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Robust Fitting of Circle Arcs

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Abstract Geometric fitting is present in different fields of sciences, engineering and astronomy. In particular, circular arc primitives are some of the most commonly employed geometric features in digital image analysis and visual pattern recognition. In this paper, a robust geometric method based on mean absolute error to fit a set of points is proposed. Most geometric and algebraic methods are sensitive to noise and outlier points and so the results are not usually acceptable. It is well known that the least absolute error criterion leads to robust estimations. However, the objective function is non differentiable and thus algorithms based on gradient cannot be applied. We propose an algorithm based on left and right side partial derivatives that is computationally efficient as an alternative to conventional algorithms, and evaluate the sensitivity of circle fits for different types of data.

Keywords Circle fitting · Absolute geometric error · Image occlusion

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1 Introduction

Fitting simple geometric primitives to image data is one of the basic problems in pattern recognition and computer vision, a problem that arises in many application areas, e.g. inspection and quality control [1–5], particle trajectories [6], computer graphics [7, 8], archaeology [9, 10], chromosome analysis [11] and metrology and microwave measurement [12]. The fitting process allows a reduction and simplification of the data to the benefit of the subsequent data manipulation stages. The simplest primitives are linear segments and circular arcs. We consider the problem of fitting circular arcs to given points on a plane. There are different approaches that can be used to find a circle or ellipse which best fits a data set: algebraic fitting and geometric fitting are two such approaches, which differ in their respective definition of the error distance involved. In many applications a parameterized curve, described by an implicit equation $\mathbf{F}(\mathbf{x}, \Theta) = 0$, is fitted to a set of experimental data $C = \{\mathbf{x}_i = (x_i, y_i) \in R^2, i = 1, 2, \ldots, n\}$ where Θ denotes the vector of unknown parameters to be estimated. A circumference can be represented by the implicit equation

$$F(\mathbf{x}, \Theta) = a_1 \mathbf{x}^T \mathbf{x} + \mathbf{a}_2^T \mathbf{x} + a_3 = 0 \tag{1}$$

where $\Theta = (a_1, \mathbf{a}_2, a_3), a_1 \neq 0$. The algebraic error, E_a, is defined by the deviations of the implicit equation from the expected value (i.e. zero) at each given point. The non equality of the equation indicates that the given point does not belong to the geometric feature. Thus, the expression $|F(\mathbf{x}, \Theta)|$ represents the *algebraic distance* or *error* [8, 12]. Given a set $\{\mathbf{x}_i\}_{i=1,2,\ldots,n}, n > 3$, we have an overdetermined linear system of n equations,

$$a_1 \mathbf{x}_i^T \mathbf{x} + \mathbf{a}_2^T \mathbf{x}_i + a_3 = r_i, \quad i = 1, 2, \ldots, n. \tag{2}$$

The problem of fitting of a circle to a set of points may be approached by minimizing the sum of squared algebraic distances

$$E_a(\Theta) = \sum_{i=1}^{n} r_i^2 \tag{3}$$

and this is called *algebraic* criterion [13]. This approach has the advantage of being simple since its solution is reduced to a simple Linear Least-Squares problem but the drawback is that we are uncertain what we are minimizing in a geometric sense.

The algebraic fitting has several disadvantages [8] such as

- Its definition of error distance does not coincide with measurement guidelines;
- The estimated fitting parameters are biased;
- The fitting parameters are not invariant to the coordinate transformation;
- The reliability of the estimated fitting parameters is very difficult to test.

On the other hand, the error distances can be defined by the shortest distance from the given points to the curve to be fitted (*geometric distance*). Let $d_i(\Theta)$ be the distance from the point (x_i, y_i) to the curve defined by $\mathbf{F}(x, y, \Theta) = 0$. If the curve is a circumference then

$$d_i(\Theta) = \left| \sqrt{(x_i - a)^2 + (y_i - b)^2} - r \right| \tag{4}$$

and the *geometric distance* or *error*, E_g, is defined by the distances from each point to the curve. The curve fitting may be approached by minimizing the sum of the squared geometric distances

$$E_g(\Theta) = \sum_{i=1}^{n} [d_i(\Theta)]^2 \tag{5}$$

This is called Orthogonal Least Squared fit (OLSF).

Moreover, geometric fitting seeks to provide the best visual fit for graphical and image applications. The OLSF is the method of choice in practice, especially for simple curves such as lines and circles. However, for more general curves the OLSF becomes intractable, because the precise distance d_i is hard to compute. For the circle fitting there are some well established methods. Calafiore [14] has used the *difference-of-squares* geometric error criterion (DOS) for points in n-dimensional space based on the expression

$$E(a, b, r) = \sum_{i=1}^{n} \left((x_i - a)^2 + (y_i - b)^2 - r^2 \right)^2 \tag{6}$$

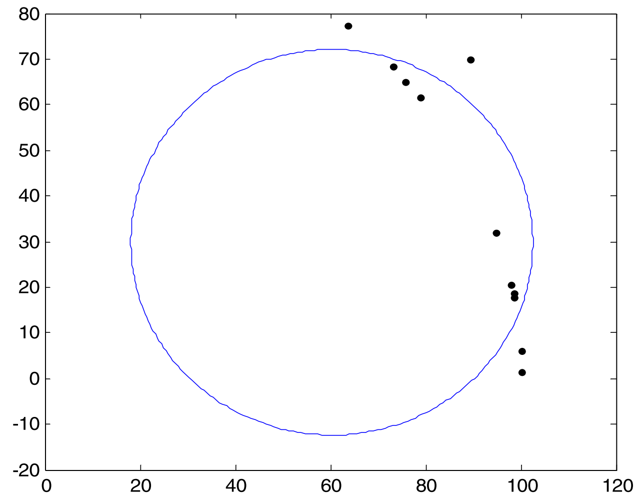


Fig. 1 Optimal closed-form solution for the DOS criterion

and has found a closed form solution for this problem. The optimal radius is given by

$$r = \left[\frac{1}{n} \sum_{i=1}^{n} \left((x_i - a)^2 + (y_i - b)^2 \right) \right]^{1/2} \tag{7}$$

and the centre (a, b) is found by building an orthonormal basis for the null space of $\bar{X}$ obtained from the singular value decomposition, where $\bar{X}$ is the matrix of centered data. However, this solution is particularly sensitive to the presence of outliers in the data. Figure 1 shows a set of ten points and the solution that is achieved by this method when the algorithm is initiated at the mean point (initial centre). Note that this solution is unacceptable and that criteria based on the sum of squared residuals are particularly sensitive to the presence of outliers in the data.

A different approach was presented in Kåsa [15] and Zelniker and Clarkson [16] where the solution for the centre (a, b) was obtained by solving linear equations. Ahn et al. [8] have presented an algorithm for the least-squares orthogonal distances fitting of circles in an n-dimensional space based on Gauss-Newton iterations with the Jacobian matrix. The initial parameter vector may be supplied from an algebraic circle fitting. However, this algorithm is based on the least-squares criterion that is sensitive to the presence of outliers in the data. Gander et al. [13] have presented several algorithms based on geometric or algebraic fitting of circles. A geometric fitting algorithm is based on minimizing the geometric distance and it solves a nonlinear least squares problem using the Gauss-Newton method. An alternative solution is a geometric fit in parametric form which is reduced to solve a nonlinear squares problem. The Landau algorithm [17] employs a simple fixed-point iterative scheme that shows remarkable stability and is widely used in practice. The Späth algorithm [18] uses an additional set of (dummy) parameters, which is based on alternating minimization, first with respect to the dummy parameters and then with respect to the main parameters, (a, b, r). At each step, one set of parameters is fixed, and a global minimum of the objective function F with respect to the other parameter set is found, which guarantees (at least theoretically) that F decreases at every iteration. The convergence of Späth's algorithm is, however, known to be slow.

Chernov and Lesort [19] have evaluated several popular circle fitting algorithms and have proposed an algorithm that accurately fits circles to data derived from short arcs. Drezner et al. [20] have considered three criteria to find the circle closest to a given set of points: least squared objective, minimax objective and minisum objective. These problems are formulated and solved as mathematical programs using standard mathematical programming software such as AMPL. However, Drezner et al. [20] assume that the points are quite close to the circle, therefore the search for the centre of the circle can be restricted to quite a small area, and a global optimization procedure can be adopted. The *Big Square Small Square* [21, 22] approach can perform an exhaustive search in the area close to the perceived centre of the circle and results in an optimal solution. Umbach and Jones [23] have discussed five methods of fitting circles to data. The reduced least-squares method (RLS) locates the centre of the circle at the point where the sum of the distances from the centre to each of the perpendicular bisectors of all chords is minimum. This method is not very stable. The modified least-squares method (MLS) is robust against measurement error and it yields closed form solutions. Caudill [24] adds the normality assumption to the problem of finding the best circle fitting and uses the BHHH algorithm of Berndt et al. [25] to estimate the parameters by maximum likelihood.

Geometric fit is commonly regarded as the most accurate, but it can only be implemented by iterative schemes that are computationally intensive and subject to occasional divergence. Algebraic fits are faster but presumably less precise. However, in order to minimize the algebraic distance, one needs to impose scale normalization on the variables. The crucial fact is that the solution depends on the choice of scale normalization. Exploiting this extra freedom Al-Sharadqah and Chernov [26] have analyzed the best scale normalization that maximizes the accuracy and have proposed a new algebraic (non-iterative) circle fitting algorithm (Hyper fit) that has no essential bias at all and that outperforms all the existing methods, including the geometric fit.

Moreover, Rangarajan and Kanatani [27] have modified the algebraic fitting algorithm "Hyper" of Al-Sharadqah and Chernov [26] and eliminated the bias of the solution up to the second order in noise and the solution outperforms the Hyper when the number of points is small and the noise is large.

However, it is well known that least-squares techniques are not robust in the sense of good tolerance to outliers in the data points. So, we should establish criteria based on robust or resistant estimators, for example, the median versus mean.

In contrast to others robust methods, the Random Sample Consensus algorithm (RANSAC) can also be applied to robust fitting of circles. It was introduced by Fischler and Bolles in 1981 [28] for fitting a model to experimental data which may contain a significant percentage of gross errors or outliers. Nevertheless, this method requires three predefined parameter (error tolerance, threshold value, number maximum of trials). So, the solution depends on these chosen parameters.

It is generally recognized that it is better to consider criteria based on the sum of the absolute values of the residual. That is, the error to minimize is given by the expression

$$E_g(\Theta) = \sum_{i=1}^{n} d_i(\Theta) \tag{8}$$

where $d_i(\Theta) = |\sqrt{(x-a)^2 + (y-b)^2} - r|$. However, this objective function is non differentiable and so it is not possible to use methods based on the gradient. In this paper, we develop a robust algorithm based on this criterion for fitting a circular arc or a circle whose circumference is as close as possible to a set of given points that use partial derivatives on the right-hand and on the left-hand. The radio r is estimated by the median of the distances from each point to the centre of the circle. Moreover, the algorithm takes into account the directions instead of the distances from each point to the centre of circle and distinguishes between inside and outside points with respect to circle.

The proposed algorithm has the following advantages:

- It seeks to provide the best visual fit for graphical and image applications
- High robustness to noise and outlier points
- Good performance when a circle is partially occluded
- Computational efficiency
- Invariance to affine transformation of the data

Section 2 describes the circle fitting problem based on the least absolute error and presents some properties of this problem. In Sect. 3, we develop an algorithm that uses partial derivatives for the geometric fitting of circles. In Sect. 4 the proposed algorithm is applied to a different data sets and the results are presented. Finally, conclusions are drawn in Sect. 5.

2 Formulation and Properties

We consider the following problem: given a set of points $\mathscr{P} = \{P_i = (x_i, y_i), i = 1, 2, \ldots, n\}$, find the centre (a, b), and the radius r of the circle that passes closest to all the points, i.e., minimize the error function

$$E(a,b,r)=\sum_{i=1}^{n}\left|\sqrt{(x_i-a)^2+(y_i-b)^2}-r\right|. \tag{9}$$

This objective function (9), called Absolute Geometric Error (AGE), is non differentiable. Unfortunately, there is no closed form solution for this problem, so we need to use an iterative method. It is well-known that given the values a and b, the optimal value of r is the *median* of all distances from the points to the centre of the circle, (a,b). Thus, the optimal value of the objective function (9) is the mean absolute deviation with respect to the median of the distances to the centre of the circle, i.e., it is a measure of dispersion of the distances to the centre of the circle.

In this paper we assume that the n fixed points are not collinear. Note that if $n<4$, any circle that intersects all the fixed points will be optimal, and the objective function will have an optimal value of zero. Next, we present some properties of this problem.

Proposition 1

(a) *The optimal radius of the circle is given by the median of the distances from each point $P_i=(x_i,y_i)$ to the centre of the circle $P=(a,b)$*
(b) *There exists an optimal circumference that contains at least a point $P_i=(x_i,y_i)$. So, the number of inside and outside points cannot be more than $n/2$*
(c) *If a circumference contains at least $[n/2]+1$ points then it is a local minimum solution of the function (9) with respect to r*

Proof

(a) Let $d_i=\sqrt{(x_i-a)^2+(y_i-b)^2}$, $i=1,2,\dots,n$, be the n distances from each point to the center, where the centre *(a, b)* is fixed.

It is well known that the value r^* that minimizes the expression $\sum_{i=1}^{n}|d_i-r|$ is the median value of the distances, $med\{d_i, i=1,2,\dots,n\}$.

(b) If the above values are sorted we have

$$d_{(i)}=\sqrt{(x_{(i)}-a)^2+(y_{(i)}-b)^2},\quad i=1,2,\dots,n \tag{10}$$

$$d_{(1)}\le d_{(2)}\le\cdots\le d_{(n)} \tag{11}$$

So, r^* is any value of the interval $[d_{(n/2)},d_{(1+n/2)}]$, when n is even, and $r^*=d_{((n+1)/2)}$ when n is odd, i.e., an optimal circumference passes through the points $(x_{((n+1)/2)},y_{((n+1)/2)})$ (or the points $(x_{(n/2)},\ y_{(n/2)})$) when n is even and it passes through the point $(x_{((n+1)/2)}, y_{((n+1)/2)})$ when n is odd.

(c) Let k and l be the numbers of the inside and outside points, respectively. Then,

$$\begin{aligned}d_{(1)}&\le d_{(2)}\le\cdots\le d_{(k)}\le d_{(k+1)}=d_{(k+2)}\\&=\cdots=d_{(k+m)}\\&\le d_{(k+m+1)}\le\cdots\le d_{(k+m+l)}\le\cdots\le d_{(n)},\end{aligned}$$

where

$$d_{(k+1)}=d_{(k+2)}=\cdots=d_{(k+m)}=r$$

By hypothesis, $k+l+(n/2)+1\le n$ if n is even, and $k+j+(n+1)/2+1\le n$ if n is odd. Thus, $k+l<n/2$ if n is even, and $k+j<(n+1)/2$ if n is odd. Consequently, the median distance corresponds to a point on the circumference. □

Note that two different circumferences can have at most two common points. Thus, there exists at most a circumference that passes through a set containing at least $[n/2]+2$ points.

Problem (9) depends on three variables, the coordinates of the centre and the radius. When the centre of the circumference is determined, the radius is given by $r(a,b)=med\{\sqrt{(x_k-a)^2+(y_k-b)^2}, k=1,2,\dots,n\}$. So, the optimization problem is reduced to find the centre (a,b) that minimizes the function

$$E(a,b)=\sum_{i=1}^{n}\left|\sqrt{(x_i-a)^2+(y_i-b)^2}-r(a,b)\right| \tag{12}$$

where

$$r(a,b)=med\{\sqrt{(x_k-a)^2+(y_k-b)^2}, k=1,2,\dots,n\} \tag{13}$$

In Fig. 2 a problem is shown with 4 points, (1, 1), (2, 4), (3, 2) and (4, 2) and level curves of the error function of the circumference. Note that the problem has several local minimal solutions and we can verify that the optimal solution (global minimum) corresponds to a circumference whose centre is the point (1.5, 2.5).

3 A Method for Geometric Fitting Circles to Data

One simple method that uses search line directions is the steepest descent method based on gradient as the search line. However, the error function (9) is not differentiable at all points, since the absolute value function is not differentiable. Moreover, the error function is not convex. Drezner et al. [20] have used AMPL [29] to solve problem (9) by nonlinear programs. They have indicated the drawbacks involved in obtaining the optimal solution, since AMPL is not

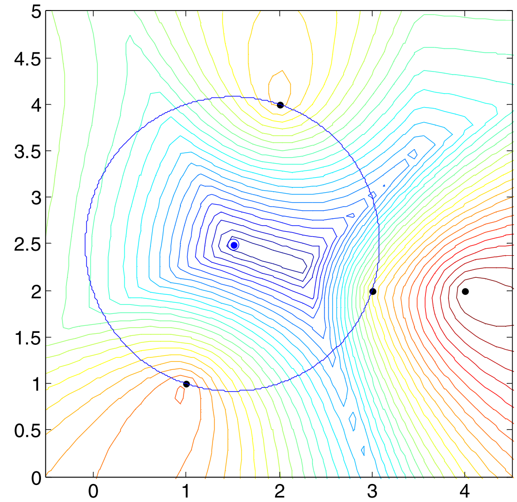


Fig. 2 Level curves of the error function

reliable for non differentiable functions. When initial solutions are random then several different solution points are reached and some are not local minimum. Drezner et al. have modified the AMPL code using $2n$ variables $u_i, v_i, i = 1, 2, \dots, n$, and have solved the following problem:

$$\text{Minimize} \quad \sum_{i=1}^{n} (u_i + v_i) \tag{14}$$

subject to

$$\sqrt{(x_i - a)^2 + (y_i - b)^2} - r = u_i - v_i, \quad i = 1, 2, \dots, n.$$
$$u_i \geq 0, \qquad v_i \geq 0, \quad i = 1, 2, \dots, n. \tag{15}$$

Note that this formulation has $3 + 2n$ variables and n non-linear constraint.

Most practical minimization methods use only first derivative information, combined with line searches in selected directions.

We consider the problem of minimize the error function $E(a, b, r)$ given by the expression (9) that is a unconstrained optimization problem. The simplest approach using line search is the steepest descent method. Given an initial point (a, b), the steepest descent algorithm proceeds by performing a line search in the direction $\mathbf{d} = -\nabla E(a, b)$. However, the first problem is the question of differentiability of the function $E(a, b, r)$. The differentiability of the criterion function (9) is not guaranteed at any given point (a, b, r). Moreover, this function is non-convex and so methods of convex optimization can not be used. Note that the function $f(a, b) = |\sqrt{(x_i - a)^2 + (y_i - b)^2} - r_0|$ is equal to zero on the circumference $\{(a, b) \in R^2 : \sqrt{(x_i - a)^2 + (y_i - b)^2} = r_0\}$, and positive in its interior with a local maximum at (x_i, y_i), so it is a non convex function.

Now, we describe our algorithm for minimization of error function (9). We know that the optimal radius is given by expression (13) for any given centre, $P = (a, b)$. Note that the median is a robust estimator. A measure of robustness is the *breakdown point*, that is, the proportion of incorrect observations that an estimator can handle before giving an arbitrarily large result. The mean estimator has a breakdown point of 0 because the mean can be arbitrarily large just by changing any observation. A breakdown point cannot exceed 50% because if more than half of the observations are contaminated, it is not possible to distinguish between the underlying distribution and the contaminating distribution. Therefore, the maximum breakdown point is 0.5. The median has a breakdown point of 0.5. So, the median is a resistant statistic.

Once the radius r is determined the points are partitioned into three sets:

Set of outside points:

$$O = \left\{(x_i, y_i) : \sqrt{(x_i - a)^2 + (y_i - b)^2} > r(a, b)\right\}. \tag{16}$$

Set of inside points:

$$I = \left\{(x_i, y_i) : \sqrt{(x_i - a)^2 + (y_i - b)^2} < r(a, b)\right\}. \tag{17}$$

Set of points on the circumference:

$$C = \left\{(x_i, y_i) : \sqrt{(x_i - a)^2 + (y_i - b)^2} = r(a, b)\right\}. \tag{18}$$

The error function is defined by expression:

$$\begin{aligned} E(a, b) = \sum_{i \in O} &\left(\sqrt{(x_i - a)^2 + (y_i - b)^2} - r\right) \\ &+ \sum_{i \in I} \left(r - \sqrt{(x_i - a)^2 + (y_i - b)^2}\right) \\ &+ \sum_{i \in C} \left|r - \sqrt{(x_i - a)^2 + (y_i - b)^2}\right|. \end{aligned} \tag{19}$$

Note that the last term is a function non differentiable at (a, b) and its value is zero. Given a centre (a, b), there exists $\varepsilon > 0$ such that for every $\Delta a \in [0, \varepsilon)$ there do not exist points (x_i, y_i) on the circumference with centre $(a + \Delta a, b)$ and radius r, since the set of points P is finite. We have

$$\begin{aligned} &E(a + \Delta a, b) \\ &\quad = \sum_{i \in O} \left(\sqrt{(x_i - a - \Delta a)^2 + (y_i - b)^2} - r\right) \\ &\quad + \sum_{i \in I} \left(r - \sqrt{(x_i - a - \Delta a)^2 + (y_i - b)^2}\right) \\ &\quad + \sum_{i \in C_O^+} \left(\sqrt{(x_i - a - \Delta a)^2 + (y_i - b)^2} - r\right) \end{aligned}$$

$$+\sum_{i\in C_I^+}\left(r-\sqrt{(x_i-a-\Delta a)^2+(y_i-b)^2}\right) \quad (20)$$

where $C_I^+=\{i\in C: x_i>a+\Delta a/2\}$ is the set of points belonging to C that become inside points and $C_O^+=C-C_I^+$ is the set of points belonging to C that become outside points. Hence,

$$\begin{aligned}E'_{a+}(a,b)&=\lim_{\Delta a\to 0}\frac{E(a+\Delta a,b)-E(a,b)}{\Delta a}\\&=\sum_{i\in I}\frac{(x_i-a)}{\sqrt{(x_i-a)^2+(y_i-b)^2}}\\&\quad-\sum_{i\in O}\frac{(x_i-a)}{\sqrt{(x_i-a)^2+(y_i-b)^2}}\\&\quad+\sum_{i\in C}\frac{|x_i-a|}{\sqrt{(x_i-a)^2+(y_i-b)^2}} \quad (21)\end{aligned}$$

Note that if the point (x_i,y_i) is equal to centre (a,b) then the corresponding term is zero.

Similarly, given a centre (a,b), there exists a $\varepsilon>0$ such that for every $\Delta a\in(-\varepsilon,0]$ there do not exist points (x_i,y_i) on the circumference with centre $(a+\Delta a,b)$ and radius r, and so

$$\begin{aligned}E'_{a-}(a,b)&=\lim_{\Delta a\to 0}\frac{E(a+\Delta a,b)-E(a,b)}{\Delta a}\\&=\sum_{i\in I}\frac{(x_i-a)}{\sqrt{(x_i-a)^2+(y_i-b)^2}}\\&\quad-\sum_{i\in O}\frac{(x_i-a)}{\sqrt{(x_i-a)^2+(y_i-b)^2}}\\&\quad-\sum_{i\in C}\frac{|x_i-a|}{\sqrt{(x_i-a)^2+(y_i-b)^2}}. \quad (22)\end{aligned}$$

In this case, $C_I^-=\{i\in C: x_i<a+\Delta a/2\}$ is the set of points belonging to C that become inside points and $C_O^-=C-C_I^-$ is the set of points belonging to C that become outside points.

Note that when the n points do not belong to the circumference then the function E is differentiable at (a,b). Similarly, the expressions of $E'_{b+}(a,b)$ and $E'_{b-}(a,b)$ can be obtained.

If θ_i is the angle defined by the vector (x_i-a,y_i-b) with the abscissa axis, we have

$$E'_{a+}(a,b)=\sum_{i\in I}\cos\theta_i-\sum_{i\in O}\cos\theta_i+\sum_{i\in C}|\cos\theta_i|, \quad (23)$$

$$E'_{a-}(a,b)=\sum_{i\in I}\cos\theta_i-\sum_{i\in O}\cos\theta_i-\sum_{i\in C}|\cos\theta_i|, \quad (24)$$

$$E'_{b+}(a,b)=\sum_{i\in I}\sin\theta_i-\sum_{i\in O}\sin\theta_i+\sum_{i\in C}|\sin\theta_i|, \quad (25)$$

$$E'_{b-}(a,b)=\sum_{i\in I}\sin\theta_i-\sum_{i\in O}\sin\theta_i-\sum_{i\in C}|\sin\theta_i|. \quad (26)$$

Note that $E'_{a+}(a,b)\ge E'_{a-}(a,b)$, $E'_{b+}(a,b)\ge E'_{b-}(a,b)$, $\forall(a,b)\in R^2$, and the right-hand derivative coincides with the left-hand derivative when C is an empty set.

A vector $\mathbf{d}=(d_1,d_2)$ is called a **descent direction** of E at (a,b) if there exists an $\varepsilon>0$ such that

$$E[(a,b)+\lambda(d_1,d_2)]<E[(a,b)],\quad \text{for each } \lambda\in[0,\varepsilon) \quad (27)$$

Many nonlinear programming algorithms proceed as follows. Given a point $\mathbf{x}_k$, find a direction vector $\mathbf{d}_k$ and then a suitable step size λ_k, yielding a new point $\mathbf{x}_{k+1}=\mathbf{x}_k+\lambda_k\mathbf{d}_k$, the process is then repeated. Here we consider the direction $\mathbf{d}=(d_1,d_2)$ given by

$$d_1=\begin{cases}-\left[\alpha E'_{a-}(a,b)+(1-\alpha)E'_{a+}(a,b)\right]\\ \quad\text{if } E'_{a+}(a,b)\le 0 \text{ or } E'_{a-}(a,b)\ge 0,\\ 0\quad\text{otherwise.}\end{cases} \quad (28)$$

$$d_2=\begin{cases}-\left[\beta E'_{b-}(a,b)+(1-\beta)E'_{b+}(a,b)\right]\\ \quad\text{if } E'_{b+}(a,b)\le 0 \text{ or } E'_{b-}(a,b)\ge 0,\\ 0\quad\text{otherwise.}\end{cases} \quad (29)$$

$\forall\alpha,\beta\in[0,1]$.

Note that if $\alpha=\beta=1$ then we use the derivative on the right-hand when it is negative and the derivative on the left-hand when it is positive. We have the following property.

Proposition 2 *There exists an* $\varepsilon>0$ *such that*

$$E(a+\lambda d_1,b)\le E(a,b), \quad (30)$$

$$E(a,b+\lambda d_2)\le E(a,b), \quad (31)$$

$\forall\lambda\in[0,\varepsilon)$.

Proof Suppose that $E'_{a+}(a,b)\le 0$. Then we have $E'_{a-}(a,b)\le 0$ and

$$\alpha E'_{a-}(a,b)+(1-\alpha)E'_{a+}(a,b)\le 0,\quad \Delta a=-\lambda d_1\ge 0.$$

Since $E'_{a+}(a,b)\le 0$, there exists an $\varepsilon>0$, such that

$$E(a+\Delta a,b)-E(a,b)\le 0\quad \forall\lambda\in[0,\varepsilon).$$

Now suppose that $E'_{a-}(a,b)\ge 0$. Then we have $E'_{a+}(a,b)\ge 0$ and so

$$\alpha E'_{a-}(a,b)+(1-\alpha)E'_{a+}(a,b)\ge 0,\quad \Delta a=-\lambda d_1\le 0.$$

Since $E'_{a-}(a,b)\ge 0$, there exists an $\varepsilon>0$ such that

$$E(a+\Delta a,b)-E(a,b)\le 0,\quad \forall\lambda\in[0,\varepsilon).$$

Finally, if $E'_{a+}(a,b) \geq 0$ and $E'_{a-}(a,b) \leq 0$, then $d_1 = 0$, and $E(a + \Delta a, b) - E(a,b) = 0$. The proof is similar for Δb. □

Note that if $E'_{a-}(a,b) \leq 0$ and $E'_{a+}(a,b) \geq 0$, then $d_1 = 0$ and $d_2 = 0$. In this case, we take

$$d_1 = -[E'_{a-}(a,b) + E'_{a+}(a,b)]/2 \quad \text{and}$$
$$d_2 = -[E'_{b-}(a,b) + E'_{b+}(a,b)]/2.$$

However, a decreasing of the error function E is not guaranteed in this case. This direction is obtained when the function E is approximated by the function

$$\hat{E}(a,b) = \sum_{i \in O}\left(\sqrt{(x_i - a)^2 + (y_i - b)^2} - r\right) + \sum_{i \in I}\left(r - \sqrt{(x_i - a)^2 + (y_i - b)^2}\right)$$

where the points on the circumference are not taken into account.

So, the direction vector $\mathbf{d} = (d_1, d_2)$ is given by the following expressions:

$$d_1 = -\left[\alpha E'_{a-}(a,b) + (1-\alpha)E'_{a+}(a,b)\right], \tag{32}$$
$$d_2 = -\left[\beta E'_{b-}(a,b) + (1-\beta)E'_{b+}(a,b)\right] \tag{33}$$

where

$$\alpha(a,b) = \begin{cases} 1 & \text{if } E'_{a-}(a,b) \geq 0, \\ 0 & \text{if } E'_{a+}(a,b) \leq 0, \\ 1/2 & \text{otherwise,} \end{cases} \tag{34}$$

$$\beta(a,b) = \begin{cases} 1 & \text{if } E'_{b-}(a,b) \geq 0, \\ 0 & \text{if } E'_{b+}(a,b) \leq 0, \\ 1/2 & \text{otherwise.} \end{cases} \tag{35}$$

The descent method applied to this problem has an obvious interpretation. It tries to move the centre toward the outside points and away from the inside points. At the same time, the points belonging to the circumference try to avoid the displacement of the centre.

Thus, we propose the following algorithm:

Algorithm

Input: A set $\chi = \{(x_i, y_i) : i = 1, 2, \ldots, n\}$.

Output: Centre (a,b) and radius r of a circumference

Method:

Step 1. Initial solution: the centre of circle, (a,b), can be determined by three data points or the centroid of the data points. The initial step size is $\lambda = \lambda_o$.

Step 2. Compute the values

$$\sqrt{(x_i - a)^2 + (y_i - b)^2}, \quad i = 1, 2, \ldots, n$$

and their median value r.

Step 3. Determine the sets

$$I = \left\{i : \sqrt{(x_i - a)^2 + (y_i - b)^2} < r\right\}$$
$$O = \left\{i : \sqrt{(x_i - a)^2 + (y_i - b)^2} > r\right\}$$
$$C = \left\{i : \sqrt{(x_i - a)^2 + (y_i - b)^2} = r\right\}$$

Step 4. Determine the values

$$\cos(\theta_i) = \frac{x_i - a}{\sqrt{(x_i - a)^2 + (y_i - b)^2}}, \quad i = 1, 2, \ldots, n$$

Step 5. Set

$$E'_{a+}(a,b) = \sum_{i \in I}\cos\theta_i - \sum_{i \in O}\cos\theta_i + \sum_{i \in C}|\cos\theta_i|$$
$$E'_{a-}(a,b) = \sum_{i \in I}\cos\theta_i - \sum_{i \in O}\cos\theta_i - \sum_{i \in C}|\cos\theta_i|$$
$$E'_{b+}(a,b) = \sum_{i \in I}\sin\theta_i - \sum_{i \in O}\sin\theta_i + \sum_{i \in C}|\sin\theta_i|$$
$$E'_{b-}(a,b) = \sum_{i \in I}\sin\theta_i - \sum_{i \in O}\sin\theta_i - \sum_{i \in C}|\sin\theta_i|$$

Step 6. Determine the new centre

$$(a,b)_{New} \leftarrow (a,b)_{Old} + \lambda(d_1, d_2)$$

where $(d_1, d_2)$is given by expressions (32) and (33).

Step 7. (k-iteration) **If** $E^{New}(a,b,r) < E^{Old}(a,b,r)$ then put $\lambda_k = 1.1\lambda_{k-1}$ and repeat from Step 2. **Otherwise** $(a,b)_{New} \leftarrow (a,b)_{Old}$ and repeat from Step 2 with $\lambda_k = 0.9\lambda_{k-1}$.

Step 8. **Stop** when the maximum number of iterations is exceeded or the requested accuracy is obtained for the solution $(a,b)_{new} \approx (a,b)_{old}$.

4 Experimental Results

In this section, we report numerical experiments for different data sets and we provide an indication of how sensitive the methods are to noise or measurement errors. First we consider the following data (Gander et al. [13]):

$$P = \begin{bmatrix} 1 & 2 & 5 & 7 & 9 & 3 \\ 7 & 6 & 8 & 7 & 5 & 7 \end{bmatrix}$$

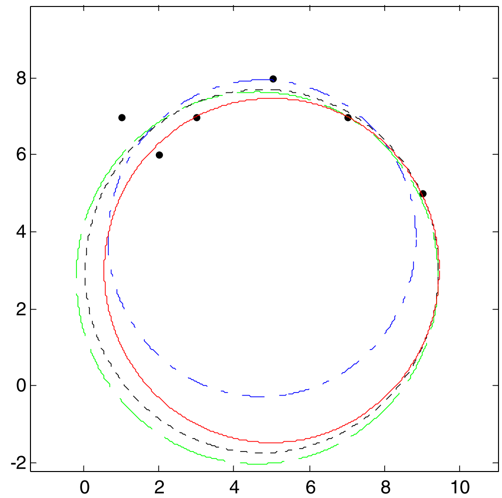
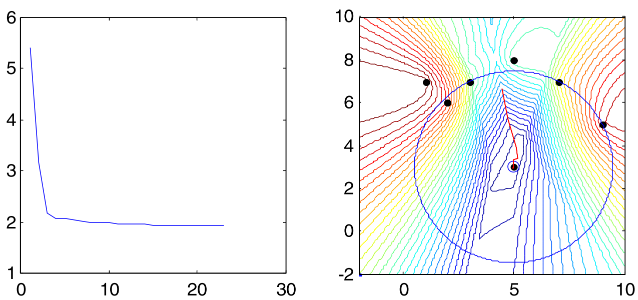


Fig. 3 Solutions for different criteria: AGE (*solid line*); OLSF (*dotted line*); DOS (*dash-dotted line*); Algebraic Error "Hyper" (*dashed line*)

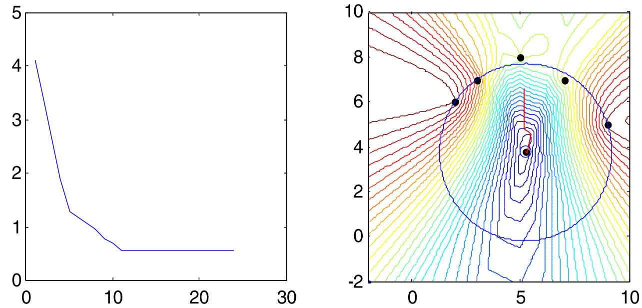


(a) (b)

Fig. 4 (**a**) Error function graphic. (**b**) Optimal circle and level curves

The optimal circle computed by the proposed algorithm based on the criterion Absolute Geometric Error (AGE) has its centre in (5, 3), and radius $r = 4.4721$, and is depicted in Fig. 3, together with the solution obtained by DOS criterion (6) which has its centre in (4.7423, 3.8551) and radius 4.1088, as reported in Calafiore [14], and the solution that minimizes the sum of squared geometric errors (5) that has its centre in (4.7398, 2.9835) and radius 4.71423. Also, the solution that minimizes the sum of squared algebraic errors with centre (4.6155, 2.8074) and radius 4.8276 is shown in Fig. 3. Note that the solution circumference based on the AGE criterion passes through three points and it is near to another point and far from two of the points, $P_1 = (1, 7)$ and $P_3 = (5, 8)$, while the other criteria reduce the maximum error. We observe that the solution obtained by the AGE criterion is close to the solution based on OLSF criterion which is not robust.

In Fig. 4(a) the error function of the proposed algorithm is shown. It can be seen that the minimum value reached is 1.942 in iteration 22. The algorithm starts in the mean value (gravity centre) of the six points and it ends if the step size is lower than 0.001. Obviously, any other criteria could be used to stop the procedure. Figure 4 (b) illustrates the path followed by the solution (a, b) that has been generated in 22 iterations, and the optimal circle with centre (5, 3) and radius $r = 4.472$. Note that the descent method works well during the early stages of the optimization process and that the zigzagging phenomena occur at later iterations in the valley as shown in Fig. 4(b) by the level curves. The zigzagging is prevented by a reduction of step size while we have $E(k+1) > E(k)$. So, several reductions and evaluations of the function E could be needed in each step. We have chosen 1.1 as incremental factor (Factor 1) and 0.9 as step size reduction factor (Factor 2). The parameter λ_o given us the initial step size and λ_k is the minimum step size considered at k-th iteration.



(a) (b)

Fig. 5 (**a**) Error function graphic. (**b**) Optimal circle and level curves

If the first point of the set of Gander et al. [13] is removed then the new set of points will be

$$P = \begin{bmatrix} 2 & 5 & 7 & 9 & 3 \\ 6 & 8 & 7 & 5 & 7 \end{bmatrix}$$

Note that the circle in Fig. 4 is similar to the circle in Fig. 5. As can be verified in Fig. 5 the removal of an outlier point does not affect significantly the fit of the curve to the points, thus our algorithm is robust. However, the circles based on criteria DOS and OLSF are quite different according to whether the set of points has 5 or 6 points (see Fig. 3 and Fig. 6).

Now, the proposed algorithm is applied to the set of points given by *Drezner* et al. [20]:

$$P = \begin{bmatrix} -9 & -11 & 2 & -1 & 4 & 9 & 7 & 7 & 10 \\ 2 & -1 & 10 & -10 & 9 & -5 & 7 & -7 & 1 \end{bmatrix}$$

The optimal solution is a circumference with centre $(a, b) = (0.143, -0.143)$, radius $r = 9.923$, and absolute error $E = 2.599$ (see Table 2). This solution passes through three points belonging to the points set. The error function, the optimal circle and the trajectory of centre are illustrated in Fig. 7.

Table 1 Parameter fitting and results using AGE algorithm with Gander points

Algorithm	Points	λ_o	λ_k	Factor1	Factor2	Time	E	Iterations	Parameters		
									a	b	r
	6 points	0.5	< 0.001	1.2	0.9	0.015	1.942	22	5.000	3.000	4.472
		0.5	< 0.001	1.2	0.95	0.015	1.942	13	5.000	3.000	4.472
Normalized		0.5	< 0.001	1.2	0.95	0.015	1.942	22	5.000	3.000	4.472
		1	< 0.001	1.2	0.9	0.015	1.942	20	5.000	3.000	4.472
Normalized		1.5	< 0.001	1.1	0.95	0.015	1.942	37	5.000	3.000	4.472
		1.5	< 0.001	1.1	0.95	0.015	1.942	26	5.000	3.000	4.472
	5 points	0.5	< 0.001	1.2	0.9	0.015	0.566	64	5.25	3.75	3.952
		0.5	< 0.001	1.2	0.95	0.015	0.566	33	5.25	3.75	3.952
		0.1	< 0.001	1.2	0.95	0.015	0.566	38	5.25	3.75	3.952
		1	< 0.001	1.2	0.9	0.015	0.566	47	5.25	3.75	3.952
		1.5	< 0.001	1.2	0.9	0.015	0.566	31	5.25	3.75	3.952
Normalized		0.5	< 0.001	1.2	0.9	0.015	0.566	32	5.25	3.75	3.952
Normalized		2	< 0.001	1.2	0.9	0.015	0.566	17	5.25	3.75	3.952
Normalized		0.5	< 0.001	1.2	0.95	0.015	0.566	30	5.25	3.75	3.952
Normalized		1	< 0.001	1.1	0.95	0.015	0.566	23	5.25	3.75	3.952
Normalized		1.5	< 0.001	1.1	0.95	0.015	0.566	22	5.25	3.75	3.952

Table 2 Parameter of fit and results on Drezner and Umbach point using the AGE algorithm

Algorithm	Points	λ_o	λ_k	Factor1	Factor2	Time	E	Iterations	Solution		
									a	b	r
AGE	Drezner	1	< 0.001	1.2	0.9	0.0150	2.599	14	0.143	-0.143	9.923
		0.5						11			
AGE	Umbach	0.5	< 0.001	1.2	0.9	0.0150	0.00005165	12	1.000	1.000	1.000

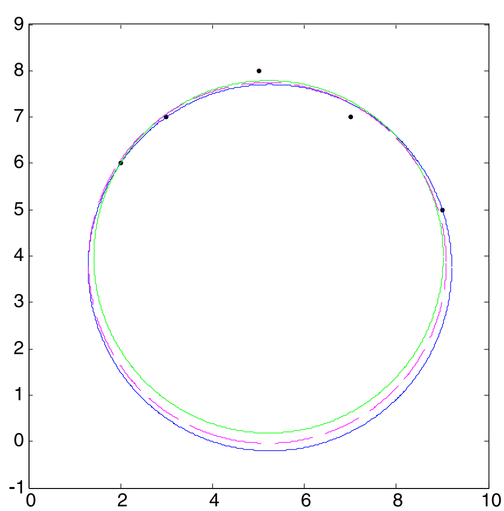


Fig. 6 Comparison methods AGE (*solid-line*), DOS (*dotted line*) y OLSF (*dashed line*)

Next, we use the set of points given by Umbach and Jones [23].

$$P = \begin{bmatrix} 0 & 2 & 1 & 1 & 0.03 \\ 1 & 1 & 0 & 2 & 1.02 \end{bmatrix}$$

The last data point was incorrectly recorded and the results of the fit to these five points are displayed in Fig. 8. However, we see that the AGE circle was not affected. The fit is close to the circle of radius 1 centered at (1, 1), (see Table 2). Nevertheless, Umbach and Jones [23] have reported that the RLS circle was seriously affected. In contrast, the FLS, Kåsa, and MLS circles are not different from our solution.

The problem of fitting a circular arc to a set of points in the plane is easily formulated as a nonlinear total least-squared problem which may be solved by the Gauss-Newton minimization algorithm. This straightforward approach is shown to be inefficient and extremely sensitive to the presence of outliers [31]. The presence of outliers can seriously affect the efficiency of the Gauss-Newton method, and usually quasi-Newton methods are used instead on a large residual problem. Now, we consider the data of Gruntz [32]:

$$P = \begin{bmatrix} 0.7 & 3.3 & 5.6 & 7.5 & 6.4 & 4.4 & 0.3 & -1.1 \\ 4.0 & 4.7 & 4.0 & 1.3 & -1.1 & -3.0 & -2.5 & 1.3 \end{bmatrix}$$

The Gruntz procedure can be used to calculate the best fit, and the values obtained for the centre and radius are

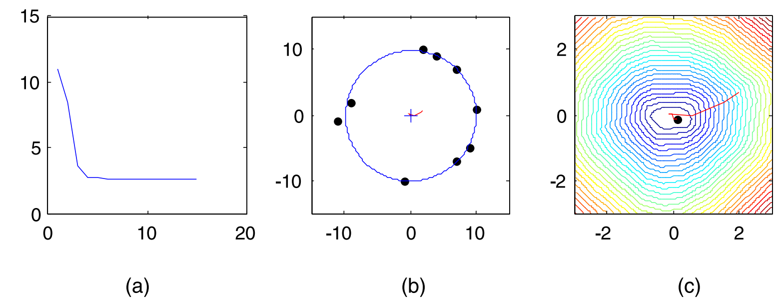


Fig. 7 (**a**) Error function. (**b**) Optimal circle. (**c**) Level curves

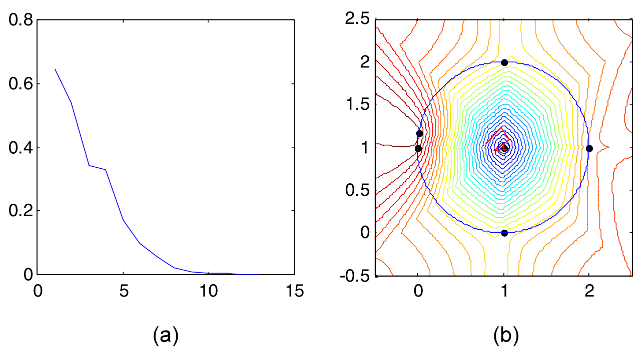


Fig. 8 (**a**) Error function. (**b**) Optimal circle and level curves

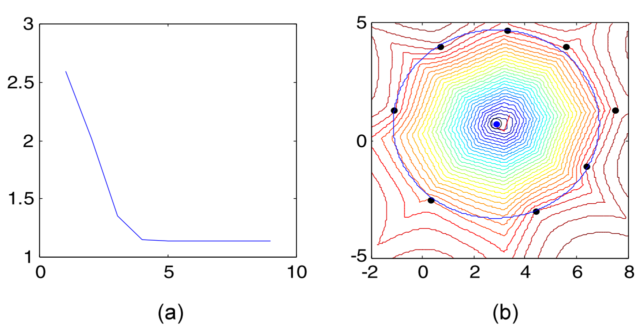


Fig. 9 (**a**) Error function. (**b**) Optimal circle and level curves for 8 Gruntz points

$a = 3.043, b = 0.746$, and $r = 4.106$. However, the addition of one more data point (3, 1) close to the previously fitted centre has a significant effect on the fitting procedure, $a = 3.255, b = 0.166, r = 3.79302$. However, the solution based on absolute geometric error is similar for 8 or 9 points, as can be seen in Figs. 9 and 10. Table 3 shows the solutions obtained for different parametric values based on AGE algorithm proposed.

The methods based on the sum of squares of the distances to the given points are particularly sensitive to the presence of outliers in the data. In this case it is better to consider criteria based on the absolute values of the residuals (AGE). To analyze this question, we consider a set of points belonging to a circular arc and an outlier point. As is shown in Fig. 11, DOS, OLSF and HYPER solutions are affected by the out-

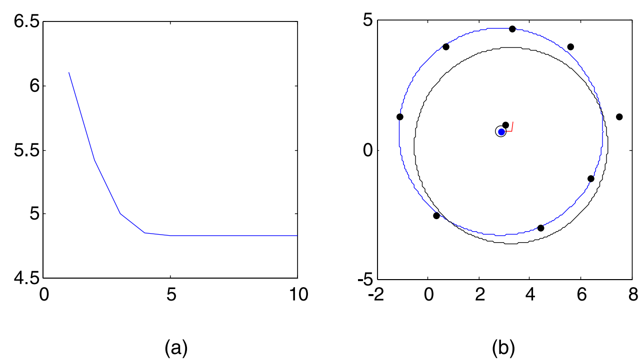


Fig. 10 (**a**) Error function. (**b**) Optimal circle for 9 Gruntz points: AGE (*solid line*) and OLSF (*dashed line*)

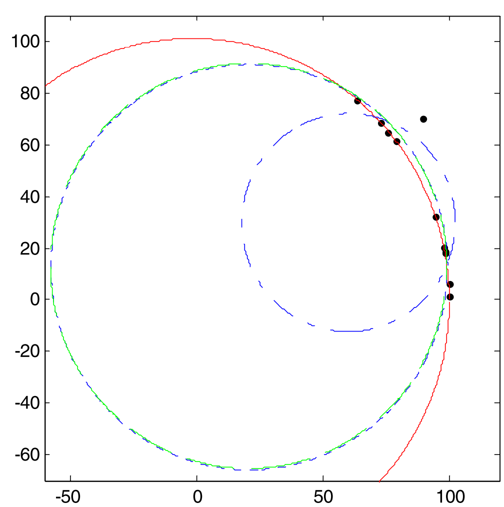


Fig. 11 Solutions for different criteria in arcs with outliers: AGE (*solid line*); OLSF (*dotted line*); DOS (*dash-dotted line*); Algebraic Error “Hyper” (*dashed line*)

lier point and so these methods are not robust, as can be seen in Table 4.

Finally, we have fitted a circle arc to 50 points (pixels) belonging to a moon image with 537×358 pixels size of the MATLAB© database, see Fig. 12(a). These points have been randomly selected (Table 5) from a contour obtained by the Canny filter [30] with a threshold value $t = 0.3$, see Fig. 12(b). We have used the proposed algorithm to deter-

Table 3 Parameters of fit and results for Gruntz points for the AGE algorithm

Algorithm	Points	λ_o	λ_k	Factor1	Factor2	E	Iterations	Solution		
								a	b	r
AGE	9 points	0.1	< 0.001	1.1	0.95	4.9487	17	2.8212	0.6349	3.9779
AGE	9 points	0.2	< 0.001	1.2	0.9	4.9487	7	2.8211	0.6348	3.9774
AGE	8 points	0.2	< 0.001	1.2	0.9	1.1337	8	2.8699	0.7098	4.0143

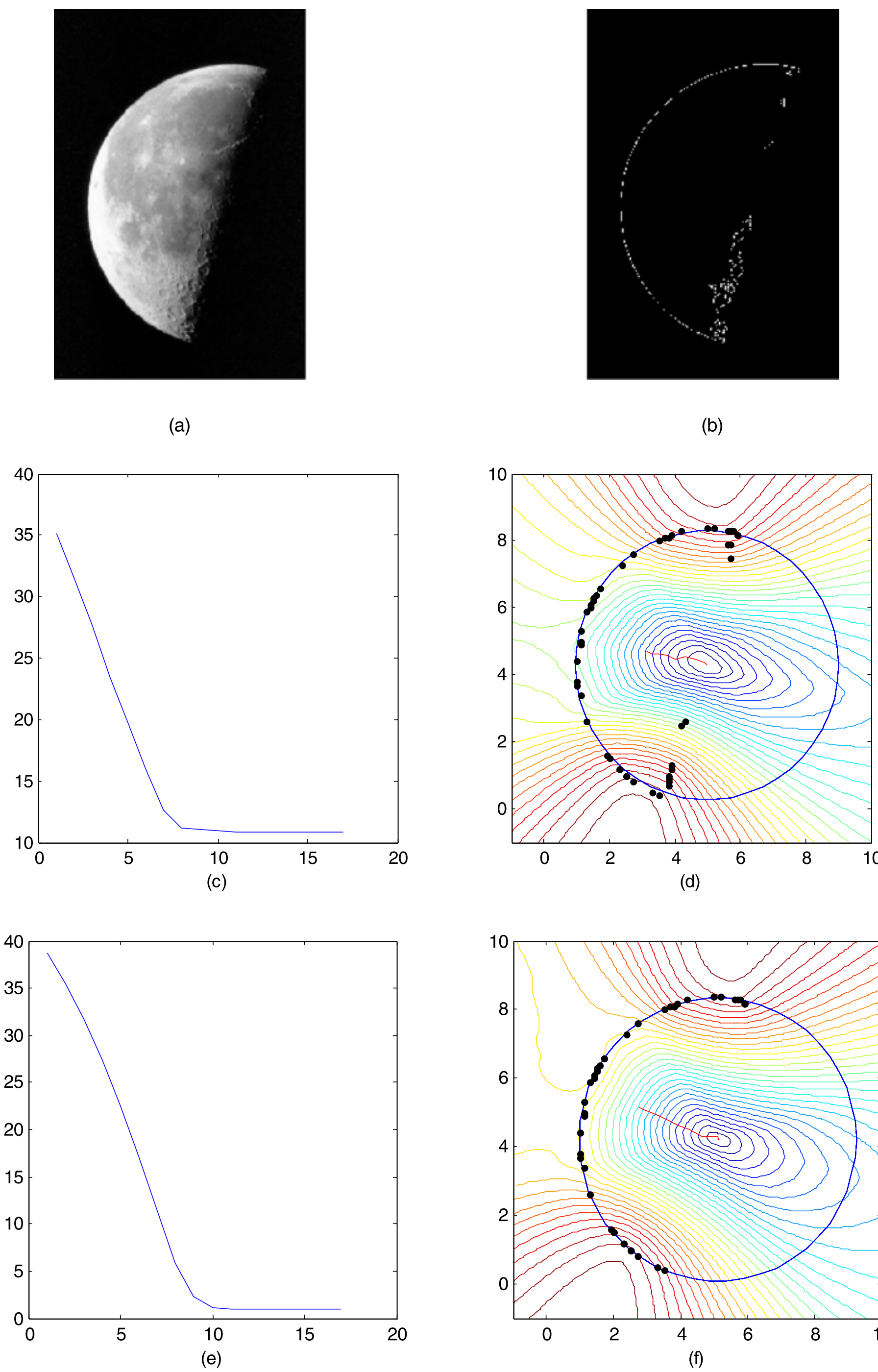


Fig. 12 (**a**) Moon Image. (**b**) Canny edges. (**c**) Error function for 50 points. (**d**) Optimal circle and level curves. (**e**) Error function for 39 points. (**f**) Optimal circle and level curves without outliers

Table 4 Solutions for AGE, OLSF, DOS and HYPER using outliers

Algorithm	Parameters		
	a	*b*	*r*
AGE	−1.5616	−0.6898	101.8012
OLSF	21.0109	12.5833	78.4706
DOS	60.3210	29.8543	42.2443
HYPER	20.7742	13.0504	78.4098

Table 5 Points of the moon image

Id	*x*	*y*	Id	*x*	*y*
1	1	3.8	26	2.5	1
2	2.5	1	**27**	**3.8**	**0.8**
3	2.4	7.3	28	1	3.7
4	2.3	1.2	29	5.7	8.3
5	1.1	4.9	**30**	**3.9**	**1.2**
6	2.5	1	31	1.1	5
7	**3.8**	**0.7**	32	4.2	8.3
8	5.9	8.2	33	2.3	1.2
9	1.4	6.1	34	1.1	5.3
10	2	1.5	35	1.3	2.6
11	1.6	6.4	36	1.5	6.2
12	1.7	6.6	37	1.4	6
13	1.3	5.9	**38**	**4.2**	**2.5**
14	1	4.4	**39**	**4.3**	**2.6**
15	2.7	0.8	40	3.5	8
16	1.5	6.3	41	3.5	0.4
17	5	8.4	42	3.9	8.2
18	3.3	0.5	43	5.9	8.2
19	**5.6**	**7.9**	**44**	**3.9**	**1.3**
20	5.8	8.3	**45**	**3.8**	**0.9**
21	3.8	8.1	46	5.6	8.3
22	1.1	3.4	47	3.7	8.1
23	**5.7**	**7.9**	48	1.9	1.6
24	**3.8**	**1**	**49**	**5.7**	**7.5**
25	5.2	8.4	50	2.7	7.6

mine the optimum circle. Note that there are 11 points (22%, points in bold type in Table 5) that have an absolute geometric error greater than 0.2, see histogram of errors in Fig. 13. If we establish an error tolerance, for example, 0.2, then we have 39 inliers and 11 outliers. We show that the final solution based on 39 points (Fig. 12(f)) is similar to the solution with 11 additional outliers (Fig. 12(d)). These results are shown in Table 6. So, our algorithm can also be used for detection of outliers when an error tolerance is fixed. On the other hand, the circle determined by "Hyper" algorithm is sensitively affected by the outlier points (see Fig. 14). So, first it is necessary to apply an outlier detection method.

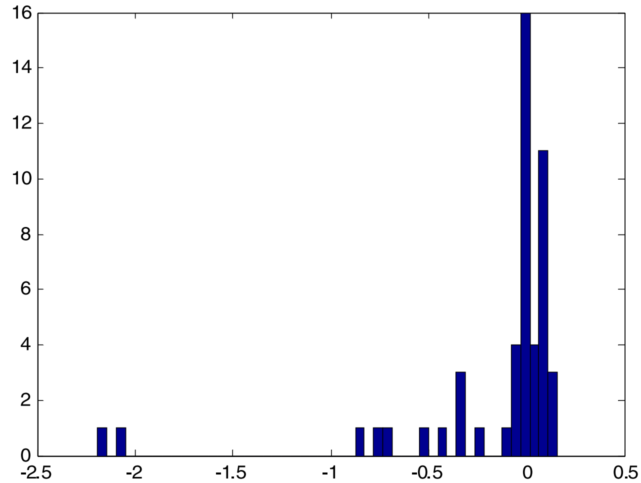


Fig. 13 Histogram of errors

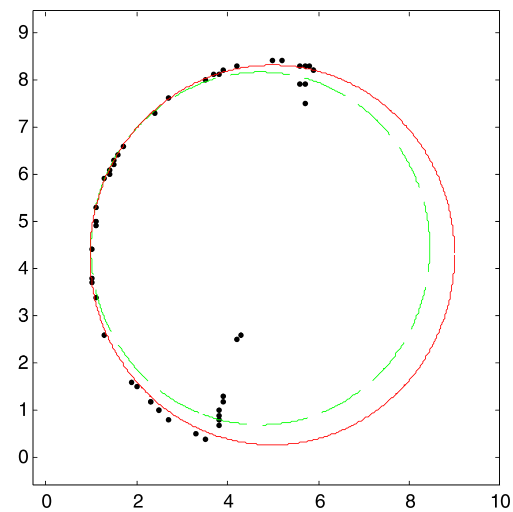


Fig. 14 Solutions for different criteria in a moon example with outliers: AGE (*solid line*); Algebraic Error "Hyper" (*dashed line*)

The RANSAC algorithm can also be applied to robust fitting of circles. For fitting a circle, RANSAC carries out the following steps:

1. First, random samples are drawn uniformly from the input points. The size of the sample is three, since three points are required to determine the circle parameters. Drawing more than three points is inefficient, since the probability of selecting a sample consisting only of inliers data points (i.e. all data points belonging to the same model) that gives a good estimate, decreases with respect to the increasing sample size.
2. Determine the three parameters of the model, (a, b, r) using drawn three points.
3. Count the number of points which lie within an *error tolerance* t to the circle (inliers).

Table 6 Results of fitting to moon points

Algorithm	Moon Points	λ_o	λ_k	Factor1	Factor2	E	Iterations	Parameters a	b	r	Fig.
AGE	50	0.01	< 0.0001	1.2	0.9	10.7886	17	4.9842	4.2914	4.0144	12(d)
AGE	39	0.01	< 0.0001	1.2	0.9	0.9344	17	5.1105	4.2108	4.1371	12(f)

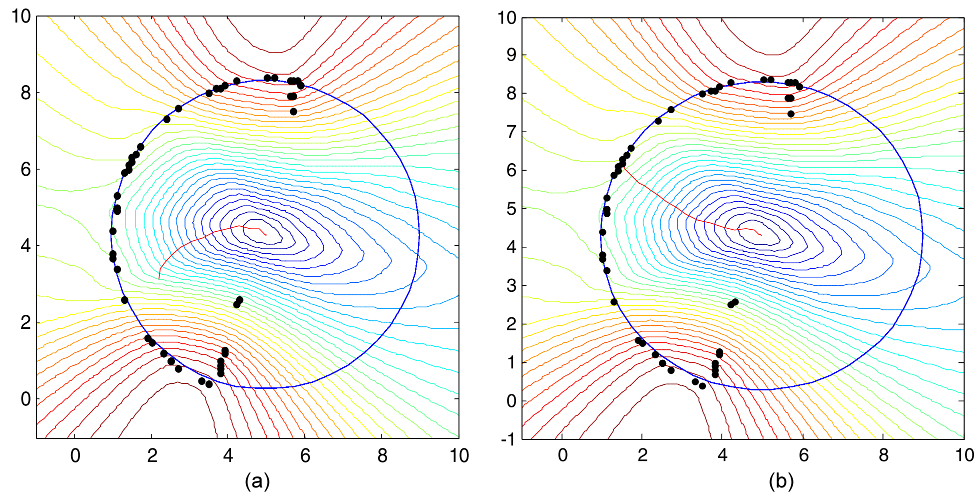


Fig. 15 Centre trajectory. (**a**) Initial centre is determined by 3 points. (**b**) Initial centre is a data point

4. If the number of inliers exceeds a predefined *threshold value* τ, do circle fitting for all inliers, else repeat steps 1 through 4 until a maximum number m of trials is reached.

The RANSAC technique requires three parameters or variables to control the model estimation process. The solution circle depends on the chosen values for these three parameters. Moreover, it also depends on three points that have been randomly drawn in step 1 to provide initial values for the parameter, a, b and r, since the inliers are determined by these parameters. In contrast, our algorithm usually achieves the optimal solution from any initial solution (a, b) determined by three points belongs to point set (inliers or outliers). Also, a data point can be used as initial centre (see centre trajectories in Fig. 15).

On the other hand, our algorithm is robust in the sense of good tolerance to outliers in the data and it does not require parameter tuning. Note that the proposed algorithm can also be used in the step 4 for circle fitting with the inliers as set of points. The step 2 of RANSAC can also be used in our algorithm to provide an initial solution.

5 Conclusions and Discussion

In this paper, we have developed an algorithm to fit a circular arc to data points by minimizing the sum of the geometric distances to the data points (L_1-distance). This objective function is not differentiable, so the proposed method uses right-hand and left-hand derivatives to determine a new direction that allows finding a new solution to improve the objective function value. Traditional geometric criteria that are used in fitting problem are based on a differentiable objective function. They give fast iterative algorithms based on gradients or second derivatives, using nonlinear programming methods. The sensitivity to an initial guess is a common issue with least squares solvers based on the Gauss-Newton method. Moreover, these criteria could be very sensitive to outliers or data noise. In the same way, algebraic criteria leads to efficient algorithms but their accuracy can be unacceptable in the presence of outliers. However, the proposed algorithm is robust since it is not sensitive to outliers and to data noise, so it allows us to identify outliers and to remove these points. A new fitting based on inliers leads to a similar solution. Although the algorithm requires an iterative search which could be sensitive to initial conditions, in our algorithm the mean of the coordinates of the points has been used as an initial centre and the median of point-centre distances as initial radius. These initial conditions have led to optimal solutions even when the points correspond to a circular arc. Also, our algorithm has a computational efficiency similar to the best algorithms that have been developed for geometric criteria in the literature. On the other hand, the optimization techniques for non differential AMPL functions have some problems since AMPL is not reliable for these functions. Finally, we emphasize the computational efficiency and the simplicity of our method, which moves the centre toward the outside points and away from the inside points.

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