Data Fusion
An Intuitive Look

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1. Introduction
2. Graphical Solution
3. Mathematical Solution
4. Programmatical Solution

Sequence : detailed

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   - Problem Statement
2. Graphical Solution
   - Start
   - Step 1
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   - Step 3
3. Mathematical Solution
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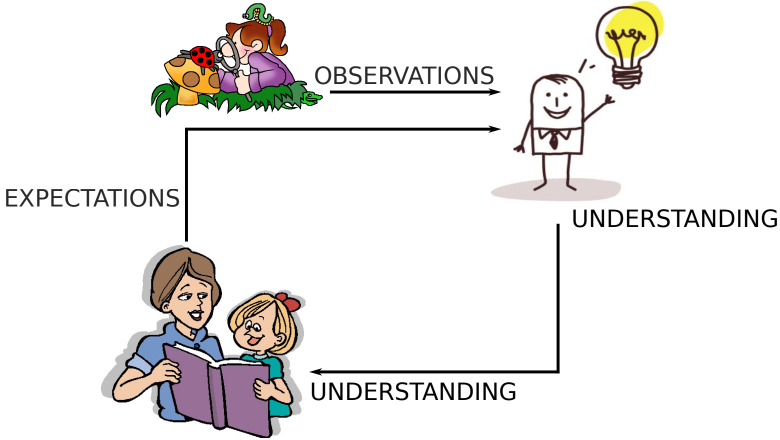
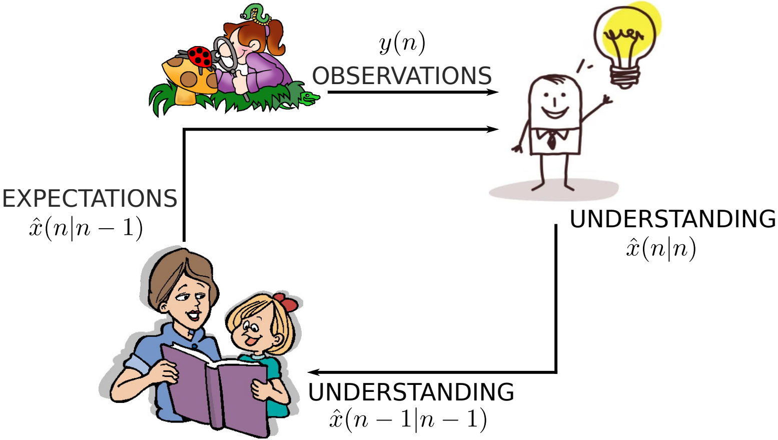
   - Comparison With Vector Case
4. Programmatical Solution

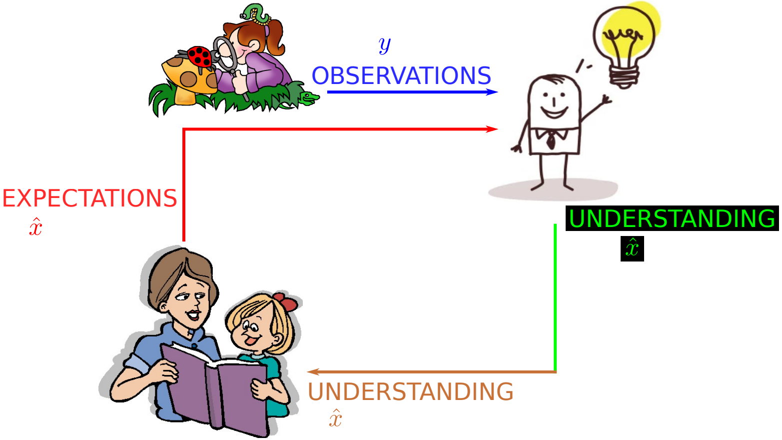
Overview


OBSERVATIONS
EXPECTATIONS
UNDERSTANDING
UNDERSTANDING

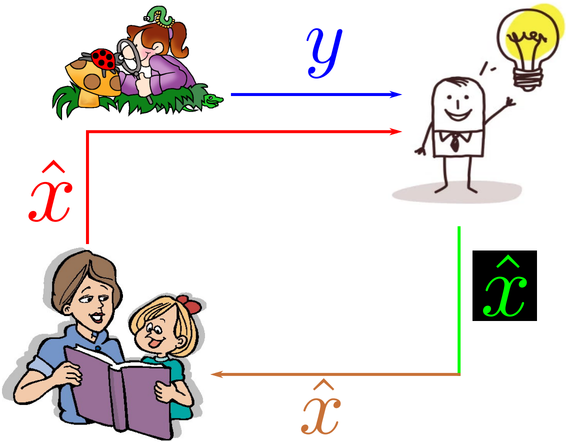
Overview


$y(n)$
OBSERVATIONS
EXPECTATIONS
$\hat{x}(n|n-1)$
UNDERSTANDING
$\hat{x}(n|n)$
UNDERSTANDING
$\hat{x}(n-1|n-1)$

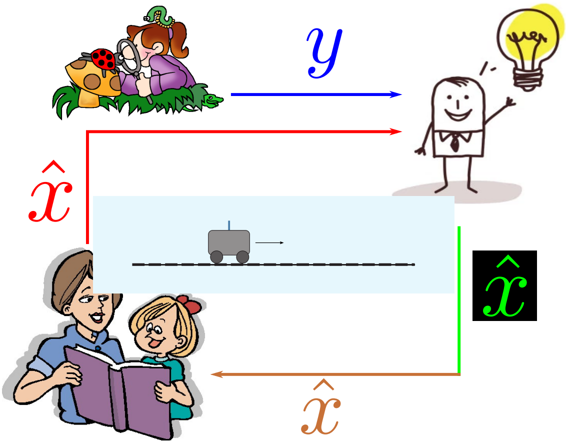
Overview


y
OBSERVATIONS
EXPECTATIONS
$\hat{x}$
UNDERSTANDING
$\hat{x}$
UNDERSTANDING
$\hat{x}$

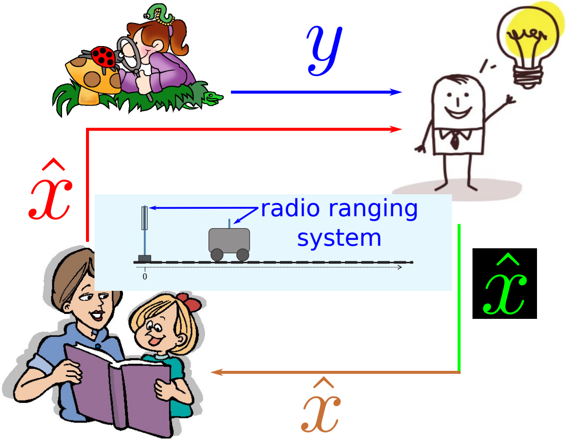
Overview



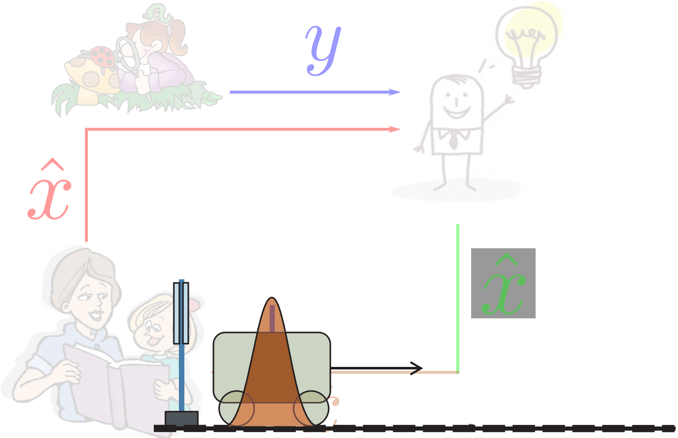
Overview



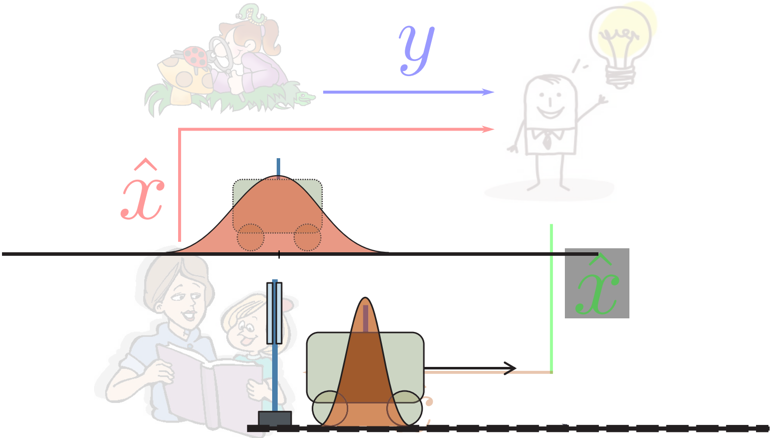
Overview



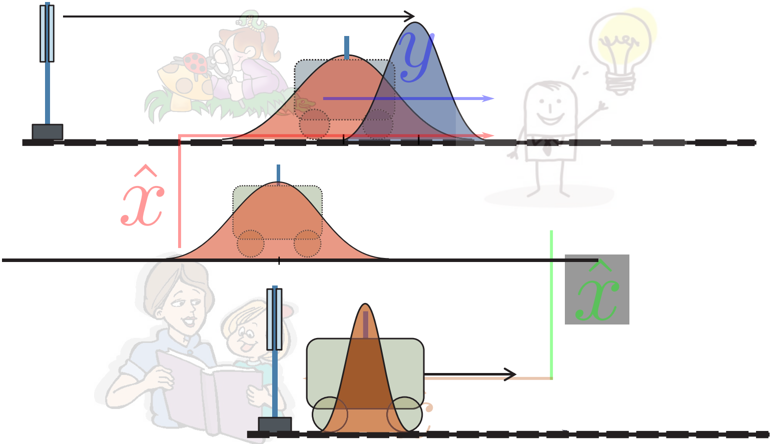
Old Understanding



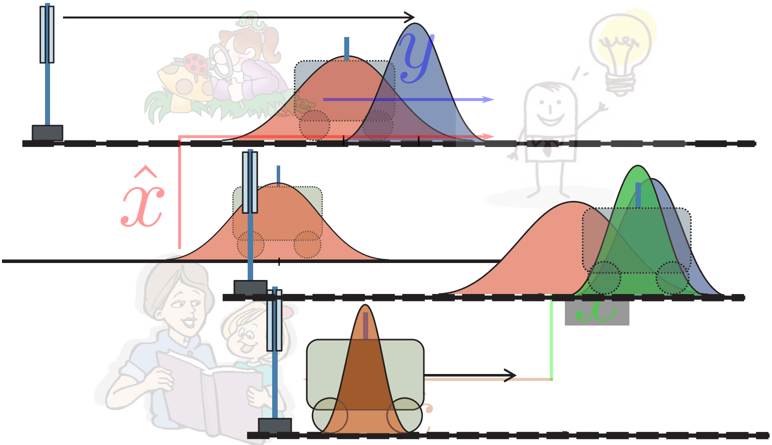
Expectations



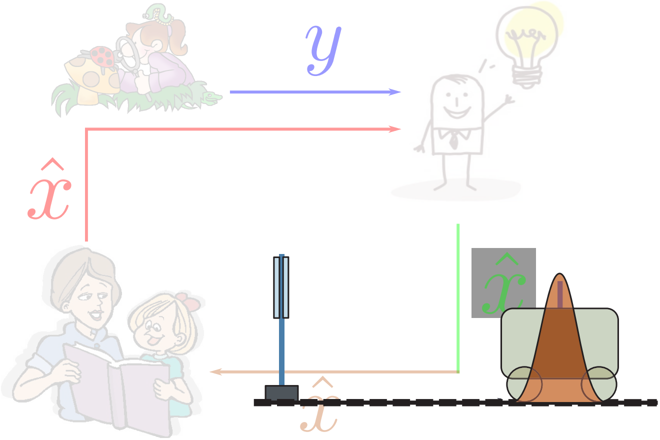
Observations



New Understanding



New Understanding



Derivation of Scalar Case

Let p_1 and p_2 be two Gaussian pdfs:

$$\begin{aligned} p_1(x;\mu,\sigma_1) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ p_2(x;\mu,\sigma_2) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \end{aligned}$$

Their product is:

$$p_1 p_2 = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $K = \frac{C\sigma^2}{C^2\sigma^2+\sigma^2}$

Derivation of Scalar Case

$p_1 p_2$ will be a Gaussian function with mean

$$\begin{aligned}
\mu &= \frac{\mu\sigma^2+\mu\sigma^2}{\sigma^2+\sigma^2} \\
&= \frac{\mu\sigma^2+\mu C\sigma^2}{C^2\sigma^2+\sigma^2} \quad \text{(C is used for units conversion)} \\
&= \mu + \frac{\sigma^2(C\mu-C^2\mu)}{C^2\sigma^2+\sigma^2} \\
&= \mu + \frac{C\sigma^2(\mu-C\mu)}{C^2\sigma^2+\sigma^2} \\
&= \mu + K(\mu - C\mu)
\end{aligned}$$

Derivation of Scalar Case

And variance

$$\begin{aligned}
\sigma^2_{fused} &= \frac{\sigma^2\sigma^2}{\sigma^2+\sigma^2} \\
&= \frac{\sigma^2\sigma^2}{C^2\sigma^2+\sigma^2} \\
&= \sigma^2 - \frac{C^2\sigma_1^4}{C^2\sigma^2+\sigma^2} \\
&= \sigma^2 - \frac{C\sigma^2}{C^2\sigma^2+\sigma^2} C\sigma^2 \\
&= (1 - KC)\sigma^2
\end{aligned}$$

where K is the same as defined in the previous slide

One-to-one Correspondence

	single state	multiple states
I. PREDICTION	from previous time	from previous time
1. state	$\mu_1 = A\mu_{fused} + Bu$	$\hat{\mathbf{x}}_{k\|k-1} = \mathbf{A}\hat{\mathbf{x}}_{k-1\|k-1} + \mathbf{B}\mathbf{u}_{k-1}$
2. state variance/covariance	$\sigma_1^2 = A\sigma_{fused}^2 A + Q = A^2\sigma_{fused}^2 + Q$	$\mathbf{P}_{k\|k-1} = \mathbf{A}\mathbf{P}_{k-1\|k-1}\mathbf{A}^T + \mathbf{Q}_k$
II. UPDATE		
3. state	$\mu_{fused} = \mu_1 + K(\mu_2 - C\mu_1)$, where $K = \sigma_1^2 C(C\sigma_1^2 C + \sigma_2^2)^{-1} = \frac{C\sigma_1^2}{C^2\sigma_1^2 + \sigma_2^2}$	$\hat{\mathbf{x}}_{k\|k} = \hat{\mathbf{x}}_{k\|k-1} + \mathbf{K}_k \big(\underbrace{\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_{k\|k-1}}_{\text{innovation: measurement}} \big)$, where $\mathbf{K}_k = \mathbf{P}_{k\|k-1}\mathbf{C}^T \big(\underbrace{\mathbf{C}\mathbf{P}_{k\|k-1}\mathbf{C}^T + \mathbf{R}_k}_{\text{innovation: covariance}} \big)^{-1}$
4. state variance/covariance	$\sigma_{fused}^2 = (1 - KC)\sigma_1^2$	$\mathbf{P}_{k\|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{C})\mathbf{P}_{k\|k-1}$

prediction
measurement
update

Matlab Code I

The code below implements 2 time steps for the train:

```
clear; clc; clf;
%initialization
x        =   -5:0.1:25;   %x axis
A        =   2;           %state transition matrix
B        =   5;           %control input matrix
C        =   1;           %transformation matrix
u        =   1;           %control input
Q        =   1.2;         %process noise variance, adds uncertainty to prediction
mu_2     =   [7 14];      %the output of the radio ranging system at next two times
var_2    =   1;           %the variance of the radio ranging system given by manufacturer
mu_f     =   0;           %the estimated location of the train at current time, k=0
var_f    =   1;           %the estimated variance of our estimate at current time, k=0
pdf_0    =   (1/sqrt(2*pi*var_f))*exp(-(0.5/var_f)*(x-mu_f).^2);
%=========================================
%run Kalman Filter
%=========================================
for k=1:2
    %predict
    mu_1      =   A*mu_f     +   B*u;
    var_1     =   A^2*var_f  +   Q;
    %update
    K         =   (C*var_1)/(C^2*var_1 + var_2);
    mu_f      =   mu_1 + K*(mu_2(k)-C*mu_1);
    var_f     =   (1-K*C)*var_1;
    pred      =   (1/sqrt(2*pi*var_1))*exp(-(0.5/var_1)*(x-mu_1).^2);
    meas      =   (1/sqrt(2*pi*var_2))*exp(-(0.5/var_2)*(x-mu_2(k)).^2);
    fused     =   (1/sqrt(2*pi*var_f))*exp(-(0.5/var_f)*(x-mu_f).^2);
```

Matlab Code II

```
            axis([-5 25 0 0.5]);
            grid on;
            hold on;
            plot(x,pred, 'r--x');
            plot(x,meas, 'b--o');
            plot(x,fused,'g--.')
            plot(x,pdf_0,'g--.');
end
legend('predicted', 'measured', 'fused')
xlabel('railway track (meters), direction along which train is traveling ->')
ylabel('belief in fused/predicted/measured position of train')
title('Data fusion using the Kalman Filter')
```

Results

The code on the previous slide produces this output:

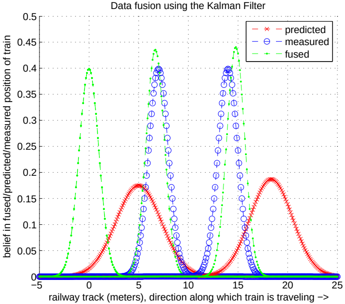


Figure: In this example, a train is moving along the x-axis. The problem begins at time $kT = 0$ sec when our initial estimate of the train is that it is standing at $x = 0$ meters. The prediction and measurements for the next two time instants, $kT = 1$ sec and $kT = 2$ sec are shown. We assume for simplicity but without loss of generality, that $T = 1$ sec, and therefore we use k (in sec) to depict time. It may be mentioned that the version of the Kalman filter for continuous time is called the Kalman-Bucy filter.

Questions