An Exercise for Mason

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Proposition 0.1 (Exercise 1). *Suppose that G is a compact, connected topological group, and U is a nonempty open subset of G. Then $U^k = G$ for some finite k.*

Proof. Consider the smallest subgroup H generated by U. Since $H = \bigcup_{n\in\mathbb{Z}} U^n$, and U^n is open for each $n \in \mathbb{Z}$, H is an open subgroup of G. Thus, H is a closed subgroup of G whence by connectedness $H = G$. It then follows that $\bigcup_{n\in\mathbb{Z}} U^n$ is an open subcover of G, so it has a finite subcover $U^{-n_1} \cup \cdots \cup U^{-n_k} \cup U^{m_1} \cup \cdots \cup U^{m_l}$. Now, both $U^{m_1} \cup \cdots \cup U^{m_l}$ and $U^{-n_1} \cup \cdots \cup U^{-n_k}$ are open, and they cover U. Thus, either they are disjoint, or they intersect.

In the former case, $U^{m_1} \cup \cdots \cup U^{m_l}$ is open and closed, and thus is equal to G. Thus, in particular, $1 \in U^m$ for some m. In the latter case, there exists some m_i, n_j such that $U^{m_i} \cap U^{-n_j}$ has an intersection. It then follows that $1 \in U^{m_i+n_j}$. Again, $1 \in U^m$ for some m.

Now, let $W = U^m \cap U^{-m}$; W is symmetric and open, so the subgroup J generated by W is equal to $\bigcup_{k\in\mathbb{Z}^+} W^k$. Since J is the union of open sets, it is open, whence by the same reasoning as earlier $J = G$. Thus, since G is compact, $\{W_k\}_{k=1}^{\infty}$ is an open cover of G, and as W^k is increasing in k, it follows that $G = W^N$ for some N. Since $W \subseteq U^m$, $W^N \subseteq U^{mN}$, whence $U^{mN} = G$, as desired. □