

Persistent Unobservables

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Wednesday 6th November, 2024

Grad IO

Persistent Unobserved Heterogeneity

Suppose we think about a model with a friction such as a switching cost.

- ▶ If $y_{it} \neq y_{i,t-1}$ you pay a switching cost F_i .
- ▶ How do we use data to tell apart large switching costs $F_i \gg 0$ from persistent tastes $Cov(\varepsilon_{i,t}, \varepsilon_{i,t-1}) > 0$?
- ▶ The **conditional independence assumption** tells us it has to be the switching cost not the autocorrelated unobservables.
- ▶ This is probably why people don't like this assumption.

Discrete Unobserved Types

- ▶ Up until now we consider models satisfying Rust's **conditional independence** assumption on the ε 's. This rules out persistence in unobservables which are economically meaningful.
- ▶ Suppose there are two types of buses good ($s_i = g$) and bad ($s_i = b$).
- ▶ Assume that this is known to HZ but not the econometrician.
- ▶ Single period utility now depends on s_i so $u(x_{it}, s_i, d_{it}; \theta)$ **unobserved state variable**.
- ▶ In case of the nested fixed point algorithm, this unobserved persistent heterogeneity is not a big problem as we can solve for the value function (and expected policy functions) given the state variables and **integrate it out** in the likelihood

Unobserved State Variables: What happened?

$$\mathbb{P}(d_{i1}, \dots, d_{iT} | x_{i1}, \dots, x_{iT}) = \sum_s \prod_{t=1}^T \mathbb{P}(d_{it} | x_{it}) p(s_i)$$

- ▶ Conditional on s_i replacement decisions are independent across t given x_{it} .
- ▶ The resulting likelihood is just a **finite mixture model**.
- ▶ These can be hard to solve when both s_i and its distribution $p(s_i)$ are unknown.
- ▶ Arcidiacono and Miller (2011) provide theoretical results for these types of problems.

A much earlier application

Pakes (1986): Patents as Options

How much are patents worth? Valuable for optimal patent length and design? Sufficient incentive for innovation?

- ▶ Q_A : value of patent at age A
- ▶ Goal of paper is to estimate Q_A using data on their renewal. Q_A is inferred from patent renewal process via **revealed preference** for patent renewal behavior.
- ▶ Treat renewal systems as exogenous (in Europe)

Timing

- ▶ For $a = 1, \dots, L$ a patent can be renewed by paying the fee c_a .
- ▶ At age $a = 1$ patent holder gets r_1 from patent
- ▶ Decide whether or not to renew (pay c_1 and go to a_2).
- ▶ At age $a = 2$ get r_2 from patent
- ▶ and so on...

Gives us the value function

$$V \equiv \max_{t \in [a, L]} \sum_{a'=1}^{L-a} \beta^{a'} R(a + a')$$
$$R(a) = \begin{cases} r_a - c_a, & \text{if } t \geq a \text{ when you hold patent} \\ 0 & \text{if } t < a \text{ after patent expires} \end{cases}$$

- ▶ t above denotes the age which allows the patent to expire and is the choice variable. Another **optimal stopping** problem.
- ▶ $R(a)$ are the profits from year a . This is a **controlled stochastic process**. It is random but affected by the actions of the agent.

Pakes (1986)

- ▶ The maximum age L is finite so it is finite-horizon DP.
- ▶ The single period revenue r_a is the state variable.
- ▶ We can solve the problem with *backward recursion*.

$$V_a(r_a) = \max \{0, Q_a \equiv r_a + \beta \mathbb{E}[V_{a+1}(r_{a+1})|\Omega_a] - c_a\}$$

- ▶ Renew iff $Q_a - c_a > 0$.
- ▶ Ω_a : history up to age $a = \{r_1, r_2, \dots, r_a\}$.
- ▶ Expectation is over $r_{a+1}|\Omega_a$. The sequence of conditional distributions $G_a \equiv F(r_{a+1}|\Omega_a)$, $a = 1, 2, \dots$ is an important component of model specification.

$$r_{a+1} = \begin{cases} 0 & \text{w. prob } \exp(-\theta r_a) \\ \max(\delta r_a, z) & \text{w. prob } 1 - \exp(-\theta r_a) \end{cases}$$

Pakes (1986)

Model has the following parameters

- ▶ density of z $q_a = \frac{1}{\sigma_a} \exp[-(\gamma + z)/\sigma_a]$ and $\sigma_a = \phi^{a-1}\sigma$, for $a = 1, \dots, L - 1$.
- ▶ $(\delta, \theta, \gamma, \phi, \sigma)$ are the structural parameters of the model
- ▶ Break down the model period by period and decide whether or not to renew if $Q_a = r_a +$ “option value”.
- ▶ Option value is about keeping the patent alive in case it pays off in the future.

Implications

- ▶ Drop out at age a if $c_a > Q_a$
- ▶ Optimal decision is characterized by cutoff points $Q_a > c_a \Leftrightarrow r_a > \bar{r}_a$ (Key assumptions is Q_a increasing /single crossing)
- ▶ Cutoff points are increasing sequence $\bar{r}_a < \bar{r}_{a+1} < \dots < \bar{r}_{L-1}$.

Estimation

Instead of using Pakes' notation r_t for the patent revenue. We will use the generic Rust notation of ϵ_t the unobserved state variable, and i_t to denote the choice (renewal).

- ▶ For a single patent \tilde{T} denotes the age at which it is allowed to expire. Let $T = \min(L - 1, \tilde{T})$ denote the period sins which the agent makes a renewal decision where we model the agent's choice.
- ▶ ϵ follows a first-order Markov process $F(\epsilon'|\epsilon)$
- ▶ Age-specific policy function by $i_t^*(\epsilon)$.

Likelihood function is

$$\ell(i_1, \dots, i_T | \epsilon_0, i_0, \theta) = \prod_{t=1}^T \mathbb{P}(i_t | i_0, \dots, x_{t-1}, i_{t-1}; \epsilon_0, \theta)$$

Serial correlation in ϵ means there is dependence among i_t, i_{t-2} even after conditioning on x_{t-1}, i_{t-1} .

- ▶ It might seem like we were stuck since it no longer has a closed form. However, we can simulate the “outer loop” of the nested fixed point routine given a guess of $i_t^*(\varepsilon, \theta)$.
- ▶ Because ε is serially correlated we need to start with an initial ε_0 (or distribution) and assume that it is known. This is the **initial conditions problem** of finite MDPs.
- ▶ Note that simulation is part of the “outer loop” of nested fixed point estimation routine. So at the point when we simulate, we already know the policy functions $i_t^*(\varepsilon, \theta)$ (How would you compute this?)

Naive Frequency Simulator (Don't do this...)

Go back to the full likelihood function (condition on initial ε_0 for serial correlation):

$$\ell(i_1, \dots, i_T | i_0, \varepsilon_0, \theta) = \mathbb{P}(i_t^*(\varepsilon_t, \theta) = i_t, \forall t = 1, \dots, T)$$

Need to take probability over distribution of $(\varepsilon_1, \dots, \varepsilon_T | \varepsilon_0)$. Let $F(\varepsilon_{t+1} | \varepsilon_t, \theta)$ then the above probability can be expressed as the integral:

$$\int \cdots \int \prod_t \mathbf{1}(i_t^*(\varepsilon_t, \theta) = i_t) \prod_t dF(\varepsilon_t | \varepsilon_{t-1}; \theta)$$

Simulate by drawing sequences of (ε_t) .

Naive Frequency Simulator (Don't do this...)

Simulate by drawing sequences of (ε_t) and for each draw $s = 1, \dots, S$ we take as initial values $(x_0, i_0, \varepsilon_0)$ then

- ▶ Generate (ε_1^s, i_1^s)
 1. Generate $\varepsilon_1^s \sim F(\varepsilon_1 | \varepsilon_0)$
 2. Compute $i_1^s = i_1^*(\varepsilon_1^s; \theta)$
- ▶ Generate (ε_2^s, i_2^s)
 1. Generate $\varepsilon_2^s \sim F(\varepsilon_2 | \varepsilon_1^s)$
 2. Subsequently compute $i_2^s = i_2^*(\varepsilon_2^s; \theta)$
- ▶ And so on, up to (ε_T^s, i_T^s) .

Naive Frequency Simulator (Don't do this...)

And for the case where (i, x) are both discrete (Rust) we can approximate:

$$\ell(i_t, \dots, i_T | \varepsilon_0, i_0; \theta) \approx \frac{1}{S} \sum_s \prod_{t=1}^T \mathbf{1}(i_t^s = i_t)$$

Frequency of simulated sequences which match observed sequence. T long or S small you're in trouble (non-smooth).

Importance Sampling: Particle Filtering

- ▶ We can use importance sampling to simulate the likelihood function.
- ▶ This is not straightforward given time dependence in (i_t, ϵ_t)
- ▶ Consider particle filtering approach from Fernandez-Villaverde and Rubio-Ramirez (2007) or Flury and Shehard (2008) (non-Gaussian Kalman filtering).
- ▶ A more up to date take: Blevins (2016) : Sequential Monte Carlo Methods for Estimating Dynamic Microeconomic Models

Importance Sampling: Particle Filtering

- ▶ Evolution of utility shocks $\epsilon_t | \epsilon_{t-1} \sim f(\epsilon' | \epsilon)$. Ignore dependence of distribution of ϵ on age t for convenience.
- ▶ As before, the policy function is $i_t = i^*(\epsilon_t)$
- ▶ Let $\epsilon^t \equiv \{\epsilon_1, \dots, \epsilon_t\}$.
- ▶ The initial values of y_0 and ϵ_0 are known

Go back to the factorized likelihood

$$\begin{aligned} l(y^T | y_0, \epsilon_0) &= \prod_{t=1}^T l(y_t | y^{t-1}, y_0, \epsilon_0) = \prod_{t=1}^T \int l(y_t | \epsilon^t, y^{t-1}) p(\epsilon^t | y^{t-1}) d\epsilon^t \\ &\approx \frac{1}{S} \sum_s l(y_t | \epsilon^{t|t-1,s}, y^{t-1}) \end{aligned}$$

We omit conditioning on (ϵ_0, y_0) for convenience, and $\epsilon^{t|t-1,s}$ is a simulated draw of $\epsilon^t \sim p(\epsilon^t | y^{t-1})$.

Importance Sampling: Particle Filtering

Let's look more closely at the last line:

- ▶ first term: $l(y_t, |\epsilon^t, y^{t-1})$ we can calculate for a value of ϵ_t

$$l(y_t|\epsilon^t, y^{t-1}) = p(i_t|\epsilon^t, y^{t-1}) = p(i_t|\epsilon_t) = \mathbf{1}(i(\epsilon_t) = i_t)$$

- ▶ the second term $p(\epsilon^t|y^{t-1})$ is generally not obtainable in closed form. So numerical integration is not feasible. Particle filtering let's us draw ϵ^t from this distribution for every period t .

Particle filtering proposes a recursive approach to draw sequences $p(\epsilon^t|y^{t-1})$ for every t

Particle Filtering Algorithm

First period: $t = 1$ In order to simulate the integral corresponding to the first period we need to draw from $p(\epsilon^1|y^0, \epsilon_0)$ (easy).

- ▶ We draw $\{\epsilon^{1|0,s}\}_{s=1}^S$ according to $f(\epsilon'|\epsilon_0)$.
- ▶ The notation $\epsilon^{1|0,s}$ makes it explicit that the ϵ is a draw from $p(\epsilon^1|y^0, \epsilon_0)$
- ▶ Use the S draws we can evaluate the period $t = 1$ likelihood.

Second period: $t = 2$. We need to draw from $p(\epsilon^2|y^1)$ factorize as:

$$p(\epsilon^2|y^1) = p(\epsilon^1|y^1) \cdot p(\epsilon_2|\epsilon^1) \text{ recall } \epsilon^2 \equiv \{\epsilon_1, \epsilon_2\}$$

Filtering Step

Getting a draw from $p(\epsilon^1|y^1)$, given that we already have draws $\{\epsilon^{1|0,s}\}$ from $p(\epsilon^1|y_0)$, from the previous period $t = 1$, is the heart of particle filtering. We use the principle of importance sampling: by Bayes' Rule

$$p(\epsilon^1|y^1) \propto p(y_1|\epsilon^1, y^0) \cdot p(\epsilon^1|y^0)$$

Hence, if our desired sampling density is $p(\epsilon^1|y^1)$, but we actually have draws $\{\epsilon^{1|0,s}\}$ from $p(\epsilon^1|y^0)$, then the importance sampling weight for the draw $\epsilon^{1|0,s}$ is proportional to

$$\tau_1^s \equiv p(y_1|\epsilon^{1|0,s}, y^0)$$

Note that this coincides with the likelihood contribution for period 1, evaluated at the shock $\epsilon^{1|0,s}$. The SIR algorithm in Rubin (1988) proposes that making S draws with replacement from samples $\{\epsilon^{1|0,s}\}_{s=1}^S$, using weights proportional τ_1^s yields draws from the desired density $p(\epsilon^1|y^1)$ which we denote $\{\epsilon^{1|0,s}\}_{s=1}^S$.

Prediction Step

For the second term in the equation: we simply draw one ϵ_2^s from $f(\epsilon' | \epsilon^{1,s})$, for each draw $\epsilon^{1,s}$ from the filtering step. This is the **prediction** step.

By combining the draws from these two terms, we have $\{\epsilon^{2|1,s}\}_{s=1}^S$. which is S drawn sequences from $p(\epsilon^2 | y^1)$. Using these S draws, we can evaluate the simulated likelihood for period 2

Prediction Step (Continued)

Third period, $t = 3$: start again by factoring

$$p(\epsilon^3|y^2) = p(\epsilon^2|y^2) \cdot p(\epsilon^3|\epsilon^2)$$

As above, drawing from requires filtering the draws $\{\epsilon^{2|1,s}\}_{s=1}^S$, from the previous period $t = 2$, to obtain draws $\{\epsilon^{2,s}\}_{s=1}^S$. Given these draws, draw $\epsilon_3^s \sim f(\epsilon'|\epsilon^{2,s})$ for each s .

And so on. By the last period $t = T$, you have

$$\left\{ \left\{ \epsilon^{t|t-1,s} \right\}_{s=1}^S \right\}_{t=1}^T$$

Prediction Step (continued)

Hence the factorized likelihood can be approximated by simulation as:

$$\prod_t \frac{1}{S} \sum_s \ell(y_t | \epsilon^{t|t-1,s}, y^{t-1})$$

As noted above, the likelihood term $\ell(y_t | \epsilon^{t|t-1,s}, y^{t-1})$ coincides with the simulation weight τ_t^s . Hence the simulated likelihood can also be constructed as:

$$\ell(y^T | y_0, \epsilon_0) = \sum_t \log \left\{ \frac{1}{S} \sum_s \tau_t^s \right\}$$

Particle Filtering (Summary)

- ▶ Start by drawing $\{\epsilon^{1|0,s}\}_{s=1}^S$ from $p(\epsilon^1|y^0, \epsilon_0)$.
- ▶ In period t , we start with $\{\epsilon^{t-1|t-2,s}\}_{s=1}^S$ draws from $p(\epsilon^{t-1}|y^{t-2}, \epsilon_0)$.
 1. **Filter step:** Calculate proportion weights $\tau_{t-1}^s \equiv p(y_{t-1}|\epsilon^{t-1|t-2,s}, y^{t-2})$ using $p(i_t|\epsilon_t)$. Draw $\{\epsilon^{t-1|t-1,s}\}_{s=1}^S$ by resampling from $\{\epsilon^{t-1|t-2,s}\}_{s=1}^S$ with weights τ_{t-1}^s .
 2. **Prediction step:** Draw ϵ_t^s from $p(\epsilon_t|\epsilon^{t-1|t-1,s})$, for $s = 1, \dots, S$. Combine to get $\{\epsilon^{t|t-1,s}\}_{s=1}^S$.
- ▶ Set $t = t + 1$ and go back to step 2. Stop when $t = T + 1$.

The difference is that the crude simulator draws S sequences and puts zero weight on those which don't match the observed sequence. In each period t we just keep sequences where predicted choices match observed choice of *that period*. This is more accurate likelihood as long as S is large enough that we don't have all the weight on a single sequence in period t .

References

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