Bluestein FFT

The DFT of a signal $x[n], n = 0, \ldots, N-1$ is given by:

$$X[k] = \sum_{n=0}^{N-1} x[n] W^{kn} \tag{1}$$

with the basic twiddle factor W defined as: $W = e^{-j\frac{2\pi}{N}}$. Multiply both sides with $W^{-\frac{1}{2}k^2}$:

$$\begin{aligned} W^{-\frac{1}{2}k^2} X[k] &= \sum_{n=0}^{N-1} x[n] W^{kn} W^{-\frac{1}{2}k^2} \\ &= \sum_{n=0}^{N-1} x[n] W^{kn-\frac{1}{2}k^2} \\ &= \sum_{n=0}^{N-1} x[n] W^{\frac{1}{2}(2kn-k^2)} \end{aligned} \tag{2}$$

Observe that $2kn - k^2 = -(n-k)^2 + n^2$ because of $(n-k)^2 = n^2 - 2kn + k^2$ - replace the $(2kn - k^2)$ term in the exponent accordingly:

$$W^{-\frac{1}{2}k^2} X[k] = \sum_{n=0}^{N-1} x[n] W^{\frac{1}{2}(-(n-k)^2+n^2)} \tag{3}$$

Split the W exponent and re-arrange:

$$\begin{aligned} W^{-\frac{1}{2}k^2} X[k] &= \sum_{n=0}^{N-1} x[n] W^{-\frac{1}{2}(n-k)^2} W^{\frac{1}{2}n^2} \\ &= \sum_{n=0}^{N-1} \underbrace{x[n] W^{\frac{1}{2}n^2}}_{y[n]} \underbrace{W^{-\frac{1}{2}(n-k)^2}}_{h[n-k]} \end{aligned} \tag{4}$$

Where the names $y[n]$ and $h[n-k]$ have been assigned to the sequences for convenience. With these definitions, we can rewrite the equation as:

$$W^{-\frac{1}{2}k^2} X[k] = \sum_{n=0}^{N-1} y[n] h[n-k] \tag{5}$$

Defining $h[n-k]$ as above implies $h[n] = W^{-\frac{1}{2}n^2}$. By substituting k for n, this is observed to be the factor in front of the DFT coefficient on the left hand side, so we can write:

$$h[k] X[k] = \sum_{n=0}^{N-1} y[n] h[n-k] \tag{6}$$

Interpretation: The right hand side of equation 6 is recognized as the convolution of the two sequences $y[n]$ and $h[n]$. The sequence $y[n] = x[n]W^{\frac{1}{2}n^2}$ represents our input signal modulated by the sequence $c[n] := W^{\frac{1}{2}n^2}$ and this modulating signal represents a complex sinusoid with linearly increasing frequency - a so called chirp signal. The impulse response in this convolution $h[n] = W^{-\frac{1}{2}n^2}$ is a chirp signal as well but rotating in the opposite direction when viewed as complex phasor. The left hand side represents the sequence of DFT-coefficients - again modulated by the chirp-signal $h[n]$. This means, we can obtain the modulated DFT for arbitrary N by computing a convolution between a properly modulated input signal with a properly chosen impulse response. The convolution itself can be carried out via a radix-2 FFT $\rightarrow$ spectral multiplication $\rightarrow$ radix-2 IFFT algorithm. This requires zero-padding the sequence $x[n]$ and the impulse response $h[n]$ to length M which has to be chosen to be a power of 2 larger or equal to $2N - 1$. The first N coefficients in this convolution product will represent the chirp-modulated DFT sequence of our original $x[n]$. By dividing them by $h[k], k = 0, \ldots N-1$ and discarding the rest of the length M DFT coefficient vector, we obtain the DFT of $x[n]$. The chirp signals $h[n]$ and $c[n]$ can be precomputed for any given DFT-size or computed on the fly in linear time. This yields an overall complexity of the algorithm of $\mathcal{O}(N\log(N))$.