Out: March 03, 2016
In: March 11–31, 2016

NTNU – Trondheim
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Exercise 9

TTK4130 Modeling and Simulation

Problem 1 (Rotation matrices)

Consider the vectors

$$\mathbf{u}^b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{w}^a = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

and the matrix

$$\mathbf{R}_b^a = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Show that $\mathbf{R}_b^a$ is a rotation matrix by showing that it is part of SO(3).

Solution: SO(3) is defined as

$$\mathrm{SO}(3) = \left\{ R | R \in \mathbb{R}^{3\times 3},\ R^\mathsf{T} R = I, \text{ and } \det R = 1 \right\}.$$

We have that

$$\left(\mathbf{R}_b^a\right)^\mathsf{T} \mathbf{R}_b^a = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}^\mathsf{T} \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,$$

and

$$\begin{aligned} \det\left(\mathbf{R}_b^a\right) &= \det \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{vmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{vmatrix} \\ &= -\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\sqrt{3}\right)\left(\frac{1}{2}\sqrt{3}\right) \\ &= 1, \end{aligned}$$

which imply that $\mathbf{R}_b^a \in \mathrm{SO}(3)$, or in other words, that $\mathbf{R}_b^a$ is a rotation matrix.

(b) What (simple) rotation does $\mathbf{R}_b^a$ represent?

Solution: We recognize the form of $\mathbf{R}_b^a$ to be the same as for a rotation about the z-axis,

$$\mathbf{R}_z(\phi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that we must have $\psi = \pi/6$, that is, $\mathbf{R}_b^a$ represent a rotation $\psi = \pi/6$ about the z-axis.

(c) What is $\mathbf{R}_a^b$?

Solution: Since $\mathbf{R}_a^b = \left(\mathbf{R}_b^a\right)^\mathsf{T}$,

$$\mathbf{R}_a^b = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) Compute $\mathbf{u}^a$ and $\mathbf{w}^b$.

Solution:

$$\mathbf{u}^a = \mathbf{R}_b^a \mathbf{u}^b = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-2+\sqrt{3}}{2} \\ \frac{1+2\sqrt{3}}{2} \\ 3 \end{pmatrix}$$

$$\mathbf{w}^b = \mathbf{R}_a^b \mathbf{w}^a = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt{3}}{2} \\ \frac{-1-\sqrt{3}}{2} \\ 2 \end{pmatrix}$$

(e) Show that

i) $(\mathbf{u}^a)^\top \mathbf{w}^a = \left(\mathbf{u}^b\right)^\top \mathbf{w}^b$

ii) $(\mathbf{u}^a)^\times \mathbf{w}^a = \mathbf{R}_b^a \left(\mathbf{u}^b\right)^\times \mathbf{w}^b$, where $(\mathbf{u}^a)^\times$ is the skew-symmetric form of $\mathbf{u}^a$. Try both a geometric proof (by drawing a figure and argue based on that) and an algebraic proof (using the coordinate transformation rule for matrix representations of a dyadic).

Solution:

i)

$$\begin{aligned} \left(\mathbf{u}^b\right)^\top \mathbf{w}^b &= \left(\mathbf{R}_a^b \mathbf{u}^a\right)^\top \mathbf{R}_a^b \mathbf{w}^a \\ &= (\mathbf{u}^a)^\top \left(\mathbf{R}_a^b\right)^\top \mathbf{R}_a^b \mathbf{w}^a \\ &= (\mathbf{u}^a)^\top \mathbf{w}^a \end{aligned}$$

ii) Geometrically: The cross product $\vec{z} = \vec{u} \times \vec{w}$ can be written on coordinate form in frame a and b as $\mathbf{z}^a = (\mathbf{u}^a)^\times \mathbf{w}^a$ and $\mathbf{z}^b = \left(\mathbf{u}^b\right)^\times \mathbf{w}^b$. See Figure 1. The equation to prove is then nothing else than a coordinate transformation of the cross product, that is, $\mathbf{z}^a = \mathbf{R}_b^a \mathbf{z}^b$.

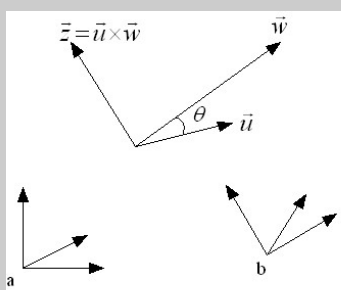

Figure 1: Illustration of cross product

Algebraically: If we use that $\left(\mathbf{u}^b\right)^\times = \mathbf{R}_a^b (\mathbf{u}^a)^\times \mathbf{R}_b^a$ (see (6.109) in book), then

$$\begin{aligned} \mathbf{R}_b^a \left(\mathbf{u}^b\right)^\times \mathbf{w}^b &= \mathbf{R}_b^a \mathbf{R}_a^b (\mathbf{u}^a)^\times \mathbf{R}_b^a \mathbf{R}_a^b \mathbf{w}^a \\ &= (\mathbf{u}^a)^\times \mathbf{w}^a \end{aligned}$$

Remark: A consequence (or reformulation) of this result, is that

$$\mathbf{R}_b^a \left(\mathbf{u}^b\right)^\times \mathbf{w}^b = \left(\mathbf{R}_b^a \mathbf{u}^b\right)^\times \mathbf{R}_b^a \mathbf{w}^b,$$

that is, the transformation of a cross product is equal to the cross product of the transformations.

(f) Let $\mathbf{R}_b^a$ be given by

$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi} \tag{1}$$

where $\mathbf{R}_{z,\psi}$, $\mathbf{R}_{y,\theta}$ and $\mathbf{R}_{x,\phi}$ are the simple rotations (defined by (6.101)–(6.103) in the book). Calculate the elements in $\mathbf{R}_b^a$ as a function of the Euler angles ψ, θ and ϕ. (You may abbreviate $\cos\psi$ to $\mathrm{c}\psi$, $\sin\psi$ to $\mathrm{s}\psi$, etc.)

Solution: From

$$\mathbf{R}_{x,\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad \mathbf{R}_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad \mathbf{R}_{z,\psi} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we get

$$\begin{aligned} \mathbf{R}_b^a &= \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi} \\ &= \begin{pmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \end{aligned}$$

(g) Given the following three rotation matrices:

$$\mathbf{R}_1 = \begin{pmatrix} * & * & * \\ * & -1 & * \\ * & * & 1 \end{pmatrix}, \quad \mathbf{R}_2 = \begin{pmatrix} * & 1 & * \\ * & * & * \\ * & * & 1 \end{pmatrix}, \quad \mathbf{R}_3 = \begin{pmatrix} * & \frac{1}{\sqrt{3}} & * \\ * & * & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & * & * \end{pmatrix}.$$

Use the fact that these are rotation matrices ($\mathbf{R}_i \in SO(3)$, that is, the matrices are orthogonal and $\det(R_i) = 1$) to find the elements marked with $*$. (It is enough to outline a procedure for $\mathbf{R}_3$, actually finding $\mathbf{R}_3$ is optional.)

Solution:

$\mathbf{R}_1$: Since columns and rows must be unit vectors, the remaining elements in the second and third column and row must be zero:

$$r_{12} = r_{13} = r_{21} = r_{23} = r_{31} = r_{32} = 0.$$

The only remaining element, r_{11} is found from $\det\mathbf{R}_1 = -r_{11} = 1$,

$$r_{11} = -1$$

which gives us

$$\mathbf{R}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{R}_2$: The requirement that rows and columns should be unit vectors leaves us with only r_{21} as unknown. That $\det(\mathbf{R}_2) = -r_{21} = 1$ implies $r_{21} = -1$, and

$$\mathbf{R}_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{R}_3$: Denote the unknown elements with x_i, as in

$$\mathbf{R}_3 = \begin{pmatrix} x_1 & \frac{1}{\sqrt{3}} & x_5 \\ x_2 & x_3 & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & x_4 & x_6 \end{pmatrix}. \tag{2}$$

Orthogonality of columns gives the following three equations:

$$\begin{aligned} ax_1 + x_2x_3 + cx_4 &= 0 \\ x_1x_5 + bx_2 + cx_6 &= 0 \\ ax_5 + bx_3 + x_4x_6 &= 0 \end{aligned}$$

where $a = 1/\sqrt{3}$, $b = -1/2$, and $c = 1/\sqrt{2}$. That columns should be unit vectors gives the following three equations:

$$\begin{aligned} \sqrt{x_1^2 + x_2^2 + c^2} &= 1 \\ \sqrt{a^2 + x_3^2 + x_4^2} &= 1 \\ \sqrt{x_5^2 + b^2 + x_6^2} &= 1 \end{aligned}$$

This is six equations in six unknowns. This is difficult to solve by hand, but we can let Matlab help us:

```matlab
a = 1/sqrt(3); b = -1/2; c = 1/sqrt(2);

f = @(x) [
    a*x(1) + x(2)*x(3) + c*x(4);
    x(1)*x(5) + x(2)*b + c*x(6);
    a*x(5) + b*x(3) + x(4)*x(6);
    1 - sqrt(x(1)^2 + x(2)^2 + c^2);
    1 - sqrt(a^2 + x(3)^2 + x(4)^2);
    1 - sqrt(x(5)^2 + b^2 + x(6)^2) ];

x = fsolve(f,ones(6,1));

R = [x(1) a x(5); x(2) x(3) b; c x(4) x(6)]
```

This gives

$$\mathbf{R}_3 = \begin{pmatrix} 0.1459 & 0.5774 & 0.8034 \\ 0.6919 & 0.5209 & -0.5000 \\ 0.7071 & -0.6288 & 0.3235 \end{pmatrix}$$

(Check, if matrix fulfils $\det(R_i) = 1$, otherwise multiply with -1)

Problem 2 (Homogenous transformation matrices, Denavit-Hartenberg convention)

The Denavit-Hartenberg (D-H) convention is used to specify the relations between the different coordinate systems used in robotic manipulators. In this convention, each homogeneous transformation $A_i = T^i_{i+1}$ is given as a product of four basic transformations

$$A_i = \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i}, \tag{3}$$

where θ_i, d_i, a_i and α_i are parameters related to joint i, and in addition

$$\begin{aligned}
\mathrm{Rot}_{z,\theta_i} &: \text{Rotation } \theta_i \text{ about } z\text{-axis} \\
\mathrm{Trans}_{z,d_i} &: \text{Translation } d_i \text{ along } z\text{-axis} \\
\mathrm{Trans}_{x,a_i} &: \text{Translation } a_i \text{ along } x\text{-axis} \\
\mathrm{Rot}_{x,\alpha_i} &: \text{Rotation } \alpha_i \text{ about } x\text{-axis}
\end{aligned}$$

See Figure 2.

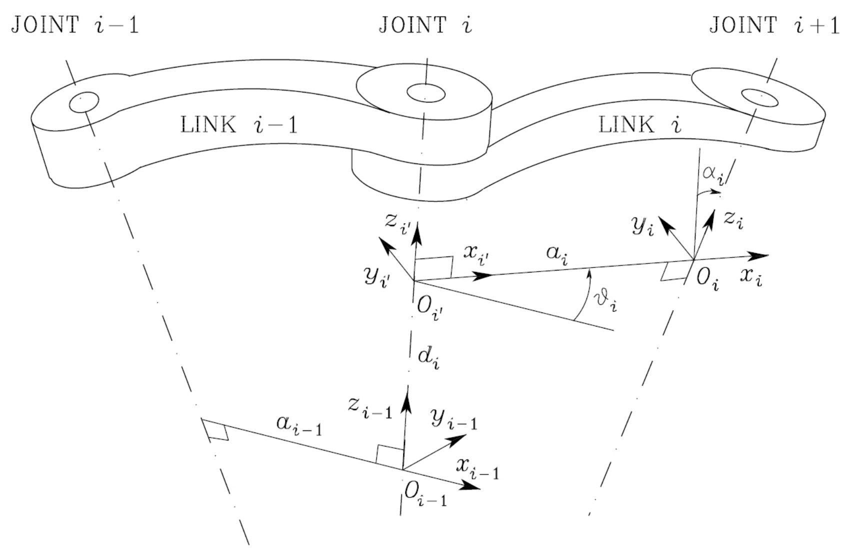

Figure 2: Illustration of transformations involved in the Denavit-Hartenberg convention (taken from Sciavicco, Siciliano, "Modeling and control of robotic manipulators).

We now want to describe the kinematics of the two manipulators in Figure 3:

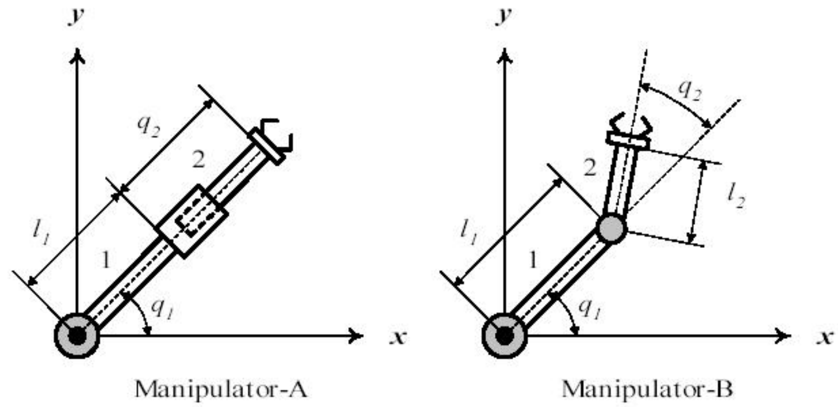

Figure 3: Two robotic manipulators

The D-H parameters for these manipulators can be tabulated as follows:

- Manipulator A, with one rotational joint and one translational (prismatic) joint. The variables q_1 and q_2 are the joint variables (degrees of freedom), while l_1 is constant:

Joint	θ_i	d_i	a_i	α_i
1	q_1	0	l_1	0
2	0	0	q_2	0

- Manipulator B, with two rotational joints. The variables q_1 og q_2 are the joint variables, while l_1 and l_2 are constants:

Joint	θ_i	d_i	a_i	α_i
1	q_1	0	l_1	0
2	q_2	0	l_2	0

(a) Find a general expression for A_i as a function of θ_i, d_i, a_i and α_i.

Solution:

$$
\begin{aligned}
A_i &= \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i} \\
&= \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

(b) Find the homogenous transformation matrices for each joint, (A_1 and A_2) for each of the manipulators.

Solution: For manipulator A:

$$A_1 = \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & q_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For manipulator B:

$$A_1 = \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & l_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & l_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) Find the overall transformation matrix T_2^0 for both manipulators.

Solution: For manipulator A:

$$\begin{aligned} T_2^0 &= A_1 A_2 \\ &= \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & q_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & (q_2 + l_1) \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & (q_2 + l_1) \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

For manipulator B:

$$\begin{aligned} T_2^0 &= A_1 A_2 \\ &= \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & l_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & l_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(q_1 + q_2) & -\sin(q_1 + q_2) & 0 & l_2 \cos(q_1 + q_2) + l_1 \cos q_1 \\ \sin(q_1 + q_2) & \cos(q_1 + q_2) & 0 & l_2 \sin(q_1 + q_2) + l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(d) Assume we have a vector

$$g = g^2 = \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}^\mathsf{T} \tag{4}$$

given in the tool frame. What is this in the base frame for the manipulators? The base frame (coordinate system) is shown in Figure 3, while the tool frame (coordinate system) is the system given by transforming the base frame with T_2^0.

Solution: We denote the base frame with 0 and the tool frame with 2. Remember that since T_2^0 transforms frame 0 to frame 2, it is the coordinate transformation matrix from frame 2 to frame 0.

- For manipulator A:

$$\begin{aligned}
g^0 &= T_2^0 g^2 \\
&= \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & (q_2 + l_1)\cos q_1 \\ \sin q_1 & \cos q_1 & 0 & (q_2 + l_1)\sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos q_1 - \sin q_1 + (q_2 + l_1)\cos q_1 \\ \sin q_1 + \cos q_1 + (q_2 + l_1)\sin q_1 \\ 1 \\ 1 \end{pmatrix}
\end{aligned}$$

- For manipulator- B:

$$\begin{aligned}
g^0 &= T_2^0 g^2 \\
&= \begin{pmatrix} \cos(q_1+q_2) & -\sin(q_1+q_2) & 0 & l_2\cos(q_1+q_2) + l_1\cos q_1 \\ \sin(q_1+q_2) & \cos(q_1+q_2) & 0 & l_2\sin(q_1+q_2) + l_1\sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(q_1+q_2) - \sin(q_1+q_2) + l_2\cos(q_1+q_2) + l_1\cos q_1 \\ \sin(q_1+q_2) + \cos(q_1+q_2) + l_2\sin(q_1+q_2) + l_1\sin q_1 \\ 1 \\ 1 \end{pmatrix}
\end{aligned}$$

Problem 3 (From rotation matrix to Euler parameters)

The rotation (or orientation) specified by a rotation matrix

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

can be represented by an angle θ about an axis $\mathbf{k}$ (angle-axis representation), and may then be written

$$\mathbf{R} = \mathbf{R}_{\mathbf{k},\theta} = \mathbf{e}^\times + \cos\theta\mathbf{I} + \mathbf{k}\mathbf{k}^\mathsf{T}(1 - \cos\theta) \tag{5}$$

where $\mathbf{e} = \mathbf{k}\sin\theta$ is the Euler rotation vector (equation (6.231) in book). When $\mathbf{R} = \mathbf{R}_b^a = \mathbf{R}_{\mathbf{k},\theta}$, then $\mathbf{k}$ is the same in both system a and b, $\mathbf{k} = \mathbf{k}^a = \mathbf{k}^b$ (why?).

When implementing control systems involving rotations (for instance for robotic manipulators or satellites), it is often desirable to find $\mathbf{k}$ and θ (or Euler parameters) from a given rotation matrix.

A good procedure for this is Shepperd's method (book 6.7.6). In this problem we will derive a procedure that is slightly simpler, but not as general as it cannot be used for $\sin\theta = 0$, and is inaccurate when $\sin\theta$ is small.

(a) Show that

$$\cos\theta = \frac{r_{11} + r_{22} + r_{33} - 1}{2},$$

using (5).

Solution: From

$$\mathbf{R} = \mathbf{k}^{\times}\sin\theta + \cos\theta\mathbf{I} + \mathbf{k}\mathbf{k}^{\mathsf{T}}(1 - \cos\theta)$$

and

$$\mathbf{k}\mathbf{k}^{\mathsf{T}} = \begin{pmatrix} k_1^2 & k_1k_2 & k_1k_3 \\ k_2k_1 & k_2^2 & k_2k_3 \\ k_3k_1 & k_3k_2 & k_3^2 \end{pmatrix}$$

for $\mathbf{k} = \begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix}^{\mathsf{T}}$. By extracting the diagonal elements and using that $\mathbf{k}$ is a unit vector, we get

$$\begin{aligned} r_{11} + r_{22} + r_{33} &= 0 + 3\cos\theta + (1 - \cos\theta)(k_1^2 + k_2^2 + k_3^2) \\ &= 3\cos\theta + (1 - \cos\theta) \\ &= 1 + 2\cos\theta \end{aligned}$$

which is easily transformed to the desired expression for $\cos\theta$.

(b) Show that

$$\mathbf{e} = \frac{1}{2}\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

Hint: See 6.7.7 in the book.

Solution: Since the two last terms of (5) are symmetrical (equivalently: independent of sign of θ), we see that

$$\mathbf{e}^{\times} = \frac{1}{2}\left(\mathbf{R}_{\mathbf{k},\theta} - \mathbf{R}_{\mathbf{k},-\theta}\right).$$

Written out, this is

$$\mathbf{R} - \mathbf{R}^{\mathsf{T}} = \begin{pmatrix} 0 & r_{12} - r_{21} & r_{13} - r_{31} \\ r_{21} - r_{12} & 0 & r_{23} - r_{32} \\ r_{31} - r_{13} & r_{32} - r_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2e_z & 2e_y \\ 2e_z & 0 & -2e_x \\ -2e_y & 2e_x & 0 \end{pmatrix}$$

which we see, by comparing elements, imply that

$$\mathbf{e} = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} r_{32} - r_{31} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

(c) We can now calculate $\mathbf{k}$ from

$$\mathbf{k} = \frac{\mathbf{e}}{\sin\theta},$$

where θ is as given in (a).

In (a) one can choose if $\theta \in [0, \pi]$ or $\theta \in [\pi, 2\pi]$, and this choice influences $\mathbf{k}$. Explain why $\mathbf{k}$ and θ from both choices represent the same rotation.

Solution: Since $\cos\theta = \cos(2\pi - \theta)$, both θ and $2\pi - \theta$ are possible angles in (a). However, since $\sin\theta = -\sin(2\pi - \theta)$, the two choices of angle give $\mathbf{k}$s that points in opposite direction.
The explanation is that a rotation θ about $\mathbf{k}$ is the same as the rotation 2π about $-\mathbf{k}$.

(d) Given the matrix

$$\mathbf{R} = \begin{pmatrix} 0.2133 & -0.2915 & 0.9325 \\ 0.9209 & -0.2588 & -0.2915 \\ 0.3263 & 0.9209 & 0.2133 \end{pmatrix}$$

Implement an algorithm in Matlab to find $\mathbf{k}$ and θ by using the formulas above. Enclose a printout of the code, and the results.

Solution: The code

```
e = 1/2*[R(3,2)-R(2,3); R(1,3)-R(3,1); R(2,1)-R(1,2)];
trace = R(1,1)+R(2,2)+R(3,3);
theta = acos((trace-1)/2);
k = e/sin(theta);
```

produce the answer $\theta = 2$, $\mathbf{k} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}^{\mathsf{T}}$ (approximately).

(e) (Optional) Implement Shepperd's method (Section 6.7.6 in book) to check your answer.

Solution: The code below should give the same answer.

```
r11 = R(1,1);
r22 = R(2,2);
r33 = R(3,3);
r00 = r11 + r22 + r33;
[rii,i] = max([r00, r11, r22, r33]); i = i-1;

if i == 0,
    z0 = sqrt(1 + 2*rii - r00);
    z(1) = (R(3,2)-R(2,3))/z0;
    z(2) = (R(1,3)-R(3,1))/z0;
    z(3) = (R(2,1)-R(1,2))/z0;
elseif i == 1,
    z(1) = sqrt(1 + 2*rii - r00);
    z0 = (R(3,2)-R(2,3))/z(1);
    z(2) = (R(2,1)+R(1,2))/z(1);
    z(3) = (R(1,3)+R(3,1))/z(1);
elseif i == 2,
    z(2) = sqrt(1 + 2*rii - r00);
    z0 = (R(1,3)-R(3,1))/z(2);
    z(1) = (R(2,1)+R(1,2))/z(2);
    z(3) = (R(3,2)+R(2,3))/z(2);
elseif i == 3,
    z(3) = sqrt(1 + 2*rii - r00);
    z0 = (R(2,1)-R(1,2))/z(3);
    z(1) = (R(1,3)+R(3,1))/z(3);
    z(2) = (R(3,2)+R(2,3))/z(3);
end
```

```
% Euler parameters
eta = z0/2;
epsilon = [z(1); z(2); z(3)]/2;

% Angle axis
theta= 2*acos(eta);
k = epsilon/(sin(theta/2));
```