

LECTURES WITH YIZHAQ: 1

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1 Bases, Hopfian, and Residually Finite Groups

In these notes, let F_n be the free group of rank n . In the case $n = 1$, $F_1 \simeq \mathbb{Z}$. For most questions, we will be interested in the less trivial cases, those for which $n \geq 3$.

Definition 1 (Base for Free Group). A *base* for F_n is an n -tuple that freely generates F_n .

Theorem 2 (Redundant Freeness). *Any n -tuple which generates F_n generates it freely.*

Definition 3 (Hopfian). A group G is said to be *Hopfian* if any surjective homomorphism $\phi : G \rightarrow G$ is also injective, i.e., it is an automorphism.

Lemma 4 (The Free Group is Hopfian). *The free group is Hopfian.*

Using this lemma, Theorem 2 follows immediately.

Proof. Suppose that $\{x_1, \dots, x_n\}$ is a free base for F_n and $\{g_1, \dots, g_n\}$ is a generating tuple. Then, consider a homomorphism $\phi : F_n \rightarrow F_n$ given by $x_n \rightarrow g_n$. Since $\langle g_1, \dots, g_n \rangle = F_n$, ϕ is surjective; since F_n is Hopfian, this establishes that ϕ is injective and thus that $\{g_1, \dots, g_n\}$ freely generate F_n . \square

To establish Lemma 4, we use the following.

Definition 5 (Residually Finite). A group G is residually finite if for all $g \in G \setminus \{1\}$ there is a finite group F and homomorphism $\phi : G \rightarrow F$ such that $\phi(g) \neq 1 \in F$.

Lemma 6 (The Free Group is Residually Finite). *The free group F_n is residually finite.*

Proof. Let x_1, \dots, x_n be the standard free basis of F_n ; then, any nontrivial element of F_n is a word w in these elements. Suppose that w is a nontrivial word. Then, there exists some letter x_i which appears to the power $k \neq 0$. Then, one may define a homomorphism $\phi : F_n \rightarrow \mathbb{Z}/2k\mathbb{Z}$ given by $x_j \mapsto \delta_{ji}$ \square

Proposition 7 (Residually Finite is Hopfian). *A residually finite finitely generated group is Hopfian.*

Proof. Let G be a finitely generated residually finite group. Let $\phi : G \twoheadrightarrow G$ be a surjective map and $N = \ker \phi$. Let a_n be the number of subgroups of G of index n ; since G is finitely generated, a_n is finite for all n . If $H \leq G$ is of index n , then $\phi^{-1}(H)$ is also of index n in G since ϕ is onto. Yet $\phi^{-1}(H) \geq N$. Thus, there are a_n subgroups of index n in G that contain N . Thus, all subgroups of finite index contain N .

Yet, if G is residually finite, for any $g \in G$ which is not the identity, there exists a finite group F and homomorphism $\psi : G \rightarrow F$ such that $\psi(g) \neq 1$. Then, $[G : \ker \psi] < \infty$, and $\ker \psi \not\ni g$, whence $g \notin N$. Thus, $N = \{1\}$. \square

Lemma 4 follows as a corollary.

2 Homomorphisms to Other Groups

Now, if $\{x_1, \dots, x_n\}$ is a base for F_n , then if G is any group. A map $\phi : F_n \rightarrow G$ is determined by $\phi(x_1), \phi(x_2), \dots, \phi(x_n)$. Therefore, $\text{Hom}(F_n, G) \simeq G^n$, but since this identification requires a basis, this isomorphism is non-canonical.

Definition 8 (Primitive Elements). An element $x \in F_n$ is called *primitive* if it belongs to any base. The set of *all* primitive elements is denoted $P_n \subseteq F_n$.

Example 9 (Primitive Elements and Automorphisms). Suppose that $x \in F_n$ is primitive. Then the orbit of x under $\text{Aut}(F_n)$, that is, $\text{Aut}(F_n)(x)$, is precisely P_n .

Example 10 (Self-Embeddings). Consider the natural embedding $F_n \triangleleft \text{Aut}(F_n)$ given by taking each element to its corresponding inner automorphism. Then, $P_n \subseteq F_n$ is a conjugacy class in $\text{Aut}(F_n)$.

Problem 11 (Learn About Primitive Elements). *Learn about the set of primitive elements P_n .*

3 Conjectures and the Product Replacement Graph

Conjecture 12 (An Open Problem). *Let $n \geq 3$, and let G be a finite simple group. Let $\phi : F_n \twoheadrightarrow G$ be a surjective map. Then, $\phi(P_n) = G$.*

Consider the group $A_{52} = G$. Consider an n -tuple (g_1, \dots, g_n) that generates G , and replace it by another n -tuple by applying a random Nielsen move (e.g. $(g_1, g_2, \dots, g_n) \mapsto (g_1, g_1 g_2, \dots, g_n)$).

Definition 13 (The Graph PRG). Let G be a finite simple group. Define a graph $\text{PRG}(G, n)$ whose vertices are an n -tuples of elements of G which generate G , and are joined by an edge if they are related by a Nielsen move.

Conjecture 14 (Weigold Conjecture). *For $n \geq 3$ and G a finite simple group, $\text{PRG}(G, n)$ is connected.*