MTH208 Coursework (S2, AY2020-21)
Buckling of a Circular Ring

Ziniu Wu (1824944)

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Problem Formation

A system of seven differential equations serves as a model for a circular ring with compressibility c, under hydrostatic pressure p coming from all directions. The model will be nondimensionalized for simplicity, and we will assume that the ring has radius 1 with horizontal and vertical symmetry in the absence of external pressure.

The model accounts for only the upper left quarter of the ring—the rest can be filled in by the symmetry assumption. The independent variable s represents arc length along the original centerline of the ring, which goes from $s = 0$ to $s = \frac{\pi}{2}$. The dependent variables at the point specified by arc length s are as follows:

$$\begin{aligned}
&y_1(s) = \text{ angle of centerline with respect to horizonta} \\
&y_2(s) = x \text{ -coordinate} \\
&y_3(s) = y \text{ -coordinate} \\
&y_4(s) = \text{ arc length along deformed centerline} \\
&y_5(s) = \text{ internal axial force} \\
&y_6(s) = \text{ internal normal force} \\
&y_7(s) = \text{ bending moment.}
\end{aligned}$$

The boundary value problem formed as Huddleston [2000]:

$$\begin{aligned}
&y_1' = -1 - cy_5 + (c+1)y_7 \\
&y_2' = (1 + c\,(y_5 - y_7))\cos y_1 \\
&y_3' = (1 + c\,(y_5 - y_7))\sin y_1 \\
&y_4' = 1 + c\,(y_5 - y_7) \\
&y_5' = -y_6\,(-1 - cy_5 + (c+1)y_7) \\
&y_6' = y_7 y_5 - (1 + c\,(y_5 - y_7))\,(y_5 + p) \\
&y_7' = (1 + c\,(y_5 - y_7))\,y_6
\end{aligned}$$

where $y_1(0) = \frac{\pi}{2}$, $y_1(\frac{\pi}{2}) = 0$, $y_2(\frac{\pi}{2}) = 0$, $y_3(0) = 0$, $y_4(0) = 0$, $y_6(0) = 0$, $y_6(\frac{\pi}{2}) = 0$.

The following circular solution (7.10) to the boundary value problem exists for any choice of parameters c and p:

$$
\begin{aligned}
&y_1(s) = \tfrac{\pi}{2} - s \\
&y_2(s) = \tfrac{c+1}{cp+c+1}(-\cos s) \\
&y_3(s) = \tfrac{c+1}{cp+c+1} \sin s \\
&y_4(s) = \tfrac{c+1}{cp+c+1} s \\
&y_5(s) = -\tfrac{c+1}{cp+c+1} p \\
&y_6(s) = 0 \\
&y_7(s) = -\tfrac{cp}{cp+c+1}
\end{aligned}
$$

The critical pressure depends on the compressibility of the ring. The smaller the parameter c, the less compressible the ring is, and the lower the critical pressure at which it changes shape instead of compressing in original shape.

Question 1

Verify that (7.10) is a solution of the BVP for each compressibility c and pressure p.

Solution:

In order to verify that (7.10) is a solution of the BVP for each compressibility c and pressure p. We need to find the first order derivative with repect to s of (7.10):

$$
\begin{aligned}
&y_1^{'}(s) = -1 \\
&y_2^{'}(s) = \tfrac{c+1}{cp+c+1} \sin s \\
&y_3^{'}(s) = \tfrac{c+1}{cp+c+1} \cos s \\
&y_4^{'}(s) = \tfrac{c+1}{cp+c+1} \\
&y_5^{'}(s) = 0 \\
&y_6^{'}(s) = 0 \\
&y_7^{'}(s) = 0
\end{aligned}
$$

Then:

$$
\begin{aligned}
&-1 - cy_5 + (c+1)y_7 := -1 = y_1^{'}(s) \\
&(1 + c\,(y_5 - y_7)) \cos y_1 := (1 + c\tfrac{-p}{cp+c+1})cos(\tfrac{\pi}{2} - s) = \tfrac{c+1}{cp+c+1} \sin s = y_2^{'}(s) \\
&(1 + c\,(y_5 - y_7)) \sin y_1 := (1 + c\tfrac{-p}{cp+c+1})sin(\tfrac{\pi}{2} - s) = \tfrac{c+1}{cp+c+1} \cos s = y_3^{'}(s) \\
&1 + c\,(y_5 - y_7) := 1 + c\tfrac{-p}{cp+c+1} = \tfrac{c+1}{cp+c+1} = y_4^{'}(s) \\
&-y_6\,(-1 - cy_5 + (c+1)y_7) := 0 = y_5^{'}(s) \\
&y_7 y_5 - (1 + c\,(y_5 - y_7))\,(y_5 + p) := -\tfrac{cp}{cp+c+1}(-\tfrac{c+1}{cp+c+1}p) - (-\tfrac{c+1}{cp+c+1})(-\tfrac{cp}{cp+c+1}) = y_6^{'}(s) \\
&(1 + c\,(y_5 - y_7))\,y_6 := 0 = y_7^{'}(s)
\end{aligned}
$$

satifying the systems of differential equations in the BVP. Then, check the boundary value conditions by taking $s = 0$ and $s = \frac{\pi}{2}$ into (7.10):

$y_1(0) = \frac{\pi}{2}$, $y_1(\frac{\pi}{2}) = 0$, $y_2(\frac{\pi}{2}) = 0$, $y_3(0) = 0$, $y_4(0) = 0$, $y_6(0) = 0$, $y_6(\frac{\pi}{2}) = 0$.

Hence, (7.10) is a solution of the BVP for each compressibility c and pressure p.

Question 2

Set compressibility to the moderate value c = 0.01. Solve the BVP by the **Shooting Method** for pressures p = 0 and 3. The function F in the Shooting Method should use the three missing initial values (y2(0),y5(0),y7(0)) as input and the three final values (y1(π/2),y2(π/2),y6(π/2)) as output. The multivariate solver **Broyden II** from Chapter 2 can be used to solve for the roots of F. Compare with the correct solution (7.10). Note that, for both values of p, various initial conditions for Broyden's Method all result in the same solution trajectory. How much does the radius decrease when p increases from 0 to 3?

Solution:

Define

$$\mathbf{s} = \begin{bmatrix} y_2(0) \\ y_5(0) \\ y_7(0) \end{bmatrix} = \begin{bmatrix} s_2 \\ s_5 \\ s_7 \end{bmatrix}, \mathbf{F}(\mathbf{s}) = \begin{bmatrix} y_1\left(\frac{\pi}{2};\mathbf{s}\right) \\ y_2\left(\frac{\pi}{2};\mathbf{s}\right) \\ y_6\left(\frac{\pi}{2};\mathbf{s}\right) \end{bmatrix}$$

with $\mathbf{y} = (t;s)$ solving the ODEs using ode45 and

$$\mathbf{y}(0;s) = \begin{bmatrix} \frac{\pi}{2} \\ s_2 \\ 0 \\ 0 \\ s_5 \\ 0 \\ s_7 \end{bmatrix}$$

Then, apply shooting method and broyden method. Therefore, we can solve the problem by the script **solveQ2.m**:

```
clc;
clear;
%% Q2 and Q3
global c p;
c = 0.01;
p = 3; % Change the value of p here

%% Init
s=[0 0 0 0 0 0 0];
s0 = [3;1;0];
sspan = [0 pi/2];
y0 = [pi/2 s(2) 0 0 s(5) 0 s(7)];
yb = [0 0 0 0 0 0 0]; %Value to be Shot
k=100;

%% Solver
% z=F(s0, c, p);

sstar = broyden2(@(s)F(s, c, p), s0, k);

if (norm(F(sstar, c, p)) < 1e-5)
    disp("sstar found")
    % Plot the Result
    y0 = [pi/2 sstar(1) 0 0 sstar(2) 0 sstar(3)]; %Initial State
    [¬,w_sol]=ode45(@(s,y) odefcn(y,c,p),sspan,y0);

    axis equal
    hold on
```

```
    plot(w_sol(:,2),w_sol(:,3),'b-o'); %LU (You may change color here)
    plot(-w_sol(:,2),w_sol(:,3),'b-o');%RU
    plot(-w_sol(:,2),-w_sol(:,3),'b-o');%RD
    plot(w_sol(:,2),-w_sol(:,3),'b-o');%LD

    w_radius = 1 - sqrt(w_sol(end,2)^2 + w_sol(end,3)^2);% Reduced Radius
    disp(["Reduced Radius=",num2str(w_radius)]);

    %c=0.01, p=3
    syms s
    f2(s) = (c+1) / (c*p+c+1) * (-cos(s));
    f3(s) = (c+1) / (c*p+c+1) * (sin(s));
    fplot(f2,f3,'-');
else
    disp("Failed")
end

%% ODEs
% This is a system of ODE (7.10)
function dyds = odefcn(y,c,p)
dyds = zeros(7,1);
dyds(1) = -1-c*y(5)+(c+1)*y(7);
dyds(2) = (1+c*(y(5)-y(7)))*cos(y(1));
dyds(3) = (1+c*(y(5)-y(7)))*sin(y(1));
dyds(4) = 1+c*(y(5)-y(7));
dyds(5) = -y(6)*(-1-c*y(5)+(c+1)*y(7));
dyds(6) = y(7)*y(5)-(1+c*(y(5)-y(7)))*(y(5)+p);
dyds(7) = (1+c*(y(5)-y(7)))*y(6);
end

%% Shooting Method
% Input: Initial state, s 3x1 vector
% Output: shot solution of z
% Example usage: z=F(s, c, p)
function z = F(s, c, p)
sspan = [0 pi/2];
y0 = [pi/2 s(1) 0 0 s(2) 0 s(3)]; %Initial State
yb = [0 0 0 0 0 0 0]; %Value to be Shot

[¬,w]=ode45(@(s,y) odefcn(y,c,p),sspan,y0);

y1bs = w(end,1)-yb(end,1);
y2bs = w(end,2)-yb(end,2);
y6bs = w(end,6)-yb(end,6);

z = [y1bs; y2bs; y6bs];
end

%% Broyden's Method II
% Input: x0 initial vector, k = max steps
% Output: solution x
% Example usage: broyden2(f,[1;2],10)
function x=broyden2(f,x0,k)
[n,m]=size(x0);
b=eye(n,n);           % initial b
for i=1:k
x=x0-b*f(x0);
del=x-x0;△=f(x)-f(x0);
b=b+(del-b*△)*del'*b/(del'*b*△);
x0=x;
end
end
```

To obtain the result of problem 2, running the script solveQ2.m by change the value of p and $s0$. we take an initial guess s0 = [3;1;0], the ma step for broyden is 100 and $c = 0.01$.

For $p = 0$, We compute the **sstar** = $\begin{bmatrix} -1.0000 \\ -8.1959e-25 \\ -1.2988e-16 \end{bmatrix}$. Then the follwoing figure show the numerial resultin figure 1:

Similarly, we can find the result when $p = 3$ and **sstar** = $\begin{bmatrix} -0.9712 \\ -2.9135 \\ -0.0288 \end{bmatrix}$, as shown in figure 1:

From the numerical solution, we found that the radius of the circle at $p = 0$ is 1, the radius of the circle at $p = 3$ is 0.9712. The radius decrease 0.028846 when p increases from 0 to 3.

Question 3

Plot the solutions in Step 2. The curve (y2(s),y3(s)) represents the upper left quarter of the ring. Use the horizontal and vertical symmetry to plot the entire ring.

Solution:

From the problem formation, the analytic solution of the BVP is (7.10). and the the numerical solution are found in question 2:

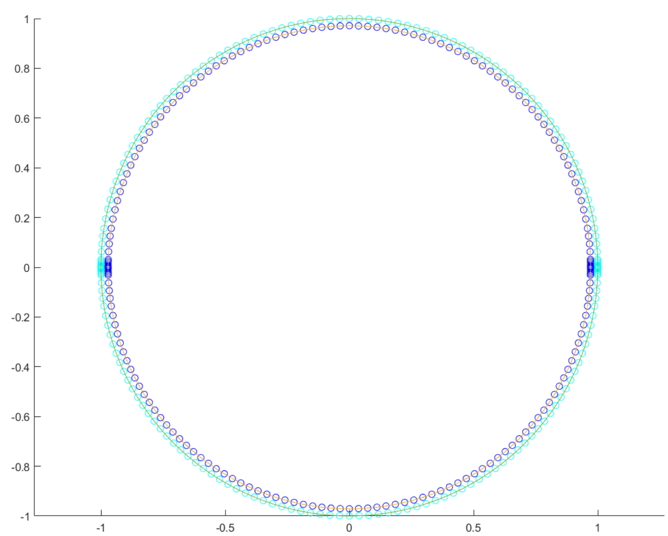

Figure 1: Result

The figure 1 shows the solutions of the BVP by analytic solution and the numerial solution in question 2. To plot the following figure, change the parameter p, then hold on the figure. Then the result is shown in figure 1. The Matlab code can be found at **solveQ2.m** file. For $p = 0$, the analytical solution is green, the numerical solution is cyan. For $p = 3$, the numerial solution is blue, the analytical solution is orange.

Question 4

Change pressure to p = 3.5, and resolve the BVP. Note that the solution obtained depends on the initial condition used for Broyden's Method II. Plot each different solution found.

Solution: The parameter are $c = 0.01$, $p = 3.5$, $k = 100$, then we need to try different sets of parameters to find the three different solutions. Hence, we tried s0 = [2;1;0]; s0 = [-1.28;1.8;0]; s0 = [-0.2;0.6;-1.2]. Then we run the script **solveQ4.m**:

```
clc;
clear;
%% Q4
global c p;
c = 0.01;
p = 3.5; % Change the value of p here

%% Init
s=[0 0 0 0 0 0 0];
s0 = [-0.2;0.6;-1.2];
% circular solution: s0 = [2;1;0];
% buckled solution 1: s0 = [-1.28;1.8;0];
% buckled solution 2: s0 = [-0.2;0.6;-1.2];
sspan = [0 pi/2];
y0 = [pi/2 s(2) 0 0 s(5) 0 s(7)];
yb = [0 0 0 0 0 0 0]; %Value to be Shot
k=100;

%% Solver
% z=F(s0, c, p);

sstar = broyden2(@(s)F(s, c, p), s0, k);

if (norm(F(sstar, c, p)) < 1e-3)
    disp("sstar found")
    % Plot the Result
    y0 = [pi/2 sstar(1) 0 0 sstar(2) 0 sstar(3)]; %Initial State
    [¬,w_sol]=ode45(@(s,y) odefcn(y,c,p),sspan,y0);

    axis equal
    hold on
    plot(w_sol(:,2),w_sol(:,3),'b-o'); %LU (You may change color here)
    plot(-w_sol(:,2),w_sol(:,3),'b-o');%RU
    plot(-w_sol(:,2),-w_sol(:,3),'b-o');%RD
    plot(w_sol(:,2),-w_sol(:,3),'b-o');%LD
else
    disp("Failed")
end
```

For different initial state, s0 = [2;1;0]; s0 = [-1.28;1.8;0]; s0 = [-0.2;0.6;-1.2]. The corresponding sstar are $\textbf{sstar} = \begin{bmatrix} -0.9665 \\ -3.3828 \\ -0.0335 \end{bmatrix}$ (blue), $\textbf{sstar} = \begin{bmatrix} -0.5879 \\ -2.0578 \\ 0.9293 \end{bmatrix}$ (Cyan), $\textbf{sstar} = \begin{bmatrix} -1.2317 \\ -4.3109 \\ -1.206 \end{bmatrix}$ (Red). Then, we hold on graph and plot 3 different sets of solution as shown in figure 2:

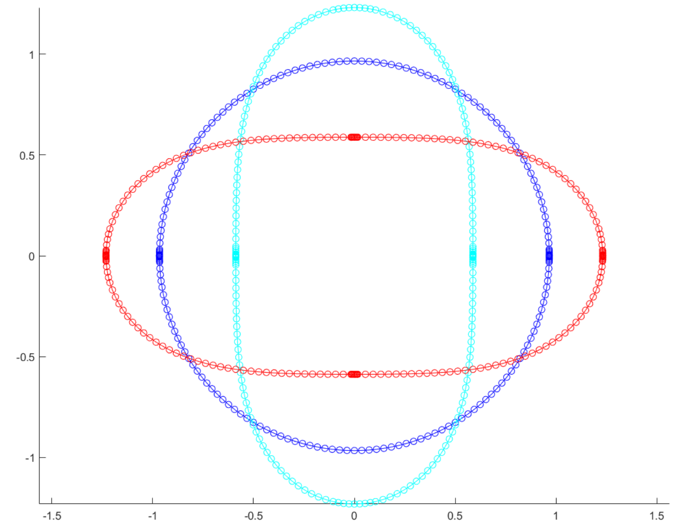

Figure 2: Result

Question 5

Find the critical pressure p_c for the compressibility c = 0.01, accurate to two decimal places. For $p > p_c$, there are three different solutions. For $p < p_c$, there is only one solution (7.10).

Solution:

For applied pressure p below the critical pressure p_c, only solution (7.10) exists. For $p > p_c$, three different solutions of the BVP exist. From the disccusion above, we believe that the $p_c \in [3, 3.5]$. The idea is that if p changes a little, then the solution may change also only a little. So, if we can find a solution at p_a, then we can use the solution as a good initial guess at $p_a + \textit{step size}$., in our case, the step size is 0.01.

Then, we run the script many time by changing the s0 and p. Finally, we found the critical pressure is in the smaller inverval $[3.18, 3.25]$.

```
clc;
clear;
%% Q4
global c p;
c = 0.01;
p = 3.5; % Change the value of p here

%% Init
s=[0 0 0 0 0 0 0];
s0 = [-0.9712;-2.9135;-0.0288];
sspan = [0 pi/2];
y0 = [pi/2 s(2) 0 0 s(5) 0 s(7)];
yb = [0 0 0 0 0 0 0]; %Value to be Shot
k=100;
h=0.001;

%% Solver
```

```
% z=F(s0, c, p);
for p = 3: +h:3.5
    sstar = broyden2(@(s)F(s, c, p), s0, k);
    y0 = [pi/2 sstar(1) 0 0 sstar(2) 0 sstar(3)];
    [¬,w_sol]=ode45(@(s,y) odefcn(y,c,p),sspan,y0);
    w_radius = 1 - sqrt(w_sol(end,2)^2 + w_sol(end,3)^2);

    sstar2 = broyden2(@(s)F(s, c, p), sstar, k);
    y0 = [pi/2 sstar2(1) 0 0 sstar2(2) 0 sstar2(3)];
    [¬,w_sol2]=ode45(@(s,y) odefcn(y,c,p),sspan,y0);
    w_radius2 = 1 - sqrt(w_sol2(end,2)^2 + w_sol2(end,3)^2);

    disp(w_radius);
    disp(w_radius2);

    if (abs(w_radius - w_radius2) > 1e-1)
        disp("not continuous", sstar);
        break;
    end
    disp(p);
end
```

Question 6

Carry out Step 5 for the reduced compressibility c = 0.001. The ring now is more brittle. Is the change in p_c for the reduced compressibility case consistent with your intuition?

Solution:

Sure, the p_c increased with my intuition.

Question 7

Carry out Step 5 for increased compressibility c = 0.05.

Solution:

The methodology is similar in step 5.

References

Huddleston, J.V. & Sivaselvan, M.V. (2000) Buckling and Postbuckling of Compressible Circular Rings Under Hydrostatic Pressure. AIAA Journal. [Online] 38 (12), 2361–2363. Available from: doi:10.2514/2.909.