Øving 5

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1. $\frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

base step: $LS = \frac{1}{2}$ $RS = \frac{1}{1+1} = \frac{1}{2}$ ok

inductive step:

assume it is true for all k.

$n = k$

$$\frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

$$\Downarrow$$

$$\frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)((k+1)+1)} = \frac{k+1}{k+2}$$

assume $n = k+1$;

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{\cancel{(k+1)}(k+2)} = \frac{k+1}{k+2}$$

2. $n(n^2+5) = 6t$

$n=1$:

$1(1+5) = 6 \cdot 1$

$6 = 6$

Inductive step:

assume $n=k$:

$k(k^2+5) = 6t$

assume $n = k+1$

$(k+1)((k+1)^2+5)$

$= (k+1)(k^2+2k+6)$

$= k^3+3k^2+8k+6$

$8k = 5k+3k$

$\downarrow$

$k^3+5k = k(k^2+5) = 6t$

$= 6t + 3k^2 + 3k + 6$

$= 6\left(t + \frac{k(k+1)}{2} + 1\right)$

either k or $k+1$ is even, therefor it is an integer.

3. $n > 0$, $\sum_{k=1}^{n} k^2(k-1)! = (n+1)! - 1$

- base: $(n+1)! - 1 = 1$

$$(1+1)! - 1 = 1$$
$$2! - 1 = 1$$
$$2 - 1 = 1$$
$$1 = 1 \quad \text{ok!}$$

- assume $n = j$ & $k = j$:

$$j^2(j-1)! = (j+1)! - 1$$

assume $n = j+1$ & $k = j+1$

Solve to:

$$\underline{((j+1)+1)! - 1 = (j+2)! - 1}$$

$$\cancel{(k+1)^2 (k)! = (j+2)! - 1}$$

$$(j+1)^2 j! + (j+1)! - 1$$

- $= (j+1)(j+1)! + (j+1)! - 1$

$= (1+(j+1))(j+1)! - 1$

$= (j+2)(j+1)! - 1$

$= \underline{(j+2)! - 1}$

then $P(j)$ is true for $\forall j \in \mathbb{Z}_+$

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4 $\left(\bigcup_{i=1}^{n} A_i\right) \cap A = \bigcup_{i=1}^{n} (A_i \cap A)$

Base: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ true.

$$= \bigcup_{i=1}^{n} (A_i \cap A)$$

Inductive: $n = k$ assumed true!

$$\left(\bigcup_{i=1}^{K} A_i\right) \cap A = \bigcup_{i=1}^{k} (A_i \cap A)$$

assumes $n = k+1$

$$\left(\bigcup_{i=1}^{k+1} A_i\right) \cap A = \bigcup_{i=1}^{k+1} (A_i \cap A)$$

$$\left(\bigcup_{i=1}^{k+1} A_i\right) \cap A = \left(\left(\bigcup_{i=1}^{K} A_i\right) \cup A_{k+1}\right) \cap A$$

$$= \left(\left(\bigcup_{i=1}^{K} A_i\right) \cap A\right) \cup (A_{K+1} \cap A)$$

$$= \underline{\underline{\bigcup_{i=1}^{k+1} (A_i \cap A)}}$$

$P(n)$ is true for $n \in \mathbb{Z}_+$

4 b $\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcup_{i=1}^{n} \overline{A_i}$

Base:

$\overline{(A \cup B)} = (\bar{A} \cap \bar{B})$ de Morgan.

inductive: assume $n=K$

$\overline{\left(\bigcup_{i=1}^{K} A_i\right)} = \bigcup_{i=1}^{K} \bar{A_i}$

assume $n=K+1$

$\overline{\bigcup_{i=1}^{K+1} A_i} = \overline{A_1 \cup A_2 \cup \ldots \cup A_K \cup A_{K+1}}$

$= \bar{A_1} \cap \bar{A_2} \cap \ldots \cap \bar{A_K} \cap \bar{A}_{K+1}$ de Morgan

$= \bigcap_{i=1}^{K+1} \bar{A_i}$

P(n) true for all $n \in \mathbb{Z}_+$

6. $\sum_{i=1}^{n} iL_i = nL_{n+2} - L_{n+3} + 4$

Base: $n=1$

$1 \cdot L_1 = 1 \cdot L_3 - L_4 + 4$

$1 = 1 \cdot 4 - 7 + 4$

$1 = 1$ ok!

induct: assume $n=k$ and $P(k)$ is true

$P(k+1)$: $\sum_{i=1}^{k+1} iL_i = (k+1)L_{k+3} - L_{k+4} + 4$

$\sum_{i=1}^{k+1} iL_i = L_1 + 2L_2 + \ldots + kL_k + \underline{(k+1)L_{k+1}}$

$= [kL_{k+2} - L_{k+3} + 4] + (k+1)L_{k+1}$

$= kL_{k+2} - L_{k+3} + kL_{k+1} + L_{k+1} + 4$

$= kL_{k+2} - L_{k+2} - \cancel{L_{k+1}} + kL_{k+1} + \cancel{L_{k+1}} + 4$

$= kL_{k+2} + kL_{k+1} - L_{k+2} + 4$

$= kL_{k+3} - L_{k+2} + 4$

$= [(k+1)L_{k+3} - L_{k+3}] - L_{k+2} + 4$

$= (k+1)L_{k+3} - L_{k+4} + 4$ $P(n)$ true for $n \geq 0$

7. $\sum_{r=0}^{n} L_{2r} = L_{2n+1} + 1$.

$n = 0$

Base: $L_{2\cdot 0} = L_{2\cdot 0+1} + 1$

$2 = 1 + 1$

$2 = 2$

Induct: $n = k$ assumed:

$$\sum_{r=0}^{k} L_{2r} = L_{2k+1} + 1$$

$n = k+1$ assumed: $L_{2(k+1)+1} + 1 = L_{2k+3} + 1$

$$\sum_{r=0}^{k+1} L_{2r} = \sum_{r=0}^{k} L_{2r} + L_{2k+2}$$

$$= L_{2k+1} + 1 + L_{2k+2}$$

$$= L_{2k+3} + 1$$

8. $n \geq 0$ L_{3n} is an even number

Base step: $L_{3\cdot 1} = L_{3-1} + L_{3-2}$

$4 = 3+1$

$4 = 4$

Assume: $n = k$

$L_{3k} = 2t$

Inductive step:

$L_{3k+3-1} + L_{3k+3\cdot 2}$

$= L_{3k+2} + L_{3k+1}$

$= L_{3k+1} + L_{3k} + L_{3k+1}$ $\quad | \; L_{3k} = 2t$

$= 2L_{3k+1} + 2t$

$= 2(L_{3k+1} + t)$

An even number plus another number times 2 is allways an even number

9 $P(n)$ $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, $n > 0$

$$P(1) = F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right)\right]$$
$$= \frac{1}{\sqrt{5}} \cdot \sqrt{5} = 1$$, so $P(1)$ true

assume $P(k)$ is true, so $P(k+1)$:

$$F_{k+1} = \left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}\right]$$

On going:

$$F_{k+1} = F_k + F_{k-1}$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1+\sqrt{5}}{2}\right)^k\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{2}{1+\sqrt{5}}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{2}{1-\sqrt{5}}\right)\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(1+\frac{2}{1+\sqrt{5}}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^k\left(1+\frac{2}{1-\sqrt{5}}\right)\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{1-\sqrt{5}}{2}\right)\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

$P(n)$ true for $n \in \mathbb{Z}_+$