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Buckling and Postbuckling of Compressible Circular Rings Under Hydrostatic Pressure

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I. Introduction

It is important for the designer of ribbed undersea vessels to understand how the buckling and postbuckling behavior of rings under external pressure is affected by two factors: 1) the shape of the ring and 2) the centerline extensibility or compressibility of the ring. In this Note, the simplest case of a circular ring with uniform cross section is studied. The method of solution used can also be applied to rings pressurized internally, such as those in aircraft or space structures. That, however, is a simpler problem to solve because there is no bifurcation point in the response of such rings. Still, the two mentioned factors also affect the response of those rings.

Figure 1 shows a circular, compressible ring subjected to a uniform external pressure p per unit length of deformed centerline (which we use because of the way hydrostatic pressure behaves physically). Such a ring will deform into a smaller circle as the pressure is increased, until a bifurcation point is reached, and then it will buckle into a noncircular shape. Figure 2 shows three possible buckled configurations. In Fig. 2a, the deformed centerline has two axes of symmetry, and integration is required from point A to point B only. In Fig. 2b, the deformed shape has only one axis of symmetry, requiring integration from A to C. Finally, in Fig. 2c, there is no symmetry, and the integration must be carried out from A all the way around and back to point A.

The only problems previously solved in the literature have assumed that the ring has uniform cross-sectional area A and uniform moment of inertia I along its centerline, and only the doubly symmetric mode of buckling has been considered. The method used in the present Note is not restricted to uniform section properties. These may vary along the centerline. Then, if the original circle has only the x axis as an axis of symmetry with respect to the section properties, the buckled shape might also be singly symmetric, in which case Fig. 2b would apply. If the original circle lacked any symmetry whatever in the section properties, then Fig. 2c would apply. The present Note, however, deals only with the doubly symmetric mode. It shows how the buckling and postbuckling behavior of symmetrical circular rings is affected by the axial compressibility of the ring.

Previous recent work on problems similar to this has been done by Wang[1] and Fu and Waas.[2] Because the ring in Wang's[1] problem contained a hinge at point C, he had to assume a singly symmetric deformed configuration similar to that in Fig. 2b. The results obtained, however, were limited to incompressible rings. Fu and Waas[2] considered thick rings, as we do in the present Note, but their results were limited to the initial postbuckling stage. Our method permits analysis of the entire postbuckling regime.

The buckling of rings under hydrostatic, constantly directed, and centrally directed pressure using classical energy approaches is discussed by El Naschie.[3] His book is also a source of references to earlier works on circular rings.

II. Theory

Table 1 shows the nonlinear boundary-value problem that describes the buckled ring for the case of doubly symmetric deformation (as well as describing the compressed circular ring before the critical pressure is reached). The equations in Table 1 are taken from the work of Huddleston[4] and are not restricted to slender rings except insofar as the underlying assumption that plane sections remain plane ultimately limits the thickness of the ring. It is difficult to say at what thickness our results are appreciably affected by this limitation because shear deformations enter the picture for thick rings and shear deformations in curved members are not well understood. The notation in Table 1 is as follows:

s_0	=	distance along original centerline measured from point A
θ	=	angle of inclination of final centerline at each point
ξ	=	final x coordinate of general point
η	=	final y coordinate of general point
s	=	distance along final centerline measured from point A
K_0	=	original curvature of centerline
N	=	normal force at general cross section
M	=	bending moment at general cross section
E	=	Young's modulus
A	=	cross-sectional area
I'	=	section property, defined by

$$\iint_A \frac{y^2}{1 - K_0 y}\,\mathrm{d}A$$

where the y in this integral is a coordinate from the centroidal axis of the cross section to the general point on the cross section.

Actually, the differential equations in Table 1 are not restricted to circular rings. They apply to rings of any original shape, as long as the original curvature at each point along the centerline is used during the integration process.

Because of the need to find N and M at each point during the integration, the four geometric differential equations in Table 1 must

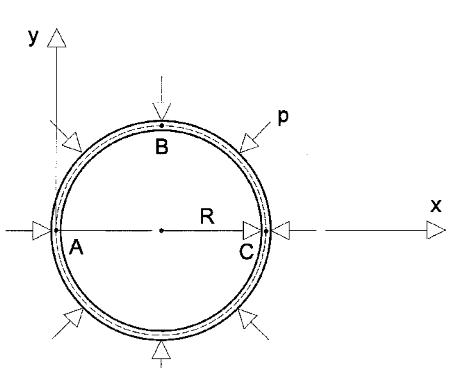

Fig. 1 Compressible circular ring under hydrostatic pressure.

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Table 1 Dimensioned boundary-value problem

Equation	Differential equations	Boundary values at A	Boundary values at B
(1)	$\frac{d\theta}{ds_0} = K_0\left(1+\frac{N}{EA}\right)+\left(\frac{1}{EI'}+\frac{K_0^2}{EA}\right)M$	$\theta(0)=\frac{\pi}{2}$	$\theta\left(\frac{\pi R}{2}\right)=0$
(2)	$\frac{d\xi}{ds_0}=\left(1+\frac{N}{EA}+\frac{K_0M}{EA}\right)\cos\theta$	$\xi(0)=0$	——
(3)	$\frac{d\eta}{ds_0}=\left(1+\frac{N}{EA}+\frac{K_0M}{EA}\right)\sin\theta$	$\eta(0)=0$	——
(4)	$\frac{ds}{ds_0}=1+\frac{N}{EA}+\frac{K_0M}{EA}$	$s(0)=0$	——

Table 2 Addition of the differential equations of equilibrium

Equation	Differential equations	Boundary values at A	Boundary values at B
(5)	$\frac{dN}{ds_0}=-\frac{d\theta}{ds_0}S$	$N(0)=?$	——
(6)	$\frac{dS}{ds_0}=\frac{d\theta}{ds_0}N-\left(1+\frac{N}{EA}+\frac{K_0M}{EA}\right)p$	$S(0)=0$	$S\left(\frac{\pi R}{2}\right)=0$
(7)	$\frac{dM}{ds_0}=\left(1+\frac{N}{EA}+\frac{K_0M}{EA}\right)S$	$M(0)=?$	——

Table 3 Dimensionless boundary-value problem

Equation	Differential equations	Boundary values at A	Boundary values at B
(1)	$\frac{d\theta}{d(s_0/R)}=-1-C\left(\frac{NR^2}{EI_0'}\right)+(1+C)\left(\frac{MR}{EI_0'}\right)$	$\theta(0)=\frac{\pi}{2}$	$\theta\left(\frac{\pi}{2}\right)=0$
(2)	$\frac{d(\xi/R)}{d(s_0/R)}=\left[1+C\left(\frac{NR^2}{EI_0'}\right)-C\left(\frac{MR}{EI_0'}\right)\right]\cos\theta$	$\frac{\xi}{R}(0)=0$	——
(3)	$\frac{d(\eta/R)}{d(s_0/R)}=\left[1+C\left(\frac{NR^2}{EI_0'}\right)-C\left(\frac{MR}{EI_0'}\right)\right]\sin\theta$	$\frac{\eta}{R}(0)=0$	——
(4)	$\frac{d(s/R)}{d(s_0/R)}=1+C\left(\frac{NR^2}{EI_0'}\right)-C\left(\frac{MR}{EI_0'}\right)$	$\frac{s}{R}(0)=0$	——
(5)	$\frac{d}{d(s_0/R)}\left(\frac{NR^2}{EI_0'}\right)=-\left(\frac{d\theta}{d(s_0/R)}\right)\left(\frac{SR^2}{EI_0'}\right)$	$\frac{NR^2}{EI_0'}(0)=?$	——
(6)	$\frac{d}{d(s_0/R)}\left(\frac{SR^2}{EI_0'}\right)=\left(\frac{d\theta}{d(s_0/R)}\right)\left(\frac{NR^2}{EI_0'}\right)$		
	$-\left[1+C\left(\frac{NR^2}{EI_0'}\right)-C\left(\frac{MR}{EI_0'}\right)\right]\left[\frac{pR^3}{EI_0'}\right]$	$\frac{SR^2}{EI_0'}(0)=0$	$\frac{SR^2}{EI_0'}\left(\frac{\pi}{2}\right)=0$
(7)	$\frac{d}{d(s_0/R)}\left(\frac{MR}{EI_0'}\right)=\left[1+C\left(\frac{NR^2}{EI_0'}\right)-C\left(\frac{MR}{EI_0'}\right)\right]\left[\frac{SR^2}{EI_0'}\right]$	$\frac{MR}{EI_0'}(0)=?$	——

be augmented by the three differential equations of equilibrium in Table 2, where S is shear force at the general cross section and p is external pressure. Looking at Tables 1 and 2, we now see that there are two unknown initial values at A, for Eqs. (5) and (7), and two known terminal values at B, for Eqs. (1) and (6). This problem thus lends itself to a shooting method with two levels of regula falsi.

For the case of uniform cross section along the centerline, we can take $A = A_0$ and $I' = I_0'$, and we can nondimensionalize the complete boundary-value problem as shown in Table 3. The compressibility C in this case is defined by the following equation:

$$C = I_0'/A_0R^2$$

where R is the original radius of the circular centerline. For an analysis of the buckling and postbuckling behavior of circular and parabolic arches with this kind of compressibility parameter nonzero, the reader is referred to the research monograph by Huddleston.[5]

III. Solution and Results

For each value of the system parameter C and the dimensionless pressure P, where

$$P = pR^3/EI_0'$$

the two unknowns in Table 3, namely, dimensionless normal force at A and dimensionless bending moment at A, are obtained by a

Table 4 Approximate bifurcation values

C	P_{cr}
0.001	3.05
0.010	3.20
0.050	4.40

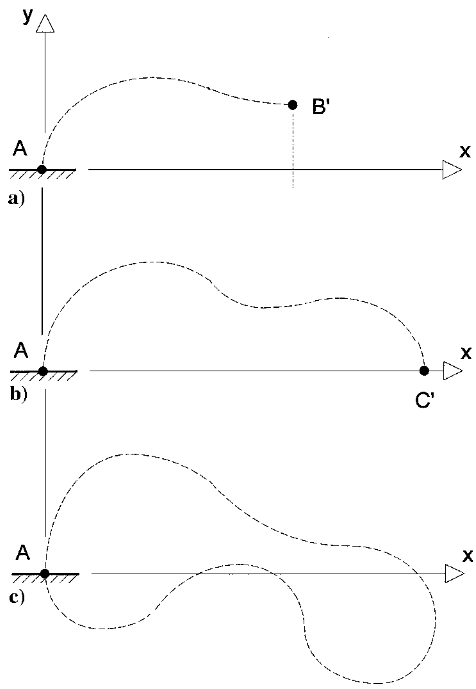

Fig. 2 Classes of deformation of the centerline.

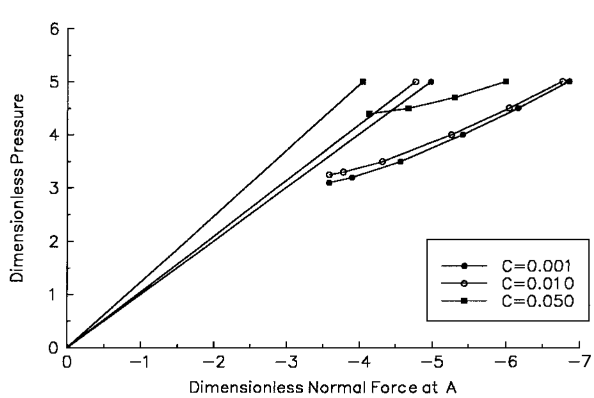

Fig. 3 Pressure vs normal force at A.

two-level shooting method using the zero angle at B and the zero dimensionless shear force at B as the two target values. Details of this method can be found in Ref. 5.

Computer runs have been made for three values of C: 0.001, representing a moderately compressible ring; 0.010, a very compressible ring; and 0.050, an extremely compressible ring. Graphs of dimensionless pressure P are plotted for the three values of C against the two unknowns from Table 3 in Figs. 3 and 4, that is, the abscissas are the dimensionless normal force at A and the dimensionless bending moment at A, respectively. The straight lines in Fig. 3 represent the unbuckled compressible ring, whereas the curved lines represent the postbuckled states. Bifurcation values of the pressure can be obtained by projecting the curves until they intersect the corresponding straight lines. Approximate values so obtained are given in Table 4. It has long been known that the value of the critical pressure for the incompressible ring is exactly 3.

In Fig. 4, there is a straight line for the case of $C = 0.050$, and this means that small secondary bending moments are set up during the compression stage even when there is no buckling. This also happens for the other two values of C, but the lines are so close to the vertical axis for those cases that no attempt was made to show them on the graph. Again, intersections of the curved lines with the straight lines confirm the critical values obtained earlier from Fig. 3.

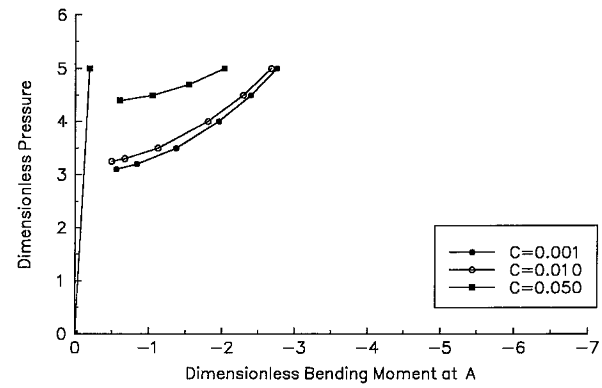

Fig. 4 Pressure vs bending moment at A.

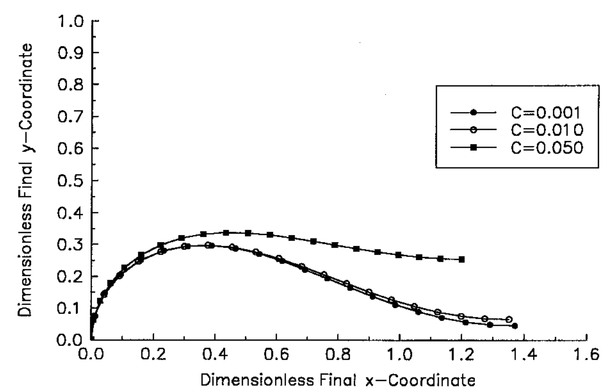

Fig. 5 Deformed shapes: $P = 5.0$.

Finally, Fig. 5 shows the deflected shapes of the buckled rings at a pressure of $P = 5.0$ for the three values of C. It is apparent from these results that the more compressible the ring is, the higher is the critical pressure, and the less deformation it suffers at a particular supercritical pressure.

All of the plotted points in the graphs of Figs. 3–5 are plotted precisely, but they are connected to each other by straight lines, which are only suggestive of the actual curved lines that go through the points.

IV. Conclusions

The critical pressure and the postbuckled shapes of compressible circular rings under hydrostatic pressure can be found by a two-level shooting method for any value of the compressibility. The larger the compressibility, the higher is the buckling pressure and the less deformation the ring suffers at a given supercritical pressure.

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