NTNU – Trondheim
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Exercise 7

TTK4130 Modeling and Simulation

Problem 1 (Solving the transmission line PDEs)

In this exercise, we will simulate the transmission line partial differential equations (PDEs). The transmission line PDEs can be written (4.5.2 in the book)

$$\frac{\partial p\left(x,t\right)}{\partial t} = -cZ_0 \frac{\partial q\left(x,t\right)}{\partial x} \tag{1a}$$

$$\frac{\partial q\left(x,t\right)}{\partial t} = -\frac{c}{Z_0}\frac{\partial p\left(x,t\right)}{\partial x} - \frac{F\left(q\left(x,t\right)\right)}{\rho_0}, \tag{1b}$$

where the sonic velocity c and the line impedance Z_0 are defined as

$$c = \sqrt{\frac{\beta}{\rho_0}},$$

$$Z_0 = \frac{\rho_0 c}{A} = \frac{\sqrt{\rho_0 \beta}}{A}$$

and β is the fluid bulk modulus, ρ_0 is the fluid density, A is the cross sectional area, and $F(q)$ is friction as a function of volume flow.

The book outlines two different routes to simulating this model:

1. The first type of methods (4.5.10-4.5.16) is based on calculating rational transfer function approximations to the irrational transfer functions that results from Laplace-transformation of the PDE model. (Remember, irrational transfer functions cannot be simulated directly in (for example) Simulink).
2. The second method is to discretize the spatial dimension into a set of "finite volumes", and use the Helmholtz resonator model inside each volume. This is explained in 4.6.

In this exercise, we will look into the second route. In addition to simulating the transmission line model, the objective is to give a glimpse into how general PDE models are solved using discretization methods (like difference methods, finite volume methods or finite element methods). The method used in this exercise is similar to the finite volume method, which is described in Chapter 15 in the book, but it is not necessary to read this before the exercise.

The "Finite Volume"-method can briefly be described by dividing the transmission line of length L into n smaller "volumes" with length $h = \frac{L}{n}$, and let the variables be spatially invariant within these volumes. Inside each volume, the time variation of the variables are given by a Helmholtz resonator model (4.6.2 and 4.6.3).

We will in this exercise assume that the flows at each end of the transmission line are the inputs, and we want to calculate the pressures in each end (that is, we will implement an impedance model version (4.6.5) of the PDE). In the first part of the exercise, we will assume a simple linear friction model in (1b), that is, $F = \rho_0 B q$ where $B = \frac{8\nu_0}{r_0^2}$.

(a) Set up the model for each volume element (hint: See Figure 4.17 and read Ch. 4.6.1-4.6.5).

Solution: The model becomes (see Ch. 4.6.5)

$$\dot{p}_i = \frac{c^2 \rho_0}{Ah}\left(q_{i-1} - q_i\right), \quad i = 1, \ldots, N$$

$$\dot{q}_{i-1} = \frac{A}{h\rho_0}\left(p_{i-1} - p_i\right) - Bq_{i-1}, \quad i = 2, \ldots, N$$

$$q_0 = q_{in}, \quad q_N = q_{out}$$

(b) We will now write the model of all the volumes as a linear state space model where the boundary conditions (inputs) are volume flow into the first volume (q_0) and volume flow out of the last volume (q_N). That is, gather all variables in a state variable $\mathbf{x}$,

$$\mathbf{x}^\top = \begin{pmatrix} p_1 & q_1 & p_2 & q_2 & \cdots & q_{N-1} & p_N \end{pmatrix},$$

and write the model as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \end{aligned}$$

where the inputs and outputs are

$$\mathbf{u} = \begin{pmatrix} q_0 \\ q_N \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} p_1 \\ p_N \end{pmatrix}.$$

Set up $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ for n volumes. Can you see a pattern in $\mathbf{A}$?

Solution: The matrices are

$$\mathbf{A} = \begin{pmatrix} 0 & -M & 0 & 0 & \dots & 0 & 0 \\ N & -B & -N & 0 & \dots & 0 & 0 \\ 0 & M & 0 & -M & \dots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & M & 0 & -M & 0 \\ 0 & 0 & \dots & 0 & N & -B & -N \\ 0 & 0 & \dots & 0 & 0 & M & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} M & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -M \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where M and N are given as

$$M = \frac{c^2\rho_0}{Ah} = \frac{\beta}{Ah}, \quad N = \frac{A}{h\rho_0}.$$

Note the band pattern of $\mathbf{A}$ (the matrix is tridiagonal). Also note how the rows repeat: $\mathbf{A}$ can be written

$$\mathbf{A} = \begin{pmatrix} 0 \;\; -M \;\; 0 \;\; 0 & \dots & 0 \;\; 0 \;\; 0 \;\; 0 \\ \mathbf{A_1} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{A_1} \\ 0 \;\; 0 \;\; 0 \;\; 0 & \dots & 0 \;\; N \;\; -B \;\; -N \\ 0 \;\; 0 \;\; 0 \;\; 0 & \dots & 0 \;\; 0 \;\; M \;\; 0 \end{pmatrix}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} N & -B & -N & 0 \\ 0 & M & 0 & -M \end{pmatrix}.$$

(c) We will now implement the model in Simulink. The parameters are

Parameter	Symbol	Value	Unit
Length	L	19.76	m
Bulk modulus	β	$1.7052 \cdot 10^9$	Pa
Density	ρ_0	870	kg/m^3
Pipe radius	r_0	$6.17 \cdot 10^{-3}$	m
Kinematic viscosity	ν_0	$8 \cdot 10^{-5}$	m^2/s

Use a state-space block in Simulink, with inputs and outputs as given in (b). Instead of inputting the matrices by hand in the state-space block, write just A under A, B under B, C under C, D under D and (if you want other initial conditions than 0) x0 under initial conditions. We will use a mask to specify these variables. This makes it also easier to vary the number of volumes.

Make a sub-system of the state-space block, as illustrated in Figure 1.

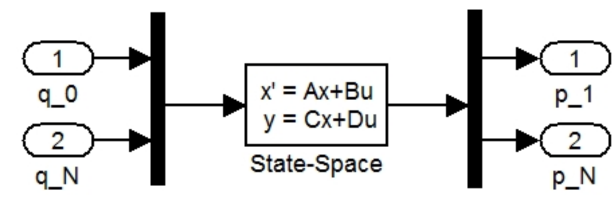

Figure 1: State-space model in Simulink

Right-click the subsystem, and choose Mask Subsystem. Add the parameters from the table above in addition to number of volumes (and possibly initial values) under Parameters, and the matlab code to calculate the **A**, **B**, **C** and **D**-matrices (and possibly $x0$) under Initialization.

Solution: The system that contains the state-space model can look something like in Figure 2.

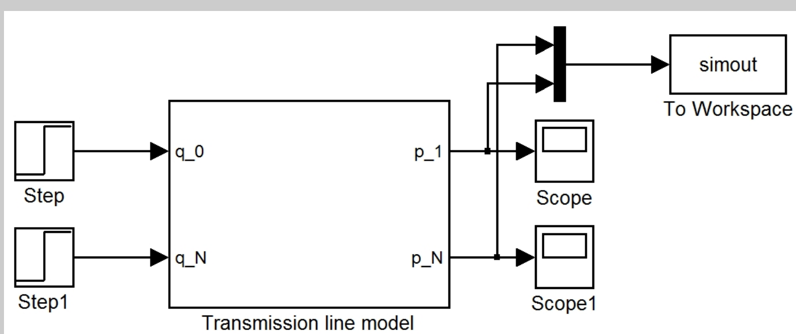

Figure 2: Simulink model that simulates the transmission line model

The code that is entered into the Initialization-part of the mask, is

```
area = r0^2*pi;
h = L/n;
M = beta/(area*h);
N = area/(h*rho0);
B = 8*nu0/r0^2;
x0 = zeros(2*n-1,1);
mat = [M 0 -M 0; 0 N -B -N];
```

```
A = [0 -M zeros(1,(2*n-3)); N -B -N zeros(1,(2*n-4))];
x0(1:2) = [p0; q0];
for i = 1:1:n-2
 addmat = [zeros(2,2*i-1) mat zeros(2,2*(n-2-i))];
 A = [A;addmat];
 x0((2*(i+1)-1):2*(i+1)) = [p0; q0];
end
x0(2*n-1) = p0;
A = [A; zeros(1, (2*n-3)) M 0];
B = [M 0 ; zeros((2*n-3),2); 0 -M];
C = [1 zeros(1, 2*n-2); zeros(1, 2*n-2) 1];
D = zeros(2,2);
```

The name of the parameters must of course match the ones entered in the parameter list.

(d) Simulate the system over 5 seconds using the inputs

$$q_0 = \begin{cases} 0 & t < 1s \\ 0,001 & t \geq 1s \end{cases}, \qquad q_N = \begin{cases} 0 & t < 2s \\ 0,001 & t \geq 2s \end{cases},$$

for $n = 5, 10$ and 50. Tip: Use the To Workspace-block to make comparing plots in Matlab. Comment.

Make Bode-plots of transfer function from q_0 to pressures on both sides (p_1 and p_N) for the same ns. Use same method as on earlier exercises to make bode-plots. Comment on plots.

Solution: The simulations are shown in Figure 3. When the inflow increases without outflow increasing, the pressure increases. Pressure stops increasing ones outflow matches inflow. In addition to these large effects, there are oscillations corresponding to pressure waves going back and forth in the transmission line. We see that using few elements, tend to smooth out the pressure waves.

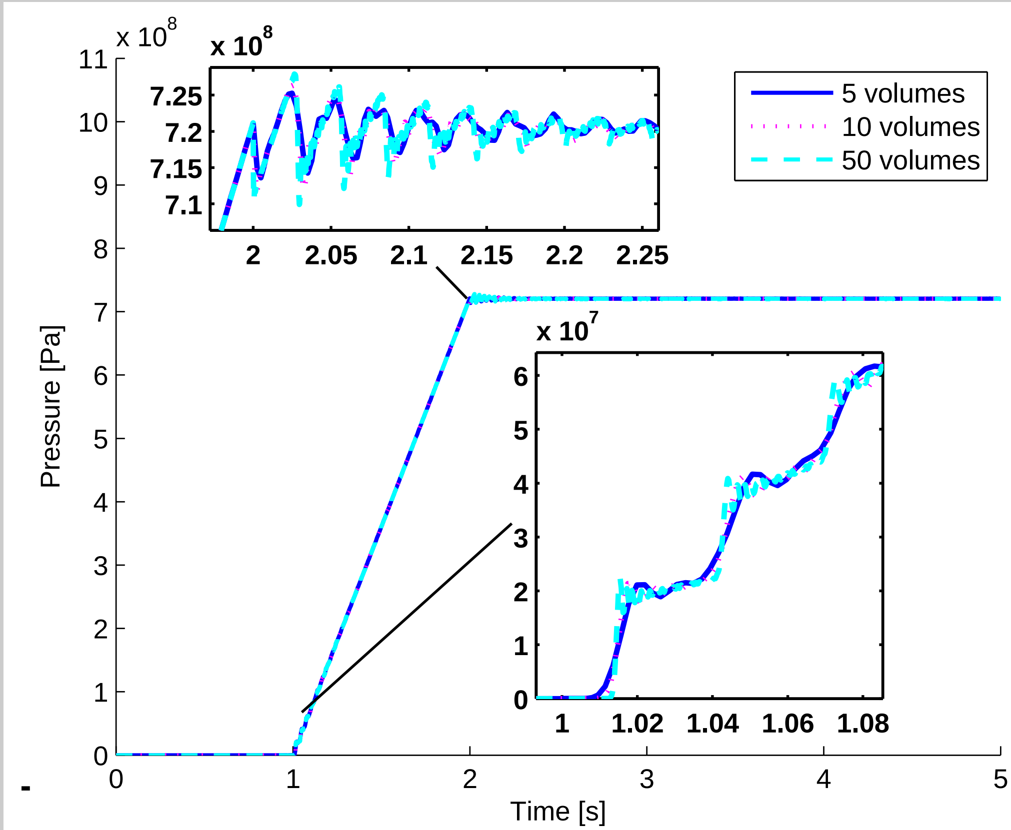

Figure 3: Simulation with zoom.

The Bode-plots are shown in Figure 4 and 5.

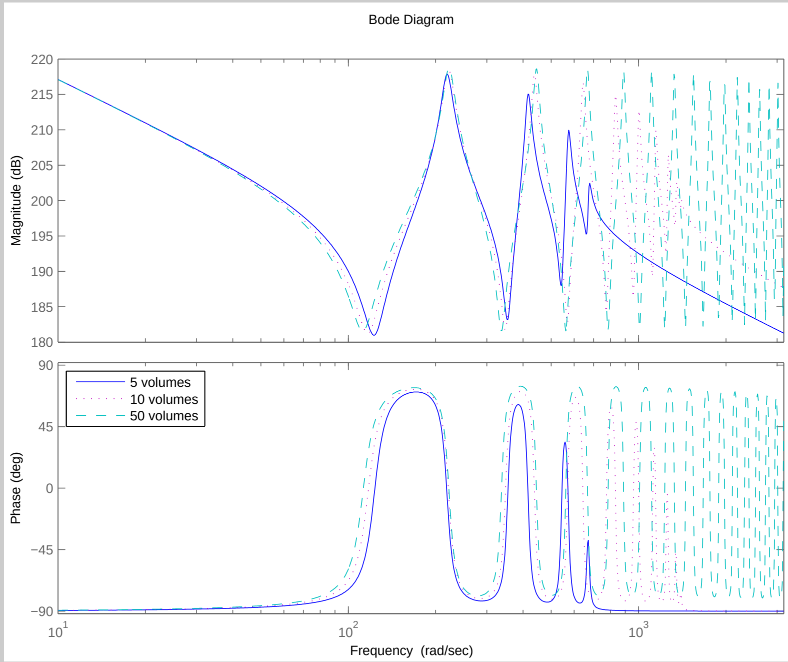

Figure 4: Frequency response from q_0 to p_1.

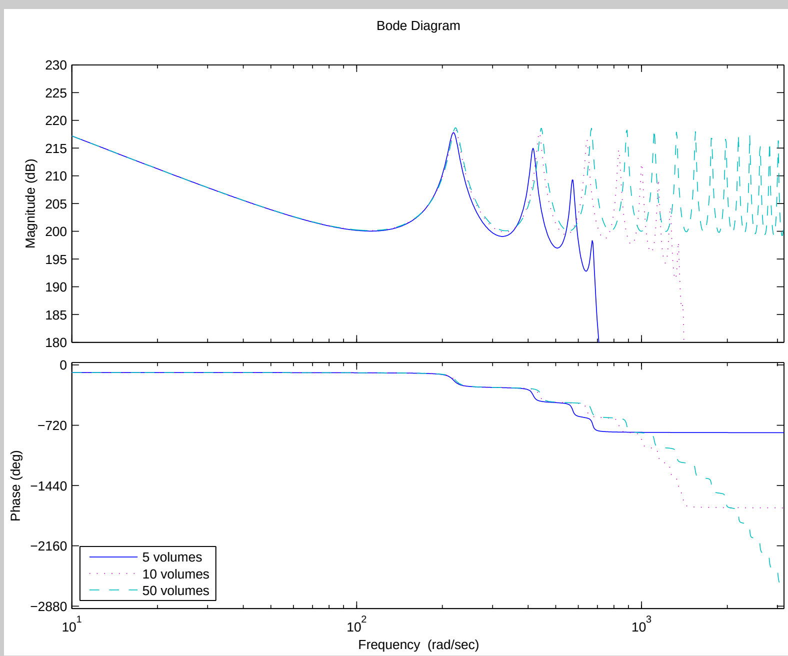

Figure 5: Frequency response from q_0 to p_N.

There is no delay (other than a phase-shift) from inflow to pressure in the first volume element, while there is a strong delay from inflow to pressure in the last volume element. The number of resonances depend on the number of elements in the model.

Problem 2 (Using another friction model)
We will now investigate another friction model (linear friction and diffusion) for the same system. The partial differential equation for the volume flow is now

$$\frac{\partial q}{\partial t} = -\frac{A}{\rho_0}\frac{\partial p}{\partial x} + \nu_0\left(\frac{\partial^2 q}{\partial x^2} - \frac{8}{r_0^2}q\right). \tag{2}$$

(a) Show that a reasonable discretization (difference approximation) of $\frac{\partial^2 q}{\partial x^2}$ is

$$\frac{\partial^2 q_i}{\partial x^2} = \frac{q_{i-1} - 2q_i + q_{i+1}}{h^2}.$$

Hint: Use forward and backward Euler approximations to the spatial derivative, twice.

Solution:

$$\begin{aligned}
\frac{\partial^2 q_i}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial q_i}{\partial x} \\
&\approx \frac{1}{h}\left(\frac{\partial q_{i+1}}{\partial x} - \frac{\partial q_i}{\partial x}\right) \\
&\approx \frac{1}{h}\left(\frac{1}{h}(q_{i+1} - q_i) - \frac{1}{h}(q_i - q_{i-1})\right) \\
&= \frac{q_{i-1} - 2q_i + q_{i+1}}{h^2}.
\end{aligned}$$

(b) Using this, discretize (2) in a similar manner as in Problem 1. Write down a linear state-space model for the discretized system.

Solution: The model becomes

$$\begin{aligned}
\dot{p}_i &= \frac{c^2\rho_0}{Ah}\left(q_{i-1} - q_i\right), \quad i = 1, \ldots, N \\
\dot{q}_{i-1} &= \frac{A}{h\rho_0}\left(p_{i-1} - p_i\right) - \left(\frac{2\nu_0}{h^2} + \frac{8\nu_0}{r_0^2}\right)q_{i-1} + \frac{\nu_0}{h^2}\left(q_{i-2} + q_i\right), \quad i = 2, \ldots, N \\
q_0 &= q_{in}, \quad q_N = q_{out}
\end{aligned}$$

This can be written on state-space form,

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}_v\mathbf{x} + \mathbf{B}_v\mathbf{u}, \\
\mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u},
\end{aligned}$$

where $\mathbf{A}_v$ is given as

$$\mathbf{A}_v = \begin{pmatrix} \begin{matrix} 0 & -M & 0 & 0 \\ N & -d & -N & k \end{matrix} & \cdots & \mathbf{0} \\ \begin{matrix} \mathbf{0} & \mathbf{A}_2 \end{matrix} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \begin{matrix} \mathbf{A}_2 & \mathbf{0} \end{matrix} \\ \mathbf{0} & \cdots & \begin{matrix} M & 0 & -M & 0 \\ k & N & -d & -N \\ 0 & 0 & M & 0 \end{matrix} \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} M & 0 & -M & 0 & 0 \\ k & N & -d & -N & k \end{pmatrix}$$

$$M = \frac{\beta}{Ah},\ N = \frac{A}{h\rho_0},\ d = \frac{2\nu_0}{h^2} + \frac{8\nu_0}{r_0^2},\ k = \frac{\nu_0}{h^2},\ \mathbf{u} = \begin{pmatrix} q_0 \\ q_N \end{pmatrix}$$

and $\mathbf{B}_v$ as

$$\mathbf{B}_v = \begin{pmatrix} M & 0 \\ k & 0 \\ \vdots & \vdots \\ 0 & k \\ 0 & -M \end{pmatrix}$$

and **C** and **D** as in Problem 1.