A Technique for Estimating a Contingency Table, Given the Marginal Totals and Some Supplementary Data

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SUMMARY

The problem considered is that of estimating the cell frequencies in a contingency table, given the marginal totals and some information about the internal structure of the table. A technique used by the General Register Office of England and Wales is examined from a theoretical point of view, and the relationship between this method and other methods proposed by Deming and Stephan is discussed.

1. INTRODUCTION

A TECHNIQUE for dealing with this problem has been in use for many years, notably by the General Register Office of England and Wales in handling demographic data. It appears that its underlying theory has never been studied and thus, for instance, it is not clear in what circumstances it is applicable. Lack of justification of it may also account for its apparently not being well known, although its use is clearly not restricted to the demographic field.

This paper describes the technique briefly and gives its basic theory. Essentially the situation considered is not probabilistic, i.e. it is not possible to define the situation in terms of probabilities. It has nevertheless seemed desirable to associate its solution with that of a probabilistic situation considered by Deming and Stephan (1940). (See also Stephan (1942) and Deming (1943, pp. 121–124). The merits of the solution proposed by Deming and Stephan and the second solution proposed by Stephan are discussed at some length by Smith (1947).)

2. THE PROBLEM

Consider a two-dimensional integral distribution where only the two marginal distributions (the sums along the rows and columns) are given, and suppose some information is available about the structure of the inner distribution. It is required to estimate the internal values.

A typical example in demography would be the estimation of a population by both age and marital condition, given the marginal distributions, by age for all marital conditions and by marital condition for all ages. Secondary information, essentially on the association between age and marital condition, i.e. relevant to age-at-marriage, might be available from a census taken a few years earlier, or from another population having the same religious or ethnic characteristics, i.e. these sources might supply values to enter in the internal cells of the table as a first approximation, notwithstanding the fact that they aggregated neither to the row nor to the column totals given.

3. THE PRACTICAL SOLUTION

Call the chosen internal distribution the first approximation. The first approximation rows will, in general, not aggregate to the appropriate given row totals. The differences may be dissipated along each row, *pro rata* to the first approximations, as first increments (positive or negative). The second approximations (= first approximations plus first increments) will now add along rows to the given row totals, but will not, in general, add down columns to the given column totals. Third approximations are similarly produced by adding second increments to the second approximations, the second increments being a dissipation down columns of the differences, distributed *pro rata* to the first approximations.

Further approximations are obtained by continuing to work alternately on rows and columns, the differences in every case being dissipated *pro rata* to the first approximations. In practice it usually happens that adequate accuracy is achieved after only a few iterations.

Alternatively, the process might be operated starting with the columns instead of the rows.

4. WORKED EXAMPLE

Consider the estimation of the number of women in England and Wales at mid-1958 by age and marital condition, given the correct marginal distributions for 1958 and using the 1957 structure as the first approximation. The data are given in Table 1.

The first approximation row for age group 15–19 adds to a total 23,000 less than the given 1958 control. This deficiency is dissipated by a positive first increment, distributed along this row in the proportions of (1,306 : 83 : 0) to 1,389, and so on for all the rows, and then for the columns.

TABLE 1

Estimated female population by age and marital condition, England and Wales

(*thousands*)

Age group	*Mid*-1957 *Official estimate* (*first approximation*)			*Mid*-1958 *As estimated by this technique*			*Mid*-1958 *Official estimate*			
	S†	*M*†	*W & D*†	*S*	*M*	*W & D*	*S*	*M*	*W & D*	*All*‡
15–19	1,306	83	—	1,326	86	—	1,326	86	—	1,412
20–24	619	765	3	614	785	3	613	787	2	1,402
25–29	263	1,194	9	254	1,187	9	250	1,192	8	1,450
30–34	173	1,372	28	164	1,350	27	163	1,356	22	1,541
35–39	171	1,393	51	172	1,456	53	174	1,457	50	1,681
40–44	159	1,372	81	147	1,309	76	145	1,309	78	1,532
45–49	208	1,350	108	203	1,352	107	199	1,354	109	1,662
50+	1,116	4,100	2,329	1,108	4,177	2,359	1,118	4,161	2,365	7,644
15 and over	4,015	11,629	2,609	3,988	11,702	2,634	3,988	11,702	2,634	18,324

Source: Registrar General.

† Single, married, widowed and divorced.

‡ All marital conditions.

It will be seen from the above example that for the relatively large groups the estimates do not diverge from the official figures by more than 2 per cent. Much worse results were obtained for smaller groups, mainly for "widowed and divorced" at the younger ages. This example shows that estimates produced with this technique should be accepted with caution.

5. Alternative Techniques

An alternative technique, apparently so similar to that of the Registrar General that it has been mistaken for it (see Grauman, 1959, p. 567), was proposed by Deming and Stephan (1940). It differs from the above by dealing with the approximations directly, instead of increments, and is tantamount to the above technique modified by dissipating the nth difference *pro rata* to the nth approximation. The theory of this method is, however, intractable. A second alternative, due to Stephan (1942) (see also Deming, 1943), will be referred to later.

6. Mathematical Theory

Notation

Denote by X_{ij} the first approximation value in the ith row and jth column ($i = 1, \ldots, n; j = 1, \ldots, m$). Define

$$\sum_j X_{ij} = X_{i\cdot}, \quad \sum_i X_{ij} = X_{\cdot j} \quad \text{and} \quad \sum_{ij} X_{ij} = X_{\cdot\cdot}.$$

Let L_i and M_j be the given marginal totals of rows and columns respectively, so that

$$\sum_i L_i = \sum_j M_j.$$

Let x_{ij} denote the required value to be estimated for the ith row and the jth column.

Solution

The nature of the first approximation implies that no mathematical statement can be made about the probability of obtaining any particular difference between the first approximation and the "correct" values. The problem may, however, be considered this way.

The values chosen for the internal cells must aggregate to the marginal totals and should compromise between lying close to the first approximation values for some cells and deviating substantially for others and *vice versa*. The selected values should then be not "too far" from the first approximation. A criterion of "how far" a set of cell values lies from the first approximation, i.e. a "distance" function, is then provided by the χ^2-type function:

$$\sum_{ij} \frac{(x_{ij} - X_{ij})^2}{X_{ij}}.$$

Deming and Stephan (1940) were examining a different problem which was capable of a mathematical formulation, and which led them to this form of χ^2. For comments, see Smith (1947, pp. 244–247). For another use of a χ^2-type function, when a mathematical formulation was not possible, see Glass and Grebenik (1954, p. 87).

A solution is then provided by minimizing the χ^2-type function under the restraints

$$\sum_j x_{ij} = L_i \quad \text{and} \quad \sum_i x_{ij} = M_j.$$

It is shown in Appendix I that the iterative process used by the General Register Office produces results that converge to the values of x_{ij} that minimize the χ^2-type function under the restraints. Stephan (1942)† also considered the solution of these equations by an iterative process but his procedure seems less direct than that used by the General Register Office, and would certainly require more expert computing staff. Stephan's solution is discussed in Appendix II.

7. DISCUSSION

There are several justifications for studying the theory of a problem that is not probabilistic.

First, consider the problem as follows. Denoting the two dimensions of the table by *A* and *B*, the data comprise information on the precise current "*A*" and "*B*" structures and less precise information on the "*AB*" interaction. The problem of finding a solution, bearing in mind these three conditions, is one typically appropriate to a mathematical solution but with the minimum additional assumptions as to the nature of the data. (See Smith (1947, pp. 249–252) for a very simple solution, requiring however the assumption of proportionality along rows and columns.) One requires a knowledge of the basic theory of the arithmetical processes employed to ensure that no inappropriate assumptions are involved.

Secondly, without the theoretical study, it might appear that the iterative solution was purely arbitrary.

Thirdly, some practical questions are immediately answered, for instance the result when starting with the columns first is seen from the theory to be the same as that starting first with the rows.

Although the theory studied is closely analogous to that of the probabilistic situation examined by Deming and Stephan, the present problem is not probabilistic and so no question arises as to the standard errors involved or whether a better arithmetical technique may be found leading to less error. Indeed, in actual applications of the technique, it is not further consideration of theoretical aspects that is important but attention to practical detail. For instance, where there is no objective criterion on which the first approximation may be chosen, great care is required in its selection subjectively; the results may be poor unless it is chosen wisely.

ACKNOWLEDGEMENT

I am indebted to Mr N. H. Carrier for drawing my attention to this problem, and suggesting the field in which a solution might lie.

APPENDIX I

General Theory and the Iterative Solution of the General Register Office

Formal Solution

The function

$$\sum_{ij} \frac{(x_{ij}-X_{ij})^2}{X_{ij}}$$

is to be minimized under the $n+m$ restraints:

$$\sum_j x_{ij} = L_i \quad \text{and} \quad \sum_i x_{ij} = M_j \quad (i = 1, \ldots, n; j = 1, \ldots, m).$$

† See also Deming's (1943) account of Stephan's method.

Introducing $n+m$ Lagrange's multipliers λ_i $(i = 1, \ldots, n)$, μ_j $(j = 1, \ldots, m)$, differentiating with respect to x_{ij}, and rearranging, we have:

$$x_{ij} = X_{ij} + \lambda_i X_{ij} + \mu_j X_{ij} \quad (i = 1, \ldots, n, j = 1, \ldots, m); \tag{1}$$

and the restraints equations:

$$\sum_j x_{ij} = L_i, \tag{2}$$

$$\sum_i x_{ij} = M_j \tag{3}$$

for determining the unknown λ_i and μ_j values.

Summing (1) over i and alternatively over j,

$$(L_i/X_{i\cdot} - 1) - \lambda_i - \sum_j \mu_j X_{ij}/X_{i\cdot} = 0, \tag{4}$$

$$(M_j/X_{\cdot j} - 1) - \sum_i \lambda_i X_{ij}/X_{\cdot j} - \mu_j = 0. \tag{5}$$

Notation

Define the following $n \times m$ matrices:

$$\mathbf{P} = (X_{ij}/X_{i\cdot}) \quad \text{and} \quad \mathbf{Q} = (X_{ij}/X_{\cdot j}),$$

$1 \times n$ matrices:

$$\boldsymbol{\lambda} = (\lambda_i) \quad \text{and} \quad \mathbf{L} = (L_i/X_{i\cdot} - 1),$$

and $1 \times m$ matrices:

$$\boldsymbol{\mu} = (\mu_j) \quad \text{and} \quad \mathbf{M} = (M_j/X_{\cdot j} - 1).$$

Let the vector formed by the kth column of any matrix $\mathbf{A}$ be denoted by $[\mathbf{A}]^{(k)}$.

Matrix Solution

Equations (4) and (5) may be rewritten:

$$\mathbf{L} - \boldsymbol{\lambda} - \boldsymbol{\mu}\mathbf{P}' = 0 \tag{6}$$

and

$$\mathbf{M} - \boldsymbol{\lambda}\mathbf{Q} - \boldsymbol{\mu} = 0. \tag{7}$$

Hence

$$(\mathbf{L} - \mathbf{M}\mathbf{P}') - \boldsymbol{\lambda}(\mathbf{I} - \mathbf{Q}\mathbf{P}') = 0, \tag{8}$$

where $\mathbf{I}$ denotes a unit matrix of the appropriate size.

The matrix $(\mathbf{I} - \mathbf{Q}\mathbf{P}')$ can be shown to be singular and therefore its reciprocal does not exist. Thus $\boldsymbol{\lambda}$ cannot be calculated directly from (8), and indeed will, in general, not be unique; nor will $\boldsymbol{\mu}$.

If the equations are consistent† (and if they are not, there is of course no solution), suppose that there are $(k+1)$ independent sets of solutions for $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. Then there exist matrices (with numbers of rows and columns as shown): $\mathbf{A}$ $(1, k)$; $\mathbf{R}$ $(1, n)$; $\mathbf{S}$ (k, n); $\mathbf{T}$ $(1, m)$; $\mathbf{U}$ (k, m), such that any solution can be expressed in the form

† It can be shown that in general the condition for consistency is that

$$\sum_i L_i = \sum_j M_j,$$

but in certain degenerate cases this condition is necessary but not sufficient, and more restrictions have to be imposed upon L_i and M_j.

$\boldsymbol{\lambda} = \mathbf{R}+\mathbf{AS}$ and $\boldsymbol{\mu} = \mathbf{T}+\mathbf{AU}$ and any $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ given by these equations will be a solution, where $\mathbf{R}, \mathbf{S}, \mathbf{T}, \mathbf{U}$ are fixed and $\mathbf{A}$ is an arbitrary set of parameters. Two sets of solutions of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ will be said to correspond if and only if they satisfy the above equations with the same value of $\mathbf{A}$.

Since any solution must satisfy equation (6) whatever the value of $\mathbf{A}$, by substituting the above parametric forms of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ into equation (6), it is seen that $\mathbf{L}-\mathbf{R}-\mathbf{TP}' = 0$ and $\mathbf{S}+\mathbf{UP}' = 0$.

Suppose that solutions of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ derived from parametric values $\mathbf{A}$ and $\mathbf{A}'$ respectively satisfy equation (6). By substitution into equation (6) and reduction it is seen that $(\mathbf{A}-\mathbf{A}')\mathbf{S} = 0$. If this is so, unless $\mathbf{A} = \mathbf{A}'$, it would be possible to reduce the parametric form of $\boldsymbol{\lambda}$ and thus express any solution of $\boldsymbol{\lambda}$ in terms of $(k-1)$ parameters, which is contrary to the hypothesis. Thus, if two solutions of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ satisfy equation (6), they must correspond.

It can be shown that, despite the indeterminacy of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, x_{ij} is uniquely determined, and that the solutions are obtained by substituting corresponding solutions of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ into equation (1). It will now be shown that the iterative process produces a solution which converges to the solution of the above system of equations.

The General Register Office Iterative Solution

Consider the first step in the iterative process. The ith row of X_{ij} aggregates to $X_{i\cdot}$ instead of to L_i. The deficiency $(L_i - X_{i\cdot})$ is dissipated by a positive increment obtained by distributing the deficiency *pro rata* to the first approximation X_{ij}.

Denoting generally the Nth increment to X_{ij} by ${}_NE_{ij}$, we have, ${}_1E_{ij} = X_{ij}[\mathbf{L}]^{(i)}$. The jth column has a deficiency, after the first increment has been added, of

$$(M_j - X_{\cdot j}) - \sum_i (L_i - X_{i\cdot})\, X_{ij}/X_{i\cdot}\,.$$

After dissipating this along columns, we have

$$\begin{aligned} {}_2E_{ij} &= X_{ij}\{(M_j/X_{\cdot j}-1) - \sum_i (L_i/X_{i\cdot}-1)\, X_{ij}/X_{\cdot j}\} \\ &= X_{ij}[\mathbf{M}-\mathbf{LQ}]^{(j)}. \end{aligned}$$

Similarly $${}_3E_{ij} = X_{ij}[-\mathbf{MP}'+\mathbf{LQP}']^{(i)},$$

and $${}_4E_{ij} = X_{ij}[\mathbf{MP}'\mathbf{Q}-\mathbf{LQP}'\mathbf{Q}]^{(j)}, \quad \text{etc.}$$

Hence the $2N$th approximation for cell (i,j) is

$$\begin{aligned} &X_{ij}+{}_1E_{ij}+{}_2E_{ij}, \ldots + {}_{2N}E_{ij} \\ &\quad = X_{ij}\{1+[\mathbf{L}-\mathbf{MP}'+\mathbf{LQP}'-\mathbf{MP}'\mathbf{QP}'\ldots-\mathbf{MP}'(\mathbf{QP}')^{(N-2)}+\mathbf{L}(\mathbf{QP}')^{(N-1)}]^{(i)} \\ &\qquad\quad +[\mathbf{M}-\mathbf{LQ}+\mathbf{MP}'\mathbf{Q}-\mathbf{LQP}'\mathbf{Q}\ldots+\mathbf{M}(\mathbf{P}'\mathbf{Q})^{(N-1)}-\mathbf{LQ}(\mathbf{P}'\mathbf{Q})^{(N-1)}]^{(j)}\} \\ &\quad = X_{ij}\{1+[(\mathbf{L}-\mathbf{MP}')\{\mathbf{I}+\mathbf{QP}'+\ldots+(\mathbf{QP}')^{(N-2)}\}+\mathbf{L}(\mathbf{QP}')^{(N-1)}]^{(i)} \\ &\qquad\quad +[(\mathbf{M}-\mathbf{LQ})\{\mathbf{I}+\mathbf{P}'\mathbf{Q}+\ldots+(\mathbf{P}'\mathbf{Q})^{(N-1)}\}]^{(j)}\}. \end{aligned} \tag{9}$$

Now, if it can be shown that, as $N \to \infty$

$$[(\mathbf{L}-\mathbf{MP}')\{\mathbf{I}+\mathbf{QP}'+\ldots+(\mathbf{QP}')^{(N-2)}\}+\mathbf{L}(\mathbf{QP}')^{(N-1)}] \to \text{a solution for } \boldsymbol{\lambda} \tag{10}$$

and $$[(\mathbf{M}-\mathbf{LQ})\{\mathbf{I}+\mathbf{P}'\mathbf{Q}+\ldots+(\mathbf{P}'\mathbf{Q})^{(N-1)}\}]\to \text{a solution for } \boldsymbol{\mu}, \tag{11}$$

then the limiting form of (9) will solve (1), providing these solutions correspond.

To be a solution for $\boldsymbol{\lambda}$, the limiting form of the left-hand side of equation (10) must satisfy equation (8). Multiply (10) by $(\mathbf{I}-\mathbf{QP}')$. We have

$$(\mathbf{L}-\mathbf{MP}')\{\mathbf{I}+\mathbf{QP}'+\ldots+(\mathbf{QP}')^{(N-2)}\}(\mathbf{I}-\mathbf{QP}')+\mathbf{L}(\mathbf{QP}')^{(N-1)}(\mathbf{I}-\mathbf{QP}')$$
$$=(\mathbf{L}-\mathbf{MP}')+\mathbf{MP}'(\mathbf{QP}')^{(N-1)}-\mathbf{L}(\mathbf{QP}')^{N}.$$

But it can be proved that, as $N\to\infty$, $(\mathbf{QP}')^N\to[c_{ij}]$, where $c_{ij}=X_{i\cdot}/X_{\cdot\cdot}$, that is, does not depend on j. (This is a property which holds generally for the Nth power, as $N\to\infty$, of any stochastic matrix, that is a matrix of non-negative elements whose columns add to unity. A detailed proof is given, for instance, by Doob (1953, p. 173).)

Thus, as $N\to\infty$, an element of $\mathbf{MP}'(\mathbf{QP}')^{(N-1)}$ will tend to

$$\sum_i\sum_j (M_j/X_{\cdot j}-1)(X_{ij}/X_{i\cdot})(X_{i\cdot}/X_{\cdot\cdot})=\sum_j (M_j/X_{\cdot j}-1)(X_{\cdot j}/X_{\cdot\cdot})$$
$$=\left(\sum_j M_j\right)/X_{\cdot\cdot}-1,$$

and similarly an element of $\mathbf{L}(\mathbf{QP}')^N$ tends to

$$\left(\sum_j L_i\right)/X_{\cdot\cdot}-1.$$

Hence, as $N\to\infty$, $$\mathbf{MP}'(\mathbf{QP})^{(N-1)}-\mathbf{L}(\mathbf{QP}')^N\to 0.$$

Therefore, the left-hand side of (10) tends to a solution for $\boldsymbol{\lambda}$ and satisfies (8).

From equations (6) and (7) we have also a set of equations in $\boldsymbol{\mu}$:

$$(\mathbf{M}-\mathbf{LQ})-\boldsymbol{\mu}(\mathbf{I}-\mathbf{P}'\mathbf{Q})=0, \tag{12}$$

and in a similar way it can be seen that, after multiplying both sides of (11) by $(\mathbf{I}-\mathbf{P}'\mathbf{Q})$, we have

$$(\mathbf{M}-\mathbf{LQ})\{1+\mathbf{P}'\mathbf{Q}+\ldots+(\mathbf{P}'\mathbf{Q})^{(N-1)}\}(1-\mathbf{P}'\mathbf{Q})$$
$$=(\mathbf{M}-\mathbf{LQ})-\mathbf{M}(\mathbf{P}'\mathbf{Q})^N+(\mathbf{LQ})(\mathbf{P}'\mathbf{Q})^N$$

and again as $N\to\infty$, an element of $\mathbf{M}(\mathbf{P}'\mathbf{Q})^N$ will tend to

$$\left(\sum_j M_j\right)/X_{\cdot\cdot}-1;$$

and of $(\mathbf{LQ})(\mathbf{P}'\mathbf{Q})^N$ will tend to

$$\left(\sum_i L_i\right)/X_{\cdot\cdot}-1$$

and hence $\mathbf{M}(\mathbf{P}'\mathbf{Q})^N$ tends to $(\mathbf{LQ})(\mathbf{P}'\mathbf{Q})^N$. Thus, as $N\to\infty$, the limiting value of the left-hand side of (11) is a solution for $\boldsymbol{\mu}$ and satisfies (12).

It remains to show that the solutions for $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ correspond. The iterative process specifically ensures that equation (2) or (3) is satisfied at each stage, and in the limit that both are satisfied. Thus equation (6) is satisfied and thus the solutions correspond.

Thus the iterative process produces results which converge to the solution of the minimum χ^2-condition.

APPENDIX II

The Relationship between Stephan's Iterative Process and that of the General Register Office

Stephan's iterative process, as described by Deming (1943), is in terms of $p_i(N)$ and $q_j(N)$, where these denote the values derived from the Nth iteration. Let p_i and q_j denote the values to which they converge as $N \to \infty$. These quantities are related to the quantities λ_i and μ_j in our solution, in the following manner.

From equation (1) of Appendix I, the final value of x_{ij} in our solution is given by:

$$x_{ij} = X_{ij}(1+\lambda_i+\mu_j).$$

In Stephan's solution, it is given by:

$$x_{ij} = X_{ij}(p_i+q_j). \tag{13}$$

Denote the $1 \times n$ matrices $[p_i(N)]$ by $\mathbf{p}(N)$, and $[p_i]$ by $\mathbf{p}$. Similarly denote the $1 \times m$ matrices $[q_j(N)]$ by $\mathbf{q}(N)$, and $[q_j]$ by $\mathbf{q}$.

As in the derivation of equation (8), we have

$$\{(\mathbf{L}+\mathbf{I})-(\mathbf{M}+\mathbf{I})\,\mathbf{P}'\}-\mathbf{p}(\mathbf{I}-\mathbf{QP}') = 0, \tag{14}$$

and a similar expression for $\mathbf{q}$.

In the present notations, Stephan's iterative process is defined by

$$\begin{aligned}
p_e(1) &= \tfrac{1}{2}L_e/X_{e.}, \\
q_f(1) &= M_f/X_{.f}-\sum_i p_i(1)\,X_{if}/X_{.f}, \\
&\quad\vdots \\
p_e(N) &= L_e/X_{e.}-\sum_j q_j(N-1)\,X_{ej}/X_{e.}, \\
q_f(N) &= M_f/X_{.f}-\sum_i p_i(N)\,X_{if}/X_{.f},
\end{aligned}$$

which appear in matrix notation as:

$$\begin{aligned}
p_e(1) &= \tfrac{1}{2}[\mathbf{L}+\mathbf{I}]^{(e)} \\
q_f(1) &= [(\mathbf{M}+\mathbf{I})-\tfrac{1}{2}(\mathbf{L}+\mathbf{I})\,\mathbf{Q}]^{(f)} \\
p_e(2) &= [(\mathbf{L}+\mathbf{I})-(\mathbf{M}+\mathbf{I})\,\mathbf{P}'+\tfrac{1}{2}(\mathbf{L}+\mathbf{I})\,\mathbf{QP}']^{(e)} \\
q_f(2) &= [(\mathbf{M}+\mathbf{I})-(\mathbf{L}+\mathbf{I})\,\mathbf{Q}+(\mathbf{M}+\mathbf{I})\,\mathbf{P}'\mathbf{Q}-\tfrac{1}{2}(\mathbf{L}+\mathbf{I})\,\mathbf{QP}'\mathbf{Q}]^{(f)} \\
&\quad\vdots
\end{aligned}$$

$$\begin{aligned}
p_e(N) &\\
&= [(\mathbf{L}+\mathbf{I})-(\mathbf{M}+\mathbf{I})\,\mathbf{P}'+(\mathbf{L}+\mathbf{I})\,\mathbf{QP}'\ldots-(\mathbf{M}+\mathbf{I})\,\mathbf{P}'(\mathbf{QP}')^{(N-2)}+\tfrac{1}{2}(\mathbf{L}+\mathbf{I})\,(\mathbf{QP}')^{(N-1)}]^{(e)} \\
&= [\{(\mathbf{L}+\mathbf{I})-(N+\mathbf{I})\,\mathbf{P}'\}\{\mathbf{I}+\mathbf{QP}'+(\mathbf{QP}')^2+\ldots+(\mathbf{QP}')^{(N-2)}+\tfrac{1}{2}(\mathbf{L}+\mathbf{I})\,(\mathbf{QP}')^{(N-1)}\}]^{(e)},
\end{aligned} \tag{15}$$

and similarly

$$q_f(N) = [\{(\mathbf{M}+\mathbf{I})-(\mathbf{L}+\mathbf{I})\,\mathbf{Q}\}\{\mathbf{I}+\mathbf{P}'\mathbf{Q}+(\mathbf{P}'\mathbf{Q})^2+\ldots+\tfrac{1}{2}(\mathbf{P}'\mathbf{Q})^{(N-1)}\}]^{(f)}. \qquad (16)$$

If (15) is multiplied by $(\mathbf{I}-\mathbf{QP}')$, we have

$$p(N)\,(\mathbf{I}-\mathbf{QP}') = \{(\mathbf{L}+\mathbf{I})-(\mathbf{M}+\mathbf{I})\,\mathbf{P}'\}-\tfrac{1}{2}(\mathbf{L}+\mathbf{I})\,(\mathbf{QP}')^{(N-1)}-\tfrac{1}{2}(\mathbf{L}+\mathbf{I})\,(\mathbf{QP}')^{N} + (\mathbf{M}+\mathbf{I})\,\mathbf{P}'(\mathbf{QP}')^{(N-1)},$$

and, as shown in Appendix I, as $N\to\infty$, the last three terms together tend to zero. Thus the right-hand side tends to $\{(\mathbf{L}+\mathbf{I})-(\mathbf{M}+\mathbf{I})\,\mathbf{P}'\}$, which is identical with the left-hand expression of (14). Thus the limiting matrix, $\mathbf{p}$, satisfies equation (14).

From equation (16), it may similarly be shown that the limiting matrix, $\mathbf{q}$, satisfies an analogous condition to (14).

It follows that $(1+\lambda_i+\mu_j) = (p_i+q_j)$.

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