International Journal of Robust and Nonlinear Control

WILEY

RESEARCH ARTICLE OPEN ACCESS

Safety Critical Control of a System With Element-Wise Estimation of Unknown Time-Varying Parameters

Emil Lykke Diget | Yitaek Kim | Christoffer Sloth

Maersk Mc-Kinney Moller Institute, University of Southern Denmark, Odense, Denmark

Correspondence: Emil Lykke Diget (eld@mmmi.sdu.dk)

Received: 28 November 2023 | **Revised:** 23 May 2025 | **Accepted:** 20 July 2025

Funding: This work was supported by EUROfusion (Grant No. 101052200) and Innovationsfonden (Grant no. 9090-00002B).

Keywords: adaptive control | parameter estimation | safety | safety-critical control

ABSTRACT

This paper presents a method for guaranteeing the safety of a system with time-varying parameters. First, we extend Dynamic Regressor Extension and Mixing to estimate time-varying parameters with a finite-time update rule, and present a bound on the estimation error. In the case of a constant parameter, we provide an improved lower bound on the update gain compared with the literature. Lastly, the parameter error bound is used to provide a Robust adaptive Control Barrier Function for systems with time-varying parameters. We conduct a numerical simulation of a system with time-varying parameters. The presented estimation method estimates the parameters with a smaller mean-square error compared with methods from the literature. The presented control barrier function can keep the system safe despite parametric uncertainties.

1 | Introduction

Parametric uncertainties often appear in modern complex dynamical systems (e.g., robotics and industry automation), ruining the performance of model-based control schemes such as model predictive control and set invariance techniques that are mainly used to ensure the system safety criteria. For instance, they might not only deteriorate the stability but also cause unsafe behaviors of the system, and these are critical problems that should be resolved.

Generally, to address the issues caused by the uncertainties, adaptive control approaches based on the regular gradient- and least-squares-based methods have been leveraged to estimate quasi-static parameters [1]. However, since time-varying parameters might be inherent in modern control systems, it is difficult to resolve the uncertainties with the above-mentioned methods. A way of handling this issue is to fit a polynomial to the parameters locally with constant polynomial coefficients that should be estimated [2–5]. Another approach is to apply fixed-time algorithms as presented in References [6] and [7], in which a bound on the parameter error is supplied based on the maximum velocity of the parameter change. However, they rely on the unnecessarily restrictive condition of persistence of excitation (PE).

Dynamic Regressor Extension and Mixing (DREM) was proposed by [8] and does not require PE for asymptotic convergence, but only the less restrictive condition of not being square-integrable, $\Delta \notin \mathcal{L}_2$. A key feature of DREM is the guaranteed performance increase over the gradient-based parameter estimator. This has recently been extended with local polynomial extension to estimate time-varying parameters [9]. Another approach is to use a fixed-time parameter update rule together with DREM, which was explored in References [10, 11]. Though in these papers, they only analyze the case of constant parameters. The paper [12] adds alertness preservation that enables the finite-time DREM to track changing parameters.

DREM-based parameter estimators reduce parametric uncertainties, which leads to the potential to be integrated on top of safety applications. In this work, we focus on the safety perspectives based on set invariance approaches in the presence of parametric uncertainties, which can be handled by DREM. One of the popular approaches of set invariance techniques, Control Barrier Functions (CBFs) [13] are used to ensure the safety of dynamical systems by using the prior knowledge of a given system model. However, the lack of parametric certainty in the system model often causes violations of safety constraints. To address this issue, most CBFs-based safe controllers first consider the constant parameters in the given system model. Adaptive Control Barrier Function (aCBF) was introduced to reduce parametric uncertainty by estimating the parameters in the model, enforcing the system to be in a safety set [14]. However, aCBF is overly conservative due to the strict condition for safety guarantees. The conservatism was resolved by introducing the maximum possible error that tightens the safety set; in the so-called Robust adaptive Control Barrier Function (RaCBF) proposed by [15]. There have also been studies with time-varying model uncertainty in References [16–18]. However, they still have conservatism in their control designs since they do not estimate the model uncertainty.

Especially, RaCBFs were considered in Reference [7] with a fixed-time stable adaptation law for systems with constant parameters. However, since DREM is not used, the parameter error is not bounded elementwise. In Reference [19], the authors presented RaCBF with DREM for the estimation of constant parameters. They used a finite-time parameter adaptation rule and presented proofs for the bounds of the learning rate and the barrier function. However, the presented lower bound on the gain is not tight, that is, the computed gain is too large.

In this paper, we provide three contributions. First, we present a tighter bound on the parameter update gain compared with [19] when the true parameter is constant. Second, we combine DREM with a finite-time update rule and present an upper bound for the parameter estimation error when the true parameter is time-varying. Lastly, the presented parameter estimation error bound is used in a CBF to ensure safety in a system with time-varying parameters. We provide two simulation models to validate our theoretical contributions and to obtain practical insights into implementations, while showing that our proposed method not only ensures the safety guarantees but also reduces model uncertainty and conservatism at the same time. To the best of our knowledge, this is the first CBF that considers systems with time-varying inherent model parameters that are directly estimated.

The paper is organized as follows. Section 2 presents the problems addressed in the paper, and Section 3 presents the necessary preliminaries on DREM and CBFs. The finite-time parameter estimation is presented in Section 4, and an RaCBF using this estimate is presented in Section 5. Results from simulations regarding the parameter estimation and the safety, respectively, are outlined in Section 6. Finally, conclusions are provided in Section 7.

2 | Problem Formulation

In this paper, we consider the problem of designing a controller that guarantees the safety of a dynamical system in the presence of time-varying parameters. We consider the control-affine system:

$$\dot{x} = f(x) + F(x)\theta(t) + g(x)u \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state, $u \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input, $f : \mathcal{X} \to \mathbb{R}^n$, $g : \mathcal{X} \to \mathbb{R}^{n\times m}$ are the known model structures, which are locally Lipschitz continuous functions, $\theta : \mathbb{R}_{\geq 0} \to \Theta \subseteq \mathbb{R}^k$ is an unknown time-varying parameter where Θ is a closed and convex set and $|\dot{\theta}_i(t)| \leq \Omega_i$ for all $t \geq 0$, where $\Omega_i > 0$, $i = 1, 2, \ldots, k$, and $F : \mathcal{X} \to \mathbb{R}^{n\times k}$ is smooth on the state space with $F(0) = 0$ and models parametric uncertainty.

We say that the control system (1) is *safe* with respect to $\mathcal{S} \subset \mathcal{X}$ if $\mathcal{S}$ is forward invariant and define the safe set $\mathcal{S}$ as the following superlevel set:

$$\mathcal{S} = \{x \in \mathcal{X} | h(x) \geq 0\} \tag{2}$$

where $h : \mathcal{X} \to \mathbb{R}$ is a continuously differentiable function.

This paper provides solutions to the following problems.

Problem 1 (Estimation of Time-Varying Parameter). Given system model (1) and measurements of x and $\dot{x}$. Provide a parameter estimate $\hat{\theta} : \mathbb{R}_{\geq 0} \to \Theta$ in finite time where the element-wise error is bounded and can be made arbitrarily small.

A solution to Problem 1 is found in Proposition 2; and for constant parameters in Proposition 3, providing a tighter bound on the gain compared with [19].

Problem 2 (Safety of System with Time-Varying Parameters). Given system model (1) and measurements of x and $\dot{x}$. Provide a control signal $u \in \mathcal{U}$ such that the system is safe with respect to $\mathcal{S}$.

A solution to Problem 2 is found in Theorem 1. This extends the Robust adaptive Control Barrier Function in Reference [15] to systems with time-varying parameters. An overview of the complete system can be seen in Figure 1.

Remark 1. If measurements of x and $\dot{x}$ are not available, one could consider the application of an extended Kalman filter to estimate the state, and perhaps the parameter in conjunction [20, 21].

3 | Preliminaries

This section presents preliminary results on Dynamic Regressor Extension and Mixing (DREM) and Robust Adaptive Control Barrier Functions (RaCBF) for systems with constant parameters.

3.1 | Dynamic Regressor Extension and Mixing

The considered control system (1) has the unknown parameter vector θ, which can be estimated by $\hat{\theta}$. In this section, the

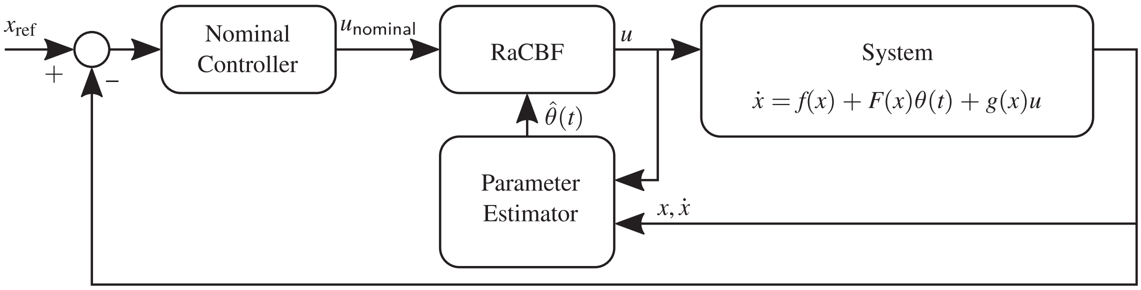

FIGURE 1 | A block diagram of the considered system. We consider a dynamic system with a parameter that varies with time, controlled by a nominal controller with safety guaranteed by an RaCBF. It is assumed, that x and $\dot{x}$ can be measured. The nominal controller uses the reference state x_{ref} and the measured state x to compute a nominal control input u_{nominal}. The time-varying parameter $\theta(t)$ is estimated by a parameter estimator, and then $\hat{\theta}(t)$ is used by the RaCBF to compute a control input u such that the system remains within a defined safety region.

parameter vector is assumed to be constant; thus the following linear regression equation (LRE) is considered

$$y(t) = \phi^{\intercal}(t)\theta \tag{3}$$

where $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\phi^{\intercal} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times k}$ and $\theta \in \Theta \subseteq \mathbb{R}^k$. The system dynamics (1) can be written as the LRE

$$\underbrace{\dot{x} - f(x) - g(x)u}_{y} = \underbrace{F(x)}_{\phi^{\intercal}}\ \theta \tag{4}$$

DREM consists of two steps; *extension* and *mixing* [8]. A linear $\mathcal{L}_\infty$-operator $\mathcal{H}$ is introduced to *extend* (3) to

$$Y(t) = \varphi(t)\theta \tag{5}$$

where

$$Y(t) \triangleq \mathcal{H}[y(t)] \in \mathbb{R}^k \tag{6}$$

$$\varphi(t) \triangleq \mathcal{H}[\phi^{\intercal}(t)] \in \mathbb{R}^{k\times k} \tag{7}$$

The operator $\mathcal{H}$ can be chosen in different ways [22–24]. Recently, it was found that using the Kreisselmeier filter with DREM can preserve the persistence of excitation (PE) of the signal ϕ [25, 26]. The Kreisselmeier filter has the following linear time-varying state-space representation:

$$\dot{x}(t) = -\ell x(t) + \phi(t)u(t) \tag{8}$$

$$y_u(t) = x(t) \tag{9}$$

where $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^N$ is the internal state vector, $\ell > 0$ is a tuning parameter, $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is an input signal, and $y_u = \mathcal{H}[u] : \mathbb{R}_{\geq 0} \to \mathbb{R}^N$ is the output signal. This filter only has one tunable parameter, making it easy to use. It can be applied to (3) and provides the following solution to (6) and (7):

$$\begin{aligned} \dot{Y} &= -\ell Y + \phi(t)y(t), \qquad && Y(0) = 0, \\ \dot{\varphi} &= -\ell\varphi + \phi(t)\phi^{\intercal}(t), \qquad && \varphi(0) = 0 \end{aligned} \tag{10}$$

The next step of DREM is to *mix* the equations to obtain k one-dimensional LREs such that each element of the parameter vector, θ_i, can be independently estimated. The known relationship $\mathrm{adj}\{M\}M = \det\{M\}I_m$, where $\mathrm{adj}\{\cdot\}$ is the adjugate matrix, $\det\{\cdot\}$ is the determinant, and $I_m \in \mathbb{R}^{m\times m}$ is the identity matrix, is used for getting k individual equations:

$$\mathcal{Y}_i(t) = \Delta(t)\theta_i \tag{11}$$

where $\mathcal{Y}_i$ is the ith element of $\mathcal{Y}$ with $i = 1, \ldots, k$ and

$$\begin{aligned} \mathcal{Y}(t) &\triangleq \mathrm{adj}\{\varphi(t)\}Y(t) \in \mathbb{R}^k, \\ \Delta(t) &\triangleq \det\{\varphi(t)\} \in \mathbb{R} \end{aligned} \tag{12}$$

The parameter θ_i can be estimated using various methods [24, 27]; including the short finite-time update law (20) presented in Section 4.

A limitation of the standard gradient estimator [1] is that it is only globally exponentially stable if and only if the regressor is PE. Since DREM converts the regression problem into scalar equations, the PE condition on the regressor Δ is relaxed to $\Delta \notin \mathcal{L}_2$ (not square-integrable), which is easier to fulfill [8].

3.2 | Safety With Control Barrier Functions

In this section, we consider how to guarantee safety in a control system. First, let us introduce the Control Barrier Function.

Definition 1 (Control Barrier Function [13]). Consider the control-affine system:

$$\dot{x} = f(x) + g(x)u \tag{13}$$

and the set $\mathcal{S}$ defined in Equation (2). The continuously differentiable function $h : \mathcal{X} \to \mathbb{R}$ is a Control Barrier Function (CBF) if there exists an extended class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ satisfying the following:

$$\sup_{u\in\mathcal{U}} \left\{ \frac{\partial h}{\partial x}(x)\left(f(x) + g(x)u\right) \right\} \geq -\alpha(h(x)) \quad \forall x \in \mathcal{X} \tag{14}$$

Any locally Lipschitz continuous control inputs u satisfying (14) renders the safety set (2) forward invariant and thus the system (13) is safe [13].

The CBF in Definition 1 cannot be directly applied to the parameterized control-affine system (1) that we are focusing on in this paper. Therefore, we leverage robust adaptive control barrier functions [15]. Let $\mathcal{S}_\theta$ be the parameterized safe set with the constant parameter θ,

$$\mathcal{S}_\theta = \left\{ x \in \mathcal{X} : h_r(x,\theta) \geq 0,\ \theta \in \Theta \right\} \tag{15}$$

In this work, we have imperfect knowledge about the value of the parameter, but assume that the parameter estimation error $\tilde{\theta} \triangleq \theta - \hat{\theta}$ is bounded; specifically, $\tilde{\theta} \in \tilde{\Theta}$ where $\tilde{\Theta}$ is a closed convex set. In addition, we define the maximum possible error as $\tilde{\vartheta} = \arg\max_{\tilde{\theta} \in \tilde{\Theta}} \tilde{\theta}^\top \Gamma^{-1} \tilde{\theta}$, where $\Gamma \in \mathbb{R}^{k\times k}$ is a positive semi-definite adaptive gain used in the RaCBF defined in the remainder of the section. We can then define a tightened safe set $\mathcal{S}_\theta^r \subseteq \mathcal{S}_\theta$ as:

$$\mathcal{S}_\theta^r = \left\{ x \in \mathcal{X} : h_r(x,\theta) \geq \frac{1}{2} \tilde{\vartheta}^\top \Gamma^{-1} \tilde{\vartheta},\ \theta \in \Theta \right\} \tag{16}$$

where $h_r : \mathcal{X} \times \Theta \to \mathbb{R}$ is a continuously differentiable function.

Assumption 1. The unknown parameters θ belong to a known closed convex set Θ. The parameter estimation error $\tilde{\theta} \triangleq \theta - \hat{\theta}$ then also belongs to a known closed convex set $\tilde{\Theta}$ and the maximum possible error for each parameter is $\tilde{\vartheta} \in \mathbb{R}^k$.

The function h_r is an RaCBF for (1) with constant parameter θ if there exists an extended class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that for any $\theta \in \Theta$:

$$\begin{aligned} &\sup_{u \in \mathcal{U}} \left\{ \frac{\partial h_r}{\partial x}(x,\hat{\theta}) \left(f(x) + F(x)\lambda(x,\hat{\theta}) + g(x)u \right) \right\} \\ &\quad \geq -\alpha \left(h_r(x,\hat{\theta}) - \frac{1}{2} \tilde{\vartheta}^\top \Gamma^{-1} \tilde{\vartheta} \right) \end{aligned} \tag{17}$$

where

$$\lambda(x,\theta) \triangleq \theta - \Gamma \left(\frac{\partial h_r}{\partial \theta}(x,\theta) \right)^\top \tag{18}$$

and the adaptation law

$$\dot{\hat{\theta}} = -\Gamma \left(\frac{\partial h_r}{\partial x}(x,\hat{\theta}) F(x) \right)^\top \tag{19}$$

where $\Gamma \in \mathbb{R}^{n\times n}$ is an adaptive gain satisfying $\lambda_{\min}(\Gamma) \geq \frac{\|\tilde{\vartheta}\|^2}{2h_r(x,\theta)}$, where $\lambda_{\min}(\Gamma)$ is the minimum eigenvalue of Γ.

Proposition 1 ([15]). *Let h_r be a RaCBF on $\mathcal{S}_\theta^r$. Any locally Lipschitz continuous controller satisfying (17) renders $\mathcal{S}_\theta^r \subseteq \mathcal{S}_\theta$ forward invariant and safe.*

The use of the maximum possible error in Equation (17) leads to the conservative behaviors of the controller. To resolve this issue, data-driven approaches such as Set Membership IDentification (SMID) [28] can be used to monotonically decrease $\tilde{\vartheta}$, but it still causes the conservatism since it does not ensure that $\tilde{\vartheta}$ converges to 0 [28]. Instead, we can directly estimate the model parameters precisely with DREM and use the updated parameters in Equation (17) to ensure safety. The following sections describe how we combine RaCBF with DREM.

4 | Finite-Time Parameter Estimation

This section provides a solution to Problem 1. Consider the LRE resulting from the DREM procedure (11) and the finite-time update rule [6]:

$$\dot{\hat{\theta}}_i(t) = \gamma_i \Delta(t) \left\lceil \mathcal{Y}_i(t) - \Delta(t)\hat{\theta}_i(t) \right\rfloor^r \tag{20}$$

where $\lceil s \rfloor^r = |s|^r \operatorname{sign}(s)$, $r \in (0,1)$ and $\gamma_i > 0$ is a tuning parameter.

The error dynamic, where $\tilde{\theta}_i(t) = \theta_i(t) - \hat{\theta}_i(t)$, is:

$$\dot{\tilde{\theta}}_i(t) = \dot{\theta}_i(t) - \gamma_i \Delta(t) \left\lceil \Delta(t)\tilde{\theta}_i(t) \right\rfloor^r \tag{21}$$

where it can be seen that the parameter error dynamics depends on the derivative of the actual parameter $\dot{\theta}_i$. Since it is unknown, we have to assume a worst-case value for it. A bound on the parameter error $\tilde{\theta}_i$ can be found by analyzing the error dynamics (21) which leads to the following Proposition:

Proposition 2. *Let $\Delta(t)$ be bounded from below such that $|\Delta(t)| \geq \Delta_{\min} > 0$ for all $t \geq 0$, and $|\dot{\theta}_i| \leq \Omega_i$ for all $t \geq 0$, where $\Omega_i > 0$. Then the system (21) for $r \in (0,1)$ and $\gamma_i > 0$, is globally ultimately bounded and its trajectories satisfy the following bound:*

$$\left|\tilde{\theta}_i(t)\right| \leq \mu_i \quad \forall t \geq T\left(\tilde{\theta}_i(0)\right) \tag{22}$$

with

$$\mu_i = \left(\frac{\Omega_i}{\gamma_i \Delta_{\min}^{r+1} \delta} \right)^{\frac{1}{r}} \tag{23}$$

$$T\left(\tilde{\theta}_i(0)\right) = \max\left(0, \frac{\left|\tilde{\theta}_i(0)\right|^{1-r} - \mu_i^{1-r}}{\Delta_{\min}^{r+1}(1-\delta)\gamma_i(1-r)} \right) \tag{24}$$

where $\delta \in (0,1)$ and $\tilde{\theta}(0) \in \mathbb{R}^n$, and

$$\left|\tilde{\theta}_i(t)\right| \leq \tilde{\zeta}_i(t) \quad \forall t < T\left(\tilde{\theta}_i(0)\right) \tag{25}$$

where

$$\tilde{\zeta}_i(t) = \sqrt{c_i} \left(c_i^{\frac{r-1}{2}} \left|\tilde{\theta}_i(0)\right|^{1-r} - \frac{(1-r)(1-\delta)c_i^{\frac{r+1}{2}} \Delta_{\min}^{r+1}}{2} t \right)^{\frac{1}{1-r}} \tag{26}$$

Proposition 2 provides an answer to Problem 1; a proof is given in the appendix. Corollary 1 in Reference [6] is similar to Proposition 2, but since it does not apply DREM to the LRE, it considers the bound on the norm of the parameter estimation error opposed to the elementwise parameter estimation error.

Remark 2. From Equation (23), the value of γ_i can be determined based on a desired maximum parameter error $\tilde{\vartheta}_i$ as

$$\gamma_i \geq \frac{\Omega_i}{\tilde{\vartheta}_i^r \Delta_{\min}^{r+1} \delta} \tag{27}$$

and μ_i can be made arbitrarily small by increasing γ_i.

Remark 3. If the parameter is constant, then the bound on the parameter error changes. This leads to the following parameter error dynamics, where $\tilde{\theta}_i(t) = \theta_i - \hat{\theta}_i(t)$:

$$\dot{\tilde{\theta}}_i(t) = -\gamma_i \Delta(t) \left[\Delta(t) \tilde{\theta}_i(t) \right]^r \tag{28}$$

It is seen that γ_i affects the rate of convergence, and Δ affects the error dynamics in a manner quantified in Proposition 3. Before the proposition, it is necessary to introduce the notion of interval excitation as follows:

Definition 2 (Interval Excitation [29]). A signal $\kappa(t)$ is IE in $(0, t_c)$, $t_c > 0$ if it satisfies

$$\int_0^{t_c} \kappa(t)\kappa^\intercal(t)dt \geq \beta I \tag{29}$$

where $\beta > 0$ is a constant.

Given a maximum parameter error $\tilde{\vartheta}_i$, the value of γ_i can be found. In what follows, we provide a tighter bound on γ_i compared to Proposition 3 in Reference [19].

Proposition 3. *Consider the error dynamics (28). If the signal $\kappa(t) = |\Delta(t)|^{(1+r)/2}$ is IE in $(0, t_c)$ at level β, then the element-wise estimation error $\tilde{\theta}_i(t)$ monotonically converges to zero within $(0, t_c)$ if*

$$\gamma_i \geq \frac{\tilde{\vartheta}_i^{1-r}}{(1-r)\beta} \tag{30}$$

where $\tilde{\vartheta}_i$ is the maximum parameter error and $r \in (0, 1)$, with a parameter estimation error satisfying

$$\left|\tilde{\theta}_i(t)\right| \leq \left(\left|\tilde{\theta}_i(0)\right|^{1-r} - (1-r)\gamma_i \int_0^t \kappa^2(\tau)\, \mathrm{d}\tau \right)^{\frac{1}{1-r}} \tag{31}$$

for all $t < t_c$, while for all $t \geq t_c$, $\tilde{\theta} = 0$.

The proof of Proposition 3 is found in the appendix.

Example 1. Consider the LRE defined as

$$\Delta(t) = (1 - e^{-t}) \sin(t) \cos(10t), \quad \theta_1 = 4 \tag{32}$$

Since there is only one parameter, the index i is ignored for clarity in the remainder of this example. The purpose of this example is to compare the gain γ_i (30), here γ_{new}, with the one presented in Reference [19]:

$$\gamma_{\text{old}} \geq \frac{2\tilde{\vartheta}^2}{(1-r)\beta_{\text{old}}} \tag{33}$$

The choice of γ, either (30) or (33) will influence the theoretical convergence time, t_c, described in Definition 2. The two gains result in the following values of β:

$$\beta_{\text{new}} \geq \frac{\tilde{\vartheta}^{1-r}}{(1-r)\gamma_{\text{new}}} \tag{34}$$

$$\beta_{\text{old}} \geq \frac{2\tilde{\vartheta}^2}{(1-r)\gamma_{\text{old}}} \tag{35}$$

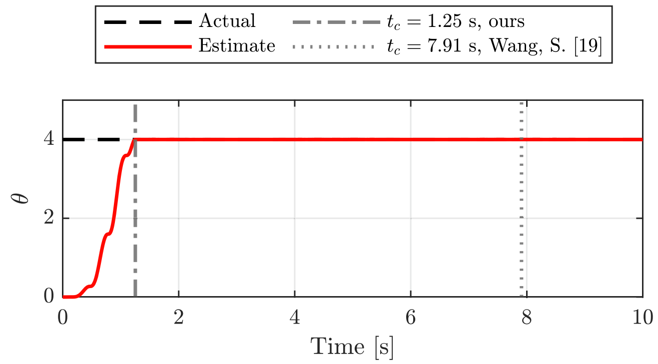

FIGURE 2 | A comparison of convergence time as follows from our formulation of the gain (30) and the gain (33) reported in Reference [19]. It can be seen that our computed convergence time corresponds to the actual one at 1.25 s and the one reported in Reference [19] is too large at 7.91 s.

respectively. Then, with the two different values of β, the integral (29) can be computed to find two values of t_c. With the hyperparameters $\gamma_{\text{old}} = \gamma_{\text{new}} = 30$, $r = 0.5$ and $\tilde{\vartheta} = 4$, the simulation can be seen on Figure 2. The parameter estimate converges to the true value in 1.25 s. The first vertical line from the left is t_c computed with β_{new} resulting from Equation (30), while the second line is the t_c resulting from β_{old} from Equation (33). It can be seen that the presented t_c corresponds to the actual convergence time, and the t_c resulting from Reference [19] is far too large at 7.91 s.

5 | Safety

In this section, we present a new RaCBF condition for the case when the true parameter varies with time. First, we analyze the ordinary RaCBF and present the result in Theorem 1. Then we present Theorem 1 where the parameter is estimated using the presented finite-time update rule.

5.1 | RaCBF With Time-Varying Parameters

This section provides a solution to Problem 2 by a safety condition for the dynamical system (1) with time-varying parameter vector $\theta(t)$; the condition is an extension of (17).

Following the procedure of the proof of Proposition 1 and with the parameter error $\tilde{\theta}_i(t) = \theta_i(t) - \hat{\theta}_i(t)$ leads to the following corollary:

Corollary 1. *Let $S_{\hat{\theta}}^r \subset \mathbb{R}^n$ be a superlevel set of a continuously differentiable function $h_r : \mathcal{X} \times \Theta \to \mathbb{R}$. If h_r is a CBF on $S_{\hat{\theta}}^r$, then any locally Lipschitz continuous controller satisfying the following:*

$$\begin{aligned} &\frac{\partial h_r}{\partial x}(x, \hat{\theta}) \left(f(x) + F(x)\lambda(x, \hat{\theta}) + g(x)u \right) \\ &\geq -\alpha \left(h_r(x, \hat{\theta}) - \frac{1}{2}\tilde{\vartheta}^\intercal \Gamma^{-1} \tilde{\vartheta} \right) \\ &+ \tilde{\vartheta}^\intercal \Gamma^{-1} \Omega \end{aligned} \tag{36}$$

renders the set $S_{\hat{\theta}}^r$ safe with adaptation law (19) and adaptation gain as $\lambda_{\min}(\Gamma) \geq \frac{||\tilde{\vartheta}||^2}{2h_r(x_r, \theta_r)}$, where $\tilde{\vartheta} \in \mathbb{R}^k$ is the maximum possible *parameter error,* $\Omega \in \mathbb{R}^k$ *is the maximum velocity of the parameter error,* $h_r(x_r, \theta_r) > 0$ *can be decided freely based on the desired conservatism, and* $\Gamma = \Gamma^\top$ *is a symmetric positive definite matrix. Furthermore, the original set* $\mathcal{S}_{\hat{\theta}}$ *is also safe for the uncertain system.*

The proof is similar to [15]; hence, omitted.

Remark 4. If the parameter is constant, that is, the maximum parameter change $\Omega = 0$, then (36) reduces to (17).

5.2 | RaCBF With Finite-Time Parameter Estimation

With the bound on the parameter error as given in Proposition 2, a CBF can be formulated in the following Theorem:

Theorem 1. *Consider a superlevel set* $\mathcal{S}^s_{\hat{\theta}} = \{x \in \mathcal{X} \mid h_r(x, \theta) \geq \Xi(t)\}$ *of a CBF* $h(x)$ *with*

$$\sup_{u \in \mathcal{U}} \left\{ \frac{\partial h_r}{\partial x}(x, \hat{\theta}) \left(f(x) + F(x)\hat{\theta}(t) + g(x)u \right) + \psi(t) + \frac{\partial h_r}{\partial \hat{\theta}}(x, \hat{\theta})\dot{\hat{\theta}}(t) \right\} \geq -\alpha \left(h_r(x, \hat{\theta}) - \Xi(t) \right) + \tilde{\zeta}^\top(t)\Gamma^{-1}|\dot{\tilde{\zeta}}(t)| \tag{37}$$

where α *is an extended class* $\mathcal{K}_\infty$ *function and*

$$\psi(t) = -\sum_{i=1}^{n}\sum_{j=1}^{k} \left| \frac{\partial h_r}{\partial x_i}(x, \hat{\theta}) F_{i,j}(x) \right| \tilde{\zeta}_j(t) \tag{38}$$

in which

$$\tilde{\zeta}_i(t) = \sqrt{c_i} \left(c_i^{\frac{r-1}{2}} \left|\tilde{\theta}_i(0)\right|^{1-r} - \frac{(1-r)(1-\delta)c_i^{\frac{r+1}{2}} \Delta_{\min}^{r+1}}{2} t \right)^{\frac{1}{1-r}} \geq 0 \tag{39}$$

where $c_i = 2\gamma_i$, $\Xi(t) = \frac{1}{2}\tilde{\zeta}^\top(t)\Gamma^{-1}\tilde{\zeta}(t)$, *and*

$$\lambda_{\min}(\Gamma) \geq \frac{||\tilde{\vartheta}||^2}{2h_r(x_r, \theta_r)} \tag{40}$$

where $\tilde{\vartheta} \in \mathbb{R}^k$ *is the maximum possible parameter error,* $h_r(x_r, \theta_r) > 0$ *can be decided freely based on the desired conservatism, and* $\Gamma = \Gamma^\top$ *is a symmetric positive definite matrix. Then for any locally Lipschitz continuous controller satisfying (37) renders* $\mathcal{S}^s_{\hat{\theta}}$ *forward invariant with finite-time adaption law (20) and learning rate (27). Then* $\mathcal{S}^s_{\hat{\theta}} \subset \mathcal{X}$ *is the safe set of systems (1) as* $|\tilde{\theta}_i(t)| \leq \mu_i \quad \forall t \geq T(\tilde{\theta}_i(0))$.

Theorem 1 provides an answer to Problem 2, and is proved in the appendix.

Remark 5. If the parameter is constant, $\dot{\theta} = 0$, then the bound on the parameter error changes.

Then the bound on the parameter estimation error is described by (31), which leads to the following corollary regarding the CBF:

Corollary 2. *Consider a superlevel set* $\mathcal{S}^s_{\hat{\theta}} = \{x \in \mathcal{X} \mid h_r(x, \theta) \geq \Xi(t)\}$ *of a CBF* $h(x)$ *with the condition (37), where*

$$\tilde{\zeta}_i(t) = \left(\left|\tilde{\theta}_i(0)\right|^{1-r} - (1-r)\gamma_i \int_0^t \kappa^2(\tau)\, \mathrm{d}\tau \right)^{\frac{1}{1-r}} \tag{41}$$

$c_i = 2\gamma_i$ *and* $\Xi(t) = \sum_{i=1}^{k} \frac{1}{2\gamma_i}\tilde{\zeta}_i^2(t)$. *Then for any locally Lipschitz continuous controller satisfying (37) renders* $\mathcal{S}^s_{\hat{\theta}}$ *forward invariant with finite-time adaption law (20) and learning rate (30). Then* $\mathcal{S}^s_{\hat{\theta}} \subset \mathcal{X}$ *is the safe set of systems (1) as* $\tilde{\theta}_i(t) = 0 \quad \forall t \geq t_c$.

The proof of Corollary 2 follows the proof of Theorem 1 in the appendix just with a different parameter estimation error bound; (41) instead of (39). Hence, it is omitted.

Remark 6. The adaptation law used in Corollary 1 is designed to keep the system safe, while the adaptation law in Theorem 1 is designed to minimize the parameter estimation error $\tilde{\theta}$. This means that by using the formulation in Theorem 1, the estimated system dynamics come closer to the true system dynamics, and the conservatism can be lowered.

6 | Simulation

In this section, we simulate two systems. First, a mass-spring-damper system controlled by a P-controller is studied. We compare the presented parameter estimation methods in isolation first, and then we compare the two presented CBFs to limit the force. Then, we simulate a second system: A circular robot with a nonlinear model controlled by an adaptive non-collocated partial feedback linearization control scheme. For this system, we compare the two presented CBFs to limit the velocity. The simulations were conducted in MATLAB and integrated using Euler's method with sample time $T_s = 1$ kHz.

6.1 | Mass-Spring-Damper System

We simulate a mass-spring-damper system with time-varying parameters and a controllable force applied to the mass. The system model is given by:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\frac{1}{m}\begin{bmatrix} 0 & 0 \\ x & \dot{x} \end{bmatrix}}_{F(\mathbf{x})} \underbrace{\begin{bmatrix} k(t) \\ b(t) \end{bmatrix}}_{\theta(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{g(\mathbf{x})} u \tag{42}$$

where $\mathbf{x} = \begin{bmatrix} x & \dot{x} \end{bmatrix}^\top \in \mathcal{X}$ is the system state representing the position and velocity of the mass, $u \in \mathcal{U}$ is the controlled force input to the system, $m = 1$ is the mass, $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the stiffness and $b : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the damping.

The nominal control u_{nominal} is a P-controller with feed-forward given as

$$u_{\text{nominal}} = -K_p(f_d - f_c) - f_d \tag{43}$$

where $K_p = 1$ is the proportional gain, f_c is the spring-damper force $f_c = -k(t)x(t) - b(t)\dot{x}(t)$ and f_d is the desired force given as

$$f_d(t) = 7.5\sin(\pi t) - 9.5 \tag{44}$$

The time-varying parameters are given by:

$$\theta_1(t) = k(t) = 100\sin(0.2\pi t) + 500$$
$$\theta_2(t) = b(t) = 5\cos(0.2\pi t) + 5 \tag{45}$$

Next, we compare the parameter estimation algorithm in Proposition 2 with methods from the literature. After that, we apply the two presented RaCBF-based safety-filters to the system.

6.1.1 | Parameter Estimation

In this section, the presented DREM with finite-time update rule (FT-DREM) (20) is compared with different methods from the literature; the gradient update rule (Gradient) [1], and the finite-time update rule (FT) [6]. The system is simulated for 30 s with the initial conditions $\mathbf{x} = [0,0]^\intercal$ and $\hat{\theta}(0) = [0,0]^\intercal$. For completeness, the parameters used in each algorithm are presented in Table 1.

The result of the simulation is shown in Figure 3, where the parameter estimation error is plotted. The root mean square error (RMSE) was calculated with

$$\mathrm{RMSE}_i = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\tilde{\theta}_{i,j}^2} \tag{46}$$

TABLE 1 | Hyperparameters for the parameter estimation methods in the simulation.

Gradient update rule [1]	$\gamma = \mathrm{diag}[1\times10^4, 1\times10^4]$
Finite-time update rule [6]	$\gamma = \mathrm{diag}[1\times10^4, 1\times10^4]$
	$r = 0.5$
Finite-time DREM	$\ell = 2$
	$r = 0.5$
	$\gamma = [5.6199\times10^{15}, 0.5130\times10^{15}]$

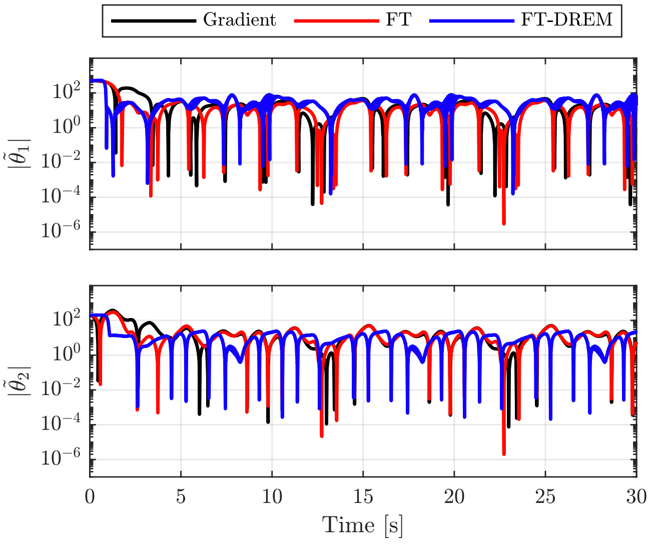

FIGURE 3 | Parameter estimation error. Notice that the biggest difference between the three methods is in the first 5 s where the gradient and FT produce a big parameter estimation error. In contrast, FT-DREM quickly reduces the parameter error within 1 s.

TABLE 2 | Root mean square error of the different parameter estimation algorithms. The lowest values are marked in bold. [Correction added on 23 August 2025, after first online publication: Table 2 has been updated in this version.]

$\tilde{\theta}_1$	Gradient	99.0526
	FT	95.6876
	FT-DREM	**91.0691**
$\tilde{\theta}_2$	Gradient	61.6369
	FT	49.8368
	FT-DREM	**37.3371**

where N is the number of samples and $\tilde{\theta}_{i,j}$ is the estimation error of the i-th parameter at sample j. The RMSE is presented in Table 2, from which it is seen that for the first parameter, the gradient rule has the highest RMSE (99.0526) and FT-DREM has the lowest (91.0691), and for the second parameter, the gradient rule has the highest RMSE (61.6369) and FT-DREM has the lowest (37.3371).

6.1.2 | Safe Control

The system is simulated for 50 s with the initial conditions $\mathbf{x} = [0,0]^\intercal$ and $\hat{\theta}(0) = \theta(0)$. Next, the performance of the controller itself is considered. We say that the system is safe if the spring-damper force satisfies the following constraint:

$$f_c \geq f_{\min} \tag{47}$$

where $f_{\min} = -15$ N is the minimum force; thus, we define the following control barrier function

$$h_r(x,\hat{\theta}) = -\mathbf{x}^\intercal\hat{\theta}(t) - f_{\min} \tag{48}$$

We are comparing the two methods presented in this paper: TV-RaCBF (36) and FT-RaCBF (37). The maximum possible parameter error is given as $\tilde{\vartheta} = [300,100]^\intercal$ and the maximum possible parameter velocity is given as $\Omega = [10,0.5]^\intercal \cdot 2\pi$. For FT-RaCBF the following hyper-parameters are used: $\ell = 2$, $r = 0.5$, $\Delta_{\min} = 5\times10^{-10}$ and $\delta = 0.1$. Both methods uses the initial value $h_r(x_r,\theta_r) = 8$.

The function $\alpha(t)$ was chosen as a linear function $\alpha(t) = at$, which is a $\mathcal{K}_\infty$ function. α was tuned individually for each method and was chosen as the highest value that ensured safety across different values of $\hat{\theta}(0)$.

The result of the simulation can be seen in Figure 4, where TV-RaCBF and FT-RaCBF are compared. The force can be seen on the top plot, and the value of the barrier function h_r can be seen in the middle plot. They closely resemble one another due to the simple formulation of h_r in Equation (48). It is clear that TV-RaCBF is more conservative than FT-RaCBF, which can adapt to the true system parameters and reduce the conservatism.

6.2 | Circular Robot

In this section, we simulate and control a two-dimensional circular robot controlled by a pendulum as seen on Figure 5. The

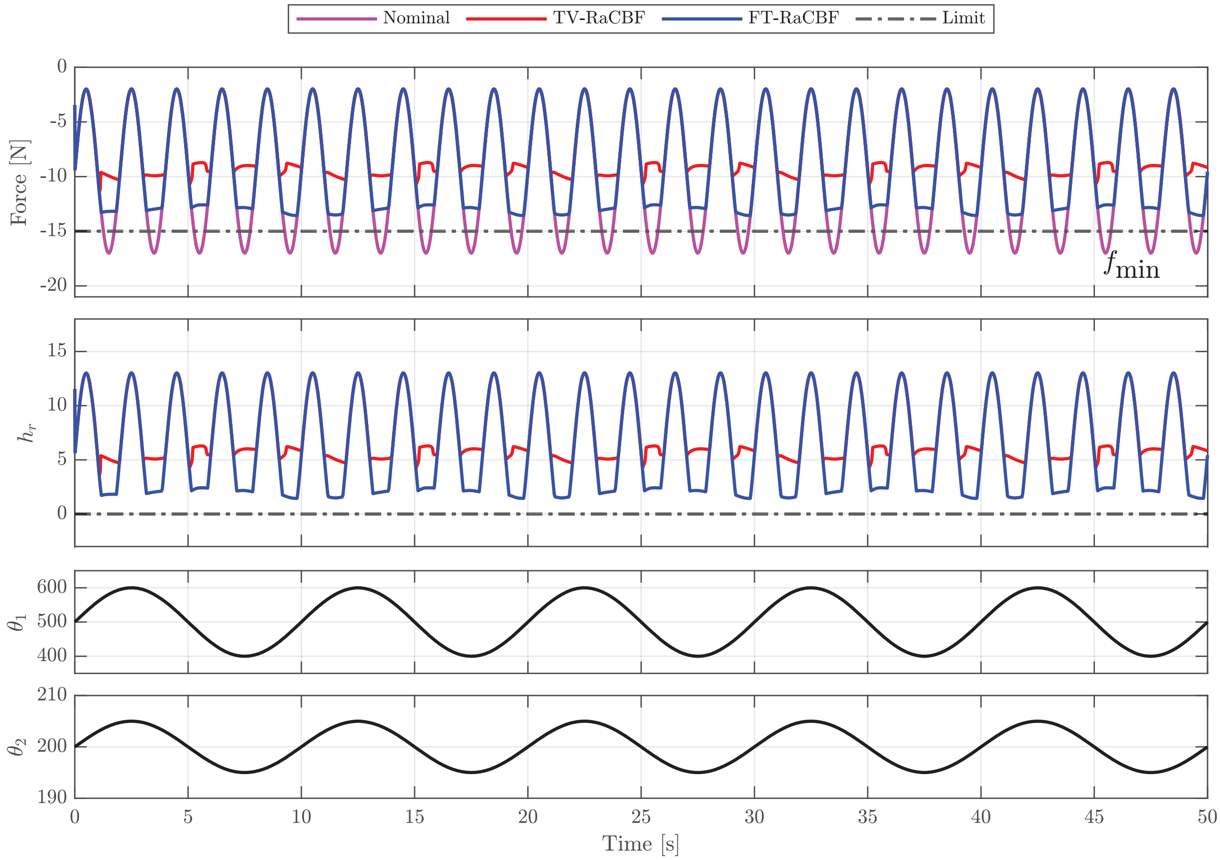

FIGURE 4 | The performance of the safe controllers TV-RaCBF and FT-RaCBF. *Top:* The desired force and the safe force controlled by TV-RaCBF and FT-RaCBF, respectively. FT-RaCBF can reduce the system conservatism compared with TV-RaCBF since it can adapt the parameter estimate to the true parameter. *Middle:* The value of the barrier function h_r. Due to the simple formulation of h_r, it closely resembles the top plot showing the force. *Bottom:* The true system parameters. Notice that the lower level of the force signal on the top plot oscillates with the same frequency as the parameters.

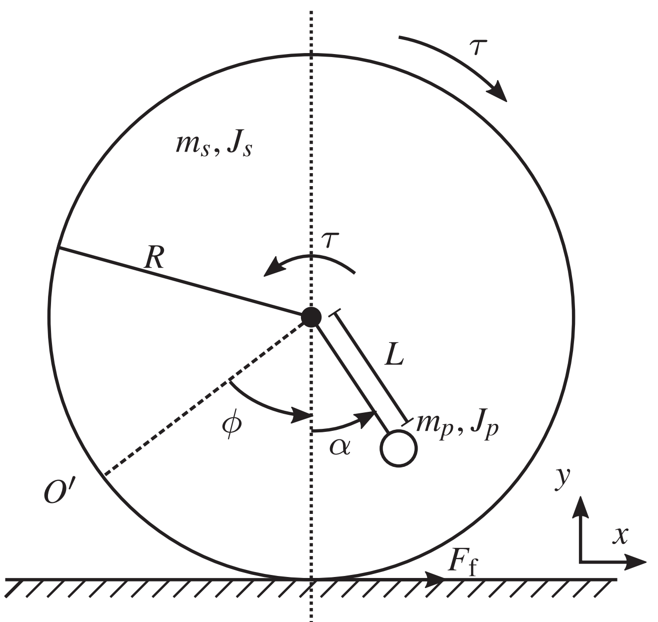

FIGURE 5 | A diagram of the circular robot used in the simulation. The robot consists of two degrees of freedom: The outer shell and the inner pendulum. The pendulum can be actuated with the control torque τ.

system dynamics can be modeled as

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + F_f = \tau \tag{49}$$

where

$$B(q) = \begin{bmatrix} m_s R^2 + m_p R^2 + J_s & m_p RL\cos(q_2) \\ m_p RL\cos(q_2) & m_p L^2 + J_p \end{bmatrix} \tag{50}$$

$$C(q,\dot{q}) = \begin{bmatrix} 0 & -m_p RL\sin(q_2)\dot{q}_2 \\ 0 & 0 \end{bmatrix} \tag{51}$$

$$g_{\text{grav}}(q) = \begin{bmatrix} 0 \\ m_p g L\sin(q_2) \end{bmatrix} \tag{52}$$

$$F_f(t) = F_v(t)\dot{q} \tag{53}$$

$$u = \begin{bmatrix} -\tau \\ \tau \end{bmatrix} \tag{54}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \phi \\ \alpha \end{bmatrix} \tag{55}$$

where m_s, J_s is the mass and the inertia of the shell, m_p, J_p is the mass and inertia of the pendulum, ϕ is the rolling angle of the robot in reference to the origin O' frame, R is the radius of the shell, L is the length of the pendulum, α is the swing angle of the pendulum, F_f is the friction force and

$$F_v = \begin{bmatrix} 100 + 10\sin(0.1t) \\ 5 \end{bmatrix} \tag{56}$$

is its coefficient.

The system can be written in the control-affine form (1)

$$\dot{x} = f(x) + F(x)\theta(t) + g(x)u \tag{57}$$

TABLE 3 | The inertial parameters of the circular robot sphere and pendulum.

Parameter	Symbol	Value
Sphere mass	m_s	1 kg
Sphere inertia	J_s	0.16 kgm^2
Sphere radius	R	0.4 m
Pendulum mass	m_p	5 kg
Pendulum inertia	J_p	0.4167 kgm^2
Pendulum arm	L	0.5 m
Gravitational acceleration	g	9.82 $\mathrm{m/s}^2$

with

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \tag{58}$$

$$f(x) = \begin{bmatrix} \dot{q} \\ B^{-1}(q)\left(-C(q,\dot{q})\dot{q} - g_{\mathrm{grav}}(q)\right) \end{bmatrix} \tag{59}$$

$$F(x) = \begin{bmatrix} -B^{-1}(q)\mathrm{diag}(\dot{q}) \end{bmatrix} \tag{60}$$

$$\theta(t) = F_v(t) \tag{61}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ B^{-1}(q)\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix} \tag{62}$$

and the input u is the control torque computed by a non-collocated partial feedback linearization control scheme [30], which adapts to the estimated parameter, where u is limited to $u_{\mathrm{lim}} = \pm g m_p L \sin(\pi/2) = \pm 24.55$ Nm, corresponding to the pendulum at 90°. The reference trajectory for $\dot{\phi}$ was:

$$\dot{\phi}_{\mathrm{ref}}(t) = -0.005\pi^2 \sin(0.01\pi t) \tag{63}$$

and the inertial parameters of the robot are given in Table 3.

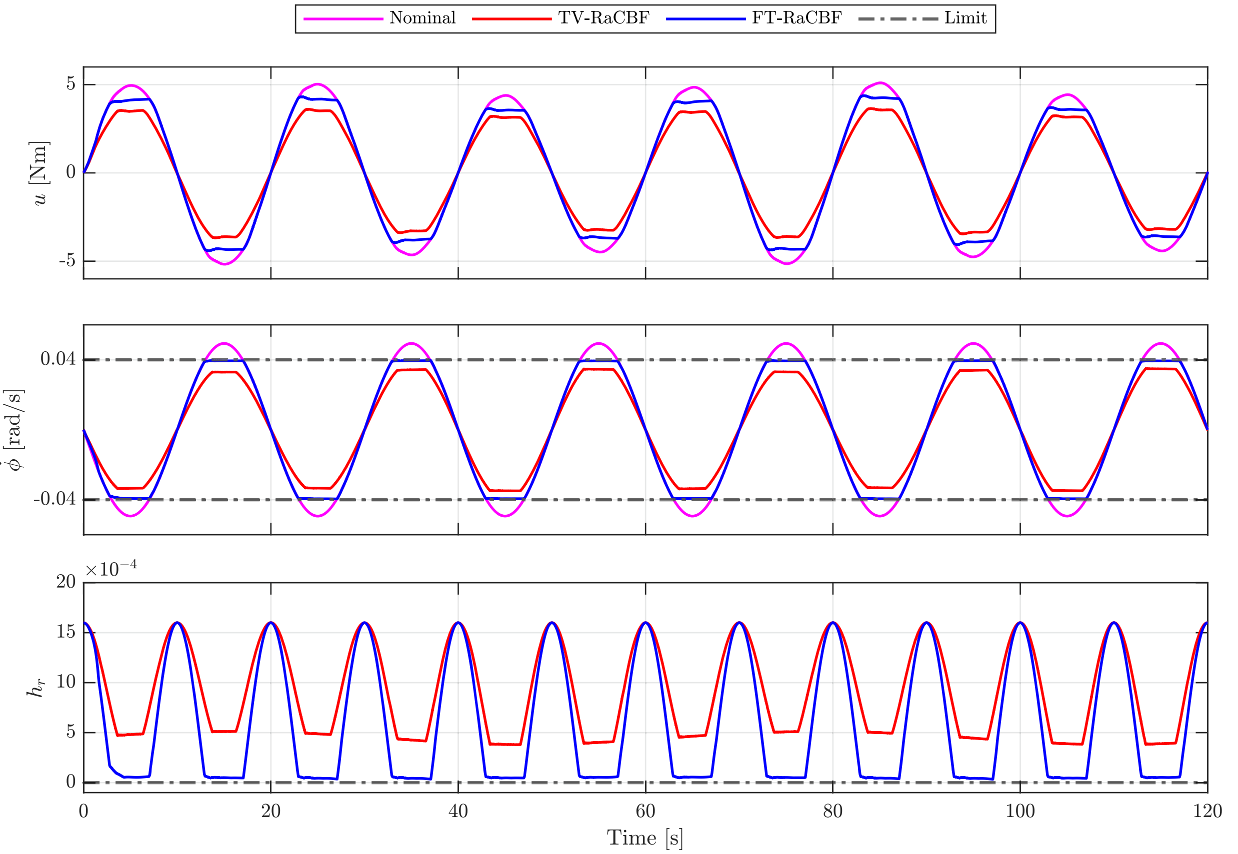

FIGURE 6 | *Top:* The control input. The black dashed line is the nominal controller, while the two full lines are the control inputs computed by TV-RaCBF and FT-RaCBF, respectively. *Middle:* The sphere velocity. The FT-RaCBF-based controller can follow the trajectory closer and has a lower conservatism compared with TV-RaCBF. Since the used controller is adaptive with respect to the estimated parameters, the FT-RaCBF-system performs better since it actually aims to estimate the true parameter, and not only remain safe, as the TV-RaCBF-system does. *Bottom:* The barrier function h_r. The FT-RaCBF-based system comes closer to 0 compared with TV-RaCBF-based system.

We put a safety constraint on the velocity of the outer shell; thus, the control barrier function is

$$h_r(x, \hat{\theta}) = v_{\max}^2 - \dot{\phi}^2 \tag{64}$$

where $v_{\max} = 0.04$ rad/s is the velocity limit. To ensure the safety of the system, the methods proposed in the paper are applied: TV-RaCBF (36) and FT-RaCBF (37). The result of the simulation can be seen in Figure 6, where the system is simulated for 120 s. The influence of the control effort can be seen in the middle plot, where the velocity of the outer shell $\dot{\phi}$ is plotted. Here it is clear that TV-RaCBF is overly conservative, only reaching ~0.034 rad/s while FT-RaCBF can come closer to the safety limit at ~0.039 rad/s. The same phenomenon is apparent when looking at the value of the barrier function h_r in the bottom plot. For FT-RaCBF the barrier reaches as low as $\sim 4.4 \times 10^{-5}$ while TV-RaCBF only reaches $\sim 5.0^{-4}$.

7 | Conclusion

In this paper, we presented a method for estimating time-varying parameters using Dynamic Regressor Extension and Mixing combined with a finite-time update law. Two simulations showed that it outperformed methods from the literature. The bound on the parameter estimation error enabled the use of a Robust adaptive Control Barrier Function to ensure the safety of systems with time-varying parameters. In both simulations, the FT-RaCBF-based system yields a smaller conservatism compared with the TV-RaCBF-based system. By giving an estimate of the true parameter, the estimated system model used in FT-RaCBF comes closer to the real model, which enables the algorithm to lower the conservatism.

In the future, we would like to study the limitations of DREM with respect to noise and system uncertainties. The safety-critical control should be implemented and tested on a real system.

Acknowledgments

Thank you to the editor and the reviewers for valuable feedback and comments that have helped to improve the manuscript.

Yitaek Kim was supported by MADE FAST, funded by Innovation Fund Denmark, Grant No. 9090-00002B.

This work has partially been carried out within the framework of the EUROfusion Consortium, funded by the European Union via the Euratom Research and Training Programme (Grant Agreement No. 101052200 – EUROfusion).

Views and opinions expressed are, however, those of the authors only and do not necessarily reflect those of the European Union or the European Commission. Neither the European Union nor the European Commission can be held responsible for them.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

1. S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence, and Robustness* (Prentice-Hall, Inc, 1989).

2. H. A. Nielsen, T. S. Nielsen, A. K. Joensen, H. Madsen, and J. Holst, "Tracking Time-Varying-Coefficient Functions," *International Journal of Adaptive Control and Signal Processing* 14 (2000): 813–828, https://doi.org/10.1002/1099-1115(200012)14:8¡813::aid-acs622¿3.0.co;2–6.

3. A. Joensen, H. Madsen, H. A. Nielsen, and T. S. Nielsen, "Tracking Time-Varying Parameters With Local Regression," *Automatica* 36 (2000): 1199–1204, https://doi.org/10.1016/s0005-1098(00)00029-7.

4. Y. Zhu and P. R. Pagilla, "Adaptive Estimation of Time-Varying Parameters in Linearly Parametrized Systems," *Journal of Dynamic Systems, Measurement, and Control* 128 (2006): 691–695, https://doi.org/10.1115/1.2234488.

5. J. Na, J. Yang, X. Ren, and Y. Guo, "Robust Adaptive Estimation of Nonlinear System With Time-Varying Parameters," *International Journal of Adaptive Control and Signal Processing* 29 (2015): 1055–1072, https://doi.org/10.1002/acs.2524.

6. H. Rios, D. Efimov, J. A. Moreno, W. Perruquetti, and J. G. Rueda-Escobedo, "Time-Varying Parameter Identification Algorithms: Finite and Fixed-Time Convergence," *IEEE Transactions on Automatic Control* 62 (2017): 3671–3678, https://doi.org/10.1109/tac.2017.2673413.

7. M. Black, E. Arabi, and D. Panagou, "A Fixed-Time Stable Adaptation Law for Safety-Critical Control Under Parametric Uncertainty," in *2021 European Control Conference (ECC)* (IEEE, 2021), https://doi.org/10.23919/ecc54610.2021.9655080.

8. S. Aranovskiy, A. Bobtsov, R. Ortega, and A. Pyrkin, "Performance Enhancement of Parameter Estimators via Dynamic Regressor Extension and Mixing," *IEEE Transactions on Automatic Control* 62 (2017): 3546–3550, https://doi.org/10.1109/tac.2016.2614889.

9. E. L. Diget and C. Sloth, "Adaptive Estimation of Time-Varying Parameters Using DREM," in *2023 62nd IEEE Conference on Decision and Control (CDC)* (IEEE, 2023), https://doi.org/10.1109/cdc49753.2023.10383699.

10. J. Wang, D. Efimov, and A. A. Bobtsov, "On Robust Parameter Estimation in Finite-Time Without Persistence of Excitation," *IEEE Transactions on Automatic Control* 65 (2020): 1731–1738, https://doi.org/10.1109/tac.2019.2932960.

11. J. Wang, D. Efimov, S. Aranovskiy, and A. A. Bobtsov, "Fixed-Time Estimation of Parameters for Non-Persistent Excitation," *European Journal of Control* 55 (2020): 24–32, https://doi.org/10.1016/j.ejcon.2019.07.005.

12. R. Ortega, A. Bobtsov, and N. Nikolaev, "Parameter Identification With Finite-Convergence Time Alertness Preservation," *IEEE Control Systems Letters* 6 (2022): 205–210, https://doi.org/10.1109/lcsys.2021.3057012.

13. A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control Barrier Functions: Theory and Applications," in *18th European Control Conference (ECC)* (IEEE, 2019), 3420–3431, https://doi.org/10.23919/ECC.2019.8796030.

14. A. J. Taylor and A. D. Ames, "Adaptive Safety With Control Barrier Functions," in *2020 American Control Conference (ACC)* (IEEE, 2020), 1399–1405, https://doi.org/10.23919/ACC45564.2020.9147463.

15. B. T. Lopez, J. J. E. Slotine, and J. P. How, "Robust Adaptive Control Barrier Functions: An Adaptive and Data-Driven Approach to Safety," *IEEE Control Systems Letters* 5, no. 3 (2020): 1031–1036.

16. M. Jankovic, "Robust Control Barrier Functions for Constrained Stabilization of Nonlinear Systems," *Automatica* 96 (2018): 359–367, https://doi.org/10.1016/j.automatica.2018.07.004.

17. S. Kolathaya and A. D. Ames, "Input-To-State Safety With Control Barrier Functions," *IEEE Control Systems Letters* 3, no. 1 (2019): 108–113, https://doi.org/10.1109/LCSYS.2018.2853698.

18. A. Alan, A. J. Taylor, C. R. He, G. Orosz, and A. D. Ames, "Safe Controller Synthesis With Tunable Input-To-State Safe Control Barrier Functions," *IEEE Control Systems Letters* 6 (2022): 908–913, https://doi.org/10.1109/LCSYS.2021.3087443.

19. S. Wang, B. Lyu, S. Wen, K. Shi, S. Zhu, and T. Huang, "Robust Adaptive Safety-Critical Control for Unknown Systems With Finite-Time Elementwise Parameter Estimation," *IEEE Transactions on Systems, Man, and Cybernetics, Systems* 53, no. 3 (2022): 1607–1617, https://doi.org/10.1109/tsmc.2022.3203176.

20. H. Cox, "On the Estimation of State Variables and Parameters for Noisy Dynamic Systems," *IEEE Transactions on Automatic Control* 9 (1964): 5–12, https://doi.org/10.1109/tac.1964.1105635.

21. L. Ljung, "Asymptotic Behavior of the Extended Kalman Filter as a Parameter Estimator for Linear Systems," *IEEE Transactions on Automatic Control* 24 (1979): 36–50, https://doi.org/10.1109/tac.1979.1101943.

22. R. Ortega, L. Praly, S. Aranovskiy, B. Yi, and W. Zhang, "On Dynamic Regressor Extension and Mixing Parameter Estimators: Two Luenberger Observers Interpretations," *Automatica* 95 (2018): 548–551, https://doi.org/10.1016/j.automatica.2018.06.011.

23. R. Ortega, V. Nikiforov, and D. Gerasimov, "On Modified Parameter Estimators for Identification and Adaptive Control. A Unified Framework and Some New Schemes," *Annual Reviews in Control* 50 (2020): 278–293, https://doi.org/10.1016/j.arcontrol.2020.06.002.

24. R. Ortega, S. Aranovskiy, A. A. Pyrkin, A. Astolfi, and A. A. Bobtsov, "New Results on Parameter Estimation via Dynamic Regressor Extension and Mixing: Continuous and Discrete-Time Cases," *IEEE Transactions on Automatic Control* 66 (2021): 2265–2272, https://doi.org/10.1109/tac.2020.3003651.

25. M. Korotina, S. Aranovskiy, R. Ushirobira, and A. Vedyakov, "On Parameter Tuning and Convergence Properties of the DREM Procedure," in *2020 European Control Conference (ECC)* (IEEE, 2020), https://doi.org/10.23919/ecc51009.2020.9143808.

26. S. Aranovskiy, R. Ushirobira, M. Korotina, and A. Vedyakov, "On Preserving-Excitation Properties of Kreisselmeier's Regressor Extension Scheme," *IEEE Transactions on Automatic Control* 68, no. 2 (2023): 1296–1302, https://doi.org/10.1109/TAC.2022.3172175.

27. A. Glushchenko and K. Lastochkin, "Robust Time-Varying Parameters Estimation Based on I-DREM Procedure," *IFAC-PapersOnLine* 55 (2022): 91–96, https://doi.org/10.1016/j.ifacol.2022.07.293.

28. R. Kosut, M. Lau, and S. Boyd, "Set-Membership Identification of Systems With Parametric and Nonparametric Uncertainty," *IEEE Transactions on Automatic Control* 37, no. 7 (1992): 929–941, https://doi.org/10.1109/9.148345.

29. G. Kreisselmeier and G. Rietze-Augst, "Richness and Excitation on an Interval-With Application to Continuous-Time Adaptive Control," *IEEE Transactions on Automatic Control* 35 (1990): 165–171, https://doi.org/10.1109/9.45172.

30. M. Spong, "Partial Feedback Linearization of Underactuated Mechanical Systems," in *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS'94)*, vol. 1 (IEEE, 1994), 314–321, https://doi.org/10.1109/iros.1994.407375.

31. H. K. Khalil, *Nonlinear Systems*, 3rd ed. (Prentice-Hall, 2002).

Appendix A

Detailed Mathematical Proofs

Proof of Proposition 2

Proof. Consider the Lyapunov candidate

$$\begin{aligned} V\left(\tilde{\theta}_i\right) &= c^{-1}\tilde{\theta}_i^2(t) \qquad \text{(A1)}\\ &= c^{-1}\left|\tilde{\theta}_i(t)\right|^2 \\ V^{\frac{r+1}{2}}\left(\tilde{\theta}_i\right) &= c^{-\frac{r+1}{2}}\left|\tilde{\theta}_i(t)\right|^{r+1} \qquad \text{(A2)}\end{aligned}$$

where $c_i = 2\gamma_i$ and $V : \mathbb{R}^k \to \mathbb{R}$. In the remainder of the proof, we omit arguments to enhance readability. The Lie derivative with respect to the vector field (28) is

$$\begin{aligned} \dot{V} &= \tilde{\theta}_i\gamma_i^{-1}\left(-\gamma_i\Delta\left\lceil\Delta\tilde{\theta}_i\right\rfloor^r + \dot{\theta}_i\right) \\ &\le -\tilde{\theta}_i\Delta\left\lceil\Delta\tilde{\theta}_i\right\rfloor^r + \gamma_i^{-1}\left|\tilde{\theta}_i\right|\Omega_i \\ &= -\left|\Delta\tilde{\theta}_i\right|^{r+1} + \frac{\Omega_i}{\gamma_i}\left|\tilde{\theta}_i\right| \\ &\le -\Delta_{\min}^{r+1}\left|\tilde{\theta}_i\right|^{r+1} + \frac{\Omega_i}{\gamma_i}\left|\tilde{\theta}_i\right| \qquad \text{(A3)}\end{aligned}$$

where $|\dot{\theta}_i(t)| \le \Omega_i$ and $\Delta(t) \ge \Delta_{\min}$ $\forall t > 0$. Introducing $\delta \in (0,1)$ leads to

$$\dot{V} \le -\Delta_{\min}^{r+1}\left|\tilde{\theta}_i\right|^{r+1}(1-\delta) - \Delta_{\min}^{r+1}\left|\tilde{\theta}_i\right|^{r+1}\delta + \frac{\Omega_i}{\gamma_i}\left|\tilde{\theta}_i\right| \qquad \text{(A4)}$$

then

$$\dot{V} \le -\Delta_{\min}^{r+1}\left|\tilde{\theta}_i\right|^{r+1}(1-\delta), \quad \forall\left|\tilde{\theta}_i\right| \ge \mu_i \qquad \text{(A5)}$$

From Equations (A5) and (A2) it follows that

$$\dot{V} \le -\Delta_{\min}^{r+1}(1-\delta)c_i^{\frac{r+1}{2}}V^{\frac{r+1}{2}}\left(\tilde{\theta}_i\right) \qquad \text{(A6)}$$

with the solution, denoting $V\left(\hat{\theta}_i(0)\right)$ as V_0, one has, following the comparison principle (see, e.g., [31]),

$$V\left(\tilde{\theta}_i\right) \le \left(V_0^{\frac{1-r}{2}} - \frac{(1-r)(1-\delta)c_i^{\frac{r+1}{2}}\Delta_{\min}^{r+1}}{2}t\right)^{\frac{2}{1-r}} \qquad \text{(A7)}$$

which ensures that $\tilde{\theta}_i$ satisfies the following bound, by inserting (A1) and isolating $|\tilde{\theta}_i|$:

$$\left|\tilde{\theta}_i\right| \le \sqrt{c_i}\left(c_i^{\frac{r-1}{2}}\left|\tilde{\theta}_i(0)\right|^{1-r} - \frac{(1-r)(1-\delta)c_i^{\frac{r+1}{2}}\Delta_{\min}^{r+1}}{2}t\right)^{\frac{1}{1-r}} \qquad \text{(A8)}$$

for all $t < T\left(\tilde{\theta}_i(0)\right)$, while for all $t \ge T(\tilde{\theta}(0))$, $\tilde{\theta}_i$ is bounded by μ_i, that is, $\left|\tilde{\theta}_i(t)\right| \le \mu_i$. The time $T\left(\tilde{\theta}_i(0)\right)$ can be found by equaling the right side of (A8) with μ_i and isolating for t and denote it $T\left(\tilde{\theta}_i(0)\right)$. □

Proof of Proposition 3

Proof. We consider the Lyapunov candidate

$$V\left(\tilde{\theta}_i(t)\right) = \frac{1}{2}\tilde{\theta}_i^2(t) \qquad \text{(A9)}$$

where $V:\mathbb{R}^k \to \mathbb{R}$. It is seen that V is positive definite. In the remainder of the proof, we omit arguments to enhance readability. The parameter is constant, $\dot{\theta}_i = 0$, thus the Lie derivative of V with respect to the vector field (28) is:

$$\begin{aligned}
\dot{V} &= \tilde{\theta}_i \dot{\tilde{\theta}}_i \\
&= -\tilde{\theta}_i \gamma_i \Delta \lceil \Delta \tilde{\theta}_i \rfloor^r \\
&= -\gamma_i |\Delta|^{1+r} \left|\tilde{\theta}_i\right|^{1+r} \\
&= -\gamma_i |\Delta|^{1+r} \left|\tilde{\theta}_i^2\right|^{\frac{1+r}{2}} \\
&= -\gamma_i |\Delta|^{1+r} \left|2V\left(\tilde{\theta}_i\right)\right|^{\frac{1+r}{2}} \\
&= -2^{\frac{1+r}{2}} \gamma_i \kappa^2 V^{\frac{1+r}{2}} \left(\tilde{\theta}_i\right)
\end{aligned} \tag{A10}$$

where $\kappa = |\Delta|^{(1+r)/2}$, $\gamma_i > 0$ and $\Delta > 0$. Solving the above equation, denoting $V\left(\tilde{\theta}_i(0)\right)$ as V_0, one has:

$$V = \left(V_0^{\frac{1-r}{2}} - (1-r)\gamma_i 2^{\frac{r-1}{2}} \int_0^t \kappa^2(\tau)\, \mathrm{d}\tau\right)^{\frac{2}{1-r}} \tag{A11}$$

Since κ is IE, $\forall t \geq t_c$, $\int_0^t \kappa^2(t)\, \mathrm{d}t \geq \beta > 0$, that is,

$$V \leq \left(V_0^{\frac{1-r}{2}} - (1-r)\gamma_i 2^{\frac{r-1}{2}} \beta\right)^{\frac{2}{1-r}} \tag{A12}$$

which implies that $V\left(\tilde{\theta}_i\right)$ converges to zero within $(0, t_c)$ if γ_i is bounded from below as

$$\gamma_i \geq \frac{\tilde{\vartheta}_i^{1-r}}{(1-r)\beta} \tag{A13}$$

where $\tilde{\vartheta}_i$ is the maximum possible parameter error, and ensures that $\tilde{\theta}_i$ satisfies the following bound:

$$\left|\tilde{\theta}_i\right| \leq \left(\left|\tilde{\theta}_i(0)\right|^{1-r} - (1-r)\gamma_i \int_0^t \kappa^2(\tau)\, \mathrm{d}\tau\right)^{\frac{1}{1-r}} \tag{A14}$$

for all $t < t_c$, while for all $t \geq t_c$, $\tilde{\theta} = 0$. □

Proof of Theorem 1

Proof. Let

$$h(x, \hat{\theta}) = h_r(x, \hat{\theta}) - \Xi(t) \tag{A15}$$

where $\Xi(t) = \frac{1}{2}\tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t)$.

Calculating the Lie derivative of h with respect to t, one has

$$\begin{aligned}
\dot{h}(x, \dot{x}, \hat{\theta}, \dot{\hat{\theta}}) &= \dot{h}_r(x, \dot{x}, \hat{\theta}, \dot{\hat{\theta}}) - \dot{\Xi}(t) \\
&= \frac{\partial h_r}{\partial x}(x, \hat{\theta})\left(f(x) + F(x)\theta(t) + g(x)u\right) \\
&\quad + \frac{\partial h_r}{\partial \hat{\theta}}(x, \hat{\theta})\dot{\hat{\theta}} - \tilde{\theta}^\top(t)\Gamma^{-1}\dot{\tilde{\theta}}(t) \\
&= \frac{\partial h_r}{\partial x}(x, \hat{\theta})\left(f(x) + F(x)\hat{\theta}(t) + g(x)u\right) \\
&\quad + \frac{\partial h_r}{\partial \hat{\theta}}(x, \hat{\theta})\dot{\hat{\theta}} - \tilde{\theta}^\top(t)\Gamma^{-1}\dot{\tilde{\theta}}(t) \\
&\quad + \frac{\partial h_r}{\partial x}(x, \hat{\theta})F(x)\tilde{\theta}(t) \\
&= \frac{\partial h_r}{\partial x}(x, \hat{\theta})\left(f(x) + F(x)\hat{\theta}(t) + g(x)u\right) \\
&\quad + \frac{\partial h_r}{\partial \hat{\theta}}(x, \hat{\theta})\dot{\hat{\theta}} - \tilde{\theta}^\top(t)\Gamma^{-1}\dot{\tilde{\theta}}(t) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{\partial h_r}{\partial x_i}(x, \hat{\theta})F_{i,j}(x)\tilde{\theta}_j(t) \\
&\geq \frac{\partial h_r}{\partial x}(x, \hat{\theta})\left(f(x) + F(x)\hat{\theta}(t) + g(x)u\right) \\
&\quad + \frac{\partial h_r}{\partial \hat{\theta}}(x, \hat{\theta})\dot{\hat{\theta}} - \tilde{\zeta}^\top(t)\Gamma^{-1}|\dot{\tilde{\zeta}}(t)| \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{k} \left|\frac{\partial h_r}{\partial x_i}(x, \hat{\theta})F_{i,j}(x)\right|\tilde{\zeta}_j(t) \\
&\geq -\alpha\left(h_r(x, \hat{\theta}) - \Xi(t)\right)
\end{aligned}$$

The second inequality comes from $0 \leq \left|\tilde{\theta}_i(t)\right| \leq \tilde{\zeta}_i(t)$ and $0 \leq \left|\dot{\tilde{\theta}}_i(t)\right| \leq \left|\dot{\tilde{\zeta}}_i(t)\right|$. From Equation (40), $h \geq 0$ and $h_r \geq h$; hence $h_r \geq \frac{1}{2}\tilde{\zeta}^\top(t)\Gamma^{-1}\tilde{\zeta}(t)$ and $\mathcal{S}_{\hat{\theta}}^s$ is forward invariant. □