TTK4115 Linear System Theory
Department of Engineering Cybernetics
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Solution to homework assignment 3

Problem 1: State feedback

a) The controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 4 & 32 \\ 1 & 5 & 25 \end{bmatrix}.$$

Because the controllability matrix has full row rank, i.e. $\text{rank}(\mathcal{C}) = 3 = n$, we conclude that the system is controllable.

b) The characteristic polynomial of $\bar{\mathbf{A}}$ is given by

$$\begin{aligned}
\det(\bar{\mathbf{A}} - \lambda\mathbf{I}) &= \det(\mathbf{A} - \mathbf{BK} - \lambda\mathbf{I}) \\
&= \det\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) \\
&= \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 4 \\ -k_1 & -k_2 & 5-k_3-\lambda \end{vmatrix} \\
&= -\lambda^3 + (9-k_3)\lambda^2 + (4k_3 - 4k_2 - 23)\lambda - 8k_1 + 4k_2 - 3k_3 + 15.
\end{aligned}$$

c) The characteristic polynomial of $\bar{\mathbf{A}}$ should be equal to

$$\begin{aligned}
\det(\bar{\mathbf{A}} - \lambda\mathbf{I}) &= (\bar{\lambda}_1 - \lambda)(\bar{\lambda}_2 - \lambda)(\bar{\lambda}_3 - \lambda) \\
&= (-1-\lambda)(-2-\lambda)(-3-\lambda) \\
&= -\lambda^3 - 6\lambda^2 - 11\lambda - 6.
\end{aligned}$$

Comparing this to the characteristic polynomial obtained in a), we obtain the equations

$$\begin{aligned}
9 - k_3 &= -6 \\
4k_3 - 4k_2 - 23 &= -11 \\
-8k_1 + 4k_2 - 3k_3 + 15 &= -6.
\end{aligned}$$

By solving these equalities, we obtain the feedback matrix $\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$, with

$$k_1 = 3, \qquad k_2 = 12 \qquad \text{and} \qquad k_3 = 15.$$

Problem 2: Linear quadratic regulator and tracking control

a) Using the differential equation, we have

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ -2y(t) - \dot{y}(t) + 2u(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -2x_1(t) - x_2(t) + 2u(t) \end{bmatrix}.$$

Therefore, we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t).$$

Note that $y(t) = x_1(t)$. Therefore, we have

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

b) From the first equation in (2), we obtain that

$$\mathbf{A}\mathbf{x}_{eq} = -\mathbf{B}u_{eq}.$$

Because $\mathbf{A}$ is invertible, we get the following expression for $\mathbf{x}_{eq}$:

$$\mathbf{x}_{eq} = -\mathbf{A}^{-1}\mathbf{B}u_{eq} = -\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} u_{eq} = -\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} u_{eq} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{eq}.$$

By substituting this in the second equation in (2), we obtain

$$r = \mathbf{C}\mathbf{x}_{eq} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{eq} = u_{eq}.$$

Hence, we have

$$u_{eq} = Gr,$$

with

$$G = 1.$$

Substituting this in the expression for $\mathbf{x}_{eq}$ leads to

$$\mathbf{x}_{eq} = \mathbf{F}r,$$

with

$$\mathbf{F} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

c) From the definition of the state error, the control input and the system equations, we have

$$\dot{\tilde{\mathbf{x}}}(t) = \dot{\mathbf{x}}(t) - \underbrace{\dot{\mathbf{x}}_{eq}}_{=\mathbf{0}} = \mathbf{A}\mathbf{x}(t) - \mathbf{B}u(t) = \mathbf{A}(\mathbf{x}_{eq} + \tilde{\mathbf{x}}(t)) + \mathbf{B}(u_{eq} + \tilde{u}(t)).$$

By combining this with the first equation in (2), we obtain

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}(t) &= \underbrace{\mathbf{A}\mathbf{x}_{eq} + \mathbf{B}u_{eq}}_{=\mathbf{0}} + \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}\tilde{u}(t) \\ &= \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}\tilde{u}(t).\end{aligned}$$

The output error is given by

$$\tilde{y}(t) = y(t) - r = \mathbf{C}\mathbf{x}(t) - r = \mathbf{C}(\mathbf{x}_{eq} + \tilde{\mathbf{x}}(t)) - r.$$

From the second equation in (2), we have that $r = \mathbf{C}\mathbf{x}_{eq}$, which implies that

$$\begin{aligned}\tilde{y}(t) &= \mathbf{C}\mathbf{x}_{eq} + \mathbf{C}\tilde{\mathbf{x}}(t) - r \\ &= \mathbf{C}\mathbf{x}_{eq} + \mathbf{C}\tilde{\mathbf{x}}(t) - \mathbf{C}\mathbf{x}_{eq} \\ &= \mathbf{C}\tilde{\mathbf{x}}(t).\end{aligned}$$

d) Let the positive-definite matrix $\mathbf{P}$ be given by

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix},$$

where the constants p_1, p_2 and p_3 are obtained from the algebraic Riccati equation $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{B}^T R^{-1}\mathbf{P} + \mathbf{Q} = \mathbf{0}$. Substituting $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$, R and $\mathbf{P}$ in the algebraic Riccati equation yields

$$\begin{aligned}&\begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2p_2 & -2p_3 \\ p_1 - p_2 & p_2 - p_3 \end{bmatrix} + \begin{bmatrix} -2p_2 & p_1 - p_2 \\ -2p_3 & p_2 - p_3 \end{bmatrix} - \begin{bmatrix} p_2^2 & p_2p_3 \\ p_2p_3 & p_3^2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4p_2 - p_2^2 + 5 & p_1 - p_2 - 2p_3 - p_2p_3 \\ p_1 - p_2 - 2p_3 - p_2p_3 & 2p_2 - 2p_3 - p_3^2 + 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Hence, we obtain the equations

$$\begin{aligned}-4p_2 - p_2^2 + 5 &= 0, \\ p_1 - p_2 - 2p_3 - p_2p_3 &= 0, \\ 2p_2 - 2p_3 - p_3^2 + 1 &= 0.\end{aligned}$$

From the first equation, it follows that

$$p_2 = -2 \pm 3.$$

From the third equation, we obtain

$$p_3 = -1 \pm \sqrt{2(p_2+1)}.$$

From the second equation, we get

$$p_1 = p_2 + 2p_3 + p_2 p_3.$$

Because there are two different solutions for p_2 and an additional two solutions for p_3, there are four solutions that satisfy the algebraic Riccati equation:

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -8 & 1 \\ 1 & -3 \end{bmatrix},$$

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -2-6\sqrt{2}j & -5 \\ -5 & -1+2\sqrt{2}j \end{bmatrix}, \quad \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -2+6\sqrt{2}j & -5 \\ -5 & -1-2\sqrt{2}j \end{bmatrix}.$$

The matrix $\mathbf{P}$ is a real and positive-definite solution of the algebraic Ricatti equation. A real matrix is positive definite if all of its leading principal minors are positive. (Positive definiteness of a real matrix can also be checked by looking at its eigenvalues, which should be real and positive.) The leading principle minors of $\mathbf{P}$ are

$$p_1 \qquad \text{and} \qquad \det(\mathbf{P}) = \begin{vmatrix} p_1 & p_2 \\ p_2 & p_3 \end{vmatrix} = p_1 p_3 - p_2^2.$$

Because $\mathbf{P}$ is positive definite, we must have that

$$p_1 > 0 \qquad \text{and} \qquad p_1 p_3 - p_2^2 > 0.$$

The only solution of the algebraic Riccati equation that is real and positive definite is $\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$. Therefore, we have

$$\mathbf{P} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

The corresponding gain matrix $\mathbf{K}$ is given by

$$\mathbf{K} = R^{-1}\mathbf{B}^T\mathbf{P} = \frac{1}{4}\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{1}$$

e) The controller is given by

$$\begin{aligned} u(t) &= u_{eq} + \tilde{u}(t) \\ &= Gr - \mathbf{K}\tilde{\mathbf{x}}(t) \\ &= Gr - \mathbf{K}(\mathbf{x}(t) - \mathbf{x}_{eq}) \\ &= Gr - \mathbf{K}(\mathbf{x}(t) - \mathbf{F}r). \end{aligned}$$

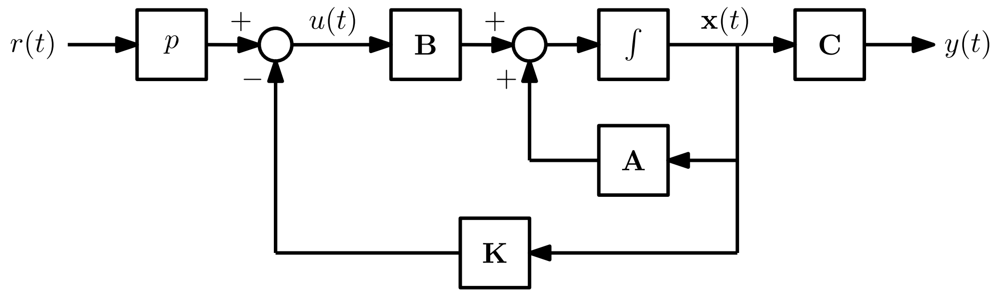
Hence, we have

$$u(t) = -\mathbf{K}\mathbf{x}(t) + pr,$$

with

$$p = G + \mathbf{K}\mathbf{F} = 1 + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{3}{2}.$$

f) The block diagram of the system with controller is given by:



g) In order to obtain a faster convergence of the output, we need to penalize the deviation of the output $y(t)$ with respect to the setpoint value r more (i.e. we need to penalize $\tilde{y}(t)$ more). We can do this by increasing the first element on the diagonal of $\mathbf{Q}$ (which currently has the value 5). This element corresponds to the state $\tilde{x}_1(t) = \tilde{y}(t)$. Alternatively, we can penalize $\tilde{u}(t)$ less by decreasing the value of R.
By increasing the effort to get a small deviation of the output $y(t)$ with respect to the setpoint value r, we decrease the effort of having a small feedback input $\tilde{u}(t)$. Therefore, the signal $\tilde{u}(t)$ will be larger. Hence, we can expect a larger input signal $u(t) = u_{eq} + \tilde{u}(t)$, especially when the deviation of $y(t)$ with respect to r is large.

Problem 3: State estimator

a) The observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}.$$

Because the observability matrix has full column rank, i.e. $\text{rank}(\mathcal{O}) = 2 = n$, we conclude that the system is observable.

b) From the state-space equation of the system, the equation of the state estimator and $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, it immediately follows that

$$\begin{aligned}
\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\
&= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) - \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}u(t) - \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t) - Du(t)) \\
&= \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) - \mathbf{L}(\mathbf{C}\mathbf{x}(t) + Du(t) - \mathbf{C}\hat{\mathbf{x}}(t) - Du(t)) \\
&= (\mathbf{A} - \mathbf{LC})(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \\
&= (\mathbf{A} - \mathbf{LC})\mathbf{e}(t).
\end{aligned}$$

c) Let the estimator-gain matrix $\mathbf{L}$ be given by

$$\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix},$$

where l_1 and l_2 are constants that have to be determined. The eigenvalues of $\mathbf{A} - \mathbf{LC}$ can be calculated from the characteristic polynomial of $\mathbf{A} - \mathbf{LC}$, which is given by

$$\det(\mathbf{A} - \mathbf{LC} - \lambda\mathbf{I}) = \begin{vmatrix} -4-\lambda & -l_1 \\ 3 & -1-l_2-\lambda \end{vmatrix} = \lambda^2 + (l_2+5)\lambda + 3l_1 + 4l_2 + 4.$$

If the eigenvalues of $\mathbf{A}-\mathbf{LC}$ are equal to -8 and -10, the characteristic polynomial is given by

$$\begin{aligned} \det(\mathbf{A} - \mathbf{LC} - \lambda\mathbf{I}) &= (-8-\lambda)(-10-\lambda) \\ &= \lambda^2 + 18\lambda + 80. \end{aligned}$$

By combining these two expressions for the characteristic polynomial, we obtain the equations

$$\begin{aligned} l_2 + 5 &= 18, \\ 3l_1 + 4l_2 + 4 &= 80. \end{aligned}$$

Solving for l_1 and l_2 yields $l_1 = 8$ and $l_2 = 13$. Hence, we obtain

$$\mathbf{L} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}.$$

d) The block diagram of the system with state estimator is given by: