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Curvature

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In general, there are two important types of curvature: [extrinsic curvature](#) and [intrinsic curvature](#). The [extrinsic curvature](#) of curves in two- and three-space was the first type of curvature to be studied historically, culminating in the [Frenet formulas](#), which describe a [space curve](#) entirely in terms of its "curvature," [torsion](#), and the initial starting point and direction.

After the curvature of two- and three-dimensional curves was studied, attention turned to the curvature of surfaces in three-space. The main curvatures that emerged from this scrutiny are the [mean curvature](#), [Gaussian curvature](#), and the [shape operator](#). [Mean curvature](#) was the most important for applications at the time and was the most studied, but Gauss was the first to recognize the importance of the [Gaussian curvature](#).

Because [Gaussian curvature](#) is "intrinsic," it is detectable to two-dimensional "inhabitants" of the surface, whereas [mean curvature](#) and the [shape operator](#) are not detectable to someone who can't study the three-dimensional space surrounding the surface on which he resides. The importance of [Gaussian curvature](#) to an inhabitant is that it controls the surface [area](#) of [spheres](#) around the inhabitant.

Riemann and many others generalized the concept of curvature to [sectional curvature](#), [scalar curvature](#), the [Riemann tensor](#), [Ricci curvature tensor](#), and a host of other [intrinsic](#) and [extrinsic curvatures](#). General curvatures no longer need to be numbers, and can take the form of a [map](#), [group](#), [groupoid](#), tensor field, etc.

The simplest form of curvature and that usually first encountered in [calculus](#) is an [extrinsic curvature](#). In two dimensions, let a [plane curve](#) be given by [Cartesian parametric equations](#) $x = x(t)$ and $y = y(t)$. Then the curvature κ , sometimes also called the "first curvature" (Kreyszig 1991, p. 47), is defined by

$$\kappa \equiv \frac{d\phi}{ds} \tag{1}$$

$$= \frac{\frac{d\phi}{dt}}{\frac{ds}{dt}} \tag{2}$$

$$= \frac{\frac{d\phi}{dt}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} \tag{3}$$

$$= \frac{\frac{d\phi}{dt}}{\sqrt{x'^2 + y'^2}}, \tag{4}$$

where ϕ is the [tangential angle](#) and s is the [arc length](#). As can readily be seen from the definition, curvature therefore has units of inverse distance. The $d\phi/dt$ derivative in the above equation can be found using the identity

$$\tan \phi = \frac{dy}{dx} \tag{5}$$

$$= \frac{dy/dt}{dx/dt} \tag{6}$$

$$= \frac{y'}{x'}, \tag{7}$$

so

$$\frac{d}{dt}(\tan \phi) = \sec^2 \phi \frac{d\phi}{dt} = \frac{x' y'' - y' x''}{x'^2} \tag{8}$$

and

$$\frac{d\phi}{dt} = \frac{1}{\sec^2 \phi} \frac{d}{dt}(\tan \phi) \tag{9}$$

$$= \frac{1}{1 + \tan^2 \phi} \frac{x' y'' - y' x''}{x'^2} \tag{10}$$

$$= \frac{1}{1 + \frac{y'^2}{x'^2}} \frac{x' y'' - y' x''}{x'^2} \tag{11}$$

$$= \frac{x' y'' - y' x''}{x'^2 + y'^2}. \tag{12}$$

Combining equations (8), (3), (10), and (12) then gives

$$\kappa = \frac{x' y'' - y' x''}{(x'^2 + y'^2)^{3/2}}. \tag{13}$$

For a two-dimensional curve written in the form $y = f(x)$, the equation of curvature becomes

$$\kappa = \frac{\frac{d^2 y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}. \tag{14}$$

If the two-dimensional curve is instead parameterized in [polar coordinates](#), then

curvature of epicycloid


- THINGS TO TRY:
- = curvature of epicycloid
 - = curvature of a circle
 - = curvature of a sphere

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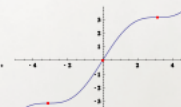
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100%

1. $\int \tan^2(x) \sec^2(x) dx = \frac{\tan^3(x)}{3} + C$

2. $\int_{-1}^1 \sqrt{9t^4 + 4t^2} dt = \frac{2}{27} (13\sqrt{13} - 8)$

3. 

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$$\kappa = \frac{r^2 + 2r_\theta^2 - r r_{\theta\theta}}{(r^2 + r_\theta^2)^{3/2}}, \quad (15)$$

where $r_\theta \equiv \partial r / \partial \theta$ (Gray 1997, p. 89). In [pedal coordinates](#), the curvature is given by

$$\kappa = \frac{1}{r} \frac{dp}{dr}. \quad (16)$$

The curvature for a two-dimensional curve given implicitly by $g(x, y) = 0$ is given by

$$\kappa = \frac{g_{xx} g_y^2 - 2 g_{xy} g_x g_y + g_{yy} g_x^2}{(g_x^2 + g_y^2)^{3/2}} \quad (17)$$

(Gray 1997).

Now consider a parameterized [space curve](#) $\mathbf{r}(t)$ in three dimensions for which the [tangent vector](#) $\hat{\mathbf{T}}$ is defined as

$$\hat{\mathbf{T}} \equiv \frac{\frac{d\mathbf{r}}{dt}}{\left| \frac{d\mathbf{r}}{dt} \right|} = \frac{\frac{d\mathbf{r}}{ds}}{\frac{ds}{dt}}. \quad (18)$$

Therefore,

$$\frac{d\mathbf{r}}{dt} = \frac{ds}{dt} \hat{\mathbf{T}} \quad (19)$$

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2s}{dt^2} \hat{\mathbf{T}} + \frac{ds}{dt} \frac{d\hat{\mathbf{T}}}{dt} \quad (20)$$

$$= \frac{d^2s}{dt^2} \hat{\mathbf{T}} + \kappa \hat{\mathbf{N}} \left(\frac{ds}{dt} \right)^2, \quad (21)$$

where $\hat{\mathbf{N}}$ is the [normal vector](#). But

$$\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} = \frac{ds}{dt} \frac{d^2s}{dt^2} (\hat{\mathbf{T}} \times \hat{\mathbf{T}}) + \kappa \left(\frac{ds}{dt} \right)^3 (\hat{\mathbf{T}} \times \hat{\mathbf{N}}) \quad (22)$$

$$= \kappa \left(\frac{ds}{dt} \right)^3 (\hat{\mathbf{T}} \times \hat{\mathbf{N}}), \quad (23)$$

so taking norms of both sides gives

$$\left| \frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} \right| = \kappa \left(\frac{ds}{dt} \right)^3 = \kappa \left| \frac{d\mathbf{r}}{dt} \right|^3. \quad (24)$$

Solving for κ then gives

$$\kappa = \frac{\left| \frac{d\hat{\mathbf{T}}}{ds} \right|}{\left| \frac{d\mathbf{r}}{dt} \right|} \quad (25)$$

$$= \frac{\left| \frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} \right|}{\left| \frac{d\mathbf{r}}{dt} \right|^3} \quad (26)$$

$$= \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} \quad (27)$$

(Gray 1997, p. 192).

The curvature of a two-dimensional curve is related to the [radius of curvature](#) of the curve's [osculating circle](#). Consider a [circle](#) specified parametrically by

$$x = a \cos t \quad (28)$$

$$y = a \sin t \quad (29)$$

which is tangent to the curve at a given point. The curvature is then

$$\kappa = \frac{x' y'' - y' x''}{(x'^2 + y'^2)^{3/2}} = \frac{a^2}{a^3} = \frac{1}{a}, \quad (30)$$

or one over the [radius of curvature](#). The curvature of a [circle](#) can also be repeated in vector notation. For the [circle](#) with $0 \leq t < 2\pi$, the [arc length](#) is

$$s(t) = \int_0^t \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt \quad (31)$$

$$= \int_0^t \sqrt{a^2 \cos^2 t + a^2 \sin^2 t} dt \quad (32)$$

$$= a t, \quad (33)$$

so $t = s/a$ and the equations of the [circle](#) can be rewritten as

$$x = a \cos \left(\frac{s}{a} \right) \quad (34)$$

$$y = a \sin \left(\frac{s}{a} \right). \quad (35)$$

The [radius vector](#) is then given by

$$\mathbf{r}(s) = a \cos \left(\frac{s}{a} \right) \hat{\mathbf{x}} + a \sin \left(\frac{s}{a} \right) \hat{\mathbf{y}}, \quad (36)$$

and the [tangent vector](#) is

$$\hat{\mathbf{T}} = \frac{d\mathbf{r}}{ds} \quad (37)$$

$$= \quad (38)$$

$$-\sin\left(\frac{s}{a}\right)\hat{\mathbf{x}} + \cos\left(\frac{s}{a}\right)\hat{\mathbf{y}},$$

so the curvature is related to the [radius of curvature](#) a by

$$\kappa = \left| \frac{d\hat{\mathbf{T}}}{ds} \right| \quad (39)$$

$$= \left| -\frac{1}{a} \cos\left(\frac{s}{a}\right)\hat{\mathbf{x}} - \frac{1}{a} \sin\left(\frac{s}{a}\right)\hat{\mathbf{y}} \right| \quad (40)$$

$$= \sqrt{\frac{\cos^2\left(\frac{s}{a}\right) + \sin^2\left(\frac{s}{a}\right)}{a^2}} \quad (41)$$

$$= \frac{1}{a}, \quad (42)$$

as expected.

Four very important derivative relations in differential geometry related to the [Frenet formulas](#) are

$$\dot{\mathbf{r}} = \mathbf{T} \quad (43)$$

$$\dot{\mathbf{T}} = \kappa \mathbf{N} \quad (44)$$

$$\dot{\mathbf{N}} = -\kappa \mathbf{T} + \tau (\mathbf{B} - \kappa \mathbf{T}) \quad (45)$$

$$[\dot{\mathbf{r}}, \dot{\mathbf{T}}, \dot{\mathbf{N}}] = \kappa^2 \tau, \quad (46)$$

where \mathbf{T} is the [tangent vector](#), \mathbf{N} is the [normal vector](#), \mathbf{B} is the [binormal vector](#), and τ is the [torsion](#) (Coxeter 1969, p. 322).

The curvature at a point on a surface takes on a variety of values as the [plane](#) through the normal varies. As κ varies, it achieves a minimum and a maximum (which are in perpendicular directions) known as the [principal curvatures](#). As shown in Coxeter (1969, pp. 352-353),

$$\kappa^2 - \sum b_i^j \kappa + \det(b_i^j) = 0 \quad (47)$$

$$\kappa^2 - 2H\kappa + K = 0, \quad (48)$$

where K is the [Gaussian curvature](#), H is the [mean curvature](#), and \det denotes the [determinant](#).

The curvature κ is sometimes called the first curvature and the [torsion](#) τ the second curvature. In addition, a [third curvature](#) (sometimes called [total curvature](#))

$$\sqrt{ds_T^2 + ds_B^2} \quad (49)$$

is also defined. A signed version of the curvature of a [circle](#) appearing in the [Descartes circle theorem](#) for the radius of the fourth of four mutually tangent circles is called the [bend](#).

SEE ALSO:

[Bend](#), [Binormal Vector](#), [Curvature Center](#), [Extrinsic Curvature](#), [Four-Vertex Theorem](#), [Gaussian Curvature](#), [Intrinsic Curvature](#), [Lancret Equation](#), [Line of Curvature](#), [Mean Curvature](#), [Multivariable Calculus](#), [Normal Curvature](#), [Normal Vector](#), [Osculating Circle](#), [Principal Curvatures](#), [Radius of Curvature](#), [Ricci Curvature Tensor](#), [Riemann Tensor](#), [Scalar Curvature](#), [Sectional Curvature](#), [Shape Operator](#), [Special Affine Curvature](#), [Soddy Circles](#), [Tangent Vector](#), [Third Curvature](#), [Torsion](#), [Total Curvature](#)

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