Q-learning as approximate Q-iteration

- Q-value iteration:

$$Q_{k+1}(s,a) \leftarrow \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \max_{a'} Q_k(s',a') \right]$$

- Approximate the expectation using samples and running average:
  - $Q(s,a) \leftarrow (1-\alpha) \cdot Q(s,a) + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a')]$

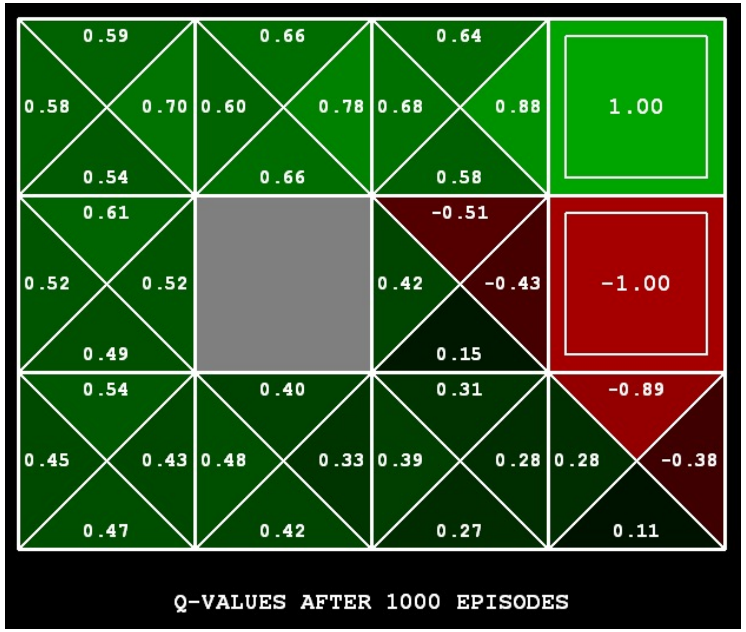
- Q-learning algorithm:
  - Receive a sample *(s,a,s',r)*
  - Get your old estimate: *Q(s,a)*
  - Construct your new sample:

    sample = $R(s,a,s') + \gamma \max_{a'} Q(s',a')$

  - Incorporate the new sample into the running average:

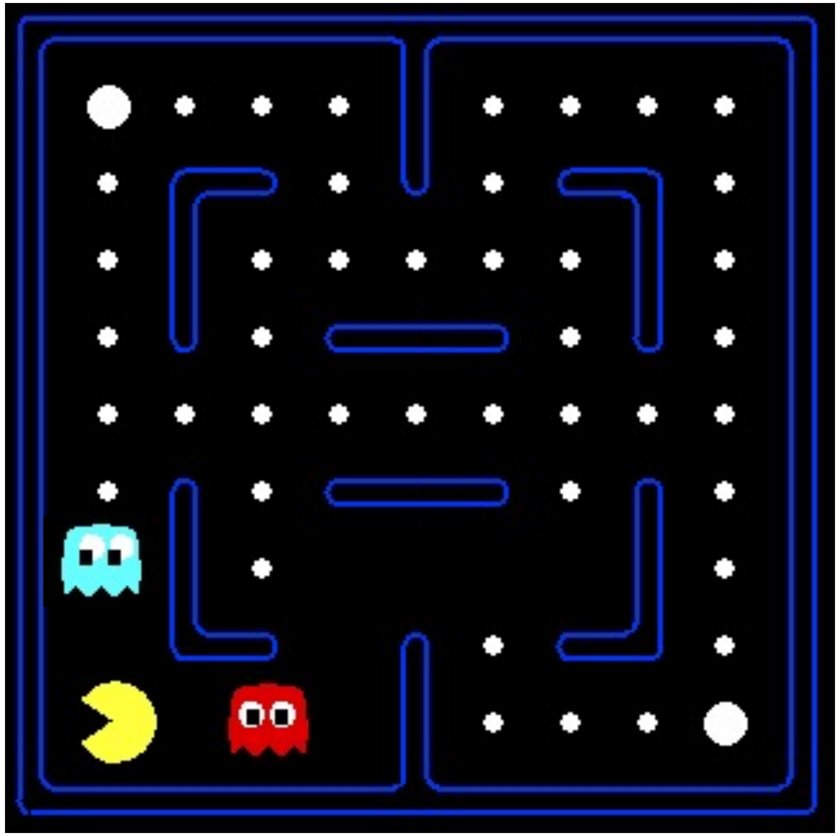
    $Q(s,a) \leftarrow (1-\alpha)\, Q(s,a) + \alpha \cdot [sample]$

Problem: *can not scale up to large state and action spaces!*



Feature-Based Representations

- Describe a state using a vector of ***features***
  - Features are functions from states to real numbers (often 0/1) that capture important properties of the state
  - Example features:
    - Distance to closest ghost f_{GST}
    - Distance to closest dot
    - Number of ghosts
    - 1 / (distance to closest dot) f_{DOT}
    - Is Pacman in a tunnel? (0/1)
    - …… etc.
  - Can also describe a q-state (s, a) with features
    - e.g., action moves closer to food, f_{DOT} gets higher



Linear Value Functions

- Express V and Q (approximately) as weighted linear functions of feature values:
  - $V_{\mathbf{w}}(s) = w_1 f_1(s) + w_2 f_2(s) + \ldots + w_n f_n(s)$
  - $Q_{\mathbf{w}}(s,a) = w_1 f_1(s,a) + w_2 f_2(s,a) + \ldots + w_n f_n(s,a)$

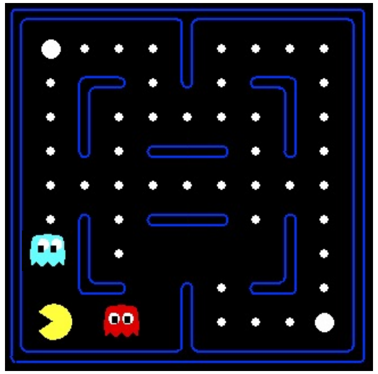
- Update linear value functions:
  - Original Q-learning directly updates Q's to reduce the error at s,a:
    - $Q(s,a) \leftarrow Q(s,a) + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)]$
  - Instead, we update the *weights* to reduce the error at s,a:
    - $w_i \leftarrow w_i + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)]\ \partial Q_{\mathbf{w}}(s,a)/\partial w_i$

      $= w_i + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)]\ f_i(s,a)$

Example: Q-Pacman

$Q(s,a) = 4.0\, f_{DOT}(s,a) - 1.0\, f_{GST}(s,a)$

s

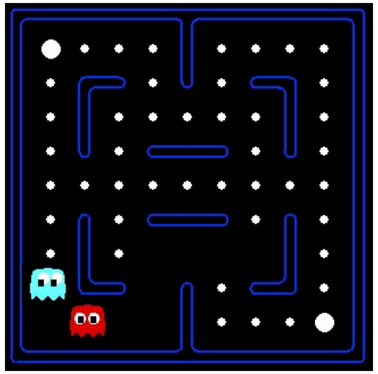


$f_{DOT}(s,\text{NORTH}) = 0.5$

$f_{GST}(s,\text{NORTH}) = 1.0$

a = NORTH
r = –500

s'



Original: $Q(s,\text{NORTH}) = +1$

$Q(s',\cdot) = 0$

Sample: $r + \gamma \max_{a'} Q(s',a') = -500 + 0$

difference = –501

$w_{DOT} \leftarrow 4.0 + \alpha[-501]0.5$

$w_{GST} \leftarrow -1.0 + \alpha[-501]1.0$

$Q(s,a) = 3.0\, f_{DOT}(s,a) - 3.0\, f_{GST}(s,a)$

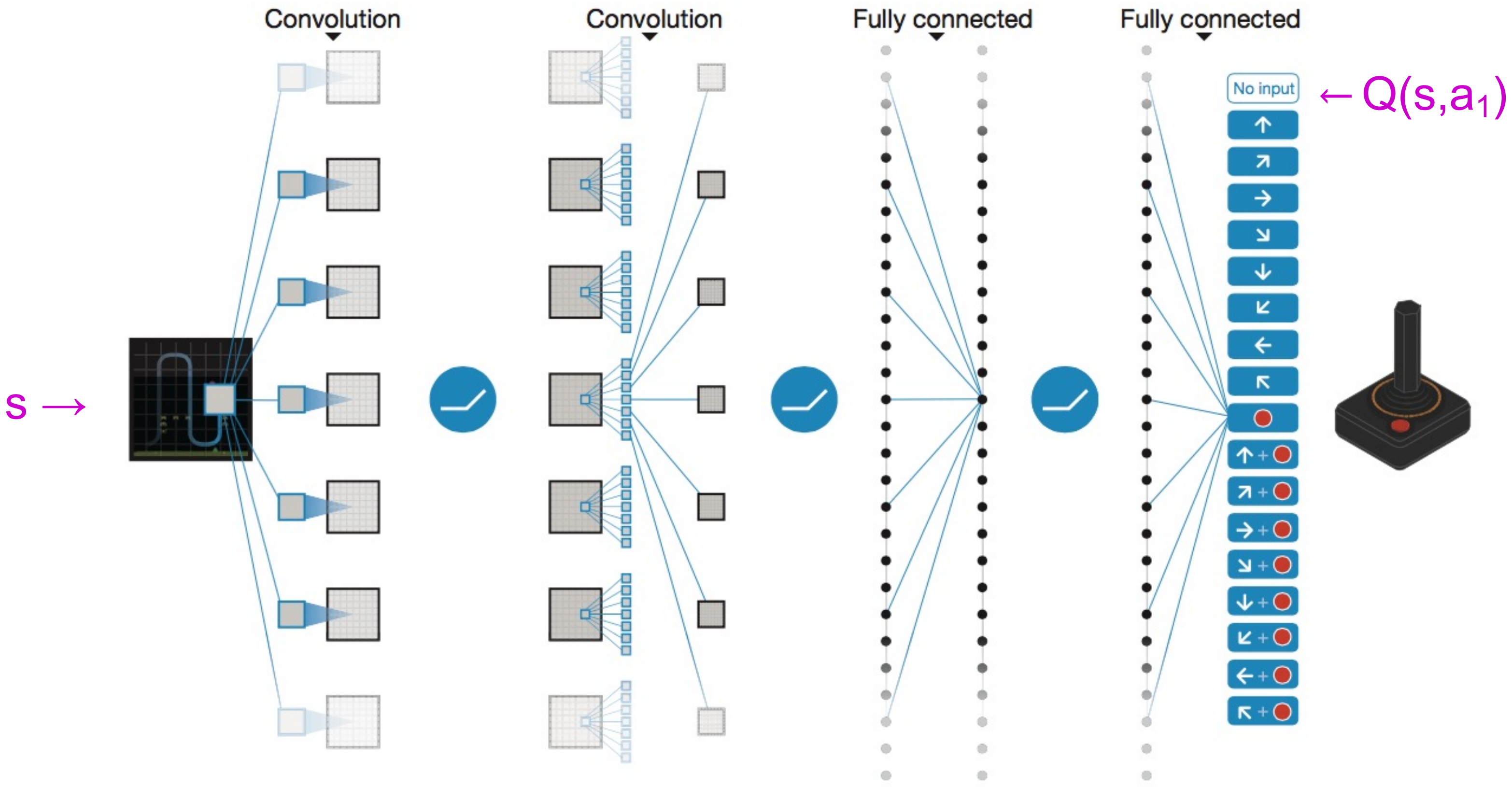
Demo Approximate Q-Learning -- Pacman

Nonlinear function approximators

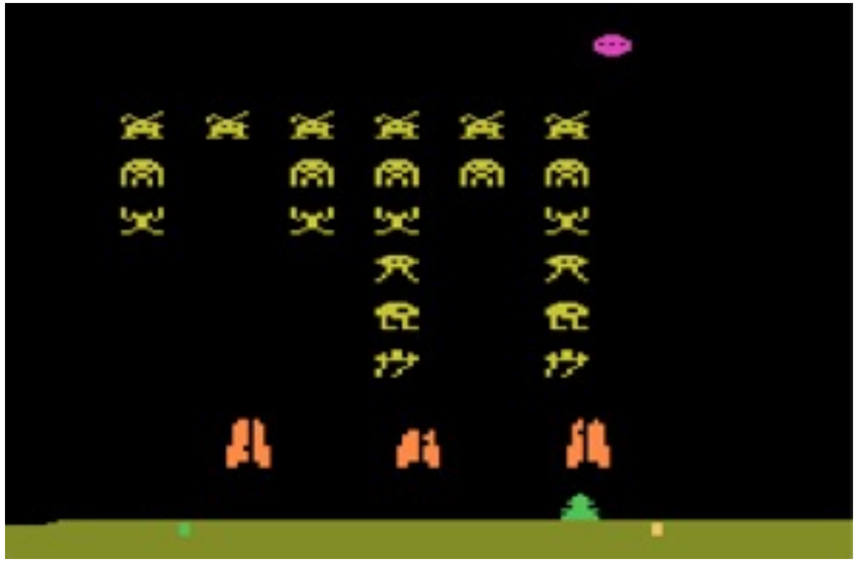
- The gradient-based update can be applied to ***any*** Q_w:
  - $w_i \leftarrow w_i + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)] \, \partial Q_w(s,a)/\partial w_i$
- Neural networks?
  - Back-propagation computes the gradient!
- **Hypothesis**: maybe we can get much better V or Q approximators using deep neural nets instead of linear functions

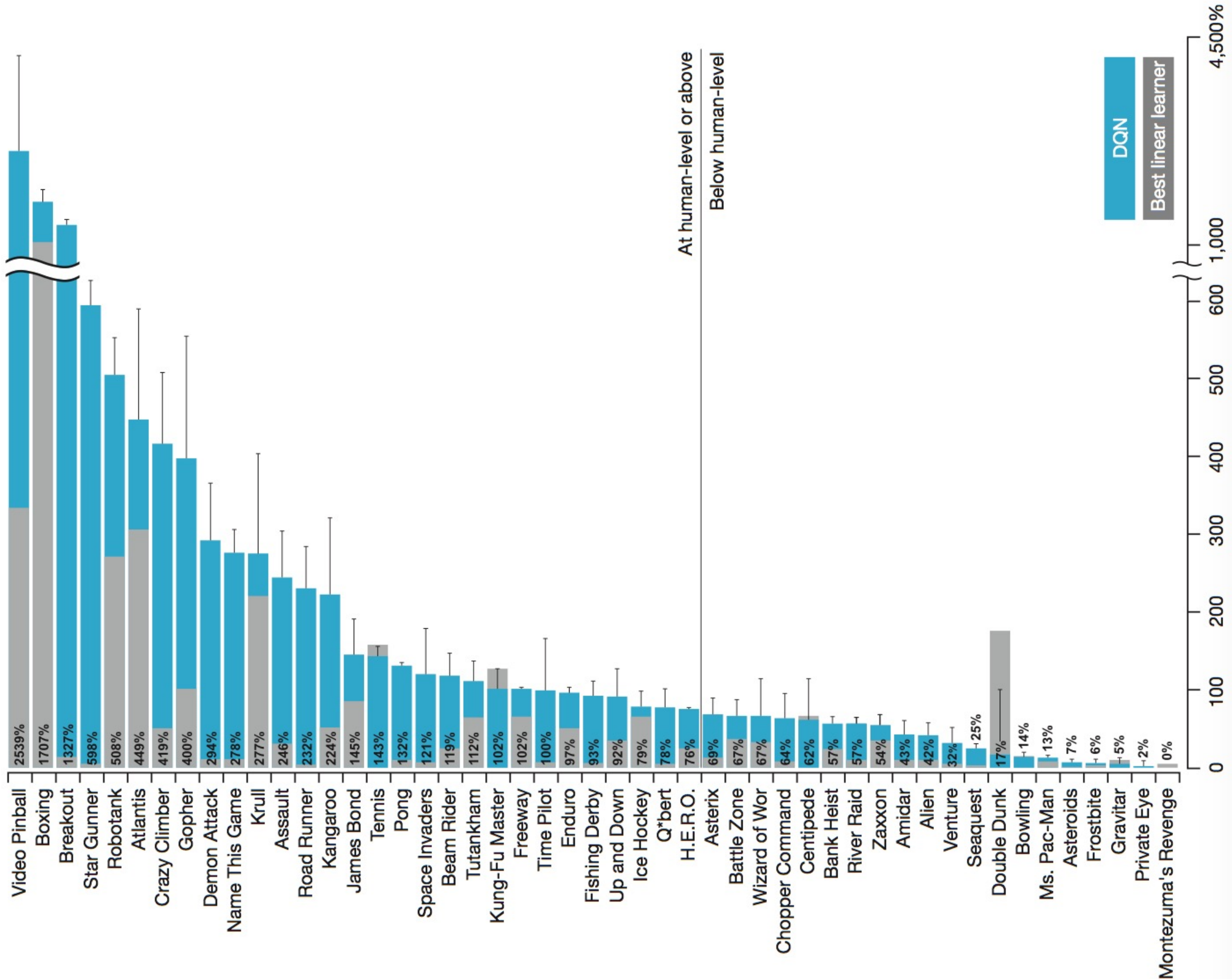
DeepMind DQN

- Used a deep neural network to represent Q:
  - Input: last 4 screen images (84x84 pixel values) + score
  - Output: Q values



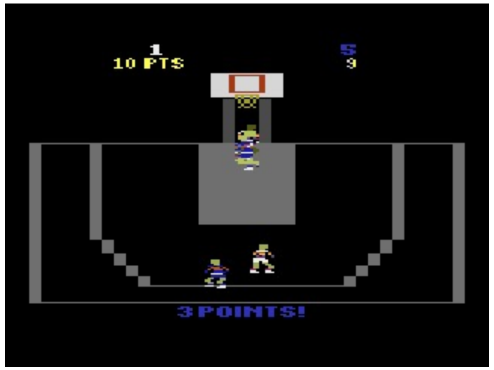
Play 49 Atari games with DQN





Video Pinball
Boxing
Breakout

Double dunk



Summary

- Exploration vs. exploitation
  - Exploration guided by unfamiliarity and potential
  - Appropriately designed bonuses tend to minimize regret
- Generalization allows RL to scale up to real problems
  - Represent V or Q with parameterized functions
  - Adjust parameters to reduce prediction error of samples

Next: Uncertainty

- The real world is rife with uncertainty!
  - E.g., if I set off 60 minutes before my flight, will I arrive in time?
- Common causes:
  - partial observability (road state, other drivers' plans, etc.)
  - noisy sensors (radio traffic reports, Google maps, etc.)
  - complexity of predicting other's behaviors (give way? compete? etc.)
  - lack of knowledge on world dynamics (need COVID test? A major event?)

Uncertainty

- Probabilistic assertions result from ***ignorance*** and ***laziness***
  - ***Ignorance***: lack sufficient information or understanding to make a definitive statement about a subject.
  - ***laziness***: avoid the overwhelming effort required to gain more precise or comprehensive information on a complex subject
  - ***Probabilistic models:*** make probabilistic statements or predictions

CS 3317: Artificial Intelligence

Probability

Instructors: **Cai Panpan**

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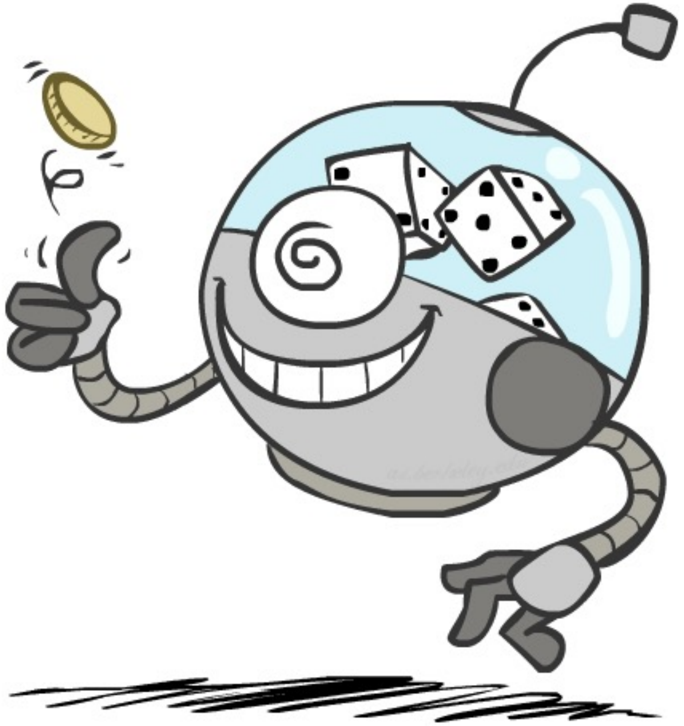
(slides adapted from UC Berkeley CS188)

Random Variables

- Random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot?
  - D = How long will it take to drive to work?
- Capital letters: Random variables
- Lowercase letters: values that the R.V. can take
  - $r \in \{+r, -r\}$
  - $t \in \{+t, -t\}$
  - $d \in [0, \infty)$



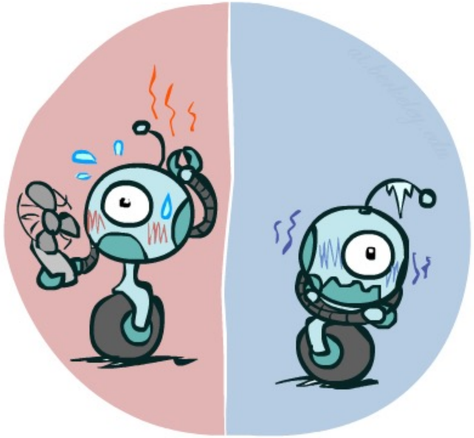
Probability Distributions

- Unobserved random variables have distributions. A distribution (for a discrete variable) is a *table* of probabilities of values:

  - Temperature:

    $P(T)$

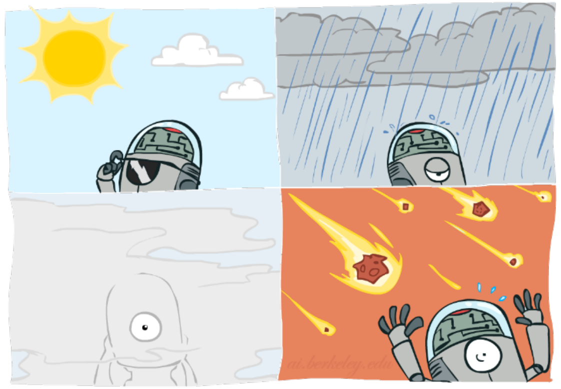
T	P
hot	0.5
cold	0.5



  - Weather:

    $P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0



- A probability is a single number $P(T=hot)=0.5$

- Must have: $\forall x \;\; P(X=x) \geq 0$ and $\sum_x P(X=x) = 1$

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots, X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

  - Must obey: $P(x_1, x_2, \ldots x_n) \geq 0$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

P(T,W)

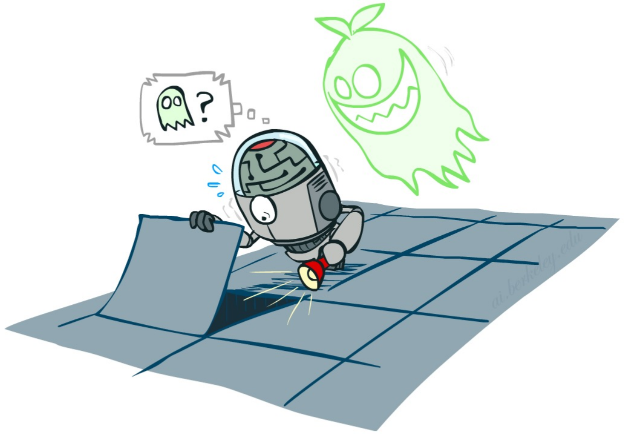
		Temperature	
		hot	cold
Weather	sun	0.45	0.15
	rain	0.02	0.08
	fog	0.03	0.27
	meteor	0.00	0.00

- Size of distribution if n variables with domain sizes d?
  - impractical to write out except for smallest ones!

$$d^n$$

Probabilistic Models

- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
  - Random variables with domains
  - Assignments are *outcomes*
  - Joint distributions: say whether assignments (outcomes) are likely
  - *Normalized:* sum to 1.0
  - Ideally: only certain variables directly interact

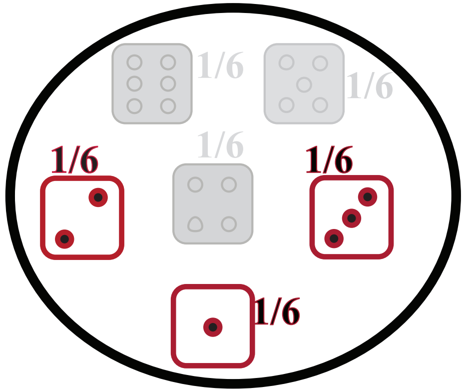


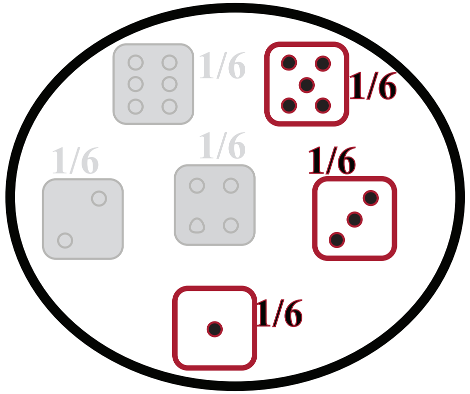
$P(T,W)$

		Temperature	
		hot	cold
Weather	sun	0.45	0.15
	rain	0.02	0.08
	fog	0.03	0.27
	meteor	0.00	0.00

Events

- An *event* is a set E of outcomes
  - E.g., for the random variable of rolling a dice, R:
    - event "*R < 4*" is the set {1,2,3}
    - event "*R is odd*" is the set {1,3,5}





- Typically, events are *partial assignments*
  - E.g., event "*T=hot*", given the joint distribution *P(T,W)*
- The *probability* of an event is the *sum* of probabilities over included outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

Quiz: Probability of Events

- The *probability* of an event is the *sum* of probabilities over included outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- Given the joint distribution *P(T,W)*, compute the probability of the following events:
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR not foggy?

- ***Joint distribution***

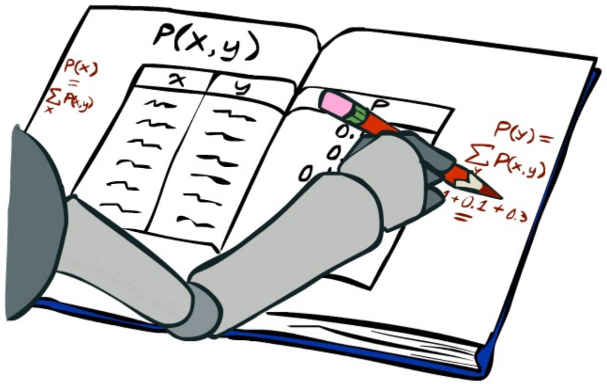
P(T,W)

		Temperature	
		hot	cold
Weather	sun	0.45	0.15
	rain	0.02	0.08
	fog	0.03	0.27
	meteor	0.00	0.00

Marginal Distributions

- ***Marginal distributions*** are sub-tables which eliminate variables
- ***Marginalization*** (***summing out***): Collapse a dimension by adding

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$



		Temperature		
		hot	cold	
Weather	sun	0.45	0.15	0.60
	rain	0.02	0.08	0.10
	fog	0.03	0.27	0.30
	meteor	0.00	0.00	0.00
		0.50	0.50	

P(*W*)

P(*T*)

$P(T)$

	P
hot	0.5
cold	0.5

$P(t) = \sum_w P(t,w)$

$P(W)$

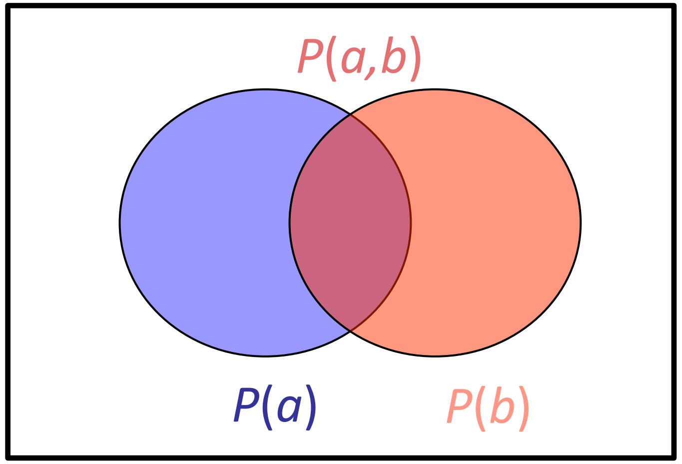
	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

$P(w) = \sum_t P(t,w)$

Conditional Probabilities

- A simple relation between joint and conditional probabilities
  - In fact, this is taken as the *definition* of a conditional probability

$$P(a \mid b) = \frac{P(a, b)}{P(b)}$$



$P(T, W)$

		Temperature	
		hot	cold
Weather	sun	0.45	0.15
	rain	0.02	0.08
	fog	0.03	0.27
	meteor	0.00	0.00

$$P(W{=}s \mid T{=}c) = \frac{P(W{=}s, T{=}c)}{P(T{=}c)} = 0.15/0.50 = 0.3$$

$= P(W{=}s, T{=}c) + P(W{=}r, T{=}c) + P(W{=}f, T{=}c) + P(W{=}m, T{=}c)$
$= 0.15 + 0.08 + 0.27 + 0.00 = 0.50$

Conditional Distributions

- Distributions over one set of variables given fixed values of another set

		Temperature	
		hot	cold
Weather	sun	0.45	0.15
	rain	0.02	0.08
	fog	0.03	0.27
	meteor	0.00	0.00

$P(W \mid T=h)$

hot
0.90
0.04
0.06
0.00

$P(W \mid T=c)$

cold
0.30
0.16
0.54
0.00

$P(W \mid T)$

hot	cold
0.90	0.30
0.04	0.16
0.06	0.54
0.00	0.00

Normalization Trick

- If we compute a conditional using definition directly:

$P(W,T)$

		Temperature	
		hot	cold
Weather	sun	0.45	0.15
	rain	0.02	0.08
	fog	0.03	0.27
	meteor	0.00	0.00

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$

$$= \frac{P(W=s,T=c)}{P(s,c) + P(r,c) + P(f,c) + P(m,c)}$$

$$= \frac{0.15}{0.5} = 0.3$$

$$P(W=r|T=c) = \ ... = 0.16$$

$$P(W=f|T=c) = \ ... = 0.54$$

$$P(W=m|T=c) = \ ... = 0.00$$

$P(W|T=c)$

0.30
0.16
0.54
0.00

Normalization Trick

- But it can be made much simpler…

P(*W*, *T*)

		Temperature	
		hot	cold
Weather	sun	0.45	0.15
	rain	0.02	0.08
	fog	0.03	0.27
	meteor	0.00	0.00

SELECT the joint probabilities matching the evidence

P(*W*, *T*=*c*)

0.15
0.08
0.27
0.00

NORMALIZE the selection (make it sum to one)

P(*W* | *T*=*c*)

0.30
0.16
0.54
0.00

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$

$$= \frac{P(W=s,T=c)}{P(s,c)\ +\ P(r,c)\ +\ P(f,c)\ +\ P(m,c)} = \frac{0.15}{0.5} = 0.3$$

$$P(W=r|T=c) = \ \ldots = 0.16$$

$$P(W=f|T=c) = \ \ldots = 0.54$$

$$P(W=m|T=c) = \ \ldots = 0.00$$

- Multiply each entry by α = 1/(sum over all entries)
- **Why does this work?**
  - Sum of selection is P(evidence)! (P(T=c) here)

Quiz: Normalization Trick

- P(X | Y=-y) ?

P(X,Y)

		Y	
		+y	-y
X	+x	0.2	0.3
	-x	0.4	0.1

SELECT the joint probabilities matching the evidence

→

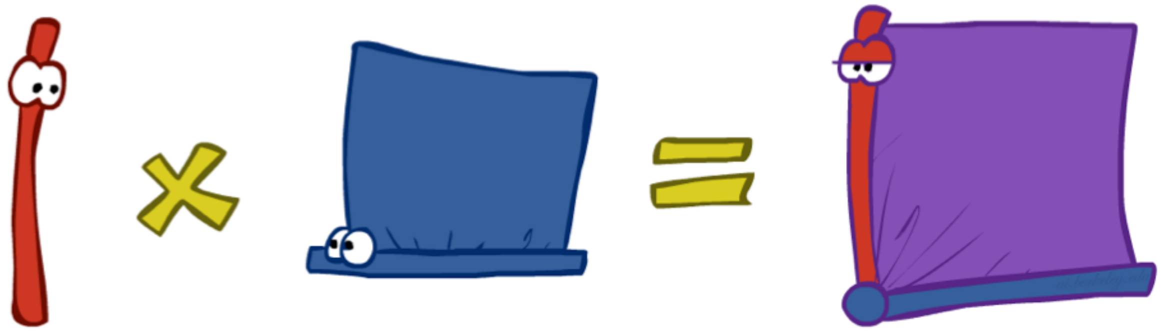
NORMALIZE the selection (make it sum to one)

→

The Product Rule

- Sometimes we have conditional distributions but want the joint

$$P(a \mid b)\, P(b) = P(a, b) \quad \Longleftrightarrow \quad P(a \mid b) = \frac{P(a, b)}{P(b)}$$



The Product Rule: Example

$$P(W \mid T)\, P(T) = P(W, T)$$

$P(W \mid T)$

	hot	cold
sun	0.90	0.30
rain	0.04	0.16
fog	0.06	0.54
meteor	0.00	0.00

$P(T)$

	P
hot	0.5
cold	0.5

$P(W, T)$

		Temperature	
		hot	cold
Weather	sun		
	rain		
	fog		
	meteor		

The Chain Rule

- A joint distribution can be written as a product of conditional distributions by repeated application of the product rule:
- $P(x_1, x_2, x_3) = P(x_3 \mid x_1, x_2)\, P(x_1, x_2) = P(x_3 \mid x_1, x_2)\, P(x_2 \mid x_1)\, P(x_1)$
- $P(x_1, x_2, \ldots, x_n) = \prod_i P(x_i \mid x_1, \ldots, x_{i-1})$

Probabilistic Inference

- Probabilistic inference: compute a desired probability from a probability model
  - Typically for a ***query variable*** given ***evidence***
  - E.g., P(airport on time | no accidents) = 0.90
  - These represent the agent's *beliefs* given the evidence

- Probabilities change with new evidence:
  - P(airport on time | no accidents, 5 a.m.) = 0.95
  - P(airport on time | no accidents, 5 a.m., raining) = 0.80
  - Observing new evidence causes ***beliefs to be updated***

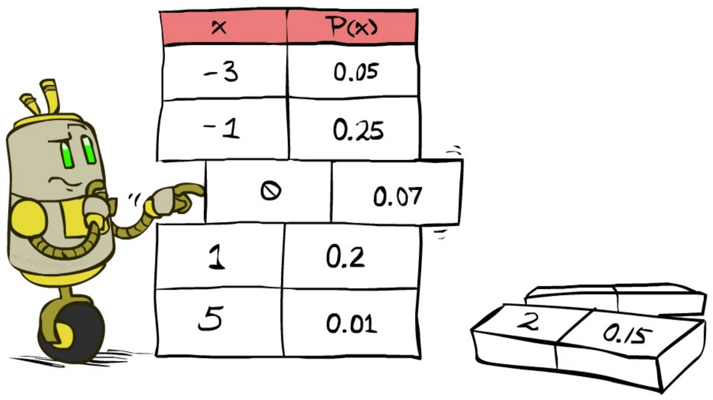
Inference by Enumeration

- Probability model $P(X_1, \ldots, X_n)$ is given
- Partition the variables $X_1, \ldots, X_n$ into sets as follows:
  - Evidence variables: $\boldsymbol{E} = \boldsymbol{e}$
  - Query variables: $\boldsymbol{Q}$
  - Hidden variables: $\boldsymbol{H}$

- We want:

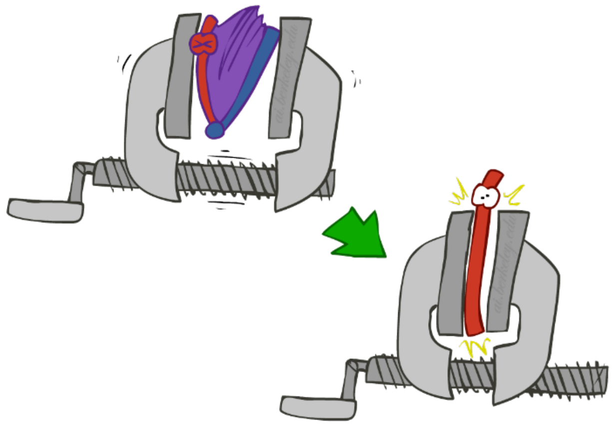
$$P(\boldsymbol{Q} \mid \boldsymbol{e})$$

- Step 1: Select the entries consistent with the evidence



- Step 2: Sum out $\boldsymbol{H}$ from model to get joint of query and evidence

$$P(\boldsymbol{Q}, \boldsymbol{e}) = \sum_h \underbrace{P(\boldsymbol{Q}, \boldsymbol{h}, \boldsymbol{e})}_{X_1, \ldots, X_n}$$



- Step 3: Normalize

$$P(\boldsymbol{Q} \mid \boldsymbol{e}) = \alpha\, P(\boldsymbol{Q}, \boldsymbol{e})$$

Inference by Enumeration

- P(W)?

Season	Temp	Weather	P
summer	hot	sun	0.26
summer	hot	rain	0.01
summer	hot	fog	0.01
summer	hot	meteor	0.00
summer	cold	sun	0.07
summer	cold	rain	0.05
summer	cold	fog	0.07
summer	cold	meteor	0.00
winter	hot	sun	0.08
winter	hot	rain	0.01
winter	hot	fog	0.02
winter	hot	meteor	0.00
winter	cold	sun	0.12
winter	cold	rain	0.15
winter	cold	fog	0.15
winter	cold	meteor	0.00

Inference by Enumeration

- P(W)?

$$P(\text{sun}) = 0.26 + 0.07 + 0.08 + 0.12 = 0.53$$
$$P(\text{rain}) = 0.01 + 0.05 + 0.01 + 0.15 = 0.22$$
$$P(\text{fog}) = 0.01 + 0.07 + 0.02 + 0.15 = 0.25$$
$$P(\text{meteor}) = 0.0 + 0.0 + 0.0 + 0.0 = 0.0$$

Season	Temp	Weather	P
summer	hot	sun	0.26
summer	hot	rain	0.01
summer	hot	fog	0.01
summer	hot	meteor	0.00
summer	cold	sun	0.07
summer	cold	rain	0.05
summer	cold	fog	0.07
summer	cold	meteor	0.00
winter	hot	sun	0.08
winter	hot	rain	0.01
winter	hot	fog	0.02
winter	hot	meteor	0.00
winter	cold	sun	0.12
winter	cold	rain	0.15
winter	cold	fog	0.15
winter	cold	meteor	0.00

Inference by Enumeration

- P(W | winter)?
  - Step 1: Select the entries consistent with the evidence

Season	Temp	Weather	P
summer	hot	sun	0.26
summer	hot	rain	0.01
summer	hot	fog	0.01
summer	hot	meteor	0.00
summer	cold	sun	0.07
summer	cold	rain	0.05
summer	cold	fog	0.07
summer	cold	meteor	0.00
winter	hot	sun	0.08
winter	hot	rain	0.01
winter	hot	fog	0.02
winter	hot	meteor	0.00
winter	cold	sun	0.12
winter	cold	rain	0.15
winter	cold	fog	0.15
winter	cold	meteor	0.00

Inference by Enumeration

- P(W | winter)?
  - Step 1: Select the entries consistent with the evidence

Season	Temp	Weather	P
summer	hot	sun	0.26
summer	hot	rain	0.01
summer	hot	fog	0.01
summer	hot	meteor	0.00
summer	cold	sun	0.07
summer	cold	rain	0.05
summer	cold	fog	0.07
summer	cold	meteor	0.00
winter	hot	sun	0.08
winter	hot	rain	0.01
winter	hot	fog	0.02
winter	hot	meteor	0.00
winter	cold	sun	0.12
winter	cold	rain	0.15
winter	cold	fog	0.15
winter	cold	meteor	0.00

Inference by Enumeration

- P(W | winter)?
  - Step 1: Select the entries consistent with the evidence
  - Step 2: Sum out ***H*** from model to get joint of query and evidence

Season	Weather	P(W, winter)
winter	sun	0.20
winter	rain	0.16
winter	fog	0.17
winter	meteor	0.00

Season	Temp	Weather	P
summer	hot	sun	0.26
summer	hot	rain	0.01
summer	hot	fog	0.01
summer	hot	meteor	0.00
summer	cold	sun	0.07
summer	cold	rain	0.05
summer	cold	fog	0.07
summer	cold	meteor	0.00
winter	hot	sun	0.08
winter	hot	rain	0.01
winter	hot	fog	0.02
winter	hot	meteor	0.00
winter	cold	sun	0.12
winter	cold	rain	0.15
winter	cold	fog	0.15
winter	cold	meteor	0.00

Inference by Enumeration

- P(W | winter)?
  - Step 1: Select the entries consistent with the evidence
  - Step 2: Sum out ***H*** from model to get joint of query and evidence
  - Step 3: Normalize (divide by sum)

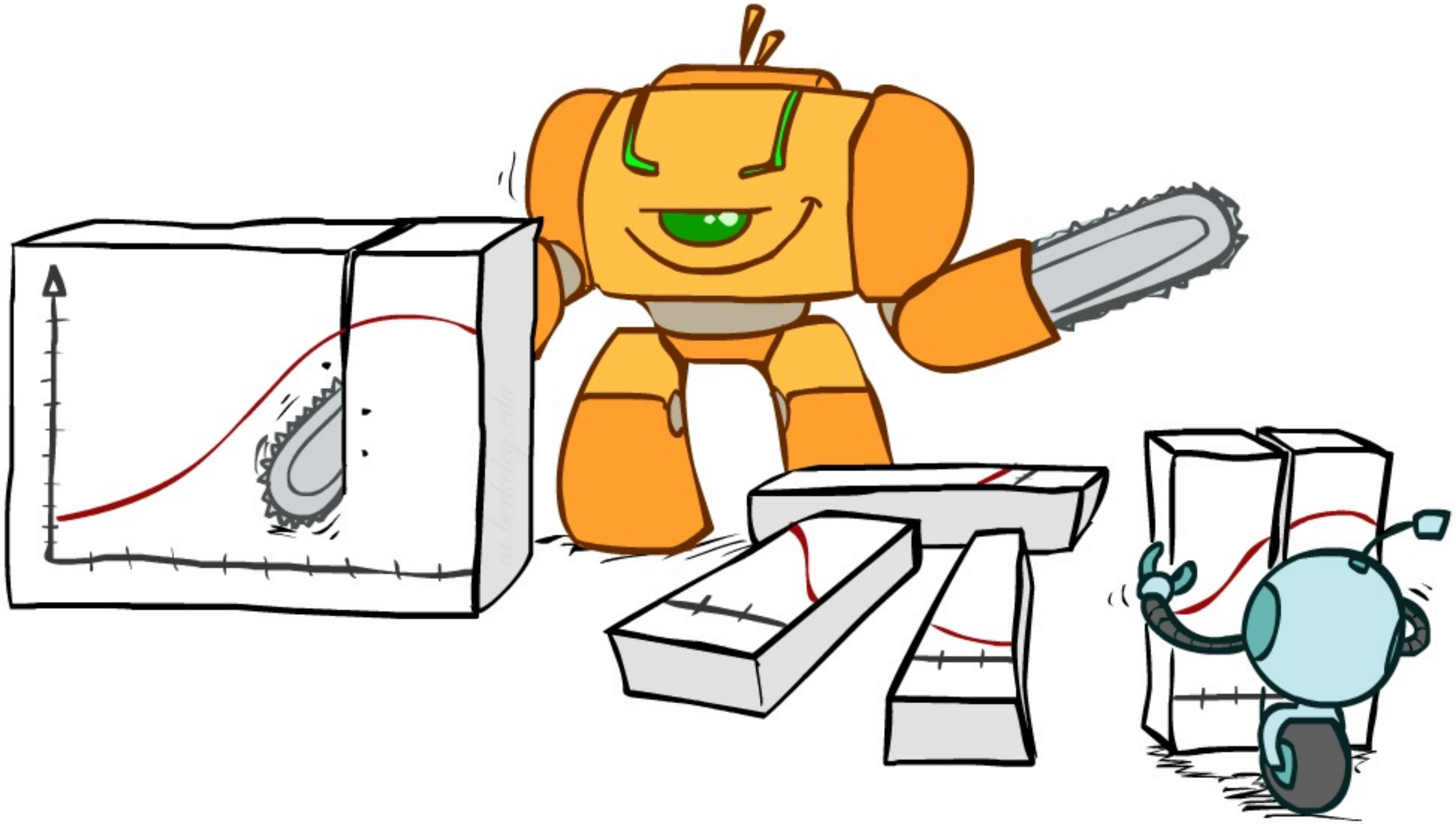
Season	Weather	P(W|winter)
winter	sun	0.38
winter	rain	0.30
winter	fog	0.32
winter	meteor	0.00

Season	Temp	Weather	P
summer	hot	sun	0.26
summer	hot	rain	0.01
summer	hot	fog	0.01
summer	hot	meteor	0.00
summer	cold	sun	0.07
summer	cold	rain	0.05
summer	cold	fog	0.07
summer	cold	meteor	0.00
winter	hot	sun	0.08
winter	hot	rain	0.01
winter	hot	fog	0.02
winter	hot	meteor	0.00
winter	cold	sun	0.12
winter	cold	rain	0.15
winter	cold	fog	0.15
winter	cold	meteor	0.00

Inference by Enumeration

- Obvious problems:
  - Worst-case time complexity $O(d^n)$ (exponential in #hidden variables)
  - Space complexity $O(d^n)$ to store the joint distribution
  - $O(d^n)$ data points to estimate the entries in the joint distribution

Bayes Rule



Bayes' Rule

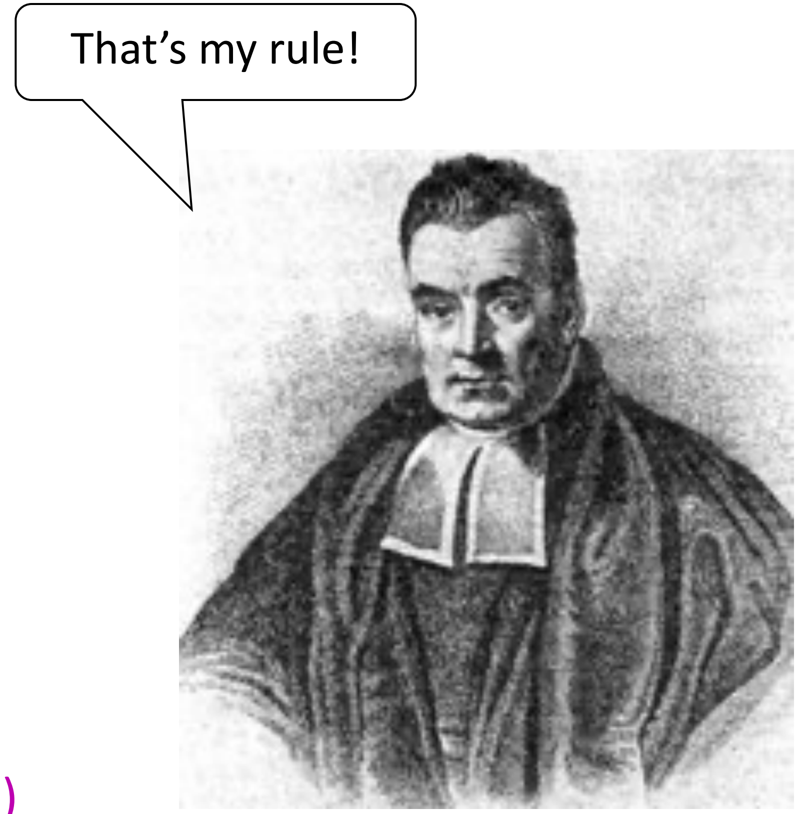
- Write the product rule both ways:

$$\mathbf{P(a \mid b)\ P(b)} = P(a, b) = \mathbf{P(b \mid a)\ P(a)}$$

- Dividing left and right expressions, we get:

$$\mathbf{P(a \mid b) = \frac{P(b \mid a)\ P(a)}{P(b)}}$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Describes an "update" step from prior $P(a)$ to posterior $P(a \mid b)$
    - Hence provides a simple, formal theory of learning



Thomas Bayes
English statistician
1763

Inference with Bayes' Rule

- Example: probabilistic diagnosis from causal probability:

$$P(cause \mid effect) = \frac{P(effect \mid cause)\, P(cause)}{P(effect)}$$

- Example:
  - M: meningitis, S: stiff neck

$$\left.\begin{aligned} P(s \mid m) &= 0.8 \\ P(m) &= 0.0001 \\ P(s) &= 0.01 \end{aligned}\right\} \text{Example givens}$$

$$P(m \mid s) = \frac{P(s \mid m)\, P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.01}$$

  - Note: posterior probability of meningitis still very small: 0.008
  - Note: you should still get stiff necks checked out! Why?

Independence

- Two variables X and Y are (absolutely) ***independent*** if

$$\forall x,y \qquad P(x, y) = P(x)\, P(y)$$

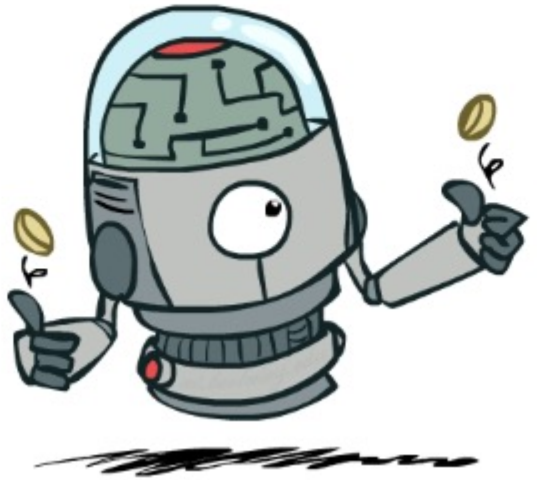
  - I.e., the joint distribution *factors* into a product of two simpler distributions

- Equivalently, via the product rule $P(x,y) = P(x|y)P(y)$,

$$P(x \mid y) = P(x) \quad \text{or} \quad P(y \mid x) = P(y)$$

- We write: $X \perp\!\!\!\perp Y$

- Independence is a simplifying *modeling assumption*
  - Joint distributions in real: at best "close" to independent
  - What could we assume for {Weather, Traffic, Cavity, Toothache}?



Example: Independence?

$P(T)$

T	P
hot	0.5
cold	0.5

$P_1(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P_2(T, W)$

T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

$P(W)$

W	P
sun	0.6
rain	0.4

Example: Independence

- n fair, independent coin flips:

$P(X_1)$

H	0.5
T	0.5

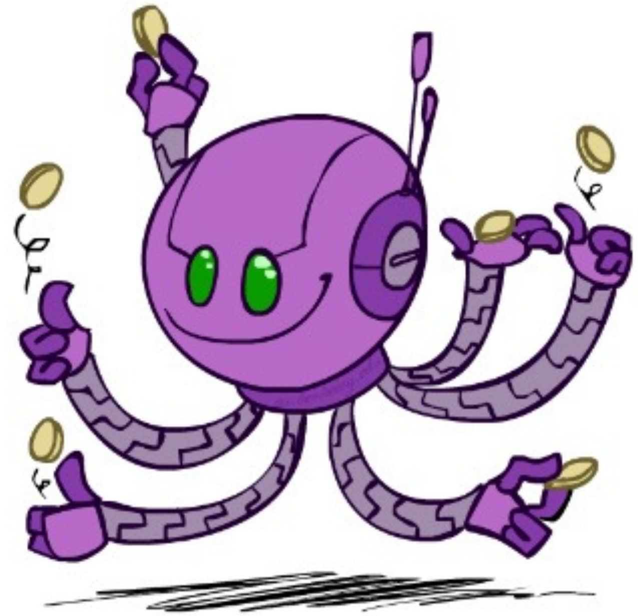
$P(X_2)$

H	0.5
T	0.5

. . .

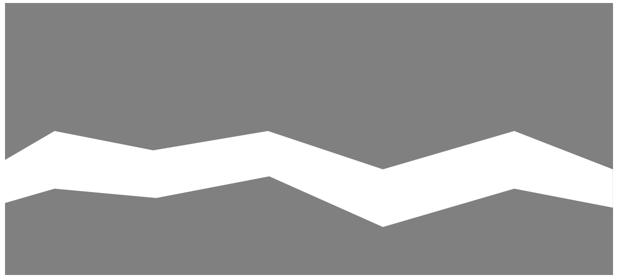
$P(X_n)$

H	0.5
T	0.5



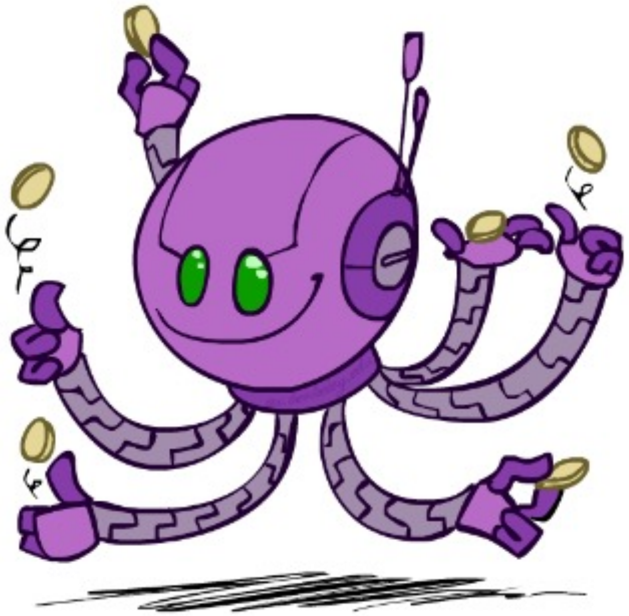
$P(X_1, X_2, \ldots, X_n)$

2^n

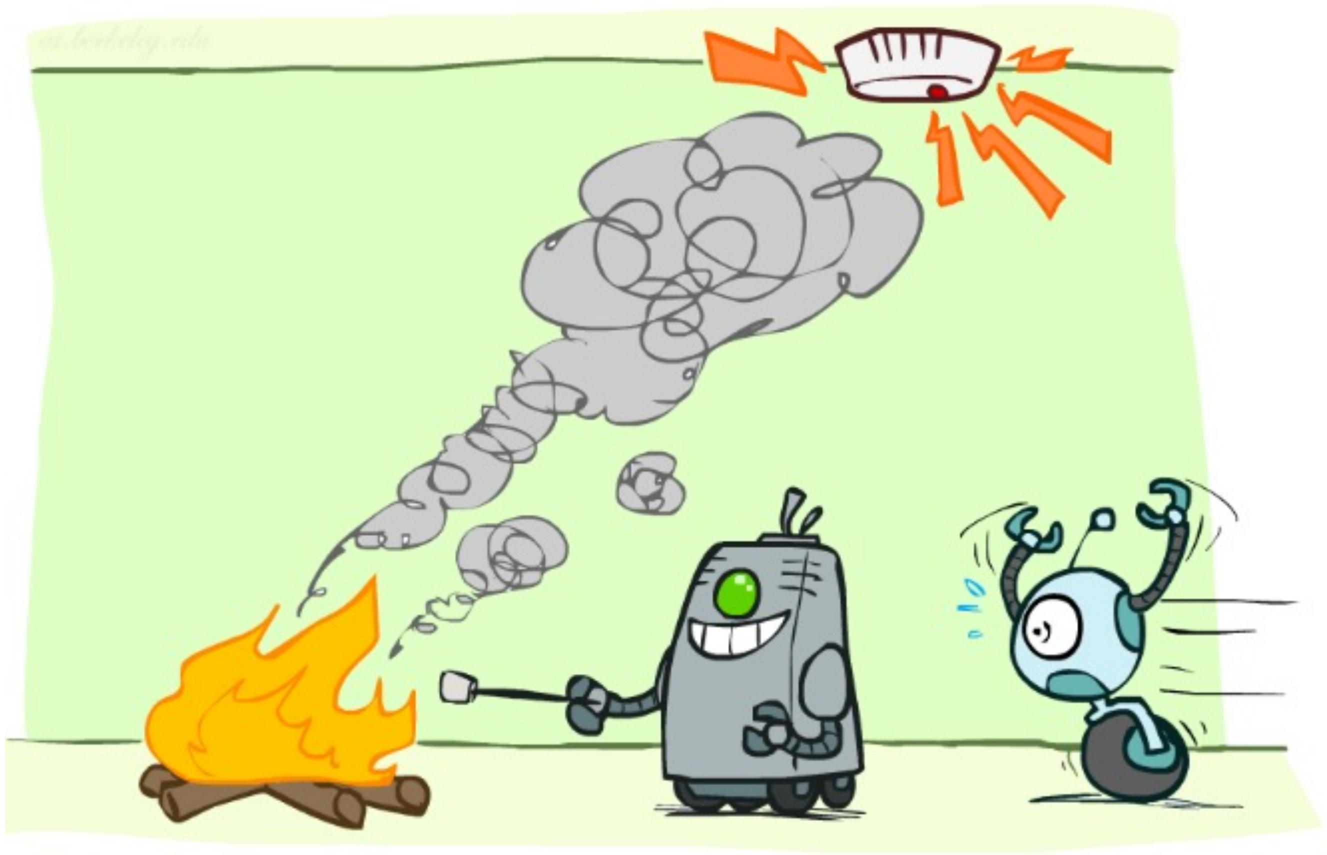


Independence, contd.

- Independence is incredibly powerful
  - Exponential reduction in representation size
- Independence is extremely rare!
- *Conditional* independence is much more common!!

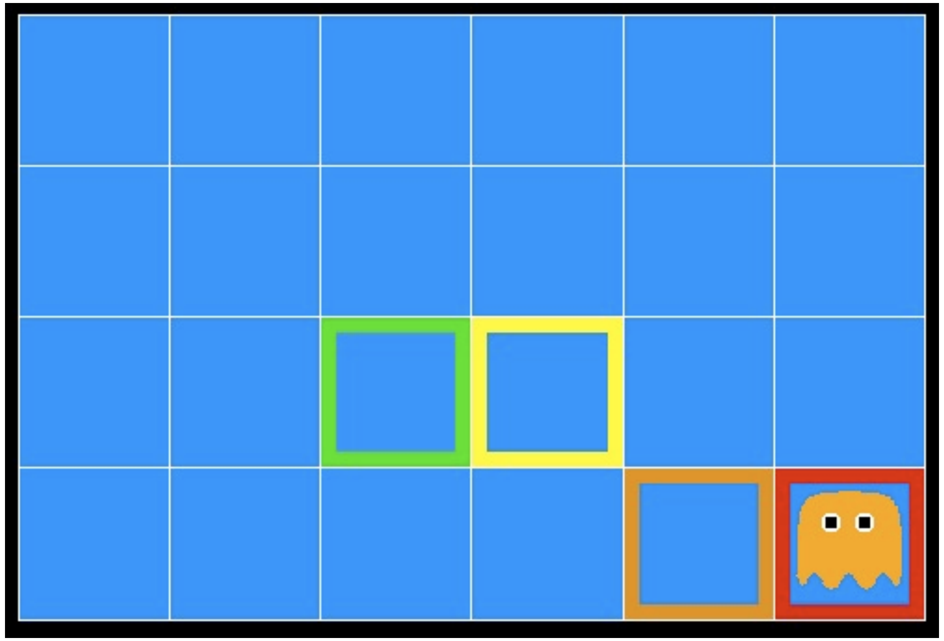


Conditional Independence



Ghostbusters

- A ghost is in the grid somewhere
- Sensor readings tell how close a square is to the ghost
  - On the ghost: usually red
  - 1 or 2 away: mostly orange
  - 3 or 4 away: typically yellow
  - 5+ away: often green
- Click on squares until confident of location, then "***bust***"



Video of Demo Ghostbusters with Probability

Ghostbusters model

- Variables and ranges:
  - G (ghost location) in $\{(1,1),\ldots,(3,3)\}$
  - $C_{x,y}$ (color measured at square x,y) in {red,orange,yellow,green}
- We have two distributions at hand:
  - *Prior distribution* over ghost location: $P(G)$
    - Let's say this is uniform
  - *Sensor model*: $P(C_{x,y} \mid G)$
    - Let's say it depends only on distance to G
      - E.g. $P(C_{1,1} = \text{red} \mid G = (1,1)) = 0.6$
      - $P(C_{1,1} = \text{orange} \mid G = (1,1)) = 0.25$
      - …

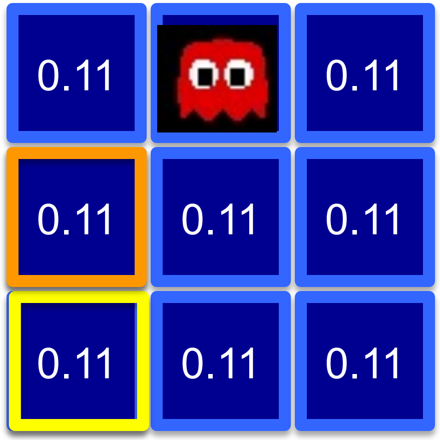
0.11	0.11	0.11
0.11	0.11	0.11
0.11	0.11	0.11

Ghostbusters model, contd.

- $P(G, C_{1,1}, \ldots C_{3,3})$ has $9 \times 4^9 = 2{,}359{,}296$ entries!!!

- Ghostbuster independence:
  - Are $C_{1,1}$ and $C_{1,2}$ independent?
    - i.e., does $P(C_{1,1} = \text{yellow}) = P(C_{1,1} = \text{yellow} \mid C_{1,2} = \text{orange})$?

- Ghostbuster physics again:
  - We know that $P(C_{x,y} \mid G)$ depends *only* on distance to G
    - So $P(C_{1,1} = \text{yellow} \mid \underline{G = (2,3)}) = P(C_{1,1} = \text{yellow} \mid \underline{G = (2,3)}, C_{1,2} = \text{orange})$
    - I.e., $C_{1,1}$ is *conditionally independent* of $C_{1,2}$ *given G*



Ghostbusters model, contd.

- Simplify the model using the conditional independence?

- Apply the chain rule to decompose the joint probability model:
  - $P(G, C_{1,1}, \ldots C_{3,3}) = P(G)\, P(C_{1,1} \mid G)\, P(C_{1,2} \mid G, C_{1,1})\, P(C_{1,3} \mid G, C_{1,1}, C_{1,2}) \ldots P(C_{3,3} \mid G, C_{1,1}, \ldots, C_{3,2})$

- Now simplify using conditional independence:
  - $P(G, C_{1,1}, \ldots C_{3,3}) = P(G)\, P(C_{1,1} \mid G)\, P(C_{1,2} \mid G)\, P(C_{1,3} \mid G) \ldots P(C_{3,3} \mid G)$

- I.e., conditional independence properties of ghostbuster physics simplify the probability model from *exponential* to *quadratic* in the number of squares

Conditional Independence

- ***Conditional independence*** is our most basic and robust form of knowledge about uncertain environments.

- X is conditionally independent of Y given Z if and only if:

$$\forall x,y,z \qquad P(x \mid y, z) = P(x \mid z)$$

  or, equivalently, if and only if

$$\forall x,y,z \qquad P(x, y \mid z) = P(x \mid z)\, P(y \mid z)$$

  we write: $X \perp\!\!\!\perp Y | Z$

Next time

- Bayes nets