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NTNU – Trondheim
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Exercise 10
TTK4130 Modeling and Simulation

Problem 1 (Double inverted pendulum)

Figure 1 shows a double inverted pendulum on a cart. The two pendulums move independently of each other. The pendulum masses are m_2 and m_3, the rods have no mass. The cart mass is m_1 and τ is a force acting on the cart.

Find the equations of motion for the system, either using Kane's method (Ch. 7., only "borderline curriculum") or Lagrange's equations (Ch. 8).

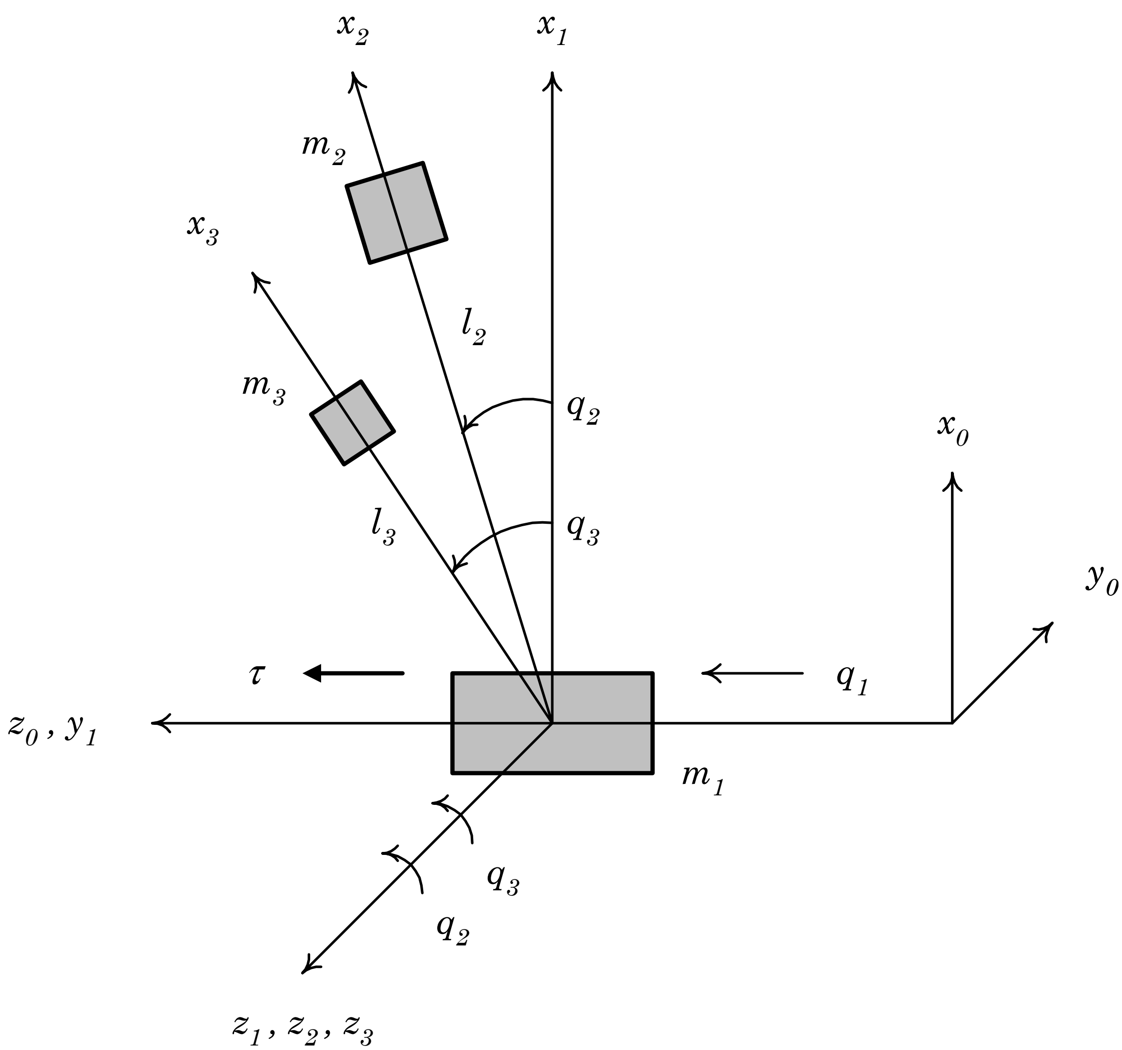

Figure 1: Double inverted pendulum

Solution: Here we use Kane's method. Using Lagrange's equations gives the same answer, perhaps a bit more straightforward.

Direction cosines:

$$\begin{aligned} &\vec{b}_3 = \vec{c}_3 = -\vec{a}_2 \\ &\vec{a}_1 \cdot \vec{b}_1 = \cos q_2, \qquad \vec{a}_1 \cdot \vec{b}_2 = -\sin q_2 \\ &\vec{a}_3 \cdot \vec{b}_1 = \sin q_2, \qquad \vec{a}_3 \cdot \vec{b}_2 = \cos q_2 \\ &\vec{a}_1 \cdot \vec{c}_1 = \cos q_3, \qquad \vec{a}_1 \cdot \vec{c}_2 = -\sin q_3 \\ &\vec{a}_3 \cdot \vec{c}_1 = \sin q_3, \qquad \vec{a}_3 \cdot \vec{c}_2 = \cos q_3 \end{aligned}$$

Angular velocities become

$$\begin{aligned} \vec{\omega}_1 &= 0 \\ \vec{\omega}_2 &= \dot{q}_2 \vec{b}_3 \\ \vec{\omega}_3 &= \dot{q}_3 \vec{c}_3 \end{aligned}$$

Mass center velocities:

$$\begin{aligned} \vec{v}_{c1} &= \dot{q}_1 \vec{a}_3 \\ \vec{v}_{c2} &= \dot{q}_1 \vec{a}_3 + \dot{q}_2 l_2 \vec{b}_2 \\ \vec{v}_{c3} &= \dot{q}_1 \vec{a}_3 + \dot{q}_3 l_3 \vec{c}_2 \end{aligned}$$

This gives

$$\begin{aligned} &\vec{\omega}_{1,1} = \vec{0}, \qquad \vec{\omega}_{1,2} = \vec{0}, \qquad \vec{\omega}_{1,3} = \vec{0} \\ &\vec{\omega}_{2,1} = \vec{0}, \qquad \vec{\omega}_{2,2} = \vec{b}_3, \qquad \vec{\omega}_{2,3} = \vec{0} \\ &\vec{\omega}_{3,1} = \vec{0}, \qquad \vec{\omega}_{3,2} = \vec{0}, \qquad \vec{\omega}_{3,3} = \vec{c}_3 \end{aligned}$$

and

$$\begin{aligned} &\vec{v}_{c1,1} = \vec{a}_3 \qquad \vec{v}_{c1,2} = \vec{0} \qquad \vec{v}_{c1,3} = \vec{0} \\ &\vec{v}_{c2,1} = \vec{a}_3 \qquad \vec{v}_{c2,2} = l_2 \vec{b}_2 \qquad \vec{v}_{c2,3} = \vec{0} \\ &\vec{v}_{c3,1} = \vec{a}_3 \qquad \vec{v}_{c3,2} = \vec{0} \qquad \vec{v}_{c3,3} = l_3 \vec{c}_2 \end{aligned}$$

The angular accelerations:

$$\begin{aligned} \vec{\alpha}_1 &= 0 \\ \vec{\alpha}_2 &= \ddot{q}_2 \vec{b}_3 \\ \vec{\alpha}_3 &= \ddot{q}_3 \vec{c}_3 \end{aligned}$$

and

$$\begin{aligned} \vec{a}_{c1} &= \ddot{q}_1 \vec{a}_3 \\ \vec{a}_{c2} &= \ddot{q}_1 \vec{a}_3 + \left(\ddot{q}_2 \vec{b}_3 \times l_2 \vec{b}_1 \right) + \left[\dot{q}_2 \vec{b}_3 \times \left(\dot{q}_2 \vec{b}_3 \times l_2 \vec{b}_1 \right) \right] \\ &= \ddot{q}_1 \vec{a}_3 + \left(\ddot{q}_2 l_2 \vec{b}_2 \right) + \left[\dot{q}_2 \vec{b}_3 \times \dot{q}_2 l_2 \vec{b}_2 \right] \\ &= \ddot{q}_1 \vec{a}_3 + \ddot{q}_2 l_2 \vec{b}_2 - \dot{q}_2^2 l_2 \vec{b}_1 \\ \vec{a}_{c3} &= \ddot{q}_1 \vec{a}_3 + (\ddot{q}_3 \vec{c}_3 \times l_3 \vec{c}_1) + [\dot{q}_3 \vec{c}_3 \times (\dot{q}_3 \vec{c}_3 \times l_3 \vec{c}_1)] \\ &= \ddot{q}_1 \vec{a}_3 + (\ddot{q}_3 l_3 \vec{c}_2) + [\dot{q}_3 \vec{c}_3 \times \dot{q}_3 l_3 \vec{c}_2] \\ &= \ddot{q}_1 \vec{a}_3 + \ddot{q}_3 l_3 \vec{c}_2 - \dot{q}_3^2 l_3 \vec{c}_1 \end{aligned}$$

Furthermore,

$$\begin{aligned} \vec{M}_{c2} &= \vec{0} \\ \vec{M}_{c3} &= \vec{0} \end{aligned}$$

We insert this into Kane's equation

$$\sum_{i=1}^{3}\left[\vec{v}_{ci,j}\cdot m_i\vec{a}_{ci}+\vec{\omega}_{i,j}\cdot\left(\vec{M}_{ci}\cdot\vec{\alpha}_i+\vec{\omega}_i\times\left(\vec{M}_{ci}\cdot\vec{\omega}_i\right)\right)\right]=\tau_j$$

giving

$$\begin{aligned}&\vec{v}_{c1,1}\cdot m_1\vec{a}_{c1}+\vec{\omega}_{1,1}\cdot\left(\vec{M}_{c1}\cdot\vec{\alpha}_1+\vec{\omega}_1\times\left(\vec{M}_{c1}\cdot\vec{\omega}_1\right)\right)\\&+\vec{v}_{c2,1}\cdot m_2\vec{a}_{c2}+\vec{\omega}_{2,1}\cdot\left(\vec{M}_{c2}\cdot\vec{\alpha}_2+\vec{\omega}_2\times\left(\vec{M}_{c2}\cdot\vec{\omega}_2\right)\right)\\&+\vec{v}_{c3,1}\cdot m_3\vec{a}_{c3}+\vec{\omega}_{3,1}\cdot\left(\vec{M}_{c3}\cdot\vec{\alpha}_3+\vec{\omega}_3\times\left(\vec{M}_{c3}\cdot\vec{\omega}_3\right)\right)\quad=\tau_1\end{aligned}$$

$$\begin{aligned}&(\vec{a}_3)\cdot m_1\left(\ddot{q}_1\vec{a}_3\right)\\&+(\vec{a}_3)\cdot m_2\left(\ddot{q}_1\vec{a}_3+\ddot{q}_2l_2\vec{b}_2-\dot{q}_2^2l_2\vec{b}_1\right)\\&+(\vec{a}_3)\cdot m_3\left(\ddot{q}_1\vec{a}_3+\ddot{q}_3l_3\vec{c}_2-\dot{q}_3^2l_3\vec{c}_1\right)\quad=\tau_1\end{aligned}$$

$$\boxed{\begin{aligned}&(m_1+m_2+m_3)\,\ddot{q}_1+m_2l_2\ddot{q}_2\cos q_2\\&-m_2l_2\dot{q}_2^2\sin q_2+m_3\ddot{q}_3l_3\cos q_3-m_3l_3\dot{q}_3^2\sin q_3\quad=\tau_1\end{aligned}}$$

$$\begin{aligned}&\vec{v}_{c1,2}\cdot m_1\vec{a}_{c1}+\vec{\omega}_{1,2}\cdot\left(\vec{M}_{c1}\cdot\vec{\alpha}_1+\vec{\omega}_1\times\left(\vec{M}_{c1}\cdot\vec{\omega}_1\right)\right)\\&+\vec{v}_{c2,2}\cdot m_2\vec{a}_{c2}+\vec{\omega}_{2,2}\cdot\left(\vec{M}_{c2}\cdot\vec{\alpha}_2+\vec{\omega}_2\times\left(\vec{M}_{c2}\cdot\vec{\omega}_2\right)\right)\\&+\vec{v}_{c3,2}\cdot m_3\vec{a}_{c3}+\vec{\omega}_{3,2}\cdot\left(\vec{M}_{c3}\cdot\vec{\alpha}_3+\vec{\omega}_3\times\left(\vec{M}_{c3}\cdot\vec{\omega}_3\right)\right)\quad=\tau_2\end{aligned}$$

$$\vec{v}_{c2,2}\cdot m_2\vec{a}_{c2}+\vec{\omega}_{2,2}\cdot\left(\vec{M}_{c2}\cdot\vec{\alpha}_2+\vec{\omega}_2\times\left(\vec{M}_{c2}\cdot\vec{\omega}_2\right)\right)\quad=\tau_2$$

$$\begin{aligned}&l_2\vec{b}_2\cdot m_2\left(\ddot{q}_1\vec{a}_3+\ddot{q}_2l_2\vec{b}_2-\dot{q}_2^2l_2\vec{b}_1\right)\\&+\vec{b}_3\cdot\left[\left(\vec{0}\right)\cdot\left(\ddot{q}_2\vec{b}_3\right)+\left(\dot{q}_2\vec{b}_3\right)\times\left(\left(\vec{0}\right)\cdot\dot{q}_2\vec{b}_3\right)\right]\quad=\tau_2\end{aligned}$$

$$\boxed{m_2\ddot{q}_1l_2\cos q_2+\left[m_2l_2^2\right]\ddot{q}_2=m_2gl_2\sin q_2}$$

In the same way, we can find

$$\boxed{m_3\ddot{q}_1l_3\cos q_3+\left[m_3l_3^2\right]\ddot{q}_3=m_3gl_3\sin q_3}$$

The mass matrix becomes

$$M=\begin{bmatrix}(m_1+m_2+m_3) & m_2l_2\cos q_2 & m_3l_3\cos q_3\\ m_2l_2\cos q_2 & m_2l_2^2 & 0\\ m_3l_3\cos q_2 & 0 & m_3l_3^2\end{bmatrix}$$

which we see is symmetrical, $M=M^\mathsf{T}$, and also positive definite, $xMx^\mathsf{T}>0\ \forall x$.

Problem 2 (Kinematic modeling of a quadrotor)

In this problem, and the next, we will develop a model for all degrees of freedom for a quadrotor, modeling the quadrotor as a rigid body. See Figure 2 for definition of coordinate systems, and a "free body diagram” with forces and moments acting on the quadrotor.

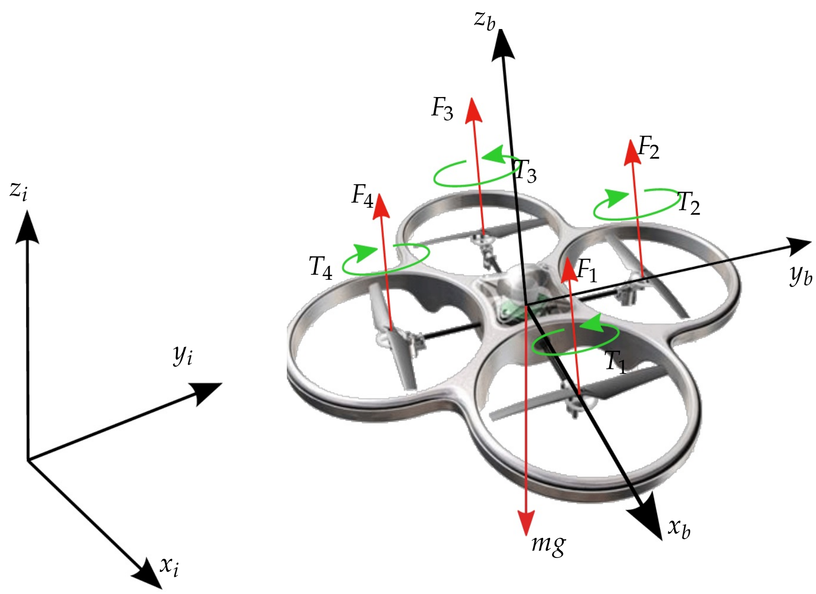

Figure 2: Coordinate systems and forces/moments.

(a) To specify the orientation of the quadrotor, the Z-X-Y Euler angles are sometimes used. These are specified by first a rotation α about the (inertial) z-axis, then β about the intermediate (rotated) x-axis, and finally γ about the body y-axis. Write up an expression for the rotation matrix $\mathbf{R}_b^i = \mathbf{R}_b^i(\boldsymbol{\phi})$ as a function of the Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)^\mathsf{T}$.

Solution: The rotation matrix is found by writing up the simple rotations in the order they appear:

$$
\begin{aligned}
\mathbf{R}_b^i &= \mathbf{R}_{z,\alpha}\mathbf{R}_{x,\beta}\mathbf{R}_{y,\gamma} \\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\alpha & \cos\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta \\ \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta \\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\cos\gamma \end{pmatrix}
\end{aligned}
$$

These Euler angles are used for instance in Vijay Kumar’s lab at University of Pennsylvania, and Raffaello D’Andrea’s lab at ETH. Other groups use other conventions.

(b) Find the kinematic differential equations for this choice of Euler angles. Assume that the angular velocity is given in body-frame. (It is not necessary to perform a matrix inversion for full score.)

Solution: The answer depends on whether the angular velocity is given in the inertial or body system. The latter is more natural, and is assumed in this problem.

The total angular velocity is the sum of the angular velocities of each rotation (6.269), but we need to transform the angular velocities to a common coordinate system when summing. In this case this common system is the body system:

$$
\begin{aligned}
\boldsymbol{\omega}_{ib}^{b} &= \mathbf{R}_{y,-\gamma}\mathbf{R}_{x,-\beta}\begin{pmatrix}0\\0\\\dot{\alpha}\end{pmatrix} + \mathbf{R}_{y,-\gamma}\begin{pmatrix}\dot{\beta}\\0\\0\end{pmatrix} + \begin{pmatrix}0\\\dot{\gamma}\\0\end{pmatrix} \\
&= \begin{pmatrix}-\sin\gamma\cos\beta\dot{\alpha} + \cos\gamma\dot{\beta}\\ \sin\beta\dot{\alpha} + \dot{\gamma}\\ \cos\gamma\cos\beta\dot{\alpha} + \sin\gamma\dot{\beta}\end{pmatrix} \\
&= \begin{pmatrix}-\sin\gamma\cos\beta & \cos\gamma & 0\\ \sin\beta & 0 & 1\\ \cos\gamma\cos\beta & \sin\gamma & 0\end{pmatrix}\begin{pmatrix}\dot{\alpha}\\\dot{\beta}\\\dot{\gamma}\end{pmatrix} \\
&= \mathbf{E}_b(\phi)\dot{\boldsymbol{\phi}}
\end{aligned}
$$

where $\dot{\boldsymbol{\phi}} = (\dot{\alpha}, \dot{\beta}, \dot{\gamma})^{\mathsf{T}}$ and

$$
\mathbf{E}_b(\boldsymbol{\phi}) = \begin{pmatrix}-\sin\gamma\cos\beta & \cos\gamma & 0\\ \sin\beta & 0 & 1\\ \cos\gamma\cos\beta & \sin\gamma & 0\end{pmatrix}.
$$

The kinematic differential equations are then

$$
\dot{\boldsymbol{\phi}} = \mathbf{E}_b^{-1}(\boldsymbol{\phi})\boldsymbol{\omega}_{ib}^{b}.
$$

Compare (6.316) for the choice of Euler angles used in the book (roll-pitch-yaw Euler angles). Not asked for: The inverse of $\mathbf{E}_b(\boldsymbol{\phi})$ is

$$
\mathbf{E}_b^{-1}(\boldsymbol{\phi}) = \frac{1}{\cos\beta}\begin{pmatrix}-\sin\gamma & 0 & \cos\gamma\\ \cos\gamma\cos\beta & 0 & \sin\gamma\cos\beta\\ \sin\beta\sin\gamma & \cos\beta & -\cos\gamma\sin\beta\end{pmatrix}
$$

and we see that we have a singularity for $\beta = \pi/2 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

Problem 3 (Complete dynamic model of a quadrotor)

In this problem, we will continue to develop the complete dynamic model of the quadrotor by modeling the kinetics. The forces and moments acting on the quadrotor are illustrated in Figure 2. The body system has origin in the center of mass, and the quadrotor has mass m and an inertia matrix $\mathbf{M}_{b/c}^{b}$. Note that the moments T_i due to rotation of the rotors give moments acting about the z_b-axis, and that the rotor forces F_i will give cause to moments about the x_b and y_b axis, with "arm" (distance from center of mass to rotor) L for all rotors. Note also that T_i has a "sign" defined in the figure, due to the default direction of rotation of the rotors.

(a) Why is it natural to use the Newton-Euler equations of motions as starting point, rather than the Lagrange equations of motion?

Solution: We will model the quadcopter in all ("six") degrees of freedom, therefore there are no "forces of constraints" to eliminate.

(b) Write up expressions for the force and torque vectors acting on the center of mass, $\mathbf{F}_{bc}^{b}$ and $\mathbf{T}_{bc}^{b}$, decomposed in the body system, as function of the forces and torques defined in Figure 2.

Solution:

$$\mathbf{F}^b_{bc} = \mathbf{R}^b_i(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix}, \quad \mathbf{T}^b_{bc} = \begin{pmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ T_1 - T_2 + T_3 - T_4 \end{pmatrix}$$

(The very observant will have noticed that the sign of T_i in the figure is wrong when you consider the propeller configuration used on this particular quadcopter.)

(c) What are the equations of motion of the quadrotor, on vector form? The components of vectors equations in the answer should amount to 12 first-order differential equations, including the answer from Problem 2(b).

Solution: There are (at least) two different correct answers here, depending on whether the velocity is expressed in body-fixed or inertial coordinates (and whether angular velocity is expressed in body-fixed or inertial coordinates, see Problem 2(b)).

First, the force balance. With velocity in inertial coordinates (which perhaps is simplest and most natural in this case), we can write down

$$\begin{aligned} m\dot{\mathbf{v}}^i_c &= \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \mathbf{R}^i_b(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix} \\ \dot{\mathbf{r}}^i_c &= \mathbf{v}^i_c \end{aligned}$$

where the latter equation is a kinematic differential equation.

Alternatively, with velocity in body-fixed coordinates, we get (since $\dot{\mathbf{v}}^i_c = \mathbf{R}^i_b(\boldsymbol{\phi}) \left(\dot{\mathbf{v}}^b_c + (\boldsymbol{\omega}^b_{ib})^\times \mathbf{v}^b_c \right)$)

$$\begin{aligned} m\dot{\mathbf{v}}^b_c &= -m(\boldsymbol{\omega}^b_{ib})^\times \mathbf{v}^b_c + \mathbf{R}^b_i(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix} \\ \dot{\mathbf{r}}^i_c &= \mathbf{R}^i_b(\boldsymbol{\phi}) \mathbf{v}^b_c. \end{aligned}$$

Second, the torque balance (Euler's equation) is

$$\mathbf{M}^b_{b/c} \dot{\boldsymbol{\omega}}^b_{ib} = \mathbf{T}^b_{bc} - \boldsymbol{\omega}^b_{ib} \times \left(\mathbf{M}^b_{b/c} \boldsymbol{\omega}^b_{ib} \right).$$

Together with

$$\dot{\boldsymbol{\phi}} = \mathbf{E}_b^{-1}(\boldsymbol{\phi}) \boldsymbol{\omega}^b_{ib}$$

from Problem 2(b), this specifies 12 ODEs.

(Problems 2 and 3 are based on the article: Daniel Mellinger, Nathan Michael and Vijay Kumar, *Trajectory generation and control for precise aggressive maneuvers with quadrotors*, The International Journal of Robotics Research 31(5):664–674, 2012.)

Problem 4 (Robotic manipulator)

We wish to model a robotic manipulator with the configuration shown in Figure 3.

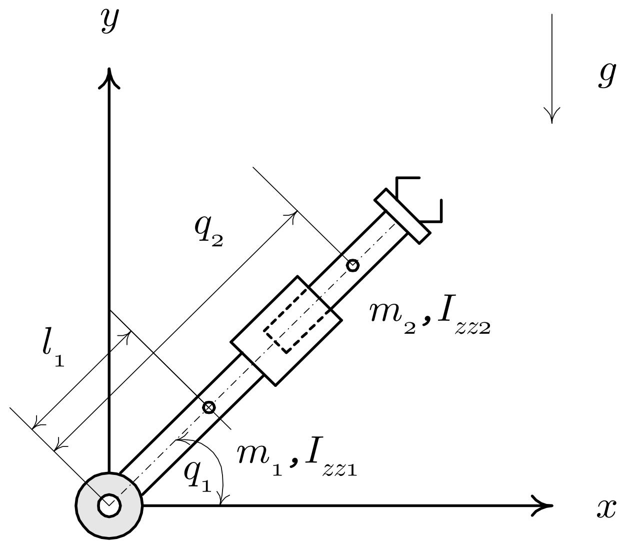

Figure 3: Manipulator

The manipulator has two degrees of freedom (that is, two generalized coordinates). We will use Lagrange's equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \qquad i = 1, 2$$

to set up the equations of motion for the manipulator, where

$$\mathcal{L} = T - U = \text{kinetic energy} - \text{potential energy} \tag{1}$$

and q_1 and q_2 are the generalized coordinates (see Figure 3). The axis x and y can be assumed fixed, that is, axes in an inertial system.

We will disregard mass and inertia of the motors in this problem. The moment of inertia of the first arm is denoted I_{zz1}, while the moment of inertia of the second arm is I_{zz2} (each referenced to the center of mass of the respective arm). The dots on the figure marks the centers of mass for each arm. The arrow marked g illustrates the direction of gravity.

(a) Find the total kinetic energy, T, for the manipulator, and show that it can be written on the form $T = \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ where $\mathbf{q} = \begin{pmatrix} q_1 & q_2 \end{pmatrix}^{\mathsf{T}}$ and

$$\mathbf{M}(\mathbf{q}) = \begin{pmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{pmatrix}.$$

Solution: The expression for kinetic energy for a rigid body (each arm) can be written

$$\frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2} \vec{w}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{w}_{ib}.$$

We are only interested in motion in the plane, so we disregard (assume zero) velocity in the z-direction, and angular velocity not about the z-axis.

The velocity of center of mass (superscript i denotes the base (inertial) system) and angular velocity (about z-axis) for each body becomes:

- Arm 1:

$$\mathbf{r}^i_{c1} = \begin{pmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \end{pmatrix}, \qquad \mathbf{v}^i_{c1} = \begin{pmatrix} -l_1 \sin q_1 \dot{q}_1 \\ l_1 \cos q_1 \dot{q}_1 \end{pmatrix}, \qquad w_{z1} = \dot{q}_1$$

$$\vec{v}_{c1} \cdot \vec{v}_{c1} = \left(\mathbf{v}^i_{c1}\right)^\mathsf{T} \mathbf{v}^i_{c1} = l_1^2 \sin^2 q_1 \dot{q}_1^2 + l_1^2 \cos^2 q_1 \dot{q}_1^2 = l_1^2 \dot{q}_1^2$$

- Arm 2:

$$\mathbf{r}^i_{c2} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \qquad \mathbf{v}^i_{c2} = \begin{pmatrix} \dot{q}_2 \cos q_1 - q_2 \sin q_1 \dot{q}_1 \\ \dot{q}_2 \sin q_1 + q_2 \cos q_1 \dot{q}_1 \end{pmatrix}, \qquad w_{z2} = \dot{q}_1$$

$$\begin{aligned} \vec{v}_{c2} \cdot \vec{v}_{c2} &= \left(\mathbf{v}^i_{c2}\right)^\mathsf{T} \mathbf{v}^i_{c2} \\ &= \dot{q}_2^2 \cos^2 q_1 - \dot{q}_2 \cos q_1 q_2 \sin q_1 \dot{q}_1 + q_2^2 \sin^2 q_1 \dot{q}_1^2 \\ &\quad + \dot{q}_2^2 \sin^2 q_1 + \dot{q}_2 \sin q_1 q_2 \cos q_1 \dot{q}_1 + q_2^2 \cos^2 q_1 \dot{q}_1^2 \\ &= \dot{q}_2^2 + q_2^2 \dot{q}_1^2 \end{aligned}$$

The kinetic energy for each body becomes

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{2} I_{zz1} \dot{q}_1^2, \\ T_2 &= \frac{1}{2} m_2 \left(q_2^2 \dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2} I_{zz2} \dot{q}_1^2, \end{aligned}$$

and the total kinetic energy for the system is

$$T = T_1 + T_2.$$

Remark: For a simple system like the first body, one can "see" directly that the velocity is $|\vec{v}_{c1}| = l_1 \dot{q}_1$. This may also be possible for the second body, but as setups become more complicated, it is easy to make mistakes when using the "see"-method.

(b) Find the potential energy, U, for the manipulator.

Solution: The potential energy for each arm is

$$\begin{aligned} U_1 &= m_1 g l_1 \sin q_1, \\ U_2 &= m_2 g q_2 \sin q_1. \end{aligned}$$

The total potential energy is

$$U = U_1 + U_2.$$

(c) Derive the equations of motion for the manipulator by use of Lagrange's equation.

Solution: The Lagrangian of the manipulator is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= T_1 + T_2 - U_1 - U_2 \\ &= \frac{1}{2} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{2} I_{zz1} \dot{q}_1^2 + \frac{1}{2} m_2 \left(q_2^2 \dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2} I_{zz2} \dot{q}_1^2 - (m_1 l_1 + m_2 q_2) g \sin q_1 \end{aligned}$$

We use Lagrange's equation,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \qquad i = 1, 2$$

where

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= m_1 l_1^2 \dot{q}_1 + I_{zz1}\dot{q}_1 + m_2 q_2^2 \dot{q}_1 + I_{zz2}\dot{q}_1 \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= m_1 l_1^2 \ddot{q}_1 + I_{zz1}\ddot{q}_1 + m_2 q_2^2 \ddot{q}_1 + I_{zz2}\ddot{q}_1 + 2 m_2 q_2 \dot{q}_2 \dot{q}_1 \\
\frac{\partial \mathcal{L}}{\partial q_1} &= -\left(m_1 l_1 + m_2 q_2\right) g \cos q_1 \\
\frac{\partial \mathcal{L}}{\partial \dot{q}_2} &= m_2 \dot{q}_2 \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} &= m_2 \ddot{q}_2 \\
\frac{\partial \mathcal{L}}{\partial q_2} &= m_2 q_2 \dot{q}_1^2 - m_2 g \sin q_1
\end{aligned}$$

which gives these equations of motion:

$$\begin{aligned}
\left(m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2\right) \ddot{q}_1 + 2 m_2 q_2 \dot{q}_2 \dot{q}_1 + \left(m_1 l_1 + m_2 q_2\right) g \cos q_1 &= \tau_1 \\
m_2 \ddot{q}_2 - m_2 q_2 \dot{q}_1^2 + m_2 g \sin q_1 &= \tau_2
\end{aligned}$$

Here, τ_1 is the generalized force corresponding to q_1, that is, a motor torque giving rotation, and τ_2 is the generalized force corresponding to q_2, a motor force giving translational motion of arm 2.

(d) In this problem you should show that the equations of motion in (c) can be written

$$\mathbf{M}\left(\mathbf{q}\right)\ddot{\mathbf{q}} + \mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right)\dot{\mathbf{q}} + \mathbf{g}\left(\mathbf{q}\right) = \boldsymbol{\tau}. \tag{2}$$

Explain how several choices are possible for $\mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right)$. Show that when you use the Christoffel symbols (cf. eq. (8.57)–(8.58) in the book), then

$$\mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right) = \begin{pmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}.$$

What is the vector $\mathbf{g}\left(\mathbf{q}\right)$?

Solution: We first find the "mass matrix" $\mathbf{M}(\mathbf{q})$ and $\mathbf{g}(\mathbf{q})$,

$$\begin{aligned}
\mathbf{M}(\mathbf{q}) &= \begin{pmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{pmatrix} \\
\mathbf{g}(\mathbf{q}) &= \begin{pmatrix} \left(m_1 l_1 + m_2 q_2\right) g \cos q_1 \\ m_2 g \sin q_1 \end{pmatrix}
\end{aligned}$$

We see that we have term containing multiplications of derivatives of q_i, that is, $\dot{q}_1 \dot{q}_2$. This term can be placed two places in $\mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right)$, depending on if we extract $\dot{q}_1$ or $\dot{q}_2$.
We find the Christoffel symbols c_{ijk} by using the equation

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial m_{kj}}{q_i} + \frac{\partial m_{ik}}{q_j} - \frac{\partial m_{ij}}{q_k}\right)$$

which gives

$$
\begin{aligned}
c_{111} &= \frac{1}{2}\left(\frac{\partial m_{11}}{\partial q_1} + \frac{\partial m_{11}}{\partial q_1} - \frac{\partial m_{11}}{\partial q_1}\right) = 0 \\
c_{112} &= \frac{1}{2}\left(\frac{\partial m_{21}}{\partial q_1} + \frac{\partial m_{12}}{\partial q_1} - \frac{\partial m_{11}}{\partial q_2}\right) = -m_2 q_2 \\
c_{122} &= \frac{1}{2}\left(\frac{\partial m_{22}}{\partial q_1} + \frac{\partial m_{12}}{\partial q_2} - \frac{\partial m_{12}}{\partial q_2}\right) = 0 \\
c_{211} &= \frac{1}{2}\left(\frac{\partial m_{11}}{\partial q_2} + \frac{\partial m_{21}}{\partial q_1} - \frac{\partial m_{21}}{\partial q_1}\right) = m_2 q_2 \\
c_{222} &= \frac{1}{2}\left(\frac{\partial m_{12}}{\partial q_2} + \frac{\partial m_{21}}{\partial q_2} - \frac{\partial m_{22}}{\partial q_1}\right) = 0 \\
c_{222} &= \frac{1}{2}\left(\frac{\partial m_{22}}{\partial q_2} + \frac{\partial m_{22}}{\partial q_2} - \frac{\partial m_{22}}{\partial q_2}\right) = 0
\end{aligned}
$$

where $c_{211} = c_{121}$ and $c_{212} = c_{122}$ due to the symmetry in the $\mathbf{M}(\mathbf{q})$ matrix. (Note that, for a fixed k, we have $c_{ijk} = c_{jik}$)

We can now find the elements in the $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ matrix by

$$
c_{ij} = \sum_{k=1}^{n=2} c_{ijk} \dot{q}_k
$$

which gives

$$
\begin{aligned}
c_{11} &= c_{111}\dot{q}_1 + c_{112}\dot{q}_2 = m_2 q_2 \dot{q}_2 \\
c_{12} &= c_{121}\dot{q}_1 + c_{122}\dot{q}_2 = m_2 q_2 \dot{q}_1 \\
c_{21} &= c_{211}\dot{q}_1 + c_{212}\dot{q}_2 = -m_2 q_2 \dot{q}_1 \\
c_{22} &= c_{221}\dot{q}_1 + c_{222}\dot{q}_2 = 0
\end{aligned}
$$

We then get

$$
\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{pmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}
$$

(e) What matrix properties do the matrices $\mathbf{M}(\mathbf{q})$ and $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ possess?

Solution: The mass matrix $\mathbf{M}(\mathbf{q})$ is

$$
\begin{aligned}
&\text{symmetric :} \quad \mathbf{M} = \mathbf{M}^\top \\
&\text{positive definite :} \quad \mathbf{x}^\top \mathbf{M}(\mathbf{q})\mathbf{x} > 0 \qquad \forall \mathbf{x} \neq 0
\end{aligned}
$$

The matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ has no specific property, but as we will see next, we can choose it so $\dot{\mathbf{M}}(\mathbf{q}) - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric.

(f) Show (using the matrices developed in this problem) that the matrix $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric when $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ has been defined by use of the Christoffel symbols.

Solution:

$$
\dot{\mathbf{M}}(\mathbf{q}) = \begin{pmatrix} 2m_2 q_2 \dot{q}_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{pmatrix} 2m_2 q_2 \dot{q}_2 & 2m_2 q_2 \dot{q}_1 \\ -2m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}
$$

which gives

$$\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right) = \begin{pmatrix} 2m_2q_2\dot{q}_2 & 2m_2q_2\dot{q}_1 \\ -2m_2q_2\dot{q}_1 & 0 \end{pmatrix} - \begin{pmatrix} 2m_2q_2\dot{q}_2 & 2m_2q_2\dot{q}_1 \\ -2m_2q_2\dot{q}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2m_2q_2\dot{q}_1 \\ 2m_2q_2\dot{q}_1 & 0 \end{pmatrix}$$

We see that $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right)$ is skew symmetric.

(g) Show that the derivative of the energy function $E(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) + U(\mathbf{q})$ is

$$\dot{E}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^\mathsf{T}\boldsymbol{\tau}.$$

Hint: Use $T = \frac{1}{2}\dot{\mathbf{q}}^\mathsf{T}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ and (2), do not insert the detailed model. Use that $\frac{\partial U}{\partial \mathbf{q}} = \mathbf{g}^\mathsf{T}(\mathbf{q})$. What can we say about passivity of the manipulator?

Solution:

$$\begin{aligned} \dot{E}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\dot{\mathbf{q}}^\mathsf{T}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{\partial U}{\partial \mathbf{q}}\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^\mathsf{T}\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^\mathsf{T}\dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \frac{\partial U}{\partial \mathbf{q}}\dot{\mathbf{q}} \end{aligned}$$

Inserting (2) and $\frac{\partial U}{\partial \mathbf{q}} = \mathbf{g}^\mathsf{T}(\mathbf{q})$ gives

$$\begin{aligned} \dot{E}(\mathbf{q}, \dot{\mathbf{q}}) &= \dot{\mathbf{q}}^\mathsf{T}\left(\boldsymbol{\tau} - \mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right)\dot{\mathbf{q}} - \mathbf{g}\left(\mathbf{q}\right)\right) + \frac{1}{2}\dot{\mathbf{q}}^\mathsf{T}\dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{g}^\mathsf{T}(\mathbf{q})\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^\mathsf{T}\boldsymbol{\tau} + \frac{1}{2}\dot{\mathbf{q}}^\mathsf{T}\left(\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\right)\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^\mathsf{T}\boldsymbol{\tau} \end{aligned}$$

where we used that $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric.

This almost implies that the manipulator is passive with applied generalized forces $\boldsymbol{\tau}$ (torque on arm 1 and force on arm 2) as input and generalized velocities $\dot{\mathbf{q}}$ as output (angular velocity of arm 1 and velocity of arm 2).

The only glitch is that the energy function that proves passivity should be positive, while the potential energy seems to be unbounded below. However, if we know that $U > U_{\mathrm{min}}$ for some constant U_{min}, we can use $V = E - U_{\mathrm{min}} = T + U - U_{\mathrm{min}}$ as energy function, which fulfill

$$\dot{V}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^\mathsf{T}\boldsymbol{\tau}.$$

The potential energy is bounded in this problem if $0 < q_2 < q_{2,\mathrm{max}}$.