Simultaneous Coordinate Calibrations by Solving the AX=YB Problem without Correspondence*

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***Abstract*— In image-guided system, relationships of hand-eye (X) and robot-world (Y) coordinates have to be calculated and simultaneous solution is useful in sensor calibration problem. Due to asynchrony of sensors timing, the correspondence between A and B is unknown. A probabilistic method is presented to solve the homogeneous matrix equations without a priori knowledge of the correspondence. Using Euclidean-Group invariants, an exact solution can be found. We illustrate the calculation in numeric simulation including various numbers of robot movements. The results show the efficiency and robustness of the proposed simultaneous method.**

I. INTRODUCTION

Image-guided system has been widely used in robotics such as robot assisted surgery, (more examples). Sensors such as a camera, a laser scanner or an ultrasound probe are usually mounted on the distal end of a robotic manipulator. For a typical "hand-eye" system as described above, the relative transformation of the sensor with respect to the end-effector should be accurately calibrated, and it is often characterized as the well known **AX=XB** problem. A variation of this problem is characterized as the **AX=YB** problem, where both the hand-eye transformation and the pose of the robot base with respect to the world frame need to be calibrated. In a typical environment setup, the relationship among the sensor frame, robot frame and world frame is variant and the uncertainties exist. Therefore, simultaneous coordinate calibrations have to be determined frequently in order to enable the robots to respond to dynamic environments.

In the **AX=YB** problem, As and Bs can be respectively obtained via different sensors. The data streams can be in an asynchronous fashion due to the different working frequencies of the sensors. The asynchrony causes a shift between the two streams of data which damages the correspondence between As and Bs. In this paper, a novel method is presented to solve for an X and Y without the need to know a priori knowledge of the correspondence between As and Bs.

The hand-eye calibration problem can be modeled as the **AX = XB**, where A and B are the homogeneous transformation matrices describing the relative motions of end-

Fig. 1. (1) The hand-eye and robot-world calibration problem which is formulated as AX=YB (The universal robot as shown in the picture is owned by professor Emad Boctor in). (2) The hand-eye calibration problem which is formulated as AX=XB.

effector and the sensor respectively. As shown in Fig. 1, $A = A^i(A^{i+1})^{-1}$ and $B = B^i(B^{i+1})^{-1}$. Given multiple pairs of $\{A, B\}$ with correspondence, many methods have been proposed to solve for X. To the best of the authors' knowledge, Shiu [1] and Tsai [2] are the first to solve the $AX = XB$ sensor calibration problem. The other methods include but are not limited to the quaternion, dual quaternion, screw theory, Lie group theory convex optimization and gradient descent methods [3]–[9]. All of the methods above assume a prior knowledge of exact correspondence between A_i and B_i. For data streams $\{A_i\}$ and $\{B_i\}$ that are asynchronous, several methods have been proposed in the literature to solve for X using data without correspondence. Theses methods assume that there is exact knowledge of the As and Bs correspondence [10]–[12].

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Simultaneous estimation of the hand-eye and robot-world transformations has been viewed as the **AX=YB** problem. As shown in Fig. 1, Y is the transformation from the robot base to the world frame, A denotes the pose of the sensor in the world frame and B is the transformation from the the end-effector to its fixed base. The A and B in **AX=YB** are different from those in **AX=XB** where the former uses absolute transformations and the latter uses relative transformations. This problem has been solved by many different methods such as kronecker product, quaternion, dual quaternion, and nonlinear optimization methods [13]–[20]. Simulatneous calibration of X and Y can be problematic in that all the methods above assume exact correspondence between $\{A_i\}$ and $\{B_j\}$, which is not the case in the real world. Another similar problem involves the calibration of multiple robots in terms of hand-eye, tool-flange and robot-robot system, and it is formulated as the **AXB=YCZ** problem [21]. Simultaneous solution for X and Y in **AX=YB** problem is an challenging issue. In the above methods, the correspondence between A and B is known a prior. In this paper, we focus on the **AX=YB** problem which does not require a priori knowledge of the correspondence of the data streams.

The rest of the paper is organized as follows. In Section II, a novel probabilistic method is presented to solve for X and Y. In Section III, an algorithm involving both correlation theorem and Euclidean group invariants is proposed to recover the correspondence between $\{A_i\}$ and $\{B_j\}$. The simulation results which deal with noisy data without correspondence are illustrated in Section IV. Finally, conclusions are drawn based on the numerical results and possible future works are pointed out.

II. Solving AX=YB using a probabilistic method on motion groups

In this section, a brief introduction to the concepts of probability density function on the special Eucledian group $SE(3)$ is presented and the probabilistic representation of $AX = YB$ are derived.

Any rigid transformation matrix can be viewed as a group element of $SE(3)$:

$$H(R,t) = \begin{pmatrix} R & t \\ 0^T & 1 \end{pmatrix} \in SE(3) \tag{1}$$

where $SO(3)$ denotes the special orthogonal group, $t \in \mathbb{R}^3$ is translational vector and H is symbol for group element.

Given a large set of pairs $(A_i, B_i) \in SE(3) \times SE(3)$ where $i = 1, \cdots, n$, the following equation is true if the correspondence is known as a priori:

$$A_i X = Y B_i. \tag{2}$$

For a matrix $H \in SE(3)$, a Dirac delta function $\delta(H)$ is defined to be finite only at the identity and zero elsewhere:

$$\delta(H) = \begin{cases} +\infty, & H = I \\ 0, & H \neq I \end{cases} \tag{3}$$

It also satisfies the identity constraint as:

$$\int_{SE(3)} \delta(H) dH = 1. \tag{4}$$

A shifted Dirac delta function can be defined as $\delta_A(H) = \delta(A^{-1}H)$. Given $K, H \in SE(3)$ and two well-defined functions $f_1, f_2 \in \left(L^1 \cap L^2\right)(SE(3))$, their convolution on $SE(3)$ is defined as:

$$(f_1 * f_2)(H) = \int_{SE(3)} f_1(K) f_2(K^{-1} \circ H) dK. \tag{5}$$

Employing the properties of δ function, it is straightforward to see that:

$$(f * \delta)(H) = \int_{SE(3)} f(K)\delta(K^{-1} \circ H) dK = f(H). \tag{6}$$

Therefore, for each A_i and B_i, the following equations can be obtained:

$$(\delta_{A_i} * \delta_X)(H) = \delta(A_i^{-1} H X^{-1}) \tag{7a}$$
$$(\delta_Y * \delta_{B_i})(H) = \delta(Y^{-1} H B_i^{-1}). \tag{7b}$$

Using Eq.(2), the above two equations can be combined into a single equation as:

$$(\delta_{A_i} * \delta_X)(H) = (\delta_Y * \delta_{B_i})(H) \tag{8}$$

Define the probability density function of $\{A_i\}$ and $\{B_i\}$ as:

$$f_A(H) = \frac{1}{n}\sum_{i=1}^{n} \delta(A_i^{-1} H) \tag{9a}$$

$$f_B(H) = \frac{1}{n}\sum_{i=1}^{n} \delta(B_i^{-1} H) \tag{9b}$$

Using the distributivity of convolution, Eq.(9) can be substituted into Eq.(8) to get:

$$(f_{A_i} * \delta_X)(g) = (\delta_Y * f_{B_i})(g) \tag{10}$$

For each of the data stream $\{A_i\}$ and $\{B_i\}$, small relative motions are calculated between consecutive reference frames. Take $\{A_i\}$ for example, one metric of the distance between A_i and A_{i+1} can be defined as:

$$d^2(A_i, A_{i+1}) = \parallel \Delta A \parallel_W^2 = trace[(\Delta A) W (\Delta A)^T] = \epsilon, \tag{11}$$

where $\Delta A = A_i - A_{i+1}$ and $0 < \epsilon \ll 1$.

The convolution of two "highly focused" probability density functions (PDF) have some interesting properties that can be used to solve for X. In particular, define the mean M and covariance Σ of a probability density function on $SE(3)$ as:

$$\int_{SE(3)} log(M^{-1}H))f(H)dH = 0 \tag{12a}$$

$$\Sigma = \int_{SE(3)} log^{\vee}(M^{-1}H)[log^{\vee}(M^{-1)}H)]^T f(H)dH \tag{12b}$$

Then the corresponding discrete version is:

$$\sum_{i=1}^{n} log(M^{-1}H)) = 0 \tag{13a}$$

$$\Sigma = \sum_{i=1}^{n} log^{\vee}(M^{-1}H)[log^{\vee}(M^{-1)}H)]^T. \tag{13b}$$

Given $\{A_i\}$ where the cloud of frames A_i clustering around M_A, an iterative formula can be used for computing M_A [22] as:

$$^{k+1}M_A =^k M_A \circ exp[\frac{1}{n}\sum_{i=1}^{n} log(^k M_A^{-1} \circ A_i)] \tag{14}$$

An initial estimate of the iterative procedure can be chosen as $^0M_A = \frac{1}{n}\sum_{i=1}^{n} log(A_i)$, then a local minimum of M_A is obtained by solving a nonlinear optimization problem with the cost function being $||\sum_{i=1}^{n} log(M_A^{-1}A_i)||^2$. A similar procedure can be used to compute M_B. Σ_A and Σ_B are then straight forward to compute given known M_A and M_B.

The mean and covariance for the convolution $(f_1 * f_2)(g)$ of two highly focused functions f_1 and f_2 are calculated as in [22]:

$$M_{1*2} = M_1 M_2 \tag{15a}$$

$$\Sigma_{1*2} = Ad(M_2^{-1})\Sigma_1 Ad^T(M_2^{-1}) + \Sigma_2. \tag{15b}$$

where

$$Ad(H) = \begin{pmatrix} R & O \\ \hat{t}R & R \end{pmatrix}.$$

Because X and Y are constant, their corresponding PDF will be $\delta_X(g)$ and $\delta_Y(g)$, of which the mean and covariance are $M_X = X$, $\Sigma_X = \mathbb{O}_{6\times 6}$ and $M_Y = Y$, $\Sigma_Y = \mathbb{O}_{6\times 6}$, respectively. Therefore, the following equations can be obtained using Eq.(15):

$$M_A X = Y M_B \tag{16a}$$

$$Ad(X^{-1})\Sigma_A Ad^T(X^{-1}) = \Sigma_B. \tag{16b}$$

To solve the above equations, Eq.(16a) is decomposed into a rotational equation and a translational equation as below:

$$R_{M_A} R_X = R_Y R_{M_B} \tag{17a}$$

$$R_{M_A} t_X + t_{M_A} = R_Y t_{M_B} + t_Y. \tag{17b}$$

Σ_A and Σ_B can be decomposed into blocks as $\begin{pmatrix} \Sigma_A^1 & \Sigma_A^2 \\ \Sigma_A^3 & \Sigma_A^4 \end{pmatrix}$ and $\begin{pmatrix} \Sigma_B^1 & \Sigma_B^2 \\ \Sigma_B^3 & \Sigma_B^4 \end{pmatrix}$, respectively. Knowing that $X^{-1} = \begin{pmatrix} R_X^T & -R_X^T t_X \\ 0 & 1 \end{pmatrix}$, then the first two blocks of Eq.(16b) can be written as follows:

$$\Sigma_{M_B}^1 = R_X^T \Sigma_{M_A}^1 R_X \tag{18a}$$

$$\Sigma_{M_B}^2 = R_X^T \Sigma_{M_A}^1 R_X (\widehat{R_X^T t_X}) + R_X^T \Sigma_{M_A}^2 R_X. \tag{18b}$$

Because Eq.(18a) is a similarity transformation between $\Sigma_{M_B}^1$ and $\Sigma_{M_A}^1$, they share the same eigenvalues and can be eigendecomposed into $\Sigma_{M_A}^1 = Q_{M_A}\Lambda Q_{M_A}^T$ and $\Sigma_{M_B}^1 = Q_{M_B}\Lambda Q_{M_B}^T$ where Λ is a diagonal matrix whose diagonal elements are the eigenvalues of $\Sigma_{M_A}^1$ ($\Sigma_{M_B}^1$), and Q_{M_A} (Q_{M_B}) is a square matrix whose columns are the corresponding eigenvectors. Following equation is obtained after substitution into Eq.(18a):

$$\Lambda = (Q_{M_A}^T R_X^T Q_{M_B})\Lambda(Q_{M_B}^T R_X Q_{M_A}) = P\Lambda P^T \tag{19}$$

where $P = Q_{M_A}^T R_X Q_{M_B}$. If Q_{M_A} and Q_{M_B} are further constrained to be rotation matrices, then a rotation matrix P that satisfies Eq.(19) can be:

$$\mathcal{P} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}. \tag{20}$$

Therefore, there are eight candidates of R_X which can be calculated via $R_X = Q_{M_A} P Q_{M_B}^T$. Then the corresponding t_X can be obtained from Eq.(18b). Given known X, Y can be solved for by $Y = M_A^{-1} X M_B^{-1}$. At last, eight candidate pairs of $\{X_k, Y_k\}$ can be obtained as:

$$X_k = \begin{pmatrix} R_{X_k} & t_{X_k} \\ \mathbf{0}^T & \mathbf{1} \end{pmatrix}, \quad Y_k = \begin{pmatrix} R_{Y_k} & t_{Y_k} \\ \mathbf{0}^T & \mathbf{1} \end{pmatrix} \tag{21}$$

where $k = 1, 2, ..., 8$.

The problem then becomes selecting the best pair of $\{X_k, Y_k\}$ from the eight candidates. Based on the screw theory (references?), it is known that a homogeneous transformation H can be expressed by the four screw parameters $(\theta, d, \mathbf{n}, \mathbf{p})$ as:

$$H = \begin{pmatrix} e^{\theta\hat{\mathbf{n}}} & (\mathbf{I}_3 - e^{\theta\hat{\mathbf{n}}})\mathbf{p} + d\mathbf{n} \\ \mathbf{0}^T & \mathbf{1} \end{pmatrix} \tag{22}$$

where θ, d, $\mathbf{n}$ and $\mathbf{p}$ are ?.

Moreover, $AX_k = Y_k B$ can be written as $AX_k = X_k(X_k^{-1}Y_k B)$. If define $B^k = X_k^{-1}Y_k B$, then we will have $AX_k = X_k B^k$. There exist two Euclidean-Group invariant relationships for each pair of $(A_i, B_i^k)(i = 1, \cdots, n; k = 1, \ldots, 8)$ as follows:

$$\theta_{A_i} = \theta_{B_i^k}, d_{A_i} = d_{B_i^k} \tag{23}$$

Among the four pairs (X_k, Y_k), one can find an optimal solution which minimizes the cost function defined as:

$$(X,Y) = \underset{(X_k,Y_k)}{\mathbf{arg}min} \frac{1}{n} \sum_{i=1}^{n} (\parallel \theta_{A_i} - \theta_{B_i^k} \parallel + \parallel d_{A_i} - d_{B_i^k} \parallel) \tag{24}$$

III. Solution with unknown correspondence between A_i and B_i^k

In most cases, the homogeneous transformations $\{A_i\}$ and $\{B_j\}$ are calculated based on the data obtained from different sensors. Due to the asynchronous timing of the sensor readings, the correspondences between $\{A_i\}$ and $\{B_j^k\}$ is usually unknown. This section deals with the case where there is a shift between $\{A_i\}$ and $\{B_j\}$, and Euclidean-Group invariants are used to recover the correspondence between the data streams. The advantage of the above probabilistic solution lies in that X and Y can be calculated even if there is not a priori knowledge of the correspondence. However, there are still eight possible candidates of (X_k, Y_k) to choose from and by using Euclidean-Group invariants, it is straightforward to determine which pair is the optimal one if the correspondence between A_i and B_i^k can be known.

The Discrete Fourier Transform (DFT) decomposes a time-domain signal into its constituent frequencies. The input is a finite list of equally spaced samples of a function. Given a discrete signal consisting of a sequence of N complex numbers $x_0, x_1, \cdots, x_{N-1}$, the DFT is denoted by $X_\kappa = \mathcal{F}(x_n)$ as:

$$X_\kappa = \sum_{n=0}^{N-1} x_n \cdot exp(-i\frac{2\pi}{N} n\kappa). \tag{25}$$

The Inverse Discrete Fourier transform (IDFT) is denoted as:

$$x_n = \frac{1}{N} \sum_{n=0}^{N-1} X_\kappa \cdot exp(i\frac{2\pi}{N} n\kappa). \tag{26}$$

The discrete convolution of two sequences f_n and g_n are defined

$$(f * g)(\tau) = \sum_{i=0}^{N} f(t_i) g(t_i - \tau). \tag{27}$$

In the convolution theorem, the Fourier transform of a convolution is the product of the Fourier transforms, namely:

$$f * g = \mathcal{F}^{-1}[\mathcal{F}(f) \cdot \mathcal{F}(g)]. \tag{28}$$

The correlation theorem indicates that the correlation function, $Corr(f,g)$, will possess a larger value for a shift vector where the two sequences f_n and g_n can share more similar features. The correlation can be obtained based on the convolution theorem. The DFT of $Corr(f,g)$ is equal to the product of the DFT of f_n and the complex conjugate $\mathcal{F}^*$ of the DFT of g_n:

$$Corr(f,g) = f \star g = \mathcal{F}^{-1}[\mathcal{F}(f) \cdot (\mathcal{F}(g))^*]. \tag{29}$$

Fig. 2. Bs are randomly generated using $B_i = B_{init} exp(\widehat{\mathbf{x}})$,$\mathbf{x} = (x_1, \ldots, x_6)^T$ where $x_i \sim \mathcal{N}(0, 0.1)$.

Compared to the standard time-domain convolution algorithm, the complexity of the convolution by multiplication in the frequency domain is significantly reduced with the help of the convolution theorem and the fast Fourier transform (FFT).

Given two sequences $\{\theta_{A_i}\}$ and $\{\theta_{B_i^k}\}$ corresponding to $\{A_i\}$ and $\{B_i^k\}$, the shift that is needed to recover the data correspondence is obtained as below. Firstly, θ_{Ai} and $\theta_{B_i^k}$ are normalized as:

$$\theta_1 = \frac{(\theta_{A_i} - \mu_{A_i})}{\sigma_{A_i}}, \theta_2 = \frac{(\theta_{B_i^k} - \mu_{B_i^k})}{\sigma_{B_i^k}} \tag{30}$$

where $\mu_{A_i}(\mu_{B_i^k})$ is the mean of $\theta_{A_i}(\theta_{B_i^k})$ and $\sigma_{A_i}(\sigma_{B_i^k})$ is the standard deviation.

Here, the correlation function $Corr(\theta_1, \theta_2)$ is the function of the time sequence index (n) which describes the probability that these two sequences are separated by this particular unit. The index corresponding to the maximum of $Corr(\theta_1, \theta_2)$ indicates the amount of shift τ_{shift} between the $\{\theta_{A_i}\}$ and $\{\theta_{B_i^k}\}$.

$$\tau_{shift} = \underset{index}{\mathbf{arg}max}(Corr(\theta_1, \theta_2)) \tag{31}$$

Therefore, the correspondence between the two sequences can be found. The data of θ_{A_i} or d_{A_i} are shifted by $-\tau_{shift}$ to obtain a sequence of new pairs $(\theta_{A_i}(i + \tau_{shift}), \theta_{B_i^k})$ and $(d_{A_i}(i + \tau_{shift}), d_{B_i^k})$, $max(i, i + \tau_{shift}) \leq i \leq min(i, i + \tau_{shift})$ TO BE REPHRASED. The data stream can be shifted back to regain correspondence once the shift is computed and the correct solution of X and Y can also be recovered by minimizing the cost function Eq.(27) using Euclidean-Group invariants as shown in Section II.

IV. Simulation Studies

In the numerical experiments in this section, the rotational and translational error for X and Y are measured as $Error(R_X) = \parallel log^\vee(R^T_{X_{Solved}} R_{X_{true}}) \parallel$,

Fig. 3. As are calculated using $AX = YB$. X and Y are assumed.

Fig. 4. The cross correlation of data streams of (A_i, B_i^k) respectively.

$Error(t_X) = \parallel (t_{X_{Solved}} - t_{X_{true}}) \parallel$, $Error(R_Y) = \parallel log^{\vee}(R^T_{Y_{Solved}} R_{Y_{true}}) \parallel$ and $Error(t_Y) = \parallel (t_{Y_{Solved}} - t_{Y_{true}}) \parallel$ respectively.

B_i are generated randomly closely around B_{init} using $B_i = B_{init}exp(\widehat{\mathbf{x}})$ and i pose measurements were employed for generating i A_i by $A_i = YB_iX^{-1}$ as shown in Fig. 2 in which a example shows A and B distribution. As a result by applying the above probabilistic method, 8 sequences $(\theta_{A_i}, \theta_{B_i^k})$ and $(d_{A_i}, d_{B_i^k})$ $(i = 5, \cdots, 100, k = 1, \ldots, 8)$ can be obtained respectively.

If the data streams of As were shifted by m units compared to the data stream Bs. The maximum of cross correlation can be used to find the corresponding shift, which is $-m$ shown in Fig. 4, representing the data stream of B_i^k has been shifted by -m units with respect to A_i. Therefore, we shift the data stream inversely to recover the correspondence for finding a correct solution satisfying Euclidean-Group invariants.

Fig. 5. Calculated rotational and translational deviation of X and Y solved using the data in Fig. 2 and Fig. 3.

Fig. 6. $Error(R_X)$ distribution as the As and Bs spread $(\sigma(x_i) = 0.1, 0.2, 0.4, 0.6, 0.8, 1, 2$ as shown in Fig. 2)

Using the minimum sum of $||\theta_{A_i} - \theta_{B_i^k}||$ and $||d_{A_i} - d_{B_i^k}||$, we can find the $B_i^k(k = 3)$ corresponding to the least sum of errors and then, only a (X_k, Y_k) is the desired solution. In Fig. 5, as the number of (A, B) pairs increase, the errors of translation and rotations are reduced but when the number comes to a certain value, the errors cannot be reduced furthermore.

In the generation of $B_i = B_{init}exp(\widehat{\mathbf{x}})$, each element x_j of $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)^T$ is Gaussian with $N \sim (\mu, \sigma)$. Small disturbances are exerted to the B_i to make the noisy $B_i^{noise} = B_iexp(\widehat{\mathbf{x}}_{noise})$, where each of Lie Algebra

Fig. 7. $Error(R_Y)$ distribution as the As and Bs spread ($\sigma(x_i) = 0.1, 0.2, 0.4, 0.6, 0.8, 1, 2$ as shown in Fig. 2)

Fig. 8. $Error(t_X)$ distribution as the As and Bs spread ($\sigma(x_i) = 0.1, 0.2, 0.4, 0.6, 0.8, 1, 2$ as shown in Fig. 2)

Fig. 9. $Error(t_Y)$ distribution as the As and Bs spread ($\sigma(x_i) = 0.1, 0.2, 0.4, 0.6, 0.8, 1, 2$ as shown in Fig. 2)

Fig. 10. The distribution of solved X.

Fig. 11. The distribution of solved Y.

element of $\mathbf{x}$ is Gaussian distribution $N \sim (\mu_{noise}, \sigma_{noise})$. In Fig. 8, 9, 10 and 11, σ_{noise} is 0.005. As σ varies from 0.1 to 2, the errors of R_X, R_Y, t_X, and t_Y are reduced as shown in the box-and-whisker plot. There are several outliers not included between the whiskers. The median data can be used as the final solved X and Y.

V. Conclusions

In this paper, we developed a probabilistic approach to simultaneously obtain X and Y in $AX = YB$ sensor calibration problem. Without a prior knowledge of the correspondence between A and B, in the algorithm the probability theory in Lie group is used to constrain the solution of X and Y to eight candidates. As for the shifted data stream of A and B, using the correlation theorem with Euclidean group invariants,the correspondence is recovered to determine the correct solution from eight candidates. In numeric simulation, the method perform well with different data samples.

APPENDIX

ACKNOWLEDGMENT

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