

# Simultaneous Hand-Eye and Robot-World Calibration by Solving the $AX=YB$ Problem without Correspondence\*

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**Abstract**—Calibration is often an important and necessary step in the use of image-guided systems. In the case of the  $AX = YB$  problem, the relative hand-eye ( $X$ ) and robot-world ( $Y$ ) transformations must be determined to provide accurate data for use in control. As an added difficulty, the exact correspondence between the streams of sensor data ( $As$  and  $Bs$ ) is typically unknown due to asynchrony in sampling rates and processing time. One common scenario is a constant shift between the two data streams. Therefore, in this paper we present a probabilistic method to simultaneously solve for  $X$  and  $Y$  without a priori knowledge of the correspondence between the streams of  $As$  and  $Bs$ . We begin by discussing probability density functions on  $SE(3)$  and then introduce Euclidean-group invariants to obtain an exact solution for  $X$  and  $Y$ . Then, we present a method to simultaneously recover  $X$  and  $Y$  and the correspondence between the shifted data sets. Finally, we numerically simulated the proposed method with shifted, asynchronous data and noise added to the stream of  $Bs$  to verify its efficiency and robustness.

## I. INTRODUCTION

Image-guided systems have been widely employed in applications throughout robotics such as robot-assisted surgery, autonomously guided vehicles, etc. Sensors such as a camera, a laser scanner, or an ultrasound probe are usually mounted on the distal end of a robotic manipulator. For a typical “hand-eye” system as described above, the relative transformation between the sensor with respect to the end-effector should be accurately calibrated, and it is often characterized as the well-known  $AX=XB$  problem. A variation of this problem is the  $AX=YB$  problem, where both the hand-eye transformation and the pose of the robot base with respect to the world frame need to be calibrated. In a typical environment, the relationships between the sensor frame, robot frame, and world frame are variant and uncertainties exist. Therefore, simultaneous coordinate calibrations have to be determined frequently in order to enable the robot to respond to dynamic environments.

In the  $AX=YB$  problem, data streams of  $As$  and  $Bs$  can be respectively obtained via different sensors. The data streams may arrive in an asynchronous fashion due to the different working frequencies of the sensors. This asynchrony causes a shift between the two data streams which removes the

Fig. 1. (1) The hand-eye and robot-world calibration problem formulated as  $AX=YB$ . (2) The hand-eye calibration problem formulated as  $AX=XB$ . Note: matrices  $A$  and  $B$  above have different physical meanings in the  $AX = XB$  and  $AX = YB$  problems. (The Universal Robot pictured above is owned by Prof. Emad Bector of Johns Hopkins University). Note: the  $A$  and  $B$  in the above two problems have different physical meanings. One is the absolute transformation and the other is relative transformation as can be seen in the figures.

correspondence between the  $As$  and  $Bs$ . In this paper, a novel method is presented to solve for  $X$  and  $Y$  without a priori knowledge of the correspondence between the  $As$  and  $Bs$ .

The hand-eye calibration problem can be modelled as  $AX = XB$ , where  $A$  and  $B$  are the homogeneous transformation matrices describing the relative motions of the end-effector and the sensor respectively. As shown in Fig. 1 part (2),  $A = A^i(A^{i+1})^{-1}$  and  $B = (B^i)^{-1}B^{i+1}$ . Given multiple pairs of  $(A_i, B_i)$  with correspondence (note that  $(A_i, B_i)$  are the relative transformations obtained from the raw data), many deterministic methods have been proposed to solve for  $X$ . To the best of the authors’ knowledge, Shiu [1] and Tsai

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[2] are the first to solve the  $AX = XB$  sensor calibration problem. The other methods include but are not limited to the quaternion, dual quaternion, screw theory, Lie group theory, motor algebra, convex optimization and gradient descent methods [3]–[11]. All of the methods above assume a priori knowledge of the exact correspondence between  $A_i$  and  $B_i$ . For data streams  $\{A_i\}$  and  $\{B_j\}$  that are asynchronous, several methods have been proposed in the literature to solve for  $X$  using data without a priori knowledge of the correspondence. These methods assume that there are some corresponding pairs of data in the streams of the  $\{A_i\}$  and  $\{B_j\}$ ; however, the exact correspondence is unknown [11]–[13].

Simultaneous estimation of the hand-eye and robot-world transformations has been viewed as the  $\mathbf{AX}=\mathbf{YB}$  problem. As shown in Fig. 1 part (1),  $Y$  is the transformation from the robot base to the world frame,  $A$  denotes the pose of the sensor in the world frame and  $B$  is the transformation from the end-effector to its fixed base. The  $A$  and  $B$  in  $\mathbf{AX}=\mathbf{YB}$  are different from those in  $\mathbf{AX}=\mathbf{XB}$  where the former uses absolute transformations and the latter uses relative transformations. This problem has been solved by many different methods such as the Kronecker product, quaternion, dual quaternion, and nonlinear optimization methods [14]–[21]. Simultaneous calibration of  $X$  and  $Y$  can be problematic in that all the methods above assume exact correspondence between  $\{A_i\}$  and  $\{B_j\}$ , which is not the case in the real world, and this is why a simultaneous solution for  $X$  and  $Y$  in the  $\mathbf{AX}=\mathbf{YB}$  problem can be a challenging issue. Another similar problem involves the calibration of multiple robots in terms of hand-eye, tool-flange, and robot-robot transformations, and it is formulated as the  $\mathbf{AXB}=\mathbf{Y CZ}$  problem [22] which will not be discussed in detail here. In the above methods, the correspondence between  $A$  and  $B$  is known a priori. In this paper, we focus on one case of the  $\mathbf{AX}=\mathbf{YB}$  problem where there is no a priori knowledge of the correspondence between the data streams.

The rest of the paper is organized as follows. In Section II, a novel probabilistic method is presented to solve for eight candidates of  $X$  and  $Y$ . In Section III, an algorithm involving both correlation theorem and Euclidean group invariants is proposed to recover the correspondence between  $\{A_i\}$  and  $\{B_j\}$ , which is used to select the optimal solution among the candidates. The simulation results obtained by taking noisy data without correspondence are illustrated in Section IV. In Section VI, we briefly discuss the case where one can obtain  $(X, Y)$  without recovering the correspondence between the data sets. Finally, conclusions are drawn based on the numerical results and possible future work is pointed out.

## II. SOLVING $\mathbf{AX}=\mathbf{YB}$ USING A PROBABILISTIC METHOD ON MOTION GROUPS

In this section, a brief introduction to the concept of probability density functions (PDFs) on the special Euclidean group  $SE(3)$  is presented and the probabilistic representation of  $AX = YB$  is derived.

Any rigid body transformation matrix can be viewed as a group element of  $SE(3)$  :

$$H(R, t) = \begin{pmatrix} R & t \\ 0^T & 1 \end{pmatrix} \in SE(3), \quad R \in SO(3) \quad (1)$$

where  $SO(3)$  denotes the special orthogonal group,  $t \in \mathbb{R}^3$  is a translational vector and  $H$  is the symbol for the group element.

Given a large set of pairs  $(A_i, B_i) \in SE(3) \times SE(3)$  where  $i = 1, \dots, n$ , the following equation is true if the correspondence is known a priori:

$$A_i X = Y B_i. \quad (2)$$

For a group element  $H \in SE(3)$ , a Dirac delta function  $\delta(H)$  is defined to be finite only at the identity and zero elsewhere:

$$\delta(H) = \begin{cases} +\infty, & H = I \\ 0, & H \neq I. \end{cases} \quad (3)$$

The Dirac delta function also satisfies the identity constraint as:

$$\int_{SE(3)} \delta(H) dH = 1. \quad (4)$$

A shifted Dirac delta function can be defined as  $\delta_A(H) = \delta(A^{-1}H)$ . Given  $K, H \in SE(3)$  and two well-defined functions  $f_1, f_2 \in (L^1 \cap L^2)(SE(3))$ , their convolution on  $SE(3)$  is defined as:

$$(f_1 * f_2)(H) = \int_{SE(3)} f_1(K) f_2(K^{-1} \circ H) dK. \quad (5)$$

where  $\circ$  denotes the group product, and in this case it is simply matrix multiplication. Employing the properties of the  $\delta$  function, it is straightforward to see that:

$$(f * \delta)(H) = \int_{SE(3)} f(K) \delta(K^{-1} \circ H) dK = f(H). \quad (6)$$

Therefore, for each  $A_i$  and  $B_i$ , the following equations can be obtained:

$$(\delta_{A_i} * \delta_X)(H) = \delta(A_i^{-1} H X^{-1}) \quad (7a)$$

$$(\delta_Y * \delta_{B_i})(H) = \delta(Y^{-1} H B_i^{-1}). \quad (7b)$$

Using Eq. (2) and Eq. (3), the above two equations can be combined into a single equation as:

$$(\delta_{A_i} * \delta_X)(H) = (\delta_Y * \delta_{B_i})(H) \quad (8)$$

Defining the PDF of  $\{A_i\}$  and  $\{B_i\}$  as:

$$f_A(H) = \frac{1}{n} \sum_{i=1}^n \delta_{A_i}(H) \quad (9a)$$

$$f_B(H) = \frac{1}{n} \sum_{i=1}^n \delta_{B_i}(H) \quad (9b)$$

then by using the distributivity of convolution, add  $n$  instances of Eq. (8), and substitute Eq. (9) into the summation, and we will have:

$$(f_{A_i} * \delta_X)(H) = (\delta_Y * f_{B_i})(H) \quad (10)$$

If each of the data streams  $\{A_i\}$  and  $\{B_i\}$  are generated using Gaussian distribution over  $SE(3)$ , then they can be viewed as “highly-focused”. The convolution of two highly-focused PDFs have some interesting properties that can be used to solve for  $X$ . For simplicity, we will drop the integral interval “ $SE(3)$ ” for the remainder of the paper, as it should be self-evident given the integration measure that is used. In particular, the mean  $M$  and covariance  $\Sigma$  of a PDF  $f(H)$  on  $SE(3)$  are defined as:

$$\int \log(M^{-1}H)f(H)dH = \mathbb{O} \quad (11a)$$

$$\Sigma = \int \log^\vee(M^{-1}H)[\log^\vee(M^{-1}H)]^T f(H)dH \quad (11b)$$

where the explicit expression of the matrix logarithm  $\log(H)$  along with its vectorized form  $\log^\vee(H)$  are given in [23] as:

$$\log(H) = \hat{\mathbf{h}} = \begin{pmatrix} 0 & -h_3 & h_2 & h_4 \\ h_3 & 0 & -h_1 & h_5 \\ -h_2 & -h_1 & 0 & h_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (12)$$

where  $\mathbf{h} = \log^\vee(H) \in \mathbb{R}^{6 \times 1}$  and  $\hat{\mathbf{h}}$  is the corresponding Lie algebra element ( $se(3)$ ) of  $H$ .

If  $f_A(H)$  is given as in Eq. (9), then the corresponding discrete version of the mean  $M_A$  and covariance  $\Sigma_A$  will be:

$$\sum_{i=1}^n \log(M_A^{-1}A_i) = \mathbb{O} \quad (13a)$$

$$\Sigma = \sum_{i=1}^n \log^\vee(M_A^{-1}A_i)[\log^\vee(M_A^{-1}A_i)]^T. \quad (13b)$$

Given  $\{A_i\}$  with the cloud of frames  $A_i$  clustering around  $M_A$ , an iterative formula can be used for computing  $M_A$  [24] as:

$$^{k+1}M_A = ^k M_A \circ \exp\left[\frac{1}{n} \sum_{i=1}^n \log(^k M_A^{-1} \circ A_i)\right] \quad (14)$$

An initial estimate of the iterative procedure can be chosen as  $^0M_A = \frac{1}{n} \sum_{i=1}^n \log(A_i)$ , then a local minimum of  $M_A$  is obtained by solving a nonlinear optimization problem with the cost function of  $\|\sum_{i=1}^n \log(M_A^{-1}A_i)\|^2$ . A similar procedure can be used to compute  $M_B$ .  $\Sigma_A$  and  $\Sigma_B$  are then straight forward to compute given known  $M_A$  and  $M_B$ .

The mean and covariance for the convolution  $(f_1 * f_2)(g)$  of two highly-focused functions  $f_1$  and  $f_2$  are calculated as in [24]:

$$M_{1*2} = M_1 M_2 \quad (15a)$$

$$\Sigma_{1*2} = Ad(M_2^{-1})\Sigma_1 Ad^T(M_2^{-1}) + \Sigma_2. \quad (15b)$$

where

$$Ad(H) = \begin{pmatrix} R & O \\ \hat{t}R & R \end{pmatrix}.$$

Because  $X$  and  $Y$  are constant, their corresponding PDF will be  $\delta_X(g)$  and  $\delta_Y(g)$ , of which the mean and covariance are  $M_X = X$ ,  $\Sigma_X = \mathbb{O}_{6 \times 6}$  and  $M_Y = Y$ ,  $\Sigma_Y = \mathbb{O}_{6 \times 6}$ , respectively. Therefore, the following equations can be obtained using Eq. 15:

$$M_A X = Y M_B \quad (16a)$$

$$Ad(X^{-1})\Sigma_A Ad^T(X^{-1}) = \Sigma_B. \quad (16b)$$

To solve the above equations, Eq. (16a) is decomposed into a rotational equation and a translational equation as follows:

$$R_{M_A} R_X = R_Y R_{M_B} \quad (17a)$$

$$R_{M_A} t_X + t_{M_A} = R_Y t_{M_B} + t_Y. \quad (17b)$$

$\Sigma_A$  and  $\Sigma_B$  can be decomposed into blocks as  $\begin{pmatrix} \Sigma_A^1 & \Sigma_A^2 \\ \Sigma_A^3 & \Sigma_A^4 \end{pmatrix}$  and  $\begin{pmatrix} \Sigma_B^1 & \Sigma_B^2 \\ \Sigma_B^3 & \Sigma_B^4 \end{pmatrix}$ , respectively. Knowing that  $X^{-1} = \begin{pmatrix} R_X^T & -R_X^T t_X \\ 0 & 1 \end{pmatrix}$ , then the first two blocks of Eq. (16b) can be written as follows:

$$\Sigma_{M_B}^1 = R_X^T \Sigma_{M_A}^1 R_X \quad (18a)$$

$$\Sigma_{M_B}^2 = R_X^T \Sigma_{M_A}^1 R_X (\widehat{R_X^T t_X}) + R_X^T \Sigma_{M_A}^2 R_X. \quad (18b)$$

Because Eq. (18a) is a similarity transformation between  $\Sigma_{M_B}^1$  and  $\Sigma_{M_A}^1$ , they share the same eigenvalues and can be eigendecomposed into  $\Sigma_{M_A}^1 = Q_{M_A} \Lambda Q_{M_A}^T$  and  $\Sigma_{M_B}^1 = Q_{M_B} \Lambda Q_{M_B}^T$  where  $\Lambda$  is a diagonal matrix whose diagonal elements are the eigenvalues of  $\Sigma_{M_A}^1$  (or  $\Sigma_{M_B}^1$ ), and  $Q_{M_A}$  (or  $Q_{M_B}$ ) is a square matrix whose columns are the corresponding eigenvectors. The following equation is obtained after substituting  $\Sigma_{M_B}^1$  and  $\Sigma_{M_A}^1$  into Eq. (18a):

$$\Lambda = (Q_{M_A}^T R_X^T Q_{M_B}) \Lambda (Q_{M_B}^T R_X Q_{M_A}) = P \Lambda P^T \quad (19)$$

where  $P = Q_{M_A}^T R_X Q_{M_B}$ . If  $Q_{M_A}$  and  $Q_{M_B}$  are further constrained to be rotation matrices, the rotation matrix  $P$  satisfies Eq. (20).

$$\begin{cases} P^T = P^{-1} \\ \det(P) = \pm 1. \end{cases} \quad (20)$$

Combing Eq. (19) and Eq. (20), then a rotation matrix  $P$  can be one of  $\mathcal{P}$  or  $-\mathcal{P}$  :

$$\mathcal{P} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}. \quad (21)$$

Therefore, there are eight candidates for  $R_X$  which can be calculated via  $R_X = Q_{M_A} P Q_{M_B}^T$ , and the corresponding  $t_X$  can be obtained from Eq. (18b). Given known  $X$ ,  $Y$  can be solved from  $Y = M_A X M_B^{-1}$ . At last, eight candidate pairs of  $\{X_k, Y_k\}$  can be obtained as:

$$X_k = \begin{pmatrix} R_{X_k} & t_{X_k} \\ \mathbf{0}^T & 1 \end{pmatrix}, \quad Y_k = \begin{pmatrix} R_{Y_k} & t_{Y_k} \\ \mathbf{0}^T & 1 \end{pmatrix} \quad (22)$$

where  $k = 1, 2, \dots, 8$ .

The problem then becomes selecting the best pair of  $\{X_k, Y_k\}$  from the eight candidates. Based on screw theory, it is known that a homogeneous transformation  $H$  can be expressed by the four screw parameters  $(\theta, d, \mathbf{n}, \mathbf{p})$  that define the *Plücker* coordinates of the screw motion as:

$$H = \begin{pmatrix} e^{\theta \hat{\mathbf{n}}} & (\mathbf{I}_3 - e^{\theta \hat{\mathbf{n}}})\mathbf{p} + d\mathbf{n} \\ \mathbf{0}^T & 1 \end{pmatrix} \quad (23)$$

where  $\theta$  is the angle of rotation,  $d$  is the translation along the rotation axis,  $\mathbf{n}$  is the unit vector representing the axis of rotation and  $\mathbf{p}$  is the position of the line to the origin with  $\mathbf{p} \cdot \mathbf{n} = 0$ .

Moreover,  $AX_k = Y_k B$  can be written as  $AX_k = X_k(X_k^{-1}Y_k B)$ . Defining  $B^k = X_k^{-1}Y_k B$ , we have  $AX_k = X_k B^k$ . As discussed in [25], for  $AX = XB$  problem, there exist two Euclidean-group invariant relationships for each pair of  $(A_i, B_i^k)(i = 1, \dots, n; k = 1, \dots, 8)$  as follows:

$$\theta_{A_i} = \theta_{B_i^k}, d_{A_i} = d_{B_i^k} \quad (24)$$

Among the eight pairs  $(X_k, Y_k)$ , one can find an optimal solution which minimizes the cost function defined as:

$$(X, Y) = \underset{(X_k, Y_k)}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^n (\|\theta_{A_i} - \theta_{B_i^k}\| + \|d_{A_i} - d_{B_i^k}\|) \quad (25)$$

Eight candidates of  $(X_k, Y_k)$  are calculated using the probabilistic method on  $SE(3)$ , which doesn't require the correspondence between  $A_i$  and  $B_j$  to be known. However, the correspondences need to be recovered to pick the optimal  $(X_k, Y_k)$ . Note that the Euclidean-group invariant relationships in the context of  $AX = YB$  problem are still unknown. Therefore,  $AX = YB$  is converted into  $AX = XB$  problem to recover the correspondence of data using invariants.

### III. SOLUTION WITH UNKNOWN CORRESPONDENCE BETWEEN $A_i$ AND $B_j^k$

In most cases, the two sets of homogeneous transformations  $\{A_i\}$  and  $\{B_j\}$  are calculated based on the data obtained from different sensors. Due to the asynchronous timing of the sensor readings, the correspondence between  $\{A_i\}$  and  $\{B_j\}$  is usually unknown. This section deals with the case where there is a shift between  $\{A_i\}$  and  $\{B_j\}$ , and the Euclidean-group invariants are used to recover the correspondence between the data streams. The advantage of the above probabilistic solution lies in that  $X$  and  $Y$  can be calculated even if there is no a priori knowledge of the correspondence. However, there are still eight possible candidates of  $(X_k, Y_k)$  to choose from and by using Euclidean-group invariants, it is straightforward to determine which is the optimal pair if the correspondence between  $A_i$  and  $B_j^k$  can be known.

The Discrete Fourier Transform (DFT) decomposes a time-domain signal into its constituent frequencies. The input is a finite list of equally spaced samples of a function. Given a discrete signal consisting of a sequence of  $N$  complex numbers  $x_0, x_1, \dots, x_{N-1}$ , the DFT is denoted by  $X_\kappa = \mathcal{F}(x_n)$  as:

$$X_\kappa = \sum_{n=0}^{N-1} x_n \cdot \exp(-i \frac{2\pi}{N} n\kappa). \quad (26)$$

where  $i$  here is the imaginary unit.

The Inverse Discrete Fourier Transform (IDFT) is denoted as:

$$x_n = \frac{1}{N} \sum_{\kappa=0}^{N-1} X_\kappa \cdot \exp(i \frac{2\pi}{N} n\kappa). \quad (27)$$

The discrete convolution of two sequences  $f_n$  and  $g_n$  is defined as:

$$(f * g)(\tau) = \sum_{j=0}^N f(t_j)g(t_j - \tau). \quad (28)$$

In the convolution theorem, the Fourier transform of a convolution is the product of the Fourier transforms, namely:

$$f * g = \mathcal{F}^{-1}[\mathcal{F}(f) \cdot \mathcal{F}(g)]. \quad (29)$$

The correlation theorem indicates that the correlation function,  $\operatorname{Corr}(f, g)$ , will be larger for a shift vector where the two sequences  $f_n$  and  $g_n$  share more similar features. The correlation can be obtained based on the convolution theorem. The DFT of  $\operatorname{Corr}(f, g)$  is equal to the product of the DFT of  $f_n$  and the complex conjugate  $\mathcal{F}^*$  of the DFT of  $g_n$ :

$$\operatorname{Corr}(f, g) = f \star g = \mathcal{F}^{-1}[\mathcal{F}(f) \cdot \mathcal{F}^*(g)]. \quad (30)$$

Compared to the standard time-domain convolution algorithm, the complexity of the convolution by multiplication in the frequency domain is significantly reduced with the help

Fig. 2. The translational and rotational errors versus the shift between data streams  $\{A_i\}$  and  $\{B_i\}$ .

of the convolution theorem and the Fast Fourier Transform (FFT).

Given two sequences  $\{\theta_{A_i}\}$  and  $\{\theta_{B_j^k}\}$  corresponding to  $\{A_i\}$  and  $\{B_j^k\}$ , the shift that is needed to recover the data correspondence is obtained as below. Firstly,  $\theta_{A_i}$  and  $\theta_{B_j^k}$  are normalized as:

$$\theta_{1,k} = \frac{(\theta_{A_i} - \mu_A)}{\sigma_A}, \theta_{2,k} = \frac{(\theta_{B_j^k} - \mu_{B^k})}{\sigma_{B^k}} \quad (31)$$

where  $\mu_A(\mu_{B^k})$  is the mean of  $\theta_{A_i}(\theta_{B_j^k})$  and  $\sigma_A(\sigma_{B^k})$  is the standard deviation.

Here, the correlation function  $Corr(\theta_{1,k}, \theta_{2,k})$  is the function of the time sequence index  $n$  which describes the probability of these two sequences being separated by this particular index. The index corresponding to the maximum of  $Corr(\theta_{1,k}, \theta_{2,k})$  indicates the amount of shift  $\tau_{shift}$  between  $\{\theta_{A_i}\}$  and  $\{\theta_{B_j^k}\}$ .

$$\tau_{shift} = \underset{index}{\operatorname{argmax}}(Corr(\theta_{1,k}, \theta_{2,k})) \quad (32)$$

Therefore, the correspondence between the two sequences can be found. The data of  $\theta_{A_i}$  or  $d_{A_i}$  are shifted by  $-\tau_{shift}$  to obtain a sequence of new pairs  $(\theta_{A_i}(i + \tau_{shift}), \theta_{B_i^k})$  and  $(d_{A_i}(i + \tau_{shift}), d_{B_i^k})$ , where  $\max(0, \tau_{shift}) \leq i \leq \min(n, n + \tau_{shift})$ . The data stream can be shifted back to regain correspondence to synchronize the data streams once the shift is computed, and the optimal solution of  $X$  and  $Y$  can also be recovered by minimizing the cost function in Eq. (25) using the Euclidean-group invariants as shown in Section II.

#### IV. SIMULATION STUDIES

For the numerical experiments in this section, the rotational and translational errors for  $X$  and  $Y$  are measured as  $Error(R_X) = \|\log^\vee(R_{X_{Solved}}^T R_{X_{true}})\|$ ,  $Error(t_X) = \|t_{X_{Solved}} - t_{X_{true}}\|$ ,  $Error(R_Y) = \|\log^\vee(R_{Y_{Solved}}^T R_{Y_{true}})\|$  and  $Error(t_Y) = \|t_{Y_{Solved}} - t_{Y_{true}}\|$  respectively.

There are multiple ways of generating the data streams  $\{A_i\}$  and  $\{B_i\}$ . One way is to first generate  $\{B_i\}$  and then map it to  $\{A_i\}$  using  $A = YBX^{-1}$ .  $\{B_i\}$  can be obtained by randomly sampling on the Lie algebra of  $B$  from a zero mean multivariate Gaussian distribution as follows:

Fig. 3. Box-and-whisker plots of translational and rotational errors versus the covariance noise on data stream  $\{B_i\}$ .

Fig. 4. (a) The solved  $X$  (in red) and the actual  $X$  (in black) for 10 simulation trials with covariance noise of 0.05 and shift of 2. (b) The solved  $Y$  (in blue) and the actual  $Y$  (in black) for 10 simulation trials with covariance noise of 0.05 and shift of 2.

$$\delta_i \in \mathcal{N}(\mathbf{0}; \Sigma) \subset \mathbb{R}^6 \quad (33a)$$

$$B_i = \exp(\hat{\delta}_i) \exp(\mu) \quad (33b)$$

where the mean  $\mu = \mathbf{0} \in se(3)$  and the covariance matrix  $\Sigma \in \mathbb{R}^{6 \times 6}$  is a diagonal matrix with same diagonal elements  $\sigma$ . The hat operator  $\hat{\delta}$  converts a 6 by 1 vector into its corresponding Lie algebra. The data stream  $\{A_i\}$  can be easily obtained as described above. After employing the proposed probabilistic method, 8 sets of sequences  $(\theta_{A_i}, \theta_{B_i^k})$  and  $(d_{A_i}, d_{B_i^k})$  can be obtained respectively where  $i = 1, \dots, 100$  and  $k = 1, \dots, 8$ .

If the data stream  $\{A_i\}$  is shifted by  $m$  units relative to  $\{B_i\}$ , then the maximum of the cross correlation can be used to recover the shift. After that, we can shift the data stream  $\{A_i\}$  back to its original position to recover the correct correspondence with  $\{B_i\}$ , which will be used to find a correct solution satisfying the Euclidean-group invariants as defined in Eq. (24). Therefore, a unique pair of  $(X_k, Y_k)$  ( $k = 1, \dots, 8$ ) can be selected to minimize the cost function. In Fig. 2, because the shift between  $\{A_i\}$  and  $\{B_i\}$  is calculated accurately, the translational and rotational errors fluctuate by only a small amount compared to the errors of the no-shift data streams.

To test the robustness of the proposed method, noises are exerted onto  $\{B_i\}$  by employing  $B_i^{noise} = B_i \exp(\hat{\mathbf{x}}_{noise})$ , where each element of the Lie Algebra  $\mathbf{x}_{noise}$  belongs to the Gaussian distribution defined as  $N \sim (\mu_{noise}, \sigma_{noise})$ . In Fig. 3, as the covariance noise  $\sigma_{noise}$  increments from 0.01 to 0.08, the errors of  $R_X$ ,  $R_Y$ ,  $t_X$ , and  $t_Y$  increase as

Fig. 5. Orientation and translation errors of  $X$  and  $Y$  versus shift using Li's and Shah's methods without correspondence.

Fig. 6. Orientation and translation errors of  $X$  and  $Y$  versus shift using Li's and Shah's methods with correspondence.

shown in the box-and-whisker plot. There are several outliers outside the whiskers, while the median is calculated as the final solved  $X$  and  $Y$ . Fig. 4 shows the solved  $(X, Y)$ s in red and blue with the actual  $(X, Y)$  in black with covariance noise of  $\sigma = 0.05$  and shift  $n = 2$ .

The probabilistic method can recover the correspondence between shifted data streams, which is useful for other sensor calibration methods. In the  $AX = YB$  problem, there have been many calibration methods developed for solving  $X$  and  $Y$  given data streams with correspondence. However, few of them considered the cases without correspondence. When data streams of  $A$  and  $B$  are shifted or asynchronous, most of these methods fail to give a valid solution. To further test the effectiveness of our method, we shift the data sequence of  $\{A_i\}$  by  $n = 0, 1, 2, 3, 4, 5$  with respect to the data sequence of  $\{B_i\}$  such that  $A_{k+n}$  “matches”  $B_k$  where  $k = 1, 2, \dots, m - n$  and  $i = 1, 2, \dots, m$ . We augment other  $AX = YB$  solvers with our probabilistic approach by recovering the correspondence between shift data sets. In Li's method [19],  $X$  and  $Y$  are solved for at the same time, while Shah [20] solved for  $X$  and  $Y$  in a separate way. As shown in Fig. 5, when dealing with the shifted data streams  $\{A_{k+n}, B_k\}$ , the errors on both rotations and translations are significant. After recovering the correspondence between data streams by using the probabilistic method, Li and Shah's methods achieve the same level of performance as shown in Fig. 6.

Fig. 7. Rotation Error in  $X$  and  $Y$  v.s. Scrambling Rate

## V. A BRIEF CASE STUDY WITH COMPLETELY SCRAMBLED DATA

In this section, we will briefly discuss the case where  $\{A_i\}$  and  $\{B_j\}$  are completely scrambled. Unlike the case of shifted data, it is extremely hard to recover the correspondence between two completely scrambled data sets  $\{A_i\}$  and  $\{B_j\}$ . The correlation theorem can't be applied because there is no shift in the scrambled data sets. Euclidean group invariants are not practical either because given  $\{A_i\}$  and  $\{B_j\}$  both of which have the size of  $m$ , there are  $m! = m \times m - 1 \times \dots \times 1$  combinations between the data sets, and it is extremely computationally intensive to test all the combinations. Without recovering the correspondence between the data sets, it is impossible to choose the optimal solution from the eight candidates of  $\{X_k, Y_k\}$ .

In the above approach, we used Eq. (16b) to calculate  $X_k$  and Eq. (16a) to obtain the corresponding  $Y_k$ . However, we now show that one can calculate the eight candidates of  $Y$  independently and employ Eq. (16a) as a constraint to filter out the optimal  $\{X, Y\}$  pair.

Given the equation  $AX = YB$ , apply an inverse on both sides of the equation and we will have  $B^{-1}Y^{-1} = X^{-1}A^{-1}$ . Following the same derivations from Eq. (6) to Eq. (16b), we have:

$$M_{B^{-1}}Y^{-1} = X^{-1}M_{A^{-1}} \quad (34a)$$

$$Ad(Y)\Sigma_{B^{-1}}Ad^T(Y) = \Sigma_{A^{-1}}. \quad (34b)$$

Similarly, Eq. (34b) can give eight candidates of  $Y^{-1}$ , or equivalently  $Y$ . Let  $X_{k1}$  where  $k1 = 1, 2, \dots, 8$  denote the  $X$ s obtained from Eq. (16b) and  $Y_{k2}$  where  $k2 = 1, 2, \dots, 8$  denote the  $Y$ s obtained from Eq. (34b), and we can use Eq. (16a) and Eq. (34a) to form a minimization problem as:

$$\min_{k1, k2} \|M_A X_{k1} - Y_{k2} M_B\|_F + \|M_B^{-1} Y_{k2}^{-1} - X_{k1}^{-1} M_A^{-1}\|_F \quad (35)$$

which can give the optimal  $(X_{k1}, Y_{k2})$  pair.

## VI. CONCLUSIONS

In this paper, we developed a probabilistic approach to simultaneously obtain  $X$  and  $Y$  in the  $AX = YB$

Fig. 8. Translation Error in X and Y v.s. Scrambling Rate

sensor calibration problem. Without a priori knowledge of the correspondence between  $\{A_i\}$  and  $\{B_j\}$ , the proposed probabilistic method on Lie groups is used to constrain the possible solutions of  $X$  and  $Y$  to eight pairs of candidates. Given shifted data streams of  $\{A_{i+s}\}$  and  $\{B_i\}$ , using the correlation theorem with Euclidean-group invariants, the correspondence is recovered to determine the correct solution among the eight candidates. In the numerical simulation, the method performs well with different sets of data samples. Future work will be to further develop the existing algorithm to solve for more general asynchronous cases, such as flipped data pairs or missing data in the data streams.

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#### REFERENCES

- [1] Y. C. Shiu and S. Ahmad, "Calibration of wrist-mounted robotic sensors by solving homogeneous transform equations of the form  $ax=xb$ ," *Robotics and Automation, IEEE Transactions on*, vol. 5, no. 1, pp. 16–29, 1989.
- [2] R. Y. Tsai and R. K. Lenz, "A new technique for fully autonomous and efficient 3d robotics hand/eye calibration," *Robotics and Automation, IEEE Transactions on*, vol. 5, no. 3, pp. 345–358, 1989.
- [3] C.-C. Wang, "Extrinsic calibration of a vision sensor mounted on a robot," *Robotics and Automation, IEEE Transactions on*, vol. 8, no. 2, pp. 161–175, 1992.
- [4] F. C. Park and B. J. Martin, "Robot sensor calibration: solving  $ax=xb$  on the euclidean group," *IEEE Transactions on Robotics and Automation (Institute of Electrical and Electronics Engineers);(United States)*, vol. 10, no. 5, 1994.
- [5] R. Horaud and F. Dornaika, "Hand-eye calibration," *The international journal of robotics research*, vol. 14, no. 3, pp. 195–210, 1995.
- [6] K. Daniilidis, "Hand-eye calibration using dual quaternions," *The International Journal of Robotics Research*, vol. 18, no. 3, pp. 286–298, 1999.
- [7] E. Bayro-Corrochano, K. Daniilidis, and G. Sommer, "Motor algebra for 3d kinematics: The case of the hand-eye calibration," *Journal of Mathematical Imaging and Vision*, vol. 13, no. 2, pp. 79–100, 2000.
- [8] J. R. Rovelo, S. H. Garcia, and E. B. Corrochano, "Geometric hand-eye calibration for an endoscopic neurosurgery system," in *Robotics and Automation, 2008. ICRA 2008. IEEE International Conference on*. IEEE, 2008, pp. 1418–1423.
- [9] I. Fassi and G. Legnani, "Hand to sensor calibration: A geometrical interpretation of the matrix equation  $ax=xb$ ," *Journal of Robotic Systems*, vol. 22, no. 9, pp. 497–506, 2005.
- [10] Z. Zhao, "Hand-eye calibration using convex optimization," in *Robotics and Automation (ICRA), 2011 IEEE International Conference on*. IEEE, 2011, pp. 2947–2952.
- [11] M. K. Ackerman, A. Cheng, E. Bector, and G. Chirikjian, "Online ultrasound sensor calibration using gradient descent on the euclidean group," in *Robotics and Automation (ICRA), 2014 IEEE International Conference on*. IEEE, 2014, pp. 4900–4905.
- [12] M. K. Ackerman and G. S. Chirikjian, "A probabilistic solution to the  $ax=xb$  problem: Sensor calibration without correspondence," in *Geometric Science of Information*. Springer, 2013, pp. 693–701.
- [13] M. K. Ackerman, A. Cheng, and G. Chirikjian, "An information-theoretic approach to the correspondence-free  $ax=xb$  sensor calibration problem," in *Robotics and Automation (ICRA), 2014 IEEE International Conference on*. IEEE, 2014, pp. 4893–4899.
- [14] H. Zhuang, Z. S. Roth, and R. Sudhakar, "Simultaneous robot/world and tool/flange calibration by solving homogeneous transformation equations of the form  $ax=yb$ ," *Robotics and Automation, IEEE Transactions on*, vol. 10, no. 4, pp. 549–554, 1994.
- [15] F. Dornaika and R. Horaud, "Simultaneous robot-world and hand-eye calibration," *Robotics and Automation, IEEE Transactions on*, vol. 14, no. 4, pp. 617–622, 1998.
- [16] R. L. Hirsh, G. N. DeSouza, and A. C. Kak, "An iterative approach to the hand-eye and base-world calibration problem," in *Robotics and Automation, 2001. Proceedings 2001 ICRA. IEEE International Conference on*, vol. 3. IEEE, 2001, pp. 2171–2176.
- [17] F. Ernst, L. Richter, L. Matthäus, V. Martens, R. Bruder, A. Schläefer, and A. Schweikard, "Non-orthogonal tool/flange and robot/world calibration," *The International Journal of Medical Robotics and Computer Assisted Surgery*, vol. 8, no. 4, pp. 407–420, 2012.
- [18] K. H. Strobl and G. Hirzinger, "Optimal hand-eye calibration," in *Intelligent Robots and Systems, 2006 IEEE/RSJ International Conference on*. IEEE, 2006, pp. 4647–4653.
- [19] A. Li, L. Wang, and D. Wu, "Simultaneous robot-world and hand-eye calibration using dual-quaternions and kronecker product," *Inter. J. Phys. Sci*, vol. 5, no. 10, pp. 1530–1536, 2010.
- [20] M. Shah, "Solving the robot-world/hand-eye calibration problem using the kronecker product," *Journal of Mechanisms and Robotics*, vol. 5, no. 3, p. 031007, 2013.
- [21] J. Heller, D. Henrion, and T. Pajdla, "Hand-eye and robot-world calibration by global polynomial optimization," in *Robotics and Automation (ICRA), 2014 IEEE International Conference on*. IEEE, 2014, pp. 3157–3164.
- [22] J. Wang, L. Wu, M. Q.-H. Meng, and H. Ren, "Towards simultaneous coordinate calibrations for cooperative multiple robots," in *Intelligent Robots and Systems (IROS 2014), 2014 IEEE/RSJ International Conference on*. IEEE, 2014, pp. 410–415.
- [23] G. S. Chirikjian, *Stochastic Models, Information Theory, and Lie Groups, Volume 2: Analytic Methods and Modern Applications*. Springer Science & Business Media, 2011, vol. 2.
- [24] Y. Wang and G. S. Chirikjian, "Nonparametric second-order theory of error propagation on motion groups," *The International journal of robotics research*, vol. 27, no. 11-12, pp. 1258–1273, 2008.
- [25] M. K. Ackerman, A. Cheng, B. Shiffman, E. Bector, and G. Chirikjian, "Sensor calibration with unknown correspondence: Solving  $ax=xb$  using euclidean-group invariants," in *Intelligent Robots and Systems (IROS), 2013 IEEE/RSJ International Conference on*. IEEE, 2013, pp. 1308–1313.