Dynamics of Automorphisms of F_n

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Contents

1 Meeting with Katy, July 7th

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} e^1 & 0 \\ 0 & e^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^1 & 0 \\ 0 & e^{-1} \end{bmatrix} = \begin{bmatrix} e & e \\ 0 & e^{-1} \end{bmatrix} - \begin{bmatrix} e & e^{-1} \\ 0 & e^{-1} \end{bmatrix} = \begin{bmatrix} 0 & e - e^{-1} \\ 0 & 0 \end{bmatrix}$$

Definition 1 (Primitive Elements)**.** Consider a free group F_n with standard basis $x_1, \ldots, x_n$. A *primitive element* is an element y such that there exists a basis $y, y_2, \ldots, y_n$ for F_n.

Example 2 (Examples of Primitive Elements)**.** In the notation given above,

1. x_1x_2 is a primitive element, since $\{x_1x_2, x_2, \ldots, x_n\}$ is a basis for F_n.

2. x_1^2 is not a primitive element. To see why, consider a group homomorphism $\phi : F_n \to G$. Then, if there existed a basis $\{x_1^2, y_2, \ldots, y_n\}$ for F_n, these elements would satisfy the universal property for free groups for F_n; that is, for any group G and list $g_1, \ldots, g_n \in G$, there exists a unique homomorphism such that $\phi(x_1^2) = g_1$ and $\phi(y_i) = g_i$.
   Yet this is not possible, for if $G = \mathbb{Z}_2$, then $\phi(x_1^2) = 2\phi(x_1) = 0$. Thus, the basis fails to satisfy the universal property for the list $g_1 = \cdots = g_n = 1$.

Problem 3 (Classifying Primitive Elements)**.** *Let $X = \{a_1, \ldots, a_n\} \subseteq F_n$ and $H = \langle a_1, \ldots, a_n \rangle = \langle X \rangle$. Does H contain a primitive element?*

The answer is related to Whitehead's algorithm, which addresses this problem:

> **Problem 4** (Automorphisms on Sets of n Words)**.** *Suppose that* $U = \{u_1, \ldots, u_m\}$ *and* $V = \{v_1, \ldots, v_m\}$ *are two sets of words in* F_n*. Then, does there exist* $\phi \in \mathrm{Aut}(F_n)$ *such that* $\phi(U) = V$*.*

Of course, this is trivial for $m = 1$, but is immediately more complicated for $m > 1$.