9

Conics and their duals

You always admire what you really don't understand.
Blaise Pascal

So far we dealt almost exclusively with situations in which only points and lines were involved. Geometry would be quite a pure topic if these were the only objects to be treated. Large parts of classical elementary geometry deal with constructions involving *circles*. Already the most elementary drawing tools treated by Euclid (the *ruler* and the *compass*) contain a tool for generating circles. In a sense so far we dealt with the ruler alone. Unfortunately, circles are not a concept of projective geometry. This can be easily seen by observing that a shape of a circle is not invariant under projective transformations. If you look at a sheet of paper on which a circle is drawn from a skew angle you will see an ellipse. In fact projective transformations of circles include ellipses, hyperbolas and parabolas. They are subsumed under the term *conic sections* or *conic*, for short. Conics are the concept of projective geometry that comes closest to the concept of circles in Euclidean geometry. It is the purpose of this section to give a purely projective treatment of conics. Later on we will see how certain specializations give interesting insights in the geometry of circles.

9.1 The equation of a conic

Let us start with the unit circle in the euclidean plane:

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

In this section we will investigate which shapes can arise if we transform this circle projectively. For this we again consider the Euclidean plane embedded at the $z = 1$ plane of $\mathbb{R}^3$ and represented by homogeneous coordinates (any other affine embedding not containing the origin would serve as well and would lead to the same result). Thus the points of the circle correspond to points with homogeneous coordinates $(x, y, 1)$ with $x^2 + y^2 = 1$. Taking into account that homogeneous coordinates are only specified up to a scalar multiple, we may rewrite this condition in a more general way as points (x, y, z) with $x^2 + y^2 = z^2$. Setting $z = 1$ we obtain the original formula. According to the fact that every term of the expression $x^2 + y^2 = z^2$ is quadratic a vector (x, y, z) satisfies this expression if $(\lambda x, \lambda y, \lambda z);\ \lambda \neq 0$ satisfies it. We may rewrite the quadratic equation as

$$(x, y, z) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Thus the homogeneous coordinate vectors that satisfy this equation are exactly those that represent points of our circle. We now want to transform these points projectively. Applying a projective transformation to the points can be carried out by replacing both occurrences of the vector $p = (x, y, z)$ by a transformed vector $M \cdot p$. Thus we obtain that a projectively transformed unit circle can be expressed algebraically as the solutions of the equation

$$((x, y, z) \cdot M^T) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot (M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}) = 0,$$

where M is a real 3×3 matrix with non-zero determinant. Multiplying the three matrices in the middle of this expression we are led to an equation

$$(x, y, z) \cdot \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0, \qquad (*)$$

for suitable parameters a, b, c, d, e, f. Observe that the matrix in the middle is necessarily symmetric. Expanding the above product expression yields the following quadratic equation

$$a \cdot x^2 + c \cdot y^2 + f \cdot z^2 + 2b \cdot xy + 2d \cdot xz + 2e \cdot yz = 0.$$

We call such an expression of the form $p^T A p$ a *quadratic form* (regardless of the matrix A being symmetric or not). The set of points that satisfies such an equation will be called a *conic*. In a sense we are dealing on three different levels when we speak about matrices, quadratic forms and conics. Before we continue we want to clarify this relation. Let A be a 3×3 matrix with entries in some field $\mathbb{K}$ the associated quadratic form is a homogeneous quadratic function $\mathcal{Q}_A : \mathbb{K}^3 \to \mathbb{K}$ defined by

$$\mathcal{Q}_A(p) = p^T A p.$$

It is important to notice that different matrices may lead to the same quadratic form. This can be seen by expanding

$$\mathcal{Q}_A = (x, y, z) \cdot \begin{pmatrix} a & b_1 & d_2 \\ b_2 & c & e_1 \\ d_1 & e_2 & f \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

to

$$\mathcal{Q}_A = a \cdot x^2 + c \cdot y^2 + f \cdot z^2 + (b_1 + b_2) \cdot xy + (d_1 + d_2) \cdot xz + (e_1 + e_2) \cdot yz.$$

Observe that, the quadratic form associated to a matrix does only depend on the diagonal entries and the sums of matrix entries $A_{ij} + A_{ji}$. For every potentially non-symmetric matrix A the matrix $(A + A^T)/2$ creates the same quadratic form:

$$\mathcal{Q}_A = \mathcal{Q}_{(A+A^T)/2}.$$

We call this process *symmetrization*. Furthermore it is also clear that for every homogeneous quadratic polynomial $f(x, y, z)$ also a (symmetric) matrix A with $\mathcal{Q}_A(x, y, z) = f(x, y, z)$.

The conics themselves are the third level with which we have to deal. They consist of all points in $\mathcal{P}_{\mathbb{K}}$ for which a quadratic form vanishes. The conic associated to a matrix (or quadratic form) is

$$\mathcal{C}_A = \{[p] \in \mathcal{P}_{\mathbb{K}} \mid p^T A p = 0\}.$$

Since the expression $p^T A p = 0$ is stable under scalar multiplication of p with a non-zero scalar we will by common abuse of notation as usual work on the level of homogeneous coordinates and omit the brackets $[\ldots]$. How much information about the matrix A is still present in the conic in a sense depends on the underlying field. Over $\mathbb{R}$ it may for instance happen that a conic (like the solution set of $x^2 + y^2 + z^2 = 0$) has no non-zero solutions at all. In such a case then there are still complex solutions. We will deal with them later.

We come back to the case of a projectively transformed circle and the quadratic form described in equation $(*)$. In this case the parameters $a, \ldots, f$ are not completely independent. *Sylvesters law of inertia* from linear algebra tells us that the signature of the eigenvalues must be the same as in the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and that this is the only restriction[1].

[1] Sylvesters law of inertia states that if M is nonsingular then A and $M^T AM$ have the same signature of eigenvalues for any symmetric matrix A. Furthermore this is the only restriction on the coefficients of the symmetric matrix $M^T AM$.

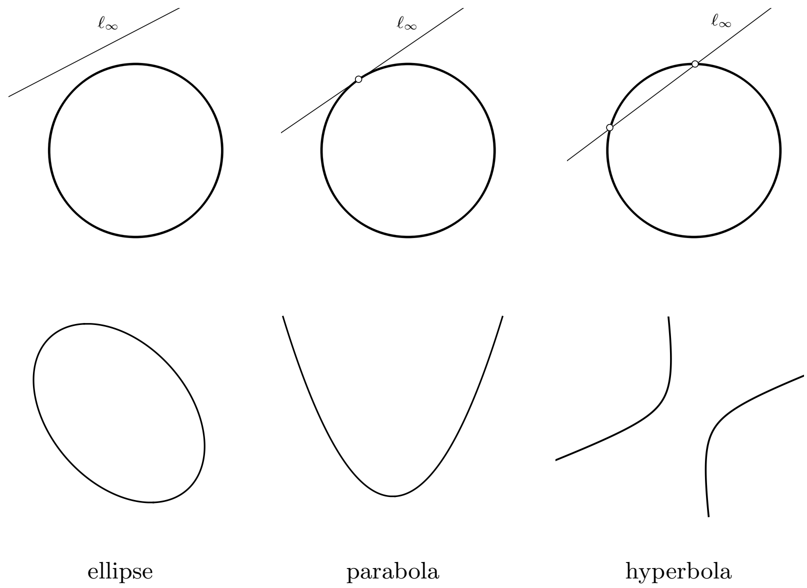


Fig. 9.1. The possible images of a circle.

If we allow general real parameters in equation $(*)$ we can classify the different cases by the signature of the eigenvalues. Since the quadratic forms of the matrices A and $-A$ describe identical point sets, we may identify a signature vector with its negative. Thus we end up with five essentially different cases of signatures.

$$(+,+,-), \quad (+,+,+), \quad ,(+,-,0), \quad (+,+,0), \quad (+,0,0).$$

Each of these cases describes a geometric situation that cannot be transformed projectively into any of the other cases. The case of a circle corresponds to the first entry in the list. The last three cases correspond to *degenerate* conics (the determinant of A is zero) and the second case corresponds to a case like $x^2 + y^2 + z^2 = 0$ in which we have no real solutions at all (but where still complex solutions exist). Before we clarify the geometric meaning of the other four cases in detail we will have a closer look at the circular case. Projectively all quadratic forms with eigenvalue signature $(+,+,-)$ have to be considered as isomorphic (they are the projective images of a unit circle). However if we fix a certain embedding of the Euclidean plane into $\mathbb{RP}^2$ (say the standard $z = 1$ embedding) and by this single out a specific line at infinity, then we may classify them also with respect to Euclidean motions. In fact there is an infinite variety of forms of conics that are in-equivalent under Euclidean transformations. Still there is a very useful (and well known) coarser classification if we just count the intersection of the conic with the line at infinity. Intersecting the quadratic form of a projectively transformed circle

$$a \cdot x^2 + c \cdot y^2 + f \cdot z^2 + 2b \cdot xy + 2d \cdot xz + 2e \cdot yz = 0.$$

with the line at infinity (i.e. setting $z = 0$) we a are left with the equation

$$a \cdot x^2 + c \cdot y^2 + 2b \cdot xy = 0.$$

This homogeneous quadratic equation may lead to zero, one or two solutions up to scalar multiples. The three cases are shown in Figure 9.1. They correspond to the well known cases of ellipses, parabola, and hyperbola. Thus one could say that a parabola is a conic that *touches the line at infinity*, while a hyperbola has two points at infinity. The three cases can be algebraically distinguished by considering the discriminant of the equation $a \cdot x^2 + c \cdot y^2 + 2b \cdot xy = 0$. This sign of the discriminant turns out to be the sign of the determinant

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

If this sign is negative we are in the hyperbolic case. If it is zero we get a parabola and if it is positive we get an ellipse.

9.2 Polars and tangents

For the moment we will stick to the case of non-degenerate conics that are projective images of a circle. Our next task will be to calculate a tangent to such a conic. For this we first need an algebraic characterization of a line being tangent to a conic. There are essentially two ways of doing this. The first one is related to concepts of differential geometry: "A line is tangent to a conic if at a point of intersection it has the same slope as the conic." The other approach would be intersection theoretic and uses the fact that we know that a conic is a quadratic curve. "A line is tangent to a conic if it has exactly one point in common with it." The first approach is slightly more general, since it also cover the case of degenerate conics. Still we want to follow the second approach, since it fits smoothly to the concepts introduced so far and we will generalize it later.

Before we will start investigating the intersection properties of lines and conics we have to recall a few facts concerning homogeneous quadratic equations. First consider the quadratic equation:

$$(\lambda, \mu) \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = a \cdot \lambda^2 + 2b \cdot \lambda\mu + c \cdot \mu^2 = 0.$$

Clearly if (λ, μ) is a solution then any scalar multiple of it is a solution as well. If we as usual consider equivalence classes $[(\lambda, \mu)]$ of solutions modulo non-zero scalars this quadratic equation will have zero, one or two solutions if at least one of the parameters a, b, c, d does not vanish. In this case the number of solutions depends on the sign of the discriminant

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

If the sign is zero we get exactly one solution, if it is negative we get two solutions if it is positive we get no real solutions but two complex solutions. If all parameters a, b, c, d are zero any $(\lambda, \mu) \in \mathbb{R}^2$ will be a solution.

The second fact we need is that a quadratic form $\mathcal{Q}_A(p)$ may factor into two linear terms. Then it has the form $\langle p, l\rangle \cdot \langle p, g\rangle$ for two suitable vectors l and g. The quadratic form may then be written as $\mathcal{Q}_{lg^T}(p)$ since we have

$$p^T(lg^T)p = (p^T l)(g^T p) = \langle p, l\rangle \cdot \langle p, g\rangle.$$

Furthermore if the corresponding conic contains a line with homogeneous coordinates l then the quadratic form must necessarily factor to the above form for suitable g.

In order to approach the calculation of a tangent we will study the possible types of intersections of a line given by a linear combination of two points a and b on it and a conic given by a symmetric matrix A. In what follows all coordinates of points and lines will be given by homogeneous coordinates and we will frequently identify the points with their coordinate representation.

Lemma 9.1. *Let A be a symmetric real 3×3 matrix and l be a line $\lambda a + \mu b$ given by two distinct points a and b. Then points q on the line that satisfy $q^T A q = 0$ correspond to the solutions of the homogeneous system*

$$(\lambda, \mu) \cdot \begin{pmatrix} (a^T A a) & (b^T A a) \\ (b^T A a) & (b^T A b) \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 0.$$

In particular, we get exactly two (real or complex) solutions (up to scalar multiples) if and only if the determinant of the above matrix does not vanish.

Proof. The proof goes simply by expansion of the formula $(\lambda a + \mu b)^T A(\lambda a + \mu b) = 0$ which describes the common points of the line and the quadratic form. Expanding it we get

$$(\lambda, \mu) \cdot \begin{pmatrix} (a^T A a) & (b^T A a) \\ (a^T A b) & (b^T A b) \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 0.$$

which is the desired formula except for one switch of a and b in the lower left entry. However, this can be done easily since A was assumed to be symmetric. The second part of the theorem is just an application of the discriminant formula. $\square$

Our next lemma states that degenerate conics must have a vanishing determinant.

Lemma 9.2. *A quadratic equation* $q^T Aq = 0$ *with symmetric* 3×3 *matrix* A *with* $\det(A) \neq 0$ *cannot hold for all points of a projective line.*

Proof. If the solution space of the quadratic equation $q^T Aq = 0$ contains a whole line l then one must be able to rewrite it as product of two linear factors $\langle p, l\rangle \cdot \langle p, g\rangle = 0$. One can rewrite this as

$$(q^T l) \cdot (g^T q) = 0.$$

This in turn can be interpreted as

$$q^T (l \cdot g^T) q = 0.$$

This is a quadratic form with a non-symmetric rank 1 matrix $M = l \cdot g^T$. We can symmetrize this matrix by replacing it with $M + M^T$. This matrix is singular since either $g = \lambda l$ and the matrix $M + M^T$ itself has rank 1 or g and l are linearly independent and

$$(M + M^T)(g \times l) = (l \cdot g^T + g \cdot l^T)(g \times l) = l\langle g, g \times l\rangle + g\langle l, g \times l\rangle = 0.$$

Since $M + M^T$ must be a multiple of A this proves the claim. □

We now define tangency in terms of numbers of intersections for the non-degenerate case of $\det(A) \neq 0$. We postpone the definition of tangents in the degenerate case a little since it is a little more subtle.

Definition 9.1. *Let* A *be a symmetric real* 3×3 *matrix with* $\det(A) \neq 0$. *A line* l *is tangent to the conic* $\mathcal{C}_A$ *if it has exactly one intersection with it.*

The following theorem gives a recipe for calculating a tangent to a conic.

Theorem 9.1. *Let* A *be a symmetric real* 3×3 *matrix with* $\det(A) \neq 0$ *and let* p *be a point on the corresponding conic. Then* $A{\cdot}p$ *gives homogeneous coordinates for the tangent at point* p *to the conic.*

Proof. We consider $p \mapsto A{\cdot}p$ as a function of the space of points to the space of lines (represented by homogeneous coordinates). Since A is non-degenerate this map is bijective. First we show that the point p and the line $l := A{\cdot}p$ coincide. For this simply observe

$$\langle p, l\rangle = \langle p, A{\cdot}p\rangle = p^T Ap = 0.$$

Now assume that there was a second point q that is as well on l as on the conic represented by A. If p and q represent different points then $\lambda p + \mu q$ must have more than one intersection with the conic. In this case the following three equations hold: $p^T Ap = 0$ (point p is on the conic), $q^T Aq = 0$ (point q is on the conic), $q^T Ap = 0$ (point q is on l). Forming a linear combination of the first and the last equation yields: $(\lambda p + \mu q)^T Ap = 0$ for arbitrary λ, μ.

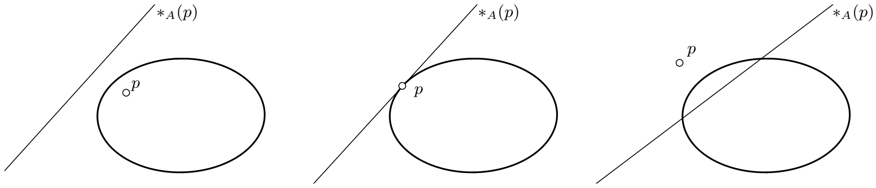


Fig. 9.2. Images under polarity.

Combining the second and the last yields $q^T A(\lambda p + \mu q) = 0$ which (by the symmetry of A) gives $(\lambda p + \mu q)^T Aq = 0$. Another combination shows that

$$(\lambda p + \mu q)^T A(\lambda p + \mu q) = 0.$$

This proofs that all linear combinations of p and q must be on the conic as well. Thus either p and q represent both the same point (and all linear combinations are also identical to this point) or the conic contains the entire line $\lambda p + \mu q$. The latter could not happen, since A was assumed to be non-degenerate. □

The last Theorem gives a very simple recipe to calculate the homogeneous coordinates of a tangent l to a point p on a conic $p^T Ap = 0$: simply calculate $l = Ap$! This function is also defined if p is not on the conic. The map $p \mapsto Ap$ is called a *polarity* and it has various interesting properties. Figure 9.2 shows three drawings of a point and its polar line with respect to an ellipse. Observe that if the the point is inside the ellipse the corresponding polar lies entirely outside. If the the point is outside the conic, then the polar does intersect the conic in two points. In the limit situation when the point is on the ellipse the polar is the tangent in this point. We want to investigate the properties of polarities a little further.

Definition 9.2. *Let A be a symmetric 3×3 matrix with $\det(A) \neq 0$. The map*

$$\begin{aligned} *_A \colon \mathcal{P}_\mathbb{R} &\to \mathcal{L}_\mathbb{R} \\ p &\mapsto Ap \end{aligned}$$

is called a polarity. Since the domain and image space of $*_A$ are disjoint we extend $*_A$ by its own inverse and define:

$$\begin{aligned} *_A \colon \mathcal{L}_\mathbb{R} &\to \mathcal{P}_\mathbb{R} \\ l &\mapsto A^{-1}l \end{aligned}$$

Polarities are very closely related to projective transformations. However, they map points to lines instead of points to points.

Theorem 9.2. *Let A be a symmetric matrix with $\det(A) \neq 0$ and let $\mathcal{C}_A$ the associated conic. The polarity $*_A$ satisfies have the following properties:*

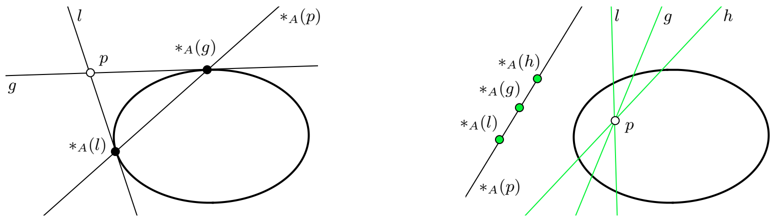


Fig. 9.3. Properties of polarities.

(i) *For any element* $a \in \mathcal{P}_\mathbb{R} \cup \mathcal{P}_\mathcal{L}$ *we have* $*_A(*_A(a)) = a$
(ii) *Three points* $a, b, c \in \mathcal{P}_\mathbb{R}$ *are collinear if and only if* $*_A(a), *_A(b), *_A(c) \in \mathcal{L}_\mathbb{R}$ *are concurrent.*
(iii) *Three lines* $a, b, c \in \mathcal{L}_\mathbb{R}$ *are concurrent if and only if* $*_A(a), *_A(b), *_A(c) \in \mathbb{R}_\mathbb{R}$ *are collinear.*
(iv) $p \in \mathcal{P}_\mathbb{R}$ *and* $l \in \mathcal{L}_\mathbb{R}$ *are incident if and only if* $*_A(p)$ *and* $*_A(l)$ *are incident.*
(v) p *and* $*_A(p)$ *are incident if and only if* p *is on* $\mathcal{C}_A$*. Then* $*_A(p)$ *is the tangent to* p *at the conic* $\mathcal{C}_A$*.*

Proof. (i) is clear since for $A^{-1}A = AA^{-1} = E$. (ii) and (iii) follow from the fact that collinearity/concurrence can be expressed by the condition $\det(a, b, c) = 0$ and that this condition is since $\det(A) \neq 0$ equivalent to $\det(Aa, Ab, Ac) = 0$. (iv) is the equivalence of $\langle p, l\rangle = 0$ and $\langle Ap, A^{-1}l\rangle = 0$. For (v) the incidence of p and Ap states simply that $p^T Ap = 0$ which means that p is on the conic. The property of being tangent is exactly the statement of Theorem 9.1. □

Polarities are in a very concrete way a representation of the dual character of projective geometry. They give a concrete dictionary of how to translate statements of projective geometry into their dual statements. For every non-degenerate matrix A we obtain such a dictionary.

Algebraically the multiplication with A (resp. A^{-1}) is the simplest way to carry out a polarity. However, property (v) of the last theorem gives us also a possibility to construct a polar of a point p geometrically if we are able to construct a tangent to a conic. This is particularly easy if the point p is "outside" the conic (i.e. the polar of p intersects the conic in two points). Then we simply have to draw the two tangents of p to the conics. The points where they touch the conic are the polars of these lines. If we join these two points then we get the polar of the original point. Figure 9.3 on the left shows this procedure. Algebraically this construction is resembled by the fact that if p is a point and l and g are two lines through this point. Then by Theorem 9.2 (iv) the line $*_A(p)$ is incident to the points $*_A(l)$ and $*_A(g)$. Since l and g

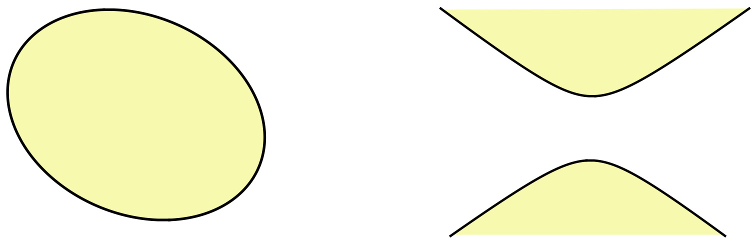


Fig. 9.4. Inside an ellipse and a inside a hyperbola.

are the tangents then by Theorem 9.2 (v) the polars are the corresponding touching points.

We can also reverse the construction in different ways. If the line $*_A(p)$ is given we can construct the polar point p by intersection of the two tangents. Furthermore, if for instance in a computer geometry system a method for calculating the polar is present (multiplication by A or A^{-1}) and if it is possible to intersect a conic with a line then this can be used to construct the tangents through a point p to a conic. Simply intersect the polar of the point with the conic and join the intersections with point p.

Figure 9.3 on the right demonstrates that concurrent lines lead to collinear polars. This property is also the key to construct the polar of a point "inside" a conic. For this one must simple draw to lines incident to the point, construct their polars and join them.

So far we have used the terms "inside" and "outside" a conic in a kind of informal way by always drawing images of an ellipse. In fact, one can formalize this concept by referring to the number of intersections of the conic with the corresponding polar.

Definition 9.3. *For a* 3×3 *symmetric matrix* A *with* $\det(A) \neq 0$ *a point* a *is inside the conic defined by* $p^T Ap = 0$ *if the polar does not intersect the polar* $*_A(p)$. *If the polar intersects the conic in two (real) points then the point* p *is outside the conic.*

9.3 Dual quadratic forms

In Chapter 2 and Chapter 3 we learned that projective geometry is a dual theory. The roles of points and lines, meets and joins, etc are interchangeable. In the last section we saw how we can make the duality explicit by using a polarity with respect to a conic. There must also be a dual counterpart of the concept of a conic. This will be defined in this section. Again we will deal with the non-degenerate case first.

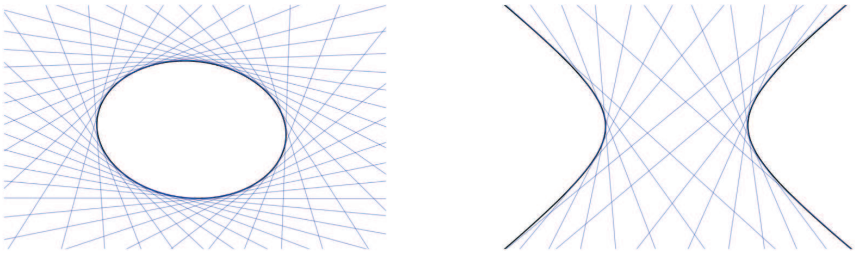


Fig. 9.5. Many tangents to conics.

A conic consists of all points p that satisfy an equation $p^T Ap = 0$. The set of all tangents to this conic, can be described as $\{Ap \mid p^T Ap = 0\}$. We can describe this set of homogeneous coordinates for lines directly as a quadratic form by the following observation

$$p^T Ap = p^T AA^{-1}Ap = p^T A^T A^{-1}Ap = (Ap)^T A^{-1}(Ap).$$

The right side of the equation explains how the set of tangent of a conic may be directly interpreted as the zero set of a quadratic form with matrix A^{-1}. Thus we obtain that the set of all tangents is described by

$$\mathcal{C}^*_A := \{[l] \in \mathcal{L}_\mathbb{R} \mid l^T A^\triangle l = 0\}.$$

In this expression we replaced the inverse A^{-1} by the adjoint $A^\triangle = A^{-1} \cdot \det(A)$ of the matrix A which is (for a symmetric matrix) defined by

$$\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}^\triangle = \begin{pmatrix} +\begin{vmatrix} c & e \\ e & f \end{vmatrix} & -\begin{vmatrix} b & e \\ d & f \end{vmatrix} & +\begin{vmatrix} b & c \\ d & e \end{vmatrix} \\ -\begin{vmatrix} b & d \\ e & f \end{vmatrix} & +\begin{vmatrix} a & d \\ d & f \end{vmatrix} & -\begin{vmatrix} a & b \\ d & e \end{vmatrix} \\ +\begin{vmatrix} b & d \\ c & e \end{vmatrix} & -\begin{vmatrix} a & d \\ b & e \end{vmatrix} & +\begin{vmatrix} a & b \\ b & c \end{vmatrix} \end{pmatrix}.$$

Compared to the inverse the adjoint has the advantage that it is also computable if A is not invertible since it avoids the division by the determinant.

Definition 9.4. *The dual of the quadratic form $p^T Ap$ is the quadratic form $l^T A^\triangle l$. (The p and the l indicate that the first has to be interpreted in the world of points, while the latter has to be interpreted in the world of lines.)*

Figure 9.5 gives a rough impression of how one could imagine a dual conic. A dual conic consists of all tangents to the original conic. While a conic is a set of points a dual conic is a set of lines.

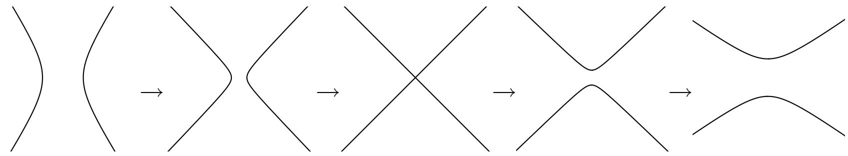


Fig. 9.6. From a hyperbola to degenerate conic, and back.

9.4 Degenerate conics

So far we considered only conics that came from invertible matrices A. In this case there is a one-to-one correspondence of conics and their duals. We will now study the degenerate case. We first study the possible cases in the world of *primal* conics that correspond to point sets. For a classification we have again to study the different signatures of eigenvalues that we met in Section 9.1.

$$(+,+,-), \quad (+,+,+), \quad ,(+,-,0), \quad (+,+,0), \quad (+,0,0).$$

The first case corresponds to a the class of real non-degenerate conics, which we studied so far. The second case is still not degenerate but the corresponding conic consists entirely of complex points. The last three cases lead to situations with vanishing determinant and have to be considered as degenerate conics. Up to projective equivalence they may be represented by the following quadratic forms.

$$x^2 - y^2 = 0, \quad x^2 + y^2 = 0, \quad x^2 = 0.$$

The first case consists of all points (x, y, z) for which $|x| = |y|$. Thus up to scalar multiple the points are either of the form $(1, 1, \alpha)$, $(1, -1, \alpha)$ or $(0, 0, 1)$. The first and second case describe two lines each with one point missing and the last case describes the missing intersection point of the two lines. Thus the conic $x^2 - y^2 = 0$ consists of two intersecting lines. In this specific case with the usual $z = 1$ embedding of the Euclidean plane these are the two lines with slope $\pm 45°$ through the origin.

We may interpret this degenerate case by considering the limit case of a hyperbola with a very sharp bend as indicated in Figure 9.6. From a projective viewpoint we can say that the case with eigenvalue signature $(+, -, 0)$ represents the situation where the conics degenerated into two real lines l and g. The corresponding quadratic form is then given by $\mathcal{Q}_{lg^T}$. Here lg^T is a non-symmetric rank one matrix. Equivalently we may consider the symmetrized matrix $lg^T + gl^T$.

The second case in our list corresponds to the equation $x^2 + y^2 = 0$. The only real point satisfying this condition is $(0, 0, 1)$. Still we get many complex

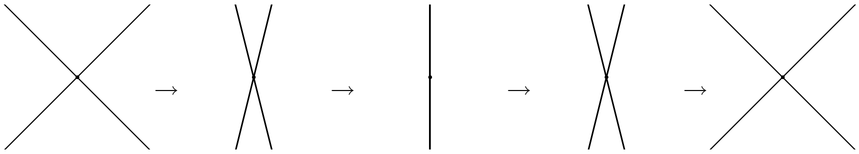


Fig. 9.7. From two single lines to a double line, and back.

solutions (and we will list them here for later usage). Treating this case similar to the above one we see that we points on this conic are either of the form $(1, i, \alpha)$, $(1, -i, \alpha)$ or $(0, 0, 1)$. The first two cases correspond to complex lines. These lines still have a real intersection namely the point $(0, 0, 1)$. In Summary the case with signature $(+, +, 0)$ can be considered as consisting of two conjugate complex lines together with their real point of intersection.

The last case to be considered has signature $(+, 0, 0)$ an corresponds to the equation $x^2 = 0$. This implies that $x = 0$ and all points on the conic are of the form $(0, \alpha, \beta)$. This is exactly the line with homogeneous coordinates $(1, 0, 0)$. In other words this case consists of a real line. In fact, it is reasonable to consider this line with a multiplicity of *two* (a double line), since the situation arises as a limit case when the two lines of a conic with signature $(+, -, 0)$ coincide. The situation is indicated in Figure 9.7.

9.5 Primal-dual pairs

What happens to tangents in the process of degeneration? Figure 9.8 shows the situation in a kind of continuous process. A hyperbola is deformed such that the deformation passes through a situation in which the conic degenerates into two real lines. The green lines in the pictures are tangents of the conic. First note that the tangents in the first picture all are “outside” the conic. When the conic becomes more degenerate the tangents seem to accumulate in the center of the conic. In the degenerate situation where the conic consists of two lines there are indeed many lines that have only one intersection with the conic. They all pass through the singular point at which the two lines

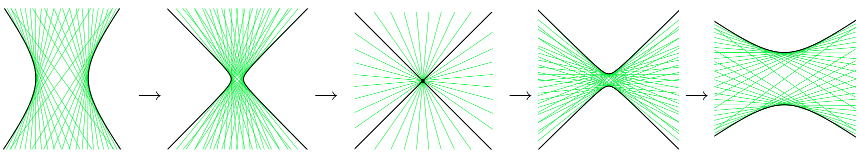


Fig. 9.8. Tangents of an hyperbola.

meet. After the degenerate situation the hyperbola "switches its branches" (left/right to top/bottom). Again the tangents are outside the hyperbola, but now occupy a different region of the projective plane. The degenerate situation is a kind of intermediate stage, in which tangents in both regions are possible. Our tangency concept of Definition 9.1 did not cover the degenerate case, so far.

For the degenerate case it is reasonable to consider *all* lines through the point of intersection as tangents also including the two lines of the conic themselves. Algebraically this can be done quite elegantly again by considering the adjoint of the corresponding matrix.

Theorem 9.3. *Let A be the symmetric 3×3 matrix corresponding to a degenerate conic $\mathcal{C}_A$ consisting of two distinct lines g and h. Then the adjoint $A^\triangle$ has the property that the lines passing through the intersection $g \times h$ are exactly those that satisfy $l^T A^\triangle l = 0$.*

Proof. This theorem can be proved by applying the following formula that holds for arbitrary 3-dimensional vectors $g = (g_1, g_2, g_3)^T$ and $h = (h_1, h_2, h_3)^T$.

$$(gh^T + hg^T)^\triangle = -(g \times h)(g \times h)^T.$$

First observe that this formula can simply be proved by expanding the terms on both sides of the expression and comparing the entries of the resulting 3×3 matrices (this is left to the patient reader).

The left side of the equation is the adjoint of the symmetric matrix representing a degenerate conic consisting of the lines g and h. If g and h represent distinct vectors then $g \times h = p$ represents their point of intersection. Thus the right side has the form $-pp^T$. Now assume that a line l is incident to p then we have $l^T(-pp^T)l = -(l^T p)(p^T l) = 0$. Conversely is aline satisfies $l^T(-pp^T)l = 0$ then we have $(l^T p)^2 = 0$ thus l must be incident to p as claimed in the theorem. □

The last theorem together with the fact that for a non-degenerate matrix A the inverse is a multiple of the adjoint allows us to nicely combine the description of polars for both cases. In both cases we can consider the dual as the following set of lines

$$\{[l] \in \mathcal{L}_\mathbb{R} \mid l^T A^\triangle l = 0\}.$$

In the case of a degenerate primal conic the dual describes all points that pass through the *double point* in which the two lines intersect.

We so far have not treated the case in which the conic degenerates into just one line (we may consider this as a double line). Algebraically this case is characterized that the symmetric matrix A has rank 1 and is of the form $A = gg^T$. It also corresponds to the eigenvalue signature $(+, 0, 0)$. In this case the adjoint is simply the zero matrix and does no longer carry any geometric

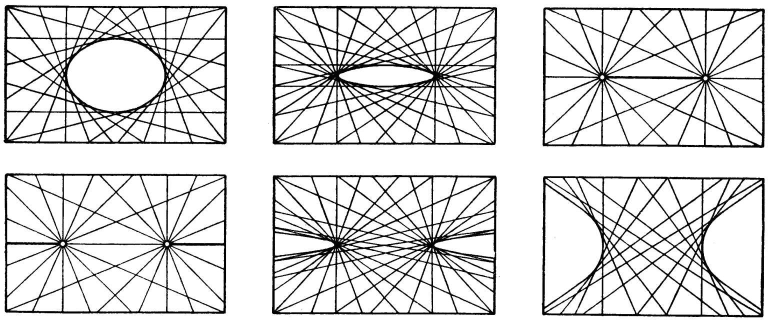


Fig. 9.9. Deformation via a double line (reproduction of a drawing of Felix Klein).

information. In fact, in this case it is reasonable (and as we will later see highly useful) to *blow-up* the situation and assign information by attaching a suitable matrix to the dual.

Since the situation is a little subtle we will approach it from several perspectives. First of all we consider a deformation of a conic that passes through the situation of a double line. For this consider the drawing in Figure 9.9 that has been reproduces from the brilliant book of Felix Klein "Vorlesungen über nicht-Euklidische Geometrie" (1925). There we see a sequence of pictures. The first shows an ellipse. This ellipse is squeezed horizontally until it becomes extremely thin. We can model this situation by considering the equation $\alpha x^2 + y^2 - \alpha z^2 = 0$ for with α moving from 0.5 to 0. In the limit situation all tangents seem to pass through the two "endpoints" of the squeezed Ellipse. If we deform α further (from 0 to -0.5) the conic becomes a hyperbola. Again close to the limit case all tangents seem to path through the two special points. In the limit case $\alpha = 0$ the conic just consists of the doubly covered x axis. On this axis two points play a special role.

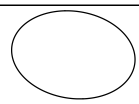
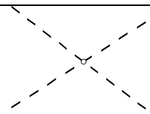
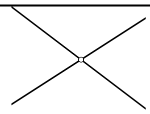
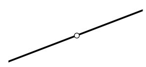
We can reinterpret this situation in terms of dual conics. What is the dual of a conic that consists of *two distinct lines* and a *double point* where they meet? Dualizing this description word by word we see that the dual of such a conic must consist of *two distinct points* and a *double line* that joins them. This is exactly what we see in Klein's drawing. Algebraically the dual conic may be represented by a single quadratic form $l^T A l = 0$, with a rank 2 symmetric matrix A its dual is described by the matrix $A^{\triangle}$. The quadratic form $p^T A^{\triangle} p = 0$ describes exactly the double line through the two points.

The considerations of the last few paragraphs suggest that it is reasonable to represent a conic by a *pair* of matrices. One of them representing the primal object and the other representing the dual object. If one of them is too degenerate the other may still carry geometric information. The concept is covered by the following definition.

Definition 9.5. *A primal/dual pair of conics is given by a pair* (A, B) *of symmetric* 3×3 *matrices such that there exist factors* λ *and* μ *with* $A^{\triangle} = \lambda B$ *and* $B^{\triangle} = \mu A$.

This definition covers all possible cases. If A is invertible then B must simply be a multiple of the inverse (and thus represent the dual conic). If A describes a conic given by two distinct lines then B must be a non-zero multiple of $A^{\triangle}$ and thus representing again the dual conic that consists of a double point. in this case A has rank 2 and B has rank 1. $B^{\triangle}$ will be the zero matrix which can be covered by setting $\mu = 0$. Conversely if B has rank 2 then B represents a dual conic consisting of a two points on a double line. A must be rank 1 and a non-zero multiple of $B^{\triangle}$. Thus A is the double line. In this case $\lambda = 0$. Finally we are left with the case that the ranks of A and B are both 1. Then we have $\mu = \lambda = 0$ and the situation consists of a double line with a double point on it.

We can still make the situation finer by not only considering the ranks of the matrices but also their eigenvalue signature. By this we can distinguish between real and complex cases. The following Table lists all possibilities cases.

A	B	type	
$(+,+,+)$	$(+,+,+)$	complex non-degenerate conic	
$(+,+,-)$	$(+,+,-)$	real non-degenerate conic	
$(+,+,0)$	$(+,0,0)$	two complex lines and a double real point	
$(+,-,0)$	$(+,0,0)$	two real lines and a double real point	
$(+,0,0)$	$(+,+,0)$	two complex points and a real double line	
$(+,0,0)$	$(+,-,0)$	two real points and a real double line	
$(+,0,0)$	$(+,0,0)$	a real double line and a real double point	

Later on these different cases will become relevant for classifying different types of metric geometries like hyperbolic geometry, Euclidean geometry, elliptic geometry and others.