TTK4115 Linear System Theory
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# Solution to homework assignment 1

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## Problem 1: State-space equation, transfer function and impulse response

a) Using the differential equation, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} - \dot{u} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -2\dot{y} + 4u \end{bmatrix} = \begin{bmatrix} \dot{y} - u + u \\ -2(\dot{y} - u) + 2u \end{bmatrix} = \begin{bmatrix} x_2 + u \\ -2x_2 + 2u \end{bmatrix}.$$

Therefore, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u.$$

Note that $y = x_1$. Therefore, we have

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Hence, we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u,$$
$$y = \mathbf{C}\mathbf{x} + Du,$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 0 \end{bmatrix}.$$

b) First, we compute $(s\mathbf{I} - \mathbf{A})^{-1}$ as follows:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2 + 2s} \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2+2s} \\ 0 & \frac{1}{s+2} \end{bmatrix}.$$

Using $\hat{g}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$, it follows that

$$\hat{g}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2+2s} \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{s} + \frac{2}{s^2 + 2s} = \frac{s+4}{s^2 + 2s}.$$

c) Applying the Laplace transform to the differential equation while assuming zero initial conditions yields

$$s^2 \hat{y}(s) + 2s\hat{y}(s) = s\hat{u}(s) + 4\hat{u}(s).$$

It follows that

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{s+4}{s^2+2s},$$

which is the same result as obtained in the previous question.

d) To compute the constants $\alpha_1$ and $\alpha_2$, note that

$$\hat{g}(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+2} = \frac{\alpha_1(s+2)}{s(s+2)} + \frac{s\alpha_2}{s(s+2)} = \frac{(\alpha_1+\alpha_2)s + 2\alpha_1}{s^2+2s} = \frac{s+4}{s^2+2s}.$$

From this, we obtain the equations

$$\alpha_1 + \alpha_2 = 1 \qquad \text{and} \qquad 2\alpha_1 = 4.$$

Solving for $\alpha_1$ and $\alpha_2$ yields $\alpha_1 = 2$ and $\alpha_2 = -1$. Therefore, we have

$$\hat{g}(s) = \frac{2}{s} - \frac{1}{s+2}.$$

The corresponding impulse response is given by

$$g(t) = \mathcal{L}^{-1}[\hat{g}(s)] = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{1}{s+2}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = 2 - e^{-2t}.$$

**Problem 2: Solutions of state-space equations**

a) First, $(s\mathbf{I} - \mathbf{A})^{-1}$ can be computed as

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+3 \end{bmatrix}^{-1} = \frac{1}{s^2+3s}\begin{bmatrix} s+3 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2+3s} \\ 0 & \frac{1}{s+3} \end{bmatrix}.$$

Note that

$$\frac{1}{s^2+3s} = \frac{1}{3s} - \frac{1}{3(s+3)}.$$

Therefore, we have

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{3s} - \frac{1}{3(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix}.$$

Taking the inverse Laplace transform leads to

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \begin{bmatrix} \mathcal{L}^{-1}\left[\frac{1}{s}\right] & \mathcal{L}^{-1}\left[\frac{1}{3s} - \frac{1}{3(s+3)}\right] \\ 0 & \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}.$$

b) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda - 3 \end{vmatrix} = \lambda^2 + 3\lambda = \lambda(\lambda+3).$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -3$. The corresponding eigenvectors $\mathbf{q}_i$ can be obtained from the kernel of the matrix $(\lambda_i \mathbf{I} - \mathbf{A})$ for $i = 1, 2$:

$$\ker(\lambda_1 \mathbf{I} - \mathbf{A}) = \ker\left(\begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \implies \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\ker(\lambda_2 \mathbf{I} - \mathbf{A}) = \ker\left(\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

c) The matrices $\hat{\mathbf{A}}$ and $\mathbf{Q}$ are given by

$$\hat{\mathbf{A}} = \operatorname{diag}\{\lambda_1, \lambda_2\} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2] = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}.$$

Note that

$$\mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} = \mathbf{A}.$$

d) The matrix $e^{\mathbf{A}t}$ can be computed as follows

$$e^{\mathbf{A}t} = \mathbf{Q} e^{\hat{\mathbf{A}}t} \mathbf{Q}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}.$$

We note that this the same result as obtained in a).

e) The output $y(t)$ is given by

$$y(t) = \mathbf{C} e^{\mathbf{A}t} \mathbf{x}(0) + \mathbf{C} \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B} u(\tau) d\tau + \mathbf{D} u(t).$$

Substitution yields

$$\begin{aligned} y(t) &= \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 3 & 0 \end{bmatrix} \int_0^t \begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3(t-\tau)} \\ 0 & e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} d\tau + 1 \\ &= \begin{bmatrix} 3 & 1 - e^{-3t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 3 & 0 \end{bmatrix} \int_0^t \begin{bmatrix} -2e^{-3(t-\tau)} \\ 6e^{-3(t-\tau)} \end{bmatrix} d\tau + 1 \\ &= 3x_1(0) + (1 - e^{-3t}) x_2(0) + \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3}e^{-3(t-\tau)} - \frac{2}{3} \\ 2 - 2e^{-3(t-\tau)} \end{bmatrix} + 1 \\ &= 3x_1(0) + (1 - e^{-3t}) x_2(0) + 2e^{-3t} - 1. \end{aligned}$$

f) From the solution of question d), we have

$$\begin{aligned} y(1) &= 3x_1(0) + (1 - e^{-3}) x_2(0) + 2e^{-3} - 1, \\ y(2) &= 3x_1(0) + (1 - e^{-6}) x_2(0) + 2e^{-6} - 1. \end{aligned}$$

By substituting $y(1) = y(2) = 4$, we obtain

$$3x_1(0) + (1 - e^{-3})x_2(0) = 5 - 2e^{-3},$$
$$3x_1(0) + (1 - e^{-6})x_2(0) = 5 - 2e^{-6}.$$

Solving for $x_1(0)$ and $x_2(0)$ yields $x_1(0) = 1$ and $x_2(0) = 2$, or equivalently $\mathbf{x}(0) = [1, 2]^T$.

**Problem 3: Linearization and discretization**

a) Using the equation of motion, we have

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = \begin{bmatrix} \dot{r} \\ \ddot{r} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ -\frac{2u_1}{\sqrt{r^2+9}}\dot{r} - 3r - 4 + \frac{r}{\sqrt{r^2+9}}u_1 + u_2 \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} x_2 \\ -\frac{2x_2u_1}{\sqrt{x_1^2+9}} - 3x_1 - 4 + \frac{x_1u_1}{\sqrt{x_1^2+9}} + u_2 \end{bmatrix}}_{\mathbf{h}(\mathbf{x},\mathbf{u})}.$$

From $y = 5\sqrt{r^2 + 9}$, it follows that

$$y = \underbrace{5\sqrt{x_1^2 + 9}}_{f(\mathbf{x},\mathbf{u})}.$$

b) The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are computed as follows

$$\mathbf{A} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0) = \left.\begin{bmatrix} 0 & 1 \\ \frac{2x_1x_2u_1}{(x_1^2+9)^{\frac{3}{2}}} - 3 + \frac{u_1}{\sqrt{x_1^2+9}} - \frac{x_1^2u_1}{(x_1^2+9)^{\frac{3}{2}}} & \frac{-2u_1}{\sqrt{x_1^2+9}} \end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ \mathbf{u}=\mathbf{u}_0}}$$
$$= \begin{bmatrix} 0 & 1 \\ \frac{80}{125} - 3 + \frac{5}{5} - \frac{80}{125} & \frac{-10}{5} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix},$$

$$\mathbf{B} = \frac{\partial \mathbf{h}}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0) = \left.\begin{bmatrix} 0 & 0 \\ \frac{-2x_2}{\sqrt{x_1^2+9}} + \frac{x_1}{\sqrt{x_1^2+9}} & 1 \end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ \mathbf{u}=\mathbf{u}_0}} = \begin{bmatrix} 0 & 0 \\ \frac{-4}{5} + \frac{4}{5} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{C} = \frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0) = \left.\begin{bmatrix} \frac{5x_1}{\sqrt{x_1^2+9}} & 0 \end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ \mathbf{u}=\mathbf{u}_0}} = \begin{bmatrix} \frac{20}{5} & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \end{bmatrix},$$

$$\mathbf{D} = \frac{\partial f}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0) = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

c) The discretized linearized system is given by

$$\begin{aligned}\bar{\mathbf{x}}[k+1] &= \mathbf{A}_d\bar{\mathbf{x}}[k] + \mathbf{B}_d\bar{\mathbf{u}}[k],\\ \bar{y}[k] &= \mathbf{C}_d\bar{\mathbf{x}}[k] + \mathbf{D}_d\bar{\mathbf{u}}[k].\end{aligned}$$

The corresponding matrices $\mathbf{A}_d$, $\mathbf{B}_d$, $\mathbf{C}_d$ and $\mathbf{D}_d$ are calculated next.

To obtain the matrix $\mathbf{A}_d = e^{\mathbf{A}T}$, we compute $e^{\mathbf{A}t}$. To compute $e^{\mathbf{A}t}$, we first determine the matrices $\hat{\mathbf{A}}$ and $\mathbf{Q}$, such that $\mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1}$, where $\hat{\mathbf{A}}$ is in Jordan form. The characteristic polynomial of $\mathbf{A}$ is given by

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = \begin{vmatrix} -\lambda & 1 \\ -2 & -\lambda - 2 \end{vmatrix} = \lambda^2 + 2\lambda + 2 = (\lambda + 1 - j)(\lambda + 1 + j).$$

From this, it is easy to see that the roots of the characteristic polynomial of $\mathbf{A}$, and therefore the eigenvalues of $\mathbf{A}$, are given by $\lambda_1 = -1 + j$ and $\lambda_1 = -1 - j$. The corresponding eigenvectors $\mathbf{q}_i$ can be obtained from the kernel of the matrix $(\lambda_i\mathbf{I} - \mathbf{A})$ for $i = 1, 2$:

$$\ker\left(\lambda_1\mathbf{I} - \mathbf{A}\right) = \ker\left(\begin{bmatrix} 1-j & 1 \\ -2 & -1-j \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 2 & 1+j \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q}_1 = \begin{bmatrix} -1-j \\ 2 \end{bmatrix},$$

$$\ker\left(\lambda_2\mathbf{I} - \mathbf{A}\right) = \ker\left(\begin{bmatrix} 1+j & 1 \\ -2 & -1+j \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 2 & 1-j \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q}_2 = \begin{bmatrix} -1+j \\ 2 \end{bmatrix}.$$

Therefore, $\hat{\mathbf{A}}$ and $\mathbf{Q}$ are given by

$$\hat{\mathbf{A}} = \begin{bmatrix} -1+j & 0 \\ 0 & -1-j \end{bmatrix}, \qquad \text{and} \qquad \mathbf{Q} = \begin{bmatrix} -1-j & -1+j \\ 2 & 2 \end{bmatrix}.$$

We note that $\mathbf{Q}$ is not unique. Next, we compute $e^{\mathbf{A}t} = \mathbf{Q}e^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1}$, with

$$e^{\hat{\mathbf{A}}t} = \begin{bmatrix} e^{(-1+j)t} & 0 \\ 0 & e^{(-1-j)t} \end{bmatrix} = \begin{bmatrix} e^{-t}(\cos(t) + j\sin(t)) & 0 \\ 0 & e^{-t}(\cos(t) - j\sin(t)) \end{bmatrix},$$

where we used $e^{(-1+j)t} = e^{-t}e^{jt}$ and $e^{(-1-j)t} = e^{-t}e^{-jt}$, with Euler's formula $e^{jt} = \cos(t) + j\sin(t)$. We obtain,

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{Q}e^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1} \\ &= \begin{bmatrix} -1-j & -1+j \\ 2 & 2 \end{bmatrix} \begin{bmatrix} e^{-t}(\cos(t) + j\sin(t)) & 0 \\ 0 & e^{-t}(\cos(t) - j\sin(t)) \end{bmatrix} \frac{1}{4} \begin{bmatrix} 2j & 1+j \\ -2j & 1-j \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}(\cos(t) + \sin(t)) & e^{-t}\sin(t) \\ -2e^{-t}\sin(t) & e^{-t}(\cos(t) - \sin(t)) \end{bmatrix}. \end{aligned}$$

Substituting $t = T = \frac{\pi}{2}$ in $e^{\mathbf{A}t}$, we have

$$\begin{aligned}
\mathbf{A}_d = e^{\mathbf{A}T} &= \begin{bmatrix} e^{-T}(\cos(T)+\sin(T)) & e^{-T}\sin(T) \\ -2e^{-T}\sin(T) & e^{-T}(\cos(T)-\sin(T)) \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{\pi}{2}}(\cos(\frac{\pi}{2})+\sin(\frac{\pi}{2})) & e^{-\frac{\pi}{2}}\sin(\frac{\pi}{2}) \\ -2e^{-\frac{\pi}{2}}\sin(\frac{\pi}{2}) & e^{-\frac{\pi}{2}}(\cos(\frac{\pi}{2})-\sin(\frac{\pi}{2})) \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{\pi}{2}} & e^{-\frac{\pi}{2}} \\ -2e^{-\frac{\pi}{2}} & -e^{-\frac{\pi}{2}} \end{bmatrix} \approx \begin{bmatrix} 0.2079 & 0.2079 \\ -0.4158 & -0.2079 \end{bmatrix}.
\end{aligned}$$

Because $\mathbf{A}$ is nonsingular, it follows that

$$\begin{aligned}
\mathbf{B}_d = \left(\int_0^T e^{\mathbf{A}\tau} d\tau\right)\mathbf{B} = \mathbf{A}^{-1}(\mathbf{A}_d - \mathbf{I})\mathbf{B} &= \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{\pi}{2}} - 1 & e^{-\frac{\pi}{2}} \\ -2e^{-\frac{\pi}{2}} & -e^{-\frac{\pi}{2}} - 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{2} - \frac{1}{2}e^{-\frac{\pi}{2}} \\ 0 & e^{-\frac{\pi}{2}} \end{bmatrix} \approx \begin{bmatrix} 0 & 0.4991 \\ 0 & 0.0019 \end{bmatrix}.
\end{aligned}$$

The matrices $\mathbf{C}_d$ and $\mathbf{D}_d$ are given by

$$\mathbf{C}_d = \mathbf{C} = \begin{bmatrix} 4 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D}_d = \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$