



TTT4120 Digital Signal Processing Suggested Solutions for Problem Set 7

Problem 1

- (a) The samples from the white noises are uncorrelated and drawn from the corresponding distribution. Since a white noise has zero mean and we are told that these noises have unit variance, we have all the parameters for their probability distributions.

For the uniform distribution we have to find the limits (which will be symmetric because of the zero mean property):

$$\frac{1}{2a} \int_{-a}^a x^2 dx = 1 \rightarrow a = \sqrt{3},$$

Matlab code:

```
g = randn(1,N);  
u = sqrt(3)*(2*rand(1,N) - 1);  
b = 2*(rand(1,N)>0.5)-1;
```

Figure 1 shows the realization of the different white noises. It is common for all the noises that the samples are uncorrelated for each of them. This means that given one sample there is no information about the value of the next one. The binary noise is limited to two different values, whereas the uniform and the gaussian noise takes values in a given interval. At first sight it is difficult to make difference between the gaussian noise and the uniform one. However it can be seen that the gaussian noise takes some values out of $(-\sqrt{3}, \sqrt{3})$. In addition, the value of the samples from the gaussian noise are concentrated in the center, where as the value of the samples of the uniform noise are equally spaced.

- (b) White binary noise:

$$P_X(x) = \begin{cases} 0.5 & \text{if } x = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

White uniform noise:

$$P_X(x) = \begin{cases} \frac{1}{2\sqrt{3}} & \text{if } x \in (-\sqrt{3}, \sqrt{3}) \\ 0 & \text{otherwise} \end{cases}$$

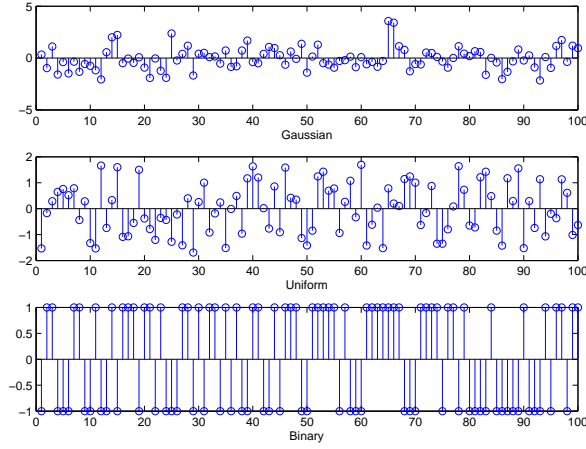


Figure 1: Realizations of different types of white noises

White gaussian noise:

$$P_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

To compute the mean of $X(n)$ we use the symmetry properties of $P_X(x)$:

$$\int x P_X(x) dx = 0$$

To compute the autocorrelation function we have to use the fact that the sample are uncorrelated (because they are white signals):

$$R_X[k] = EX[n]X[n-k] = \sigma^2 \delta(k)$$

The power density spectrum in the DTFT of the autocorrelation function:

$$S_X(w) = DTFT \{R_X[k]\} = \sigma_X^2$$

- (c) The mean estimates should be quite accurate (less than 0.001) Figure 2 shows that the estimated correlations are very accurate. They were found using the function `xcorr` so that we get a normalized estimate.

Problem 2

The mean value of the input $w(n)$ is zero, because it is a white noise process.

The autocorrelation function of the input, $w(n)$, is given by

$$\gamma_{ww}(m) = E[w(n)w(n+m)] = \sigma_w^2 \delta(m) = \frac{3}{4} \delta(m),$$

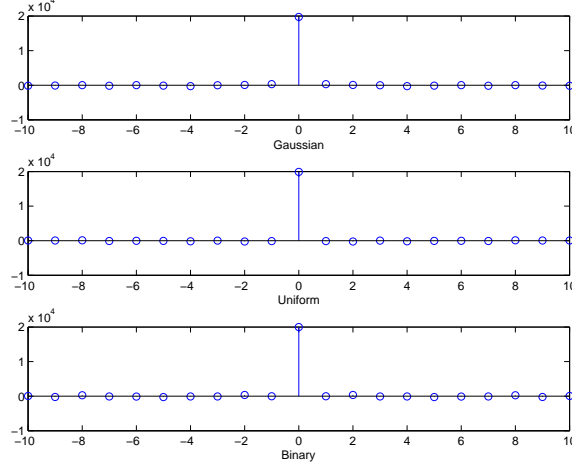


Figure 2: Estimates of the autocorrelation function

since the samples of white noise are uncorrelated. The power density spectrum of the input is given by

$$\begin{aligned}\Gamma_{ww}(f) &= DTFT\{\gamma_{ww}(m)\} = \sum_{m=-\infty}^{\infty} \gamma_{ww}(m)e^{-j2\pi fm} \\ &= \sum_{m=-\infty}^{\infty} \sigma_w^2 \delta(m)e^{-j2\pi fm} = \sigma_w^2 = \frac{3}{4}\end{aligned}$$

(a) The mean of the signal $x(n)$ is given by

$$m_x = E[x(n)] = m_w \sum_{k=-\infty}^{\infty} h(k) = 0, \quad \text{since } m_w = 0.$$

The autocorrelation function of the signal $x(n)$ is given by

$$\gamma_{xx}(m) = \gamma_{ww}(m) * r_{hh}(m),$$

where $r_{hh}(m)$ is the autocorrelation function of the unit sample response $h(n)$ of the filter (which is a deterministic signal), i.e.

$$r_{hh}(m) = \sum_{n=-\infty}^{\infty} h(n)h(n+m)$$

The unit sample response for the filter is given by

$$h(n) = \mathcal{Z}^{-1}\{H(z)\} = \left(-\frac{1}{2}\right)^n u(n).$$

We first find the autocorrelation function $r_{hh}(m)$ for $m \geq 0$,

$$\begin{aligned} r_{hh}(m) &= \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) \left(-\frac{1}{2}\right)^{n+m} u(n+m) \\ &= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{2n+m} = \left(-\frac{1}{2}\right)^m \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \\ &= \left(-\frac{1}{2}\right)^m \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \left(-\frac{1}{2}\right)^m \end{aligned}$$

For $m < 0$, we have that $r_{hh}(m) = r_{hh}(-m) = \frac{4}{3} \left(-\frac{1}{2}\right)^{-m}$. It follows that

$$r_{hh}(m) = \frac{4}{3} \left(-\frac{1}{2}\right)^{|m|}, \text{ for all } m$$

The autocorrelation function of the signal $x(n)$ is now given by

$$\gamma_{xx}(m) = \gamma_{ww}(m) * r_{hh}(m) = \frac{3}{4} \delta(m) * \frac{4}{3} \left(-\frac{1}{2}\right)^{|m|} = \left(-\frac{1}{2}\right)^{|m|}.$$

The power density spectrum of the signal $x(n)$ is given by

$$\Gamma_{xx}(\omega) = \Gamma_{ww}(\omega) |H(\omega)|^2$$

We can compute $|H(\omega)|^2$ from the transfer function $H(z)$ by remembering that $H(\omega) = H(z)|_{z=e^{j\omega}}$.

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = \frac{1}{1 + \frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1 + \frac{1}{2}e^{j\omega}} = \frac{4}{5 + 4\cos\omega}.$$

It follows that

$$\Gamma_{xx}(\omega) = \frac{3}{4} \frac{4}{5 + 4\cos\omega} = \frac{3}{5 + 4\cos\omega}.$$

The power of the signal $x(n)$ is given by

$$P_x = E[x^2(n)] = \gamma_{xx}(0) = 1,$$

and variance

$$\sigma_x^2 = E[(x(n) - m_x)^2] = E[x(n)^2] = P_x = 1, \quad \text{since } m_x = 0$$

(b) We have the following expressions for the estimates:

Mean:

$$\hat{m}_x = \frac{1}{N} \sum_{n=0}^{N-1} x(n)$$

Power:

$$\hat{P}_x = \frac{1}{N} \sum_{n=0}^{N-1} x^2(n)$$

Autocorrelation function:

$$\hat{\gamma}_{xx}(l) = \begin{cases} \frac{1}{N} \sum_{n=0}^{N-1} x(n)x(n+l) & \text{for } 0 \leq |l| \leq N-1 \\ 0 & \text{for } N \leq |l| \end{cases}$$

Power spectral density:

$$\hat{\Gamma}_{xx} = \text{DFT}(\hat{\gamma}_{xx}(l))$$

- (c) The signal $x(n)$ can be estimated by first generating a segment of white Gaussian noise using the Matlab command `randn`, and then filtering it through the filter $H(z)$ using Matlab command `filter`.

The plots of $\gamma_{xx}(m)$ and $\hat{\gamma}_{xx}(m)$ are shown in Figure 3.

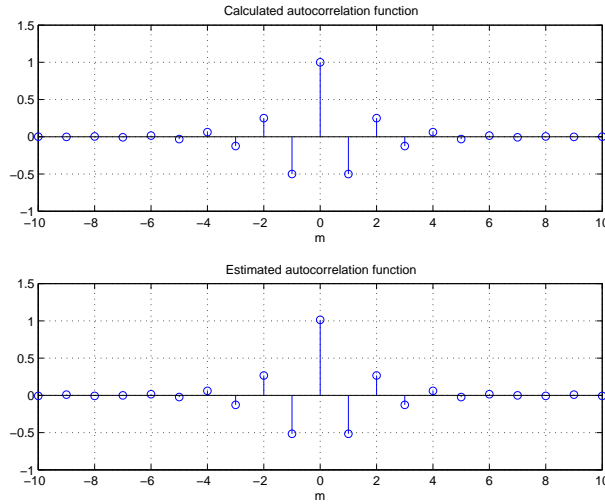


Figure 3: Autocorrelation function and its unbiased estimate

(d)

Problem 3

The histograms for $K=20, 40$ and 100 are shown in Figure 4. We observe that the variance of the mean estimate is reduced when the segment length is increased. This is confirmed when the variances is computed using Matlab command `var`.

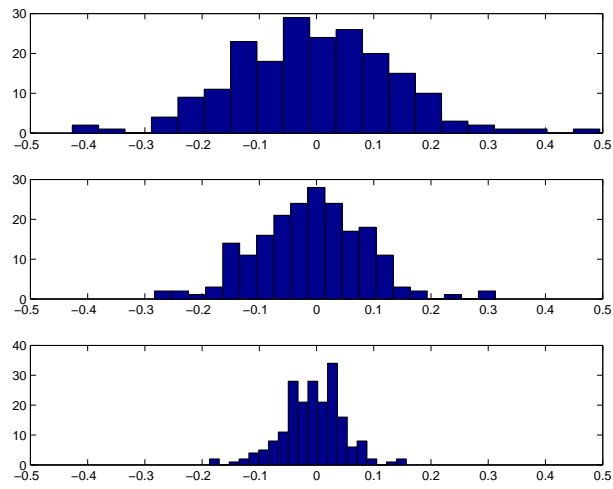


Figure 4: Histograms of the mean value computed on segments of length 20, 40 and 100