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TTT4120 Digital Signal Processing
Suggested Solutions for Problem Set 11

Problem 1

(a) $\boxed{\uparrow \text{I}}$ - The upsampler increases the sampling frequency of the signal I times by insering $I-1$ zero-samples between the samples of $x(n)$.

$\boxed{h(m)}$ - A lowpass filter that performs smoothing on the signal $w(m)$ so that the new zero-samples are replaced by correct signal values. The effect of the filter in the frequency domain is that it removes unwanted replica of the spectrum.

(b) The signal $w(m)$ is given by

$$w(m) = \begin{cases} x(m/I) & m = 0, \pm I, \pm 2I, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $f_x = F/F_{sx}$ and $f_y = F/F_{sy}$. Then $f_x = If_y$. The DTFT of $w(m)$ can be calculated as follows.

$$W(f_y) = \sum_{m=-\infty}^{\infty} w(m)e^{-j2\pi f_y m}$$

Since $w(m)$ can only be non-zero when $m = kI$, where k is an integer, we can change the summation such that all zero terms are excluded.

$$\begin{aligned} W(f_y) &= \sum_{k=-\infty}^{\infty} w(kI)e^{-j2\pi f_y kI} \\ &= \sum_{k=-\infty}^{\infty} x(k)e^{-j2\pi f_x k} \\ &= X(f_x) \end{aligned}$$

Thus, we have that

$$W\left(\frac{F}{F_{sy}}\right) = X\left(\frac{F}{F_{sx}}\right).$$

This means that there is no change in the spectrum as a function of physical frequency F. However, as a function of digital frequency we have $W(f_y) = X(If_y)$, which means that the spectrum has been compressed I times.

(c) We have that

$$X\left(\frac{F}{F_{sx}}\right) = F_{sx} \sum_{k=-\infty}^{\infty} X_a(F - kF_{sx}),$$

i.e. the spectrum of $x(n)$ is a periodic extension of the spectrum of $x_a(t)$ with period F_{sx}, and scaled by F_{sx}. Furthermore, as shown in (b), the spectra of $w(m)$ and $x(n)$ are equal when seen as functions of physical frequency F. They are illustrated in the first two graphs of Figure 1.

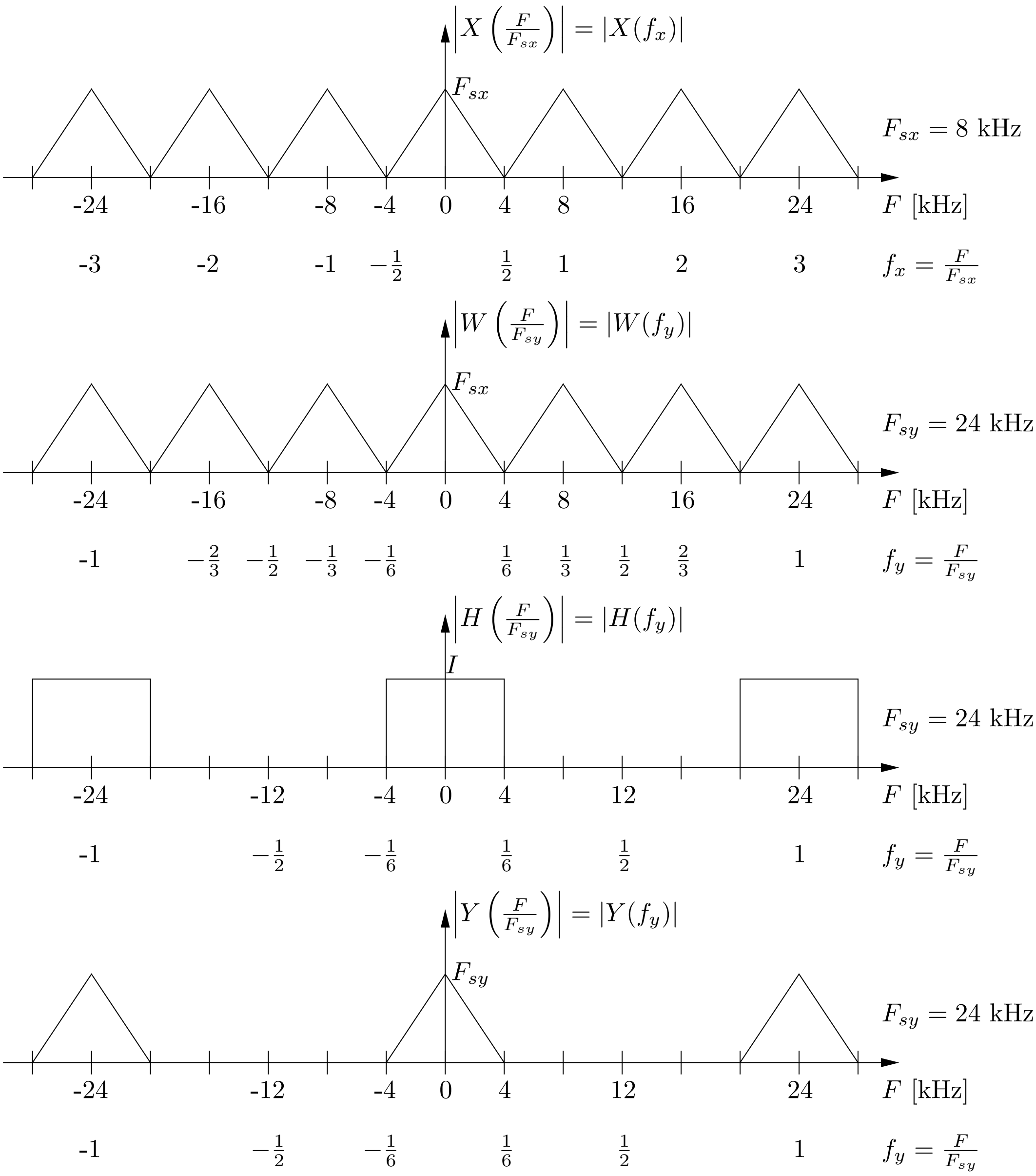

Figure 1: Magnitude spectra for an interpolator by factor I=3.

The spectrum of the output signal is a periodic extension of the spectrum of $x_a(t)$ with period F_{sy}, scaled by F_{sy}

$$Y\left(\frac{F}{F_{sy}}\right) = F_{sy} \sum_{k=-\infty}^{\infty} X_a(F - kF_{sy}),$$

It is illustrated in the last graph of Figure 1.

The filter should be designed to remove the unwanted replica of the original spectrum from $W(f_y)$, and to provide the correct scaling of the output spectrum. Thus, the magnitude response of the filter should be defined as

$$H(f_y) = \begin{cases} I & |f_y| \leq \frac{1}{6} \\ 0 & \frac{1}{6} < |f_y| < \frac{1}{2}. \end{cases}$$

Problem 2

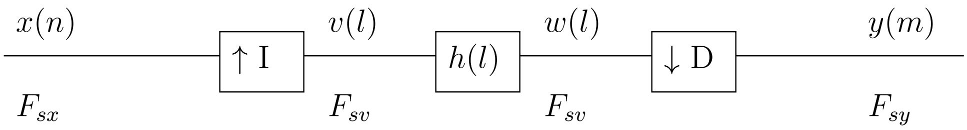

Figure 2: Block diagram for the system

(a) The block diagram is shown in Figure 2. It contains the following elements:

- $\boxed{\uparrow \text{I}}$ - The upsampler increases the sampling frequency of the signal I times by insering $I-1$ zero-samples between the samples of $x(n)$.
- $\boxed{h(l)}$ - A digital lowpass filter that removes all frequency contents above a $F_{sy}/2$ in order to avoid aliasing after decimation and to remove unwanted replica of the spectrum introduced by insertion of zeros.
- $\boxed{\downarrow \text{D}}$ - A downsampler that reduces the sampling frequency by a factor D by retaining only every Dth sample.

(b) The sampling frequency is to be reduced by a factor $F_{sy}/F_{sx} = 6/8 = 3/4$. To achieve this, we first increase the sampling frequency by a factor 3 and then reduce it by a factor 4. Thus $I = 3$ and $D = 4$.

Specification of the filter can be found as follows. The filter must remove the frequency content above $F_c = F_{sy}/2 = 3$kHz in order to avoid aliasing after decimation. This also ensures that unwanted spectral replica introduced by insertion of zeros are removed. The filter $h(l)$ operates at sampling frequency $F_{sv} = 3 \cdot 8$kHz. Thus, its (digital) cut-off frequency should be $f_c = F_c/F_{sv} = 3/(3 \cdot 8) = 1/8$, i.e.

$$|H(f)| = \begin{cases} 3 & |f| \leq \frac{1}{8} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Since the frequency components above 3kHz have been removed during filtering, it is not possible to reconstruct $x_a(t)$ from $y(m)$. (Some information is lost during filtering.)

(d) Sketches of the magnitude spectra of the signals and the filter are shown in Figure 3.

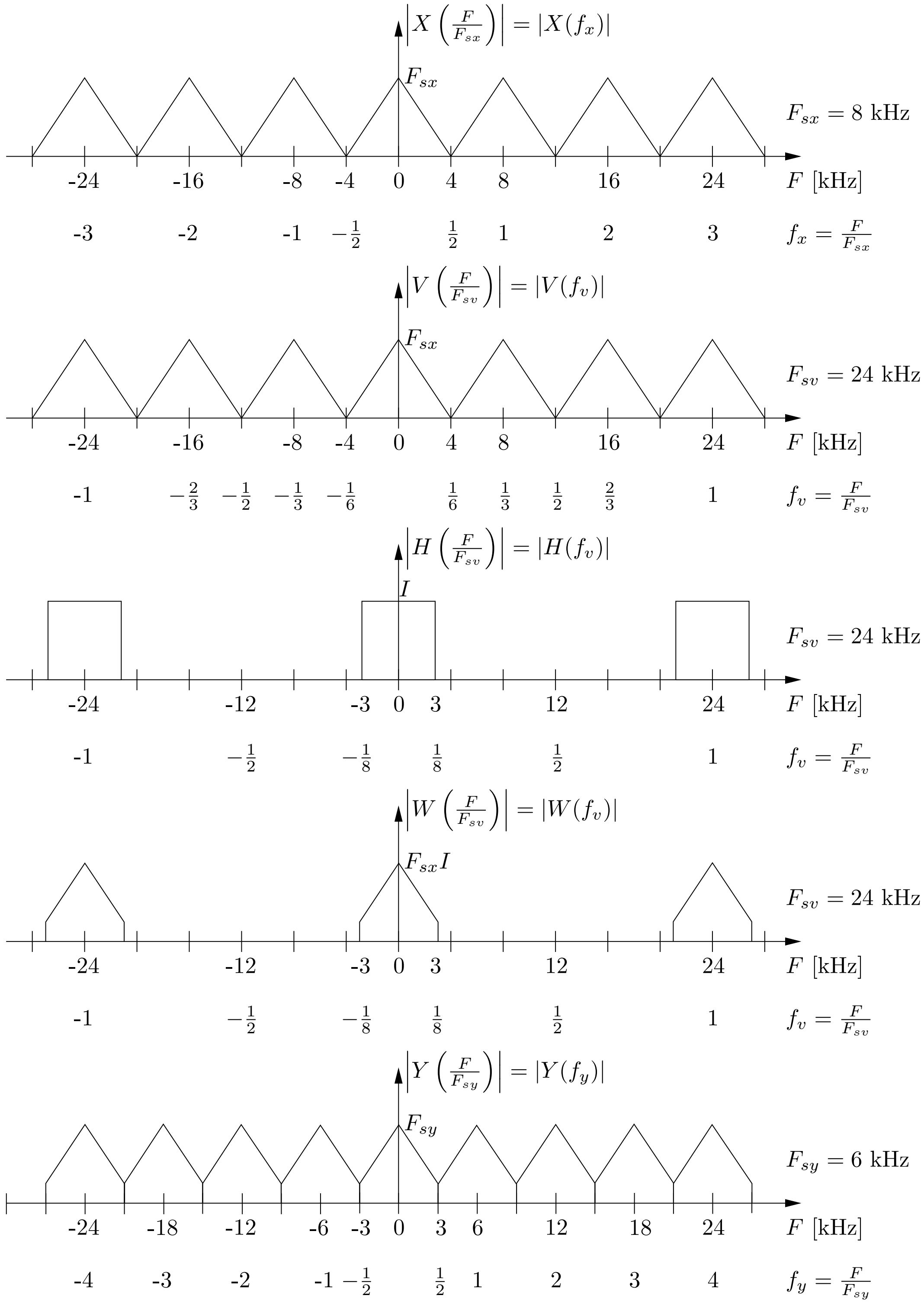

Figure 3: Magnitude spectra for $\frac{I}{D} = \frac{3}{4}$.

(e) The sampling frequency should now be increased by a factor $F_{sy}/F_{sx} = 12/8 = 3/2$. This can be achieved by the system in Figure 2 with $I = 3$ and $D = 2$. The magnitude spectra are shown in Figure 4.

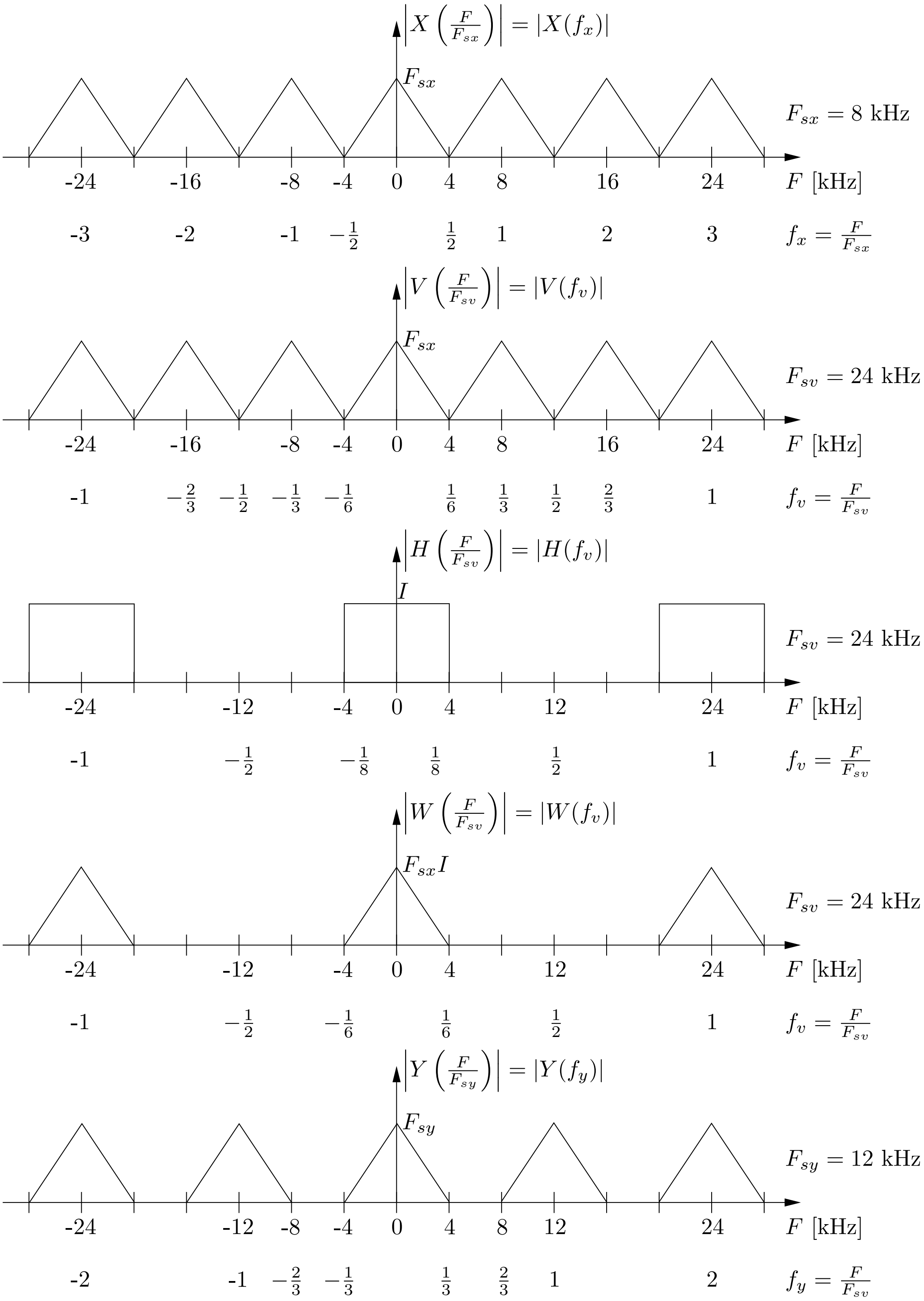

Figure 4: Magnitude spectra for $\frac{I}{D} = \frac{3}{2}$.

The filter should now remove frequency content above $F_c = F_{sx}/2 = 4$kHz in order to remove unwanted replica of the spectrum after upsampling. This also ensures that no aliasing will appear after downsampling. Since the filter operates at sampling frequency

$F_{sv} = 3 \cdot 8$kHz, we get $f_c = F_c/F_{sv} = 4/(3 \cdot 8) = 1/6$, i.e.

$$|H(f)| = \begin{cases} 3 & |f| \leq \frac{1}{6} \\ 0 & \text{otherwise.} \end{cases}$$

The above procedure does not influence the frequency range $|F| < 4$kHz of the original signal. Thus, no information is lost, and it is possible to reconstruct $x_a(t)$ from $y(m)$.

Problem 3

(a) The IDTFT of $X_1(f)$ can be found as follows.

$$\begin{aligned} x_1(n) &= \text{IDTFT}\{X_1(f)\} \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} X_1(f)\, e^{j2\pi fn} df \\ &= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} (\delta(f-f_1)+\delta(f+f_1))e^{j2\pi fn} df \\ &= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(f-f_1)e^{j2\pi fn} df + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(f+f_1)e^{j2\pi fn} df \end{aligned}$$

Since $\delta(f) = 0$ for all $f \neq 0$, and $\int_{-\infty}^{\infty} \delta(f)df = 1$, it follows that

$$x_1(n) = \frac{1}{2}(e^{-j2\pi f_1 n} + e^{j2\pi f_1 n}) = \cos(2\pi f_1 n)$$

(b) The spectra for the case when filtering is performed before downsampling are shown in Figure 5. In order to avoid aliasing after downsampling, the filter should remove all frequency components above $F_s/(2D) = 1500$ Hz. We see that since the frequency of $x_2(n)$ is greater than 1500Hz, this component is removed by the filter.

The output spectrum when the filter is removed is shown in Figure 6. In this case we get aliasing, since the highest frequency component has not been removed. This causes the appearance of an alias of the high frequency component. This new component appears at $F = 1000$ Hz.

(c) When listening to the signal $x(n)$ one can hear that two frequency components are present. After filtering and downsampling, only the lowest frequency remains. If the filter is not used, one can hear that there are still two components present, but the signal sounds different from $x(n)$. This is because there is now an aliased component at $F = 1000$ Hz, in addition to the component at $F = 900$ Hz.

By listening carefully to the original music signal, one can hear some high frequency "plings". After filtering and downsampling, these plings are gone. However, if the filter is not used, the plings are still present, but they are at a much lower frequency than they were originally. This

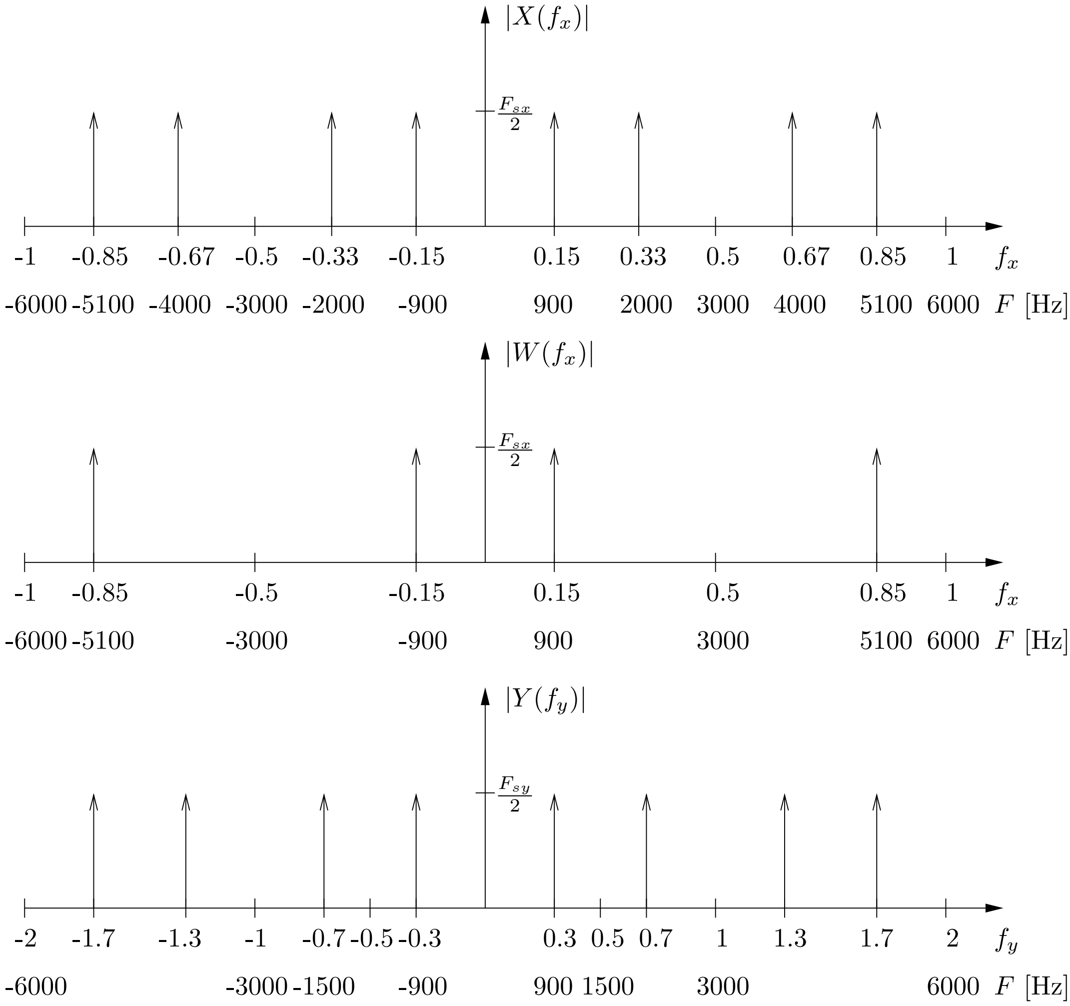

Figure 5: Magnitude spectra when the filter is used.

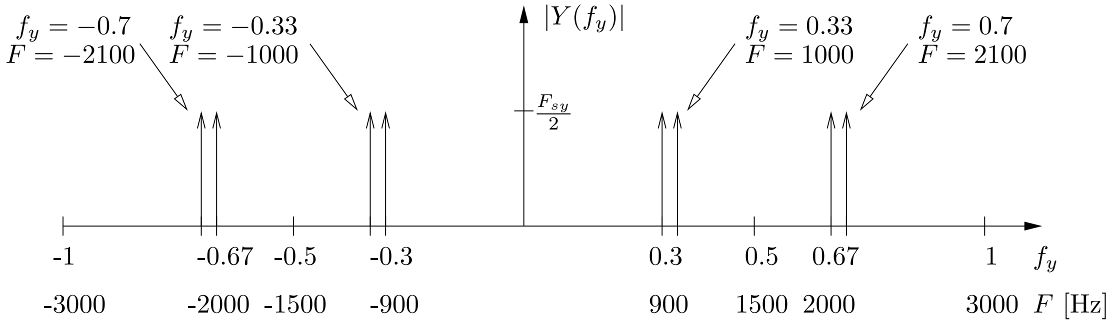

Figure 6: Magnitude spectrum of the output signal when the filter is not used.

is because they are aliases of the original plings. One can also hear that the sounds s, sh and ch, that contain high frequency components, are distorted more when there is aliasing than when the filter is used.

Note that if we use a filter to avoid aliasing, we remove frequency components above $F_s/(2D)$. If we remove the filter we will in addition cause distortion of frequency components below $F_s/(2D)$ (between $\frac{F_s}{D} - B$ and $\frac{F_s}{2D}$), due to aliasing.