

Planning with Attitude

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I. INTRODUCTION

Many useful robotic systems—including quadrotors, airplanes, satellites, autonomous underwater vehicles, and quadrupeds—can perform arbitrarily large three-dimensional translations and rotations as part of their normal operation. While simply representing the attitude of these systems is nontrivial, generating and tracking motion plans for floating-base systems is an even more challenging problem.

Many approaches **TODO: add some citations here** have been taken to address the problem of motion planning and control on the $SE(3)$ group; most of these rely heavily on ideas and notation from differential geometry. Despite some impressive results, many of these ideas have not been widely adopted by the robotics community, and many practitioners continue to ignore the group structure of rotations. One issue preventing wider adoption is that accounting for this structure requires low-level changes to the underlying optimization algorithms, which is difficult or impossible when relying on existing off-the-shelf solvers.

TODO: I'm worried that $SE(3)$ and $SO(3)$ results are getting confused/conflated here. I would explicitly talk about both, emphasize that we're handling $SO(3)$ with the quaternion stuff, and that $SE(3) = SO(3) \times \mathbb{R}^3$ is a straightforward extension.

To address this lack of solver support, we formulate a straightforward, generic method for adapting existing Newton and gradient-based algorithms to correctly account for the group structure of rotations. Based entirely on basic mathematical tools from linear algebra and calculus, our method is computationally efficient and should be both accessible and familiar to most roboticists. We apply this method to the ALTRO solver [1] and demonstrate its performance on several constrained trajectory optimization problems.

II. NOTATION AND CONVENTIONS

We begin by defining some useful conventions and notation. Attitude is defined as the rotation from the robot's body frame to a global inertial frame. We also define gradients to be row vectors, that is, for $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, $\frac{\partial f}{\partial x} \in \mathbb{R}^{1 \times n}$.

We represent—following the Hamilton convention—a quaternion $\mathbf{q} \in \mathbb{H}$ as a standard vector $q \in \mathbb{R}^4 := [q_s \ q_v^T]^T$ where $q_s \in \mathbb{R}$ and $q_v \in \mathbb{R}^3$ are the scalar and vector part of the quaternion, respectively. We denote the composition of two quaternions as $q_3 = q_2 \otimes q_1$.

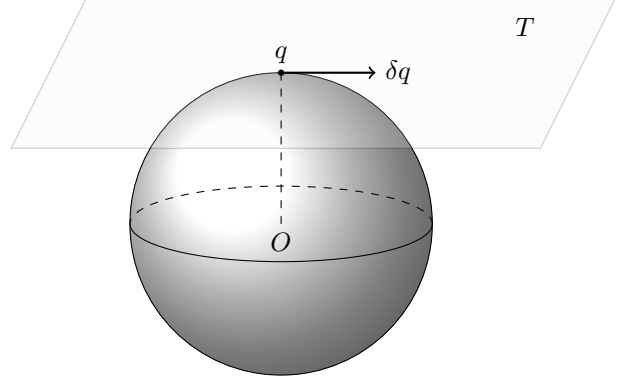


Fig. 1. When linearizing about a point q on an sphere \mathbb{S}^{n-1} in n -dimensional space, the tangent space T is a hyper-plane \mathbb{R}^{n-1} , illustrated here with $n = 3$. Therefore, when linearizing about a unit quaternion $q \in \mathbb{S}^3$, the space of differential rotations lives in \mathbb{R}^3 .

III. QUATERNION DIFFERENTIAL CALCULUS

Most modern methods for planning and control require derivatives with respect to the state vector. We present a simple but powerful method for taking derivatives of quaternion-valued state vectors. The extension of this method to more general state vectors that contain Cartesian products of $SO(3)$, $SE(3)$, and \mathbb{R}^n is straightforward.

The key idea of this section is that differential quaternions live in three-dimensional space instead of the four-dimensional space of quaternions. This can be seen graphically (see Figure 1) or deduced intuitively, since rotations are inherently three-dimensional and differential rotations should live in the same space as angular velocity, i.e. \mathbb{R}^3 .

In practice, we have found Rodrigues Parameters to be a very effective representation for three-dimensional differential rotations since they are computationally efficient and do not inherit the sign ambiguity of quaternions.

The mapping between a vector of Rodrigues parameters $\phi \in \mathbb{R}^3$ and a unit quaternion q is known as the Cayley map:

$$q = \varphi(\phi) = \frac{1}{\sqrt{1 + \|\phi\|^2}} \begin{bmatrix} 1 \\ \phi \end{bmatrix}. \quad (1)$$

We will also make use of the inverse Cayley map:

$$\phi = \varphi^{-1}(q) = \frac{q_v}{q_s}. \quad (2)$$

A. Jacobian of Vector-Valued Functions

When taking derivatives with respect to quaternions, we must take into account both the composition rule and the nonlinear mapping between the space of unit quaternions and our chosen three-parameter error representation.

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Let $\phi \in \mathbb{R}^3$ be a differential rotation applied to a function with unit quaternion inputs $y = h(q) : \mathbb{S}^3 \rightarrow \mathbb{R}^p$, such that

$$y + \delta y = h(q \otimes \varphi(\phi)) \approx h(q) + \nabla h(q)\phi. \quad (3)$$

We can calculate the Jacobian $\nabla h(q) \in \mathbb{R}^{p \times 3}$ by differentiating (3) with respect to ϕ , evaluated at $\phi = 0$:

$$\nabla h(q) = \frac{\partial h}{\partial q} \begin{bmatrix} -q_v^T \\ sI_3 + [q_v]^\times \end{bmatrix} := \frac{\partial h}{\partial q} G(q) \quad (4)$$

where $G(q) \in \mathbb{R}^{4 \times 3}$ is referred to as the *attitude Jacobian*, which essentially becomes a “conversion factor” allowing us to apply results from standard vector calculus to the space of unit quaternions. This form is particularly useful in practice since $\partial h / \partial q \in \mathbb{R}^{p \times 4}$ can be obtained using finite difference or automatic differentiation. As an aside, although we have used Rodrigues parameters, $G(q)$ is actually the same (up to a constant scaling factor) for any choice of three-parameter attitude representation.

B. Other derivatives

Using similar techniques, we find useful expressions for the Hessian of a scalar-valued function (i.e. $p = 1$):

$$\nabla^2 h(q) = G(q)^T \frac{\partial^2 h}{\partial q^2} G(q) + I_3 \frac{\partial h}{\partial q} q, \quad (5)$$

and the Jacobian of a quaternion-valued function $q' = f(q) : \mathbb{S}^3 \rightarrow \mathbb{S}^3$:

$$\nabla f(q) = G(q')^T \frac{\partial f}{\partial q} G(q). \quad (6)$$

IV. FAST CONSTRAINED TRAJECTORY OPTIMIZATION ON $SE(3)$

A. Algorithmic Modifications

Here we briefly outline the changes to the ALTRO solver [1] to efficiently solve trajectory optimization problems for systems with 3D rotations in the state vector. The linearization of the nonlinear discrete dynamics function $x_{k+1} = f(x_k, u_k)$ is “corrected” using 6:

$$A = G(f(x, u))^T \frac{\partial f}{\partial x} G(x); \quad B = G(f(x, u))^T \frac{\partial f}{\partial u}. \quad (7)$$

We can similarly compute the linearization of a nonlinear constraint function $c(x, u)$ using 4:

$$c_x = \frac{\partial c}{\partial x} G(x); \quad c_u = \frac{\partial c}{\partial u}. \quad (8)$$

Lastly, we adjust the computation of the quadratic expansion of the augmented Lagrangian **TODO: I would leave out the AL stuff and just talk about a generic cost function here to make it more general** cost function $J(x, u) + \lambda^T c(x, u) +$

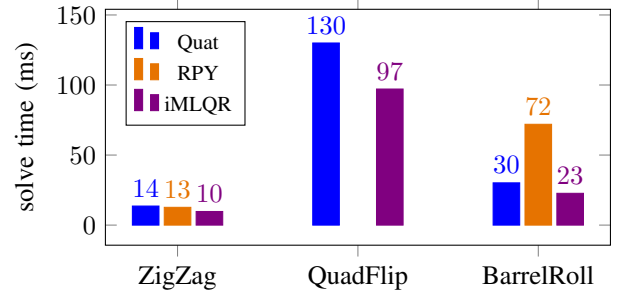


Fig. 2. Timing results for the airplane barrel roll, quadrotor flip, and quadrotor zig-zag examples using iLQR with roll-pitch-yaw Euler angles (RPY), quaternions, and the current method. The timing result for the Quadrotor flip with Euler angles is omitted since it failed to converge.

$\frac{1}{2}c(x, u)^T I_\mu c(x, u)$ using 4 and 5:

$$Q = G(x)^T \frac{\partial^2 J}{\partial x^2} G(x) + c_x^T I_\mu c_x + I_3 \frac{\partial J}{\partial x} x \quad (9a)$$

$$R = \frac{\partial^2 J}{\partial u^2} + c_u^T I_\mu c_u \quad (9b)$$

$$H = \frac{\partial^2 J}{\partial u \partial x} G(x) + c_u^T I_\mu c_x \quad (9c)$$

$$q = G(x)^T \frac{\partial J}{\partial x} + c_x^T (I_\mu c(x, u) + \lambda) \quad (9d)$$

$$r = \frac{\partial J}{\partial u} + c_u^T (I_\mu c(x, u) + \lambda). \quad (9e)$$

Here we define the *state attitude Jacobian* $G(x)$ to be a block-diagonal matrix where the block is an identity matrix for any vector-valued state and $G(q)$ for any quaternion in the state vector. We refer the reader to the original ALTRO paper [1] for further details on the algorithm.

These adjustments effectively modify the optimization so that it is performed on the quaternion error state (or Rodrigues Parameters) instead of the space of unit quaternions. In practice, this is analogous to the Multiplicative Extended Kalman Filter [2] in the state estimation community.

The algorithm only needs one small modification to the “forward pass,” where the trajectory is modified by simulating the system forward using the optimal feedback policy, $\delta u = K\delta x + d$, computed during the “backward” pass. Instead of computing δx using simple subtraction, we now compute the error state using the inverse Cayley map (2) for the quaternion states. The rest of the forward pass is unmodified. **TODO: I think you can safely omit some of these details and just refer to the ALTRO paper.**

B. Examples

To demonstrate the effectiveness of the proposed approach, we ran the modified version of ALTRO on several problems involving arbitrary 3D orientations. One required a quadrotor to do an aggressive “zig-zag” pattern, another to do a flip, and the last tasked a model airplane model to do a barrel roll. The timing results are shown in Figure 2.

TODO: It would be great to compare to both the minimal iLQR stuff and SNOPT or IPOPT with quaternion norm

constraints + slack variables (referred to as the “embedding approach”).

V. CONCLUSION

We have proposed a general, accessible, and computationally efficient approach to doing planning and control for systems with one or more unit quaternions in their state vectors. The approach allows for straightforward adaptation of many gradient-based methods for optimal control and motion planning, which we demonstrated on the ALTRO solver. We have demonstrated that correctly leveraging the group structure of rotations results in lower solve times and improved robustness of the optimization algorithm.

REFERENCES

- [1] Taylor A Howell, Brian E Jackson, and Zachary Manchester. “ALTRO: A fast solver for constrained trajectory optimization”. In: *2019 IEEE International Conference on Intelligent Robots and Systems, IEEE*. 2019.
- [2] F Landis Markley and John L Crassidis. *Fundamentals of spacecraft attitude determination and control*. Vol. 33. Springer, 2014.