TTK4115 Linear System Theory

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Solution to homework assignment 4

Problem 1: Output-feedback controllers

a) Substituting the output equation $y(t) = \mathbf{C}\mathbf{x}(t)$ in the equation for the controller $u(t) = -k_p y(t)$, we obtain

$$u(t) = -k_p \mathbf{C}\mathbf{x}(t).$$

Substituting this in the equation for the system dynamics yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}k_p\mathbf{C}\mathbf{x}(t) = (\mathbf{A} - \mathbf{B}k_p\mathbf{C})\mathbf{x}(t).$$

Hence, we have

$$\dot{\mathbf{x}}(t) = \mathbf{A}_p\mathbf{x}(t),$$

with

$$\mathbf{A}_p = \mathbf{A} - \mathbf{B}k_p\mathbf{C} = \begin{bmatrix} 4 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} k_p \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 - 2k_p & -2 \end{bmatrix}.$$

b) The eigenvalues of $\mathbf{A}_p$ can be calculated from the roots of the characteristic polynomial of $\mathbf{A}_p$:

$$\begin{aligned} \det(\mathbf{A}_p - \lambda\mathbf{I}) = \begin{vmatrix} 4 - \lambda & 3 \\ -1 - 2k_p & -2 - \lambda \end{vmatrix} &= \lambda^2 - 2\lambda + 6k_p - 5 \\ &= (1 + \sqrt{6 - 6k_p} - \lambda)(1 - \sqrt{6 - 6k_p} - \lambda). \end{aligned}$$

Hence, we obtain the eigenvalues

$$\lambda_{1,2} = 1 \pm \sqrt{6 - 6k_p}.$$

From this, we conclude that there exists no value of k_p such that both eigenvalues of $\mathbf{A}_p$ have a negative real part. Therefore, there exists no value of k_p such that the closed-loop system is asymptotically stable.

c) First, we compute the time-derivative of the output:

$$\dot{y}(t) = \mathbf{C}\dot{\mathbf{x}}(t) = \mathbf{C}\mathbf{A}\mathbf{x}(t) + \mathbf{C}\mathbf{B}u(t).$$

We substitute this equation and the output equation $y(t) = \mathbf{C}\mathbf{x}(t)$ in the expression for the PD-controller:

$$u(t) = -k_p\mathbf{C}\mathbf{x}(t) - k_d\mathbf{C}\mathbf{A}\mathbf{x}(t) - k_d\mathbf{C}\mathbf{B}u(t).$$

From this, it follows that

$$(1 + k_d\mathbf{CB})u(t) = -(k_p\mathbf{C} + k_d\mathbf{CA})\mathbf{x}(t),$$

which implies that

$$u(t) = -\frac{k_p\mathbf{C} + k_d\mathbf{CA}}{1 + k_d\mathbf{CB}}\mathbf{x}(t).$$

By substituting this in the equation for the system dynamics, we obtain

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}(k_p\mathbf{C} + k_d\mathbf{CA})\mathbf{x}(t) = \left(\mathbf{A} - \mathbf{B}\frac{k_p\mathbf{C} + k_d\mathbf{CA}}{1 + k_d\mathbf{CB}}\right)\mathbf{x}(t).$$

Hence, we have

$$\dot{\mathbf{x}}(t) = \mathbf{A}_{pd}\mathbf{x}(t),$$

with

$$\begin{aligned}
\mathbf{A}_{pd} &= \mathbf{A} - \mathbf{B}\frac{k_p\mathbf{C} + k_d\mathbf{CA}}{1 + k_d\mathbf{CB}} \\
&= \begin{bmatrix} 4 & 3 \\ -1 & -2 \end{bmatrix} - \frac{1}{1 + k_d\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\left(k_p\begin{bmatrix} 1 & 0 \end{bmatrix} + k_d\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & -2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 4 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 2k_p + 8k_d & 4k_d \end{bmatrix} \\
&= \begin{bmatrix} 4 & 3 \\ -1 - 2k_p - 8k_d & -2 - 6k_d \end{bmatrix}.
\end{aligned}$$

d) The characteristic polynomial of $\mathbf{A}_{pd}$ is given by

$$\det(\mathbf{A}_{pd} - \lambda\mathbf{I}) = \begin{vmatrix} 4 - \lambda & 3 \\ -1 - 2k_p - 8k_d & -2 - 6k_d - \lambda \end{vmatrix} = \lambda^2 + (6k_d - 2)\lambda + 6k_p - 5.$$

If the eigenvalues of $\mathbf{A}_{pd}$ are given by $\lambda_{1,2} = -1 \pm i$, the characteristic polynomial of $\mathbf{A}_{pd}$ is given by

$$\det(\mathbf{A}_{pd} - \lambda\mathbf{I}) = (-1 + i - \lambda)(-1 - i - \lambda) = \lambda^2 + 2\lambda + 2.$$

By comparing both expression for the characteristic polynomial of $\mathbf{A}_{pd}$, we obtain the equations

$$\begin{aligned} 6k_d - 2 &= 2, \\ 6k_p - 5 &= 2. \end{aligned}$$

Solving for k_p and k_d yields $k_p = \frac{7}{6}$ and $k_d = \frac{2}{3}$.

Problem 2: Separation principle

a) By combining the equations $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and $\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$, we obtain

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t).$$

From $\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$, it follows that

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}(\mathbf{e}(t) + \mathbf{x}(t)) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{e}(t).$$

Taking the time derivative of $\mathbf{e}(t)$ yields

$$\dot{\mathbf{e}}(t) = \dot{\hat{\mathbf{x}}}(t) - \dot{\mathbf{x}}(t).$$

By substituting the equations $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and $\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t))$, we get

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t)) - \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{u}(t) \\ &= \mathbf{A}\mathbf{e}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t)).\end{aligned}$$

By combining this with $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$, we obtain

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}\mathbf{e}(t) + \mathbf{L}(\mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t)) \\ &= \mathbf{A}\mathbf{e}(t) + \mathbf{L}\mathbf{C}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).\end{aligned}$$

From $\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{e}(t)$ and $\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t)$, we get

$$\begin{bmatrix}\dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} - \mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t) \\ \mathbf{e}(t)\end{bmatrix}.$$

b) The characteristic polynomial of the matrix $\mathbf{H}$ is given by

$$\begin{aligned}\det(\mathbf{H} - \lambda\mathbf{I}) &= \det\left(\begin{bmatrix}\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I} & -\mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}\end{bmatrix}\right) \\ &= \det(\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I})\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}).\end{aligned}$$

For any eigenvalue λ of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$, we have $\det(\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I}) = 0$ or $\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}) = 0$. This implies that $\det(\mathbf{H} - \lambda\mathbf{I}) = \det(\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I})\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}) = 0$. Because $\det(\mathbf{H} - \lambda\mathbf{I})$ is zero, λ must be an eigenvalue of $\mathbf{H}$. Hence, any eigenvalue of the matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$ or the matrix $\mathbf{A} - \mathbf{L}\mathbf{C}$ is an eigenvalue of the matrix $\mathbf{H}$. Moreover, because $\det(\mathbf{H} - \lambda\mathbf{I})$ is only zero if λ is an eigenvalue of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$, we have that all eigenvalues of $\mathbf{H}$ are eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$. Hence, the eigenvalues of $\mathbf{H}$ are the union of the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$.

c) If the system is controllable, then the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ can be assigned arbitrarily by choosing $\mathbf{K}$. Moreover, if the system is observable, then the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ can be assigned arbitrarily by choosing $\mathbf{L}$. Because the poles of the closed-loop system (i.e. the eigenvalues of $\mathbf{H}$) are the union of the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$, we conclude that the poles of the closed-loop system can be assigned arbitrarily.

Problem 3: Process classification

a) The probability density function of the variable Φ is given by

$$f_\Phi(\phi) = \begin{cases} \frac{1}{2\pi}, & \text{if } -\pi \leq \phi < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

The mean $\mu_X(t) = E[X(t)]$ is calculated as follows:

$$\begin{aligned}
\mu_X(t) &= E[X(t)] = E[a\sin(\omega t + \Phi)] = aE[\sin(\omega t + \Phi)] \\
&= a\int_{-\infty}^{\infty} \sin(\omega t + \phi) f_\Phi(\phi) d\phi = \frac{a}{2\pi}\int_{-\pi}^{\pi} \sin(\omega t + \phi) d\phi \\
&= \frac{a}{2\pi}\left[-\cos(\omega t + \phi)\right]_{-\pi}^{\pi} = \frac{a}{2\pi}\left[-\cos(\omega t + \pi) + \cos(\omega t + -\pi)\right] \\
&= \frac{a}{2\pi}\left[\cos(\omega t) - \cos(\omega t)\right] = 0.
\end{aligned}$$

b) The variance $\sigma_X^2(t) = E[X^2(t)]$ is given by

$$\begin{aligned}
\sigma_X^2(t) &= E[X^2(t)] = E[(a\sin(\omega t + \Phi))^2] = a^2 E[\sin^2(\omega t + \Phi)] \\
&= a^2 E\left[\frac{1 - \cos(2\omega t + 2\Phi)}{2}\right] = \frac{a^2}{2}\left(1 - E\left[\cos(2\omega t + 2\Phi)\right]\right) \\
&= \frac{a^2}{2}\left(1 - \int_{-\infty}^{\infty} \cos(2\omega t + 2\phi) f_\Phi(\phi) d\phi\right) \\
&= \frac{a^2}{2}\left(1 - \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\omega t + 2\phi) d\phi\right) = \frac{a^2}{2}\left(1 - \frac{1}{2\pi}\left[\frac{\sin(2\omega t + 2\phi)}{2}\right]_{-\pi}^{\pi}\right) \\
&= \frac{a^2}{2}\left(1 - \frac{1}{4\pi}\left[\sin(2\omega t + 2\pi) - \sin(2\omega t - 2\pi)\right]\right) \\
&= \frac{a^2}{2}\left(1 - \frac{1}{4\pi}\left[\sin(2\omega t) - \sin(2\omega t)\right]\right) = \frac{a^2}{2},
\end{aligned}$$

where we used the probability density function f_Φ in a).

c) Using the probability density function f_Φ in a), we obtain the following autocorrelation function $R_X(t_1,t_2) = E[X(t_1)X(t_2)]$:

$$\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)X(t_2)] = E[(a\sin(\omega t_1+\Phi))(a\sin(\omega t_2+\Phi))] \\
&= a^2 E[\sin(\omega t_1+\Phi)\sin(\omega t_2+\Phi)] \\
&= a^2 E\left[\frac{1}{2}\cos(\omega t_1+\Phi-(\omega t_2+\Phi)) - \frac{1}{2}\cos(\omega t_1+\Phi+(\omega t_2+\Phi))\right] \\
&= \frac{a^2}{2}E\left[\cos(\omega(t_1-t_2)) - \cos(\omega(t_1+t_2)+2\Phi)\right] \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - E[\cos(\omega(t_1+t_2)+2\Phi)]\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \int_{-\infty}^{\infty}\cos(\omega(t_1+t_2)+2\phi)f_\Phi(\phi)d\phi\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega(t_1+t_2)+2\phi)d\phi\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{2\pi}\left[\frac{\sin(\omega(t_1+t_2)+2\phi)}{2}\right]_{-\pi}^{\pi}\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{4\pi}\left[\sin(\omega(t_1+t_2)+2\pi) - \sin(\omega(t_1+t_2)-2\pi)\right]\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{4\pi}\left[\sin(\omega(t_1+t_2)) - \sin(\omega(t_1+t_2))\right]\right) \\
&= \frac{a^2}{2}\cos(\omega(t_1-t_2)).
\end{aligned}$$

Substituting $t_1 = t$ and $t_2 = t+\tau$, we get

$$R_X(\tau) = E[X(t)X(t+\tau)] = \frac{a^2}{2}\cos(\omega(t-(t+\tau))) = \frac{a^2}{2}\cos(-\omega\tau) = \frac{a^2}{2}\cos(\omega\tau).$$

d) The process is a deterministic random process. With $\Phi = \Phi_1$ the process becomes $X(t,\Phi_1) = a\sin(\omega t + \Phi_1)$. Knowledge about the process for $t \leq t_0$ makes identification of Φ_1, ω and A possible, and the process is uniquely defined $\forall t > t_0$.

e) Because the mean $\mu_X(t)$ is not dependent on the time origin (i.e. $\mu_X(t)$ is independent of t, see a)) and the autocorrelation function $R_X(t_1,t_2)$ in c) is only dependent on the time difference between sample points (i.e. $R_X(t_1,t_2)$ is dependent only on the time difference $t_2 - t_1$, since we can write $R_X(t_1,t_2) = R_X(\tau)$ for $t_1 = t$ and $t_2 = t+\tau$, see c)), the process is wide-sense stationary. In fact, it can be shown that all density functions associated with the process are independent of time, which implies that the process is stationary, which is a stronger property than wide-sense stationarity.

f) While ergodicity applies to all density functions associated with the process, ergodicity in wide sense only applies to the mean and autocorrelation function of the process. For a process to be ergodic in wide sense, the time mean and the time autocorrelation function must be equivalent to the ensemble mean (i.e. μ_X) and the ensemble autocorrelation function (i.e. $R_X(\tau)$), respectively.

The time mean is given by

$$\begin{aligned}
\mathfrak{m}_X &= \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T a\sin(\omega t + \Phi)dt \\
&= \lim_{T\to\infty} \frac{a}{T} \left[\frac{-\cos(\omega t + \Phi)}{\omega} \right]_0^T = \lim_{T\to\infty} \frac{a}{\omega T} \left[-\cos(\omega T + \Phi) + \cos(\Phi) \right] = 0.
\end{aligned}$$

The time autocorrelation function is given by

$$\begin{aligned}
\mathfrak{R}_X(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)X(t+\tau)dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T (a\sin(\omega t + \Phi))(a\sin(\omega(t+\tau) + \Phi))dt \\
&= \lim_{T\to\infty} \frac{a^2}{T} \int_0^T \sin(\omega t + \Phi)\sin(\omega(t+\tau) + \Phi)dt \\
&= \lim_{T\to\infty} \frac{a^2}{T} \int_0^T \bigg(\frac{1}{2}\cos(\omega t + \Phi - (\omega(t+\tau) + \Phi)) \\
&\qquad - \frac{1}{2}\cos(\omega t + \Phi + \omega(t+\tau) + \Phi) \bigg) dt \\
&= \lim_{T\to\infty} \frac{a^2}{2T} \int_0^T \left(\cos(-\omega\tau) - \cos(2\omega t + \omega\tau + 2\Phi) \right) dt \\
&= \lim_{T\to\infty} \frac{a^2}{2T} \left[\cos(\omega\tau)t - \frac{\sin(2\omega t + \omega\tau + 2\Phi)}{2\omega} \right]_0^T \\
&= \lim_{T\to\infty} \frac{a^2}{2T} \left[\cos(\omega\tau)T - \frac{\sin(2\omega T + \omega\tau + 2\Phi)}{2\omega} + \frac{\sin(\omega\tau + 2\Phi)}{2\omega} \right] \\
&= \frac{a^2}{2}\cos(\omega\tau).
\end{aligned}$$

Because the time mean $\mathfrak{m}_X$ and time autocorrelation function $\mathfrak{R}_X(\tau)$ are equal to the ensemble mean μ_X in a) and the ensemble autocorrelation function $R_X(\tau)$ in c), respectively, we conclude that the process is ergodic in wide sense. In fact, it can be shown that process is ergodic (not only in wide sense).