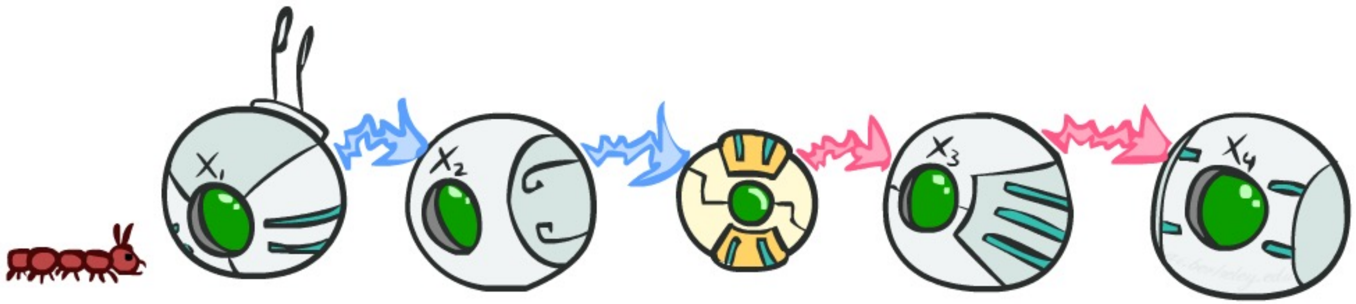
Project Presentation Arrangements

- Alpha Chess: December 25
  - Ordered by group ID
- Alpha Robot: December 27
  - Ordered by group ID
- Guidelines:
  - Slides must be written in English
  - 9 mins for each group
  - +5 points (total) if present in English

CS 3317: Artificial Intelligence

Markov Models



Instructor: Panpan Cai

[Slides adapted from UC Berkeley CS188]

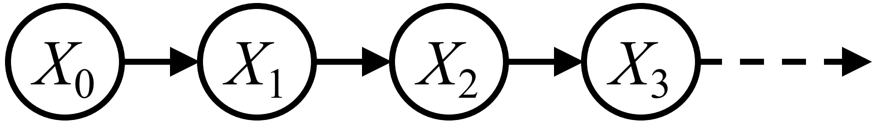
Uncertainty and Time

- Often, we want to reason about a *sequence* of observations where the state of the underlying system is *changing*
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
  - Global climate

- Need to introduce time into our models

Markov Models (aka Markov chain/process)

- Value of X at a given time is called the *state* (usually discrete, finite)



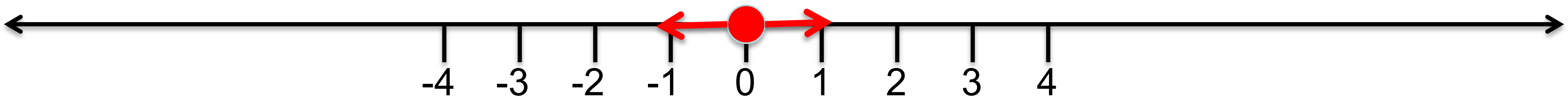
$P(X_0)$ $P(X_t \mid X_{t-1})$

- The *transition model* $P(X_t \mid X_{t-1})$ specifies how the state evolves over time
- *Stationarity* assumption: transition probabilities are the same at all times
- *Markov* assumption: “future is independent of the past given the present”
  - X_{t+1} is independent of $X_0,\ldots, X_{t-1}$ given X_t
  - This is a *first-order* Markov model (a *k*th-order model allows dependencies on *k* earlier steps)
- Joint distribution $P(X_0,\ldots, X_T) = P(X_0) \prod_t P(X_t \mid X_{t-1})$

Quiz: are Markov models just Bayes nets?

- Yes and no!
- Yes:
  - Directed acyclic graph
  - Joint = product of conditionals
- Not standard:
  - Infinitely many variables
    - joint probabilities become zero
  - Repetition of transition model
    - not part of standard Bayes net syntax

Example: Random walk in one dimension



- State: location on the unbounded integer line
- Initial probability: starts at 0
- Transition model: $P(X_t = k \pm 1 \mid X_{t-1} = k) = 0.5$
- Applications: particle motion in crystals, stock prices, gambling, genetics, etc.
- Interesting facts:
  - How far does it get as a function of t?
    - Expected distance is $O(\sqrt{t})$
  - Will it get back to 0 or can it never come back?
    - In 1D and 2D, returns w.p. 1; in 3D, returns w.p. 0.34053733

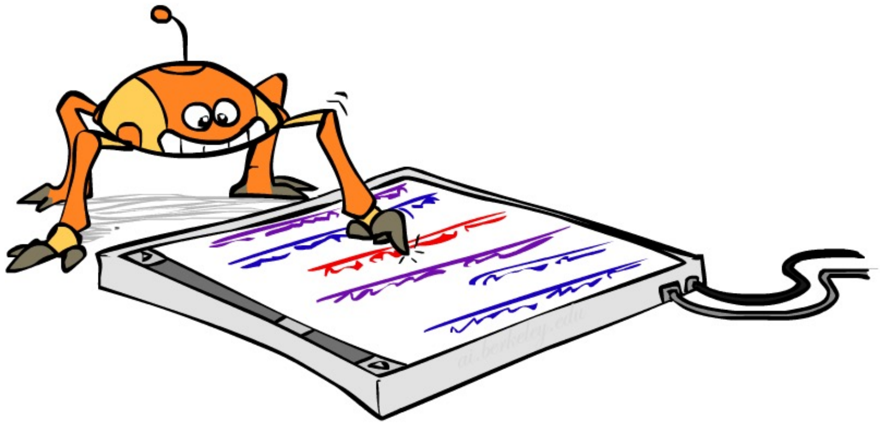
Example: n-gram models

Text: *"We call ourselves Homo sapiens—man the wise—because our intelligence is so important to us. For thousands of years, we have tried to understand how we think; that is, how a mere handful of matter can perceive, understand, predict, and manipulate a world far larger and more complicated than itself. …."*

- State: word at position t in text (can also be letters or *tokens*)
- Transition model (probabilities come from empirical frequencies):
  - Unigram (zero-order): $P(Word_t = i)$
    - *"logical are as are confusion a may right tries agent goal the was . . ."*
  - Bigram (first-order): $P(Word_t = i \mid Word_{t-1} = j)$
    - *"systems are very similar computational approach would be represented . . ."*
  - Trigram (second-order): $P(Word_t = i \mid Word_{t-1} = j, Word_{t-2} = k)$
    - *"planning and scheduling are integrated the success of naive bayes model is . . ."*
- Applications: text classification, spam detection, author identification, language classification, speech recognition

Example: Web browsing

- State: URL visited at step *t*
- Transition model:
  - With probability *p*, choose an outgoing link at random
  - With probability (1-*p*), choose an arbitrary new page
- Question: What is the *stationary distribution* over pages?
  - I.e., if the process runs forever, what fraction of time does it spend in any given page?
- Application: Google page rank



Example: Weather

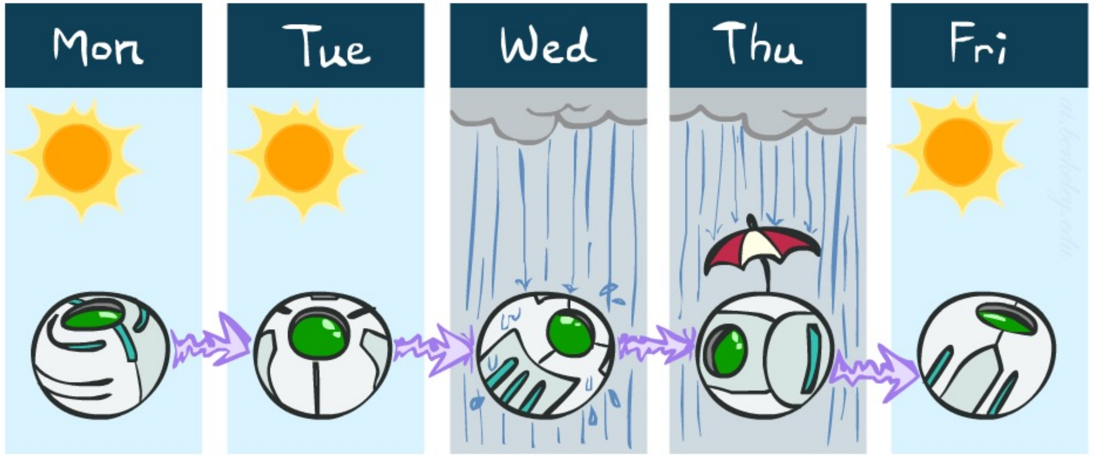
- States {rain, sun}

- Initial distribution $P(X_0)$

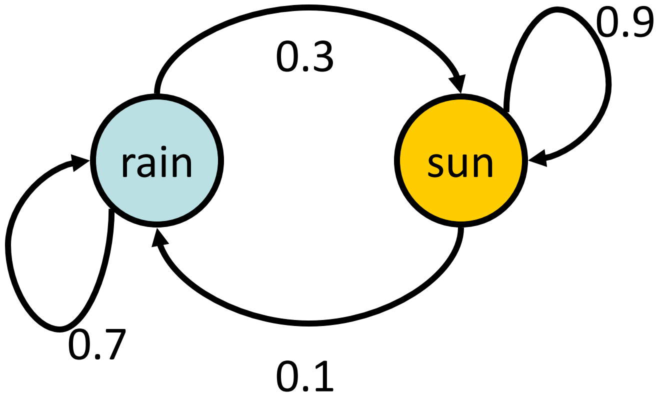
$P(X_0)$	
sun	rain
0.5	0.5

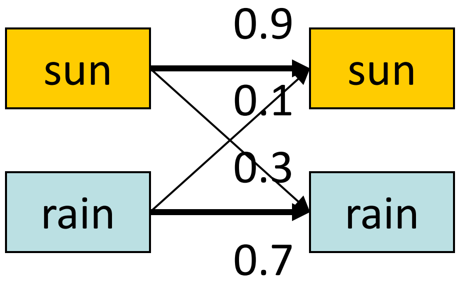
- Transition model $P(X_t \mid X_{t-1})$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



Two new ways of representing the same CPT

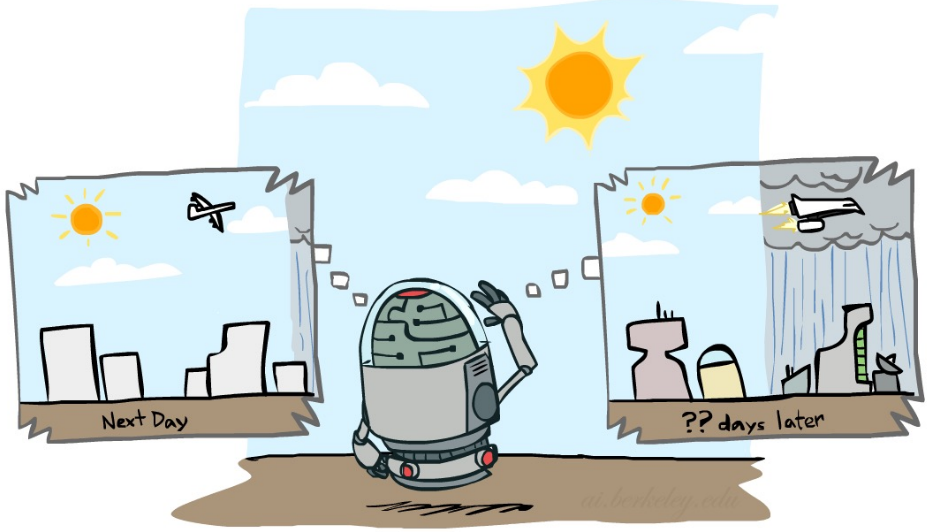




Weather prediction

- Time 0: <0.5,0.5>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



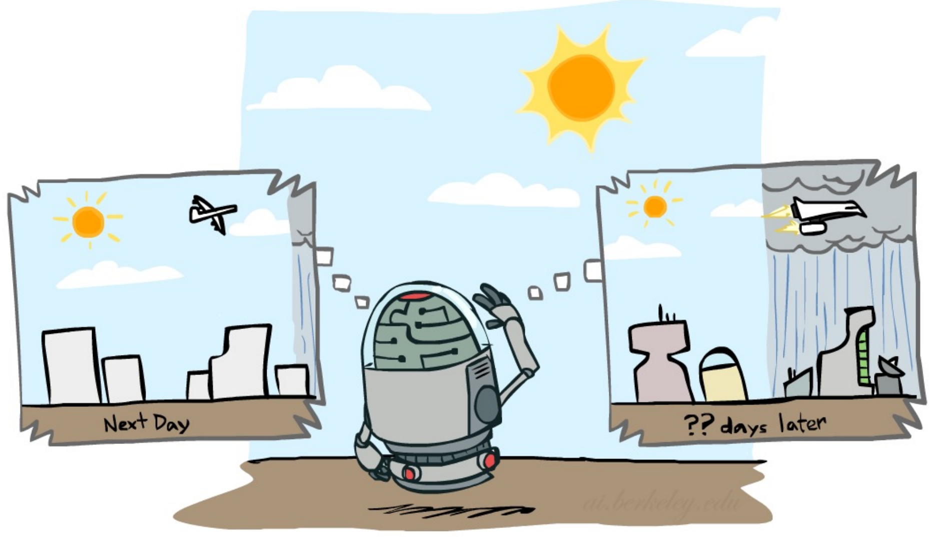
- What is the weather like at time 1?

$$P(X_1) = \sum_{x_0} P(X_1, X_0 = x_0)$$
$$= \sum_{x_0} P(X_0 = x_0)\, P(X_1 \mid X_0 = x_0)$$
$$= 0.5\langle 0.9, 0.1\rangle + 0.5\langle 0.3, 0.7\rangle = \langle 0.6, 0.4\rangle$$

Weather prediction, contd.

- Time 1: <0.6,0.4>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



- What is the weather like at time 2?

$P(X_2) = \sum_{x_1} P(X_2, X_1 = x_1)$

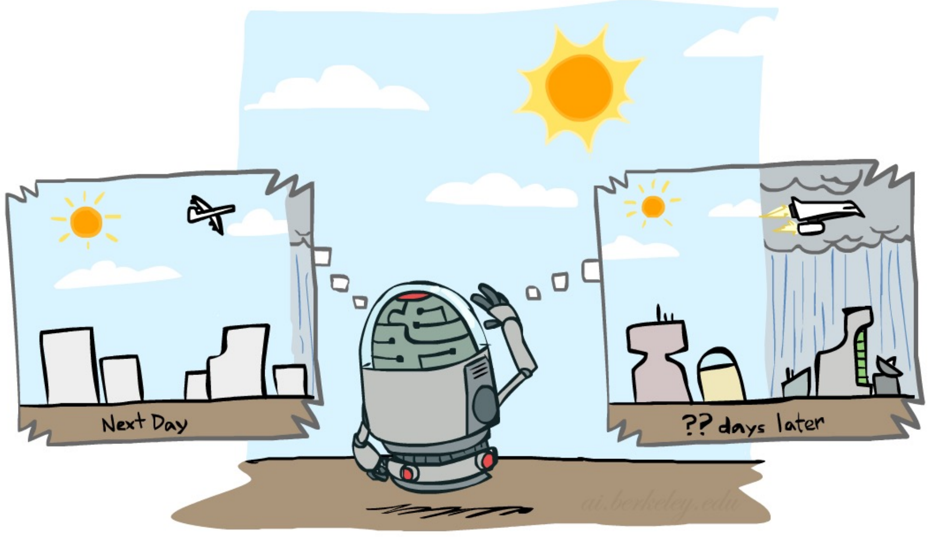
$= \sum_{x_1} P(X_1 = x_1)\, P(X_2 \mid X_1 = x_1)$

$= 0.6\langle 0.9, 0.1\rangle + 0.4\langle 0.3, 0.7\rangle = \langle 0.66, 0.34\rangle$

Weather prediction, contd.

- Time 2: <0.66,0.34>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



- What is the weather like at time 3?

$P(X_3) = \sum_{x_2} P(X_3, X_2=x_2)$

$= \sum_{x_2} P(X_2=x_2)\, P(X_3 \mid X_2=x_2)$

$=$ 0.66<0.9,0.1> + 0.34<0.3,0.7> $=$ <0.696,0.304>

Forward algorithm (simple form)

Probability from previous iteration

Transition model

- What is the state at time t?

  $P(X_t) = \sum_{x_{t-1}} P(X_t, X_{t-1}=x_{t-1})$

  $= \sum_{x_{t-1}} P(X_{t-1}=x_{t-1})\, P(X_t \mid X_{t-1}=x_{t-1})$

- Iterate this update starting at t=0
  - This is called a *recursive* update: $P_t = g(P_{t-1}) = g(g(g(g(\ldots P_0))))$

Write in linear algebra

- What is the weather like at time 2?
  - $P(X_2) = 0.6\langle 0.9, 0.1\rangle + 0.4\langle 0.3, 0.7\rangle = \langle 0.66, 0.34\rangle$

- In matrix-vector form:
  - $P(X_2) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix}$

- I.e., multiply by T^T, *transpose* of transition matrix

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

Stationary Distributions

- The limiting distribution is called the *stationary distribution* P_∞ of the chain
- It satisfies $P_\infty = P_{\infty+1} = T^{\top} P_\infty$
- Solving for P_∞ in the example:

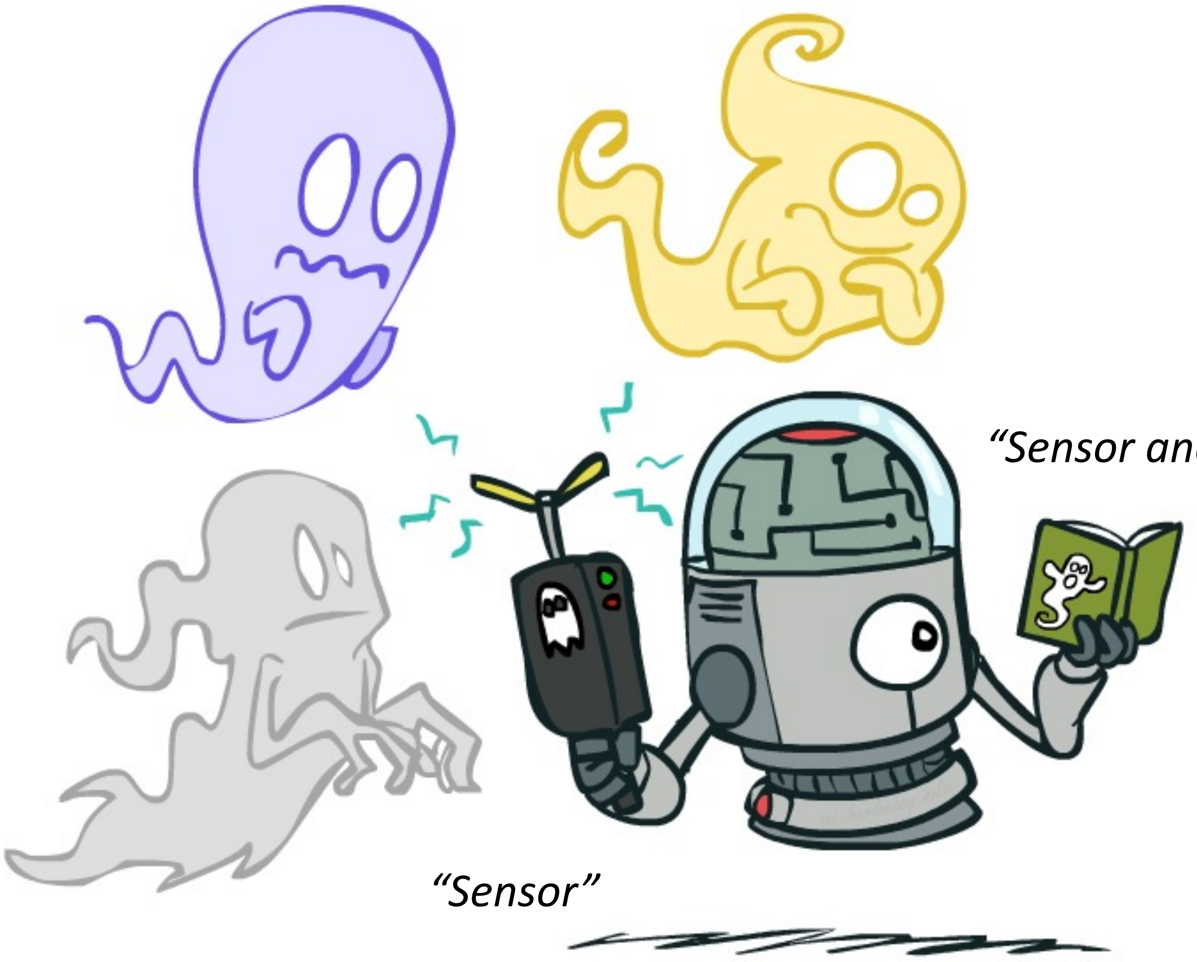
$$\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

$0.9p + 0.3(1-p) = p$

$p = 0.75$

Stationary distribution is <0.75,0.25> *regardless of starting distribution*

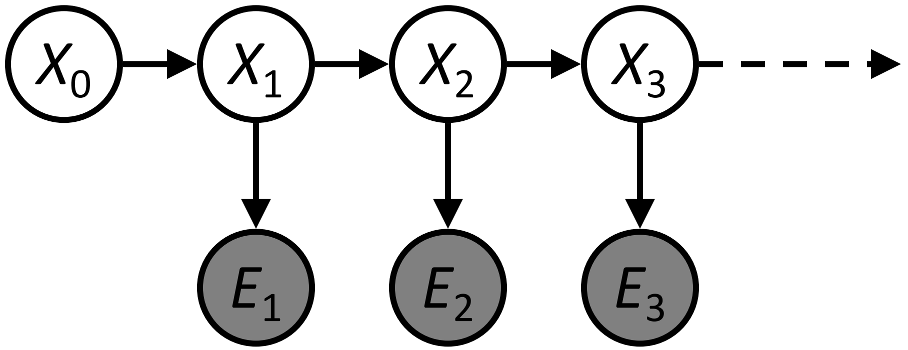
Hidden Markov Models



Hidden Markov Models

- Usually the true state is not observed directly

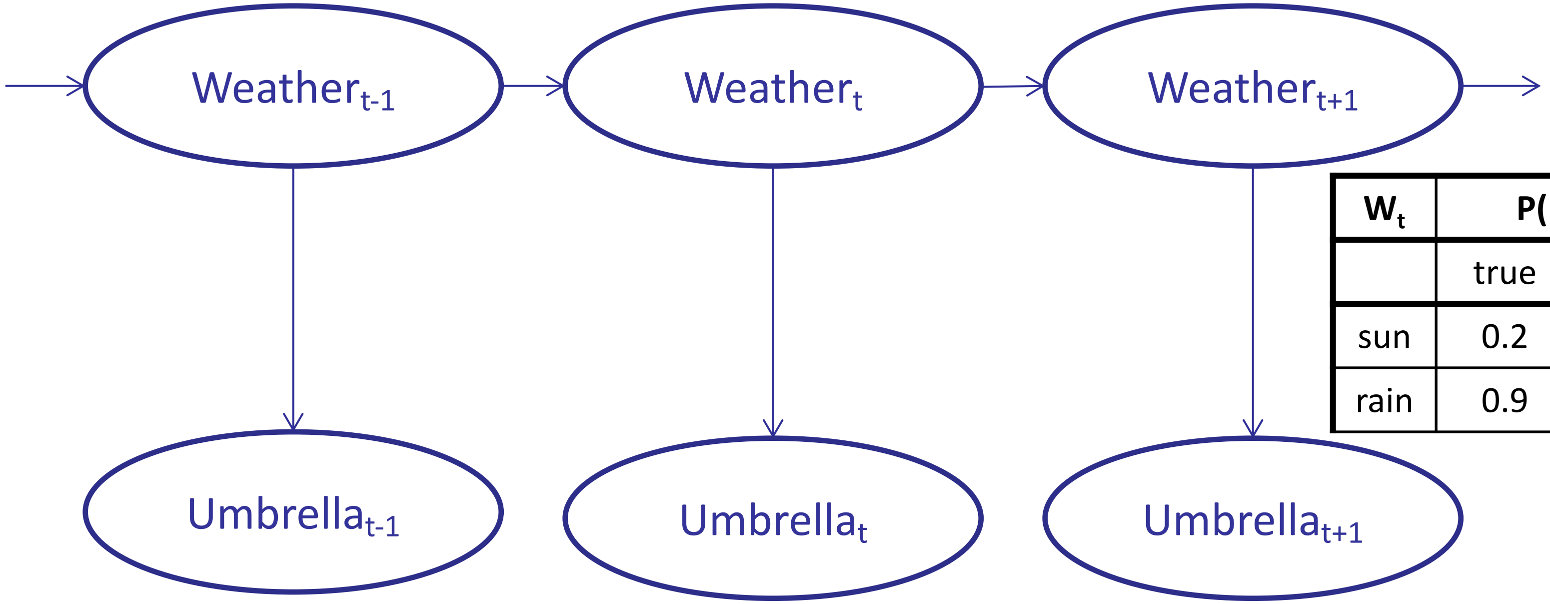
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states *X*
  - You observe evidence *E* at each time step
  - X_t is a *single discrete variable*
  - E_t may be continuous and may consist of several variables



Example: Weather HMM

- An HMM is defined by:
  - Initial distribution: $P(X_0)$
  - Transition model: $P(X_t \mid X_{t-1})$
  - Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

HMM as probability model

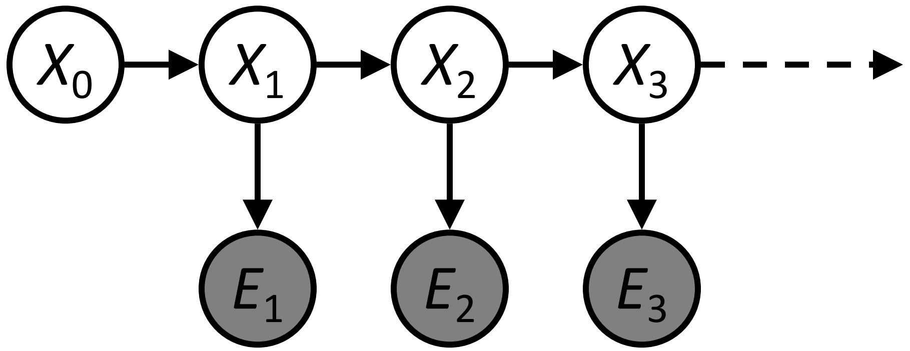
- Joint distribution for Markov model:

$$P(X_0,\ldots, X_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})$$

- Joint distribution for hidden Markov model:

$$P(X_0,X_1,\ldots, X_T,E_{1:T}) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})\, P(E_t \mid X_t)$$

- Conditional independences:
  - Future states are independent of the past given the present
  - Current evidence is independent of everything else given the current state
  - Are evidence variables independent of each other?



Useful notation:

$$X_{a:b} = X_a\,, X_{a+1}, \ldots, X_b$$

Real HMM Examples

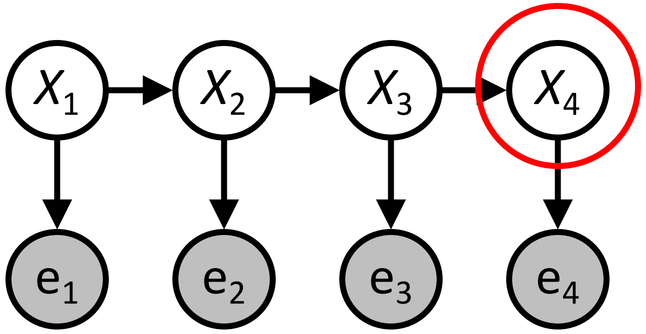
- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)
- Molecular biology:
  - Observations are nucleotides ACGT
  - States are coding/non-coding/start/stop/splice-site etc.

Inference tasks

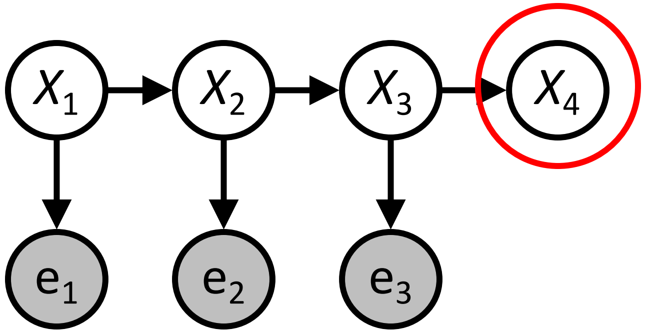
- *Filtering*: $P(X_t | e_{1:t})$
  - outputs a *belief state*—input to the decision process of a rational agent
  - POMDP planners often use a Bayesian filter as the "belief tracker"
- *Prediction*: $P(X_{t+k} | e_{1:t})$ for $k > 0$
  - predict future world states
  - like filtering without the evidence
- *Smoothing*: $P(X_k | e_{1:t})$ for $0 \leq k < t$
  - estimate past states better with hindsight
  - useful for collecting data for learning
- *Most likely explanation*: $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  - speech recognition, translation, …

Inference tasks

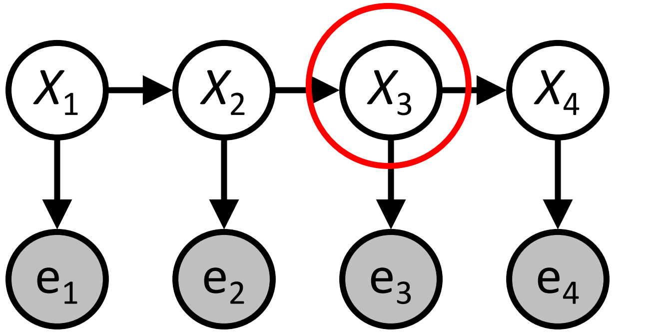
Filtering: $P(X_t | e_{1:t})$



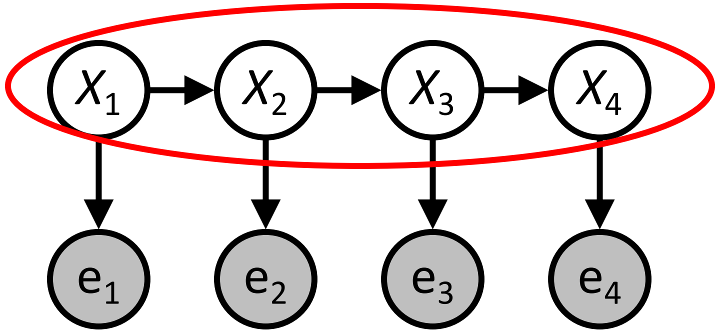
Prediction: $P(X_{t+k} | e_{1:t})$



Smoothing: $P(X_k | e_{1:t})$, $k<t$



Explanation: $P(X_{1:t} | e_{1:t})$



Filtering / Belief Tracking

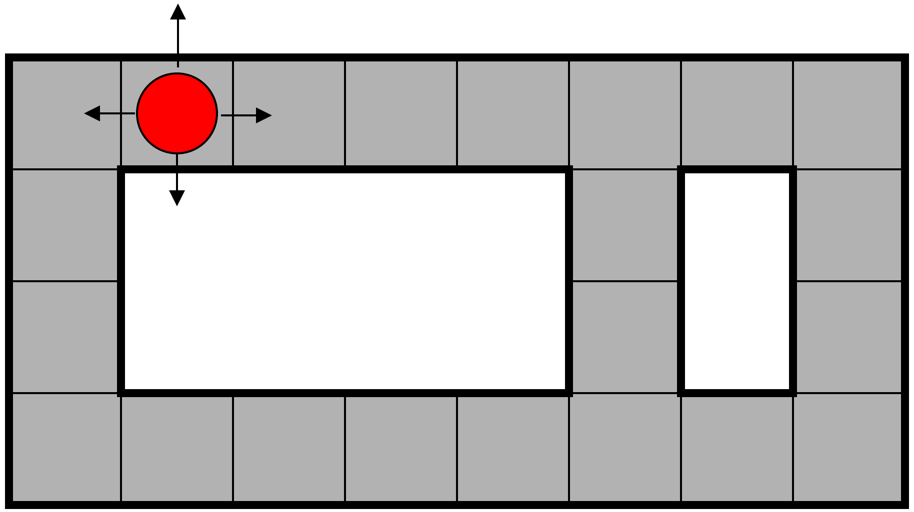
- Filtering, or belief tracking, or state estimation, is the task of maintaining the distribution $b_t = P(X_t | e_{1:t})$ over time

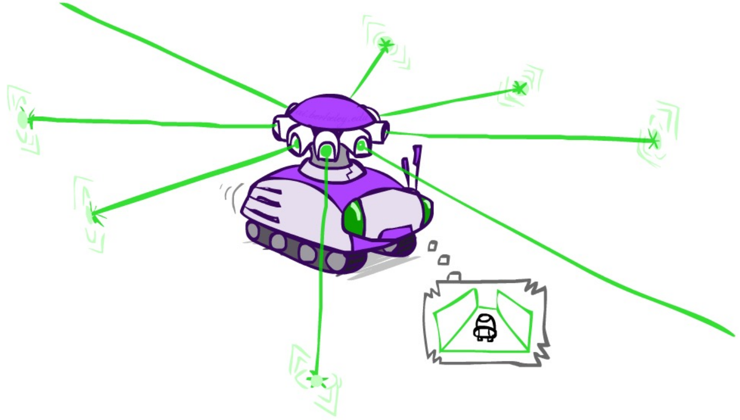
- We start with b_0 in an initial setting, usually uniform

- Filtering is a fundamental task in engineering and science
  - The Kalman filter (continuous variables, linear dynamics, Gaussian noise) was invented in 1960 and used for trajectory estimation in the Apollo program

Example: Robot Localization

Example from
Michael Pfeiffer



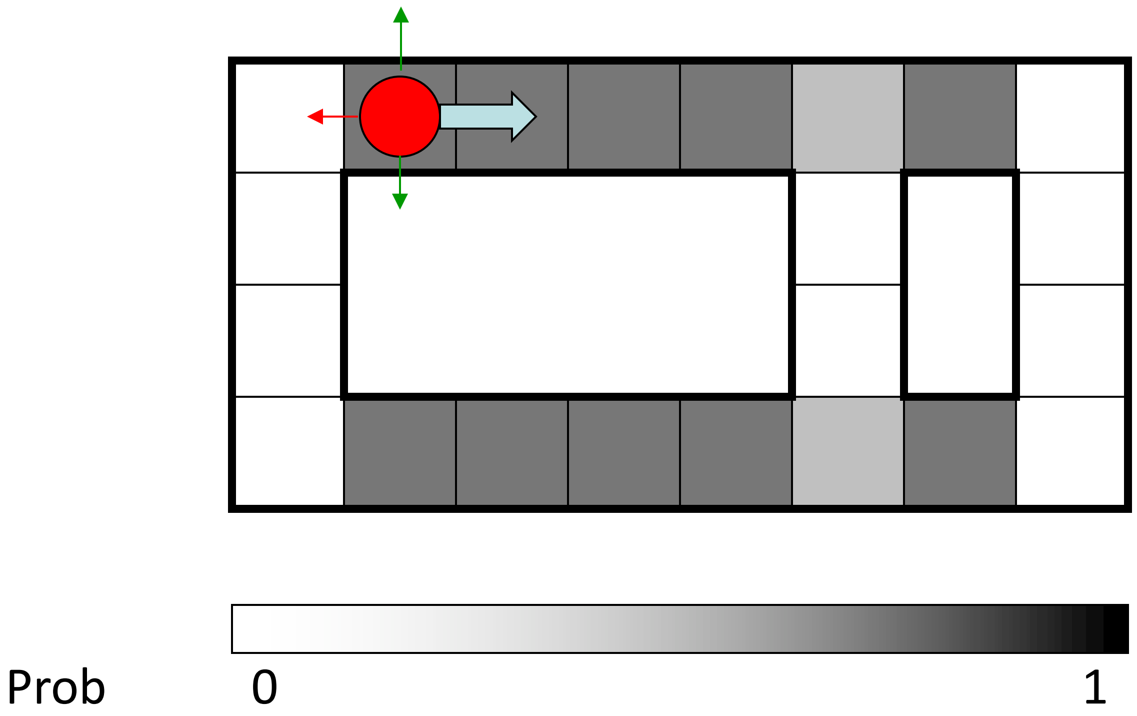


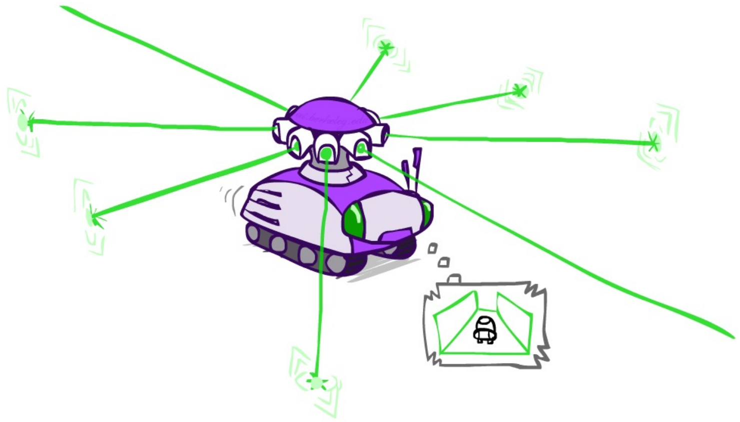
t=0

Sensor model: four bits for wall/no-wall in each direction, no more than 1 mistake

Transition model: action may fail with small prob.

Example: Robot Localization

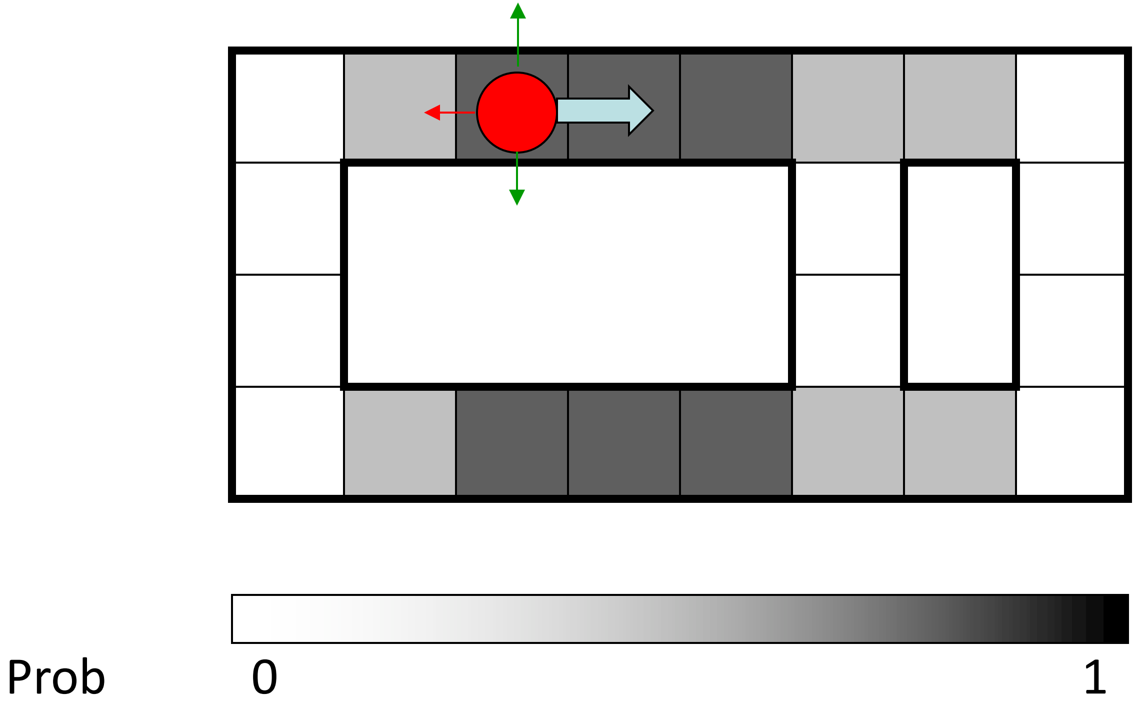


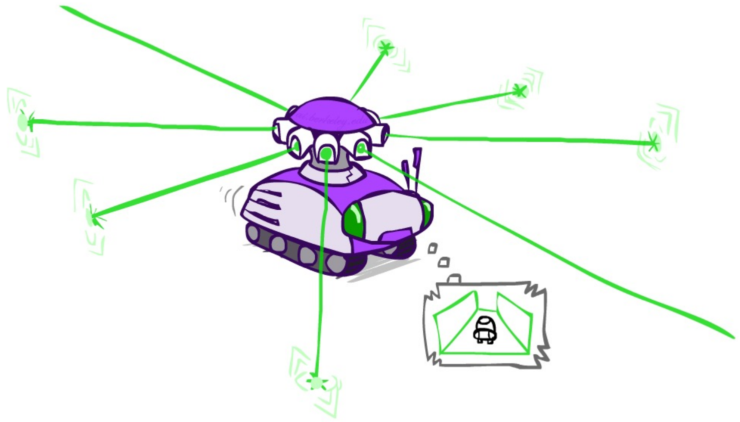


t=1

Lighter grey: was ***possible*** to get the reading, but ***less likely*** (required 1 mistake)

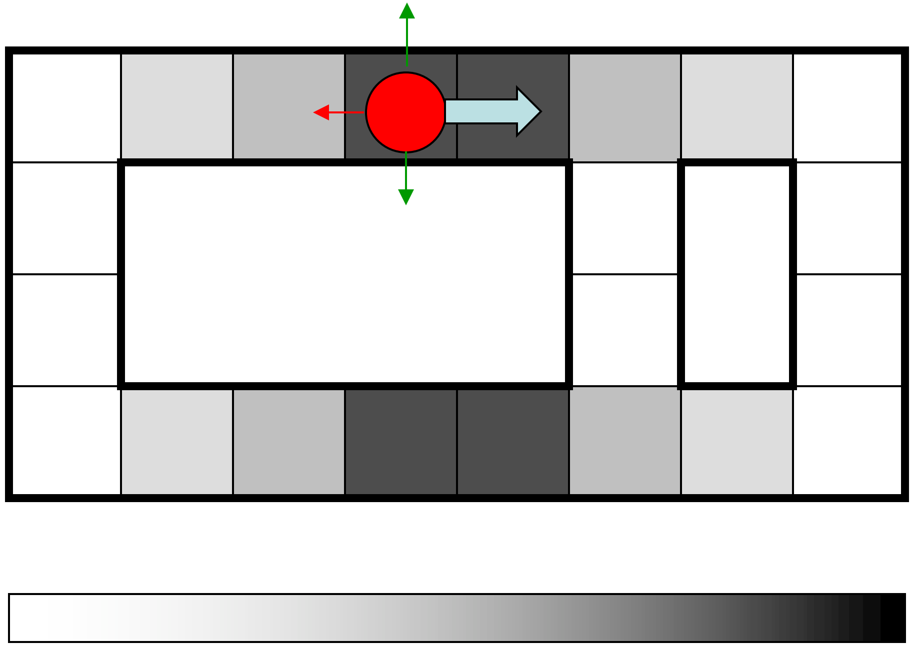
Example: Robot Localization

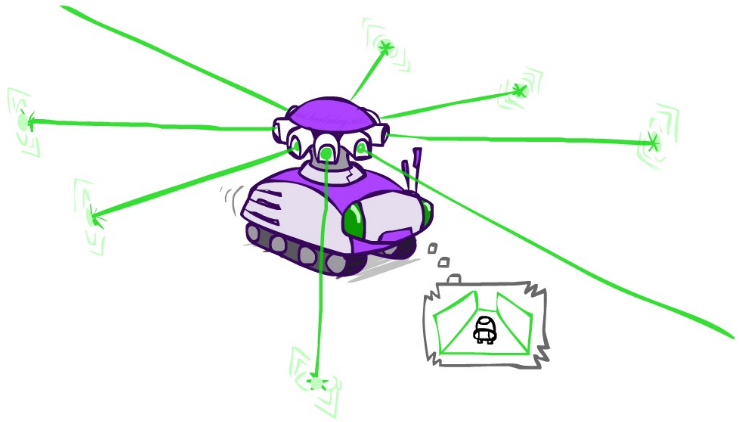




t=2

Example: Robot Localization



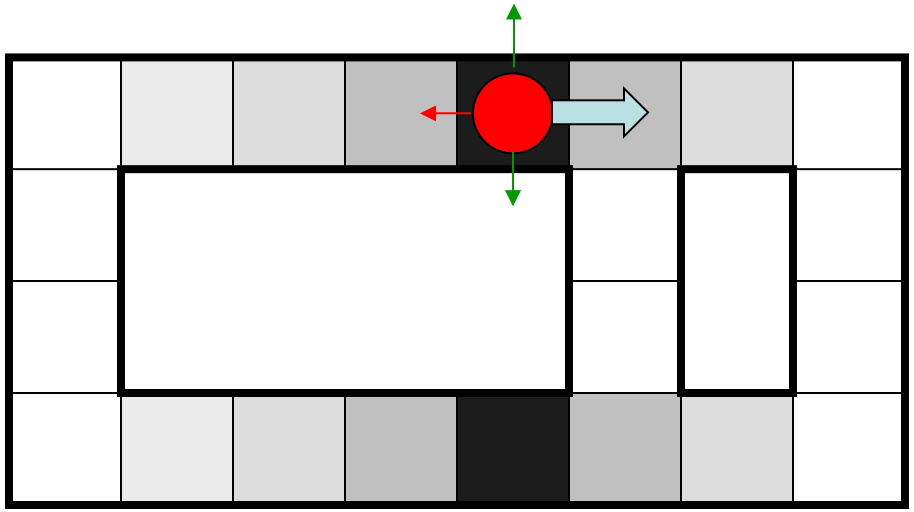


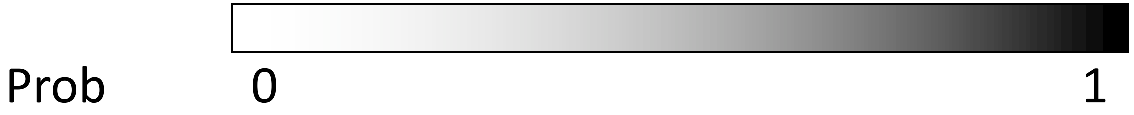
Prob

0 1

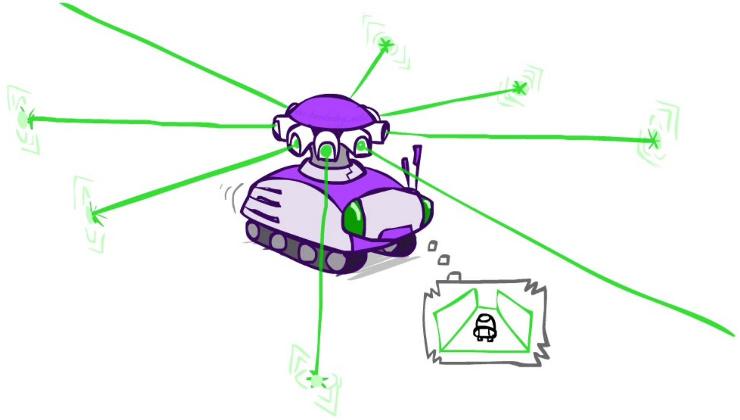
t=3

Example: Robot Localization

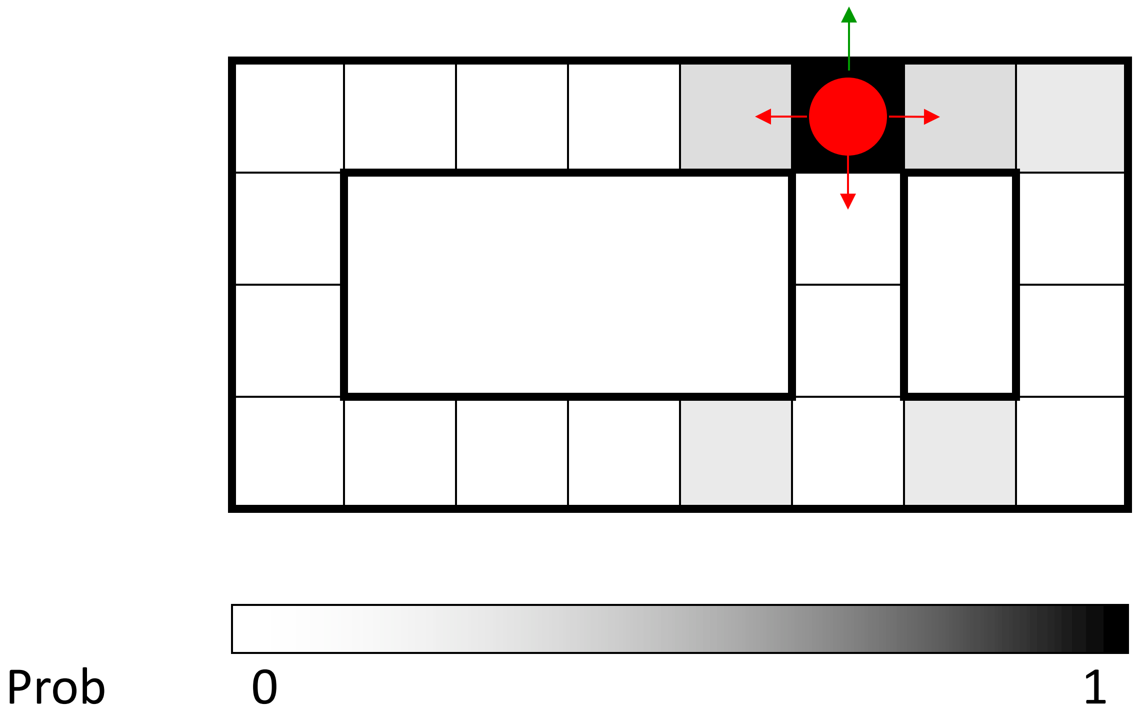


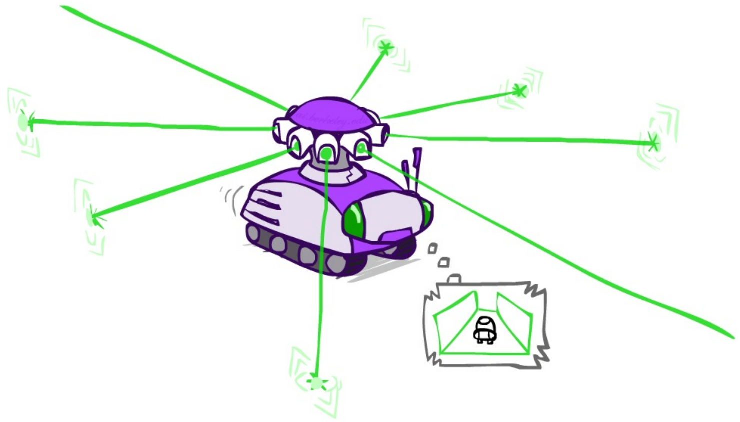


t=4



Example: Robot Localization





t=5

Filtering algorithm

- Derive a *recursive filtering* algorithm of the form:

  $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$

- $P(X_{t+1} | e_{1:t+1}) =$

Filtering algorithm

- Derive a *recursive filtering* algorithm of the form:

  $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$

- $P(X_{t+1} | e_{1:t+1}) = P(X_{t+1} | e_{t+1}, e_{1:t})$ — *Apply Bayes' rule*
- $= \alpha\, P(e_{t+1} | X_{t+1}, e_{1:t})\, P(X_{t+1} | e_{1:t})$ — *Apply conditional independence*
- $= \alpha\, P(e_{t+1} | X_{t+1})\, P(X_{t+1} | e_{1:t})$ — *Condition on X_t*
- $= \alpha\, P(e_{t+1} | X_{t+1}) \sum_{x_t} P(X_{t+1} | x_t, e_{1:t})\, P(x_t | e_{1:t})$
- $= \alpha\, P(e_{t+1} | X_{t+1}) \sum_{x_t} P(X_{t+1} | x_t)\, P(x_t | e_{1:t})$

Normalize

Update

Predict

Sensor model *Transition model* *Last belief*

Filtering algorithm

- $P(X_{t+1}|e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \underbrace{P(e_{t+1}|X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(X_{t+1}|\, x_t)\, P(x_t \mid e_{1:t})}_{\text{Predict}}$

- $\boldsymbol{b}_{t+1} = \text{FORWARD}(\boldsymbol{b}_t\, , e_{t+1})$
  - Cost per time step: $O(|X|^2)$ where $|X|$ is the number of states.
  - Time and space costs are *constant*, independent of t. (^_^)
  - $O(|X|^2)$ is infeasible for models with large state spaces. (T_T)
  - We will get to approximate filtering algorithms later. (^_^)

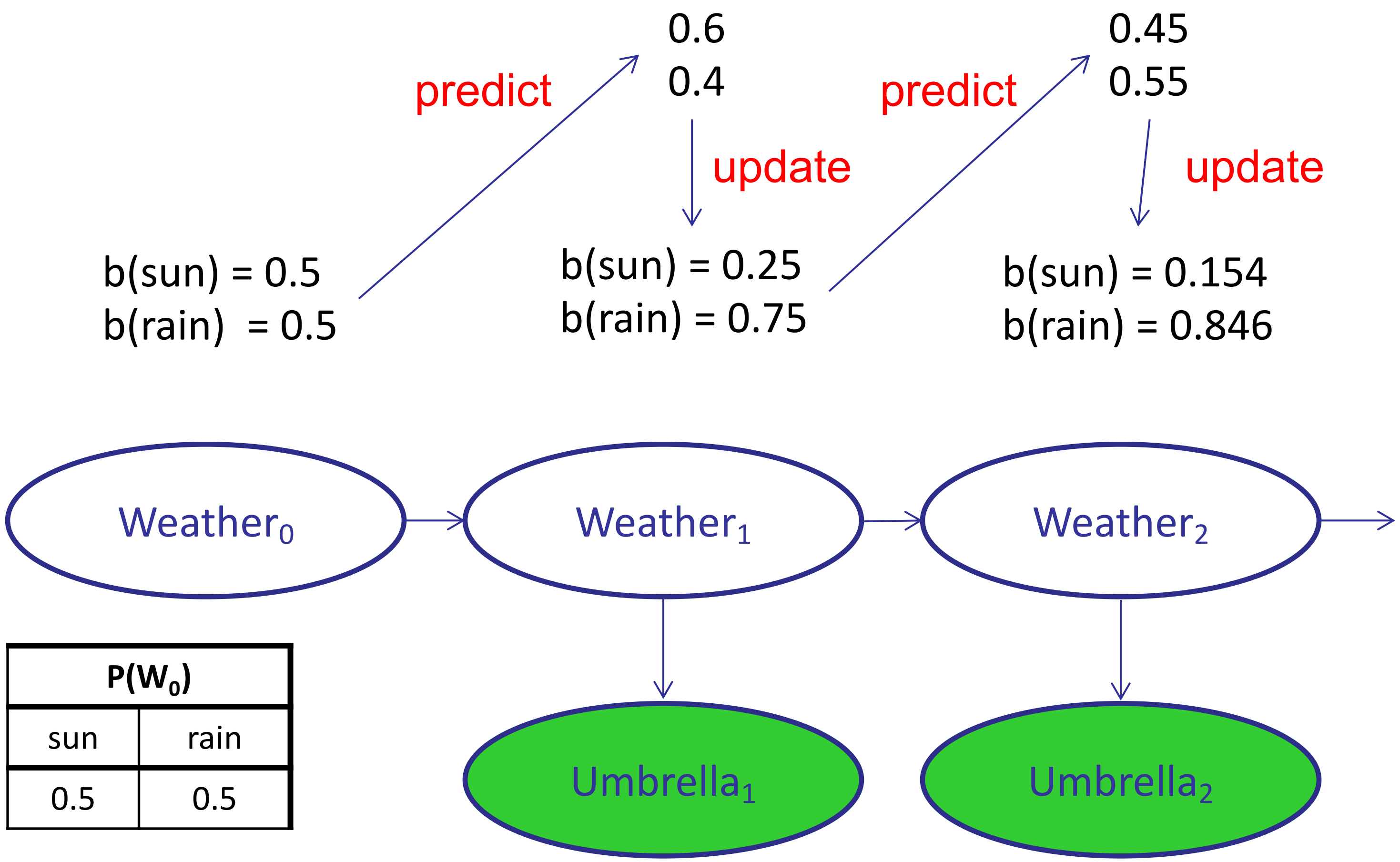
Write in linear algebra

- Transition matrix T, observation matrix O_t
  - Observation matrix contains likelihoods for E_t along its diagonal
  - E.g., for U_1 = true, $O_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.9 \end{pmatrix}$
- Filtering algorithm becomes
  - $\boldsymbol{b}_{t+1} = \alpha\, O_{t+1} T^{\mathsf{T}}\, \boldsymbol{b}_t$
  - easy to implement in Python or MATLAB

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Example: Weather HMM

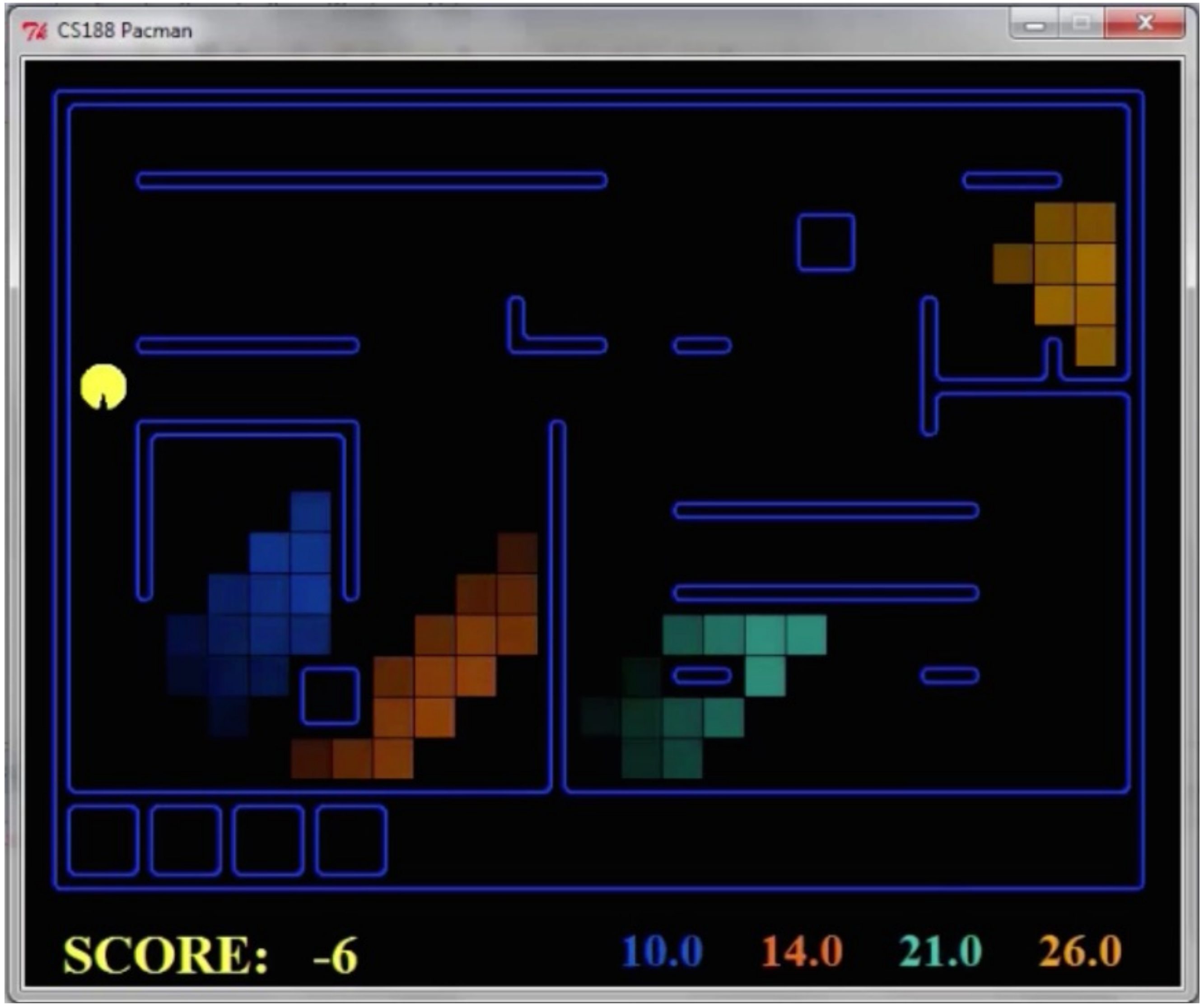


$P(W_0)$	
sun	rain
0.5	0.5

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Pacman – Hunting Invisible Ghosts with Sonar



[Demo: Pacman – Sonar – No Beliefs(L14D1)]

Video of Demo Pacman – Sonar