Simultaneous Coordinate Calibrations by Solving the AX=YB Problem without Correspondence*

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***Abstract*— In image-guided system, relationships of hand-eye (X) and robot-world (Y) coordinates have to be calculated and simultaneous solution is useful in sensor calibration problem. Due to asynchrony of sensors timing, the correspondence between A and B is unknown. A probabilistic method is presented to solve the homogeneous matrix equations without a priori knowledge of the correspondence. Using Euclidean-Group invariants, an exact solution can be found. We illustrate the calculation in numeric simulation including various numbers of robot movements. The results show the efficiency and robustness of the proposed simultaneous method.**

I. Introduction

Image-guided system has been widely used in robotics such as robot assisted surgery. A sensor such as camera, laser scanner or ultrasound probe is usually mounted on the hand of a robotic manipulator. To use the "hand-eye" system, the problem of determining the coordination relationship between the sensor frame with respect to the end-effector frame must be solved, which is widely known **AX=XB** problem for the hand-eye calibration. With an extension of this problem, **AX=YB** problem without knowing the relationship between the world frame and the robot base frame is proposed to achieve the robot-world calibration as well as hand-eye calibration. In a dynamic setup environment, the relationship among the sensor frame, robot frame and world frame is variant and the uncertainties exist. Therefore, simultaneous coordinate calibrations have to be determined frequently in order to enable the robots to respond to changing environments.

In the **AX=YB** problem, As and Bs can be respectively calculated using different sensors. The data streams are in an asynchronous fashion due to different timing sequences. The asynchrony causes a shift between the two streams of data that results in missing the correspondences between As and Bs. In this paper, we present a method to solve for an X and Y without needing know a priori knowledge of correspondence between As and Bs.

The hand-eye calibration problem can be generalized as the **AX = XB** formulation. A and B are respectively the homogeneous transformation matrices of end-effector and

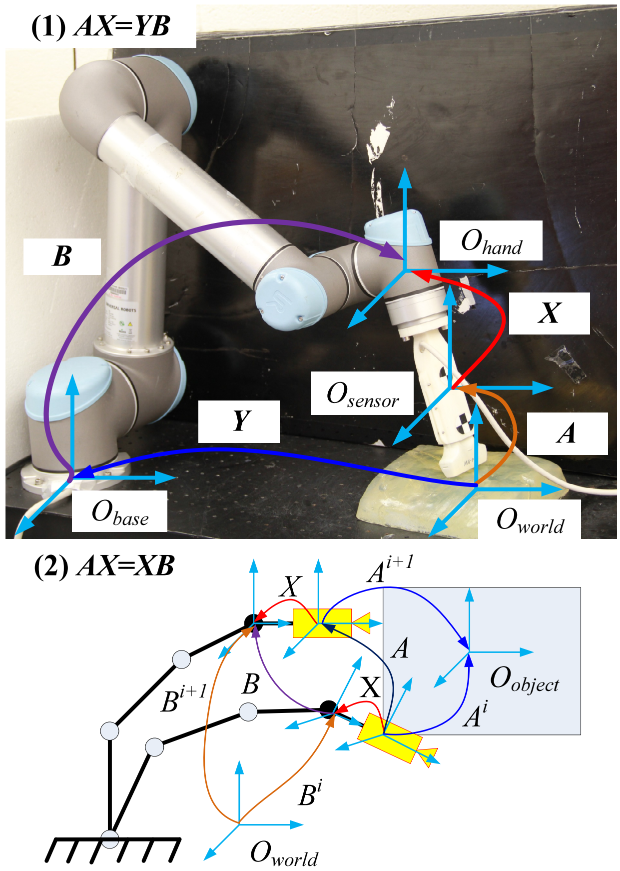

Fig. 1. (1) The hand-eye and robot-robot calibration problem which is formulated as AX=YB. (2) The hand-eye calibration problem which is formulated in a matrix as AX=XB.

sensor movement using $A = A^i(A^{i+1})^{-1}; B = B^i(B^{i+1})^{-1}$ as shown in Fig. 1. The homogeneous transformation matrix can be described with the form.

$$g(R,t) = \begin{pmatrix} R & t \\ 0^T & 1 \end{pmatrix} \quad (1)$$

where $R \in SO(3)$ is a rotation matrix and $t \in R^3$ is a translation vector.

According to multiple pairs $(A, B) \in SE(3) \times SE(3)$ which are known a prior, many methods for the only X have been proposed including decoupling of rotation and translation, least squares fitting, singular value decomposition (*SVD*), screw theory, nonlinear optimization, quaternion, gradient descent and interactive approaches [1], [2], [3], [4], [5], [6], [7], [8], [9]. Theses methods assume that there is exact knowledge of the As and Bs correspondence.

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Considering data streams containing the A and B will be asynchronous that are discussed in many instances in the literature, several methods regardless of the correspondence or recovering the correspondence between two data sets are presented [10], [11], [12].

Simultaneous estimation of the hand-eye transformation and robot-world one has been viewed as **AX=YB** problem. As shown in Fig. 1 (1), Y is the transformation of the robot base relative to the world, A is the sensor to the world transformation, and B is the hand/end-effector to the robot base rigid transformation. The A and B in **AX=YB** calibration is different from ones in **AX=XB** from a physical view. This problem has been studied in different methods such as SVD, closed-form, quaternion and nonlinear optimization [13], [14], [15], [16], [17], [18], [19], [20]. Another approach integrate multiple robots to calibrate hand-eye, tool-flange and robot-robot transformation in **AXB=YCZ** problem [21]. Simultaneous solution for X and Y in **AX=YB** problem is an challenging issue. In the above methods, the correspondence between A and B is known a prior. In this paper, our solution for **AX=YB** doesn't require a priori knowledge of correspondences.

The rest of the paper is organized as follows. In Section II we present the probabilistic theory to solve for X and Y. In Section III a algorithmic solution involving correlation theorem and Euclidean group invariants is posed to recover the correspondence. The simulation results, including unknown correspondence and noisy data stream, are illustrated in Section IV. Finally, we draw some conclusions.

II. Solving AX=YB using a probabilistic theory on motion groups

Given a large set of pairs $(A_i, B_i) \in SE(3) \times SE(3)$ for $i = 1, \cdots, n$ that are acquired by measurements and satisfy the following equations

$$A_i X = Y B_i \tag{2}$$

In the case of $SE(3)$, a Dirac delta function, or δ function, is thought of as a function which is zero everywhere except at the identity where it is infinite.

$$\delta(H) = \begin{cases} +\infty, & H = I \\ 0, & H \neq I \end{cases} \tag{3}$$

Dirac delta function is also constrained to satisfy the identity.

$$\int_{SE(3)} \delta(H) dH = 1 \tag{4}$$

A shifted Dirac delta function can be defined as $\delta_A(H) = \delta(A^{-1}H)$. Given two functions $f_1(g)$ and $f_2(g)$, their convolution in Lie group is defined as follows,

$$(f_1 * f_2)(g) = \int_{SE(3)} f_1(h) f_2(h^{-1} \circ g) dh \tag{5}$$

Considering the properties of δ function, the following equation is built.

$$(f * \delta)(g) = \int_{SE(3)} f(h)\delta(h^{-1} \circ g)dh = f(g) \tag{6}$$

Therefore, for each A_i and B_i, we can get

$$(\delta_{A_i} * \delta_X)(g) = \delta(A_i^{-1} g X^{-1}) \tag{7a}$$

$$(\delta_Y * \delta_{B_i})(g) = \delta(Y^{-1} g B_i^{-1}) \tag{7b}$$

Together with $g = A_i X = Y B_i$, we can obtain the convolution equation

$$(\delta_{A_i} * \delta_X)(g) = (\delta_Y * \delta_{B_i})(g) \tag{8}$$

Convolution provides a linear operation on functions with additional properties. We can add up n instances into a single function.

$$f_A(g) = \frac{1}{n} \sum_{i=1}^{n} \delta(A_i^{-1} g) \tag{9a}$$

$$f_B(g) = \frac{1}{n} \sum_{i=1}^{n} \delta(B_i^{-1} g) \tag{9b}$$

Therefore,

$$(f_{A_i} * \delta_X)(g) = (\delta_Y * f_{B_i})(g) \tag{10}$$

In each transformation set A_is and B_is, we are using small relative motions between consecutive reference frames. Given a measure of distance between reference frames, e.g.

$$d^2(A_i, A_j) = \| \Delta A \|_W^2 = trace[(\Delta A) W (\Delta A)^T] = \epsilon, \tag{11}$$

we have that $\Delta A = A_i - A_j$ and $0 < \epsilon \ll 1$

The convolution of "highly focused" distributions corresponding to closely clumped sets of reference frames have some interesting properties that we can exploit to solve for X. In particular, let the mean and covariance of a probability density, $f(g)$(e.g. $f_A(g), f_B(g)$), be defined by the conditions.

$$\int_{SE(3)} log(M^{-1}g)) f(g) dg = 0 \tag{12a}$$

$$\Sigma = \int_{SE(3)} log^{\vee}(M^{-1}g)[log^{\vee}(M^{-1})g)]^T f(g) dg \tag{12b}$$

A discrete version as for $f_A(g)$ is

$$\sum_{i=1}^{n} log(M^{-1}g)) = 0 \tag{13a}$$

$$\Sigma_A = \sum_{i=1}^{n} log^{\vee}(M^{-1}g)[log^{\vee}(M^{-1})g)]^T \tag{13b}$$

While the cloud of frames A_i is clustered around M_A, an iterative formula can be used for computing M_A [22].

$$ {}^{k+1}M_A = {}^k M_A \circ exp[\frac{1}{n}\sum_{i=1}^{n} log({}^k M_A^{-1} \circ A_i)] \quad (14) $$

An initial estimate of the iterative procedure can be ${}^0M_A = \frac{1}{n}\sum_{i=1}^{n} log(A_i)$, then the process iterates until the cost function, $\| \sum_{i=1}^{n} log(M_A^{-1}A_i) \|^2$ falls below a predefined threshold, where the cost function is minimized and the minimum defines M_A. A similar procedure is used for computing M_B.

The mean and covariance for the convolution $(f_1 * f_2)(g)$ of two highly focused functions, f_1 and f_2 can be computed as

$$ M_{1*2} = M_1 M_2 \quad (15a) $$
$$ \Sigma_{1*2} = Ad(M_2^{-1})\Sigma_1 Ad^T(M_2^{-1}) + \Sigma_2 \quad (15b) $$

where

$$ Ad(g) = \begin{pmatrix} R & O \\ \hat{t}R & R \end{pmatrix} $$

Because X and Y is fixed, as for $\delta_X(g)$ and $\delta_Y(g)$, mean and covariance are $M_X = X$, $\Sigma_X = 0$ and $M_Y = Y$, $\Sigma_Y = 0$, respectively, therefore we can obtain

$$ M_A X = Y M_B \quad (16a) $$
$$ Ad(X^{-1})\Sigma_A Ad^T(X^{-1}) = \Sigma_B \quad (16b) $$

From (16a), we can obtain

$$ R_{M_A} R_X = R_Y R_{M_B} \quad (17a) $$
$$ R_{M_A} t_X + t_{M_A} = R_Y t_{M_B} + t_Y \quad (17b) $$

Σ_{M_A} and Σ_{M_B} can be decomposed into blocks as $\begin{pmatrix} \Sigma^1_{M_A} & \Sigma^2_{M_A} \\ \Sigma^3_{M_A} & \Sigma^4_{M_A} \end{pmatrix}$ and $\begin{pmatrix} \Sigma^1_{M_B} & \Sigma^2_{M_B} \\ \Sigma^3_{M_B} & \Sigma^4_{M_B} \end{pmatrix}$, respectively. Using $X^{-1} = X^T = \begin{pmatrix} R_X^T & -R_X^T t_X \\ 0 & 1 \end{pmatrix}$, then we can write the first two blocks of (16b) as follows,

$$ \Sigma^1_{M_B} = R_X^T \Sigma^1_{M_A} R_X \quad (18a) $$
$$ \Sigma^2_{M_B} = R_X^T \Sigma^1_{M_A} R_X (\widehat{R_X^T t_X}) + R_X^T \Sigma^2_{M_A} R_X \quad (18b) $$

The first blocks (18a) is eigendecomposed with the same diagonal matrix due to matrix similarity $\Sigma^1_{M_A} = Q_{M_A} \wedge Q^T_{M_A}$, $\Sigma^1_{M_B} = Q_{M_B} \wedge Q^T_{M_B}$. Then,

$$ \wedge = (Q^T_{M_A} R_X Q_{M_B}) \wedge (Q^T_{M_B} R_X^T Q_{M_A}) = P \wedge P^T \quad (19) $$

In $P = Q^T_{M_A} R_X Q_{M_B}$, Q_{M_A} and Q_{M_B} are constrained to be a rotation matrix and therefore $P \in \Omega \bigcup (-\Omega)$,

$$ \Omega = \left(\begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{array} \right) \quad (20) $$

Therefore, there are eight possibilities of $R_X, R_X = Q_{M_A} P Q^T_{M_B}$. Then, from 18b, eight t_X corresponding to R_X can be directly found. Furthermore, eight candidate $R_Y andt_Y$ can be found. At last, there are eight possibilities of solution $(X_i, Y_i), i = 1, \ldots, 8$. where,

$$ X_i = \begin{pmatrix} R_X & t_X \\ \mathbf{0}^T & \mathbf{1} \end{pmatrix}, \quad Y_i = \begin{pmatrix} R_Y & t_Y \\ \mathbf{0}^T & \mathbf{1} \end{pmatrix} \quad (21) $$

Based on the screw theory, it is known that a homogeneous transformation H can be written in the form with four parameters $(\theta, d, \mathbf{n}, \mathbf{p})$.

$$ H = \begin{pmatrix} e^{\theta\hat{\mathbf{n}}} & (\mathbf{I}_3 - e^{\theta\hat{\mathbf{n}}})\mathbf{p} + d\mathbf{n} \\ \mathbf{0}^T & \mathbf{1} \end{pmatrix} \quad (22) $$

$AX = YB$ can be written as $AX = X(X^{-1}YB)$ and let $B' = X^{-1}YB$. In the form $AX = XB'$, there exit two Euclidean-Group invariant relationships for one of eight groups of $(A_i, B_i^k)(i = 1, \cdots, n; k = 1, \ldots, 8), B_i^k = X^{-1}YB_i$ as follows,

$$ \theta_{A_i} = \theta_{B_i^k}, d_{A_i} = d_{B_i^k} \quad (23) $$

From among the four pairs (X_i, Y_i), we can find a correct solution to minimize the absolute deviations,

$$ (X, Y) = \underset{(X_i,Y_i)}{\mathbf{arg}min} \frac{1}{n}\sum_{i=1}^{n}(\| \theta_{A_i} - \theta_{B_i^k} \| + \| d_{A_i} - d_{B_i^k} \|) \quad (24) $$

III. SOLUTION WITH UNKNOWN CORRESPONDENCE OF A_i AND B_i^k

In most cases, the homogeneous transformations with As and Bs are given based on the data from different sensors. Due to asynchronous timing of the measurement transmissions, the correspondences between A_i and B_i^k is unknown. The advantage of the above probabilistic solution lies on that X and Y can be calculated even if without any a priori knowledge of the correspondence. However, there are still eight possible candidate results (X_i, Y_i). Using Euclidean-Group invariants, it is straightforward to determine which pair is the correct one if the correspondence between A_i and B_i^k can be known.

The Discrete Fourier Transform (DFT) decomposes a time-domain signal into its constituent frequencies. The input is a finite list of equally spaced samples of a function. Given a discrete signal consisting of a sequence of N complex numbers $x_0, x_1, \cdots, x_{N-1}$, the DFT is denoted by $X_\kappa = \mathcal{F}x_n$

$$X_\kappa = \sum_{n=0}^{N-1} x_n \cdot exp(-i\frac{2\pi}{N}n\kappa) \tag{25}$$

The Inverse Discrete Fourier transform (IDFT) denoted by

$$x_n = \frac{1}{N}\sum_{n=0}^{N-1} X_\kappa \cdot exp(i\frac{2\pi}{N}n\kappa) \tag{26}$$

The discrete convolution of two sequences f_n and g_n are defined

$$(f * g)(\tau) = \sum_{i=0}^{N} f(t_i)g(t_i - \tau) \tag{27}$$

In convolution theorem, the Fourier transform of a convolution is the product of the Fourier transforms, namely,

$$f * g = \mathcal{F}^{-1}[\mathcal{F}(f) \cdot \mathcal{F}(g)] \tag{28}$$

The correlation theorem indicates that the correlation function, $Corr(f, g)$, will have a large value at a shift vector if the two sequences f and g contain similar features. The correlation can be obtained based on the convolution theorem. The DFT of the correlation $Corr(f, g)$ is equal to the product of the DFT of a sequence f_n and the complex conjugate $\mathcal{F}^*$ of the DFT of the other sequence g_n.

$$Corr(f, g) = f \star g = \mathcal{F}^{-1}[\mathcal{F}(f) \cdot (\mathcal{F}(g))^*] \tag{29}$$

Compared with the standard time-domain convolution algorithm, the complexity of the convolution by multiplication in the frequency domain is significantly reduced with the help of the convolution theorem and the fast Fourier transform (FFT).

There are two sequences θ_{A_i} and $\theta_{B_i^k}$ from each pair (A_i, B_i^k). For homogeneous transformations from which the range of θ can vary, two sequences θ_{Ai} and $\theta_{B_i^k}$ can be first normalized.

$$\theta_1 = \frac{(\theta_{A_i} - \mu_{A_i})}{\sigma_{A_i}}, \theta_2 = \frac{(\theta_{B_i^k} - \mu_{B_i^k})}{\sigma_{B_i^k}} \tag{30}$$

where $\mu_{A_i}(\mu_{B_i^k})$ is the average of $\theta_{A_i}(\theta_{B_i^k})$ and $\sigma_{A_i}(\sigma_{B_i^k})$ is the standard deviation.

Here, the correlation function $Corr(\theta_1, \theta_2)$ is the function of the time sequence index (n) which describes the probability that these two sequences are separated by this particular unit. The location of the function maximum indicates the amount of shift, τ_{shift}, between the two sequence θ_{A_i} and $\theta_{B_i^k}$.

$$\tau_{shift} = \underset{index}{\mathbf{arg}max}(Corr(\theta_1, \theta_2)) \tag{31}$$

Therefore, the correspondence between the two sequences can be found. The data of θ_{A_i} or d_{A_i} are shifted by $-\tau_{shift}$ to obtain a sequence of new pairs $(\theta_{A_i}(i+\tau_{shift}), \theta_{B_i^k})$ and $(d_{A_i}(i+\tau_{shift}), d_{B_i^k})$, $max(i, i+\tau_{shift}) \leq i \leq min(i, i+\tau_{shift})$. The data stream can be shifted to reach correspondence once the shift is found and the correct solution can also be found by minimizing the absolute deviations based on Euclidean-Group invariants relations using the method in Section II.

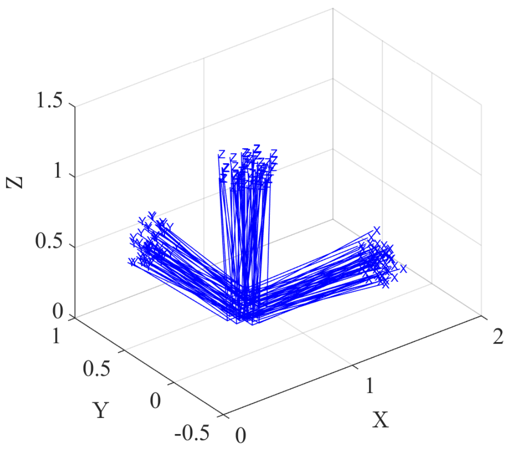

Fig. 2. Bs are randomly generated using $B_i = B_{init}exp(\hat{\mathbf{x}})$,$\mathbf{x} = (x_1, \dots, x_6)^T$ $x_i \sim \mathcal{N}(0, 0.1)$.

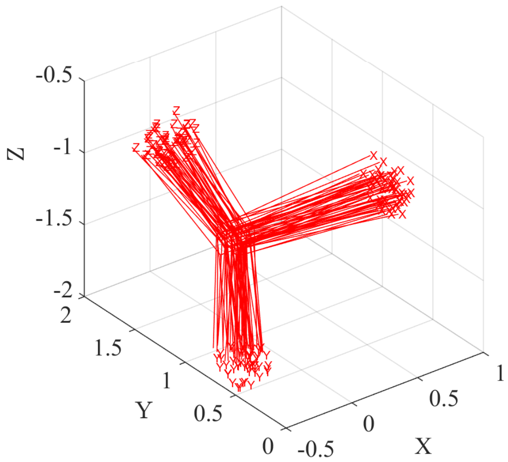

Fig. 3. As are calculated using $AX = YB$. X and Y are assumed.

IV. Simulation Studies

In the numerical experiments in this section, the rotational and translational error for X and Y are measured as $Error(R_X) = \| log^\vee(R_{X_{Solved}}^T R_{X_{true}}) \|$, $Error(t_X) = \| (t_{X_{Solved}} - t_{X_{true}}) \|$, $Error(R_Y) = \| log^\vee(R_{Y_{Solved}}^T R_{Y_{true}}) \|$ and $Error(t_Y) = \| (t_{Y_{Solved}} - t_{Y_{true}}) \|$ respectively.

B_i are generated randomly closely around B_{init} using $B_i = B_{init}exp(\hat{\mathbf{x}})$ and i pose measurements were employed for generating i A_i by $A_i = YB_iX^{-1}$ as shown in Fig. 2 in which a example shows A and B distribution. As a result

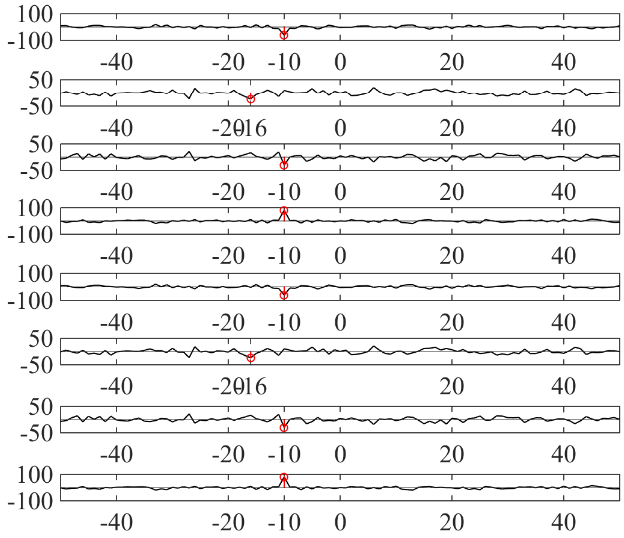

Fig. 4. The cross correlation of data streams of (A_i, B_i^k) respectively.

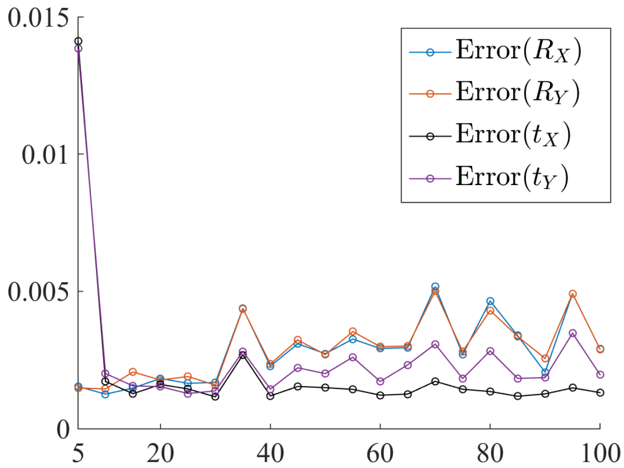

Fig. 5. Calculated rotational and translational deviation of X and Y solved using the data in Fig. 2 and Fig. 3.

by applying the above probabilistic method, 8 sequences $(\theta_{A_i}, \theta_{B_i^k})$ and $(d_{A_i}, d_{B_i^k})$ $(i = 5, \cdots, 100, k = 1, \ldots, 8)$ can be obtained respectively.

If the data streams of As were shifted by m units compared to the data stream Bs. The maximum of cross correlation can be used to find the corresponding shift, which is $-m$ shown in Fig. 4, representing the data stream of B_i^k has been shifted by -m units with respect to A_i. Therefore, we shift the data stream inversely to recover the correspondence for finding a correct solution satisfying Euclidean-Group invariants.

Using the minimum sum of $||\theta_{A_i} - \theta_{B_i^k}||$ and $||d_{A_i} - d_{B_i^k}||$, we can find the $B_i^k (k = 3)$ corresponding to the least sum of errors and then, only a (X_k, Y_k) is the desired solution. In Fig. 5, as the number of (A, B) pairs increase, the errors of translation and rotations are reduced but when the number comes to a certain value, the errors cannot be reduced furthermore.

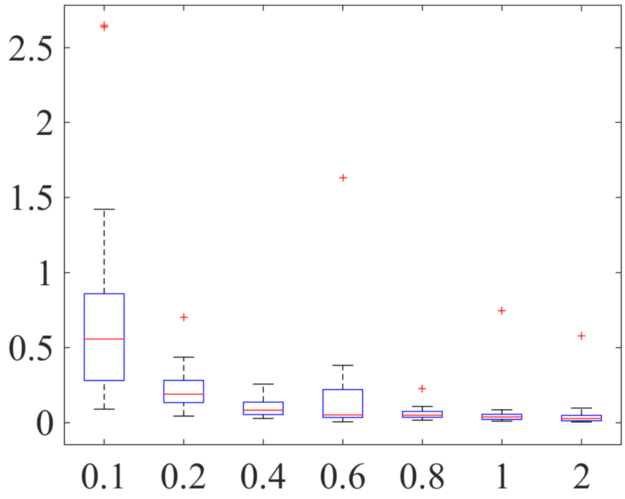

Fig. 6. $Error(R_X)$ distribution as the As and Bs spread $(\sigma(x_i) = 0.1, 0.2, 0.4, 0.6, 0.8, 1, 2$ as shown in Fig. 2)

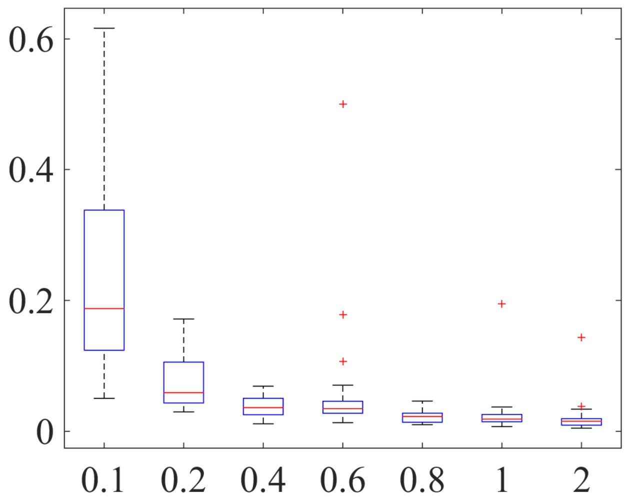

Fig. 7. $Error(R_Y)$ distribution as the As and Bs spread $(\sigma(x_i) = 0.1, 0.2, 0.4, 0.6, 0.8, 1, 2$ as shown in Fig. 2)

In the generation of $B_i = B_{init} exp(\widehat{\mathbf{x}})$, each element x_j of $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)^T$ is Gaussian with $N \sim (\mu, \sigma)$. Small disturbances are exerted to the B_i to make the noisy $B_i^{noise} = B_i exp(\widehat{\mathbf{x}}_{noise})$, where each of Lie Algebra element of $\mathbf{x}$ is Gaussian distribution $N \sim (\mu_{noise}, \sigma_{noise})$. In Fig. 8, 9, 10 and 11, σ_{noise} is 0.005. As σ varies from 0.1 to 2, the errors of R_X, R_Y, t_X, and t_Y are reduced as shown in the box-and-whisker plot. There are several outliers not included between the whiskers. The median data can be used as the final solved X and Y.

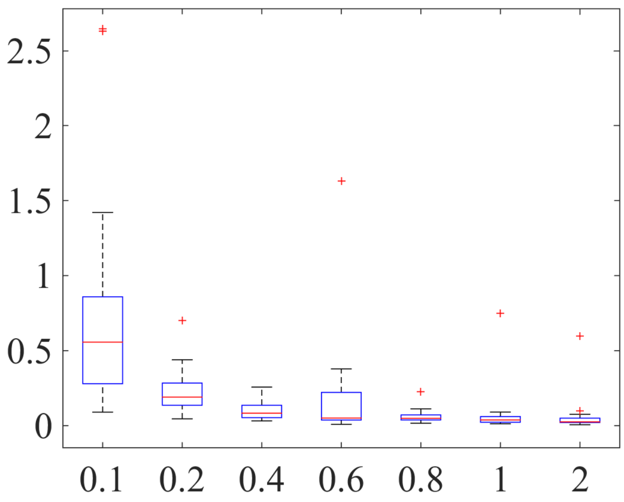

Fig. 8. $Error(t_X)$ distribution as the As and Bs spread ($\sigma(x_i)$ = $0.1, 0.2, 0.4, 0.6, 0.8, 1, 2$ as shown in Fig. 2)

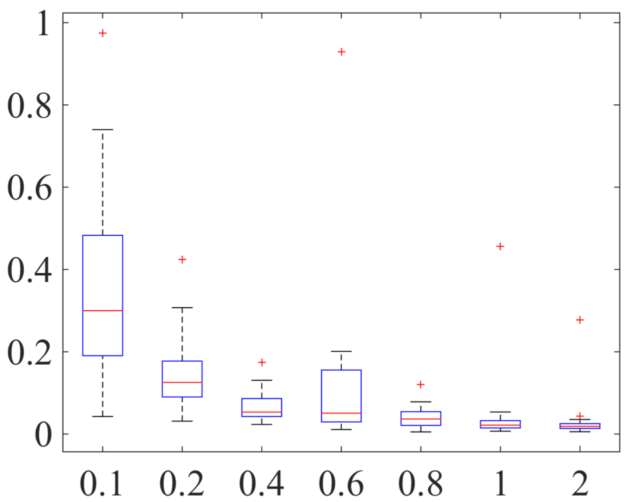

Fig. 9. $Error(t_Y)$ distribution as the As and Bs spread ($\sigma(x_i)$ = $0.1, 0.2, 0.4, 0.6, 0.8, 1, 2$ as shown in Fig. 2)

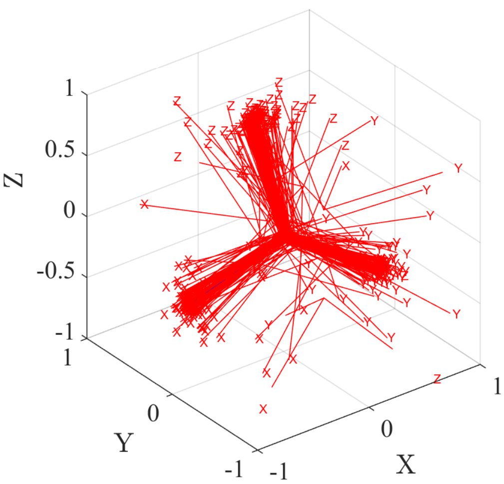

Fig. 10. The distribution of solved X.

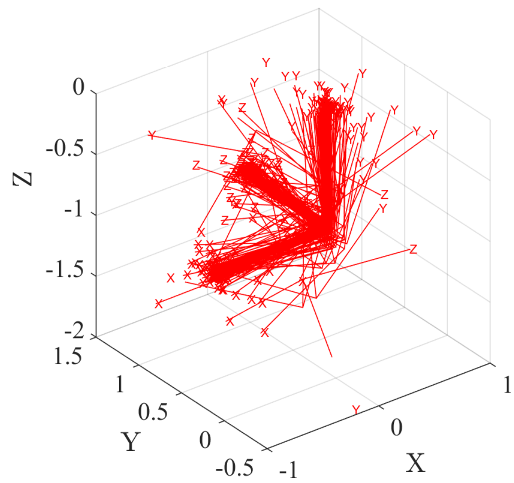

Fig. 11. The distribution of solved Y.

V. Conclusions

In this paper, we developed a probabilistic approach to simultaneously obtain X and Y in $AX = YB$ sensor calibration problem. Without a prior knowledge of the correspondence between A and B, in the algorithm the probability theory in Lie group is used to constrain the solution of X and Y to eight candidates. As for the shifted data stream of A and B, using the correlation theorem with Euclidean group invariants,the correspondence is recovered to determine the correct solution from eight candidates. In numeric simulation, the method perform well with different data samples.

APPENDIX

ACKNOWLEDGMENT

References

[1] Y. C. Shiu and S. Ahmad, "Calibration of wrist-mounted robotic sensors by solving homogeneous transform equations of the form ax= xb," *Robotics and Automation, IEEE Transactions on*, vol. 5, no. 1, pp. 16–29, 1989.

[2] R. Y. Tsai and R. K. Lenz, "A new technique for fully autonomous and efficient 3d robotics hand/eye calibration," *Robotics and Automation, IEEE Transactions on*, vol. 5, no. 3, pp. 345–358, 1989.

[3] C.-C. Wang, "Extrinsic calibration of a vision sensor mounted on a robot," *Robotics and Automation, IEEE Transactions on*, vol. 8, no. 2, pp. 161–175, 1992.

[4] F. C. Park and B. J. Martin, "Robot sensor calibration: solving ax= xb on the euclidean group," *IEEE Transactions on Robotics and Automation (Institute of Electrical and Electronics Engineers);(United States)*, vol. 10, no. 5, 1994.

[5] R. Horaud and F. Dornaika, "Hand-eye calibration," *The international journal of robotics research*, vol. 14, no. 3, pp. 195–210, 1995.

[6] K. Daniilidis, "Hand-eye calibration using dual quaternions," *The International Journal of Robotics Research*, vol. 18, no. 3, pp. 286–298, 1999.

[7] I. Fassi and G. Legnani, "Hand to sensor calibration: A geometrical interpretation of the matrix equation ax= xb," *Journal of Robotic Systems*, vol. 22, no. 9, pp. 497–506, 2005.

[8] Z. Zhao, "Hand-eye calibration using convex optimization," in *Robotics and Automation (ICRA), 2011 IEEE International Conference on*. IEEE, 2011, pp. 2947–2952.

[9] M. K. Ackerman, A. Cheng, E. Boctor, and G. Chirikjian, "Online ultrasound sensor calibration using gradient descent on the euclidean group," in *Robotics and Automation (ICRA), 2014 IEEE International Conference on*. IEEE, 2014, pp. 4900–4905.

[10] M. K. Ackerman and G. S. Chirikjian, “A probabilistic solution to the ax= xb problem: Sensor calibration without correspondence,” in *Geometric Science of Information*. Springer, 2013, pp. 693–701.

[11] M. K. Ackerman, A. Cheng, B. Shiffman, E. Boctor, and G. Chirikjian, “Sensor calibration with unknown correspondence: Solving ax= xb using euclidean-group invariants,” in *Intelligent Robots and Systems (IROS), 2013 IEEE/RSJ International Conference on*. IEEE, 2013, pp. 1308–1313.

[12] M. K. Ackerman, A. Cheng, and G. Chirikjian, “An information-theoretic approach to the correspondence-free ax= xb sensor calibration problem,” in *Robotics and Automation (ICRA), 2014 IEEE International Conference on*. IEEE, 2014, pp. 4893–4899.

[13] H. Zhuang, Z. S. Roth, and R. Sudhakar, “Simultaneous robot/world and tool/flange calibration by solving homogeneous transformation equations of the form ax= yb,” *Robotics and Automation, IEEE Transactions on*, vol. 10, no. 4, pp. 549–554, 1994.

[14] F. Dornaika and R. Horaud, “Simultaneous robot-world and hand-eye calibration,” *Robotics and Automation, IEEE Transactions on*, vol. 14, no. 4, pp. 617–622, 1998.

[15] R. L. Hirsh, G. N. DeSouza, and A. C. Kak, “An iterative approach to the hand-eye and base-world calibration problem,” in *Robotics and Automation, 2001. Proceedings 2001 ICRA. IEEE International Conference on*, vol. 3. IEEE, 2001, pp. 2171–2176.

[16] F. Ernst, L. Richter, L. Matthäus, V. Martens, R. Bruder, A. Schlaefer, and A. Schweikard, “Non-orthogonal tool/flange and robot/world calibration,” *The International Journal of Medical Robotics and Computer Assisted Surgery*, vol. 8, no. 4, pp. 407–420, 2012.

[17] K. H. Strobl and G. Hirzinger, “Optimal hand-eye calibration,” in *Intelligent Robots and Systems, 2006 IEEE/RSJ International Conference on*. IEEE, 2006, pp. 4647–4653.

[18] A. Li, L. Wang, and D. Wu, “Simultaneous robot-world and hand-eye calibration using dual-quaternions and kronecker product,” *Inter. J. Phys. Sci*, vol. 5, no. 10, pp. 1530–1536, 2010.

[19] M. Shah, “Solving the robot-world/hand-eye calibration problem using the kronecker product,” *Journal of Mechanisms and Robotics*, vol. 5, no. 3, p. 031007, 2013.

[20] J. Heller, D. Henrion, and T. Pajdla, “Hand-eye and robot-world calibration by global polynomial optimization,” in *Robotics and Automation (ICRA), 2014 IEEE International Conference on*. IEEE, 2014, pp. 3157–3164.

[21] J. Wang, L. Wu, M. Q.-H. Meng, and H. Ren, “Towards simultaneous coordinate calibrations for cooperative multiple robots,” in *Intelligent Robots and Systems (IROS 2014), 2014 IEEE/RSJ International Conference on*. IEEE, 2014, pp. 410–415.

[22] Y. Wang and G. S. Chirikjian, “Nonparametric second-order theory of error propagation on motion groups,” *The International journal of robotics research*, vol. 27, no. 11-12, pp. 1258–1273, 2008.