TTK4115 Linear System Theory
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Solution to homework assignment 5

Problem 1: Linear system with white noise

a) White noise processes have a zero mean. Because the disturbance $w(t)$ is a white noise process, we have $\mu_w = 0$.

The reasoning behind this follows next. Let $v(t)$ be a white noise process. By definition, white noise has a flat spectrum. Therefore, the power spectrum density function associated with $v(t)$ is given by $S_v(j\omega) = \alpha_v$, where α_v is a nonnegative constant. Using the inverse Fourier transform, we obtain the corresponding autocorrelation function

$$R_v(\tau) = \mathcal{F}^{-1}\{S_v(j\omega)\} = \alpha_v \delta(\tau),$$

where $\delta(\tau)$ is the Dirac delta function. We can define the zero-mean white-noise process $\bar{v}(t) = v(t) - \mu_v$, where $\mu_v = E[v(t)]$ is the mean of $v(t)$. Note that because $\bar{v}(t)$ is a white noise process, we have $S_{\bar{v}}(j\omega) = \alpha_{\bar{v}}$ for some nonnegative constant $\alpha_{\bar{v}}$. Similar as for $v(t)$, the autocorrelation function associated with $\bar{v}(t)$ is given by

$$R_{\bar{v}}(\tau) = \mathcal{F}^{-1}\{S_{\bar{v}}(j\omega)\} = \alpha_{\bar{v}} \delta(\tau).$$

Now, note that from the definition of the autocorrelation function, it follows that

$$\begin{aligned}
R_{\bar{v}}(\tau) &= E[\bar{v}(t)\bar{v}(t+\tau)] = E[(v(t) - \mu_v)(v(t+\tau) - \mu_v)] \\
&= E[v(t)v(t+\tau) - \mu_v v(t) - \mu_v v(t+\tau) + \mu_v^2] \\
&= E[v(t)v(t+\tau)] - \mu_v E[v(t)] - \mu_v E[v(t+\tau)] + \mu_v^2 \\
&= R_v(\tau) - \mu_v^2 - \mu_v^2 + \mu_v^2 = R_v(\tau) - \mu_v^2.
\end{aligned}$$

Substituting $R_v(\tau) = \alpha_v \delta(\tau)$ and $R_{\bar{v}}(\tau) = \alpha_{\bar{v}} \delta(\tau)$, we obtain

$$\alpha_{\bar{v}} \delta(\tau) = \alpha_v \delta(\tau) - \mu_v^2.$$

This is only valid for all τ if $\alpha_{\bar{v}} = \alpha_v$ and $\mu_v = 0$. Because the mean μ_v of $v(t)$ is equal to zero and $v(t)$ is an arbirtrary white noise process, we conclude that all white noise processes must have a zero mean.

b) The variance σ_w^2 can directly be obtained from the autocorrelation function $R_w(\tau)$:

$$\sigma_w^2 = E[w^2(t)] = R_w(0) = 4\delta(0) = \infty.$$

c) The power spectral density function $S_w(j\omega)$ of the disturbance $w(t)$ is obtained by taking the Fourier transform of the autocorrelation function $R_w(\tau)$:

$$S_w(j\omega) = \mathcal{F}\{R_w(\tau)\} = \mathcal{F}\{4\delta(\tau)\} = 4\mathcal{F}\{\delta(\tau)\} = 4.$$

d) The transfer function $\hat{g}(s) = \frac{\hat{y}(s)}{\hat{w}(s)}$ can be obtained from $\hat{g}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, where $\mathbf{I}$ is the identity matrix. Hence, we get

$$\begin{aligned}\hat{g}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2+6s+8} \begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{s+8}{s^2+6s+8}.\end{aligned}$$

e) The poles of the system are equal to the roots of the denominator polynomial of the transfer function $\hat{g}(s)$ (i.e. the roots of $s^2 + 6s + 8$) and are given by $\lambda_1 = -4$ and $\lambda_2 = -2$. Given that $\hat{g}(s) = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2}$, we obtain

$$\begin{aligned}\hat{g}(s) &= \frac{\alpha_1}{s+4} + \frac{\alpha_2}{s+2} = \frac{\alpha_1(s+2)}{(s+2)(s+4)} + \frac{\alpha_2(s+4)}{(s+2)(s+4)} \\ &= \frac{(\alpha_1+\alpha_2)s + 2\alpha_1 + 4\alpha_2}{s^2+6s+8} = \frac{s+8}{s^2+6s+8}.\end{aligned}$$

From this, we conclude that

$$\alpha_1 + \alpha_2 = 1 \qquad \text{and} \qquad 2\alpha_1 + 4\alpha_2 = 8.$$

Solving for α_1 and α_2 yields $\alpha_1 = -2$ and $\alpha_2 = 3$. Hence, the transfer function $g(s)$ can be written as

$$\hat{g}(s) = \frac{-2}{s+4} + \frac{3}{s+2}.$$

By taking the inverse Laplace transform of the transfer function $g(s)$, we obtain the impulse response $g(t)$, which is given by

$$\begin{aligned}g(t) &= \mathcal{L}^{-1}\{\hat{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{-2}{s+4} + \frac{3}{s+2}\right\} \\ &= -2\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = -2e^{-4t} + 3e^{-2t}.\end{aligned}$$

f) Using $y(t) = \int_0^t g(\tau)w(t-\tau)d\tau$, the mean $\mu_y(t)$ is calculated as follows:

$$\begin{aligned}\mu_y(t) &= E[y(t)] = E\left[\int_0^t g(\tau)w(t-\tau)d\tau\right] = \int_0^t g(\tau)E[w(t-\tau)]d\tau \\ &= \int_0^t g(\tau)\mu_w d\tau = \mu_w \int_0^t g(\tau)d\tau = \mu_w \int_0^t (-2e^{-4\tau} + 3e^{-2\tau})d\tau \\ &= \mu_w \left[\frac{1}{2}e^{-4\tau} - \frac{3}{2}e^{-2\tau}\right]_0^t = \mu_w \left(\frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t} - \frac{1}{2} + \frac{3}{2}\right) \\ &= \mu_w \left(\frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t} + 1\right).\end{aligned}$$

The stationary mean $\bar{\mu}_y$ is given by

$$\bar{\mu}_y = \lim_{t\to\infty} \mu_y(t) = \lim_{t\to\infty} \mu_w \left(\frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t} + 1 \right) = \mu_w.$$

From a), we have $\mu_w = 0$. Hence, we obtain $\bar{\mu}_y = \mu_w = 0$.

g) Note that the variance $\sigma_y^2(t)$ is equal to the mean-square value of $y(t)$, i.e. $\sigma_y^2(t) = E[y^2(t)]$. It follows that

$$\begin{aligned}
\sigma_y^2(t) = E[y^2(t)] &= E\left[\int_0^t g(\tau_1)w(t-\tau_1)d\tau_1 \int_0^t g(\tau_2)w(t-\tau_2)d\tau_2 \right] \\
&= E\left[\int_0^t g(\tau_2) \int_0^t g(\tau_1)w(t-\tau_1)w(t-\tau_2)d\tau_1 d\tau_2 \right] \\
&= \int_0^t g(\tau_2) \int_0^t g(\tau_1)E\left[w(t-\tau_1)w(t-\tau_2)\right] d\tau_1 d\tau_2 \\
&= \int_0^t g(\tau_2) \int_0^t g(\tau_1)R_w(\tau_2-\tau_1)d\tau_1 d\tau_2 \\
&= 4\int_0^t g(\tau_2) \int_0^t g(\tau_1)\delta(\tau_2-\tau_1)d\tau_1 d\tau_2 \\
&= 4\int_0^t g(\tau_2)g(\tau_2)d\tau_2 = 4\int_0^t g^2(\tau_2)d\tau_2 \\
&= 4\int_0^t (-2e^{-4\tau_2} + 3e^{-2\tau_2})^2 d\tau_2 = 4\int_0^t (4e^{-8\tau_2} - 12e^{-6\tau_2} + 9e^{-4\tau_2})d\tau_2 \\
&= 4\left[-\frac{1}{2}e^{-8\tau_2} + 2e^{-6\tau_2} - \frac{9}{4}e^{-4\tau_2} \right]_0^t \\
&= 4\left(-\frac{1}{2}e^{-8t} + 2e^{-6t} - \frac{9}{4}e^{-4t} + \frac{1}{2} - 2 + \frac{9}{4} \right) \\
&= -2e^{-8t} + 8e^{-6t} - 9e^{-4t} + 3.
\end{aligned}$$

The stationary variance $\bar{\sigma}_y^2$ is given by

$$\bar{\sigma}_y^2 = \lim_{t\to\infty} \sigma_y^2(t) = \lim_{t\to\infty} \left(-2e^{-8t} + 8e^{-6t} - 9e^{-4t} + 3 \right) = 3.$$

h) The power spectral density function $S_y(j\omega)$ of the output $y(t)$ is given by

$$S_y(j\omega) = |g(j\omega)|^2 S_w(j\omega) = g(j\omega)g(-j\omega)S_w(j\omega).$$

From c), we have that $S_w(j\omega) = 4$. In addition, using the transfer function $g(s) = \frac{s+8}{s^2+6s+8}$ in d), we obtain

$$\begin{aligned}
S_y(j\omega) &= \frac{j\omega + 8}{(j\omega)^2 + 6(j\omega) + 8} \cdot \frac{(-j\omega) + 8}{(-j\omega)^2 + 6(-j\omega) + 8} \cdot 4 \\
&= \frac{j\omega + 8}{-\omega^2 + 6j\omega + 8} \cdot \frac{-j\omega + 8}{-\omega^2 - 6j\omega + 8} \cdot 4 \\
&= \frac{4\omega^2 + 256}{\omega^4 + 20\omega^2 + 64} = \frac{20}{\omega^2 + 4} - \frac{16}{\omega^2 + 16}.
\end{aligned}$$

Problem 2: Kalman-filter derivation

a) From the output equation in (1), it follows that

$$\begin{aligned}\mathbf{y}_k^- &= E[\mathbf{y}_k] \\ &= E[\mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k + \mathbf{H}\mathbf{v}_k] \\ &= \mathbf{C}E[\mathbf{x}_k] + \mathbf{D}\mathbf{u}_k + \mathbf{H}E[\mathbf{v}_k] \\ &= \mathbf{C}\hat{\mathbf{x}}_k^- + \mathbf{D}\mathbf{u}_k.\end{aligned}$$

b) By substituting the expression for $\hat{\mathbf{x}}_k$ in (2), the *a posteriori* estimation error $\mathbf{x}_k - \hat{\mathbf{x}}_k$ can be written as

$$\mathbf{x}_k - \hat{\mathbf{x}}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k^- - \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k^-).$$

From the output equation in (1) and from (2), it follows that

$$\begin{aligned}\mathbf{y}_k - \hat{\mathbf{y}}_k^- &= \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k + \mathbf{H}\mathbf{v}_k - \mathbf{C}\hat{\mathbf{x}}_k^- - \mathbf{D}\mathbf{u}_k \\ &= \mathbf{C}\mathbf{x}_k + \mathbf{H}\mathbf{v}_k - \mathbf{C}\hat{\mathbf{x}}_k^- \\ &= \mathbf{C}(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) + \mathbf{H}\mathbf{v}_k.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\mathbf{x}_k - \hat{\mathbf{x}}_k &= \mathbf{x}_k - \hat{\mathbf{x}}_k^- - \mathbf{K}_k(\mathbf{C}(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) + \mathbf{H}\mathbf{v}_k) \\ &= (\mathbf{I} - \mathbf{K}_k\mathbf{C})(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) - \mathbf{K}_k\mathbf{H}\mathbf{v}_k.\end{aligned}$$

By substituting this in the definition of $\mathbf{P}_k$, we get

$$\begin{aligned}\mathbf{P}_k &= E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T] \\ &= E[((\mathbf{I} - \mathbf{K}_k\mathbf{C})(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) - \mathbf{K}_k\mathbf{H}\mathbf{v}_k)((\mathbf{I} - \mathbf{K}_k\mathbf{C})(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) - \mathbf{K}_k\mathbf{H}\mathbf{v}_k)^T] \\ &= E[(\mathbf{I} - \mathbf{K}_k\mathbf{C})(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T] \\ &\quad - E[\mathbf{K}_k\mathbf{H}\mathbf{v}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T] - E[(\mathbf{I} - \mathbf{K}_k\mathbf{C})(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\mathbf{v}_k^T\mathbf{H}^T\mathbf{K}_k^T] \\ &\quad + E[\mathbf{K}_k\mathbf{H}\mathbf{v}_k\mathbf{v}_k^T\mathbf{H}^T\mathbf{K}_k^T] \\ &= (\mathbf{I} - \mathbf{K}_k\mathbf{C})E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T](\mathbf{I} - \mathbf{K}_k\mathbf{C})^T \\ &\quad - \mathbf{K}_k\mathbf{H}E[\mathbf{v}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T](\mathbf{I} - \mathbf{K}_k\mathbf{C})^T - (\mathbf{I} - \mathbf{K}_k\mathbf{C})E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\mathbf{v}_k^T]\mathbf{H}^T\mathbf{K}_k^T \\ &\quad + \mathbf{K}_k\mathbf{H}E[\mathbf{v}_k\mathbf{v}_k^T]\mathbf{H}^T\mathbf{K}_k^T \\ &= (\mathbf{I} - \mathbf{K}_k\mathbf{C})\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T + \mathbf{K}_k\mathbf{H}\mathbf{R}\mathbf{H}^T\mathbf{K}_k^T.\end{aligned}$$

c) The *a posteriori* error covariance matrix $\mathbf{P}_k$ in (4) can be written as

$$\begin{aligned}\mathbf{P}_k &= (\mathbf{I} - \mathbf{K}_k\mathbf{C})\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T + \mathbf{K}_k\mathbf{H}\mathbf{R}\mathbf{H}^T\mathbf{K}_k^T \\ &= \mathbf{P}_k^- - \mathbf{K}_k\mathbf{C}\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{C}^T\mathbf{K}_k^T + \mathbf{K}_k\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T\mathbf{K}_k^T + \mathbf{K}_k\mathbf{H}\mathbf{R}\mathbf{H}^T\mathbf{K}_k^T \\ &= \mathbf{P}_k^- - \mathbf{K}_k\mathbf{C}\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{C}^T\mathbf{K}_k^T + \mathbf{K}_k(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T.\end{aligned}$$

d) By taking the derivative of $\mathbf{P}_k$ with respect to $\mathbf{K}_k$, we obtain

$$\begin{aligned}\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k} &= -\mathbf{C}\mathbf{P}_k^- - \mathbf{C}\mathbf{P}_k^- + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^T)\mathbf{K}_k^T \\ &= -\mathbf{C}\mathbf{P}_k^- - \mathbf{C}\mathbf{P}_k^- + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^T\mathbf{K}_k^T \\ &= -\mathbf{C}\mathbf{P}_k^- - \mathbf{C}\mathbf{P}_k^- + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T + (\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T \\ &= -2\mathbf{C}\mathbf{P}_k^- + 2(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T.\end{aligned}$$

e) From d) and $\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k} = \mathbf{0}$, we have that

$$\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k} = -2\mathbf{C}\mathbf{P}_k^- + 2(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T = \mathbf{0}.$$

By taking the transposed, it follows that

$$-2\mathbf{P}_k^-\mathbf{C}^T + 2\mathbf{K}_k(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^T = -2\mathbf{P}_k^-\mathbf{C}^T + 2\mathbf{K}_k(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T) = \mathbf{0}.$$

From this, we get

$$\mathbf{K}_k(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T) = \mathbf{P}_k^-\mathbf{C}^T.$$

By postmultiplying both sides of the equation by $(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^{-1}$, we obtain

$$\mathbf{K}_k = \mathbf{P}_k^-\mathbf{C}^T(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^{-1}.$$

f) From the state equation in (1), we get

$$\begin{aligned}\hat{\mathbf{x}}_{k+1}^- &= E[\mathbf{x}_{k+1}] \\ &= E[\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{G}\mathbf{w}_k] \\ &= \mathbf{A}E[\mathbf{x}_k] + \mathbf{B}\mathbf{u}_k + \mathbf{G}E[\mathbf{w}_k] \\ &= \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k.\end{aligned}$$

g) From equations (1) and (6), it follows that

$$\begin{aligned}\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^- &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{G}\mathbf{w}_k - \mathbf{A}\hat{\mathbf{x}}_k - \mathbf{B}\mathbf{u}_k \\ &= \mathbf{A}(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{G}\mathbf{w}_k.\end{aligned}$$

Substituting this in the definition of $\mathbf{P}_{k+1}^-$ yields

$$\begin{aligned}\mathbf{P}_{k+1}^- &= E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)^T] \\ &= E[(\mathbf{A}(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{G}\mathbf{w}_k)(\mathbf{A}(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{G}\mathbf{w}_k)^T] \\ &= E[\mathbf{A}(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T\mathbf{A}^T] + E[\mathbf{A}(\mathbf{x}_k - \hat{\mathbf{x}}_k)\mathbf{w}_k^T\mathbf{G}^T] \\ &\quad + E[\mathbf{G}\mathbf{w}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T\mathbf{A}^T] + E[\mathbf{G}\mathbf{w}_k\mathbf{w}_k^T\mathbf{G}^T] \\ &= \mathbf{A}E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T]\mathbf{A}^T + \mathbf{A}E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)\mathbf{w}_k^T]\mathbf{G}^T \\ &\quad + \mathbf{G}E[\mathbf{w}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T]\mathbf{A}^T + \mathbf{G}E[\mathbf{w}_k\mathbf{w}_k^T]\mathbf{G}^T \\ &= \mathbf{A}\mathbf{P}_k\mathbf{A}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T.\end{aligned}$$

h) To find the values for $\hat{\mathbf{x}}_0$, $\hat{\mathbf{x}}_1^-$ and $\hat{\mathbf{x}}_1$, we use the formulas in the assignment as follows:

$$\hat{y}_0^- = \mathbf{C}\hat{\mathbf{x}}_0^- + Du_0 = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} + 4\cdot 1 = 4,$$

$$\begin{aligned}
\mathbf{K}_0 &= \mathbf{P}_0^-\mathbf{C}^T(\mathbf{C}\mathbf{P}_0^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^{-1}\\
&= \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}\right)^{-1}\\
&= \begin{bmatrix}2\\0\end{bmatrix}(2+2)^{-1}\\
&= \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} = \begin{bmatrix}0.5\\0\end{bmatrix},
\end{aligned}$$

$$\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}_0^- + \mathbf{K}_0(y_0 - \hat{y}_0^-) = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(3-4) = \begin{bmatrix}-\frac{1}{2}\\0\end{bmatrix} = \begin{bmatrix}-0.5\\0\end{bmatrix},$$

$$\begin{aligned}
\mathbf{P}_0 &= (\mathbf{I} - \mathbf{K}_0\mathbf{C})\mathbf{P}_0^-(\mathbf{I} - \mathbf{K}_0\mathbf{C})^T + \mathbf{K}_0\mathbf{H}\mathbf{R}\mathbf{H}^T\mathbf{K}_0^T\\
&= \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}-\frac{1}{2}\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}-\frac{1}{2}\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)^T\\
&\quad + \begin{bmatrix}-\frac{1}{2}\\0\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & 0\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{2} & 0\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & 0\\0 & 0\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{2} & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & 0\\0 & 0\end{bmatrix}\\
&= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},
\end{aligned}$$

$$\hat{\mathbf{x}}_1^- = \mathbf{A}\hat{\mathbf{x}}_0 + \mathbf{B}u_0 = \begin{bmatrix}-1 & 1\\0 & -3\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}1 = \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}0.5\\1\end{bmatrix},$$

$$\begin{aligned}
\mathbf{P}_1^- &= \mathbf{A}\mathbf{P}_0\mathbf{A}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T\\
&= \begin{bmatrix}-1 & 1\\0 & -3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\1 & -3\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}2\begin{bmatrix}1 & 0\end{bmatrix}\\
&= \begin{bmatrix}2 & -3\\-3 & 9\end{bmatrix} + \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\\
&= \begin{bmatrix}4 & -3\\-3 & 9\end{bmatrix},
\end{aligned}$$

$$\hat{y}_1^- = \mathbf{C}\hat{\mathbf{x}}_1^- + Du_1 = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix} + 4\cdot -1 = -\frac{7}{2} = -3.5,$$

$$
\begin{aligned}
\mathbf{K}_1 &= \mathbf{P}_1^- \mathbf{C}^T (\mathbf{C}\mathbf{P}_1^- \mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^{-1} \\
&= \begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} 4 \\ -3 \end{bmatrix} (4+2)^{-1} \\
&= \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{2} \end{bmatrix} \approx \begin{bmatrix} 0.6667 \\ -0.5 \end{bmatrix},
\end{aligned}
$$

$$
\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_1^- + \mathbf{K}_1 (y_1 - \hat{y}_1^-) = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{2} \end{bmatrix} \left(-4 + \frac{7}{2} \right) = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{4} \end{bmatrix} \approx \begin{bmatrix} 0.1667 \\ 1.25 \end{bmatrix}.
$$

Hence, we obtain

$$
\hat{\mathbf{x}}_0 = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{x}}_1^- = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \hat{\mathbf{x}}_1 = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{4} \end{bmatrix}.
$$