

# INFOF-409 — Learning Dynamics

## Assignment 1 — Game Theory

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### Introduction

Subsections 1.1 and 2.1 do not contain answers to the explicit questions in the instructions. They are only used to expose the used notations and the assertions that will serve to answer the questions.

## 1 The Hawk-Dove Game

### 1.1 Modeling and notations

The Hawk-Dove game can be modeled as in Table 1 with parameters  $V, D, T \in \mathbb{R}^{*+}$ . Let us denote by  $\mathcal{A}^{(i)} := \{H, D\}$  the available actions for player  $i \in \mathcal{I}$  with  $\mathcal{I} = \{1, 2\}$ , the finite ordered set of players.<sup>1</sup> We consider (without loss of generality) that  $\mathcal{I}$  is ordered only to define a set product iterated on  $\mathcal{I}$  in a unique way so that the order of the tuples of such a product is induced by the order relation on  $\mathcal{I}$  (mathematical formalism helps increasing sanity).

Let us also denote:

$$\mathcal{A} := \prod_{i \in \mathcal{I}} \mathcal{A}^{(i)} = \mathcal{A}^{(1)} \times \mathcal{A}^{(2)} = \{H, D\}^2,$$

the set of all profiles.

Let us use the standard notations  $\mathcal{A}^{-(i)}$  for  $\prod_{j \in \mathcal{I} \setminus \{i\}} \mathcal{A}^{(j)}$  the set of all profiles omitting player  $i$ 's choice,  $a_{-i}$  for an element of  $\mathcal{A}^{-(i)}$  and  $u^{(i)} : \mathcal{A} \rightarrow \mathbb{R}$  a utility function for each player  $i \in \mathcal{I}$ .

|   | H               | D                 |
|---|-----------------|-------------------|
| H | $\frac{V-D}{2}$ | V                 |
| D | 0               | $\frac{V}{2} - T$ |

Table 1: Payoff of player 1 according to his actions (rows) and his opponent's actions (columns).

The payoff values in Table 1 can be viewed as a payoff matrix. To justify this, for  $i \in \mathcal{I}$ , consider somehow naturally a (quite arbitrary) bijection  $\pi_i : \mathcal{A}^{(i)} \rightarrow \{1, \dots, n_i\}$  with  $n_i := |\mathcal{A}^{(i)}|$ , the number of available actions

<sup>1</sup>Even if  $\mathcal{I}$  is not a set of numbers, as long as it is finite, we can consider it to be fully ordered by mapping it to  $\{1, \dots, N\}$  with  $N := \mathcal{I}$  the number of players by a projection principle explained a bit further.

for player  $i$ . Extend it to  $\pi : \mathcal{A} \rightarrow \prod_{i \in \mathcal{J}} \{1, \dots, n_i\}$  by:

$$\forall \mathbf{a} \in \mathcal{A} : \pi(\mathbf{a}) = \pi((a_i)_{i \in \mathcal{J}}) := (\pi_i(a_i))_{i \in \mathcal{J}},$$

which then associates each profile to an integer tuple.

Then for every player  $i \in \mathcal{J}$ , there naturally exists a *matrix*  $U^{(i)} \in \mathbb{R}^{\prod_{j \in \mathcal{J}} n_j}$  containing the utility value of each action profile. Note that the name *matrix* is used only because  $N = 2$  but the notion generalizes easily to any finite value of  $N$ .

$$\forall i \in \mathcal{J} : \forall (k_1, \dots, k_N) \in \prod_{\ell \in \mathcal{J}} \{1, \dots, n_\ell\} : U_{(k_1, \dots, k_N)}^{(i)} = u^{(i)}(\pi^{-1}(k_1, \dots, k_N)).$$

This notion of utility matrix needs to be explicit because in the case of this Hawk-Dove game, one can easily observe that the users' matrices are transpose of one another, i.e.  $U^{(1)} = U^{(2)'}$  and both players have the same available actions. This *symmetry* is important because it is used as an argument for proofs below.

Note also that to avoid selecting such an arbitrary bijection  $\pi$ , the utility matrix can be directly indexed by the actions instead of numbers attached to them.

Consider  $A$  a discrete random variable on  $\mathcal{A}$  with probability law  $\mathbb{P}$ . By independence:

$$\forall \alpha \in \mathcal{A} : \mathbb{P}[A = \alpha] = \prod_{i \in \mathcal{J}} \mathbb{P}[A_i = \alpha_i].$$

For player  $i \in \{1, 2\}$ , let us denote by  $\mathbb{E}_i[A] := \mathbb{E}[u^{(i)}(A)]$  the player  $i$ 's expected utility:

$$\begin{aligned} \mathbb{E}_i[A] &= \sum_{\mathbf{a} \in \mathcal{A}} \mathbb{P}[A = \mathbf{a}] u^{(i)}(\mathbf{a}) = \sum_{\mathbf{a}_i \in \mathcal{A}^{(i)}} \sum_{\mathbf{a}_{-i} \in \mathcal{A}^{-(i)}} \mathbb{P}[A = (\mathbf{a}_i, \mathbf{a}_{-i})] u^{(i)}(\mathbf{a}_i, \mathbf{a}_{-i}) \\ &= \sum_{\mathbf{a}_i \in \mathcal{A}^{(i)}} \sum_{\alpha \in \mathcal{A}} \mathbb{P}[A = \alpha \wedge A_i = \mathbf{a}_i] u^{(i)}(\alpha) = \sum_{\mathbf{a}_i \in \mathcal{A}^{(i)}} \sum_{\alpha \in \mathcal{A}} \mathbb{P}[A_i = \mathbf{a}_i] \mathbb{P}[A = \alpha | A_i = \mathbf{a}_i] u^{(i)}(\alpha) \\ &= \sum_{\mathbf{a}_i \in \mathcal{A}^{(i)}} \mathbb{P}[A_i = \mathbf{a}_i] \mathbb{E}_i[A | A_i = \mathbf{a}_i]. \end{aligned}$$

For all  $\mathbf{a}_i \in \mathcal{A}^{(i)}$ , let us then denote  $\mathbb{E}_i^{\mathbf{a}_i}[A] := \mathbb{E}_i[A | A_i = \mathbf{a}_i] := \mathbb{E}[u^{(i)}(A) | A_i = \mathbf{a}_i]$  for sake of convenience.

Define  $p, q \in [0, 1]$  as follows:

- $p := \mathbb{P}[A_1 = H]$  (therefore  $\mathbb{P}[A_1 = D] = 1 - p$ ),
- $q := \mathbb{P}[A_2 = H]$  (therefore  $\mathbb{P}[A_2 = D] = 1 - q$ ).

## 1.2 Game Equilibria

### 1.2.1 Pure Strategy Nash Equilibria

We can split the problem in 3 cases according to the sign of  $V - D$ .

**First case** Let's start by assuming that  $V > D$  and let  $\delta$  be defined as  $\delta := \frac{V-D}{2}$ . Therefore  $u^{(1)}(H, H) = u^{(2)}(H, H) = \delta > 0$ , which shows that  $(H, H) \in \mathcal{A}$  is a Nash equilibrium because if player 1 switches to Dove, then  $u^{(1)}(D, H) = 0 < \delta$  and by symmetry, we have  $u^{(2)}(H, D) = 0 < \delta$  then player 2 wouldn't switch to Dove either.

Under this assumption, there is no other equilibrium because Hawk strongly (strictly) dominates Dove. Let show prove this statement for player 1. By symmetry, this proof is valid for player 2 as well.

If player 2 plays Dove, then  $u^{(1)}(D, D) = \frac{V}{2} - T < V = u^{(1)}(H, D)$  because  $T > 0$ . If player 2 plays Hawk, then  $u^{(1)}(H, D) = 0 < \delta = u^{(1)}(H, H)$ . We therefore deduce that no equilibrium can contain a player playing Dove.  $(H, H)$  is then the only possible Nash equilibrium.

**Second case** Let's assume that  $V = D$ . We have then  $\delta = 0$ . In this case,  $(H, H)$  stays a Nash equilibrium because  $u^{(1)}(H, H) = 0 = u^{(1)}(D, H)$  and  $u^{(2)}(H, H) = 0 = u^{(2)}(H, D)$ , thus no player would strictly increase their payoff by switching to Dove.

But this action is not the only Nash equilibrium: Hawk doesn't strongly dominate Dove anymore.  $(H, D)$  and  $(D, H)$  are 2 new Nash equilibria of the game. Let us show that  $(D, H)$  is one,  $(H, D)$  is one as well by symmetry.

$u^{(1)}(D, H) = 0 = u^{(1)}(H, H)$  then player 1 wouldn't strictly increase his payoff by switching to Hawk, and  $u^{(2)}(D, H) = V > \frac{V}{2} - T = u^{(2)}(D, D)$  then player 2 wouldn't switch to Dove because it would lower his payoff.

In this case, there are then 3 Nash equilibria:  $\{(H, H), (D, H), (H, D)\}$ . It is easy to show that  $(D, D)$  is not a Nash equilibrium because both players would want to switch to Hawk if their opponent plays Dove since  $u^{(1)}(H, D) = V > \frac{V}{2} - T = u^{(1)}(D, D)$  (same for  $u^{(2)}$ ).

**Third case** Let's finally assume that  $V < D$ , and then  $\delta < 0$ . This implies that  $(H, H)$  is no longer a Nash equilibrium because both players would want to switch to Dove to strictly increase their payoff from  $\delta < 0$  to 0. Nonetheless,  $(D, H)$  and  $(H, D)$  are still Nash equilibria since  $u^{(1)}(D, H) = 0 > \delta = u^{(1)}(H, H)$  and  $u^{(2)}(D, H) = V > \frac{V}{2} - T = u^{(2)}(D, D)$ .

**Remark** We can see that  $T$  does not play an important role in this analysis: whether it is lower, equal or greater than  $\frac{V}{2}$  doesn't affect the Nash equilibria, and never makes  $(D, D)$  a Nash equilibrium.

### 1.2.2 Mixed Strategy Nash Equilibrium

Let us evaluate the following values:  $\mathbb{E}_1^H[A]$ ,  $\mathbb{E}_1^D[A]$ ,  $\mathbb{E}_2^H[A]$  and  $\mathbb{E}_2^D[A]$ :

- $\mathbb{E}_1^H[A] = \frac{q}{2}(V - D) + (1 - q)V$ ,
- $\mathbb{E}_1^D[A] = (1 - q)\left(\frac{V}{2} - T\right)$ .

By symmetry, we find that:

- $\mathbb{E}_2^H[A] = \frac{p}{2}(V - D) + (1 - p)V$ ,
- $\mathbb{E}_2^D[A] = (1 - p)\left(\frac{V}{2} - T\right)$ .

We find easily that  $\mathbb{E}_1^D[A] - \mathbb{E}_1^H[A] = q\left(T + \frac{D}{2}\right) - \left(\frac{V}{2} + T\right)$  and is then a linear increasing function in  $q$ . Therefore,  $\mathbb{E}_1^D[A] = \mathbb{E}_1^H[A] \iff q = \frac{V+2T}{D+2T} =: P$ .

By symmetry one more time, we get the exact same result for  $p$ , i.e.  $\mathbb{E}_2^D[A] = \mathbb{E}_2^H[A] \iff p = P$ . There exists therefore a Mixed Strategy Nash Equilibrium where each player plays Hawk with probability  $p = q = P$  if and only if this value is between 0 and 1, for it to be a probability.

This condition  $P \in [0, 1]$  is equivalent to  $V \leq D$  since all these parameters are  $\geq 0$ .<sup>2</sup>

We can also deduce from these calculations that under this  $V < D$  assumption, if a player ( $i$ ) chooses Hawk with a probability lower than  $P$ , then the other player ( $j$ ) optimal strategy becomes playing Hawk all the time because  $\mathbb{E}_j^H[A] > \mathbb{E}_j^D[A]$ . The opposite also stand: if player  $i$  chooses Hawk with probability higher than  $P$ , then  $\mathbb{E}_j^H[A] < \mathbb{E}_j^D[A]$  and player  $j$ 's optimal strategy becomes playing Dove.

## 2 Which Social Dilemma?

### 2.1 Extension of Notations and Preliminary Results

First of all, to stay consistent with previous notations (and by a remark in the first section), we will consider that players are identified by numbers rather than letters. Just remark that  $A \mapsto 1$  and  $B \mapsto 2$  forms a bijection between  $\{A, B\}$  and  $\{1, 2\}$  (that we will denote by  $\mathcal{J}$  to stay consistent).

Let  $\mathcal{G}$  be set of games  $\{PD, SH, SD\}$ . Let's denote (arbitrarily) each of these games by a  $g_j$  with  $j \in \mathcal{J}$ , with  $\mathcal{J} = \{1, 2, 3\}$  the ordered set of game indices so that we can refer to  $\mathcal{G}$  as  $\mathcal{G} = \{g_i\}_{i \in \mathcal{J}}$ .

Note that the remark on the order relation on  $\mathcal{J}$  above also stands here for the order relation on  $\mathcal{J}$ .

Let us also extend the utility function notation. The utility function of player  $i \in \mathcal{J}$  will be denoted by:

$$u^{(i)} : \mathcal{A} = \prod_{i \in \mathcal{J}} \mathcal{A}^{(i)} := \prod_{i \in \mathcal{J}} \prod_{j \in \mathcal{J}_i} \mathcal{A}_j^{(i)} \rightarrow \mathbb{R},$$

with  $\mathcal{J}_i \subset \mathcal{J}$  the set of playable games by player  $i$ , and  $\mathcal{A}_j^{(i)}$ , the set of available actions for player  $i \in \mathcal{J}$  in game  $g_j$ . In this precise case, player 1 only plays one game and has therefore only one set of possible actions ( $\{C, D\}$ ) whereas player 2 has 3 games has then strategies composed of three actions: one for each game ( $|\mathcal{J}_1| = 1$  and  $|\mathcal{J}_2| = 3$  since  $\mathcal{J}_2 = \mathcal{J}$ ). Note that  $j \in \mathcal{J}$  such that  $\mathcal{J}_1 = \{j\}$  can be chosen arbitrarily.

This  $u^{(i)}$  function requires  $u_{(j_k)_k}^{(i)} : \prod_{k \in \mathcal{J}} \mathcal{A}_{j_k}^{(k)} \rightarrow \mathbb{R}$  to be defined for every  $(j_k)_k \in \prod_{k \in \mathcal{J}} \mathcal{J}_k$ , which correspond to the utility function of player  $i$  when each player  $k$  is playing game  $j_k \in \mathcal{J}_k \subset \mathcal{J}$ .

This extension also applies to the  $U^{(i)}$  matrices. One can define:

$$u_{((a_{ij})_{j \in \mathcal{J}_i})_{i \in \mathcal{J}}}^{(i)} := \sum_{(j_1, \dots, j_N) \in \prod_{k \in \mathcal{J}} \mathcal{J}_k} \left( \prod_{k \in \mathcal{J}} \mathbb{P}[\mathcal{G}_k = g_{j_k}] \right) u_{(j_1, \dots, j_N)}^{(i)}(a_{1j_1}, \dots, a_{Nj_N}).$$

We will focus here on the expected payoff of each player according to the probability distribution of the played game according to each player. To formalize this, for every  $i \in \mathcal{J}$ , take  $\mathcal{G}_i$  a discrete random variable

<sup>2</sup>Note that  $P$  is not defined if  $D = T = 0$  but as  $0 \leq V \leq D = 0$ , we have  $V = D = T = 0$  and then both players have a payoff of 0, no matter what action they choose. Then any value  $P \in [0, 1]$  suits. In particular, if  $P \in \{0, 1\}$ , then the strategy is a pure strategy, and thus is a Nash equilibrium since in constant utility games (a pathological situation because there is no interesting preference relation left due to the fact that  $\forall a, b \in \mathcal{A} : \forall i \in \mathcal{J} : (a \succsim_i b) \wedge (b \succsim_i a)$ ), every pure strategy is a Nash equilibrium.

on  $\{g_j\}_{j \in \mathcal{J}_i} \subset \mathcal{G}$  with probability law  $\mathbb{P}$ .<sup>3</sup>

Now, let us remark that if there exist a player  $i \in \mathcal{I}$  and  $\tilde{j}_i \in \mathcal{J}_i$  such that the game  $g_{\tilde{j}_i} \in \mathcal{G}$  contains an action  $a^-$  strongly dominated by another action  $a^+$  for player  $i$ , not matter what games  $(g_{j_k})_{k \neq i}$  are played by other players, then any strategy of player  $i$  that contains  $a^-$  as action for game  $g_{\tilde{j}_i}$  cannot be a Nash equilibrium. This follows easily from the fact that the expected payoff of a strategy  $a$  for player  $B$  with  $a_i = a^-$  will be strictly less than the expected payoff of the strategy  $(a^+, a_{-i})$ .

To proof this statement, take  $a^{(1)} = (a_{ij})_{ij} = ((a_{1j})_j, \dots, (a_{Nj})_j)$  such that  $a_{i\tilde{j}_i} = a^-$ , and take  $a^{(2)} = (a_{-i\tilde{j}_i}^{(1)}, a^+)$ , which is then the same action profile as  $a^{(1)}$  except that player  $i$  now plays  $a^+$  instead of  $a^-$  in game  $g_{\tilde{j}_i}$ . Then:

$$u_{a^{(1)}}^{(i)} - u_{a^{(2)}}^{(i)} = \sum_{(j_1, \dots, j_N) \text{ s.t. } j_i = \tilde{j}_i} \underbrace{\left( \prod_{k=1}^N \mathbb{P}[\mathcal{G}_k = g_{j_k}] \right)}_{\geq 0} \underbrace{\left( u_{(j_1, \dots, j_N)}^{(i)}(a^{(1)}) - u_{(j_1, \dots, j_N)}^{(i)}(a^{(2)}) \right)}_{< 0} \leq 0.$$

This value is indeed negative since it is a sum of negative values. The second term of the product in the sum is always negative since this is exactly what being strongly dominated means. Therefore, if  $a^{(1)}$  was a played strategy profile, player  $i$  would prefer to switch from  $a^-$  to  $a^+$ , which implies that  $a^{(1)}$  cannot be a Nash equilibrium.

## 2.2 Resolution

### 2.2.1 Pure Strategy Nash Equilibria

The previous statement about strictly dominated actions is used here. One can observe that in the prisoners dilemma, the *Cooperate* action is strictly dominated by the *Defect* action. Therefore, the situations where player 2 plays C on first game won't be relevant in the study of Nash equilibria and can therefore be discarded.

Table 2 shows the expected payoff of each player according to their strategy, without player 2 strategies containing C for the prisoners dilemma.

|   | DCC           | DCD            | DDC           | DDD           |
|---|---------------|----------------|---------------|---------------|
| C | (7, 12)       | <b>(6, 13)</b> | (2, 9)        | (1, 10)       |
| D | <b>(8, 2)</b> | (3, 1)         | <b>(7, 3)</b> | <b>(2, 2)</b> |

Table 2: Payoff of each player according to the strategies. Best responses are written in boldface, which allows to easily find out the Nash equilibria. Payoffs are multiplied by 3 to avoid handling fractions. This is still the same payoff table since the magnitude of the value of a utility function isn't important, what is important is the order relation on these values. Therefore, if  $u^{(i)}$  is a utility function for player  $i$ , then for any non decreasing function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the function  $f \circ u^{(i)}$  is still a utility function for player  $i$  because it represents the same preference relation.

2 Pure Strategy Nash Equilibria are present in this table: (C, DCD) and (D, DDC).

<sup>3</sup>It is precised in the instructions that  $\mathcal{G}_2$  follows a discrete finite uniform distribution, i.e.  $\forall j_2 \in \mathcal{J}_2 : \mathbb{P}[\mathcal{G}_2 = g_{j_2}] = |\mathcal{J}_2|^{-1} = \frac{1}{3}$  and that  $\mathcal{G}_1$  has values in a singleton and then for  $j \in \mathcal{J}$  such that  $\mathcal{J}_1 = \{j\}$ ,  $\mathbb{P}[\mathcal{G}_1 = g_j] = 1$ . Nevertheless, the notations detailed below are extensible to any distribution  $\mathbb{P}$  for the  $\mathcal{G}_i$ 's. Note also that these probability measure are not the same as the one for  $A$  defined previously because they are not defined on the same set. Yet the context is clear enough to know which measure is used, there is no need to introduce new notations.

### 2.2.2 Mixed Strategy Nash Equilibria

In order to find Mixed Strategy Nash Equilibria, let's consider that player 1 plays C with probability  $p \in [0, 1]$ , and that player 2 plays C in the PD with probability  $q_1 \in [0, 1]$ , plays C in the SH with probability  $q_2 \in [0, 1]$  and plays C in SD with probability  $q_3 \in [0, 1]$ . Let again define  $A$  a discrete random variable on  $\mathcal{A}$  with probability law  $\mathbb{P}$ .

One can not simply discard PD by stating that player 2 would always choose D because C is strongly dominated by D because in a Mixed Strategy Nash Equilibrium, a dominated action can be played with strictly positive probability.

Yet by computing the expected values  $\mathbb{E}_2^a[A]$  for  $a \in \mathcal{A}^{(2)}$ :

- $\mathbb{E}_2^{CCC}[A] = 1 + 8p,$
- $\mathbb{E}_2^{CCD}[A] = 10p,$
- $\mathbb{E}_2^{CDC}[A] = 2 + 4p,$
- $\mathbb{E}_2^{CDD}[A] = 1 + 6p,$
- $\mathbb{E}_2^{DCC}[A] = 2 + 10p,$
- $\mathbb{E}_2^{DCD}[A] = 1 + 12p,$
- $\mathbb{E}_2^{DDC}[A] = 3 + 6p,$
- $\mathbb{E}_2^{DDD}[A] = 2 + 8p;$

one can observe that:

- $\mathbb{E}_2^{CDD}[A] \leq \mathbb{E}_2^{CCC}[A] \leq \mathbb{E}_2^{DCD}[A],$
- $\mathbb{E}_2^{CCD}[A] \leq \mathbb{E}_2^{DDC}[A],$
- and  $\mathbb{E}_2^{CDC}[A] \leq \mathbb{E}_2^{DCC}[A].$

Therefore any strategy for player 2 with action C in PD has lower expected payoff than the same strategy with D instead. We can conclude (in our equilibria analysis) that  $q_1 = 0$  because no Mixed Strategy Nash Equilibrium could contain player 2 playing C in PD.

Although stating that the first game (PD) can be discarded (in the equilibria analysis) because C is dominated by D is not true, in this study, the first game can be discarded. Only strategies DCC, DCD, DDC, and DDD need to be considered to appear in an equilibrium.

Player 2's best response function varies according to the value of  $p$  and gives the strategy (strategies) maximizing the expected payoff. As  $\mathbb{E}_2^{DDD}[A] \leq \mathbb{E}_2^{DCC}[A]$ , only DCC, DCD, and DDC should appear in the best response function yields. One can find:

- if  $p < \frac{1}{4}$ , then player 2 should play DDC,
- if  $p = \frac{1}{4}$ , then player 2 should play DDC or DCC with any probability distribution,
- if  $\frac{1}{4} < p < \frac{1}{2}$ , then player 2 should play DCC,
- if  $p = \frac{1}{2}$ , then player 2 should play DCC or DCD with any probability distribution,
- and if  $p > \frac{1}{2}$ , then player 2 should play DCD.

**Remark** Note that, as expected, extreme cases of this result are still consistent with the Pure Strategy Nash Equilibria found before: if  $p$  is sufficiently low, then player 1 would play D most of the time, and then player 2 should play DDC ((D, DDC) is a Nash Equilibrium), and if  $p$  is large enough, then player 1 would play C most of the time, and then player 2 should play DCD ((C, DCD) is a Nash Equilibrium).

Let's now find player 1's best response function in order to find Mixed Strategy Nash Equilibria of the system.

Let's compute the values of  $\mathbb{E}_1^C[A]$  and  $\mathbb{E}_1^D[A]$ :

- $\mathbb{E}_1^C[A] = 5q_2 + q_3 + 1$ ,
- and  $\mathbb{E}_1^D[A] = q_2 + 5q_3 + 2$ .

Therefore  $\mathbb{E}_1^C[A] = \mathbb{E}_1^D[A] \iff q_2 = q_3 + \frac{1}{4}$ .

- If  $q_2 - q_3 \leq \frac{1}{4}$ , then  $\mathbb{E}_1^C[A] < \mathbb{E}_1^D[A]$  which implies that player 1 has no interest in playing C and that he shall play D instead (i.e.  $p = 0$ ),
- if  $q_2 - q_3 = \frac{1}{4}$ , then player 1 has no strict preference between C and D and that both can be played in a given probability distribution,
- otherwise, if  $q_2 - q_3 > \frac{1}{4}$ , then  $\mathbb{E}_1^D[A] < \mathbb{E}_1^C[A]$  which implies that player 1 has no interest in playing D and that he shall play C instead (i.e.  $p = 1$ ).

To determine possible equilibria, just check mixed strategies of player 2 found above, and see if they can intersect player 1 mixed strategies found right above:

- a possible Mixed Strategy Nash Equilibrium would be to have  $p = \frac{1}{4}$  and  $q_2, q_3$  set such that  $q_2 - q_3 = \frac{1}{4}$ . Yet this is impossible since  $p = \frac{1}{4}$  implies that  $q_3$  is set to 1 for player 2 to choose between DCC and DDC. In this case  $q_2 - q_3 \leq 0$ , in particular  $q_2 - q_3 \not\geq \frac{1}{4}$ .
- The other would be to have  $p = \frac{1}{2}$  and  $q_2, q_3$  set such that  $q_2 - q_3 = \frac{1}{4}$ . But  $p = \frac{1}{2}$  implies that  $q_2$  is set to 1 for player 2 to choose between DCC and DCD. Therefore, the only triple  $(p, q_2, q_3)$  that fulfills these requirements is  $(\frac{1}{2}, 1, \frac{3}{4})$ .

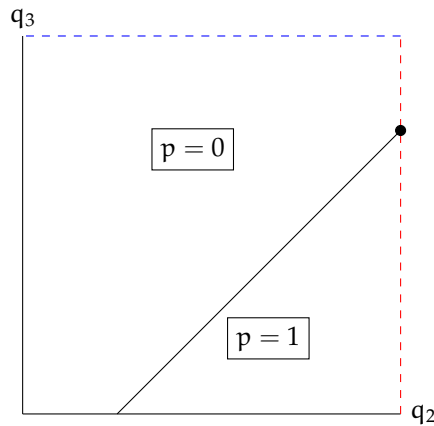


Figure 1: Mixed Strategy Diagram. The red dashed line represents all the probability distribution allowing player 2 to play either DCC or DCD ( $q_2$  set to 1). The blue dashed line represents all the probability distribution allowing player 2 to play either DDC or DCC ( $q_3$  set to 1). The black filled dot represents the only possibly Mixed Strategy Nash Equilibrium.

Looking at Figure 1 makes obvious that the only area where  $p \in (0, 1)$  is allowed is the line between the points  $(.25, 0)$  and  $(1, .75)$ , and therefore a Mixed Strategy Nash Equilibrium needs to sit on it. Also an equilibrium should sit on one of the dashed lines since these are the only mixed strategies that player 2 can play rationally. Therefore, only one point fulfills both these criteria: the equilibrium described above.

One can then deduce that in addition to the two Pure Strategy Nash Equilibria described above, a Mixed Strategy Nash Equilibrium exists for this system.

### 3 Sequential Truel

#### 3.1 Modelling and Notations

Figure 2 shows the game tree of the truel. Chance moves are shown as nodes in the graph but are distinguishable from *choice* nodes because such chance moves are represented by a circle node.

One assumption is made in the resolution of this problem: as player C is the last one to play, no matter who he aims, player C will always survive, and then have a payoff of one if he is allowed to *play*. Therefore, in order to evaluate the expected payoff of a subgame, if both actions that a player can take lead to identical payoffs, then the expected payoff of the players will be made by a simple arithmetic average of the expected payoffs of each available action.

Otherwise, the expected payoff of an action made by player  $i \in \mathcal{I}$  for each player is dependent on the action that maximizes player  $i$ 's expected payoff.

Also, the expected payoff of a chance move is the average of the payoffs of each possible outcome of the chance move, weighted by the probabilities of each outcome occurring (usual definition of an expected value).





Figure 2: Sequential diagram of the Truel. Circle nodes represent chance moves (occurrences of probability). Leaves of the tree (payoffs) are written in red.

### 3.2 Resolution

This game has to be solved by Backward Induction. To do this, the expected payoff of each player must be extracted of each subgame, starting from the bottom. The first step is then to start from the *leaves* of the tree (which correspond to the moment where players get their payoff, or more precisely, the moment where each player still alive is sure to remain such way, and already dead player are also sure to remain this way) which are all preceded by a chance move, and to average those as described above.

Second step is to average situations where last playing player has no preference on the actions that he may take. This leads to a reduced version of the tree of Figure 2, shown on Figure 3.

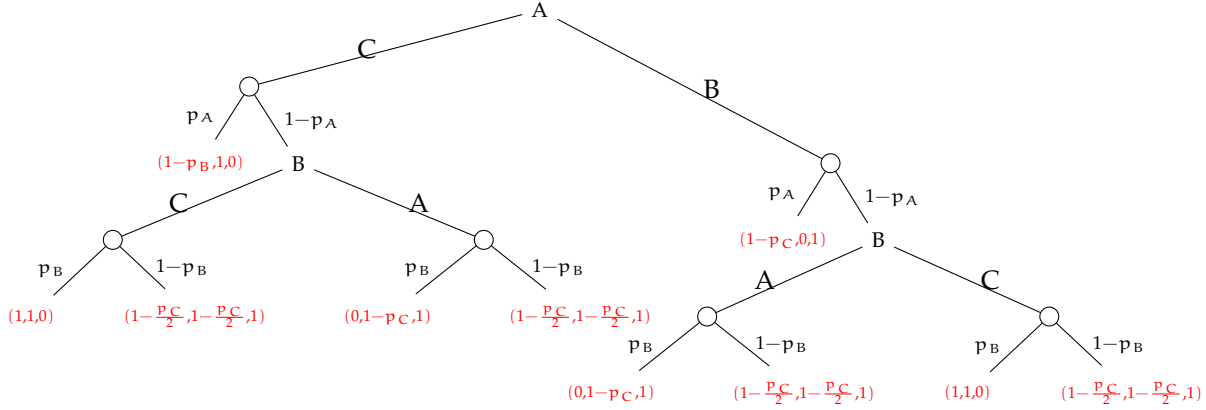
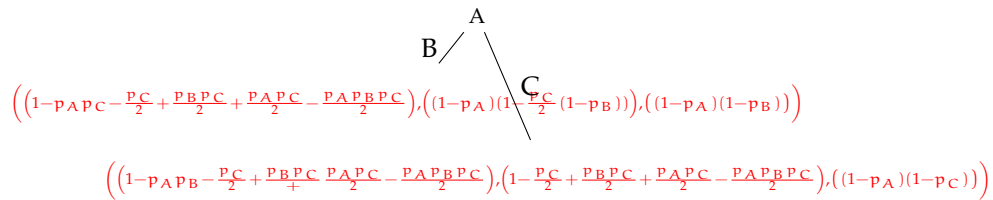


Figure 3: Game tree of the Truel with last player's actions replaced by expected payoffs.

By weight-averaging again on this tree, one can clearly see that the expected payoff of player B by shooting A is  $p_B p_C$  smaller than his expected payoff by shooting C. This is obvious since by shooting C, player B allows himself to live longer if he aims correctly, whereas shooting A won't prevent C from shooting him.

By averaging values one more time (still from bottom to top), A needs to choose between shooting B or C, and the game tree becomes:



Therefore, player A's action will be to aim B rather than C if and only if  $p_B > p_C$  since player A's expected payoff for shooting C minus player A's expected payoff for shooting B is given by:

$$\left(1 - p_A p_B - \frac{p_C}{2} + \frac{p_B p_C}{2} + \frac{p_A p_C}{2} - \frac{p_A p_B p_C}{2}\right) - \left(1 - p_A p_C - \frac{p_C}{2} + \frac{p_B p_C}{2} + \frac{p_A p_C}{2} - \frac{p_A p_B p_C}{2}\right) = p_A p_C - p_A p_B = p_A (p_C - p_B).$$

If  $p_B > p_C$ , then shooting C yields a lower expected payoff than shooting B.

There are then two possibilities:

- either  $p_B > p_C$ , and then player A shoots B. If A misses, then B shoots C, and if B misses, then C shoots either A or B. If A didn't miss, then C shoots A.
- or  $p_B < p_C$ , and then player A shoots C, then B shoots C as well. If both missed, then C shoots either A or B.

This implies that if  $p_C > p_B$ , then A would shoot C, and B would also shoot C if still alive. Then, C's probability to stay alive is  $(1 - p_A)(1 - p_B)$ , and if  $p_C < p_B$ , then C's probability to stay alive is  $p_A + (1 - p_A)(1 - p_B)$ . Therefore, C's odds to stay alive are higher if  $p_C < p_B$ . C is thus more likely to survive if he is a worse shooter than B is.

### 3.3 Remark

Note that if  $p_A = p_B = p_C = 1$ , then A's best action, even though not stated as possible action, would be to shoot in the air so that B would be forced to kill C, otherwise, if A would shoot either B or C, then the alive one would shoot him dead.

Note also that the game has been modeled with payoffs of either 0 or 1. This reflects the assumption that players only want to live. Another modelling could be to have payoffs of 0 if the player dies when shot by A, 1 when shot in second (thus either by B if still alive or by C if A killed B), 2 if player is killed by third shot (thus by C if still alive), and 4 if player doesn't die.

The strategy would probably differ since for example player A would prefer to be shot by C by third shoot than by B in second shoot, whereas in the studied model above, player A has no *preference* between being killed in second or third turn.