The Loss Landscape of Dense Associative Memory

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October 31, 2024

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Review

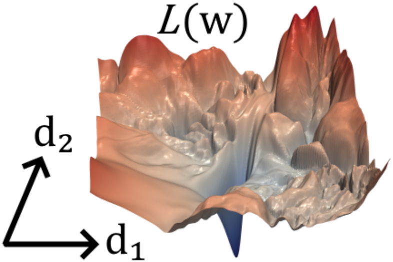
Purpose of (generative) ML: fit data $\mathbf{x}$ and labels y with $\mathrm{P}\left(\mathbf{x}, y \middle| \mathbf{w}\right)$.

[Lecun et al., 1998]: MNIST digits.

One typically minimizes a loss $L\left(\mathbf{w}; \mathbf{x}, y\right)$ to find the parameters $\mathbf{w}$.

Neural Network Loss Landscape

Previous works investigated the loss $L(\mathbf{w})$ of neural networks (NN) as a function of parameters $\mathbf{w}$ to study how they learn.



[Li et al., 2018]: visualizing the loss landscape.

- [Choromanska et al., 2015]: local mins almost as good as global.
- [Dauphin et al., 2014]: large NNs have much more saddles than local mins.

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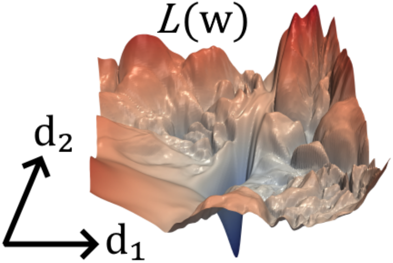
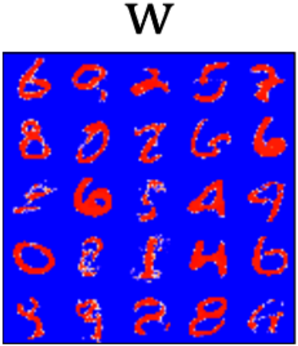


[Li et al., 2018]: visualizing the loss landscape.

Characterizing and classifying critical points is still an open problem. See [Zhang et al., 2021] for recent progress.

Dense Associative Memory (DAM)

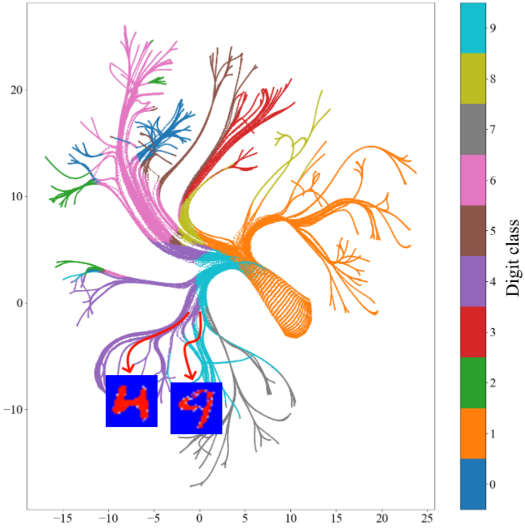
A way forward: dense associative memory (DAM), related to transformers [Ramsauer et al., 2020].
DAM learns data archetypes or *memories* as min of $L(\mathbf{w})$.



[Krotov and Hopfield, 2016]

Can the other critical points be characterized as well?

Dense Associative Memory (DAM)

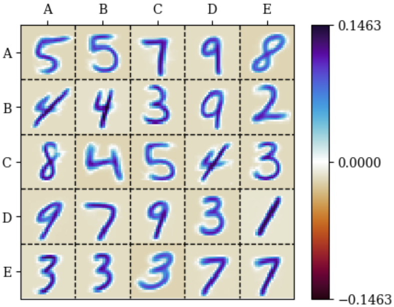


[Boukacem et al., 2024]: Parameters during training

Our DAM

Our work: characterize DAM saddle points.

- We design a new DAM, different form [Krotov and Hopfield, 2016, Boukacem et al., 2024].
- Easier to study using statistical mechanics.
- Verify it learns similar memories $\mathbf{w}$ as the previous version:



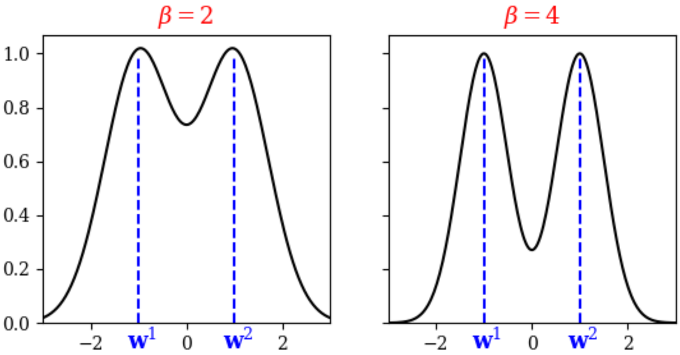
Our DAM

Recall (generative) ML fits data $\mathbf{x}$ and labels y with $\mathrm{P}(\mathbf{x}, y|\mathbf{w})$.

Inverse temperature

$$\text{Our DAM: } \mathrm{P}(\mathbf{x}, y|\mathbf{w}, \mathbf{g}) = \sum_{\mu=1}^{P} \mathbf{g}_y^\mu \exp\left(\beta \sum_{i=1}^{N} \mathbf{w}_i^\mu \mathbf{x}_i\right)$$

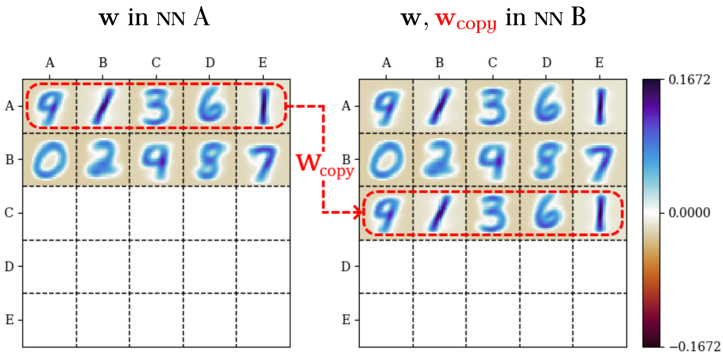
Memories



Fixed points

Let $\mathbf{w}_{\text{copy}}$ be a subset of the memories $\mathbf{w}$ of NN A, and let NN B have memories $\mathbf{w}, \mathbf{w}_{\text{copy}}$.

We show that, if $\mathbf{w}$ is a fixed point of $L(\mathbf{w})$ in NN A, then $\mathbf{w}, \mathbf{w}_{\text{copy}}$ is a fixed point of $L\left(\mathbf{w}, \mathbf{w}_{\text{copy}}\right)$ in NN B.



When β is large enough, $\mathbf{w}, \mathbf{w}_{\text{copy}}$ is unstable, i.e. a saddle.

Split Memories to Accelerate Training

Exploit saddles to accelerate training.

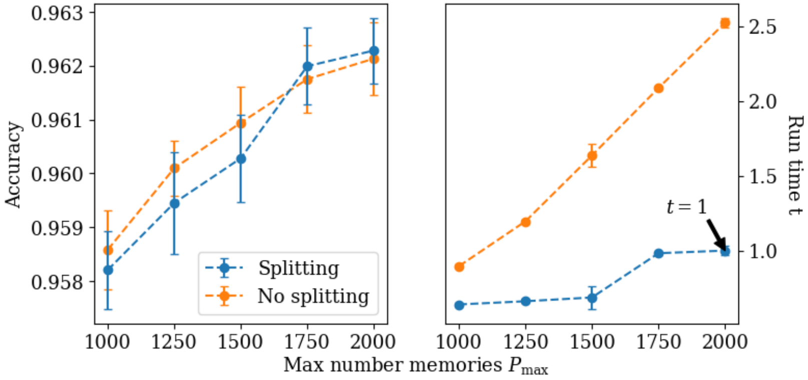
Splitting steepest descent [Wu et al., 2019, Wang et al., 2019]

1: Let P be the no. of memories $\mathbf{w}$
2: min $L(\mathbf{w})$
3: **while** $P_{\text{cur}} < P_{\text{max}}$ **do**
4: $\quad \mathbf{w} \leftarrow \mathbf{w}, \mathbf{w}_{\text{copy}}$
5: $\quad$ min $L(\mathbf{w})$
6: **end while**

While $\mathbf{w}, \mathbf{w}_{\text{copy}}$ is a saddle, each min step improves $L(\mathbf{w})$.

Split Memories to Accelerate Training

- Trains faster than using $P_{\max}$ memories from scratch.
- The speedup scales with $P_{\max}$.



Could be useful to train transformers [Ramsauer et al., 2020].

Summary

- We derive a dense associative memory (DAM) model amenable to statistical mechanics calculations.
- Show splitting memories transforms DAM minima into saddles.
- Exploit splitting to accelerate training by 2 times or more, which could be applied to transformers [Ramsauer et al., 2020].

Saddle points are often seen as a hindrance, but here they are useful. Perhaps they are misunderstood.

Summary

- We derive a dense associative memory (DAM) model amenable to statistical mechanics calculations.
- Show splitting memories transforms DAM minima into saddles.
- Exploit splitting to accelerate training by two times or more, which could be applied to transformers [Ramsauer et al., 2020].

The saddle points

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Saddle-point equations

$$\bar{\mathbf{x}}_i^{\mu} = \frac{1}{P^*}\sum_{\gamma_*} \mathbf{x}_i^{\gamma_*}\sigma_\mu\left(\beta m^{\gamma_*} - s^{\gamma_*}\right)$$

$$\bar{\mathbf{y}}_y^{\gamma} = \frac{1}{P^*}\sum_{\gamma_*} \mathbf{y}_y^{\gamma_*}\sigma_\gamma\left(\beta m^{\gamma_*} - s^{\gamma_*}\right)$$

$$m^{\mu_*\mu} = \varsigma\left(2\beta\alpha\sqrt{\sum_i \left[\bar{\mathbf{x}}_i^{\mu}\right]^2}\right)\frac{\sum_i \mathbf{x}_i^{\mu_*}\bar{\mathbf{x}}_i^{\mu}}{\sqrt{\sum_i \left[\bar{\mathbf{x}}_i^{\mu}\right]^2}}$$

$$s^{\gamma_*\gamma} = -\sum_y \mathbf{y}_y^{\gamma_*}\log\left[\frac{\mathrm{C}\left(\gamma\right)\bar{\mathbf{y}}_y^{\gamma}}{\mathrm{P}\left(y\right)\sum_{y'}\bar{\mathbf{y}}_{y'}^{\gamma}}\right]$$

Additional plots