

Computing the principal pivoting transform in the context of Lemke's Algorithm for solving Linear Complementarity Problems

Hongkai Dai*

1 Problem statement

Given tuples $w = \{w_1, \dots, w_n\}$ and $z = \{z_1, \dots, z_n, z_{n+1}\}$, which can be arranged into vectors as follows:

$$\mathbf{w} = [w_1 \quad \dots \quad w_n]^T \in \mathbb{R}^n \quad (1)$$

$$\mathbf{z} = [z_1 \quad \dots \quad z_n \quad z_{n+1}]^T \in \mathbb{R}^{n+1}, \quad (2)$$

we seek to permute the relationship:

$$\mathbf{w} = \mathbf{q} + \mathbf{M}\mathbf{z} \quad (3)$$

where $\mathbf{q} \in \mathbb{R}^n$, $\mathbf{M} \in \mathbb{R}^{n \times (n+1)}$ are a given vector/matrix (note that boldface is used to help distinguish vectors and matrices from sets). z_{n+1} will hereforth be referred to as the “artificial variable” by swapping some entries in the “dependent variable tuple” w with some entries in the “independent variable” tuple z , yielding two new sets z' and w' . Similarly to the matrix/vector relationship above, vectors $\mathbf{z}' \in \mathbb{R}^{n+1}$ and $\mathbf{w}' \in \mathbb{R}^n$ can be defined that correspond to variable indices of z and w . *The problem that this document seeks to solve is the vector \mathbf{q}' and the matrix \mathbf{M}' (actually, just a single column of it) such that the relationship:*

$$\mathbf{w}' = \mathbf{q}' + \mathbf{M}'\mathbf{z}' \quad (4)$$

follows from the ordering in the vectors corresponding to the positions of the individual w and z variables in the tuples w' and z' . Recall that pivoting takes the value $\mathbf{z}' = \mathbf{0}$, which allows \mathbf{w}' to be determined simply by setting it equal to \mathbf{q}' , via the equation above. This rearrangement is known as a *principal pivoting transform*.

One of the variables in the independent variable tuple z' will be identified as the *driving variable*. We will denote this variable as z'_{driving} . Our goal is to

*Toyota Research Institute

find the vector \mathbf{q}' , together with the column in \mathbf{M}' that will be multiplied (i.e., via the inner product operation) with z'_{driving} . We will denote this column as $\mathbf{M}'_{\text{driving}}$.

1.0.1 Running example

We consider the LCP from Example 4.4.7 in [1]:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -3 \\ 6 \\ -1 \end{bmatrix}}_q + \underbrace{\begin{bmatrix} 0 & -1 & 2 & 1 \\ 2 & 0 & -2 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix}}_M \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \quad (5)$$

After some number of pivoting operations, assume that the “dependent” tuple w' and “independent” tuple z' consist of:

$$w' = \{z_4, w_2, z_3\}, z' = \{w_1, w_3, z_2, z_1\} \quad (6)$$

1.1 Terminology

We now introduce some terminology:

- **INDEPENDENT W**: the tuple of variables from w that (partially) comprise z' in the order in which they are found in w (i.e., the elements of **INDEPENDENT W** appear in ascending order, e.g., $\{w_1, w_2\}$). In the running example, **INDEPENDENT W** is $\{w_1, w_3\}$.
 - α : the positions in w of the elements from **INDEPENDENT W**. In the running example, $\alpha = \{1, 3\}$. From the definition of **INDEPENDENT W**, these w elements of α appear in ascending order.
 - α' : the respective indices in z' of the variables from **INDEPENDENT W**. In the running example, $\alpha' = \{1, 2\}$. These w elements of α' *elements will not necessarily be present in ascending order*.

Note that the elements in the “vanilla” tuple correspond to variable indices, while elements in the primed tuple correspond to tuple indices. Also, **observe** the invariant $w_{\alpha_i} = z'_{\alpha'_i}$.

- **DEPENDENT Z**: the tuple of variables from z that (partially) comprise w' in the order in which they are found in z (i.e., the elements of **DEPENDENT Z** appear in ascending order, e.g., $\{z_3, z_4\}$). In the running example, **DEPENDENT Z** is $\{z_3, z_4\}$.
 - β : one greater, respectively, than the positions in z of the elements from **DEPENDENT Z**. In the running example, $\beta = \{3, 4\}$ (corresponding to z_3 and z_4). From the definition of **DEPENDENT Z**, these z elements of β appear in ascending order.

- β' : the respective indices in w' of the variables from DEPENDENT Z. In the running example, $\beta' = \{3, 1\}$. These z elements of α' *elements will not necessarily be present in ascending order*.

Again, note that the elements in the “vanilla” tuple correspond to variable indices, while elements in the primed tuple correspond to tuple indices. Also, **observe** the invariant $z_{\beta_i} = w'_{\beta'_i}$.

- DEPENDENT W: the tuple of variables from w that (partially) comprise w' in the order in which they are found in w (i.e., the elements of DEPENDENT W appear in ascending order, e.g., $\{w_1, w_2\}$). In the running example, DEPENDENT W is $\{w_2\}$.
 - $\bar{\alpha}$: the positions in w of the elements from DEPENDENT W. In the running example, $\bar{\alpha} = \{2\}$. From the definition of DEPENDENT W, these w elements of $\bar{\alpha}$ appear in ascending order.
 - $\bar{\alpha}'$: the respective indices in w' of the variables from DEPENDENT W. In the running example, $\bar{\alpha}' = \{2\}$. These w elements of $\bar{\alpha}'$ *elements are not necessarily present in ascending order*.

Again, note that the elements in the “vanilla” tuple correspond to variable indices, while elements in the primed tuple correspond to tuple indices. Also, **observe** the invariant $w_{\bar{\alpha}_i} = w'_{\bar{\alpha}'_i}$.

- INDEPENDENT Z: the tuple of variables from z that (partially) comprise z' in the order in which they are found in z (i.e., the elements of INDEPENDENT Z appear in ascending order, e.g., $\{z_1, z_2\}$). In the running example, INDEPENDENT Z is $\{z_1, z_2\}$. The elements in INDEPENDENT Z are present in ascending order (e.g., $\{z_1, z_2\}$).
 - $\bar{\beta}$: one greater, respectively, than the positions in z of the elements from INDEPENDENT Z. In the running examples, $\bar{\beta} = \{1, 2\}$. From the definition of INDEPENDENT Z, the elements of $\bar{\beta}$ ’s elements appear in ascending order.
 - $\bar{\beta}'$: the respective indices in z' of the variables from INDEPENDENT Z. In the running example, $\bar{\beta}' = \{4, 3\}$. These z elements of $\bar{\beta}'$ *elements are not necessarily present in ascending order*.

Once more, note that the elements in the “vanilla” tuple correspond to variable indices, while elements in the primed tuple correspond to tuple indices. Also, **observe** the invariant $z_{\bar{\beta}_i} = z'_{\bar{\beta}'_i}$.

2 Approach

Lemma 1 *The length of INDEPENDENT W is the same as that of DEPENDENT Z.*

We will denote the length of INDEPENDENT W as m . In the running example, $m = 2$.

2.1 Computing q'

We first define a *square* matrix $\mathbf{M}^{\alpha\beta} \in \mathbb{R}^{m \times m}$ as being comprised of entries $i \in 1, \dots, m$ and $j \in 1, \dots, m$:

$$\mathbf{M}_{ij}^{\alpha\beta} = \mathbf{M}_{\alpha_i\beta_j} \quad (7)$$

and a rectangular matrix $\mathbf{M}^{\bar{\alpha}\beta} \in \mathbb{R}^{(n-m) \times m}$ as being comprised of entries $i \in 1, \dots, n-m$ and $j \in 1, \dots, m$:

$$\mathbf{M}_{ij}^{\bar{\alpha}\beta} = \mathbf{M}_{\bar{\alpha}_i\beta_j} \quad (8)$$

Likewise, we define vectors $\mathbf{q}^\alpha \in \mathbb{R}^m$ and $\mathbf{q}^{\bar{\alpha}} \in \mathbb{R}^{(n-m)}$ as being comprised of entries $i \in 1, \dots, m$ and $j \in 1, \dots, n-m$:

$$q_i^\alpha = q_{\alpha_i} \quad (9)$$

$$q_j^{\bar{\alpha}} = q_{\bar{\alpha}_j} \quad (10)$$

In the running example—recall that $\alpha = \{1, 3\}, \beta = \{3, 4\}, \bar{\alpha} = \{2\}$ —we highlight the parts of \mathbf{q} and \mathbf{M} that correspond to $\mathbf{q}^\alpha, \mathbf{M}^{\alpha\beta}$ in red, and $\mathbf{q}^{\bar{\alpha}}, \mathbf{M}^{\bar{\alpha}\beta}$ in blue. Now putting the example into “tableaux form”:

		z_1	z_2	z_3	z_4
w_1	-3	0	-1	2	1
w_2	6	2	0	-2	1
w_3	-1	-1	-1	0	1

thereby yielding the following vectors and matrices:

$$\mathbf{q}^\alpha = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \mathbf{q}^{\bar{\alpha}} = [6], \mathbf{M}^{\alpha\beta} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{M}^{\bar{\alpha}\beta} = \begin{bmatrix} -2 & 1 \end{bmatrix} \quad (11)$$

With the index sets $\alpha', \bar{\alpha}'$ defined in Subsection 1.1, and Equation 10 from Page 71 of [1], we can write \mathbf{q}' as:

$$\mathbf{q}'^{\beta'} = -(\mathbf{M}^{\alpha\beta})^{-1} \mathbf{q}^\alpha \quad (12)$$

$$\mathbf{q}'^{\bar{\alpha}'} = \mathbf{q}^{\bar{\alpha}} + \mathbf{M}^{\bar{\alpha}\beta} \mathbf{q}'^{\beta'} \quad (13)$$

where $\mathbf{q}'^{\alpha'}, \mathbf{q}'^{\bar{\alpha}'}$ are “views” of the vector \mathbf{q}' , as $\mathbf{q}'^{\alpha'}_i = \mathbf{q}'_{\alpha'_i}, \mathbf{q}'^{\bar{\alpha}'}_i = \mathbf{q}'_{\bar{\alpha}'_i}$. In the running example $\mathbf{q}'^{\alpha'} = [1 \ 1]^\top, \mathbf{q}'^{\bar{\alpha}'} = [5]$. \mathbf{q}' is not composed by stacking $\mathbf{q}'^{\alpha'}, \mathbf{q}'^{\bar{\alpha}'}$ as is done in similar procedures in [1].

2.2 Computing $\mathbf{M}'_{\text{driving}}$

This computation requires considering two cases for the driving variable, which is an entry in the new independent variable z' . The driving variable can be either:

1. a dependent variable from w .
2. an independent variable z .

in the running example, if the driving variable is w_1 or w_3 , then it belongs to the first case; if the driving variable is z_2 or z_1 , then it belongs to the second case. Let's discuss the two cases separately.

2.2.1 Driving variable is a dependent variable from w

If the driving variable z'_{driving} is a dependent variable from w , then it also belongs to the vector INDEPENDENT w . We denote the index of z'_{driving} in INDEPENDENT w as γ , namely:

$$w_\gamma^\alpha = w_{\alpha\gamma} = z'_{\text{driving}} \quad (14)$$

where w^α is the vector INDEPENDENT w . In the running example, INDEPENDENT w is $\{w_1, w_3\}$, thus $\alpha = \{1, 3\}$. If $z'_{\text{driving}} \equiv w_1$, then the index $\gamma \equiv 1$. If $z'_{\text{driving}} \equiv w_3$, then the index $\gamma \equiv 2$. To compute the column vector $\mathbf{M}'_{\text{driving}}$, we need to first compute the γ^{th} column of the matrix $(\mathbf{M}^{\alpha\beta})^{-1}$. To this end, we define a unit vector $\mathbf{e} \in \mathbb{R}^m$ as:

$$e_\gamma = 1 \quad (15)$$

$$e_i = 0 \quad \text{if } i \neq \gamma \quad (16)$$

and we can compute $\mathbf{M}'_{\text{driving}}$ as:

$$\mathbf{M}'_{\text{driving}}^{\beta'} = (\mathbf{M}^{\alpha\beta})^{-1} \mathbf{e} \quad (17)$$

$$\mathbf{M}'_{\text{driving}}^{\bar{\alpha}'} = \mathbf{M}^{\bar{\alpha}\beta} \mathbf{M}'_{\text{driving}\beta'} \quad (18)$$

Notice that $\mathbf{M}'_{\text{driving}\alpha'}$ is the γ^{th} column of $\mathbf{M}_{\alpha\beta}^{-1}$ according to Equation (17).

2.2.2 Driving variable is an independent variable from z

If the driving variable z'_{driving} is an independent variable from z , we denote the position in z of z'_{driving} as ζ . In the running example, if the driving variable is z_2 , then $\zeta \equiv 2$; if the driving variable is z_1 , then $\zeta \equiv 1$.

To compute the column vector $\mathbf{M}'_{\text{driving}}$ in \mathbf{M}' , we need the ζ^{th} column of \mathbf{M} , denoted \mathbf{g} , which will be decomposed into two sub-vectors. One sub-vector \mathbf{g}^α contains the entries with row indices in α while the other, $\mathbf{g}^{\bar{\alpha}}$, contains the entries with row indices from $\bar{\alpha}$. Namely:

$$g_i^\alpha = M_{\alpha_i\zeta} \quad (19)$$

$$g_i^{\bar{\alpha}} = M_{\bar{\alpha}_i\zeta} \quad (20)$$

In the running example, w_1, w_3 will be pivoted to z' , and w_2 will remain in w' . If the driving variable is z_2 , we highlight the vector $g^\alpha, g^{\bar{\alpha}}$ in \mathbf{M} as:

	z_1	z_2	z_3	z_0
w_1	0	-1	2	1
w_2	2	0	-2	1
w_3	-1	1	0	1

so $g^\alpha \equiv [-1 \ 1]^\top$, $g^{\bar{\alpha}} = [0]$.

with the vector $g^\alpha, g^{\bar{\alpha}}$, we can then compute $\mathbf{M}'_{\text{driving}}$ as

$$\mathbf{M}'_{\text{driving}}{}^{\beta'} = -(\mathbf{M}^{\alpha\beta})^{-1} g^\alpha \quad (21)$$

$$\mathbf{M}'_{\text{driving}}{}^{\bar{\alpha}'} = g^{\bar{\alpha}} + \mathbf{M}^{\bar{\alpha}\beta} \mathbf{M}'_{\text{driving}}{}^{\beta'} \quad (22)$$

References

- [1] R. W. Cottle, J.-S. Pang, and R. Stone. *The Linear Complementarity Problem*. Academic Press, Boston, 1992.