Dynamics of Automorphisms of F_n

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Lemma 1 (Ping-Pong Lemma)**.** *Let Γ be a group acting on a set X. Suppose there exist "rackets" $x, y \in \Gamma$, and nonempty disjoint "balls" $A, B, C, D \subseteq X$, such that*

$$\begin{aligned} x \cdot (A \cup C \cup D) &\subseteq A \\ x^{-1} \cdot (B \cup C \cup D) &\subseteq B \\ y \cdot (A \cup B \cup C) &\subseteq C \\ y^{-1} \cdot (A \cup B \cup D) &\subseteq D \end{aligned}$$

Then, $\langle x, y \rangle = F_2$, the free group on 2 elements.

Proof. It must only be shown that any nonempty reduced (no consecutive elements are inverses) sequence (word) of rackets is not the identity. The key idea is that any racket x, y, x^{-1}, y^{-1} collapses two of these nonempty balls into a single ball, and any following racket hit *except the first racket's inverse* sends that ball into another, single, ball. Thus, any nonempty reduced word collapses two of these balls into a single ball, and therefore cannot be the identity. The result follows, then, from the element-wise definition of the free group. $\square$

Now, there is a generalization of this result which may prove useful:

Lemma 2 (Ping-Pong Lemma 2). *Let Γ be a group acting on a set X. Suppose that there exist subgroups $G, H \leq \Gamma$, and disjoint nonempty $A, B \subseteq X$ such that*

1. $|G| \geq 3$,
2. $(G \setminus \{1\})B \subseteq A$,
3. $(H \setminus \{1\})A \subseteq B$.

Then, $\langle G, H \rangle \simeq G * H$ (the free product of G and H).

Proof. Let $a_1 \cdots a_n$ be a reduced word. By collapsing any elements in the same group, we may assume that the word is further reduced so that $a_i \in G \Rightarrow a_{i+1} \notin G$, and $a_i \in H \Rightarrow a_{i+1} \notin H$ for all i. Then, suppose $a_1, a_n \in G$. Then, $a_1 \cdots a_n$ sends B to A, so it is not the identity. On the other hand, suppose that at least one of $a_1, a_n \notin G$. Then, since G has order at least 3, there exists an element $a \in G$ distinct from a_1^{-1}, a_n^{-1} and 1, so that $a^{-1}a_1 \cdots a_n a$ is still reduced. By the previous case, the expanded word is not the identity; thus, it cannot be the case that $a_1 \cdots a_n$ is the identity. $\square$

Corollary 3 (Ping-Pong Lemma). *See Lemma 1.*

Proof. The result follows because the given assumption immediately implies that both x and y have infinite order; for example, if $x^m = 1$ for any $m > 0$, we get a contradiction by considering the action of each side on C (the right-hand side sends it to A, and the left-hand side sends it to C). Then one can choose $G = \langle x \rangle$, $H = \langle y \rangle$ (where infinite order implies G is large enough), $A' = A \cup B$, and $B' = C \cup D$, and then the result follows from Lemma 2. $\square$

Example 4 (The Special Linear Group). Consider the action of the special linear group $\mathrm{SL}_2(\mathbb{R})$ on the projective line $\mathbb{R}\P^1$ given by sending the line L through the origin to the line AL through the origin. Then, let

$$x = \begin{bmatrix} 100 & 0 \\ 0 & \frac{1}{100} \end{bmatrix} \qquad y = R_{\pi/4} \begin{bmatrix} 100 & 0 \\ 0 & \frac{1}{100}. \end{bmatrix} R_{-\pi/4}.$$

Notice that any line which is not close to the y-axis is mapped to a line which is close to the x-axis. Thus, one may define those lines which are close to the x-axis to be A, and those lines which are close to the y-axis to be B. On the other hand, because of the conjugation by a rotation of angle $\frac{\pi}{4}$, a similar result holds for y regarding the lines $x = y$ (lines near which are defined to be C) and $x = -y$ (lines near which are defined to be D). The required rules of Lemma 1 follow, so that x and y generate F_2, as desired.