central tenets: hedging, no-arbitrage, and risk-neutral valuation. Together, they articulate a singular truth: the fair price of a derivative is the one that precludes any riskless profit. Even though the underlying price follows a random path, it is theoretically possible to construct a continuously hedged portfolio that completely eliminates this randomness. Once such a riskless combination is established, economic logic dictates that it must earn the risk-free rate of return; otherwise, an arbitrage opportunity would exist.

The Black-Scholes model formalizes this intuition mathematically. By applying Itô's Lemma to the derivative value $V(S_t, t)$, substituting the GBM dynamics for S_t, and enforcing the no-arbitrage condition (that the hedged portfolio earns the risk-free rate r), we derive the Black-Scholes partial differential equation—the mathematical core of modern option pricing theory.

IV.2.1 Deriving the Black-Scholes Equation

We begin with the option value $V(S_t, t)$, which depends on both the underlying price S_t and time t. Here, $V(S_t, t)$ denotes the derivative's value, but it is often denoted as $C(S_t, t)$ for a call option or $P(S_t, t)$ for a put option. Since S_t follows a stochastic process, the option value will also move randomly. By applying Itô's Lemma, we can express its infinitesimal change as:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS_t)^2 \tag{2.1}$$

The underlying price evolves according to the geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.2}$$

where μ is the expected return, σ the volatility, and dW_t the random shock (the increment of a Wiener process). Substituting this into the expression for dV gives:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} (\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (\mu S_t dt + \sigma S_t dW_t)^2 \tag{2.3}$$

Recalling the stochastic calculus rule that $(dW_t)^2 = dt$, we simplify the squared term:

$$dV = \frac{\partial V}{\partial t}dt + \mu S_t \frac{\partial V}{\partial S}dt + \sigma S_t \frac{\partial V}{\partial S}dW_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 dt \tag{2.4}$$

Then, collecting terms by dt and dW_t yields:

$$dV = \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S_t \frac{\partial V}{\partial S}dW_t \tag{2.5}$$

Both S_t and $V(S_t, t)$ depend on the same random term dW_t. This common dependence exposes both the stock and the option to the same underlying source of uncertainty.

IV.2.2 Constructing a Riskless Portfolio

To neutralize this randomness, we construct a portfolio that combines the derivative and the underlying stock in such proportions that the random parts cancel out. Let this portfolio be defined as:

$$\Pi_t = V(S_t, t) - \Delta S_t \tag{2.6}$$

where Δ represents the number of shares of the underlying held short (if $\Delta > 0$) or long (if $\Delta < 0$). The infinitesimal change in the portfolio value over time dt is given by:

$$d\Pi_t = dV - \Delta dS_t \tag{2.7}$$

Substituting the earlier derived expressions for dV and dS_t gives:

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S_t \frac{\partial V}{\partial S}dW_t - \Delta(\mu S_t dt + \sigma S_t dW_t) \tag{2.8}$$

Rearranging the terms to group deterministic and stochastic components yields:

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - \Delta\mu S_t\right) dt + \left(\sigma S_t \frac{\partial V}{\partial S} - \Delta\sigma S_t\right) dW_t \tag{2.9}$$

The second bracket contains the random term multiplied by dW_t. We can make the portfolio risk-free by choosing Δ such that this random part vanishes. To eliminate it, we set:

$$\Delta = \frac{\partial V}{\partial S} \tag{2.10}$$

This choice makes the entire coefficient of dW_t equal to zero. In words, by holding exactly $\frac{\partial V}{\partial S}$ shares of the underlying stock for each option, we perfectly offset the instantaneous risk of small price movements.

Substituting this value of Δ back into the expression for $d\Pi_t$, we obtain:

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - \mu S_t \frac{\partial V}{\partial S}\right) dt + \left(\sigma S_t \frac{\partial V}{\partial S} - \sigma S_t \frac{\partial V}{\partial S}\right) dW_t \tag{2.11}$$

Which simplifies to:

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt \tag{2.12}$$

The randomness has been completely removed—the portfolio now evolves deterministically. It is locally risk-free over an infinitesimal time step.

IV.2.3 Applying the No-Arbitrage Condition

A portfolio that carries no risk must earn the same rate of return as any other risk-free investment, otherwise arbitrage opportunities would arise. This means that the return on the hedged portfolio must equal the risk-free rate r:

$$d\Pi_t = r\Pi_t dt \tag{2.13}$$

Since the portfolio's value is $\Pi_t = V - \Delta S_t$ substituting this in gives:

$$d\Pi_t = r(V - \Delta S_t)dt \tag{2.14}$$

Finally, we substitute $\Delta = \frac{\partial V}{\partial S}$ and the earlier deterministic expression for $d\Pi_t$:

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - S_t \frac{\partial V}{\partial S}\right) dt \tag{2.15}$$

Since dt appears on both sides, we can divide through by it to obtain the differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + rS_t \frac{\partial V}{\partial S} - rV = 0 \tag{2.16}$$

This is the Black-Scholes partial differential equation, the mathematical backbone of modern option pricing. It expresses the precise balance required to prevent arbitrage in a continuously hedged market.

The beauty of this result is that the stock's expected return μ—which depends on investors' risk preferences—has disappeared entirely. Once risk is eliminated through hedging, the only rate that matters is the risk-free rate r. This is why the equation is universal: it prices all derivatives that can be perfectly hedged, independent of subjective expectations.

IV.2.4 Interpreting the Equation's Terms

Each term in Equation has a clear financial meaning:

— $\frac{\partial V}{\partial t}$ represents the time decay of the derivative.
— $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$ captures the effect of curvature—how the option's sensitivity changes as the underlying moves.
— $rS\frac{\partial V}{\partial S}$ reflects the risk-neutral drift of the underlying required by replication, growing at the risk-free rate under the risk-neutral measure.
— $-rV$ represents the opportunity cost of capital—the return that could be earned by investing the same amount in a risk-free bond.

The Black-Scholes equation expresses a fundamental equilibrium: the instantaneous expected return on the derivative must equal the return from a risk-free investment. If this balance were violated, investors could construct a self-financing arbitrage strategy.

IV.2.5 Solving the Equation for European Options

Having established the Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \tag{2.17}$$

we can now apply it to value a specific derivative. Since our primary interest lies in the European call option, we denote the value function by $C(S, t)$. The contract pays $\max(S_T - K, 0)$ at maturity, establishing the specific terminal condition:

$$C(S_T, T) = \max(S_T - K, 0) \tag{2.18}$$

This payoff serves as the boundary condition from which we must solve the differential equation backwards in time to determine the fair price at any prior moment $t < T$.

The solution strategy involves a sequence of transformations: time is reversed via $\tau = T - t$, the stock price is expressed logarithmically, and variables are rescaled. Through these steps, the Black-Scholes PDE is mapped onto the classical heat equation. This transformation is mathematically elegant, yielding a solution in the form of a discounted risk-neutral expectation. Converting back to the original variables provides the closed-form price of the European call:

$$C(S_t, t) = S_t N(p_+) - K e^{-r(T-t)} N(p_-) \tag{2.19}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. The quantities p_+ and p_- emerge directly from the change of variables:

$$p_+ = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \qquad p_- = p_+ - \sigma\sqrt{T-t} \tag{2.20}$$

The terms p_+ and p_- originate from the probability distribution of the log-price under the risk-neutral measure. In the transformed coordinates, the terminal payoff becomes a kinked function of a normally distributed variable; reversing the transformations yields exactly these combinations.

The two main components of the formula possess intuitive economic interpretations:

— $S_t N(p_+)$ represents the expected value, under the risk-neutral measure, of receiving the stock at maturity if the option finishes in the money.

— $K e^{-r(T-t)} N(p_-)$ represents the present value of the cash outflow (paying the strike price K) upon exercise.

Their difference provides the fair, arbitrage-free value of the call option at any time $t < T$. This solution captures how every key parameter $(S_t, K, T-t, r, \sigma)$ influences the price.

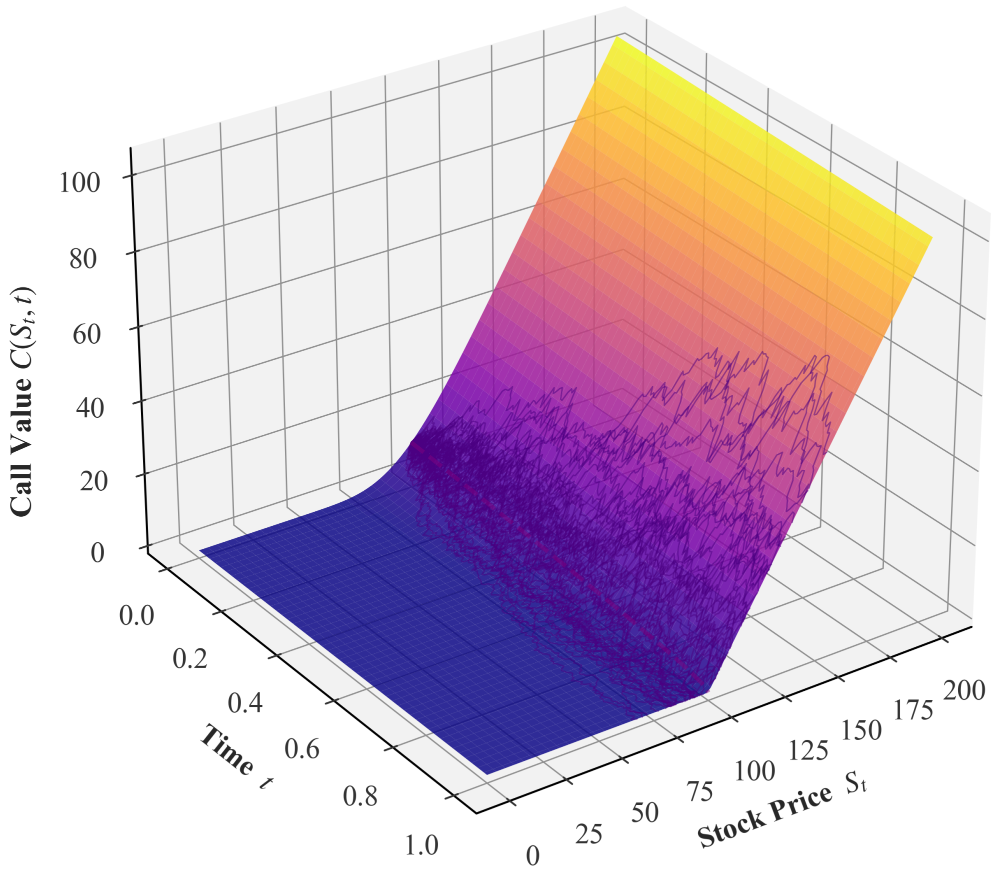

Figure IV.2.1 – **Option Value Surface under the Black-Scholes Model.** The surface depicts the arbitrage-free price of a European call option as a function of the underlying price S_t and time t. The simulated paths (black lines) show possible random evolutions of the asset. However, the option's value evolves deterministically across the smooth surface, dictated by the no-arbitrage condition. The red curve highlights the strike region, showing how the transition into the payoff becomes sharp as $t \to T$.

6. The European Put Price via Put-Call Parity

Having derived the expression for the call price $C(S_t, t)$, we do not need to solve the PDE again for the put option. Instead, the price of a European put option follows directly from the put-call parity relationship:

$$C(S_t, t) - P(S_t, t) = S_t - Ke^{-r(T-t)} \tag{2.21}$$

We can solve for the put price by rearranging the terms:

$$P(S_t, t) = C(S_t, t) - S_t + Ke^{-r(T-t)} \tag{2.22}$$

Substituting the closed-form call solution into this expression, we obtain:

$$P(S_t, t) = Ke^{-r(T-t)}N(-p_-) - S_tN(-p_+) \tag{2.23}$$

The same quantities p_+ and p_- appear because the underlying price process and the transformed normal variable governing the distribution of $\ln(S_T)$ are identical for both calls and puts. The only difference lies in the direction of the payoff asymmetry. Thus, a single PDE and solution technique consistently yield the prices for both European call and put options.

The System of Option Greeks: Sensitivity Analysis

V.1 The Option Greeks

Having established the analytical expressions for the prices of European call and put options, we can now proceed to differentiate these prices with respect to the model's various inputs. These partial derivatives—collectively known as the "Greeks"—quantify how the option's value responds to infinitesimal changes in the underlying parameters. They effectively translate abstract mathematical sensitivity into concrete economic meaning. Although derived formally via calculus, each Greek encapsulates a specific concept of risk and market behavior.

Recall the closed-form solutions for the call price:

$$C(S,t) = SN(p_+) - Ke^{-r(T-t)}N(p_-) \tag{1.1}$$

and the put price:

$$P(S,t) = Ke^{-r(T-t)}N(-p_-) - SN(-p_+) \tag{1.2}$$

By differentiating these formulas, we can derive all the Greeks directly. These measures are more than just formal derivatives; they are fundamental descriptors of risk, hedging requirements, and market dynamics.

V.1.1 Delta (Δ): Directional Sensitivity

Delta measures the sensitivity of the option price to a small change in the underlying asset's price. It answers the question: "If the stock price moves by \$1, how much does my option value move?".

Mathematically, it is defined as:

$$\Delta_{\text{call}} = \frac{\partial C}{\partial S} \quad \Delta_{\text{put}} = \frac{\partial P}{\partial S} \tag{1.3}$$

For a European call option, performing the differentiation yields a remarkably simple result:

$$\Delta_{\text{call}} = N(p_+) \tag{1.4}$$

This result carries a dual interpretation. $N(p_+)$ represents the risk-neutral probability that the option will finish in the money (assuming the asset serves as the numéraire), but it also defines the slope of the option price curve with respect to the underlying. Geometrically, Delta tells us how steeply the option value rises at the current price S.

— When the call is deep out-of-the-money, Delta is close to 0.
— When deep in-the-money, it approaches 1.
— In between, it transitions smoothly, capturing the degree to which the option behaves "stock-like".

For the put option, the relationship is:

$$\Delta_{\text{put}} = N(p_+) - 1 = -N(-p_+) \tag{1.5}$$

Consequently, the Delta of a put is always negative: as the asset price rises, the put option loses value.

Economically, Delta is the most critical Greek for hedging: it is exactly the quantity Δ used to construct the riskless portfolio Π_t in the derivation of the Black-Scholes equation.

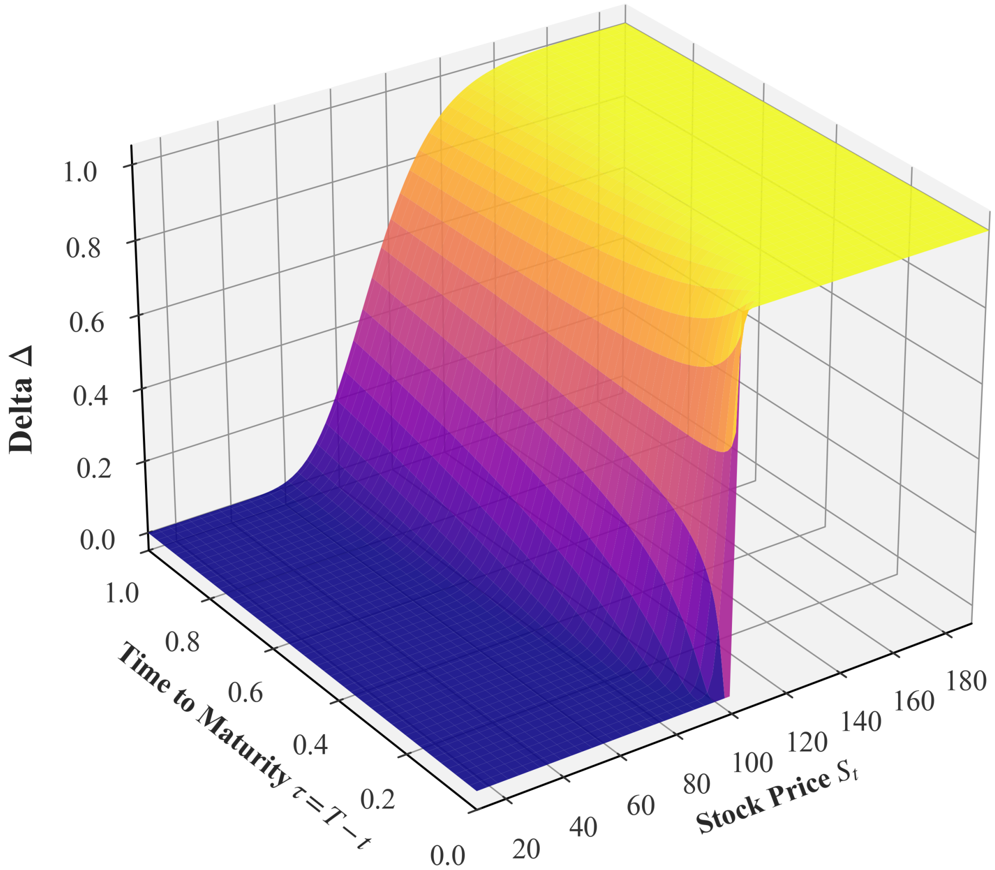

Figure V.1.1 – **Delta Surface $\Delta_{\text{call}}(S, \tau)$ of a European Call Option.** The surface illustrates the smooth transition of the call option's Delta from 0 to 1 as the stock price increases. The ridge along higher S_t values reflects the option becoming increasingly stock-like. Notably, as the time to maturity shrinks ($\tau \to 0$), this transition becomes sharper, illustrating how Delta collapses into a binary step function at expiration.

V.1.2 Gamma (Γ): Convexity and Hedging Stability

Gamma serves as a second-order sensitivity measure, quantifying how Delta itself evolves as the underlying asset price changes. Mathematically, it is the second derivative of the option value with respect to the underlying price:

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 P}{\partial S^2} \tag{1.6}$$

Within the Black-Scholes framework, the analytical expression for Gamma is identical for both European calls and puts:

$$\Gamma = \frac{N'(p_+)}{S\sigma\sqrt{T-t}} \tag{1.7}$$

where N' denotes the standard normal probability density function. Since the density is strictly positive, Gamma is always positive for long positions in both calls and puts. This positivity reflects the convexity of option prices: holders benefit from volatility and the curvature of the payoff structure.

A high Gamma implies that Delta is highly sensitive to small price movements. Consequently, the option's exposure changes rapidly, requiring frequent rebalancing of the hedge to maintain neutrality. Gamma peaks significantly when the option is "at-the-money" and close to expiration, because in this region, small price changes determine whether the option finishes in or out of the money.

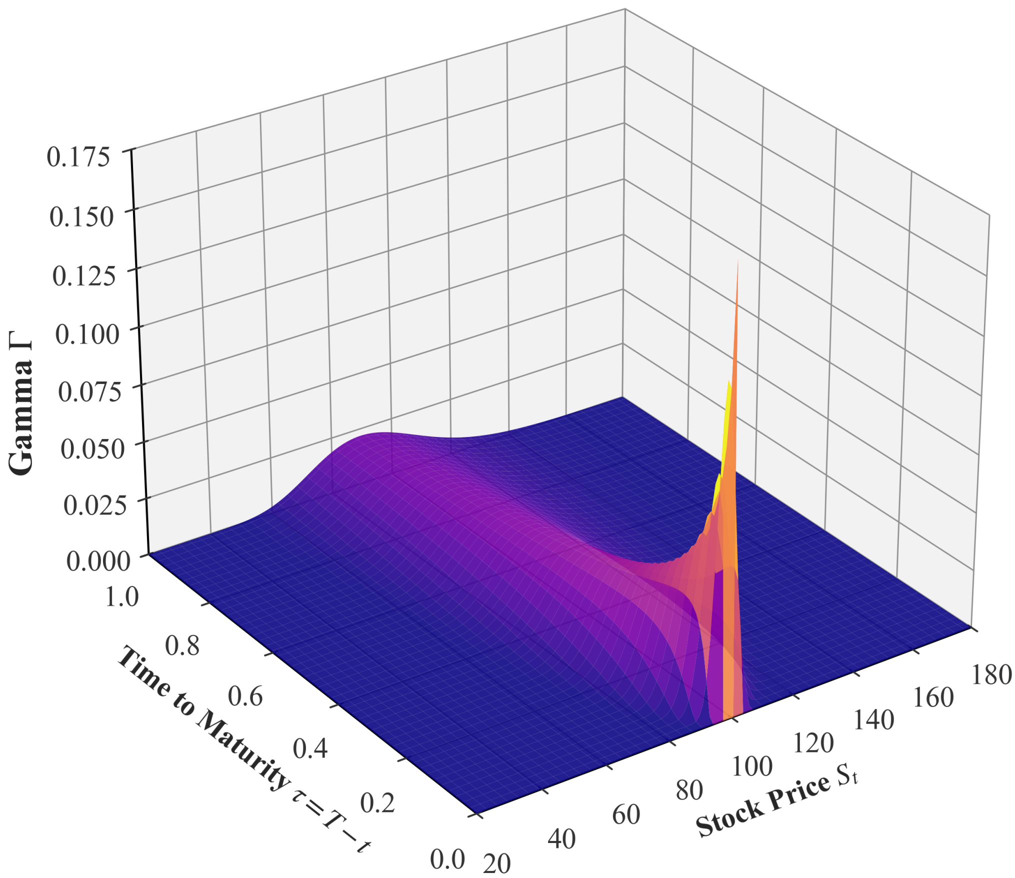

Figure V.1.2 – **Gamma Surface $\Gamma(S, \tau)$ of a European Call/Put Option.** The surface demonstrates that Gamma peaks sharply when the option is near the strike price and close to maturity, forming a distinct, thin ridge centered around the at-the-money region. This visualization highlights how the option's curvature—and thus the sensitivity of Delta—becomes extremely concentrated as expiry approaches. Conversely, deep in-the-money and out-of-the-money regions appear flat, exhibiting almost zero Gamma.

V.1.3 Theta (Θ): Time Decay

Theta measures the sensitivity of the option's price to the passage of time, often referred to as "time decay." It is defined as the partial derivative with respect to time t:

$$\Theta = \frac{\partial C}{\partial t} \quad \Theta = \frac{\partial P}{\partial t} \tag{1.8}$$

For a European call option, the formula is given by:

$$\Theta_{\text{call}} = -\frac{SN'(p_+)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(p_-) \tag{1.9}$$

This expression is composed of two distinct terms:

- The first term reflects the "decay" of the option's time value due to volatility. As maturity approaches, the potential for the asset to move into a profitable region diminishes.
- The second term captures the effect of discounting the strike price.

Theta is generally negative for long positions in calls and puts, meaning that options lose value as time elapses, assuming all other factors remain constant. However, exceptions exist, such as deep in-the-money puts which can exhibit positive Theta. In practical trading terms, Theta represents the daily "cost" of holding optionality.

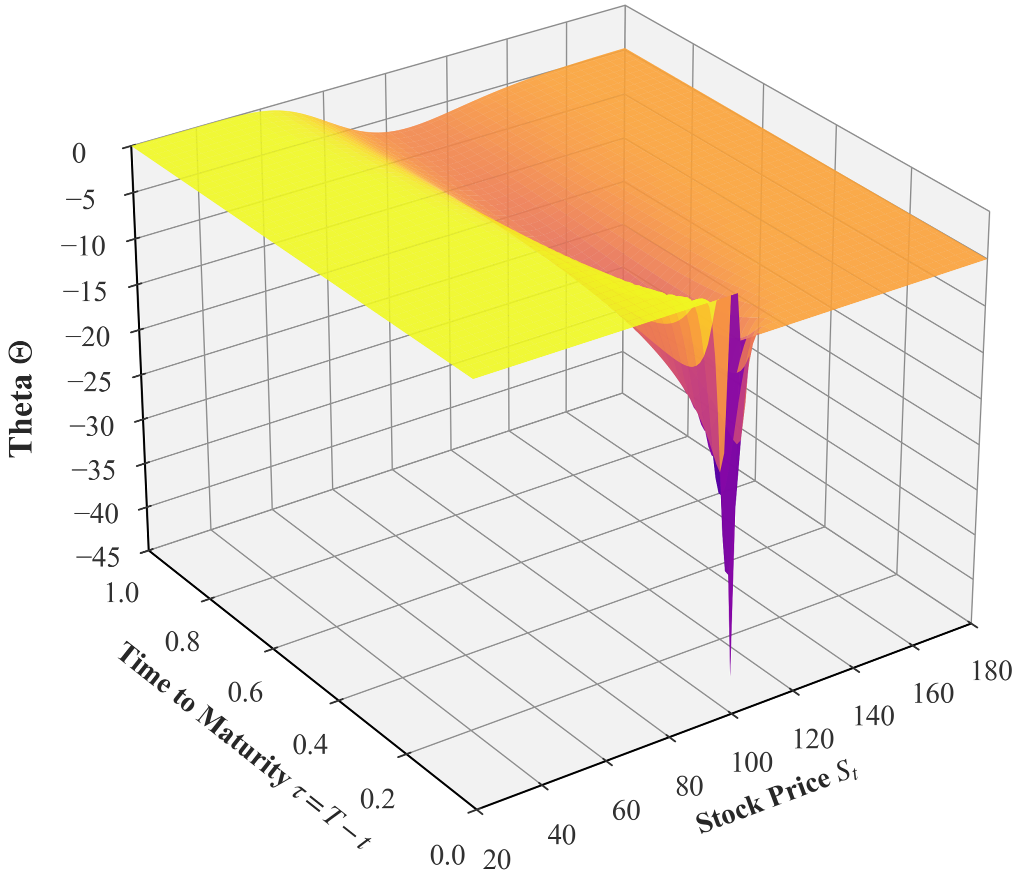

Figure V.1.3 – **Theta Surface $\Theta_{\text{call}}(S, \tau)$ of a European Call Option.** Theta is most negative when the option is near the strike price and approaching expiration, creating a deep downward spike in the surface. This visualizes the rapid erosion of time value for near-the-money options as maturity looms. In contrast, for options that are deep in-the-money or out-of-the-money, the value erosion occurs much more slowly.

V.1.4 Vega (ν): Volatility Exposure

Vega (denoted by ν, although not a Greek letter, the name stems from a misreading of the Greek ν) quantifies the sensitivity of the option's value to variations in the underlying asset's volatility. Unlike Delta or Gamma which relate to price movements, Vega isolates the impact of uncertainty itself.

Mathematically, it is defined as the partial derivative with respect to σ:

$$\nu = \frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} \tag{1.10}$$

A distinctive feature of Vega is that it is identical for both European calls and puts. The closed-form expression is:

$$\nu = SN'(p_+)\sqrt{T-t} \tag{1.11}$$

This formula confirms the intuitive notion that higher volatility increases the value of all options. Greater volatility expands the probability distribution of future prices, increasing the likelihood of extreme outcomes (deep in-the-money), while the downside is capped at the premium paid.

Vega peaks when the option is "at-the-money"—precisely where the outcome is most uncertain—and diminishes as the option moves deep in-the-money or out-of-the-money.

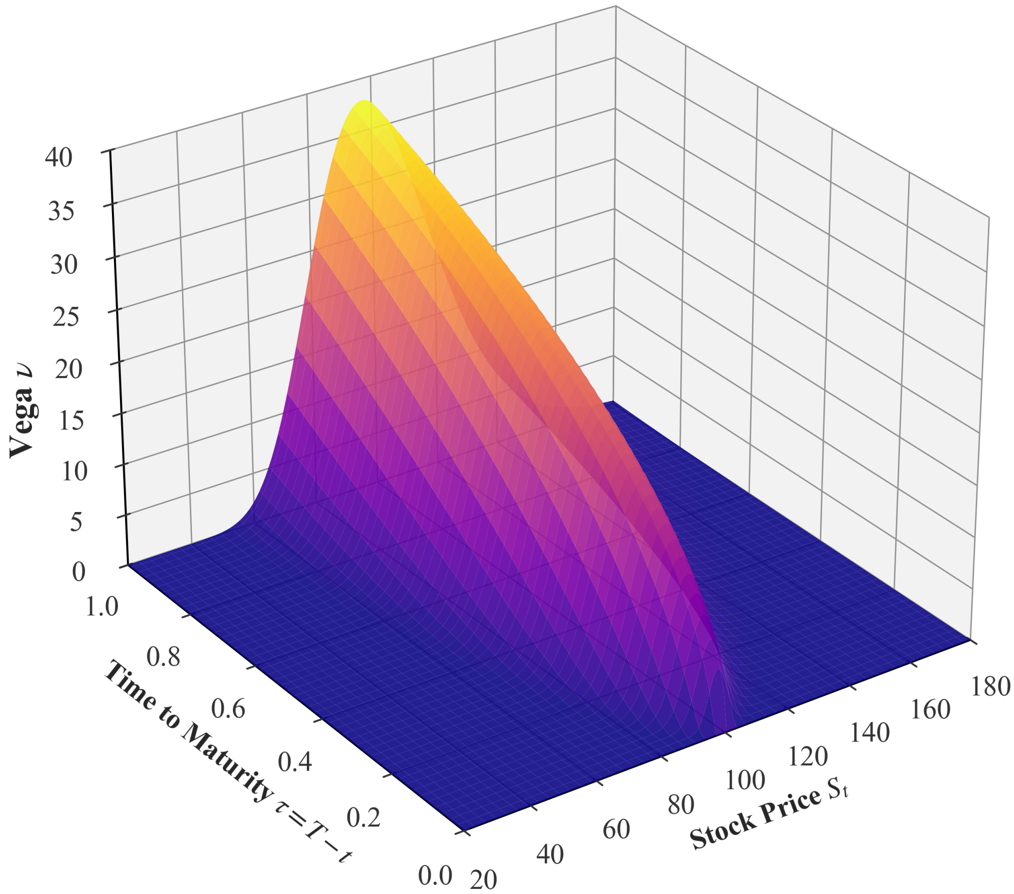

Figure V.1.4 – **Vega Surface $\nu(S, \tau)$ of a European Call/Put Option.** The surface displays a smooth, bell-shaped ridge that maximizes along the at-the-money strike price. The height of this ridge increases with time to maturity, indicating that long-dated options are far more sensitive to changes in volatility estimates than short-dated ones. Conversely, for options that are deeply in-the-money or out-of-the-money, Vega approaches zero.

V.1.5 Rho (ρ): Interest Rate Sensitivity

Rho measures the sensitivity of the option price to changes in the risk-free interest rate. While often less critical for short-term trading than Delta or Vega, it becomes significant for long-dated contracts (LEAPS) or in high-interest environments.

It is defined as:

$$\rho_{\text{call}} = \frac{\partial C}{\partial r} \quad \rho_{\text{put}} = \frac{\partial P}{\partial r} \tag{1.12}$$

For a European call option, the formula is:

$$\rho_{\text{call}} = K(T-t)e^{-r(T-t)}N(p_-) \tag{1.13}$$

This value is always positive. Higher interest rates reduce the present value of the strike price K (which the call holder pays in the future), thereby increasing the call's current value.

For a European put option, the relationship is inverted:

$$\rho_{\text{put}} = -K(T-t)e^{-r(T-t)}N(-p_-) \tag{1.14}$$

Rho is negative for puts because higher interest rates lower the present value of the cash received from selling the asset at the strike price.

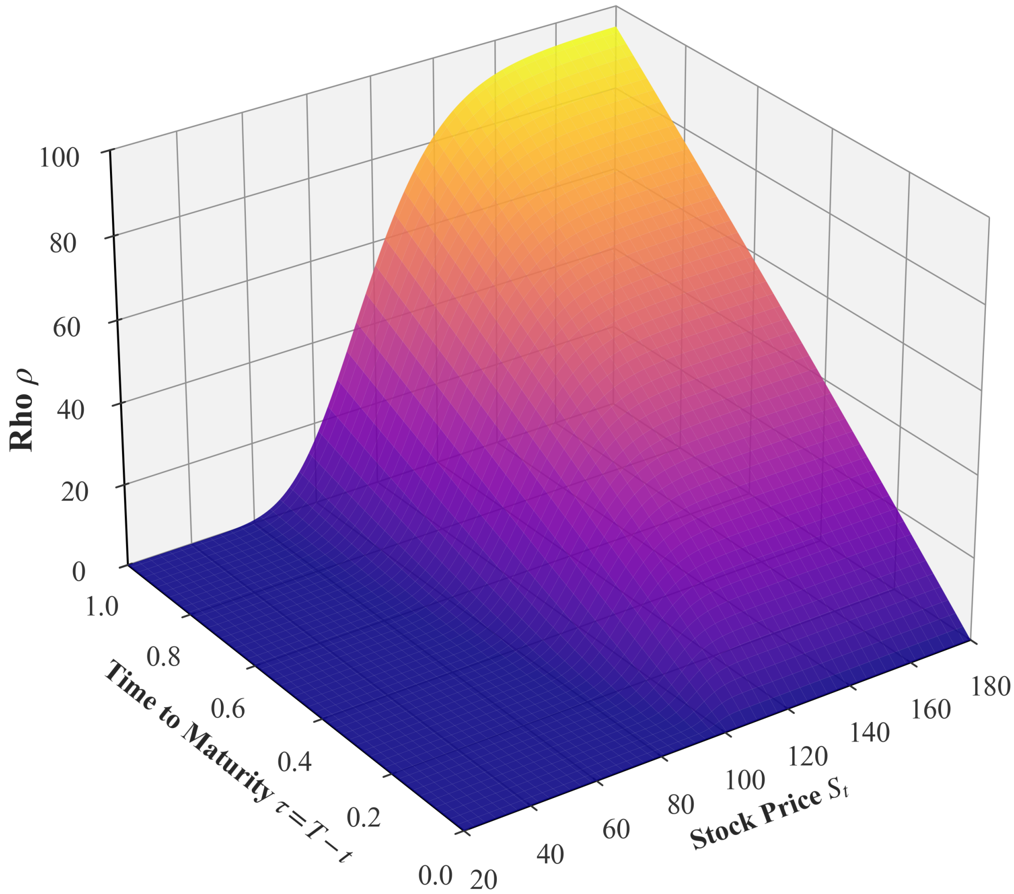

Figure V.1.5 – **Rho Surface $\rho_{\text{call}}(S, \tau)$ of a European Call Option.** The surface shows that Rho increases with both the underlying stock price and the time to maturity. It is highest for deep-in-the-money, long-dated calls, reflecting their high sensitivity to the cost of carry. For short maturities (near $\tau = 0$), the surface is flat, indicating that interest rate changes have a negligible impact on imminent expiration values.

V.1.6 The Greeks as a System

Taken together, the Greeks provide a comprehensive "dictionary" for interpreting how an option's price reacts to the shifting market environment. They decompose the complex pricing formula into distinct dimensions of risk:

— **Delta** (Δ): Measures directional sensitivity (exposure to price moves).
— **Gamma** (Γ): Measures the curvature or stability of that sensitivity (exposure to convexity).
— **Theta** (Θ): Captures the inevitable erosion of value due to the passage of time.
— **Vega** (ν): Captures exposure to changes in market volatility expectations.
— **Rho** (ρ): Captures exposure to shifts in interest rates.

Each Greek addresses a specific question about risk management. In theoretical continuous hedging, Delta and Gamma are paramount for replication. However, in practical trading portfolios, monitoring Theta (time decay) and Vega (volatility risk) is equally essential. This system transforms the Black-Scholes model from a theoretical pricing tool into a practical framework for risk management.

Conclusion

This project has aimed to provide a comprehensive reconstruction of the Black-Scholes-Merton framework, bridging the gap between fundamental probability theory and advanced financial engineering. By systematically developing the mathematical tools and applying them to option pricing, we can draw two main categories of conclusions regarding the theoretical foundations and their practical implications.

Theoretical Synthesis: From Discrete Randomness to Stochastic Calculus

The first major achievement of this study was to demonstrate that continuous-time financial models are not arbitrary constructions but rigorous limits of discrete processes. Starting from the simple Bernoulli random walk, we established that the Wiener process emerges naturally as the time step approaches zero, provided that price fluctuations scale with the square root of time. This transition highlighted the fundamental limitation of ordinary calculus in dealing with financial assets: because Brownian paths are nowhere differentiable and possess non-zero quadratic variation, the standard chain rule fails.

Consequently, the introduction of Itô's Lemma proved to be the cornerstone of the framework. It allowed us to model the dynamics of complex derivatives and laid the groundwork for the stochastic differential equations governing asset prices. This theoretical progression underscores that the complexity of modern finance is built upon simple, observable statistical properties of market returns, formalized through the lens of stochastic calculus.

Financial Application: Dynamic Hedging and Risk Sensitivity

In the second part of the analysis, we moved from mathematical description to financial valuation. The derivation of the Black-Scholes-Merton Partial Differential Equation (PDE) revealed the profound insight of the model: the price of a derivative is independent of the investors' risk preferences and depends solely on the risk-free rate and volatility. By constructing a risk-free portfolio through dynamic replication, we showed that it is theoretically possible to eliminate market risk entirely.

Furthermore, the analytical derivation and 3D visualization of the "Greeks" (Delta, Gamma, Theta, Vega, Rho) provided a concrete toolkit for risk management. These sensitivities demonstrate that pricing is not a static exercise but a dynamic process requiring constant rebalancing. Ultimately, while the Black-Scholes model relies on idealized assumptions—such as constant volatility and frictionless markets—it remains the benchmark for understanding how market variables interact to determine value and risk in financial derivatives.

Appendix

Appendix A: Mathematical Derivation of Itô's Lemma

The objective of this appendix is to derive Itô's Lemma, the fundamental theorem of stochastic calculus. Unlike ordinary calculus, where second-order terms vanish in the limit, stochastic calculus requires retaining specific second-order terms due to the non-differentiable nature of Brownian motion.

The Taylor Expansion Setting

Let X_t be a stochastic process governed by the following Stochastic Differential Equation (SDE):

$$dX_t = \mu dt + \sigma dW_t \tag{1.1}$$

where μ is the drift, σ is the volatility, and W_t is a standard Wiener process.

Consider a smooth function $f(X_t, t)$ that depends on both the stochastic process X_t and time t. We wish to determine the differential df. According to the bivariate Taylor expansion up to the second order, the infinitesimal change in f is given by:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}dt dX_t \tag{1.2}$$

Analyzing the Quadratic Variation

The crucial step distinguishes stochastic calculus from ordinary calculus: evaluating the squared increment $(dX_t)^2$. Squaring the expression for dX_t:

$$(dX_t)^2 = (\mu dt + \sigma dW_t)^2 \tag{1.3}$$

Expanding this square yields:

$$(dX_t)^2 = \mu^2(dt)^2 + 2\mu\sigma dt dW_t + \sigma^2(dW_t)^2 \tag{1.4}$$

We now apply the multiplication rules of stochastic differentials as $dt \to 0$. We retain terms of order dt and discard higher-order terms (such as $(dt)^2$ or $dt^{3/2}$).

— $(dt)^2 \to 0$: This term is negligible (higher order).
— $dt dW_t \to 0$: Since dW_t scales as $\sqrt{dt}$, the product scales as $dt^{3/2}$, which is negligible relative to dt.
— $(dW_t)^2 \to dt$: This is the defining property of Brownian motion (quadratic variation). The variance of the increment over time dt is exactly dt.

Substituting these limits back into the expression for $(dX_t)^2$, the only surviving term is:

$$(dX_t)^2 = \sigma^2 dt \tag{1.5}$$

Final Derivation

We substitute $(dX_t)^2 = \sigma^2 dt$ back into the Taylor expansion. We also discard the terms involving $(dt)^2$ and $dt dX_t$ (which contains $dt^{3/2}$ and $(dt)^2$) as they vanish in the limit. The expansion simplifies to:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\sigma^2 dt) \tag{1.6}$$

Now, we substitute the explicit form of $dX_t = \mu dt + \sigma dW_t$:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu dt + \sigma dW_t) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}dt \tag{1.7}$$

Finally, we group the terms by dt (deterministic/drift component) and dW_t (stochastic/diffusion component):

$$df = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma \frac{\partial f}{\partial x} dW_t \tag{1.8}$$

This concludes the derivation of Itô's Lemma. The presence of the term $\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}$ represents the convexity adjustment required when taking the expectation of a function of a random variable, a feature absent in classical calculus.

References

[1] F. Black, M. Scholes, *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 81(3), pp. 637-654, 1973.

[2] R. C. Merton, *Theory of Rational Option Pricing*, The Bell Journal of Economics and Management Science, 4(1), pp. 141-183, 1973.

[3] J. C. Hull, *Options, Futures, and Other Derivatives (10th ed.)*, Pearson, 2018.

[4] S. E. Shreve, *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*, Springer, 2004.

[5] S. E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer, 2004.

[6] T. Björk, *Arbitrage Theory in Continuous Time (3rd ed.)*, Oxford University Press, 2009.

[7] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications (6th ed.)*, Springer, 2003.

[8] P. Wilmott, S. Howison, J. Dewynne, *The Mathematics of Financial Derivatives*, Cambridge University Press, 1995.