INFOF-409 — Learning Dynamics
Assignment 2 — Evolutionary Game Theory

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Contents

1 Part I

Figure 1 shows the cooperation proportion for different grid (lattice) sizes, for different neighbourhood types, and for different payoffs. In this first part, only the first two subfigures are of interest: the upper left one presents the proportion of cooperation level for different lattice sizes $n \times n$ (respectively $n = 4, 8, 12, 20, 50, 100, 500$) when playing Prisoner Dilemma with best neighbour replication rule and with Moore neighbourhood, and the upper right one is the same configuration bu with Von Neumann neighbourhood instead.

1.1 Moore Neighbourhood Analysis

1.1.1 Cooperation Level

What one can see is that, first of all, with Moore neighbourhood, the larger the lattice, the higher the cooperation level is in the lattice. A hypothesis that one could express is that for small lattices such as 8 x 8 or
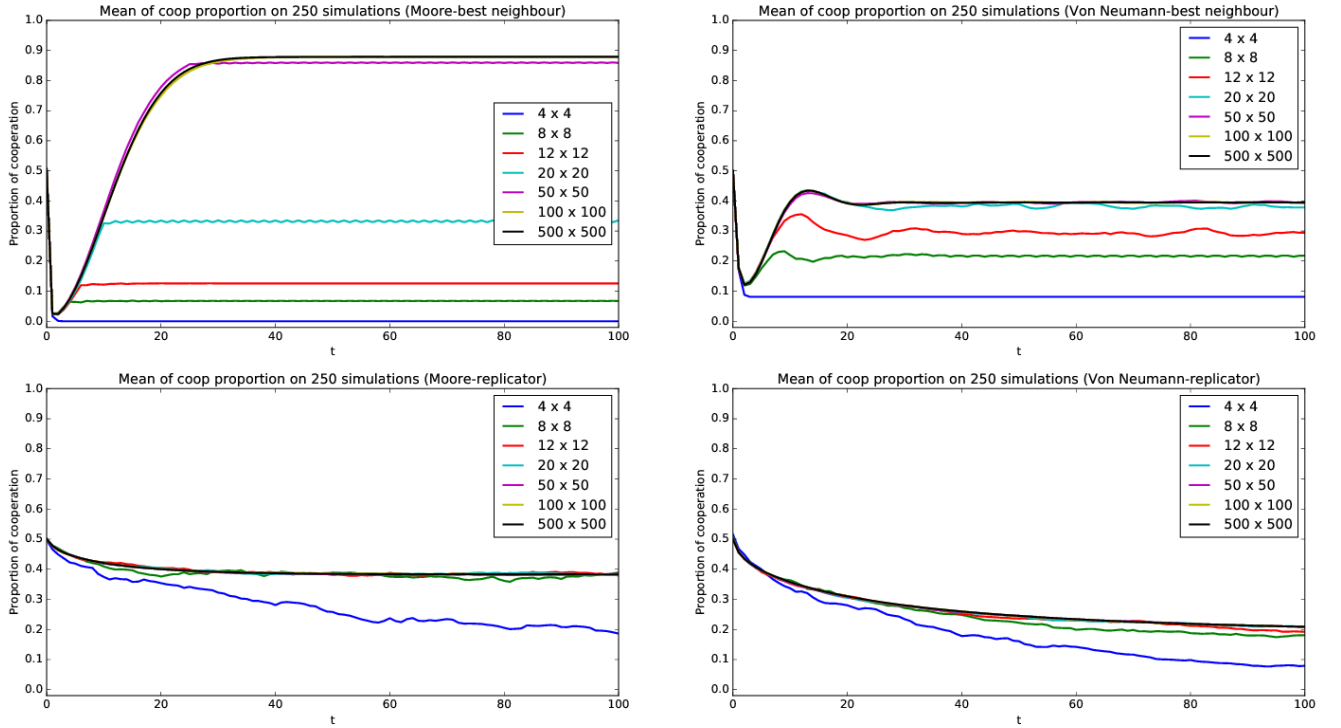
Figure 1: Proportion of cooperation level according to lattice size, neighbourhood size, and replication rule (coupled with payoff matrix) on 250 evolution steps. First plot represents the cooperation proportion of square lattice (in the sense of square dimension: $n \times n$ since lattices are considered as placed on the surface of a torus) playing weak Prisoner Dilemma with Best Neighbour replication rule and Moore neighbourhood. Second plot is the same as the first one with Von Neumann neighbourhood instead of Moore neighbourhood. Third plot represents the cooperation proportion of square lattices playing Snowdrift Game with stochastic replication rule and Moore neighbourhood. Finally, the last plot is the same as the third one with Von Neumann neighbourhood instead of Moore neighbourhood.

12 x 12, the lattice reaches some sort of steady state very quickly since there is not much distance overall that would allow more complex patterns to emerge.

Also, we see that for a 4x4 lattice, the cooperative players tend to vanish in only two periods. This is due to the fact that in such a small space, there cannot be enough of cooperative players clustered together, which would allow them to increase their individual payoff, they are surrounded by too many defecting players. Therefore, in a 4 x 4 lattice with Moore neighbourhood, the cooperation is not viable.

Another interesting observation is that even though instructions required to test lattice sizes up to 50 x 50, when plotting the mean cooperation level for lattices even bigger (e.g. 100 x 100 and 500 x 500), we observe that the cooperation level asymptotes to around 88%. A possible interpretation is that as the lattice is sufficiently large, cooperative players have the possibility to expand on the lattice, but will not cover it entirely because a defecting player still have an advantage over cooperative players in some circumstances. For instance, a defecting player surrounded by cooperative players will have a payoff of 56. Then, if one of its cooperative neighbours has less than six cooperative neighbours, his payoff will be lower or equal to 50. In this situation, it is clear that the defecting method is preferable. Also, the high level of cooperation at *stability* can be explained by this same interpretation: in a large lattice, the majority tends to cooperate, but defecting only represents an advantage for a player with many cooperative neighbours. Therefore, a small portion defects, but this proportion cannot be too big otherwise defecting stops being an advantage.

We can also observe that at time $t = 1$, the cooperation level of the lattice crashes down to around 2%. The explanation only requires the update method, and no the neighbourhood type, which is why we will make the same observation with Von Neumann neighbourhood. In this first part, the update method is *Best neighbour*, meaning that at each period, every player (agent) will change its action by replacing it by the action of its neighbour having the best payoff (if this payoff is bigger than their own payoff, otherwise they don't change action). Also, the lattice is initialized randomly with a 50% proportion of cooperative players and 50% of defecting players. Therefore, one can assume that in average (more and more reasonable assumption as the lattice size increases) the cooperative and defecting players are scatter along the lattice, and not clumped together. This reduces highly cooperative players' payoff, and then makes cooperative players change to defecting for next period, whereas in general defecting players' wouldn't change to cooperation since their payoff is bigger.

Yet, despite all of the differences in behaviour according to the lattice size, all curves on the first subplot all share one property: in average, they all converge to a stable cooperation proportion, and the time required to reach this stability seems to be dependent on the value of the equilibrium. This can be explained by the sudden drop that we observe at first step: the cooperation level on the lattice crashes down, and then has to climb up to the stable point. So, the higher the stability, the longer the lattice must *wait* to reach that stability.

1.1.2 Spatial Representation

Figure 2 shows the spatial representation of the lattice at times $t = 0, 1, 5, 10, 50, 100$. On Figure 2a, we see that at $t = 0$, the players are scattered randomly, and that at $t = 1$, the cooperative players proportion suddenly drops to 2%, and then increase slowly to 84%.
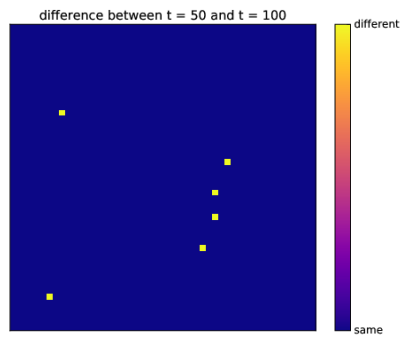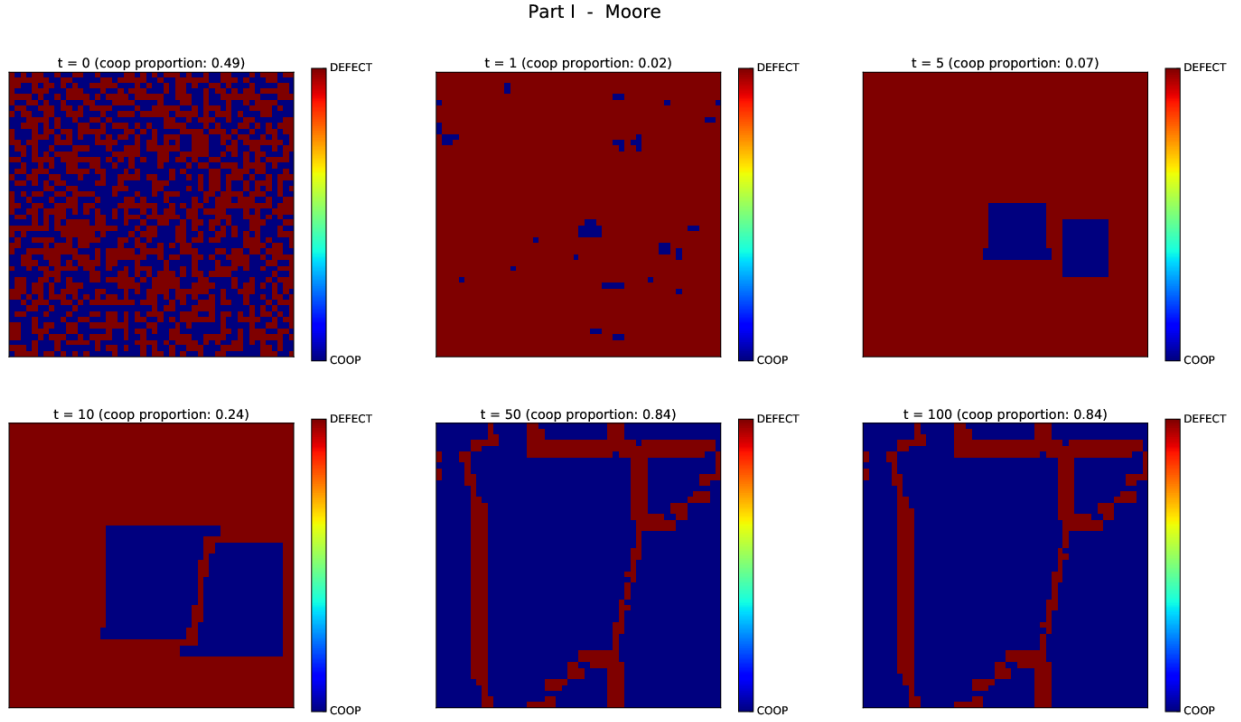
We can see that at time $t = 1$, there are only a few cooperative players left and that, in the most part have less than 4 cooperative neighbours. By the discussion above, this leads them to change action to defection. At time $t = 5$, we see clearly that only two blocks of cooperative players were big enough to not change to defection, and to expand. At time $t = 10$, we can see that these blocks are the same, but have just expanded (meaning that their payoff was sufficiently big to force defecting neighbours become cooperative).

Still at $t = 10$, we observe a clear boundary of defecting players between the two observable blocks of cooperative players. This phenomenon is explained above in the discussion on the convergence of cooperation

level. For the defecting players in this region, they have between 3 and 6 cooperative neighbours, leading to a high payoff.

These boundaries take the form of *strips* of defecting players between cooperative players, and can be observed even further with $t > 10$: for $t = 50$ and $t = 100$.

Also, even though the lattice looks identical between times $t = 50$ and $t = 100$, it has actually changed, but very slightly, which is shown in Figure 2b. We can interpret this as stability, especially when looking at the difference in Figure 2b: only 6 players have changed action. Note that more can have changed their action between these two periods, but in this case, they need to have settled back to their action of time

(a) Spatial view of the lattice at different periods.

(b) Difference of actions in lattice between periods 50 and 100.

Figure 2: Spatial representation of Part I with Moore neighbourhood

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$t = 50$. Therefore, the behaviour can either be complete stability: players don't change actions anymore, or periodicity.

What can be concluded from this analysis is that in the configuration of the first part, with a Moore neighbourhood, a cooperative strategy is not viable if the amount of defecting neighbours is too high, but if enough cooperative players are clumped together in a region of space, then they will *convert* their defecting neighbours since their payoff will be high enough.

1.2 Von Neumann Neighbourhood Analysis

1.2.1 Cooperation Level

Looking back at Figure 1, on the second subplot we can see that a lot of observations made in the previous subsection still hold: there is a sudden drop in the cooperation level at time $t = 1$; the larger the lattice, the higher the steady value; the higher the steady value, the more time is required to reach *stability*. Note that "stability" is written in italic because the so called *stability* studied here is an average pseudo-stability: an individual lattice situation could either completely miss the equilibrium we are discussing, or miss any equilibrium by having seemingly chaotic behaviour. Yet, when considering the behaviour as an average of many different lattices, it may be considered stable.

Despite sharing many common features with the first study case, this second one has a few differences. First of all, we can see that there is a lot less variance in the stability values reached by the different lattice sizes: grids smaller than 20 x 20 have a higher average cooperation level stability than in the previous study, and grids bigger than 20 x 20 have a smaller average cooperation level stability than in the previous case. The 20 x 20 lattice also increased a bit.

But not only are these values closer to one another, we especially observe that the asymptote value is more than halved. We can hypothesize that as neighbourhood is reduced compared to previous case, cooperative players have a harder time *spreading* their strategy since only fewer defecting neighbours may change their strategy to cooperation.

Also, we can see that on the 4 x 4 lattice, cooperation doesn't die out as it did with Moore neighbourhood. This could either be due to the averaging of 250 simulations or be a general persistence of cooperative players. It seems reasonable to consider this to not be due to the averaging: in a Von Neumann neighbourhood, it is viable for a cooperative player to have a single cooperative neighbour, whereas it is not in a Moore neighbourhood.

D	D	D	D
D	D	**C**	**C**
D	D	D	D
D	D	D	D

(a) Action of each player on the lattice

0	0	7	7
7	7	**10**	**10**
0	0	7	7
0	0	0	0

(b) Payoff of each player on the lattice for Von Neumann neighbourhood

7	7	14	14
7	7	**10**	**10**
7	7	14	14
0	0	0	0

(c) Payoff of each player on the lattice for Moore neighbourhood

Table 1: Example of situation on a 4 x 4 lattice where only two cooperative players is viable (stable) with Von Neumann neighbourhood but is not with Moore neighbourhood. Cooperative players are marked in boldface to make them easier to locate. Borders are marked in dashed lines only to emphasize the torus topology of the lattice: *borders* are **not** borders but the lattice wraps back on itself.

As can be seen in Table 1, it is possible to find a non-decreasing situation (in the sense of the cooperation level) with only two cooperative players on a 4 x 4 lattice with a Von Neumann neighbourhood. This is

impossible with a Moore neighbourhood. The smallest non-decreasing situation on a 4 x 4 lattice with Moore neighbourhood contains 4 cooperative players arranged as follows: one placed arbitrarily and the three others placed in its Von Neumann neighbourhood (i.e. non diagonal neighbours). This is due to the fact that Von Neumann neighbourhood forces two neighbours to have an empty common neighbourhood, whereas Moore neighbourhood forces two neighbours to have exactly a common neighbourhood of length 4.

Note that as the lattice borders are identified by parallel pairs, i.e. as the lattice is a torus, modulo rotations, this situation is the only one possible.

One last thing to note is that after the sudden drop in the cooperation proportion, this cooperation level reaches a local maximum and decreases slightly afterwards. This is easy to see for grids 20 x 20 and bigger. This phenomenon may be perceived for 8 x 8 and 12 x 12 but is very tiny, and the curves are less smooth, so it is to interpret with care.

This double-bump effect is hard to interpret because it appears on the averaged plot. Therefore, it would be needed to explore many lattices individually to see if this phenomenon appears every time, or if it is due to the averaging of several different behaviours.
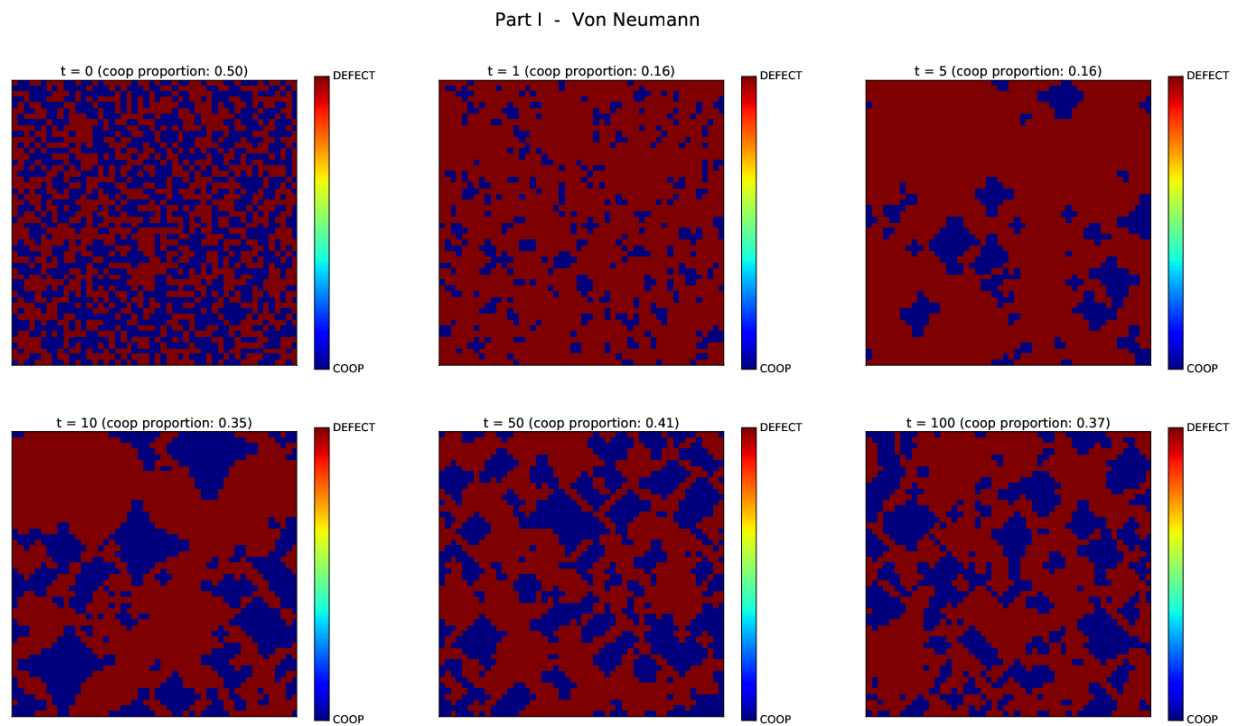
1.2.2 Spatial Representation

Figure 3 shows the spatial representation of the 50 x 50 lattice at times $t = 0, 1, 5, 10, 50, 100$. The first thing to note is that at time $t = 1$, the cooperative players have heavily dropped (reduced to around a third than the cooperation level at time $t = 0$) just like before, but the magnitude of the phenomenon is completely different: divided by 25 for Moore neighbourhood and by 3 for Von Neumann neighbourhood. This can also be seen in Figure 1: the drop is not as strong on the second plot, and also it takes longer to fully drop. With Moore neighbourhood, there is a sudden drop at time $t = 1$, and the proportion level stays approximately constant at time $t = 2$ (an equal amount switches from cooperation to defection than from defection to cooperation) whereas with Von Neumann neighbourhood, the drop takes two step to reach around 12%.

Compared to Figure 2a, we clearly see that in this case, even though cooperative players get scattered along the lattice, small clusters still remain whereas in first study, cooperative players only remained in quite large blocks (at least large enough to have a high enough payoff).

In the end, we see that the amount of cooperative players is lower than in Figure 2a, as expected by looking at Figure 1, and also that defecting players don't form strips as before but form somehow large clusters, probably because cooperative players aren't able to break through the defecting borders.

We can also observe that at time $t = 10$, the cooperative players seem to form diamond shapes versus rectangles in Figure 2a, which is quite clear due to the neighbourhood shape, and then due to the propagation vectors.

In comparison with Figure 2b, Figure 3b shows that the lattice is way less stable in this second study case than in the first one: even though the proportion of cooperative players stays roughly constant, the lattice itself is not static at all.

(a) Spatial view of the lattice at different periods.
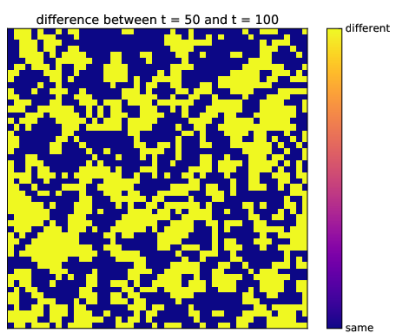
(b) Difference of actions in lattice between periods 50 and 100.

Figure 3: Spatial representation of Part I with Von Neumann neighbourhood.

2 Part II

In this second part, the update method has changed: for each player, one of the neighbours is chosen randomly among all of the 8 if Moore, 4 if Von Neumann neighbours, and the player's action is changed to the neighbour's action with probability $p_{ij} = 1/2 + \frac{W_j - W_i}{2 \cdot 10N}$ where N is the number of neighbours, j represents the randomly chosen neighbour and i represents the player. The 10 in the denominator comes from the difference between $T = 10$ and $P = 0$, respectively the highest and the lowest payoffs.

This value p_{ij} is indeed a probability because it is between between 0 and 1. It is easy to see that the smallest possible value is when W_j is null, and when W_i is as high as possible, which occurs if the player i has a payoff of 10 against each of the N neighbours, leading to a payoff of $10N$. In this case, $p_{ij} = .5 + \frac{-10N}{20N} = .5 - .5 = 0$. Similarly, by setting player i's payoff to 0, and player j's payoff to be maximal, the probability becomes $p_{ij} = .5 + .5 = 1$.

This probability is then well defined. This formula makes sense to be used because it is linear in the difference of payoff. Therefore, the higher the neighbour's payoff relative to the player's payoff, the higher the probability to change the player's action. On the other hand, the higher the player's payoff relative to the neighbour's payoff, the smaller probability to change. Also, if both the player and its neighbour have the same payoff, the probability to change is a half, thus allowing to explore more possible patterns in the lattice.

Note that allowing a player to change its action to the action of a neighbour having a lower payoff allows to explore a broader portion of the state space, reducing the chances of converging, but reducing the chances of finding a local maximum (considering that the proportion of cooperating players is a value to maximize for instance).

A possible improvement to the analysis, which will not be performed in this document would be to keep this probability p_{ij} of changing the action, but changing the way of picking the neighbour. Instead of having a uniform distribution over the neighbours, one could think of a distribution giving more probability mass to the neighbours having a high payoff. This leads to a very wide class of possible distributions where many distributions could be tested. For instance, the probability of picking the neighbour j would be W_j/W, for W the sum of the payoffs of each neighbour (and a uniform distribution if all neighbours have payoff 0). This method would lead to a uniform distribution with probability $1/N$ for each neighbour in the situation where all neighbours have the same payoff but could lead to a totally different behaviour when the neighbours have different payoffs.

For example, considering a Moore neighbourhood with a player having 7 neighbours with a null payoff, and one with a strictly positive payoff. If the distribution over neighbours is uniform, the player has probability $7/8 = 87.5\%$ to pick a neighbour with null payoff, whereas with the possible formula provided above only the neighbour with non null payoff would be selected.

2.1 Moore Neighbourhood Analysis

2.1.1 Cooperation Level

This subsubsection concerns the third plot of Figure 1.

There are several major differences between this situation and the previous ones. Let's focus the analysis on the difference between this situation and its homologue of Part I for now. First of all, we can observe that the cooperation proportion is clearly decreasing with time. A hypothesis would be to say that co-operative players have a higher probability to change to defect than defecting players have to change to cooperation.

From time t = 0 to time t = 1
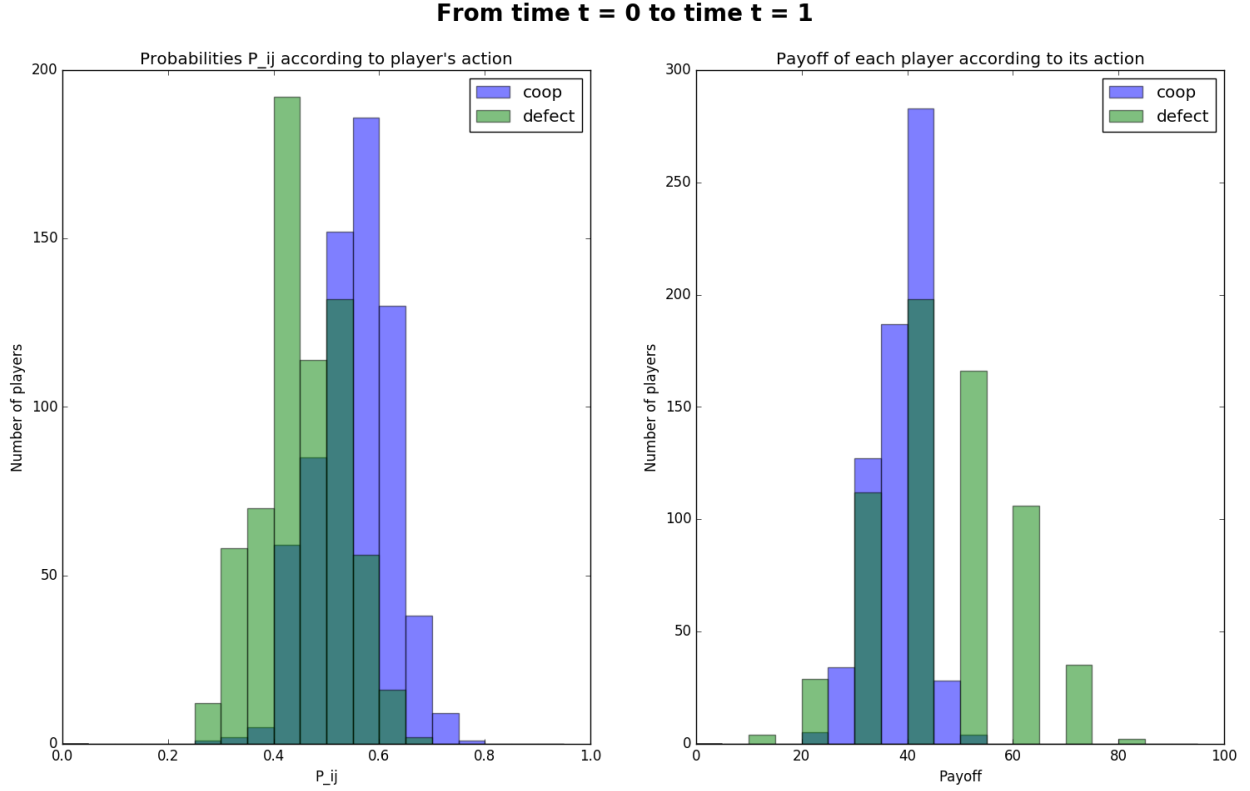
Figure 4: Payoff and probability to change action for players on a 50 x 50 lattice at time $t = 0$. Every player considered in the first histogram had either cooperating or defecting strategy, and had picked a neighbour at random with the opposite strategy; players with a 100% chance of keeping their strategy were not included. Second histogram represents the payoff distribution of these same players, still separated by their strategy.

This hypothesis seems confirmed by Figure 4 which shows a significant difference between the probability distribution of changing from cooperation to defection and from defection to cooperation. Such a difference must have to do with the initial random attribution of strategies on the lattice, and with the random choice that is performed.

Therefore, the cooperation level decreases with time until these probabilities reach an equilibrium. This can be observed in Figure 5: distribution of probabilities p_{ij} don't significantly differ anymore between times $t = 99$ and $t = 100$. This means that the expected amount of players changing from defection to cooperation is around the same as the number of players changing from cooperation to defection, thus leading to a stable cooperation proportion.

As a second major observation, the average cooperation level doesn't depend on the lattice size (except for the 4 x 4 lattice which will be discussed right after): from 8 x 8 up to 500 x 500, at a given time t, the cooperation proportion is the same. In order to explain this result, we can assume that the discussion above applies to any lattice size, it only depends on the direct neighbourhood. Also, as opposed to the first analysis, the fact that a neighbour is chosen at random and that it's not always the best one can explain why the cooperation level is smaller than in first part: for the cooperating strategy to expand, a high concentration of cooperative players must be encountered, which has a very lower probability to appear with the stochastic replicator rule than with the best neighbour replicator rule.

About the 4 x 4 lattice, there are only $2^{16} = 2^{4\times4} = 65\,536$ different configurations possible, thus it is possible

From time t = 99 to time t = 100
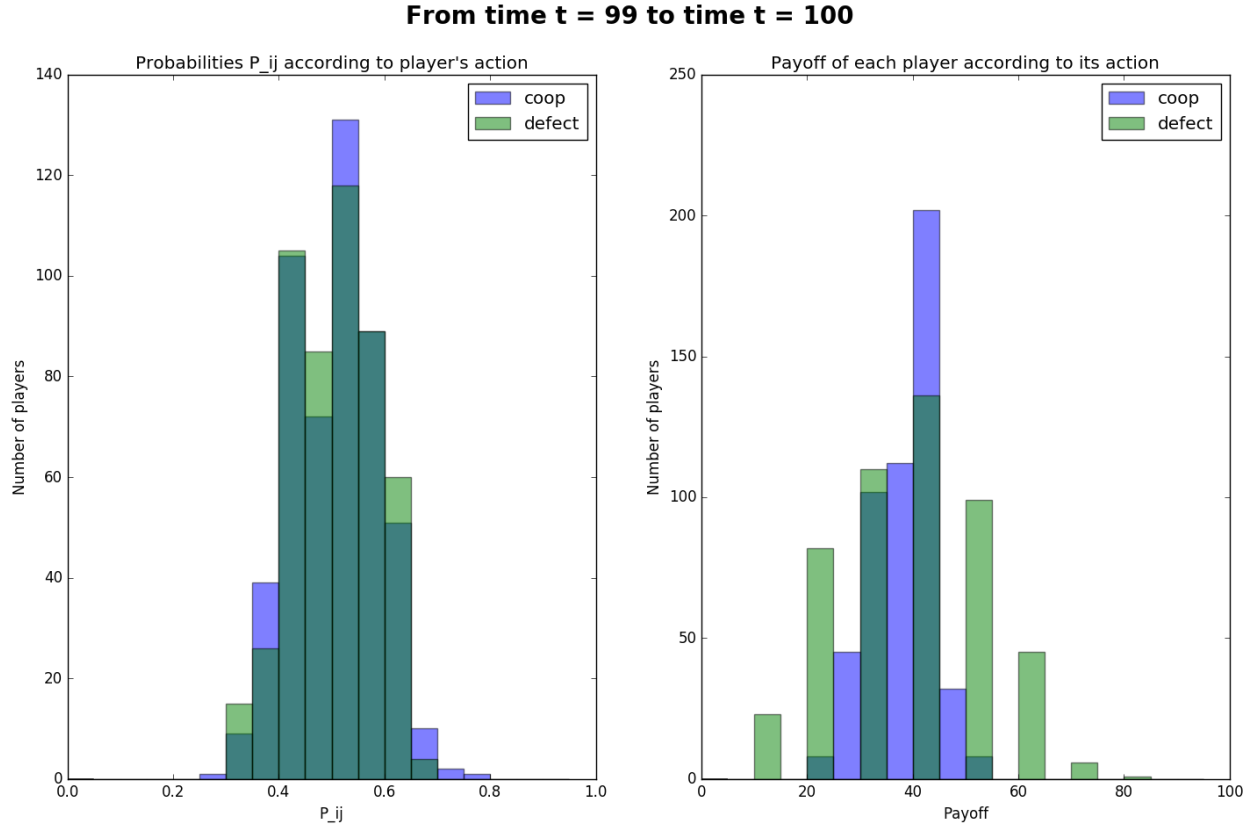
Figure 5: Adaptation of Figure 4 at time $t = 99$.

to test them all by enumeration, which is not really feasible for a 50 x 50 lattice since there are $\left(\frac{50}{4}\right)^2$ more orders of magnitude:

$$\log_{10}\left(2^{50^2}\right) = \log_{10}\left(\left(2^{4^2}\right)^{\left(\frac{50}{4}\right)^2}\right) = \left(\frac{50}{4}\right)^2 \log_{10}\left(2^{4^2}\right).$$

By enumerating them all, and by enumerating all the action changes possible in each player's neighbourhood and plotting the same histogram as in Figures 4 and 5, we obtain what is shown in Figure 6: a lower expected probability to update from defection to cooperation than to update from cooperation to defection. This can explain why 4 x 4 lattices don't follow the tendency of the other lattices.

The major conclusion about this part is that regarding the cooperation level, size doesn't matter, but too small doesn't get us very far...
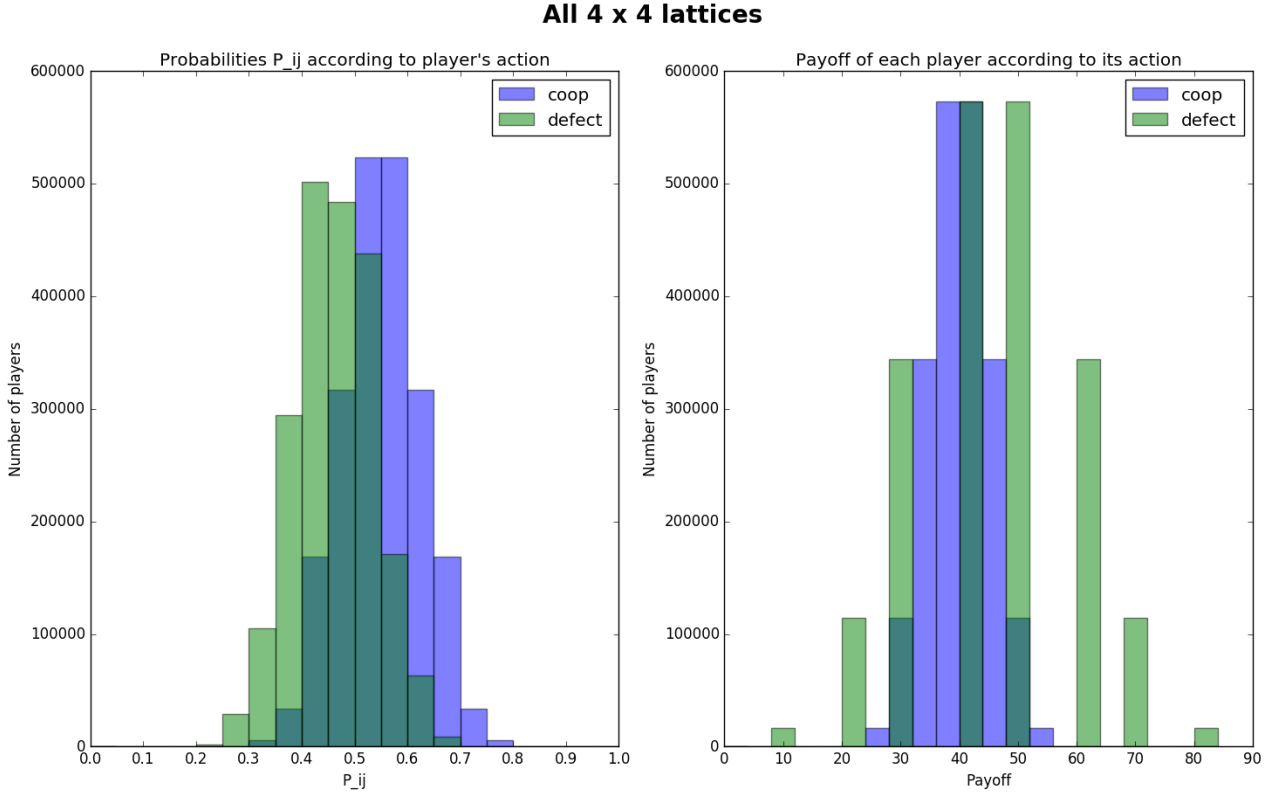
Figure 6: Distribution of the probability p_{ij} and of the payoff of each player according to the action, computed on all of possible the 2^{16} 4 x 4 lattice configurations.

2.1.2 Spatial Representation

Looking at the spatial representation of the 50 x 50 lattice (show in Figure 7), we observe that indeed, the cooperation level is slowly decreasing, but the more visible phenomenon is that as opposed to the Part I, cooperative and defecting players don't form distinct clusters as strongly: even though we can see at some places that defecting players form small red areas, in each subplot, it is possible to find single cooperative players surrounded by defecting players and vice versa. This is impossible to see in a Best Neighbour simulation (at least not after a few steps following the random initialization) because a single cooperative player surrounded by defecting would have a null payoff (in part I, $P = 0$) whereas its neighbours would have a payoff of at least T, leading to an update from cooperation to defection.

Also, a single defecting player surrounded by cooperative players would have a payoff of $N \times T$, which is maximal, therefore spreading the defecting strategy. See Figure 2a: at time $t = 1$, some single cooperative players surrounded by defecting players are visible at time $t = 1$, but have changed to defection before time $t = 5$.

Furthermore, no pattern (to not say *area* due to the lack of an absolute reference point on a torus) present on the lattice seems to be particularly stable with time: the small clusters are not big enough. This leads to a lattice that is highly changing, especially on the borders between defecting and cooperative small modules.

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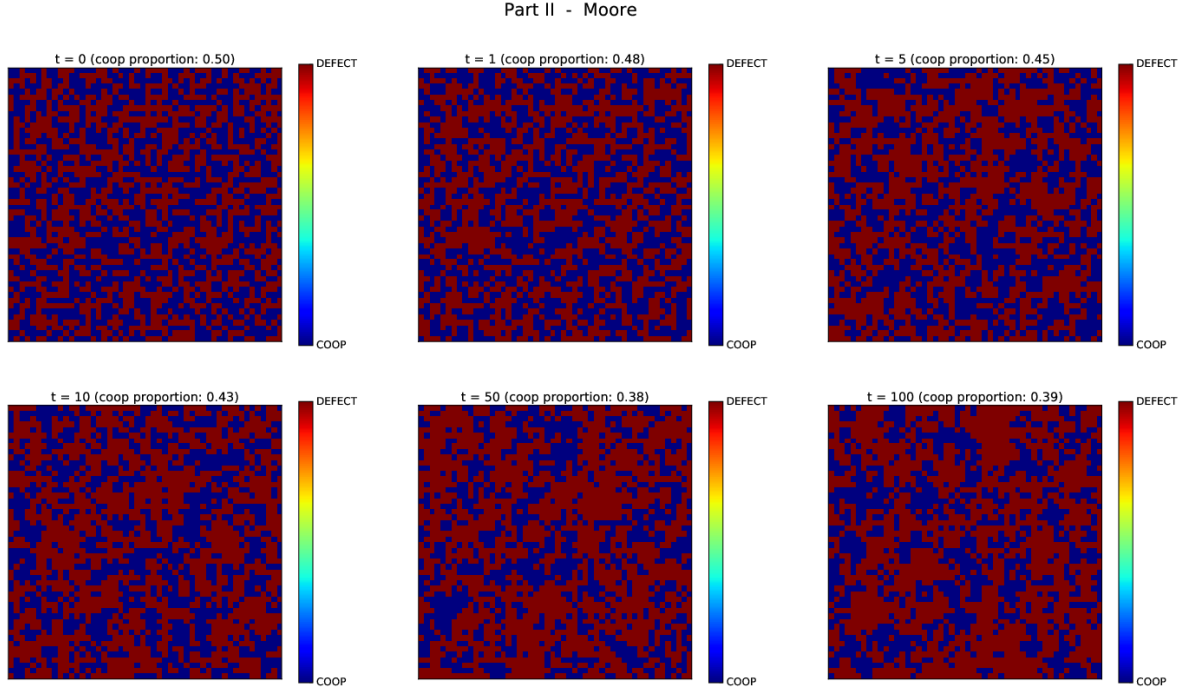
Figure 7: Spatial representation of a 50 x 50 lattice with agents playing Snowdrift Game at times $t = 0, 1, 5, 10, 50, 100$.

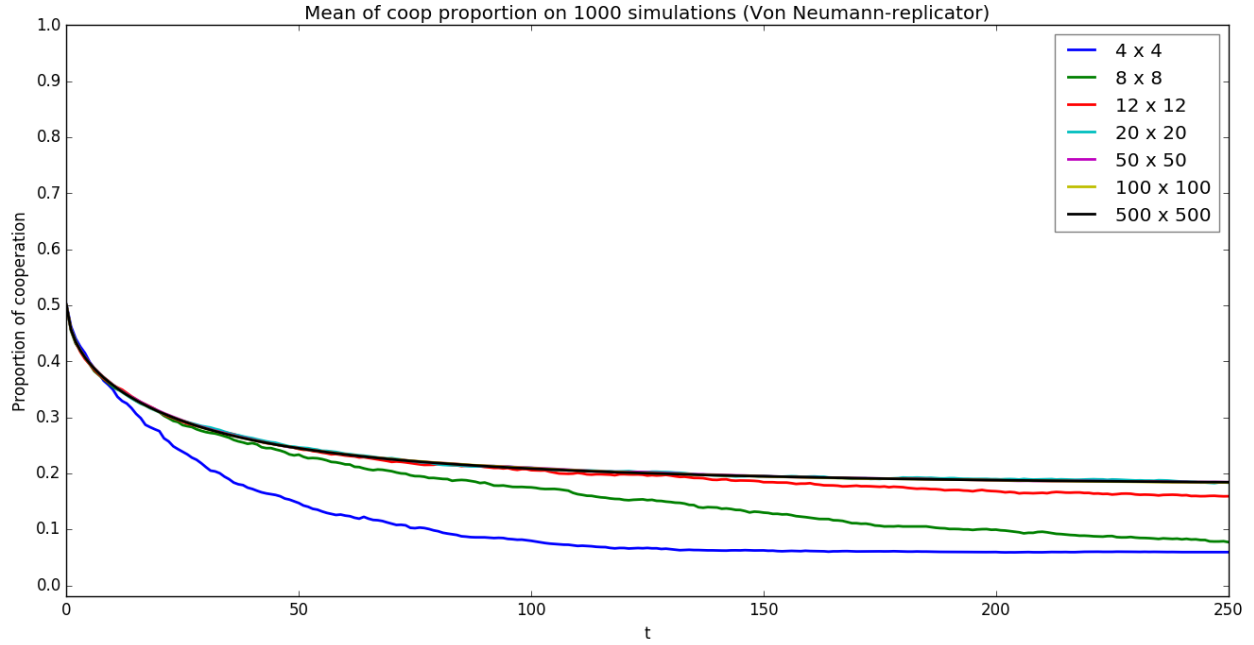## 2.2 Von Neumann Neighbourhood Analysis

2.2.1 Cooperation Level

This last study case is the only one for which the average cooperation level is not stabilized after 100 steps: we can see that as for the Moore Neighbourhood analysis of this second part, the curves are slowly decreasing (faster in this case than in the previous one, but still way slower than the cooperation level drop that has been observed in this first part), but don't reach a steady state. At time $t = 100$, we see that the cooperation level is around 20%, but seems to keep decreasing.

When looking at Figure 8, we see that the cooperation level converges to around 18% for lattices with size ≥ 20 x 20 (the 12 x 12 is slightly below), the 4 x 4 lattice reaches convergence around 6%, and the 8 x 8 keeps decreasing. Therefore, even though it takes more time, the cooperation level reaches a steady state as well in this configuration.

Like in the Moore situation of this part, without considering the 4 x 4 and 8 x 8 lattices, the achieved steady state does not depend on the lattice dimensions, and the discussion above still stands since it only implies the replication rule, not the neighbourhood.

Yet, the reached steady state is lower than the previous one. This was expected since neighbourhood change from Moore to Von Neumann brought a hard decrease in the cooperation level in first part; thus it is not surprising that in this situation, the neighbourhood change also lead to a decrease. Note also that in first part, this change of neighbourhood lead to a new steady state around 2.2 times lower, and in this second part, the division factor is again roughly 2.2 ($\simeq 40/18$).

For completeness, Figures 9a and 9b represent the distribution of payoffs and p_{ij} probabilities in the 50 x 50

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Figure 8: Average cooperation level of several square lattices (of dimension $n = 4, 8, 12, 20, 50, 100, 500$) computed on 10^4 simulations for 250 steps.
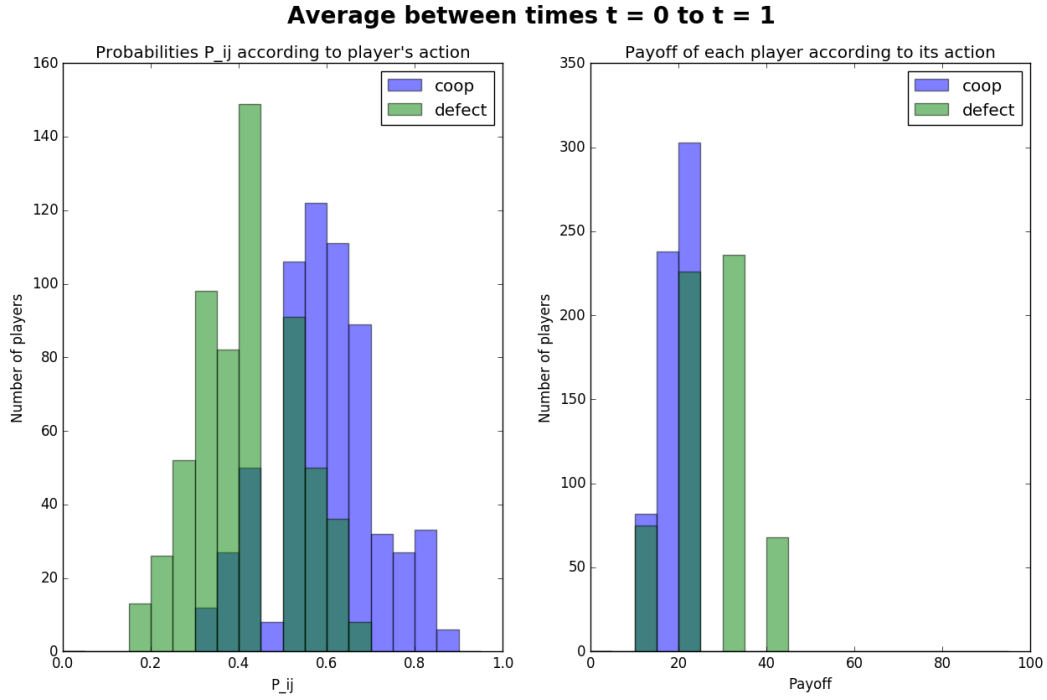
lattice with Von Neumann neighbourhood.

What can be observed is that at time $t = 0$, the distinction between the two distribution is even more visible than with the Moore neighbourhood: there seems to be an offset of around 20% in the distributions (considering that they are alike). Also, we can see that payoffs are really smaller, which is logical since the neighbourhood size is divided by two, which implies that the maximum payoff is also divided by two.

Also, the p_{ij} probability distribution has a higher variance and takes more possible values in this situation than with a Moore neighbourhood.

The discussion about these distributions would be roughly identical to the one above, with the exception that between times $t = 99$ and $t = 100$, is is hard to say that the difference between these distributions is negligible. Yet, this is coherent with the cooperation level that is observed: the cooperation level on the lattice is not stable at this time.

The p_{ij} probability distribution and the payoff distribution of the 4 x 4 lattice are shown in Figure 10 (by comparison with Figure 6). We see that the distribution is again broader, and that the distinction between probabilities to change from cooperation to defection is on average bigger than the probability to change from defection to cooperation.
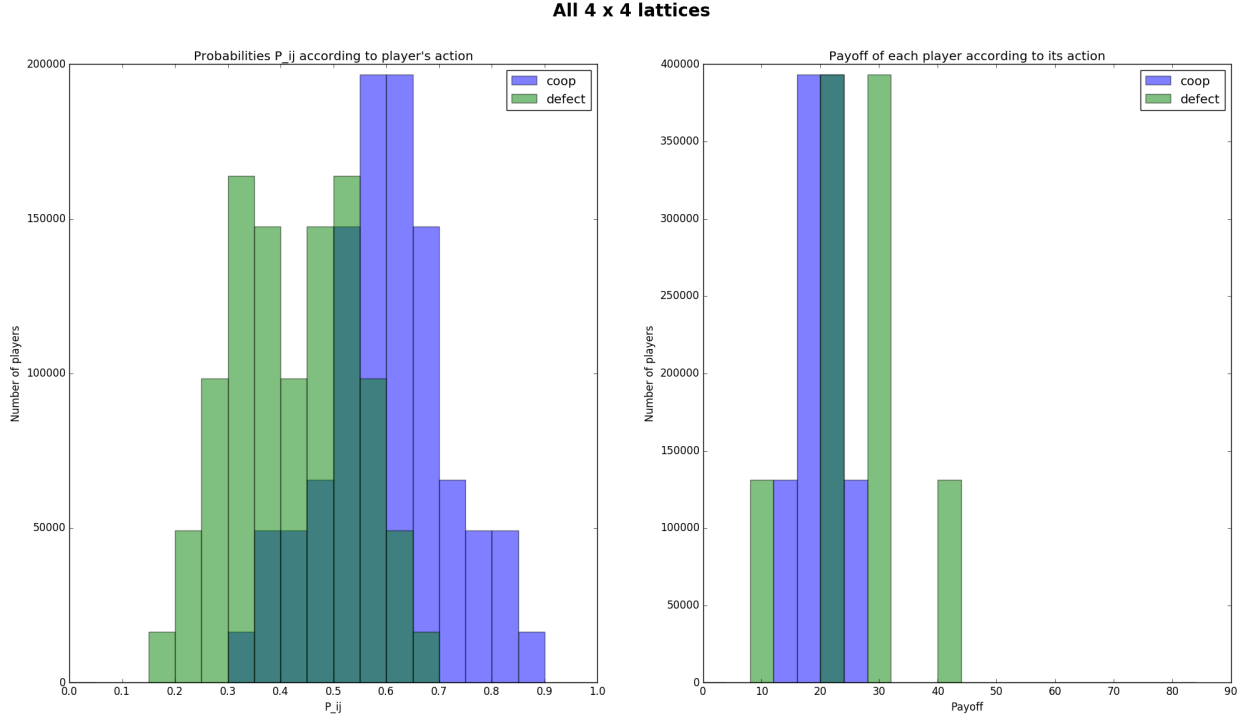
(a) Adaptation of Figure 4 for Von Neumann Neighbourhood.
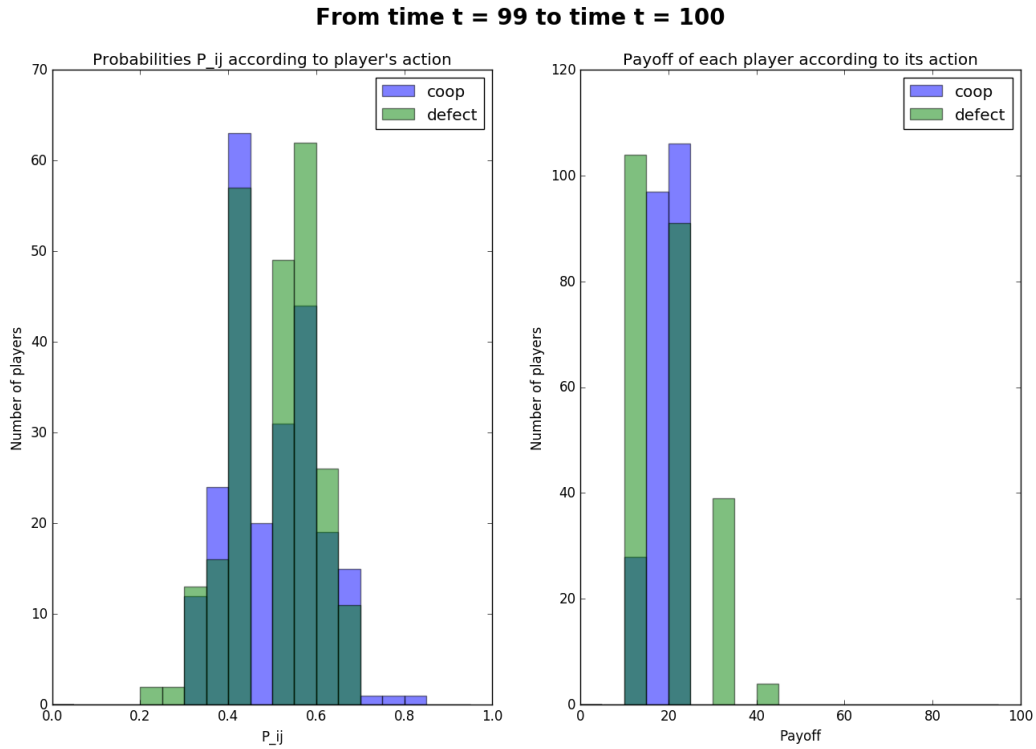
(b) Adaptation of Figure 5 for Von Neumann Neighbourhood.

Figure 9: Probability p_{ij} and payoff of players in a 50 x 50 lattice when changing strategy.

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Figure 10: Distribution of both p_{ij} probability and payoff with a Von Neumann neighbourhood by taking all the 4 x 4 lattices into account.
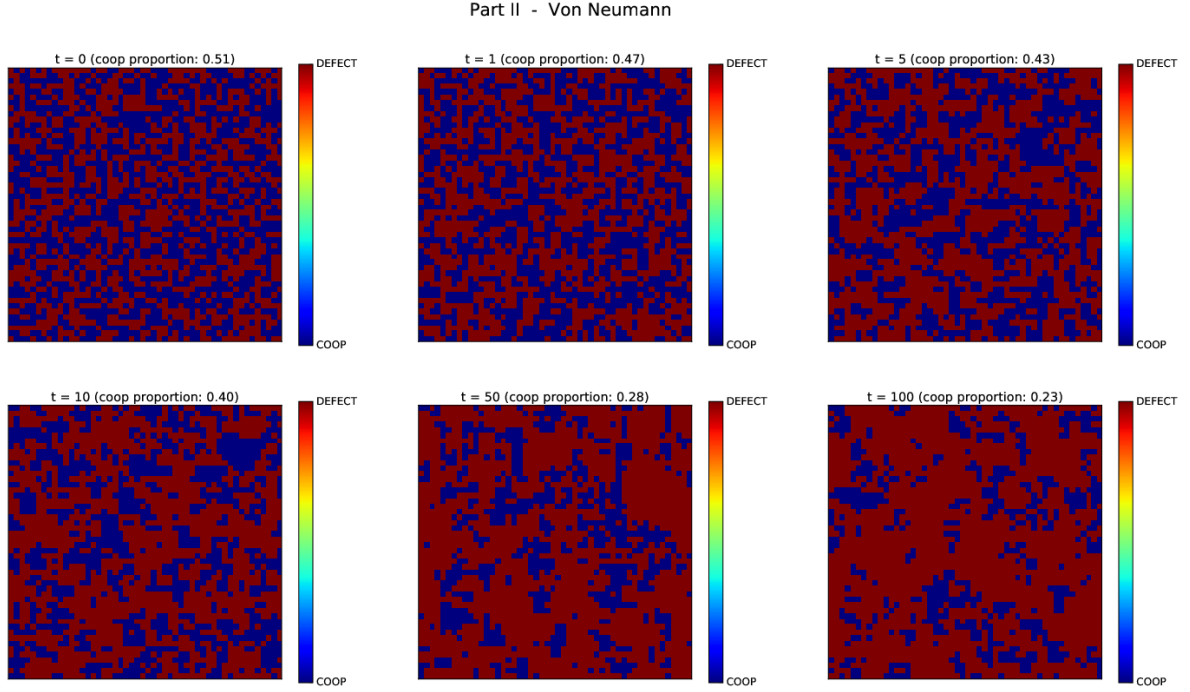
2.2.2 Spatial Representation

On Figure 11, one can see the spatial representation of the lattice with time. As discussed above, the replication rule used in this second part allows players to be surrounded by only players with the opposite strategy (and to not change their own strategy). This can be observed in all of the steps shown.

Note that for such *lonely* players to be present in a quite large cluster, it is required that they either have been stable there for several periods, or that they got *stuck* in the merging of two different clusters *expanding*. These words written in italic are used because they are convenient to describe what *seems* to be happening, but keep in mind that each element of the lattice is considered as a single agent that changes strategy with time, according to their neighbours' behaviour.

Also, in this situation, we can see that defecting players tend to form clusters, as opposed to the previous case with Moore neighbourhood. Yet, even though some high density defection patterns tend to appear, they don't come in a particular shape (such as rectangles as in first study case or diamonds as in second study case).

But as mentioned previously, the cooperation level stabilizes to a certain level (see previous subsubsection), yet this doesn't mean that the lattice is static! It is quite remarkable when looking at times $t = 50$ and $t = 100$ (note that even if the cooperation level still hasn't stabilized at this point, it decreases sufficiently slow to notice that consider it stabilized when looking at the *movement* of the density).

Figure 11: Spatial representation of the 50 x 50 lattice playing Snowdrift with Von Neumann neighbourhood on times $t = 0, 1, 5, 10, 50, 100$.

3   Conclusion

We have experimented on two types of neighbourhood in this document: Moore and Von Neumann neighbourhood, both in two distinct situations: Prisoner Dilemma with Best neighbour replication rule, and Snowdrift with stochastic replication rule.

For both these neighbourhood, the common observation was that a smaller neighbourhood lead to a smaller average cooperation level on the lattice. Also, for Best Neighbour (first part), the change in neighbourhood shape lead to a change in the cooperative clusters shape: from rectangles for Moore to diamonds for Von Neumann.

Also, when introducing stochastic behaviour (from first part to second part), the difference in cooperation level related to the lattice size disappeared: no matter what is the dimension of the lattice, it converges to a given steady state only depending on the neighbourhood (and the payoff matrix but which is not a changing factor in this study).

With all of the differences that have been observed between each of these 4 study cases, we can say that both these factors clearly change the behaviour of the lattice dynamics.