Pre–Course Homework

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The following exercises should be review for many, but will help to cement concepts that may be unfamiliar to some.

1 Convolution

For continuous signals, the convolution of an input signal, $x(t)$, with a filter, $h(t)$ is defined as,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau. \tag{1}$$

For notation's sake, I will note this as

$$y(t) = h(t) \otimes x(t). \tag{2}$$

1. Given this definition, prove that convolution is linear

$$h(t) \otimes [\alpha x(t) + \beta y(t)] = \alpha h(t) \otimes x(t) + \beta h(t) \otimes y(t) \tag{3}$$

2. Prove that convolution is commutative

$$h(t) \otimes x(t) = x(t) \otimes h(t) \tag{4}$$

2 Fourier Transforms

We'll be working a lot with the Fourier Transform of continuous signals. This transform is given by,

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi f t} dt. \tag{5}$$

Further, the inverse Fourier Transform is also given by,

$$h(t) = \int_{-\infty}^{\infty} H(t)\, e^{j2\pi f t} df. \tag{6}$$

1. Suppose the Fourier transform of $h_o(t)$ is known. What is the Fourier transform of $h_o e^{j2\pi f_c t}$?

2. What is the Fourier transform of $h_o(t-\tau)$?

3. What is the Fourier transform of $h_o^*(-t)$?

4. What is the Fourier transform of $h_o\left(\frac{t}{T_s}\right)$?

5. Calculate the Fourier transform of the following waveforms:

   (a)

$$x(t) = \begin{cases} \frac{1}{\sqrt{T_s}} & |t| < \frac{T_s}{2} \\ 0 & \text{Otherwise} \end{cases} \tag{7}$$

   (b)

$$x(t) = \begin{cases} \frac{1}{\sqrt{T_s}} \cos\left(\frac{\pi}{2}\frac{t}{T_s}\right) & |t| < T_s \\ 0 & \text{Otherwise} \end{cases} \tag{8}$$

   You may find it useful to use the relationship that,

$$\cos(\omega) = \frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega}. \tag{9}$$

   (c)

$$x(t) = \begin{cases} \frac{1}{\sqrt{T_s}} \left[1 + \cos\left(\pi\frac{t}{T_s}\right)\right] & |t| < T_s \\ 0 & \text{Otherwise} \end{cases} \tag{10}$$

   (d)

$$x(t) = \begin{cases} \sqrt{\frac{3}{2T_s}} \left[1 - \frac{|t|}{T_s}\right] & |t| < T_s \\ 0 & \text{Otherwise} \end{cases} \tag{11}$$

(e)

$$x(t) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{t^2}{2}} \tag{12}$$

6. **Bonus:** Show that the convolution of two waveforms is equal to the inverse Fourier transform of their product. Hint, start with

$$h(t) \otimes x(t) = \int h(\tau)\, x(t-\tau)\, d\tau, \tag{13}$$

and take the Fourier transform of both sides.

7. **Bonus:** Show that the Fourier transform conserves energy. That is, prove the following:

$$\int_{-\infty}^{\infty} |h(t)|^2\, dt = \int_{-\infty}^{\infty} |H(f)|^2\, df. \tag{14}$$

3 Autocorrelation

The autocorrelation of a random waveform is defined by the expected value of the product of two points from that waveform,

$$R_x(t,\tau) = \mathcal{E}\left\{ x^*\left(t - \frac{\tau}{2}\right) x\left(t + \frac{\tau}{2}\right) \right\}. \tag{15}$$

1. Given $R(t,\tau)$ for $\tau > 0$, what is $R(t,\tau)$ when $\tau < 0$?

2. Suppose $y(t) = h(t) \otimes x(t)$. What is the autocorrelation of $y(t)$ given the autocorrelation of $x(t)$, $R_x(t,\tau)$?