Computing the principal pivoting transform in the context of Lemke's Algorithm for solving Linear Complementarity Problems

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1 Problem statement

Given tuples $w = \{w_1, \ldots, w_n\}$ and $z = \{z_1, \ldots, z_n, z_{n+1}\}$, which can be arranged into vectors as follows:

$$\boldsymbol{w} = \begin{bmatrix} w_1 & \ldots & w_n \end{bmatrix}^\mathsf{T} \in \mathbb{R}^n \tag{1}$$

$$\boldsymbol{z} = \begin{bmatrix} z_1 & \ldots & z_n & z_{n+1} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{n+1}, \tag{2}$$

we seek to permute the relationship:

$$\boldsymbol{w} = \boldsymbol{q} + \mathbf{M}\boldsymbol{z} \tag{3}$$

where $\boldsymbol{q} \in \mathbb{R}^n, \mathbf{M} \in \mathbb{R}^{n \times (n+1)}$ are a given vector/matrix (note that boldface is used to help distinguish vectors and matrices from sets). z_{n+1} will hereforth be referred to as the "artificial variable" by swapping some entries in the "dependent variable tuple" w with some entries in the "independent variable" tuple z, yielding two new sets z' and w'. Similarly to the matrix/vector relationship above, vectors $\boldsymbol{z}' \in \mathbb{R}^{n+1}$ and $\boldsymbol{w}' \in \mathbb{R}^n$ can be defined that correspond to variable indices of z and w. *The problem that this document seeks to solve is the vector* $\boldsymbol{q}'$ *and the matrix* $\mathbf{M}'$ (actually, just a single column of it) *such that the relationship*:

$$\boldsymbol{w}' = \boldsymbol{q}' + \mathbf{M}'\boldsymbol{z}' \tag{4}$$

follows from the ordering in the vectors corresponding to the positions of the individual w and z variables in the tuples w' and z'. Recall that pivoting takes the value $\boldsymbol{z}' = \mathbf{0}$, which allows $\boldsymbol{w}'$ to be determined simply be setting it equal to $\boldsymbol{q}'$, via the equation above. This rearrangement is known as a *principal pivoting transform*.

One of the variables in the independent variable tuple z' will be identified as the *driving variable*. We will denote this variable as z'_{driving}. Our goal is to find the vector $\boldsymbol{q}'$, together with the column in $\mathbf{M}'$ that will be multiplied (i.e., via the inner product operation) with z'_{driving}. We will denote this column as $\mathbf{M}'_{\text{driving}}$.

1.0.1 Running example

We consider the LCP from Example 4.4.7 in [1]:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -3 \\ 6 \\ -1 \end{bmatrix}}_{q} + \underbrace{\begin{bmatrix} 0 & -1 & 2 & 1 \\ 2 & 0 & -2 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix}}_{M} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \tag{5}$$

After some number of pivoting operations, assume that the "dependent" tuple w' and "independent" tuple z' consist of:

$$w' = \{z_4, w_2, z_3\}, z' = \{w_1, w_3, z_2, z_1\} \tag{6}$$

1.1 Terminology

We now introduce some terminology:

- INDEPENDENT W: the tuple of variables from w that (partially) comprise z' in the order in which they are found in w (i.e., the elements of INDEPENDENT W appear in ascending order, e.g., $\{w_1, w_2\}$). In the running example, INDEPENDENT W is $\{w_1, w_3\}$.
  - α: the positions in w of the elements from INDEPENDENT W. In the running example, $\alpha = \{1, 3\}$. From the definition of INDEPENDENT W, these w elements of α appear in ascending order.
  - α': the respective indices in z' of the variables from INDEPENDENT W. In the running example, $\alpha' = \{1, 2\}$. These w elements of α' *elements will not necessarily be present in ascending order.*

  Note that the elements in the "vanilla" tuple correspond to variable indices, while elements in the primed tuple correspond to tuple indices. Also, **observe** the invariant $w_{\alpha_i} = z'_{\alpha'_i}$.

- DEPENDENT Z: the tuple of variables from z that (partially) comprise w' in the order in which they are found in z (i.e., the elements of DEPENDENT Z appear in ascending order, e.g., $\{z_3, z_4\}$). In the running example, DEPENDENT Z is $\{z_3, z_4\}$.
  - β: the positions in z of the elements from DEPENDENT Z. In the running example, $\beta = \{3, 4\}$ (corresponding to z_3 and z_4). From the definition of DEPENDENT Z, these z elements of β appear in ascending order.
  - β': the respective indices in w' of the variables from DEPENDENT Z. In the running example, $\beta' = \{3, 1\}$. These z elements of α' *elements will not necessarily be present in ascending order.*

Again, note that the elements in the "vanilla" tuple correspond to variable indices, while elements in the primed tuple correspond to tuple indices. Also, **observe** the invariant $z_{\beta_i} = w'_{\beta'_i}$.

- DEPENDENT W: the tuple of variables from w that (partially) comprise w' in the order in which they are found in w (i.e., the elements of DEPENDENT W appear in ascending order, e.g., $\{w_1, w_2\}$). In the running example, DEPENDENT W is $\{w_2\}$.

  - $\bar{\alpha}$: the positions in w of the elements from DEPENDENT W. In the running example, $\bar{\alpha} = \{2\}$. From the definition of DEPENDENT W, these w elements of $\bar{\alpha}$ appear in ascending order.
  - $\bar{\alpha}'$: the respective indices in w' of the variables from DEPENDENT W. In the running example, $\bar{\alpha'} = \{2\}$. These w elements of $\bar{\alpha}$'s *elements are not necessarily present in ascending order.*

  Again, note that the elements in the "vanilla" tuple correspond to variable indices, while elements in the primed tuple correspond to tuple indices. Also, **observe** the invariant $w_{\bar{\alpha}_i} = w'_{\bar{\alpha'}_i}$.

- INDEPENDENT Z: the tuple of variables from z that (partially) comprise z' in the order in which they are found in z (i.e., the elements of INDEPENDENT Z appear in ascending order, e.g., $\{z_1, z_2\}$). In the running example, INDEPENDENT Z is $\{z_1, z_2\}$. The elements in INDEPENDENT Z are present in ascending order (e.g., $\{z_1, z_2\}$).

  - $\bar{\beta}$: one greater, respectively, than the positions in z of the elements from INDEPENDENT Z. In the running examples, $\bar{\beta} = \{1, 2\}$. From the definition of INDEPENDENT Z, the elements of $\bar{\beta}$'s elements appear in ascending order.
  - $\bar{\beta}'$: the respective indices in z' of the variables from INDEPENDENT Z. In the running example, $\bar{\beta}' = \{4, 3\}$. These z elments of $\bar{\beta}'$ *elements are not necessarily present in ascending order.*

  Once more, note that the elements in the "vanilla" tuple correspond to variable indices, while elements in the primed tuple correspond to tuple indices. Also, **observe** the invariant $z_{\bar{\beta}_i} = z'_{\bar{\beta}'_i}$.

2 Approach

Lemma 1 *The length of* INDEPENDENT W *is the same as that of* DEPENDENT Z*.*

We will denote the length of INDEPENDENT W as m. In the running example, $m = 2$.

2.1 Computing q'

We first define a *square* matrix $\mathbf{M}^{\alpha\beta} \in \mathbb{R}^{m\times m}$ as being comprised of entries $i \in 1,\ldots,m$ and $j \in 1,\ldots,m$:

$$\mathbf{M}^{\alpha\beta}_{ij} = \mathbf{M}_{\alpha_i\beta_j} \tag{7}$$

and a rectangular matrix $\mathbf{M}^{\bar{\alpha}\beta} \in \mathbb{R}^{(n-m)\times m}$ as being comprised of entries $i \in 1,\ldots,n-m$ and $j \in 1,\ldots,m$:

$$\mathbf{M}^{\bar{\alpha}\beta}_{ij} = \mathbf{M}_{\bar{\alpha}_i\beta_j} \tag{8}$$

Likewise, we define vectors $\boldsymbol{q}^{\alpha} \in \mathbb{R}^m$ and $\boldsymbol{q}^{\bar{\alpha}} \in \mathbb{R}^{(n-m)}$ as being comprised of entries $i \in 1,\ldots,m$ and $j \in 1,\ldots,n-m$:

$$q^{\alpha}_i = q_{\alpha_i} \tag{9}$$
$$q^{\bar{\alpha}}_j = q_{\bar{\alpha}_j} \tag{10}$$

In the running example—recall that $\alpha = \{1,3\}, \beta = \{3,4\}, \bar{\alpha} = \{2\}$—we highlight the parts of $\boldsymbol{q}$ and $\mathbf{M}$ that correspond to q^{α}, $M^{\alpha\beta}$ in red, and $q^{\bar{\alpha}}$, $M^{\bar{\alpha}\beta}$ in blue. Now putting the example into "tableaux form":

		z_1	z_2	z_3	z_4
w_1	-3	0	-1	2	1
w_2	6	2	0	-2	1
w_3	-1	-1	-1	0	1

thereby yielding the following vectors and matrices:

$$q^{\alpha} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, q^{\bar{\alpha}} = \begin{bmatrix} 6 \end{bmatrix}, M^{\alpha\beta} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, M^{\bar{\alpha}\beta} = \begin{bmatrix} -2 & 1 \end{bmatrix} \tag{11}$$

With the index sets $\alpha', \bar{\alpha}'$ defined in Subsection 1.1, and Equation 10 from Page 71 of [1], we can write $\boldsymbol{q}'$ as:

$$\boldsymbol{q}'^{\beta'} = -(\mathbf{M}^{\alpha\beta})^{-1}\boldsymbol{q}^{\alpha} \tag{12}$$
$$\boldsymbol{q}'^{\bar{\alpha}'} = \boldsymbol{q}^{\bar{\alpha}} + \mathbf{M}^{\bar{\alpha}\beta}\boldsymbol{q}'^{\beta'} \tag{13}$$

where $\boldsymbol{q}'^{\alpha'}, \boldsymbol{q}'^{\bar{\alpha}'}$ are "views" of the vector q', as $\boldsymbol{q}'^{\alpha'}_i = \boldsymbol{q}'_{\alpha'_i}, \boldsymbol{q}'^{\bar{\alpha}'}_i = \boldsymbol{q}'_{\bar{\alpha}'_i}$. In the running example $\boldsymbol{q}'^{\alpha'} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathsf{T}}, \boldsymbol{q}'^{\bar{\alpha}'} = [5]$. **$\boldsymbol{q}'$ is not composed by stacking $\boldsymbol{q}'^{\alpha'}, \boldsymbol{q}'^{\bar{\alpha}'}$ as is done in similar procedures in [1].**

2.2 Computing $\mathbf{M}'_{\textbf{driving}}$

This computation requires considering two cases for the driving variable, which is an entry in the new independent variable z'. The driving variable can be either:

1. a dependent variable from w.

2. an independent variable z.

in the running example, if the driving variable is w_1 or w_3, then it belongs to the first case; if the driving variable is z_2 or z_1, then it belongs to the second case. Let's discuss the two cases separately.

2.2.1 Driving variable is a dependent variable from w

If the driving variable z'_{driving} is a dependent variable from w, then it also belongs to the vector INDEPENDENT W. We denote the index of z'_{driving} in INDEPENDENT W as γ, namely:

$$w^{\alpha}_{\gamma} = w_{\alpha_\gamma} = z'_{\text{driving}} \tag{14}$$

where w^{α} is the vector INDEPENDENT W. In the running example, INDEPENDENT W is $\{w_1, w_3\}$, thus $\alpha = \{1, 3\}$. If $z'_{\text{driving}} \equiv w_1$, then the index $\gamma \equiv 1$. If $z'_{\text{driving}} \equiv w_3$, then the index $\gamma \equiv 2$. To compute the column vector $\mathbf{M}'_{\text{driving}}$, we need to first compute the γ^{th} column of the matrix $(\mathbf{M}^{\alpha\beta})^{-1}$. To this end, we define a unit vector $\boldsymbol{e} \in \mathbb{R}^m$ as:

$$e_\gamma = 1 \tag{15}$$
$$e_i = 0 \quad \text{if } i \neq \gamma \tag{16}$$

and we can compute M'_{driving} as:

$$\mathbf{M}'^{\beta'}_{\text{driving}} = (\mathbf{M}^{\alpha\beta})^{-1}\boldsymbol{e} \tag{17}$$
$$\mathbf{M}'^{\bar{\alpha}'}_{\text{driving}} = \mathbf{M}^{\bar{\alpha}\beta}\mathbf{M}'_{\text{driving}_{\beta'}} \tag{18}$$

Notice that $\mathbf{M}'_{\text{driving}_{\alpha'}}$ is the γ^{th} column of $\mathbf{M}^{-1}_{\alpha\beta}$ according to Equation (17).

2.2.2 Driving variable is an independent variable from z

If the driving variable z'_{driving} is an independent variable from z, we denote the position in z of z'_{driving} as ζ. In the running example, if the driving variable is z_2, then $\zeta \equiv 2$; if the driving variable is z_1, then $\zeta \equiv 1$.

To compute the column vector $\mathbf{M}'_{\text{driving}}$ in $\mathbf{M}'$, we need the ζ^{th} column of $\mathbf{M}$, denoted $\boldsymbol{g}$, which will be decomposed into two sub-vectors. One sub-vector $\boldsymbol{g}^{\alpha}$ contains the entries with row indices in α while the other, $\boldsymbol{g}^{\bar{\alpha}}$, contains the entries with row indices from $\bar{\alpha}$. Namely:

$$g^{\alpha}_i = M_{\alpha_i \zeta} \tag{19}$$
$$g^{\bar{\alpha}}_i = M_{\bar{\alpha}_i \zeta} \tag{20}$$

In the running example, w_1, w_3 will be pivoted to z', and w_2 will remain in w'. If the driving variable is z_2, we highlight the vector $g^{\alpha}, g^{\bar{\alpha}}$ in $\mathbf{M}$ as:

	z_1	z_2	z_3	z_0
w_1	0	-1	2	1
w_2	2	0	-2	1
w_3	-1	1	0	1

so $g^{\alpha} \equiv \begin{bmatrix}-1 & 1\end{bmatrix}^{\mathsf{T}}, g^{\bar{\alpha}} = [0]$.

with the vector $\boldsymbol{g}^{\alpha}, \boldsymbol{g}^{\bar{\alpha}}$, we can then compute $\mathbf{M}'_{\text{driving}}$ as

$$\mathbf{M}'^{\beta'}_{\text{driving}} = -(\mathbf{M}^{\alpha\beta})^{-1}\boldsymbol{g}^{\alpha} \tag{21}$$

$$\mathbf{M}'^{\bar{\alpha}'}_{\text{driving}} = \boldsymbol{g}^{\bar{\alpha}} + \mathbf{M}^{\bar{\alpha}\beta}\mathbf{M}'^{\beta'}_{\text{driving}} \tag{22}$$

References

[1] R. W. Cottle, J.-S. Pang, and R. Stone. *The Linear Complementarity Problem.* Academic Press, Boston, 1992.