NTNU – Trondheim
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Exercise 11

TTK4130 Modeling and Simulation

Problem 1 (Sliding stick (Exam 2010))

Consider a stick of length ℓ with uniformly distributed mass m. It has center of mass/gravity C, about which it has a moment of inertia I_z. The stick is in contact with a frictionless horizontal surface, and moves due to the influence of gravity. See Figure 1.

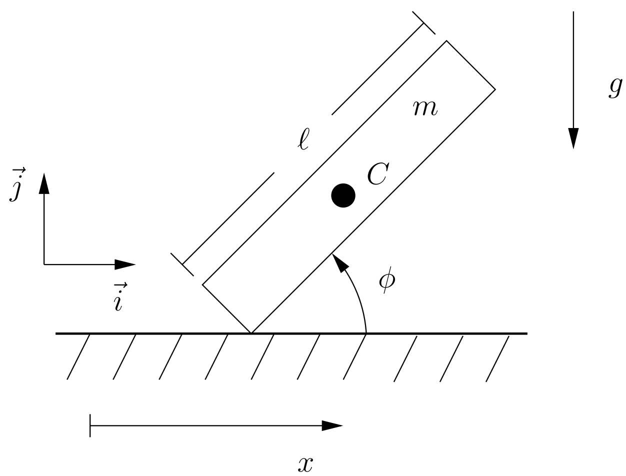

Figure 1: Stick sliding on frictionless surface

(a) Choose appropriate generalized coordinates (the figure should give you some hints). What are the corresponding generalized (actuator) forces?

> **Solution:** A natural choice for generalized coordinates are the horizontal position of the center of mass/gravity (denoted x), and ϕ, the angle between the stick and the surface. An alternative to x could be the contact point between the stick and the surface.
>
> There are no (generalized) actuator forces corresponding to these coordinates. (A candidate answering 'gravity' might get full score if he/she uses it correctly in the rest of the Problem.)

(b) What are the position, velocity, and angular velocity of the center of mass, as function of your chosen generalized coordinates (and/or their derivatives)?

> **Solution:**
>
> $$\begin{aligned} \vec{r}_c &= x\vec{i} + \frac{\ell}{2}\sin\phi\vec{j} \\ \vec{v}_c &= \dot{x}\vec{i} + \frac{\ell}{2}\dot{\phi}\cos\phi\vec{j} \\ \vec{w}_{ib} &= \dot{\phi}\vec{k} \end{aligned}$$
>
> (Coordinate vectors also accepted for the position and velocity, scalar accepted for angular velocity.)

(c) Write up the kinetic and potential energy of the stick, as function of your chosen generalized coordinates (and/or their derivatives).

Solution: The kinetic energy for the rigid body is

$$
\begin{aligned}
T &= \frac{1}{2}m\vec{v}_c \cdot \vec{v}_c + \frac{1}{2}\vec{w}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{w}_{ib} \\
&= \frac{1}{2}m\left(\dot{x}^2 + \frac{\ell^2}{4}\dot{\phi}^2\cos^2\phi\right) + \frac{1}{2}I_z\dot{\phi}^2.
\end{aligned}
$$

The potential energy due to gravity is

$$
U = mg\frac{\ell}{2}\sin\phi.
$$

(d) Derive the equations of motion for the stick.

Solution: It is probably easiest to use Lagrange's equation of motion. Define the Lagrangian $L = T - U$. Then the first equation of motion is

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0,
$$

which reduces to

$$
\ddot{x} = 0.
$$

The second equation of motion is

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0
$$

which gives

$$
\left(\frac{m\ell^2}{4}\cos^2\phi + I_z\right)\ddot{\phi} - \frac{m\ell^2}{4}\dot{\phi}^2\cos\phi\sin\phi + mg\frac{\ell}{2}\cos\phi = 0.
$$

Problem 2 (Tank with liquid)

A tank with area A is filled with an incompressible liquid with (constant) density ρ and level h. The liquid volume is then $V = Ah$ and the mass of the liquid in the tank is $m = V\rho$. Liquid enters the tank through a pipe with mass flow $w_i = \rho A_i v_i$, where A_i is the pipe cross section, and v_i is the velocity (constant over the cross section). Liquid leaves the tank through a second pipe with mass flow $w_u = \rho A_u v_u$ where A_u is the cross section of the pipe and v_u is the velocity.

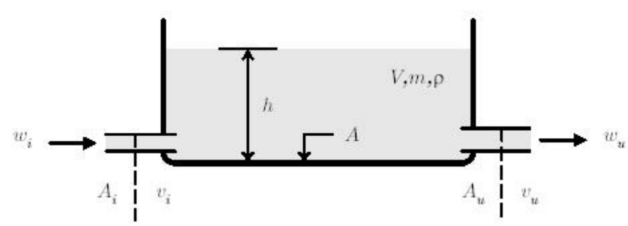

Figure 2: Tank with liquid

Use a mass balance for the tank to set up a differential equation for the level h.

Solution: The principle of mass conservation is

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint_V \rho dV = 0,$$

that is, the mass is constant in a material volume. Using eq. (10.90) in the book, the liquid mass balance for a fixed volume V_f (the total volume of the tank) becomes (eq. (11.8) in the book):

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_f} \rho \mathrm{d}V}_{\substack{\text{rate of change}\\ \text{of liquid mass}\\ \text{in } V_f}} = - \underbrace{\iint_{\partial V_f} \rho \mathbf{v}^\mathsf{T}\mathbf{n}\mathrm{d}A}_{\substack{\text{net increase of mass}\\ \text{by flow in and out}\\ \text{of } V_f}}$$

(Alternatively, one could assume the liquid volume as "control volume" and use eq. (11.10).)

We have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_f} \rho \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\rho V = \rho A\dot{h},$$

(that is, the total mass of liquid in the tank), and

$$-\iint_{\partial V_f} \rho \mathbf{v}^\mathsf{T}\mathbf{n}\mathrm{d}A = \rho v_i A_i - \rho v_u A_u$$

(flow in and out of the tank).

Inserted into the mass balance above, this gives

$$\begin{aligned} A\rho\dot{h} &= \rho v_i A_i - \rho v_u A_u \\ \dot{h} &= \frac{A_i}{A}v_i - \frac{A_u}{A}v_u. \end{aligned}$$

Problem 3 (Stirred tank (Exam 2015))

In this problem, we consider a stirred tank that cools an inlet stream, see Figure 3. The tank is cooled by a "jacket" that contains a fluid of (presumably) lower temperature than the tank. The inlet stream to the tank has density ρ, temperature T_1, and massflowrate w_1. The outflow from the tank is

$$w_2 = Cu\sqrt{h},$$

where C is a constant and u is the valve opening. The liquid level is h. You can assume that the outflow is controlled such that the level does not exceed the height of the jacket.

The inlet and outlet massflowrates for the jacket is matched such that the jacket is always filled with fluid ($w_3 = w_4$). The cooling fluid has density ρ_c, and the inlet stream to the jacket has temperature T_3. Since the tank is stirred, we assume homogenous conditions, that is, the temperature T is the same everywhere in the tank. Similarly, we assume that the temperature T_c is the same everywhere in the jacket.

The cross-sectional area of the tank is A. The volume of the jacket is V_c.

The heat transfer from the tank to the jacket is

$$Q = Gh(T - T_c),$$

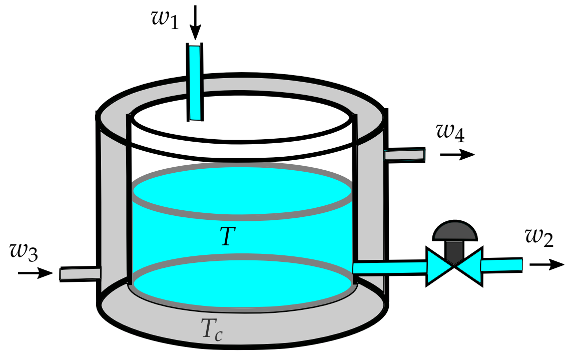

Figure 3: Tank with cooling jacket.

where h is the height of the liquid in the tank, and G a (constant) heat transfer coefficient. We assume that the jacket (and tank) is well insulated from the surroundings, meaning there are no other heat losses.

We assume both fluids incompressible, meaning that specific internal energy and enthalpy both can be assumed equal and proportional to temperature, with constant of proportionality being c_p and c_{pc} for the two fluids, respectively.

(a) Set up differential equations for the temperatures T in the tank and T_c in the jacket, and the level h in the tank.

Solution: We must first set up mass balances. For the tank, the mass balance

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho A h) = w_1 - w_2$$

gives

$$\boxed{\dot{h} = \frac{1}{\rho A}\left(w_1 - Cu\sqrt{h}\right)}$$

For later use, the mass balance for the jacket:

$$\frac{\mathrm{d}}{\mathrm{d}t} m = w_3 - w_4 = 0.$$

(*The volumn could be balanced instead of the mass. However, in that case it has to be explicitly mentioned that the density is constant and, therefore, it is possible to balance volumn.*)

Then, the energy balance for the tank gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho c_p T A h\right) &= w_1 c_p T_1 - w_2 c_p T - Gh(T - T_c) \\
\rho c_p A h \frac{\mathrm{d}}{\mathrm{d}t} T + \rho c_p A T \frac{\mathrm{d}}{\mathrm{d}t} h &= w_1 c_p T_1 - w_2 c_p T - Gh(T - T_c) \\
\rho c_p A h \frac{\mathrm{d}}{\mathrm{d}t} T + \rho c_p A T \frac{1}{\rho A}\left(w_1 - w_2\right) &= w_1 c_p T_1 - w_2 c_p T - Gh(T - T_c) \\
\rho c_p A h \frac{\mathrm{d}}{\mathrm{d}t} T &= w_1 c_p (T_1 - T) - Gh(T - T_c)
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} T = \frac{w_1}{\rho A h}(T_1 - T) - \frac{G}{\rho c_p A}(T - T_c)}$$

For the jacket:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_c c_{p,c} T_c V_c\right) = w_3 c_{p,c} T_3 - w_4 c_{p,c} T_c + Gh(T - T_c)$$
$$\rho_c c_{p,c} V_c \frac{\mathrm{d}}{\mathrm{d}t} T_c = w_3 c_{p,c} T_3 - w_4 c_{p,c} T_c + Gh(T - T_c)$$

Since $w_3 = w_4$:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} T_c = \frac{w_3}{\rho_c V_c}(T_3 - T_c) + \frac{Gh}{\rho_c c_{p,c} V_c}(T - T_c)}$$

Problem 4 (Compressor, momentum balance, Bernoulli's equation)

A compressor takes in air with pressure p_0 and velocity $v_0 = 0$ from the surroundings. The air flows through a duct into the compressor. For control, it would be beneficial to have a measurement of the mass flow into the compressor. However, this measurement is not available.

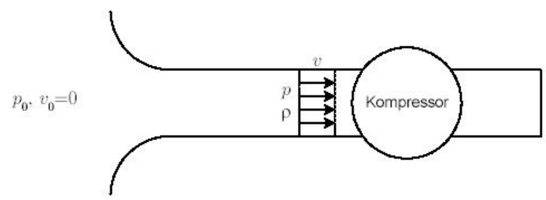

Figure 4: Compressor

Instead, there is a pressure measurement in the duct, giving a measurement p. How can the mass flow w and velocity v be found from this measurement? Assume that the density ρ in the duct is constant and known, there is no friction, and that the velocity is uniform over the cross-section where the pressure transmitter is located.

Hint: Use (the stationary) Bernoulli's equation.

Solution: Bernoulli's equation for frictionless, incompressible flow along a streamline relates pressure, velocity and elevation in two points:

$$\frac{p_1 - p_0}{\rho} + \frac{1}{2}\left(v_1^2 - v_0^2\right) + (z_1 - z_0)g = 0.$$

Choosing point 1 to be the location of the pressure transmitter ($p_1 = p$, $v_1 = v$ and $z_1 = 0$) and point 0 to be the duct inlet ($p_0 = p_0$, $v_0 = 0$ and $z_0 = 0$), we get

$$p = p_0 - \frac{1}{2} v^2 \rho$$

that can be solved to

$$v = \sqrt{\frac{2\left(p_0 - p\right)}{\rho}}$$

(note that p must be smaller than p_0). This gives mass flow

$$w = A\rho v = A\sqrt{2\rho\left(p_0 - p\right)}$$

Problem 5 (Mixing, reactions (Exam 2010))

An incompressible liquid of substance C enters a perfectly mixed tank (a continuous stirred tank reactor, CSTR) with mass flow w_C and temperature T_C. In the tank, the substance reacts (e.g. due to the presence of a catalyst) to form the substance D with a rate JV, where J is the reaction rate per unit volume, and $V = Ah$ is the volume of the tank. The tank then consists of a mixture of C and D, which leaves the tank with mass flow w and temperature T. The mass of substance C in the tank is denoted m_C, and the mass of substance D is denoted m_D.

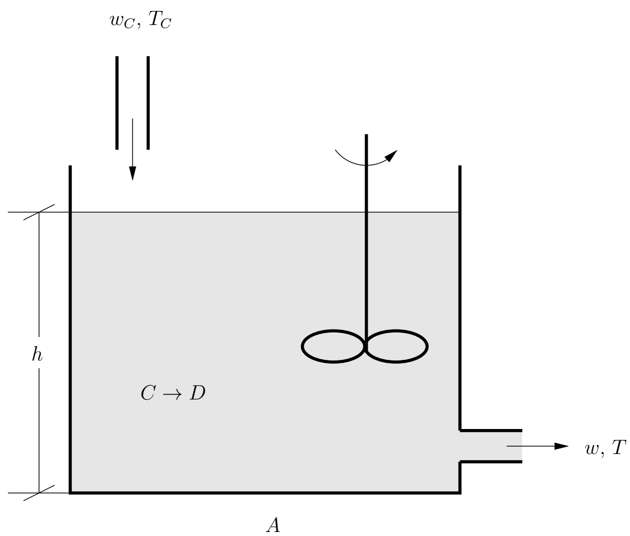

Figure 5: Tank reactor

(a) Set up a differential equation for the level of the tank. (Hint: Use the ordinary overall mass balance. Assume that the average density ρ is constant.)

Solution: The mass balance equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_f} \rho \mathrm{d}V = -\iint_{\partial V_f} \rho \vec{v} \cdot \vec{n} \mathrm{d}A$$

$$\frac{\mathrm{d}}{\mathrm{d}t} (\rho A h) = w_C - w$$

$$\frac{\mathrm{d}}{\mathrm{d}t} h = \frac{w_C - w}{\rho A}$$

(b) In a material volume V_m, the following holds:

$$\frac{\mathrm{D}}{\mathrm{D}t} \iiint_{V_m} \rho_C \mathrm{d}V = -\iiint_{V_m} J \mathrm{d}V.$$

Use this together with the appropriate form of the transport theorem to explain that the mass balance for substance C on integral form in a fixed control volume V_f is

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_f} \rho_C \mathrm{d}V = -\iiint_{V_f} J \mathrm{d}V - \iint_{\partial V_f} \rho_C \vec{v} \cdot \vec{n} \mathrm{d}A.$$

(In this particular case, the natural control volume, the volume of liquid in the tank, is not fixed, but this can be ignored since $\rho_C \vec{v}_c \cdot \vec{n} = 0$ – expansion of the volume does not accumulate more of substance C.)

Solution: Take eq. (10.90) in the book, set $\phi = \rho_C$ and insert the first equation to obtain the result.

(c) Use this to write up the mass balance for the mass of substance C in the tank ($\frac{\mathrm{d}}{\mathrm{d}t} m_C = \ldots$). Assume here, and for the rest of the problem, that J is proportional to the density of substance C, $J = k\frac{m_C}{V}$, and that the outflow of substance C is proportional to the mass ratio of substance C to the total mass in the tank, and the total outflow, $w_{C,out} = \frac{m_C}{m_C+m_D} w$.

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_C A h) &= w_C - JV - w_{C,out} \\ \frac{\mathrm{d}}{\mathrm{d}t} m_C &= w_C - k m_C - \frac{m_C}{m_C + m_D} w\end{aligned}$$

(d) What is the mass balance equation on integral form for substance D (in a fixed volume)? Use this to write up the mass balance of substance D.

Solution: For substance D, we have

$$\frac{\mathrm{D}}{\mathrm{D}t} \iiint_{V_m} \rho_D \mathrm{d}V = \iiint_{V_m} J \mathrm{d}V.$$

Insertion into (10.90) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_f} \rho_D \mathrm{d}V = \iiint_{V_f} J \mathrm{d}V - \iint_{\partial V_f} \rho_D \vec{v} \cdot \vec{n} \mathrm{d}A.$$

Solving the integrals, give

$$\frac{\mathrm{d}}{\mathrm{d}t} m_D = JV - w_D = k m_C - \frac{m_D}{m_C + m_D} w.$$

(e) Check that the solution in (c) and (d) agrees with the answer in (a).

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} m &= \frac{\mathrm{d}}{\mathrm{d}t} m_C + \frac{\mathrm{d}}{\mathrm{d}t} m_D \\ &= w_C - k m_C - \frac{m_C}{m_C + m_D} w + k m_C - \frac{m_D}{m_C + m_D} w \\ &= w_C - w.\end{aligned}$$

This agrees with the solution to (a).

The final question is optional:

(f) Set up a differential equation for the temperature in the tank. Assume that the heat generated by the reaction is proportional to J, with proportionality constant c. Disregard kinetic energy, potential energy and pressure work. Assume no 'heat flux' (the tank is well insulated). Assume the internal energy is $u = c_p T$.

Solution: The book does not treat energy balances with "internally generated" energy. We must therefore derive the energy balance on integral form for this (as we did for the mass balance above).

Under the assumptions made, (11.164) takes the form

$$\frac{\mathrm{D}}{\mathrm{D}t} \iiint\limits_{V_m} \rho u \mathrm{d}V = \iiint\limits_{V_m} cJ \mathrm{d}V$$

($e = u$, pressure work and heat flux ignored, but heat from reaction added.) Insertion into (11.169) (for a fixed volume) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V_f} \rho u \mathrm{d}V = \iiint\limits_{V_f} cJ \mathrm{d}V - \iint\limits_{\partial V_f} \rho u \vec{v} \cdot \vec{n} \mathrm{d}A.$$

Inserting $u = c_p T$ and resolving the integrals, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho c_p T V\right) &= cJV + w_C c_p T_C - w c_p T \\ \rho c_p V \frac{\mathrm{d}}{\mathrm{d}t} T + \rho c_p T A \frac{\mathrm{d}}{\mathrm{d}t} h &= cJV + w_C c_p T_C - w c_p T \end{aligned}$$

Insertion of the result in (a), and using $JV = k m_C$,

$$\begin{aligned} \rho c_p A h \frac{\mathrm{d}}{\mathrm{d}t} T + c_p T (w_C - w) &= c k m_C + w_C c_p T_C - w c_p T \\ \frac{\mathrm{d}}{\mathrm{d}t} T &= \frac{c k m_C + c_p w_C (T_C - T)}{\rho c_p A h} \end{aligned}$$

Correct result (without derivation of the energy balance) will give full score.