227-0689-00L System Identification **29 September 2020**

Exercise 2: Least Squares

Background reading

The background material for this exercise is Sections 10.1, 13.2 and Appendix II of Ljung (*System Identification; Theory for the User*, 2nd Ed., Prentice-Hall, 1999).

Problem 1:

Consider the least-squares (LS) estimation problem:

$$Y = \Phi\theta + \epsilon,$$

where Φ is the regressor matrix and θ is the parameter vector to be estimated

$$Y := \begin{bmatrix} y(0) \\ \vdots \\ y(N-1)) \end{bmatrix}, \qquad \Phi := \begin{bmatrix} \varphi^\top(0) \\ \vdots \\ \varphi^\top(N-1) \end{bmatrix}, \qquad \theta := \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_p \end{bmatrix}.$$

Assume that the noise, ϵ, is zero-mean Gaussian and correlated with $E\left\{\epsilon\epsilon^\top\right\} = R$. In this exercise we look for a linear estimator $\hat{\theta}$ of the form,

$$\hat{\theta} = Z^\top Y, \tag{2.1}$$

which is unbiased and minimizes its variance. For a given Φ show the following:

1. For a linear estimator of the form (2.1) to be unbiased we require that $Z^\top\Phi = I$.
2. The covariance matrix of any linear unbiased estimator of the form (2.1) is $\operatorname{cov}\left\{\hat{\theta}\right\} = Z^\top R Z$.
3. The covariance matrix of the best linear unbiased estimator (BLUE) $\hat{\theta}_Z$ with $\hat{\theta}_Z = \left(\Phi^\top R^{-1}\Phi\right)^{-1}\Phi^\top R^{-1} Y$ is $\operatorname{cov}\left\{\hat{\theta}_Z\right\} = \left(\Phi^\top R^{-1}\Phi\right)^{-1}$.
4. The best linear unbiased estimator $\hat{\theta}_Z$ exhibits the smallest variance in the class of all unbiased estimators, i.e. $\operatorname{cov}\left\{\hat{\theta}_Z\right\} \leq \operatorname{cov}\left\{\hat{\theta}\right\}$.

Hint: All covariance matrices are positive semi-definite and in our case we can assume that R is positive definite. The inverse of a positive definite matrix is also positive definite.

Problem 2:

Show that the least squares solution for the problem of fitting the coeficients of a polynomial of degree k to n i.i.d. observations can be written as:

$$\hat{a} = (X^T X)^{-1} X^T y, \tag{2.2}$$

where

$$\hat{a} := \begin{bmatrix} \hat{a}_0 \\ \vdots \\ \hat{a}_k \end{bmatrix}, \qquad y := \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}, \qquad X := \begin{bmatrix} 1 & x_1 & \dots & x_1^k \\ & \vdots & & \\ 1 & x_n & \dots & x_n^k \end{bmatrix}$$

MATLAB **exercises:**

Problem 3:

An austronaut is in the vicinity of Saturn when a technical problem forces an automated emergency landing in a nearby moon. Unfortunately the space positioning system (SPS) of the spaceship is damaged. The color of the surface indicates this is either Iapetus ($g_I = 0.223 \pm 0.001$ ms^{-2}) or Rhea ($g_R = 0.264 \pm 0.001$ ms^{-2}). The values presented for the acceleration are in the format $g = \mu \pm \sigma$, i.e. the value of g follows a normal distribution with mean μ and standard deviation σ: $g \sim \mathcal{N}(\mu, \sigma^2)$.

Using a metric scale printed on the spaceship, a watch and a SysID book, an experiment is conducted to determine the local gravity at the surface, the results were recorded in the file "experiment1.dat". This file contains three columns: 1$^{\text{st}}$) the time since the beginning of the experiment in seconds, 2$^{\text{nd}}$) the measured position of the object in meters, 3$^{\text{rd}}$) the standard deviation of the measurement noise in meters.

Recall that the position of a SysID book moving in a gravitational field is given by

$$y(t) = y_0 + v_0 t + \frac{1}{2} a t^2 \tag{2.3}$$

1. Determine the local gravity on the surface using Least Squares. Plot the experimental data points along with the fit. Hint: Estimate the vector $\theta = [y_0, v_0, \frac{1}{2}a]$.
2. Provide the standard deviation of the estimates for $[y_0, v_0, a]$. What moon is this one?
3. After closer inspection the astronaut found that the scale had been damaged by the landing, so the measurements were more accurate in some parts of the scale than others. The results from the experiment were updated with the information of the non constant uncertainty of each measurement (see file "experiment2.dat"). Determine the local gravity on the surface using weighted Least Squares. Plot the experimental data points along with the fit.
4. Provide the standard deviation of the new estimates for $[y_0, v_0, a]$. What moon is this one?

Problem 4:

The emission spectrum of an incandescent lightbulb can be aproximated by the one of a black body. By changing the operating temperature of the filament one can change the spectrum of emitted radiation as shown in figure 2.1.

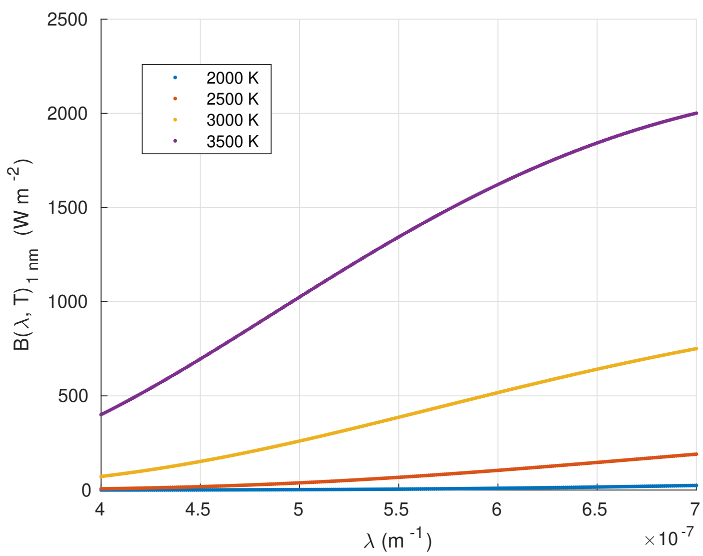

Figure 2.1: Black body spectral radiance B as a function of wavelenght λ in 1 nm bins. The plot shows the intensity of light emitted for four different temperatures, in Kelvin, of a heated tungsten filament.

In an experiment a sample is illuminated exclusively with the collimated light from an incandescent bulb. The temperature of the bulb filament is precisely controlled with a closed loop control system, and the radiation transmitted by the sample is captured by the CCD sensor inside a conventional camera. The sensitivity of the sensor for each of the colors (Red, Green and Blue) as a function of the wavelenght is shown in figure 2.2.

Aperture and ISO were manually adjusted initially and then fixed for the entirety of the experiment. Shutter speed was adjusted such that the total amount of emitted radiation with $\lambda = 700$ nm was for each temperature always the same (normalisation).

The file ”experiment3.mat” contains the camera measurements $(y_R,\ y_G,\ y_B)$ for one pixel over different temperatures T of the light bulb, as well as the sensitivity $(s_R(\lambda), s_G(\lambda), s_B(\lambda))$ of the camera and the normalised black body emission of radiation $e(T, \lambda)$. Assuming linearity of the sensor, the transmittance of the sample can be used to predict the camera measurement for each colour channel

$$y_R(T) = \sum_{\lambda} s_R(T, \lambda)\ e(T, \lambda)\ t(\lambda), \tag{2.4}$$

where y_R is the output of the red channel of the camera. An identical formula can be written for the output of the green y_G and blue channels y_B. In this context:

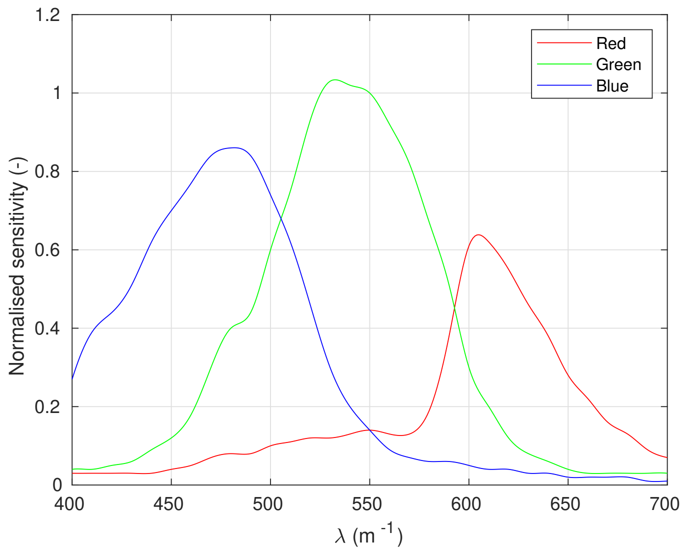

Figure 2.2: Leica M8 sensitivity as a function of wavelenght [1]

- transmittance $t(\lambda) \in [0, 1]$ is an adimensional measure of the ratio of the incident light in a specific wavelength that goes through a sample without being absorbed, reflected or scattered. This is a property of the sample to be analised, and the objective of the experiment is to estimate it.
- sensitivity $s(\lambda) \in [0, 1]$ is an adimensional measure of the ratio of photons of a specific wavelength that are detected by the CCD sensor. Each channel (Red, Green and Blue) has a specific sensitivity $(s_R(\lambda), s_G(\lambda), s_B(\lambda))$. The sensitivity of the camera is know and given in the file "experiment3.mat", table "Camera_sensitivity".
- normalised emission $e(T, \lambda)$ is an adimensional measure of the proportion of photons within a specific wavelength range emitted from the tungsten filament inside the light bulb. The temperature of the filament is an independent variable. The emission of the filament is considered to be the same of the known formula for a black body. Its values are given in the file "experiment3.mat", table "BlackBody_emission".
- camera measurements $(y_R,\ y_G,\ y_B)$ are the RGB values of one pixel, over different photographs. 151 were taken, one for each different temperature of the filament.
- Use the temperatures T found in the "experiment3.mat" file, i.e. $T = [2000, 2010, \ldots, 3500]$ K.
- Use the wavelengths λ found in the "experiment3.mat" file, i.e. $\lambda = [400, 420, \ldots, 700]$ m^{-1}.

1. Compute the reconstructed multi-spectral transmittance in the window $\lambda \in [400, 700]$ nm in 20 nm steps, and plot it.

Hint 1: Standard least squares might not be enough. You might find useful to use regularisation or constrained least squares.

Hint 2: You can ignore the physics of the problem and focus on rewriting (2.4) and the information provided in the file ”experiment3.mat” into the standard least squares formulation.

References

[1] Christian Mauer and Dietmar Wueller. Measuring the spectral response with a set of interference filters. In *Digital Photography V*, volume 7250, page 72500S. International Society for Optics and Photonics, 2009. 2-4