# Multinomial Discrete Choice: Mixed Logit

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Grad IO

## Mixed/ Random Coefficients Logit

An alternative is to allow for individuals to have random coefficients $\beta_i$:

$$U_{ij} = \beta_i x_j + \varepsilon_{ij}, \quad \beta_i \sim f(\beta_i|\theta), \quad \varepsilon_{ij} \sim \text{Type I EV}$$

As an alternative, we could have specified an error components structure on $\varepsilon_i$.

$$U_{ij} = \beta x_{ij} + \underbrace{\nu_i z_{ij} + \varepsilon_{ij}}_{\tilde{\varepsilon}_{ij}}$$

- $x_{ij}$ are observed per usual and $\varepsilon_{ij}$ is IID Type I EV.
- The key is that $\nu_i$ is unobserved and mean zero (often normally distributed).
- This allows for a heteroskedastic structure on $\varepsilon_i$, but only one which we can project down onto the space of $x$.

## Mixed Logit

We relax the IIA property by mixing over various logits:

$$u_{ijt} = x_j\beta_i + \varepsilon_{ij} \qquad\qquad = x_j\beta + \mu_{ij} + \varepsilon_{ij}$$

$$s_{ij}(\theta) = \int \frac{\exp[x_j\beta_i]}{1 + \sum_k \exp[x_k\beta_i]} f(\boldsymbol{\beta_i}|\theta) \qquad\qquad = \int \frac{\exp[x_j\beta + \mu_{ij}]}{1 + \sum_k \exp[x_k\beta + \mu_{ik}]} f(\boldsymbol{\mu_i}|\theta)$$

- Each individual draws a vector $\beta_i/\mu_i$ (separately from $\varepsilon_i$).
- Conditional on $\beta_i/\mu_i$ each person follows an IIA logit model.
- However we integrate (or mix) over many such individuals giving us a mixed logit or heirarchical model (if you are a statistician)
- In practice these are not that different from linear random effects models you have learned about in econometrics.
- It helps to think about fixing $\beta_i/\mu_i$ first and then integrating out over $\varepsilon_i$

## Kinds of heterogeneity

$$s_{ij}(\theta) = \int \frac{\exp[x_j\beta_i]}{1 + \sum_k \exp[x_k\beta_i]} f(\boldsymbol{\beta_i}|\theta) \approx \sum_{s=1}^{S} w_i^s \frac{\exp[x_j\beta_i^s]}{1 + \sum_k \exp[x_k\beta_i^s]}$$

▸ We can only approximate the integral with weights $w_i^s$ and nodes $\beta_i^s$
  ▸ We can allow for there to be two types of $\beta_i$ in the population (high-type, low-type). latent class model.
  ▸ We can allow $\beta_i$ to follow an independent normal distribution for each component of $x_{ij}$ such as $\beta_i = \overline{\beta} + \nu_i\sigma$.
  ▸ We can allow for correlated normal draws $\beta_i \sim N(\mu_\beta, \Sigma_\beta)$.
  ▸ Can allow for non-normal distributions too (lognormal, exponential).
  ▸ Why is normal so easy?

## Adding Heterogeneity

- The structure is extremely flexible but at a cost.
- We generally must perform the integration numerically.
- High-dimensional numerical integration is difficult. In fact, integration in dimension 8 or higher makes me very nervous.
- We need to be parsimonious in how many variables have unobservable heterogeneity.
- Again observed heterogeneity does not make life difficult so the more of that the better!
  - If we see individual income, education, distance to hospital, etc. we can always interact that with observed characteristics without doing any additional integration.

## Mixed Logit

How does it work?

- Well we are mixing over individuals who conditional on $\beta_i$ or $\mu_i$ follow logit substitution patterns, however they may differ wildly in their $s_{ij}$ and hence their substitution patterns.
- For example if we are buying cameras: I may care a lot about price, you may care a lot about megapixels, and someone else may care mostly about zoom.
- The basic idea is that we need to explain the heteroskedasticity of $Cov(\varepsilon_i, \varepsilon_j)$ what random coefficients do is let us use a basis from our $X$'s.
- If our $X$'s are able to span the space effectively, then an RC logit model can approximate any arbitrary RUM (such as probit) (McFadden and Train 2002).
- Of course if you have 1000 products and two random coefficients, you are asking for a lot.

## Elasticities

At the level of an individual substitution patterns follow a plain logit:

$$\epsilon_{s_j, p_j} = \frac{\partial s_j}{\partial p_j} \cdot \frac{p_j}{s_j} = \frac{p_j}{s_j} \cdot \int \frac{\partial s_{ij}}{\partial p_j} \, dF_i = \frac{p_j}{s_j} \int \beta_i \cdot s_{ij} \cdot (1 - s_{ij}) \, dF_i$$

$$\epsilon_{s_j, p_k} = \frac{\partial s_j}{\partial p_k} \cdot \frac{p_k}{s_j} = \frac{p_k}{s_j} \cdot \int \frac{\partial s_{ij}}{\partial p_k} \, dF_i = -\frac{p_k}{s_j} \int \beta_i \cdot s_{ij} \cdot s_{ik} \, dF_i$$

Notation: here $s_j \equiv s_j(x_i)$ denotes an individual choice probability (we re-use the $i$ subscript) and $s_{ij}(\boldsymbol{\mu_i}) = s_{ij}$.

**Substitution (Conlon Mortimer 2020)**

At the level of an individual substitution patterns follow a plain logit:

$$D_{jk,i}(x) = \frac{\frac{\partial s_{ik}}{\partial p_j}}{\left|\frac{\partial s_{ij}}{\partial p_j}\right|} = \frac{s_{ik}(x)}{1 - s_{ij}(x)}$$

But at the aggregate level, we mix over heterogeneous individuals:

$$D_{jk}(x) = \int D_{jk,i}(x)\, w_i(z_j, z_j', x)\, dF_i \quad \text{where} \quad w_i(z_j, z_j', x) = \frac{q_{ij}(z_j', x) - q_{ij}(z_j, x)}{q_j(z_j', x) - q_j(z_j, x)}$$

Different interventions (price changes, product removals, quality changes, etc.) give different weighting schemes.

## Estimation Details: MSL/MSLE

How do we estimate these models? (Derived from multinomial log-likelihood):

$$\theta_{MLE} = \arg \min_\theta - \sum_{i=1}^{N} y_{ij} \ln \widehat{s}_{ij}(\theta)$$

▸ We need to perform numerical integration to get $\widehat{s}_{ij}(\theta)$ and its derivative: $\widehat{\frac{\partial s_{ij}}{\partial \theta}}$.

▸ Consistency
  ▸ Technically for fixed number of MC draws the MSLE estimator is inconsistent. Why?
  ▸ Very accurate integration is even more important!

## Estimation Alternatives: MSM

Can also do a simulated GMM estimator with moment conditions:

$$E\left[(y_{ij} - \widehat{s}_{ij}(\theta)) | z_{ij}\right] = 0$$

- We need to perform numerical integration to get $\widehat{s}_{ij}(\theta)$ and its derivative: $\widehat{\frac{\partial s_{ij}}{\partial \theta}}$.
- The optimal instruments in the sense of Chamberlain (1987) or Amemiya (1977) are the derivative of moments with respect to parameters (scores): $z_{ij} = \frac{\partial \log s_{ij}}{\partial \theta}(\theta)$.
- The true scores are infeasible (they depend on $\theta_0$).
  - At true scores: same FOC as MLE.
  - Simulated scores have same consistency issue as MSLE.

**Estimation Alternatives: MSS**

As an alternative we could work with the score function directly:

$$\sum_{i=1}^{n} \widehat{\frac{\partial \log s_{ij}}{\partial \theta}}(\theta) = \sum_{i=1}^{N} \frac{1}{\widehat{s_{ij}}(\theta)} \cdot \widehat{\frac{\partial s_{ij}}{\partial \theta}}(\theta) = 0$$

- We can get unbiased estimated of derivative (linearity)
- We can't get unbiased estimate of $\frac{1}{\widehat{s_{ij}}(\theta)}$
  - Train's book suggests Accept-Reject
  - Maybe importance sampling would work?

## Estimation Details: Hints

How bad is the simulation error?

- Depends how small your shares are.
- Since you care about $\log s_{jt}$ when shares are small, tiny errors can be enormous.
- Often it is pretty bad.

## Estimation Details: Recommendations

You should read these separate notes, but my recommendations are:

1. Numerical integration:
   - Use Gauss-Hermite quadrature rules to integrate over normal densities.
   - Follow Heiss and Winschel (JoE) and do sparse grids in medium dimensions
   - Try http://sparse-grids.de.
2. Nonlinear optimization: with mixtures the problem is non-convex (ie: very hard)
   - You must provide analytic gradients.
   - You should use a quasi-Newton solver and may need to try several.
   - Try multiple starting values.
3. Generalized Method of Moments
   - You can read more about optimal IV in these settings.