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Algorithmic Perspectives to Collective Natural Phenomena

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Abstract

The capabilities of distributed systems in stochastic environments are inherently limited by the difficulty of efficiently circulating information within the system. To better understand this limitation, the thesis presents and analyses several idealised models, in which agents are required to disseminate information despite noisy and constrained communications. We focus on identifying natural, robust and efficient protocols, that do not assume agents to have high computational abilities, and that do not rely on initial global organization. First, we consider the self-stabilizing *information spread* problem, inspired by decision-making processes in distributed biological ensembles. We propose a novel algorithm that uses minimal communications, and provide theoretical guarantees on its performance. Furthermore, we explore the role of memory by identifying and analysing a memory-less protocol on the one hand, and by presenting a strong lower bound on the other hand. Then, motivated by flocking scenarios, we investigate the feasibility of alignment in the presence of drift noise. We show that a simple weighted-average decision rule is optimal, even when compared to centralized systems endowed with unrestricted communication abilities.

In addition to fully cooperative systems, the thesis studies systems composed of self-interested players, focusing on the emergence of cooperation. Using a game-theoretical perspective, we describe several counter-intuitive phenomena within contexts inspired by biological and sociological scenarios involving selfish agents. First, we establish a counter-intuitive results that is reminiscent of Braess' paradox, albeit in the context of social foraging rather than network flows. Then, we emphasize the possibility that tolerance towards hypocritical behaviours might help mitigating the tragedy-of-the-commons in spatial public good games.

Overall, this thesis aims to offer fresh insights into the capabilities of distributed and competitive biological systems, shedding new light on their potential and limitations.

Keywords: Distributed Computing, Stochastic Processes, Opinion Dynamics, Consensus Protocols, Collective Animal Behavior, Free Rider Problem, Game Theory

Aperçu

Les aptitudes des systèmes distribués qui évoluent dans des environnements stochastiques sont limitées par la difficulté de faire circuler efficacement l'information en leur sein. L'objectif de cette thèse est de mieux comprendre cette limitation en analysant différents modèles mathématiques, dans lesquels des agents doivent partager des informations et atteindre un consensus, malgré des communications restreintes et bruitées. Nous cherchons à identifier des protocoles naturels, robustes et efficaces, qui ne nécessitent pas de grandes capacités de calcul, et ne font pas l'hypothèse d'une quelconque organisation initiale du système. Dans un premier temps, nous étudions un problème de diffusion "auto-stabilisante" d'information, modélisant la prise de décision collective par des entités biologiques. Nous proposons un algorithme nouveau, utilisant un minimum de communications, et nous donnons des garanties théoriques sur ses performances. En outre, nous démontrons l'utilité de la mémoire dans ce problème en caractérisant les performances des protocoles n'utilisant pas de mémoire, qui s'avèrent particulièrement limitées. Dans un deuxième temps, nous étudions la possibilité, pour des agents perpétuellement en train de dériver, de s'aligner les uns avec les autres. Nous montrons qu'un algorithme réalisant une simple moyenne pondérée est optimal dans ce modèle, même lorsqu'il est comparé à des systèmes centralisés dotés de capacités de communication supérieures.

Au-delà des systèmes entièrement coopératifs, la thèse s'intéresse également aux systèmes composés d'acteurs en concurrence les uns avec les autres, et plus particulièrement à l'émergence de la coopération dans de tels systèmes. Dans le cadre de la théorie des jeux, nous décrivons plusieurs phénomènes contre-intuitifs impliquant des agents rationnels et égoïstes. Premièrement, nous établissons des résultats en apparence contradictoires qui rappellent le paradoxe de Braess, dans le contexte de la recherche collective de nourriture par des animaux sociaux. Deuxièmement, nous montrons comment la tolérance envers les comportements hypocrites peut atténuer la tragédie des biens communs dans des jeux des bien public joués sur des réseaux.

Dans l'ensemble, cette thèse vise à offrir de nouvelles perspectives sur les capacités des systèmes biologiques, distribués d'une part et concurrentiels d'autre part, apportant un nouvel éclairage sur leur potentiel et leurs limites.

Mots clés: Calcul Distribué, Processus Stochastiques, Dynamiques d'Opinion, Protocoles de Consensus, Comportement Animal Collectif, Problème du Passager Clandestin, Théorie des Jeux

Résumé en français

Cette thèse est construite autour de 4 travaux relativement indépendants, mais suivant tous les mêmes grands principes : ils s'inspirent de systèmes naturels ; ils impliquent de multiples entités ; enfin, ils s'efforcent de mettre à jour des propriétés contre-intuitives. Ici, je commence par décrire ces principes en détails afin d'illustrer mes motivations. Dans un deuxième temps, je résume séparément ces 4 travaux. Ils donnent lieu chacun à un chapitre différent, organisés en 2 parties.

Fils directeurs

Inspirés de phénomènes naturels. De nombreux problèmes naturels sont intrinsèquement *algorithmiques*, dans le sens où ils nécessitent le traitement d'informations. Par conséquent, l'informatique théorique peut fournir des outils appropriés pour les comprendre. Cette idée, notamment développée par Karp sous le nom de *lentille algorithmique* [101], a gagné en popularité ces dernières années [122]. En particulier, plusieurs tentatives prometteuses ont été faites d'utiliser la théorie du calcul distribué et les algorithmes en général pour comprendre les comportements des animaux. Leur lien avec la biologie varie, du plus théorique [69, 63, 77, 114, 44] au plus empirique [2, 70, 84, 87]. Dans cette thèse, j'essaie de contribuer à cette tendance émergente. Bien que tous les résultats présentés ici soient ancrés dans l'informatique théorique et ne soient pas directement basés sur des données expérimentales, chacun d'entre eux vise à améliorer notre compréhension des processus naturels correspondants. Pour ce faire, j'examine des modèles et j'aborde des questions inspirés de scénarios réels.

La Partie I s'inspire principalement de systèmes biologiques, tels que les nuées d'oiseaux, les bancs de poissons, les bactéries, les insectes, etc. Les résultats présentés dans la Partie II sont plus pertinents dans le cadre de la sociologie et l'économie, bien qu'ils puissent aussi s'appliquer au monde animal, par exemple en ce qui concerne la recherche collective de nourriture.

Systèmes multi-agents. Les entités qui composent les ensembles biologiques, parmi lesquels les sociétés humaines, présentent toujours un certain niveau d'indépendance et d'autonomie. Par conséquent, tous les systèmes étudiés dans cette thèse sont constitués de multiples *agents* qui prennent des décisions sur la base de leurs propres informations et objectifs. Cette absence de centralisation a deux conséquences.

La première conséquence est que la plupart des défis algorithmiques qui se posent relèvent du domaine du calcul distribué. Contrairement à d'autres domaines de l'informatique, le calcul distribué met moins l'accent sur les raisonnements internes effectués par les agents que sur leurs interactions mutuelles. Ainsi, les tâches impliquant de propager efficacement l'information au sein du groupe, et de rechercher un consensus entre les agents, revêtent une importance particulière [100, 13]. Dans la Partie I, je tente de caractériser les algorithmes optimaux pour ces tâches, dans divers contextes coopératifs impliquant des communications limitées et bruitées [66, 20, 21, 53].

Bien que l'hypothèse de comportements coopératifs soit facilement justifiée dans le cas des espèces eusociales – comme les fourmis, les abeilles ou les termites –, on ne s'attend pas à ce que tous les ensembles biologiques se comportent de manière totalement coopérative. Ainsi, une deuxième conséquence de la nature décentralisée des systèmes biologiques est la tension potentielle entre les intérêts individuels et collectifs. Il est peu probable qu'un protocole conduisant au meilleur résultat (collectivement parlant) soit respecté par des entités intéressées, si celles-ci peuvent obtenir un bénéfice quelconque en s'en écartant. En outre, un tel protocole serait d'autant plus irréaliste que seules les stratégies offrant les meilleure chances de survie sont censées résister à la sélection naturelle. Dans la Partie II, je m'éloigne des considérations algorithmiques pour me concentrer sur le conflit entre les comportements coopératifs et égoïstes, ce que l'on appelle parfois les *dilemmes sociaux*. Dans le cadre de la théorie des jeux, j'étudie les conditions dans lesquelles la coopération peut émerger dans des environnements compétitifs, apportant ainsi une modeste contribution à la littérature existante [14, 123, 5]. Une fois de plus, cette contribution est façonnée par l'informatique théorique : le Chapitre 4 s'inspire du paradoxe de Braess, tandis que le Chapitre 5 s'appuie sur la théorie des mécanismes d'incitation.

Propriétés contre-intuitives. Enfin, les recherches menées dans le cadre de cette thèse sont motivées par une fascination pour les phénomènes en apparence paradoxaux. Elles mettent l'accent sur la découverte de propriétés qui sont non seulement difficiles à deviner à partir d'une simple description du contexte (comme c'est souvent le cas en informatique théorique), mais qui semblent de plus invalides à première vue. Ces propriétés ont des mérites objectifs ; en effet, les résultats contre-intuitifs peuvent être révélateurs de mécanismes sous-jacents qui n'ont pas encore été découverts ou entièrement compris. En outre, ils ont plus de chances de susciter la réflexion et d'élargir les perspectives des chercheurs dans diverses disciplines.

Dans la Partie I, des propriétés contre-intuitives apparaissent en raison des mécanismes complexes par lesquels la performance collective émerge du comportement individuel. La Partie II, en particulier, a sa part de résultats paradoxaux, pour lesquels la théorie des jeux est un terrain fertile. Ici, ils se produisent lorsqu'un changement de cadre, a priori préjudiciable aux agents, s'avère renforcer la coopération de manière si significative qu'il améliore en fin de compte le bien-être social.

Résumé par chapitre

Dans le Chapitre 2, nous examinons le problème de *bit-dissemination* introduit dans [22], sur un réseau entièrement connecté de n agents, dans le modèle $\mathcal{PULL}$ de communication. Les agents doivent choisir entre deux *opinions* possibles. À chaque activation, ils observent un petit nombre d'autres agents choisis uniformément aléatoirement ; après quoi, ils sont libres de changer d'opinion. L'un d'entre eux (la *source*) détient initialement l'opinion *correcte*, et la conserve tout au long de l'exécution. L'opinion correcte représente une décision préférable à l'autre – par exemple, le meilleur itinéraire de migration, la zone de recherche de nourriture la plus sûre, etc. – et la source représente un individu informé. Logiquement, l'objectif des autres agents est d'adopter rapidement l'opinion correcte. Motivés par des scénarios biologiques, nous faisons l'hypothèse que les communications sont "passives", c'est-à-dire que la seule information que l'on peut obtenir en observant un agent est l'opinion de ce dernier. En particulier, nous supposons que les agents ne sont pas en mesure d'identifier la source. Enfin, conformément au cadre d'auto-stabilisation, nous nous intéressons uniquement aux protocoles capable de converger indépendamment de la configuration initiale.

Nous étudions deux schémas d'activation : le schéma *parallèle*, dans lequel tous les agents sont activés simultanément, et le schéma *séquentiel*, dans lequel un seul agent est activé à la fois. Afin d'explorer le rôle de la mémoire, nous comparons les protocoles *sans mémoire*, avec les protocoles pour lesquels les agents mémorisent quelques bits d'information entre deux activations.

D'abord, nous proposons un algorithme simple et élégant qui, en utilisant une petite quantité de mémoire, charge les agents d'estimer et de suivre la tendance émergente de la dynamique. Nous montrons qu'il peut résoudre le problème en un temps poly-logarithmique en n dans le schéma d'activation parallèle, et nous effectuons des simulations qui suggèrent qu'il est également efficace pour le schéma séquentiel.

En outre, nous montrons que la mémoire est nécessaire pour obtenir une convergence rapide dans le schéma séquentiel. Plus précisément, nous prouvons que tout protocole sans mémoire nécessite un temps linéaire en n pour résoudre notre problème dans le schéma séquentiel, même lorsque les agents peuvent observer la configuration complète du système. En analysant le temps de convergence du modèle "Voter", nous montrons que cette borne inférieure est quasiment atteinte.

En somme, nous mettons à jour deux résultats de séparation. Premièrement, les protocoles sans mémoire sont plus lents dans le schéma séquentiel que dans le schéma parallèle d'un facteur exponentiel (du moment que le nombre d'agents observés à chaque activation n'est pas limité). De plus, dans le schéma séquentiel, les protocoles sans mémoire sont plus lents que les protocoles sans restriction de mémoire, également d'un facteur exponentiel. Enfin, nous expliquons comment généraliser certains de nos résultats au cas où les agents peuvent choisir parmi un nombre arbitraire d'opinions possibles.

Dans le Chapitre 3, nous étudions un problème d'*alignement*, sous l'hypothèse que les agents peuvent percevoir approximativement l'orientation générale du groupe. Plus précisément, nous considérons n agents, chacun étant associé à un nombre réel. Ce nombre, que nous appelons *position*, peut représenter l'orientation de l'agent dans le contexte du mouvement collectif, ou bien une notion de temps dans le contexte de la synchronisation d'horloge. Nous définissons le *centre de masse* comme la position moyenne du groupe, et la *déviation* d'un agent comme sa distance par rapport au centre de masse. L'objectif des agents est d'être le plus proche possible du centre de masse, c'est-à-dire de minimiser leur déviation, à tout instant. Dans ce but, à chaque tour, ils reçoivent une mesure bruitée de leur déviation, et peuvent mettre à jour leur position sur la base de cette mesure. Cependant, leur position est perturbée par une dérive aléatoire à la fin de chaque tour. Nous supposons que le bruit sur les mesures, ainsi que la dérive, sont tous deux gaussiens.

Nous montrons qu'un simple algorithme de moyenne pondérée minimise de manière optimale l'écart de chaque agent par rapport au centre de masse. Notre résultat principal est cet algorithme reste optimal dans le cas où un agent "maître" est capable de rassembler toutes les mesures des agents, avant d'ordonner un déplacement à chacun d'entre eux. En d'autres termes, nous montrons que même lorsque les agents sont autorisés à communiquer leurs mesures entre eux sans limites, ils leur est impossible d'obtenir de meilleurs résultats que lorsque aucune communication n'est permise ! Nous trouvons ce résultat surprenant, car l'ensemble des mesures obtenues par tous les agents s'avère contenir strictement plus d'informations sur la déviation de l'agent i que la seule mesure obtenue par l'agent i. Bien que cette information soit pertinente pour l'agent i, elle n'est pas traitée par lui lors de l'exécution d'un algorithme de moyenne pondérée, ce dernier étant pourtant optimal. En plus de démontrer ce résultat, nous fournissons des explications informelles et des preuves supplémentaires afin de permettre une compréhension complète du problème.

Enfin, nous analysons également la dérive du centre de masse dans le temps. Nous montrons qu'autoriser les communications permet d'obtenir une dérive légèrement plus faible, soulignant ainsi le rôle de la communication dans le maintien d'une orientation stable du groupe.

Dans le Chapitre 4, nous analysons un jeu de type producteur-profiteur, dans lequel les joueurs (représentant des animaux) doivent choisir entre une recherche intensive de nourriture en tant que *producteurs*, et une recherche modérée en s'appuyant sur les découvertes des autres membres du groupe en tant que *profiteurs*. Dans la version originale bien connue du modèle producteur-profiteur [153], il est prévu que la population atteigne un équilibre mixte, où la proportion exacte de producteurs et de profiteurs dépend des paramètres du problème. Nous montrons que, dans certaines circonstances, l'augmentation de la quantité de nourriture disponible peut modifier cette proportion à tel point qu'elle conduit paradoxalement à une réduction de la consommation des animaux. À l'aide de simulations numériques, nous illustrons et quantifions ce phénomène pour une large gamme de paramètres.

Nous trouvons un phénomène similaire dans un autre jeu modélisant le parasitisme parmi les travailleurs d'une entreprise. Nous montrons que, pour certains mécanismes de récompense, l'augmentation des capacités de production des travailleurs peut involontairement déclencher une hausse des comportements de parasitisme, entraînant à la fois une diminution de la productivité du groupe et une réduction des gains individuels.

Dans les deux jeux, l'apparition de ce phénomène contre-intuitif est extrêmement sensible aux détails du modèle. Afin de mieux comprendre les circonstances dans lesquelles il peut se produire, nous identifions finalement une condition nécessaire.

Nos résultats rappellent le paradoxe de Braess bien connu, mais dans le contexte des groupes productifs, et bien que nos modèles ne puissent pas être décrits comme des jeux de congestion. Dans l'ensemble, ils permettent de mieux comprendre la dynamique complexe en jeu dans les groupes productifs, en soulignant la relation complexe entre les performances individuelles et collectives, ainsi que l'impact préjudiciable des comportements de parasitisme. En outre, nous illustrons les conséquences subtiles des facteurs contextuels dans la compréhension et la prévision de l'impact de l'augmentation (ou de la diminution) de la disponibilité des ressources sur les résultats individuels et collectifs.

Dans le Chapitre 5, nous étudions l'émergence de la coopération dans des jeux de biens publics spatiaux. Pour de tels jeux, les modèles théoriques prédisent généralement un effondrement de la coopération d'une manière qui rappelle la métaphore de la "tragédie des biens communs" [133], à moins qu'une forte pression sociale ne soit exercée sur les tricheurs (ou, au contraire, que des récompenses importantes soient accordées aux coopérateurs). Nous illustrons le fait que la coopération peut émerger lorsque la pression sociale est faible. Pour ce faire, nous avons recours à un comportement supplémentaire, l'*hypocrisie*, qui apparaît coopératif du point de vue de l'observateur externe, mais qui, en réalité, ne contribue pas vraiment au bien-être social.

Plus précisément, nous considérons un ensemble de n joueurs organisés sur un réseau quelconque, les arêtes représentant les liens réciproques entre les joueurs. L'utilité d'un joueur dépend de sa propre stratégie et de celle de ses voisins, et combine un coût énergétique associé à la coopération et un coût de pression sociale induit par les voisins. Plus précisément, nous supposons que les individus apparemment coopératifs – les coopérateurs et les hypocrites – exercent une pression sociale modeste sur individus non coopératifs – les tricheurs et les hypocrites. L'ampleur de la pression sociale exercée peut varier d'un comportement à l'autre et constitue un paramètre du modèle.

Nous montrons que l'émergence de la coopération dépend fortement de l'ampleur de la pression sociale exercée sur les joueurs hypocrites. De manière peut-être surprenante, nous montrons que lorsqu'elle est trop élevée, le système reste bloqué dans une configuration dégénérée. À l'inverse, le fait de la fixer à un niveau intermédiaire (inférieur à celui employé contre les tricheurs) permet à un système composé presque exclusivement de tricheurs de se transformer rapidement en un système entièrement coopératif. En somme, nos résultats suggèrent que l'hypocrisie peut jouer un rôle social en permettant d'échapper à la tragédie des biens communs.

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Contents

Chapter 1

Introduction

1.1 Motivation

This document is a compilation of 4 independent works. Despite their independence, these works share significant features: they are inspired by natural systems, involve multiple entities, and exhibit counter-intuitive properties. In this section, I describe these features in order to help the reader understand my motivations.

Driven by natural phenomena. Many natural systems are inherently *computational*, in the sense that they involve processing information. As a result, theoretical computer science may provide suitable tools to understand them. This idea, notably developed by Karp as the *computational lens* [101], has gained attention in recent years [122]. In particular, several promising attempts have been made to use the theory of distributed computing and algorithms in general to understand animal behaviours. Their connection to biology varies from more theoretical [69, 63, 77, 114, 44] to more empirical [2, 70, 84, 87]. In this thesis, I try to contribute to this emerging trend. Although all the findings presented here are rooted in theoretical computer science and not directly based on empirical data, each of them aims to improve our understanding of corresponding natural processes. Towards this, I examine models and address questions inspired by real-life scenarios. Part I draws inspiration mainly from biological systems, such as flocks of birds, schools of fish, bacteria, insects, and more. In contrast, the results presented in Part II are more relevant to sociology and economy, with a connection to the animal world through social foraging.

I must mention that pursuing this goal complicates the task of identifying the right problems to study. Indeed, besides aligning with natural scenarios, a good problem must satisfy the typical requirements of formal sciences: being simultaneously technically non-trivial and analytically tractable. Non trivial, because a theoretical solution can only be insightful when the underlying model has sufficient mathematical depth; and analytically tractable, for us to make significant statements about it. I invested considerable effort in exploring numerous settings and their variations to find ones that meet all these criteria. Some had to be eventually discarded and are not presented here.

Multi-agent systems. The entities making up biological ensembles, including human groups, always exhibit a certain level of independence and autonomy. Accordingly, all the systems considered in this thesis consist of multiple *agents* who make individual decisions based on their own local information and objectives. This lack of centralization leads to two consequences.

The first consequence is that some of the algorithmic challenges that arise fall under the domain of *distributed computing*. Unlike other fields of computer science, distributed computing places less emphasis on the internal calculations performed by agents, and more on their interactions with one another. As a result, tasks involving efficient information propagation and reaching agreement among agents hold significant importance [100, 13]. In Part I, I focus on characterizing optimal algorithms for these tasks, in various cooperative settings that involve noisy and constrained communications [66, 20, 21, 53].

Cooperative settings are especially relevant to eusocial species, like ants, bees or termites – however, not all biological ensembles are expected to behave fully cooperatively. In general, another consequence of the multi-agent nature of biological systems is the potential tension between individual and collective interests. The protocol that leads to the best collective outcome is unlikely to be followed by self-interested entities, if they can gain personal benefits by deviating from it. Moreover, such a protocol would be all the more unrealistic, as only strategies offering the highest fitness are expected to survive through evolution. In Part II, I shift my focus from algorithmic considerations to the conflict between cooperative and selfish behaviours, which is sometimes referred to as *social dilemma*. Within the framework of game theory, I investigate the conditions under which cooperation can emerge in competitive environments, making a modest contribution to the existing literature [14, 123, 5]. Once again, this contribution is shaped by theoretical computer science: Chapter 4 draws inspiration from the Braess paradox in network theory, while Chapter 5 is influenced by mechanism design principles.

Counter-intuitive. Finally, the research conducted for this thesis was motivated by a fascination for seemingly paradoxical phenomena. I am particularly interested in properties that are not only hard to guess from a simple description of the setting (as is often the case in theoretical computer science), but also that appear to be invalid at first glance. Although this is primarily a personal preference, I believe they hold objective merits. Indeed, counter-intuitive findings can be revealing of underlying mechanisms that are yet to be uncovered or fully understood. Moreover, they have, in my opinion, more potential to be thought-provoking and broaden the perspectives of researchers across various disciplines. In Part I, counter-intuitive statements arise due to the complex mechanisms through which collective performance emerges from individual behaviour. Part II, especially, has its share of paradoxical results, for whom game theory is a fertile ground. Here, they happen when a change in the setting, a priori detrimental to the agents, turns out to enhance cooperation so significantly that it ultimately improves social welfare.

1.2 Other Works with an Algorithmic Perspective to Biology

In this section, I further describe some of the works mentioned in Section 1.1, in order to give a better idea of the context of this thesis. What follows is not a survey – each chapter discusses its own related works –, but rather a non-exhaustive overview of the most successful applications of the computational lens.

One notable example is the theory of *natural algorithms*, formulated by Chazelle [39]. Designed to study "influence systems" like flocks of birds, it provides suitable tools for their analysis, and includes celebrated formal statements, for instance on the convergence process of flock members to a common direction and speed. Nevertheless, to the best of my knowledge, no attempt was made to connect this theory to empirical data.

In the distributed computing community, among the many works analysing the computational power of multi-agent systems relying on unreliable and constrained interactions, a few make assumptions that are specifically relevant to biological ensembles. In a seminal paper, Afek et al. derive a solution to the Maximal Independent Set (MIS) problem from a biological process (the development of the fly's nervous system) which uses few, short messages and where nodes do not need to know their own degree [2]. Later, they show how to solve MIS in another weak setting, coined the *beeping* model [1]. In parallel, Emek and Wattenhofer introduce *stone age* distributed computing, designed to capture decentralized graph protocols that can be implemented by entities with extremely limited computation and communication abilities [63].

Other examples can be found within the *swarm intelligence* literature, which focuses on the ability of self-organised decentralized systems to solve complex tasks. While most of it is directed towards applications in artificial intelligence or swarm robotics, some noteworthy articles explain how species like ants [31], or slime molds [25], could manage to compute shortest paths.

Speaking of ant navigation, recent works successfully combine formal analysis and experiments. Fonio et al. demonstrate how a group of ants carrying a large object can overcome misleading information by randomly deviating from the guiding path [70]. Chandrasekhar et al. study the trails formed by Turtle ant over a network made of branches and vegetation [38]. They observe that these ants usually opt for paths of minimal nodes number, at the expanse of total edge lengths – suggesting that ease of navigation within the network is worth the additional distance travelled. A follow up work by Garg et al. [73] identifies necessary conditions for creating such shortest paths in an idealised model, and suggest a robust distributed protocol.

In the remainder of the section, I present two additional examples in more details to better illustrate the algorithmic approach. This is a short and personal summary, and I suggest readers who are interested to look them up directly.

House-Hunting in Ant Colonies

When their nest gets damaged, *Temnothorax* ants must find a new one quickly; how fast this happens and how good the new nest is matter a lot for the colony's survival. The challenge is to reach an agreement on the new spot, keeping the colony together, while considering each ant's limited knowledge about candidate nests quality. In this regard, the *house-hunting* task is a typical consensus problem as studied by distributed computing.

Many existing field observations of ants' social behaviour during house-hunting helped Ghaffari, Musco, Radeva and Lynch [77] design a seemingly realistic mathematical model for this problem. The model is simple enough to be analytically tractable, yet rich enough to allow a large class of behaviours compatible with the observations. They identify a robust and efficient algorithm to solve the house-hunting problem in this setting. This informs us about ants insofar as an efficient algorithm is more likely to mirror their real behaviour, which is expected to be optimized by evolution (given their biological constraints). In addition, they show a lower bound (logarithmic in the size of the colony) on the time needed to complete this task.

In a later work, Zhao, Lynch and Pratt [163] conduct in-depth simulations on a different but similar model. Their objective was to assess whether performance is increased when ants are able to estimate the number of individuals at a candidate site. They show that, indeed, this feature allows significantly faster protocols, which also exhibit a smaller probability of failing to reach consensus – prompting biologists to test this capability empirically.

This series of works demonstrates the collaborative potential between biology and distributed computing: observations help create a simple model, which, in turn, provides insights into animal behaviour through formal analysis and simulations. This is in contrast with traditional "ant colony optimization" algorithms [61], where inspiration from biology is fully directed towards computer science applications, with typically no attempt to understand the underlying processes.

The main challenges of house-hunting – reaching agreement and processing information in noisy environments – are also at the heart of the work presented in Part I of this thesis, and are discussed further in the corresponding Background and Motivation section.

Lévy Flight Foraging Hypothesis

The search patterns of many animals, and even the human eye, resemble *Lévy walks*, a family of random walks in which the step-lengths follow an heavy-tailed distribution. In 1999, inspired by this observation, Viswanathan et al. [156] showed that a specific instance of Lévy walk, the inverse-square Lévy walk (sometimes referred to as the *Cauchy walk*) is particularly effective at solving an idealized search problem. This work gave rise to the "Lévy flight foraging hypothesis", according to which the prevalence of Lévy walks (and especially the Cauchy walk) in nature must be the consequence of their inherent efficiency at finding a target in certain circumstances. As a consequence, empirical studies looked for the Cauchy walk in various scenarios, and found it in many cases.

The result published in 1999 highly relies on some details of the model, and hence its validity was later disproved [110]. However, recent works have provided a new, solid support for the Lévy flight foraging hypothesis. Taking a purely algorithmic perspective, Guinard and Korman identify a two-dimensional setting in which the Cauchy walk is optimal [87]. On the one hand, they assume that the "effective" size of the target is unpredictable; in other words, they require the search to be efficient regardless of the size of the area from which the target can be detected. On the other hand, they assume that the search is intermittent, i.e., that the searcher cannot detect the target while moving. Overall, their result suggests that Cauchy walks are more likely to be found in biological scenarios featuring varying target sizes and weak detection. Other works explore an additional explanation: Lévy walks seem to outperform other search patterns also when multiple searchers are involved [108, 44], although this explanation is not supported by empirical evidence so far.

In this line of work, theoretical statements, whether they come from physics or computer science, happened to precede and guide empirical observations. The fact that they are sometimes able to make valid predictions is a strong argument in favour of the effectiveness of algorithmic tools for understanding natural systems.

1.3 Our Results

In this section, I briefly present the contributions of this thesis.

Bit-Dissemination With Passive Communication

In Chapter 2, we consider the *bit-dissemination* problem introduced in [22], on a fully-connected network of n agents, in the basic $\mathcal{PULL}$ model of communication. Agents have to choose between two possible opinions. Upon *activation*, they observe a few other agents chosen uniformly at random, after what they may update their current opinion. One *source* agent initially holds the *correct* opinion and remains with it throughout the execution. The correct opinion represents the preferable decision – e.g., the best migration route, the safest foraging area, etc. – and the source agent represents an informed individual. Accordingly, the goal of the remaining agents is to quickly adopt the correct opinion. Following the self-stabilization framework, we require that convergence happens regardless of the initial configuration. Motivated by biological scenarios, we focus on the constrained model of passive communication, which assumes that when observing another agent, the only information that can be extracted is the opinion of that agent. In particular, we assume that uninformed individuals are not able to distinguish the source from others.

We investigate two activations patterns: the *parallel* setting, in which all agents act simultaneously, and the *sequential* setting, in which only one agent is activated at a time. In order to explore the role of memory, we compare the capabilities of *memory-less* dynamics with the capabilities of dynamics where agents are allowed to store a small number of bits between two activations.

We propose a simple and elegant algorithm which, using a modest amount of memory, lets agents estimate the current tendency direction of the dynamics and adapt to the emerging trend. We show that it can solve the bit-dissemination problem in time poly-logarithmic in n in the parallel setting, and run simulations suggesting that it is also suitable for the sequential setting.

In addition, we show that memory is necessary to achieve fast convergence in the sequential setting. Specifically, we prove that any memory-less dynamics requires linear time to solve our problem in the sequential setting, even when agents have access to the full configuration of the system. By analysing the convergence time of the *voter* dynamics for our problem, we show that this lower bound is almost tight.

Overall, we uncover two separations results. First, memory-less dynamics are slower in the sequential setting than in the parallel setting by an exponential factor, as long as the sample size is unbounded. Moreover, within the sequential setting, memory-less dynamics are slower than dynamics that have no memory restrictions also by an exponential factor.

Finally, we explain how to generalize some of our results to the case in which an arbitrary (fixed) number of opinions are available to the agents.

Alignment With Noisy Samples of Group Average

In Chapter 3, we study a stochastic alignment problem under the assumption that agents can sense the general tendency of the system. More specifically, we consider n agents, each being associated with a real number. This number, that we refer to as *position*, may represent the orientation of the agent in group navigation contexts, or a notion of time in a clock synchronization context. We define the *center of mass* as the average position of the group, and the *stretch* of an agent as its distance from the center of mass. The goal of the agents is to be as close as possible to the center of mass, i.e., to minimize their stretch, at all times. To this aim, in each round, they receive a noisy measurement of their stretch, and can update their position based on this measurement. However, their position is perturbed by random drift at the end of each round. We assume that both measurement noise and drift are Gaussian.

We prove that a distributed weighted-average algorithm optimally minimizes the deviation of each agent from the center of mass, and for every round. Our main result is that its optimality holds even in the *centralized* setting, where a master agent can gather all the agents' measurements and instruct a move to each one. In other words, we show that even when agents are able to fully communicate their measurements with each-others, they are not able to perform better than when no communication is allowed. We find this result surprising since the set of measurements obtained by all agents contains strictly more information about the stretch of Agent i, than the information contained in the measurements obtained by Agent i alone. Although this information is relevant for Agent i, it is not processed by it when running a weighted-average algorithm. In addition to the main proofs, we provide informal explanations and side results in order to build a full understanding of the problem.

Finally, we also analyse the drift of the center of mass. We show that the centralized setting allows to achieve a slightly smaller drift, highlighting the role of communication in maintaining a stable group position.

Food Availability can Deteriorate Fitness

In Chapter 4, we conduct a game-theoretic analysis of a basic producer-scrounger model, in which animals must choose between intensive food searching as producers or moderate searching while relying on group members as scroungers. In the well-known original version of the producer-scrounger model [153], the population is predicted to reach a mixed equilibrium, where the exact proportion of producers and scroungers depends on the parameters. We show that, under certain circumstances, increasing food availability can modify this proportion to such an extent that it paradoxically leads to a reduction in animals' food consumption, compared to scenarios with limited food availability. Using numerical computations, we illustrate and quantify this phenomenon over a broad range of parameters.

We find a similar phenomenon in another model capturing free-riding dynamics among workers in a company. We demonstrate that, under certain reward mechanisms, enhancing workers' production capacities can inadvertently trigger a surge in free-riding behaviour, leading to both diminished group productivity and reduced individual payoffs.

In both games, the occurrence of this counter-intuitive phenomenon is extremely sensitive to the details of the setting. Aiming to better understand the circumstances under which it can arise, we finally identify one necessary condition for it to happen.

Our findings are reminiscent of the Braess' paradox in transportation networks, albeit in the context of productive groups, and although our models do not fit the framework of congestion games. Overall, we give valuable insights into the complex dynamics at play in producer-scrounger games, underscoring the intricate relationship between individual and group performances, as well as the detrimental impact of free-riding behaviour. Moreover, we highlight the nuanced consequences of contextual factors in understanding and predicting the impact of increased (or decreased) resource availability on both individual and collective outcomes.

On the Role of Hypocrisy

In Chapter 5, we study the emergence of cooperation in large spatial public goods games. For such games, theoretical models typically predict a system collapse in a way that is reminiscent of the "tragedy-of-the-commons" metaphor [133], unless severe social-pressure is employed against *defectors* (or, alternatively, significant rewards are granted to *cooperators*). Drawing on a dynamic network model, we demonstrate how cooperation can emerge when the social-pressure is mild. This is achieved with the aid of an additional behaviour called *hypocrisy*, which appears to be cooperative from the external observer's perspective by pretending to participate, but in fact hardly contributes to the social-welfare.

More precisely, we consider a set of n players organized over a fixed arbitrary network, with the edges representing reciprocal links between players. The utility of a player depends on its own strategy and the strategy of its neighbours, and combines an *energetic cost* associated with cooperation, with a *social-pressure* cost induced by the neighbours. Specifically, we assume that seemingly cooperative individuals – cooperators and hypocritical players – induce a modest social-pressure over both defectors and hypocritical players. The extent of the social-pressure given and received may differ from behaviour to behaviour, and is left as a parameter of the model.

We show that the emergence of cooperation highly depends on how much social-pressure is applied against hypocritical players. Perhaps surprisingly, we show that when it is too high, the system remains locked in a degenerate configuration. Conversely, setting it to be at some intermediate range below the one employed against defectors allows a system composed almost exclusively of defectors to quickly transform into a fully cooperative one.

Overall, our results suggest that hypocrisy may play a social role in escaping the tragedy-of-the-commons.

Part I

Cooperative Information Dissemination in Stochastic Environments

Background and Motivation

Multi-agent systems, from computer networks and mobile sensor systems to neural networks and animal groups, have to address two interconnected challenges: achieving consensus among the agents, and aggregating local information about their environment. On the one hand, reaching agreement, or approximate agreement, is often a necessary condition for collective decision-making. On the other hand, the quality of the decisions and the performance of the group depend heavily on how efficiently local information is processed.

In the natural world, a fascinating illustration occurs during *cooperative transport* in ants, in which a group of ants join forces to carry a large or heavy food load [74, 75, 116]. Consensus on the direction of forces applied by the ants leads to faster movement of the carried load, since less forces are cancelling each other. In addition, information about the location of the nest and the preferable routes to bypass obstacles – which may be distributed unevenly among the ants – must be processed to enable the group to navigate in the right direction [15]. Another beautiful example is collective motion, by, e.g., flocking birds or schooling fishes [154]. Here, individuals need to align their directions of movement, for the sake of group cohesion; at the same time, each group member might have its own preferred direction, depending on specific knowledge which might concern a better migration route [113, 97], or the location of a food source [48]. For yet another example, see house-hunting in ants, discussed in Section 1.2.

These tasks become particularly challenging when the system is prone to faults, or when the interactions between the agents are noisy and constrained. In particular, in many biological ensembles such as ants, birds, bats, etc., the internal computational abilities of individuals are impressively diverse, whereas the communication capacity is highly limited [134, 68]. Communication limitations also arise in a variety of other systems, including chemical reaction networks [40], and mobile sensor networks [160]. An extreme situation, often referred to as *passive communication* [159], is when information can be gained only by observing the behaviour of other animals, which, in some cases, may not even intend to communicate [51]. In other words, passive communications are limited to "behavioural cues", such as the position in space, direction, speed, etc., of other group members. While they are hardly sufficient to distinguish directly which group members have pertinent information [49], they can still improve fitness greatly when used properly [54].

From a theoretical perspective, these tasks are often explored through "consensus problems" in which all agents are required to converge on the same opinion, or approximately the same opinion. The unreliable aspect of communications is modelled by assuming random interactions between the agents. The resulting framework is referred to as *opinion dynamics*. Amongst others, this framework include the *gossip* model[1], where in each round, agents get access to the opinion of one or more other random agents, and can then update their own opinion by employing a simple rule [100, 46]. Perhaps the most famous opinion dynamics is the *voter* dynamics [112], introduced to study conflicts between species in biology and in interacting particle systems [45]. Other notorious update rules known to rapidly achieve consensus include the majority rule and variants of it [19, 121], originated from the study of consensus processes in spin systems [107], such as the *2-choices* and *3-majority* dynamics [60, 76]. Computer science typically focuses on analysing the convergence time of these processes; see [18] for a review. Some works also explore whether opinion formation leads to stable consensus or clustering equilibria [126]. In parallel to the gossip model, the *population protocols* model considers agents with limited memory interacting randomly by pairs, and has attracted significant attention

[1] The gossip model is also known as *rumour spreading*, $\mathcal{PULL}$ or $\mathcal{PUSH}$, or *epidemic* protocols.

in recent years [13, 10, 4]. It comes with its own well-known consensus protocols, such as the *undecided-state* strategy [11].

By now, we understand the computational power of such systems rather well, as long as they are restricted to non-faulty scenarios. When investigating faults, a typical assumption is that the population contains a small numbers of *byzantine* agents, and protocols are required to converge regardless of their behaviour [129, 3]. Some other lines of research investigate the case that messages may be perturbed by noise [53, 67, 23]. Finally, another fault-tolerance framework is that of *self-stabilization* [59], where the system must eventually reach its goal regardless of its initial configuration. In principle, a self-stabilizing system is able to effectively recover from faults of varying duration, type, and severity, as long as the faults occur only temporarily.

In Part I of this thesis, we wish to take the following approach. (1) We start by considering a distributed problem inspired by a given biological scenario, (2) we try to find the dynamics which are the most efficient at solving the problem at hand. Sometimes, this might involve designing new protocols, tailored specifically for our problem; and (3) we use lower-bound techniques to pin down the minimal resources (communication, memory, etc.) required to solve the problem efficiently.

The goal is to use our understanding of the limits of computation – a traditional focus in computer science – in order to identify which specific algorithms are more likely to be employed by biological systems. Specifically, we hope that these real algorithms might be related to the ones that we have found in step (2). Through step (3), we can eliminate settings where the lower bounds do not align with the observed performance, thereby reducing the likelihood of considering unrealistic models. Moreover, this step aids in gaining a better understanding of the scenario's critical parameters, regarding the environment or individuals capabilities. Overall, we hope to characterize natural phenomena that might be harder to uncover using more conventional approaches, e.g., simulation-based or differential equations techniques.

Our approach turns out to deviate from the typical assumptions made in distributed computing. There, consensus is traditionally studied on a fixed network, with byzantine agents and without noise in communication [129, 3]. In contrast, biological ensembles often lack structure and are stochastic in nature. Communications, and sometimes even internal computations, can be expected to be highly noisy; whereas adversarial faults are less likely to appear.

The limitations of our approach, particularly in step (2), is that we are restricted to simple dynamics, that must be analytically tractable. Although a high degree of abstraction may increase the generality of our results and the chances that they find relevance, they may end up too "clean" to perfectly capture any realistic setting. Nonetheless, what follows consists in a detailed description of our attempts. Chapter 2 considers an information spread problem, where the goal is to converge to the preference of a few knowledgeable individuals. Chapter 3 considers an alignment problem, inspired by collective motion.

Chapter 2

Bit-Dissemination with Passive Communication

Chapter Contents

This chapter is based on

[105] Amos Korman and R.V. Early adapting to trends: Self-stabilizing information spread using passive communication. In *Proceedings of the 2022 ACM Symposium on Principles of Distributed Computing*, PODC'22, pages 235–245, New York, NY, USA, July 2022.

and

[17] Luca Becchetti, Andrea Clementi, Amos Korman, Francesco Pasquale, Luca Trevisan, and R.V. On the role of memory in robust opinion dynamics, *arXiv preprint arXiv:2302.08600*, February 2023.

The latter is accepted for publication in IJCAI 2023.

2.1 Introduction

In this chapter, we focus on the problem of information spread, in which few individuals have pertinent information about the environment, and other agents wish to learn this information.

Animals by the pond. Consider the following scenario that serves as an inspiration for our model. A group of n animals is scattered around a pond to drink water from it. Assume that one side of the pond, either the northern or the southern side, is preferable (e.g., because the risk of having predators there is reduced). However, the preferable side is known to a few individuals only. These knowledgeable animals will therefore spend most of their time in the preferable side of the pond. The remaining group members would like to exploit the knowledge held by the knowledgeable animals, but they are unable to distinguish them from others. Instead, what they can do, is to scan the area in order to roughly estimate how many animals are on each side of the pond, and, if they wish, move from side to side. We consider the following questions:

> *Can the group of non-knowledgeable animals manage to locate themselves on the preferable side relatively fast, despite being spread initially in an arbitrary way, and lacking global organization?*

The scenario above illustrates the notion of passive communication. At any given time, an animal must decide on which side of the pond to stay. This choice is visible by others, and is in fact the only information that it can reveal. Moreover, it cannot avoid revealing it: in particular, even the knowledgeable animals, who do not necessarily wish to communicate intentionally, cannot avoid displaying their optimal decision. Provided that these animals do not actively try to harm others, they would simply remain on the preferable side, and promote this choice passively. Uninformed animals, on the other hand, have a clear incentive to cooperate. They could, in principle, manipulate their decisions in order to collectively process the information hold by the knowledgeable animals. However, such manipulation has to be limited in time, since they need to converge towards the desirable side of the pond as fast as possible. Ultimately, our goal is to identify the minimal computational resources (memory, communications, etc.) that are necessary for information to be disseminated efficiently using passive communication.

The bit-dissemination problem. In order to capture the challenges posed by the aforementioned scenario, we consider the self-stabilizing bit-dissemination problem, introduced in [22] by Boczkowski, Korman, and Natale. The problem consists in a fully-connected network of n agents, each of which holds an *opinion* in $\{0,1\}$. One of these opinions is called *correct*. The population contains one *source agent* which knows which opinion is correct[1], and remains with it throughout the execution. The goal of non-source agents is to converge on the correct opinion as fast as possible. We adopt the basic $\mathcal{PULL}$ model of communication. Following the passive communication assumption, we assume that the only information that can be obtained when pulling an agent is its opinion bit. In particular, the source is undistinguishable from other agents. Finally, we require the opinion dynamics to be self-stabilizing, i.e., that it converges from any initial configuration. A formal and complete description of the problem will be given in Section 2.1.1.

On the difficulties resulting from using passive communication. To illustrate the difficulty of self-stabilizing information spread under passive communication, let us consider a more generalized problem than bit-dissemination called *majority bit-dissemination*. In this problem, the population contains $k \geq 1$ source agents which may not necessarily agree on which opinion is correct. Specifically, in addition to its opinion, each source-agent stores a *preference* $\in \{0,1\}$. Let k_i be the number of source agents whose preference is i. Assume that sufficiently more source agents share a preference i over $1-i$ (e.g., at least twice as many), and call i the *correct bit*. Then, the problem requires that w.h.p., all agents (including the sources that might have the opposite preference) should converge their opinions on the correct bit in poly-logarithmic time, and remain with that opinion for polynomial

[1] All our results can be extended to a constant number of source agents.

time[2]. The authors of [22] showed that the self-stabilizing majority bit-dissemination problem can be solved in logarithmic time, using messages of size 3 bits, and the authors of [16] showed how to reduce the message size to 1. However, as illustrated on Figure 2.1 below, this problem could not be solved in poly-logarithmic time under the model of passive communication, even if the sample size was arbitrarily large (i.e., all agents are being observed in each round)!

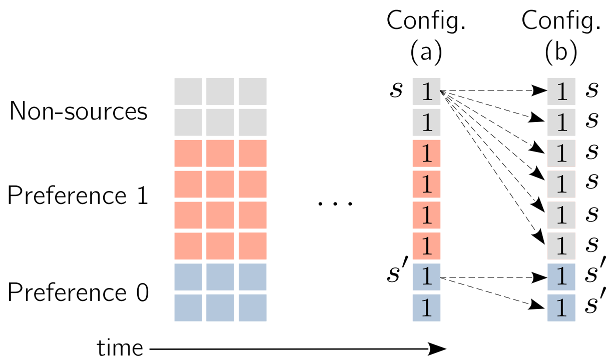


Figure 2.1: **Informal proof that majority bit-dissemination cannot be solved with passive communications.** Assume by contradiction that there exists a self-stabilizing algorithm that solves the majority bit-dissemination problem using passive communication. Let us run this algorithm on a scenario with $k_1 = n/2$ and $k_0 = n/4$. Since $k_1 \gg k_0$, after a poly-logarithmic time, w.h.p., the system reaches a configuration (a) where all agents have opinion 1. Let s be the internal state of a non-source agent in (a), and s' the state of source agent with preference 0 in (a). By assumption, w.h.p., all agents in configuration (a) remain with opinion 1 for a polynomially long period, including the agents with state s and s'. Now, consider a configuration (b), in which $3n/4$ agents have state s, and $n/4$ agents have state s'. Because of the passive communications, and since all opinions are initially equal to 1, we can prove using a union bound argument that w.h.p., all agents remain with opinion 1 for polynomial time also in configuration (b). However, there the agents should converge on opinion 0 in poly-logarithmic time, since $k_1 = 0$ and $k_0 = n/4$ in (b) – which concludes the proof.

Note that this impossibility result does not preclude the possibility of solving the self-stabilizing bit dissemination problem in the passive communication model, which does not involve a conflict between sources.

2.1.1 Preliminaries

Consider $n \in \mathbb{N}$, a set of agents indexed by $I = \{1, \ldots, n\}$, and a binary space of opinions $\mathcal{Y} = \{0, 1\}$. The indices are used for analysis purposes, and it should be clear that an agent is not aware of its index. We refer to Agent 1 as the *source*. A *protocol* for the bit-Dissemination problem in the passive communication model consists in an *internal state space* Σ, an integer ℓ, and a random *transition function*

$$f : (\mathcal{Y} \times \Sigma) \times \mathcal{Y}^{\ell} \mapsto \mathcal{Y} \times \Sigma.$$

Specifically, f is a random map specifying how an agent i with opinion $Y^{(i)} \in \mathcal{Y}$ and internal state $\sigma^{(i)} \in \Sigma$, which sees some sample $S \in \mathcal{Y}^{\ell}$, computes its new opinion and internal state. The same transition function f is executed by all non-source agents. In contrast, the source agent remains fixed throughout the execution and does not update its state.

For a protocol $P = (\Sigma, \ell, f)$, let $\mathcal{C}(P) = (\mathcal{Y} \times \Sigma)^I$ denote the set of *configurations* associated with P. A configuration $C \in \mathcal{C}(P)$ specifies the opinion bit $Y^{(i)} \in \mathcal{Y}$ and the internal state $\sigma^{(i)} \in \Sigma$ of each agent i. For the special case of the source, i.e., $i = 1$, the opinion bit $Y^{(1)}$ is called *correct*, and the internal state $\sigma^{(1)}$ is insignificant.

Following the $\mathcal{PULL}$ model of communication, an *opinion sample* on a configuration C is a random vector $\mathrm{Sample}(C) \in \mathcal{Y}^{\ell}$, whose elements correspond to the opinion bits (in C) of ℓ agents drawn

[2]In fact, the case $k = 1$ consists of a slightly different version than the bit-dissemination problem. Indeed, in the latter, the opinion of the source must always be equal to its preference. In addition, the configuration must remain correct forever, instead of a polynomial time.

independently and uniformly at random with replacement. When *activated* within a configuration C, an agent i receives a new (random) state (Y', σ') obtained from C as

$$(Y', \sigma') = f\left(C^{(i)}, \mathrm{Sample}(C)\right).$$

Accordingly, an *execution* of a protocol $P = (\Sigma, \ell, f)$ on an initial configuration $C \in \mathcal{C}(P)$ is a random sequence of configurations $(C_t)_{t \in \mathbb{N}}$, such that $C_0 = C$, defined inductively as follows.

- In a *parallel* execution, C_{t+1} is obtained from C_t by activating *all* non-source agents simultaneously.
- In a *sequential* execution, C_{t+1} is obtained from C_t by activating *only one* non-source agent, drawn uniformly at random.

For a protocol $P = (\Sigma, \ell, f)$, we say that P uses $\log|\Sigma|$ bits of *memory*. We say that P is *memory-less* if $|\Sigma| = 1$.

Definition 2.1. The convergence time of an execution of P w.r.t. an initial configuration C for the *bit-dissemination problem* is a random variable corresponding to the first round for which all agents have the same opinion and remain with it forever. Since the source always hold the correct opinion, it corresponds to the first round from which all agents have the correct opinion. Formally, the convergence time is defined as

$$\inf\left\{T \in \mathbb{N} \mid \forall t \geq T, \forall i \geq 2, Y_t^{(i)} = Y^{(1)}\right\}.$$

Note that we do not require agents to irrevocably commit on their final opinion, but rather that they eventually converge on the correct opinion without necessarily being aware that convergence has happened. For any $T \in \mathbb{N}$, we say that a protocol P solves the bit-dissemination problem in time T if for any initial configuration $C \in \mathcal{C}(P)$, the convergence time of an execution of P on C is less than T with high probability.

Following [52], we use the term *parallel round* to denote one round of a parallel execution, and n rounds of a sequential execution (so that it always amounts to $\Theta(n)$ activations). This will make it easier to compare the convergence time between the settings.

Finally, for a configuration C_t, we write X_t (resp. x_t) the number (resp. proportion) of agents with opinion 1 in C_t (including the source), i.e.,

$$X_t = |\{i \in I \mid Y_t^{(i)} = 1\}|, \quad x_t = \frac{X_t}{n}. \tag{2.1}$$

The voter dynamics

As an illustration, we briefly describe the *voter dynamics*, a popular dynamics in which the activated agent simply adopts the opinion of another random agent. Formally, the internal state space $\Sigma = \{\sigma\}$ is a singleton. The sample size ℓ is equal to 1. The transition function f, given an agent i with opinion Y_t, state $\sigma_t = \sigma$, and sampling opinion S, would simply output S. In this chapter, we will describe transition functions by giving their pseudocode, such as Algorithm 1 below. For memory-less dynamics, we will omit the internal state σ_t.

Algorithm 1: The voter dynamics

Input: Current opinion $Y_t \in \mathcal{Y}$, opinion sample $S \in \mathcal{Y}$

$Y_{t+1} \leftarrow S$;

Output: Y_{t+1}

2.1.2 Our Results

The reader is invited to refer to Table 2.1 at the end of this section for an overview of the results.

Fast dissemination with memory. We propose a simple algorithm, called *Follow the Trend (FtT)*, that efficiently solves the self-stabilizing bit-dissemination problem in the passive communication model, using a moderate amount of memory. The algorithm has a natural appeal as it is based on letting agents estimate the current tendency direction of the dynamics, and then adapt to the emerging trend. Informally, each non-source agent counts the number of agents with opinion 1 it observes in the current round and compares it to the number observed in the previous round. If more 1's are observed now, then the agent adopts the opinion 1, and similarly, if more 0's are observed now, then it adopts the opinion 0. If the same number of 1's is observed in both rounds, then the agent does not change its opinion. Formally, our algorithm is defined as Algorithm 2. Intuitively, on the global level, this behaviour creates a persistent movement of the average opinion of the non-source agents towards either 0 or 1, which "bounces" back when hitting the wrong opinion.

Algorithm 2: Follow the Trend (core idea) [3]

1 **Input**: Current opinion $Y_t \in \mathcal{Y}$ and internal state $\sigma_t \in \{0, \ldots, \ell\}$, opinion sample $S \in \mathcal{Y}^\ell$

2 $\sigma_{t+1} \leftarrow$ number of 1-opinions in S ;

3 **if** $\sigma_{t+1} > \sigma_t$ **then** $Y_{t+1} \leftarrow 1$;

4 **else if** $\sigma_{t+1} < \sigma_t$ **then** $Y_{t+1} \leftarrow 0$;

5 **else** $Y_{t+1} \leftarrow Y_t$;

6 **Output**: Y_{t+1}, σ_{t+1}

For technical reasons, we never analyze Algorithm 2 directly, but some slightly modified versions instead. First, we show that one such version can solve the bit-dissemination problem efficiently in the parallel setting, with a relatively small sample size.

> **Theorem 2.2.** *There exists a protocol based on Algorithm 2 that solves the bit-dissemination problem in the parallel setting in $O(\log^{5/2} n)$ parallel rounds with high probability, while relying on $\ell = \Theta(\log n)$ samples in each round and using $\Theta(\log \ell)$ bits of memory.*

This dynamics, and its analysis, are discussed in detail in Section 2.2. In addition, we observe in simulations that the Follow The Trend approach can also be used in the sequential setting. We identify a protocol based on Algorithm 2 that empirically appears to solve the bit-dissemination problem with an estimated convergence time compatible with $\log^{O(1)} n$ parallel rounds. We do not, however, provide any mathematical analysis to support this claim. The protocol and its simulations are discussed in Section 2.3.

Memory-less dynamics. We prove that every memory-less dynamics in the sequential setting needs linearly many parallel rounds to solve the bit-dissemination problem. A bit surprisingly, our analysis holds for arbitrarily large sample size ℓ – or, equivalently, it holds even when the activated agent has access to the current opinions of *all* agents in the system.

> **Theorem 2.3.** *The expected convergence time of any memory-less dynamics in the sequential setting is at least $\Omega(n)$ parallel rounds, even when each sample consists of all opinions in the system.*

For comparison, in standard "symmetric" consensus problems, in which agents are required to achieve consensus on *any* of the opinions independently of any source, convergence is achieved in $O(\log n)$ parallel rounds with high probability by a large class of majority-like dynamics, and using samples of constant size [139]. We thus have an exponential gap between the two settings.

[3] Note that, although we used time indices for clarity, the protocol does not require the agents to know t.

We further show that our lower bound is essentially tight. Interestingly, we prove that the standard voter model achieves almost optimal performance, despite using samples of size $\ell = 1$.

Theorem 2.4. *The (memory-less) voter dynamics solves the bit-dissemination problem in the sequential setting in $O(n \log n)$ parallel rounds in expectation, while relying on $\ell = 1$ sample in each round.*

This result, together with Theorem 2.3, suggests that sample size cannot be a key ingredient in achieving fast consensus to the correct opinion when it comes to sequential, memory-less dynamics. Their proofs are deferred to Section 2.4. In contrast, we prove in Section 2.4.4 that memory is not required in the parallel setting, as long as the sample size is sufficiently large.

Observation 2.5. *There is a memory-less protocol that solves the bit-dissemination problem in the parallel setting in $O(1)$ parallel rounds, when each sample consists of all opinions in the system.*

Many opinions. Finally, in Section 2.5, we show how to generalize some of our results in the case of more than 2 opinions, i.e., when $\mathcal{Y} = \{0, \ldots, k-1\}$ for some integer k. We prove that Theorems 2.3 and 2.4 can be generalized easily for any fixed integer k. In addition, we describe a version of Algorithm 2 that can deal with any any fixed number of opinions, yielding a generalization of Theorem 2.2. However, as we will explain in Section 2.5, this last generalization is quite weak, because it actively exploits the unrealistic fact that the ordering of the opinions $(0 < 1 < \ldots < k-1)$ is a common knowledge.

	parallel	sequential
ℓ arbitrarily large memory-less	$O(1)$ Observation 2.5	$\Omega(n), O(n \log n)$ Theorems 2.3 and 2.4
$\ell = \Theta(\log n)$ $\Theta(\log \ell)$ bits of memory	$O(\text{polylog}n)$ Theorem 2.2	$O(\text{polylog}n)$ Section 2.3 (empirical)

Table 2.1: **Summary of the results.** The cells depict bounds on the convergence time in various settings, expressed in parallel rounds. Two separation results appear. (1) Regarding memory-less dynamics with unbounded sample size, there is an exponential gap between the convergence times in the parallel setting, and in the sequential setting. (2) In the sequential setting, there is an exponential gap between the convergence times of memory-less dynamics, and dynamics that are allowed to use a modest amount of memory.

2.1.3 Related Works

The problem we consider spans a number of areas of potential interest across several communities. The corresponding literature is vast and providing an exhaustive review is infeasible here. In the following paragraphs, we discuss previous contributions that most closely relate to the present work.

Opinion dynamics. In most of the opinion dynamics literature, consensus must be reached either on an arbitrary value, or on the majority (or plurality) opinion, as evident in the initial configuration. However, in many natural settings, as well as in the problem we consider here, the group must converge on a particular consensus value that is a function of the environment. Moreover, agents have different levels of knowledge about the desired value, and the system must utilize the information held by the more knowledgeable individuals [145, 15, 103, 131]. In this direction, we note an increasing interest in the recent past for biased variants of opinion dynamics [8, 19, 50, 109], where the "bias" can be interpreted in terms of preference for a particular opinion. In general, the focus of this line of work is different from ours, mostly being on the sometimes complex interplay between bias and convergence to an equilibrium. In contrast, our focus is on how quickly dynamics can converge to the (unknown) correct opinion. For reference, it is easy to verify that "majority like" rules fail, in general, to complete the bit-dissemination task.

Information spread. Propagating information from one or more sources to the rest of the population has been the focus of a myriad of works in distributed computing. This dissemination problem has been studied under various models taking different names, including *rumor spreading*, *information spreading*, *gossip*, *broadcast*, and others, see e.g., [78, 36, 56, 42, 79]. A classical algorithm spreads the opinion of the source to all others in a logarithmic number of rounds, by letting each uninformed agent copy the opinion of an informed agent whenever seeing one for the first time [100]. Unfortunately, this elegant algorithm does not suit all realistic scenarios, since its soundness crucially relies on the absence of misleading information. For example, it fails in the self-stabilization setting, in which non-source agents may be initialized to "think" that they have already been informed by the correct opinion, while they actually hold the wrong opinion.

Consensus in the presence of zealot agents. A large body of work considers opinion dynamics in the presence of zealot agents, i.e., agents (generally holding heterogeneous opinions) that never depart from their initial opinion [53, 115, 137, 161] and may try to influence the rest of the agent population. In this case, the process resulting from a certain dynamics can result in equilibria characterized by multiple opinions. To the best of our knowledge, the main focus there is different from ours, mostly concerning the impact of the number of zealots, their positions in the network and the topology of the network itself on such equilibria [115, 137, 161, 72].

Previous works on the self-stabilizing bit-dissemination problem. The bit-dissemination problem was introduced in [22], with the aim of minimizing the message size. As mentioned therein, if all agents share the same notion of global time, then convergence can be achieved in $O(\log n)$ time w.h.p., even under passive communication. The idea is that agents divide the time horizon into phases of length $T = 4 \log n$, and that each phase is further subdivided into 2 subphases of length $2 \log n$ each. In the first subphase of each phase, if a non-source agent observes an opinion 0, then it copies it as its new opinion, but if it sees 1 it ignores it. In the second subphase, it does the opposite, namely, it adopts the output bit 1 if and only if it sees an opinion 1. Now, consider the first phase. If the source supports opinion 0 then at the end of the first subphase, every output bit would be 0 w.h.p., and the configuration would remain that way forever. Otherwise, if the source supports 1, then at the end of the second subphase all output bits would be 1 w.h.p., and remain 1 forever.

The aforementioned protocol indicates that the self-stabilizing bit-dissemination problem could be solved efficiently by running a self-stabilizing clock synchronization protocol in parallel. This parallel execution amounts to adding one bit to the message size of the clock synchronization protocol. The main technical contribution of [22], as well as the focus of subsequent work in [16], was solving the self-stabilizing clock-synchronization using as few as possible bits per message. In fact, the authors in [16] managed to do so using 1-bit messages. This construction thus implies a solution to the self-stabilizing bit-dissemination problem using 2 bits per message. A recursive procedure, similar to the one established in [22], then allowed to further compress the 2 bits into 1-bit messages. Importantly, however, the 1-bit message revealed by an agent is different from its opinion bit: instead, the opinion is considered as an internal variable, and hence, do not fit the framework of passive communication.

At first glance, to adhere to the passive communication model, one may suggest that agents simply choose their opinion to be this 1-bit message used in [16], just for the purpose of communication, until a consensus is reached, and then switch the opinion to be the correct bit, once it is identified. There are, however, two difficulties to consider regarding this approach. First, in our setting, the source agent does not change its opinion (which, in the case of [16], may prevent the protocol from reaching a consensus at all). Second, even assuming that the protocol functions properly despite the source having a stable opinion, it is not clear how to transition from the "communication" phase (where agents use their opinion to operate the protocol, e.g., for synchronizing clocks) to the "consensus" phase (where all opinions must be equal to the correct bit at every round). For instance, the first agents to make the transition may disrupt other agents still in the first phase.

Moreover, these works use complex recursive algorithms with refined clocks that are unlikely to be used by biological entities. Instead, we are interested in identifying algorithms that have a more natural appeal.

2.2 Follow the Trend in the Parallel Setting

The goal of this section is to prove Theorem 2.2. As it turns out, one feature of Algorithm 2 will make the analysis difficult: that is, that Y_{t+1} depends not only on Y_t and on the configuration in round $t-1$ and t, but also on the opinion sample S received in round $t-1$. This is because S is used to compute both Y_t and Y_{t+1}: for example, if S happens to contain more 1's, then Y_t has a higher chance of being 1, and Y_{t+1} has a higher chance of being 0. In order to solve this dependence issue, we introduce a modified version of the protocol (Algorithm 3). The idea is to partition the opinion sample received at round t into 2 subsets of equal size. One subset will be used to compare with a subset of round $t-1$, and the other subset will be used to compare with a subset of round $t+1$. Note that this implies that the sample size of the algorithm becomes 2ℓ rather than ℓ. However, since we are interested in the case $\ell = O(\log n)$, this does not really matter. This modified protocol is the one we will actually consider in the rest of the section.

Algorithm 3: Follow the Trend without dependencies

Input: Current opinion $Y_t \in \mathcal{Y}$ and internal state $\sigma_t \in \{0, \dots, \ell\}$, opinion sample $S \in \mathcal{Y}^\ell$

Partition Sample into two sets $\mathrm{Sample}', \mathrm{Sample}''$ of equal size uniformly at random ;

$\sigma_{t+1} \leftarrow$ number of 1-opinions in S' ;

if $\sigma_{t+1} > \sigma_t$ **then** $Y_{t+1} \leftarrow 1$;

else if $\sigma_{t+1} < \sigma_t$ **then** $Y_{t+1} \leftarrow 0$;

else $Y_{t+1} \leftarrow Y_t$;

$\sigma_{t+1} \leftarrow$ number of 1-opinions in S'' ;

Output: Y_{t+1}, σ_{t+1}

2.2.1 General Overview of the Proof

In this section, we briefly sketch the proof of Theorem 2.2, for which we consider Algorithm 3.

The $O(\log \ell)$ bits upper bound on the memory complexity clearly follows from the fact that the internal memory state σ_t is only used to count the number of 1's in a sample (of size ℓ).

Since the protocol is symmetric with respect to the opinion of the source, we may assume without loss of generality that the source has opinion 1. Our goal would therefore be to show that the FtT algorithm converges to 1 fast, w.h.p., regardless of the initial configuration of non-source agents. Note that in order to achieve running time of $O(T)$ w.h.p guarantee, is it sufficient to show that the algorithm stabilizes in T rounds with probability at least $1 - 1/n^\epsilon$, for some $\epsilon > 0$. Indeed, because of the self-stabilizing property of the algorithm, the probability that the algorithm does not stabilize within $2T/\epsilon$ rounds is at most $(1/n^\epsilon)^{2/\epsilon} = 1/n^2$.

For the sake of analysis, let x_t denote the fraction of agents with opinion 1 at round t among the whole population of agents (including the source), when Algorithm 3 is used. We shall extensively use the two dimensional grid $\mathcal{G} := \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}^2$. When analyzing what happens at round $t+2$, the x-axis of $\mathcal{G}$ would represent x_t, and the y-axis would represent x_{t+1}.

For every pair X, Y of independent random variables following binomial distributions of parameter (k, p) and (ℓ, q) respectively, and any $* \in \{=, <, >\}$, we will write

$$\mathbb{P}\left(B_k\left(p\right) * B_\ell\left(q\right)\right) := P(X * Y).$$

Observation 2.6. *For any t, conditioning on $(x_t, x_{t+1}) = (\mathbf{x_t}, \mathbf{x_{t+1}})$ and on $Y^{(i)}_{t+1} = \mathbf{Y^{(i)}_{t+1}}$, a non-source agent i has opinion 1 on round $t+2$ w.p.*

$$\mathbb{P}\left(Y^{(i)}_{t+2} = 1 \;\middle|\; \begin{array}{c} x_t = \mathbf{x_t} \\ x_{t+1} = \mathbf{x_{t+1}} \\ Y^{(i)}_{t+1} = \mathbf{Y^{(i)}_{t+1}} \end{array}\right) = \mathbb{P}\left(B_\ell\left(\mathbf{x_{t+1}}\right) > B_\ell\left(\mathbf{x_t}\right)\right) + \mathbb{1}_{\{\mathbf{Y^{(i)}_{t+1}}=1\}} \cdot \mathbb{P}\left(B_\ell\left(\mathbf{x_{t+1}}\right) = B_\ell\left(\mathbf{x_t}\right)\right). \tag{2.2}$$

Moreover, there are independent binary random variables[a] $X_1, \ldots, X_n$ such that x_{t+2} is distributed as $\frac{1}{n}\sum X_i$. Eventually,

$$\mathbb{E}\left(x_{t+2} \;\middle|\; \begin{array}{c} x_t = \mathbf{x_t} \\ x_{t+1} = \mathbf{x_{t+1}} \end{array}\right) = \mathbb{P}\left(B_\ell\left(\mathbf{x_{t+1}}\right) > B_\ell\left(\mathbf{x_t}\right)\right) + \mathbf{x_{t+1}} \cdot \mathbb{P}\left(B_\ell\left(\mathbf{x_{t+1}}\right) = B_\ell\left(\mathbf{x_t}\right)\right) + \frac{1}{n}\left(1 - \mathbb{P}\left(B_\ell\left(\mathbf{x_{t+1}}\right) \geq B_\ell\left(\mathbf{x_t}\right)\right)\right). \tag{2.3}$$

[a] In fact, in general, $\{Y^{(i)}_{t+2}\}, i \in I$ are *not* independent conditioned on (x_t, x_{t+1}). This is not a problem since we mainly care about their sum, but it forces us to introduce variables $X_1, \ldots, X_n$ in order to use classical concentration results on their sum.

The proof of Observation 2.6 is deferred to Section 2.2.2. A consequence of Observation 2.6, is that the execution of the algorithm induces a Markov chain on $\mathcal{G}$. This Markov chain has a unique absorbing state, $(1, 1)$, since we assumed the source to have opinion 1. To prove Theorem 2.2 we therefore only need to bound the time needed to reach $(1, 1)$.

2.2.1.1 Partitioning the grid into domains

Let us fix $0 < \delta < 1/2$ (δ should be though of as a very small quantity), and $\lambda_n = \frac{1}{\log^{1/2+\delta} n}$. We partition $\mathcal{G}$ into domains as follows (see illustration on Figure 2.2a).

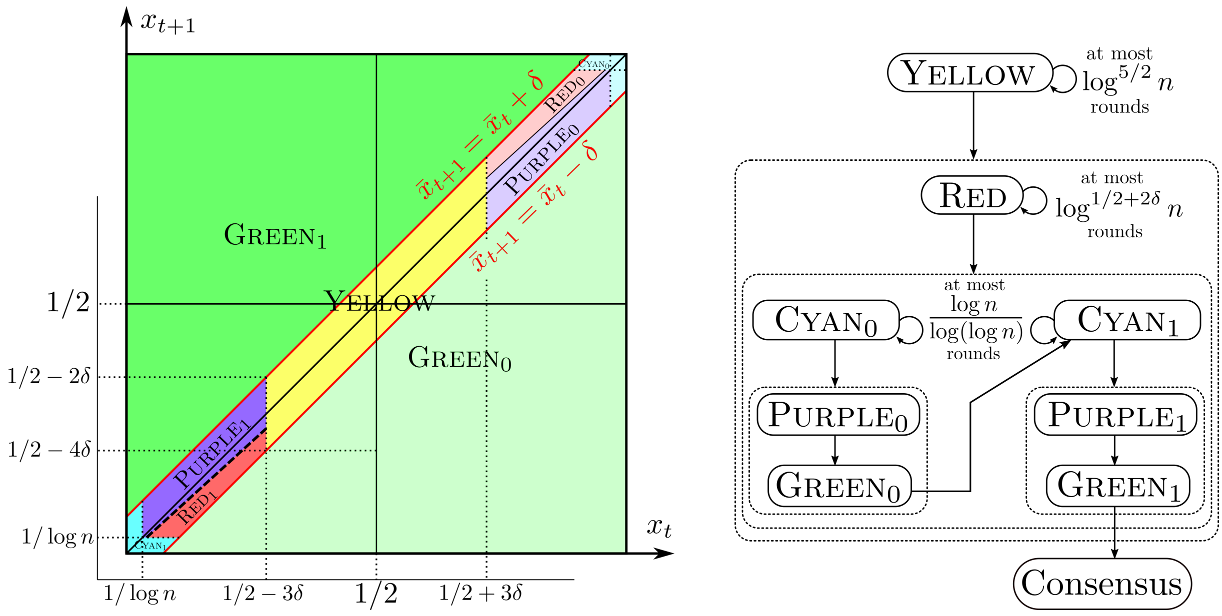
$$\begin{aligned}
\mathsf{GREEN}_1 &= \Big\{(x_t, x_{t+1}) \mid x_{t+1} \geq x_t + \delta\Big\}, \\
\mathsf{PURPLE}_1 &= \Big\{(x_t, x_{t+1}) \mid \frac{1}{\log n} \leq x_t < \frac{1}{2} - 3\delta \text{ and } (1 - \lambda_n) \cdot x_t \leq x_{t+1} < x_t + \delta\Big\}, \\
\mathsf{RED}_1 &= \Big\{(x_t, x_{t+1}) \mid \frac{1}{\log n} \leq x_{t+1} \text{ and } x_t < \frac{1}{2} - 3\delta \text{ and } x_t - \delta \leq x_{t+1} < (1 - \lambda_n) \cdot x_t\Big\}, \\
\mathsf{CYAN}_1 &= \Big\{(x_t, x_{t+1}) \mid 0 \leq \min(x_t, x_{t+1}) < \frac{1}{\log n} \text{ and } x_t - \delta < x_{t+1} < x_t + \delta\Big\}, \\
\mathsf{YELLOW} &= \Big\{(x_t, x_{t+1}) \mid \tfrac{1}{2} - 3\delta \leq x_t < \tfrac{1}{2} + 3\delta,\ \tfrac{1}{2} - 4\delta \leq x_{t+1} \leq \tfrac{1}{2} + 4\delta \text{ and } |x_{t+1} - x_t| < \delta\Big\}.
\end{aligned}$$

Similarly, for the former 4 domains, we define $\mathsf{GREEN}_0, \mathsf{PURPLE}_0, \mathsf{RED}_0$ and CYAN_0 to be their symmetric equivalents (w.r.t the point $(\frac{1}{2}, \frac{1}{2})$), and finally define: $\mathsf{GREEN} = \mathsf{GREEN}_0 \cup \mathsf{GREEN}_1$, $\mathsf{PURPLE} = \mathsf{PURPLE}_0 \cup \mathsf{PURPLE}_1$, $\mathsf{RED} = \mathsf{RED}_0 \cup \mathsf{RED}_1$, and $\mathsf{CYAN} = \mathsf{CYAN}_0 \cup \mathsf{CYAN}_1$. We shall analyze each area separately, conditioning on the Markov chain to be at any point in that area, and focusing on the number of rounds required to escape the area, and the probability that this escape is made to a particular other area. Figure 2.2b represents a sketch of the proof of Theorem 2.2, which may help to navigate the intermediate results.

As it happens, the dynamics starting from a point (x_t, x_{t+1}) highly depends on the difference between x_t and x_{t+1}. Roughly speaking, the larger $|x_{t+1} - x_t|$ is the faster is the convergence. For this reason, we refer to $|x_{t+1} - x_t|$ as the *speed* of the point (x_t, x_{t+1}). (This could also be viewed as the "derivative" of the process at time t.)

2.2.1.2 Analyzing the Markov chain at different domains

Let us give an overview of the intermediate results. We will assume that the sample size is $\ell = c \log n$, where c is a sufficiently large constant. First we consider GREEN, in which the speed of points is



(a) (b)

Figure 2.2: (a) Partitioning the state space into domains. The x-axis (resp., y-axis) represents the proportion of agents with opinion 1 in round t (resp., $t+1$). The thick dashed line at the frontier between PURPLE$_1$ and RED$_1$ is defined by $x_{t+1} = (1-\lambda_n)x_t$. (b) Sketch of the proof of Theorem 2.2. The process stays in a domain for as many rounds as indicated on the corresponding self-loop w.p. at least $1 - 1/n^{\Omega(1)}$, and at most a constant number of rounds when no self-loop is represented. The source is assumed to have opinion 1.

large. In Lemma 2.7 we show that from points in that domain, non-source agents reach a consensus in just one round w.h.p. In particular, if the Markov chain is at some point in GREEN$_1$, then the consensus will be on 1, and we are done. If, on the other hand, the Markov chain is in GREEN$_0$, then the consensus of non-source agents would be on 0. As we show later, in that case the Markov chain would reach CYAN$_1$ in one round w.h.p.

Lemma 2.7 (Green area)**.** *Assume that $\ell = c\log n$, where c is sufficiently large. If $(x_t, x_{t+1}) \in$ GREEN$_1$, then w.h.p., for every non-source agent i, $Y_i^{(t+2)} = 1$. Similarly, if $(x_t, x_{t+1}) \in$ GREEN$_0$, then w.h.p., for every non-source agent i, $Y_i^{(t+2)} = 0$.*

The proof of Lemma 2.7 follows from a simple application of Hœffding's inequality, and is deferred to Section 2.2.3.1. Next, we consider the area PURPLE, and show that the population goes from PURPLE to GREEN in just one round, w.h.p. In PURPLE, the speed is relatively low, and x_t and x_{t+1} are quite far from $1/2$. On the next round, we expect x_{t+2} to be close to $1/2$, thus gaining enough speed in the process to join GREEN. The proof of the following lemma is rather straightforward, and is deferred to Section 2.2.3.2.

Lemma 2.8 (Purple area)**.** *Assume that $\ell = c\log n$, where c is sufficiently large. If $(x_t, x_{t+1}) \in$ PURPLE$_1$, then $(x_{t+1}, x_{t+2}) \in$ GREEN$_1$ w.h.p. Similarly, if $(x_t, x_{t+1}) \in$ PURPLE$_0$, then $(x_{t+1}, x_{t+2}) \in$ GREEN$_0$ w.h.p.*

Next, we bound the time that can be spent in RED, by using the fact that as long as the process is in RED$_1$ (resp., RED$_0$), x_t (resp., $(1-x_t)$) decreases (deterministically) by at least a multiplicative factor of $(1-\lambda_n)$ at each round. After a poly-logarithmic number of rounds, the Markov chain must leave RED and in this case, we can show that it cannot reach YELLOW right away. The proof of the following lemma is again relatively simple, and is deferred to Section 2.2.3.3.

Lemma 2.9 (Red area)**.** *Assume that $\ell = c\log n$, where c is sufficiently large. Consider the case that $(x_{t_0}, x_{t_0+1}) \in$ RED for some round t_0, and let $t_1 = \min\{t \geq t_0, (x_t, x_{t+1}) \notin \text{RED}\}$.*

Then $t_1 < t_0 + \log^{1/2+2\delta} n$*, and* $(x_{t_1}, x_{t_1+1}) \notin \mathsf{YELLOW} \cup \mathsf{RED}$.

Next, we bound the time that can be spent in CYAN_1. (A similar result holds for CYAN_0.) Roughly speaking, this area corresponds to the situation in which, over the last two consecutive rounds, the population is in an almost-consensus over the wrong opinion. In this case, many agents (a constant fraction) see only 0 in their corresponding samples in the latter round. As a consequence, everyone of them who will see at least one opinion 1 in the next round, will adopt opinion 1. We can expect this number to be of order $\ell = O(\log n)$. This means that, as long as the Markov chain is in CYAN_1, the value of x_t would grow by a logarithmic factor in each round. This implies that within $\log(n)/\log(\log n)$ rounds, the Markov chain will leave the CYAN_1 area and go to $\mathsf{GREEN}_1 \cup \mathsf{PURPLE}_1$. Informally, this phenomenon can be viewed as a form of "bouncing" — the population of non-sources reaches an almost consensus on the wrong opinion, and "bounces back", by gradually increasing the fraction of agents with the correct opinion, up to an extent that is sufficient to enter $\mathsf{GREEN}_1 \cup \mathsf{PURPLE}_1$. The proof of the following lemma is given in Section 2.2.3.4.

Lemma 2.10 (Cyan area)**.** *Assume that* $\ell = c \log n$*, where* c *is sufficiently large. Consider the case that* $(x_{t_0}, x_{t_0+1}) \in \mathsf{CYAN}_1$ *for some round* t_0*, and let* $t_1 = \min\{t \geq t_0, (x_t, x_{t+1}) \notin \mathsf{CYAN}_1\}$*. Then with probability at least* $1 - 1/n^{\Omega(1)}$ *we have (1)* $t_1 < t_0 + O(\log(n)/\log(\log n))$*, and (2)* $(x_{t_1}, x_{t_1+1}) \in \mathsf{GREEN}_1 \cup \mathsf{PURPLE}_1$*. Moreover, the same holds symmetrically for* CYAN_0.

Eventually, we consider the central area, namely, YELLOW, where the speed is very low, and bound the time that can be spent there. The proof of the following lemma is more complex than the previous ones, and it appears in Section 2.2.4.

Lemma 2.11 (Yellow area)**.** *Assume that* $\ell = c \log n$*, where* c *is sufficiently large. Consider the case that* $(x_{t_0}, x_{t_0+1}) \in \mathsf{YELLOW}$*. Then, w.h.p.,*

$$\min\{t > t_0 \textit{ s.t. } (x_t, x_{t+1}) \notin \mathsf{YELLOW}\} < t_0 + O(\log^{5/2} n).$$

2.2.1.3 Assembling the lemmas

Given the aforementioned lemmas, we have everything we need to prove our main result.

Proof of Theorem 2.2. Recall that without loss of generality, we assumed the source to have opinion 1. The reader is strongly encouraged to refer to Figure 2.2b to follow the ensuing arguments more easily.

- Let $t_1 = \min\{t \geq 0, (x_t, x_{t+1}) \notin \mathsf{YELLOW}\}$. If $(x_0, x_1) \in \mathsf{YELLOW}$, we apply Lemma 2.11 to get that

$$t_1 < O(\log^{5/2} n) \text{ w.h.p. and } (x_{t_1}, x_{t_1+1}) \in \mathsf{RED} \cup \mathsf{CYAN} \cup \mathsf{PURPLE} \cup \mathsf{GREEN}. \tag{2.4}$$

  Else, $(x_0, x_1) \notin \mathsf{YELLOW}$ so $t_1 = 0$, and Eq. (2.4) also holds.

- Let $t_2 = \min\{t \geq t_1, (x_t, x_{t+1}) \notin \mathsf{RED}\}$. If $(x_{t_1}, x_{t_1+1}) \in \mathsf{RED}$, we apply Lemma 2.9 to get that

$$t_2 < t_1 + \log^{1/2+2\delta} n \text{ w.h.p. and } (x_{t_2}, x_{t_2+1}) \in \mathsf{CYAN} \cup \mathsf{PURPLE} \cup \mathsf{GREEN}. \tag{2.5}$$

  Else, $(x_{t_1}, x_{t_1+1}) \notin \mathsf{RED}$ so $t_1 = t_2$, and by Eq. (2.4), it implies that Eq. (2.5) also holds.

- Let $t_3 = \min\{t \geq t_2, (x_t, x_{t+1}) \notin \mathsf{CYAN}\}$. If $(x_{t_2}, x_{t_2+1}) \in \mathsf{CYAN}$, we apply Lemma 2.10 to get that

$$\begin{cases} t_3 < t_2 + \log(n)/\log(\log n), \text{ and} \\ (x_{t_3}, x_{t_3+1}) \in \mathsf{PURPLE} \cup \mathsf{GREEN} \end{cases} \text{w.p. } 1 - 1/n^{\Omega(1)}. \tag{2.6}$$

  Else, $(x_{t_2}, x_{t_2+1}) \notin \mathsf{CYAN}$ so $t_2 = t_3$, and by Eq. (2.5), it implies that Eq. (2.6) also holds.

- Let $t_4 = \min\{t \geq t_3, (x_t, x_{t+1}) \in \mathsf{GREEN}\}$. By Lemma 2.8, and by Eq. (2.6), we have that $t_4 = t_3$ or $t_4 = t_3 + 1$ w.h.p.

If $(x_{t_4}, x_{t_4+1}) \in \mathsf{GREEN}_1$, then by Lemma 2.7 the consensus is reached on round $t_4 + 1$. Otherwise, if $(x_{t_4}, x_{t_4+1}) \in \mathsf{GREEN}_0$, by Lemma 2.7, we obtain that $x_{t_4+2} = 1/n$ w.h.p. (meaning that all agents have opinion 0 except the source). Therefore, in this case, either $(x_{t_4+1}, x_{t_4+2}) \in \mathsf{GREEN}_0$ or $(x_{t_4+1}, x_{t_4+2}) \in \mathsf{CYAN}_1$ (because for a point (x_t, x_{t+1}) to be in any other area, it must be the case that $x_{t+1} \geq 1/\log(n)$, by definition). In the former case, we apply Lemma 2.7 again to get that $x_{t_4+3} = 1/n$ w.h.p., which implies that $(x_{t_4+2}, x_{t_4+3}) = (1/n, 1/n) \in \mathsf{CYAN}_1$. As we did before, we apply Lemmas 2.7, 2.8 and 2.10 to show that, with probability at least $1 - 1/n^{\Omega(1)}$, the system goes successively to $\mathsf{PURPLE}_1 \cup \mathsf{GREEN}_1$, then to GREEN_1, and eventually reaches the absorbing state $(1, 1)$ in less than $\log(n)/\log(\log n) + 2$ rounds.

Altogether, the convergence time is dominated by t_1, and is hence $O(\log n)^{5/2}$ with probability at least $1 - 1/n^{\epsilon}$, for some $\epsilon > 0$. As mentioned, this implies that for any given $c > 1$, the algorithm reaches consensus in $O(\log n)^{5/2}$ time with probability at least $1 - 1/n^{c}$. This concludes the proof of Theorem 2.2. □

2.2.2 Preliminary Observations

2.2.2.1 Proof of Observation 2.6

Proof of Observation 2.6. Let I be the set of agents (including the source). Let $I_t^1 \subset I$ be the set of all *non-source* agents with opinion 1 at round t. Recall that we condition on $x_t = \mathbf{x_t}$ and $x_{t+1} = \mathbf{x_{t+1}}$ (although we avoid writing this conditioning). In addition, the proof will proceed by conditioning on $I_{t+1}^1 = \mathbf{I_{t+1}^1}$. Since we shall show that the statements are true for every $\mathbf{I_{t+1}^1}$, the lemma will hold without this latter conditioning.

By definition of the protocol, and because it operates under the $\mathcal{PULL}$ model, $\mathrm{Sample}'^{(i)}$ and $\mathrm{Sample}''^{(i)}$ are obtained by sampling ℓ agents uniformly at random in the population (with replacement) and counting how many have opinion 1. Therefore, conditioning on (x_t, x_{t+1}) and I_{t+1}^1,

(i) variables $(\mathrm{Sample}'^{(i)}_{t+1})_{i \in I}$ and $(\mathrm{Sample}''^{(i)}_{t})_{i \in I}$ are mutually independent, thus variables $(Y_{t+2}^{(i)})_{i \in I}$ are mutually independent.

(ii) for every $i \in I$, $\mathrm{Sample}'^{(i)}_{t+1} \sim \mathcal{B}_\ell(x_{t+1})$, and $\mathrm{Sample}''^{(i)}_{t} \sim \mathcal{B}_\ell(x_t)$, so we can write for every non-source agent $i \in I_{t+1}^1$,

$$\mathbb{P}\left(Y_{t+2}^{(i)} = 1\right) = \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \geq B_\ell\left(x_t\right)\right),$$

and for every non-source agent $i \notin I_{t+1}^1$,

$$\mathbb{P}\left(Y_{t+2}^{(i)} = 1\right) = \mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right).$$

This establishes Eq. (2.2). Now, let us define independent binary random variables $(X_j)_{1 \leq j \leq n}$, taking values in $\{0, 1\}$, as follows;

- $X_1 = 1$,
- for every j s.t. $1 < j \leq n \cdot x_{t+1}$, $\mathbb{P}\left(X_j = 1\right) = \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \geq B_\ell\left(x_t\right)\right)$,
- for every j s.t. $n \cdot x_{t+1} < j \leq n$, $\mathbb{P}\left(X_j = 1\right) = \mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right)$.

We assumed the source agent to have opinion 1, so there are $nx_t - 1$ non-source agents with opinion 1 and $n(1 - x_t)$ non-source agents with opinion 0. Therefore, by (i) and (ii) and by construction of the $(X_j)_{1\le j\le n}$, $x_{t+2} = \frac{1}{n}\sum_{i\in I} Y_{t+2}^{(i)}$ is distributed as $\frac{1}{n}\sum_{j=1}^{n} X_j$, which establishes the second statement in Observation 2.6. Computing the expectation (still conditioning on x_t, x_{t+1}) is straightforward and does not depend on I_{t+1}^1:

$$\begin{aligned}
&\mathbf{E}\left(x_{t+2}\right) \\
&= \left(x_{t+1} - \frac{1}{n}\right) \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \ge B_\ell\left(x_t\right)\right) + \left(1 - x_{t+1}\right) \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) + \frac{1}{n} \\
&= x_{t+1} \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \ge B_\ell\left(x_t\right)\right) + \left(1 - x_{t+1}\right) \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) \\
&\qquad\qquad + \frac{1}{n}\left(1 - \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \ge B_\ell\left(x_t\right)\right)\right) \\
&= \mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) + x_{t+1} \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) = B_\ell\left(x_t\right)\right) + \frac{1 - \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \ge B_\ell\left(x_t\right)\right)}{n}.
\end{aligned}$$

This establishes Eq. (2.3), and concludes the proof of Observation 2.6. □

From Observation 2.6, we obtain the following straightforward bounds.

Claim 2.12. *For every non-source agent i,*

$$\mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) \quad \le \quad \mathbb{P}\left(Y_{t+2}^{(i)} = 1\right) \quad \le \quad \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \ge B_\ell\left(x_t\right)\right), \tag{2.7}$$

and

$$\mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) - \frac{1}{n} \quad \le \quad \mathbf{E}\left(x_{t+2}\right) \quad \le \quad \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \ge B_\ell\left(x_t\right)\right) + \frac{1}{n}. \tag{2.8}$$

Because of the source agent having opinion 1, the left hand side in Eq. (2.8) is loose (specifically, $-1/n$ is not necessary). Nevertheless, we will use this equation in the proofs, because it has a symmetric equivalent (w.r.t. to the center of $\mathcal{G}$, $(\frac{1}{2}, \frac{1}{2})$) which will allow our statements about x_{t+2} to hold symmetrically for $1 - x_{t+2}$. Furthermore, Eq. (2.3) in Observation 2.6 implies the following convenient bounds.

Claim 2.13. *For any round t,*

$$\begin{aligned}
&\mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) + x_{t+1} \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) = B_\ell\left(x_t\right)\right) - \frac{1}{n} \\
&\qquad < \quad \mathbf{E}\left(x_{t+2}\right) \quad < \\
&\mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) + x_{t+1} \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) = B_\ell\left(x_t\right)\right) + \frac{1}{n}.
\end{aligned} \tag{2.9}$$

2.2.2.2 Effects of noise

When it comes to the central area, YELLOW, we will need the following result to break ties.

Lemma 2.14. *There exists a constant $\beta > 0$ s.t. for n large enough, and if $\mathbf{E}(x_{t+2}) \in [1/3, 2/3]$, then*

$$\mathbb{P}\left(x_{t+2} \le \mathbf{E}(x_{t+2}) - 1/\sqrt{n}\right), \mathbb{P}\left(x_{t+2} \ge \mathbf{E}(x_{t+2}) + 1/\sqrt{n}\right) \ge \beta.$$

Proof. Consider $X_1, \ldots, X_n$ from the statement of Observation 2.6. We have (see the proof of Observation 2.6)

- $X_1 = 1$,
- for every j s.t. $1 < j \le n \cdot x_{t+1}$, $\mathbb{P}\left(X_j = 1\right) = \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \ge B_\ell\left(x_t\right)\right)$,
- for every j s.t. $n \cdot x_{t+1} < j \le n$, $\mathbb{P}\left(X_j = 1\right) = \mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right)$.

Let $p = \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \geq B_\ell\left(x_t\right)\right)$ and $q = \mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right)$. By Observation 2.6,

$$\mathbf{E}(x_{t+2}) = \mathbf{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = x_{t+1}\cdot p + (1-x_{t+1})\cdot q + \frac{1}{n}(1-p).$$

By assumption on $\mathbf{E}(x_{t+2})$, this implies that

$$x_{t+1}\cdot p + (1-x_{t+1})\cdot q \in \left[\frac{1}{3} - \frac{1}{n}, \frac{2}{3}\right].$$

Moreover, we have that $p - q = \mathbb{P}\left(B_\ell(x_{t+1} = B_\ell(x_t)\right)$ which tends to 0 as n tends to infinity, i.e., p and q are arbitrarily close. Hence, for n large enough, the last equation implies that $p \in [1/4, 3/4]$ and $q \in [1/4, 3/4]$. Let $Y_p = \sum_{i=2}^{n\cdot x_{t+1}} X_i$ and $Y_q = \sum_{i=n\cdot x_{t+1}+1}^{n} X_i$. These two variables are binomially distributed, and since $p, q \in [1/4, 3/4]$, there is a constant probability that $Y_p \geq \mathbf{E}(Y_p)$, and there is a constant probability that $Y_q \geq \mathbf{E}(Y_q)$ as well. Without loss of generality, we assume that $x_{t+1} \geq 1/2$ and focus on Y_p (if $x_{t+1} < 1/2$, then we could consider Y_q instead). Let $m = n\cdot x_{t+1} - 1$ be the number of samples of Y_p. In this case, $m \geq n/2 - 1$ tends to $+\infty$ as n tends to $+\infty$. Let $\sigma_p = \sqrt{p(1-p)}$. By the central limit theorem (Theorem A.3), the random variable

$$\frac{\sqrt{m}}{\sigma_p}\left(\frac{1}{m}Y_p - p\right) = \frac{Y_p - \mathbf{E}(Y_p)}{\sigma_p\sqrt{m}}$$

converges in distribution to $\mathcal{N}(0,1)$. Moreover, $\mathrm{Var}(Y_p) = m\sigma_p^2 = (n\cdot x_{t+1} - 1)p(1-p) \geq (n/2 - 1)p(1-p) \geq np(1-p)/3$, so for any $\epsilon > 0$ and n large enough,

$$\begin{aligned}
\mathbb{P}\left(Y_p \geq \mathbf{E}\left(Y_p\right) + \sqrt{n}\right) &= \mathbb{P}\left(\frac{Y_p - \mathbf{E}\left(Y_p\right)}{\sigma_p\sqrt{m}} \geq \frac{\sqrt{n}}{\sigma_p\sqrt{m}}\right) \\
&\geq \mathbb{P}\left(\frac{Y_p - \mathbf{E}\left(Y_p\right)}{\sigma_p\sqrt{m}} \geq \sqrt{\frac{3}{p(1-p)}}\right) \\
&\geq 1 - \Phi\left(\sqrt{\frac{3}{p(1-p)}}\right) - \epsilon.
\end{aligned}$$

For ϵ small enough, and because p is bounded, this probability is bounded away from zero. This concludes the proof of Lemma 2.14 (the other inequality can be obtained symmetrically). ☐

2.2.3 Analyzing Domains

2.2.3.1 Green area

Proof of Lemma 2.7. Let us prove the first part and assume $(x_t, x_{t+1}) \in \mathsf{GREEN}_1$ (the proof of the second part is analogous). By Eq. (2.7) in Claim 2.12, we have for every agent i

$$\mathbb{P}\left(Y_i^{(t+2)} = 0\right) \leq \mathbb{P}\left(B_\ell\left(x_{t+1}\right) \leq B_\ell\left(x_t\right)\right).$$

By Lemma A.5, we have

$$\mathbb{P}\left(B_\ell\left(x_{t+1}\right) \leq B_\ell\left(x_t\right)\right) \leq \exp\left(-\frac{1}{2}\ell(x_{t+1} - x_t)^2\right) \leq \exp\left(-\frac{1}{2}\ell\delta^2\right) = \exp\left(-\frac{c\delta^2}{2}\log n\right).$$

Then, by the union bound,

$$\mathbb{P}\left(\bigcup_{i\in I\setminus\{\mathsf{source}\}}\left(Y_i^{(t+2)} = 0\right)\right) \leq (n-1)\cdot\exp\left(-\frac{c\delta^2}{2}\log n\right),$$

which is $o(n^{-\epsilon})$ for some $\epsilon > 0$ provided that $c > 2/\delta^2$. □

2.2.3.2 Purple area

Proof of Lemma 2.8. Let us prove the first part and assume $(x_t, x_{t+1}) \in \text{PURPLE}_1$ (the proof of the second part is analogous). By Eq. (2.8) in Claim 2.12,

$$\mathbf{E}(x_{t+2}) \geq \mathbb{P}(B_\ell(x_{t+1}) > B_\ell(x_t)) - \frac{1}{n}.$$

Since $(x_t, x_{t+1}) \in \text{PURPLE}_1$, and since in this area $x_{t+1} \geq (1-\lambda_n)x_t$, we have

$$\mathbb{P}(B_\ell(x_{t+1}) > B_\ell(x_t)) \geq \mathbb{P}(B_\ell((1-\lambda_n)x_t) > B_\ell(x_t)).$$

Let

$$\sigma = \sqrt{x_t(1-x_t) + (1-\lambda_n)x_t(1-(1-\lambda_n)x_t)} > \sqrt{x_t(1-x_t)} > \sqrt{\frac{x_t}{2}}, \tag{2.10}$$

where the last inequality is by the fact that $x_t < 1/2$ which follows from the definition of PURPLE_1. By Lemma A.9,

$$\mathbb{P}(B_\ell((1-\lambda_n)x_t) > B_\ell(x_t)) > 1 - \Phi\left(\frac{\sqrt{\ell}\lambda_n x_t}{\sigma}\right) - \frac{C}{\sigma\sqrt{\ell}}.$$

We have (Eq. (2.10) and definition of PURPLE_1)

$$\sigma > \sqrt{\frac{x_t}{2}} > \sqrt{\frac{1}{2\log n}}$$

so

$$\frac{C}{\sigma\sqrt{\ell}} < \frac{\sqrt{2}C}{\sqrt{c}}.$$

If c is large enough (specifically, if $c > 32C^2/\delta^2$), we obtain

$$\mathbb{P}(B_\ell((1-\lambda_n)x_t) > B_\ell(x_t)) > 1 - \Phi\left(\frac{\sqrt{\ell}\lambda_n x_t}{\sigma}\right) - \frac{\delta}{4}.$$

We have

$$0 \leq \frac{\sqrt{\ell}\lambda_n x_t}{\sigma} \leq \sqrt{\ell}\lambda_n\sqrt{2x_t} \leq \sqrt{\ell}\lambda_n = \frac{\sqrt{c}}{\log^\delta n} \to 0.$$

where the second inequality is by Eq. (2.10), and the third is because $x_t < 1/2$. So, for n large enough

$$1 - \Phi\left(\frac{\sqrt{\ell}\lambda_n}{\sigma}\right) - \frac{\delta}{4} > 1 - \Phi(0) - \frac{\delta}{2} = \frac{1-\delta}{2}.$$

Overall, we have proved that if n is large enough, then $\mathbf{E}(x_{t+2}) > (1-\delta)/2$. By Observation 2.6, we can apply Chernoff's inequality (Theorem A.1) to get that $x_{t+2} > 1/2 - \delta$ w.h.p. Since by definition of PURPLE_1 we have $1/2 - \delta > x_{t+1} + \delta$, we obtain $x_{t+2} > x_{t+1} + \delta$ w.h.p., which concludes the proof of the lemma. □

2.2.3.3 Red area

Proof of Lemma 2.9. Without loss of generality, we assume that $t_0 = 0$, and that $(x_0, x_1) \in \text{RED}_1$ (the proof in the case that $(x_0, x_1) \in \text{RED}_0$ is the same). First we note that for every round t, by definition, if $(x_t, x_{t+1}) \in \mathsf{RED}_1$ then $x_{t+1} < (1-\lambda_n)x_t$. So, we can prove by induction on t that for every $1 \le t \le t_1$,

$$x_t < x_0(1-\lambda_n)^t.$$

In particular, we have that $x_{t_1} < x_{t_0} < 1/2 - 3\delta$, and so $(x_{t_1}, x_{t_1+1}) \notin \mathsf{YELLOW}$ by definition of YELLOW.

Also by definition, $x_0 < 1/2$ and $x_t > 1/\log(n)$ for every $0 \le t \le t_1$, hence, we obtain from the last equation that

$$\frac{1}{\log n} < \frac{1}{2}(1-\lambda_n)^t.$$

Taking the logarithm and rearranging, we get

$$\log\left(\frac{1}{2}\right) + \log(\log n) > t \cdot \log\left(\frac{1}{1-\lambda_n}\right).$$

We know that $\log(1-\lambda_n) < -\lambda_n$ and thus $t \cdot \log\left(1/(1-\lambda_n)\right) > t\lambda_n$. Together with the above equation, this gives

$$t < \frac{1}{\lambda_n}\left(\log\left(\frac{1}{2}\right) + \log(\log n)\right) = o\left(\log^{1/2+2\delta} n\right),$$

which concludes the proof. ◻

2.2.3.4 Cyan area

The goal of this section is to prove Lemma 2.10. We only prove the result for CYAN_1, but the same arguments apply to CYAN_0 symmetrically. We distinguish between two cases.

Case 1. $x_{t_0} \ge 1/\log(n)$. In this case, by definition of CYAN_1, we must have $x_{t_0+1} < 1/\log(n)$. Note that in this case, for n large enough, $x_{t_0+1} - \delta < 1/\log(n) - \delta < 0$. Then,

- either $x_{t_0+2} < x_{t_0+1} + \delta$. In this case, $x_{t_0+1} - \delta < 0 < x_{t_0+2} < x_{t_0+1} + \delta$, and so $(x_{t_0+1}, x_{t_0+2}) \in \mathsf{CYAN}_1$ (but this time Case 2 applies).
- or $x_{t_0+2} \ge x_{t_0+1} + \delta$, and so $(x_{t_0+1}, x_{t_0+2}) \in \mathsf{GREEN}_1$.
- (We can't have $x_{t_0+2} = 0$ because the source is assumed to have opinion 1.)

Case 2. $x_{t_0} < 1/\log(n)$. Let $\gamma = \gamma(c) = (1-1/e) \cdot \exp(-2c)/2$ and let $K = K(c) = c \cdot \exp(-2c)/2$. We will study separately three ranges of value for x_{t+1}. Claim 2.15 below concerns small values of x_{t+1}, Claim 2.16 concerns intermediate values of x_{t+1}, and Claim 2.17 concerns large values of x_{t+1}. Their proofs follow from simple applications of Chernoff's bound.

Claim 2.15. *If $x_t < 1/\log(n)$, and if $0 < x_{t+1} \le 1/\ell$, then $\mathbb{P}\left(x_{t+2} > \frac{K}{2} x_{t+1}\log n\right) > 1 - \exp\left(-\frac{K}{8}\log n\right)$.*

Proof. We note that, since $x_t < 1/\log(n)$, the probability that an agent does not see a 1 in round t is

$$\mathbb{P}\left(B_\ell\left(x_t\right) = 0\right) = (1-x_t)^\ell > \left(1 - \frac{1}{\log n}\right)^\ell = \exp\left(c\log(n)\log\left(1-\frac{1}{\log n}\right)\right) > e^{-2c},$$

for n large enough. Moreover,

$$(1-x_{t+1})^\ell < 1 - \ell x_{t+1} + \frac{1}{2}\ell^2 x_{t+1}^2,$$

so the probability that an agent sees at least a 1 in round $t+1$ is

$$\mathbb{P}(B_\ell(x_{t+1}) \geq 1) = 1 - (1 - x_{t+1})^\ell > \ell x_{t+1}\left(1 - \frac{1}{2}\ell x_{t+1}\right) > \frac{1}{2}\ell x_{t+1},$$

where the last inequality comes from the assumption that $x_{t+1} \leq 1/\ell$. Eventually, we can write

$$\begin{aligned}
\mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) &\geq \mathbb{P}\left(B_\ell(x_t) = 0\right) \cdot \mathbb{P}\left(B_\ell(x_{t+1}) \geq 1\right) \\
&\geq \frac{c}{2} \cdot e^{-2c} \cdot x_{t+1} \log n \\
&= K x_{t+1} \log n.
\end{aligned}$$

Hence, by Eq. (2.8) in Claim 2.12, $\mathbf{E}\left(x_{t+2}\right) \geq K x_{t+1} \log n - 1/n$. By Observation 2.6, we can apply Chernoff's inequality (Theorem A.1) to conclude the proof of Claim 2.15. □

Claim 2.16. *If $x_t < 1/\log(n)$, and if $1/\ell < x_{t+1} \leq \gamma$, then*

$$\mathbb{P}\left(x_{t+2} > \gamma\right) > 1 - \exp\left(-\frac{\gamma n}{8}\right) > 1 - \exp\left(-\frac{K}{8}\log n\right).$$

Proof. The proof follows along similar lines as the proof of Claim 2.15. We note that, since $x_t < 1/\log(n)$, the probability that an agent does not see a 1 in round t is

$$\mathbb{P}\left(B_\ell\left(x_t\right) = 0\right) = (1 - x_t)^\ell > \left(1 - \frac{1}{\log n}\right)^\ell = \exp\left(c \log n \log\left(1 - \frac{1}{\log n}\right)\right) > e^{-2c},$$

for n large enough. Moreover, the probability that an agent sees at least one 1 in round $t+1$ is

$$\mathbb{P}(B_\ell(x_{t+1}) \geq 1) = 1 - (1 - x_{t+1})^\ell \geq 1 - \left(1 - \frac{1}{\ell}\right)^\ell > 1 - \frac{1}{e}.$$

Eventually, we can write

$$\mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) \geq \mathbb{P}\left(B_\ell(x_{t+1}) \geq 1\right) \cdot \mathbb{P}\left(B_\ell(x_t) = 0\right) \geq e^{-2c} \cdot \left(1 - \frac{1}{e}\right) = 2\gamma.$$

Hence, by Eq. (2.8) in Claim 2.12, $\mathbf{E}\left(x_{t+2}\right) \geq 2\gamma - 1/n$. By Observation 2.6, we can apply Chernoff's inequality (Theorem A.1) to conclude the proof of Claim 2.16. □

Claim 2.17. *If $x_t < 1/\log(n)$, and if $x_{t+1} > \gamma$, then $\mathbb{P}\left(x_{t+2} > \frac{1}{2}\right) > 1 - \exp\left(-\frac{n}{18}\right) > 1 - \exp\left(-\frac{K}{8}\log n\right)$.*

Proof. By assumption, $x_{t+1} - x_t \geq \gamma - 1/\log(n)$, and so by Lemma A.5,

$$\mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) \geq 1 - \exp\left(-\frac{1}{2}\ell\left(\gamma - \frac{1}{\log n}\right)^2\right) > \frac{3}{4}$$

for n large enough. Hence, by Eq. (2.8) in Claim 2.12, $\mathbf{E}\left(x_{t+2}\right) \geq 3/4 - 1/n$. By Observation 2.6, we can apply Chernoff's inequality (Theorem A.1) to conclude the proof of Claim 2.17. □

Proof of Lemma 2.10. We say that a round t is *successful* if $(x_t, x_{t+1}) \in \mathsf{CYAN}_1$, and either

$$\begin{cases} x_{t+1} \le 1/\ell \text{ and } x_{t+2} > K x_{t+1} \log n \text{ (corresponding to Claim 2.15), or} \\ 1/\ell < \ x_{t+1} \le \gamma \text{ and } x_{t+2} > \gamma \text{ (corresponding to Claim 2.16), or} \\ \gamma < \ x_{t+1} \text{ and } (x_{t+1}, x_{t+2}) \notin \mathsf{CYAN}_1 \text{ (corresponding to Claim 2.17).} \end{cases}$$

Let X be the number of successful rounds starting from t_0. This definition implies that necessarily,

$$X < \frac{\log(n/\ell)}{\log(K \cdot \log n/2)} + 2 := X_{\max}.$$

Indeed, since $x_{t_0+1} > 1/n$ (by definition of CYAN_1),

- $\log(n/\ell)/\log(K \cdot \log(n)/2)$ rounds are always enough to get $x_{t+1} > 1/\ell$
- one more round is enough to get $x_{t+1} > \gamma$
- one more round is enough to get $x_{t+1} > 1/2$, in which case $(x_t, x_{t+1}) \notin \mathsf{CYAN}_1$.

Therefore, by Claims 2.15 to 2.17, the probability that, starting from t_0, all rounds are successful until the system is out of CYAN_1 is at least $\left(1 - \exp\left(-\frac{K}{8}\log n\right)\right)^{X_{\max}} \ge 1 - X_{\max} \cdot \exp\left(-\frac{K}{8}\log n\right) = 1 - 1/n^{\Omega(1)}$. Moreover, for any successful round t, $x_{t+2} > x_{t+1}$ (by definition of a successful round) and $x_{t+1} < \delta + 1/\log(n)$ (this is a straightforward consequence of the definition of CYAN_1). Thus, by construction of the partition, we must have $(x_{t+1}, x_{t+2}) \in \mathsf{CYAN}_1 \cup \mathsf{GREEN}_1 \cup \mathsf{PURPLE}_1$. This implies that $(x_{t_1}, x_{t_1+1}) \in \mathsf{GREEN}_1 \cup \mathsf{PURPLE}_1$, which concludes the proof of Lemma 2.10. $\square$

2.2.4 The Yellow Area

The goal of this section is to prove Lemma 2.11. It might be easier for the reader to think of the Yellow area as a square. Formally, let us define YELLOW' as the following square bounding box around YELLOW:

$$\mathsf{YELLOW}' = \left\{ (x_t, x_{t+1}) \text{ s.t. } 1/2 - 4\delta \le x_t, x_{t+1} \le 1/2 + 4\delta \right\}.$$

Obviously, $\mathsf{YELLOW} \subset \mathsf{YELLOW}'$, so in order to prove Lemma 2.11 it suffices to prove Lemma 2.18 below.

Lemma 2.18. *Consider that $(x_{t_0}, x_{t_0+1}) \in \mathsf{YELLOW}'$. Then, w.h.p.,*

$$\min\{t > t_0 \text{ s.t. } (x_t, x_{t+1}) \notin \mathsf{YELLOW}'\} < t_0 + O(\log^{5/2} n).$$

In this section, we can use a previous result about the effects of noise, to show that the Markov process (x_t, x_{t+1}) is never too likely to be in any specific region.

Lemma 2.19. *There is a constant $c_1 = c_1(c) > 0$ (recall that the sample size is $\ell = c \cdot \log n$), such that for any $a \in [1/2 - 4\delta, 1/2 + 4\delta]$, and any round t s.t. $(x_t, x_{t+1}) \in \mathsf{YELLOW}'$, we have either*

$$\mathbb{P}\left(|x_{t+2} - a| > \frac{1}{\sqrt{n}}\right) > c_1,$$

or $(x_{t+1}, x_{t+2}) \notin \mathsf{YELLOW}'$ w.h.p.

Proof. If $\mathbb{E}(x_{t+2}) \notin [1/3, 2/3]$, then $(x_{t+1}, x_{t+2}) \notin \mathsf{YELLOW}'$ w.h.p. Otherwise, let $a \in [1/2 -$

$4\delta, 1/2 + 4\delta]$. If $a > \mathbb{E}(x_{t+2})$, then by Lemma 2.14,

$$\mathbb{P}\left(|x_{t+2} - a| > \frac{1}{\sqrt{n}}\right) \geq \mathbb{P}\left(x_{t+2} \leq \mathbb{E}(x_{t+2}) - \frac{1}{\sqrt{n}}\right) \geq \beta.$$

Similarly, if $a \leq \mathbb{E}(x_{t+2})$, Lemma 2.14 gives

$$\mathbb{P}\left(|x_{t+2} - a| > \frac{1}{\sqrt{n}}\right) \geq \mathbb{P}\left(x_{t+2} \geq \mathbb{E}(x_{t+2}) + \frac{1}{\sqrt{n}}\right) \geq \beta,$$

which concludes the proof of Lemma 2.19. $\square$

2.2.4.1 General structure of the proof

In order to prove Lemma 2.18, we first partition YELLOW′, as follows (for an illustration, see Figure 2.3):

$$\begin{aligned}
\mathbf{A}_1 &= \{(x_t, x_{t+1}) \mid \text{ (i) } x_{t+1} \geq 1/2 \text{ and (ii) } x_{t+1} - x_t \geq x_t - 1/2\} \bigcap \text{YELLOW}',\\
\mathbf{B}_1 &= \{(x_t, x_{t+1}) \mid \text{ (i) } x_{t+1} \geq x_t \text{ and (ii) } x_{t+1} - x_t < x_t - 1/2\} \bigcap \text{YELLOW}',\\
\mathbf{C}_1 &= \{(x_t, x_{t+1}) \mid \text{ (i) } x_{t+1} < 1/2 \text{ and (ii) } x_{t+1} \geq x_t\} \bigcap \text{YELLOW}'.
\end{aligned}$$

Similarly, we define $\mathbf{A}_0, \mathbf{B}_0, \mathbf{C}_0$ their symmetric equivalents (w.r.t the point $(\frac{1}{2}, \frac{1}{2})$), and $\mathbf{A} = \mathbf{A}_0 \cup \mathbf{A}_1$, $\mathbf{B} = \mathbf{B}_0 \cup \mathbf{B}_1$, and $\mathbf{C} = \mathbf{C}_0 \cup \mathbf{C}_1$.

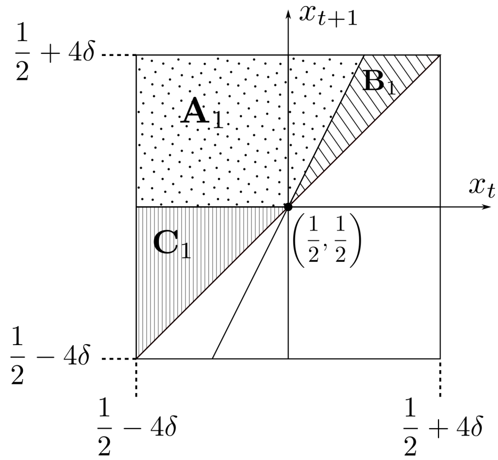


Figure 2.3: Partitioning the YELLOW′ domain.

In Lemma 2.20, we study the distribution of the future location of any point $(x_t, x_{t+1}) \in \mathbf{A}$. This area happens to be ideal to escape YELLOW′, because it allows the Markov chain to quickly build up "speed". Item (a) in the next lemma says that, with some probability that depends on the current speed the following occur: (1) the speed in the following round increases by a factor of two, and (2) the process in the next round either remains in $\mathbf{A}$, or goes outside of YELLOW′. Note that when the current speed is not too low, that is, when $x_{t+1} - x_t > 1/\sqrt{n}$, this combined event happens with constant probability. Item (b) says that with constant probability, (1) the speed in the next round would not be too low, and (2) the process either remains in $\mathbf{A}$, or goes outside of YELLOW′.

Lemma 2.20. *If $(x_t, x_{t+1}) \in \mathbf{A}$, and provided that δ is small enough and n is large enough,*

(a) We have

$$\mathbb{P}\left((x_{t+1}, x_{t+2}) \notin \text{YELLOW}' \setminus \mathbf{A} \bigcap |x_{t+2} - x_{t+1}| > 2|x_{t+1} - x_t|\right) > 1 - \exp\left(-3n \cdot (x_{t+1} - x_t)^2\right).$$

(b) *There exists a constant* $c_2 = c_2(c) > 0$ *s.t.*

$$\mathbb{P}\left((x_{t+1}, x_{t+2}) \notin \mathsf{YELLOW}' \setminus \mathbf{A} \bigcap |x_{t+2} - x_{t+1}| > 1/\sqrt{n}\right) > c_2.$$

Now, we can iteratively use the previous result to prove that any state in $\mathbf{A}$ has a reasonable chance to escape YELLOW′. The proofs of both Lemmas 2.20 and 2.21 are deferred to Section 2.2.4.2.

Lemma 2.21. *There is a constant* $c_3 = c_3(c)$ *s.t. if* $(x_{t_0}, x_{t_0+1}) \in \mathbf{A}$, *then*

$$\mathbb{P}\left(\exists t_1 < t_0 + \log n, (x_{t_1}, x_{t_1+1}) \notin \mathsf{YELLOW}'\right) > c_3.$$

We are left with proving that the system cannot be stuck in $\mathbf{B}$ or $\mathbf{C}$ for too long. We start with $\mathbf{B}$. The analysis of this area is relatively complex, because it is difficult to rule out the possibility that the Markov chain remains there at a low speed. We prove that any state in $\mathbf{B}$ must either make a step towards escaping YELLOW′, or have a good chance of leaving $\mathbf{B}$. The proof of the following lemma is given in Section 2.2.4.3.

Lemma 2.22. *There are constants* $c_4, c_5 > 0$ *such that if* $(x_t, x_{t+1}) \in \mathbf{B}$, *then either*

(a) $|x_{t+1} - 1/2| > \left(1 + c_4/\sqrt{\ell}\right) |x_t - 1/2|$, *or*

(b) $\mathbb{P}\left((x_{t+1}, x_{t+2}) \notin \mathbf{B}\right) > c_5$.

Now, we can iteratively use the previous result to prove that any state in $\mathbf{B}$ either leaves $\mathbf{B}$ or escapes YELLOW' in a reasonable amount of time. The proof of Lemma 2.23 is deferred to Section 2.2.4.3.

Lemma 2.23. *If* $(x_{t_0}, x_{t_0+1}) \in \mathbf{B}$, *then, w.h.p.,* $\min\{t \geq t_0, (x_t, x_{t+1}) \notin \mathbf{B}\} < t_0 + \frac{\sqrt{c}}{c_4} \cdot \log^{3/2} n$.

We are left with proving that the system cannot stay in $\mathbf{C}$ for too long. Fortunately, from this area, the Markov chain is naturally pushed towards $\mathbf{A}$, which makes the analysis simple. The proof of Lemma 2.24 is deferred to Section 2.2.4.4.

Lemma 2.24. *There is a constant* $c_6 > 0$ *such that if* $(x_t, x_{t+1}) \in \mathbf{C}$, *then*

$$\max\left\{\mathbb{P}\left((x_{t+1}, x_{t+2}) \notin \mathsf{YELLOW}' \setminus \mathbf{A}\right), \mathbb{P}\left((x_{t+2}, x_{t+3}) \notin \mathsf{YELLOW}' \setminus \mathbf{A}\right)\right\} > c_6.$$

Eventually, we have all the necessary results to conclude the proof regarding the Yellow area.

Proof of Lemma 2.18. By Lemma 2.21, if $(x_{t_0}, x_{t_0+1}) \in \mathbf{A}$, then

$$\mathbb{P}\left(\exists t_1 < t_0 + \log n, (x_{t_1}, x_{t_1+1}) \notin \mathsf{YELLOW}'\right) > c_3 > 0.$$

By Lemma 2.24, this implies that if $(x_{t_0}, x_{t_0+1}) \in \mathbf{A} \cup \mathbf{C}$,

$$\mathbb{P}\left(\exists t_1 < t_0 + \log n + 2, (x_{t_1}, x_{t_1+1}) \notin \mathsf{YELLOW}'\right) > \min(c_3, c_3 \cdot c_6) = c_3 \cdot c_6 > 0. \tag{2.11}$$

By Lemma 2.23, w.h.p., whenever the process is at $\mathbf{B}$, it does not spend more than $(\sqrt{c}/c_4) \cdot \log^{3/2} n$ consecutive rounds there. This means, that for any constant $c' > 0$, during $c' \log^{5/2} n$ consecutive rounds, w.h.p., we must either leave YELLOW′ or be at $\mathbf{A} \cup \mathbf{C}$ on at least

$$\frac{c' \log^{5/2} n}{(\sqrt{c}/c_4) \cdot \log^{3/2} n} = \frac{c' c_4}{\sqrt{c}} \cdot \log n$$

distinct rounds. By Eq. (2.11), the probability that the system fails to escape YELLOW′ in each of these occasions is at most $(1 - c_3 \cdot c_6)^{(c' c_4/\sqrt{c}) \cdot \log n}$. Taking c' to be sufficiently large concludes the proof of Lemma 2.18. □

2.2.4.2 Area A

Proof of Lemma 2.20. Without loss of generality, we assume that $(x_t, x_{t+1}) \in \mathbf{A}_1$ (the same arguments apply to $\mathbf{A}_0$ symmetrically). We have, provided that δ is small enough and n is large enough,

$$
\begin{aligned}
\mathbf{E}(x_{t+2}) &> \mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) + x_{t+1} \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) = B_\ell\left(x_t\right)\right) - \frac{1}{n} && \text{(by Claim 2.13)} \\
&> \frac{1}{2} + 6(x_{t+1} - x_t) + \left(x_{t+1} - \frac{1}{2}\right) \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) = B_\ell\left(x_t\right)\right) && \text{(by Lemma A.6)} \\
&> \frac{1}{2} + 6(x_{t+1} - x_t). && \text{(definition of } \mathbf{A}_1\text{)}
\end{aligned}
$$

More precisely, in the second inequality, the term $-1/n$ disappears within the $6(x_{t+1} - x_t)$ lower bound. Indeed, we can take $\lambda \gg 6$ (from Lemma A.6) for this purpose. Moreover, we can assume $x_{t+1} - x_t \geq 1/n$, by definition of $\mathbf{A}_1$, and ruling out the case $x_t = x_{t+1}$ with Lemma 2.19. Hence,

$$
\mathbf{E}(x_{t+2}) - x_{t+1} > \frac{1}{2} - x_t + 5(x_{t+1} - x_t) = (x_{t+1} - (2x_t - \frac{1}{2})) + 4(x_{t+1} - x_t),
$$

and by definition of $\mathbf{A}_1$, $(x_{t+1} - (2x_t - 1/2)) \geq 0$ and so

$$
\mathbf{E}(x_{t+2}) > 4(x_{t+1} - x_t) + x_{t+1}. \tag{2.12}
$$

By Observation 2.6, we can apply Chernoff's inequality (Theorem A.1). Taking

$$
\epsilon = \frac{2(x_{t+1} - x_t)}{4(x_{t+1} - x_t) + x_{t+1}},
$$

we have by by Eq. (2.12)

$$
\begin{aligned}
\mathbb{P}\left(x_{t+2} - x_{t+1} \leq 2(x_{t+1} - x_t)\right) &= \mathbb{P}\left(x_{t+2} \leq (1-\epsilon)\left(4(x_{t+1} - x_t) + x_{t+1}\right)\right) \\
&\leq \mathbb{P}\left(n x_{t+2} \leq (1-\epsilon)\mathbf{E}(n x_{t+2})\right).
\end{aligned}
$$

By the Chernoff bound (Theorem A.1), this is upper bounded by

$$
\exp\left(-\frac{\epsilon^2}{2}\mathbf{E}(n x_{t+2})\right) \leq \exp\left(-\frac{2x_{t+1}}{(4(x_{t+1} - x_t) + x_{t+1})^2}(x_{t+1} - x_t)^2 n\right),
$$

where the last inequality is by Eq. (2.12) and definition of ϵ. Since x_t and x_{t+1} are close to $1/2$, we have for δ small enough

$$
\mathbb{P}\left(x_{t+2} - x_{t+1} > 2(x_{t+1} - x_t)\right) \geq 1 - \exp\left(-3(x_{t+1} - x_t)^2 n\right). \tag{2.13}
$$

Now, we show that the event "$x_{t+2} - x_{t+1} > 2(x_{t+1} - x_t)$" suffices for (x_{t+1}, x_{t+2}) to remain in $\mathbf{A}_1$ or leave YELLOW′.

Claim 2.25. *If $(x_t, x_{t+1}) \in \mathbf{A}_1$ and $x_{t+2} - x_{t+1} > 2(x_{t+1} - x_t)$, then $(x_{t+1}, x_{t+2}) \in \mathbf{A}_1$ or $(x_{t+1}, x_{t+2}) \notin$ YELLOW′.*

Proof. If $(x_{t+1}, x_{t+2}) \notin \text{YELLOW}'$, the result holds. Otherwise, $(x_{t+1}, x_{t+2}) \in \text{YELLOW}'$ and we have to prove that (x_{t+1}, x_{t+2}) satisfies $\mathbf{A}_1$.(i) and $\mathbf{A}_1$.(ii). First we prove that (x_{t+1}, x_{t+2}) satisfies $\mathbf{A}_1$.(i):

$$\begin{aligned} x_{t+2} &> x_{t+1} + 2(x_{t+1} - x_t) && \text{(by assumption in the claim)} \\ &\geq x_{t+1} && \text{(because } (x_t, x_{t+1}) \in \mathbf{A}_1 \Rightarrow x_{t+1} \geq x_t) \\ &\geq 1/2. && \text{(because } (x_t, x_{t+1}) \in \mathbf{A}_1 \text{ and by } \mathbf{A}_1.\text{(i))} \end{aligned}$$

Then we prove that (x_{t+1}, x_{t+2}) satisfies $\mathbf{A}_1$.(ii):

$$\begin{aligned} x_{t+2} - x_{t+1} &> 2(x_{t+1} - x_t) && \text{(by assumption in the claim)} \\ &> (x_{t+1} - x_t) + (x_t - 1/2) && \text{(because } (x_t, x_{t+1}) \in \mathbf{A}_1 \text{ and by } \mathbf{A}_1.\text{(ii))} \\ &= x_{t+1} - 1/2, \end{aligned}$$

which concludes the proof of Claim 2.25. □

Next, we apply Claim 2.25 to Eq. (2.13) to establish (a). Finally, $x_{t+2} > x_{t+1} + 4(x_{t+1} - x_t) + 1/\sqrt{n}$ implies $x_{t+2} - x_{t+1} > +2(x_{t+1} - x_t)$ so we can use Claim 2.25,

$$\begin{aligned} &\mathbb{P}\left((x_{t+1}, x_{t+2}) \notin \text{YELLOW}' \setminus \mathbf{A}_1 \bigcap x_{t+2} > x_{t+1} + 4(x_{t+1} - x_t) + 1/\sqrt{n}\right) \\ &= \mathbb{P}\left(x_{t+2} > x_{t+1} + 4(x_{t+1} - x_t) + 1/\sqrt{n}\right) && \text{(by Claim 2.25)} \\ &> \mathbb{P}\left(x_{t+2} > \mathbf{E}(x_{t+2}) + 1/\sqrt{n}\right) && \text{(by Eq. (2.12))} \\ &> c_2 > 0, \end{aligned}$$

where the existence of c_2 is guaranteed by Lemma 2.14. This establishes (b). □

Proof of Lemma 2.21. Without loss of generality, we assume that $(x_t, x_{t+1}) \in \mathbf{A}_1$ (the same arguments apply to $\mathbf{A}_0$ symmetrically). Let us define event H_{t_0+1}, that the system is either in A_1 or out of YELLOW′ in round $t_0 + 1$, and that the "gap" $(x_{t_0+2} - x_{t_0+1})$ is not too small. Formally,

$$H_{t_0+1}: \quad (x_{t_0+1}, x_{t_0+2}) \notin \text{YELLOW}' \setminus \mathbf{A}_1 \bigcap x_{t_0+2} - x_{t_0+1} > 1/\sqrt{n}.$$

For $t > t_0 + 1$, we define event H_t, that the system is either in A_1 or out of YELLOW′ in round t, and that the gap $(x_{t+1} - x_t)$ doubles. Formally,

$$H_t: \quad (x_t, x_{t+1}) \notin \text{YELLOW}' \setminus \mathbf{A}_1 \bigcap x_{t+1} - x_t > 2(x_t - x_{t-1}).$$

We start with the following observation, which results directly from the definition of H_t for $t \geq t_0 + 1$:

$$\bigcap_{s=t_0+1}^{t-1} H_s \Rightarrow (x_t - x_{t-1}) > 2^{(t-t_0-2)}(x_{t_0+2} - x_{t_0+1}) \Rightarrow (x_t - x_{t-1}) > 2^{(t-t_0-2)}/\sqrt{n}. \tag{2.14}$$

For every $t > t_0 + 1$ such that $(x_t, x_{t+1}) \in \mathbf{A}_1$,

$$\begin{aligned} \mathbb{P}\left(H_t \,\middle|\, \bigcap_{s=t0+1}^{t-1} H_s\right) &> 1 - \exp\left(-3n \cdot (x_t - x_{t-1})^2\right) && \text{(By Lemma 2.20)} \\ &> 1 - \exp\left(-\frac{3}{4} \cdot 4^{(t-t_0-1)}\right). && \text{(by Eq. (2.14))} \end{aligned}$$

By Lemma 2.20 (b), $(x_{t_0+1}, x_{t_0+2}) \in \mathbf{A}_1$ and $x_{t_0+2} - x_{t_0+1} > 1/\sqrt{n}$ w.p. $c_2 > 0$. Together with the last equation and using the union bound, we get

$$\mathbb{P}\left(\bigcap_{t=t0+1}^{t_1} H_t\right) > c_2 \cdot \left(1 - \sum_{t=t0+2}^{t_1} \exp\left(-\frac{3}{4} \cdot 4^{(t-t_0-1)}\right)\right).$$

We have the following very rough upper bounds

$$\begin{aligned}\sum_{t=t0+2}^{t_1} \exp\left(-\frac{3}{4} \cdot 4^{(t-t_0-1)}\right) &< \sum_{t=t0+2}^{t_1} \exp\left(-\frac{3}{4} \cdot 4 \cdot (t-t_0-1)\right) \\ &< 2 \cdot e^{-3}.\end{aligned}$$

Hence, we have proved that for every $t_1 > t_0 + 1$ such that $(x_{t_1}, x_{t_1+1}) \in \mathbf{A}_1$,

$$\mathbb{P}\left(\bigcap_{t=t0+1}^{t_1} H_t\right) > c_2 \cdot \left(1 - 2 \cdot e^{-3}\right) := c_3 > 0.$$

By Eq. (2.14), it implies that for every $t_1 > t_0 + 1$ such that $(x_{t_1}, x_{t_1+1}) \in \mathbf{A}_1$,

$$\mathbb{P}\left((x_{t_1} - x_{t_1-1}) > 2^{(t_1-t_0-2)}/\sqrt{n}\right) > c_3.$$

For t_1 large enough (e.g., $t_1 = t_0 + \log n$), this implies that $(x_{t_1-1}, x_{t_1}) \notin$ YELLOW′, otherwise the gap $(x_{t_1} - x_{t_1-1})$ would be greater than 8δ which is the diameter of YELLOW′. This concludes the proof of Lemma 2.21. □

2.2.4.3 Area B

The goal of this section is to prove Lemmas 2.22 and 2.23. We start by stating a technical result whose proof can be found in Appendix A.2.2.

Lemma 2.26. *Let*

$$g(x,y) = \mathbb{P}\left(B_\ell(y) > B_\ell(x)\right) + y \cdot \mathbb{P}\left(B_\ell(y) = B_\ell(x)\right) + \frac{1}{n}\left(1 - \mathbb{P}\left(B_\ell(y) \geq B_\ell(x)\right)\right).$$

Let $x \in [1/2 + 4/n, 1/2 + 4\delta]$, and let $h_x : y \mapsto g(x,y) - y$.

(i) h_x is strictly increasing on $[x, x + 1/\sqrt{\ell}]$.

(ii) h_x has at most one zero on $[x, x + 1/\sqrt{\ell}]$.

Let $f(x)$ be this fixed point if it exists, and let it be equal to $x + 1/\sqrt{\ell}$ otherwise.

(iii) We have that $h_x(f(x)) \leq 0$.

(iv) There exists a constant α such that

$$\left(f(x) - \frac{1}{2}\right) > \left(1 + \frac{1}{4\alpha\sqrt{\ell}}\right) \cdot \left(x - \frac{1}{2}\right).$$

Proof of Lemma 2.22. Without loss of generality, we may assume that $(x_t, x_{t+1}) \in \mathbf{B}_1$ (the same arguments apply to $\mathbf{B}_0$ symmetrically). Note that, by Observation 2.6, and conditioning on (x_t, x_{t+1}), function g is defined such that $\mathbf{E}(x_{t+2}) = g(x_t, x_{t+1})$. Let $c_4 = 1/4\alpha$.

- If $x_t \in [1/2, 1/2 + 4/n]$, then by definition of $\mathbf{B}$, $x_{t+1} \in [1/2, 1/2 + 8/n]$. For the same reason, for (x_{t+1}, x_{t+2}) to be in $\mathbf{B}$, it is necessary that $x_{t+2} \in [1/2, 1/2 + 16/n]$. By Lemma 2.19 (see Section 2.2.2.2), there is a constant probability that it is not the case, and so *(b)* (in the statement of the Lemma 2.22) holds.

- Otherwise, if $x_t \in [1/2 + 4/n, 1/2 + 4\delta]$ and $x_{t+1} > f(x_t)$, then by (iv) in Lemma 2.26,
$$x_{t+1} - \frac{1}{2} > f(x_t) - \frac{1}{2} > \left(1 + \frac{c_4}{\sqrt{\ell}}\right)\left(x_t - \frac{1}{2}\right),$$
and so *(a)* (in the statement of the Lemma 2.22) holds.

- Else, $f(x_t) \geq x_{t+1}$. Moreover, by the definitions of f and $\mathbf{B}_1$, we have the following relation:
$$x_t + 1/\sqrt{\ell} \geq f(x_t) \geq x_{t+1} \geq x_t. \tag{2.15}$$
By (iii) in Lemma 2.26, $g(x_t, f(x_t)) - f(x_t) \leq 0$. Moreover, Eq. (2.15) ensures that x_{t+1} and $f(x_t)$ are within interval $[x_t, x_t + 1/\sqrt{\ell}]$, so by (i) in Lemma 2.26, $g(x_t, x_{t+1}) - x_{t+1} \leq g(x_t, f(x_t)) - f(x_t) \leq 0$, i.e., $g(x_t, x_{t+1}) \leq x_{t+1}$. Recall that $\mathbf{E}(x_{t+2}) = g(x_t, x_{t+1})$, so $\mathbf{E}(x_{t+2}) \leq x_{t+1}$. By Lemma 2.14 (see Section 2.2.2.2), there is a constant probability c_5 that $x_{t+2} < x_{t+1}$. If this the case, since $x_{t+1} > 1/2$ and by the definition of $\mathbf{B}$, we get that $(x_{t+1}, x_{t+2}) \notin \mathbf{B}$ and *(b)* (in the statement of the Lemma 2.22) holds.

This concludes the proof of Lemma 2.22. □

Proof of Lemma 2.23. Without loss of generality, we assume that $(x_t, x_{t+1}) \in \mathbf{B}_1$ (the same arguments apply to $\mathbf{B}_0$ symmetrically). For any round t, let H_t the event that $(x_t, x_{t+1}) \in \mathbf{B}$ and *(a)* of Lemma 2.22 holds. Let $t_{\max} = t_0 + (\sqrt{c}/c_4) \cdot \log^{3/2} n$, and let X be the number of rounds between t_0 and $t_{\max}$ for which H_t does not happen. Each time (a) in Lemma 2.22 doesn't hold, (b) of Lemma 2.22 holds so there is a constant probability to leave $\mathbf{B}$, so

$$\mathbb{P}\left(\text{for every } t \text{ such that } t_0 \leq t \leq t_{\max}, (x_t, x_{t+1}) \in \mathbf{B} \quad \bigcap \quad X = \mathbf{x}\right) \leq (1 - c_5)^{\mathbf{x}}. \tag{2.16}$$

Note that
$$(1 - c_5)^{(t_{\max} - t_0)/4} = \exp\left(\log(1 - c_5) \cdot \frac{\sqrt{c}}{4c_4} \cdot \log^{3/2} n\right).$$

This, together with Eq. (2.16), implies that either (i) $X < (t_{\max} - t_0)/4$, or w.h.p. (ii) there is a time $t_0 \leq t \leq t_{\max}$ such that $(x_t, x_{t+1}) \notin \mathbf{B}$ (in which case Lemma 2.23 holds).

Now, consider case (i). Let $u_t = x_t - 1/2$. By Lemma 2.19, after $O(\log n)$ rounds, we'll have $u_t > 1/\sqrt{n}$ w.h.p., so up to waiting a logarithmic number of rounds, we assume that $u_{t_0} > 1/\sqrt{n}$.

Note now, that whenever $(x_t, x_{t+1}) \in \mathbf{B}_1$, by definition $x_{t+1} \geq x_t > 1/2$ and so (x_{t+1}, x_{t+2}) cannot be in $\mathbf{B}_0$. This implies that the system must remain in $\mathbf{B}_1$ until it leaves $\mathbf{B}$. Also by definition of $\mathbf{B}_1$, if $(x_t, x_{t+1}) \in \mathbf{B}_1$ then $x_t \leq x_{t+1}$, and so

$$u_t \leq u_{t+1}. \tag{2.17}$$

Moreover, by the fact that we are in case (i), we have that the number of rounds $t_0 \leq t \leq t_{\max}$ such that H_t happens is at least

$$k := \frac{3(t_{\max} - t_0)}{4} = \frac{3\sqrt{c}}{4c_4} \cdot \log^{3/2} n.$$

Note that at each such round, by definition of H_t and (a) in Lemma 2.22,

$$u_{t+1} > u_t \cdot \left(1 + \frac{c_4}{\sqrt{\ell}}\right).$$

Hence, by Eq. (2.17),

$$
\begin{aligned}
u_{t_{\max}} > u_{t_0} \cdot \left(1 + \frac{c_4}{\sqrt{\ell}}\right)^k &= u_{t_0} \cdot \exp\left(k \log\left(1 + \frac{c_4}{\sqrt{\ell}}\right)\right) \\
&> u_{t_0} \cdot \exp\left(\frac{4}{5} \cdot \frac{k \cdot c_4}{\sqrt{\ell}}\right) && \text{(for } n \text{ large enough)} \\
&= u_{t_0} \cdot \exp\left(\frac{4}{5} \cdot \frac{3}{4} \log n\right) && \text{(by definition of } k \text{ and } \ell\text{)} \\
&= \cdot u_{t_0} \cdot n^{3/5} \\
&> n^{1/10}. && \text{(since we assumed } u_{t_0} > 1/\sqrt{n}\text{)}
\end{aligned}
$$

When n is large, this quantity is larger than 1, hence (i) is impossible unless $(x_t, x_{t+1}) \notin \mathbf{B}$. This concludes the proof of Lemma 2.23. □

2.2.4.4 Area C

Proof of Lemma 2.24. Without loss of generality, we assume that $(x_t, x_{t+1}) \in \mathbf{C}_1$ (the same arguments apply to $\mathbf{C}_0$ symmetrically). By Observation 2.6, we have

$$
\mathbf{E}(x_{t+2}) = \mathbb{P}\left(B_\ell\left(x_{t+1}\right) > B_\ell\left(x_t\right)\right) + x_{t+1} \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) = B_\ell\left(x_t\right)\right) - \frac{1}{n}.
$$

By Lemma A.6 (taking $\lambda > 2$), this becomes

$$
\mathbf{E}(x_{t+2}) > \frac{1}{2} + 2 \cdot (x_{t+1} - x_t) - \left(\frac{1}{2} - x_{t+1}\right) \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) = B_\ell\left(x_t\right)\right). \tag{2.18}
$$

Case 1. If $(x_{t+1} - x_t) > 1/2 - x_{t+1}$, then Eq. (2.18) implies

$$
\mathbf{E}(x_{t+2}) > \frac{1}{2} + 2 \cdot (x_{t+1} - x_t) - \left(\frac{1}{2} - x_{t+1}\right) > \frac{1}{2} + (x_{t+1} - x_t) > \frac{1}{2},
$$

so with constant probability $x_{t+2} > 1/2$ and $(x_{t+1}, x_{t+2}) \in \mathbf{A}_1$ or is not in YELLOW′.

Case 2. Else, if $(x_{t+1} - x_t) \leq 1/2 - x_{t+1}$, Eq. (2.18) rewrites

$$
\begin{aligned}
\mathbf{E}(x_{t+2}) > &\frac{1}{2}\left(\frac{1}{2} + x_{t+1}\right) \\
&+ \frac{1}{2}\left(\frac{1}{2} + 4 \cdot (x_{t+1} - x_t) - 2\left(\frac{1}{2} - x_{t+1}\right) \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) = B_\ell\left(x_t\right)\right) - x_{t+1}\right),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\mathbf{E}(x_{t+2}) = &\frac{1}{2}\left(\frac{1}{2} + x_{t+1}\right) \\
&+ \frac{1}{2}\left(4 \cdot (x_{t+1} - x_t) + \left(\frac{1}{2} - x_{t+1}\right)\left(1 - 2 \cdot \mathbb{P}\left(B_\ell\left(x_{t+1}\right) = B_\ell\left(x_t\right)\right)\right)\right).
\end{aligned}
$$

Since $(x_t, x_{t+1}) \in \mathbf{C}_1$, we have $x_{t+1} \geq x_t$ and $1/2 > x_{t+1}$. Moreover, for ℓ large enough, $1 - 2 \cdot \mathbb{P}\left(B_\ell(x_{t+1}) = B_\ell(x_t)\right) > 0$. Hence,

$$
\mathbf{E}(x_{t+2}) > \frac{1}{2}\left(\frac{1}{2} + x_{t+1}\right).
$$

If $\mathbb{E}(x_{t+2}) \notin [1/3, 2/3]$, then $(x_{t+1}, x_{t+2}) \notin \mathbf{A}_1$ w.h.p. Otherwise, we can apply Lemma 2.14: with constant probability $x_{t+2} > (1/2 + x_{t+1})/2$, i.e., $x_{t+2} - x_{t+1} > 1/2 - x_{t+2}$. If so, Case 1 applies and with constant probability, $(x_{t+2}, x_{t+3}) \in \mathbf{A}_1$ or is not in YELLOW′.

This concludes the proof of Lemma 2.24. □

2.3 Follow the Trend in the Sequential Setting

In this section, we give experimental evidence suggesting that, with the help of a modest amount of memory, the Follow The Trend approach can solve the bit-dissemination problem in an almost linear number of parallel rounds in the sequential setting.

We start by giving an informal description of the dynamics that we use in the simulations. It is based on the "Follow The Trend" approach (Algorithm 2). To adapt this approach to the sequential setting, we need the agents *not* to execute the protocol every time that they are activated. Indeed, because of the randomness and asynchrony of the activations that are inherent to the sequential setting, some agents will be activated more often than others, undermining the convergence of the dynamics. Instead, we decide that only one every ℓ activations will be *busy*. We will have the agents execute one round of Algorithm 2 only on busy activations, and do nothing otherwise. This behavior can be implemented by adding a clock modulo-ℓ to the internal state of the protocol. As a consequence, the memory size required is doubled, but remains $O(\log \ell)$ bits. In addition, the convergence is expected to be slowed down by a factor $\ell = \Theta(\log n)$ compared to Algorithm 2; yet, it would remain poly-logarithmic in n (in terms of parallel rounds). Formally, the protocol used in the simulations is described as Algorithm 4.

Algorithm 4: Follow the Trend in the sequential setting

Input: Current opinion $Y_t \in \mathcal{Y}$ and internal state $(\sigma_t, \texttt{clock}_t) \in \{0, \dots, \ell\}^2$, sample $S \in \mathcal{Y}^\ell$

if $\texttt{clock} = 0$ **then**
  $\texttt{clock}_{t+1} \leftarrow \ell$;
  $(Y_{t+1}, \sigma_{t+1}) \leftarrow$ Algorithm 2 (Y_t, σ_t, S)
else
  $\texttt{clock}_{t+1} \leftarrow \texttt{clock}_t - 1$;
  $(Y_{t+1}, \sigma_{t+1}) \leftarrow (Y_t, \sigma_t)$;

Output: $Y_{t+1}, (\sigma_{t+1}, \texttt{clock}_{t+1})$

Experimental results. Simulations were performed for $n = 2^i$, where $i \in \{3, \dots, 17\}$ for Algorithm 4, and repeated 100 times each. For comparison purposes, we also simulate the voter dynamics for $i \in \{3, \dots, 10\}$. For Algorithm 4, we took $\ell = 10 \log n$[4]. Every population contains a single source agent.

In self-stabilizing settings, it is not clear what are the worst initial configurations for a given dynamics. Here, we looked at two different ones:

- a configuration in which all opinions (including the one of the source agent) are independently and uniformly distributed in $\{0, 1\}$,
- a configuration in which the source agent holds opinion 0, while all other agents hold opinion 1.

In addition, we simulate Algorithm 6, described in Section 2.5, when a total of 10 opinions (instead of 2) are available to the agents, from a uniformly random configuration.

[4]The factor 10 in the sample size can be replaced by any sufficiently large constant. We did not try to optimize it.

Results are summed up in Figure 2.4, where they are depicted in terms of parallel rounds. The empirical convergence times of the voter dynamics are consistent with the insights of Section 2.4.3. Moreover, our results suggest that the expected convergence time of Algorithm 4 is about $\Theta(\text{polylog}n)$ parallel rounds.

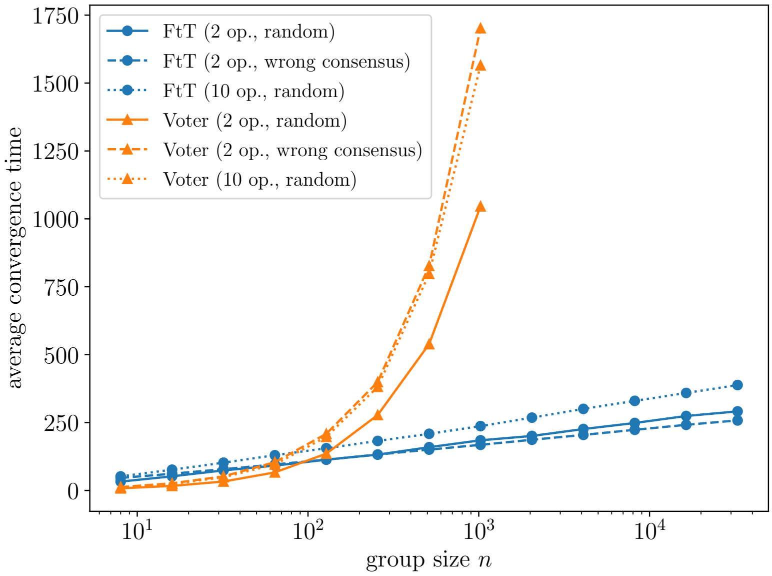


Figure 2.4: **"Follow the Trend" versus the voter model.** Average convergence time (in parallel rounds) is depicted for different values of n, over 100 iterations each. Blue lines with circular markers correspond to Algorithm 4. Orange lines with triangular markers correspond to the voter model. Full lines depict initial configurations in which all opinions are drawn uniformly at random from $\{0, 1\}$. Dashed lines depict initial configurations in which the correct opinion is 0 and all other opinions are 1. Dotted lines depict initial configurations in which all opinions are drawn uniformly at random from $\{0, \dots, 9\}$. The algorithm that is used to deal with more than 2 opinions is derived from Algorithm 6, described in Section 2.5.

2.4 Memory-less Dynamics in the Sequential Setting

The goal of this section is to prove Theorems 2.3 and 2.4.

2.4.1 Notations and Preliminaries

Since we focus on memory-less dynamics and the sequential setting in this section, we start by simplifying the notations defined in Section 2.1.1, and also introduce some useful related concepts. In the remainder, we will assume without loss of generality that the population contains $z \geq 1$ source agents, that have opinion 1.

Transition functions for memory-less dynamics. For memory-less dynamics, we would like to omit the internal state space as defined in Section 2.1.1, which must anyway consist of a singleton: $\Sigma = \{\sigma\}$. Moreover, we are not interested in the order in which the 1-opinions appear in the opinion sample. Therefore, from the transition function f of a given protocol P, we define for $y \in \{0, 1\}$ and $s \in \{0, \dots, \ell\}$,

$$g_y(s) = \mathbb{P}\big(f((y, \sigma), S) = (1, \sigma) \mid \text{the number of 1-opinions in } S \text{ is } s\big).$$

In words, $g_y(s)$ is equal to the probability that an agent, holding opinion y and presented with a opinion sample containing s 1-opinions, adopts opinion 1.

In the context of consensus problems, including the bit-dissemination problem that we consider, g_y must assign probability zero to opinions that are not equal to the current opinion, and have no support in the sample, i.e., $g_0(0) = 0$ and $g_1(\ell) = 1$. Indeed, dynamics which do not meet this constraint cannot enforce consensus.

As an example, with this notation, the voter dynamics is defined by $\ell = 1$, and

$$\begin{cases} g_0(0) = g_1(0) = 0, \\ g_0(1) = g_1(1) = 1. \end{cases}$$

The 3-majority dynamics is defined by $\ell = 3$, and

$$\begin{cases} g_0(0) = g_1(0) = g_0(1) = g_1(1) = 0, \\ g_0(2) = g_1(2) = g_0(3) = g_1(3) = 1. \end{cases}$$

Note that these two dynamics satisfy $g_0(s) = g_1(s)$ for every $s \in \{0, \dots, \ell\}$, since they do not depend on the current opinion of the agents. Conversely, the 2-choice dynamics is defined by $\ell = 2$, and

$$\begin{cases} g_0(0) = g_1(0) = g_0(1) = 0, \\ g_1(2) = g_0(2) = g_1(1) = 1. \end{cases}$$

Similarly, this notation can be used to describe a large class of majority-like dynamics [18].

Markov chains and birth-death chains. The configuration $C_t(P)$ of a memory-less protocol P in round t can be entirely described by the number X_t of agents with opinion 1. This is because, contrary to the more general framework described in Section 2.1.1, there is no need to keep track of the internal state of each agent. Thus, the process is always a Markov chain X_t on $\{0, \dots, n\}$. Moreover, since we place ourselves in the sequential setting in which only one agent is activated at a time, the process is in fact always a *birth-death* chain on $\{0, \dots, n\}$, meaning that the only possible transitions from a given state i are to the states i, $i+1$ (if $i \leq n-1$) and $i-1$ (if $i \geq 1$). Finally, since we assumed the z source agents to have opinion 1, the configuration space can be reduced to $\{z, \dots, n\}$. Following [111], we denote by p_i and q_i respectively the probability of moving to $i+1$ and the probability of moving to $i-1$ when the chain is in state i. Note that $p_n = 0$ and $q_z = 0$. Finally, $r_i = 1 - p_i - q_i$ denotes the probability that, when in state i, the chain remains in that state in the next step. Specifically, the transition probabilities of the birth-death chain $\mathcal{C}$ corresponding to a memory-less dynamics with transition function g, can be computed as follows: for every $i \in \{z, \dots n-1\}$,

$$\begin{aligned} p_i &= \mathbb{P}\left(X_{t+1} = i+1 \mid X_t = i\right) \\ &= \frac{n-i}{n} \sum_{s=0}^{\ell} g_0(s) \cdot \mathbb{P}\left(|S| = s \mid X_t = i\right) \\ &= \frac{n-i}{n} \mathbf{E}_i\left(g_0(|S|)\right), \end{aligned} \tag{2.19}$$

where, following the notation of [111], for a random variable V defined over some Markov chain $\mathcal{C}$, we denote by $\mathbf{E}_i(V)$ the expectation of V when $\mathcal{C}$ starts in state i. Eq. (2.19) follows from the law of total probability applied to the possible values for $|S|$ and observing that: (1) the transition $i \to i+1$ can only occur if an agent holding opinion 0 is activated, which happens with probability $(n-i)/n$, and (2) if such an agent observes s agents with opinion 1 in its sample, it will adopt that opinion with probability $g_0(s)$. Likewise, for $i \in \{z+1, \dots n\}$:

$$q_i = \mathbb{P}\left(X_{t+1} = i-1 \mid X_t = i\right) = \frac{i-z}{n}\left(1 - \mathbf{E}_i\left(g_1(|S|)\right)\right), \tag{2.20}$$

with the only caveat that, differently from the previous case, the transition $i+1 \to i$ can only occur if an agent with opinion 1 is selected for update and *this agent is not a source*. For this chain, in addition to $p_n = 0$ and $q_z = 0$ we also have $q_n = 0$, which follows since $g_1(\ell) = 1$.

The *hitting time* [111, Section 1] of state i is the first time the chain is in state i, namely:

$$\tau_i = \min\{t \geq 0 : X_t = i\}.$$

Therefore, the expected convergence time of P starting in a configuration with $i \geq z$ agents holding opinion 1 is simply $\mathbf{E}_i\left(\tau_n\right)$.

2.4.2 Lower Bound

In this section, we prove a lower bound on the convergence time of memoryless dynamics (Theorem 2.3). We show that this negative result holds in a very-strong sense: any dynamics must take $\Omega(n)$ parallel rounds in expectation, even if the agents have full knowledge of the current system configuration.

To account for the fact that agents have access to the exact configuration of the system, we slightly modify the notation introduced in Section 2.4.1, so that here $g_{x_u} : \{0, \ldots, n\} \to [0, 1]$ assigns a probability to the number of ones that appear in the population, rather than in a random sample of size ℓ. Before we prove our main result, we need the following two technical lemmas.

Lemma 2.27. *For every $N \in \mathbb{N}$, for every $x \in \mathbb{R}^N$ s.t. for every $i \in \{1, \ldots, N\}$, $x_i > 0$, we have either $\sum_{i=1}^{N} x_i \geq N$ or $\sum_{i=1}^{N} \frac{1}{x_i} \geq N$.*

Proof. Consider the case that $\sum_{i=1}^{n} x_i \leq N$. Using the inequality of arithmetic and geometric means, we can write

$$1 \geq \frac{1}{N} \sum_{i=1}^{N} x_i \geq \left(\prod_{i=1}^{N} x_i \right)^{\frac{1}{N}}.$$

Therefore,

$$1 \leq \left(\prod_{i=1}^{N} \frac{1}{x_i} \right)^{\frac{1}{N}} \leq \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x_i},$$

which concludes the proof of Lemma 2.27. □

Lemma 2.28. *Consider any birth-death chain on $\{0, \ldots, n\}$. For $1 \leq i \leq j \leq n$, let $a_i = q_i/p_{i-1}$ and $a(i:j) = \prod_{k=i}^{j} a_k$. Then, $\mathbf{E}_0(\tau_n) \geq \sum_{1 \leq i < j \leq n} a(i:j)$.*

Proof. Let $w_0 = 1$ and for $i \in \{1, \ldots, n\}$, let $w_i = 1/a(1:i)$. The following result is well-known [111, Eq. (2.13)]: for every $\ell \in \{1, \ldots, n\}$,

$$\mathbf{E}_{\ell-1}(\tau_\ell) = \frac{1}{q_\ell w_\ell} \sum_{i=0}^{\ell-1} w_i.$$

Thus,

$$\mathbf{E}_{\ell-1}(\tau_\ell) = \frac{1}{q_\ell} \sum_{i=0}^{\ell-1} \frac{a(1:\ell)}{a(1:i)} = \frac{1}{q_\ell} \sum_{i=1}^{\ell} a(i:\ell) \geq \sum_{i=1}^{\ell} a(i:\ell).$$

Eventually, we can write

$$\mathbf{E}_0(\tau_n) = \sum_{\ell=1}^{n} \mathbf{E}_{\ell-1}(\tau_\ell) \geq \sum_{1 \leq i < j \leq n} a(i:j),$$

which concludes the proof of Lemma 2.28. □

Proof of Theorem 2.3. Fix $z \in \mathbb{N}$. Let $n \in \mathbb{N}$, s.t. $n > 4z$, and let P be any memoryless dynamics. The idea of the proof is to show that the birth-death chain associated with P, as described in Section 2.4.1, cannot be "fast" in both directions at the same time. We restrict the analysis to the subset of states $\chi = \{n/4, \ldots, 3n/4\}$. More precisely, we consider the two following birth-death chains: (i) $\mathcal{C}$ with state space χ, whose states represent the number of agents with opinion 1, and assuming that the source agents hold opinion 1; and (ii) $\mathcal{C}'$ with state space χ, whose states represent the number of agents with opinion 0, and assuming that the source agents hold opinion 0. Let $\tau_{3n/4}$ (resp. $\tau'_{3n/4}$) be the hitting time of the state $3n/4$ of chain $\mathcal{C}$ (resp. $\mathcal{C}'$). We will show that

$$\max\left(\mathbf{E}_{n/4}\left(\tau_{3n/4}\right), \mathbf{E}_{n/4}\left(\tau'_{3n/4}\right)\right) = \Omega(n^2).$$

Let $g_0, g_1 : \chi \to [0,1]$ be the transition functions of P, restricted to χ. Following Eqs. (2.19) and (2.20), the transition probabilities for $\mathcal{C}$ are

$$p_i = \frac{n-i}{n} g_0(i), \qquad q_i = \frac{i-z}{n}(1 - g_1(i)).$$

Note that the expectations have been removed as a consequence of agents having "full knowledge" of the configuration. Similarly, for $\mathcal{C}'$, the transition probabilities are

$$p'_i = \frac{n-i}{n}(1 - g_1(n-i)), \qquad q'_i = \frac{i-z}{n} g_0(n-i).$$

Following the definition in the statement of Lemma 2.28, we define a_i and a'_i for $\mathcal{C}$ and $\mathcal{C}'$ respectively. We have

$$a_i = \frac{q_i}{p_{i-1}} = \frac{i-z}{n-i+1} \cdot \frac{1-g_1(i)}{g_0(i-1)},$$

and

$$a'_i = \frac{q'_i}{p'_{i-1}} = \frac{i-z}{n-i+1} \cdot \frac{g_0(n-i)}{1-g_1(n-i+1)}.$$

Observe that we can multiply these quantities by pairs to cancel the factors on the right hand side:

$$a_{n-i+1} \cdot a'_i = \frac{i-z}{i} \cdot \frac{n-i+1-z}{n-i+1}. \tag{2.21}$$

$(i-z)/i$ is increasing in i, so it is minimized on χ for $i = n/4$. Similarly, $(n-i+1-z)/(n-i+1)$ is minimized for $i = 3n/4$. Hence, we get the following coarse lower bound from Eq. (2.21): for every $i \in \chi$,

$$a_{n-i+1} \cdot a'_i \geq \left(1 - \frac{4z}{n}\right)^2. \tag{2.22}$$

Following the definition in the statement of Lemma 2.28, we define $a(i:j)$ and $a'(i:j)$ for $\mathcal{C}$ and $\mathcal{C}'$ respectively. From Eq. (2.22), we get for any $i, j \in \chi$ with $i \leq j$:

$$\begin{aligned} a'(i:j) &\geq \left(1 - \frac{4z}{n}\right)^{2(j-i+1)} \frac{1}{a(n-j+1 : n-i+1)} \\ &\geq \left(1 - \frac{4z}{n}\right)^{n} \frac{1}{a(n-j+1 : n-i+1)}. \end{aligned}$$

Let $c = c(z) = \exp(-4z)/2$. For n large enough,

$$a'(i:j) \geq \frac{c}{a(n-j+1 : n-i+1)}. \tag{2.23}$$

Let $N = n^2/8 + n/4$. By Lemma 2.27, either

$$\sum_{\substack{i,j \in \chi \\ i<j}} a(i:j) \geq N,$$

or (by Eq. (2.23))

$$\sum_{\substack{i,j \in \chi \\ i<j}} a'(i:j) \geq c \sum_{\substack{i,j \in \chi \\ i<j}} \frac{1}{a(i:j)} \geq cN.$$

By Lemma 2.28, it implies that either

$$\mathbf{E}_{n/4}\left(\tau_{3n/4}\right) \geq N, \quad \text{or} \quad \mathbf{E}_{n/4}\left(\tau'_{3n/4}\right) \geq cN.$$

In both cases, there exists an initial configuration for which at least $\Omega(n^2)$ rounds, i.e., $\Omega(n)$ parallel rounds, are needed to achieve consensus, which concludes the proof of Theorem 2.3. □

2.4.3 Upper Bound

In this section, we prove that the voter dynamics achieves consensus within $O(n \log n)$ parallel rounds in expectation (Theorem 2.4). We prove the result for $z = 1$, noting that the upper bound can only improve for $z > 1$.

The modified chain $\mathcal{C}'$. In principle, we could study convergence of the voter model using the chain $\mathcal{C}$ introduced in Section 2.4.1. Unfortunately, $\mathcal{C}$ has one absorbing state (the state n corresponding to consensus), hence it is not reversible, so that we cannot leverage known properties of reversible birth-death chains [111, Section 2.5] that would simplify the proof. Note, however, that we are only interested in τ_n, the number of rounds to reach state n under the voter model. For this purpose, it is possible to consider a second chain $\mathcal{C}'$ that is almost identical to $\mathcal{C}$, but reversible. In particular, the transition probabilities p_i and q_i of $\mathcal{C}'$ are the same as in $\mathcal{C}$, for $i \in \{z, \dots, n-1\}$. Moreover, we have $p_n = 0$ (as in $\mathcal{C}$) but $q_n = 1$ [5]. Obviously, for any initial state $i \leq n-1$, τ_n has exactly the same distribution in $\mathcal{C}$ and $\mathcal{C}'$. For this reason, in the remainder of this section we consider the chain $\mathcal{C}'$, unless otherwise stated.

Proof of Theorem 2.4. We first compute the general expression for $\mathbf{E}_z(\tau_n)$, i.e., the expected time to reach state n (thus, consensus) in $\mathcal{C}'$ when the initial state is z, corresponding to the system starting in a state in which only the source agents hold opinion 1. We then give a specific upper bound when $z = 1$. First of all, we recall that, for z source agents we have that

$$\mathbf{E}_z(\tau_n) = \sum_{k=z+1}^{n} \mathbf{E}_{k-1}(\tau_k). \tag{2.24}$$

As was already mentioned in Section 2.4.1, for the voted dynamics, we have $g_0 = g_1 = g$, since the output does not depend on the opinion of the agent. By Eqs. (2.19) and (2.20), since $\mathbf{E}\left(g(|S|)\right) = i/n$ whenever the number of agent with opinion 1 in the system is i, we have for $\mathcal{C}'$

$$\begin{aligned} p_i &= \begin{cases} \frac{(n-i)i}{n^2} & \text{for } i \in \{z, \dots, n-1\}, \\ 0 & \text{for } i = n, \end{cases} \\ q_i &= \begin{cases} 0 & \text{for } i = z, \\ \frac{(n-i)(i-z)}{n^2} & \text{for } i \in \{z+1, \dots, n-1\}, \\ 1 & \text{for } i = n. \end{cases} \end{aligned} \tag{2.25}$$

The proof now proceeds along the following steps.

General expression for $\mathrm{E}_{k-1}(\tau_k)$. It is not difficult to see that

$$\mathbf{E}_{k-1}(\tau_k) = \frac{1}{q_k w_k} \sum_{j=z}^{k-1} w_j\,, \tag{2.26}$$

[5] Setting $q_n = 1$ is only for the sake of simplicity, any positive value will do.

where $w_0 = 1$ and $w_k = \prod_{i=z+1}^{k} \frac{p_{i-1}}{q_i}$, for $k = z+1, \ldots, n$. Indeed, the w_k's satisfy the detailed balanced conditions $p_{k-1}w_{k-1} = q_k w_k$ for $k = z+1, \ldots, n$, since

$$p_{k-1}w_{k-1} = p_{k-1}\frac{q_k}{p_{k-1}} \prod_{i=z+1}^{k} \frac{p_{i-1}}{q_i} = q_k w_k.$$

and Eq. (2.26) follows proceeding like in [111, Section 2.5].

Computing $\mathbf{E}_{k-1}(\tau_k)$ for $\mathcal{C}'$. Considering the expressions of p_i and q_i in Eq. (2.25), for $k \in \{z+1, \ldots, n-1\}$, we have

$$\begin{aligned} w_k &= \prod_{i=z+1}^{k} \frac{(n-i+1)(i-1)}{(i-z)(n-i)} \\ &= \prod_{i=z+1}^{k} \frac{n-i+1}{n-i} \cdot \prod_{i=z+1}^{k} \frac{i-1}{i-z} = \frac{n-z}{n-k} \cdot \prod_{i=z+1}^{k} \frac{i-1}{i-z}. \end{aligned}$$

Hence

$$w_k = \begin{cases} \frac{n-z}{n-k} f(k) & \text{for } k \in \{z+1, \ldots, n-1\}, \\ \frac{(n-z)(n-1)}{n^2} f(n-1) & \text{for } k = n, \end{cases}$$

where $f(k) = \prod_{i=z+1}^{k} \frac{i-1}{i-z}$.

The case $z = 1$. In this case, the formulas above simplify and, for $k \in \{z+1, \ldots, n-1\}$, we have

$$\mathbf{E}_{k-1}(\tau_k) = \frac{n^2}{(k-1)f(k)} \sum_{j=1}^{k-1} \frac{f(j)}{n-j} = \frac{n^2}{k-1} \sum_{j=1}^{k-1} \frac{1}{n-j},$$

where the last equality follows from the fact that $f(z) = f(z+1) = \cdots = f(k) = 1$, whenever $z = 1$. Moreover, for $k = n$ we have

$$\begin{aligned} \mathbf{E}n - 1\tau_n &= \frac{1}{q_n w_n} \sum_{j=1}^{n-1} w_j = \left(\frac{n}{n-1}\right)^2 \sum_{j=1}^{n-1} \frac{n-1}{n-j} \\ &= \frac{n}{n-1} H_{n-1} = O(\log n), \end{aligned}$$

where H_k denotes the k-th harmonic number. Hence, for $z = 1$ we have

$$\begin{aligned} \mathbf{E}_1(\tau_n) &= \sum_{k=2}^{n} \mathbf{E}_{k-1}(\tau_k) \\ &= n^2 \sum_{k=2}^{n-1} \frac{1}{k-1} \sum_{j=1}^{k-1} \frac{1}{n-j} + O(\log n), \end{aligned} \tag{2.27}$$

where in the second equality we took into account that $\mathbf{E}_{n-1}(\tau_n) = O(\log n)$. Finally, it is easy to see that

$$\sum_{k=2}^{n-1} \frac{1}{k-1} \sum_{j=1}^{k-1} \frac{1}{n-j} = O(\log n) \tag{2.28}$$

Indeed, if we split the sum at $\lfloor n/2 \rfloor$, for $k \leq \lfloor n/2 \rfloor$ we have

$$\sum_{k=2}^{\lfloor n/2 \rfloor} \frac{1}{k-1} \sum_{j=1}^{k-1} \frac{1}{n-j} \leq \sum_{k=2}^{\lfloor n/2 \rfloor} \frac{1}{k-1} \sum_{j=1}^{k-1} \frac{2}{n} = O(1) \tag{2.29}$$

and for $k > \lfloor n/2 \rfloor$ we have

$$\sum_{k=\lfloor n/2\rfloor+1}^{n-1} \frac{1}{k-1} \sum_{j=1}^{k-1} \frac{1}{n-j} \leq \sum_{k=\lfloor n/2\rfloor+1}^{n-1} \frac{2}{n} \sum_{j=0}^{n-1} \frac{1}{n-j}$$
$$= \sum_{k=\lfloor n/2\rfloor+1}^{n-1} \frac{2}{n} H_n = O(\log n)\,.$$

From Eqs. (2.29) and (2.30) we get Eq. (2.28), and Theorem 2.4 follows by using in Eq. (2.27) the bound in Eq. (2.28).

□

2.4.4 Faster Dissemination in the Parallel Setting

In this section, we briefly prove Observation 2.5. To this aim, we define the *minority* dynamics, which has an interest in itself. Informally, if all samples are equal to 0 (resp. 1), then the minority dynamics adopts opinion 0 (resp. 1); otherwise, it adopts the opinion with less samples (breaking ties randomly). In other words, it consists in choosing the opinion with minimal number of samples, among the opinions that appear at least once. The latter description is valid even when $\mathcal{Y}$ contains more than 2 opinions, and corresponds to Algorithm 5 below.

Algorithm 5: Minority dynamics

1 **Input**: Current opinion $Y_t \in \mathcal{Y}$, opinion sample $S \in \mathcal{Y}^\ell$

2 $Y_{t+1} \leftarrow \underset{\substack{y\in\mathcal{Y} \\ y \text{ appears in } S}}{\operatorname{argmin}} \{\text{number of occurrences of } y \text{ in } S\}$ `// Breaking ties randomly`

3 **Output**: Y_{t+1}

Proof of Observation 2.5. Assume that each sample consists of all opinions in the system, and that agents run Algorithm 5 in the parallel setting. Consider the case that the source has opinion 1. We distinguish between 3 cases. **(1)** If $1 \leq X_t < n/2$, then all non-source agents will adopt opinion 1, and consensus will be achieved in the next round. **(2)** If $n/2 < X_t < n$, then all non-source agents will adopt opinion 0, so $X_{t+1} = 1$ and it reduces to case (1). **(3)** If $X_t = n/2$, the system will be in case (1) or (2) in the next round with high probability. Overall, the expected convergence time is $O(1)$ parallel rounds, which concludes the proof. □

Note that Algorithm 5 can also solve the bit-dissemination problem in logarithmic time when ℓ is "only" linear in n, e.g., when $\ell = n/3$. This claim can be proved easily, with the same techniques as in Section 2.2. However, finding the minimal ℓ for which the minority dynamics succeeds is an interesting and probably challenging open question.

2.5 The Case of Many Opinions

In this section, we consider the more general case with k opinions, for an arbitrary $k \in \mathbb{N}$. In this case, the opinion space (as defined in Section 2.1.1) rewrites $\mathcal{Y} = \{0, \dots, k-1\}$. We assume that k is fixed with respect to n.

We will also assume that an agent can only adopt an opinion if it is present in the current opinion sample, or if it is the current opinion of the agent. This is to prevent protocols from using a larger set of opinions as a way to artificially increase their memory, or the number of bits of information revealed with each interaction.

An unrealistic consequence of our model is that agents can theoretically agree, in advance, on a total order over the opinion space. This can be leveraged by algorithms to obtain better convergence times, especially when more than 2 opinions are concerned. However, it is a form of "global organization" that we want to avoid in our model, since it rarely holds in biological scenarios. Nonetheless, we will keep the definitions that we adopted so far, for the sake of convenience[6]. We note that the generalization of Theorems 2.3 and 2.4 described in Section 2.5.1 would also apply if opinions were unordered. The results of Section 2.5.2 on the other hand, rely heavily on this property, and should therefore be considered as weaker.

2.5.1 Memory-less Dynamics with Many Opinions

We note that the lower bound on the convergence time stated in Theorem 2.3 already applies to any number of opinions, by restricting attention to the binary case.

Generalization of Theorem 2.3. In the self-stabilizing framework, the convergence time of a protocol is defined as the maximum expected convergence time, over all initial configurations (Definition 2.1). This include configurations in which the opinions of all agents belong to $\{0, 1\}$. On such initial configurations, the opinions will always remain binary, since we assume in this section that an agent cannot adopt an opinion that is not already present in the population. Therefore, we can also use the lower-bound of Theorem 2.3 in the case of k opinions. □

It is quite straightforward to extend the proof of Theorem 2.4 as well, using the following standard argument.

Generalization of Theorem 2.4. When it comes to the voter dynamics, we note that we can collapse all wrong opinions into a single class without loss of generality. Indeed, the Markov chain corresponding to the number of agents with the correct opinion will have the same transition probabilities as the chain $\mathcal{C}'$ defined by Eq. (2.25) in the proof of Theorem 2.4. Therefore, conditioning on the number of agents with the correct opinion in the initial configuration, the analysis of Section 2.4.3 carries over seamlessly, which implies that the expected convergence time is the same regardless of k. □

2.5.2 Follow the Trend with Many Opinions

In this section, we explain how to modify Algorithm 2 in order to deal with more than 2 opinions. We start by describing a natural generalization. Upon activation, agents simply adopt one of the opinions maximizing the difference between the number of occurrences in the current sample, and the number of occurrences in the previous sample – and prioritizing the current opinion, if possible. Formally, this is described by Algorithm 6. One can check that, when $k = 2$, it is strictly equivalent to Algorithm 2.

Algorithm 6: Follow the Trend with many opinions

Input: Current opinion $Y_t \in \mathcal{Y}$, current internal state $\sigma_t \in \{0, \ldots, \ell\}^{\mathcal{Y}}$, opinion sample $S \in \mathcal{Y}^\ell$

for $y \in \mathcal{Y}$ **do**
  $\sigma_{t+1}[y] \leftarrow$ number of occurences of y in S ;

$A \leftarrow \underset{y \in \mathcal{Y}}{\operatorname{argmax}} \left(\sigma_{t+1}[y] - \sigma_t[y]\right)$;

if $Y_t \in A$ **then** $Y_{t+1} \leftarrow Y_t$;

else $Y_{t+1} \leftarrow$ any $y \in A$;

Output: Y_{t+1}, σ_{t+1}

[6] A possible but cumbersome workaround is the following: (1) set, for each Agent i, a permutation σ_i on $\mathcal{Y}$, that Agent i is not aware of, representing his *perception* of the opinions, (2) when Agent i observes Agent j holding opinion y, set the corresponding sample to be $\sigma_i \circ \sigma_j^{-1}(y)$, and (3) define the convergence time of a protocol as the maximum over all possible choices of $(\sigma_i)_{i \in I}$.

Empirically, Algorithm 6 appears to converge fast on the correct opinion. For an illustration, see Figure 2.4, that displays the estimated convergence time of a version of Algorithm 6 adapted to the sequential setting. Unfortunately, its mathematical analysis remains for future work.

However, we can imagine another version of Algorithm 2, whose analysis can be reduced to the proof described in Section 2.2. Contrary to Algorithm 6, this protocol utilizes the fact that the opinions are taken from a total order, and that all agents agree on this order, as discussed in the introduction of Section 2.5.

The idea behind the reduction is to let the agents operate Algorithm 2 successively on each bit of their opinion. More precisely, let $m = \lceil \log_2 k \rceil \in \mathbb{N}$. Given an opinion $y \in \mathcal{Y}$, consider the binary representation $[y]_2$ of y (as an m-bit string). At any given round, a non-source agent starts by collecting the set of opinions that correspond to its current sample and the sample obtained in the previous round. It then identifies the most significant bit i on which at least 2 opinions disagree. To set a value for this bit, the "Follow the Trend" strategy is used. The remaining bits, $j \neq i$, are then set such that the final adopted opinion matches one of the sampled opinions. This is to ensure that the opinion chosen by our agent already exists in the population.

Formally, the internal state space is $\Sigma = \{0, \dots, \ell\}^m$. In the state $\sigma_t^{(i)} = (\sigma_t^{(i)}[1], \dots, \sigma_t^{(i)}[m])$ of Agent i in round t, $\sigma_t^{(i)}[j]$ represents the number of samples received in the previous round, for which the j^{th} bit is equal to 1. The protocol is described by Algorithm 7.

Algorithm 7: Follow the Trend bit by bit

Input: Current opinion $Y_t \in \mathcal{Y}$, current internal state $\sigma_t \in \{0, \dots, \ell\}^m$, opinion sample $S \in \mathcal{Y}^\ell$

for $j \in \{1, \dots, m\}$ **do**
    $\sigma_{t+1}[j] \leftarrow$ number of opinions in S whose j^{th}-bit is equal to 1 ;

`// Find all bit indices for which at least 2 samples disagree`

$B \leftarrow \{j \in \{1, \dots, m\} \mid \sigma_{t+1}[j] \notin \{0, \ell\}$ or $\sigma_t[j] \neq \sigma_{t+1}[j]\}$;

if $B = \emptyset$ **then** $Y_{t+1} \leftarrow$ any $y \in S$;

else
    `// Apply FET on the most significant bit for which at least 2 samples disagree`
    $j_0 \leftarrow \max(B)$;
    if $\sigma_{t+1}[j_0] > \sigma_t[j_0]$ **then** $Y_{t+1} \leftarrow$ any $y \in S$ s.t. the j_0^{th} bit of y is 1 ;
    else if $\sigma_{t+1}[j_0] < \sigma_t[j_0]$ **then** $Y_{t+1} \leftarrow$ any $y \in S$ s.t. the j_0^{th} bit of y is 0 ;
    else $Y_{t+1} \leftarrow Y_t$;

Output: Y_{t+1}, σ_{t+1}

We show that Algorithm 7 solves the bit-dissemination problem by applying the analysis of Section 2.2 to each bit successively.

Theorem 2.29. *When $\mathcal{Y} = \{0, \dots, k-1\}$, Algorithm 7 solves the bit-dissemination problem in the parallel setting in $O(\log k \log^{5/2} n)$ parallel rounds with high probability, while relying on $\ell = \Theta(\log n)$ samples in each round and using $\Theta(k \log \ell)$ bits of memory.*

A useful observation is that Algorithm 7 never introduces opinions that are not already present in the population. Formally,

Lemma 2.30. *Assume that all agents run Algorithm 7. Let $y \in \mathcal{Y}$ and $t \in \mathbb{N}$. If for every $i \in I$, $Y_t^{(i)} \neq y$, then for all $t' > t$, and for every $i \in I$, $Y_{t'}^{(i)} \neq y$.*

However, one undesirable feature of Algorithm 7 is that sometimes it does not exactly behave like Algorithm 2 w.r.t. every bit. This happens when there is $j > j_0$ s.t. all samples (in round $t-1$ and t) have the same j^{th} bit, but it is different from the j^{th} bit of the opinion. In this situation:

- Algorithm 2 keeps the same opinion, because of the absence of trend;

- Algorithm 7 outputs one of the samples, resulting in a change of the opinion.

In fact, deviating from Algorithm 2 in this way is necessary to avoid creating a new opinion. Before proceeding to the reduction, we need to show that this modification does not prevent Algorithm 2 from stabilizing.

Lemma 2.31. *The modified version of Algorithm 2, in which an agent observing only 1 (resp. 0) in two consecutive rounds adopts opinion 1 (resp 0) despite the absence of trend, also solves the bit-dissemination problem with 2 opinions in the parallel setting in $O(\log^{5/2} n)$ parallel rounds with high probability.*

Proof. Consider the case that all non-source agents run the aforementioned modified version of Algorithm 2. We say that Agent i is in a *degenerate state* in round t if (1) all samples received in round $t-1$ and t are the same, and (2) its opinion is different. We claim the following:

(i) If Agent i is not in a degenerate state in round t, then it will never be in a degenerate state again, for every round $s > t$.

(ii) If Agent i is not in a degenerate state in round t, then it will behave exactly as Algorithm 2.

The proof of Lemma 2.31 follows from (i) and (ii) by applying Theorem 2.2, since after one round, the protocol is undistinguishable from Algorithm 2. Statement (ii) follows directly from the definition of Algorithms 2 and 7, so all that remains is to prove Statement (i).

Assume that Agent i is in a degenerate state in round $t > 1$, i.e., $\sigma_t^{(i)} = \ell \cdot (1 - Y_t^{(i)})$. We will show that Agent i must be in a degenerate state also in round $t-1$. Without loss of generality, we assume that $Y_t^{(i)} = 1$ and $\sigma_t^{(i)} = 0$. If $\sigma_{t-1}^{(i)} > 0$, the protocol would have set $Y_t^{(i)} \leftarrow 0$, so necessarily we have $\sigma_{t-1}^{(i)} = 0$. Then $\sigma_{t-1}^{(i)} = \sigma_t^{(i)}$, so the protocol must have set $Y_t^{(i)} \leftarrow Y_{t-1}^{(i)}$. Hence $Y_{t-1}^{(i)} = Y_t^{(i)} = 1$ and Agent i in a degenerate state in round $t-1$. By taking the contrapositive, we obtain that if Agent i is not in a degenerate state in round $t-1$, then it is not in a degenerate state in round t and we conclude the proof of Lemma 2.31 by induction on t. □

For $y \in \mathcal{Y}$, we denote by $y[j] \in \{0,1\}$ the j^{th} bit of y. Let $\mathcal{P}_t(j,y)$ be the property that in round t, all agents agree on all bit values of opinion y down to the j^{th}. Formally,

$$\mathcal{P}_t(j,y) \iff \text{for every } i \in I \text{ and every } k' \geq j, Y_t^{(i)}[j'] = y[j'].$$

Claim 2.32. *If there is $t \in \mathbb{N}$, $j \in \{1,\dots,m\}$ and $y \in \mathcal{Y}$ s.t. $\mathcal{P}_t(j,y)$ holds, then for every $t' > t$, $\mathcal{P}_{t'}(j,y)$ holds.*

Proof. This is a direct consequence of Lemma 2.30. □

Let $\mathcal{T}(j) = \inf\{t \mid \exists y \in \mathcal{Y} \text{ s.t. } \mathcal{P}_t(j,y) \text{ holds}\}$. In other words, $\mathcal{T}(j)$ is the smallest round for which all agents agree on all bit values down to the j^{th}. To prove the theorem, it is enough to show that $\mathcal{T}(1) \leq m \cdot O(\log^{5/2} n)$ w.h.p. For this purpose, we will bound the time between $\mathcal{T}(j+1)$ and $\mathcal{T}(j)$.

Claim 2.33. *For every $j \geq 1$, $\mathcal{T}(j) < \mathcal{T}(j+1) + O(\log^{5/2} n)$ w.p. at least $1 - 1/n^2$.*

Proof. Let $j \geq 0$. By definition, from round $\mathcal{T}(j+1)$ onward, we can find $y \in \mathcal{Y}$ s.t. $\mathcal{P}_{\mathcal{T}(j+1)}(j+1,y)$ holds. Let $t \geq \mathcal{T}(j+1)+1$, and $i \in I$. Consider the execution of Algorithm 7 by Agent i in round $t+1$, as specified by the pseudo-code. By Claim 2.32, $\mathcal{P}_{t-1}(j+1,y)$ and $\mathcal{P}_t(j+1,y)$ hold. Thus, all samples received by Agent i in round $t-1$ and t share the same bit values for every index $j' \geq j+1$. Hence, in round $t+1$, $B = \emptyset$ or $\max(B) \leq j$, and we can restrict the analysis to the two following cases.

- **Case** $j = \max(B)$. This happens if and only if at least two samples, in $S_{t-1} \cup S_t$, disagree. In this case Algorithm 7 behaves explicitly as Algorithm 2 w.r.t. the j^{th} bit.
- **Case** $j \notin B$. This happens if and only if all samples in $S_{t-1} \cup S_t$ have the same j^{th} bit, that is, there is $b \in \{0,1\}$ s.t. $\sigma_t^{(j)} = \sigma_{t+1}^{(j)} = \ell \cdot b$. Algorithm 7 always outputs one of the samples, so $Y_{t+1}[j] = b$, and this time Algorithm 7 behaves as the modified version of Algorithm 2 mentioned in lemma 2.31 w.r.t. the i^{th} bit.

By Lemma 2.31, this concludes the proof of Claim 2.33. □

Proof of Theorem 2.29. By definition, $\mathcal{T}(m+1) = 0$. By applying Claim 2.33 inductively and taking the union bound, we obtain that $\mathcal{T}(0) \leq m \cdot O(\log^{5/2} n)$ w.p. at least $1 - m/n^2 > 1 - \log(n)/n^2$, which concludes the proof of Theorem 2.29. □

2.6 Discussion and Future Work

This chapter considers a natural problem of information spreading in a self-stabilizing context, where it is assumed that a source agent has useful knowledge about the environment, and others would like to learn this information without being able to distinguish the source from non-source agents. Motivated by biological scenarios, our focus is on solutions that utilize passive communication.

We identify a natural algorithm, called Follow The Trend (Algorithm 2), which turns out to be efficient for this problem, and provide an bound on its convergence time in the parallel setting. In addition, we investigate the role played by memory. Under the sequential setting, we prove that incorporating past observations in the current decision is necessary for achieving fast convergence, regardless of the sample size. Two separations arise from our results. On one hand, for memory-less protocols with unbounded sample size, we identify an exponential gap in terms of convergence times between the parallel and the sequential setting. On the other hand, in the sequential setting, we find another exponential gap between memory-less dynamics, and dynamics that are allowed to use a small amount of memory.

Regarding our analysis of Algorithm 2, different performance parameters may be further optimized in future work. For example, our analysis uses $O(\log n)$ samples per round, and it would be interesting to see whether the problem can be solved in poly-logarithmic time w.h.p, by using only a constant number of samples per round. In addition, we do not exclude the possibility that a tighter analysis would reduce our bound on the running time. Finally, the analysis has yet to be extended to the sequential setting, or to the case of more than 2 opinions, for which we only provide empirical clues.

We note that our protocols and analyses could be adapted, in the future, to any process that is required to alternate consensus between different opinions, periodically. Such oscillating behaviour is fundamental to sequential decision making processes [15].

To conclude, we recall that the ultimate goal of our line of research is to reflect on biological processes. Still, more work must be done to obtain concrete biological conclusions. Conducting an experiment that fully adheres to our model, or refining our results to apply to more realistic settings, remains for future work. The goal would be to derive a lower bound on a biological parameter, or to check empirically whether the "Follow The Trend" strategy exists in animal groups. Promising experimental settings include fish schooling [49, 48, 145], collective sequential decision making in ants [15], and recruitment in ants [134, 131]. Any successful outcome would be highly pioneering from a methodological perspective.

Chapter 3

Alignment with Noisy Samples of Group Average

Chapter Contents

This chapter is based on

[106] Amos Korman and R.V. Distributed Alignment Processes With Samples of Group Average. *IEEE Transactions on Control of Network Systems*, 10(2):960–971, June 2023.

3.1 Introduction

In several contexts, consensus algorithms can be designed by letting each agent repeatedly execute the two following steps.

1. obtain an estimate regarding the average opinion of other agents in the system;
2. shift the opinion by a certain extent towards the estimated average.

In what follows, we will abstract away the first step, and focus instead on the second step. Specifically, we will see how agents can use noisy information about the average opinion in order to reach, and maintain, approximate agreement. Before that, we will see some motivating examples and describe these steps in more details.

First step: examples of global parameter samplings. To begin with, we provide a few examples of scenarios in which agents are somehow able to sense the general behaviour of the system. We classify these examples into three categories.

- *Using an external entity as a relay of information.* In several contexts, agents may use an external entity (like a "leader" agent) to gather information, compute a certain function of it (e.g., the average value), and then communicate this information to all others [155].

  Indirectly, this happens during the process of cooperative transport. In this process, the carried object itself serves as a medium entity through which the global tendency of the group can be sensed by each carrying agent. In ants for example, the authors of [74, 75, 116, 15] assume that each individual can feel the sum of the forces exerted by all carrying ants. After measuring this sum (which corresponds to the velocity of the object), each ant decides whether, and by how much, it should align its force with it.

  Other natural examples include the behavior of a swarm of flying midges [85], and firefly synchronization [86]. In such swarms, each midge (respectively, firefly) is affected by the acoustic (respectively, visual) field produced by all others. In these cases, the environment plays the role of relaying group-related information to each agent.

- *Broadcast in clique networks.* Obtaining measurements of the whole system is also relevant for relatively packed systems, for which communications are essentially being broadcasted over a clique. In the natural world, the clique abstraction can find relevance in small flocks of birds, or in other small animal groups [147, 47].

- *Well-mixed populations.* The assumption that agents sense global parameters of the system also finds relevance in highly mobile systems. In particular, when the population is *well-mixed*, the neighbours of every agent are constantly being replaced by new agents chosen uniformly at random. In such cases, at all time, interactions with neighbours can serve as a fresh random sample from the global population. This perspective is adopted by population protocols and opinion dynamics (although the set of opinions is typically discrete in these works). For example, the *majority dynamics* can be viewed as being composed of two parts: (1) obtain a noisy sample of the majority opinion in the population, and (2) align to this majority.

Second step: updating the opinion. The most natural way to achieve consensus for agents that can estimate the average opinion, is to get as close as possible to the most recent estimation. However, there are situations in which this "fully responsive" behaviour is not optimal. This happens, for instance, when some agents have a pre-existing preference based on their own knowledge, as in Chapter 2. It is also the case when the estimation is noisy and imperfect, in which case incorporating past observations into the decision is expected to improve its accuracy. In this chapter, we focus on the latter.

Informally, the alignment problem we study, considers a group of n agents positioned on the real line, aiming to be located as close as possible to one another. The position of an agent on the line does not necessarily represent its physical locations, which may belong to a domain of higher dimension (e.g., a two or three dimensional Euclidean space). Instead, it aims to capture its "opinion" with respect to the consensus that the group is trying to achieve. For example, in view of general alignment problems, the position of an agent models its direction. A one-dimensional direction is sufficient anytime the navigation considered remains two-dimensional. This is the case for cooperative transport by ants, but also flocking by birds, as long as their altitude remains stable. In view of clock-synchronization [146, 142] or firefly synchronization [86], the position of an agent models its clock (i.e., time), which again can be viewed as a one-dimensional space.

Depending on the application, the actual domain may be bounded, or periodic. For example, when modeling directions, the domain is $[-\pi, \pi]$ and when modeling clock synchronization, the domain may be $[0, T]$ for some phase duration T. Since we are interested in the cases where agents are more or less aligned, approximating an interval domain with the real line is not expected to reduce the generality of our results.

Initially, agents' positions are sampled from a Gaussian distribution around 0. Execution proceeds in discrete rounds, where in each round, each agent receives a noisy measurement of its current deviation from the average position of others. Then, governed by the rules of its algorithm, each agent performs a *move* to re-adjust its position. Subsequently, before the next round begins, the position of each agent is perturbed following random drift. The drift component may model unreliability due to external conditions (e.g., perturbation of direction by wind), or the inability of an agent to precisely adjust itself as it wishes. Both noises in measurements and random drifts are governed by Gaussian distributions. We are mostly interested in the following questions:

- Which re-adjustment rule should agents adopt if their goal is to minimize expected distance of each agent from the average position?
- Could further communication between agents (e.g., by sharing measurements) help?
- Which protocols minimize the drift of the center of mass?

Importantly, we assume that agents are unaware of the actual value of their current positions, and of the realizations of the random drifts, and instead, must base their movement decisions only on noisy measurements of relative positions. This lack of global "sense of orientation" prevents the implementation of the trivial distributed protocol in which all agents simply move to a predetermined point, say 0.

As mentioned before, one trivial algorithm is the "fully responsive" protocol, where in each round, each agent moves all the way to its current measurement of the average position of others. When drift is large, measurement noise is negligible, and the number of agents is large, this protocol can become highly efficient. However, when measurement noise is non-negligible, it is expected that incorporating past measurements could enhance the cohesion, even though drift may have changed the configuration slightly.

Perhaps the simplest class of algorithms that take such past information into account are weighted-average algorithms. By weighing the current position against the measured position in a non-trivial way, such algorithms can potentially exploit the fact that the current position implicitly encodes information from past measurements. Indeed, in a centralized setting, when a single agent aims to estimate a fixed target relying on noisy Gaussian measurements, a weighted-average algorithm is known to be optimal, in the sense that it minimizes the expected distance between the agent and the target [9]. However, here the setting is more complex since it is distributed, and the objective goal is to estimate (and get closer to) the average position, which is a function of the agents decisions.

3.1.1 Preliminaries

In this section, we define the *alignment problem* more formally. We consider n agents located on the real line $\mathbb{R}$. Let $I = \{1, \ldots, n\}$ be the set of agents. We denote by $\theta_i^{(t)} \in \mathbb{R}$ the position of Agent i at round t, where initially, the position of each agent is chosen independently according to a normal distribution around 0, with variance σ_0^2, that is, for each Agent i, $\theta_i^{(0)} \sim \mathcal{N}\left(0, \sigma_0^2\right)$. As mentioned, we assume that an agent is unaware of the actual value of its current position.

Execution proceeds in discrete rounds. At round t, each agent i receives a noisy measurement of the deviation from the current average position of all other agents. Specifically, denote the average of the positions of all agents except i by:

$$\langle\theta_{-i}^{(t)}\rangle = \frac{1}{n-1}\sum_{j\in I\setminus i}\theta_i^{(t)}.$$

Let $\overline{\theta}_i^{(t)} = \langle\theta_{-i}^{(t)}\rangle - \theta_i^{(t)}$ denote the *stretch* of Agent i. At any round t, for every $i \in I$, a noisy measurement of the stretch of Agent i is sampled:

$$Y_i^{(t)} = \overline{\theta}_i^{(t)} + N_{m,i}^{(t)}, \tag{3.1}$$

where $N_{m,i}^{(t)} \sim \mathcal{M}(0,\sigma_m^2)$. In response, Agent i makes a move $d\theta_i^{(t)}$ and may update its memory state (if it has any). Finally, the position of Agent i at the next round is obtained by adding a drift:

$$\theta_i^{(t+1)} = \theta_i^{(t)} + d\theta_i^{(t)} + N_{d,i}^{(t)}, \tag{3.2}$$

where $N_{d,i}^{(t)} \sim \mathcal{M}(0,\sigma_d^2)$. All random perturbations $\left(N_{m,i}^{(t)}\right)_{i\in I}$ and $\left(N_{d,i}^{(t)}\right)_{i\in I}$ are mutually independent, and we assume that $\sigma_m, \sigma_d > 0$.

Definition 3.1. The *cost* of Agent i at a given time t is the absolute value of its expected stretch at that time [a], i.e.,

$$\mathcal{C}_i^{(t)} := \mathbf{E}\left(\left|\overline{\theta}_i^{(t)}\right|\right).$$

[a]Another natural cost measure is the expected deviation from the average position of all agents (including the agent), i.e., $\mathcal{C}_i^{(t)\prime} := \mathbf{E}(|\frac{1}{n}\sum_{j\in I}\theta_j^{(t)} - \theta_i^{(t)}|)$. These two measures are effectively equivalent. Indeed, $\mathcal{C}_i^{(t)\prime} = \frac{n-1}{n}\mathcal{C}_i^{(t)}$, thus an algorithm minimizing one measure will also minimize the other.

Note that the cost depends on the algorithm used by i and on the algorithms used by others. As these algorithms will be clear from the context, we omit mentioning them in notations.

Definition 3.2. We say that an algorithm is *optimal* if, for every $i \in I$ and every round t, no algorithm can achieve a strictly smaller cost $\mathcal{C}_i^{(t)}$. Note that the definition refers to a very strong notion of optimality — it implies that if Algorithm A is optimal, then for any given round t and for any given Agent i, no algorithm B can achieve better performances for Agent i at round t than the performances guaranteed by A, regardless of what B did in previous rounds.

Definition 3.3. A *weighted-average* algorithm is a distributed algorithm that is characterized by a responsiveness parameter $\rho^{(t)}$ for each round t, indicating the weight given to the measurement at that round. Formally, an agent i following the weighted-average algorithm $\mathcal{W}(\rho^{(t)})$ at round t, sets

$$d\theta_i^{(t)} = \rho^{(t)} Y_i^{(t)}. \tag{3.3}$$

Full communication (or centralized) model. When executing a weighted-average algorithm, an agent bases its decisions solely on its own measurements. A main question we ask is whether, and if so to what extent, can performances be improved if agents could communicate with each other to share their measurements. Therefore, in order to study the impact of communication, we compare the performances of the best weighted-average algorithm to the performances of the best algorithm in the *full-communication model*, where agents are free to share their measurements with all other agents at no cost. In the case that agents have identities, this setting is essentially equivalent to the following centralized setting.

Definition 3.4 (*Centralized setting*)**.** Consider a *master* agent that is external to the system. The master agent receives, at any round t, the stretch measurements of all agents, i.e., the collection $\{Y_j^{(t)}\}_{j\in I}$, where these measurements are noisy in the same manner as described in Eq. (3.1). Analyzing these measurements at round t, the master agent then instructs each agent i to move by a quantity $d\theta_i^{(t)}$. After moving, the agents are subject to drift, as described in

Eq. (3.2). Note that the master agent is unable to "see" the positions of the agents, and its information regarding their locations is based only on the measurements it gathers from the agents, and on the movements it instructs. The goal of the master agent is to minimize the maximal cost of an agent, (where the maximum is over the agents), per round. In particular, an algorithm is said to be optimal in the centralized setting if it satisfies Definition 3.2, where optimality is with respect to all algorithms in the centralized setting.

At a first glance, it may appear that weighted-average algorithms may be sub-optimal in the centralized setting. This is because the measurements made by Agent i alone contain strictly less information about the agent's stretch $\overline{\theta}_i^{(t)}$, than the information contained in the collection of all measurements $\{Y_j^{(t)}\}_{j\in I}$. For example, it can be shown that

$$-\sum_{j\in I\setminus i} Y_j^{(t)} = \overline{\theta}_i^{(t)} - \sum_{j\in I\setminus i} N_{m,j}^{(t)}, \tag{3.4}$$

thus representing an additional "fresh" estimation of $\overline{\theta}_i^{(t)}$. Therefore, the stretches could potentially be reduced in the centralized setting by letting the master agent process all measurements.

Finally, we will write $\Delta_t = \langle\theta^{(t)}\rangle - \langle\theta^{(0)}\rangle$ the drift of the center of mass of the group, in round t.

3.1.2 Our Results

We prove that a distributed weighted-average algorithm, termed $\mathcal{W}^\star$, optimally minimizes the expected stretch of all agents at all rounds, in the strong sense of Definition 3.2. The optimality of this weighted-average algorithm holds even in the centralized (full-communication) model. We find this result surprising since, as mentioned, the measurements obtained by all agents contain strictly more information about the stretch of any given Agent i, than the measurements obtained by Agent i alone (Eq. (3.4)). Although this information is relevant for Agent i, it is not processed by it when running $\mathcal{W}^\star$. Nevertheless, this weighted-average algorithm is optimal, since other agents manage to fully process this information in a way that also benefits Agent i, by shifting the center of mass towards it. Finally, we also analyze the drift of the center of mass and show that no distributed algorithm can achieve as small of a drift as can be achieved by the best centralized algorithm. In light of this, we also show that the drift associated with $\mathcal{W}^\star$ incurs a relatively small overhead over the best possible drift in the centralized setting.

3.1.2.1 Intuition: the case of two agents in one round

In order to obtain preliminary understanding, let us first examine the case of two agents operating in a single round, and compare the situation between the centralized and the distributed setting. This simple example is informative since it already demonstrates the aforementioned surprising phenomenon.

To really limit this example to "one round", we assume that the initial distribution of the stretches has a very large variance (formally, $\sigma_0^2 \gg \sigma_m^2$). Therefore, we can put aside the Bayesian conflict between the a priory knowledge about the stretches, and the information contained in the new measurement – and only work with the latter.

When considering two agents, the stretch $\overline{\theta}_i$ of Agent $i \in \{1,2\}$ is simply its relative distance to the other agent. Assume that their positions are initially set to $\theta_1 = 0$ and $\theta_2 = 1$, so that $\overline{\theta}_1 = +1$ and $\overline{\theta}_2 = -1$. Assume further that the respective measurement noises are $N_{m,1} = 0.1$ and $N_{m,2} = 0.2$. In this case, the measurements of the stretches are:

$$\begin{cases} Y_1 = \overline{\theta}_1 + N_{m,1} = +1.1 \\ Y_2 = \overline{\theta}_2 + N_{m,2} = -0.8. \end{cases} \tag{3.5}$$

Centralized setting. Here, a central entity gathers both measurements Y_1 and Y_2, and, based on these measurements, instructs each agent of its move. Since we assumed that the initial distribution of the stretches has a very large variance, the estimates of the relative stretches between the agents depends only on the measurements

$$\begin{cases} \hat{\theta}_1^{\text{cent}} = \frac{1}{2}(Y_1 - Y_2) = \frac{+1.9}{2} = +0.95 \\ \hat{\theta}_2^{\text{cent}} = \frac{1}{2}(Y_2 - Y_1) = \frac{-1.9}{2} = -0.95. \end{cases} \tag{3.6}$$

Recall that the goal of the central entity is to minimize the absolute stretch between the agents. Note that for this purpose, it has a degree of freedom, since, once it moves each of them, shifting both by the same amount would not change their stretch. However, if the central entity also wishes to minimize the drift of the center of mass, then it should instruct the agents to move towards the estimated center, shifting them by $\hat{\theta}_i/2$ towards each-other. (This type of algorithm will later be called "meet at the center".) Specifically, the instructed moves would be $d\theta_1^{\text{cent}} = \frac{1}{2}\,\hat{\theta}_1^{\text{cent}} = +0.475$ and $d\theta_2^{\text{cent}} = \frac{1}{2}\,\hat{\theta}_2^{\text{cent}} = -0.475$. By definition, the new positions of the agents (before drift applies) would be:

$$\begin{cases} \theta_1'^{\text{cent}} = \theta_1 + d\theta_1^{\text{cent}} = 0.475 \\ \theta_2'^{\text{cent}} = \theta_2 + d\theta_2^{\text{cent}} = 0.525 \end{cases} \text{so} \begin{cases} \overline{\theta}_1'^{\text{cent}} = +0.05, \\ \overline{\theta}_2'^{\text{cent}} = -0.05. \end{cases} \tag{3.7}$$

As can be seen, the center of mass is unchanged. In fact, this is no coincidence and would be the case for any measurement noises (before drift takes place).

Distributed setting. In the *distributed* setting, an agent must base its estimate of the stretch on its measurement alone. In this case, we have:

$$\begin{cases} \hat{\theta}_1^{\text{dist}} = Y_1 = +1.1, \\ \hat{\theta}_2^{\text{dist}} = Y_2 = -0.8. \end{cases}$$

Of course, the quality of these estimates is less than in the centralized setting (compare with Eq. (3.6)). Once again, in attempting to "meet at the center", the best that the agents can do is to go half way to their respective estimate of the location of the other agent. That is, $d\theta_1^{\text{dist}} = \frac{1}{2}\,\hat{\theta}_1^{\text{dist}} = \frac{1}{2}\cdot Y_1 = +0.55$ and $d\theta_2^{\text{dist}} = \frac{1}{2}\,\hat{\theta}_2^{\text{dist}} = \frac{1}{2}\cdot Y_2 = -0.4$. By definition, their new locations (before drift applies) would be:

$$\begin{cases} \theta_1'^{\text{dist}} = \theta_1 + d\theta_1^{\text{dist}} = 0.55 \\ \theta_2'^{\text{dist}} = \theta_2 + d\theta_2^{\text{dist}} = 0.6 \end{cases}, \text{ yielding} \begin{cases} \overline{\theta}_1'^{\text{dist}} = +0.05 \\ \overline{\theta}_2'^{\text{dist}} = -0.05. \end{cases}$$

Evidently, in contrast to the centralized case (Eq. (3.7)), here the center of mass moved as a result of the agents movements. Importantly, however, the relative positions of the agents (and therefore, their costs) are identical to the ones obtained in the centralized protocol. Again, this is no coincidence, and holds for any measurement noises. The reason is, that the two locations of the agents in the distributed setting differ by the same shift, compared to their location in the centralized setting. Indeed, we have:

$$\begin{cases} \theta_1'^{\text{dist}} - \theta_1'^{\text{cent}} = d\theta_1^{\text{dist}} - d\theta_1^{\text{cent}} = \frac{1}{4}(Y_1 + Y_2) \\ \theta_2'^{\text{dist}} - \theta_2'^{\text{cent}} = d\theta_2^{\text{dist}} - d\theta_2^{\text{cent}} = \frac{1}{4}(Y_1 + Y_2). \end{cases}$$

This shift, i.e., $\frac{1}{4}(Y_1 + Y_2)$, represents the "mistake" made by distributed agents in attempting to meet at the center, compared with the centralized algorithm. However, since both agents in the distributed setting are shifted by the same quantity compared to their respective positions in the centralized setting, their relative stretch remains the same in both settings. A main challenge of this paper is to show that this phenomenon holds also in the multi-round scenario, and with arbitrary number of agents.

3.1.2.2 Weighted-average algorithms with fixed responsiveness

As a warm-up towards understanding multi-round scenarios with multiple agents, we first investigate weighted-average algorithms in which all agents have the same responsiveness ρ, that furthermore remains fixed throughout the execution (see Eq. (3.3)). The proofs of both Theorems 3.5 and 3.6 below are given in Section 3.2.

Theorem 3.5. *Assume that all agents execute $\mathcal{W}(\rho)$, for a fixed $0 \le \rho \le 1$. Then for every $i \in I$ and every $t \in \mathbb{N}$, the stretch $\overline{\theta}_i^{(t)}$ is normally distributed, and*

$$\lim_{t\to+\infty} \mathrm{Var}\left(\overline{\theta}_i^{(t)}\right) = \frac{\frac{n}{n-1}(\rho^2\sigma_m^2 + \sigma_d^2)}{1-(1-\frac{n}{n-1}\rho)^2},$$

with the convention that $\lim_{t\to+\infty} \mathrm{Var}(\overline{\theta}_i^{(t)}) = +\infty$ if the denominator $1-(1-\frac{n}{n-1}\rho)^2 = 0$.

If all agents run $\mathcal{W}(\rho)$, then Agent i's cost at steady state is captured by

$$\mathsf{Var}(\rho) := \lim_{t\to+\infty} \mathrm{Var}\left(\overline{\theta}_i^{(t)}\right).$$

Indeed, for every i, since $\overline{\theta}_i^{(t)}$ is normally distributed, $\lim_{t\to+\infty} \mathbf{E}\left(\left|\overline{\theta}_i^{(t)}\right|\right) = \sqrt{\frac{2}{\pi}\mathsf{Var}(\rho)}$. The minimal value of this is achieved when taking $\mathrm{argmin}_\rho \mathsf{Var}(\rho)$ as the responsiveness parameter.

Theorem 3.6. *The weighted-average algorithm that minimizes Var(ρ) among all weighted-average algorithms $\mathcal{W}(\rho)$ (that use the same responsiveness parameter ρ at all rounds) is $\mathcal{W}(\rho^\star)$, where*

$$\rho^\star = \frac{\sigma_d\sqrt{4\sigma_m^2 + \left(\frac{n}{n-1}\sigma_d\right)^2} - \frac{n}{n-1}\sigma_d^2}{2\sigma_m^2}. \tag{3.8}$$

When n is large, Eq. (3.8) becomes $\rho^\star \approx \frac{\sigma_d\sqrt{4\sigma_m^2+\sigma_d^2}-\sigma_d^2}{2\sigma_m^2}$. Note that the role played by the measurement noise is very different than the role played by the drift. For example, if $\sigma_m \gg \sigma_d$, then $\rho^\star \approx 0$. However, if $\sigma_m \ll \sigma_d$ then $\rho^\star \approx 1$. Interestingly, if $\sigma_m = \sigma_d$ then $\rho^\star \approx \frac{\sqrt{5}-1}{2}$, which is reminiscent of the golden ratio. Moreover, for large n, the minimal $\mathrm{Var}(\rho)$ is

$$\mathrm{Var}(\rho^\star) = \frac{1}{2}\sigma_d\left(\sqrt{4\sigma_m^2+\sigma_d^2} + \sigma_d\right). \tag{3.9}$$

Note that when the measurements are perfect, i.e., $\sigma_m = 0$, we have $\mathrm{Var}(\rho^\star) = \sigma_d^2$, which is the best achievable value that an agent can hope for, since no algorithm can overcome the drift-noise.

3.1.2.3 The impact of communication

Our next goal is to understand whether, and if so, to what extent, can performances be improved if further communication between agents is allowed. For this purpose, we compare the performances of $\mathcal{W}(\rho^\star)$ to the performances of the best algorithm in the centralized (full-communication) setting.

A natural candidate for an optimal algorithm in the centralized setting is the "meet at the center" algorithm. This algorithm first obtains, for each agent, the best possible estimate of the distance from the agent's position to the average position $\langle\theta\rangle$, based on all measurements, and then instructs the agent to move by this quantity (towards the estimated average position). However, it is not immediate to figure out the distances to the average position, and furthermore, quantify the performances of this algorithm. To this end, we make use of the Kalman filter tool [9], by adapting it to our setting. Solving the Kalman filter system associated with the centralized version of our alignment problem, we obtain an estimate of the relative distance of each agent i from the average position (based on all measurements). To describe these estimates we first define the following.

Definition 3.7 (Uncertainty)**.** We inductively define the sequence $(\alpha_t)_{t=0}^{\infty}$. Let $\alpha_0 = n\sigma_0^2/(n-1)$, and for every t, let

$$\alpha_{t+1} = \frac{\sigma_m^2\alpha_t}{\frac{n}{n-1}\alpha_t + \sigma_m^2} + \frac{n}{n-1}\sigma_d^2. \tag{3.10}$$

The value of α_t aims to represent the variance of the (random) error made when estimating the stretch of the agents at round t. When thinking of n as very large, the formula for α_{t+1} is reminiscent of the resistance formula in electrical circuits, where we add in parallel to the current resistor with resistance α_t a resistor with resistance σ_m^2 (standing for the measurement noise), and another resistor in a series, with resistance σ_d^2 (standing for the drift).

Definition 3.8 (Optimal weight)**.** For every integer t, let

$$\rho_\star^{(t)} = \frac{\alpha_t}{\frac{n}{n-1}\alpha_t + \sigma_m^2}.$$

The value of $\rho_\star^{(t)}$ aims to indicate by how much to weigh the measurements of round t in order to move optimally: if $\sigma_m^2 \ll \alpha_t$, then $\rho_\star^{(t)} \approx 1$, meaning that the agent should entirely rely on the measurement. Conversely, if $\sigma_m^2 \gg \alpha_t$, then $\rho_\star^{(t)} \approx 0$, and the agent should not take the measurement into account.

At each round t, the Kalman filter returns an estimate of the relative distance of each agent i from the average position, which turns out to be

$$\rho_\star^{(t)} \left(\frac{n-1}{n} Y_i^{(t)} - \frac{1}{n} \sum_{j \neq i} Y_j^{(t)} \right).$$

As guaranteed by the properties of the Kalman filter, these estimates minimize the expected sum of square-errors, which can be translated to our desired measure of minimizing the agents' costs. The "meet at the center" algorithm is given by Algorithm 8 below, and the following theorem stating its optimality is proved in Section 3.3.2.1.

Algorithm 8: Meet at the center

foreach *round* t **do**
    Consider all measurements at round t, $\{Y_j^{(t)} \mid 1 \le j \le n\}$;
    foreach *agent* i **do**
        Set $d\theta_i^{(t)} = \rho_\star^{(t)} \left(\frac{n-1}{n} Y_i^{(t)} - \frac{1}{n} \sum_{j \neq i} Y_j^{(t)} \right)$;   /* Estimate of $\langle\theta\rangle - \theta_i$ */
    end
end

Theorem 3.9. *Algorithm 8 is optimal in the centralized setting.*

Note that in the centralized setting, once we have an optimal algorithm A, we can derive another optimal algorithm B by simply shifting all agents, at each round t, by a fixed quantity λ_t. Indeed, such shifts do not influence the relative positions between the agents. Conversely, we show in Section 3.4.1 that all (optimal) deterministic algorithms in the centralized setting are shifts of one another (note that the shifts λ_t are not necessarily the same for all rounds t).

As it happens, another solution that follows from the Kalman filter estimations can be described as a distributed weighted-average algorithm (see Algorithm 9). As such, this algorithm, henceforth called $\mathcal{W}^\star$, also yields optimal stretches. The proof of the following theorem is given in Section 3.3.2.2.

Theorem 3.10. *The weighted-average algorithm $\mathcal{W}^\star = \mathcal{W}(\rho_\star^{(t)})$ (Algorithm 9) is optimal in the centralized setting.*

Since both the "meet at the center" algorithm and $\mathcal{W}^\star$ are optimal deterministic algorithms, they must be shifts of one another. Indeed, as validated in Section 3.4.2, by adding the shift $\lambda_t = \frac{1}{n}\rho_\star^{(t)} \sum_{i \in I} Y_i^{(t)}$ to the agents in Algorithm 8 we obtain Algorithm $\mathcal{W}^\star$.

Note that in contrast to $\mathcal{W}(\rho^\star)$, Algorithm $\mathcal{W}^\star$ uses a different responsiveness $\rho_\star^{(t)}$ in each round. We next argue that the sequences (α_t) and $(\rho_\star^{(t)})$ converge. Not surprisingly, at the limit, we recover

Algorithm 9: The weighted-average algorithm $\mathcal{W}^{\star}$

```
foreach round t do
    Consider all measurements at round t, {Y_j^(t) | 1 ≤ j ≤ n} ;
    foreach agent i do
        Set dθ_i^(t) = ρ_⋆^(t) Y_i^(t) ;
    end
end
```

the optimal fixed responsiveness as stated in Theorem 3.6. The proof of the following result can be found in Section 3.4.3.

Observation 3.11. *The sequence α_t converges to*

$$\alpha_{\infty} := \frac{1}{2}\left(\sigma_d\sqrt{4\sigma_m^2 + \left(\frac{n}{n-1}\sigma_d\right)^2} + \frac{n}{n-1}\sigma_d^2\right).$$

Moreover, $\lim_{t\to+\infty}\rho_\star^{(t)} = \rho_\star$ as defined in Eq. (3.8).

3.1.2.4 Drift of the center of mass

Finally, we analyze the drift of the center of mass. Essentially, we show that with respect to this drift (and in contrast to the measure of stretch as used in Definition 3.1), there is a gap between the centralized and the distributed settings. Recall that we denote by $\Delta_t = \langle\theta^{(t)}\rangle - \langle\theta^{(0)}\rangle$ the drift of the center of mass in round t. First, we note that no algorithm can achieve zero drift.

Theorem 3.12. *For every protocol in the centralized setting and for every round t, the drift of the center of mass satisfies $\mathbf{E}\left(\Delta_t^2\right) \geq \frac{t+1}{n}\sigma_d^2$.*

As a consequence, we introduce the following notion of drift-optimality, and show that it is satisfied by the "meet at the center" algorithm (Algorithm 8).

Definition 3.13. An algorithm is called *drift-optimal*, if for every round t, the drift of the center of mass satisfies $\mathbf{E}\left(\Delta_t^2\right) = \frac{t+1}{n}\sigma_d^2$.

Theorem 3.14. *The "meet at the center" algorithm (Algorithm 8) is drift-optimal.*

In contrast, we prove that there is no distributed algorithm that can achieve as small of a drift as the one obtained by the "meet at the center" algorithm.

Claim 3.15. *No algorithm in the distributed setting can be simultaneously optimal and drift-optimal.*

Consistent with this result, the (distributed) weighted-average algorithm $\mathcal{W}^{\star}$ (Algorithm 9) is not drift-optimal. Yet, we show that the overhead it incurs in terms of this drift is relatively small, especially when n is large.

Theorem 3.16. *When Algorithm $\mathcal{W}^{\star}$ is used, the drift of the center of mass satisfies*

$$\mathbf{E}\left(\Delta_t^2\right) = \frac{t+1}{n}\sigma_d^2 + \frac{1}{n}\sum_{s=0}^{t}{\rho_\star^{(s)}}^2\sigma_m^2.$$

The proofs of all these results are deferred to Section 3.5.

3.1.3 Related Works

The Kalman filter algorithm is a prolific tool commonly used in control theory and statistics, with numerous technological applications [9]. In this chapter, we use the Kalman filter to investigate the centralized setting, where all relative measurements are gathered at one master agent that processes them and instructs agents how to move. Fusing relative measurements of multiple agents is often referred to in the literature as *distributed Kalman filter*, see surveys in [34, 125, 135]. However, there, the typical setting is that each agent invokes a separate Kalman filter to process its measurements, and then the resulted outputs are integrated using communication. Moreover, works in that domain often assume that observations are attained by static observers (i.e., the agents are residing on a fixed underlying graph) and that the measured target is external to the system. In contrast, here we consider a self-organizing system, with mobile agents that measure a target (average position) that is a function of their positions. Closer works to our setting where a group of mobile agents aim to gather together are [148, 96]. However, among other differences, these papers assume that there is no drift and that agents communicate their absolute position, rather than their relative position.

Similarly to our findings, the authors of [103] also emphasize the effectiveness of performing a weighted-average between the opinion of the individual and the sample it receives. Yet, we find this result less surprising under their model, which features one-to-one communications and no drift.

Another related self-organization problem is *clock synchronization*, where the goal is to maintain a common notion of time in the absence of a global time source. The main difficulty is handling the fact that clocks count time at slightly different rates and that messages arrive with some delays. Variants of this problem were studied in wireless network contexts, mostly by control theoreticians [160, 146, 142]. A common technique uses oscillating models [118, 95].

3.2 Warm-up Proofs

The goal of this section is to prove Theorems 3.5 and 3.6. These theorems assume all agents execute a weighted-average algorithm with the same responsiveness parameter ρ, that furthermore remains fixed throughout the execution (see Eq. (3.3)). We start with the following observation.

Lemma 3.17. *For any integer t, and whatever the positions of the agents, we have*

$$\sum_{i\in I}\overline{\theta}_i^{(t)} = 0.$$

Proof.

$$\begin{aligned}\sum_{i\in I}\overline{\theta}_i^{(t)} &= \sum_{i\in I}\left(\langle\theta_{-i}^{(t)}\rangle - \theta_i^{(t)}\right)\\ &= \frac{1}{n-1}\sum_{i\in I}\sum_{j\in I\setminus i}\theta_i^{(t)} - \sum_{i\in I}\theta_i^{(t)} = \frac{1}{n-1}\cdot(n-1)\sum_{i\in I}\theta_i^{(t)} - \sum_{i\in I}\theta_i^{(t)} = 0.\end{aligned}$$

□

Next, we compute how the stretch $\overline{\theta}_i^{(t)}$ of Agent i changes when all agents perform weighted-average moves with constant responsiveness parameter ρ.

Lemma 3.18. *Assume that all agents execute $\mathcal{W}(\rho)$, for some $0\le\rho\le 1$. Let $\mathcal{E}_j^{(t)} = \rho N_{m,j}^{(t)} + N_{d,j}^{(t)}$. Then for every $i\in I$ and every $t\in\mathbb{N}$,*

$$\overline{\theta}_i^{(t+1)} = \left(1-\frac{n}{n-1}\rho\right)\overline{\theta}_i^{(t)} + \frac{1}{n-1}\sum_{j\in I\setminus i}\mathcal{E}_j^{(t)} - \mathcal{E}_i^{(t)}.$$

Proof. Note that the $\{\mathcal{E}_j^{(t)}\}_{j\in I}$ are independent and normally distributed by definition. The stretch of Agent i at round $t+1$ is given by:

$$\begin{aligned}
\overline{\theta}_i^{(t+1)} &= \langle\theta_{-i}^{(t+1)}\rangle - \theta_i^{(t+1)} \\
&= \langle\theta_{-i}^{(t+1)}\rangle - \left(\theta_i^{(t)} + d\theta_i^{(t)} + N_{d,i}^{(t)}\right) \\
&= \left(\langle\theta_{-i}^{(t+1)}\rangle - \langle\theta_{-i}^{(t)}\rangle\right) + \left(\langle\theta_{-i}^{(t)}\rangle - \theta_i^{(t)}\right) - d\theta_i^{(t)} - N_{d,i}^{(t)} \\
&= \left(\langle\theta_{-i}^{(t+1)}\rangle - \langle\theta_{-i}^{(t)}\rangle\right) + \overline{\theta}_i^{(t)} - d\theta_i^{(t)} - N_{d,i}^{(t)},
\end{aligned} \tag{3.11}$$

where the first equality is by definition, and the second equality is a consequence of (3.2). Let us break down the first term:

$$\langle\theta_{-i}^{(t+1)}\rangle - \langle\theta_{-i}^{(t)}\rangle = \frac{1}{n-1}\sum_{j\in I\setminus i}\left(d\theta_j^{(t)} + N_{d,j}^{(t)}\right) = \frac{1}{n-1}\sum_{j\in I\setminus i}\left(\rho\left(\overline{\theta}_j^{(t)} + N_{m,j}^{(t)}\right) + N_{d,j}^{(t)}\right),$$

where the second equality is because

$$d\theta_j^{(t)} = \rho Y_j^{(t)} = \rho\left(\overline{\theta}_j^{(t)} + N_{m,j}^{(t)}\right).$$

By Lemma 3.17, $\sum_{j\in I\setminus i}\overline{\theta}_j^{(t)} = -\overline{\theta}_i^{(t)}$, so we can rewrite the last equation as

$$\begin{aligned}
\langle\theta_{-i}^{(t+1)}\rangle - \langle\theta_{-i}^{(t)}\rangle &= \frac{-\rho}{n-1}\overline{\theta}_i^{(t)} + \frac{1}{n-1}\sum_{j\in I\setminus i}\left(\rho N_{m,j}^{(t)} + N_{d,j}^{(t)}\right) \\
&= \frac{-\rho}{n-1}\overline{\theta}_i^{(t)} + \frac{1}{n-1}\sum_{j\in I\setminus i}\mathcal{E}_j^{(t)}.
\end{aligned} \tag{3.12}$$

Plugging Eq. (3.12) into Eq. (3.11) gives

$$\overline{\theta}_i^{(t+1)} = \left(1 - \frac{\rho}{n-1}\right)\overline{\theta}_i^{(t)} + \frac{1}{n-1}\sum_{j\in I\setminus i}\mathcal{E}_j^{(t)} - d\theta_i^{(t)} - N_{d,i}^{(t)}. \tag{3.13}$$

By assumption, $d\theta_i^{(t)} = \rho Y_i^{(t)} = \rho\left(\overline{\theta}_i^{(t)} + N_{m,i}^{(t)}\right)$. Plugging this expression into Eq. (3.13) gives the result. □

Now we can prove that the stretch of each agent is normally distributed at every round, and compute its variance.

Lemma 3.19. *Assume that all agents execute $\mathcal{W}(\rho)$, for some $0 \le \rho \le 1$. Then, for every $i \in I$ and every $t \in \mathbb{N}$, the stretch $\overline{\theta}_i^{(t)}$ is normally distributed, $\mathbf{E}\left(\overline{\theta}_i^{(t)}\right) = 0$, and*

$$\mathrm{Var}\left(\overline{\theta}_i^{(t+1)}\right) = \left(1 - \frac{n\rho}{n-1}\right)^2 \cdot \mathrm{Var}\left(\overline{\theta}_i^{(t)}\right) + \frac{n(\rho^2\sigma_m^2 + \sigma_d^2)}{n-1}.$$

Proof. We prove that the stretch is normally distributed with mean 0 by induction on t. By construction, for every i, $\overline{\theta}_i^{(0)}$ is normally distributed with mean 0. Let us assume that $\overline{\theta}_i^{(t)}$ is normally distributed with mean 0 for some round t, and consider round $t+1$. Recall that Lemma 3.18 gives

$$\overline{\theta}_i^{(t+1)} = \left(1 - \frac{n}{n-1}\rho\right)\overline{\theta}_i^{(t)} + \frac{1}{n-1}\sum_{j\in I\setminus i}\mathcal{E}_j^{(t)} - \mathcal{E}_i^{(t)}. \tag{3.14}$$

Since by definition, for every j, $\mathcal{E}_j^{(t)}$ is normally distributed around 0, and by induction $\overline{\theta}_i^{(t)}$ is normally distributed around 0, then $\overline{\theta}_i^{(t+1)}$ is also normally distributed around 0. This concludes the induction.

Moreover, note that $\mathrm{Var}\left(\mathcal{E}_j^{(t)}\right) = \rho^2\sigma_m^2 + \sigma_d^2$, so

$$\mathrm{Var}\left(\frac{1}{n-1}\sum_{j\in I\setminus i}\mathcal{E}_j^{(t)} - \mathcal{E}_i^{(t)}\right) = \frac{1}{(n-1)^2}\sum_{j\in I\setminus i}\mathrm{Var}\left(\mathcal{E}_j^{(t)}\right) + \mathrm{Var}\left(\mathcal{E}_i^{(t)}\right) = \tfrac{n}{n-1}\left(\rho^2\sigma_m^2 + \sigma_d^2\right).$$

Hence, by Eq. (3.14),

$$\mathrm{Var}\left(\overline{\theta}_i^{(t+1)}\right) = \left(1 - \frac{n\rho}{n-1}\right)^2\mathrm{Var}\left(\overline{\theta}_i^{(t)}\right) + \frac{n\left(\rho^2\sigma_m^2 + \sigma_d^2\right)}{n-1},$$

which concludes the proof. □

Before proving Theorem 3.5, we need a small technical result.

Claim 3.20. *Let $a, b \geq 0$. Consider the sequence $\{u_n\}_{n=0}^{\infty}$ defined by letting $u_0 \in \mathbb{R}$ and for every integer n, $u_{n+1} = au_n + b$. If $a < 1$, then $\{u_n\}_{n=0}^{\infty}$ converges and $\lim_{n\to+\infty} u_n = b/(1-a)$. If $a = 1$, and $b > 0$, then $\lim_{n\to+\infty} u_n = +\infty$.*

Proof. First, consider the case that $a < 1$. Let $\lambda = b/(a-1)$. Consider the sequence defined by $v_n = u_n + \lambda$. We have

$$\begin{aligned} v_{n+1} = u_{n+1} + \lambda &= au_n + b + \lambda \\ &= au_n + (a-1)\lambda + \lambda \\ &= au_n + a\lambda = a(u_n + \lambda) = av_n. \end{aligned}$$

Since $0 \leq a < 1$, $\lim_{n\to+\infty} v_n = 0$, and so $\lim_{n\to+\infty} u_n = -\lambda = b/(1-a)$.

Now, if $a = 1$, then we have for every $n \in \mathbb{N}$, $u_n = u_0 + nb$. If $b > 0$ then $\lim_{n\to+\infty} u_n = +\infty$. □

Proof of Theorem 3.5. First, apply Lemma 3.19. Hence, by Claim 3.20, $\mathrm{Var}(\overline{\theta}_i^{(t)})$ converges to the limit as stated. Note that the variance is infinite if either (1) the responsiveness is equal to 0, in which case the drift adds up endlessly, or (2) the responsiveness is equal to 1 and $n = 2$, in which case the agents "swap" at each round, producing the same result. This completes the proof of Theorem 3.5. □

Proof of Theorem 3.6. Consider the function

$$\mathrm{Var}(\rho) = \frac{\frac{n}{n-1}(\rho^2\sigma_m^2 + \sigma_d^2)}{1 - (1 - \frac{n}{n-1}\rho)^2} = \frac{\rho^2\sigma_m^2 + \sigma_d^2}{2\rho - \frac{n}{n-1}\rho^2}.$$

Note that this function evaluates to $+\infty$ when $\rho = 0$, or when $\rho = 1$ and $n = 2$. Computing its derivative yields:

$$\mathrm{Var}'(\rho) = \frac{2\rho\sigma_m^2\left(2\rho - \frac{n}{n-1}\rho^2\right) - (\rho^2\sigma_m^2 + \sigma_d^2)\left(2 - 2\frac{n}{n-1}\rho\right)}{\left(2\rho - \frac{n}{n-1}\rho^2\right)^2}.$$

Hence, $\mathrm{Var}'(\rho) = 0$ if and only if:

$$\rho^2\sigma_m^2\left(2 - \tfrac{n}{n-1}\rho\right) - (\rho^2\sigma_m^2 + \sigma_d^2)\left(1 - \frac{n}{n-1}\rho\right) = 0 \iff \rho^2\sigma_m^2 - \sigma_d^2\left(1 - \frac{n}{n-1}\rho\right) = 0.$$

This equation has a unique solution in the interval $[0, 1]$, which is $\rho = \frac{\sigma_d\sqrt{4\sigma_m^2 + \left(\frac{n}{n-1}\sigma_d\right)^2} - \frac{n}{n-1}\sigma_d^2}{2\sigma_m^2}$. Checking that this is a minimum concludes the proof. □

3.3 Solving the Alignment Problem

The goal of this section is to prove Theorems 3.9 and 3.10. We denote $\mathcal{M}(\mu, \Sigma)$ the multivariate normal distribution with mean vector $\mu \in \mathbb{R}^n$ and co-variance matrix $\Sigma \in \mathbb{R}^{n\times n}$, and by I the identity matrix. Letting $\mathbb{1}$ be the matrix whose all coefficients are equal to 1, we denote

$$\mathcal{N}(a, b) = b\mathbb{1} + (a - b)I,$$

the matrix having diagonal coefficients equal to a, and other coefficients equal to b. We first recall the following well-known property.

Claim 3.21. *If $X \sim \mathcal{M}(\mu, \Sigma)$, then for every $c \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times n}$,*

$$c + BX \sim \mathcal{M}\left(c + B\mu, B\Sigma B^\top\right).$$

In addition, we give two useful results about matrices of the form $\mathcal{N}(a, b)$.

Claim 3.22. *For every $a, b, a', b' \in \mathbb{R}$,*

$$\mathcal{N}(a, b)\mathcal{N}(a', b') = \mathcal{N}(aa' + (n-1)bb', ab' + a'b + (n-2)bb').$$

In particular, $\mathcal{N}(a, b)\mathcal{N}(a', b') = \mathcal{N}(a', b')\mathcal{N}(a, b)$.

Claim 3.23. *For every $a, b \in \mathbb{R}$ such that $a \neq b$ and $a \neq -(n-1)b$, the matrix $\mathcal{N}(a, b)$ is invertible, and*

$$\mathcal{N}(a, b)^{-1} = \frac{\mathcal{N}(a + (n-2)b, -b)}{(a - b)(a + (n-1)b)}.$$

Proof. The claim is a consequence of the Sherman–Morrison formula[a], but we prove it here anyway. Note that $\mathbb{1}^2 = n\mathbb{1}$. Let $A = \mathcal{N}(a, b)$. We have

$$\begin{aligned}
A^2 &= (b\mathbb{1} + (a-b)I)^2 \\
&= b^2\mathbb{1}^2 + 2b(a-b)\mathbb{1} + (a-b)^2 I \\
&= (nb + 2(a-b))(b\mathbb{1}) + (a-b)^2 I \\
&= (2a + (n-2)b)(b\mathbb{1} + (a-b)I) \\
&\quad - (nb + 2(a-b))(a-b)I + (a-b)^2 I \\
&= (2a + (n-2)b)A + (a-b)(-nb - 2(a-b) \\
&\quad + (a-b))I \\
&= (2a + (n-2)b)A - (a-b)(a + (n-1)b)I.
\end{aligned}$$

Hence

$$A(A - (2a + (n-2)b)I) = -(a-b)(a + (n-1)b)I,$$

from which we conclude (provided that $a \neq b$ and $a \neq (n-1)b$),

$$A^{-1} = \frac{(2a+(n-2)b)I - A}{(a-b)(a+(n-1)b)}.$$

□

[a] Thanks to Alain Jean-Marie for pointing this out to me.

3.3.1 Linear Filtering

Before stating the proofs of Theorems 3.9 and 3.10, we explain how to adapt the Kalman filter tool to our problem.

3.3.1.1 Rephrasing the alignment problem as a linear filtering problem

The goal of this section is to write Eqs. (3.1) and (3.2) in a matrical form. Let $\overline{\theta}^{(t)}$, $d\theta^{(t)}$, $Y^{(t)}$, $N_m^{(t)}$, and $N_d^{(t)}$ denote the vectors (indexed by the set of agents) corresponding to each variable. (E.g., $\overline{\theta}^{(t)} = (\overline{\theta}_1^{(t)}, \ldots, \overline{\theta}_n^{(t)})$, and similarly for the other vectors.)

Measurement rule Recall that, as stated in Eq. (3.1), the measurement of Agent i at time t is given by $Y_i^{(t)} = \overline{\theta}_i^{(t)} + N_{m,i}^{(t)}$. We simply rewrite this equation as:

$$Y^{(t)} = \overline{\theta}^{(t)} + N_m^{(t)}, \tag{3.15}$$

where, by definition, $N_m^{(t)} \sim \mathcal{M}\left(0, \sigma_m^2 I\right)$.

Update rule We recall the update equation of the stretch, which follows from Eq. (3.2):

$$\overline{\theta}_i^{(t+1)} = \overline{\theta}_i^{(t)} - d\theta_i^{(t)} - N_{d,i}^{(t)} + \left(\langle\theta_{-i}^{(t+1)}\rangle - \langle\theta_{-i}^{(t)}\rangle\right) = \overline{\theta}_i^{(t)} - d\theta_i^{(t)} - N_{d,i}^{(t)} + \frac{1}{n-1}\sum_{j\in I\setminus i}\left(d\theta_j^{(t)} + N_{d,j}^{(t)}\right). \tag{3.16}$$

We define the matrix

$$M_n = \mathcal{N}\left(-1, \frac{1}{n-1}\right).$$

Let $\tilde{N}_d^{(t)} = M_n N_d^{(t)}$. With these definitions and Eq. (3.16), we write Eq. (3.2) in the following vector notation:

$$\overline{\theta}^{(t+1)} = \overline{\theta}^{(t)} + M_n d\theta^{(t)} + \tilde{N}_d^{(t)}. \tag{3.17}$$

By definition, $N_d^{(t)} \sim \mathcal{M}\left(0, \sigma_d^2 I\right)$, so by Claim 3.21, $\tilde{N}_d^{(t)} \sim \mathcal{M}\left(0, Q\right)$ where $Q = \sigma_d^2 \cdot M_n I M_n^\top = \sigma_d^2 M_n^2$.

3.3.1.2 Applying the Kalman filter

We use the Kalman filter as a tool for analysis purposes of the process. We stress, however, that the Kalman filter is not actually applied by any agent.

We denote by $\hat{\theta}_t$ the estimate of $\theta^{(t)}$ produced by the Kalman filter *after* the measurements at round t were obtained (Eq. (3.15)) and before the update of round t occurs (Eq. (3.17)). Let P_t denote the error co-variance matrix associated with this estimate. Specifically,

$$P_t = \mathbf{E}\left((\theta^{(t)} - \hat{\theta}_t)(\theta^{(t)} - \hat{\theta}_t)^\top\right).$$

We add the superscript "-" to these notations (for example P_t^-), to denote the same quantities *before* the measurement obtained at round t.

So, in a sense, round t can be divided into the following four consecutive time steps: (a) the filter produces an estimation $\hat{\theta}_t^-$ of $\theta^{(t)}$, (b) a new measurement vector $Y^{(t)}$ of $\theta^{(t)}$ is obtained, (c) the filter produces an estimation $\hat{\theta}_t$ of $\theta^{(t)}$ given the new measurement, and (d) $\theta^{(t)}$ is updated to $\theta^{(t+1)}$. Recall that at round 0, i.e., at the initialization stage, the agents are normally distributed around 0. For technical reasons, we define the Kalman filter estimate at round 0 to be zero, that is, $\hat{\theta}_0^- = 0$.

Measurement update In order to produce (c), the filter incorporates the measurement in (b) to the estimation in (a). Specifically, the filter computes the "Kalman gain":

$$K_t = P_t^- \left(P_t^- + \sigma_m^2 I\right)^{-1}. \tag{3.18}$$

Then, it produces the following estimate, as required in (c):

$$\hat{\theta}_t = \hat{\theta}_t^- + K_t \left(Y^{(t)} - \hat{\theta}_t^-\right). \tag{3.19}$$

The new error co-variance matrix can then be written as:

$$P_t = (I - K_t) P_t^-. \tag{3.20}$$

Time update The estimation for (a) for the following round is then given by:

$$\hat{\theta}_{t+1}^- = \hat{\theta}_t + M_n d\theta^{(t)}. \tag{3.21}$$

Finally, the new error co-variance matrix that is used in the Kalman gain corresponding to round $t+1$ (Eq. (3.18)) is:

$$P_{t+1}^- = P_t + \sigma_d^2 \cdot M_n^2. \tag{3.22}$$

3.3.1.3 Optimality

Definition 3.24. A *history* $\mathcal{H}_t$ is a realization of all measurements and all moves up to time $(t-1)$.

The following is a well-known result, see e.g., [9].

Theorem 3.25. *For every round t, every history $\mathcal{H}_t$ and every vector $x \in \mathbb{R}^n$,*

$$\sum_{i\in I} \mathbf{E}\left((\theta_i^{(t)} - \hat{\theta}_{t,i}^-)^2 \mid \mathcal{H}_t\right) \le \sum_{i\in I} \mathbf{E}\left((\theta_i^{(t)} - x_i)^2 \mid \mathcal{H}_t\right).$$

In particular, for every round t, every history $\mathcal{H}_t$, every $i \le n$ and every scalar $x \in \mathbb{R}$,

$$\mathbf{E}\left((\theta_i^{(t)} - \hat{\theta}_{t,i}^-)^2 \mid \mathcal{H}_t\right) \le \mathbf{E}\left((\theta_i^{(t)} - x)^2 \mid \mathcal{H}_t\right).$$

Definition 3.26. We say that an algorithm for the alignment problem is *Kalman-perfect* if it always produces a sequence of moves $(d\theta^{(t)})_{t\ge 0}$, such that for every integer $t \ge 1$, $\hat{\theta}_t^- = 0$.

Observation 3.27. *If there exists a Kalman-perfect algorithm for the alignment problem, then this algorithm is optimal in the centralized setting (in the sense of Definition 3.2). Moreover, any other optimal (deterministic) algorithm is Kalman-perfect.*

Observation 3.27 is a direct consequence of the following result.

Lemma 3.28. *Fix $i \in \{1, \ldots, n\}$ and an integer $t \ge 1$, and consider a history $\mathcal{H}_t$ such that the Kalman filter produces $\hat{\theta}_{t,i}^- = 0$ on $\mathcal{H}_t$. Let $d\vartheta^{(t-1)}$ be an alternative move for the agents at round $(t-1)$, and consider history $\mathcal{H}_t'$ obtained from $\mathcal{H}_t$ by replacing $d\theta^{(t-1)}$ by $d\vartheta^{(t-1)}$. Then,*

$$\mathbf{E}\left({\theta_i^{(t)}}^2 \mid \mathcal{H}_t'\right) \ge \mathbf{E}\left({\theta_i^{(t)}}^2 \mid \mathcal{H}_t\right).$$

Proof. Let c be the ith coordinate of the vector $M_n(d\theta^{(t)} - d\vartheta^{(t)})$. By Eq. (3.2), c is the difference between the position of Agent i in the beginning of round t if it moves by $d\theta^{(t)}$ instead of moving by $d\vartheta^{(t)}$ (conditioning on having the same drift at the end of round $t-1$). In other words,

$$\mathbf{E}\left({\theta_i^{(t)}}^2 \mid \mathcal{H}_t'\right) = \mathbf{E}\left((\theta_i^{(t)} - c)^2 \mid \mathcal{H}_t\right). \tag{3.23}$$

By Theorem 3.25,

$$\mathbf{E}\left((\theta_i^{(t)} - c)^2 \mid \mathcal{H}_t\right) \geq \mathbf{E}\left((\theta_i^{(t)} - \hat{\theta}_{t,i})^2 \mid \mathcal{H}_t\right). \tag{3.24}$$

By assumption, $\hat{\theta}_{t,i} = 0$, so

$$\mathbf{E}\left((\theta_i^{(t)} - \hat{\theta}_{t,i})^2 \mid \mathcal{H}_t\right) = \mathbf{E}\left({\theta_i^{(t)}}^2 \mid \mathcal{H}_t\right). \tag{3.25}$$

Combining Equations (3.23), (3.24) and (3.25) establishes the result. □

3.3.1.4 Kalman gain and the error co-variance matrix

Recall the sequences $(\alpha_t)_{t\geq 0}$ and $(\rho_\star^{(t)})_{t\geq 0}$ introduced in Definitions 3.7 and 3.8. This section is dedicated to proving the following lemma, which provides a formula for the error co-variance matrix associated with the estimate $\hat{\theta}_t^-$, as well as for the Kalman gain.

Lemma 3.29. *For every integer t, $P_t^- = \mathcal{N}\left(\alpha_t, \frac{-\alpha_t}{n-1}\right) = -\alpha_t M_n$, and $K_t = \frac{-\frac{n-1}{n}\alpha_t M_n}{\frac{n}{n-1}\sigma_m^2 + \alpha_t} = -\rho_\star^{(t)} M_n$.*

Proof. We prove the first part of the claim by induction, and prove that for every round t, the second part of the claim (regarding K_t) follows from the first part (regarding P_t^-).

By Claim 3.21, $P_0^- = \sigma_0^2 M_n^2$. By Claim 3.22,

$$M_n^2 = \mathcal{N}\left(\frac{n}{n-1}, -\frac{n}{(n-1)^2}\right) = \frac{-n}{n-1} M_n.$$

Therefore, $P_0^- = -\frac{n}{n-1}\sigma_0^2 M_n$, and so the first part of the claim holds at round 0 since $\alpha_0 = \frac{n}{n-1}\sigma_0^2$.

Now, let us assume that the first part of the claim holds for some $t \in \mathbb{N}$. It follows that $P_t^- + \sigma_m^2 I = \mathcal{N}\left(\alpha_t + \sigma_m^2, \frac{-\alpha_t}{n-1}\right)$. By Claim 3.23, since $\sigma_m > 0$, we have

$$\left(P_t^- + \sigma_m^2 I\right)^{-1} = \frac{\mathcal{N}\left(\alpha_t + \sigma_m^2 - (n-2)\cdot\frac{\alpha_t}{n-1}, \frac{\alpha_t}{n-1}\right)}{\left(\alpha_t + \sigma_m^2 + \frac{\alpha_t}{n-1}\right)\left(\alpha_t + \sigma_m^2 - (n-1)\cdot\frac{\alpha_t}{n-1}\right)} = \frac{\mathcal{N}\left(\frac{\alpha_t}{n-1} + \sigma_m^2, \frac{\alpha_t}{n-1}\right)}{\sigma_m^2\left(\frac{n}{n-1}\alpha_t + \sigma_m^2\right)}.$$

By Claim 3.22 again, we can compute the "Kalman gain":

$$K_t = P_t^- \left(P_t^- + \sigma_m^2 I\right)^{-1}$$
$$= \frac{1}{\sigma_m^2\left(\frac{n}{n-1}\alpha_t + \sigma_m^2\right)} \cdot \mathcal{N}\left(\frac{\alpha_t^2}{n-1} + \sigma_m^2\alpha_t - \frac{(n-1)\cdot\alpha_t^2}{(n-1)^2},\right.$$
$$\left.\frac{-\alpha_t^2}{(n-1)^2} - \frac{\sigma_m^2\alpha_t}{n-1} + \frac{\alpha_t^2}{n-1} - (n-2)\cdot\frac{\alpha_t^2}{(n-1)^2}\right)$$
$$= \frac{\mathcal{N}\left(\sigma_m^2\alpha_t, \frac{-\sigma_m^2\alpha_t}{n-1}\right)}{\sigma_m^2\left(\frac{n}{n-1}\alpha_t + \sigma_m^2\right)} = \frac{-\alpha_t M_n}{\frac{n}{n-1}\alpha_t + \sigma_m^2},$$

This proves that the second part of the claim holds at round t. Next, we compute the error co-variance matrix after the measurement:

$$P_t = (I - K_t)\cdot P_t^- = \frac{\mathcal{N}\left(\frac{\alpha_t}{n-1} + \sigma_m^2, \frac{\alpha_t}{n-1}\right)}{\frac{n}{n-1}\alpha_t + \sigma_m^2}\cdot P_t^-$$
$$= \frac{1}{\frac{n}{n-1}\alpha_t + \sigma_m^2}\cdot\mathcal{N}\left(\frac{\alpha_t^2}{n-1} + \sigma_m^2\alpha_t - (n-1)\cdot\frac{\alpha_t^2}{(n-1)^2},\right.$$
$$\left. - \frac{\alpha_t^2}{(n-1)^2} - \frac{\sigma_m^2\alpha_t}{n-1} + \frac{\alpha_t^2}{n-1} - (n-2)\cdot\frac{\alpha_t^2}{(n-1)^2}\right)$$
$$= \frac{\mathcal{N}\left(\sigma_m^2\alpha_t, \frac{-\sigma_m^2\alpha_t}{n-1}\right)}{\frac{n}{n-1}\alpha_t + \sigma_m^2} = -\frac{\frac{n-1}{n}\sigma_m^2\alpha_t}{\alpha_t + \frac{n-1}{n}\sigma_m^2}M_n.$$

Eventually, we compute the error co-variance matrix at round $t+1$, before the measurement:

$$P_{t+1}^- = P_t + \sigma_d^2 M_n^2 = P_t - \frac{n}{n-1}\sigma_d^2 M_n.$$

Plugging in the expression of P_t, we get

$$P_{t+1}^- = -\frac{\frac{n-1}{n}\sigma_m^2\alpha_t}{\alpha_t + \frac{n-1}{n}\sigma_m^2}M_n - \frac{n}{n-1}\sigma_d^2 M_n = -\left(\frac{\frac{n-1}{n}\sigma_m^2\alpha_t}{\frac{n-1}{n}\sigma_m^2 + \alpha_t} + \frac{n}{n-1}\sigma_d^2\right)M_n = -\alpha_{t+1}M_n,$$

which concludes the induction proof. □

3.3.2 Main Proofs

3.3.2.1 Meet at the center

The goal of this section is to prove Theorem 3.9 stating that the "meet at the center" algorithm (Algorithm 8) is optimal in the centralized setting (in the sense of Definition 3.2). The choice of the name comes from the fact that this algorithm minimizes the drift of the center of mass (see Theorem 3.14).

Proof of Theorem 3.9. By Observation 3.27, is it sufficient to prove that Algorithm 8 is Kalman-perfect, that is, that for every integer $t \geq 1$, the Kalman filter associated with the moves produced by Algorithm 8 gives the estimate $\hat{\theta}_t^- = 0$.

For this purpose, assume that all agents run Algorithm 8. We prove by induction that for every integer $t \geq 0$, the Kalman filter produces the estimate $\hat{\theta}_t^- = 0$. This holds at $t = 0$ since we assumed that the Kalman filter estimates 0 at round 0, i.e., $\hat{\theta}_0^- = 0$. Next, let us assume that $\hat{\theta}_t^- = 0$ for some integer $t \geq 0$. We have by definition,

$$\hat{\theta}_{t+1}^- = \hat{\theta}_t + M_n d\theta^{(t)}, \qquad (3.26)$$

and

$$\hat{\theta}_t = \hat{\theta}_t^- + K_t\left(Y^{(t)} - \hat{\theta}_t^-\right) = K_t Y^{(t)}, \tag{3.27}$$

where the second equality in Eq. (3.27) is by the induction hypothesis. By Lemma 3.29, $K_t = -\rho_\star^{(t)} M_n$. Note that, by definition of Algorithm 8, $d\theta^{(t)} = -\frac{n-1}{n}\rho_\star^{(t)} M_n Y^{(t)} = \frac{n-1}{n}\hat{\theta}_t$. Finally, by the equations above, we rewrite Eq. (3.26) as:

$$\hat{\theta}_{t+1}^- = \left(I + \frac{n-1}{n}M_n\right)\hat{\theta}_t = \left(I + \frac{n-1}{n}M_n\right)K_t Y^{(t)} = -\left(M_n + \frac{n-1}{n}M_n^2\right)\rho_\star^{(t)} Y^{(t)}.$$

Since $M_n^2 = \frac{-n}{n-1}M_n$, we have $\hat{\theta}_{t+1}^- = 0$. This concludes the induction, and completes the proof of the theorem. □

3.3.2.2 Optimal weighted-average algorithm

The goal of this section is to prove Theorem 3.10. The proof follows the same line of arguments as the proof of Theorem 3.9.

Proof of Theorem 3.10. Our goal is to prove that Algorithm $\mathcal{W}^\star$ is Kalman-perfect, that is, for every integer $t \geq 1$, the Kalman filter associated with the moves produced by Algorithm $\mathcal{W}^\star$ gives the estimate $\hat{\theta}_t^- = 0$. This would conclude the proof of the theorem, by Observation 3.27.

For this purpose, assume that all agents run Algorithm $\mathcal{W}^\star$. We prove by induction that for every integer $t \geq 0$, the Kalman filter produces the estimate $\hat{\theta}_t^- = 0$.

The base case, where $t = 0$, holds since we assumed that the Kalman filter estimates zero at round zero, i.e., $\hat{\theta}_0^- = 0$. Next, let us assume that $\hat{\theta}_t^- = 0$ for some integer $t \geq 0$, and consider $t+1$. We have by definition,

$$\hat{\theta}_{t+1}^- = \hat{\theta}_t + M_n d\theta^{(t)}, \tag{3.28}$$

and

$$\hat{\theta}_t = \hat{\theta}_t^- + K_t\left(Y^{(t)} - \hat{\theta}_t^-\right) = K_t Y^{(t)}, \tag{3.29}$$

where the second equality in Eq. (3.29) is by induction hypothesis. By Lemma 3.29, $K_t = -\rho_\star^{(t)} M_n$. Hence, Eq. (3.28) rewrites

$$\hat{\theta}_{t+1}^- = M_n\left(-\rho_\star^{(t)} Y^{(t)} + d\theta^{(t)}\right).$$

Finally, by the definition of $\mathcal{W}^\star$, we have $d\theta^{(t)} = \rho_\star^{(t)} Y^{(t)}$, so $\hat{\theta}_{t+1}^- = 0$, concluding the induction step. □

3.4 Additional Claims

3.4.1 All Optimal Algorithms are Shifts of One Another

In this section, we characterize all optimal (deterministic) algorithms for the alignment problem in the centralized setting. We show that each of these algorithms can be obtained from $\mathcal{W}^\star$, by shifting all the agents by the same quantity λ_t, though we stress that shifts λ_t are not necessarily the same for all rounds t.

Theorem 3.30. *A deterministic algorithm is optimal in the centralized setting if and only if for every round t, there exists λ_t such that for every $i \in \{1, \ldots, n\}$, $d\theta_i^{(t)} = \rho_\star^{(t)} Y_i^{(t)} + \lambda_t$.*

Proof. We have already established that the (deterministic) weighted-average algorithm $\mathcal{W}^\star$ is a Kalman-perfect algorithm. Therefore, by Observation 3.27, any other deterministic algorithm is optimal in the centralized setting if and only if it is Kalman-perfect. In other words, it is optimal if and only if it produces a sequence of moves such that for every round t, the Kalman-filter estimator operating on the corresponding process yields

$$\begin{aligned}\hat{\theta}_{t+1} = 0 &\iff \hat{\theta}_t + M_n d\theta^{(t)} = 0 \\ &\iff K_t Y^{(t)} + M_n d\theta^{(t)} = 0 \\ &\iff -\rho_\star^{(t)} M_n Y^{(t)} + M_n d\theta^{(t)} = 0 \\ &\iff M_n \left(-\rho_\star^{(t)} Y^{(t)} + d\theta^{(t)}\right) = 0 \\ &\iff -\rho_\star^{(t)} Y^{(t)} + d\theta^{(t)} \in \ker(M_n).\end{aligned}$$

Writing $\mathbf{1}$ to denote the vector whose coefficients are all equal to 1, we observe that $\mathbf{1} \in \ker(M_n)$. Since $\text{rank}(M_n) = n-1$, $\dim(\ker(M_n)) = 1$, so for every round t,

$$\begin{aligned}-\rho_\star^{(t)} Y^{(t)} + d\theta^{(t)} \in \ker(M_n) &\iff \exists \lambda_t \in \mathbb{R},\ -\rho_\star^{(t)} Y^{(t)} + d\theta^{(t)} = \lambda_t \cdot \mathbf{1} \\ &\iff \exists \lambda_t \in \mathbb{R},\ d\theta^{(t)} = \lambda_t \cdot \mathbf{1} + \rho_\star^{(t)} Y^{(t)}.\end{aligned}$$

□

3.4.2 Computing the shifts between $\mathcal{W}^\star$ and Algorithm 8

In this section, we consider one execution of the process when $\mathcal{W}^\star$ is used, and one execution when Algorithm 8 (meet at the center) is used. The variables involved in the execution of $\mathcal{W}^\star$ are denoted with $[\cdot]^{\mathcal{W}^\star}$, while the variables involved in the execution of Algorithm 8 are denoted with $[\cdot]^{\mathsf{MatC}}$.

We assume that the randomness is the same for both algorithms, that is, the initialization of agents is the same, and for every round t, and every $i \in I$, we have $\left[N_{m,i}^{(t)}\right]^{\mathcal{W}^\star} = \left[N_{m,i}^{(t)}\right]^{\mathsf{MatC}}$, $\left[N_{d,i}^{(t)}\right]^{\mathcal{W}^\star} = \left[N_{d,i}^{(t)}\right]^{\mathsf{MatC}}$ and $\left[\overline{\theta}_i^{(0)}\right]^{\mathcal{W}^\star} = \left[\overline{\theta}_i^{(0)}\right]^{\mathsf{MatC}}$.

Claim 3.31. *For every round t, and for every $i \in \{1, \dots, n\}$,*

- $\left[\overline{\theta}_i^{(t)}\right]^{\mathcal{W}^\star} = \left[\overline{\theta}_i^{(t)}\right]^{\mathsf{MatC}}$,
- $\left[d\theta_i^{(t)}\right]^{\mathcal{W}^\star} - \left[d\theta_i^{(t)}\right]^{\mathsf{MatC}} = \frac{1}{n}\rho_\star^{(t)} \sum_{i \in I} \left[Y_i^{(t)}\right]^{\mathcal{W}^\star}$.

Proof. The proof proceeds by induction. More precisely, we prove the first item in the claim by induction on t. For any round t, we prove that the second item follows from the first one. Then, in the induction step, when proving that the first item regarding time $t+1$ holds, we use the second item regarding the previous time t.

The base case for the first item in the claim, i.e., $\left[\overline{\theta}_i^{(0)}\right]^{\mathcal{W}^\star} = \left[\overline{\theta}_i^{(0)}\right]^{\mathsf{MatC}}$ for every $i \in I$, holds by assumption. Next, let us assume that for some round t, we have for every $i \in I$, $\left[\overline{\theta}_i^{(t)}\right]^{\mathcal{W}^\star} = \left[\overline{\theta}_i^{(t)}\right]^{\mathsf{MatC}}$. Since the measurement noises are equal, then by the induction hypothesis, the measurement are also equal, that is, for every $i \in I$,

$$\left[Y_i^{(t)}\right]^{\mathcal{W}^\star} = \left[\overline{\theta}_i^{(t)} + N_{m,i}^{(t)}\right]^{\mathcal{W}^\star} = \left[\overline{\theta}_i^{(t)} + N_{m,i}^{(t)}\right]^{\mathsf{MatC}} = \left[Y_i^{(t)}\right]^{\mathsf{MatC}} = Y_i^{(t)}.$$

Thus, by definition,

$$\left[d\theta_i^{(t)}\right]^{\mathcal{W}^\star} - \left[d\theta_i^{(t)}\right]^{\mathsf{MatC}} = \rho_\star^{(t)} Y_i^{(t)} - \frac{n-1}{n}\rho_\star^{(t)}\left(Y_i^{(t)} - \frac{1}{n-1}\sum_{j\in I\setminus i} Y_j^{(t)}\right)$$
$$= \frac{1}{n}\rho_\star^{(t)} Y_i^{(t)} + \frac{1}{n}\rho_\star^{(t)}\sum_{j\in I\setminus i} Y_j^{(t)} = \frac{1}{n}\rho_\star^{(t)}\sum_{i\in I} Y_i^{(t)} := \lambda_t.$$

Finally, by Eq. (3.16),

$$\left[\overline{\theta}_i^{(t+1)}\right]^{\mathcal{W}^\star} = \left[\overline{\theta}_i^{(t)} - d\theta_i^{(t)} - N_{d,i}^{(t)} + \frac{1}{n-1}\sum_{j\neq i}\left(d\theta_j^{(t)} + N_{d,j}^{(t)}\right)\right]^{\mathcal{W}^\star}$$
$$= \left[\overline{\theta}_i^{(t)} - N_{d,i}^{(t)}\right]^{\mathsf{MatC}} - \left[d\theta_i^{(t)}\right]^{\mathcal{W}^\star} + \frac{1}{n-1}\sum_{j\neq i}\left[d\theta_j^{(t)}\right]^{\mathcal{W}^\star} + \left[N_{d,j}^{(t)}\right]^{\mathsf{MatC}}$$
$$= \left[\overline{\theta}_i^{(t)} - (d\theta_i^{(t)} + \lambda_t) - N_{d,i}^{(t)} + \frac{1}{n-1}\sum_{j\neq i} d\theta_j^{(t)} + \lambda_t + N_{d,j}^{(t)}\right]^{\mathsf{MatC}}$$
$$= \left[\overline{\theta}_i^{(t)} - d\theta_i^{(t)} - N_{d,i}^{(t)} + \frac{1}{n-1}\sum_{j\neq i} d\theta_j^{(t)} + N_{d,j}^{(t)}\right]^{\mathsf{MatC}} = \left[\overline{\theta}_i^{(t+1)}\right]^{\mathsf{MatC}}.$$

This concludes the induction step, and establishes the proof. □

3.4.3 Limit of the sequence α_t

Proof of Observation 3.11. For $a, b > 0$, define $f_{a,b} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f_{a,b}(x) = a\frac{x}{x+a} + b$. Solving $f_{a,b}(\ell) = \ell$ on $\mathbb{R}^+$ gives $\ell = \frac{1}{2}\left(\sqrt{b}\sqrt{4a+b} + b\right)$. For every $x \in \mathbb{R}^+$:

$$f_{a,b}(x) - f_{a,b}(\ell) = a\left(\frac{x}{x+a} - \frac{\ell}{\ell+a}\right) = a\cdot\frac{x(\ell+a) - \ell(x+a)}{(x+a)(\ell+a)} = a^2\frac{x-\ell}{(x+a)(\ell+a)}.$$

Thus, and since $x \geq 0$,

$$|f_{a,b}(x) - f_{a,b}(\ell)| = \left|a^2\frac{x-\ell}{(x+a)(\ell+a)}\right| \leq \frac{a^2}{a(a+\ell)}|x-\ell| = \frac{a}{a+\ell}|x-\ell|. \tag{3.30}$$

Claim 3.32. *Let $(u_t)_{t\in\mathbb{N}}$ be a sequence defined by $u_0 \in \mathbb{R}^+$ and for every integer t, $u_{t+1} = f_{a,b}(u_t)$. Then, (u_t) converges, and $\lim_{t\to\infty} u_t = \ell$.*

Proof. Let $k = \frac{a}{a+\ell}$. Since $a, b > 0$ then $\ell > 0$, and so $k < 1$. Let us show by induction that for every t, $|u_t - \ell| \leq k^t \cdot |u_0 - \ell|$. This equality is trivial for $t = 0$. Assuming that it holds for some $t \in \mathbb{N}$, we have

$$\begin{aligned} |u_{t+1} - \ell| &= |f_{a,b}(u_t) - f_{a,b}(\ell)| && \text{(by definition of } u_t \text{ and } \ell) \\ &\leq k \cdot |u_t - \ell| && \text{(by Eq.(3.30))} \\ &\leq k^{t+1} \cdot |u_0 - \ell|, && \text{(by induction hypothesis)} \end{aligned}$$

concluding the induction. Since $k < 1$, it implies that $\lim_{t\to\infty}|u_t - \ell| = 0$, and so $\lim_{t\to\infty} u_t = \ell$. □

Applying Claim 3.32 to $(\alpha_t)_{t\in\mathbb{N}}$ with $a = \frac{n-1}{n}\sigma_m^2$ and $b = \frac{n}{n-1}\sigma_d^2$ gives $\lim_{t\to\infty}\alpha_t = \alpha_\infty$, as stated.

Next, we show that $\lim_{t\to\infty} \rho_\star^{(t)} = \rho_\star$. By letting t tend to $+\infty$ in Definition 3.7, we obtain

$$\alpha_\infty = \frac{\frac{n-1}{n}\sigma_m^2 \alpha_\infty}{\frac{n-1}{n}\sigma_m^2 + \alpha_\infty} + \frac{n}{n-1}\sigma_d^2. \tag{3.31}$$

Doing the same in Definition 3.8, we get

$$\lim_{t\to+\infty} \rho_\star^{(t)} = \frac{\frac{n-1}{n}\alpha_\infty}{\frac{n-1}{n}\sigma_m^2 + \alpha_\infty} = \frac{1}{\sigma_m^2} \frac{\frac{n-1}{n}\sigma_m^2 \alpha_\infty}{\frac{n-1}{n}\sigma_m^2 + \alpha_\infty}.$$

By Eq. (3.31), this gives

$$\lim_{t\to+\infty} \rho_\star^{(t)} = \frac{1}{\sigma_m^2}\left(\alpha_\infty - \frac{n}{n-1}\sigma_d^2\right).$$

Plugging in the expression of α_∞ mentioned in the Lemma, we find that

$$\lim_{t\to+\infty} \rho_\star^{(t)} = \frac{\sigma_d \sqrt{4\sigma_m^2 + \left(\frac{n}{n-1}\sigma_d\right)^2} - \frac{n}{n-1}\sigma_d^2}{2\sigma_m^2} = \rho_\star,$$

which establishes the proof. □

3.5 Drift of the Center of Mass

The goal of this section is to prove the results of Section 3.1.2.4. By definition in Eq. (3.2), for a given algorithm, the center of mass changes in round t as follows:

$$\langle \theta^{(t+1)} \rangle - \langle \theta^{(t)} \rangle = \frac{1}{n}\sum_{i\in I}\left(d\theta_i^{(t)} + N_{d,i}^{(t)}\right). \tag{3.32}$$

The drift of the center of mass after t rounds is therefore

$$\Delta_t = \frac{1}{n}\sum_{s=0}^{t}\sum_{i\in I}\left(d\theta_i^{(s)} + N_{d,i}^{(s)}\right).$$

Let us define *inherent-drift* as

$$\Delta_t^{\mathsf{inh}} := \frac{1}{n}\sum_{s=0}^{t}\sum_{i\in I} N_{d,i}^{(s)},$$

and *relative-drift* as

$$\Delta_t^{\mathsf{rel}} := \frac{1}{n}\sum_{s=0}^{t}\sum_{i\in I} d\theta_i^{(s)}.$$

With these notations, we have:

$$\Delta_t = \Delta_t^{\mathsf{rel}} + \Delta_t^{\mathsf{inh}}. \tag{3.33}$$

Note that, while Δ_t^{rel} depends on the protocol being used, Δ_t^{inh} has always the same distribution, regardless of the protocol used. However, this fact doesn't necessarily mean that the two variables are independent, since for some protocols, Δ_t^{rel} could, in principle, depend on Δ_t^{inh}. Indeed, under some (non-Gaussian) distributions of drift, it is possible to find a protocol that makes the two variables dependent. However, as the following lemma shows, when the underlying distributions are Gaussian (as assumed in this paper), these two variables are always independent.

Lemma 3.33. *For every protocol and for every round t, Δ_t^{inh} and Δ_t^{rel} are independent.*

Proof. First we apply Lemma A.17, whose statement and proof can be found in Appendix A.2.3. Specifically, for every $s \in \mathbb{N}$, we can find normal random variables $Z^{(s)}, X_1^{(s)}, \ldots, X_n^{(s)}$ such that $\sum_{i=1}^n X_i^{(s)} = 0$, $Z^{(s)} \sim \mathcal{M}(0, \sigma_d^2/n)$, $Z^{(s)}$ is independent from $(X_1^{(s)}, \ldots, X_n^{(s)})$, and $(Z^{(s)} + X_i^{(s)}) \sim \mathcal{M}(0, \sigma_d^2 I)$. In our analysis, for every $s \in \mathbb{N}$, we can replace the noise vector $(N_{d,i}^{(s)})_{i\in I}$ by $(Z^{(s)} + X_i^{(s)})_{i\in I}$, since these two vectors follow the same distribution. The inherent drift then rewrites

$$\Delta_t^{\mathsf{inh}} = \frac{1}{n}\sum_{s=0}^{t}\sum_{i=1}^{n} X_i^{(s)} + \frac{1}{n}\sum_{s=0}^{t}\sum_{i=1}^{n} Z^{(s)} = \sum_{s=0}^{t} Z^{(s)}. \tag{3.34}$$

We have that the drift for Agent i at round t is $Z^{(t)} + X_i^{(t)}$. Moreover, while its stretch $\overline{\theta}_i^{(t+1)}$ may depend on $X_i^{(t)}$, we know that the variables $X_i^{(t)}$ and $Z^{(t)}$ are independent. Since adding the same quantity to the position of each agent does not change the stretches, we have that the stretch $\overline{\theta}_i^{(t+1)}$ is independent from the particular realization of $Z^{(t)}$. Hence, for every $t \in \mathbb{N}$, the stretch vector $(\overline{\theta}_1^{(t)}, \ldots, \overline{\theta}_n^{(t)})$ is independent from $(Z^{(0)}, \ldots, Z^{(t)})$, and thus independent from Δ_t^{inh}, by Eq. (3.34). By definition, this implies that the measurement vector $(Y_1^{(t)}, \ldots, Y_n^{(t)})$ is also independent from Δ_t^{inh}. Since the agents' moves are based solely on their measurements, we conclude that the vector of moves $(d\theta_1^{(t)}, \ldots, d\theta_n^{(t)})$ is independent from Δ_t^{inh} as well. This concludes the proof of Lemma 3.33. □

A direct consequence of Lemma 3.33 is that

$$\mathbf{E}\left(\Delta_t^2\right) = \mathbf{E}\left({\Delta_t^{\mathsf{rel}}}^2\right) + \mathrm{Var}\left(\Delta_t^{\mathsf{inh}}\right) \geq \mathrm{Var}(\Delta_t^{\mathsf{inh}}), \tag{3.35}$$

justifying the choice of the term "inherent". Since $\mathrm{Var}(\Delta_t^{\mathsf{inh}}) = \mathrm{Var}\left(\frac{1}{n}\sum_{s=0}^{t}\sum_{i\in I} N_{d,i}^{(s)}\right) = \frac{t+1}{n}\sigma_d^2$, we can characterize the drift of Algorithms 8 and 9.

Proof of Theorem 3.14. When the agents follow the "meet at the center" centralized algorithm (Algorithm 8), for every $t \in \mathbb{N}$

$$\begin{aligned}\frac{1}{n}\sum_{i\in I} d\theta_i^{(t)} &= \frac{1}{n}\sum_{i\in I}\rho_\star^{(t)}\left(\frac{n-1}{n}Y_i^{(t)} - \frac{1}{n}\sum_{j\in I\setminus i} Y_j^{(t)}\right) \\ &= \frac{\rho_\star^{(t)}}{n}\left(\frac{n-1}{n}\sum_{i\in I} Y_i^{(t)} - \frac{1}{n}\sum_{i\in I}(n-1)\cdot Y_i^{(t)}\right) = 0.\end{aligned}$$

Hence, by definition, $\Delta_t^{\mathsf{rel}} = 0$ and by Eq. (3.35), the drift of the center of mass in round t is limited to Δ_t^{inh}. By definition, this implies that Algorithm 8 is drift-optimal, which concludes the proof of Theorem 3.14. □

Proof of Theorem 3.16. Under Algorithm $\mathcal{W}^\star$ (Algorithm 9), we have for every $t \in \mathbb{N}$

$$\frac{1}{n}\sum_{i\in I} d\theta_i^{(t)} = \frac{1}{n}\sum_{i\in I}\rho_\star^{(t)}\left(\overline{\theta}_i^{(t)} + N_{m,i}^{(t)}\right) = \frac{1}{n}\sum_{i\in I}\rho_\star^{(t)} N_{m,i}^{(t)},$$

where the last equality is by Lemma 3.17. Using Eq. (3.35), and summing over the rounds $s = 0, \ldots, t$, we conclude the proof of Theorem 3.16. □

Finally, we argue that the sub-optimality of Algorithm $\mathcal{W}^\star$ (in terms of the drift of the center of mass) is not coincidental.

Proof of Claim 3.15. Let A be an optimal algorithm different than Algorithm 8. Again, we denote the variables involved in the execution of A by $[\cdot]^A$. We have shown in Section 3.4.1 that all optimal algorithms are shifts of one another, i.e., for all t we can find a random variable λ_t s.t. for all i,

$$\left[d\theta_i^{(t)}\right]^A = \left[d\theta_i^{(t)}\right]^{\mathsf{MatC}} + \lambda_t.$$

It implies that

$$\frac{1}{n}\sum_{i\in I}\left[d\theta_i^{(t)}\right]^A = \lambda_t + \frac{1}{n}\sum_{i\in I}\left[d\theta_i^{(t)}\right]^{\mathsf{MatC}} = \lambda_t,$$

where the last equality follows from the definition of Algorithm 8. Therefore,

$$\left[\Delta_t^{\mathsf{rel}}\right]^A = \frac{1}{n}\sum_{s=0}^{t}\sum_{i\in I}\left[d\theta_i^{(s)}\right]^A = \sum_{s=0}^{t}\lambda_s.$$

Since A is different than Algorithm 8, we can define $t_0 := \min\{t \in \mathbb{N}, \mathbf{E}\left(\lambda_t^2\right) > 0\}$, and have

$$\mathbf{E}\left(\left[\Delta_{t_0}^{\mathsf{rel}}\right]^{A2}\right) = \mathbf{E}\left(\left(\sum_{s=0}^{t_0}\lambda_s\right)^2\right) \geq \sum_{s=0}^{t_0}\mathbf{E}\left(\lambda_s^2\right) > 0.$$

By Eq. (3.35), we conclude that, among optimal algorithm, "meet at the center" is the only one minimizing Δ_t for every t. Note that it is not possible for the agents to implement the "meet at the center" algorithm in the distributed setting, because it requires for each agent to know all measurements – and this observation concludes the proof of Claim 3.15. □

3.6 Discussion and Future Work

We introduced the distributed alignment problem, aiming to study basic algorithmic aspects that govern approximate agreement processes in unreliable stochastic environments, while relying on measurements of the overall group tendency. We proposed a weighted-average algorithm, termed $\mathcal{W}^\star$, and proved that it achieves optimal stretches, that is, it minimizes the expected distance of each agent from the center of mass of others, at each round. In fact, its optimality holds also in the full-communication (or centralized) model, where communication of internal states is unrestricted.

In our setting, the cost of an individual is defined as its expected distance from the average position of other agents. Minimizing this quantity is equivalent to minimizing the expected distance from the average position of all agents (see Footnote *a*). Another interesting measure is the expected *diameter* of the group, defined as the maximal distance between any two agents, at steady state. It would not come as a surprise if Algorithm $\mathcal{W}^\star$ would turn out to be optimal also with respect to this measure, however, analyzing its expected diameter would require handling further dependencies between agents, and remains for future work.

It would also be interesting to see how our work could be extended to scenarios in which each agent can observe a subset of randomly chosen agents at each round. As mentioned in the introduction, averaging the values of the agents in such a subset can serve as a noisy sample of average value in the population. However, in contrast to our model, the noise in this sample is not fixed, and depends on the distribution of values in the population and on the size of the sampled subset. In many cases, one can expect that sampling a subset containing a logarithmic number of agents could suffice to obtain a relatively good sample of the population. We find this encouraging towards generalizing our work to such scenarios.

Part II

Cooperation in Competitive Environments

Background and Motivation

In contrast to Part I, sometimes, the behaviour that leads to the best overall result (when adopted by all individuals) differs from the behaviour that's best for each player individually. These situations, often referred to as *social dilemmas*, are quite common in productive groups, be it in human societies or the animal realm. Although social dilemmas are irrelevant when it comes to fully cooperative ensembles, like eusocial species (ants, bees, termites...), they are likely to affect the actions of individuals that tend to prioritize their personal gains. Therefore, taking social dilemmas into consideration is imperative when researching collective behaviour of selfish agents.

Individual against collective interests. One of the most well-known examples of social dilemma is the metaphor of the *tragedy-of-the-commons*, made popular by Hardin in 1968 [88]. It involves a shared resource that each person can use up or damage for their own gain. The "tragedy" happens when the common good is destroyed as a result. In the original scenario of Hardin, a group of herders sharing a pasture keeps adding animals to their herd until the pasture becomes overgrazed and can't support their activities any longer. To fully grasp the issue, one must understand that at the individual level, fully taking advantage of the shared resource is actually optimal. To illustrate this, let us consider an imaginary herder named Joe. Compared to the size of Joe's herd, the pasture is so vast that Joe's actions barely affect it. Instead, the fate of the land relies on the average behaviour in the whole group of herders – a factor over which Joe has almost no control. Therefore, regardless of others' choices, Joe is always better off expanding his herd to make a profit, which clarifies why rational agents struggle to avoid the tragedy. The tragedy-of-the-commons metaphor applies to many crucial 21st-century challenges: water and air pollution (including greenhouse gas emissions), overfishing, antibiotic resistance, and more.

The tragedy-of-the-commons can also arise at a higher level, when the shared resource is a social good achieved through cooperation, rather than an existing external resource [133]. In this analogy, self-interested individuals spoil the common good by not contributing their fair share, instead of "overusing" it. Since they escape the costs and risks linked to cooperation while still enjoying its benefits, they're often referred to as *free-riders* [94, 24]. Productive groups, including students or researchers collaborating on a project, companies, or even sports teams, are especially prone to encountering some form of free-riding behaviour. Free-riders are also notably common in animal foraging scenarios, where individuals opportunistically scrounge or engage in kleptoparasitism, feeding off prey discovered or caught by others [29, 127, 57].

Sustaining cooperation. In spite of social dilemmas, cooperation is well-documented in both human and animal groups, even within competitive environments where only the most effective strategies endure over the long term [144]. This raises two crucial questions that have been the central focus of several studies in recent decades, although not in theoretical computer science.

(1) When is altruism the optimal behaviour for an individual, i.e., the one that maximises its fitness or welfare?

(2) Under what conditions can cooperation emerge in a competitive environment?

These questions are not as similar as they might seem; specifically, it is possible to imagine scenarios in which cooperation arises even if it is not strictly the optimal behaviour, as we will see later.

On the empirical side, apart from direct field observations, data can be gathered from social experiments [158, 140, 43]. These experiments involve groups of human participants who, driven by financial incentives, compete in simple games. On the theoretical side, the framework of game theory, assuming self-interested and rational agents, can be used to address these questions. The possible behaviours are typically narrowed down to two stereotyped strategies: *cooperation* and *defection*. The prevailing understanding is that the best possible average payoff is reached when every single player is a cooperator; however, each individual may increase its own payoff by being a defector, albeit at the expanse of others. Games are analysed by identifying *equilibriums* – strategic configurations in which no player can improve their payoff by altering their behaviour.

Question (1) is answered negatively in the well-known *prisoner's dilemma* and analogous games for more than two players, often referred to as *public goods* games – illustrating the tragedy-of-the-commons. However, even slight modifications to the setting can lead to circumstances that can overturn or refine this conclusion. A critical factor in this context is *reciprocity* [123]. When the game spans multiple rounds, a player's choice in one round might influence not just their immediate payoff, but also subsequent decisions by other players. Remarkably, even when every two players interact at most once, indirect forms of reciprocity are feasible, as long as some information about their past choices is publicly available, e.g., through a reputation system [117]. Reciprocity can amplify cooperation by linking a contingent benefit to it, such as gaining future rewards or evading penalties from other players. This effect has been examined in several of its aspects. Some works compare the impact of rewards and punishments [141, 132, 41]. Another line of research delves into the *second-order* cooperation problem, where imposing costly penalties on free-riders (or conversely, offering costly rewards to cooperators) becomes another form of "social good", facing a higher-level version of the tragedy-of-the-commons [90, 128, 62].

Question (2) is a focal point in evolutionary biology and evolutionary game theory. The premise there is that behaviours resulting in the greatest payoff tend to proliferate, either through imitation by others or through genetic or cultural transmission to offspring. Evolutionary mechanisms might facilitate the emergence of cooperation even when classical game theory rules it out. For instance, while altruism involves an inherent net cost for the individual, the benefits given to other altruistic agents could compensate this cost in the overall success of the altruistic behaviour. This phenomenon, known as *kin selection* [123], is better understood by thinking of behaviours themselves as strategic agents. Some studies investigate the influence of population structure [130, 124, 7]. Here, the standard assumption is that players are organized over a fixed network, with the edges representing reciprocal relations between neighbours. This arrangement can amplify kin selection, given that proximate nodes are more likely to share the same strategy. Note that it can also enhance reciprocity, since neighbours interact repeatedly.

Exploring the impact of parameter changes. Drawing inspiration from mechanism design principles, we aim to understand the influence of altering specific parameters on the prevalence of cooperation in the population. These parameter changes can signify modifications in the environment, as well as in individual capabilities, such as those resulting from genetic or cultural evolution. Our objective is to gain insights on underlying evolutionary mechanisms, or at the very least, challenge their apparent simplicity. This methodology could also help in predicting and comprehending the repercussions of climate change, which introduces new environments for numerous species.

We are particularly interested in scenarios in which changes yield counter-intuitive outcomes – instances where seemingly improved conditions lead to reduced payoff for all players. For this phenomenon to happen, the modifications to the setting must favour free-riders and hence undermine cooperation. Moreover, this effect must be substantial enough to offset their inherent advantage. In what follows, the main challenge was to identify simple and credible models exhibiting this property. The analyses, although non-trivial, do not introduce especially novel or noteworthy techniques; instead, their contribution is mainly conceptual.

Chapter 4 demonstrates how increasing food availability within a basic producer-scrounger model can eventually reduce food consumption. In Chapter 5, we show that increasing tolerance towards hypocritical behaviours may allow the emergence of cooperation in large spatial public good games.

Chapter 4

Food Availability can Deteriorate Fitness

Chapter Contents

Acknowledgments This chapter is based on

[152] R.V. and Amos Korman. Enhanced Food Availability can Deteriorate Fitness through Excessive Scrounging, *arXiv preprint arXiv:2307.04920*, July 2023.

4.1 Introduction

The framework of *Producer-Scrounger* (PS) games is a widely used mathematical framework for studying free-riding in foraging contexts [153, 82, 83, 81, 89]. In a PS game, players are faced with a choice between two strategies: producer and scrounger (corresponding to the more traditional cooperator and defector). The exact interpretation of these strategies varies according to the context, but generally, a producer invests efforts in order to produce or find more resources, whereas a scrounger invests less in producing or finding resources, and instead relies more on exploiting resources produced or found by others. Based on the rules of the particular PS game, specifying the production and rewarding mechanisms, each animal chooses a strategy and the system is assumed to converge into an equilibrium state, where each animal cannot improve its own calorie intake by changing its strategy [143].

In this chapter, we examine the impact of individual production capacity on the resulting payoffs in equilibrium configurations. The first PS game we consider aims to model a scenario consisting of a group of foraging animals, with each animal striving to maximize its own food intake. Intuitively, as long as the group size remains unchanged, one may expect that, even if it may trigger more opportunistic behaviour [30, 32], increasing food abundance should ultimately improve consumption rather than diminish it. Likewise, within a productivity-based reward system in a company, one may expect that enhancing individual productivity levels would boost group productivity and subsequently increase workers' payoffs, despite a possible increase in free-riding behaviour. However, our findings uncover a more nuanced reality, unveiling a remarkably pronounced detrimental effect of free-riding behaviour. We demonstrate how improvement in underlying conditions, which may initially seem beneficial, can actually lead to degraded performance.

Our results are reminiscent of Braess' paradox, a thought-provoking result in game theory, demonstrating that in certain transportation networks, adding a road can paradoxically increase traffic latency at equilibrium [28, 138, 35].

4.1.1 Preliminaries

Given a PS game, let $\pi_{q,p}$ denote the expected payoff of a player if it chooses to be a producer with probability q, in the case that all $n-1$ remaining players are producers with probability p.

Definition 4.1 (Maynard Smith and Price, [143]). We say that $p_\star \in [0,1]$ is an *Evolutionary Stable Strategy* (ESS) if and only if for every $q \in [0,1]$ such that $q \neq p_\star$, either (i) $\pi_{p_\star,p_\star} > \pi_{q,p_\star}$, or (ii) $\pi_{p_\star,p_\star} = \pi_{q,p_\star}$ and $\pi_{p_\star,q} > \pi_{q,q}$.

We say that a PS game exhibits a *Reverse-Correlation* (RC) phenomenon if an increase in individuals' production capacities leads to a decrease in the players' payoff, when evaluated at equilibrium configurations. More precisely, we consider a parameter γ that is positively correlated to the expected production capacities of both producers and scroungers. To check what happens as individual capabilities improve, we increase γ and observe how the payoff and total production measures change, for configurations at equilibria.

To facilitate comparisons across different parameter settings, we ensure that the games we examine have unique equilibrium configurations, i.e., we make sure that for every value of γ, the game we consider always has a unique ESS, termed $p_\star(\gamma)$. In such a case, we write $\pi_\star(\gamma) = \pi_{p_\star(\gamma),p_\star(\gamma)}$ the payoff at the ESS, and omit the parameter γ when clear from the context. Formally, we say that the system incurs a Reverse-Correlation phenomenon if increasing γ over a certain interval yields decreased payoff when evaluated at (the unique) ESS. In other words, this means that $\pi_\star(\gamma)$ is a decreasing function of γ over this interval.

Throughout this chapter, we will combine analytic investigations with computer simulations. To determine the ESS in these simulations, we utilize simple procedures that take the values of p and q as inputs and calculate $\pi_{q,p}$. Then, we search for the specific value of p that satisfies (i) $\pi_{1,p} = \pi_{0,p}$, (ii) for every $q < p$, $\pi_{1,q} > \pi_{0,q}$ and (iii) for every $q > p$, $\pi_{1,q} < \pi_{0,q}$, which together are sufficient conditions for p to be the unique ESS (see Lemma 4.4). For further details and access to the Python code, please refer to [151].

4.1.2 Our Results

We investigate two types of PS games: a foraging game involving animals searching for food and a company game involving a group of workers in a company. Our main objective is to analyze the effects of changes in individual production capabilities on players' payoffs, evaluated at equilibrium configurations.

The *Foraging game* is a generalization of the classical PS game in [153, 81]. The main difference with the classical model is that our model considers two types of food, instead of a single type as previously assumed.

As an illustration, consider a scenario involving a group of animals engaged in fruit picking from trees (see Figure 4.1a). Each animal aims to maximize its fitness, which is determined by the amount of food it consumes. The trees in this scenario contain both low-hanging fruit, accessible to both producers and scroungers, and high-hanging fruit, which can only be reached by producers. When an animal picks fruit, it retains a portion for its own consumption (let's say 70%), while the remaining fruit falls to the ground. Scroungers, instead of picking high-hanging fruit, focus on scanning the ground for fallen fruit. Fallen fruit is distributed equally among all scroungers and the animal that originally obtained it.

More precisely, consider $n \geq 2$ animals, where each of which needs to choose to be either a *producer* or a *scrounger*. We assume that a producer finds an amount of food corresponding to $F_{\mathcal{P}} = 1 + \gamma$ calories, where, adhering to the trees example above, 1 corresponds to the amount of high-hanging fruit and γ is a parameter that governs the animal's access to low-hanging fruit. In contrast, a scrounger directly finds only low-hanging fruit, corresponding to $F_{\mathcal{S}} = \gamma$ calories. After finding food (of any type) consisting of F calories, the animal (either producer or scrounger) consumes a fraction $s \in [0, 1]$ of what it found (called the "finder's share") and shares the remaining $(1-s)F$ calories equally with all scroungers. See Figure 4.1b for a schematic illustration of structure of the foraging game.

The *payoff* of a player is defined as the capacity of its calorie intake. Hence, for each $0 \leq k \leq n$, the payoff of each pure strategy in the presence of exactly k producers in the population is:

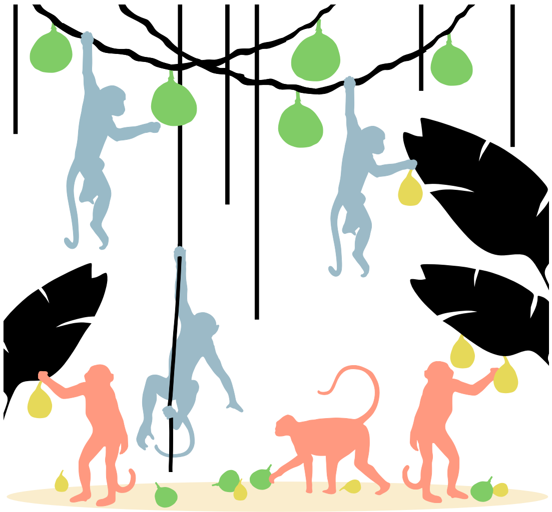
$$\pi_{\mathcal{P}}^{(k)} = s(1+\gamma) + (1-s)\frac{1+\gamma}{1+n-k}, \quad \text{and} \quad \pi_{\mathcal{S}}^{(k)} = \gamma + k(1-s)\frac{1+\gamma}{1+n-k}, \tag{4.1}$$

where the second equation follows since scrounger-to-scrounger interactions compensate each-others, and hence, can be ignored in the expression of the payoff. Note that the classical model [153, 81] is retrieved by setting $\gamma = 0$, which essentially implies that there is only one type of food.

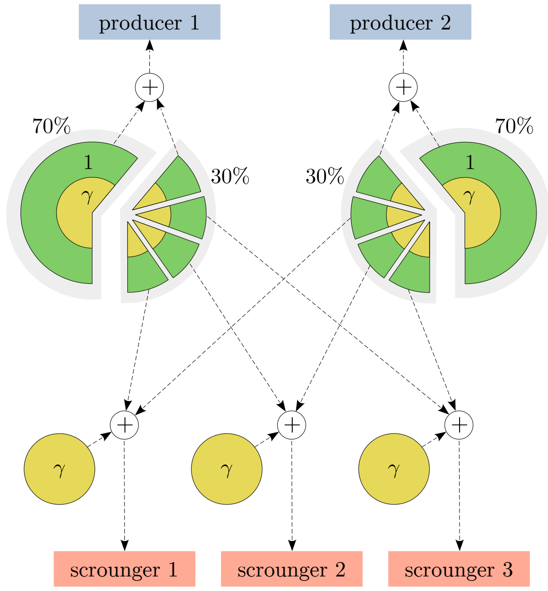
We study what happens to the payoffs of players at equilibria configurations, denoted by $\pi_\star$, as we let γ increase. This increase aims to capture the case that the low-hanging fruit becomes more abundant in the environment.

Note that for each fixed k, both $\pi_{\mathcal{P}}^{(k)}$ and $\pi_{\mathcal{S}}^{(k)}$ are increasing in γ. Hence, simply increasing γ, without changing the strategy profile, necessarily results in improved payoffs. However, allowing players to modify their strategies after such a change may potentially lead to enhanced scrounging at equilibrium, which can have a negative impact on the payoffs. Nevertheless, as mentioned earlier, one might expect that this negative effect would be outweighed by the overall improvement in fruit availability, resulting in an increase in consumption rather than a decrease. This intuition becomes apparent when comparing the scenarios with $\gamma = 0$ and $\gamma = 1$. As γ increases from 0 to 1, we can expect an increase in the proportion of scroungers due to the rising ratio of $F_{\mathcal{S}}/F_{\mathcal{P}} = \gamma/(1+\gamma)$. However, even if the system with $\gamma = 1$ ends up consisting entirely of scroungers, the average food consumption of a player (which equals 1) would still be at least as large as that of any strategy profile in the $\gamma = 0$ case. Nonetheless, as shown here, closer examination within the interval $\gamma \in [0, 1]$ reveals a Reverse-Correlation phenomenon. The proof of the following result can be found in Section 4.3.

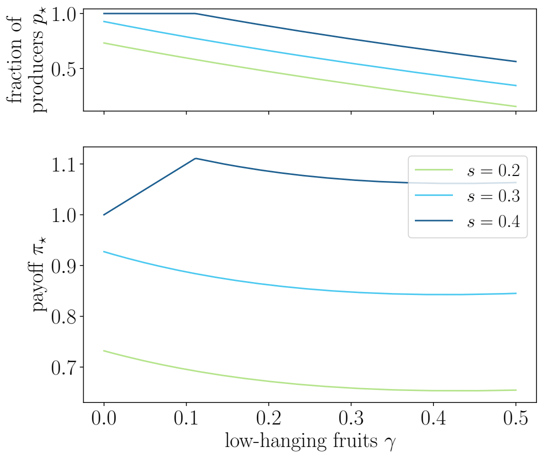
> **Theorem 4.2.** *Consider the Foraging game with $\gamma \geq 0$ and $s < 1$. If $n = 3$, then for every $\gamma \geq 0$, there is a unique ESS. Moreover, for every $s < 1/2$, there exist $\gamma_{\min}, \gamma_{\max} > 0$ such that the payoff $\pi_\star(\gamma)$ (and hence, also the total production) at ESS is strictly decreasing in γ on the interval $[\gamma_{\min}, \gamma_{\max}]$.*



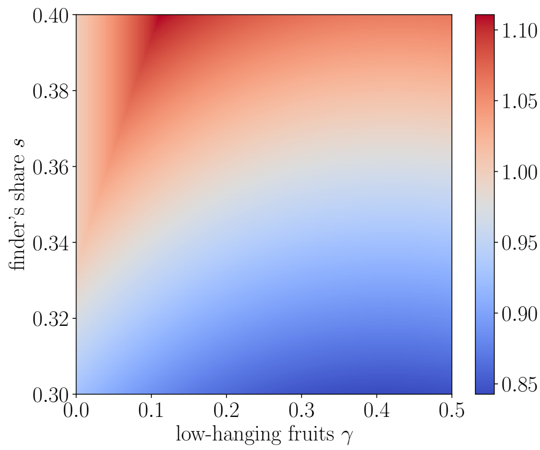
(a) An illustration of the underlying scenario.



(b) Schematic structure of the foraging game.



(c) Payoff at equilibrium $\pi_\star$ $(n = 4)$.



(d) Payoff at equilibrium $\pi_\star$ $(n = 4)$.

Figure 4.1: **The Foraging game. (a)** An illustration of the biological scenario that serves as an inspiration. Producers (in blue) can grab high-hanging fruits (in green) as well as low-hanging fruits (in yellow). Scroungers (in red) are restricted to low-hanging fruits. Whenever some fruit is picked, a fraction is eaten directly and the remaining falls to the ground where it is shared between the animal and the scroungers. **(b)** Each scrounger finds food consisting of γ calories (yellow, corresponding to low-hanging fruit) and each producer finds food of $1 + \gamma$ calories, where 1 (green) corresponds to the high-hanging fruit and γ to low-hanging fruit. A portion $s = 70\%$ is directly consumed by the finding animal, and the remaining $1 - s = 30\%$ is equally shared between the animal and all scroungers. Scrounger-to-scrounger food exchanges cancel one another and their representations are omitted. **(c)** $n = 4$. The probability of being a producer $p_\star(\gamma)$ and the payoff $\pi_\star(\gamma)$, at equilibria, for three values of the finder's share s. $\pi_\star$ is decreasing over a large interval of γ, effectively illustrating the Reverse-Correlation phenomenon. **(d)** $n = 4$. The relationship between the payoff at equilibrium $\pi_\star$ (color scale), the abundance of low-hanging fruits γ, and the finder's share s, highlighting the independence of the RC phenomenon from specific values of s.

The *Company game* aims to model a scenario with a group of $n \geq 2$ workers of equal capabilities who collaborate to produce a common product for a company. Alternatively, by replacing the salary received by a worker with a notion of prestige, the game can also capture a scenario where a group of researchers collaborate in a research project.

Each worker is assigned a specific part of the project and can choose between two pure behavioural strategies. A *producer* pays an *energetic cost* of $c > 0$ units and with probability p produces a product of quality γ (otherwise, with probability $1-p$, it produces nothing). In contrast, a *scrounger* pays no energetic cost and with probability p produces a product of lower quality $\gamma' = a\gamma$ for some given $0 \leq a < 1$ (similarly, with probability $1-p$, nothing is produced). Let $I = \{1, 2, \ldots, n\}$, and let q_i denote the quality of the product made by worker i, for $i \in I$, with $q_i = 0$ if no product is made by this player. We define the *total production* as:

$$\Gamma = \sum_{i \in I} q_i. \tag{4.2}$$

We assume that the salary σ_i of player i is proportional to a weighted average between the quality of the products made by the workers, with more weight given to q_i. If fact, by appropriately scaling the system, we may assume without loss of generality that the salary is equal to this weighted average. Formally, we set:

$$\sigma_i = sq_i + \frac{1-s}{n-1} \sum_{j \in I \setminus i} q_j, \tag{4.3}$$

for some $s \in [1/n, 1]$. Note that $s = 1$ implies that the salary each worker receives is identical to the quality of its own production, and $s = 1/n$ represents equally sharing the quality of the global product between the workers.

Next, we aim to translate the income salary of a player into his *payoff* using a *utility function*, denoted by $\phi(\cdot)$. These quantities are expected to be positively correlated, however, the correlation may in fact be far from linear. Indeed, this is supported by the seminal work by Kahneman and Deaton [99] which found that the emotional well-being of people increases relatively fast as their income rises from very low levels, but then levels off at a certain salary threshold. To capture such a correlation, we assume that ϕ is both monotonically non-decreasing, concave and bounded. In addition, the payoff of worker i is decreased by its energetic investment. Finally,

$$\pi_i := \phi(\sigma_i) - c_i, \tag{4.4}$$

where the energetic investment $c_i = c > 0$ if i is a producer and $c_i = 0$ if i is a scrounger. See Figure 4.2a for an illustration of the semantic structure of the game.

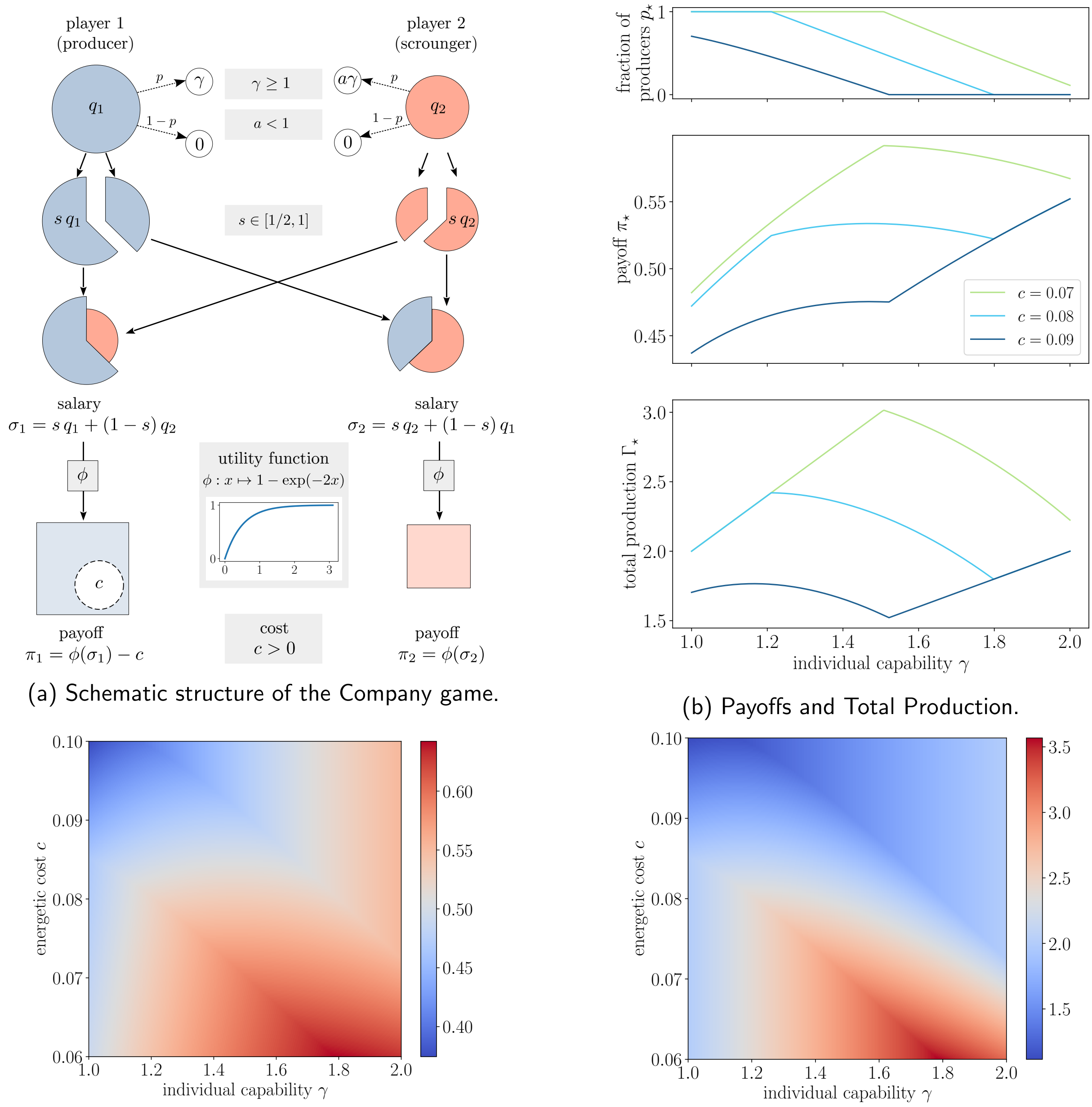
The question of whether or not the system incurs a Reverse-Correlation phenomenon turns out to depend on the model's parameters, and, in particular, on the function $\phi(x)$. For example, when $\phi : x \mapsto x$ (i.e., the case that the salary is converted entirely into payoff), there is no Reverse-Correlation phenomenon, as we will see in Section 4.4.2. However, for some concave and bounded functions $\phi(x)$ the situation is different.

Combining mathematical analysis with simulations, we identify a RC phenomenon for the case of 2 workers, when (see inset in Figure 4.2a):

$$\phi(x) = 1 - \exp(-2\,x).$$

Theorem 4.3. *Consider the Company game with $n = 2$, $\phi : x \mapsto 1 - \exp(-2\,x)$, $p = 1/2$, $a = 1/2$. For every $s < 1$, there exist $c_0 > 0$, and $\gamma_{\min}, \gamma_{\max} > 1$, for which there is a unique ESS such that $\pi_\star$ is decreasing in γ on the interval $[\gamma_{\min}, \gamma_{\max}]$.*

The proof of Theorem 4.3 can be found in Section 4.4.3. Interestingly, this result holds for every $s < 1$, demonstrating that the Reverse-Correlation phenomenon can occur even when the payoffs of individuals are substantially biased towards their own production compared to the production of others.



(a) Schematic structure of the Company game.

(b) Payoffs and Total Production.

(c) Payoffs.

(d) Total Production.

Figure 4.2: **The Company game**. **(a)** Illustration of the game's definition for the case of 2 players. The production q_i of player i is 0 with probability $1-p$, and otherwise, it is γ if the player is a producer, and $a\gamma < \gamma$, otherwise. The salary σ_i is a weighted average of the production of both players with more weight given to q_i. The utility function ϕ (inset) maps salary into the payoff, from which the energetic cost is withdrawn. Finally, $\pi_i = \phi(\sigma_i) - c_i$. **(b)—(d)** Simulating a scenario with $n = 4$ players, with the utility function $\phi : x \mapsto 1 - \exp(-2x)$, and assuming $a = p = \frac{1}{2}$, and $s = 0.6$. **(b)** The graph depicts the payoff $\pi_\star(\gamma)$ and the total production $\Gamma_\star(\gamma)$, as well as the probability of being a producer $p_\star(\gamma)$ at equilibria, displaying the Reverse-Correlation phenomenon over a certain interval that depends on c. **(c)** and **(d)** The relationship between **(c)** payoff $\pi_\star$, and **(d)** total production $\Gamma_\star$ at equilibrium, as a function (color scale) of γ and c.

Simulations In our simulations, which focus on $n = 4$ players, a noticeable decline is observed in the payoffs at equilibrium as γ increases over a relatively large sub-interval of $[0, 1]$, indicating a Reverse-Correlation phenomenon (Figures 4.1c, 4.1d and 4.2b to 4.2d). Moreover, as γ increases over a range of values we also observe a substantial reduction in total production at equilibria in the case of the Company game.

For the Company game, while the general shape of the utility function $\phi(x) = 1 - \exp(-2\,x)$ is justifiable, the function itself was chosen somewhat arbitrarily. To strengthen the generality of our results, we also provide in Section 4.4.3 (Figure 4.4) simulations supporting the Reverse-Correlation phenomenon under another type of non-decreasing, concave, and bounded utility function, specifically,

$$\phi(x) = \min(1, x).$$

A necessary condition. Finally, we identify a necessary condition for the emergence of a Reverse-Correlation phenomenon in arbitrary PS models. Specifically, we prove in Section 4.5 that a Reverse-Correlation phenomenon can occur only if the definition of the producer's payoff is sensitive to the fraction of scroungers in the population.

An interesting consequence of this condition is that a seemingly minor change in the definition of the Foraging game can prevent the occurrence of the Reverse-Correlation phenomenon. Recall that in this game, when an animal finds food, it consumes a fraction s of it (the finder's share), and the remaining $1 - s$ fraction falls to the ground and is then equally shared between the animal and all scroungers. If the game is changed such that when a producer finds food, it only consumes the finder's share and does not eat at all from the food that falls on the ground (i.e., only the scroungers eat from it), then the game stops satisfying the aforementioned necessary condition. Indeed, in this case, the payoff of a producer would always be $s(1 + \gamma)$ irrespective of the number of scroungers. Hence, the modified game does not exhibit a Reverse-Correlation phenomenon, regardless of the parameters involved.

4.2 Uniqueness of ESS

The following sufficient condition for the existence and uniqueness of an ESS is well-known. We state and prove it below for the sake of completeness.

Lemma 4.4. *If $p_\star \in [0, 1]$ is such that (i) $\pi_{1,p_\star} = \pi_{0,p_\star}$, (ii) for every $q < p_\star$, $\pi_{1,q} > \pi_{0,q}$ and (iii) for every $q > p_\star$, $\pi_{1,q} < \pi_{0,q}$, then $p_\star$ is a unique ESS.*

Proof. By assumption (i), we have for every $q \in [0, 1]$:

$$\pi_{q,p_\star} = q\,\pi_{1,p_\star} + (1 - q)\,\pi_{0,p_\star} = p_\star\,\pi_{1,p_\star} + (1 - p_\star)\,\pi_{0,p_\star} = \pi_{p_\star,p_\star}.$$

Thus, to show that $p_\star$ is an ESS, we need to check the second condition in the definition. We start by considering the case that $q < p_\star$. By assumption (ii), it implies that $\pi_{1,q} > \pi_{0,q}$, so

$$\pi_{p_\star,q} = p_\star\,\pi_{1,q} + (1 - p_\star)\,\pi_{0,q} > q\,\pi_{1,q} + (1 - q)\,\pi_{0,q} = \pi_{q,q}.$$

Similarly, in the case that $q > p_\star$, assumption (iii) implies that $\pi_{1,q} < \pi_{0,q}$, so

$$\pi_{p_\star,q} = p_\star\,\pi_{1,q} + (1 - p_\star)\,\pi_{0,q} > q\,\pi_{1,q} + (1 - q)\,\pi_{0,q} = \pi_{q,q}.$$

By the second condition in the definition, this implies that $p_\star$ is an ESS.

Finally, we prove the unicity property. Let $p \neq p_\star$. If $p < p_\star$, then $\pi_{1,p} > \pi_{0,p}$ by assumption (ii), and $p < 1$. Therefore,

$$\pi_{1,p} > p\,\pi_{1,p} + (1 - p)\,\pi_{0,p} = \pi_{p,p}.$$

If $p > p_\star$, then $\pi_{1,p} < \pi_{0,p}$ by assumption (iii), and $p > 0$. Therefore,

$$\pi_{0,p} > p\,\pi_{1,p} + (1-p)\,\pi_{0,p} = \pi_{p,p}.$$

In both cases, p is not an ESS, which concludes the proof of Lemma 4.4. □

4.3 Analysis of the Foraging Game

The goal of this section is to prove Theorem 4.2. To this aim, we will prove the more general Theorem 4.5 stated below, which considers $n = 2, 3$. For the case of 3 players, the theorem states that as long as the finder's share satisfies $s < 1/2$, there exists an interval of values for γ over which the Reverse-Correlation phenomenon occurs. In contrast, in the case of 2 players, the Reverse-Correlation phenomenon does not happen over an interval, and instead, there exists a critical value of γ at which $\pi_\star$ decreases locally. In fact, this happens even when the finder's share is close to 1.

Theorem 4.5. *Consider the Foraging game with $\gamma \geq 0$ and $s < 1$.*

- *If $n = 3$, then for every $\gamma \geq 0$, there is a unique ESS. Moreover, for every $s < 1/2$, there exist $\gamma_{\min}, \gamma_{\max} > 0$ such that the payoff $\pi_\star(\gamma)$ (and hence, also the total production) at ESS is strictly decreasing in γ on the interval $[\gamma_{\min}, \gamma_{\max}]$.*
- *If $n = 2$, then for every $\gamma \neq \gamma_s$, where $\gamma_s = \frac{1+s}{1-s}$, there is a unique ESS. Moreover, $\pi_\star(\gamma)$ is increasing on $[0, \gamma_s)$ and on $(\gamma_s, +\infty]$. However, for every $\epsilon \in (0, 1/2)$, $\pi_\star(\gamma_s - \epsilon) > \pi_\star(\gamma_s + \epsilon)$.*

Towards proving the theorem, we first establish the following lemma, which quantifies the expected payoffs of the two pure strategies, conditioning on other agents choosing to be producers with probability p.

Lemma 4.6. *For every $0 \leq p < 1$,*

$$\pi_{1,p} = sF_{\mathcal{P}} + (1-s)F_{\mathcal{P}} \cdot \frac{1-p^n}{n(1-p)},$$

and

$$\pi_{0,p} = F_{\mathcal{S}} + (1-s)F_{\mathcal{P}} \cdot p \cdot \frac{n(1-p) + p^n - 1}{n(1-p)^2}.$$

These expressions can be extended by continuity at $p = 1$, giving $\pi_{1,1} = F_{\mathcal{P}}$ and $\pi_{0,1} = F_{\mathcal{S}} + (n-1)(1-s)F_{\mathcal{P}}/2$.

Proof. Fix a player i. Consider the case that Player i is a producer, and that each player $j \neq i$ is a producer with probability p. Let X_p be the random variable indicating the number of scroungers in the population. By Eq. (4.1),

$$\pi_{1,p} = sF_{\mathcal{P}} + (1-s)F_{\mathcal{P}} \cdot \mathbf{E}\left(\frac{1}{1+X_p}\right).$$

By definition, $X_p \sim \mathcal{B}(n-1, 1-p)$. The first part of the claim, concerning $\pi_{1,p}$, now follows using Claim A.21 in Appendix A.2.4, that implies that $\mathbf{E}(1/(1+X_p)) = \frac{1-p^n}{n(1-p)}$.

Now, consider the case that Player i is a scrounger, and that each player $j \neq i$ is a producer with probability p. Let Y_p be the random variable indicating the number of producers in the population. By Eq. (4.1),

$$\pi_{0,p} = F_{\mathcal{S}} + (1-s)F_{\mathcal{P}} \cdot \mathbf{E}\left(\frac{Y_p}{1+n-Y_p}\right).$$

By definition, $Y_p \sim \mathcal{B}(n-1, p)$. The second part of the claim, concerning $\pi_{0,p}$, now follows using Claim A.22 in Appendix A.2.4, that implies that

$$\mathbf{E}\left(\frac{Y_p}{1+n-Y_p}\right) = \mathbf{E}\left(\frac{Y_p}{2+(n-1)-Y_p}\right) = p \cdot \frac{n(1-p)+p^n-1}{n(1-p)^2}.$$

This completes the proof of Lemma 4.6. ☐

In order to characterize the (unique) ESS, we first define the following quantities:

$$A(\gamma) = \frac{n(F_{\mathcal{S}} - sF_{\mathcal{P}})}{(1-s)F_{\mathcal{P}}} = \frac{n(\gamma - s(1+\gamma))}{(1-s)(1+\gamma)}, \quad \gamma_1 = \frac{2}{(n-1)(1-s)} - 1, \quad \gamma_2 = \frac{n}{(n-1)(1-s)} - 1.$$

Claim 4.7. *We have $A(\gamma_1) = -n(n-3)/2$ and $A(\gamma_2) = 1$.*

Proof. First, we rewrite

$$A(\gamma) = \frac{n(\gamma - s(1+\gamma))}{(1-s)(1+\gamma)} = n \cdot \frac{\gamma(1-s)-s}{(1-s)(1+\gamma)} = n \cdot \left(1 - \frac{1}{(1-s)(1+\gamma)}\right). \tag{4.5}$$

Plugging in the definition of γ_1 and γ_2, we obtain

$$A(\gamma_1) = n \cdot \left(1 - \frac{1}{\frac{2}{n-1}}\right) = -\frac{n(n-3)}{2} \quad \text{and} \quad A(\gamma_2) = n \cdot \left(1 - \frac{1}{\frac{n}{n-1}}\right) = 1,$$

as stated. ☐

Next, for every γ, the following result identifies the unique ESS.

Lemma 4.8.

(a) For every $n \geq 2$, for every $\gamma \in [0, \gamma_1) \cup (\gamma_2, +\infty)$, there is unique ESS, termed $p_\star(\gamma)$, that satisfies $p_\star(\gamma) = 1$ on $[0, \gamma_1)$ and $p_\star(\gamma) = 0$ on $(\gamma_2, +\infty)$.

(b) for every $n \geq 3$, for every $\gamma \in [\gamma_1, \gamma_2]$, there is unique ESS, termed $p_\star(\gamma)$. Moreover, $p_\star$ is continuously differentiable on $[\gamma_1, \gamma_2]$, $p_\star(\gamma_1) = 1$ and $p_\star(\gamma_2) = 0$.

Proof. Define the following function for $0 \leq p < 1$.

$$f(p) = \frac{1}{1-p}\left(\frac{1-p^n}{1-p} - np\right).$$

We next identify $\lim_{p \to 1} f(p)$.

Observation 4.9. *Function f can be extended to a continuous function at $p = 1$ by setting $f(1) = -n(n-3)/2$.*

Proof. Let $x = 1-p$. Using Taylor expansion at $x = 0$, we have:

$$\begin{aligned} f(x) &= \frac{1}{x}\left(\frac{1-(1-x)^n}{x} - n(1-x)\right) = \frac{1}{x}\left(\frac{nx - \frac{n(n-1)}{2}x^2 + o(x^3)}{x} - n(1-x)\right) \\ &= \frac{1}{x}\left(n\left(1 - \frac{n-1}{2}x + o(x^2)\right) - n(1-x)\right) \\ &= \frac{n}{x}\left(-\frac{n-3}{2}x + o(x^2)\right) = -\frac{n(n-3)}{2} + o(x). \end{aligned}$$

Therefore, $\lim_{p \to 1} f(p) = -\frac{n(n-3)}{2}$, which concludes the proof of the observation. ☐

To compute the ESS, we need to compare $\pi_{1,p}$ and $\pi_{0,p}$.

Claim 4.10. *For every $p \in [0,1]$, $\pi_{1,p} > \pi_{0,p} \iff f(p) > A(\gamma)$ and $\pi_{1,p} < \pi_{0,p} \iff f(p) < A(\gamma)$.*

Proof. By Lemma 4.6, for every $p \in [0,1]$,

$$\pi_{1,p} > \pi_{0,p} \iff sF_{\mathcal{P}} + (1-s)F_{\mathcal{P}} \cdot \frac{1-p^n}{n(1-p)} > F_{\mathcal{S}} + (1-s)F_{\mathcal{P}} \cdot p \cdot \frac{n(1-p)+p^n-1}{n(1-p)^2}$$
$$\iff \frac{1-p^n}{1-p} - p \cdot \frac{n(1-p)+p^n-1}{(1-p)^2} > \frac{n(F_{\mathcal{S}} - sF_{\mathcal{P}})}{(1-s)F_{\mathcal{P}}}.$$

By definition, the right hand side is equal to $A(\gamma)$. Let us rewrite the left hand side:

$$\frac{1-p^n}{1-p} - p \cdot \frac{n(1-p)+p^n-1}{(1-p)^2} = \frac{1}{1-p}\left((1-p)\frac{1-p^n}{1-p} - np + p\frac{1-p^n}{1-p}\right)$$
$$= \frac{1}{1-p}\left(\frac{1-p^n}{1-p} - np\right) = f(p),$$

which concludes the proof of the first equivalence in Claim 4.10. The second equivalence is obtained similarly. □

Claim 4.11. *f is non-increasing in p. Moreover, if $n \geq 3$, then f is strictly decreasing in p.*

Proof. First, note that in the case $n = 2$, $f(p) = 1$, so f is non-increasing. Then, consider the case that $n \geq 3$. Let us write $f(p) = u(p)/v(p)$, with

$$u(p) = \frac{1-p^n}{1-p} - np, \quad v(p) = 1 - p.$$

We have

$$u'(p) = \frac{-np^{n-1}(1-p) + (1-p^n)}{(1-p)^2} - n, \quad v'(p) = -1,$$

so

$$u'(p) \cdot v(p) = -np^{n-1} + \frac{1-p^n}{1-p} - n(1-p), \quad u(p) \cdot v'(p) = -\frac{1-p^n}{1-p} + np.$$

Therefore,

$$u'(p) \cdot v(p) - u(p) \cdot v'(p) = -np^{n-1} + 2\frac{1-p^n}{1-p} - n = 2\frac{1-p^n}{1-p} - n(1+p^{n-1}).$$

Finally,

$$f'(p) = \frac{u'(p) \cdot v(p) - u(p) \cdot v'(p)}{v(p)^2} = -\frac{n(1-p)(1+p^{n-1}) - 2(1-p^n)}{(1-p)^3}. \tag{4.6}$$

Let us define the ratio:

$$g_0(p) = \frac{n(1-p)(1+p^{n-1})}{2(1-p^n)}.$$

Next, we show that g_0 is strictly greater than 1. To this aim, we study g_0 by differentiating it several times. Define:

$$\begin{cases} g_1(p) = (n-1)p^{n-2}(1-p^2) + p^{2n-2} - 1, \\ g_2(p) = 2p^n - np^2 + n - 2, \\ g_3(p) = -2np(1-p^{n-2}). \end{cases}$$

Since $n \geq 3$, $g_3(p) < 0$. We have $g_2'(p) = g_3(p) < 0$, so g_2 is strictly decreasing, and hence $g_2(p) > g_2(1) = 0$. Moreover, $g_1'(p) = (n-1)p^{n-3}g_2(p) > 0$, so g_1 is strictly increasing, and $g_1(p) < g_1(1) = 0$. Eventually, we have:

$$\begin{aligned}
g_0'(p) &= \frac{n}{2} \cdot \frac{\left(-1 - p^{n-1} + (n-1)p^{n-2}(1-p)\right) \cdot (1-p^n) + (1-p)(1+p^{n-1}) \cdot n\, p^{n-1}}{(1-p^n)^2} \\
&= \frac{n}{2(1-p^n)^2} \cdot \Big(-1 - p^{n-1} + (n-1)p^{n-2}(1-p) + p^n + p^{2n-1} \\
&\qquad - (n-1)p^{2n-2}(1-p) + n(1-p)p^{n-1} + n(1-p)p^{2n-2} \Big) \\
&= \frac{n}{2} \cdot \frac{p^{n-2}\left(-p + (n-1)(1-p) + p^2 + np(1-p)\right) + p^{2n-2} - 1}{(1-p^n)^2} \\
&= \frac{n}{2} \cdot \frac{g_1(p)}{(1-p^n)^2} < 0,
\end{aligned}$$

so g_0 is strictly decreasing, and $g_0(p) > g_0(1) = 1$. Therefore, $n(1-p)(1+p^{n-1}) > 2(1-p^n)$. By Eq. (4.6), this implies that $f'(p) < 0$, which concludes the proof of Claim 4.11. ☐

Claim 4.12. *Function A is (strictly) increasing in γ.*

Proof. Using Eq. (4.5), we obtain

$$\frac{dA(\gamma)}{d\gamma} = \frac{n}{(1-s)(1+\gamma)^2} > 0,$$

from which Claim 4.12 follows. □

See Figure 4.3 for an overview of the following arguments.

Proof of (a). Assume $n \geq 2$, and consider the case that $\gamma < \gamma_1$ (Figure 4.3a). By Claims 4.7, 4.11 and 4.12, and Observation 4.9, for every $p \in [0,1]$,

$$f(p) \geq f(1) = -\frac{n(n-3)}{2} = A(\gamma_1) > A(\gamma).$$

By Claim 4.10, this implies that $\pi_{1,p} > \pi_{0,p}$. Thus, for every $p < 1$ and every $q \in [0,1]$,

$$\pi_{p,q} = p\,\pi_{1,q} + (1-p)\,\pi_{0,q} < \pi_{1,q}. \tag{4.7}$$

On the one hand, Eq. (4.7) implies that for every $p < 1$, p cannot satisfy neither condition (i) nor (ii) in the definition of ESS. On the other hand, Eq. (4.7) implies that $p_\star = 1$ will always satisfy condition (i) in the definition of ESS. Finally, we conclude that on $[0, \gamma_1)$, $p_\star(\gamma) = 1$ is the only ESS.

Next, consider the case that $\gamma > \gamma_2$ (Figure 4.3c). By Claims 4.7, 4.11 and 4.12, for every $p \in [0,1]$,

$$f(p) \leq f(0) = 1 = A(\gamma_2) < A(\gamma).$$

By Claim 4.10, this implies that $\pi_{1,p} < \pi_{0,p}$. Similarly, we conclude that on $(\gamma_2, +\infty]$, $p_\star(\gamma) = 0$ is the only ESS.

Proof of (b). Consider the case that $n \geq 3$ and $\gamma_1 \leq \gamma \leq \gamma_2$ (Figure 4.3b). By Claim 4.11, $f : [0,1] \mapsto [f(1), f(0)]$ is a bijection, and we can consider the inverse function

$$f^{-1} : [f(1), f(0)] \mapsto [0,1];.$$

Moreover, by Claims 4.7, 4.11 and 4.12, and Observation 4.9,

$$f(1) = A(\gamma_1) \leq A(\gamma) \leq A(\gamma_2) = f(0).$$

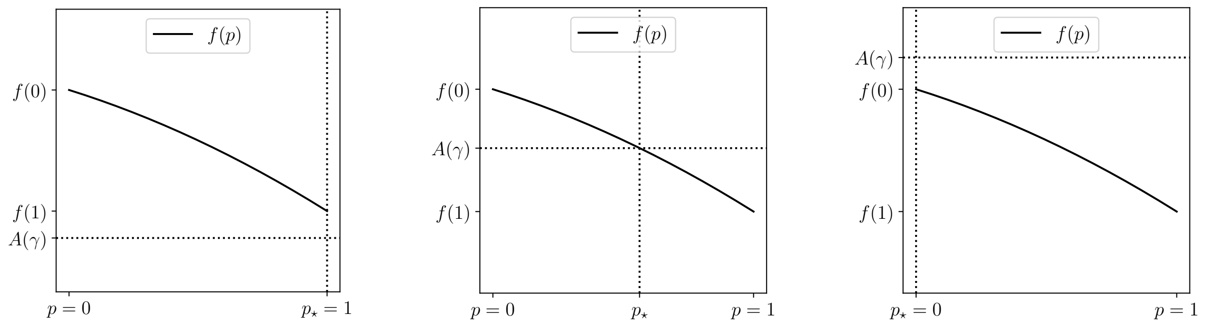
Therefore, there is a unique $p_\star \in [0,1]$ such that $f(p_\star) = A(\gamma)$. By Claims 4.10 and 4.11, we have

$$\begin{aligned}
f(p_\star) = A(\gamma) &\implies \pi_{1,p_\star} = \pi_{0,p_\star}, \\
\text{for every } q < p_\star, f(q) > f(p_\star) &\implies \pi_{1,p_\star} > \pi_{0,p_\star}, \\
\text{for every } q > p_\star, f(q) < f(p_\star) &\implies \pi_{1,p_\star} < \pi_{0,p_\star}.
\end{aligned}$$

By Lemma 4.4, this implies that $p_\star$ is the unique ESS.

As a function of γ on the interval $[\gamma_1, \gamma_2]$, $p_\star$ satisfies $p_\star(\gamma) = f^{-1}(A(\gamma))$. Function f is continuously differentiable, and the derivative is non-zero by Claim 4.11, so f^{-1} is continuously differentiable. Moreover, A is also continuously differentiable. Therefore, $p_\star$ is continuously differentiable. Finally, $p_\star$ verifies $p_\star(\gamma_1) = f^{-1}(A(\gamma_1)) = f^{-1}(f(1)) = 1$, and $p_\star(\gamma_2) = f^{-1}(A(\gamma_2)) = f^{-1}(f(0)) = 0$. □

Lemma 4.13. *If* $3 \leq n < \min\left\{1 + \frac{1}{s}, 1 + \frac{2}{1-s}\right\}$, *then there exists* $0 \leq \gamma_{\min} < \gamma_{\max}$ *such that* $\pi_\star(\gamma)$ *is decreasing on the interval* $[\gamma_{\min}, \gamma_{\max}]$.



(a) $\gamma = 0 < \gamma_1$. (b) $\gamma = 1/2 \in [\gamma_1, \gamma_2]$. (c) $\gamma = 2 > \gamma_2$.

Figure 4.3: Visualization of Functions $f(p)$, $A(\gamma)$, and $p_\star(\gamma)$ illustrating their roles in the proof. The X-axis denotes the variable p, while each subfigure corresponds to a distinct value of γ. These figures were generated for $n = 4$ and $s = 2/5$.

Proof. Since $n \geq 3$, by definition, $\gamma_1 < \gamma_2$ and so $[\gamma_1, \gamma_2]$ is a non-empty interval. We have that

$$n \leq 1 + \frac{2}{1-s} \iff \frac{2}{(n-1)(1-s)} - 1 \geq 0 \iff \gamma_1 \geq 0.$$

Moreover,

$$n < 1 + \frac{1}{s} \iff \frac{2}{(n-1)(1-s)} > \frac{n}{(n-1)(1-s)} - 1 \iff 1 + \gamma_1 > \gamma_2.$$

Next, note that by definition:

$$\pi_\star(\gamma) = p_\star(\gamma) \cdot \pi_{1,p_\star(\gamma)}(\gamma) + (1 - p_\star(\gamma)) \cdot \pi_{0,p_\star(\gamma)}(\gamma),$$

By assumption on n, we know that $\gamma_1 \geq 0$ and $1 + \gamma_1 > \gamma_2$. Since both $\pi_{1,p}(\gamma)$ and $\pi_{0,p}(\gamma)$ are continuously differentiable in p and in γ (from their expression in Lemma 4.6), and since $p_\star(\gamma)$ is continuously differentiable in γ on $[\gamma_1, \gamma_2]$ (by statement (b) in Lemma 4.8), then $\pi_\star(\gamma)$ is continuously differentiable in γ on $[\gamma_1, \gamma_2]$. Moreover, it satisfies $\pi_\star(\gamma_1) = F_{\mathcal{P}} = 1 + \gamma_1$ (since $p_\star(\gamma_1) = 1$), and $\pi_\star(\gamma_2) = F_{\mathcal{S}} = \gamma_2 < \pi_\star(\gamma_1)$ (since $p_\star(\gamma_2) = 0$). Therefore, we can find an interval $[\gamma_{\min}, \gamma_{\max}] \subseteq [\gamma_1, \gamma_2]$ on which $\pi_\star(\gamma)$ is decreasing, which concludes the proof of Lemma 4.13. □

Proof of Theorem 4.5. When $n = 3$,

$$n < \min\left\{1 + \frac{1}{s}, 1 + \frac{2}{1-s}\right\} \iff 0 < s < \frac{1}{2},$$

and the first item in Theorem 4.5 follows as a special case of Lemma 4.13.

When $n = 2$, $\gamma_1 = \gamma_2 = \gamma_s = \frac{1+s}{1-s}$. By statement (a) in Lemma 4.8, for every $\gamma < \gamma_s$, there is a unique ESS satisfying $p_\star(\gamma) = 1$ and so $\pi_\star(\gamma) = 1 + \gamma$. Similarly, for every $\gamma > \gamma_s$, there is a unique ESS satisfying $p_\star(\gamma) = 0$ and so $\pi_\star(\gamma) = \gamma$. Therefore, $\pi_\star$ is increasing on $[0, \gamma_s)$ and on $(\gamma_s, +\infty)$. Moreover, let $\epsilon \in (0, 1/2)$. Since $s \geq 0$, we have $\gamma_s \geq 1$, and

$$\pi_\star(\gamma_s - \epsilon) = 1 + \gamma_s - \epsilon > \gamma_s + \frac{1}{2} > \gamma_s + \epsilon = \pi_\star(\gamma_s + \epsilon),$$

which establishes the second item in Theorem 4.5, and thus concludes the proof of theorem. □

4.4 Analysis of the Company Game

4.4.1 Preliminaries

Following classical notations from game theory, we define, for $n = 2$ players:

$$
\begin{array}{crcl}
\textbf{(Reward)} & R(\gamma, s, c, p, a) & = & \pi_{1,1} \\
\textbf{(Sucker)} & S(\gamma, s, c, p, a) & = & \pi_{1,0} \\
\textbf{(Temptation)} & T(\gamma, s, c, p, a) & = & \pi_{0,1} \\
\textbf{(Punishment)} & P(\gamma, s, c, p, a) & = & \pi_{0,0}
\end{array}
$$

For simplicity, we do not mention (γ, s, c, p, a) when there is no risk of confusion.

The following result is well-known in game theory folklore. However, we provide a proof here for the sake of completeness.

Theorem 4.14 (Game of Chicken). *If $n = 2$ and $T > R > S > P$, then there is a unique ESS, that satisfies*

$$\pi_\star = \frac{ST - RP}{S + T - R - P}. \tag{4.8}$$

Proof. We have, by definition $\pi_{1,p} = p\,R + (1-p)\,S$ and $\pi_{0,p} = p\,T + (1-p)\,P$. Note that

$$\pi_{1,p} = \pi_{0,p} \iff p = \frac{S - P}{S + T - R - P}. \tag{4.9}$$

Define

$$p_\star = \frac{S - P}{S + T - R - P} = \frac{1}{1 + \frac{T-R}{S-P}}$$

with $T - R > 0$, $S - P > 0$, so $p_\star \in [0, 1]$. Let

$$\pi_\star = \pi_{1,p_\star} = \pi_{0,p_\star}. \tag{4.10}$$

By assumption in the theorem,

$$\frac{d}{dp}\pi_{1,p} = R - S < T - P = \frac{d}{dp}\pi_{0,p},$$

and hence, by Eq. (4.10), for every $q < p_\star$, $\pi_{1,q} > \pi_{0,q}$, and for every $q > p_\star$, $\pi_{1,q} < \pi_{0,q}$. By Lemma 4.4, this, together with Eq. (4.10), implies that $p_\star$ is a unique ESS. To conclude the proof of Theorem 4.14, we just check that $\pi_\star$ satisfies Eq. (4.8). □

4.4.2 A Case with No Reverse-Correlation Phenomenon

Observation 4.15. *Consider the case that there are $n = 2$ players and that $\phi : x \mapsto x$. Let $c \geq 0$, $s \in [1/2, 1]$, $p, a \in [0, 1]$, and set $\gamma_0 = c/(ps(1-a))$. Then for every $\gamma \neq \gamma_0$, there is a unique ESS. Moreover, the payoff $\pi_\star$ and the total production $\Gamma_\star$ corresponding to the ESS are both strictly increasing functions of γ.*

Proof. Recall that in the Company game,

$$\pi_1 = \phi(sq_1 + (1-s)q_2) - c_1. \tag{4.11}$$

Consider the case that $\phi : x \mapsto x$. For every $s \in [1/2, 1]$,

$$\mathbf{E}(\pi_1) = s\mathbf{E}(q_1) + (1-s)\mathbf{E}(q_2) - c_1.$$

Therefore,

$$\begin{aligned}
R(\gamma, s, c, p, a) &= s \cdot (p\gamma) + (1-s) \cdot (p\gamma) - c = p\gamma - c, &(4.12)\\
S(\gamma, s, c, p, a) &= s \cdot (p\gamma) + (1-s) \cdot (pa\gamma) - c = p\gamma(s + a - sa) - c, &(4.13)\\
T(\gamma, s, c, p, a) &= s \cdot (pa\gamma) + (1-s) \cdot (p\gamma) = p\gamma(1 - s + sa), &(4.14)\\
P(\gamma, s, c, p, a) &= s \cdot (pa\gamma) + (1-s) \cdot (pa\gamma) = pa\gamma. &(4.15)
\end{aligned}$$

Recall that $\gamma_0 = c/(ps(1-a))$.

- If $\gamma < \gamma_0$, then $T > R$ and $P > S$, in which scrounger is a dominant strategy. Therefore, there is a unique ESS, and we have:

  $$\pi_\star = \Gamma_\star = P = pa\gamma.$$

  In particular, these values are increasing in γ.

- If $\gamma > \gamma_0$, then $R > T$ and $S > P$, and hence producer is a dominant strategy. Therefore, there is a unique ESS, and

  $$\pi_\star = R = p\gamma - c, \quad \text{and} \quad \Gamma_\star = p\gamma.$$

  In particular, both these values are increasing in γ.

- If $\gamma = \gamma_0$, then $R = T$ and $P = S$, which implies that no player can unilaterally change its payoff. Indeed, for every $p, q \in [0, 1]$,

  $$\pi_{p,q} = pqR + (1-p)qT + p(1-q)S + (1-p)(1-q)P = qR + (1-q)S,$$

  so for every $p, p', q \in [0, 1]$, $\pi_{p,q} = \pi_{p',q}$. In this degenerate case, neither condition (i) nor (ii) in the definition of ESS can be satisfied, so there is no ESS.

To conclude the proof of Observation 4.15, we only need to show that $\pi_\star$ and $\Gamma_\star$ do not decrease at the discontinuity point $\gamma = \gamma_0$. Since $\gamma_0 \geq c/(p(1-a))$, we have

$$\lim_{\epsilon \to 0^+} \Gamma_\star(\gamma_0 - \epsilon) = \lim_{\epsilon \to 0^+} \pi_\star(\gamma_0 - \epsilon) = ap\gamma_0 \leq p\gamma_0 - c = \lim_{\epsilon \to 0^+} \pi_\star(\gamma_0 + \epsilon) \leq \lim_{\epsilon \to 0+} \Gamma_\star(\gamma_0 + \epsilon).$$

Thus, overall, the payoffs of players at equilibrium, $\pi_\star$, and the total production, $\Gamma_\star$, are both increasing in γ. □

4.4.3 Proof of Theorem 4.3

In this section, we demonstrate the Reverse-Correlation phenomenon in the Company game for two utility functions. Specifically, we first prove that the Reverse-Correlation phenomenon can occur when assuming the utility function $\phi : x \mapsto 1 - \exp(-2\,x)$. Then, in Figure 4.4 we provide simulations that demonstrate the Reverse-Correlation phenomenon assuming the utility function $\phi : x \mapsto \min(1, x)$.

Let

$$\gamma_i = \begin{cases} \gamma & \text{if Player } i \text{ is a producer,} \\ \gamma/2 & \text{otherwise.} \end{cases}$$

By definition, if player i succeeds in producing a product (which happens with probability $p = 1/2$) then the quality of its product is γ_i. Hence, Eq. (4.11) gives

$$\mathbf{E}(\pi_1) = \frac{1}{4}\left(\phi(s\gamma_1 + (1-s)\gamma_2) + \phi(s\gamma_1) + \phi((1-s)\gamma_2)\right) - c_1.$$

Plugging in $\phi(x) = 1 - \exp(-2x)$, we obtain

$$R(\gamma, s, c) = \frac{1}{4}\left(3 - e^{-2\gamma} - e^{-2s\gamma} - e^{-2(1-s)\gamma}\right) - c, \tag{4.16}$$

$$S(\gamma, s, c) = \frac{1}{4}\left(3 - e^{-(1+s)\gamma} - e^{-2s\gamma} - e^{-(1-s)\gamma}\right) - c, \tag{4.17}$$

$$T(\gamma, s, c) = \frac{1}{4}\left(3 - e^{-(2-s)\gamma} - e^{-s\gamma} - e^{-2(1-s)\gamma}\right), \tag{4.18}$$

$$P(\gamma, s, c) = \frac{1}{4}\left(3 - e^{-\gamma} - e^{-s\gamma} - e^{-(1-s)\gamma}\right). \tag{4.19}$$

Note that R, S, T and P are all increasing functions of γ. Consequently, if the strategies of Players 1 and 2 remain unchanged, then $\mathbf{E}(\pi_1)$ is also increasing in γ. However, we will show that at equilibrium, the tendency of the player to be a scrounger increases in γ to such an extent that ultimately reduces $\mathbf{E}(\pi_1)$.

The next step towards proving Theorem 4.3 is to show that for some specific values of s and c, the Company game is in fact a game of chicken.

Lemma 4.16. *For every $s < 1$, there exists a value $c_0 = c_0(s)$ and an interval $[\gamma_{\min}, \gamma_{\max}]$ such that for every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$,*

$$T(\gamma, s, c_0) > R(\gamma, s, c_0) > S(\gamma, s, c_0) > P(\gamma, s, c_0). \tag{4.20}$$

In particular, by Theorem 4.14, this implies that for every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$, there is a unique ESS satisfying

$$\pi_\star(\gamma, s, c_0) = \frac{ST - RP}{S + T - R - P}.$$

Before proving Lemma 4.16, we need two preliminary technical results. The next claim implies, in particular, that if $c = 0$ then a producer is a dominant strategy.

Claim 4.17. *For all γ, s such that $s < 1$,*

$$R(\gamma, s, 0) > S(\gamma, s, 0), T(\gamma, s, 0) \quad \text{and} \quad S(\gamma, s, 0), T(\gamma, s, 0) > P(\gamma, s, 0).$$

Proof. By pairwise comparison of the terms in Eqs. (4.16) to (4.19). □

The next claim implies that $T - P > R - S$, or in other words, that scroungers lose more than producers when the other player switches from producer to scrounger.

Claim 4.18. *For all γ, s, c such that $s < 1$,*

$$S(\gamma, s, c) - P(\gamma, s, c) > R(\gamma, s, c) - T(\gamma, s, c).$$

Proof. We have

$$4(R - S) = \left(e^{-(1+s)\gamma} - e^{-2\gamma}\right) + \left(e^{-(1-s)\gamma} - e^{-2(1-s)\gamma}\right),$$

and

$$4(T - P) = \left(e^{-\gamma} - e^{-(2-s)\gamma}\right) + \left(e^{-(1-s)\gamma} - e^{-2(1-s)\gamma}\right).$$

Factoring by $e^{-\gamma}$, this gives

$$(T - P) - (R - S) = \frac{e^{-\gamma}}{4}\left(1 + e^{-\gamma} - e^{-s\gamma} - e^{-(1-s)\gamma}\right). \tag{4.21}$$

Factoring again by $e^{-\gamma}$, and using the convexity of the function $e^x + e^{\gamma - x}$, we obtain

$$(T-P)-(R-S) = \frac{e^{-2\gamma}}{4}\left(e^{\gamma}+1-\left(e^{s\gamma}+e^{(1-s)\gamma}\right)\right) > 0,$$

which concludes the proof of Claim 4.18. □

Proof of Lemma 4.16. Let us fix $\gamma_0 > \max(1, \ln(1+\sqrt{2})/s)$. By Claims 4.17 and 4.18, we can take $c_0 = c_0(s)$ such that

$$0 < R(\gamma_0, s, 0) - T(\gamma_0, s, 0) < c_0 < S(\gamma_0, s, 0) - P(\gamma_0, s, 0).$$

As a consequence,

$$R(\gamma_0, s, c_0) = R(\gamma_0, s, 0) - c_0 < T(\gamma_0, s, 0) = T(\gamma_0, s, c_0),$$

and

$$S(\gamma_0, s, c_0) = S(\gamma_0, s, 0) - c_0 > P(\gamma_0, s, 0) = P(\gamma_0, s, c_0).$$

Finally, by Claim 4.17, we have that

$$T(\gamma_0, s, c_0) > R(\gamma_0, s, c_0) > S(\gamma_0, s, c_0) > P(\gamma_0, s, c_0).$$

By continuity, there exist $\gamma_{\min}, \gamma_{\max}$ such that $\max(1, \ln(1+\sqrt{2})/s) < \gamma_{\min} < \gamma_0 < \gamma_{\max}$ and for every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$, Eq. (4.20) holds, which concludes the proof of Lemma 4.16. □

Claim 4.19. *For every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$,*

$$\pi_\star(\gamma, s, c_0) = 1 - c_0\, e^{s\gamma} \cdot \left(\frac{e^{s\gamma}+1}{e^{s\gamma}-1}\right) = 1 - c_0\, e^{s\gamma} \coth\left(\frac{s\gamma}{2}\right).$$

Proof. We start from the expression of $\pi_\star(\gamma, s, c_0)$ given by Lemma 4.16. First, we compute $ST - RP$ using Eqs. (4.16) to (4.19). For that purpose, we expand ST and RP separately, and then simplify. We have (each line corresponds to one term of S multiplied by all the terms of T):

$$\begin{aligned}
16 \cdot ST &= \left(3 - e^{-(1+s)\gamma} - e^{-2s\gamma} - e^{-(1-s)\gamma} - 4c_0\right) \cdot \left(3 - e^{-(2-s)\gamma} - e^{-s\gamma} - e^{-2(1-s)\gamma}\right) \\
&= 9 - 3e^{-(2-s)\gamma} - 3e^{-s\gamma} - 3e^{-2(1-s)\gamma} \\
&- 3e^{-(1+s)\gamma} + e^{-3\gamma} + e^{-(1+2s)\gamma} + e^{-(3-s)\gamma} \\
&- 3e^{-2s\gamma} + e^{-(2+s)\gamma} + e^{-3s\gamma} + e^{-2\gamma} \\
&- 3e^{-(1-s)\gamma} + e^{-(3-2s)\gamma} + e^{-\gamma} + e^{-3(1-s)\gamma} \\
&- 12c_0 + 4c_0 e^{-(2-s)\gamma} + 4c_0 e^{-s\gamma} + 4c_0 e^{-2(1-s)\gamma}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
16 \cdot RP &= \left(3 - e^{-2\gamma} - e^{-2s\gamma} - e^{-2(1-s)\gamma} - 4c_0\right) \cdot \left(3 - e^{-\gamma} - e^{-s\gamma} - e^{-(1-s)\gamma}\right) \\
&= 9 - 3e^{-\gamma} - 3e^{-s\gamma} - 3e^{-(1-s)\gamma} - 3e^{-2\gamma} + e^{-3\gamma} + e^{-(2+s)\gamma} + e^{-(3-s)\gamma} \\
&- 3e^{-2s\gamma} + e^{-(1+2s)\gamma} + e^{-3s\gamma} + e^{-(1+s)\gamma} - 3e^{-2(1-s)\gamma} + e^{-(3-2s)\gamma} + e^{-(2-s)\gamma} \\
&+ e^{-3(1-s)\gamma} - 12c_0 + 4c_0 e^{-\gamma} + 4c_0 e^{-s\gamma} + 4c_0 e^{-(1-s)\gamma}.
\end{aligned}$$

When computing the difference, many terms disappear, leaving us with:

$$16 \cdot (ST - RP) =$$
$$4\left(e^{-\gamma} + e^{-2\gamma} - e^{-(1+s)\gamma} - e^{-(2-s)\gamma} - c_0\left(e^{-\gamma} + e^{-(1-s)\gamma} - e^{-(2-s)\gamma} - e^{-2(1-s)\gamma}\right)\right).$$

Factoring the right hand side by $e^{-\gamma}$, we obtain

$$ST - RP = \frac{e^{-\gamma}}{4}\left(1 + e^{-\gamma} - e^{-s\gamma} - e^{-(1-s)\gamma} - c_0\left(1 + e^{s\gamma} - e^{-(1-s)\gamma} - e^{-(1-2s)\gamma}\right)\right).$$

Using Eq. (4.21), we obtain

$$\frac{ST - RP}{S + T - R - P} = 1 - c_0 \cdot \frac{1 + e^{s\gamma} - e^{-(1-s)\gamma} - e^{-(1-2s)\gamma}}{1 + e^{-\gamma} - e^{-s\gamma} - e^{-(1-s)\gamma}}. \tag{4.22}$$

Factoring the numerator of the fraction by $e^{s\gamma}$ and rearranging, we get

$$1 - c_0 e^{s\gamma} \cdot \frac{1 - e^{-(1-s)\gamma} + (e^{-s\gamma} - e^{-\gamma})}{1 - e^{-(1-s)\gamma} - (e^{-s\gamma} - e^{-\gamma})}.$$

Dividing both the numerator and denominator by $(e^{-s\gamma} - e^{-\gamma})$, and using the fact that

$$\frac{1 - e^{-(1-s)\gamma}}{e^{-s\gamma} - e^{-\gamma}} = e^{s\gamma} \cdot \left(\frac{e^{-s\gamma} - e^{-\gamma}}{e^{-s\gamma} - e^{-\gamma}}\right) = e^{s\gamma},$$

we finally get

$$\frac{ST - RP}{S + T - R - P} = 1 - c_0 e^{s\gamma} \cdot \frac{e^{s\gamma} + 1}{e^{s\gamma} - 1},$$

which concludes the proof of Claim 4.19. □

Claim 4.20. *For every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$,*

$$\frac{\partial}{\partial \gamma} \pi_\star(\gamma, s, c_0) = \frac{c_0 s\, e^{s\gamma}}{2} \cdot \frac{1 - \sinh(s\gamma)}{\sinh\left(\frac{s\gamma}{2}\right)^2}.$$

Proof. We start from the expression of Claim 4.19, and derive using the fact that

$$\frac{d}{dx} \coth(x) = \frac{-1}{\sinh(x)^2}.$$

More precisely, by Claim 4.19,

$$\begin{aligned}
\frac{\partial}{\partial \gamma} \pi_\star(\gamma, s, c_0) &= \frac{\partial}{\partial \gamma}\left(1 - c_0\, e^{s\gamma} \coth\left(\frac{s\gamma}{2}\right)\right) \\
&= -c_0 s e^{s\gamma} \coth\left(\frac{s\gamma}{2}\right) + \frac{c_0 s e^{s\gamma}}{2\sinh\left(\frac{s\gamma}{2}\right)^2} \\
&= \frac{c_0 s\, e^{s\gamma}}{2\sinh\left(\frac{s\gamma}{2}\right)^2} \cdot \left(1 - 2\coth\left(\frac{s\gamma}{2}\right)\sinh\left(\frac{s\gamma}{2}\right)^2\right).
\end{aligned}$$

Then, we observe that for every $x \in \mathbb{R}$,

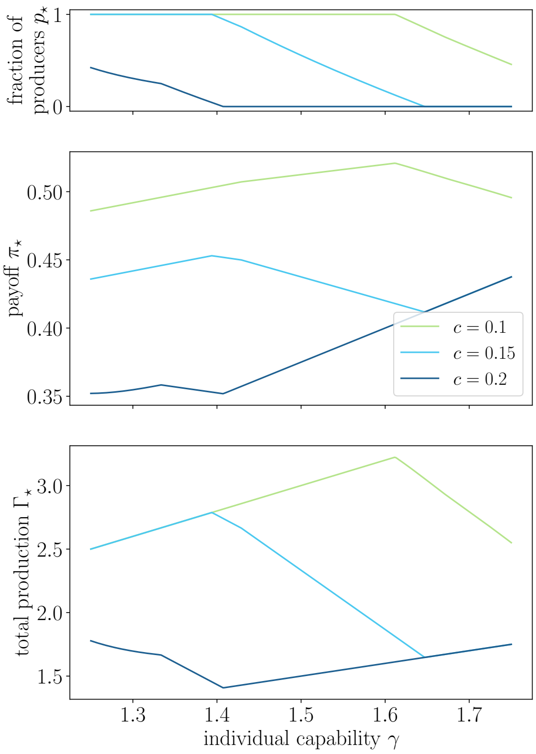
$$\begin{aligned}
2\coth(x)\sinh(x)^2 &= 2\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)\left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} = \frac{e^{2x} - e^{-2x}}{2} = \sinh(2x).
\end{aligned}$$

Plugging this in the last equation concludes the proof of Claim 4.20. ☐

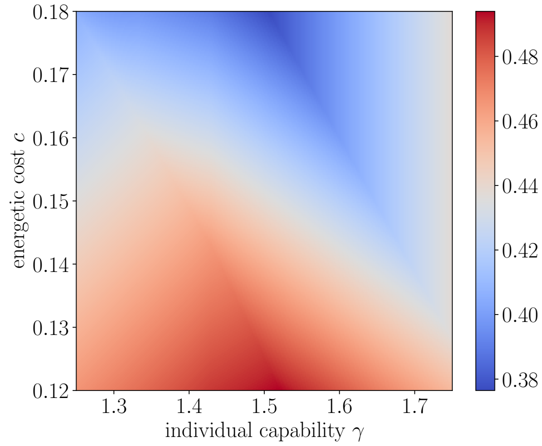
Proof of Theorem 4.3. By definition,

$$\gamma > \gamma_{\min} > \frac{\ln(1+\sqrt{2})}{s} = \frac{\sinh^{-1}(1)}{s},$$

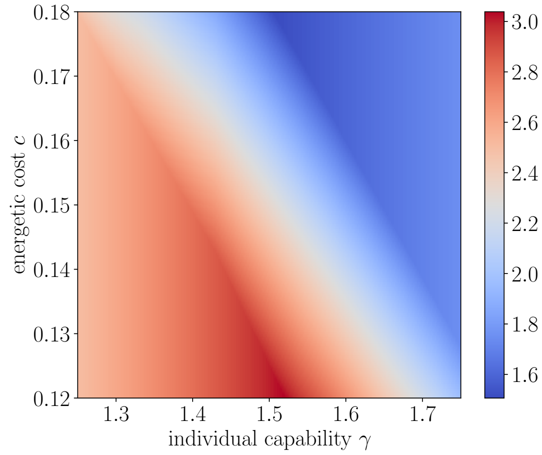
so $\sinh(s\gamma) > 1$. By Claim 4.20, this implies that $\frac{\partial}{\partial\gamma}\pi_\star(\gamma, s, c_0) < 0$ on the interval $[\gamma_{\min}, \gamma_{\max}]$. ☐



(a) Payoffs and Total Production. $c = 0.15$



(b) Payoffs.



(c) Total Production.

Figure 4.4: **The Company game with the utility function** $\phi : x \mapsto \min(1, x)$**.** Considering a scenario with $n = 4$ players, assuming $a = p = \frac{1}{2}$, and $s = 0.7$. **(a)** The graph presents the payoff $\pi_\star$ and the total production $\Gamma_\star(\gamma)$ as well as the probability of being a producer $p_\star(\gamma)$ at equilibria, as a function of individual capabilities γ and for several values of c. The declines in the corresponding plots (of the payoff and the total production) display the Reverse-Correlation phenomenon. **(b)** and **(c)** The relationship between **(b)** the payoff $\pi_\star$, and **(c)** the total production $\Gamma_\star$, as a function (color scale) of individual capabilities γ and the cost c for production.

4.5 A Necessary Condition for the RC Phenomenon

We assume that the payoffs are positively correlated with the number of producers in the group, that is,

$$\text{for every } q \in [0,1] \text{ and every } \gamma \geq 0,\ p \mapsto \pi_{q,p}(\gamma) \text{ is non-decreasing in } p. \tag{4.23}$$

In addition, we assume that the payoffs are positively correlated with the parameter γ, that is,

$$\text{for every } q \in [0,1] \text{ and every } p \in [0,1],\ \gamma \mapsto \pi_{q,p}(\gamma) \text{ is non-decreasing in } \gamma. \tag{4.24}$$

Under these assumptions, we identify the following necessary condition for the emergence of a Reverse-Correlation phenomenon.

Theorem 4.21. *For any PS model in which the payoff of producers does not depend on the strategies of other players, there is no Reverse-Correlation phenomenon. More precisely, if there are two values γ_1, γ_2 such that $\gamma_1 < \gamma_2$ and two ESS denoted $p_\star(\gamma_1)$ and $p_\star(\gamma_2)$, then the corresponding payoffs satisfy $\pi_\star(\gamma_1) \leq \pi_\star(\gamma_2)$.*

Proof. Fix a PS model. By assumption in Theorem 4.21,

$$\text{For every } \gamma \geq 0,\ p \mapsto \pi_{1,p}(\gamma) \text{ does not depend on } p. \tag{4.25}$$

In what follows, we will simply write $\pi_{1,p}(\gamma) = \pi_{\mathcal{P}}(\gamma)$. By definition of ESS, and by Eq. (4.25), we have for every $i \in \{1,2\}$:

$$p_\star(\gamma_i) = 0 \implies \pi_\star(\gamma_i) = \pi_{0,0}(\gamma_i) \geq \pi_{\mathcal{P}}(\gamma_i), \tag{4.26.a}$$

$$p_\star(\gamma_i) = 1 \implies \pi_\star(\gamma_i) = \pi_{\mathcal{P}}(\gamma_i) \geq \pi_{0,1}(\gamma_i), \tag{4.26.b}$$

$$p_\star(\gamma_i) \notin \{0,1\} \implies \pi_\star(\gamma_i) = \pi_{\mathcal{P}}(\gamma_i) = \pi_{0,p_\star(\gamma_i)}(\gamma_i), \tag{4.26.c}$$

where Eq. (4.26.a) holds because $\pi_{0,0}(\gamma_i) \geq \pi_{1,0}(\gamma_i) = \pi_{\mathcal{P}}(\gamma_i)$.

As a consequence of Eqs. (4.26.b) and (4.26.c), we have

$$p_\star(\gamma_i) \neq 0 \implies \pi_\star(\gamma_i) = \pi_{\mathcal{P}}(\gamma_i) \geq \pi_{0,p_\star(\gamma_i)}(\gamma_i). \tag{4.27}$$

Now, let us show that $\pi_\star(\gamma_1) \leq \pi_\star(\gamma_2)$.

- If $p_\star(\gamma_1) = p_\star(\gamma_2) = 0$, then

$$\pi_\star(\gamma_1) \underset{(4.26.a)}{=} \pi_{0,0}(\gamma_1) \underset{(4.24)}{\leq} \pi_{0,0}(\gamma_2) \underset{(4.26.a)}{=} \pi_\star(\gamma_2).$$

- If $p_\star(\gamma_1) \neq 0$ and $p_\star(\gamma_2) \neq 0$, then

$$\pi_\star(\gamma_1) \underset{(4.27)}{=} \pi_{\mathcal{P}}(\gamma_1) \underset{(4.24)}{\leq} \pi_{\mathcal{P}}(\gamma_2) \underset{(4.27)}{=} \pi_\star(\gamma_2).$$

- If $p_\star(\gamma_1) \neq 0$ and $p_\star(\gamma_2) = 0$, then

$$\pi_\star(\gamma_1) \underset{(4.27)}{=} \pi_{\mathcal{P}}(\gamma_1) \underset{(4.24)}{\leq} \pi_{\mathcal{P}}(\gamma_2) \underset{(4.26.a)}{\leq} \pi_\star(\gamma_2).$$

- If $p_\star(\gamma_1) = 0$ and $p_\star(\gamma_2) \neq 0$, then

$$\pi_\star(\gamma_1) \underset{(4.26.a)}{=} \pi_{0,0}(\gamma_1) \underset{(4.23)}{\leq} \pi_{0,p_\star(\gamma_2)}(\gamma_1) \underset{(4.24)}{\leq} \pi_{0,p_\star(\gamma_2)}(\gamma_2) \underset{(4.27)}{\leq} \pi_\star(\gamma_2).$$

This concludes the proof of Theorem 4.21. □

4.6 Discussion

In foraging contexts, it is commonly anticipated that an increase in food abundance would result in higher consumption, which, in turn, would lead to population growth over time. In contrast, this paper introduces the intriguing possibility of a reversed scenario: that under certain producer-scrounger conditions, if animals have sufficient time to update their strategy and reach a stable configuration before reproducing [12, 102, 120], then an increase in food abundance can paradoxically result in reduced consumption, which, in turn, can lead to a decline in population size! Note that this idea can also be viewed from the opposite perspective, namely, that by reducing food abundance, the inclination to scrounge can decrease, resulting in improved food consumption, ultimately leading to an increase in population size.

The Reverse-Correlation phenomenon corresponds to a decrease in payoffs as underlying conditions improve. The counter-intuitive aspect of it stems from the fact that players aim to maximize their payoffs, yet when conditions improve, they are driven to perform worse. Another measure of interest is the total production, defined as the sum of production over all players (Eq. (4.2)). Observe that in the Foraging game, since the animals eventually consume all food found by the group, the total production (i.e., the total food found) at equilibrium is proportional to the payoff $\pi_\star$, and hence their dynamics are similar. This implies that whenever an increase in γ results in a decrease in payoff at equilibrium (indicating a Reverse-Correlation phenomenon), the same increase in γ also leads to a decrease in total production at equilibrium. In contrast, in the Company game, production is not fully represented in the payoffs, since some of it is "lost" when translating salaries into utilities. Additionally, the distinction between payoffs and production is further emphasized due to the energetic cost incurred by producers, which is reflected in their payoffs. Despite this distinction, as observed in Figures 4.2b and 4.2d, the measure of total production also exhibits a decrease across a range of γ values. This phenomenon may carry particular importance for system designers, such as the company's principal, as it challenges a fundamental assumption underlying bottom-up approaches, namely, that as long as the system naturally progresses without external disruptions, improving individual performances should lead to enhanced group performances.

As evident by these games, the occurrence of this counter-intuitive phenomenon is highly contingent on the specific details of the game. For example, the Foraging game considers two types of food: low-hanging and high-hanging fruit (instead of just one type as consider in the classical game in [153, 81]). Only producers have access to high-hanging fruit, while both producers and scroungers can access low-hanging fruit. Similarly to the classical model, when an animal finds food, it consumes a portion s of it and the remaining $1-s$ portion is equally shared between this animal and all scroungers. The Reverse-Correlation phenomenon emerges as the abundance of low-hanging fruit increases. However, as we showed, if one modifies the model so that the remaining $1-s$ portion is shared only between the scroungers, then the system no longer exhibits a Reverse-Correlation phenomenon. Hence, while at first glance this change may appear minor, it has a profound impact on the dynamics. In the Company game, a key aspect of the model concerns the choice of the utility function, which captures the relationship between salary and payoff. Inspired by the work of Kahneman and Deaton [99], we focused on non-decreasing, concave, and bounded utility functions. Within this family of functions, we identified two that exhibit a Reverse-Correlation phenomenon. However, we note that not all utility functions in this family enable this phenomenon.

Chapter 5

On the Role of Hypocrisy

Chapter Contents

This chapter is based on

[104] Amos Korman and R.V. On the role of hypocrisy in escaping the tragedy of the commons. *Scientific Reports*, 11(1):17585, September 2021.

5.1 Introduction

As discussed in the Background and Motivation section, it is well-known that global cooperation can emerge when players severely punish their neighbouring defectors – or, alternatively, significantly reward their cooperating neighbours [141, 123, 133, 14, 65, 41]. However, inducing severe punishments on others may be costly, and hence reaching high levels of social-pressure is by itself a non-trivial problem, often referred to in the literature as the *second-order free riders* problem [27, 26, 62, 92, 93, 128]. A crucial parameter in the second-order problem is the cost of punishing, which may be correlated to the extent of punishment [136]. Clearly, when the cost exceeds a certain threshold, people would avoid punishing non-cooperators. However, when the cost is low, other factors, such as reputation considerations, can subsume the cost, ultimately making punishing beneficial [98, 158, 162]. It is therefore of interest to study the emergence of cooperation in the presence of moderate punishments or mild social-pressure.

In this chapter, we are interested in a regime of social-pressure that is high enough to maintain an already cooperative system, but is insufficient to transform a system that initially includes a large number of defectors into a cooperative one. To illustrate this, let us consider once again an imaginary person named Joe, this time in the context of recycling. When almost all of Joe's neighbours are recycling (i.e., cooperating), the social-pressure cost they induce on him can accumulate to overshadow the burden cost of recycling and incentivize him to also recycle. Conversely, when almost all of Joe's neighbours are not recycling (i.e., defecting), the burden of recycling may exceed the overall social-pressure, effectively driving Joe to defect. This raises a natural question:

How can a system that utilizes mild social-pressure
escape the tragedy-of-the-commons when it is
initially composed mostly of defectors?

The aforementioned recycling abstraction includes two extreme behaviours: defection and cooperation. Another type of generic behaviour is *hypocrisy* [150, 37, 90, 91, 92, 62], which was also experimentally studied in [64, 140]. In our interpretation, a hypocritical player pretends to be cooperative in order to reduce the social-pressure that it might experience as a defector, and, at the same time, avoids the high energetic cost incurred by a cooperator. To pretend to be a cooperator, a hypocritical player must invest a small amount of energy in contributing to the social welfare, as well as mimic the behaviour of cooperators towards their peers. This means that such players, similarly to cooperators, also induce mild social-pressure. In other words, and in contrast to *disguising* players as in [157], hypocritical players actively "demand" cooperation from others, as part of their strategy to hide their low investment in the social-welfare.

It was previously suggested that hypocritical behaviour can incentivize global cooperation [92, 93]. However, in these works, similarly to many other papers on the emergence or evolution of cooperation based on reciprocity [124, 123, 149, 6, 5], the dynamics heavily relies on the assumption that players gain substantially from the presence of nearby cooperators. Yet, this assumption is hardly justifiable in large-scale public goods scenarios such as the ones we consider.

5.1.1 Preliminaries

We consider public goods games played iteratively over a fixed connected network. The vertices of the network represent the players and the edges represent neighbouring connections [130, 124, 7, 55]. The dynamics evolve in discrete rounds. The cost of a player depends on its own behaviour and on the behaviour of its neighbours. All costs are evaluated at the beginning of each round, and then, before the next round starts, each player chooses a behaviour that minimizes its cost (breaking ties randomly), given the current behaviour of its neighbours. In other words, we assume that players greedily choose their best behaviour, given the current configuration. In our simulations, we also consider a relaxed version, where players choose the best behaviour with high probability, and with small probability choose an arbitrary behaviour.

Let us recall few standard definitions in graph-theory [58]. Let G be a connected, undirected network. In the context of our models, we often refer to the vertices of G as players. Given a player u, we write $N(u)$ the set of *neighbours* of u. Similarly, given a subset A of players, we write $N(A)$ the set of neighbours of A, that is

$$N(A) = \bigcup_{u \in A} N(u).$$

The *degree* of a player u is the number of its neighbours $|N(u)|$. We say that G has minimal degree Δ if every player has degree at least Δ. A network is called Δ-*regular* if all the vertices have degree precisely Δ.

For two players u and v in G, let $d_G(u, v)$ denote the *distance* between u and v, that is, the number edges on the shortest path linking u to v in G. Similarly, given a subset A of players, we write $d_G(u, A)$ the distance between u and A, i.e.,

$$d_G(u, A) = \min_{v \in A} d_G(u, v).$$

The maximal distance between any pair of players, i.e., the *diameter*, is denoted by

$$\mathsf{diam}(G) = \max_{u,v \in G} d_G(u, v).$$

A *bipartite network* is a network G whose set of vertices can be divided into two disjoint sets U and V, such that every edge connects a player in U to a player in V. It is a well-known fact that a network is bipartite if and only if it does not contain any odd-length cycles [58, Proposition 1.6.1, page 17].

5.1.2 Our Results

Our main model includes three behaviour types, in which those who hardly contribute to the social welfare face the risk of being caught and punished by their neighbours. The level of risk together with the extent of punishment is captured by the notion of *social-pressure*. In addition, we consider a generalized model, called the *two-order model*, which, consistent with previous work on the second-order problem [27, 62, 71, 91, 92, 93], distinguishes between first-order cooperation, that corresponds to actions that directly contribute to the social welfare, and second-order cooperation, that corresponds to applying (costly) social-pressure, or punishments, on others.

5.1.2.1 The Main Model.

The model considers two extreme behaviours, namely, *cooperation* (c) and *defection* (d), and an additional intermediate behaviour, called *hypocrisy* (h). The system starts in a configuration in which almost all players, e.g., 99%, are defectors. As mentioned, in contrast to many previous works on cooperation in networks [124, 123, 149, 6, 5], we assume that benefits from altruistic acts are negligible, so that a player does not gain anything when others cooperate. We say that the *marginal per-capita return gain* (MPCR) is zero.

The cost of a player u with a behaviour type $i \in \{d, h, c\}$ is composed of two components: the *energetic cost* E_i associated with the contribution to the social welfare, and the *social-pressure cost* $S_i(u)$ it faces, that is:

$$\mathcal{C}_i(u) = E_i + S_i(u).$$

We assume that the energetic cost of a defector is 0, and the energetic cost of a cooperator is 1, where the value of 1 is chosen for normalization:

$$E_d = 0 \text{ and } E_c = 1.$$

A hypocritical player produces the minimal social welfare required to pretend to be cooperative. Hence, we assume that

$$0 < E_h < 1,$$

thinking of E_h as closer to 0 than to 1.

As mentioned above, we focus on relatively mild social-pressure induced by cooperative players, aiming to improve their social status. Since hypocritical players aim to appear similar to cooperators from the perspective of an external observer, we assume that they too induce social-pressure on their neighbours. Defectors, on the other hand, do not induce any social-pressure since such an enhancement of the social status is not justified for them. In principle, cooperators and hypocritical players might induce different levels of social-pressure, yet, for the sake of simplicity, we assume that they induce the same extent of social-pressure. This assumption is further justified by the fact that a player u cannot distinguish its hypocritical neighbours from its cooperative neighbours, hence, u's calculation of the social-pressure is evaluated assuming all of its non-defector neighbours are cooperators.

Formally, we assume that the possible social-upgrade gain associated with cooperators or hypocritical players as a result of applying social-pressure is already taken into account when calculating the energetic costs E_c and E_h. Since we assume that this gain is small, it hardly perturbs the cost, keeping the energy consumption as the dominant component.

Implicitly, we think of the social-pressure cost incurred by a player u as the product of two factors: (1) the risk of being caught, which is assumed to be proportional to the number of u's neighbours inducing social-pressure, and (2) a fixed penalty paid when caught, which depends on u's behaviour. The product of the risk and penalty represents the expected punishment in the next round, if behaviours remain the same. Cooperators are assumed to pay zero penalty, and are hence effectively immune to social-pressure:

$$S_c(u) = 0.$$

Conversely, the social-pressure induced over defectors and hypocritical players is non-zero. For a given round, let $\Delta_{\bar{d}}(u)$ denote the number of neighbours of u which are non-defectors at that round. The social-pressure cost induced over a defector, and respectively, a hypocritical, player u is:

$$S_d(u) = \rho_d \cdot \Delta_{\bar{d}}(u), \quad \text{respectively,} \quad S_h(u) = \rho_h \cdot \Delta_{\bar{d}}(u),$$

where $\rho_d > 0$, respectively $\rho_h > 0$, represents the social-pressure induced over a defector, respectively a hypocritical, from one neighbouring non-defector. Note that when comparing the social-pressure incurred by defectors versus hypocritical players, both the risk of being caught and the extent of punishment are expected to be different. Indeed, since hypocritical players pretend to be cooperators, their risk of being caught is expected to be lower than that of defectors. Moreover, after being caught, the respected punishment of a defector might be different than that of a hypocritical player, depending on the social norms. Altogether, here we focus on the regime where $\rho_h < \rho_d$, since otherwise, becoming a defector is always more beneficial than becoming a hypocritical.

To sum up, at a given round, the total cost incurred by a player u is:

$$\mathcal{C}(u) = \begin{cases} 1 & \text{if } u \text{ is a cooperator,} \\ \rho_d \cdot \Delta_{\bar{d}}(u) & \text{if } u \text{ is a defector,} \\ E_h + \rho_h \cdot \Delta_{\bar{d}}(u) & \text{if } u \text{ is hypocritical.} \end{cases}$$

Our main result is that adjusting the level of social-pressure employed against hypocritical players compared to the one employed against defectors can have a dramatic impact on the dynamics of the system.

Theorem 5.1. *Consider a Δ-regular network G with n players. Assume that*

$$(1 - E_h)/\Delta < \rho_h < \rho_d - E_h. \tag{5.1}$$

Then, with probability at least $1 - \frac{1}{c^n}$, for some constant $c > 1$, in at most $3 \cdot \text{diam}(G) + 1$ rounds, the system will be in a configuration in which all players are cooperative, and will remain in this configuration forever.

Theorem 5.1 assumes that the underlying network is Δ-regular. In Section 5.4, we prove it and explain how to generalize it to arbitrary networks with minimal degree Δ.

Intuitively, the main idea behind the proof is as follows. When the extent of social-pressure against hypocritical players is moderate, that is, when ρ_h satisfies Eq. (5.1), the transition process can be divided into two stages. At the first stage, since the punishments of hypocritical players are

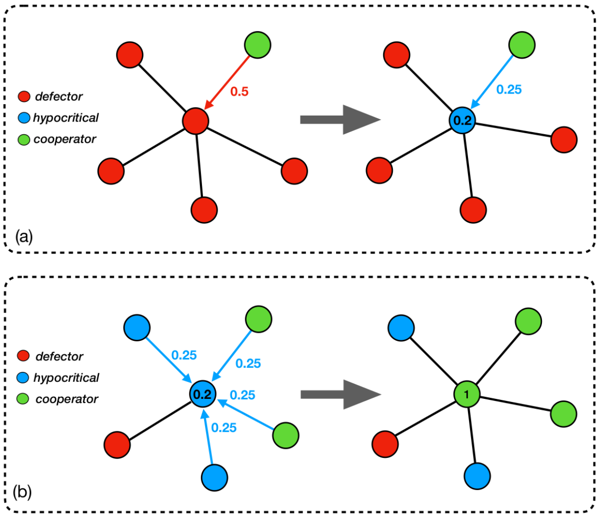
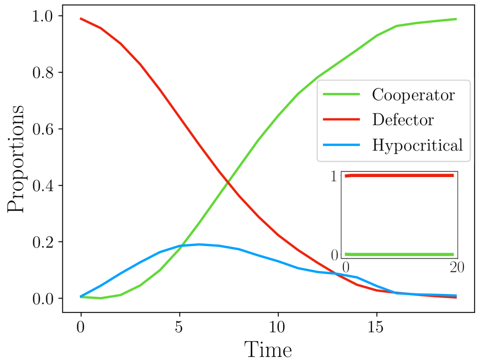


Figure 5.1: **The two stages of the dynamics.** The direction of the red and blue arrows indicates the direction of the social-pressure applied on the player occupying the central vertex. Cooperative players pay an energetic cost of $E_c = 1$ and are immune to social-pressure. A defector player pays a social-pressure cost of $\rho_d = 0.5$ per non-defector neighbour. A hypocritical player pays social-pressure cost of $\rho_h = 0.25$ per non-defector neighbour, and an energetic cost of $E_h = 0.2$. (a) First stage: defectors become hypocritical players. A defector player (central vertex on the left) has one non-defector neighbour (in this case, a cooperator), implying that its social-pressure cost is $\rho_d = 0.5$. Therefore, that player would prefer to be hypocritical (right), paying only $0.25 + 0.2 = 0.45$. (b) Second stage: hypocritical players become cooperators. Here, a hypocritical player (central vertex on the left) is surrounded by four non-defector neighbours. In this case, the social-pressure accumulates to favor cooperation (right).

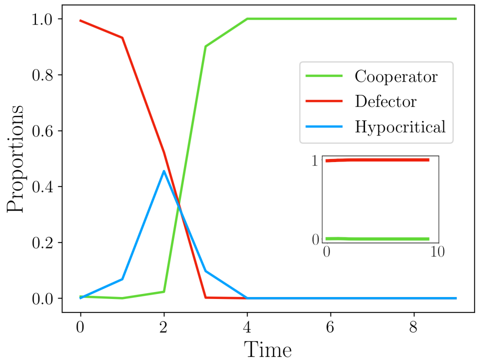
sufficiently lower than those of defectors, specifically, $\rho_h < \rho_d - E_h$, or equivalently $\rho_h + E_h < \rho_d$, the presence of at least one neighbouring non-defector u makes a hypocritical player pay less than a defector. In this case, u's neighbours would become non-defectors at the next round (Figure 5.1a). Although this does not necessarily imply that u itself remains a non-defector in the next round, it is nevertheless possible to show that the proportion of hypocritical players gradually increases on the expense of defectors. Note that at this point, the social welfare may still remain low, since hypocritical players hardly contribute to it. However, the abundance of non-defectors increases the overall social-pressure in the system. In particular, since the social-pressure on hypocritical players is also not too mild, specifically $(1 - E_h)/\Delta < \rho_h$, or equivalently $1 < \rho_h\Delta + E_h$, the presence of many neighbouring non-defectors can magnify it up to the point that the total cost incurred by a hypocritical player surpasses the energetic cost of being a cooperator (Figure 5.1b). At this second stage, cooperators prevail over both defectors and hypocritical players, and so the system converges to a cooperative configuration.

Conversely, severely punishing hypocritical players diminishes the prevalence of such players, preventing the system from escaping the initial degenerate configuration. Contrariwise, incurring too mild social-pressure towards hypocritical players would prevent the second stage of the dynamics. In particular, if $\rho_h < (1 - E_h)/\Delta$, or equivalently, if $E_h + \rho_h\Delta < 1$, then a player would always prefer to be hypocritical over being cooperative (even when all its neighbours induce social-pressure). In this case, the system would remain degenerative since the population would consist of a combination of defectors and hypocritical players.

To illustrate the dynamics we conducted simulations over several types of networks; see Section 5.2 for details on how the simulations were performed. Figure 5.2 shows how the population evolves over time, when considering a grid network (Figure 5.2a) and a random 10-regular network (Figure 5.2b). The chosen parameters satisfy the assumption in Eq. (5.1). In both dynamics, the role of hypocritical behaviour as a transitory state, essential to achieving cooperation, is well illustrated by the initial peak of hypocritical players, preceding the rise of cooperative players. Moreover, if hypocritical behaviour is disabled, then the system is unable to escape the defective state.



(a) Time evolution on a 50×50 grid

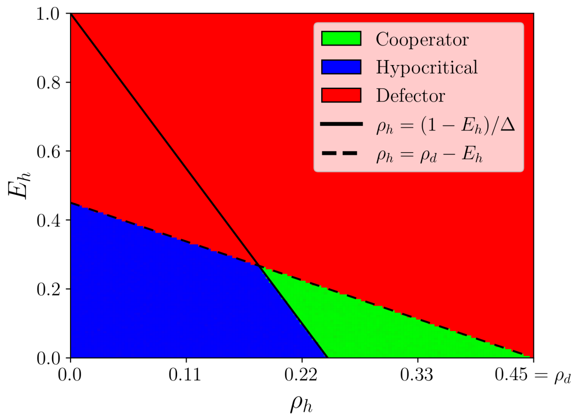


(b) Time evolution on a 10-regular network

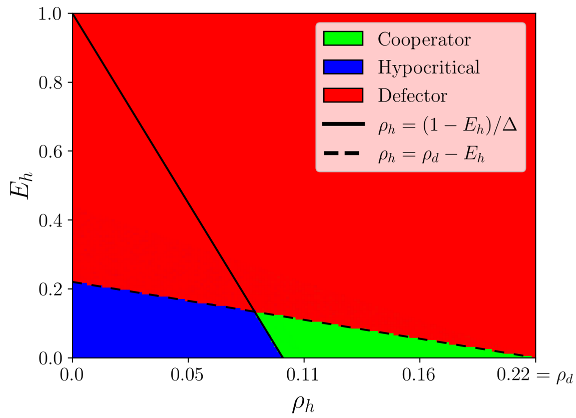
Figure 5.2: **Evolution of cooperation in grids and random 10-regular networks.** Figure (a) corresponds to a 50×50 grid network, and Figure (b) corresponds to a random 10-regular network with 1000 vertices. Both simulations start with a configuration in which 99% of players are defectors. Figures (a) and (b) show how the population evolves over time (number of rounds). The chosen parameters satisfy the assumption in Eq. (5.1). The insets show the population's evolution when hypocritical behaviour is not available to the agents. See Section 5.2 for more details.

Figure 5.3 depicts the steady-state configuration, when hypocritical players experience different levels of energetic cost (E_h) and social-pressure (ρ_h). This is illustrated on a grid network (Figure 5.3a), random 10-regular networks (Figure 5.3b), Erdös-Rényi networks with average degree 10 (Figure 5.3c), and a Barabási-Albert networks with median degree 7 (Figure 5.3d). The figures indicate that for small values of ρ_h and E_h, hypocritical behaviour is, unsurprisingly, dominant: punishments deter defectors, but are insufficient to incentivize cooperation. For moderate values of E_h, this phenomenon changes when ρ_h enters the range specified in Theorem 5.1. Then, when ρ_h increases further, the system remain defective. The correspondence to Theorem 5.1 is striking in Figures 5.3a to 5.3c, whereas it is slightly more moderate in Figure 5.3d. Recall that Theorem 5.1 considers Δ-regular networks, and therefore directly applies to grid networks and random regular networks, as simulated in Figures 5.3a and 5.3b, respectively. Moreover, although a typical Erdös-Rényi network is not regular, the degrees of its vertices are relatively concentrated around the average degree, justifying the similarity between the results in Figures 5.3b and 5.3c. For Barabási-Albert networks (Figure 5.3d) the average degree is not a good representative for the typical degree since these networks are power-law. Hence, we drew the line corresponding to $\rho_h = (1 - E_h)/\Delta$, taking Δ to be the median degree, which was in this case roughly 7. Even though many vertices in the network have a smaller degree than the median degree, high levels of cooperation emerge in the region specified by Theorem 5.1.

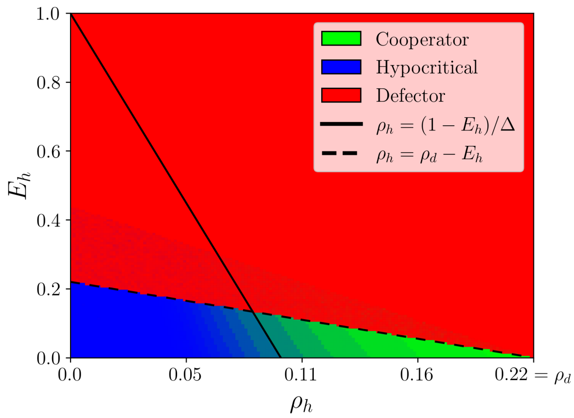
Consistent with Theorem 5.1, Figure 5.3 considers the case that players behave in a fully greedy fashion while having perfect knowledge regarding their costs. To check if this assumption is impactful, we also simulated a more noisy variant of our model, in which each player chooses the behaviour that minimizes its cost with probability 0.95, and otherwise chooses a behaviour uniformly at random. This relaxed model yields more mixed populations at steady-state, as indicated in Figure 5.4a regarding a grid network and in Figure 5.4b regarding random 10-regular networks. As another relaxation, we also simulated the case that the initial configuration is not overwhelmingly composed of defectors. Specifically, in Figure 5.4c (grid network) and Figure 5.4d (random 10-regular networks) we assumed that initially 80% of the players are defectors, instead of 99% as used in Figure 5.3. Not surprisingly, this relaxation enhances cooperation. Indeed, comparing Figure 5.4c to Figure 5.3a, and comparing Figure 5.4d to Figure 5.3b, we observe that for each of these networks, the corresponding regime of cooperation includes the one that emerges when there are more defectors initially. Overall, in all the relaxed versions in Figure 5.4 we see that the necessity of the condition $\rho_h > (1 - E_h)/\Delta$ to the emergence of cooperation is still respected. However, the other condition mentioned in Theorem 5.1, namely, $\rho_h < \rho_d - E_h$ appears to be more sensitive to randomness. Indeed, and especially for the cases of random Δ-regular networks, cooperation emerges also for larger values of ρ_h.



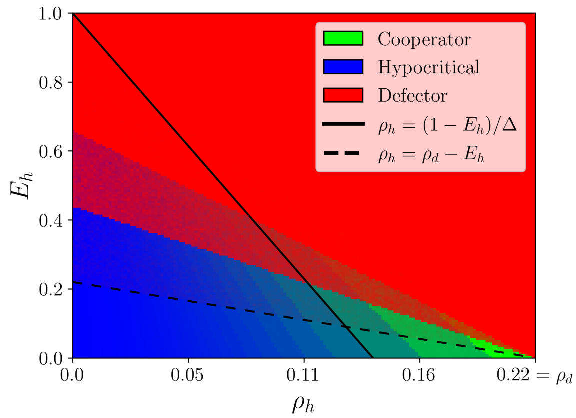
(a) Cooperation level on a 50×50 grid



(b) Cooperation level on 10-regular networks

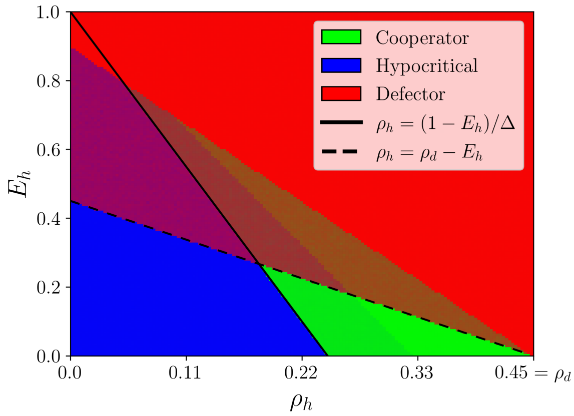


(c) Cooperation level on Erdös-Rényi networks with average degree $\Delta = 10$

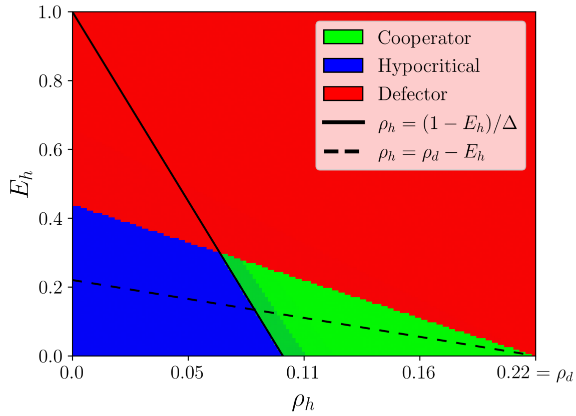


(d) Cooperation level on Barabási-Albert networks with median degree $\Delta = 7$

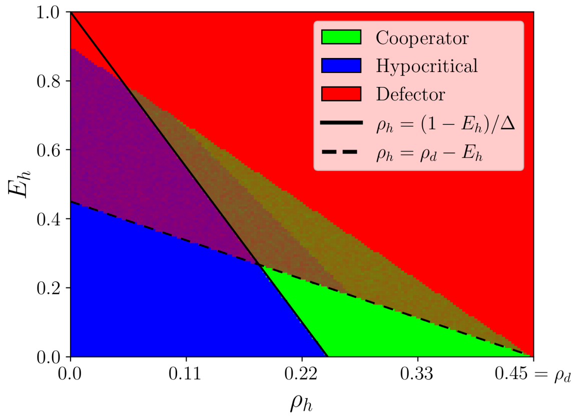
Figure 5.3: **Emergence of cooperation on various networks.** The figure depicts the steady-state levels of cooperation on different network families. (a) corresponds to a 50×50 grid network, (b) corresponds to random 10-regular networks with 1000 vertices, (c) corresponds to Erdös-Rényi networks with 1000 vertices and parameter $p = 1/100$, and (d) corresponds to Barabási-Albert networks with 1000 vertices and parameter $m = 5$. All simulations start with a configuration in which 99% of players are defectors. In all figures, for each couple (ρ_h, E_h), a pixel is being drawn, whose red (resp. green, blue) component corresponds to the average proportion of defectors (resp. cooperators, hypocrites) at steady state. See Methods for more details.



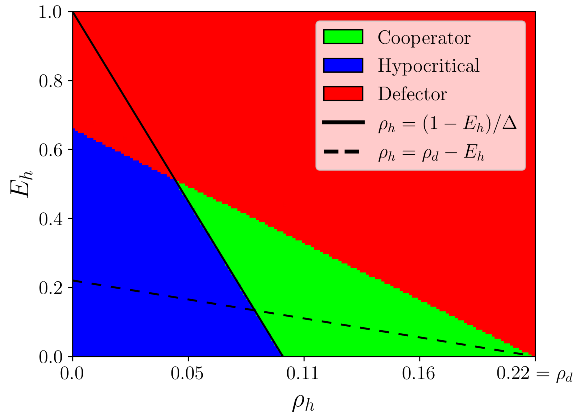
(a) Relaxed model on a 50×50 grid



(b) Relaxed model on 10-regular networks



(c) Cooperation level on a 50×50 grid with 20% non-defectors initially



(d) Cooperation level on 10-regular networks with 20% non-defectors initially

Figure 5.4: **Relaxed model on grids and random 10-regular networks.** The figure depicts simulation results using relaxed versions of the main model. In (a) and (b) the greediness assumption in the decision making process is relaxed, allowing for some "irrationality" (see Methods for more details). In (c) and (d), the initial configuration contains 80% defectors, instead of 99% as used in Figure 5.3. The plots in (a) and (c) correspond to a 50×50 grid network and should be compared with Figure 5.3a, whereas (b) and (d) correspond to random 10-regular networks with 1000 vertices and should be compared with Figure 5.3b.

5.1.2.2 A Generalized Model with Costly Punishments.

We next describe a different, more general model, termed the *two-order model*, that includes costly punishments. We then show how the second-order problem is solved in this model for a certain regime of parameters.

As the name suggests, the *two-order model* includes two levels of cooperation. Players engaged in *first-order cooperation* incur an energetic cost to produce a benefit for other individuals, whereas players engaged in *second-order cooperation* induce costly punishments on other individuals whenever they fail to cooperate (on any order). The two orders of cooperation are not mutually exclusive, that is, a player can cooperate (or not cooperate) on one of the two orders or on both.

Similarly to the main model, players are organized over a connected network G. A *behaviour* for Player u is defined as a couple of indicator functions $(\chi_1(u), \chi_2(u))$, with the convention that $\chi_1(u) = 1$ if u cooperates on the first-order (and 0 if it defects), and $\chi_2(u) = 1$ if u cooperates on the second-order (and 0 if it defects).

The cost incurred by a player is divided into two components. We denote by $\alpha_1 > 0$ the cost associated with first-order cooperation (this is analogues to the energetic cost in the main model), while $\alpha_2 > 0$ refers to the cost of second-order cooperation, that is, the cost of incurring punishments. A player u such that $\chi_2(u) = 1$ induces a *social-pressure cost* on each of its neighbours, whenever these fail to cooperate, at any order. As in the main model, the extent of this social-pressure may differ depending on whether it is applied against first-order defectors or second-order defectors. Specifically, we denote by β_1 the social-pressure cost paid by a first-order defector, and by β_2 the social-pressure cost paid by a second-order defector (fully defecting players pay both). Formally, denoting by $\Delta_2(u)$ the number of neighbours of u which are cooperating on the second-order, that is, $\Delta_2(u) = |\{v \text{ is a neighbour of } u, \chi_2(v) = 1\}|$, the total cost paid by u equals:

$$\mathcal{C}(u) = \chi_1(u)\alpha_1 + \chi_2(u)\alpha_2 + (1 - \chi_1(u))\Delta_2(u)\beta_1 + (1 - \chi_2(u))\Delta_2(u)\beta_2. \tag{5.2}$$

Let us name each of the four behaviours, and recap their cost:

- *a cooperator* $(\chi_1(u) = 1, \chi_2(u) = 1)$ pays $\alpha_1 + \alpha_2$,
- *a defector* $(\chi_1(u) = 0, \chi_2(u) = 0)$ pays $\Delta_2(u)(\beta_1 + \beta_2)$,
- *a hypocritical* $(\chi_1(u) = 0, \chi_2(u) = 1)$ pays $\alpha_2 + \Delta_2(u)\beta_1$,
- *a private cooperator* $(\chi_1(u) = 1, \chi_2(u) = 0)$ pays $\alpha_1 + \Delta_2(u)\beta_2$.

As in the main model, the system starts in a configuration in which almost all players, e.g., 99%, are defectors (see Methods). The execution proceeds in discrete synchronous rounds. The costs of each player are evaluated at the beginning of each round, and then, before the next round starts, each player chooses a behaviour that minimizes its cost (breaking ties randomly), given the current behaviour of its neighbours.

Theorem 5.2. *Consider a Δ-regular network G with n players undergoing the two-order model. Assume that the following two conditions hold:*

- Condition (i) $\alpha_2 < \beta_2$, *and*
- Condition (ii) $\alpha_1 < \Delta\beta_1$.

Then, with probability at least $1 - \frac{1}{c^n}$, for some constant $c > 1$, in at most $3 \cdot \text{diam}(G) + 1$ rounds, the system will be in a configuration in which all players are cooperative, and will remain in this configuration forever.

The proof of Theorem 5.2 can be found in Section 5.5. As in the case of Theorem 5.1, it is generalized to arbitrary networks with minimal degree Δ.

Intuitively, the proof starts by showing that for the regime of parameters satisfying Conditions (i) and (ii), after the first round, no player ever chooses to be a private cooperator. The proof proceeds by showing that for this regime of parameters, the dynamics of the two-order model can be translated to the dynamics of the main model for the regime of parameters satisfying Eq. (5.1). In other words, the proof of Theorem 5.2 is based on a reduction to Theorem 5.1.

5.2 More Details about the Simulations

The initial configuration is governed by a given fixed $0 < \epsilon < 1$, which is independent from the underlying network. In the main model, each player is initially set to be a defector with probability $1 - \epsilon$, a hypocritical with probability $\epsilon/2$, and a cooperative with probability $\epsilon/2$. Similarly, in the two-order model, each player is initially chosen to be a defector, with probability $1 - \epsilon$, and, otherwise, with probability ϵ it chooses one of the three remaining behaviours with equal probability, i.e., $\epsilon/3$. To demonstrate the strength of the emergence of cooperation, we consider ϵ as very small; for example, in each of our simulations (except the ones corresponding to Figures 5.4c and 5.4d), we took $\epsilon = 0.01$, which means that initially, 99% of the population were defectors, 0.5% were hypocritical, and 0.5% were cooperators.

We simulated the dynamics of the main model using the C++ language. Figures were obtained using the Python library "Matplotlib". In Figures 5.2a, 5.3a, 5.4a and 5.4c, we used a 50×50, 4-regular, torus grid. In Figures 5.2b, 5.3b, 5.4b and 5.4d, we used random 10-regular networks with 1000 vertices. To sample such a network, we gradually increased the number of edges, by pairing the vertices of degree less than 10 uniformly at random, until it became not possible anymore; then we discarded the few "left-overs" if necessary. As a consequence, the sampled networks have sometimes slightly less than 1000 vertices, but are always 10-regular by construction.

For Figure 5.3c, we constructed Erdös-Rényi networks with 1000 vertices, taking each edge with probability $p = 0.01$. For Figure 5.3d, we constructed Barabási-Albert networks with 1000 vertices using the parameter $m = 5$. To sample such a network, we started with an m-clique, and then added each new vertex by attaching it to m existing vertices chosen at random, with a probability proportional to their current degree.

When running the time-simulations on the grid in Figure 5.2a, we took $E_h = 0.1$, $\rho_d = 0.45$, and $\rho_h = 0.23$. In Figure 5.2b, the time-simulation was executed on a single random 10-regular network, using the parameters $E_h = 0.1$, $\rho_d = 0.22$, and $\rho_h = 0.11$. For both cases these parameters satisfy the constraints in Eq. (5.1). The insets show the evolution of the population when hypocritical behaviour is disabled. This means that each player must choose between cooperation and defection only, and that in the initial configuration, each player is a defector with probability $1 - \epsilon$, and a cooperator with probability ϵ. The setting remains otherwise unchanged.

In both Figures 5.3 and 5.4, the results of the simulations are presented for 150 values of E_h and 150 values of ρ_h, with $E_h \in [0, 1]$ and $\rho_h \in [0, \rho_d]$. For each couple (E_h, ρ_h), a pixel is drawn at the appropriate location, whose RGB color code corresponds to the proportions of defectors (red), cooperators (green), and hypocritical players (blue) in steady-state – that is, after T rounds. These proportions have been averaged over N repetitions, with each time a new starting configuration, and, a new network. For the grid, we set $T = 20, N = 10$, whereas for the other networks, we took $T = 10, N = 100$.

Figures 5.4a and 5.4b were obtained similarly to Figures 5.3a and 5.3b, respectively, except that players did not choose greedily their behaviours for the next round. Instead, at each round, each player chose a behaviour that minimizes its cost (breaking ties randomly) with probability 0.95, and otherwise chose a behaviour uniformly at random. Figures 5.4c and 5.4d were obtained similarly to Figures 5.3a and 5.3b, respectively, except that the initial proportion of non-defectors was 20% (instead of 1%), i.e., we took $\epsilon = 0.2$ (instead of 0.01).

5.3 An Intermediate Result in Graph Theory

The following lemma (mentioned also in [119]) appears to be a basic result in graph theory, however, we could not find a formal proof for it. We therefore provide a proof here for the sake of completeness.

Lemma 5.3. *The shortest odd-length cycle of any non-bipartite network G is of length at most $2\text{diam}(G) + 1$.*

Proof. Consider a non-bipartite network G. Such a network necessarily has an odd-length cycle. Let $2k+1$ be the shortest length among the odd-length cycles in G, where k is an integer, and let $C = (u_1, \dots, u_{2k+1})$ be such a cycle.

Claim 5.4. *For every $i, j \in \{1, \dots, 2k+1\}$ such that $d_C(u_i, u_j) \geq 2$, there exist $\ell \neq i, j$ and a shortest path P between u_i and u_j such that P contains u_ℓ.*

Proof. Assume by way of contradiction that we can find $i < j$ such that no shortest path between u_i and u_j has any intermediate vertex among $\{u_1, \dots, u_{2k+1}\}$. Up to re-indexing the vertices of the cycle, we can assume that $j - i \leq k$. Let $(u_i = v_1, v_2, \dots, v_s, v_{s+1} = u_j)$ be a shortest path between u_i and u_j. By assumption, $\{v_2, \dots, v_s\} \cap \{u_1, \dots, u_{2k+1}\} = \emptyset$, and $s < j - i$ (otherwise $(u_i, u_{i+1}, \dots, u_{j-1}, u_j)$ is a shortest path). Consider two cases:

- If s and $j - i$ have different parities, then $s + j - i$ is odd. Moreover, $s + j - i \leq 2(j - i) \leq 2k$, so

  $$(v_1 = u_i, u_{i+1}, \dots, u_{j-1}, u_j = v_{s+1}, v_s, \dots, v_2)$$

  is an odd-length cycle shorter than C, which is a contradiction.

- If s and $j - i$ have the same parity, then $2k + 1 + s - (j - i)$ is odd. Moreover, $2k + 1 + s - (j - i) < 2k + 1$, so

  $$(u_i = v_1, v_2, \dots, v_s, v_{s+1} = u_j, u_{j+1}, \dots, u_{2k+1}, u_1, \dots, u_{i-1})$$

  is again an odd-length cycle shorter than C, which is a contradiction.

This concludes the proof of Claim 5.4. □

Claim 5.5. *For every $i, j \in \{1, \dots, 2k+1\}$, there exist a shortest path P between u_i and u_j such that P contains only vertices of C – in other words, $d_C(u_i, u_j) = d_G(u_i, u_j)$.*

Proof. We prove the claim by induction on $d_C(u_i, u_j)$, the distance between u_i and u_j in C. When $d_C(u_i, u_j) = 1$, (u_i, u_j) is a path of length 1 between u_i and u_j containing only vertices of C. Next, let us assume that the claim holds for every pair of vertices whose distance in C is at most $1 \leq d \leq k$. Consider i and j such that $d_C(u_i, u_j) = d + 1$. By Claim 5.4, we can find ℓ and a shortest path P between u_i and u_j such that P contains u_ℓ. By the induction hypothesis, we can find shortest paths P_1 between u_i and u_ℓ, and P_2 between u_ℓ and u_j, such that P_1 and P_2 contain only vertices of C. By merging P_1 and P_2, we obtain a shortest path between u_i and u_j containing only vertices of C, which establishes the induction step. This concludes the proof of Claim 5.5. □

By Claim 5.5, $k = d_C(u_1, u_{k+1}) = d_G(u_1, u_{k+1}) \leq \mathrm{diam}(G)$, where the last inequality is by the definition of diameter. Hence, $2k + 1 \leq 2\mathrm{diam}(G) + 1$. This concludes the proof of Lemma 5.3. □

5.4 Proof of Theorem 5.1

The goal of this section is to prove Theorem 5.1. In fact, we prove the more general theorem below.

Theorem 5.6. *Consider a network G with n players and minimal degree Δ. Assume that the following conditions hold.*

- Condition (i) $E_h + \rho_h < \rho_d$, *and*
- Condition (ii) $E_h + \rho_h \cdot \Delta > 1$.

Then, for some constant $c > 1$ (that depends only on ϵ and not on G) the following holds.

- If G is not bipartite then with probability at least $1 - \frac{1}{c^n}$, in at most $3 \cdot \text{diam}(G) + 1$ rounds, the system will be in a configuration in which all players are cooperative, and will remain in this configuration forever.
- If G is bipartite and Δ-regular then with probability at least $1 - \frac{1}{c^n}$, in at most $\text{diam}(G) + 1$ rounds, the system will be in a configuration in which all players are cooperative, and will remain in this configuration forever.
- If G is bipartite then with probability at least $1 - \frac{1}{c^\Delta}$, in at most $\text{diam}(G) + 1$ rounds, the system will be in a configuration in which all players are cooperative, and will remain in this configuration forever.

Before we prove Theorem 5.6 we note that in the third item, the probability bound of $1 - \frac{1}{c^\Delta}$ is tight for bipartite graphs, up to replacing c with another constant. Indeed, consider the bipartite graph which is constructed by having Δ players in U, each of which is connected to each of the remaining $n - \Delta$ players in V. Then, with probability $\frac{1}{c^\Delta}$, for some constant c, all players in U are defectors initially. In this case, it is possible to show that, regardless of the relationships between ρ_d, ρ_h and E_h, but as long as being a defector is the best choice when all neighbours are defectors, the system keeps alternating forever, so that on even rounds all players in U are defectors, and on odd rounds all players in V are defectors.

Proof of Theorem 5.6. We start with defining $\bar{D}_t$ as the set of non-defector players at round t. The following lemma describes the propagation of the non-defector state in the network. It says that a player u is a non-defector at round $t + 1$ if and only if at least one of its neighbours v is a non-defector in round t. Note, however, that this does not imply that the neighbour v remains a non-defector in the next round as well.

Lemma 5.7. *Under Condition* (i), $\bar{D}_{t+1} = N(\bar{D}_t)$.

Proof. First, we prove that $N(\bar{D}_t) \subseteq \bar{D}_{t+1}$. Let $u \in N(\bar{D}_t)$. By definition, there exists a neighbour v of u such that v is a non-defector at round t. We claim that for u, being a hypocritical in round $t + 1$ is strictly more beneficial than being a defector. Indeed, as a hypocritical it will pay $E_h + \rho_h \cdot \Delta_{\bar{d}}(u)$, and as a defector it will pay $\rho_d \cdot \Delta_{\bar{d}}(u)$. Since v is non-defector then $\Delta_{\bar{d}}(u) \geq 1$, and hence:

$$E_h + \rho_h \cdot \Delta_{\bar{d}}(u) \leq (E_h + \rho_h) \cdot \Delta_{\bar{d}}(u) < \rho_d \cdot \Delta_{\bar{d}}(u),$$

where we used Condition *(i)* in the last inequality. Therefore, the cost of u as a defector is strictly higher than its cost as a hypocritical. This implies that in the next round u will be either a hypocritical or a cooperative player, i.e., $u \in \bar{D}_{t+1}$.

To prove the other inclusion, $\bar{D}_{t+1} \subseteq N(\bar{D}_t)$, consider a player $u \notin N(\bar{D}_t)$, i.e., having only defectors as neighbours at round t, or in other words, at round t, we have $\Delta_{\bar{d}}(u) = 0$. If u chooses to be a defector at round $t+1$, then it would pay $\Delta_{\bar{d}}(u)\rho_d = 0$, which is less than what it would pay as a hypocritical ($E_h + \Delta_{\bar{d}}(u)\rho_h = E_h$) or cooperator (1). Hence, $u \notin \bar{D}_{t+1}$. □

Lemma 5.8. *Assume that Conditions* (i) *and* (ii) *hold, and assume that for some round t_0 all players are non-defectors. Then, at round $t_0 + 1$, all players will be cooperative, and will remain cooperative forever.*

Proof. Assume that at round t_0 all players are non-defectors. By Lemma 5.7, we know that every player will remain non-defector for every round after t_0. It therefore remains to show, that at the end of round t, for any $t \geq t_0$, being a cooperative is strictly more beneficial than being a hypocritical.

Observe that since each player has at least Δ neighbours, and since all neighbours are non-defectors at round t, then for every player u, we have $\Delta_{\bar{d}}(u) \geq \Delta$ at round t. Therefore, being a hypocritical costs $E_h + \rho_h \cdot \Delta_{\bar{d}}(u) \geq E_h + \rho_h \cdot \Delta$. By Condition *(ii)*, this quantity is strictly greater than 1, hence more than what a cooperative player would pay. It follows that, at the end of round t, being a cooperative is strictly more beneficial than being a hypocritical, implying that all players would be cooperators at round $t+1$. This completes the proof of Lemma 5.8. □

Lemma 5.9. *Assume that Conditions* (i) *and* (ii) *hold, and assume that for some round* t_0*, we have* $\bar{D}_{t_0} \cap N(\bar{D}_{t_0}) \neq \emptyset$*, that is, there are at least two neighbouring non-defectors. Then in at most* diam$(G)+1$ *rounds as of round* t_0*, the system will be in the configuration in which all players are cooperative, and will remain in this configuration forever.*

Proof. By assumption, there exists two neighbours $u_0, u_0' \in \bar{D}_{t_0}$. We define inductively a sequence of sets $\{U_j\}_{j \geq 0}$, setting $U_0 = \{u_0, u_0'\}$, and for every j, defining $U_{j+1} = N(U_j)$.

Claim 5.10. *For every integer* $j \geq 0$*,*

$$U_j \subseteq N(U_j) \tag{5.3}$$

(each player in U_j *has at least one neighbour in* U_j*), and*

$$U_j \subseteq \bar{D}_{t_0+j} \tag{5.4}$$

(each player in U_j *is non-defector at round* $t_0 + j$*).*

Proof. The proof proceeds by induction. The base of the induction, corresponding to $j = 0$, is true by the assumption on u_0 and u_0'. Next, let us assume that the claim holds for some integer $j \geq 0$. By the induction hypothesis with respect to Eq. (5.3), $U_j \subseteq N(U_j)$, so $N(U_j) \subseteq N(N(U_j))$, and hence, by definition, $U_{j+1} \subseteq N(U_{j+1})$. In other words, we have proved that Eq. (5.3) holds at round $j+1$. Next, by the induction hypothesis with respect to Eq. (5.4), we have $U_j \subseteq \bar{D}_{t_0+j}$, so $N(U_j) \subseteq N(\bar{D}_{t_0+j})$. By definition of U_{j+1}, and by Lemma 5.7, we can rewrite this as $U_{j+1} \subseteq \bar{D}_{t_0+j+1}$, establishing Eq. (5.4) at round $j+1$. This completes the induction step and concludes the proof of Claim 5.10. □

A direct consequence of Eq. (5.3) in Claim 5.10 and the definition of the sequence $\{U_j\}_j$ is that $U_{j+1} = U_j \cup N(U_j)$, and so, U_{j+1} is equal to U_j together with all the neighbours of players in U_j. As a consequence, for every $j \geq$ diam(G), the set U_j contains all players. By Eq. (5.4) of Claim 5.10, this implies that from round t_0+diam(G) onward, all players are non-defectors.

By Lemma 5.8, we conclude that from round $t_0 +$ diam$(G) + 1$ onward, all players are cooperative. This completes the proof of Lemma 5.9. □

Lemma 5.11. *Assume that Conditions* (i) *and* (ii) *hold, and that* G *is not bipartite. If* $\bar{D}_0 \neq \emptyset$*, i.e., if initially there is at least one non-defector player, then in at most* $3 \cdot$ diam$(G) + 1$ *rounds, the system will be in the configuration in which all players are cooperative, and will remain in this configuration forever.*

Proof. By assumption, G is not bipartite, or equivalently, G contains at least one odd-length cycle. Let $(u_1, \ldots, u_{2k+1})$ be a shortest odd-length cycle of G. Given $s = d_G(u_1, \bar{D}_0)$, let $(v_0 \in \bar{D}_0, v_1, \ldots, v_{s-1}, v_s = u_1)$ be a shortest path from $\bar{D}_0$ to u_1. By Lemma 5.7, it follows by induction that for every $t \in \{0, \ldots, s\}$, $v_t \in \bar{D}_t$, and hence, $u_1 \in \bar{D}_s$ (note that, although $v_{t-1} \in \bar{D}_{t-1}$, it could be that $v_{t-1} \notin \bar{D}_t$). Similarly, for every $t \in \{1, \ldots, k\}$, $u_{1+t} \in \bar{D}_{s+t}$ and $u_{2k+2-t} \in \bar{D}_{s+t}$. Hence, $u_{k+1} \in \bar{D}_{s+k}$ and $u_{k+2} \in \bar{D}_{s+k}$. In other words, we have just showed that in round $s + k$, we have two non-defector neighbours.

By the definition of diameter, $s \leq \text{diam}(G)$. By Lemma 5.3, we also have $k \leq \text{diam}(G)$. By Lemma 5.9, the system needs at most $\text{diam}(G) + 1$ rounds after round $s + k$ to reach full cooperation. We conclude that it reaches cooperation in at most $3 \cdot \text{diam}(G) + 1$ rounds, as stated. □

Lemma 5.12. *Assume that Conditions* (i) *and* (ii) *hold, and that G is bipartite. The set of players can be partitioned into U and V such that $U \cap N(U) = V \cap N(V) = \emptyset$. If $\bar{D}_0 \cap U \neq \emptyset$ and $\bar{D}_0 \cap V \neq \emptyset$, then in at most $T = \text{diam}(G) + 1$ rounds, the system will be in the configuration in which all players are cooperative, and will remain in this configuration forever.*

Proof. By assumption, G is bipartite. We define inductively a sequence of subsets of the set of players, $U_0 = U \cap \bar{D}_0$, and for every $k \geq 0$, $U_{k+1} = N(N(U_t))$ – that is, U_{k+1} contains the neighbours (in U) of the neighbours (in V) of the players in U_k. Note that, as a consequence of this definition, $U_k \subseteq U_{k+1}$. Let $k_0 = \lfloor \text{diam}(G)/2 \rfloor$.

Let us show that $U_{k_0} = U$ (and hence, that for every $k \geq k_0$, $U_k = U$). For this purpose, consider a player $u \in U$. Let $(u_0, v_0, u_1, v_1, \ldots, u_{s-1}, v_{s-1}, u_s)$, where $u_0 \in U_0$ and $u_s = u$ be a shortest path from U_0 to u. This path is of length $2s \leq \text{diam}(G)$, so $s \leq k_0$. Since $u_{\ell+1} \in N(N(u_\ell))$ for every $\ell \leq s$, it follows by induction on ℓ that for every $\ell \leq s$, $u_\ell \in U_\ell$, and hence that $u \in U_s$. As we have seen, the sequence $\{U_k\}_k$ is non-decreasing and since $s \leq k_0$, we obtain $u \in U_{k_0}$. This establishes that $U_{k_0} = U$.

Next, we prove by induction that for every k, $U_k \subseteq \bar{D}_{2k}$. This is true for $k = 0$ by definition. Assume that this is true for some integer $k \geq 0$. We have

$$U_{k+1} = N(N(U_k)) \subseteq N(N(\bar{D}_{2k})) = \bar{D}_{2k+2},$$

where the second transition is by the induction hypothesis and the last transition is due to Lemma 5.7. This concludes the induction proof. Since we have already proved that $U_{k_0} = U$, we conclude that $U \subseteq \bar{D}_{2k_0}$.

We can apply the same reasoning to V and obtain that $V \subseteq \bar{D}_{2k_0}$. Thus, $\bar{D}_{2k_0}$ contains all players. By Lemma 5.8, from round $2k_0 + 1 \leq 2\text{diam}(G) + 1$ onward, the system will be in a configuration in which all players are cooperative and will remain in this configuration forever. This concludes the proof of Lemma 5.12. □

Finally, we wrap the aforementioned lemmas to prove the theorem with respect to different networks. Recall that initially, each player is set to be a defector with probability $1 - \epsilon$, a hypocritical with probability $\epsilon/2$, and a cooperative with probability $\epsilon/2$, for some fixed $0 < \epsilon < 1$ independent of n. We consider three families of networks.

- If G is not bipartite, then Lemma 5.11 guarantees that the system converges to full cooperation in $3 \cdot \text{diam}(G) + 1$ rounds, provided that the initial configuration contains at least one non-defector. This happens with overwhelmingly high probability, specifically, $1 - (1 - \epsilon)^n = 1 - \frac{1}{c^n}$, for some constant $c > 1$. This completes the proof of the first item in Theorem 5.6.

- If G is bipartite, then the set of players in G can be split into two disjoint sets U and V such that all edges are between U and V. Lemma 5.12 guarantees that the system converges to full cooperation $\text{diam}(G) + 1$ rounds, provided that there is at least one non-defector in U and at least one non-defector in V. Let us see what is the probability that the initial configuration satisfies this.

– If G is Δ-regular, then both U and V contain precisely $n/2$ players. This follows from the fact that the number of edges outgoing from U, respectively V, is precisely $\Delta|U|$, respectively $\Delta|V|$, and these numbers are equal. In this case the probability that there is at least one non-defector in U and at least one non-defector in V is $\left(1-(1-\epsilon)^{n/2}\right)^2 \geq 1-2(1-\epsilon)^{n/2} > 1-\frac{1}{c^n}$, for some constant $c > 1$. This completes the proof of the second item in Theorem 5.6.

– For general bipartite G with minimal degree Δ, we have that both $|U|$ and $|V|$ are greater or equal to Δ. Hence, the probability that there is at least one non-defector in U and at least one non-defector in V is at least $\left(1-(1-\epsilon)^{\Delta}\right)^2 \geq 1-2(1-\epsilon)^{\Delta} > 1-\frac{1}{c^{\Delta}}$, for some constant $c > 1$. This completes the proof of the third item in Theorem 5.6.

□

5.5 Proof of Theorem 5.2

The goal of this section is to prove Theorem 5.2. In fact, we prove the more general theorem below.

Theorem 5.13. *Consider a network G with n players and minimal degree Δ undergoing the two-order model, so that the following conditions hold.*

- Condition (i) $\alpha_2 < \beta_2$, *and*
- Condition (ii) $\alpha_1 < \Delta\beta_1$.

Then the following holds for some constant $c > 1$.

- *If G is not bipartite then with probability at least $1-\frac{1}{c^n}$, in at most $3 \cdot \text{diam}(G) + 1$ rounds, the system will be in a configuration in which all players are cooperative, and will remain in this configuration forever.*
- *If G is bipartite and Δ-regular then with probability at least $1-\frac{1}{c^n}$, in at most $\text{diam}(G)+1$ rounds, the system will be in a configuration in which all players are cooperative, and will remain in this configuration forever.*
- *If G is bipartite then with probability at least $1-\frac{1}{c^{\Delta}}$, in at most $\text{diam}(G)+1$ rounds, the system will be in a configuration in which all players are cooperative, and will remain in this configuration forever.*

Before we prove the theorem, we note that Condition (ii) is necessary for the emergence of cooperation on Δ-regular graphs, since having $\alpha_1 > \Delta\beta_1$ would imply that it is always beneficial to defect on the first level.

Proof of Theorem 5.13. Consider a network G, and parameters α_1, α_2, β_1, and β_2, satisfying Conditions (i) and (ii) in Theorem 5.13. We first observe that under Condition *(i)*, at any round $t \geq 1$, no player ever chooses to be a private cooperator in G. Indeed, if $\Delta_2(u) \geq 1$, then a private cooperator would pay $\alpha_1 + \Delta_2(u)\beta_2 > \alpha_1 + \Delta_2(u)\alpha_2 \geq \alpha_1 + \alpha_2$, hence, more than the cost of cooperating, and when $\Delta_2(u) = 0$, a private cooperator would pay $\alpha_1 > 0$, hence, more than the cost of defecting. It follows that, although the initial configuration may include private cooperators, this behaviour completely disappears from the system after the first round.

Next, we aim to prove Theorem 5.13 by reducing it to Theorem 5.6. Let G' be a network identical to G, undergoing the main model (for which Theorem 5.6 applies), taking the parameters:

$$E_h = \frac{\alpha_2}{\alpha_1+\alpha_2}, \quad \rho_h = \frac{\beta_1}{\alpha_1+\alpha_2}, \quad \text{and} \quad \rho_d = \frac{\beta_1+\beta_2}{\alpha_1+\alpha_2}. \tag{5.5}$$

A *configuration* $\mathcal{C}$ on G is an assignment of behaviours, namely, either defectors, cooperators, hypocritical, or private cooperators, to the players in G. Recall that the initial configuration on G is sampled according to the distribution $\psi(\epsilon)$, so that each player is initially chosen to be a defector with probability $1-\epsilon$, and any of the three remaining behaviours with probability $\epsilon/3$.

We next define a mapping f, transforming each initial configuration $\mathcal{C}$ on G to an initial configuration $\mathcal{C}'$ on G'. The mapping is very simple: All players in G' remain with the same behaviour as in G except that private cooperators are turned into defectors. It is easy to see that given the distribution $\psi(\epsilon)$, this mapping induces the distribution $\psi'(\epsilon')$ on the initial configurations in G', where $\epsilon' = \frac{2}{3}\epsilon$. Indeed, under this mapping, a player in G' is initially chosen to be a defector with probability $1-\epsilon+\epsilon/3 = 1-\epsilon'$, a cooperator with probability $\epsilon/3 = \epsilon'/2$, and hypocritical with probability $\epsilon/3 = \epsilon'/2$.

At this point, we address a technicality that concerns the randomness involved in breaking ties. That is, recall that at any round t, if the minimal cost is attained by several behaviours then the player chooses one of them uniformly at random. One way to implement this is by considering a certain order between the behaviours, and sampling a number uniformly at random $r \in [0,1]$. For instance, consider the following ordering: cooperator $>$ hypocritical $>$ defector (as we saw, in the regime of parameters we consider, a private cooperator in the two-order model never attains the minimal cost, and hence it is never considered as an option). If a player needs to choose, say, between being a cooperator or a defector, then it chooses to be a cooperator if r is in $[0, 0.5]$, and otherwise, it chooses to be a defector. This means that given a sequence of random numbers $\{r_i\}_{i=1}^{\infty}$, where $r_i \in [0,1]$, the behaviour of a player is deterministically described by the behaviours of its neighbours at each round.

Consider a fixed sequence of random numbers $\{r_i\}_{i=1}^{\infty}$, where $r_i \in [0,1]$. Let $\mathcal{C}_0$ be an initial configuration in G, and let $\mathcal{C}_t$ denote the configuration $\mathcal{C}_0$ at round t, with the costs defined according to the two-order model on G, and using the sequence $\{r_i\}_{i=1}^{\infty}$ to break ties if necessary. Let $\mathcal{C}'_0 = f(\mathcal{C}_0)$ be the mapped configuration on G', and let $\mathcal{C}'_t$ be the corresponding configuration at round $t \geq 1$, with the costs defined according to the parameters mentioned in Eq. (5.5), and using the same sequence $\{r_i\}_{i=1}^{\infty}$ to break ties if necessary.

Claim 5.14. *For every $t \geq 1$, we have*

$$\mathcal{C}'_t = \mathcal{C}_t,$$

Proof. Let us prove this claim by induction. By definition, the claim holds for $t = 0$. Assume that it holds for some integer $t \geq 0$. By the induction hypothesis, for every player u, the set $\Delta_2(u)$ in G is equal to $\Delta_{\bar{d}}(u)$ in G'. Hence, with our definitions of E_h, ρ_h and ρ_d in Eq. (5.5), we argue that the cost in $\mathcal{C}'_t$ is always $\frac{1}{\alpha_1+\alpha_2}$ times the cost in $\mathcal{C}_t$, as made clear in the following table.

	Cooperator	Hypocritical	Defector
Cost in $\mathcal{C}_t$	$\alpha_1+\alpha_2$	$\alpha_2 + \Delta_2(u)\cdot\beta_1$	$\Delta_2(u)\cdot(\beta_1+\beta_2)$
Cost in $\mathcal{C}'_t$	1 $= \frac{1}{\alpha_1+\alpha_2}\cdot(\alpha_1+\alpha_2)$	$E_h + \Delta_{\bar{d}}(u)\cdot\rho_h$ $= \frac{\alpha_2}{\alpha_1+\alpha_2} + \Delta_2(u)\frac{\beta_1}{\alpha_1+\alpha_2}$	$\Delta_{\bar{d}}(u)\cdot\rho_d$ $= \Delta_2(u)\cdot\frac{\beta_1+\beta_2}{\alpha_1+\alpha_2}$

Moreover, recall that no player in G ever chooses to be a private cooperator in rounds $t \geq 1$. Hence, the behaviour that minimizes the cost in G is the same as in G'. It follows that at round $t+1$, all players choose the same behaviour in G as they would in G'. ☐

Next, we prove that with our choices of E_h, ρ_h and ρ_d i Eq. (5.5), Conditions *(i)* and *(ii)* in Theorem 5.13 imply Conditions *(i)* and *(ii)* in Theorem 5.6:

$$\begin{aligned}\alpha_2 < \beta_2 &\iff \alpha_2 + \beta_1 < \beta_1 + \beta_2 \\ &\iff \frac{\alpha_2}{\alpha_1 + \alpha_2} + \frac{\beta_1}{\alpha_1 + \alpha_2} < \frac{\beta_1 + \beta_2}{\alpha_1 + \alpha_2} \\ &\iff E_h + \rho_h < \rho_d,\end{aligned}$$

and

$$\begin{aligned}\Delta \cdot \beta_1 > \alpha_1 &\iff \alpha_2 + \Delta \cdot \beta_1 > \alpha_1 + \alpha_2 \\ &\iff \frac{\alpha_2}{\alpha_1 + \alpha_2} + \Delta \cdot \frac{\beta_1}{\alpha_1 + \alpha_2} > 1 \\ &\iff E_h + \rho_h \cdot \Delta > 1.\end{aligned}$$

Hence, we can apply Theorem 5.6 to the mapped process on G'. It follows that in the number of rounds and probability guarantees as stated in Theorem 5.6, G' converges to the configuration in which all players are cooperators, and remains in that configuration forever. By Claim 5.14, this holds for G as well, concluding the proof of Theorem 5.13. □

5.6 Discussion

We propose a simple idealized network model that demonstrates how cooperation can emerge, even when the MPCR is zero, and even when the extent of social-pressure is low. Our results highlight the possible social role that might be played by hypocritical behaviour in escaping the tragedy-of-the-commons. The main finding is that setting the level of social-pressure towards this behaviour to be at a specific intermediate range allows to quickly transform an almost completely defective system into a fully cooperative one. Our model, like any model, neglects many of the real-life complexity parameters. Nevertheless, the insight we discovered sheds new light on the possibility of emergent cooperation. In particular, our results suggest that those who wish to influence others in the context of environmental preservation should rethink their relation to their hypocritical acquaintants.

Chapter 6

Conclusion

6.1 Summary

Information Dissemination in Stochastic Environments. In Part I of this thesis, we tried to understand how a group of cooperative entities can achieve efficient information spread, in order to eventually reach agreement. We focused on scenarios featuring noisy, constrained communications, and lacking global organization.

In Chapter 2, we considered the bit-dissemination problem, which aims to capture the challenge of information spread by requiring the group to converge on a specific *correct* opinion, held by a single source agent during the whole execution. Here, communications take the form of random interactions, where, following the passive communication assumption, only a single bit corresponding to the opinion of the agents can be revealed. The lack of global organization is modelled by the self-stabilizing framework, in which protocols must converge regardless of the initial configuration.

First, we proposed the *Follow the Trend* strategy (Algorithm 2). Informally, it consists in comparing the number of times each opinion is observed in two consecutive rounds, and then adopting the opinion for which this number grows by the maximal amount – even if it represents only a minority of the samples. We showed that when activations happen in parallel, this protocol can solve the bit-dissemination problem in at most $O(\log^{5/2} n)$ rounds in expectation, as long as the sample size is at least $c \log n$ for a sufficiently large constant c (Theorem 2.2). We did not focus on optimizing the bound on the running time, and hence, it could probably be reduced by a tighter analysis. Empirically, we observed that Algorithm 2 can be adapted to solve the bit-dissemination problem in several other settings: the sequential setting, in which only one agent is activated at a time, and the case where more than two opinions are available to the agents. Overall, our simulations hint at the robustness of the strategy, although rigorous analysis remains for future work.

Then, we investigated memory-less protocols in the sequential setting. We showed that their expected running time is always $\Omega(n)$ parallel rounds, even in the more permissive setting where the activated agents can see all other opinions, which is equivalent to having an unbounded sample size (Theorem 2.3). By adapting classical arguments to the bit-dissemination problem, we proved that this lower bound is reached (up to a $\log n$ factor) by the voter dynamics (Theorem 2.4). Finally, we showed that the lower bound does not hold in the parallel setting, where the memory-less *minority* dynamics (Algorithm 5) can achieve fast convergence (Section 2.4.4). Identifying the minimal sample size for which Algorithm 5 converges fast is left as an open problem.

In Chapter 3, we shifted from a finite opinion space to a continuous one. Here, the opinion (or *position*) of an agent aims to represent its orientation in a navigation context, or its internal notion of time in a clock synchronization context. The goal of the group is, once again, to reach and maintain agreement. More precisely, each agent is required to minimize its stretch, i.e., its distance from the average position (or *center of mass*). Communications take the form of a noisy measurement of the stretch, obtained by every agent in every round. Global organization is prevented by a random drift perturbing all positions in each round; and by the fact that agents can never estimate their actual position, but only their relative distance to the center of mass.

We quantified the performance of a simple class of protocols, called weighted-average protocols, in which agents move to a linear combination of their current position and the latest estimate of the center of mass (Theorem 3.5). We identified the optimal weighted-average protocol, Algorithm 9. To check whether costs can be decreased further by allowing more communications, we investigated the permissive centralized setting, where all measurements are publicly available. We showed that, even in the centralized setting, no algorithm can perform better than Algorithm 9 (Theorem 3.10).

Finally, we considered an other cost measure, the drift of the center of mass over time. We identified a protocol, Algorithm 8, achieving the smallest possible drift (Theorem 3.14), while being also optimal in the centralized setting for our main cost measure (Theorem 3.9). In addition, we showed that protocols that are not allowed to share measurements (including Algorithm 9) cannot achieve an optimal drift (Claim 3.15). Solving the problem for yet another possible cost measure, the diameter of the group, remains for future work.

Cooperation in Competitive Environments. In Part II, we tried to identify the conditions for cooperation to emerge in various game-theoretic settings.

In Chapter 4, we considered resource availability (or, equivalently, individual capabilities) as a parameter affecting cooperation. Specifically, a game is said to feature a reverse-correlation (RC) phenomenon when increasing resource availability leads to less cooperation, to such an extent that the payoff of every player, evaluated at equilibrium configurations, is decreased. We demonstrated the occurrence of RC phenomenons in two different games.

Our first game is a natural generalization of a classical Producer-Scrounger model studied in the context of foraging [82]. To illustrate the model, consider a group of animals gathering fruit from trees. The trees have both low-hanging fruit, which can be easily accessed by both producers and scroungers, and high-hanging fruit, reachable only by producers. When an animal picks fruit from a tree, it can directly consume a certain fraction of it while the remaining falls to the ground. The fallen fruits are shared between the animal that caused them to fall and the scroungers, who specialize in scanning the ground instead of investing efforts in foraging high-hanging fruit. We showed (Theorem 4.2) that when the abundance of low-hanging fruit increases, the tendency to scrounge can become so extreme that it ultimately results in reduced food consumption.

Moreover, we exhibited an RC phenomenon in social settings by considering a group of workers in a company collaborating towards a common goal (e.g., a team of researchers aiming to write a paper together). Similarly to the foraging example, we distinguish between two strategies: a producer that invests more efforts to increase its expected production, and a scrounger that does not invest this extra effort, and hence produces less. We showed (Theorem 4.3) that under certain plausible conditions, improving the production capability of all workers (e.g., by replacing them with more skilled workers, or by improving working condition) can result in steady-state configurations with degraded overall production, as well as smaller payoffs for workers.

In Chapter 5, we considered a spatial public good game played on a network. In this model, in addition to the two typical stereotyped behaviours – cooperation and defection –, a third one is allowed, that we refer to as *hypocrisy*. Hypocritical players pretend to be cooperative, but avoid energetic costs by hardly contributing to the social welfare. Assuming negligible MPCR, we studied how the amount of social-pressure directed towards hypocritical players affects cooperation.

More precisely, we considered that seemingly cooperative individuals (cooperators and hypocritical players) induce a mild social pressure on defectors and hypocritical players. According to our main result (Theorem 5.1), starting from a network composed almost exclusively of defectors, a fully cooperative configuration can be reached only when the amount of social-pressure induced on hypocritical neighbours is set within some interval below the one employed against defectors.

To give intuition on why employing a high social-pressure against hypocrisy is eventually detrimental to the system, we demonstrated a similar result on a equivalent model called the two-order model (Theorem 5.2). Here, behaviours are described by specifying their degree of cooperation at the *first-order* (i.e., their direct contribution to the common good) and at the *second-order* (i.e., the extent to which they punish their defecting neighbours).

6.2 Perspectives

Finally, in this section, we discuss potential future research directions.

Bringing our consensus problems together. Although the models considered in Chapters 2 and 3 share similar objectives, they have otherwise very few features in common. It would be interesting to see if their respective results still hold when mixing the settings a little bit.

One possibility is to design a variant of the bit-dissemination problem with a continuous opinion space, e.g., $\mathcal{Y} = [0,1]$. For example, this could represent an alignment problem in which one individual knows the preferable direction. It is unclear whether the lower bound on memory-less dynamics (Theorem 2.3) can be adapted to this case.

Conversely, the stretch measurement in Chapter 3 can be replaced by noisy samples of the relative position w.r.t. a random subset of agents, in order to mimic the interaction pattern of the $\mathcal{PULL}$ model. An estimation of the stretch can be computed from these samples, and one could verify the optimality of weighted-average algorithms in this slightly different setting.

The bit-dissemination problem with bounded sample size. Among the possible continuation works of Chapter 2 (as already listed in the Discussion and Future Work section or in Section 6.1), the most challenging one in my opinion is the case $\ell = O(1)$. Specifically, the goal is to identify an algorithm achieving a convergence time poly-logarithmic in n for the bit-dissemination problem, while relying only on a *fixed* number of samples (with respect to n), in contrast to Algorithm 2 for which $\ell = \Theta(\log n)$. It is highly unlikely that Algorithm 2 converges with such a small sample size, without modifications. A possible modification is to memorize the $\ell = O(1)$ samples over $\log n$ rounds, and then apply the original protocol on the resulting set. However, bounding the running time in this case appears to be beyond the reach of currently known techniques. Some algorithms mentioned in the supplementary information of [22] are also empirically promising, but similarly difficult to analyse.

Synchronous oscillations. While traditional consensus problems aim for stability, in some scenarios the system can benefit from alternating between different configurations [33]. However, this must be done without compromising the typical requirements: namely, maintaining a high degree of consensus for the majority of the time, and incorporating local information into the decision-making process. For example, in the case of cooperative transport in ants, individuals might have conflicting opinions about the location of the nest, that the group need to try consecutively to reach the destination [15]. To maximise efficiency, a possible strategy would be to spend more time exploring the options that have more support among group members. Several distributed systems exhibiting periodic behaviours have already been investigated, such as firefly swarms or neural networks. Yet, to the best of my knowledge, few of these systems attempt to integrate individual preferences into the final periodic pattern. A possible research direction would be to consider variants of the bit-dissemination problem in which the system must alternate between different consensus configurations, depending on arbitrary internal preferences.

Towards more general communication graphs. In Part I of this thesis, it is always assumed that the underlying communication network is fully connected, and more generally, that all pairs of agents are equally likely to interact. This somewhat unrealistic assumption is hard to relax, as even simple dynamics become analytically intractable when the communication graph is arbitrary. A first step towards discarding this assumption is to consider random Erdős–Rényi graphs $\mathcal{G}(n,p)$. Indeed, when p is large enough, random communications are expected to behave similarly on $\mathcal{G}(n,p)$ and on a clique, and hence one can hope to adapt existing analyses. Although some recent works already managed to study simple decision processes on random graphs [80], many simple and natural dynamics, such as 3-majority, remain to be analysed in this setting. Successfully bounding their convergence time for relatively sparse random graphs, like $\mathcal{G}(n,p)$ with $p = \Theta(\text{polylog } n)$, would pave the way to extending the results of Chapter 2 and improving their generality.

Looking for empirical support. Finally, while this thesis aimed to obtain concrete biological insights, no connection of its theoretical results with empirical data was made so far. Which biological entities actually "Follow the Trend"? Which change their alignment by performing a weighted-average depending on the amount of noise in their environment? Is it possible to enhance production by decreasing resource availability in the real world? The execution of an experiment aligned with one of our models, or the refinement of our findings to suit more practical scenarios, are definitely interesting directions for the future.

Promising experimental settings to apply the insights of Chapter 2 include fish schooling [49, 48, 145], collective sequential decision making in ants [15], and recruitment in ants [134, 131]. Regarding Chapter 4, the first step would be to try to create a reverse-correlation phenomenon in a controlled setting, such as a social experiment with humans, possibly on the internet. If successful, the next step would be to try to observe it in nature, for instance in a social foraging context.

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Appendix

Appendix A

Probabilistic Tools

A.1 Some well-known theorems

Theorem A.1. *[Multiplicative Chernoff's Bound] Let $X_1, \ldots, X_n$ be independent binary random variables, let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}(X)$. Then it holds for all $\delta > 0$ that*

$$\mathbb{P}\left(X \geq (1+\delta)\mu\right) \leq \exp\left(-\min\{\delta, \delta^2\} \cdot \frac{\mu}{3}\right),$$

and for all $0 < \epsilon < 1$,

$$\mathbb{P}\left(X \leq (1-\epsilon)\mu\right) \leq \exp\left(-\epsilon^2 \cdot \frac{\mu}{2}\right).$$

Theorem A.2. *[Hœffding's bound] Let $X_1, \ldots, X_n$ be independent random variables such that for every $1 \leq i \leq n$, $a_i \leq X_i \leq b_i$ almost surely. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}(X)$. Then it holds for all $\delta > 0$ that*

$$\mathbb{P}\left(X - \mu \geq \delta\right) \leq \exp\left(-\frac{2\delta^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$

Theorem A.3. *[Central Limit] Let $X_1, \ldots, X_n$ be i.i.d. random variables with $\mathbb{E}(X_1) = \mu$ and $\mathrm{Var}(X_1) = \sigma^2 < +\infty$. As n tends to infinity, the random variables $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n X_i - \mu\right)$ converges in distribution to $\mathcal{M}(0, \sigma^2)$.*

Let Φ be the cumulative distribution function (c.d.f.) of the standard normal distribution:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

Theorem A.4. *[Berry-Esseen] Let $X_1, \ldots, X_n$ be i.i.d. random variables, with $\mathbb{E}(X_1) = 0$, $\mathrm{Var}(X_1) = \mathbb{E}(X_1^2) > 0$, and $\mathbb{E}(|X_1|^3) = \rho < +\infty$. Let $X = \sum_{i=1}^n X_i$ and F be the c.d.f. of $X/(\sigma\sqrt{n})$. Then for, e.g., $C = 0.4748$, it holds that*

$$|F(x) - \Phi(x)| \leq \frac{C\rho}{\sigma^3\sqrt{n}},$$

A.2 Technical Results

A.2.1 Competition Between Coins

Consider two coins such that one coin has a greater probability of yielding "heads", and toss them k times each.

A.2.1.1 Lower bounds on the probability that the best coin wins

In Lemmas A.5 and A.6, we aim to lower bound the probability that the more likely coin yields more "heads", or in other words, we lower bound the probability that the favorite coin wins. Lemma A.5 is particularly effective when the difference between p and q is sufficiently large. Its proof is based on a simple application of Hœffding's inequality.

Lemma A.5. *For every $p, q \in [0,1]$ s.t. $p < q$ and every integer k, we have*

$$\mathbb{P}\left(B_k\left(p\right) < B_k\left(q\right)\right) \geq 1 - \exp\left(-\frac{1}{2}k(q-p)^2\right).$$

Proof. Let Y_i, $i \in \{1, \ldots, k\}$ be i.i.d. random variables with

$$Y_i = \begin{cases} 1 & \text{w.p. } p(1-q), \\ 0 & \text{w.p. } pq + (1-p)(1-q), \\ -1 & \text{w.p. } (1-p)q. \end{cases}$$

Then

$$\mathbb{P}\left(B_k\left(p\right) \geq B_k\left(q\right)\right) = \mathbb{P}\left(\sum_{i=1}^{k} Y_i \geq 0\right) = \mathbb{P}\left(\frac{1}{k}\sum_{i=1}^{k}\left(Y_i - (p-q)\right) \geq (q-p)\right).$$

Since each Y_i is bounded, and $\mathbb{E}(Y_i) = (p-q)$, we can apply Hœffding's inequality (Theorem A.2) to get

$$\mathbb{P}\left(B_k\left(p\right) \geq B_k\left(q\right)\right) \leq \exp\left(-\frac{2k^2(q-p)^2}{4k}\right) = \exp\left(-\frac{1}{2}k(q-p)^2\right).$$

□

Lemma A.5 is not particularly effective when p and q are close to each other. For such cases, we shall use the following lemma.

Lemma A.6. *Let $\lambda > 0$. There exist $\epsilon = \epsilon(\lambda)$ and $K = K(\lambda)$, s.t. for every $p, q \in [1/2 - \epsilon, 1/2 + \epsilon]$ with $p < q$, and every $k > K$,*

$$\mathbb{P}\left(B_k\left(p\right) < B_k\left(q\right)\right) > \frac{1}{2} + \lambda \cdot (q-p) - \frac{1}{2}\mathbb{P}\left(B_k(p) = B_k(q)\right).$$

Proof. For every $p, q \in [0,1]$, we have

$$\begin{aligned}
&\mathbb{P}\left(B_k\left(q\right) < B_k\left(p\right)\right) \\
&= \sum_{d=1}^{k} \mathbb{P}\left(B_k(q) = B_k(p) - d\right) \\
&= \sum_{d=1}^{k} \mathbb{P}\left(|B_k(q) - B_k(p)| = d\right) \cdot \frac{\mathbb{P}\left(B_k(q) = B_k(p) - d\right)}{\mathbb{P}\left(|B_k(q) - B_k(p)| = d\right)} \\
&= \sum_{d=1}^{k} \mathbb{P}\left(|B_k(q) - B_k(p)| = d\right) \cdot \frac{\mathbb{P}\left(B_k(q) = B_k(p) - d\right)}{\mathbb{P}\left(B_k(q) = B_k(p) - d\right) + \mathbb{P}\left(B_k(p) = B_k(q) - d\right)},
\end{aligned}$$

so

$$\begin{aligned}
&\mathbb{P}\left(B_k\left(p\right) < B_k\left(q\right)\right) - \mathbb{P}\left(B_k\left(q\right) < B_k\left(p\right)\right) \\
&\quad = \sum_{d=1}^{k} \mathbb{P}\left(\left|B_k(q) - B_k(p)\right| = d\right) \cdot \frac{\mathbb{P}\left(B_k(p) = B_k(q) - d\right) - \mathbb{P}\left(B_k(q) = B_k(p) - d\right)}{\mathbb{P}\left(B_k(p) = B_k(q) - d\right) + \mathbb{P}\left(B_k(q) = B_k(p) - d\right)}.
\end{aligned} \tag{A.1}$$

Let us compute $\mathbb{P}\left(B_k(q) = B_k(p) - d\right)$:

$$\begin{aligned}
\mathbb{P}\left(B_k(q) = B_k(p) - d\right) &= \sum_{i=0}^{k-d} \mathbb{P}\left(B_k(q) = i\right) \cdot \mathbb{P}\left(B_k(p) = i + d\right) \\
&= \sum_{i=0}^{k-d} \binom{k}{i} \binom{k}{i+d} q^i (1-q)^{k-i} p^{i+d} (1-p)^{k-i-d} \\
&= \left(p(1-q)\right)^d \sum_{i=0}^{k-d} \binom{k}{i} \binom{k}{i+d} (qp)^i \left((1-q)(1-p)\right)^{k-i-d} \\
&:= \left(p(1-q)\right)^d A_{k,d,p,q},
\end{aligned}$$

where

$$A_{k,d,p,q} := \sum_{i=0}^{k-d} \binom{k}{i} \binom{k}{i+d} (qp)^i \left((1-q)(1-p)\right)^{k-i-d}.$$

Since $A_{k,d,p,q}$ is symmetric w.r.t. p, q, i.e., $A_{k,d,p,q} = A_{k,d,q,p}$, we can simplify Eq. (A.1) as

$$\begin{aligned}
&\mathbb{P}\left(B_k(p) < B_k(q)\right) - \mathbb{P}\left(B_k(q) < B_k(p)\right) \\
&\qquad\qquad = \sum_{d=1}^{k} \mathbb{P}\left(\left|B_k(q) - B_k(p)\right| = d\right) \cdot \frac{\left(q(1-p)\right)^d - \left(p(1-q)\right)^d}{\left(q(1-p)\right)^d + \left(p(1-q)\right)^d}.
\end{aligned} \tag{A.2}$$

Intuitively, the quantity

$$\frac{\left(q(1-p)\right)^d - \left(p(1-q)\right)^d}{\left(q(1-p)\right)^d + \left(p(1-q)\right)^d}$$

can be seen as the "advantage" given by playing with the better coin (q) in a k-coin-tossing contest, knowing that one coin hit "head" d times more than the other. Before we continue, we need the following simple claim.

Claim A.7. *For every $a, b \in [0, 1]$ with $a > b$, the sequence*

$$\left(\frac{a^n - b^n}{a^n + b^n}\right), n \in \mathbb{N}$$

is increasing in n.

Proof. Rewrite

$$\frac{a^n - b^n}{a^n + b^n} = \frac{2a^n}{a^n + b^n} - 1 = 2 \cdot \frac{1}{1 + (b/a)^n} - 1,$$

and notice that, since $a > b$, $((b/a)^n)$, $n \in \mathbb{N}$ is a decreasing sequence. □

Claim A.8. *Let $0 < \gamma < 1$ and $d \in \mathbb{N}$. There exists $\epsilon = \epsilon(\gamma, d)$, such that for every $p, q \in [1/2 - \epsilon, 1/2 + \epsilon]$ with $p < q$,*

$$\frac{\left(q(1-p)\right)^d - \left(p(1-q)\right)^d}{\left(q(1-p)\right)^d + \left(p(1-q)\right)^d} > (q - p) \cdot 2d\gamma.$$

Proof. First, by a telescopic argument:

$$(q(1-p))^d - (p(1-q))^d = (q(1-p) - p(1-q)) \sum_{i=0}^{d-1} (q(1-p))^{d-1-i} \cdot (p(1-q))^i$$
$$= (q-p) \sum_{i=0}^{d-1} (q(1-p))^{d-1-i} \cdot (p(1-q))^i .$$

Note that

$$\lim_{p,q \to 1/2} \sum_{i=0}^{d-1} (q(1-p))^{d-1-i} \cdot (p(1-q))^i = \sum_{i=0}^{d-1} \left(\frac{1}{4}\right)^{d-1-i} \cdot \left(\frac{1}{4}\right)^i = d \cdot \left(\frac{1}{2}\right)^{2d-2},$$

and that

$$\lim_{p,q \to 1/2} (q(1-p))^d + (p(1-q))^d = \left(\frac{1}{4}\right)^d + \left(\frac{1}{4}\right)^d = \left(\frac{1}{2}\right)^{2d-1}.$$

Hence, since $\gamma < 1$, and provided that p, q are close enough to $1/2$, we obtain

$$\frac{(q(1-p))^d - (p(1-q))^d}{(q(1-p))^d + (p(1-q))^d} > (q-p) \cdot 2d\gamma,$$

which completes the proof of Claim A.8. □

Next, let $\lambda > 0$ as in the Lemma's statement, and let $\lambda' = \lambda + 1$. Denote $D = \lceil \lambda' \rceil + 1 > \lambda'$ and $\gamma = \lambda'/D < 1$. By Claim A.8, there exists $\epsilon = \epsilon(\gamma, D) = \epsilon(\lambda)$, s.t. for $p, q \in [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$,

$$\frac{(q(1-p))^D - (p(1-q))^D}{(q(1-p))^D + (p(1-q))^D} > (q-p) \cdot 2\lambda'. \tag{A.3}$$

Now we derive a lower bound on Eq. (A.2):

$$\begin{aligned}
&\mathbb{P}(B_k(p) < B_k(q)) - \mathbb{P}(B_k(q) < B_k(p)) \\
&\quad = \sum_{d=1}^{k} \mathbb{P}(|B_k(q) - B_k(p)| = d) \cdot \frac{(q(1-p))^d - (p(1-q))^d}{(q(1-p))^d + (p(1-q))^d} && \text{(Eq. (A.2))} \\
&\quad \geq \sum_{d=D}^{k} \mathbb{P}(|B_k(q) - B_k(p)| = d) \cdot \frac{(q(1-p))^d - (p(1-q))^d}{(q(1-p))^d + (p(1-q))^d} \\
&\quad \geq \sum_{d=D}^{k} \mathbb{P}(|B_k(q) - B_k(p)| = d) \cdot \frac{(q(1-p))^D - (p(1-q))^D}{(q(1-p))^D + (p(1-q))^D} && \text{(by Claim A.7)} \\
&\quad > (q-p) \cdot 2\lambda' \sum_{d=D}^{k} \mathbb{P}(|B_k(q) - B_k(p)| = d) && \text{(by Eq. (A.3))} \\
&\quad = (q-p) \cdot 2\lambda' \cdot (1 - \mathbb{P}(|B_k(q) - B_k(p)| < D)).
\end{aligned}$$

Since $\lambda' > \lambda$, and since $\mathbb{P}(|B_k(q) - B_k(p)| < D)$ tends to 0 as k tends to $+\infty$, there exists $K = K(\lambda)$ s.t. for all $k > K$,

$$\mathbb{P}(B_k(p) < B_k(q)) - \mathbb{P}(B_k(q) < B_k(p)) > (q-p) \cdot 2\lambda. \tag{A.4}$$

Eventually, we write

$$\begin{aligned}\mathbb{P}\left(B_k(p) < B_k(q)\right) &= 1 - \mathbb{P}\left(B_k(q) < B_k(p)\right) - \mathbb{P}\left(B_k(p) = B_k(q)\right) \\ &> 1 - \mathbb{P}\left(B_k(p) < B_k(q)\right) + 2\lambda(q-p) - \mathbb{P}\left(B_k(p) = B_k(q)\right). \quad \text{(Eq. (A.4))}\end{aligned}$$

Hence,

$$\mathbb{P}\left(B_k(p) < B_k(q)\right) > \frac{1}{2} + \lambda(q-p) - \frac{1}{2}\mathbb{P}\left(B_k(p) = B_k(q)\right),$$

which concludes the proof of Lemma A.6. □

A.2.1.2 Lower bounds on the probability that the worse coin wins

We now deal with the opposite problem, that is, to lower bound the probability that the underdog coin wins. Formally,

Lemma A.9. *For every $p, q \in [0,1]$ s.t. $p < q$ and every integer k, we have*

$$\mathbb{P}\left(B_k(p) > B_k(q)\right) \geq 1 - \Phi\left(\frac{\sqrt{k}(q-p)}{\sigma}\right) - \frac{C}{\sigma\sqrt{k}},$$

where $C = 0.4748$ and $\sigma = \sqrt{p(1-p) + q(1-q)}$.

Proof. Let Y_i, $i \in \{1, \ldots, k\}$ be i.i.d. random variables with

$$Y_i = \begin{cases} 1 & \text{w.p. } p(1-q), \\ 0 & \text{w.p. } pq + (1-p)(1-q), \\ -1 & \text{w.p. } (1-p)q. \end{cases}$$

Let $\mu = \mathbb{E}(Y_1)$, $\sigma = \sqrt{\mathrm{Var}(Y_1)}$, and $\rho = \mathbb{E}(|Y_1 - \mu|^3)$. Writing the definitions and simplifying, we obtain

$$\mu = p - q, \quad \sigma^2 = p(1-p) + q(1-q), \quad \rho = (2p^3 - 3p^2 + p) - (2q^3 - 3q^2 + q). \tag{A.5}$$

We have

$$\begin{aligned}\mathbb{P}\left(B_k(p) > B_k(q)\right) = \mathbb{P}\left(\sum_{i=1}^{k} Y_i > 0\right) &= \mathbb{P}\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}(Y_i - (p-q)) > \sqrt{k}(q-p)\right) \\ &= \mathbb{P}\left(\frac{1}{\sigma\sqrt{k}}\sum_{i=1}^{k}(Y_i - (p-q)) > \frac{\sqrt{k}(q-p)}{\sigma}\right) \\ &= \mathbb{P}\left(Z > \frac{\sqrt{k}(q-p)}{\sigma}\right),\end{aligned}$$

where

$$Z = \frac{1}{\sigma\sqrt{k}}\sum_{i=1}^{k}(Y_i - (p-q)).$$

By the Berry-Esseen theorem (Theorem A.4),

$$\left|\mathbb{P}\left(Z < \frac{\sqrt{k}(q-p)}{\sigma}\right) - \Phi\left(\frac{\sqrt{k}(q-p)}{\sigma}\right)\right| < \frac{C\rho}{\sigma^3\sqrt{k}},$$

implying that

$$\left| \left(1 - \Phi\left(\frac{\sqrt{k}(q-p)}{\sigma}\right)\right) - \mathbb{P}\left(Z > \frac{\sqrt{k}(q-p)}{\sigma}\right) \right| < \frac{C\rho}{\sigma^3\sqrt{k}},$$

and so

$$\mathbb{P}\left(Z > \frac{\sqrt{k}(q-p)}{\sigma}\right) > 1 - \Phi\left(\frac{\sqrt{k}(q-p)}{\sigma}\right) - \frac{C\rho}{\sigma^3\sqrt{k}},$$

where, e.g., $C = 0.4748$.

Claim A.10. *We have that $\rho < \sigma^2$.*

Proof. Let $f(p) = 2p^3 - p/2$ and $g(p) = 1/4 - p^2$. We start by proving that for every $p \in [-1/2, 1/2]$, $|f(p)| \le g(p)$. Since f is anti-symmetric, $|f|$ is symmetric, and g is symmetric, we can restrict the analysis to $[0, 1/2]$. On this interval, $|f(p)| = p/2 - 2p^3$, and

$$g(p) - |f(p)| = \frac{1}{4} - \frac{p}{2} - p^2 + 2p^3 = \frac{1}{4}(1-2p)^2(1+2p) \ge 0.$$

We can rewrite Eq. (A.5) as

$$\sigma^2 = g(p + \tfrac{1}{2}) + g(q + \tfrac{1}{2}) \quad \text{and} \quad \rho = f(p + \tfrac{1}{2}) + f(q + \tfrac{1}{2}).$$

Therefore,

$$|\rho| \le |f(p + \tfrac{1}{2})| + |f(q + \tfrac{1}{2})| \le g(p + \tfrac{1}{2}) + g(q + \tfrac{1}{2}) = \sigma^2,$$

which concludes the proof of Claim A.10. □

By Claim A.10, we end up with

$$\mathbb{P}\left(B_k(p) > B_k(q)\right) \ge 1 - \Phi\left(\frac{\sqrt{k}(q-p)}{\sigma}\right) - \frac{C}{\sigma\sqrt{k}},$$

which concludes the proof of Lemma A.9. □

Just as Lemma A.6 was a version of Lemma A.5 optimized for cases where p and q are close to each other, Lemma A.11 complements Lemma A.9 in such situations.

Lemma A.11. *There exists a constant $\alpha > 1$, s.t. for every integer k, every $p, q \in [1/3, 2/3]$ with $p < q$ and $q - p \le 1/\sqrt{k}$, we have*

$$\mathbb{P}\left(B_k(p) < B_k(q)\right) < \frac{1}{2} + \alpha(q-p)\sqrt{k} - \frac{1}{2}\mathbb{P}\left(B_k(p) = B_k(q)\right).$$

Proof. Recall that (see Eq. (A.2) in the proof of Lemma A.6):

$$\begin{aligned}\mathbb{P}\left(B_k(p) < B_k(q)\right) &- \mathbb{P}\left(B_k(q) < B_k(p)\right) \\ &= \sum_{d=1}^{k} \mathbb{P}\left(|B_k(q) - B_k(p)| = d\right) \cdot \frac{(q(1-p))^d - (p(1-q))^d}{(q(1-p))^d + (p(1-q))^d}. \qquad \text{(A.6)}\end{aligned}$$

The following claim is analogous to Claim A.8, but this time we are looking for an upper bound (instead of a lower bound) on the same quantity.

Claim A.12. *There exists a constant $\alpha > 1$, s.t. for every integer k, every $p, q \in [1/3, 2/3]$ with $p < q$, and all $d \in \mathbb{N}$,*

$$\frac{(q(1-p))^d - (p(1-q))^d}{(q(1-p))^d + (p(1-q))^d} < \alpha d \cdot (q-p).$$

Proof. As in the proof of Claim A.8, we have

$$\begin{aligned}
(q(1-p))^d - (p(1-q))^d &= (q(1-p) - p(1-q)) \sum_{i=0}^{d-1} (q(1-p))^{d-1-i} \cdot (p(1-q))^i \\
&= (q-p) \sum_{i=0}^{d-1} (q(1-p))^{d-1-i} \cdot (p(1-q))^i \\
&\leq d \cdot (q-p)\, (q(1-p))^{d-1} \\
&\leq \alpha d \cdot (q-p)\, (q(1-p))^d,
\end{aligned}$$

where α is any upper bound on $1/(q(1-p))$, e.g., $\alpha = 9$. Hence,

$$\frac{(q(1-p))^d - (p(1-q))^d}{(q(1-p))^d + (p(1-q))^d} \leq \alpha d \cdot (q-p) \cdot \frac{(q(1-p))^d}{(q(1-p))^d + (p(1-q))^d} \leq \alpha d \cdot (q-p),$$

which concludes the proof of Claim A.12. □

Using Claim A.12 on Eq. (A.6), we obtain

$$\mathbb{P}\left(B_k(p) < B_k(q)\right) - \mathbb{P}\left(B_k(q) < B_k(p)\right) \leq \alpha \cdot (q-p) \sum_{d=1}^{k} d \cdot \mathbb{P}\left(|B_k(q) - B_k(p)| = d\right). \quad \text{(A.7)}$$

Claim A.13. *For every $p, q \in [1/3, 2/3]$ with $p < q$, and every integer k,*

$$\mathbb{E}\left(|B_k(p) - B_k(q)|\right) \leq \sqrt{2kq(1-q)} + k \cdot (q-p).$$

Proof. For $i \in \{1, \dots, k\}$, let $X_i^{(1)}, X_i^{(2)} \sim \mathcal{B}(q)$ and $Y_i \sim \mathcal{B}(1 - p/q)$ be independent random variables. Let

$$X^{(1)} = \sum_{i=1}^{k} X_i^{(1)}, \qquad X^{(2)} = \sum_{i=1}^{k} X_i^{(2)}, \qquad Z = \sum_{i=1}^{k} X_i^{(2)} \cdot Y_i,$$

and

$$\tilde{X}^{(2)} = \sum_{i=1}^{k} X_i^{(2)} \cdot (1 - Y_i) = X^{(2)} - Z.$$

Clearly, $X^{(1)} \sim \mathcal{B}_k(q)$ and $X^{(2)} \sim \mathcal{B}_k(q)$. Since for every i,

$$X_i^{(2)} \cdot (1 - Y_i) = \begin{cases} 1 & \text{if } X_i^{(2)} = 1 \text{ and } Y_i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain that $\tilde{X}^{(2)} \sim \mathcal{B}_k(q \cdot (1 - (1 - p/q))) = \mathcal{B}_k(p)$. Similarly, for every i,

$$X_i^{(2)} \cdot Y_i = \begin{cases} 1 & \text{if } X_i^{(2)} = 1 \text{ and } Y_i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

hence, we obtain that $Z \sim \mathcal{B}_k(q \cdot (1 - p/q)) = \mathcal{B}_k(q - p)$. We notice that $(X^{(1)}, X^{(2)})$ are independent, as well as $(X^{(1)}, \tilde{X}^{(2)})$. Hence

$$\begin{aligned}
\mathbb{E}\left(|B_k(q) - B_k(p)|\right) &= \mathbb{E}\left(|X^{(1)} - \tilde{X}^{(2)}|\right) \\
&= \mathbb{E}\left(|X^{(1)} - X^{(2)} + Z|\right) \\
&\leq \mathbb{E}\left(|X^{(1)} - X^{(2)}| + Z\right) \\
&= \mathbb{E}\left(|X^{(1)} - X^{(2)}|\right) + \mathbb{E}(Z).
\end{aligned}$$

We have $\mathbb{E}(Z) = k(q - p)$, and

$$\begin{aligned}
\mathbb{E}\left(|X^{(1)} - X^{(2)}|\right) &= \mathbb{E}\left(\sqrt{\left(X^{(1)} - X^{(2)}\right)^2}\right) \\
&\leq \sqrt{\mathbb{E}\left(\left(X^{(1)} - X^{(2)}\right)^2\right)} && \text{(Jensen inequality, } x \mapsto \sqrt{x} \text{ being concave)} \\
&= \sqrt{\mathrm{Var}\left(X^{(1)} - X^{(2)}\right)} && \text{(since } \mathbb{E}\left(X^{(1)} - X^{(2)}\right) = 0\text{)} \\
&= \sqrt{2kq(1-q)}, && (X^{(1)}, X^{(2)} \sim \mathcal{B}_k(q) \text{ are independent)}.
\end{aligned}$$

which concludes the proof of Claim A.13. □

We note that

$$\begin{aligned}
\sum_{d=1}^{k} d \cdot \mathbb{P}\left(|B_k(q) - B_k(p)| = d\right) &= \mathbb{E}\left(|B_k(q) - B_k(p)|\right) \\
&\leq \sqrt{2kq(1-q)} + k \cdot (q - p) && \text{(by Claim A.13)} \\
&\leq \sqrt{2kq(1-q)} + \sqrt{k} && \text{(since } q - p \leq 1/\sqrt{k}\text{)} \\
&\leq 2\sqrt{k}.
\end{aligned}$$

Eventually, Eq. (A.7) becomes

$$\mathbb{P}(B_k(p) < B_k(q)) - \mathbb{P}(B_k(q) < B_k(p)) \leq 2\alpha \cdot (q-p)\sqrt{k}. \tag{A.8}$$

To conclude, we write

$$\begin{aligned}\mathbb{P}(B_k(p) < B_k(q)) &= 1 - \mathbb{P}(B_k(q) < B_k(p)) - \mathbb{P}(B_k(p) = B_k(q)) \\ &< 1 - \mathbb{P}(B_k(p) < B_k(q)) + 2\alpha \cdot (q-p)\sqrt{k} - \mathbb{P}(B_k(p) = B_k(q)),\end{aligned}$$

where the inequality is by Eq. (A.8). Hence,

$$\mathbb{P}(B_k(p) < B_k(q)) < \frac{1}{2} + \alpha \cdot (q-p)\sqrt{k} - \frac{1}{2}\mathbb{P}(B_k(p) = B_k(q)),$$

which concludes the proof of Lemma A.11. □

A.2.2 Analysis of a Function

The goal of this section is to prove Lemma 2.26, which contains several property of function $g : [0,1]^2 \mapsto [0,1]$, whose definition we recall:

$$g(x,y) = \mathbb{P}(B_\ell(y) > B_\ell(x)) + y \cdot \mathbb{P}(B_\ell(y) = B_\ell(x)) + \frac{1}{n}(1 - \mathbb{P}(B_\ell(y) \geq B_\ell(x))). \tag{A.9}$$

In addition, we recall that for $x \in [0,1]$, we define $h_x : y \mapsto g(x,y) - y$.

First, we observe that the derivative, w.r.t. x, of $\mathbb{P}(B_k(x) > B_k(p))$ in the neighbourhood of p is relatively high. The following claim formalizes this idea.

Claim A.14. *There exists a constant $\beta' > 0$ such that for every k large enough, and every $p, x \in [1/3, 2/3]$ satisfying $p \leq x \leq p + 1/\sqrt{k}$,*

$$\frac{d}{dx}\mathbb{P}(B_k(x) > B_k(p)) \geq \beta' \cdot \sqrt{k}.$$

Proof. Let $h > 0$. We will proceed by using a coupling argument. Let X_i, $i \in \{1,\dots,k\}$, be i.i.d. random variables uniformly distributed over the interval $[0,1]$. Let $Y_1 = |\{i \text{ s.t. } X_i \leq x\}|$ and $Y_2 = |\{i \text{ s.t. } X_i \leq x+h\}|$. By construction, $Y_1 \sim \mathcal{B}_k(x)$ and $Y_2 \sim \mathcal{B}_k(x+h)$. Next, let $H = |\{i \text{ s.t. } x < X_i \leq x+h\}|$. By construction, $Y_2 = Y_1 + H \geq Y_1$. Let $Z \sim \mathcal{B}_k(p)$ be a binomially distributed random variable, independent from Y_1 and Y_2. Now, we have:

$$\begin{aligned}&\mathbb{P}(B_k(x+h) > B_k(p)) - \mathbb{P}(B_k(x) > B_k(p)) \\ &= \mathbb{P}(Y_2 > Z) - \mathbb{P}(Y_1 > Z) && \text{(by definition of } Y_1, Y_2 \text{ and } Z) \\ &= \mathbb{P}\left(Y_1 \leq Z \bigcap Y_2 > Z\right) && \text{(because } Y_1 > Z \Rightarrow Y_2 > Z) \\ &= \sum_{j=0}^{k} \mathbb{P}(Z = j) \cdot \mathbb{P}\left(Y_1 \leq j \bigcap Y_2 > j\right).\end{aligned}$$

Let $J = \{j \in \mathbb{N} \text{ s.t. } kp \leq j \leq kp + \sqrt{k}\}$. We can rewrite the last equation as

$$\mathbb{P}(B_k(x+h) > B_k(p)) - \mathbb{P}(B_k(x) > B_k(p)) \geq \sum_{j \in J} \mathbb{P}(Z = j) \cdot \mathbb{P}\left(Y_1 \leq j \bigcap Y_2 > j\right). \tag{A.10}$$

The following result is a well-known fact.

Observation A.15. *There exists a constant $\beta > 0$ such that for every k large enough, every $p \in [1/3, 2/3]$, and every i satisfying $|i - kp| \leq \sqrt{k}$, we have $\mathbb{P}(B_k(p) = i) \geq \frac{\beta}{\sqrt{k}}$.*

Proof. By the De Moivre-Laplace theorem, for any i in $\{kp - \sqrt{k}, \ldots, kp + \sqrt{k}\}$,

$$\mathbb{P}\left(B_k(p) = i\right) = \binom{k}{i} p^i (1-p)^{k-i} \approx \frac{1}{\sqrt{2kp(1-p)}} \exp\left(-\frac{(i-kp)^2}{2kp(1-p)}\right), \qquad \text{(A.11)}$$

where we used $\approx$ in the sense that the ratio between the left-hand side and the right-hand side tends to 1 as k tends to infinity. Since $|i - kp| \leq \sqrt{k}$,

$$\frac{1}{\sqrt{2kp(1-p)}} \exp\left(-\frac{(i-kp)^2}{2kp(1-p)}\right) \geq \frac{1}{\sqrt{2kp(1-p)}} \exp\left(-\frac{1}{2p(1-p)}\right) := \frac{f(p)}{\sqrt{k}}.$$

By Eq. (A.11), we can conclude the proof of Observation A.15 for k large enough by taking, e.g.,

$$\beta = \frac{1}{2} \cdot \min_{p \in [1/3, 2/3]} f(p).$$

□

For $j \in J$, by Observation A.15, $\mathbb{P}\left(Z = j\right) \geq \beta/\sqrt{k}$, for some constant $\beta > 0$. Moreover,

$$\begin{aligned} \mathbb{P}\left(Y_1 \leq j \bigcap Y_2 > j\right) &\geq \mathbb{P}\left(Y_1 = j \bigcap Y_2 > j\right) \\ &= \mathbb{P}\left(Y_1 = j \bigcap H \geq 1\right) \qquad \text{(because } Y_2 = Y_1 + H\text{)} \\ &= \mathbb{P}\left(Y_1 = j\right) \cdot \mathbb{P}\left(H \geq 1 \mid Y_1 = j\right). \end{aligned}$$

By the assumption in the lemma, $p \leq x \leq p + 1/\sqrt{k}$, and so $kp \leq kx \leq kp + \sqrt{k}$. Therefore, for $j \in J$, $|j - kx| \leq \sqrt{k}$, and by Observation A.15, we get that $\mathbb{P}\left(Y_1 = j\right) \geq \beta/\sqrt{k}$. Hence, we can rewrite Eq. (A.10) as

$$\mathbb{P}\left(B_k\left(x+h\right) > B_k\left(p\right)\right) - \mathbb{P}\left(B_k\left(x\right) > B_k\left(p\right)\right) \geq \frac{\beta^2}{k} \sum_{j \in J} \mathbb{P}\left(H \geq 1 \mid Y_1 = j\right). \qquad \text{(A.12)}$$

Now, let us find a lower bound on $\mathbb{P}\left(H \geq 1 \mid Y_1 = j\right)$, for $j \in J$. Note that, by definition, $Y_1 = j$ if and only if $|\{i \text{ s.t. } X_i > x\}| = k - j$. Since $X_i, 1 \leq i \leq k$, is uniformly distributed over $[0, 1]$,

$$\mathbb{P}\left(x < X_i \leq x + h \mid X_i > x\right) = \frac{h}{1-x}.$$

Therefore, for every $j \in J$

$$\mathbb{P}\left(H = 0 \mid Y_1 = j\right) = \left(1 - \frac{h}{1-x}\right)^{k-j} \leq \left(1 - \frac{h}{1-x}\right)^{k-kp-\sqrt{k}}.$$

This implies that

$$\sum_{j \in J} \mathbb{P}\left(H \geq 1 \mid Y_1 = j\right) \geq \sqrt{k} \cdot \left(1 - \left(1 - \frac{h}{1-x}\right)^{k-kp-\sqrt{k}}\right).$$

We have

$$\begin{aligned} \lim_{h \to 0} \frac{1}{h} \cdot \sum_{j \in J} \mathbb{P}\left(H \geq 1 \mid Y_1 = j\right) &\geq \lim_{h \to 0} \frac{\sqrt{k}}{h} \cdot \left(1 - \left(1 - \frac{h}{1-x}\right)^{k-kp-\sqrt{k}}\right) \\ &= \frac{\sqrt{k}\left(k - kp - \sqrt{k}\right)}{1-x}. \end{aligned}$$

Eventually, we get from Eq. (A.12)

$$\frac{d}{dx}\mathbb{P}\left(B_k\left(x\right) > B_k\left(p\right)\right) \geq \frac{\beta^2(1-p)}{1-x}\cdot\sqrt{k}+o_k\left(\sqrt{k}\right).$$

We can conclude the proof of Claim A.14 for k large enough by taking, e.g., $\beta' = \frac{\beta^2(1-p)}{2(1-x)}$. □

Now, we are ready to start proving Lemma 2.26.

Proof of (i) in Lemma 2.26. We can rewrite Eq. (A.9) as

$$g(x,y) = \left(y-\frac{1}{n}\right)\mathbb{P}\left(B_\ell\left(y\right) \geq B_\ell\left(x\right)\right) + (1-y)\cdot\mathbb{P}\left(B_\ell\left(y\right) > B_\ell\left(x\right)\right) + \frac{1}{n}.$$

Hence,

$$\begin{aligned}\frac{d}{dy}g(x,y) \quad &= \quad \Big[\mathbb{P}\left(B_\ell\left(y\right) \geq B_\ell\left(x\right)\right) \;-\; \mathbb{P}\left(B_\ell\left(y\right) > B_\ell\left(x\right)\right)\Big] \\ &\quad + \left(y-\frac{1}{n}\right)\cdot\frac{d}{dy}\mathbb{P}\left(B_\ell\left(y\right) \geq B_\ell\left(x\right)\right) \\ &\quad + (1-y)\cdot\frac{d}{dy}\mathbb{P}\left(B_\ell\left(y\right) > B_\ell\left(x\right)\right).\end{aligned} \tag{A.13}$$

The first term in Eq. (A.13) is equal to $\mathbb{P}\left(B_\ell\left(y\right) = B_\ell\left(x\right)\right)$, which is positive. Moreover, $\mathbb{P}\left(B_\ell\left(y\right) \geq B_\ell\left(x\right)\right)$ is obviously increasing in y, so the second term is also non-negative. By Claim A.14, the third term in Eq. (A.13) satisfies

$$(1-y)\cdot\frac{d}{dy}\mathbb{P}\left(B_\ell\left(y\right) > B_\ell\left(x\right)\right) \geq (1-y)\cdot\beta'\cdot\sqrt{\ell} \geq \frac{\beta'}{4}\cdot\sqrt{\ell},$$

where the last inequality comes from the fact that $x \in [1/3, 2/3]$ and $y \in [x, x+1/\sqrt{\ell}] \subseteq [1/4, 3/4]$. For ℓ large enough, this implies that

$$\frac{d}{dy}g(x,y) \geq \frac{\beta'}{4}\cdot\sqrt{\ell} > 1,$$

which concludes the proof of (i) in Lemma 2.26. □

Proof of (ii) and (iii) in Lemma 2.26. First, we claim that $g(x,x) < x$. Let $p = \mathbb{P}(B_\ell(x) > B_\ell(x))$ and $q = \mathbb{P}(B_\ell(x) = B_\ell(x))$. We rearrange the definition of g slightly to obtain $g(x,x) = p+x\cdot q+\frac{p}{n}$. Moreover, $x = x\cdot(2p+q) \geq (1+8/n)\cdot p+x\cdot q > g(x,x)$, where the first inequality is because $x \geq 1/2+4/n$. Function h_x is continuous, and what we just showed implies $h_x(x) < 0$. Moreover, by (i) in Lemma 2.26, we know that h_x is strictly increasing on $[x, x+1/\sqrt{\ell}]$. Therefore, either $h_x(x+1/\sqrt{\ell}) \geq 0$, in which case there is a unique $y^\star \in [x, x+1/\sqrt{\ell}]$ such that $h_x(y^\star) = 0$; or $h_x(x+1/\sqrt{\ell}) < 0$, i.e., for all y on the interval, $h_x(y) < 0$, which proves Statement (ii) in Lemma 2.26. Writing $f(x)$ the zero of h_x in the interval $[x, x+1/\sqrt{\ell}]$ if it exists, and $f(x) = x+1/\sqrt{\ell}$ otherwise, we also get that $h_x(f(x)) \geq 0$, which proves Statement (iii) in Lemma 2.26. □

Claim A.16. *For any $x \in [1/2+4/n, 1/2+4\delta]$, it holds that*

$$f(x) - x > \frac{1}{2\alpha\sqrt{\ell}}\left(x-\frac{1}{2}\right),$$

where $\alpha > 1$ is the constant stated in Lemma A.11.

Proof. If $f(x)$ is not a solution to $y = g(x, y)$, then by definition $f(x) = x + 1/\sqrt{\ell}$, i.e.,

$$f(x) - x = \frac{1}{\sqrt{\ell}} > \frac{1}{2\alpha\sqrt{\ell}}\left(x - \frac{1}{2}\right),$$

and so the statement holds. Otherwise, then $f(x) = g(x, f(x))$ and belongs to $[x, x + 1/\sqrt{\ell}]$. By Lemma A.11, there exists $\alpha > 0$ s.t.

$$\mathbb{P}\left(B_\ell\left(f(x)\right) > B_\ell\left(x\right)\right) < \frac{1}{2} + \alpha(f(x) - x)\sqrt{\ell} - \frac{1}{2}\mathbb{P}\left(B_\ell\left(f(x)\right) = B_\ell\left(x\right)\right).$$

This can be plugged into the definition of f (Eq. (A.9)) to give

$$f(x) < \frac{1}{2} + \alpha(f(x) - x)\sqrt{\ell} + \left(f(x) - \frac{1}{2}\right)\mathbb{P}\left(B_\ell\left(f(x)\right) = B_\ell\left(x\right)\right) + \frac{1}{n}$$

which we can rewrite,

$$\left(1 - \mathbb{P}\left(B_\ell\left(f(x)\right) = B_\ell\left(x\right)\right)\right)\left(f(x) - \frac{1}{2}\right) < \alpha(f(x) - x)\sqrt{\ell} + \frac{1}{n}.$$

This gives

$$f(x) - x > \frac{1 - \mathbb{P}\left(B_\ell\left(f(x)\right) = B_\ell\left(x\right)\right)}{\alpha\sqrt{\ell}}\left(f(x) - \frac{1}{2}\right) - \frac{1}{\alpha \cdot n\sqrt{\ell}} > \frac{1}{2\alpha\sqrt{\ell}}\left(x - \frac{1}{2} - \frac{2}{n}\right),$$

where the last inequality comes from the upper bound $\mathbb{P}\left(B_\ell(f(x)) = B_\ell(x)\right) < 1/2$ (which is true when ℓ is large enough), and from the fact that $f(x) > x$. Since $(x - 1/2) \geq 4/n$, this implies

$$f(x) - x > \frac{1}{4\alpha\sqrt{\ell}}\left(x - \frac{1}{2}\right),$$

as desired. This completes the proof of Claim A.16. □

Proof of (iv) in Lemma 2.26. Rewriting $f(x) - x = (f(x) - 1/2) - (x - 1/2)$, we get from Claim A.16 that for every $x \in [1/2 + 4/n, 1/2 + 4\delta]$,

$$\left(f(x) - \frac{1}{2}\right) > \left(1 + \frac{1}{4\alpha\sqrt{\ell}}\right) \cdot \left(x - \frac{1}{2}\right).$$

□

A.2.3 Extracting the Average of Normally Distributed Variables

Lemma A.17. *For any $\sigma \geq 0$, there exists normal r.v. $Z, X_1, \ldots, X_n$ such that $\sum_{i \in I} X_i = 0$,*

$$(Z + X_1, \ldots, Z + X_n) \sim \mathcal{M}(0, \sigma^2 I),$$

Z is independent from $(X_1, \ldots, X_n)$ and $Z \sim \mathcal{M}(0, \sigma^2/n)$.

Proof. The first step of the proof is to explain how to construct the variables of the statement. We will apply the following idea recursively: to sample $k+1$ identically distributed normal variables $A_1, \ldots, A_{k+1}$ with variance σ^2 summing to 0, we first sample $A_1 \sim \mathcal{M}(0, \sigma^2)$, then we sample $A'_2, \ldots, A'_{k+1}$ with variance σ'^2 summing to 0, and finally, for $i \geq 2$ we take $A_i = -A_1/k + A'_i$. By construction, $\sum_{i=1}^{k+1} A_i = 0$ and for $A_i, i \geq 2$ have the desired variance, provided that we choose $\sigma'^2 = (1 - 1/k^2)\sigma^2$.

Let $Z \sim \mathcal{M}(0, \sigma^2/n)$. Following the aforementioned construction, let $\sigma_1^2 = (1-1/n)\sigma^2$, and for $i \in \{1, \ldots, n-1\}$, let

$$\sigma_{i+1}^2 = \left(1 - \frac{1}{(n-i)^2}\right)\sigma_i^2.$$

For $i, j \in \{1, \ldots, n\}$, let $W_i \sim \mathcal{M}(0, \sigma_i^2)$ s.t. $(W_i)_{1 \le i \le n}$ are independent, and

$$\lambda_{i,j} = \begin{cases} -1/(n-j) & \text{if } i > j, \\ 1 & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases}$$

Eventually, let $X_i = \sum_{j=1}^n \lambda_{i,j} W_j$ and $Y_i = Z + X_i$.

The second step of the proof is to check that $\sum_{i=1}^n X_i = 0$. Note that, by definition, for any $j \in \{1, \ldots, n\}$, we have

$$\sum_{i=1}^n \lambda_{i,j} = \sum_{i=1}^{j-1} \lambda_{i,j} + \lambda_{j,j} + \sum_{i=j+1}^n \lambda_{i,j} = 1 - (n-j) \cdot \frac{1}{n-j} = 0.$$

Therefore,

$$\sum_{i=1}^n X_i = \sum_{i=1}^n \sum_{j=1}^n \lambda_{i,j} W_j = \sum_{j=1}^n W_j \sum_{i=1}^n \lambda_{i,j} = 0.$$

The third step of the proof is to compute the distribution of each Y_i individually. For every $i \in \{1, \ldots, n\}$, Y_i is normally distributed with mean 0 (as a sum of independent normal random variables with mean 0). The variance of Y_i is, by definition,

$$\operatorname{Var}(Y_i) = \frac{\sigma^2}{n} + \sum_{j=1}^n \lambda_{i,j}^2 \sigma_j^2. \tag{A.14}$$

Claim A.18. *For every $k \in \{1, \ldots, n\}$, and for every $i \ge k$,*

$$\sum_{j=k}^n \lambda_{i,j}^2 \sigma_j^2 = \sigma_k^2. \tag{A.15}$$

Proof. The proof proceeds by a backwards induction on k. First, let us consider the base case $k = n$. In this case, we only need to consider $i = k = n$. By definition, $\lambda_{n,n} = 1$, and indeed

$$\sum_{j=n}^{n} \lambda_{i,j}^2 \sigma_j^2 = \lambda_{n,n}^2 \sigma_n^2 = \sigma_n^2.$$

Next, let us fix $k < n$, and assume by induction that the statement holds for $k+1$. Our goal is to show that it also holds for k. By definition, $\lambda_{k,k} = 1$ and for $j > k$, $\lambda_{k,j} = 0$, so Eq. (A.15) holds trivially for $i = k$. Let $i > k$. We have

$$\sum_{j=k}^{n} \lambda_{i,j}^2 \sigma_j^2 = \lambda_{i,k}^2 \sigma_k^2 + \sum_{j=k+1}^{n} \lambda_{i,j}^2 \sigma_j^2 = \lambda_{i,k}^2 \sigma_k^2 + \sigma_{k+1}^2,$$

where the second equality is by induction hypothesis. By definition, this is equal to (since $k < i$)

$$\frac{1}{(n-k)^2} \cdot \sigma_k^2 + \left(1 - \frac{1}{(n-k)^2}\right) \sigma_k^2 = \sigma_k^2,$$

and so Eq. (A.15) holds for $i > k$, which concludes the induction and the proof of Claim A.18. □

By Claim A.18, Eq. (A.14) becomes

$$\mathrm{Var}(Y_i) = \frac{\sigma^2}{n} + \sigma_1^2 = \frac{\sigma^2}{n} + \left(1 - \frac{1}{n}\right)\sigma^2 = \sigma^2,$$

and we can conclude that $Y_i \sim \mathcal{M}(0, \sigma^2)$.

The last step of the proof is to show that the $(Y_i)_{1 \le i \le n}$ are mutually independent. Since they are normally distributed, it is enough to show that they are pairwise independent. For this purpose, let us fix $i, k \in \{1, \ldots, n\}$ s.t. $i \neq k$, and show that for every $t, s \in \mathbb{R}$,

$$M_{Y_i, Y_k}(t, s) = M_{Y_i}(t) \cdot M_{Y_k}(s),$$

where M_X stands for the moment generating function of the random variable X. Recall that, since $Y_i \sim \mathcal{M}(0, \sigma^2)$, we have

$$M_{Y_i}(t) = \exp\left(\frac{1}{2}\sigma^2 t^2\right).$$

For the sake of simplicity, from now on we will denote $W_0 = Z$, and $\lambda_{i,0} = 1$, so that

$$Y_i = Z + \sum_{j=1}^{n} \lambda_{i,j} W_j = \sum_{j=0}^{n} \lambda_{i,j} W_j.$$

Moreover, we will note $\sigma_0^2 = \sigma^2/n = \mathrm{Var}(W_0)$, so that Eq. (A.14) translates nicely to

$$\mathrm{Var}(Y_i) = \sum_{j=0}^{n} \lambda_{i,j}^2 \sigma_j^2.$$

For every $t, s \in \mathbb{R}$, we have

$$\begin{aligned}
M_{Y_i,Y_k}(t,s) &= \mathbb{E}\left(\exp\left(tY_i + sY_k\right)\right) \\
&= \mathbb{E}\left(\exp\left(t\sum_{j=0}^{n}\lambda_{i,j}W_j + s\sum_{j=0}^{n}\lambda_{k,j}W_j\right)\right) \\
&= \mathbb{E}\left(\exp\left(\sum_{j=0}^{n}(t\lambda_{i,j} + s\lambda_{k,j})\cdot W_j\right)\right) \\
&= \mathbb{E}\left(\prod_{j=0}^{n}\exp\left((t\lambda_{i,j} + s\lambda_{k,j})\cdot W_j\right)\right) \\
&= \prod_{j=0}^{n}\mathbb{E}\left(\exp\left((t\lambda_{i,j} + s\lambda_{k,j})\cdot W_j\right)\right) \\
&= \prod_{j=0}^{n}M_{W_j}\left(t\lambda_{i,j} + s\lambda_{k,j}\right) \\
&= \prod_{j=0}^{n}\exp\left(\frac{1}{2}\sigma_j^2\left(t\lambda_{i,j} + s\lambda_{k,j}\right)^2\right),
\end{aligned}$$

so eventually,

$$M_{Y_i,Y_k}(t,s) = \exp\left(\frac{1}{2}\sum_{j=0}^{n}\sigma_j^2\left(t\lambda_{i,j} + s\lambda_{k,j}\right)^2\right). \tag{A.16}$$

To conclude, we need the following technical results.

Claim A.19. *For every $1 \le k < n$,*

$$\sum_{i=1}^{k}\frac{1}{(n-i)(n-i+1)} = \frac{k}{n(n-k)}.$$

Proof. Let $n \in \mathbb{N}$. We proceed by induction on k. The statement holds trivially for $k = 1$. Assuming the statement holds for some $k < n-1$, by the induction hypothesis,

$$\begin{aligned}
\sum_{i=1}^{k+1}\frac{1}{(n-i)(n-i+1)} &= \frac{k}{n(n-k)} + \frac{1}{(n-k)(n-k-1)} \\
&= \frac{k(n-k-1)+n}{n(n-k)(n-k-1)} = \frac{k+1}{n(n-k-1)},
\end{aligned}$$

which concludes the induction and the proof of Claim A.19. □

Claim A.20. *For every $i, k \in \{1, \ldots, n\}$ s.t. $i \neq k$,*

$$\sum_{j=0}^{n}\lambda_{i,j}\lambda_{k,j}\sigma_j^2 = 0.$$

Proof. Without loss of generality, we consider the case that $k < i$. By definition, we have $\lambda_{i,j} = \lambda_{k,j} = -1/(n-j)$ for $1 \le j < k$, $\lambda_{k,k} = 1$, $\lambda_{i,k} = -1/(n-k)$, and $\lambda_{k,j} = 0$ for $k < j \le n$. Hence,

$$\sum_{j=0}^{n} \lambda_{i,j}\lambda_{k,j}\sigma_j^2 = \sigma_0^2 + \sum_{j=1}^{k-1} \frac{\sigma_j^2}{(n-j)^2} - \frac{\sigma_k^2}{n-k}. \tag{A.17}$$

Next, we claim that for every $j \in \{1, \dots, n\}$,

$$\frac{\sigma_j^2}{(n-j)^2} = \frac{n}{(n-j)(n-j+1)} \cdot \frac{\sigma_1^2}{n-1}. \tag{A.18}$$

We prove Eq. (A.18) by induction on j. It holds trivially for $j = 1$. Assume it holds for some $j < n$. We write

$$\sigma_{j+1}^2 = \left(1 - \frac{1}{(n-j)^2}\right)\sigma_j^2 = \frac{(n-j)^2 - 1}{(n-j)^2}\sigma_j^2 = \frac{(n-j+1)(n-j+1)}{(n-j)^2}\sigma_j^2.$$

Therefore,

$$\frac{\sigma_{j+1}^2}{(n-j-1)^2} = \frac{n-j+1}{n-j-1} \cdot \frac{\sigma_j^2}{(n-j)^2} = \frac{n}{(n-j)(n-j-1)} \cdot \frac{\sigma_1^2}{n-1},$$

where the second equality is by the induction hypothesis. This concludes the induction and the proof of Eq. (A.18).

By Eq. (A.18), Eq. (A.17) becomes

$$\sum_{j=0}^{n} \lambda_{i,j}\lambda_{k,j}\sigma_j^2 = \sigma_0^2 + \frac{\sigma_1^2}{n-1}\left(\sum_{j=1}^{k-1} \frac{n}{(n-j)(n-j+1)} - (n-k)\frac{n}{(n-k)(n-k+1)}\right).$$

We can rearrange this expression using Claim A.19, to obtain

$$\sigma_0^2 + \frac{n\sigma_1^2}{n-1}\left(\frac{k-1}{n(n-k+1)} - \frac{1}{n-k+1}\right) = \sigma_0^2 - \frac{\sigma_1^2}{n-1}.$$

By definition,

$$\sigma_0^2 - \frac{\sigma_1^2}{n-1} = \frac{\sigma^2}{n} - \left(1 - \frac{1}{n}\right)\frac{\sigma^2}{n-1} = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0,$$

which concludes the proof of Claim A.20. □

Eventually, by Claim A.20, Eq. (A.16) becomes

$$\begin{aligned}
M_{Y_i,Y_k}(t,s) &= \exp\left(\frac{1}{2}\sum_{j=0}^{n} \sigma_j^2 \left(t\lambda_{i,j} + s\lambda_{k,j}\right)^2\right) \\
&= \exp\left(\frac{t^2}{2}\sum_{j=0}^{n} \lambda_{i,j}^2\sigma_j^2 + \frac{s^2}{2}\sum_{j=0}^{n} \lambda_{k,j}^2\sigma_j^2 + st\sum_{j=0}^{n} \lambda_{i,j}\lambda_{k,j}\sigma_j^2\right) \\
&= \exp\left(\frac{t^2}{2}\mathrm{Var}(Y_i) + \frac{s^2}{2}\mathrm{Var}(Y_k)\right) \\
&= \exp\left(\frac{t^2}{2}\mathrm{Var}(Y_i)\right) \cdot \exp\left(\frac{s^2}{2}\mathrm{Var}(Y_k)\right) \\
&= M_{Y_i}(t) \cdot M_{Y_k}(s),
\end{aligned}$$

which concludes the proof of Lemma A.17. □

A.2.4 Expectation of Some Binomial Random Variables

Claim A.21. *Let $X \sim \mathcal{B}(n,p)$. If $0 < p \leq 1$, then*

$$\mathbb{E}\left(\frac{1}{1+X}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$

Moreover, if $p = 0$, then $\mathbb{E}\left(\frac{1}{1+X}\right) = 1$.

Proof. The claim holds trivially for $p = 0$. Consider the case that $p > 0$.

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{1+X}\right) &= \sum_{k=0}^{n} \frac{1}{1+k}\mathbb{P}(X = k) \\
&= \sum_{k=0}^{n} \frac{1}{1+k} \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{1}{(n+1)p} \sum_{k=0}^{n} \binom{n+1}{k+1} p^{k+1} (1-p)^{(n+1)-(k+1)},
\end{aligned}$$

where the last equality was obtained using $\binom{n}{k} = \binom{n+1}{k+1} \cdot \frac{k+1}{n+1}$. By setting $k' = k+1$, we can rewrite the sum

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n+1}{k+1} p^{k+1}(1-p)^{(n+1)-(k+1)} &= \sum_{k'=0}^{n+1} \binom{n+1}{k'} p^{k'} (1-p)^{(n+1)-k'} - (1-p)^{n+1} \\
&= 1 - (1-p)^{n+1},
\end{aligned}$$

which concludes the proof of Claim A.21. $\square$

Claim A.22. *Let $X \sim \mathcal{B}(n,p)$. If $0 \leq p < 1$, then*

$$\mathbb{E}\left(\frac{X}{2+n-X}\right) = p \cdot \frac{(n+1)(1-p) + p^{n+1} - 1}{(n+1)(1-p)^2}.$$

Moreover, if $p = 1$, then $\mathbb{E}\left(\frac{X}{2+n-X}\right) = \frac{n}{2}$.

Proof. The claim holds trivially for $p = 1$. Consider the case that $p < 1$. Let $q = 1 - p > 0$.

We have

$$
\begin{aligned}
&\mathbb{E}\left(\frac{X}{2+n-X}\right) \\
&= \sum_{k=1}^{n} \frac{k}{n-k+2} \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n-1} \frac{n-k}{k+2} \cdot \binom{n}{k} q^k (1-q)^{n-k} && k \mapsto n-k,\ q = 1-p \\
&= \sum_{k=1}^{n} \frac{n-k+1}{k+1} \cdot \binom{n}{k-1} q^{k-1} (1-q)^{n-k+1} && k \mapsto k-1 \\
&= \sum_{k=1}^{n} \frac{k}{k+1} \cdot \binom{n}{k} q^{k-1} (1-q)^{n-k+1} && \text{using } \binom{n}{k-1} = \binom{n}{k} \cdot \frac{k}{n-k+1} \\
&= \sum_{k=1}^{n} \frac{k}{n+1} \cdot \binom{n+1}{k+1} q^{k-1} (1-q)^{n-k+1} && \text{using } \binom{n}{k} = \binom{n+1}{k+1} \cdot \frac{k+1}{n+1} \\
&= \sum_{k=2}^{n+1} \frac{k-1}{n+1} \cdot \binom{n+1}{k} q^{k-2} (1-q)^{n-k+2} && k \mapsto k-1 \\
&= \frac{1-q}{q^2(n+1)} \cdot \sum_{k=1}^{n+1} (k-1) \binom{n+1}{k} q^k (1-q)^{(n+1)-k}.
\end{aligned}
$$

By expectation of the binomial distribution,

$$
\sum_{k=1}^{n+1} k \binom{n+1}{k} q^k (1-q)^{(n+1)-k} = (n+1)q.
$$

Moreover, by the binomial theorem,

$$
\sum_{k=1}^{n+1} \binom{n+1}{k} q^k (1-q)^{(n+1)-k} = 1 - (1-q)^{n+1}.
$$

Putting every equation together, we obtain

$$
\mathbb{E}\left(\frac{X}{2+n-X}\right) = (1-q) \cdot \frac{(n+1)q + (1-q)^{n+1} - 1}{(n+1)q^2},
$$

which concludes the proof of Claim A.22. □