Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4120 Digital Signal Processing
Suggested Solutions for Problem Set 10

Problem 1

For $l \geq 0$, using the fact that the signal $s(n)$ is not correlated with future samples of the noise $v(n)$, a recursive expression for the autocorrelation function can be found as

$$\begin{aligned}
\gamma_{ss}(l) &= E\left[s(n)s(n+l)\right] \\
&= E\left[(0.9s(n-1)+v(n))(0.9s(n-1+l)+v(n+l))\right] \\
&= 0.81E\left[s(n-1)s(n-1+l)\right] + 0.9E\left[s(n-1)v(n+l)\right] \\
&+ 0.9E\left[s(n+l-1)v(n)\right] + E\left[v(n)v(n+l)\right] \\
&= 0.81\gamma_{ss}(l) + 0 + E\left[0.9s(n+l-1)(s(n)-0.9s(n-1))\right] + \gamma_{vv}(l) \\
&= 0.81\gamma_{ss}(l) + 0.9\gamma_{ss}(l-1) - 0.81\gamma_{ss}(l) + \gamma_{vv}(l) \\
&= 0.9\gamma_{ss}(l-1) + \sigma_v^2\delta(l) \\
&= 0.9\gamma_{ss}(l-1) + 0.09\delta(l)
\end{aligned}$$

Using this recursive expression and the symmetry of the autocorrelation function:

$$\begin{aligned}
\gamma_{ss}(0) &= 0.9\gamma_{ss}(-1) + 0.09 = 0.9\gamma_{ss}(1) + 0.09 \\
\gamma_{ss}(1) &= 0.9\gamma_{ss}(0) \\
\gamma_{ss}(0) &= 0.9 \cdot 0.9\gamma_{ss}(0) + 0.09 \\
\gamma_{ss}(0) &= \frac{9}{19} \\
\gamma_{ss}(1) &= 0.9\gamma_{ss}(0) \\
\gamma_{ss}(2) &= 0.81\gamma_{ss}(0).
\end{aligned}$$

If we continue the same line of reasoning, we can show that
$\gamma_{ss}(l) = \gamma_{ss}(0)0.9^{|l|} = \frac{9}{19}0.9^{|l|}$.

The normal equations for $M = 3$ are:

$$\begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(-1) & \gamma_{xx}(-2) \\ \gamma_{xx}(-1) & \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(-2) & \gamma_{xx}(-1) & \gamma_{xx}(0) \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \gamma_{dx}(0) \\ \gamma_{dx}(1) \\ \gamma_{dx}(2) \end{bmatrix}$$

For our case,

$$\gamma_{xx}(l) = \gamma_{ss}(l) + \gamma_{ww}(l) = \frac{9}{19}0.9^{|l|} + \delta(l)$$
$$\gamma_{dx}(l) = \gamma_{ss} = \frac{9}{19}0.9^{|l|}$$

Thus, we have the equations

$$\begin{bmatrix} 1.4737 & 0.4263 & 0.3837 \\ 0.4263 & 1.4737 & 0.4263 \\ 0.3837 & 0.4263 & 1.4737 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 0.4737 \\ 0.4263 \\ 0.3837 \end{bmatrix}$$

Solving for h:

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 1.4737 & 0.4263 & 0.3837 \\ 0.4263 & 1.4737 & 0.4263 \\ 0.3837 & 0.4263 & 1.4737 \end{bmatrix}^{-1} \begin{bmatrix} 0.4737 \\ 0.4263 \\ 0.3837 \end{bmatrix} = \begin{bmatrix} 0.2309 \\ 0.1796 \\ 0.1483 \end{bmatrix}$$

We also can assume that $s(n)$ is the output of a system with unit sample response $h(n)$ and input $v(n)$, then $\gamma_{ss}(l)$ can be computed:

$$\begin{aligned}
H(z) &= \frac{1}{1 - 0.9z^{-1}} \\
h(n) &= 0.9^n, & n = 0, 1, 2, ... \\
r_{hh}(l) &= \sum_{n=0}^{\infty} 0.9^n \cdot 0.9^{n+l}, & l \geq 0 \\
r_{hh}(l) &= 0.9^{|l|} \frac{1}{1 - 0.9^2} \\
&= \frac{1}{0.19} \cdot 0.9^{|l|} \\
\gamma_{ss}(l) &= \gamma_{vv}(l) * r_{hh}(l) \\
&= 0.09 \cdot \delta(l) * \frac{1}{0.19} \cdot 0.9^{|l|} \\
&= \frac{9}{19} 0.9^{|l|}
\end{aligned}$$

Problem 2

(a) The filter can be decomposed as

$$H(z) = H_1(z)H_2(z) = \frac{z^{-1} - \frac{1}{2}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + \frac{1}{2}z^{-1}},$$

where

$$A = H(z)(1 - \frac{1}{2}z^{-1})\Big|_{z=\frac{1}{2}} = \frac{z^{-1} - \frac{1}{2}}{1 + \frac{1}{2}z^{-1}}\Big|_{z=\frac{1}{2}} = \frac{3}{4}$$

and

$$B = H(z)(1 + \frac{1}{2}z^{-1})\Big|_{z=-\frac{1}{2}} = \frac{z^{-1} - \frac{1}{2}}{1 - \frac{1}{2}z^{-1}}\Big|_{z=-\frac{1}{2}} = -\frac{5}{4}$$

Hence,

$$H(z) = \frac{\frac{3}{4}}{1 - \frac{1}{2}z^{-1}} + \frac{-\frac{5}{4}}{1 + \frac{1}{2}z^{-1}},$$

(b) In order to sketch the DF2 structure of $H(z)$ it is useful to first find the difference equation for the filter. We have.

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{z^{-1} - \frac{1}{2}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} \\ Y(z) - \frac{1}{4}z^{-2}Y(z) &= z^{-1}X(z) - \frac{1}{2}X(z) \\ y(n) - \frac{1}{4}y(n-2) &= x(n-1) - \frac{1}{2}x(n) \end{aligned}$$

Starting from the difference equation it is easy to draw the direct form 1 (DF1) of the filter. See figure 1.

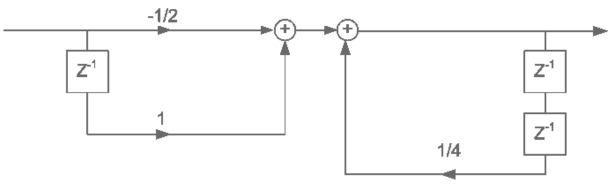


Figure 1: Direct form 1

From the DF1 we can get the DF2 structure by exchanging the left and right parts, and then merge the delay components. We then get the DF2 structure shown in figure 2.

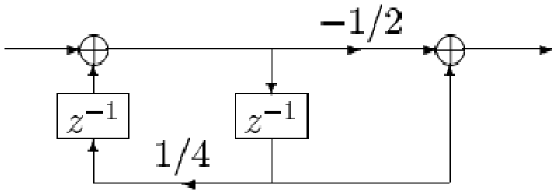


Figure 2: Direct form 2

The parallel realization of $H(z)$ can be found from equation 4 in the problem text. The result is in figure 3.

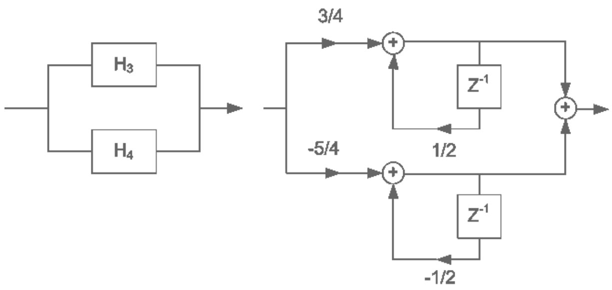


Figure 3: Parallel realization

A cascade realization of $H(z)$ can be found from equation 1 in the problem text. The result is sketched in figure 4.

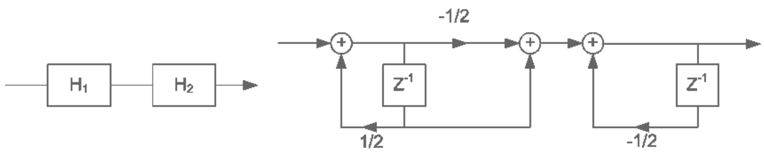
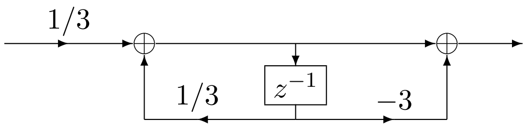


Figure 4: Cascade realization

Problem 3



(a)

(b)

$$H(z) = \frac{1}{3} \cdot \frac{1 - 3z^{-1}}{1 - 1/3z^{-1}}$$

$$G(z) = \frac{1}{1 - 1/3z^{-1}} \longrightarrow g(n) = \begin{cases} (1/3)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

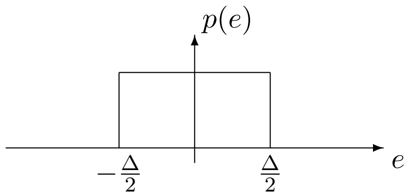
$$H(z) = \frac{1}{3}G(z) - z^{-1}G(z)$$

$$\Longrightarrow h(n) = \frac{1}{3}g(n)u(n) - g(n-1)u(n-1)$$

$$h(0) = \frac{1}{3}g(0) = \frac{1}{3}$$

$$h(n) = \begin{cases} \frac{1}{3}(1/3)^n - (1/3)^{n-1}, & n > 1 \\ 1/3, & n = 0 \\ 0, & n < 0 \end{cases}$$

where $\frac{1}{3}(\frac{1}{3})^n - (\frac{1}{3})^{n-1} = [(\frac{1}{3})^2 - 1](\frac{1}{3})^{n-1} = -\frac{8}{9}(\frac{1}{3})^{n-1}$.

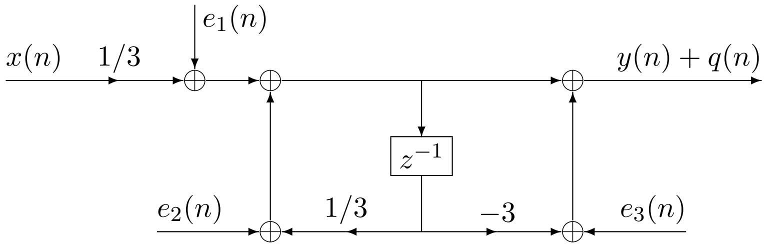


(c)

$$\sigma_e^2 = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 p(e) de = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 de \quad and \quad \Delta = 2^{-B}$$

$$\sigma_e^2 = \frac{1}{\Delta}\left[\frac{1}{3}e^3\right]_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{1}{3\Delta}\left[\frac{\Delta^3}{8} - \left(\frac{-\Delta^3}{8}\right)\right] = \frac{\Delta^2}{12} = \frac{2^{-2B}}{12}$$

(d)



$$q(n) = h_1(n) * e_1(n) + h_2(n) * e_2(n) + h_3(n) * e_3(n)$$

where $h_1(n) = h_2(n) = 3h(n), \qquad h_3(n) = \delta(n) \quad$ (short-circuiting)

$$\sigma_q^2 = 2\sigma_e^2 9 r_{hh}(0) + \sigma_e^2$$

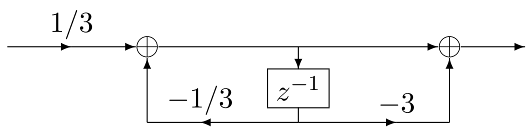
$$\begin{aligned} r_{hh}(0) &= \sum_{n=0}^{\infty} h^2(n) = \left(\frac{1}{3}\right)^2 + \sum_{n=1}^{\infty} h^2(n) \\ &= \frac{1}{9} + \sum_{n=1}^{\infty} \left(\frac{8}{9}\right)^2 \left[\left(\frac{1}{3}\right)^{n-1}\right]^2 \end{aligned}$$

$l = n - 1 \longrightarrow$

$$\begin{aligned} r_{hh}(0) &= \frac{1}{9} + \left(\frac{8}{9}\right)^2 \sum_{l=0}^{\infty} \left(\frac{1}{3}\right)^{2l} = \frac{1}{9} + \left(\frac{8}{9}\right)^2 \frac{1}{1 - 1/9} \\ &= \frac{1}{9} + \left(\frac{8}{9}\right)^2 \frac{9}{8} = \frac{1}{9} + \frac{8}{9} = \underline{1} \end{aligned}$$

$$\Longrightarrow \sigma_q^2 = 2 \cdot 9 \cdot 1 \cdot \sigma_e^2 + \sigma_e^2 = \underline{19\sigma_e^2}$$

(e) •



$$h_1(n) = \begin{cases} \frac{1}{3}\left(\frac{1}{3}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

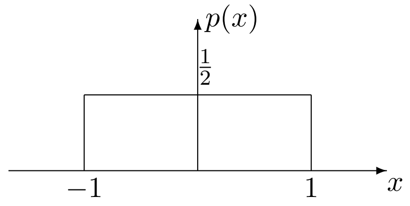
$$h_2(n) = h(n)$$

where $h_1(n)$ and $h_2(n)$ are respectively the unit sample response of the filter by assuming that the output is the signal just after first and second adder.

$$\sum |h_1(n)| = \frac{1}{3} \sum \left|\frac{1}{3}\right|^n = \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{\underline{2}}$$

$$\sum |h(n)| = \frac{1}{3} + \frac{8}{9} \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} = \frac{1}{3} + \frac{8}{9} \sum_{l=0}^{\infty} \left(\frac{1}{3}\right)^{l} = \frac{1}{3} + \frac{8}{9} \cdot \frac{3}{2} = \frac{5}{\underline{3}} > \frac{1}{2}$$

Overflow takes place just at the output of second adder. So we have to scale the input in order to prevent overflow at this point. To do so we need scaling factor $s = \frac{3}{5}$ at the input.



- $$\sigma_x^2 = E[x^2] = \int_{-\infty}^{+\infty} x^2 p(x)dx = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{3}$$
$$\sigma_e^2 = 2^{-2B}/12 = 2^{-14}/12 \qquad (B = 7)$$

the output power without applying scaling factor is:

$$\sigma_y^2 = \sigma_x^2 r_{hh}(0) = \sigma_x^2$$

SNR at the output without scaling factor is:

$$\longrightarrow \frac{\sigma_x^2}{19\sigma_e^2} \approx \underline{35\text{dB}}$$

by applying scaling factor s the SNR at the output becomes:

$$\longrightarrow \frac{s^2\sigma_x^2}{19\sigma_e^2} = \frac{(\frac{3}{5})^2}{19} \cdot \frac{\sigma_x^2}{\sigma_e^2} = \frac{9}{25 \cdot 19} \cdot \frac{\sigma_x^2}{\sigma_e^2} \approx \underline{31\text{dB}}$$