Norwegian University of Science and Technology
Department of Engineering Cybernetics

Exercise 1
Solution

Problem 1 (25 %)

Optimization problem:

$$\min x_1 + 2x_2 \qquad \text{s.t.} \qquad 2 - x_1^2 - x_2^2 \geq 0, \quad x_2 \geq 0 \tag{1}$$

a) The optimal point is $x^* = (-\sqrt{2}, 0)^\top$, this can be found by inspection.

b) The Lagrangean for the problem is

$$\mathcal{L}(x, \lambda) = x_1 + 2x_2 - \lambda_1 c_1(x) - \lambda_2 c_2(x) \tag{2}$$

with

$$c_1(x) = 2 - x_1^2 - x_2^2 \tag{3a}$$
$$c_2(x) = x_2 \tag{3b}$$
$$\mathcal{I} = \{1, 2\} \tag{3c}$$

Hence,

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 1 + 2\lambda_1^* x_1^* \\ 2 + 2\lambda_1^* x_2^* - \lambda_2 \end{bmatrix} = 0 \Rightarrow \lambda^* = \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ 2 \end{bmatrix} \tag{4}$$

and all KKT conditions are satisfied.

c) The gradients of the active constraints and the objective function at the solution are illustrated in Figure 1.

d) A positive number means any small change of x results in an increase of the objective function.

e) This is a convex problem, since the objective function is convex (all linear functions are linear) and the feasible are is convex. For a problem with only inequality constraints, the feasible are is convex if all inequalities are concave functions. Here, both $c_1(x)$ (a paraboloid with a unique maximizer) and $c_2(x)$ (a linear function) are concave functions. (Note that a linear function is both concave and convex).

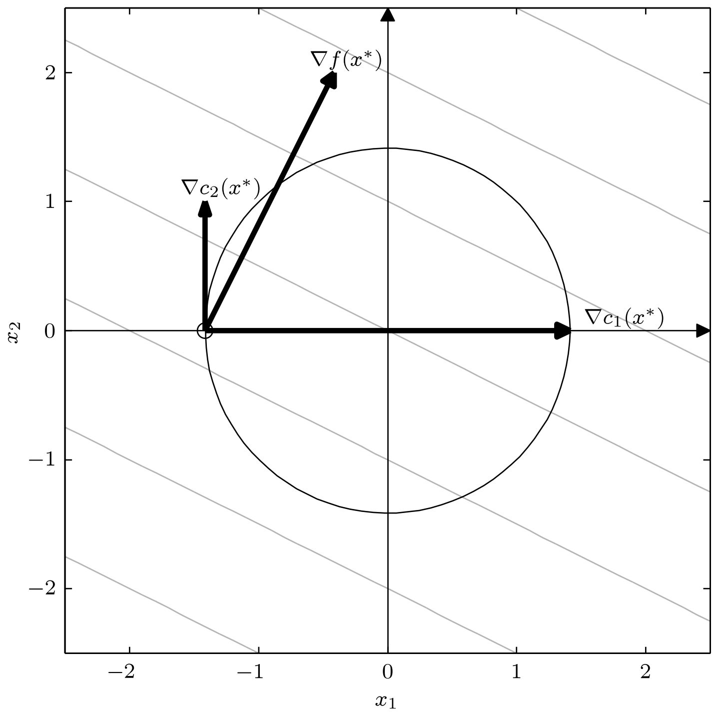

Figure 1: Gradients at the optimal point in Problem 1.

Problem 2 (30 %)

Optimization problem:

$$\min 2x_1 + x_2 \qquad \text{s.t.} \qquad x_1^2 + x_2^2 - 2 = 0 \tag{5}$$

a) The extreme points are

$$x = \left(-2\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}\right)^\top \tag{6a}$$

and

$$x = \left(2\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}\right)^\top \tag{6b}$$

(Use, for instance, $\nabla f = \lambda_1 \nabla c_1$ and $x_1^2 = 2 - x_2^2$ to find these.)

b) The Lagrangian for the problem is

$$\mathcal{L}(x, \lambda) = 2x_1 + x_2 - \lambda_1 c_1(x) \tag{7}$$

with

$$c_1(x) = x_1^2 + x_2^2 - 2,\ \mathcal{E} = \{1\} \tag{8}$$

Hence,

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 2 - 2\lambda_1^* x_1^* \\ 1 - 2\lambda_1^* x_2^* \end{bmatrix} = 0 \Rightarrow \lambda^* = \pm\frac{\sqrt{10}}{4} \tag{9}$$

c) The values of the multiplier above mean that the KKT conditions are satisfied at both extreme points.

d) The gradients of the active constraint and the objective function at the optimal point are illustrated in Figure 2.

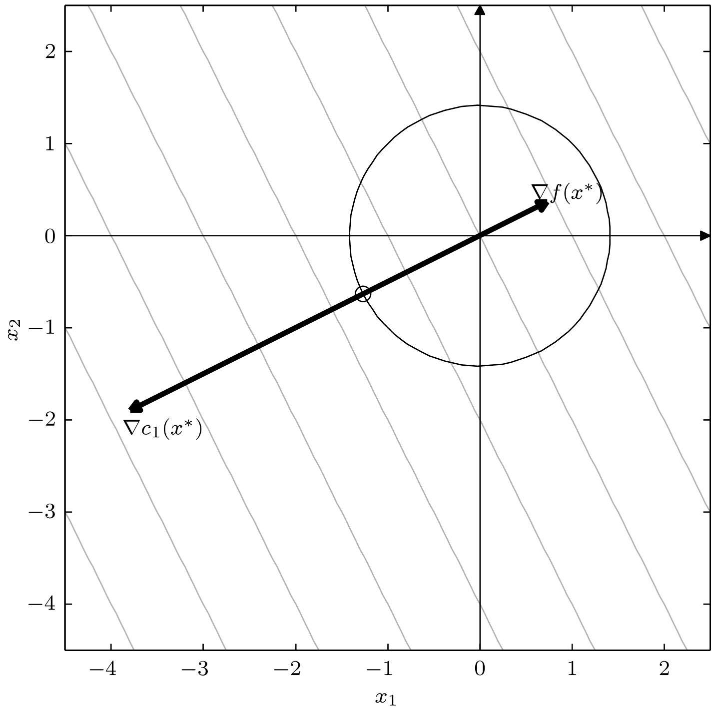

Figure 2: Gradients at the optimal point in Problem 2.

e) We have that

$$\nabla^2_{xx}\mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} -2\lambda^* & 0 \\ 0 & -2\lambda^* \end{bmatrix} > 0 \tag{10}$$

holds for $\lambda^* = -\frac{\sqrt{10}}{4}$. Then,

$$w^\top \nabla^2_{xx}\mathcal{L}(x^*, \lambda^*) w > 0, \qquad \text{for all } w \in \mathcal{C}(x^*, \lambda^*),\ w \neq 0 \tag{11}$$

which means that both the necessary and sufficient second-order conditions in Chapter 12.5 hold (Theorems 12.5 and 12.6, respectively).

f) The problem is nonconvex due to the nonlinear equality constraint.

Problem 3 (20 %)

Optimization problem:

$$\min_{x \in \mathbb{R}^2} f(x) = -2x_1 + x_2 \qquad \text{s.t.} \qquad \begin{cases} c_1(x) = (1 - x_1)^3 - x_2 \geq 0 \\ c_2(x) = x_2 + 0.25x_1^2 - 1 \geq 0 \end{cases} \tag{12}$$

a) The set of active constraint gradients at the solution,

$$\nabla c_1(x^*) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \quad \nabla c_2(x^*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{13}$$

is linearly independent. Hence, LICQ hold.

b) The Lagrangian for the problem is

$$\mathcal{L}(x, \lambda) = -2x_1 + x_2 - \lambda_1 c_1(x) - \lambda_2 c_2(x) \tag{14}$$

with

$$c_1(x) = (1 - x_1)^3 - x_2 \tag{15a}$$
$$c_2(x) = x_2 + 0.25x_1^2 - 1 \tag{15b}$$
$$\mathcal{I} = \{1, 2\} \tag{15c}$$

Hence,

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} -2 + 3\lambda_1^*(1 - x_1^*)^2 - 0.5\lambda_2^* x_1^* \\ 1 + \lambda_1 - \lambda_2 \end{bmatrix} = 0 \Rightarrow \lambda^* = \begin{bmatrix} 2/3 \\ 5/3 \end{bmatrix} \tag{16}$$

and all KKT conditions are satisfied.

d) The Hessian of the Lagrangian at the solution,

$$\nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} -6\lambda_1^*(1 - x_1^*) - 0.5\lambda_2^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -29/6 & 0 \\ 0 & 0 \end{bmatrix} \tag{17}$$

is negative semidefinite. However, the critical cone $\mathcal{C}(x^*, \lambda^*)$ contains only the vector $w^\top = [0, \ 0]$. Hence, the second-order necessary conditions are satisfied, whereas the second-order sufficient conditions are not.

Problem 4 (25 %)

Finding the maximizer for $f(x) = x_1 x_2$ is equivalent to finding the minimizer for the function $\bar{f} = -x_1 x_2$. We therefore state the optimization problem as

$$\min -x_1 x_2 \qquad \text{s.t.} \qquad 1 - x_1^2 - x_2^2 \geq 0 \tag{18}$$

Note that the objective function represents a saddle, and it is therefore clear that the minimizer(s) exist(s) on the boundary of the unit disk. The Lagrangian is given by

$$\mathcal{L}(x, \lambda) = -x_1 x_2 - \lambda_1 (1 - x_1^2 - x_2^2) \tag{19}$$

From the KKT conditions,

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} -x_2^* + 2\lambda_1^* x_1^* \\ -x_1^* + 2\lambda_1^* x_2^* \end{bmatrix} = 0 \Rightarrow \lambda_1^* = \pm\frac{1}{2} \tag{20}$$

Since λ_1^* has to be nonnegative, we have

$$x_1^* = x_2^* = \pm\frac{1}{\sqrt{2}} \tag{21}$$

Hence, $f(x) = x_1 x_2$ has two maximizers, $x^* = \left[-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right]$ and $x^* = \left[\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right]$, both with $\lambda_1^* = +\frac{1}{2}$. The gradients at the optimal point $x^* = \left[-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right]$ are illustrated in Figure 3, while the gradients at the optimal point $x^* = \left[\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right]$ are illustrated in Figure 4.

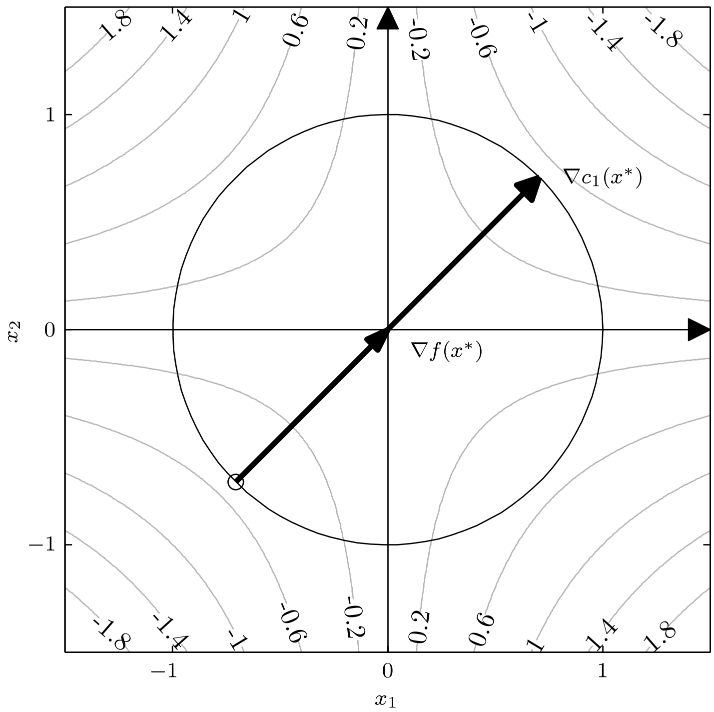

Figure 3: Gradients at the solution $x^* = \left[-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right]$ of Problem 4.

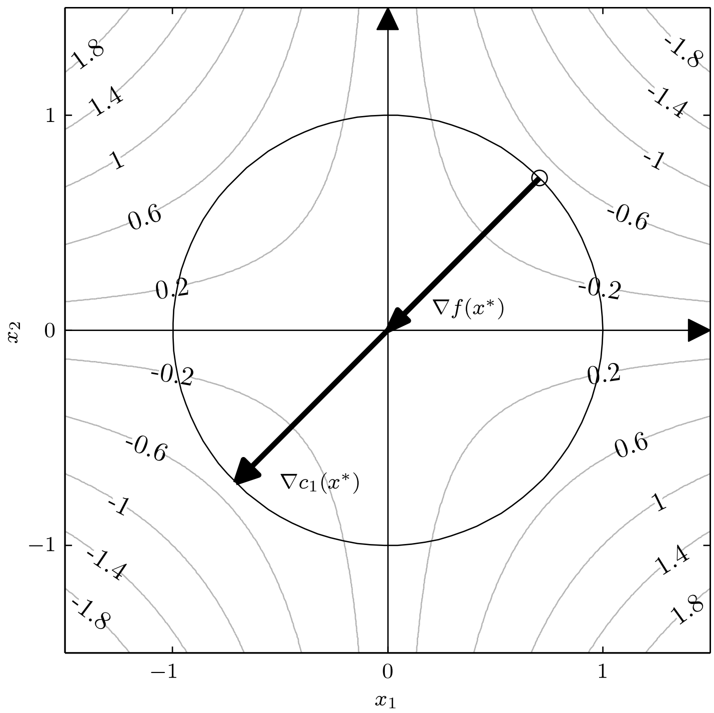

Figure 4: Gradients at the solution $x^* = \left[\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right]$ of Problem 4.