Calculus III - Notes

Curvature

In this section we want to briefly discuss the **curvature** of a smooth curve (recall that for a smooth curve we require $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$). The curvature measures how fast a curve is changing direction at a given point.

There are several formulas for determining the curvature for a curve. The formal definition of curvature is,

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

where $\vec{T}$ is the unit tangent and s is the arc length. Recall that we saw in a previous section how to reparameterize a curve to get it into terms of the arc length.

In general the formal definition of the curvature is not easy to use so there are two alternate formulas that we can use. Here they are.

$$\kappa = \frac{\left\| \vec{T}'(t) \right\|}{\left\| \vec{r}'(t) \right\|} \qquad\qquad \kappa = \frac{\left\| \vec{r}'(t) \times \vec{r}''(t) \right\|}{\left\| \vec{r}'(t) \right\|^3}$$

These may not be particularly easy to deal with either, but at least we don’t need to reparameterize the unit tangent.

Example 1 Determine the curvature for $\vec{r}(t) = \langle t, 3\sin t, 3\cos t \rangle$.

Solution

Back in the section when we introduced the tangent vector we computed the tangent and unit tangent vectors for this function. These were,

$$\vec{r}'(t) = \langle 1, 3\cos t, -3\sin t \rangle$$

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle$$

The derivative of the unit tangent is,

$$\vec{T}'(t) = \left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle$$

The magnitudes of the two vectors are,

$$\left\| \vec{r}'(t) \right\| = \sqrt{1 + 9\cos^2 t + 9\sin^2 t} = \sqrt{10}$$

$$\left\| \vec{T}'(t) \right\| = \sqrt{0 + \frac{9}{10}\sin^2 t + \frac{9}{10}\cos^2 t} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

The curvature is then,

$$\kappa = \frac{\left\| \vec{T}'(t) \right\|}{\left\| \vec{r}'(t) \right\|} = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$$

In this case the curvature is constant. This means that the curve is changing direction at the same rate at every point along it. Recalling that this curve is a helix this result makes sense.

Example 2 Determine the curvature of $\vec{r}(t) = t^2\,\vec{i} + t\,\vec{k}$.

Solution

In this case the second form of the curvature would probably be easiest. Here are the first couple of derivatives.

$$\vec{r}'(t) = 2t\,\vec{i} + \vec{k} \qquad \vec{r}''(t) = 2\,\vec{i}$$

Next, we need the cross product.

$$\begin{aligned}\vec{r}'(t)\times\vec{r}''(t) &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2t & 0 & 1\\ 2 & 0 & 0\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ 2t & 0\\ 2 & 0\end{matrix}\\ &= 2\,\vec{j}\end{aligned}$$

The magnitudes are,

$$\left\|\vec{r}'(t)\times\vec{r}''(t)\right\| = 2 \qquad \left\|\vec{r}'(t)\right\| = \sqrt{4t^2+1}$$

The curvature at any value of t is then,

$$\kappa = \frac{2}{\left(4t^2+1\right)^{\frac{3}{2}}}$$

There is a special case that we can look at here as well. Suppose that we have a curve given by $y = f(x)$ and we want to find its curvature.

As we saw when we first looked at vector functions we can write this as follows,

$$\vec{r}(x) = x\,\vec{i} + f(x)\,\vec{j}$$

If we then use the second formula for the curvature we will arrive at the following formula for the curvature.

$$\kappa = \frac{\left|f''(x)\right|}{\left(1+\left[f'(x)\right]^2\right)^{\frac{3}{2}}}$$