Intro to Geometric Algebra

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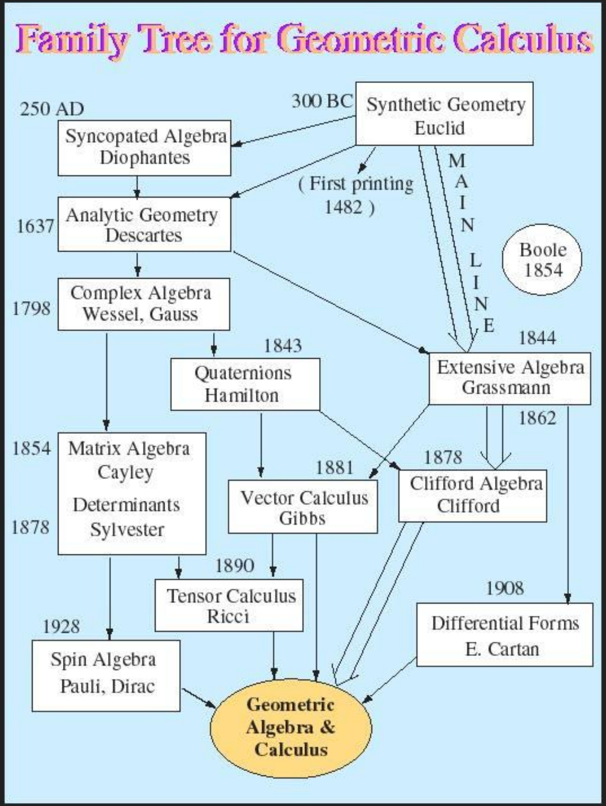
Outline

1. What is Geometric Algebra?
2. Applications
3. Geometric Algebra Introduction
4. Examples
   a. Rotors
   b. Automatic Differentiation
5. Software implementation (TFGA presentation)

1. What is Geometric Algebra?

1. What is Geometric Algebra?

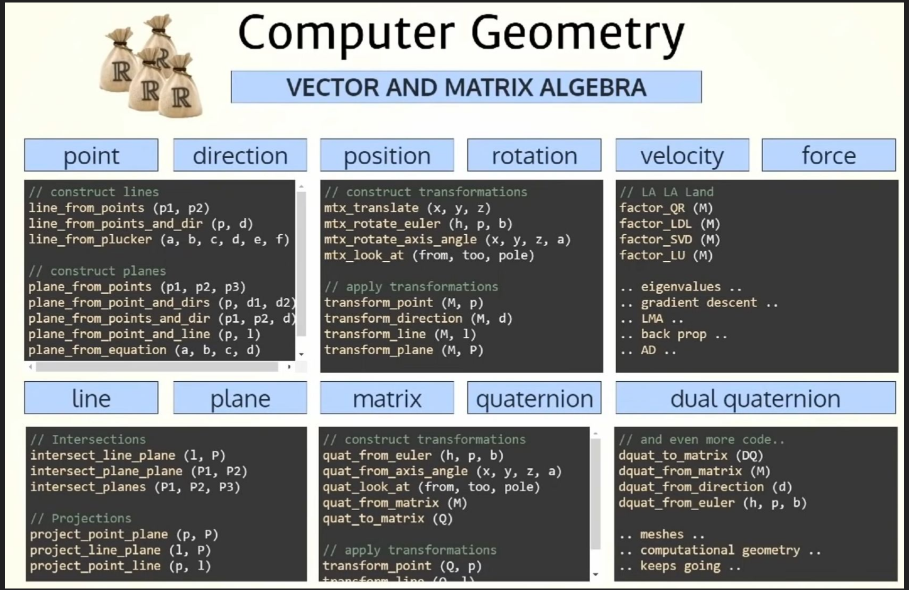
- Mathematical framework
  - …Unifying traditionally distinct concepts
  - …Simplifying difficult concepts
  - …Understanding abstract concepts
  - …Generalizing existing concepts
- Invented in 19th century by Grassmann, Clifford among others
- Somewhat forgotten in favor of Vector algebra until later (~1960s by David Hestenes)
- “Geometry without algebra is dumb! Algebra without geometry is blind!” - Hestenes

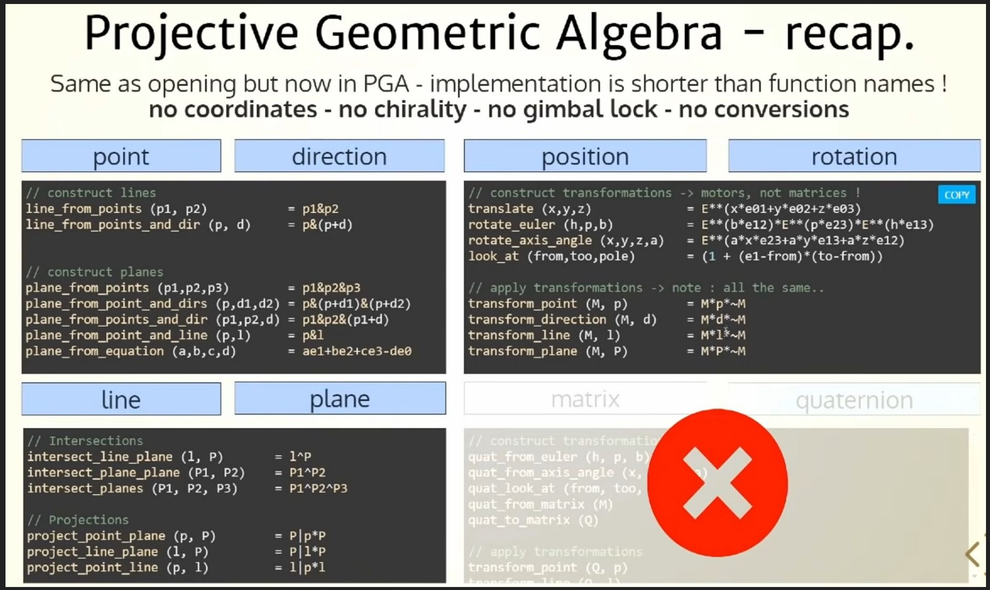


http://geocalc.clas.asu.edu/html/Evolution.html

2. Applications

2. Applications - Computer Graphics





Geometric Algebra at SIGGRAPH 2019 - https://www.youtube.com/watch?v=tX4H_ctggYo

Traditional approach

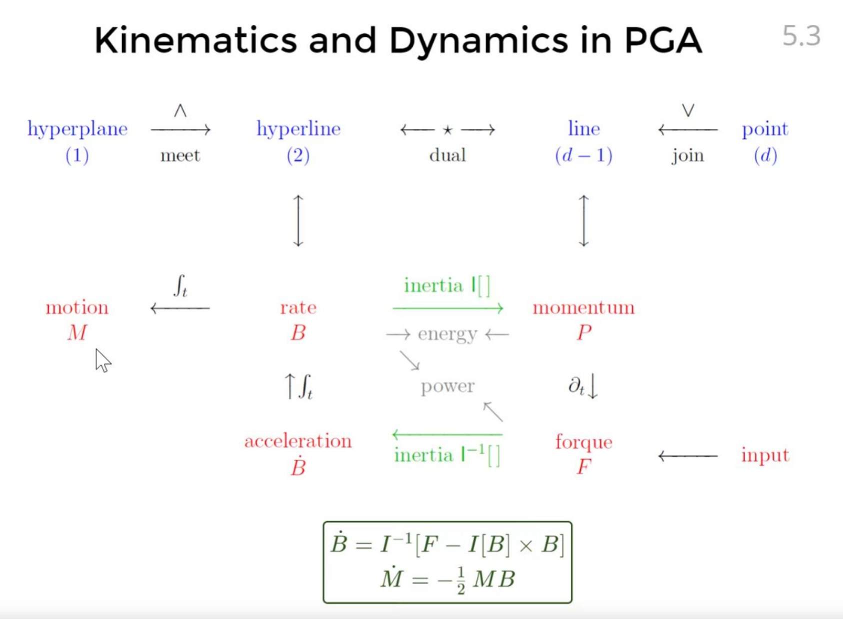
- Many types of objects: Vector, Matrix, Quaternion, Dual-Quaternion, …
- Implementations require a lot of work

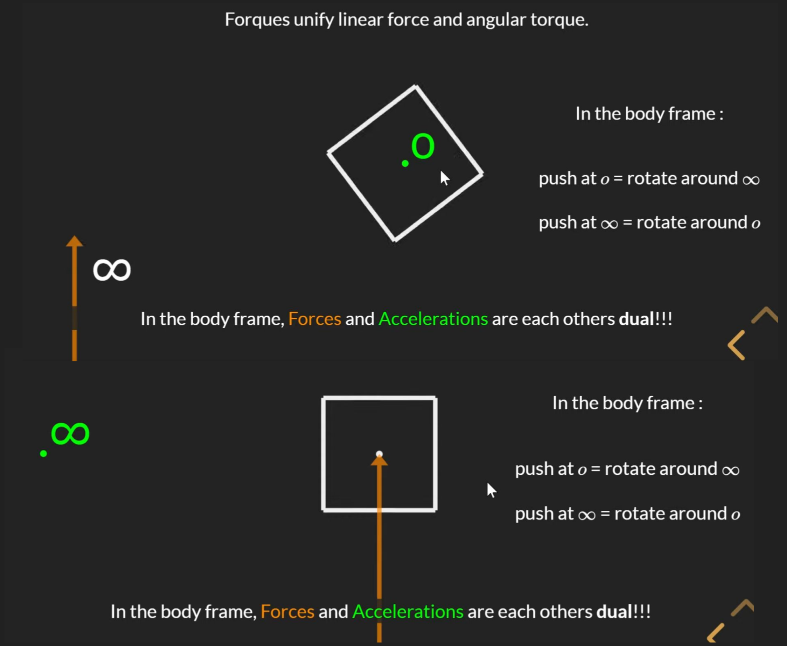
(Projective) Geometric Algebra approach

- Only one type of object: Multivector
- 1-line implementations with simple GA operations

2. Applications - Physics - Classical Mechanics

- Projective Geometric Algebra unifies rotation and translation
  - Force / Torque → “Forque”,
- Gives insight on origin of Quantum Spin
- Works in any dimension, equations stay the same (https://youtu.be/5R2sv9GCwz0?t=674)





2. Applications - Physics - Electromagnetism

- Electromagnetism
  - Maxwell's 4 equations reduced to a single equation
  - Special Relativity calculations become much easier (eg. how a moving observer sees an E-field)

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

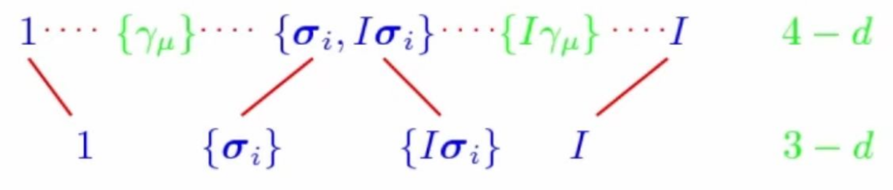
$$\nabla F = \frac{J}{c\varepsilon_0}$$

(F = E + I B)

2. Applications - Physics - QM & GR

- Quantum Mechanics - becomes more interpretable
    - 4 Gamma matrices → 4 basis vectors of spacetime (not abstract!)
    - 3 Pauli matrices → 3 basis vectors of space (and can be made from the 4 spacetime vectors!)

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



- General Relativity (Gravity), two approaches:
    - Usual curved-space approach but with GA simplifying calculations
    - Gauge-Theory Gravity (https://arxiv.org/abs/gr-qc/0405033)
        - Space is flat, introduce gauge fields instead
        - Predictions almost all identical as ordinary approach
        - Lead to theoretical discovery of gravitational wave memory-effect

3. Geometric Algebra Introduction

You could have invented Geometric Algebra

- Multiply two vectors $a = a_1e_1 + a_2e_2 + a_3e_3$ $b = b_1e_1 + b_2e_2 + b_3e_3$

You could have invented Geometric Algebra

- Multiply two vectors $a = a_1e_1 + a_2e_2 + a_3e_3$ $b = b_1e_1 + b_2e_2 + b_3e_3$

$e_ie_i = 1$ $e_ie_j = -e_je_i$

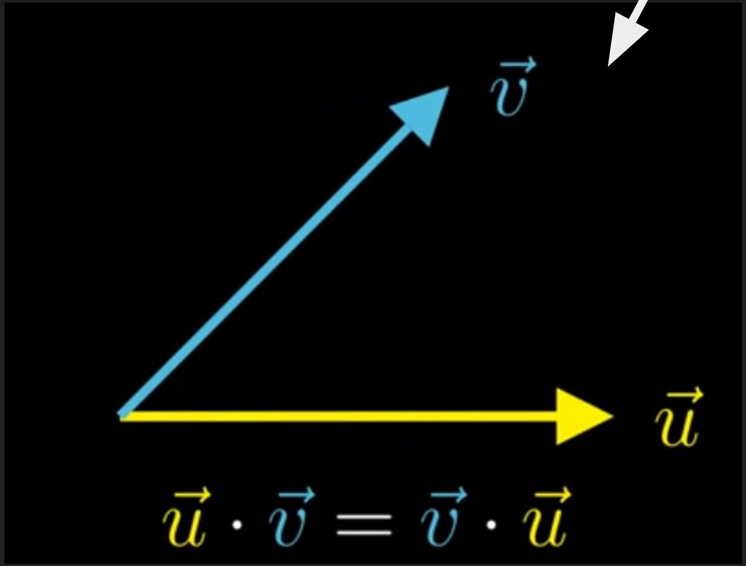
$$
\begin{aligned}
ab &= (a_1e_1 + a_2e_2 + a_3e_3)(b_1e_1 + b_2e_2 + b_3e_3) \\
&= a_1b_1e_1e_1 + a_1b_2e_1e_2 + a_1b_3e_1e_3 + \\
&\quad a_2b_1e_2e_1 + a_2b_2e_2e_2 + a_2b_3e_2e_3 + \\
&\quad a_3b_1e_3e_1 + a_3b_2e_3e_2 + a_3b_3e_3e_3 \\
&= a_1b_1 + a_2b_2 + a_3b_3 + \\
&\quad (a_1b_2 - a_2b_1)e_1e_2 + \\
&\quad (a_1b_3 - a_3b_1)e_1e_3 + \\
&\quad (a_2b_3 - a_3b_2)e_2e_3 \\
&= a \cdot b + a \wedge b
\end{aligned}
$$

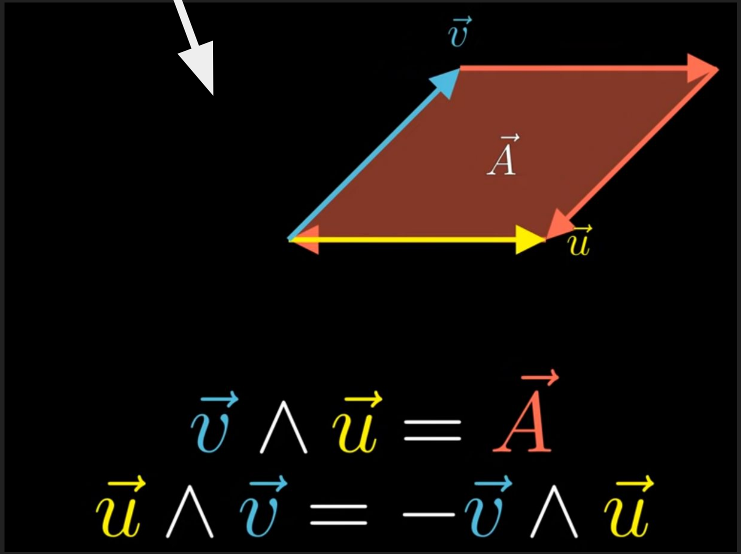
- Result has two different parts:
  - Scalar part from Dot product
  - “Bivector” part from “Wedge product”
  - Wedge product in 3D looks like Cross product
  - But: it’s a bivector and not a vector
  - Only in 3D:
    - For every plane, there is a vector orthogonal to it

What does this mean geometrically?

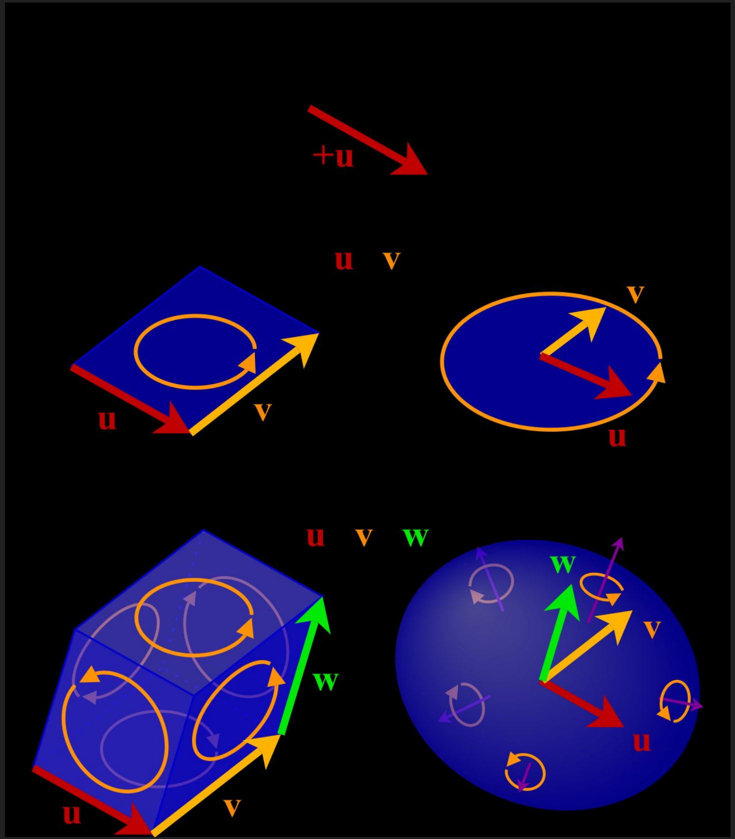
- Multiply two vectors
- Often called Geometric Product
  but really is just the ordinary product

$$ab = a \cdot b + a \wedge b$$





Geometry directly representable



Vectors: Line elements

$$e_1, e_2, e_3$$

Bivectors: Plane / Area elements

$$e_1e_2, e_2e_3, e_1e_3$$

Trivectors: Volume elements

$$e_1e_2e_3$$

4. Examples
Rotors

4. Examples - Rotors

- Multiplying two vectors gave us a bivector part (e_1 e_2)
- Can be interpreted as plane area element
- Let’s try to square a basis bivector

$$e_i e_j = -e_j e_i$$

$$e_i e_i = 1$$

$$(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1$$

- The basis bivector is like the Imaginary Unit from Complex Numbers!

4. Examples - Rotors

- Traditionally
  - A complex number is the sum of a real and imaginary part
  - Vectors and complex numbers are seen as separate concepts
  - Complex numbers can be used to represent and compose 2D rotation

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

$$e^{i\varphi_1} e^{i\varphi_2} = e^{i(\varphi_1+\varphi_2)}$$

4. Examples - Rotors

- In 3D, we have 3 basis bivectors → 3 Imaginary numbers → Quaternions

$$e_1e_2, e_2e_3, e_1e_3$$

- 3 planes of rotation
- Quaternions rotate in them just like Complex Numbers did in 2D in 1 plane of rotation
- Formula for applying rotor in 2D to a vector (R v) was a special case, in general (any dim):

$$Rv\widetilde{R}$$

"Reverse"

$$\widetilde{e_1e_2} = e_2e_1 = -e_1e_2$$

4. Examples
Automatic Differentiation

Example - Automatic Differentiation

- So far we had basis vectors squaring to +1
- Can also choose different number, eg. square to 0
- Can be used for forward-mode automatic differentiation:

$$e_1^2 = 0, f(x) = x^2$$

$$\begin{aligned} f(x + e_1) &= (x + e_1)^2 = x^2 + 2xe_1 + e_1^2 \\ &= x^2 + 2xe_1 \end{aligned}$$

- Can be extended to work for multiple inputs and higher order derivatives
- More details in my writeup: https://discourse.bivector.net/t/automatic-differentiation/289

5. Software Implementation (TFGA Presentation)

https://tfgap.warlock.ai/#/6

More resources on Geometric Algebra

- Online
  - Sudgy Lacmoe's "A Swift Introduction to Geometric Algebra": https://youtu.be/60z_hpEAtD8
  - Videos on Youtube channel Bivector: https://www.youtube.com/channel/UCZZ3MA6ChVTViq8ORmFfCPA
  - Geometric Algebra Discord community: https://discord.gg/vGY6pPk
  - Bivector website: https://bivector.net/
  - My own GA tutorials: https://geometricalgebratutorial.com/
  - Coffeeshop, lots of interactive GA examples: https://enkimute.github.io/ganja.js/examples/coffeeshop.html

- Literature
  - Geometric Algebra for Physicists: https://www.cambridge.org/core/books/geometric-algebra-for-physicists/FB8D3ACB76AB3AB10BA7F27505925091
  - Geometric Algebra for Computer Science: https://geometricalgebra.org/
  - Geometric Algebra by Eric Chisolm: https://arxiv.org/abs/1205.5935
  - Linear and Geometric Algebra: http://www.faculty.luther.edu/~macdonal/laga/index.html