Simultaneous Coordinate Calibrations by Solving the AX=YB Problem without Correspondence *

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***Abstract*—**

***Index Terms*— AX=YB, Sensor Calibration, Probabilistic Modeling**

I. INTRODUCTION

Image-guided system has been widely used in robotics such as robot assisted surgery. Sensors such as a camera, a laser scanner and an ultrasound probe are usually mounted as the end-effector of a robotic manipulator. For a typical "hand-eye" system as described above, the relative transformation of the sensor with respect to the end-effector should be accurately calibrated, and it is often characterized as the well known **AX=XB** problem. A variation of this problem is characterized as the **AX=YB** problem, where both the hand-eye transformation and the pose of the robot base with respect to the world frame need to be calibrated. In a typical environment setup, the relationship among the sensor frame, robot frame and world frame either is subject to change or can be disturbed by uncertainties. Therefore, simultaneous coordinate calibrations have to be performed frequently in order to enable the robot to respond to dynamic environments.

In the **AX=YB** problem, As and Bs can be respectively obtained via different sensors. The data streams can be in an asynchronous fashion due to different working frequencies of the sensors. The asynchrony causes a shift between the two streams of data which damages the correspondence between As and Bs. In this paper, we present a method to solve for X and Y without the need to know a priori knowledge of the correspondence between As and Bs.

The hand-eye calibration problem can be modeled as the **AX = XB**, where As and Bs are the homogeneous transformation matrices describing the relative motion of the end-effector and the sensor respectively. As shown in Fig. 1, $A_i = A^i(A^{i+1})^{-1}$ and $B_i = B^i(B^{i+1})^{-1}$. The

Fig. 1. (1) The hand-eye calibration problem formulated in a matrix as AX=XB. (2) The hand-eye and robot-robot calibration problem formulated as AX=YB.

homogeneous transformation matrix can be described as:

$$g(R,t) = \begin{pmatrix} R & t \\ 0^T & 1 \end{pmatrix} \tag{1}$$

where $R \in SO(3)$ is a rotation matrix and $t \in R^3$ is a translation vector.

Given multiple pairs of $\{A_i, B_i\} \in SE(3) \times SE(3)$ with correspondence, many methods have been proposed to solve for X. To the best of the authors' knowledge, Shiu [1] and Tsai [2] are the first to solve the $AX = XB$ sensor calibration problem. The other methods include but are not limited to quaternion, dual quaternion, screw theory, Lie group theory, convex optimization and gradient descent methods [3]–[9]. All of the methods above assume a prior knowledge of exact correspondence between A_i and B_i. For data streams $\{A_i\}$ and $\{B_j\}$ that are asynchronous, several methods have been proposed in the literature to solve for X using data without correspondence [10]–[12].

Simultaneous estimation of the hand-eye and robot-world

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transformations has been viewed as the **AX=YB** problem. As shown in Fig.1, Y is the transformation of the robot base relative to the world frame, A denotes the sensor's pose in the world frame, and B is the transformation of the end-effector with respect to its fixed base. The A_i and B_i in **AX=YB** are different from those in **AX=XB** which can be seen in Fig.1. The **AX=YB** problem has been solved by different methods such as kronecker product, quaternion, dual quaternion, and nonlinear optimization method [13]–[20]. Simultaneous calibration of X and Y can be problematic in that all the methods above assume exact correspondence between $\{A_i\}$ and $\{B_j\}$, which is not the case in real experiments. Another similar problem involves the calibration of multiple robots in terms of hand-eye, tool-flange and robot-robot system, and it is formulated as the **AXB=YCZ** problem [21]. In this paper, we focus on the **AX=YB** problem which does not require a priori knowledge of correspondences of the data streams.

The rest of the paper is organized as follows. In Section II, we present a probabilistic method to solve for X and Y. In Section III, a algorithmic solution involving both the correlation theorem and Euclidean group invariants is posed to recover the correspondence between $\{A_i\}$ and $\{B_j\}$. The simulation results which deal with the data streams with both known and unknown correspondence are illustrated in Section IV. Finally, conclusions are drawn based on the numerical results and possible future works are pointed out.

II. Solving AX=YB using a probabilistic theory on motion groups

Given a large set of pairs $\{A_i, B_i\} \in SE(3) \times SE(3)$ for $i = 1, \cdots, n$ that are acquired by measurements and satisfy the following equation:

$$A_i X = Y B_i. \tag{2}$$

For a matrix $H \in SE(3)$, a Dirac delta function $\delta(H)$ function is defined to be finite only at the identity and zero everywhere else:

$$\delta(H) = \begin{cases} +\infty, & H = I \\ 0, & H \neq I \end{cases} \tag{3}$$

It also satisfies the identity constraint as:

$$\int_{SE(3)} \delta(H) dH = 1. \tag{4}$$

A shifted Dirac delta function can be defined as $\delta_A(H) = \delta(A^{-1}H)$. Given $K, H \in SE(3)$ and two functions $f_1, f_2 \in (L^1 \cap L^2)(SE(3))$, their convolution on $SE(3)$ is defined as:

$$(f_1 * f_2)(H) = \int_{SE(3)} f_1(K) f_2(K^{-1}H) dK \tag{5}$$

where $h \in SE(3)$. Employing the properties of δ function, it is straightforward to see that:

$$(f * \delta)(H) = \int_{SE(3)} f(K)\delta(K^{-1}H)dK = f(H). \tag{6}$$

Therefore, for each A_i and B_i, we have:

$$(\delta_{A_i} * \delta_X)(H) = \delta(A_i^{-1} H X^{-1}) \tag{7a}$$
$$(\delta_Y * \delta_{B_i})(H) = \delta(Y^{-1} H B_i^{-1}) \tag{7b}$$

Together with $A_i X = Y B_i$, we can obtain the convolution equation:

$$(\delta_{A_i} * \delta_X)(H) = (\delta_Y * \delta_{B_i})(H) \tag{8}$$

Convolution provides a linear operation on functions with addition properties. After adding up n instances of $\delta_{A_i}(H)$ and $\delta_{B_i}(H)$, we obtained the following equations:

$$f_A(H) = \frac{1}{n} \sum_{i=1}^{n} \delta(A_i^{-1} H) \tag{9a}$$

$$f_B(H) = \frac{1}{n} \sum_{i=1}^{n} \delta(B_i^{-1} H) \tag{9b}$$

Therefore, we have:

$$(f_A * \delta_X)(H) = (\delta_Y * f_B)(H). \tag{10}$$

For each of the data stream $\{A_i\}$ and $\{B_i\}$, we are using small relative motions between consecutive reference frames. Take $\{A_i\}$ for example, one metric of the distance between A_i and A_j can be defined as:

$$d^2(A_i, A_{i+1}) = \| \Delta A \|_W^2 = trace[(\Delta A) W (\Delta A)^T] = \epsilon, \tag{11}$$

where $\Delta A = A_i - A_j$ (or $\Delta A = A_i^{-1} A_j$) and $0 < \epsilon \ll 1$

The convolution of two "highly focused" probability density functions (pdf) has some interesting properties that can be used to solve for X. In particular, firstly, define the mean M and covariance Σ of a probability density function on $SE(3)$ as:

$$\int_{SE(3)} log(M^{-1}H)) f(H) dH = 0 \tag{12a}$$

$$\Sigma = \int_{SE(3)} log^{\vee}(M^{-1}H)[log^{\vee}(M^{-1}H)]^T f(H) dH \tag{12b}$$

Then the corresponding discrete version of $f_A(H)$ will be:

$$\sum_{i=1}^{n} log(M_A^{-1} H)) = 0 \tag{13a}$$

$$\Sigma_A = \sum_{i=1}^{n} log^{\vee}(M_A^{-1} H)[log^{\vee}(M_A^{-1)} H)]^T \tag{13b}$$

For the case where $\{A_i\}$ is clustered around M_A, an iterative formula can be used for computing M_A [22] as:

$$ {}^{k+1}M_A = {}^{k} M_A \circ exp[\frac{1}{n}\sum_{i=1}^{n} log({}^{k}M_A^{-1} \circ A_i)]. \tag{14} $$

An initial estimate of M_A can be calculated as ${}^0M_A = \frac{1}{n}\sum_{i=1}^{n} log(A_i)$, then iterations will be performed according to Eq.(14) until the cost function $\| \sum_{i=1}^{n} log(M_A^{-1}A_i) \|^2$ falls below a predefined threshold. A similar procedure is used for computing M_B.

The mean and covariance for the convolution of two highly focused pdf $(f_1 * f_2)(H)$ can be computed as:

$$ M_{1*2} = M_1 M_2 \tag{15a} $$
$$ \Sigma_{1*2} = Ad(M_2^{-1})\Sigma_1 Ad^T(M_2^{-1}) + \Sigma_2 \tag{15b} $$

where

$$ Ad(H) = \begin{pmatrix} R & O \\ \hat{t}R & R \end{pmatrix}. $$

Because X and Y is fixed, the means and covariances for $\delta_X(H)$ and $\delta_Y(H)$ are $M_X = X$, $\Sigma_X = 0$ and $M_Y = Y$, $\Sigma_Y = 0$ respectively. Therefore, we can obtain

$$ M_A X = Y M_B \tag{16a} $$
$$ Ad(X^{-1})\Sigma_A Ad^T(X^{-1}) = \Sigma_B \tag{16b} $$

From (16a), we can obtain

$$ R_{M_A} R_X = R_Y R_{M_B} \tag{17a} $$
$$ R_{M_A} t_X + t_{M_A} = R_Y t_{M_B} + t_Y \tag{17b} $$

Σ_{M_A} and Σ_{M_B} can be decomposed into blocks as $\begin{pmatrix} \Sigma^1_{M_A} & \Sigma^2_{M_A} \\ \Sigma^3_{M_A} & \Sigma^4_{M_A} \end{pmatrix}$ and $\begin{pmatrix} \Sigma^1_{M_B} & \Sigma^2_{M_B} \\ \Sigma^3_{M_B} & \Sigma^4_{M_B} \end{pmatrix}$, respectively. Using $X^{-1} = X^T = \begin{pmatrix} R_X^T & -R_X^T t_X \\ 0 & 1 \end{pmatrix}$, then we can write the first two blocks of (16b) as follows,

$$ \Sigma^1_{M_B} = R_X^T \Sigma^1_{M_A} R_X \tag{18a} $$
$$ \Sigma^2_{M_B} = R_X^T \Sigma^1_{M_A} R_X (\widehat{R_X^T t_X}) + R_X^T \Sigma^2_{M_A} R_X \tag{18b} $$

The first blocks(18a) is eigendecomposed with the same diagonal matrix due to matrix similarity $\Sigma^1_{M_A} = Q_{M_A} \wedge Q^T_{M_A}$, $\Sigma^1_{M_B} = Q_{M_B} \wedge Q^T_{M_B}$. Then,

$$ \wedge = (Q^T_{M_A} R_X Q_{M_B}) \wedge (Q^T_{M_B} R_X^T Q_{M_A}) = P \wedge P^T \tag{19} $$

In $P = Q^T_{M_A} R_X Q_{M_B}$, Q_{M_A} and Q_{M_B} are constrained to be a rotation matrix and therefore $P \in \Omega$,

$$ \mathcal{Q} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} \tag{20} $$

Therefore, there are four possibilities of $R_X, R_X = Q_{M_A} P Q^T_{M_B}$. Then, from 18b,four t_X corresponding to R_X can be directly found. Furthermore,four candidate $R_Y and t_Y$ can be found. At last, there are four possibilities of solution $(X_i, Y_i), i = 1, 2, 3, 4$. where,

$$ X_i = \begin{pmatrix} R_X & t_X \\ \mathbf{0}^T & \mathbf{1} \end{pmatrix}, \quad Y_i = \begin{pmatrix} R_Y & t_Y \\ \mathbf{0}^T & \mathbf{1} \end{pmatrix} \tag{21} $$

Based on the screw theory, it is known that a homogeneous transformation H can be written in the form with four parameters $(\theta, d, \mathbf{n}, \mathbf{p})$.

$$ H = \begin{pmatrix} e^{\theta\hat{\mathbf{n}}} & (\mathbf{I}_3 - e^{\theta\hat{\mathbf{n}}})\mathbf{p} + d\mathbf{n} \\ \mathbf{0}^T & \mathbf{1} \end{pmatrix} \tag{22} $$

$AX = YB$ can be written as $AX = X(X^{-1}YB)$ and let $B' = X^{-1}YB$. In the form $AX = XB'$, there exit two Euclidean-Group Invariant relationships for one of four groups of $(A_i, B_i^k)(i = 1, \cdots, n; k = 1, 2, 3, 4)$ as follows,

$$ \theta_{A_i} = \theta_{B_i^k}, d_{A_i} = d_{B_i^k} \tag{23} $$

From among the four pairs (X_i, Y_i), we can find a correct solution to minimize the leat absolute deviations,

$$ (X, Y) = \underset{(X_i,Y_i)}{\mathbf{arg}min} \frac{1}{n}\sum_{i=1}^{n}(\| \theta_{A_i} - \theta_{B_i^k} \| + \| d_{A_i} - d_{B_i^k} \|) \tag{24} $$

III. Solution with unknown correspondence of A_i and B_i^k

In most cases, the homogeneous transformations with A and B are given based on the data from different sensors. Due to asynchronous timing of the measurement transmissions, the correspondences between A_i and B_i^k is unknown. The advantage of the above probabilistic solution lie that X and Y can be calculated even if without any a priori knowledge of the correspondence. However, there are still four possible candidate results (X_i, Y_i). Using Euclidean-Group Invariants, it is straightforward to determine which pair is the correct one if the correspondence between A_i and B_i^k can be known.

The Discrete Fourier transform (DFT) decomposes a time-domain signal into its constituent frequencies. The input is a finite list of equally spaced samples of a function. Given a discrete signal consisting of a sequence of N complex numbers $x_0, x_1, \cdots, x_{N_1}$, the DFT is denoted by $X_\kappa = \mathcal{F}x_n$

$$X_\kappa = \sum_{n=0}^{N-1} x_n \cdot exp(-i\frac{2\pi}{N}n\kappa) \tag{25}$$

And the Inverse Discrete Fourier transform (IDFT) denoted by

$$X_n = \frac{1}{N}\sum_{n=0}^{N-1} X_\kappa \cdot exp(i\frac{2\pi}{N}n\kappa) \tag{26}$$

The discrete convolution of two sequences f_n and g_n are defined

$$(f * g)(\tau) = \sum_{i=0}^{N} f(t_i)g(t_i - \tau) \tag{27}$$

In convolution theorem, the Fourier transform of a convolution is the product of the Fourier transforms, namely,

$$f * g = \mathcal{F}^{-1}[\mathcal{F}(f) \cdot \mathcal{F}(g)] \tag{28}$$

The correlation theorem indicates that the correlation function, $Corr(f, g)$, will have a large value at a shift vector if the two sequences f and g contain similar features. The correlation can be obtained based on the convolution theorem. The DFT of the correlation $Corr(f, g)$ is equal to the product of the DFT of a sequence f_n and the complex conjugate $\mathcal{F}^*$ of the DFT of the other sequence g_n.

$$Corr(f, g) = f \star g = \mathcal{F}^{-1}[\mathcal{F}(f) \cdot (\mathcal{F}(g))^*] \tag{29}$$

Compared with the standard time-domain convolution algorithm, the complexity of the convolution by multiplication in the frequency domain is significantly reduced with the help of the convolution theorem and the fast Fourier transform (FFT).

There are two sequences θ_{A_i} and $\theta_{B_i^k}$ from each pair (A_i, B_i^k). For homogeneous transformations from which the range of θ can vary, two sequences θ_{Ai} and $\theta_{B_i^k}$ can be first normalized.

$$\theta_1 = \frac{(\theta_{A_i} - \mu_{A_i})}{\sigma_{A_i}}, \theta_2 = \frac{(\theta_{B_i^k} - \mu_{B_i^k})}{\sigma_{B_i^k}} \tag{30}$$

where $\mu_{A_i}(\mu_{B_i^k})$ is the average of $\theta_{A_i}(\theta_{B_i^k})$ and $\sigma_{A_i}(\sigma_{B_i^k})$ is the standard deviation.

Here, the correlation function $Corr(\theta_1, \theta_2)$ is the function of the time sequence index (n) which describes the probability that these two sequences are separated by this particular unit. The location of the function maximum indicates the amount of shift, τ_{shift}, between the two sequence θ_{A_i} and $\theta_{B_i^k}$.

$$\tau_{shift} = \underset{index}{\mathbf{arg}} max(Corr(\theta_1, \theta_2)) \tag{31}$$

Therefore, the correspondence between the two sequences can be found. The data of θ_{A_i} or d_{A_i} are shifted by $-\tau_{shift}$ to obtain a sequence of new pairs $(\theta_{A_i}(i + \tau_{shift}), \theta_{B_i^k})$ and $(d_{A_i}(i + \tau_{shift}), d_{B_i^k})$, $max(i, i + \tau_{shift}) \leq i \leq min(i, i + \tau_{shift})$. The data stream can be shifted to reach correspondence once the shift is found and the correct solution can also be found by minimizing the least absolute deviations based on Euclidean-Group Invariants relations using the method in Section II.

IV. Simulation Studies

In the numerical experiments in this section, a homogeneous matrix is generated from a PUMA 560 robotic manipulator. X and Y are chosen from reference .

A. Results of solution with known correspondence

100 pose measurements with B_i closely around B_{start} was employed for generating 100 A_i. As a result by applying the above probabilistic method, four sequences $(\theta_{A_i}, \theta_{B_i^k})$ and $(d_{A_i}, d_{B_i^k})$ $(i = 1, \cdots, 100, k = 1, 2, 3, 4)$ can be obtained respectively, as shown in Fig. 2 and Fig. 3. Using $\theta_{A_i} - \theta_{B_i^k}$ and $d_{A_i} - d_{B_i^k}$ in Fig. 4 and Fig. 5, we can find the $B_i^k (k = 3)$ corresponding to the least sum of errors and then, (X_3, Y_3) is the desired solution.

From 1 measurements to 500 measurements, the rotational and translational error for X and Y are measured as $\| \; log^\vee(R^T_{X_{Solved}} R_{X_{true}}) \; \|$, $\| \; (t_{X_{Solved}} - t_{X_{true}}) \; \|$, $\| \, log^\vee(R^T_{Y_{Solved}} R_{Y_{true}}) \, \|$ and $\| \, (t_{Y_{Solved}} - t_{Y_{true}}) \, \|$ respectively as shown in Fig. 6.

Fig. 2. Calculated four pairs of rotational angles $(\theta_{A_i}, \theta_{B_i^k})(k = 1, 2, 3, 4)$ respectively from 100 measurements

Fig. 3. Calculated four pairs of translational displacement $(d_{A_i}, d_{B_i^k})(k = 1, 2, 3, 4)$ respectively from 100 measurements

Fig. 4. Calculated rotational angle deviation $\theta_{A_i} - \theta_{B_i^k}(k = 1, 2, 3, 4)$ from Fig. 2

Fig. 5. Calculated translational deviation $d_{A_i} - d_{B_i^k}(k = 1, 2, 3, 4)$ from Fig. 3

Fig. 6. Solution error with increasing pairs (A_i, B_i^k) from 1 measurements to 500 measurements

B. *Results of solution without known correspondence*

As shown in Fig. 7, the data streams of A were shifted by 10 units. The maximum of cross correlation can be used to find the corresponding shift, which is -10 shown in Fig. 8, representing the data stream of B_i^k has been shifted by -10 units respective to A_i . Therefore, we shift the data stream inversely to recover the correspondence for finding a correct solution satisfying Euclidean-Group Invariants.

Fig. 7. Shifted Data Streams of A and calculated B_i^k.

Fig. 8. The cross correlation of data streams of (A_i, B_i^k) respectively.

V. CONCLUSIONS

Conclusions

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