Lectures with Yizhaq: 1

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1 Bases, Hopfian, and Residually Finite Groups

In these notes, let F_n be the free group of rank n. In the case $n = 1$, $F_1 \simeq \mathbb{Z}$. For most questions, we will be interested in the less trivial cases, those for which $n \geq 3$.

Definition 1 (Base for Free Group)**.** A *base* for F_n is an n-tuple that freely generates F_n.

Theorem 2 (Redundant Freeness)**.** *Any n-tuple which generates F_n generates it freely.*

Definition 3 (Hopfian)**.** A group G is said to be *Hopfian* if any surjective homomorphism $\phi : G \to G$ is also injective, i.e., it is an automorphism.

Lemma 4 (The Free Group is Hopfian)**.** *The free group is Hopfian.*

Using this lemma, Theorem 2 follows immediately.

Proof. Suppose that $\{x_1, \ldots, x_n\}$ is a free base for F_n and $\{g_1, \ldots, g_n\}$ is a generating tuple. Then, consider a homomorphism $\phi : F_n \to F_n$ given by $x_n \to g_n$. Since $\langle g_1, \ldots, g_n \rangle = F_n$, ϕ is surjective; since F_n is Hopfian, this establishes that f is injective and thus that $\{g_1, \ldots, g_n\}$ *freely generate* F_n. $\square$

To establish Lemma 4, we use the following.

Definition 5 (Residually Finite)**.** A group G is residaully finite if for all $g \in G \setminus \{1\}$ there is a finite group F and homomorphism $\phi : G \to F$ such that $\phi(g) \neq 1 \in F$.

Lemma 6 (The Free Group is Residually Finite)**.** *The free group F_n is residually finite.*

Proof. Let $x_1, \ldots, x_n$ be the standard free basis of F_n; then, any nontrivial element of F_n is a word w in these elements. Suppose that w is a nontrivial word. Then, there exists some letter x_i which appears to the power $k \neq 0$. Then, one may define a homomorphism $\phi : F_n \to \mathbb{Z}/2k\mathbb{Z}$ given by $x_j \mapsto \delta_{ji}$ $\square$

Proposition 7 (Residually Finite is Hopfian)**.** *A residually finite finitely generated group is Hopfian.*

Proof. Let G be a finitely generated residually finite group. Let $\phi : G \twoheadrightarrow G$ be a surjective map and $N = \ker \phi$. Let a_n be the number of subgroups of G of index n; since G is finitely generated, a_n is finite for all n. If $H \leq G$ is of index n, then $\phi^{-1}(H)$ is also of index n in G since ϕ is onto. Yet $\phi^{-1}(H) \geq N$. Thus, there are a_n subgroups of index n in G that contain N. Thus, all subgroups of finite index contain N.

Yet, if G is residually finite, for any $g \in G$ which is not the identity, there exists a finite group F and homomorphism $\psi : G \to F$ such that $\psi(g) \neq 1$. Then, $[G : \operatorname{Ker} \psi] < \infty$, and $\operatorname{Ker} \psi \not\ni g$, whence $g \notin N$. Thus, $N = \{1\}$. $\square$

Lemma 4 follows as a corollary.

2 Homomorphisms to Other Groups

Now, if $\{x_1, \ldots, x_n\}$ is a base for F_n, then if G is any group. A map $\phi : F_n \to G$ is determined by $\phi(x_1), \phi(x_2), \ldots, \phi(x_n)$. Therefore, $\operatorname{Hom}(F_n, G) \simeq G^n$, but since this identification requires a basis, this isomorphism is non-canonical.

Definition 8 (Primitive Elements)**.** An element $x \in F_n$ is called *primitive* if it belongs to any base. The set of *all* primitive elements is denoted $P_n \subseteq F_n$.

Example 9 (Primitive Elements and Automorphisms)**.** Suppose that $x \in F_n$ is primitive. Then the orbit of x under $\operatorname{Aut}(F_n)$, that is, $\operatorname{Aut}(F_n)(x)$, is precisely P_n.

Example 10 (Self-Embeddings)**.** Consider the natural embedding $F_n \lhd \operatorname{Aut}(F_n)$ given by taking each element to its corresponding inner automorphism. Then, $P_n \subseteq F_n$ is a conjugacy class in $\operatorname{Aut}(F_n)$.

Problem 11 (Learn About Primitive Elements)**.** *Learn about the set of primitive elements* P_n*.*