Dense Hopfield networks in the teacher-student setting

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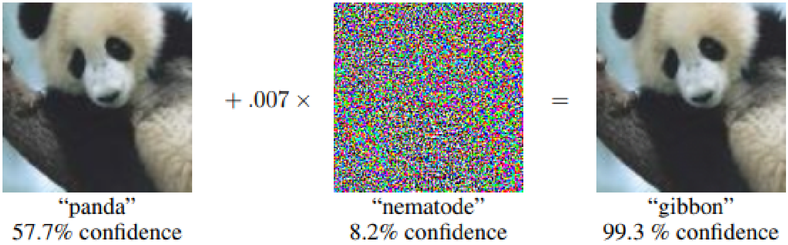
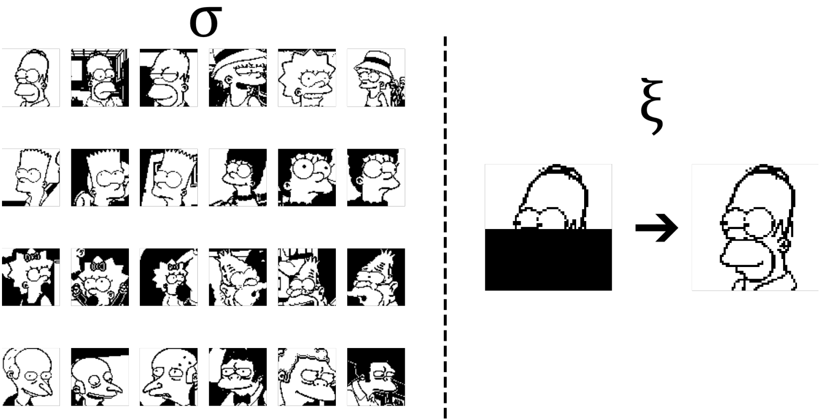
Adversarial attack

Neural networks are fooled by small adversarial perturbations.



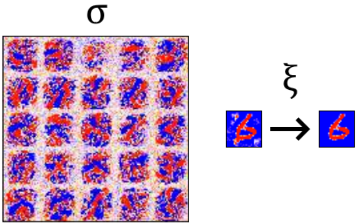
- Task: restore incomplete pattern ξ with N pixels using stored memories σ^a.
- Robust to adversarial perturbations.



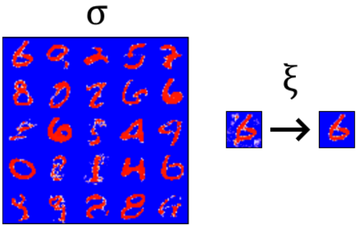
Dense Hopfield Network

Memories can be learned from data.

Feature learning



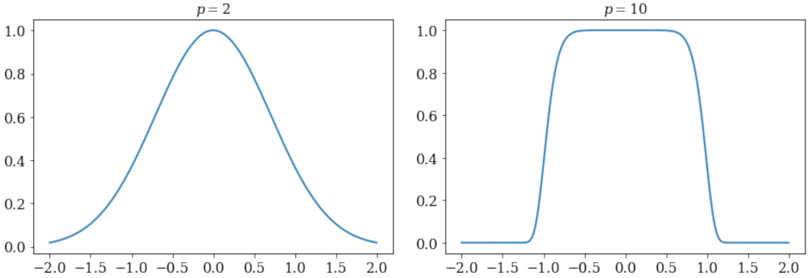
Example retrieval



Example retrieval empirically known to be robust.

Direct problem

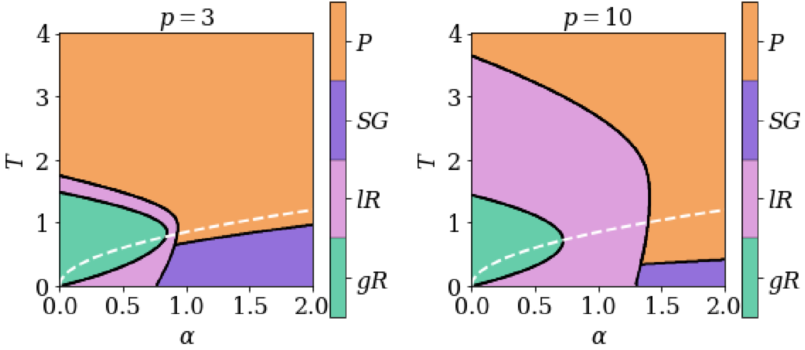
Model task: retrieve memories σ^a by sampling
$\xi \sim P(\xi|\sigma; p, T) = \mathcal{Z}^{-1} \exp(-E[\xi]/T)$ where
$E[\xi] = -\frac{p!}{N^{p-1}} \sum_a \sum_{i_1<\ldots<i_p} \xi_{i_1} \ldots \xi_{i_p} \sigma^a_{i_1} \ldots \sigma^a_{i_p}$.



- Well-studied theoretically.
- Capacity of $M \sim \mathcal{O}\left(N^{p-1}\right)$ i.i.d. random memories.

Direct problem

Parameters: temperature T and memory load $\alpha = \frac{Mp!}{N^{p-1}}$.



Caveats:

- Does not explain adversarial robustness.
- Does not capture the feature regime.
- Does not say how much data a dense net needs.
- SG transition hard to obtain exactly.

Teacher-student setting

Teacher-student setting:

- Sample examples $\sigma = \{\sigma^a\}_{a=1}^M$ from teacher with pattern ξ^*: $\sigma^a \sim P\left(\sigma^a | \xi^*; p_{\text{teacher}}, T_{\text{teacher}}\right)$.
- Sample pattern ξ from student with memories $\sigma = \{\sigma^a\}_{a=1}^M$: $\xi \sim P\left(\xi | \sigma; p_{\text{student}}, T_{\text{student}}\right)$.

Student task: recover ξ^*.

Our goal: predict student performance as a function of p_{teacher}, T_{teacher}, p_{student}, T_{student} and α (normalized M).

Teacher-student setting

Compute the order parameters m, q and q^*, where

- $m = \lim_{N\to\infty} \left\langle \frac{1}{N} \sum_i \xi_i \sigma_i^a \right\rangle_{\xi^*,\sigma,\xi}$,
- $q = \lim_{N\to\infty} \left\langle \frac{1}{N} \sum_i \xi_i^1 \xi_i^2 \right\rangle_{\xi^*,\sigma,\xi}$,
- $q^* = \lim_{N\to\infty} \left\langle \frac{1}{N} \sum_i \xi_i^* \xi_i \right\rangle_{\xi^*,\sigma,\xi}$.

m measures the proximity of the student pattern and memories.

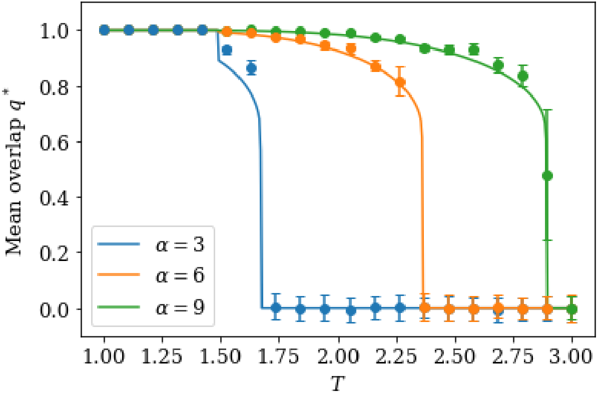
q measures the tendency of the student to stay frozen in specific configurations.

q^* measures the student performance.

Phase diagrams

Consider $p_{\text{teacher}} = p_{\text{student}}$ and $T_{\text{teacher}} = T_{\text{student}}$.

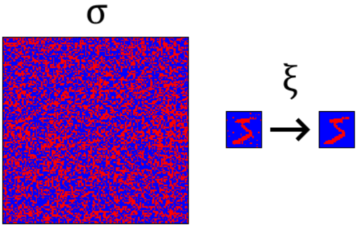
- Study the performance q^*.
- Plot: calculations (lines) consistent with simulations (dots).



Phase diagrams

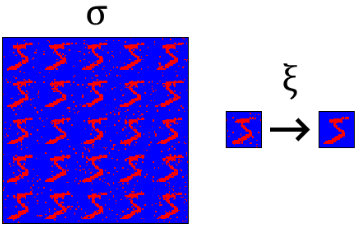
Obtain feature learning (gR/lR) and example retrieval (eR) phases.

gR/lR



$q^* = q \neq 0$ and $m = 0$

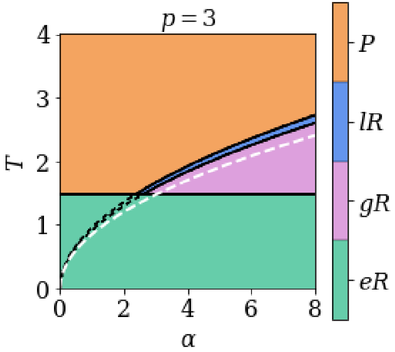
eR



$q^* = q \neq 0$ and $m \neq 0$

Phase diagrams

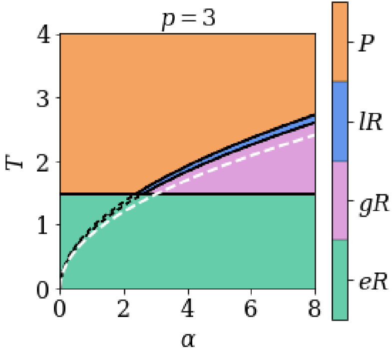
Obtain feature learning (gR/lR) and example retrieval (eR) phases.



- P: $q^* = q = 0$ and $m = 0$
- gR/lR: $q^* = q \neq 0$ and $m = 0$
- eR: $q^* = q \neq 0$ and $m \neq 0$

Phase diagrams

Obtain feature learning (gR/lR) and example retrieval (eR) phases.



The gR transition overlaps with the direct model SG transition.
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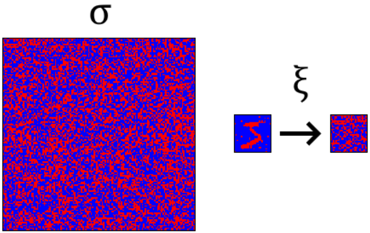
Phase diagrams

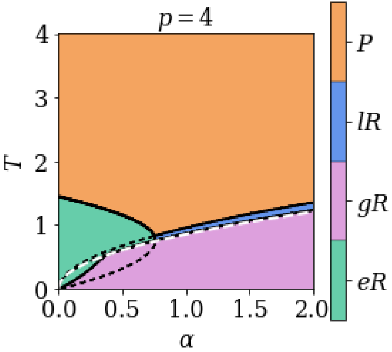
- Next: $p_{\text{teacher}} = 2$ and $p_{\text{student}} \geq 3$.
- In this regime, teacher examples are noisy, so eR is inaccurate.



Phase diagrams

Case 1:

- $M \sim \mathcal{O}\left(N^{p-1}\right)$
- $T_{\text{teacher}} \sim \mathcal{O}\left(N^{1-2/p}\right)$, i.e. can infer ξ^* despite large T_{teacher}.



Extensive tolerance to teacher noise.

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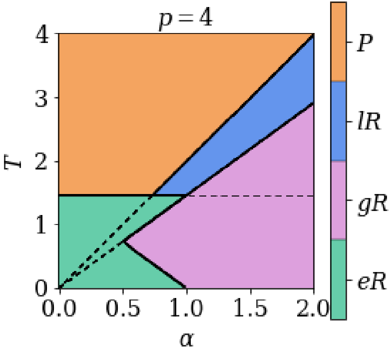
Case 2:

- $T_{\text{teacher}} \sim \mathcal{O}(1)$
- $M \sim \mathcal{O}\left(N^{p/2}\right)$, i.e. much smaller M than in the first case.



Avoids pattern interference.
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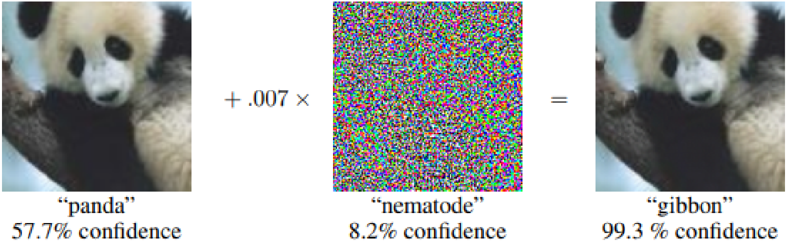
Adversarial attacks

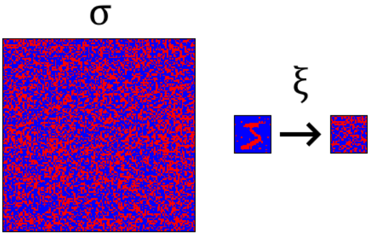
Neural networks are fooled by small adversarial perturbations.



Neural networks with more parameters are more robust.

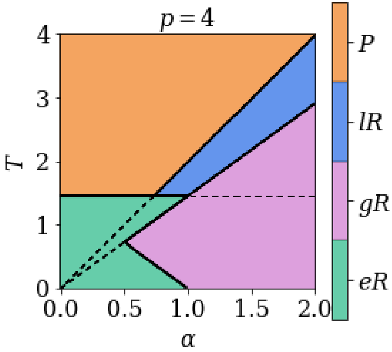
Adversarial attacks

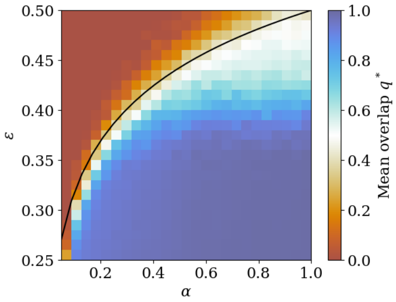
When $p_{\text{teacher}} = 2$ and $p_{\text{student}} \geq 3$, memories σ^a are noisy, so eR is inaccurate.



Adversarial attacks

Study adversarial attacks of size ε at $T_{\text{student}} = 0$.





- Right plot: calculations (line) agree with simulations (image).
- Adversarial robustness increases with the memory load α, in line with empirical observations of neural networks.

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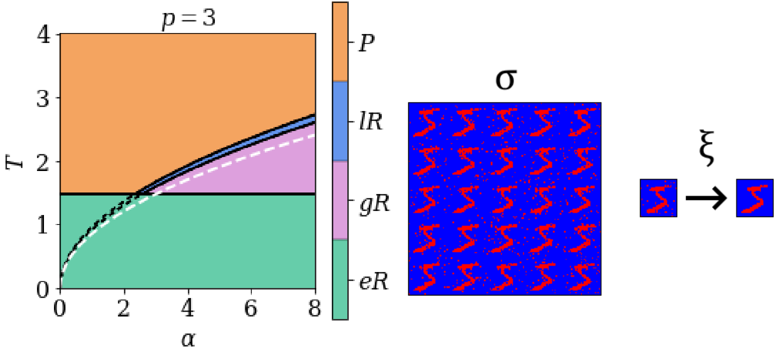
Adversarial attacks

Adversarial attacks fail when $p_{\text{teacher}} = p_{\text{student}}$ and $T_{\text{teacher}} = T_{\text{student}}$ because eR is accurate.



Clarifies why accurate example retrieval is adversarially robust.
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Summary of results

- The teacher-student feature learning phase transition overlaps the direct problem spin-glass phase transition.
- Feature learning can resist extensive teacher noise and pattern interference.
- As seen in neural networks, feature learning becomes more adversarially robust as the memory load increases.
- Our model clarifies why the example retrieval phase of dense Hopfield networks is adversarially robust.

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Dense Hopfield model

Sampling: $\xi \sim P\left(\xi|\sigma;p,T\right) = \mathcal{Z}^{-1}\exp\left(-E\left[\xi\right]/T\right)$ where

- $E\left[\xi\right] = -\frac{p!}{N^{p-1}}\sum_a \sum_{i_1<\ldots<i_p} \xi_{i_1}\ldots\xi_{i_p}\sigma^a_{i_1}\ldots\sigma^a_{i_p}$,
- $P\left(\xi|\sigma;p,T\right)$ is sampled with a Monte-Carlo simulation.
- $E\left[\xi\right]$ reduces to $E\left[\xi\right] = -\frac{1}{N}\sum_a \sum_{i\neq j} \xi_i\xi_j\sigma^a_i\sigma^a_j$ when $p=2$,
- MC reduces to $\xi_i = \text{sign}\left(\sum_a \sigma^a_i \sum_j \xi_j\sigma^a_j\right)$ when $T=0$.

Overlapping gR and SG lines

Sampling: $\xi \sim P\left(\xi|\sigma;p,T\right) = \mathcal{Z}^{-1}\left(\sigma\right)\exp\left(-E\left[\xi|\sigma\right]/T\right)$ where $\mathcal{Z}\left(\sigma\right)$ is a normalization constant.

The gR line overlaps with the SG line because

- $P\left(\sigma\right) = \frac{1}{2^{MN}}\frac{\mathcal{Z}(\sigma)}{\langle\mathcal{Z}\rangle}$,
- $\lim_{N\to\infty}\left\{\frac{\log\mathcal{Z}-\log\langle\mathcal{Z}\rangle}{N}\right\} = 0$ in the paramagnetic phase.

Adversarial robustness formula

Adversarial boundary: $\varepsilon^* = \frac{[\eta\alpha]^{\frac{1}{p-1}}}{[\eta\alpha]^{\frac{1}{p-1}}+1}$ where $\eta = \frac{2[\beta^*]^2}{(1-2\beta^*)^2}$ when $p = 4$.

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