Homogeneous coordinates in the plane

- A line in the plane $ax + by + c = 0$ is represented as $(a, b, c)^\top$.
- A line is a subset of points in the plane.
- All vectors $(ka, kb, kc)^\top = k(a, b, c)^\top, k \neq 0$ represent the same line as $(a, b, c)^\top$.
- Two vectors $k_1(a, b, c)$ and $k_2(a, b, c)$, $k_1 \neq 0, k_2 \neq 0$ are said to be *equivalent*.
- The equivalence class $k(a, b, c)$, $k \neq 0$ is called *homogeneous* vectors.
- The vector $(0, 0, 0)^\top$ does not represent any line.
- The set of homogeneous vectors $\mathcal{R}^3 - (0, 0, 0)^\top$ forms the *projective space* $\mathcal{P}^2$.

Homogeneous coordinates in the plane

- A point $\mathbf{x} = (x, y)^\top$ is on the line $\mathbf{l} = (a, b, c)^\top$ iff $ax + by + c = 0$.
- The line equation may be written as $(x, y, 1)(a, b, c)^\top = (x, y, 1)\mathbf{l} = 0$, where the vector $(x, y, 1)$ corresponds to the 2D Cartesian point (x, y).
- If $(x, y, 1)\mathbf{l} = 0$ and $k \neq 0$ then $k(x, y, 1)\mathbf{l} = 0$.
- Any vector $(kx, ky, k), k \neq 0$ is a homogeneous representation of the 2D point (x, y).
- An arbitrary homogeneous vector $\mathbf{x} = (x_1, x_2, x_3)^\top$, $x_3 \neq 0$, represents the point $(x_1/x_3, x_2/x_3)$ in $\mathcal{R}^2$.
- A homogeneous vector has 2 degrees of freedom, it is represented by 3 elements but has arbitrary scale.

Lines and points

- A point $\mathbf{x}$ is on a line $\mathbf{l}$ iff $\mathbf{x}^\top \mathbf{l} = 0$.
- A line $\mathbf{l}$ intersects a point $\mathbf{x}$ iff $\mathbf{l}^\top \mathbf{x} = 0$.
- An intersection point $\mathbf{x}$ between two lines $\mathbf{l}$ and $\mathbf{l}'$ satisfies $\mathbf{x}^\top \mathbf{l} = 0$ and $\mathbf{x}^\top \mathbf{l}' = 0$, i.e. $\mathbf{x}$ is orthogonal to $\mathbf{l}$ and $\mathbf{l}'$.
- Ex. The lines $\mathbf{l} = (1, 1, -2)$ and $\mathbf{l}' = (1, 0, -1)^T$ have intersection $\mathbf{x} = (1, 1, 1)^T$ since $\mathbf{x}^T \mathbf{l} = 0$ and $\mathbf{x}^T \mathbf{l}' = 0$.
- Similarly, a line $\mathbf{l}$ through two points $\mathbf{x}$ and $\mathbf{x}'$ satisfies $\mathbf{l}^T \mathbf{x} = 0$ and $\mathbf{l}^T \mathbf{x}' = 0$.
- Ex. The points $\mathbf{x} = (1, 1, 1)^T$ and $\mathbf{x}' = (2, 0, 1)^T$ are intersected by the line $\mathbf{l} = (1, 1, 2)^T$, $x + y - 2 = 0$ since $\mathbf{l}^T \mathbf{x} = 0$ and $\mathbf{l}^T \mathbf{x}' = 0$.

Lines and points

- Let $\mathtt{L} = [\mathbf{l}\,\mathbf{l}']$. The null-space $\mathcal{N}$ of $\mathtt{L}^\top$ is defined as $\mathcal{N}(\mathtt{L}^\top) = \{\mathbf{x} : \mathtt{L}^\top \mathbf{x} = 0\}$.
- For $\mathbf{l}, \mathbf{l}' \in \mathcal{R}^3$ this is satisfied by e.g. $\mathbf{v} = \mathbf{l} \times \mathbf{l}'$, where *the cross product* $\times$ is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = [\mathbf{a}]_\times \mathbf{b},$$

where

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

- Similarly, a line through the points $\mathbf{x}$ and $\mathbf{x}'$ may be calculated as

$$\mathbf{l} = \mathbf{x} \times \mathbf{x}'.$$

The intersection of parallel lines

- Two parallel lines $\mathbf{l} = (a, b, c)$ and $\mathbf{l}' = (a, b, c')$ intersect each other at the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}' = (c' - c)(b, -a, 0)^\top$.
- The point $(b, -a, 0)^\top$ does not have a finite representation since $(b/0, -a/0)^\top$ is not defined. This corresponds to the interpretation that parallel lines have no intersection in the Euclidean plane.
- However, if we study

$$\lim_{k \to 0} (b, -a, k)^\top$$

with Cartesian representation

$$\lim_{k \to 0} (b/k, -a/k)^\top,$$

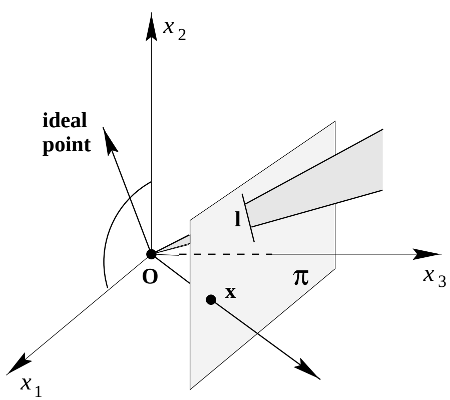
we may interpret the vector $(b, -a, 0)^\top$ as being infinitely far away in the direction of $(b, -a)^\top$.

The line at infinity

- Homogeneous vectors $\mathbf{x} = (x_1, x_2, x_3)^\top$ with $x_3 \neq 0$ correspond to finite points in the real space $\mathcal{R}^2$ or "the set of intersections between non-parallel lines".
- If we extend $\mathcal{R}^2$ with points having $x_3 = 0$ (but $(x_1, x_2)^\top \neq (0, 0)^\top$) we get the *projective space* $\mathcal{P}^2$. Points with $x_3 = 0$ are called *ideal* point or points "at infinity".
- All ideal points $(x_1, x_2, 0)^\top$ are on the *line at infinity* $\mathbf{l}_\infty = (0, 0, 1)^\top$, since $(x_1, x_2, 0)(0, 0, 1)^\top = 0$.
- In the projective plane $\mathcal{P}^2$ two distinct lines have exactly one intersection point, independently of if they are parallel or not.
- The geometry of $\mathcal{P}^n$ is called *projective geometry*.

Interpretation of the projective plane

- A useful interpretation of $\mathcal{P}^2$ is as a set of rays in $\mathcal{R}^3$.
- A homogeneous vector $k(x_1, x_2, x_3)^\top, k \neq 0$ corresponds to a ray though the origin.
- The inhomogeneous representation is given from its intersection with the plane $x_3 = 1$.
- Rays for ideal points lie within the plane $x_3 = 0$ and have no (Euclidean) intersection with the plane $x_3 = 1$.

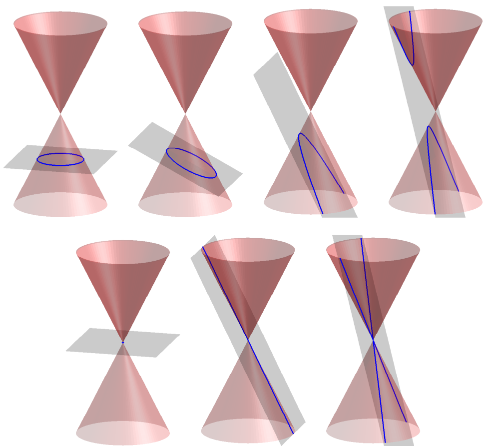


Duality

- Since $\mathbf{x}^\top \mathbf{l} = \mathbf{l}^\top \mathbf{x}$, the meaning of lines and points are interchangeable. Thus, for any relation in $\mathcal{P}^2$ there is a *dual* relation where the meaning of lines and points are interchanged.
- The equation $\mathbf{x}^\top \mathbf{l} = 0$ may be interpreted as that the point $\mathbf{x}$ is on the line $\mathbf{l}$, but also that the point $\mathbf{l}$ is on the line $\mathbf{x}$.
- The equations $\mathbf{x}^\top \mathbf{l} = \mathbf{x}^\top \mathbf{l}' = 0$ may be interpreted as that the point $\mathbf{x}$ is on the lines $\mathbf{l}$ and $\mathbf{l}'$, but also that the line $\mathbf{x}$ intersects the points $\mathbf{l}$ and $\mathbf{l}'$.

Conics

- A conic (section) is a second order curve in the plane. In Euclidean space there are three types of conics: ellipses, parabolas, and hyperbolas. Degenerate conics consist of a point or one or two lines.



The conic equation

- The equation for a conic in Euclidean coordinates is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

- In homogeneous coordinate $x = x_1/x_3, y = x_2/x_3$ it becomes

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathbf{x}^\top \mathtt{C}\mathbf{x} = 0,$$

where

$$\mathtt{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}.$$

- A conic has 5 degrees of freedom since it defined by 6 parameters but has arbitrary scale.

Five points define a conic

- Every point on a conic gives one equation for the coefficients since any point (x_i, y_i, z_i) intersected by the conic has to satisfy

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_iz_i + ey_iz_i + fz_i^2 = 0.$$

- This may be written as

$$\begin{bmatrix} x_i^2 & x_iy_i & y_i^2 & x_iz_i & y_iz_i & z_i^2 \end{bmatrix} \mathbf{c} = 0,$$

where $\mathbf{c} = (a, b, c, d, e, f)^\top$ is the conic $\mathtt{C}$ as a 6-vector.

- With 5 points we get

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1z_1 & y_1z_1 & z_1^2 \\ x_2^2 & x_2y_2 & y_2^2 & x_2z_2 & y_2z_2 & z_2^2 \\ x_3^2 & x_3y_3 & y_3^2 & x_3z_3 & y_3z_3 & z_3^2 \\ x_4^2 & x_4y_4 & y_4^2 & x_4z_4 & y_4z_4 & z_4^2 \\ x_5^2 & x_5y_5 & y_5^2 & x_5z_5 & y_5z_5 & z_5^2 \end{bmatrix} \mathbf{c} = \mathtt{X}\mathbf{c} = 0,$$

where $\mathbf{c}$ is obtained as a null-vector to the 5×6 matrix $\mathtt{X}$.

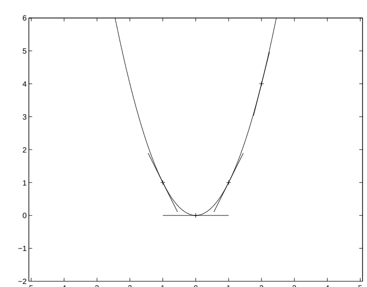
Five points define a conic

- Given the points $\mathbf{x}_1 = (0, 0, 1)$, $\mathbf{x}_2 = (-1, 1, 1)$, $\mathbf{x}_3 = (1, 1, 1)$, $\mathbf{x}_4 = (2, 4, 1)$, $\mathbf{x}_5 = (0, 1, 0)$ we get

$$\mathtt{X} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 8 & 16 & 2 & 4 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

with null-vector

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \text{ and conic } \mathtt{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$$



or $x^2 - y = 0$.

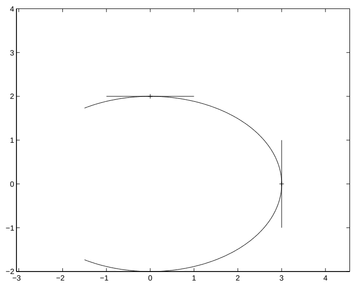
Conic tangents

- The tangent $\mathbf{l}$ to a conic $\mathtt{C}$ in a point $\mathbf{x}$ on $\mathtt{C}$ is given by

$$\mathbf{l} = \mathtt{C}\mathbf{x}.$$

- Example: The conic $(x/3)^2 + (y/2)^2 = 1$ or

$$\mathtt{C} = \begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



intersects the points $\mathbf{x}_1 = (3, 0, 1)^\top$ and $\mathbf{x}_2 = (0, 2, 1)^\top$. The tangents are

$$\mathbf{l}_1 = \mathtt{C}\mathbf{x}_1 = (1/3, 0, -1)^\top \text{ or } x = 3,$$

and

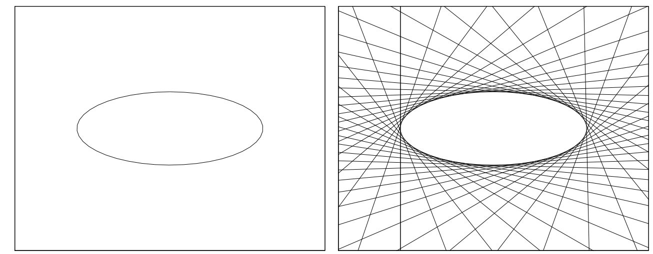
$$\mathbf{l}_2 = \mathtt{C}\mathbf{x}_2 = (0, 1/2, -1)^\top \text{ or } y = 2.$$

Dual conics

- The equation $\mathbf{x}^\top \mathtt{C}\mathbf{x} = 0$ defines a subset of points in $\mathcal{P}^2$. The conic $\mathtt{C}$ is called a *point conic*.
- There is a corresponding second order expression for lines. A line conic (dual conic) is denoted $\mathtt{C}^*$, where $\mathtt{C}^*$ is the *adjoint* matrix to $\mathtt{C}$ and the equation

$$\mathbf{l}^\top \mathtt{C}^* \mathbf{l} = 0$$

define the subset of all lines in $\mathcal{P}^2$ that are *tangent* to the point conic $\mathtt{C}$.



$\mathbf{x}^\top \mathtt{C}\mathbf{x} = 0$ $\qquad$ $\mathbf{l}^\top \mathtt{C}^* \mathbf{l} = 0$

Dual conics

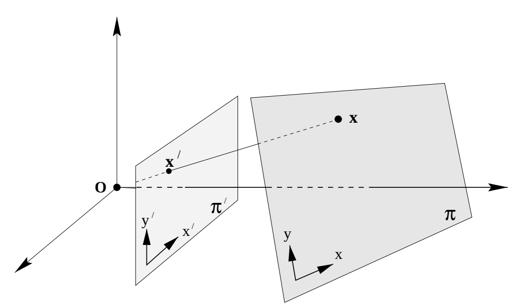
- If $\mathtt{C}$ is symmetric with full rank then $\mathtt{C}^{-1} = \mathtt{C}^*$ up to scale. This means that all points $\mathbf{x}$ on $\mathtt{C}$ have unique tangents $\mathbf{l} = \mathtt{C}\mathbf{x}$ and all tangents $\mathbf{l}$ have unique tangency points $\mathbf{x} = \mathtt{C}^{-1}\mathbf{l}$.
- In this case the the point conic $\mathbf{x}^\top \mathtt{C}\mathbf{x} = 0$ corresponds to the line conic $\mathtt{C}^{-1}$ since

$$0 = \mathbf{x}^\top \mathtt{C}\mathbf{x} = (\mathtt{C}^{-1}\mathbf{l})^\top \mathtt{C}(\mathtt{C}^{-1}\mathbf{l}) = \mathbf{l}^\top \mathtt{C}^{-1}\mathbf{l} = 0.$$

- Should the matrix $\mathtt{C}$ be rank deficient the conic is *degenerate*.
- Degenerate point conics include two lines (rank 2) and one line (rank 1).
- Ex. the point conic $\mathtt{C} = \mathbf{l}\mathbf{m}^\top + \mathbf{m}\mathbf{l}^\top$ consists of the lines $\mathbf{l}$ and $\mathbf{m}$. The null-vector $\mathbf{x} = \mathbf{l} \times \mathbf{m}$ on both lines $\mathbf{l}$ and $\mathbf{m}$ does not have a unique tangent.
- Degenerate line conics include two points (rank 2) and one point (rank 2).
- Ex. The line conic $\mathtt{C}^* = \mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top$ has rank 2 and consists of all lines intersecting $\mathbf{x}$ and/or $\mathbf{y}$. All lines have unique tangency points except $\mathbf{l} = \mathbf{x} \times \mathbf{y}$.

Projective transformations

- Definition: A *projectivity* (or *projective transformation* or *homography*) is an invertible mapping h from $\mathcal{P}^2$ onto itself such that three points $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are collinear iff $h(\mathbf{x}_1)$, $h(\mathbf{x}_2)$, and $h(\mathbf{x}_3)$ also are collinear.
- Thus: Lines are mapped onto lines.



- All projective transformations of homogeneous points $\mathbf{x}$ may be written as $\mathbf{x}' = h(\mathbf{x}) = \mathtt{H}\mathbf{x}$, where $\mathtt{H}$ is a non-singular 3×3-matrix.
- The matrix $\mathtt{H}$ has 8 degrees of freedom (9 elements, arbitrary scale).

Rectification of plane perspective

- If the coordinates for 4 points $\mathbf{x}_i$ and their mappings $\mathbf{x}'_i$ in the image are known, we may calculate the homography $\mathtt{H}$.
- Each point pair $\mathbf{x} = (x, y)$ and $\mathbf{x}' = (x', y')$ has to satisfy

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$
$$y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

or

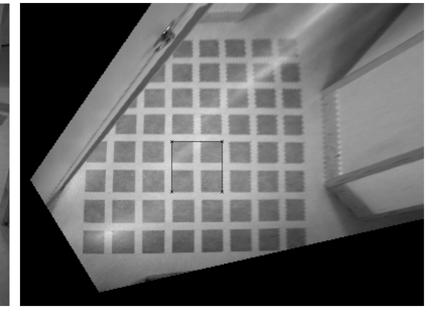
$$x'(h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13}$$
$$y'(h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23}.$$

The latter equations are linear in h_{ij}.

- Given 4 points we get 8 equations, enough to uniquely determine $\mathtt{H}$ assuming the points are in “standard position”, i.e. no 3 points are collinear.

Rectification of plane perspective

- Given $\mathtt{H}$ we may apply $\mathtt{H}^{-1}$ to remove the effect of the perspective transformation.

normhomo.m and lines2pt.m

```
function X=homonorm(X)
%HOMONORM Normalize homogenous points.
%
%X=homonorm(X);

% v1.0  2002-03-19. Niclas Borlin, niclas@cs.umu.se.

[m,n]=size(X);

X=X./X(m*ones(1,n),:);


function x=lines2pt(l1,l2)
%LINES2PT Find homogenous intersection of two homogenous lines.
%
%x=lines2pt(l1,l2)
%l1,l2 - lines in homogenous coordinates.
%x     - intersection in homogenous coordinates.

% v1.0  2002-03-19. Niclas Borlin, niclas@cs.umu.se.

x=null([l1,l2]');
```


drawhomoline.m

```
function h=drawhomoline(L,varargin)
%DRAWHOMOLINE Draw homogenous line.
%
%h=drawhomoline(L[,line attributes])
%L - matrix with homogenous lines in
%    each column.
%h - graphic handles.

% v1.0  2002-03-24. Niclas Borlin,
%                   niclas@cs.umu.se.

% Get axes scaling.
ax=axis;
xlim=ax(1:2);
ylim=ax(3:4);

% Construct lines for each side of
% the axis.
axL=[1,1,0,0;
     0,0,1,1;
     -xlim,-ylim];
```

```
% Preallocate handle vector for lines.
h=zeros(size(L,2),1);
for i=1:size(L,2)
  l=L(:,i);
  % Determine if line is more vertical or
  % horizontal.
  if (abs(l(1))<abs(l(2)))
    % More horizontal. Calculate intersection to
    % left/right sides.
    x1=normhomo(lines2pt(l,axL(:,1)));
    x2=normhomo(lines2pt(l,axL(:,2)));
  else
    % More vertical. Calculate intersection to
    % upper/lower sides.
    x1=normhomo(lines2pt(l,axL(:,3)));
    x2=normhomo(lines2pt(l,axL(:,4)));
  end
  h(i)=line([x1(1),x2(1)],[x1(2),x2(2)],varargin{:
end
```

ansformation of points, lines, and conics

- Consider a point homography $\mathbf{x}' = \mathtt{H}\mathbf{x}$. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are on the line $\mathbf{l}$ then the points $\mathbf{x}_1'$ and $\mathbf{x}_2'$ will also be on a line $\mathbf{l}' = \mathtt{H}^{-\top}\mathbf{l}$ since

$$\mathbf{l}'^\top \mathbf{x}_i' = (\mathtt{H}^{-\top}\mathbf{l})^\top \mathbf{x}_i' = \mathbf{l}^\top \mathtt{H}^{-1}\mathtt{H}\mathbf{x}_i = \mathbf{l}^\top \mathbf{x}_i = 0.$$

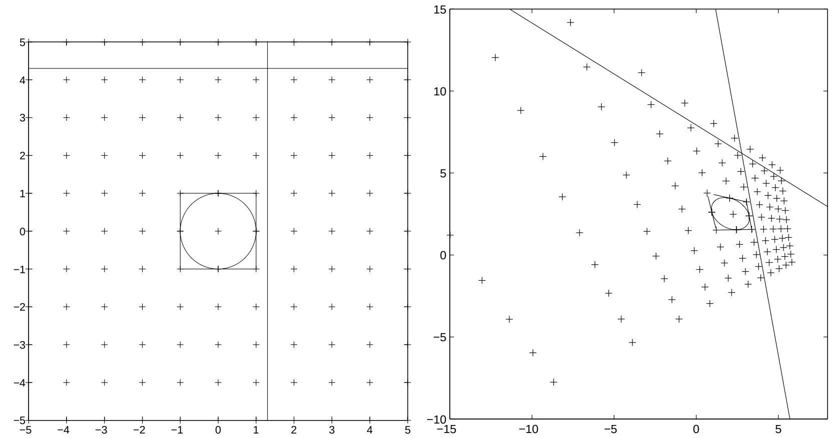
- Under the same point mapping $\mathbf{x}' = \mathtt{H}\mathbf{x}$ a conic $\mathtt{C}$ is mapped to $\mathtt{C}' = \mathtt{H}^{-\top}\mathtt{C}\mathtt{H}^{-1}$ since

$$\mathbf{x}^\top \mathtt{C}\mathbf{x} = \mathbf{x}'^\top \left(\mathtt{H}^{-1}\right)^\top \mathtt{C}\mathtt{H}^{-1}\mathbf{x}' = \mathbf{x}'^\top \underbrace{\mathtt{H}^{-\top}\mathtt{C}\mathtt{H}^{-1}}_{\mathtt{C}'}\mathbf{x}'.$$

- A line conic $\mathtt{C}^*$ is mapped to $\mathtt{C}^{*\prime} = \mathtt{H}\mathtt{C}^*\mathtt{H}^\top$ since

$$\mathbf{l}^\top \mathtt{C}^*\mathbf{l} = (\mathtt{H}^\top \mathbf{l}')^\top \mathtt{C}^*(\mathtt{H}^\top \mathbf{l}') = \mathbf{l}'^\top \underbrace{\mathtt{H}\mathtt{C}^*\mathtt{H}^\top}_{\mathtt{C}^{*\prime}}\mathbf{l}'.$$

Transformations of points, lines, and con



$\mathbf{x}, \mathbf{l}, \mathtt{C}$ $\qquad$ $\mathbf{x}' = \mathtt{H}\mathbf{x},\ \mathbf{l}' = \mathtt{H}^{-\top}\mathbf{l},\ \mathtt{C}' = \mathtt{H}^{-\top}\mathtt{C}\mathtt{H}^{-1}$

Line conics and angles

- Line conics are needed to describe angles between lines in projective geometry.
- In Euclidean geometry, angles between lines are calculated from the inner product between their normals, e.g. if $\mathbf{l} = (l_1, l_2, l_3)^\top$ and $\mathbf{m} = (m_1, m_2, m_3)^\top$ with normals $(l_1, l_2)^\top$ and $(m_1, m_2)^\top$, then the angle θ between the lines is calculated from

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}.$$

- The corresponding well-defined expression in projective geometry is

$$\cos\theta = \frac{\mathbf{l}^\top \mathtt{C}^*_\infty \mathbf{m}}{\sqrt{(\mathbf{l}^\top \mathtt{C}^*_\infty \mathbf{l})(\mathbf{m}^\top \mathtt{C}^*_\infty \mathbf{m})}}, \text{ where } \mathtt{C}^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Especially we have that $\mathbf{l}$ and $\mathbf{m}$ are orthogonal if $\mathbf{l}^\top \mathtt{C}^*_\infty \mathbf{m} = 0$.

Line conics and angles

- The expression $\mathbf{l}^\top \mathtt{C}^*_\infty \mathbf{m} = 0$ is invariant under a homography $\mathbf{x}' = \mathtt{H}\mathbf{x}$ since $\mathbf{l}' = \mathtt{H}^{-\top}\mathbf{l}$ and $\mathtt{C}^{*\prime}_\infty = \mathtt{H}\mathtt{C}^*_\infty\mathtt{H}^\top$ means that

$$\mathbf{l}'^\top \mathtt{C}^{*\prime}_\infty \mathbf{m}' = (\mathtt{H}^{-\top}\mathbf{l})^\top \mathtt{H}\mathtt{C}^*_\infty \mathtt{H}^\top \mathtt{H}^{-\top}\mathbf{m} = \mathbf{l}^\top \mathtt{H}^{-1}\mathtt{H}\mathtt{C}^*_\infty \mathtt{H}^\top \mathtt{H}^{-\top}\mathbf{m} = \mathbf{l}^\top \mathtt{C}^*_\infty \mathbf{m}.$$

- Thus, if we know the projection $\mathtt{C}^{*\prime}_\infty$ of $\mathtt{C}^*_\infty$ in an image, we can determine if two lines $\mathbf{l}'$ and $\mathbf{m}'$ in the image are orthogonal by calculating $\mathbf{l}'^\top \mathtt{C}^{*\prime}_\infty \mathbf{m}'$.

A transformation hierarchy for $\mathcal{P}^2$

- Homographies may be divided into different subgroups with different level of generality.
- The four subgroups we will talk about are the following, in order of increasing level of generality
  - Isometry.
  - Similarity.
  - Affinity.
  - Projectivity.

Class I: Isometry

- An isometry is a transformation of the plane $\mathcal{R}^2$ that preserves the Euclidean distance.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

  where $\epsilon = \pm 1$. If $\epsilon = +1$, the transformation is called *orientation preserving* and is a *Euclidean transformation* composed by a rotation and a translation. If $\epsilon = -1$, the transformation contains a mirroring.
- An isometry may be written as

$$\mathbf{x}' = \mathtt{H}_\mathtt{E}\mathbf{x} = \begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}\mathbf{x},$$

  where $\mathtt{R}$ is an orthogonal 2×2 matrix and $\mathbf{t}$ is a 2-vector.
- An isometry has 3 degrees of freedom; rotation (1) and translation (2).
- Invariants: lengths, angles, areas, etc.

Class II: Similarity

- A similarity is an isometry plus isotropic scaling.
- For orientation-preserving isometries, the similarity has the matrix form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

  or

$$\mathbf{x}' = \mathtt{H}_\mathtt{S}\mathbf{x} = \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}\mathbf{x},$$

  where the scalar s represents the scaling.
- A similarity has 4 degrees of freedom; rotation (1), translation (2) and scaling (1).
- Invariants: angles, parallelity, length ratios, area ratios, “shape”.
- A similarity is also called a *metric* transformation.

Class III: Affinity

- An affine transformation (affinity) is a non-singular transformation followed by a translation and is represented by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

  or

$$\mathbf{x}' = \mathtt{H}_\mathtt{A}\mathbf{x} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}\mathbf{x},$$

  where $\mathtt{A}$ is a non-singular 2×2 matrix.
- An affinity has 6 degrees of freedom; the elements of $\mathtt{A}$ and $\mathbf{t}$.
- Invariants: parallelity, length ration for parallel lines, area ratios.

terpretation of an affine transformation

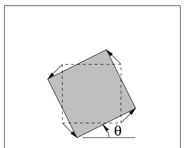
- If we factor the transformation matrix $\mathtt{A}$ into

$$\mathtt{A} = \mathtt{R}(\theta)\mathtt{R}(-\phi)\mathtt{D}\mathtt{R}(\phi),$$

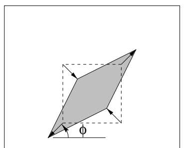
where $\mathtt{R}(\theta)$ and $\mathtt{R}(\phi)$ are rotation matrices and $\mathtt{D}$ is a diagonal matrix

$$\mathtt{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

then the transformation $\mathtt{A}$ may be interpreted as a sequence of rotations and (anisotropic) scaling.



rotation



deformation

- Thus the two extra degrees of freedom may be interpreted as the scaling λ_1/λ_2 and the "anisotropy angle" ϕ.
- This kind of factorization is always possible from the singular value decomposition (SVD)

$$\begin{aligned} \mathtt{A} &= \mathtt{U}\mathtt{D}\mathtt{V}^\top = (\mathtt{U}\mathtt{V}^\top)(\mathtt{V}\mathtt{D}\mathtt{V}^\top) \\ &= \mathtt{R}(\theta)(\mathtt{R}(-\phi)\mathtt{D}\mathtt{R}(\phi)). \end{aligned}$$

Class IV: Projectivity

- A projective transformation (projectivity) is a general linear mapping of homogeneous coordinates and is written

$$\mathbf{x}' = \mathtt{H}_\mathtt{P}\mathbf{x} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}\mathbf{x},$$

where $\mathbf{v} = (v_1, v_2)^\top$ and v is a scalar.

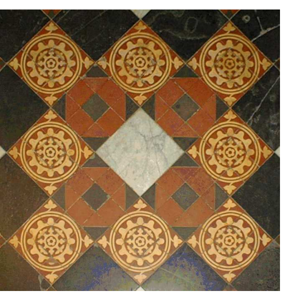
- A projectivity has 8 degrees of freedom; 9 elements in $\mathtt{H}_\mathtt{P}$ and arbitrary scale.
- Invariants: Cross ratios of line lengths.
- The most important different between a projectivity and an affinity is the vector $\mathbf{v}$ and its effect on the mapping of ideal points.
- Study the mapping of an ideal point $(x_1, x_2, 0)^\top$:

$$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathtt{A}\begin{bmatrix} x \\ x \end{bmatrix} \\ v_1x_1 + \end{bmatrix}$$

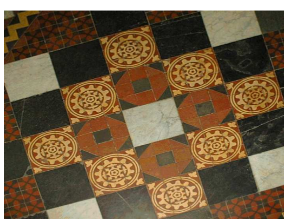
- For an affinity, $\mathbf{v} = \mathbf{0}$ and all ideal points are mapped to ideal points. For a projectivity with $\mathbf{v} \neq \mathbf{0}$, some ideal points are mapped to finite points.

The effect of different transformations

Similarity



Affinity



Projectivity

Decomposition of a projective transforma

- A projective transformation may be decomposed into a sequence of transformations on different levels in the hierarchy:

$$\mathtt{H} = \mathtt{H}_\mathtt{S}\mathtt{H}_\mathtt{A}\mathtt{H}_\mathtt{P} = \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}\begin{bmatrix} \mathtt{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}\begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{v}^\top & v \end{bmatrix} = \begin{bmatrix} \mathtt{A} & v\mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix},$$

where $\mathtt{A} = s\mathtt{R}\mathtt{K} + \mathbf{t}\mathbf{v}^\top$ is a non-singular matrix and $\mathtt{K}$ is an upper triangular matrix with $|\mathtt{K}| = 1$.

- The decomposition is valid if $v \neq 0$ and unique if s is chosen to be positive.
- Since $\mathtt{H}^{-1} = \mathtt{H}_\mathtt{P}^{-1}\mathtt{H}_\mathtt{A}^{-1}\mathtt{H}_\mathtt{S}^{-1}$ and $\mathtt{H}_\mathtt{P}^{-1}$, $\mathtt{H}_\mathtt{A}^{-1}$, $\mathtt{H}_\mathtt{P}^{-1}$ are projective, affine, and similar, respectively, it is possible to decompose the transformation in the opposite direction, i.e. there exists also a factorization such that

$$\mathtt{H} = \mathtt{H}_\mathtt{P}\mathtt{H}_\mathtt{A}\mathtt{H}_\mathtt{S} = \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{v}^\top & v \end{bmatrix}\begin{bmatrix} \mathtt{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

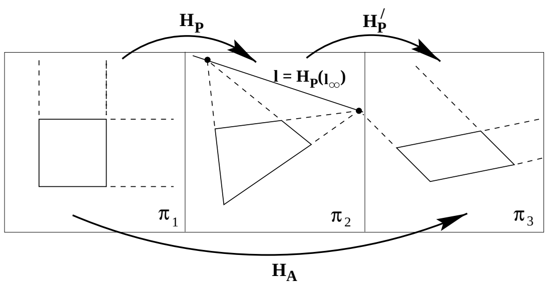
with different values for $\mathtt{K}$, $\mathtt{R}$, $\mathbf{t}$ and $\mathbf{v}$.

Reconstruction of affine properties

- An affine transformation maps the line at infinity onto itself since

$$\mathbf{l}'_\infty = \mathtt{H}_\mathtt{A}^{-\top}\mathbf{l}_\infty = \begin{bmatrix} \mathtt{A}^{-\top} & \mathbf{0} \\ -\mathbf{t}^\top\mathtt{A}^{-\top} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{l}_\infty.$$

- If we know the projection $\mathbf{l}'_\infty$ of $\mathbf{l}_\infty$ in a projective mapping of a plane we may perform affine measurements. E.g. parallel lines in the plane should intersect on $\mathbf{l}'_\infty$.



Affine rectification

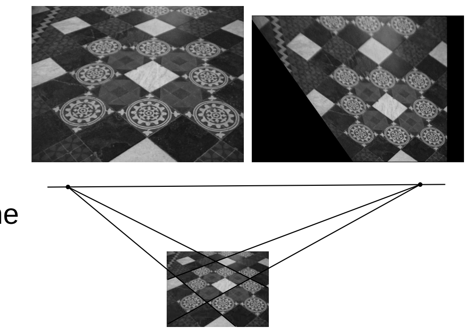
- We may also transform the image such that $\mathbf{l}'_\infty$ is transformed back to $\mathbf{l}_\infty$.
- If $\mathbf{l}'_\infty$ is the line $\mathbf{l} = (l_1, l_2, l_3)^\top$ we may (assuming $l_3 \neq 0$) construct the following transformation

$$\mathtt{H} = \mathtt{H}_\mathtt{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix},$$

where $\mathtt{H}_\mathtt{A}$ is an arbitrary affine transformation.

- If we apply $\mathtt{H}$ on the image, the line at infinity will be mapped to its canonical positions since

$$\mathtt{H}^{-\top}(l_1, l_2, l_3)^\top = (0, 0, 1)^\top = \mathbf{l}_\infty.$$



The circular points and their dual

- There are two points on $\mathbf{l}_\infty$ that are mapped onto each other under a similarity. They are called *circular* or *absolute* points and are denoted

$$\mathbf{I} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}.$$

- Under an orientation-preserving similarity

$$\begin{aligned} \mathbf{I}' &= \mathtt{H}_\mathtt{S}\mathbf{I} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} s\cos\theta - si\sin\theta \\ s\sin\theta + si\cos\theta \\ 0 \end{bmatrix} = s \begin{bmatrix} (\cos\theta - i\sin\theta)\cdot 1 \\ (\cos\theta - i\sin\theta)\cdot i \\ 0 \end{bmatrix} = se^{-i\theta} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \mathbf{I}. \end{aligned}$$

The circular points

- The circular points are intersection points between a circle and the line at infinity.
- The equation for a circle has $a = c$ and $b = 0$

$$ax_1^2 + ax_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

and intersects $\mathbf{l}_\infty$ where $x_3 = 0$ or

$$a(x_1^2 + x_2^2) = 0,$$

with solution $\mathbf{I} = (1, i, 0)^\top$ and $\mathbf{J} = (1, -i, 0)^\top$.

- Since $\mathbf{I}$ are $\mathbf{J}$ are on all circles, we only need 3 more points to uniquely determine the equation of the circle, something already known in Euclidean geometry.

Calculation of the circle equation

- Which circle intersects the points $\mathbf{x}_1 = (0,0,1)^\top$, $\mathbf{x}_2 = (1,0,1)^\top$ and $\mathbf{x}_3 = (1,1,1)$?
- If we add the circular points, the 5-point algorithm gives us the matrix

$$\mathtt{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & i & -1 & 0 & 0 & 0 \\ 1 & -i & -1 & 0 & 0 & 0 \end{bmatrix}$$

with null-space

$$\mathbf{c} = [1, 0, 1, -1, -1, 0]^\top$$

and conic

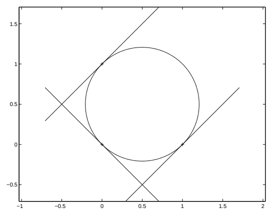
$$\mathtt{C} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

corresponding to

$$x^2 + y^2 - x - y = 0$$

or

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 - \frac{1}{2} = 0.$$



The dual conic to the circular points

- The line conic $\mathtt{C}^*_\infty = \mathbf{I}\mathbf{J}^\top + \mathbf{J}\mathbf{I}^\top$ is dual to the circular points.
- In Euclidean coordinates it is given by

$$\mathtt{C}^*_\infty = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} [1 \ -i \ 0] + \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} [1 \ i \ 0] = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}$$

- The conic $\mathtt{C}^*_\infty$ is invariant under a similarity transformation $\mathbf{x}' = \mathtt{H}_\mathtt{S}\mathbf{x}$ since

$$\begin{aligned} \mathtt{C}^{*\prime}_\infty &= \mathtt{H}_\mathtt{S}\mathtt{C}^*_\infty\mathtt{H}_\mathtt{S}^\top = \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix} \begin{bmatrix} s\mathtt{R}^\top & \mathbf{0} \\ \mathbf{t}^\top & 1 \end{bmatrix} \\ &= \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} s\mathtt{R}^\top & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix} = s^2 \begin{bmatrix} \mathtt{R}\mathtt{R}^\top & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix} = \mathtt{C}^*_\infty. \end{aligned}$$

- The conic $\mathtt{C}^*_\infty$ has 4 degrees of freedom (symmetric 3×3 matrix with arbitrary scale and $|\mathtt{C}^*_\infty| = 0$).

The information content of $\mathtt{C}^*_\infty$

- Study the line conic $\mathtt{C}^*_\infty$ under a projective transformation:

$$\begin{aligned} \mathtt{C}^{*\prime}_\infty &= (\mathtt{H}_\mathtt{P}\mathtt{H}_\mathtt{A}\mathtt{H}_\mathtt{S})\,\mathtt{C}^*_\infty\,(\mathtt{H}_\mathtt{P}\mathtt{H}_\mathtt{A}\mathtt{H}_\mathtt{S})^\top = (\mathtt{H}_\mathtt{P}\mathtt{H}_\mathtt{A})\left(\mathtt{H}_\mathtt{S}\mathtt{C}^*_\infty\mathtt{H}_\mathtt{S}^\top\right)\left(\mathtt{H}_\mathtt{A}^\top\mathtt{H}_\mathtt{P}^\top\right) \\ &= \begin{bmatrix} \mathtt{K}\mathtt{K}^\top & \mathtt{K}\mathtt{K}^\top\mathbf{v} \\ \mathbf{v}^\top\mathtt{K}\mathtt{K}^\top & \mathbf{v}^\top\mathtt{K}\mathtt{K}^\top\mathbf{v} \end{bmatrix}. \end{aligned}$$

- The (projection of the) conic $\mathtt{C}^*_\infty$ contains all information needed to perform a metric rectification, i.e. to determine the affine and projective properties of the transformation.

The effect of a homography on $\mathtt{C}^*_\infty$

- Assume we have two lines $\mathbf{l} = (1, 0, -1.3)^\top$, $\mathbf{m} = (0, 1, -4.3)^\top$ and the transformation

$$\begin{aligned} s &= 0.75,\ \mathtt{R} = \begin{bmatrix} \cos 15^\circ & -\sin 15^\circ \\ \sin 15^\circ & \cos 15^\circ \end{bmatrix},\ \mathtt{K} = \begin{bmatrix} 1.25 & 0.1 \\ 0 & 0.8 \end{bmatrix}, \\ \mathbf{v} &= \begin{bmatrix} 0.1 \\ 0 \end{bmatrix},\ v = 0.5, \text{ or} \\ \mathtt{H} &= \mathtt{H}_\mathtt{P}\mathtt{H}_\mathtt{A}\mathtt{H}_\mathtt{S} = \begin{bmatrix} 1.434 & -0.264 & 2.248 \\ 0.241 & 0.899 & 2.481 \\ 0.143 & -0.026 & 1 \end{bmatrix}. \end{aligned}$$

- Then

$$\mathbf{l}' = \mathtt{H}^{-\top}\mathbf{l} = \begin{bmatrix} -0.257 \\ -0.046 \\ 1 \end{bmatrix},\ \mathbf{m}' = \mathtt{H}^{-\top}\mathbf{m} = \begin{bmatrix} -0.079 \\ -0.126 \\ 1 \end{bmatrix},$$

and

$$\mathtt{C}^{*\prime}_\infty = \mathtt{H}\mathtt{C}^*_\infty\mathtt{H}^\top = \begin{bmatrix} 100 & 5.087 & 10 \\ 5.087 & 40.700 & 0.509 \\ 10 & 0.509 & 1 \end{bmatrix}$$

Determination of orthogonality with C^*_∞

- If we know the image (projection) of C^*_∞

$$C^{*\prime}_\infty = \begin{bmatrix} 100 & 5.087 & 10 \\ 5.087 & 40.700 & 0.509 \\ 10 & 0.509 & 1 \end{bmatrix}$$

we are able to determine if the lines

$$\mathbf{l}' = \begin{bmatrix} -0.257 \\ -0.046 \\ 1 \end{bmatrix}, \ \mathbf{m}' = \begin{bmatrix} -0.079 \\ -0.126 \\ 1 \end{bmatrix}$$

in the image are orthogonal in the world plane.

$$\mathbf{l}'^\top C^{*\prime}_\infty \mathbf{m} = \begin{bmatrix} -0.257 \\ -0.046 \\ 1 \end{bmatrix}^\top \begin{bmatrix} 100 & 5.0874 & 10 \\ 5.0874 & 40.6995 & 0.5087 \\ 10 & 0.5087 & 1 \end{bmatrix} \begin{bmatrix} -0.079 \\ -0.126 \\ 1 \end{bmatrix} = 0,$$

Metric rectification with $C^{*\prime}_\infty$

- Given $C^{*\prime}_\infty$ we can calculate the affine and projective properties K and $\mathbf{v}$:

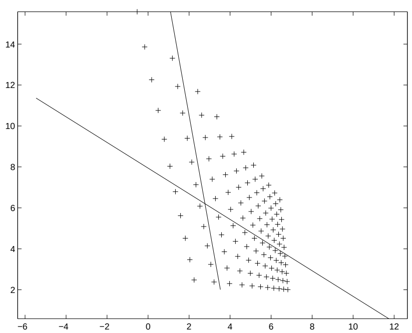
$$KK^\top = \begin{bmatrix} 100 & 5.0874 \\ 5.0874 & 40.6995 \end{bmatrix} \Rightarrow K = \begin{bmatrix} 9.9682 & 0.7975 \\ 0 & 6.3796 \end{bmatrix} = \begin{bmatrix} 1.25 & 0.1 \\ 0 & 0.8 \end{bmatrix}$$

and

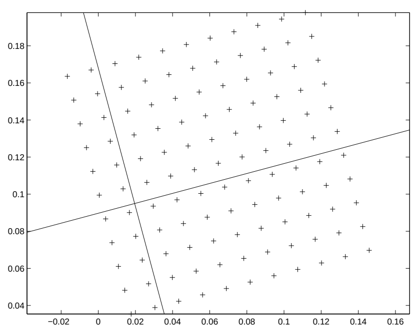
$$KK^\top \mathbf{v} = \begin{bmatrix} 10 \\ 0.5087 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 100 & 5.0874 \\ 5.0874 & 40.6995 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 0.5087 \end{bmatrix} = \begin{bmatrix} 0. \\ 0 \end{bmatrix}$$

Metric rectification with C^*_∞

- Given K and $\mathbf{v}$ we may eliminate the affine and projective component of the transformation by applying $H_M = (H_P H_A)^{-1}$ on the points and $H_M^{-\top}$ on the lines.



$\mathbf{x}'$



$H_M\mathbf{x}'$

Metric rectification by orthogonal line

- If an images has been affinely rectified we need 2 equations to determine the 2 degrees of freedom in K. We may get these equation from pairs of imaged orthogonal lines.
- Assume $\mathbf{l}'$ and $\mathbf{m}'$ in the affinely rectified image correspond to two orthogonal lines $\mathbf{l}$ and $\mathbf{m}$ in the world plane.
- Since $\mathbf{v} = \mathbf{0}$ we have

$$\begin{bmatrix} l'_1 \\ l'_2 \\ l'_3 \end{bmatrix} \begin{bmatrix} KK^\top & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix} \begin{bmatrix} m'_1 \\ m'_2 \\ m'_3 \end{bmatrix} = 0$$

- which is a linear equation in

$$S = KK^\top = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix},$$

$$[l'_1 m'_1, l'_1 m'_2 + l'_2 m'_1, l'_2 m'_2]\, \mathbf{s} = 0,$$

where $\mathbf{s} = (s_{11}, s_{12}, s_{22})^\top$.

- Given the image of two pairs of orthogonal lines we may determine $\mathbf{s}$ and therefore K and H_A up to an unknown scale.
- The application of H_A^{-1} on the image will do a metric rectification.