Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4120 Digital Signal Processing
Suggested Solutions for Problem Set 1

Problem 1

(a) The signals $x(n)$ and $y(n)$ are shown in Figure 1.

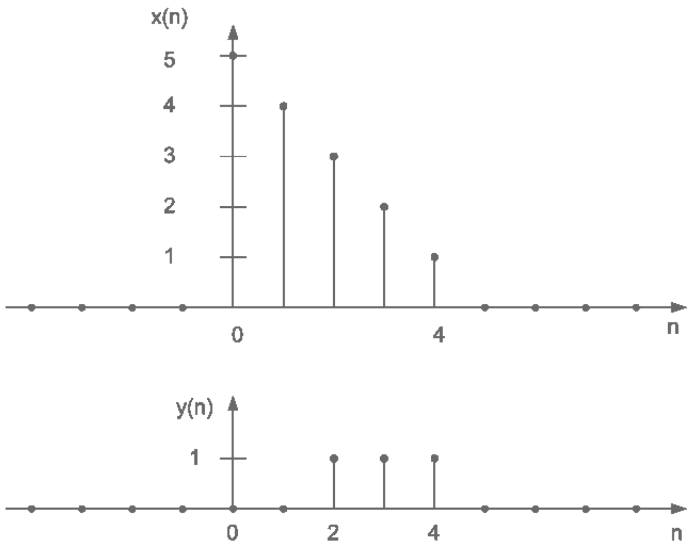

Figure 1: The signals $x(n)$ and $y(n)$.

(b) When k is positive, the signal will be shifted to the right, and for negative k, the signal will be shifted left. Thus, we get the sketches shown in Figure 2.

(c) The signal $x(-n)$ will be $x(n)$ flipped about $n = 0$. The resulting sketch is shown in Figure 3.

(d) The signal $x(5 - n)$ will be a flipped version of $x(n)$ shifted to the right. The sketch is shown in Figure 4.

(e) The signal $y(n)$ is a window signal. When multiplying $x(n)$ by $y(n)$, the two first samples of $x(n)$ will be removed. Thus, we get

$$z(n) = \begin{cases} 5 - n & 2 \leq n \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

The sketch of the resulting signal $z(n)$ is shown in Figure 5.

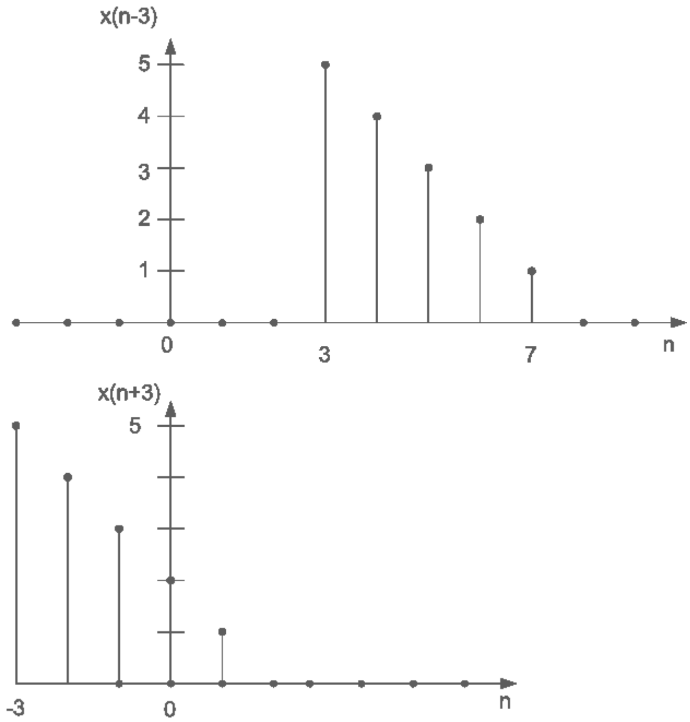

Figure 2: Shifted signals, $x(n-3)$ and $x(n+3)$.

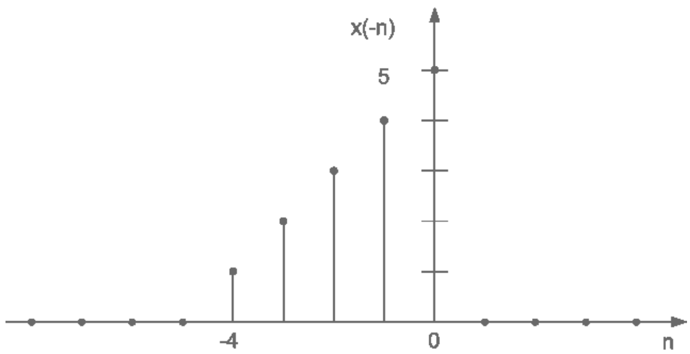

Figure 3: Flipped signal, $x(-n)$.

(f) The signal $x(n)$ can be expressed as follows.

$$x(n) = 5\delta(n) + 4\delta(n-1) + 3\delta(n-2) + 2\delta(n-3) + \delta(n-4)$$

(g) $y(n)$ can be expressed as the difference between two unit step signals as shown in Figure 6. Thus, we get

$$y(n) = u(n-2) - u(n-5).$$

(h) The energy of $x(n)$ can be found as:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = 25 + 16 + 9 + 4 + 1 = 55.$$

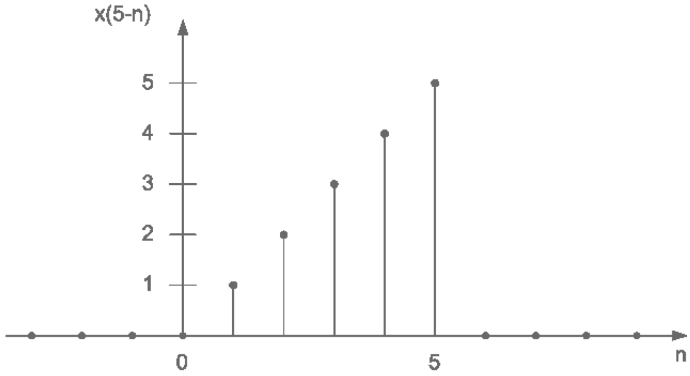

Figure 4: Flipped and shifted signal, $x(5-n)$.

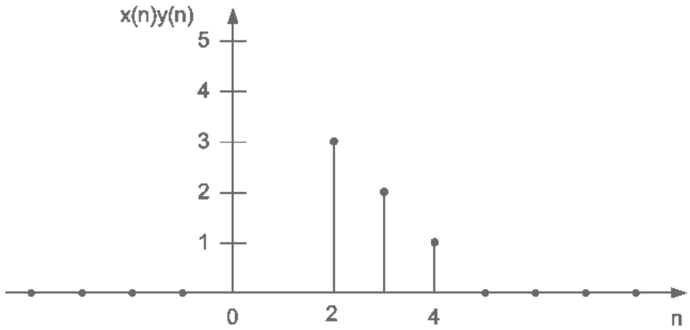

Figure 5: Signal $x(n)y(n)$.

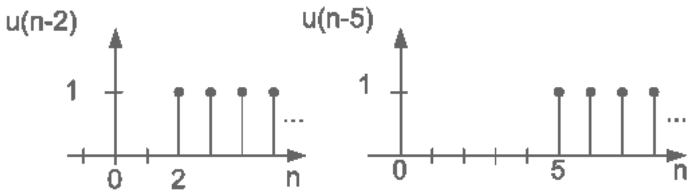

Figure 6: Signals, $u(n-2)$ and $u(n-5)$.

Problem 2

(a) The normalized frequency is used to represent discrete time signals in the frequency domain. Discrete time signals have a periodic structure in the frequency domain. The period is $[-.5, 0.5)$ (or $[0, 1)$). Using the first alternative we must have that $f_1 \in [-0.5, 0.5)$ which corresponds to $F_1 = F_s * f_1 \in [-3000, 3000)$ Hz for $F_s = 6000$ Hz.

(b) A sampled cosine signal of duration $N = 4$ seconds with normalized frequency $f_1 = 0.1$ can be generated in Matlab as:

```
F_s = 6000;
N = 4;
```

```
n = 0 : (F_s*N - 1);
f_1 = 0.4;
signal = cos(2*pi*f_1*n);
```

The resulting signal can be played with Matlab as:

```
soundsc(signal,Fs)
```

(c) For $F_s = 1000/3000/12000$ Hz the normalized frequency $f_1 = 0.3$ corresponds to $F_1 = f_1 * F_s = 300/900/3600$ Hz. Thus we will hear a higher tone when we increase the sampling rate. Thus a constant normalized frequency can correspond to any physical frequency depending on the chosen sampling rate. Especially for filter design we will see that this is an advantage.

(d) Now we use the formula $f_1 = F_1/F_s$. Thus for a sampling rate of $F_2 = 8000$ Hz the physical frequencies $F_1 = 1000/3000/6000$ Hz correspond to $f_1 = F_1/F_s = 0.125/0.375/0.75$. Logically one should expect a higher tone as the physical frequency F_1 increases. However, for $F_1 > F_2/2 = 4000$ Hz we violate the Nyquist sampling theorem. This applies for $F_1 = 6000$ Hz, i.e. $f_1 = 0.75 > 0.5$. Due to the periodicity of one this frequency will be converted to $1 - f_1 = 0.25$ which corresponds to that we hear the physical frequency $F_1 = 0.25 * 8000 = 2000$ Hz.

Problem 3

(a) Since this system involves the quadratic term $x^2(n-1)$, it is not linear. However, since the difference equation has constant coefficients (independent of n), the system is time-invariant. It is also causal, since $y(n)$ only depends on present and past samples of $x(n)$.

To show the time-invariance property from the definition, we excite the system with a delayed signal $x_1(n) = x(n-k)$, and find the output signal $y_1(n)$. If $y_1(n) = y(n-k)$, the system is time-invariant.

$$\begin{aligned} y_1(n) &= x_1(n-k) - x_1^2(n-k-1) \\ &= y(n-k) \end{aligned}$$

Thus, we have shown that the system is time-invariant.

Now, to show that it is not linear from the definition, we excite the system with two different signals $x_1(n)$ and $x_2(n)$. We call the output signals $y_1(n)$ and $y_2(n)$ respectively.

$$\begin{aligned} y_1(n) &= x_1(n) - x_1(n-1)^2 \\ y_2(n) &= x_2(n) - x_2(n-1)^2 \end{aligned}$$

Then, we excite the system with another signal, $x_3(n) = a_1x_1(n) + a_2x_2(n)$. If the system is linear then the corresponding output signal should be

$y_3(n) = a_1 y_1(n) + a_2 y_2(n)$.

$$\begin{aligned}
y_3(n) &= x_3(n) - x_3^2(n-1) \\
&= a_1 x_1(n) + a_2 x_2(n) - (a_1 x_1(n-1) + a_2 x_2(n-1))^2 \\
&= a_1 x_1(n) + a_2 x_2(n) \\
&\quad - ((a_1 x_1(n-1))^2 + 2a_1 a_2 x_1(n-1) x_2(n-1) + (a_2 x_2(n-1))^2) \\
&= a_1 y_1(n) + a_2 y_2(n) - 2a_1 a_2 x_1(n) x_2(n-1) \\
&\neq a_1 y_1(n) + a_2 y_2(n)
\end{aligned}$$

Thus, we have shown that the system is not linear.

(b) Since $y(n)$ is now a linear combination of samples from $x(n)$, this system is linear. However, since one of the coefficients is dependent on n, the system is not time-invariant. Finally, since $y(n)$ only depends on present and past samples of $x(n)$, the system is causal.

We now check time-invariance and linearity by the definitions. First time-invariance. Let $x_1(n) = x(n-k)$. Then

$$\begin{aligned}
y_1(n) &= nx_1(n) + 2x_1(n-2) \\
&= nx(n-k) + 2x(n-k-2) \\
&\neq y(n-k) \\
&= (n-k)x(n-k) + 2x(n-k-2)
\end{aligned}$$

Now, we check linearity. Let $x_3(n) = a_1 x_1(n) + a_2 x_2(n)$

$$\begin{aligned}
y_1(n) &= nx_1(n) + 2x_1(n-2) \\
y_2(n) &= nx_2(n) + 2x_2(n-2) \\
y_3(n) &= nx_3(n) + 2x_3(n-2) \\
y_3(n) &= a_1(nx_1(n) + 2x_1(n-2)) + a_2(nx_2(n) + x_2(n-2)) \\
&= a_1 y_1(n) + a_2 y_2(n)
\end{aligned}$$

Thus, the system is linear.

(c) In this system $y(n)$ is a simple linear combination of present and past samples of $x(n)$ with constant coefficients. Thus, this system is time-invariant, linear, and causal.

Again, we can check this by the definitions.

$$\begin{aligned}
y_1(n) &= x_1(n) - x_1(n-1) \\
&= x(n-k) - x(n-k-1) \\
&= y(n-k)
\end{aligned}$$

Thus, we have shown time-invariance. Then we show that the system is linear.

$$\begin{aligned}
y_1(n) &= x_1(n) - x_1(n-1) \\
y_2(n) &= x_2(n) - x_2(n-1) \\
y_3(n) &= x_3(n) - x_3(n-1) \\
&= a_1x_1(n) + a_2x_2(n) - a_1x_1(n-1) - a_2x_2(n-1) \\
&= a_1y_1(n) + a_2y_2(n)
\end{aligned}$$

(d) This system is both linear and time-invariant for the same reasons as the system in (c). However, in this system $y(n)$ depends on a future sample of $x(n)$. Thus, the system is not causal.

Problem 4

(a) The unit sample response is obtained at the output of the system when the system is excited by a unit sample $\delta(n)$. Thus, if we replace the signal $x(n)$ in the difference equation by the δ signal, we can replace the output signal $y(n)$ by the unit sample response $h(n)$. For the first system, we get

$$\begin{aligned}
h(n) &= \delta(n) + 2\delta(n-1) + \delta(n-2) \\
&= \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 1 & n = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

For the second system we have

$$h(n) = -0.9h(n-1) + \delta(n)$$

In this case we have a recursive equation. An iterative method can be used to find the unit sample response. Note that $h(n) = 0$ for $n < 0$ since the system is causal. So we only have to find $h(n)$ for $n \geq 0$. We start by determining $h(0)$.

$$h(0) = 0.9h(-1) + 1 = -0.9 \cdot 0 + 1 = 1$$

Now, for $n \neq 0$, we have

$$h(n) = -0.9h(n-1).$$

Now, we do some iterations.

$$\begin{aligned}
h(1) &= -0.9h(0) = -0.9 \\
h(2) &= -0.9h(1) = (-0.9)^2 \\
h(3) &= -0.9h(2) = (-0.9)^3 \\
&\vdots \\
h(n) &= (-0.9)^n \;\text{ for } n \geq 0 \\
&= (-0.9)^n u(n)
\end{aligned}$$

(b) As we saw in (a), the first system has a finite length unit sample response, while the unit sample response of the other system was of infinite length. Thus, the two systems are FIR and IIR, respectively.

(c) To check whether the systems are stable, we need to check whether

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

For the first system, we get

$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 2 + 1 = 4$$

so this system is stable. Note that all FIR systems are stable. For the second system we get

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |h(n)| &= \sum_{n=0}^{\infty} |(-0.9)^n| \\ &= \sum_{n=0}^{\infty} 0.9^n \\ &= \frac{1}{1-0.9} \\ &= 10\end{aligned}$$

so this system is also stable.

(d) The filters are represented in Figure 7 and Figure 8.

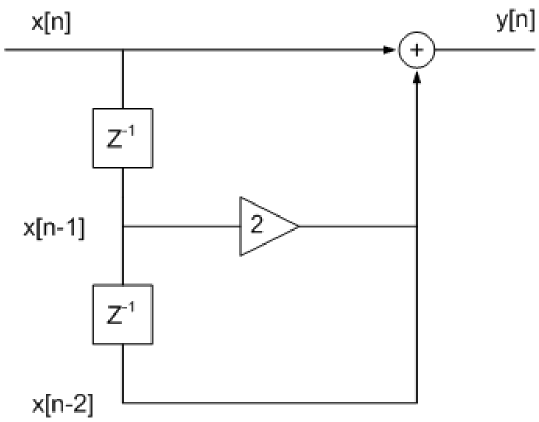

Figure 7: Filter structure of the first system

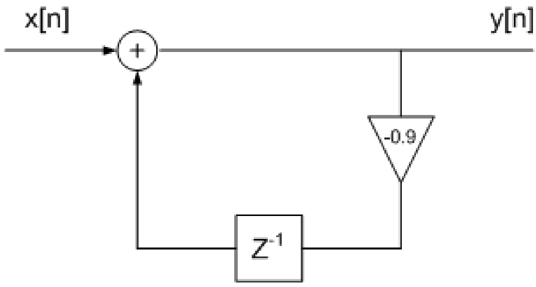

Figure 8: Filter structure of the second system

Problem 5

(a) The signal $y_1(n)$ can be computed as follows

$$\begin{aligned} y_1(n) &= x(n) * h(n) = x(n) * [\delta(n) + \delta(n-1) + \delta(n-2)] \\ &= x(n) * \delta(n) + x(n) * \delta(n-1) + x(n) * \delta(n-2)] \\ &= x(n) + x(n-1) + x(n-2) \end{aligned}$$

To get the final result we can use a graphical computation method, which is displayed in Figure 9.

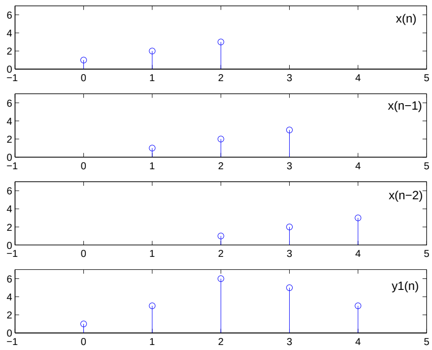

Figure 9: Computation of $y_1(n)$

(b) The second ouput is shown in Figure 10 and it can be computed with Matlab as follows:

```
y_1 = [1 3 6 5 3];
n = 0:10;
h_2 = (0.9).^n;
```

```
y_2 = conv(h_2, y_1);
n = 0:length(y_2)-1;
stem(n, y_2);
```

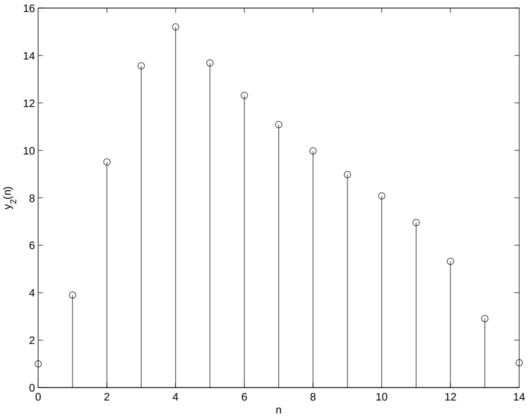

Figure 10: Output signal after filtering by both $h_1(n)$ and $h_2(n)$

(c) The length of an output signal $y(n)$ is $L_x + L_h - 1$, where L_x and L_h are the length of the input signal and the unit sample response of the filter. In our problem, $y_1(n)$ has length $3 + 3 - 1 = 5$ and $y_2(n)$ has length $5 + 11 - 1 = 15$.

(d) Since the convolution operation is commutative, it does not matter which filter comes first. Thus, the plot of the output signal after the second filter, $h_1(n)$ in this case, is exactly equal to the one in Figure 10. However, the output of the first filter, $h_2(n)$ in this case, is different than before and it is shown in Figure 11.

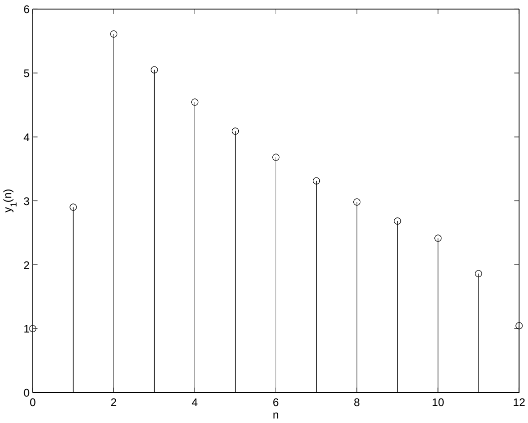

Figure 11: Output signal after filtering by $h_2(n)$