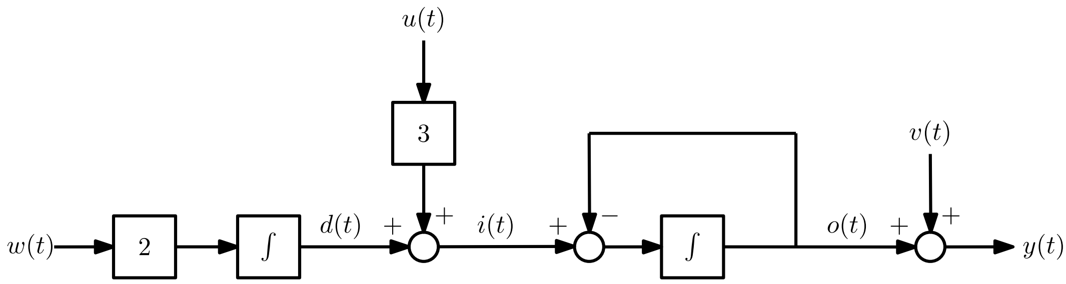
TTK4115 Linear System Theory
Department of Engineering Cybernetics
NTNU

Solution to homework assignment 6

Problem 1: Stationary Kalman filter

a) A block diagram is given by



b) From the transfer function $g(s) = \frac{o(s)}{i(s)} = \frac{1}{s+1}$, it follows that

$$(s+1)o(s) = i(s).$$

By taking the inverse Laplace transform, the following dynamics are obtained:

$$\dot{o}(t) + o(t) = i(t).$$

This can be written as

$$\dot{o}(t) = -o(t) + i(t).$$

Substituting $i(t) = 3u(t) + d(t)$, we get

$$\dot{o}(t) = -o(t) + d(t) + 3u(t).$$

From $d(t) = 2\int_0^t w(\tau)d\tau$, it follows that

$$\dot{d}(t) = 2w(t).$$

Combining these two differential equations and the output equation $y(t) = o(t) + v(t)$, we obtain the following system

$$\begin{bmatrix}\dot{o}(t)\\ \dot{d}(t)\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}o(t)\\ d(t)\end{bmatrix} + \begin{bmatrix}3\\ 0\end{bmatrix}u(t) + \begin{bmatrix}0\\ 2\end{bmatrix}w(t)$$
$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}o(t)\\ d(t)\end{bmatrix} + v(t).$$

This can be written as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t) + Hv(t),\end{aligned}$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} o(t) \\ d(t) \end{bmatrix}$ and matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad H = 1.$$

c) Because the matrix

$$\mathbf{P}(t) = \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix}$$

is time invariant, we have

$$\dot{\mathbf{P}}(t) = \mathbf{0}.$$

Therefore, to show that $\mathbf{P}(t) = \mathbf{P}$ is a solution of the Riccati differential equation, we must show that

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \mathbf{P}\mathbf{C}^T(HRH^T)^{-1}\mathbf{C}\mathbf{P} + \mathbf{G}Q\mathbf{G}^T = \mathbf{0}.$$

By substituting the values of the various matrices, we obtain

$$\begin{aligned}
&\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \mathbf{P}\mathbf{C}^T(HRH^T)^{-1}\mathbf{C}\mathbf{P} + \mathbf{G}Q\mathbf{G}^T \\
&= \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} + \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}(1 \cdot 4 \cdot 1)^{-1}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} 1 \begin{bmatrix} 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 4(2-\sqrt{3}) & 4(\sqrt{3}-1) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 4(2-\sqrt{3}) & 0 \\ 4(\sqrt{3}-1) & 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} 4(\sqrt{3}-1) \\ 4 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 4(\sqrt{3}-1) & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 8(2-\sqrt{3}) - 4(\sqrt{3}-1)^2 & 4(\sqrt{3}-1) - 4(\sqrt{3}-1) \\ 4(\sqrt{3}-1) - 4(\sqrt{3}-1) & -4+4 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Hence, the matrix $\mathbf{P}(t)$ is a solution of the Riccati differential equation.

d) The corresponding Kalman gain is given by

$$\begin{aligned}\mathbf{K}(t) &= \mathbf{P}(t)\mathbf{C}^T(HRH^T)^{-1} \\ &= \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1 \cdot 4 \cdot 1)^{-1} \\ &= \begin{bmatrix} 4(\sqrt{3}-1) \\ 4 \end{bmatrix} \frac{1}{4} \\ &= \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix}.\end{aligned}$$

e) From the definition of the estimation error $\mathbf{e}(t)$ and the differential equation of $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$, it follows that

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t) - (\mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{K}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))) \\ &= \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) + \mathbf{G}w(t) - \mathbf{K}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))) \\ &= \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) + \mathbf{G}w(t) - \mathbf{K}(\mathbf{C}\mathbf{x}(t) + Hv(t) - \mathbf{C}\hat{\mathbf{x}}(t))) \\ &= (\mathbf{A} - \mathbf{K}\mathbf{C})(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) + \mathbf{G}w(t) - \mathbf{K}Hv(t) \\ &= (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e}(t) + \mathbf{G}w(t) - \mathbf{K}Hv(t).\end{aligned}$$

f) The poles of the state estimator are equal to the eigenvalues of the matrix $\mathbf{A}-\mathbf{K}\mathbf{C}$, which is given by

$$\mathbf{A} - \mathbf{K}\mathbf{C} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{3}-1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A} - \mathbf{K}\mathbf{C}$ can be calculated from the characteristic polynomial of $\mathbf{A} - \mathbf{K}\mathbf{C}$, which is given by

$$\begin{aligned}\det(\mathbf{A} - \mathbf{K}\mathbf{C} - \lambda\mathbf{I}) &= \begin{vmatrix} -\sqrt{3}-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + \sqrt{3}\lambda + 1 \\ &= \left(\lambda + \frac{1}{2}\sqrt{3} + \frac{1}{2}i\right)\left(\lambda + \frac{1}{2}\sqrt{3} - \frac{1}{2}i\right).\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_{1,2} = -\frac{1}{2}\sqrt{3} \pm \frac{1}{2}i$. Therefore, the estimator poles are given by $-\frac{1}{2}\sqrt{3} \pm \frac{1}{2}i$.

g) From the answer in e), we have that the dynamics of the estimation error dynamics are perturbed by the disturbances $w(t)$ and $v(t)$:

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e}(t) + \mathbf{G}w(t) - \mathbf{K}Hv(t).$$

We note that the last term in the right-hand side of this equation implies that the contribution of the disturbance $v(t)$ is proportional to the Kalman gain $\mathbf{K}$

(while the contribution of the disturbance $w(t)$ is not). If the covariance of the disturbance $v(t)$ increases by a factor ten, we can expect a larger contribution of $v(t)$ in the error $\mathbf{e}(t)$. To limit the effect of this increase, we should lower the values of the elements in $\mathbf{K}$.

Problem 2: Extended Kalman filter

a) The matrix $\mathbf{C}_0$ as calculated as follows:

$$\mathbf{C}_0 = \left.\frac{dh}{d\mathbf{x}_k}\right|_{\mathbf{x}_k=\hat{\mathbf{x}}_0^-} = \left.\begin{bmatrix}1 & 2x_{2,k}\end{bmatrix}\right|_{\mathbf{x}_k=\hat{\mathbf{x}}_0^-} = \begin{bmatrix}1 & -2\end{bmatrix}.$$

Substituting this in the expression for the Kalman gain K_0 gives

$$\begin{aligned}\mathbf{K}_0 &= \mathbf{P}_0^-\mathbf{C}_0^T(\mathbf{C}_0\mathbf{P}_0^-\mathbf{C}_0^T + R)^{-1}\\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix}\left(\begin{bmatrix}1 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix} + 3\right)^{-1}\\ &= \frac{1}{8}\begin{bmatrix}1\\ -2\end{bmatrix} = \begin{bmatrix}\frac{1}{8}\\ -\frac{1}{4}\end{bmatrix} = \begin{bmatrix}0.125\\ -0.25\end{bmatrix}.\end{aligned}$$

b) The updated state estimate $\hat{\mathbf{x}}_0$ is given by

$$\begin{aligned}\hat{\mathbf{x}}_0 &= \hat{\mathbf{x}}_0^- + \mathbf{K}_0(y_0 - h(\hat{\mathbf{x}}_0^-))\\ &= \begin{bmatrix}0\\ -1\end{bmatrix} + \begin{bmatrix}\frac{1}{8}\\ -\frac{1}{4}\end{bmatrix}(17-1)\\ &= \begin{bmatrix}0\\ -1\end{bmatrix} + \begin{bmatrix}2\\ -4\end{bmatrix} = \begin{bmatrix}2\\ -5\end{bmatrix}.\end{aligned}$$

c) The updated error covariance matrix $\mathbf{P}_0$ is calculated as follows:

$$\begin{aligned}\mathbf{P}_0 &= (\mathbf{I} - \mathbf{K}_0\mathbf{C}_0)\mathbf{P}_0^-(\mathbf{I} - \mathbf{K}_0\mathbf{C}_0)^T + \mathbf{K}_0 R\mathbf{K}_0^T\\ &= \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}\frac{1}{8}\\ -\frac{1}{4}\end{bmatrix}\begin{bmatrix}1 & -2\end{bmatrix}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}\frac{1}{8}\\ -\frac{1}{4}\end{bmatrix}\begin{bmatrix}1 & -2\end{bmatrix}\right)^T\\ &\quad + \begin{bmatrix}\frac{1}{8}\\ -\frac{1}{4}\end{bmatrix}3\begin{bmatrix}\frac{1}{8} & -\frac{1}{4}\end{bmatrix}\\ &= \begin{bmatrix}\frac{7}{8} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{7}{8} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{2}\end{bmatrix} + \begin{bmatrix}\frac{3}{64} & -\frac{3}{32}\\ -\frac{3}{32} & \frac{1}{16}\end{bmatrix} = \begin{bmatrix}\frac{53}{64} & \frac{11}{32}\\ \frac{11}{32} & \frac{5}{16}\end{bmatrix} + \begin{bmatrix}\frac{3}{64} & -\frac{3}{32}\\ -\frac{3}{32} & \frac{3}{16}\end{bmatrix}\\ &= \begin{bmatrix}\frac{7}{8} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}0.875 & 0.25\\ 0.25 & 0.5\end{bmatrix}.\end{aligned}$$

d) The (a priori) state estimate $\hat{\mathbf{x}}_1^-$ is given by

$$\hat{\mathbf{x}}_1^- = \mathbf{f}(\hat{\mathbf{x}}_0, u_0) = \begin{bmatrix}-\frac{1}{2}\cdot 2^2 + 5\\ 5-5\end{bmatrix} = \begin{bmatrix}3\\ 0\end{bmatrix}.$$

Before we calculate the (a priori) covariance matrix $\mathbf{P}_1^-$, we compute the matrix $\mathbf{A}_0$, which is given by

$$\mathbf{A}_0 = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}_k}\right|_{\substack{\mathbf{x}_k=\hat{\mathbf{x}}_0 \\ u_k=u_0}} = \left.\begin{bmatrix} -x_{1,k} & -1 \\ 0 & -1 \end{bmatrix}\right|_{\substack{\mathbf{x}_k=\hat{\mathbf{x}}_0 \\ u_k=u_0}} = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix}.$$

The (a priori) covariance matrix $\mathbf{P}_1^-$ is calculated as

$$\begin{aligned}
\mathbf{P}_1^- &= \mathbf{A}_0 \mathbf{P}_0 \mathbf{A}_0^T + \mathbf{Q} \\
&= \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{7}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 1 \\ 1 & 0.5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 1 & 1.5 \end{bmatrix}.
\end{aligned}$$

e) Using the algorithm in the assignment, we obtain

$$\mathbf{C}_1 = \left.\frac{dh}{d\mathbf{x}_k}\right|_{\mathbf{x}_k=\hat{\mathbf{x}}_1^-} = \left.\begin{bmatrix} 1 & 2x_{2,k} \end{bmatrix}\right|_{\mathbf{x}_k=\hat{\mathbf{x}}_1^-} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

$$\begin{aligned}
\mathbf{K}_1 &= \mathbf{P}_1^- \mathbf{C}_1^T (\mathbf{C}_1 \mathbf{P}_1^- \mathbf{C}_1^T + R)^{-1} \\
&= \begin{bmatrix} 7 & 1 \\ 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \right)^{-1} \\
&= \frac{1}{10} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{10} \\ \frac{1}{10} \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.1 \end{bmatrix}.
\end{aligned}$$

$$\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_1^- + \mathbf{K}_1 (y_1 - h(\hat{\mathbf{x}}_1^-)) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{10} \\ \frac{1}{10} \end{bmatrix} (3 - 3) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

$$\begin{aligned}
\mathbf{P}_1 &= (\mathbf{I} - \mathbf{K}_1 \mathbf{C}_1) \mathbf{P}_1^- (\mathbf{I} - \mathbf{K}_1 \mathbf{C}_1)^T + \mathbf{K}_1 R \mathbf{K}_1^T \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{7}{10} \\ \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} 7 & 1 \\ 1 & \frac{3}{2} \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{7}{10} \\ \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right)^T \\
&\quad + \begin{bmatrix} \frac{7}{10} \\ \frac{1}{10} \end{bmatrix} 3 \begin{bmatrix} \frac{7}{10} & \frac{1}{10} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{10} & 0 \\ -\frac{1}{10} & 1 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{147}{100} & \frac{21}{100} \\ \frac{21}{100} & \frac{3}{100} \end{bmatrix} = \begin{bmatrix} \frac{63}{100} & \frac{9}{100} \\ \frac{9}{100} & \frac{137}{100} \end{bmatrix} + \begin{bmatrix} \frac{147}{100} & \frac{21}{100} \\ \frac{21}{100} & \frac{3}{100} \end{bmatrix} \\
&= \begin{bmatrix} \frac{21}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{14}{10} \end{bmatrix} = \begin{bmatrix} 2.1 & 0.3 \\ 0.3 & 1.4 \end{bmatrix}.
\end{aligned}$$

$$\hat{\mathbf{x}}_2^- = \mathbf{f}(\hat{\mathbf{x}}_1, u_1) = \begin{bmatrix} -\frac{1}{2} \cdot 3^2 \\ 0 - 1 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} -4.5 \\ -1 \end{bmatrix}.$$

$$\mathbf{A}_1 = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}_k}\right|_{\substack{\mathbf{x}_k=\hat{\mathbf{x}}_1 \\ u_k=u_1}} = \left.\begin{bmatrix} -x_{1,k} & -1 \\ 0 & -1 \end{bmatrix}\right|_{\substack{\mathbf{x}_k=\hat{\mathbf{x}}_1 \\ u_k=u_1}} = \begin{bmatrix} -3 & -1 \\ 0 & -1 \end{bmatrix}.$$

$$\begin{aligned}
\mathbf{P}_2^- &= \mathbf{A}_1\mathbf{P}_1\mathbf{A}_1^T + \mathbf{Q} \\
&= \begin{bmatrix} -3 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{21}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{14}{10} \end{bmatrix} \begin{bmatrix} -3 & 0 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{221}{10} & \frac{23}{10} \\ \frac{23}{10} & \frac{14}{10} \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{241}{10} & \frac{23}{10} \\ \frac{23}{10} & \frac{24}{10} \end{bmatrix} = \begin{bmatrix} 24.1 & 2.3 \\ 2.3 & 2.4 \end{bmatrix}.
\end{aligned}$$

$$\mathbf{C}_2 = \left.\frac{dh}{d\mathbf{x}_k}\right|_{\mathbf{x}_k=\hat{\mathbf{x}}_2^-} = \begin{bmatrix} 1 & 2x_{2,k} \end{bmatrix}\Big|_{\mathbf{x}_k=\hat{\mathbf{x}}_2^-} = \begin{bmatrix} 1 & -2 \end{bmatrix}.$$

$$\begin{aligned}
\mathbf{K}_2 &= \mathbf{P}_2^-\mathbf{C}_2^T(\mathbf{C}_2\mathbf{P}_2^-\mathbf{C}_2^T + R)^{-1} \\
&= \begin{bmatrix} \frac{241}{10} & \frac{23}{10} \\ \frac{23}{10} & \frac{24}{10} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \left(\begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{241}{10} & \frac{23}{10} \\ \frac{23}{10} & \frac{24}{10} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 3 \right)^{-1} \\
&= \frac{10}{275} \begin{bmatrix} \frac{195}{10} \\ -\frac{25}{10} \end{bmatrix} = \begin{bmatrix} \frac{39}{55} \\ -\frac{1}{11} \end{bmatrix} \approx \begin{bmatrix} 0.7091 \\ -0.0909 \end{bmatrix}.
\end{aligned}$$

$$\hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_2^- + \mathbf{K}_2(y_2 - h(\hat{\mathbf{x}}_2^-)) = \begin{bmatrix} -\frac{45}{10} \\ -1 \end{bmatrix} + \begin{bmatrix} \frac{39}{55} \\ -\frac{1}{11} \end{bmatrix} \left(2 + \frac{7}{2}\right) = \begin{bmatrix} -\frac{3}{5} \\ -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -0.6 \\ -1.5 \end{bmatrix}.$$

Hence, we get

$$\hat{\mathbf{x}}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{x}}_2^- = \begin{bmatrix} -4.5 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{x}}_2 = \begin{bmatrix} -0.6 \\ -1.5 \end{bmatrix}.$$