2. Carry out the steps of Computer Problem 1 for the BVPs.

$$\text{(a)} \begin{cases} 9y'' + \pi^2 y = 0 \\ y(0) = -1 \\ y\left(\frac{3}{2}\right) = 3 \end{cases} \qquad \text{(b)} \begin{cases} y'' = 3y - 2y' \\ y(0) = e^3 \\ y(1) = 1 \end{cases}$$

3. Apply the Shooting Method to the nonlinear BVPs. Find a bracketing interval $[s_0, s_1]$ and apply an equation solver to find and plot the solution.

$$\text{(a)} \begin{cases} y'' = 18y^2 \\ y(1) = \frac{1}{3} \\ y(2) = \frac{1}{12} \end{cases} \qquad \text{(b)} \begin{cases} y'' = 2e^{-2y}(1 - t^2) \\ y(0) = 0 \\ y(1) = \ln 2 \end{cases}$$

4. Carry out the steps of Computer Problem 3 for the nonlinear BVPs.

$$\text{(a)} \begin{cases} y'' = e^y \\ y(0) = 1 \\ y(1) = 3 \end{cases} \qquad \text{(b)} \begin{cases} y'' = \sin y' \\ y(0) = 1 \\ y(1) = -1 \end{cases}$$

5. Apply the Shooting Method to the nonlinear systems of boundary value problems. Follow the method of Example 7.7.

$$\text{(a)} \begin{cases} y_1' = 1/y_2 \\ y_2' = t + \tan y_1 \\ y_1(0) = 0 \\ y_2(1) = 2 \end{cases} \qquad \text{(b)} \begin{cases} y_1' = y_1 - 3y_1y_2 \\ y_2' = -6(ty_2 + \ln y_1) \\ y_1(0) = 1 \\ y_2(1) = -\frac{2}{3} \end{cases}$$

Reality Check 7 Buckling of a Circular Ring

Boundary value problems are natural models for structure calculations. A system of seven differential equations serves as a model for a circular ring with compressibility c, under hydrostatic pressure p coming from all directions. The model will be nondimensionalized for simplicity, and we will assume that the ring has radius 1 with horizontal and vertical symmetry in the absence of external pressure. Although simplified, the model is useful for the study of the phenomenon of **buckling**, or collapse of the circular ring shape. This example and many other structural boundary value problems can be found in Huddleston [2000].

The model accounts for only the **upper left quarter of the ring**—the rest can be filled in by the symmetry assumption. The independent variable s represents arc length along the original centerline of the ring, which goes from $s = 0$ to $s = \pi/2$. The dependent variables at the point specified by arc length s are as follows:

$y_1(s)$ = angle of centerline with respect to horizontal
$y_2(s)$ = x-coordinate
$y_3(s)$ = y-coordinate
$y_4(s)$ = arc length along deformed centerline
$y_5(s)$ = internal axial force
$y_6(s)$ = internal normal force
$y_7(s)$ = bending moment.

Figure 7.5(a) shows the ring and the first four variables. The boundary value problem (see, for example, Huddleston [2000]) is

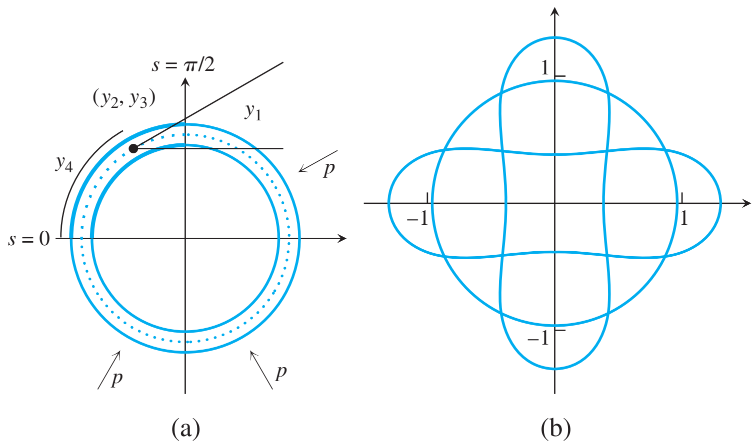

Figure 7.5 Schematics for Buckling Ring. (a) The s variable represents arc length along the dotted centerline of the top left quarter of the ring. (b) Three different solutions for the BVP with parameters $c=0.01$, $p=3.8$. The two buckled solutions are stable.

$$
\begin{array}{lll}
y_1' = -1 - cy_5 + (c+1)y_7 & y_1(0) = \frac{\pi}{2} & y_1(\frac{\pi}{2}) = 0 \\
y_2' = (1 + c(y_5 - y_7))\cos y_1 & & y_2(\frac{\pi}{2}) = 0 \\
y_3' = (1 + c(y_5 - y_7))\sin y_1 & y_3(0) = 0 & \\
y_4' = 1 + c(y_5 - y_7) & y_4(0) = 0 & \\
y_5' = -y_6(-1 - cy_5 + (c+1)y_7) & & \\
y_6' = y_7 y_5 - (1 + c(y_5 - y_7))(y_5 + p) & y_6(0) = 0 & y_6(\frac{\pi}{2}) = 0 \\
y_7' = (1 + c(y_5 - y_7))y_6. & &
\end{array}
$$

Under no pressure ($p = 0$), note that $y_1 = \pi/2 - s$, $(y_2, y_3) = (-\cos s, \sin s)$, $y_4 = s$, $y_5 = y_6 = y_7 = 0$ is a solution. This solution is a perfect quarter-circle, which corresponds to a perfectly circular ring with the symmetries.

In fact, the following circular solution to the boundary value problem exists for any choice of parameters c and p:

$$
\begin{aligned}
y_1(s) &= \frac{\pi}{2} - s \\
y_2(s) &= \frac{c+1}{cp+c+1}(-\cos s) \\
y_3(s) &= \frac{c+1}{cp+c+1}\sin s \\
y_4(s) &= \frac{c+1}{cp+c+1}s \\
y_5(s) &= -\frac{c+1}{cp+c+1}p \\
y_6(s) &= 0 \\
y_7(s) &= -\frac{cp}{cp+c+1}.
\end{aligned}
\tag{7.10}
$$

As pressure increases from zero, the radius of the circle decreases. As the pressure parameter p is increased further, there is a **bifurcation**, or change of possible states, of the ring. The circular shape of the ring remains mathematically possible, but unstable, meaning that small perturbations cause the ring to move to another possible configuration (solution of the BVP) that is stable.

For applied pressure p below the bifurcation point, or **critical pressure** p_c, only solution (7.10) exists. For $p > p_c$, three different solutions of the BVP exist, shown in Figure 7.5(b). Beyond critical pressure, the role of the circular ring as an unstable state is similar to that of the inverted pendulum (Computer Problem 6.3.6) or the bridge without torsion in Reality Check 6.

The critical pressure depends on the compressibility of the ring. The smaller the parameter c, the less compressible the ring is, and the lower the critical pressure at which it changes shape instead of compressing in original shape. Your job is to use the Shooting Method paired with Broyden's Method to find the critical pressure p_c and the resulting buckled shapes obtained by the ring.

Suggested activities:

1. Verify that (7.10) is a solution of the BVP for each compressibility c and pressure p.
2. Set compressibility to the moderate value $c = 0.01$. Solve the BVP by the Shooting Method for pressures $p = 0$ and 3. The function F in the Shooting Method should use the three missing initial values $(y_2(0), y_5(0), y_7(0))$ as input and the three final values $(y_1(\pi/2), y_2(\pi/2), y_6(\pi/2))$ as output. The multivariate solver Broyden II from Chapter 2 can be used to solve for the roots of F. Compare with the correct solution (7.10). Note that, for both values of p, various initial conditions for Broyden's Method all result in the same solution trajectory. How much does the radius decrease when p increases from 0 to 3?
3. Plot the solutions in Step 2. The curve $(y_2(s), y_3(s))$ represents the upper left quarter of the ring. Use the horizontal and vertical symmetry to plot the entire ring.
4. Change pressure to $p = 3.5$, and resolve the BVP. Note that the solution obtained depends on the initial condition used for Broyden's Method. Plot each different solution found.
5. Find the critical pressure p_c for the compressibility $c = 0.01$, accurate to two decimal places. For $p > p_c$, there are three different solutions. For $p < p_c$, there is only one solution (7.10).
6. Carry out Step 5 for the reduced compressibility $c = 0.001$. The ring now is more brittle. Is the change in p_c for the reduced compressibility case consistent with your intuition?
7. Carry out Step 5 for increased compressibility $c = 0.05$. ✔

7.2 FINITE DIFFERENCE METHODS

The fundamental idea behind finite difference methods is to replace derivatives in the differential equation by discrete approximations, and evaluate on a grid to develop a system of equations. The approach of discretizing the differential equation will also be used in Chapter 8 on PDEs.

7.2.1 Linear boundary value problems

Let $y(t)$ be a function with at least four continuous derivatives. In Chapter 5, we developed discrete approximations for the first derivative

$$y'(t) = \frac{y(t+h) - y(t-h)}{2h} - \frac{h^2}{6}y'''(c) \tag{7.11}$$