Albert-Ludwigs-Universität Freiburg, Institut für Informatik
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Lecture: Robot Mapping
Winter term 2013

Sheet 3

Topic: Extended Kalman Filter SLAM

Submission deadline: Nov. 11

Submit to: `robotmappingtutors@informatik.uni-freiburg.de`

Exercise 1: Bayes Filter and EKF

(a) Describe briefly the two main steps of the Bayes filter in your own words.

(b) Describe briefly the meaning of the following probability density functions: $p(x_t \mid u_t, x_{t-1})$, $p(z_t \mid x_t)$, and $\text{bel}(x_t)$, which are processed by the Bayes filter.

(c) Specify the (normal) distributions that correspond to the above mentioned three terms in EKF SLAM.

(d) Explain in a few sentences all of the components of the EKF SLAM algorithm, i. e., μ_t, Σ_t, g, G_t^x, G_t, R_t^x, R_t, h, H_t, Q_t, K_t and why they are needed. Specify the dimensionality of these components.

a) Step 1: Prediction Step.

- $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$
- This step estimates best estimation w/ motion model.

Step 2: Correction Step

- $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
- With measurement (sensor) model, estimates optimal estimation

b) ① $p(x_t \mid u_t, x_{t-1})$: State propagation w/ motion model.

② $p(z_t \mid x_t)$: pdf of sensor model.

③ $bel(x_t)$: pdf of current state using all the information

eg.) motion model, sensor model

(c) Motion model: $x_t = A_t x_{t-1} + B_t u_t + R_t$

Sensor model: $z_t = C_t x_t + Q_t$

① $p(x_t \mid u_t, x_{t-1})$

$$= \det(2\pi R_t)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right)$$

② $p(z_t \mid x_t)$

$$= \det(2\pi Q_t)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1} (z_t - C_t x_t)\right)$$

③ $bel(x_t) = \eta\, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$

$$= \det(2\pi \Sigma_t)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x_t - \mu_t)^T \Sigma_t^{-1} (x_t - \mu_t)\right)$$

where, i) $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$

ii) $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

iii) $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$

☆ iv) $\mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t)$

☆ v) $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

(d) ① $\mu_t \in \mathbb{R}^{3+2N}$: mean of current state estimation

② $\Sigma_t \in \mathbb{R}^{(3+2N)\times(3+2N)}$: covariance "

③ g : nonlinear function of motion model

④ $G_t^x \in \mathbb{R}^{3\times3}$: Jacobian of robot position state.

⑤ $G_t \in \mathbb{R}^{(3+2N)\times(3+2N)}$: Jacobian of whole state.

⑥ $R_t^x \in \mathbb{R}^{3\times3}$: robot pose noise

⑦ $R_t \in \mathbb{R}^{(3+2N)\times(3+2N)}$: whole state noise

⑧ h : nonlinear function of sensor model

⑨ $H_t \in \mathbb{R}^{m\times(3+2N)}$: Jacobian of sensor measurement

⑩ $Q_t \in \mathbb{R}^{m\times m}$: sensor noise

⑪ $K_t \in \mathbb{R}^{(3+2N)\times m}$: Kalman gain.

Exercise 2: Jacobians

(a) Derive the Jacobian matrix G_t^x of the noise-free motion function g with respect to the pose of the robot. Use the odometry motion model as in exercise sheet 1:

$$\begin{pmatrix} x_t \\ y_t \\ \theta_t \end{pmatrix} = \begin{pmatrix} x_{t-1} \\ y_{t-1} \\ \theta_{t-1} \end{pmatrix} + \begin{pmatrix} \delta_{trans}\cos(\theta_{t-1} + \delta_{rot1}) \\ \delta_{trans}\sin(\theta_{t-1} + \delta_{rot1}) \\ \delta_{rot1} + \delta_{rot2} \end{pmatrix}.$$

Do not use Octave for this part of the exercise.

(b) Derive the Jacobian matrix ${}^{\text{low}}H_t^i$ of the noise-free sensor function h corresponding to the i^{th} measurement:

$$h(\bar{\mu}_t, j) = z_t^i = \begin{pmatrix} r_t^i \\ \phi_t^i \end{pmatrix} = \begin{pmatrix} \sqrt{(\bar{\mu}_{j,x} - \bar{\mu}_{t,x})^2 + (\bar{\mu}_{j,y} - \bar{\mu}_{t,y})^2} \\ \operatorname{atan2}(\bar{\mu}_{j,y} - \bar{\mu}_{t,y}, \bar{\mu}_{j,x} - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix},$$

where $(\bar{\mu}_{j,x}, \bar{\mu}_{j,y})^T$ is the pose of the j^{th} landmark, $(\bar{\mu}_{t,x}, \bar{\mu}_{t,y}, \bar{\mu}_{t,\theta})^T$ is the pose of the robot at time t, and r_t^i and ϕ_t^i are respectively the observed range and bearing of the landmark. Do not use Octave for this part of the exercise.

Hint: use $\frac{\partial}{\partial x}\operatorname{atan2}(y, x) = \frac{-y}{x^2+y^2}$, and $\frac{\partial}{\partial y}\operatorname{atan2}(y, x) = \frac{x}{x^2+y^2}$.

(a) $$G_t^x = \begin{bmatrix} 1 & 0 & -\delta_{trans}\sin(\theta_{t-1} + \delta_{rot1}) \\ 0 & 1 & \delta_{trans}\cos(\theta_{t-1} + \delta_{rot1}) \\ 0 & 0 & 1 \end{bmatrix}$$

(b) i) $$\frac{\partial r_t^i}{\partial \bar{\mu}_{t,x}} = \frac{1}{2} \cdot \frac{1}{r_t^i} 2(\bar{\mu}_{j,x} - \bar{\mu}_{t,x}) \times (-1)$$
$$= -\frac{1}{r_t^i}(\bar{\mu}_{j,x} - \bar{\mu}_{t,x})$$

ii) $$\frac{\partial r_t^i}{\partial \bar{\mu}_{t,y}} = -\frac{1}{r_t^i}(\bar{\mu}_{j,y} - \bar{\mu}_{t,y})$$

iii) $$\frac{\partial r_t^i}{\partial \bar{\mu}_{t,\theta}} = 0$$

iv) $$\frac{\partial r_t^i}{\partial \mu_{j,x}} = \frac{1}{r_t^i}(\bar{\mu}_{j,x} - \bar{\mu}_{t,x})$$ / v) $$\frac{\partial r_t^i}{\partial \mu_{j,y}} = \frac{1}{r_t^i}(\bar{\mu}_{j,y} - \bar{\mu}_{t,x})$$

vi) $\frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,x}} = -\frac{1}{r_t^{i2}}(\bar{\mu}_{j,y} - \bar{\mu}_{t,y}) \times (-1) = \frac{1}{r_t^{i2}}(\bar{\mu}_{j,y} - \bar{\mu}_{t,y})$

vii) $\frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,y}} = \frac{1}{r_t^{i2}}(\bar{\mu}_{j,x} - \bar{\mu}_{t,x}) \times (-1) = -\frac{1}{r_t^{i2}}(\bar{\mu}_{j,x} - \bar{\mu}_{t,x})$

viii) $\frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,\theta}} = -1$

ix) $\frac{\partial \phi_t^i}{\partial \bar{\mu}_{j,x}} = -\frac{1}{r_t^{i2}}(\bar{\mu}_{j,y} - \bar{\mu}_{t,y})$

x) $\frac{\partial \phi_t^i}{\partial \bar{\mu}_{j,y}} = \frac{1}{r_t^{i2}}(\bar{\mu}_{j,x} - \bar{\mu}_{t,x})$

$$\boxed{\begin{aligned} &\therefore \text{low } H_t^i = \frac{1}{q}\begin{pmatrix} -\sqrt{q}\,\delta_x & -\sqrt{q}\,\delta_y & 0 & +\sqrt{q}\,\delta_x & \sqrt{q}\,\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix} \\ &\text{where,} \quad \delta_x := \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ &\qquad\qquad \delta_y := \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \\ &\qquad\qquad q := \delta_x^2 + \delta_y^2 \end{aligned}}$$