

# Persistent Unobservables

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C.Conlon - Adapted from M. Shum

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# Persistent Unobserved Heterogeneity

Suppose we think about a model with a friction such as a switching cost.

- ▶ If  $y_{it} \neq y_{i,t-1}$  you pay a switching cost  $F_i$ .
- ▶ How do we use data to tell apart large switching costs  $F_i \gg 0$  from persistent tastes  $Cov(\varepsilon_{i,t}, \varepsilon_{i,t-1}) > 0$  ?
- ▶ The **conditional independence assumption** tells us it has to be the switching cost not the autocorrelated unobservables.
- ▶ This is probably why people don't like this assumption.

# Discrete Unobserved Types

- ▶ Up until now we consider models satisfying Rust's **conditional independence** assumption on the  $\varepsilon$ 's. This rules out persistence in unobservables which are economically meaningful.
- ▶ Suppose there are two types of buses good ( $s_i = g$ ) and bad ( $s_i = b$ ).
- ▶ Assume that this is known to HZ but not the econometrician.
- ▶ Single period utility now depends on  $s_i$  so  $u(x_{it}, s_i, d_{it}; \theta)$  **unobserved state variable**.
- ▶ In case of the nested fixed point algorithm, this unobserved persistent heterogeneity is not a big problem as we can solve for the value function (and expected policy functions) given the state variables and **integrate it out** in the likelihood

# Unobserved State Variables: What happened?

$$\mathbb{P}(d_{i1}, \dots, d_{iT} | x_{i1}, \dots, x_{iT}) = \sum_s \prod_{t=1}^T \mathbb{P}(d_{it} | x_{it}) p(s_i)$$

- ▶ Conditional on  $s_i$  replacement decisions are independent across  $t$  given  $x_{it}$ .
- ▶ The resulting likelihood is just a **finite mixture model**.
- ▶ These can be hard to solve when both  $s_i$  and its distribution  $p(s_i)$  are unknown.
- ▶ Arcidiacono and Miller (2011) provide theoretical results for these types of problems.

# A much earlier application

## Pakes (1986): Patents as Options

How much are patents worth? Valuable for optimal patent length and design? Sufficient incentive for innovation?

- ▶  $Q_A$ : value of patent at age  $A$
- ▶ Goal of paper is to estimate  $Q_A$  using data on their renewal.  $Q_A$  is inferred from patent renewal process via **revealed preference** for patent renewal behavior.
- ▶ Treat renewal systems as exogenous (in Europe)

## Timing

- ▶ For  $a = 1, \dots, L$  a patent can be renewed by paying the fee  $c_a$ .
- ▶ At age  $a = 1$  patent holder gets  $r_1$  from patent
- ▶ Decide whether or not to renew (pay  $c_1$  and go to  $a_2$ ).
- ▶ At age  $a = 2$  get  $r_2$  from patent
- ▶ and so on...

Gives us the value function

$$V \equiv \max_{t \in [a, L]} \sum_{a'=1}^{L-a} \beta^{a'} R(a + a')$$
$$R(a) = \begin{cases} r_a - c_a, & \text{if } t \geq a \text{ when you hold patent} \\ 0 & \text{if } t < a \text{ after patent expires} \end{cases}$$

- ▶  $t$  above denotes the age which allows the patent to expire and is the choice variable. Another **optimal stopping** problem.
- ▶  $R(a)$  are the profits from year  $a$ . This is a **controlled stochastic process**. It is random but affected by the actions of the agent.

## Pakes (1986)

- ▶ The maximum age  $L$  is finite so it is finite-horizon DP.
- ▶ The single period revenue  $r_a$  is the state variable.
- ▶ We can solve the problem with *backward recursion*.

$$V_a(r_a) = \max \{0, Q_a \equiv r_a + \beta \mathbb{E}[V_{a+1}(r_{a+1})|\Omega_a] - c_a\}$$

- ▶ Renew iff  $Q_a - c_a > 0$ .
- ▶  $\Omega_a$ : history up to age  $a = \{r_1, r_2, \dots, r_a\}$ .
- ▶ Expectation is over  $r_{a+1}|\Omega_a$ . The sequence of conditional distributions  $G_a \equiv F(r_{a+1}|\Omega_a)$ ,  $a = 1, 2, \dots$  is an important component of model specification.

$$r_{a+1} = \begin{cases} 0 & \text{w. prob } \exp(-\theta r_a) \\ \max(\delta r_a, z) & \text{w. prob } 1 - \exp(-\theta r_a) \end{cases}$$

# Pakes (1986)

Model has the following parameters

- ▶ density of  $z$   $q_a = \frac{1}{\sigma_a} \exp[-(\gamma + z)/\sigma_a]$  and  $\sigma_a = \phi^{a-1}\sigma$ , for  $a = 1, \dots, L - 1$ .
- ▶  $(\delta, \theta, \gamma, \phi, \sigma)$  are the structural parameters of the model
- ▶ Break down the model period by period and decide whether or not to renew if  $Q_a = r_a +$  “option value”.
- ▶ Option value is about keeping the patent alive in case it pays off in the future.

Implications

- ▶ Drop out at age  $a$  if  $c_a > Q_a$
- ▶ Optimal decision is characterized by cutoff points  $Q_a > c_a \Leftrightarrow r_a > \bar{r}_a$  (Key assumptions is  $Q_a$  increasing /single crossing )
- ▶ Cutoff points are increasing sequence  $\bar{r}_a < \bar{r}_{a+1} < \dots < \bar{r}_{L-1}$ .



# Estimation

Instead of using Pakes' notation  $r_t$  for the patent revenue. We will use the generic Rust notation of  $\epsilon_t$  the unobserved state variable, and  $i_t$  to denote the choice (renewal).

- ▶ For a single patent  $\tilde{T}$  denotes the age at which it is allowed to expire. Let  $T = \min(L - 1, \tilde{T})$  denote the period sins which the agent makes a renewal decision where we model the agent's choice.
- ▶  $\epsilon$  follows a first-order Markov process  $F(\epsilon'|\epsilon)$
- ▶ Age-specific policy function by  $i_t^*(\epsilon)$ .

Likelihood function is

$$\ell(i_1, \dots, i_T | \epsilon_0, i_0, \theta) = \prod_{t=1}^T \mathbb{P}(i_t | i_0, \dots, x_{t-1}, i_{t-1}; \epsilon_0, \theta)$$

Serial correlation in  $\epsilon$  means there is dependence among  $i_t, i_{t-2}$  even after conditioning on  $x_{t-1}, i_{t-1}$ .

- ▶ It might seem like we were stuck since it no longer has a closed form. However, we can simulate the “outer loop” of the nested fixed point routine given a guess of  $i_t^*(\varepsilon, \theta)$ .
- ▶ Because  $\varepsilon$  is serially correlated we need to start with an initial  $i_0$  (or distribution) and assume that it is known. This is the **initial conditions problem** of finite MDPs.
- ▶ Note that simulation is part of the “outer loop” of nested fixed point estimation routine. So at the point when we simulate, we already know the policy functions  $i_t^*(\varepsilon, \theta)$  (How would you compute this?)

## Naive Frequency Simulator (Don't do this...)

Go back to the full likelihood function (condition on initial  $\varepsilon_0$  for serial correlation):

$$\ell(i_1, \dots, i_T | i_0, \varepsilon_0, \theta) = \mathbb{P}(i_t^*(\varepsilon_t, \theta) = i_t, \forall t = 1, \dots, T)$$

Need to take probability over distribution of  $(\varepsilon_1, \dots, \varepsilon_T | \varepsilon_0)$ . Let  $F(\varepsilon_{t+1} | \varepsilon_t, \theta)$  then the above probability can be expressed as the integral:

$$\int \cdots \int \prod_t \mathbf{1}(i_t^*(\varepsilon_t, \theta) = i_t) \prod_t dF(\varepsilon_t | \varepsilon_{t-1}; \theta)$$

Simulate by drawing sequences of  $(\varepsilon_t)$ .

# Naive Frequency Simulator (Don't do this...)

Simulate by drawing sequences of  $(\varepsilon_t)$  and for each draw  $s = 1, \dots, S$  we take as initial values  $(x_0, i_0, \varepsilon_0)$  then

- ▶ Generate  $(\varepsilon_1^s, i_1^s)$ 
  1. Generate  $\varepsilon_1^s \sim F(\varepsilon_1 | \varepsilon_0)$
  2. Compute  $i_1^s = i_1^*(\varepsilon_1^s; \theta)$
- ▶ Generate  $(\varepsilon_2^s, i_2^s)$ 
  1. Generate  $\varepsilon_2^s \sim F(\varepsilon_2 | \varepsilon_1^s)$
  2. Subsequently compute  $i_2^s = i_2^*(\varepsilon_2^s; \theta)$
- ▶ And so on, up to  $(\varepsilon_T^s, i_T^s)$ .

## Naive Frequency Simulator (Don't do this...)

And for the case where  $(i, x)$  are both discrete (Rust) we can approximate:

$$\ell(i_t, \dots, i_T | \varepsilon_0, i_0; \theta) \approx \frac{1}{S} \sum_s \prod_{t=1}^T \mathbf{1}(i_t^s = i_t)$$

Frequency of simulated sequences which match observed sequence.  $T$  long or  $S$  small you're in trouble (non-smooth).

# Importance Sampling: Particle Filtering

- ▶ We can use importance sampling to simulate the likelihood function.
- ▶ This is not straightforward given time dependence in  $(i_t, \epsilon_t)$
- ▶ Consider particle filtering approach from Fernandez-Villaverde and Rubio-Ramirez (2007) or Flury and Shehard (2008) (non-Gaussian Kalman filtering).
- ▶ A more up to date take: Blevins (2016) : Sequential Monte Carlo Methods for Estimating Dynamic Microeconomic Models

# Importance Sampling: Particle Filtering

- ▶ Evolution of utility shocks  $\epsilon_t | \epsilon_{t-1} \sim f(\epsilon' | \epsilon)$ . Ignore dependence of distribution of  $\epsilon$  on age  $t$  for convenience.
- ▶ As before, the policy function is  $i_t = i^*(\epsilon_t)$
- ▶ Let  $\epsilon^t \equiv \{\epsilon_1, \dots, \epsilon_t\}$ .
- ▶ The initial values of  $y_0$  and  $\epsilon_0$  are known

Go back to the factorized likelihood

$$\begin{aligned} l(y^T | y_0, \epsilon_0) &= \prod_{t=1}^T l(y_t | y^{t-1}, y_0, \epsilon_0) = \prod_{t=1}^T \int l(y_t | \epsilon^t, y^{t-1}) p(\epsilon^t | y^{t-1}) d\epsilon^t \\ &\approx \frac{1}{S} \sum_s l(y_t | \epsilon^{t|t-1,s}, y^{t-1}) \end{aligned}$$

We omit conditioning on  $(\epsilon_0, y_0)$  for convenience, and  $\epsilon^{t|t-1,s}$  is a simulated draw of  $\epsilon^t \sim p(\epsilon^t | y^{t-1})$ .

# Importance Sampling: Particle Filtering

Let's look more closely at the last line:

- ▶ first term:  $l(y_t, |\epsilon^t, y^{t-1})$  we can calculate for a value of  $\epsilon_t$

$$l(y_t|\epsilon^t, y^{t-1}) = p(i_t|\epsilon^t, y^{t-1}) = p(i_t|\epsilon_t) = \mathbf{1}(i(\epsilon_t) = i_t)$$

- ▶ the second term  $p(\epsilon^t|y^{t-1})$  is generally not obtainable in closed form. So numerical integration is not feasible. Particle filtering let's us draw  $\epsilon^t$  from this distribution for every period  $t$ .

Particle filtering proposes a recursive approach to draw sequences  $p(\epsilon^t|y^{t-1})$  for every  $t$



# Particle Filtering Algorithm

**First period:**  $t = 1$  In order to simulate the integral corresponding to the first period we need to draw from  $p(\epsilon^1|y^0, \epsilon_0)$  (easy).

- ▶ We draw  $\{\epsilon^{1|0,s}\}_{s=1}^S$  according to  $f(\epsilon'|\epsilon_0)$ .
- ▶ The notation  $\epsilon^{1|0,s}$  makes it explicit that the  $\epsilon$  is a draw from  $p(\epsilon^1|y^0, \epsilon_0)$
- ▶ Use the  $S$  draws we can evaluate the period  $t = 1$  likelihood.

**Second period:**  $t = 2$ . We need to draw from  $p(\epsilon^2|y^1)$  factorize as:

$$p(\epsilon^2|y^1) = p(\epsilon^1|y^1) \cdot p(\epsilon_2|\epsilon^1) \text{ recall } \epsilon^2 \equiv \{\epsilon_1, \epsilon_2\}$$

## Filtering Step

Getting a draw from  $p(\epsilon^1|y^1)$ , given that we already have draws  $\{\epsilon^{1|0,s}\}$  from  $p(\epsilon^1|y_0)$ , from the previous period  $t = 1$ , is the heart of particle filtering. We use the principle of importance sampling: by Bayes' Rule

$$p(\epsilon^1|y^1) \propto p(y_1|\epsilon^1, y^0) \cdot p(\epsilon^1|y^0)$$

Hence, if our desired sampling density is  $p(\epsilon^1|y^1)$ , but we actually have draws  $\{\epsilon^{1|0,s}\}$  from  $p(\epsilon^1|y^0)$ , then the importance sampling weight for the draw  $\epsilon^{1|0,s}$  is proportional to

$$\tau_1^s \equiv p(y_1|\epsilon^{1|0,s}, y^0)$$

Note that this coincides with the likelihood contribution for period 1, evaluated at the shock  $\epsilon^{1|0,s}$ . The SIR algorithm in Rubin (1988) proposes that making  $S$  draws with replacement from samples  $\{\epsilon^{1|0,s}\}_{s=1}^S$ , using weights proportional  $\tau_1^s$  yields draws from the desired density  $p(\epsilon^1|y^1)$  which we denote  $\{\epsilon^{1|0,s}\}_{s=1}^S$ .

## Prediction Step

For the second term in the equation: we simply draw one  $\epsilon_2^s$  from  $f(\epsilon'|\epsilon^{1,s})$ , for each draw  $\epsilon^{1,s}$  from the filtering step. This is the **prediction** step.

By combining the draws from these two terms, we have  $\{\epsilon^{2|1,s}\}_{s=1}^S$ . which is  $S$  drawn sequences from  $p(\epsilon^2|y^1)$ . Using these  $S$  draws, we can evaluate the simulated likelihood for period 2

## Prediction Step (Continued)

**Third period,  $t = 3$ :** start again by factoring

$$p(\epsilon^3|y^2) = p(\epsilon^2|y^2) \cdot p(\epsilon^3|\epsilon^2)$$

As above, drawing from requires filtering the draws  $\{\epsilon^{2|1,s}\}_{s=1}^S$ , from the previous period  $t = 2$ , to obtain draws  $\{\epsilon^{2,s}\}_{s=1}^S$ . Given these draws, draw  $\epsilon_3^s \sim f(\epsilon'|\epsilon^{2,s})$  for each  $s$ .

And so on. By the last period  $t = T$ , you have

$$\left\{ \left\{ \epsilon^{t|t-1,s} \right\}_{s=1}^S \right\}_{t=1}^T$$

## Prediction Step (continued)

Hence the factorized likelihood can be approximated by simulation as:

$$\prod_t \frac{1}{S} \sum_s l(y_t | \epsilon^{t|t-1,s}, y^{t-1})$$

As noted above, the likelihood term  $l(y_t | \epsilon^{t|t-1,s}, y^{t-1})$  coincides with the simulation weight  $\tau_t^s$ . Hence the simulated likelihood can also be constructed as:

$$\log l(y^T | y_0, \epsilon_0) = \sum_t \log \left\{ \frac{1}{S} \sum_s \tau_t^s \right\}$$

# Particle Filtering (Summary)

- ▶ Start by drawing  $\{\epsilon^{1|0,s}\}_{s=1}^S$  from  $p(\epsilon^1|y^0, \epsilon_0)$ .
- ▶ In period  $t$ , we start with  $\{\epsilon^{t-1|t-2,s}\}_{s=1}^S$  draws from  $p(\epsilon^{t-1}|y^{t-2}, \epsilon_0)$ .
  1. **Filter step:** Calculate proportion weights  $\tau_{t-1}^s \equiv p(y_{t-1}|\epsilon^{t-1|t-2,s}, y^{t-2})$  using  $p(i_t|\epsilon_t)$ . Draw  $\{\epsilon^{t-1|t-1,s}\}_{s=1}^S$  by resampling from  $\{\epsilon^{t-1|t-2,s}\}_{s=1}^S$  with weights  $\tau_{t-1}^s$ .
  2. **Prediction step:** Draw  $\epsilon_t^s$  from  $p(\epsilon_t|\epsilon^{t-1|t-1,s})$ , for  $s = 1, \dots, S$ . Combine to get  $\{\epsilon^{t|t-1,s}\}_{s=1}^S$ .
- ▶ Set  $t = t + 1$  and go back to step 2. Stop when  $t = T + 1$ .

The difference is that the crude simulator draws  $S$  sequences and puts zero weight on those which don't match the observed sequence. In each period  $t$  we just keep sequences where predicted choices match observed choice of *that period*. This is more accurate likelihood as long as  $S$  is large enough that we don't have all the weight on a single sequence in period  $t$ .

- ▶ Fernandez-Villaverde, J., and J. Rubio-Ramirez (2007): “Estimating Macroeconomic Models: A Likelihood Approach,” *Review of Economic Studies*, 74, 1059-1087.
- ▶ Flury, T., and N. Shephard (2008): “Bayesian inference based only on simulated likelihood: particle filter analysis of dynamic economic models,” manuscript, Oxford University
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- ▶ Rubin, D. (1988): “Using the SIR Algorithm to Simulate Posterior Distributions,” in *Bayesian Statistics 3*, ed. by J. Bernardo, M. DeGroot, D. Lindley, and A. Smith. Oxford University Press.