Towards Conjugate Cosets of $\mathrm{SL}_2(\mathbb{C})$ in $\mathrm{SL}_n(\mathbb{C})$

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1 Lie Groups and Lie Algebras

Definition 1 (Lie Group)**.** A *Lie group* G is a group and a smooth n-manifold so that multiplication $G \times G \to G$ and $G \to G$ are both smooth maps.

Definition 2 (Homomorphism of Lie Groups)**.** A *homomorphism of Lie groups* $\phi : G \to H$ is a map which is both differentiable and a group homomorphism.

Example 3 (General Linear Group)**.** The *general linear group* $\mathrm{GL}_n(\mathbb{C})$ is the set of all invertible $n \times n$ complex matrices under conjugation. It inherits a smooth n-manifold structure from $\mathbb{C}^{n\times n}$, of which it is an open subset; multiplication is differentiable as it is a polynomial action, and invertibility is differentiable via Cramer's formula.

Definition 4 (Representation of a Lie Group)**.** A *representation* of a Lie group on V is a Lie group homomorphism

$$G \to \mathrm{GL}(V).$$

Example 5 (Other Examples of Matrix Groups)**.**

1. $\mathrm{SL}_n(\mathbb{C})$, the set of matrices with determinant 1.
2. $B_n(\mathbb{C})$, the set of upper-triangular matrices.
3. $N_n(\mathbb{C})$, the set of upper-triangular unipotent matrices (i.e. 1s on the diagonal).

1.1 Properties of Lie Groups

Proposition 6 (Generation by Neighborhoods)**.** *Suppose that G is a connected Lie group, and $U \subseteq G$ is any neighborhood of the identity. Then, U generates G.*

Proof. First, notice that by replacing U with U □

Example 7 (Special Linear Group)**.** An example of a linear algebraic group is the *special linear group* $\mathrm{SL}_n(\mathbb{C})$, the set of all complex $n \times n$ matrices of determinant 1.

Definition 8 (Lie Algebra)**.** A *Lie algebra* over $\mathbb{C}$ is a $\mathbb{C}$-vector space V with a skew-symmetric $\mathbb{C}$-bilinear form $[\cdot,\cdot] : V \times V \to V$ which satisfies the *Jacobi identity*:

$$[X, Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in V$.

Definition 9 (Vector Field)**.** A vector field X on a Lie group G is called *left invariant* if $d(L_g)_h(X(h)) = X(gh)$ for all $g, h \in G$, or for short $(L_g)_*(X) = X$.

2 Lie Algebras of Linear Algebraic Groups