TTK4115 Linear System Theory
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# Solution to homework assignment 2

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## Problem 1: Jordan forms

a) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -\lambda & -9 \\ 1 & -6-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2.$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalue $\lambda = -3$ with multiplicity 2. The corresponding eigenvectors can be obtained from the kernel of the matrix $(\mathbf{A}-\lambda\mathbf{I})$:

$$\ker(\mathbf{A}-\lambda\mathbf{I}) = \ker\left(\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q} = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

where $\mathbf{q}$ is the corresponding eigenvector. Note that $\mathbf{A}$ has only one eigenvector associated with $\lambda$.

b) Because the eigenvalue $\lambda = -3$ of $\mathbf{A}$ has multiplicity 2, and $\mathbf{A}$ has only one eigenvector associated with $\lambda$, the eigenvalues of $\mathbf{A}$ are not (all) distinct. Therefore, the system cannot be transformed into a diagonal form using a similarity transformation.

c) In order to transform the system into a Jordan form, we have to find the generalized eigenvectors of $\mathbf{A}$. The chain of generalized eigenvectors satisfies the following equalities:

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0},$$
$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the generalized eigenvectors. Note that we can choose $\mathbf{v}_1 = \mathbf{q} = [3, 1]^T$, since $\mathbf{q}$ is an eigenvector associated with $\lambda$, and therefore $(\mathbf{A}-\lambda\mathbf{I})\mathbf{q} = \mathbf{0}$. The generalized eigenvector $\mathbf{v}_2$ can be obtained from the second equality:

$$\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Note that the choice for $\mathbf{v}_2$ is not unique. We define

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using the similarity transformation $\mathbf{x} = \mathbf{Q}\hat{\mathbf{x}}$, the system is transformed to Jordan form:

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t),$$
$$y(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}\mathbf{u}(t),$$

with matrices

$$\hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & -9 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix},$$
$$\hat{\mathbf{B}} = \mathbf{Q}^{-1}\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix},$$
$$\hat{\mathbf{C}} = \mathbf{C}\mathbf{Q} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix},$$
$$\hat{\mathbf{D}} = \mathbf{D} = 2.$$

**Problem 2: Realizations**

a) A transfer matrix is realizable if and only if it is proper and rational.

- **Proper:** A transfer function $\hat{G}(s) = \frac{n(s)}{d(s)}$ is proper if the degree of its denominator $d(s)$ is larger than or equal to the degree of its numerator $n(s)$, i.e. $\deg d(s) \geq \deg n(s)$. A transfer matrix is proper if all its elements (i.e. transfer functions) are proper.
- **Rational:** A transfer function $\hat{G}(s) = \frac{n(s)}{d(s)}$ is rational if the degrees of the numerator $n(s)$ and the denominator $d(s)$ are finite. A transfer matrix is rational if all its elements (i.e. transfer functions) are rational.

b) The transfer matrix $\hat{\mathbf{G}}(s)$ can be written as

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \hat{G}_{11}(s) & \hat{G}_{12}(s) \\ 0 & \hat{G}_{22}(s) \end{bmatrix},$$

with

$$\hat{G}_{11}(s) = \frac{n_{11}(s)}{d_{11}(s)} = \frac{s^2 + 4s + 2}{s^2 + 2s},$$
$$\hat{G}_{12}(s) = \frac{n_{12}(s)}{d_{12}(s)} = \frac{3}{s + 2},$$
$$\hat{G}_{22}(s) = \frac{n_{22}(s)}{d_{22}(s)} = \frac{2s^2}{s^2 - 4}.$$

The degrees of the numerator and denominator polynomials of the transfer functions are

$$\deg n_{11}(s) = 2, \qquad \deg d_{11}(s) = 2,$$
$$\deg n_{12}(s) = 0, \qquad \deg d_{12}(s) = 1,$$
$$\deg n_{22}(s) = 2, \qquad \deg d_{22}(s) = 2.$$

Because the degrees of the denominators of the transfer functions $\hat{G}_{11}(s)$, $\hat{G}_{12}(s)$ and $\hat{G}_{22}(s)$ are larger than or equal to the degrees of the corresponding numerators, the transfer matrix is proper. Moreover, because the degrees of the numerators and denominators of each transfer function are finite, the transfer matrix is rational. Hence, it follows that the transfer matrix is realizable.

c) The constant matrix $\mathbf{D}$ is given by

$$\mathbf{D} = \lim_{s\to\infty} \hat{\mathbf{G}}(s) = \begin{bmatrix} \lim\limits_{s\to\infty} \frac{s^2+4s+2}{s^2+2s} & \lim\limits_{s\to\infty} \frac{3}{s+2} \\ 0 & \lim\limits_{s\to\infty} \frac{2s^2}{s^2-4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now, the strictly proper transfer matrix $\hat{\mathbf{G}}_{sp}(s)$ can be calculated as

$$\begin{aligned} \hat{\mathbf{G}}_{sp}(s) = \hat{\mathbf{G}}(s) - \mathbf{D} &= \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} - 1 & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} - 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} - \frac{s^2+2s}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} - \frac{2s^2-8}{s^2-4} \end{bmatrix} = \begin{bmatrix} \frac{2s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{8}{s^2-4} \end{bmatrix}. \end{aligned}$$

Hence, we obtain $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}_{sp}(s) + \mathbf{D}$, with $\hat{\mathbf{G}}_{sp}(s)$ and $\mathbf{D}$ defined above.

d) For notational convenience, we write

$$\hat{\mathbf{G}}_{sp}(s) = \begin{bmatrix} \hat{G}_{sp11}(s) & \hat{G}_{sp12}(s) \\ 0 & \hat{G}_{sp22}(s) \end{bmatrix},$$

with transfer functions

$$\begin{aligned} \hat{G}_{sp11}(s) &= \frac{n_{sp11}(s)}{d_{sp11}(s)} = \frac{2s+2}{s^2+2s}, \\ \hat{G}_{sp12}(s) &= \frac{n_{sp12}(s)}{d_{sp12}(s)} = \frac{3}{s+2}, \\ \hat{G}_{sp22}(s) &= \frac{n_{sp22}(s)}{d_{sp22}(s)} = \frac{8}{s^2-4}. \end{aligned}$$

To find the least common denominator for the transfer functions of the transfer matrix $\hat{\mathbf{G}}_{sp}(s)$, we write the denominator of each transfer function as a product of first-order factors:

$$\begin{aligned} d_{sp11}(s) &= s^2 + 2s = s(s+2), \\ d_{sp12}(s) &= s + 2, \\ d_{sp22}(s) &= s^2 - 4 = (s+2)(s-2). \end{aligned}$$

The least common denominator is given by

$$d(s) = s(s+2)(s-2) = s^3 - 4s.$$

From this, we obtain that

$$d(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3,$$

with $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = -4$.

Next, the transfer matrix $\hat{\mathbf{G}}_{sp}(s)$ is written as

$$\begin{aligned}
\hat{\mathbf{G}}_{sp}(s) &= \begin{bmatrix} \frac{2s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{8}{s^2-4} \end{bmatrix} = \begin{bmatrix} \frac{2s+2}{s(s+2)} & \frac{3}{s+2} \\ 0 & \frac{8}{(s+2)(s-2)} \end{bmatrix} = \begin{bmatrix} \frac{2s+2}{s(s+2)} \cdot \frac{s-2}{s-2} & \frac{3}{s+2} \cdot \frac{s(s-2)}{s(s-2)} \\ 0 & \frac{8}{(s+2)(s-2)} \cdot \frac{s}{s} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2s^2-2s-4}{s^3-4s} & \frac{3s^2-6s}{s^3-4s} \\ 0 & \frac{8s}{s^3-4s} \end{bmatrix} = \frac{1}{s^3-4s} \begin{bmatrix} 2s^2-2s-4 & 3s^2-6s \\ 0 & 8s \end{bmatrix} \\
&= \frac{1}{s^3-4s} \left\{ \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} -2 & -6 \\ 0 & 8 \end{bmatrix} s + \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \right\}.
\end{aligned}$$

Hence, we obtain

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)} \left[ \mathbf{N}_1 s^2 + \mathbf{N}_2 s + \mathbf{N}_3 \right],$$

with

$$\mathbf{N}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{N}_2 = \begin{bmatrix} -2 & -6 \\ 0 & 8 \end{bmatrix} \qquad \text{and} \qquad \mathbf{N}_3 = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}.$$

e) Substitution of $\mathbf{D}$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ yields

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}(t) \\
\mathbf{y}(t) &= \begin{bmatrix} 2 & 3 & -2 & -6 & -4 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{u}(t).
\end{aligned}$$

## Problem 3: Similarity transforms and equivalent state-space equations

a) Using the equations of the coordinate transformation (2) and the system (1), we obtain

$$\dot{\bar{\mathbf{x}}} = \mathbf{T}\dot{\mathbf{x}} = \mathbf{T}\mathbf{A}\mathbf{x} + \mathbf{T}\mathbf{B}u = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\bar{\mathbf{x}} + \mathbf{T}\mathbf{B}u$$

and

$$y = \mathbf{C}\mathbf{x} + Du = \mathbf{C}\mathbf{T}^{-1}\bar{\mathbf{x}} + Du.$$

Hence, we get

$$\begin{aligned}
\dot{\bar{\mathbf{x}}} &= \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}u \\
y &= \bar{\mathbf{C}}\bar{\mathbf{x}} + \bar{D}u,
\end{aligned}$$

with

$$\bar{\mathbf{A}} = \mathbf{TAT}^{-1}, \quad \bar{\mathbf{B}} = \mathbf{TB}, \quad \bar{\mathbf{C}} = \mathbf{CT}^{-1} \quad \text{and} \quad \bar{D} = D.$$

Substituting the values for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, D and $\mathbf{T}$ yields

$$\begin{aligned}
\bar{\mathbf{A}} &= \mathbf{TAT}^{-1} = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} \\
\bar{\mathbf{B}} &= \mathbf{TB} = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} \\
\bar{\mathbf{C}} &= \mathbf{CT}^{-1} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \\
\bar{D} &= D = 2.
\end{aligned}$$

b) Because $\bar{\mathbf{x}} = \mathbf{Tx}$ is a similarity transformation, the systems (1) and (3) are algebraically equivalent. Because the systems (1) and (3) are algebraically equivalent, they are also zero-state equivalent.

c) Because the dimensions of the states of the systems (1) and (4) are different, there exists no similarity transform for the systems, i.e. there exists no invertible matrix $\mathbf{S}$ such that $\tilde{x} = \mathbf{Sx}$. Therefore, the systems (1) and (4) are not algebraically equivalent. To check if the systems (1) and (4) are zero-state equivalent, we have to check if the systems have the same transfer function (or impulse response). The transfer function of system (1) is given by

$$\begin{aligned}
\hat{G}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s+2 & -4 \\ 1 & s-3 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 2 \end{bmatrix} + 2 \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{s-3}{s^2-s-2} & \frac{4}{s^2-s-2} \\ \frac{-1}{s^2-s-2} & \frac{s+2}{s^2-s-2} \end{bmatrix} \begin{bmatrix} 8 \\ 2 \end{bmatrix} + 2 \\
&= \frac{6}{s+1} + 2 = \frac{2s+8}{s+1}.
\end{aligned}$$

The transfer function of system (4) is given by

$$\hat{\tilde{G}}(s) = \tilde{C}(s - \tilde{A})^{-1}\tilde{B} + \tilde{D} = 3(s+1)^{-1}2 + 2 = \frac{6}{s+1} + 2 = \frac{2s+8}{s+1}.$$

Hence, because the systems (1) and (4) have the same transfer function, they are zero-state equivalent.

## Problem 4: Controllability tests

a) The controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 & -6 \\ 2 & 2 & -10 & -10 \end{bmatrix}.$$

Because the controllability matrix has full row rank, i.e. $\text{rank}(\mathcal{C}) = 2 = n$, we conclude that the system is controllable.

b) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 2-\lambda & -3 \\ 4 & -5-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2).$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

c) For $\lambda = \lambda_1 = -1$, we have

$$\text{rank}\begin{bmatrix} \mathbf{A} - \lambda_1 \mathbf{I} & \mathbf{B} \end{bmatrix} = \text{rank}\begin{bmatrix} 3 & -3 & 0 & 0 \\ 4 & -4 & 2 & 2 \end{bmatrix} = 2.$$

Similarly, for $\lambda = \lambda_2 = -2$, we have

$$\text{rank}\begin{bmatrix} \mathbf{A} - \lambda_2 \mathbf{I} & \mathbf{B} \end{bmatrix} = \text{rank}\begin{bmatrix} 4 & -3 & 0 & 0 \\ 4 & -3 & 2 & 2 \end{bmatrix} = 2.$$

Because the matrix $\begin{bmatrix} \mathbf{A} - \lambda \mathbf{I} & \mathbf{B} \end{bmatrix}$ has full row rank for every eigenvalue $\lambda$ of $\mathbf{A}$, i.e. $\text{rank}\begin{bmatrix} \mathbf{A} - \lambda \mathbf{I} & \mathbf{B} \end{bmatrix} = n = 2$ for every eigenvalue $\lambda$ of $\mathbf{A}$, we conclude that the system is controllable.

d) To check that the controllability Gramian is nonsingular for all $t > 0$, we start by computing the integral given in the assignment. The first step on the way is to find $e^{\mathbf{A}\tau}$.

$$e^{\mathbf{A}\tau} = Qe^{\mathbf{D}\tau}Q^{-1}$$

Here $\mathbf{D}$ is the diagonal matrix of eigenvalues, which we have already found, and $\mathbf{Q}$ is the matrix of eigenvectors. We therefore have to find the eigenvectors $q_1$ and $q_2$ by solving $(\mathbf{A} - \lambda_i \mathbf{I})\, q_i = \mathbf{0}$ for $\lambda_1$ and $\lambda_2$.

$\lambda_1 = -1$:

$$(\mathbf{A} - (-1)\mathbf{I})\, q_1 = \mathbf{0}$$

$$\begin{bmatrix} 2+1 & -3 \\ 4 & -5+1 \end{bmatrix} q_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 4 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_1 = -2$:

$$(\mathbf{A} - (-1)\mathbf{I})\, q_2 = \mathbf{0}$$

$$\begin{bmatrix} 2+2 & -3 \\ 4 & -5+2 \end{bmatrix} q_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \left[\begin{array}{cc|c} 4 & -3 & 0 \\ 4 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies q_2 = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

This results in

$$\mathbf{Q} = \begin{bmatrix} 1 & \frac{3}{4} \\ 1 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

We then have

$$e^{\mathbf{A}\tau} = Qe^{\mathbf{D}\tau}Q^{-1} = \begin{bmatrix} 1 & \frac{3}{4} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-\tau} & 0 \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 4 & -4 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 4\,\mathrm{e}^{-\tau} - 3\,\mathrm{e}^{-2\tau} & -3\,\mathrm{e}^{-\tau} + 3\,\mathrm{e}^{-2\tau} \\ 4\,\mathrm{e}^{-\tau} - 4\,\mathrm{e}^{-2\tau} & -3\,\mathrm{e}^{-\tau} + 4\,\mathrm{e}^{-2\tau} \end{bmatrix}$$

From the hint it is given that there is no need to calculate the eigenvectors of $\mathbf{A}^T$ to find $e^{\mathbf{A}^T\tau}$. Following the hint, we find that

$$\begin{aligned} e^{\mathbf{A}^T\tau} &= \left(\mathbf{Q}^{-1}\right)^T e^{\mathbf{D}\tau}\mathbf{Q}^T \\ &= \begin{bmatrix} 4 & -3 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} e^{-\tau} & 0 \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{3}{4} & 1 \end{bmatrix} = \begin{bmatrix} 4\,\mathrm{e}^{-\tau} - 3\,\mathrm{e}^{-2\tau} & 4\,\mathrm{e}^{-\tau} - 4\,\mathrm{e}^{-2\tau} \\ -3\,\mathrm{e}^{-\tau} + 3\,\mathrm{e}^{-2\tau} & -3\,\mathrm{e}^{-\tau} + 4\,\mathrm{e}^{-2\tau} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{W}_c &= \int_0^\infty e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau \\ &= \int_0^\infty \begin{bmatrix} 4\,\mathrm{e}^{-\tau} - 3\,\mathrm{e}^{-2\tau} & 4\,\mathrm{e}^{-\tau} - 4\,\mathrm{e}^{-2\tau} \\ -3\,\mathrm{e}^{-\tau} + 3\,\mathrm{e}^{-2\tau} & -3\,\mathrm{e}^{-\tau} + 4\,\mathrm{e}^{-2\tau} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4\,\mathrm{e}^{-\tau} - 3\,\mathrm{e}^{-2\tau} & -3\,\mathrm{e}^{-\tau} + 3\,\mathrm{e}^{-2\tau} \\ 4\,\mathrm{e}^{-\tau} - 4\,\mathrm{e}^{-2\tau} & -3\,\mathrm{e}^{-\tau} + 4\,\mathrm{e}^{-2\tau} \end{bmatrix} d\tau \\ &= \int_0^\infty \begin{bmatrix} 72\left(e^{-4\tau} - 2e^{-3\tau} + e^{-2\tau}\right) & 24\left(4e^{-4\tau} - 7e^{-3\tau} + 3e^{-2\tau}\right) \\ 24\left(4e^{-4\tau} - 7e^{-3\tau} + 3e^{-2\tau}\right) & 8\left(16e^{-4\tau} - 24e^{-3\tau} + 9e^{-2\tau}\right) \end{bmatrix} d\tau \\ &= \begin{bmatrix} 6\left(1 - 6e^{-2\tau} + 8e^{-3\tau} - 3e^{-4\tau}\right) & 4\left(1 - 9e^{-2\tau} + 14e^{-3\tau} - 6e^{-4\tau}\right) \\ 4\left(1 - 9e^{-2\tau} + 14e^{-3\tau} - 6e^{-4\tau}\right) & 4\left(1 - 9e^{-2\tau} + 16e^{-3\tau} - 8e^{-4\tau}\right) \end{bmatrix}_0^\infty \\ &= \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix} \end{aligned}$$

For the system to be controllable, the controllability Gramian $\mathbf{W}_c$ has to be nonsingular. A square matrix is nonsingular if and only if its determinant is nonzero. So we have to consider the determinant of $\mathbf{W}_c$.

$$det(\mathbf{W}_c) = \begin{vmatrix} 6 & 4 \\ 4 & 4 \end{vmatrix} = 6 \cdot 4 - 4 \cdot 4 = 8 \neq 0$$

The determinant is nonzero, which implies that $\mathbf{W}_c$ is nonsingular. Thus the system is controllable.