RRC SUMMER SCHOOL

LECTURE 1

INTRODUCTION TO RIGID BODY TRANSFORMATIONS

BOLLIMUNTHA SHREYA

WHAT IS A ROBOT?

- A ROBOT IS AN AUTONOMOUS MACHINE CAPABLE OF SENSING ITS ENVIRONMENT, CARRYING OUT COMPUTATIONS TO MAKE DECISIONS, AND PERFORMING ACTIONS IN THE REAL WORLD.

WHAT IS ROBOTICS?

- ROBOTICS IS THE SCIENCE STUDYING THE INTELLIGENT CONNECTION OF PERCEPTION TO ACTION.
  - PERCEPTION: SENSORY SYSTEM
  - COGNITION: PLANNING AND DECISION MAKING
  - ACTION: MECHANICAL SYSTEM (LOCOMOTION & MANIPULATION)
- ROBOTICS IS AN INTERDISCIPLINARY SUBJECT CONCERNING MECHANICS, ELECTRONICS, INFORMATION THEORY, CONTROL THEORY, ETC.

MATHEMATICAL FOUNDATION

GROUPS

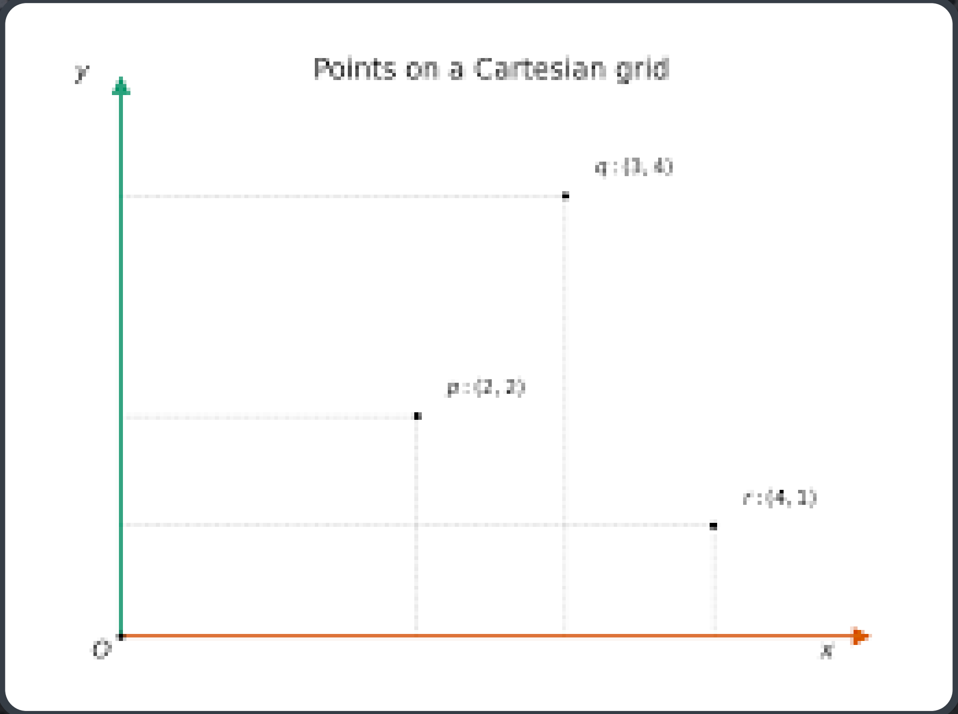
- A GROUP G IS A FINITE OR INFINITE SET OF ELEMENTS TOGETHER WITH A BINARY OPERATION $\circ$ (ADDITION AND MULTIPLICATION) THAT SATISFY THE FOUR FUNDAMENTAL PROPERTIES OF
  - CLOSURE
  - ASSOCIATIVITY
  - IDENTITY PROPERTY
  - INVERSE PROPERTY

HOMOMORPHISM

- A GROUP HOMOMORPHISM IS A FUNCTION f BETWEEN TWO GROUPS $G1$, $G2$ THAT IDENTIFIES SIMILARITIES BETWEEN THEM.
- IT IS A STRUCTURE-PRESERVING MAP BETWEEN TWO ALGEBRAIC STRUCTURES OF THE SAME TYPE (SUCH AS TWO GROUPS, TWO RINGS, OR TWO VECTOR SPACES).

COORDINATE FRAMES

- TO SPECIFY THE LOCATION OF A PARTICLE OR POINT, A COORDINATE FRAME IS NEEDED.
- THE COORDINATE FRAME UNIQUELY DESCRIBES THE LOCATION OF THE POINT
  - $p = (x\ ,\ y) \in \mathbb{R}^2$
  - NO. OF COORDINATES = 2

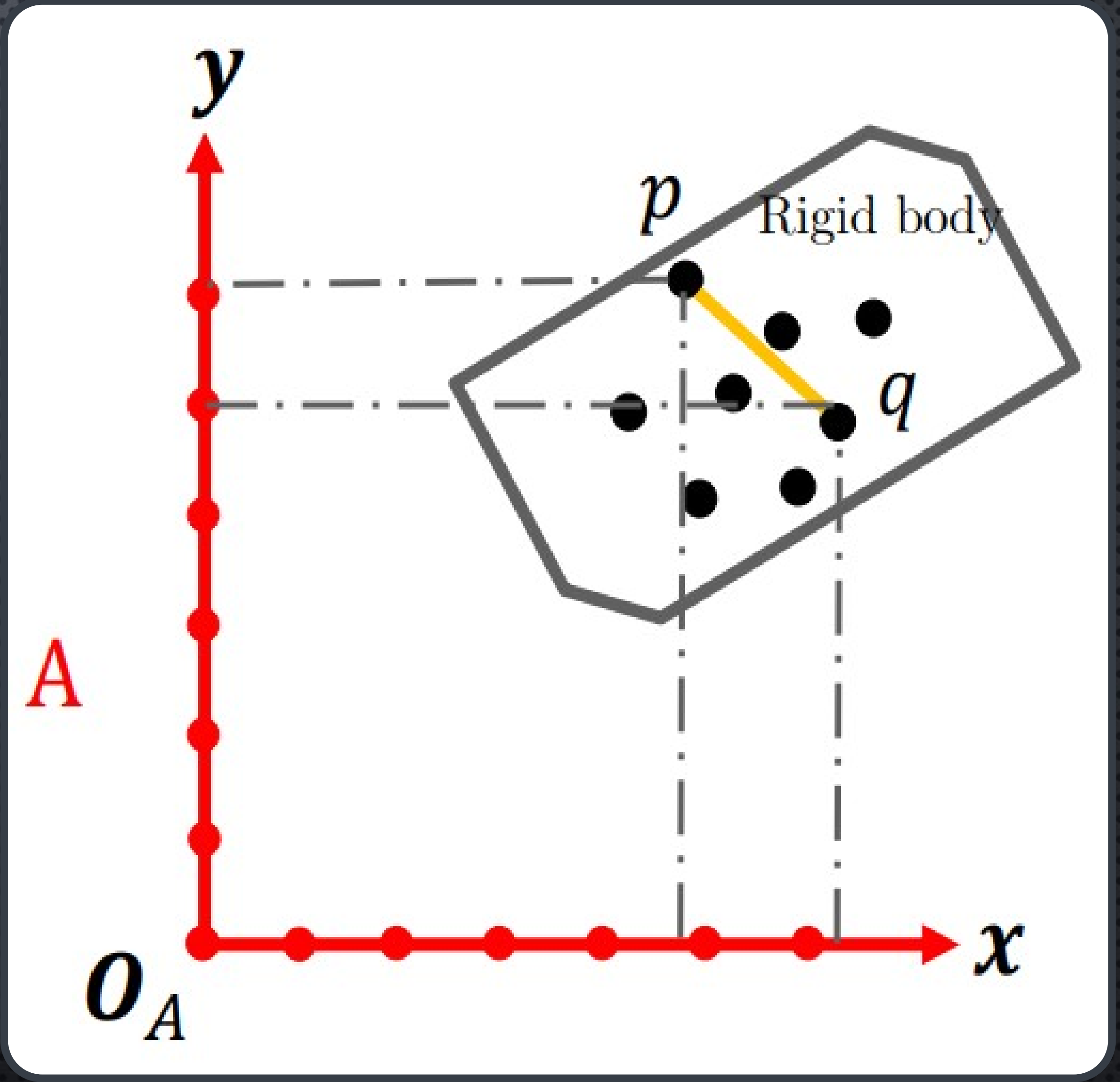


MOTION OF A PARTICLE

- THE MOTION OF A PARTICLE MOVING IN A EUCLIDEAN SPACE IS REPRESENTED BY A PARAMETERIZED CURVE $p(t)$
  - $p(t) = (x(t)\ , y(t)) \in \mathbb{R}^2$
  - COORDINATE IS A FUNCTION OF TIME
  - NO. OF COORDINATES = 2
  - AND TIME T

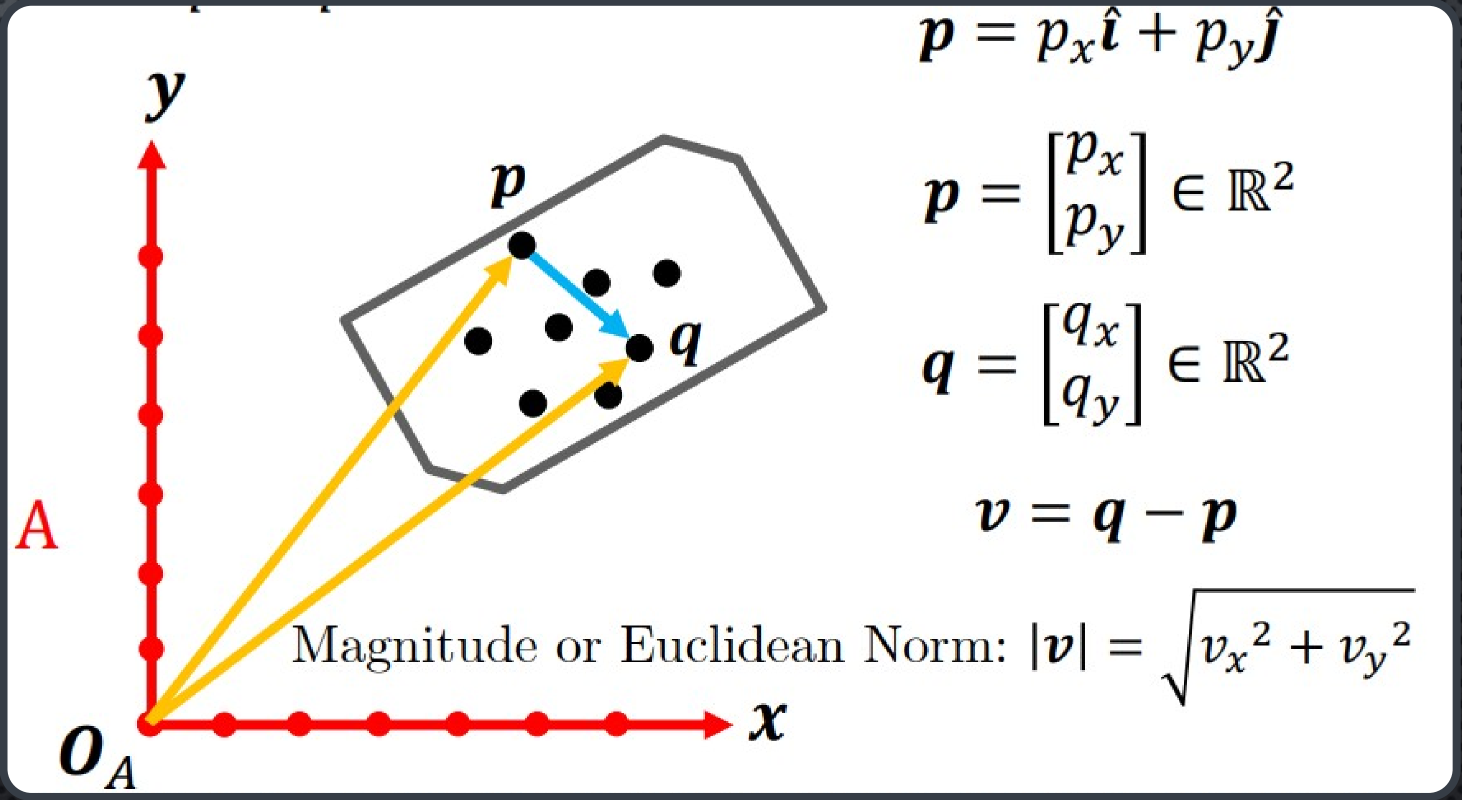
RIGID BODY

- RIGID BODY IS A COLLECTION OF PARTICLES SUCH THAT THE DISTANCE BETWEEN ANY TWO PARTICLES REMAINS FIXED, REGARDLESS OF ANY MOTION OR APPLICATION OF FORCES.
- LENGTH IS PRESERVED
- $||p - q||$ = CONSTANT



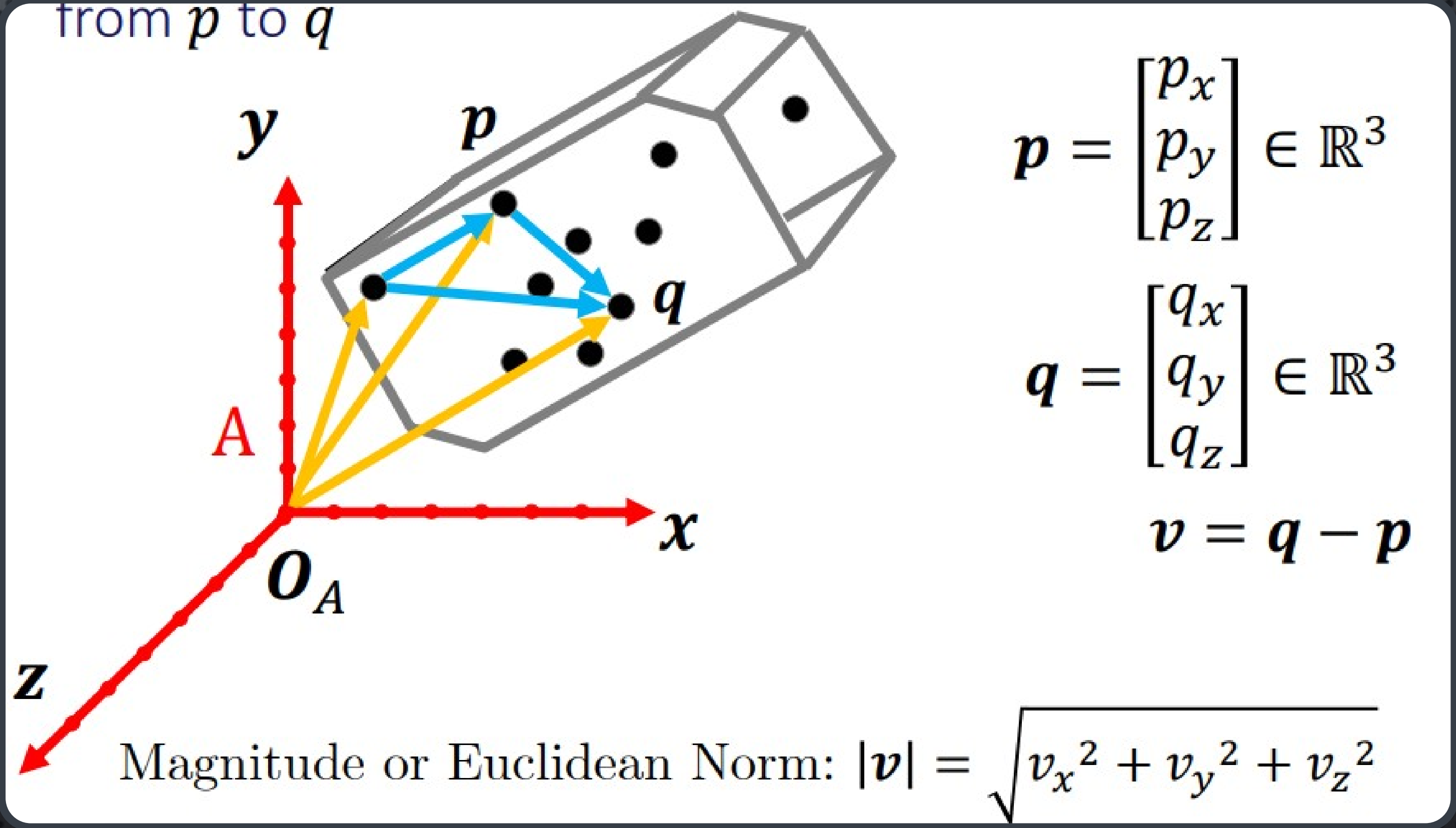
VECTORS IN $\mathbb{R}^2$

- GIVEN TWO POINTS $p, q \in O$, THE VECTOR $\boldsymbol{v} \in \mathbb{R}^2$ IS DEFINED TO BE THE DIRECTED LINE SEGMENT CONNECTING FROM p TO q.



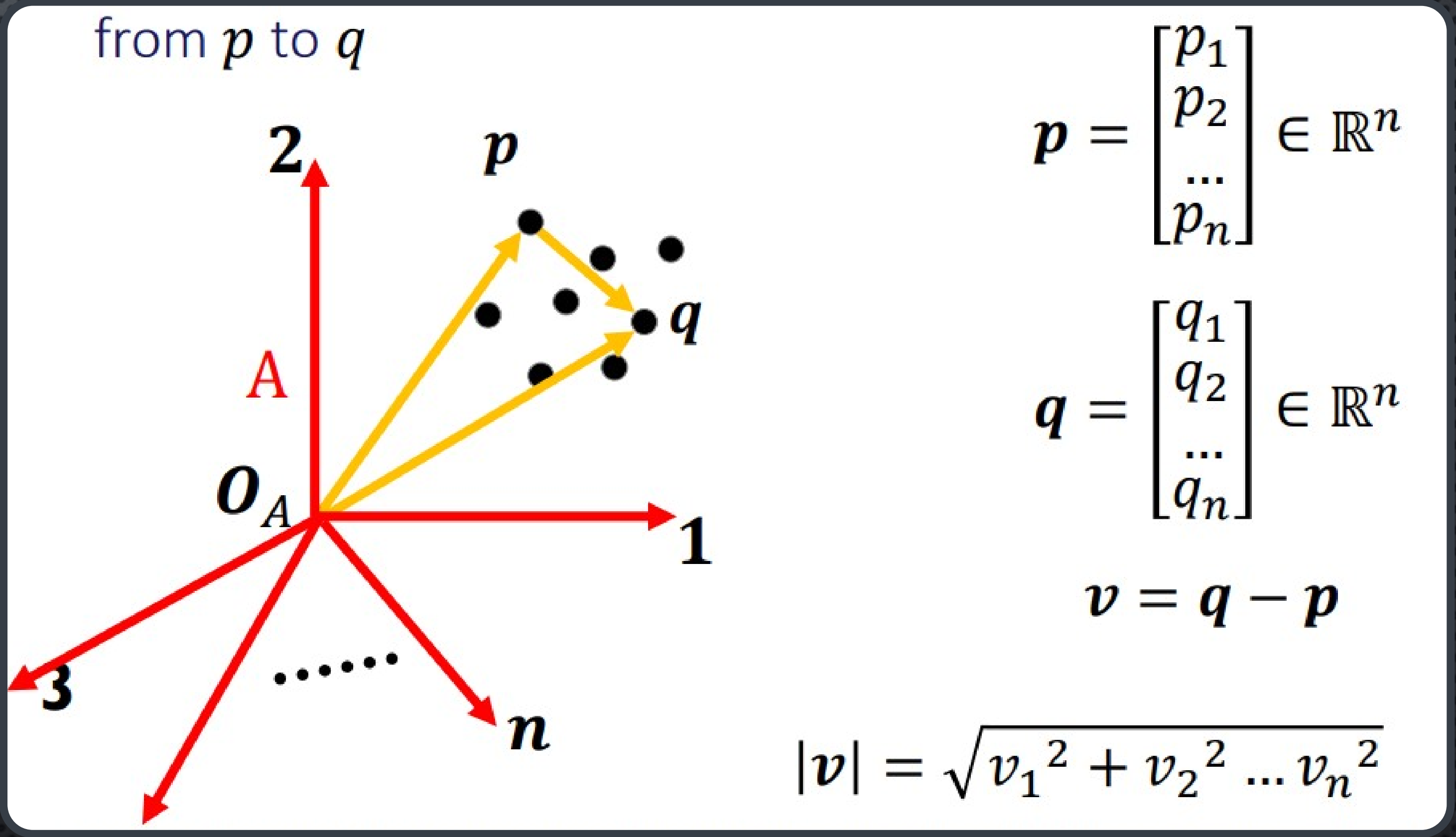
VECTORS IN $\mathbb{R}^3$

- GIVEN TWO POINTS $p, q \in O$, THE VECTOR $v \in \mathbb{R}3$ IS DEFINED TO BE THE DIRECTED LINE SEGMENT CONNECTING FROM p TO q



VECTORS IN $\mathbb{R}^n$

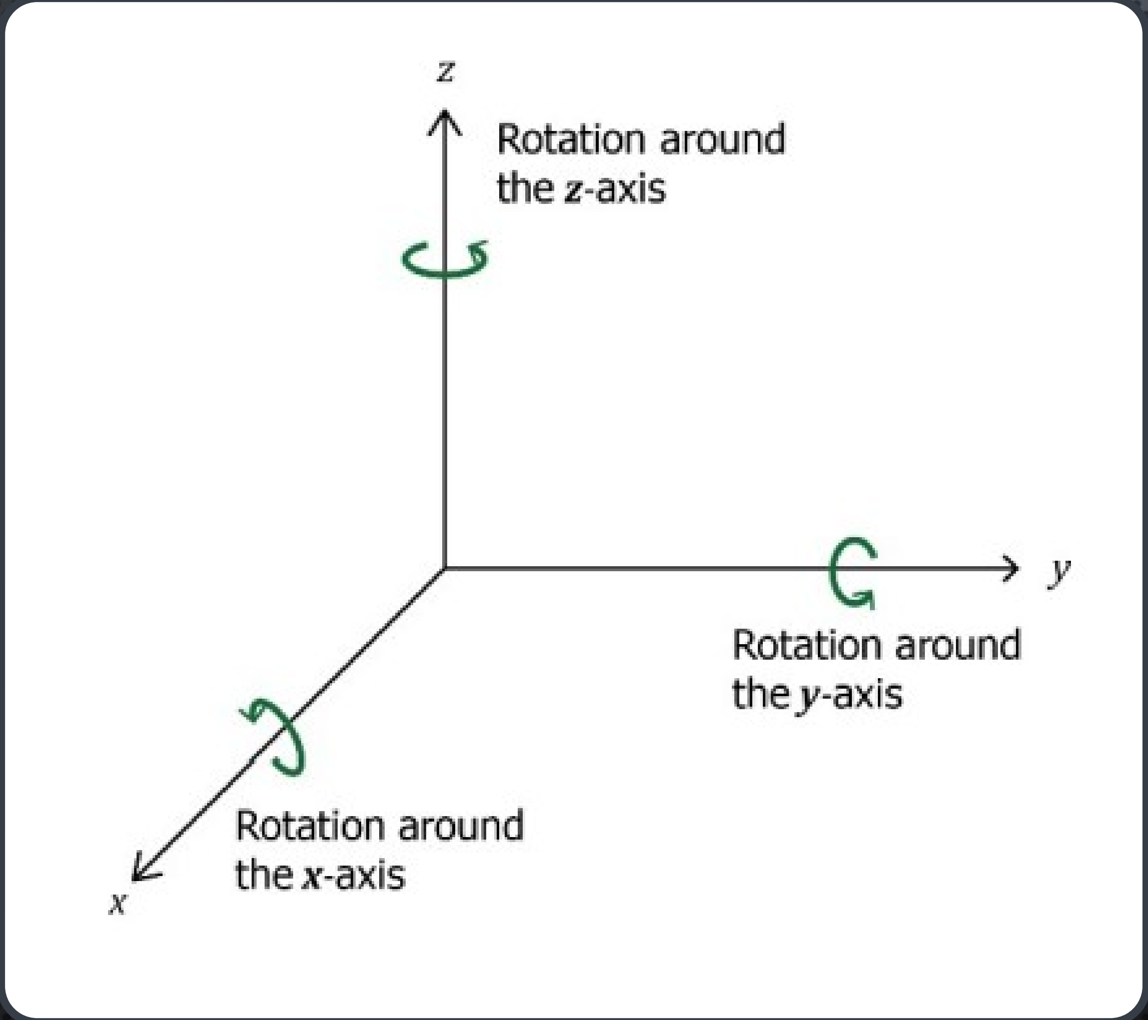
- GIVEN TWO POINTS $p, q \in O$, THE VECTOR $v \in \mathbb{R}n$ IS DEFINED TO BE THE DIRECTED LINE SEGMENT CONNECTING FROM p TO q



ROTATION MATRICES

- LINEAR TRANSFORMATIONS

ROTATION MATRICES IN 3 DIMENSIONS



- A BASIC ROTATION OF A VECTOR IN 3-DIMENSIONS IS A ROTATION AROUND ONE OF THE COORDINATE AXES. WE CAN ROTATE A VECTOR COUNTERCLOCKWISE THROUGH AN ANGLE Θ AROUND THE X–AXIS, THE Y–AXIS, OR THE Z–AXIS.

- WE WANT TO ROTATE A VECTOR [X , Y , Z] AROUND ONE OF THE AXES BY AN ANGLE TO THE NEW POSITION GIVEN BY ANOTHER VECTOR [X' , Y' , Z'].
- WE WOULD NEED ONE OF THE THREE ROTATION MATRICES.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

QUESTION

- FIND THE VECTOR [X′ Y′ Z′] THAT RESULTS WHEN THE VECTOR [X Y Z] = [1 2 3] IS ROTATED 90° COUNTERCLOCKWISE AROUND X-AXIS.

SOLUTION

Using the rotation formula $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\theta = 90^\circ$, we get

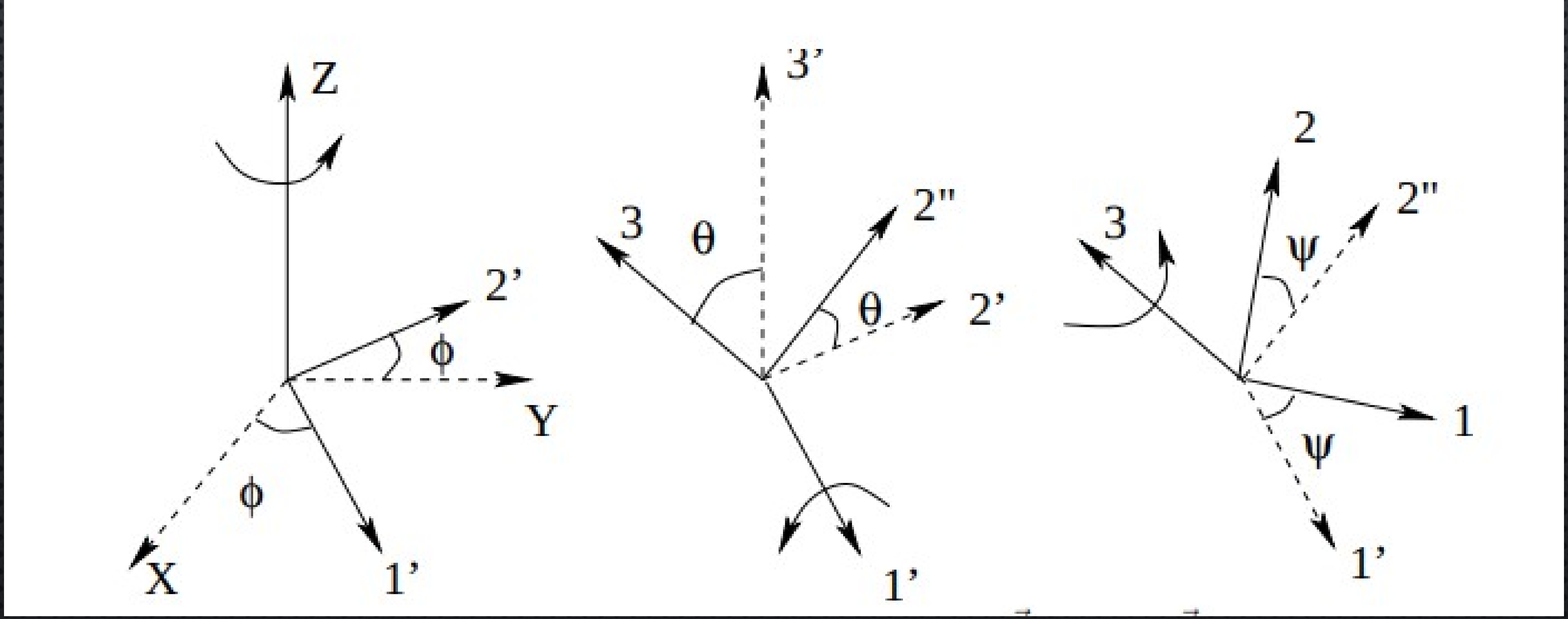
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ \\ 0 & \sin 90^\circ & \cos 90^\circ \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 2 + 0 \cdot 3 \\ 0 \cdot 1 + 0 \cdot 2 + (-1) \cdot 3 \\ 0 \cdot 1 + 1 \cdot 2 + 0 \cdot 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

When rotated counterclockwise 90° around the x–axis, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ becomes $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$.

ROTATION MATRICES FOR EULER ANGLES

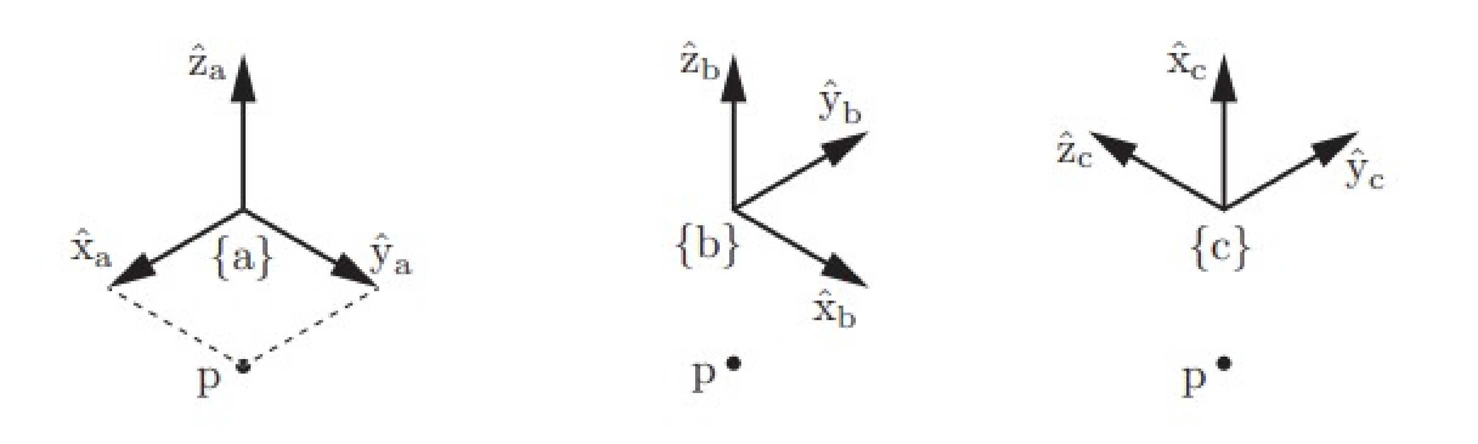


$$\hat{R}(\phi,\theta,\psi) = \hat{R}_3(\psi)\cdot\hat{R}_1(\theta)\cdot\hat{R}_3(\phi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\cdot\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{R}(\phi,\theta,\psi) = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

PROPERTIES OF ROTATION MATRICES

- SUBSCRIPT CANCELLATION



$$R_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$R_{ac} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad R_{ca} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

$$R_{ac} = R_{ab} R_{bc}.$$

$$R_{ac} = R_{ab} R_{bc} = \text{change\_reference\_frame\_from\_\{b\}\_to\_\{a\}}\,(R_{bc}).$$

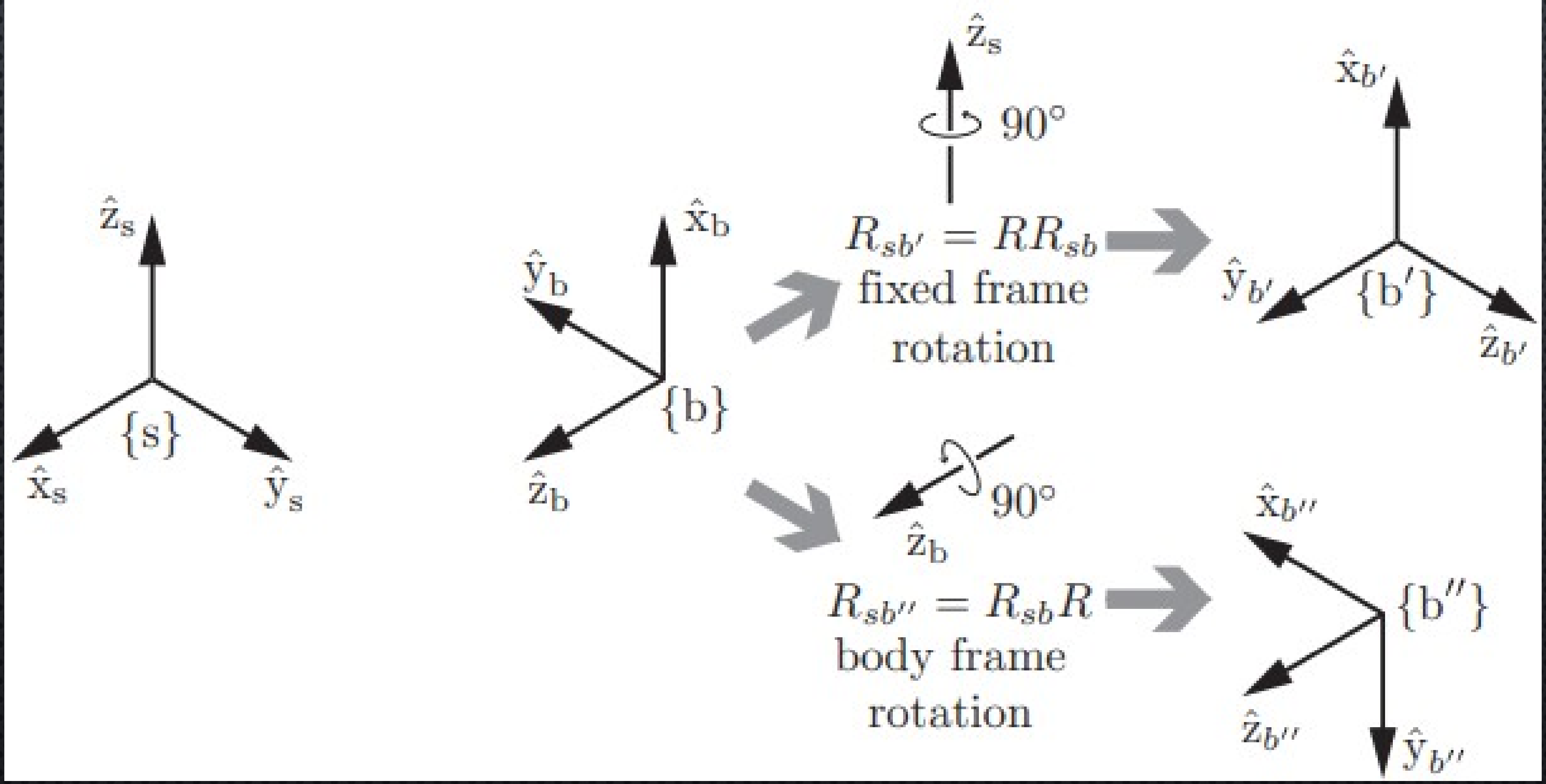
$$R_{ab} R_{bc} = R_{a\cancel{b}} R_{\cancel{b}c} = R_{ac}.$$

$$R_{ab} p_b = R_{a\cancel{b}} p_{\cancel{b}} = p_a.$$

OTHER PROPERTIES OF ROTATION MATRICES

- ROTATION MATRICES ARE ORTHOGONAL.
- INVERSE OF A ROTATION MATRIX IS EQUAL TO ITS TRANSPOSE
  - $R^T = R^{-1}$
- DET (R) = 1

FIXED AND BODY FRAME ROTATIONS



COMPOSITE TRANSFORMATIONS

HOMOGENOUS TRANSFORMATION MATRICES

- THE SPECIAL EUCLIDEAN GROUP SE(3), ALSO KNOWN AS THE GROUP OF RIGID-BODY MOTIONS OR HOMOGENEOUS TRANSFORMATION MATRICES IN R 3 , IS THE SET OF ALL 4 × 4 REAL MATRICES T OF THE FORM

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

PROPERTIES OF TRANSFORMATION MATRICES

- INVERSE OF A TRANSFORMATION MATRIX IS :

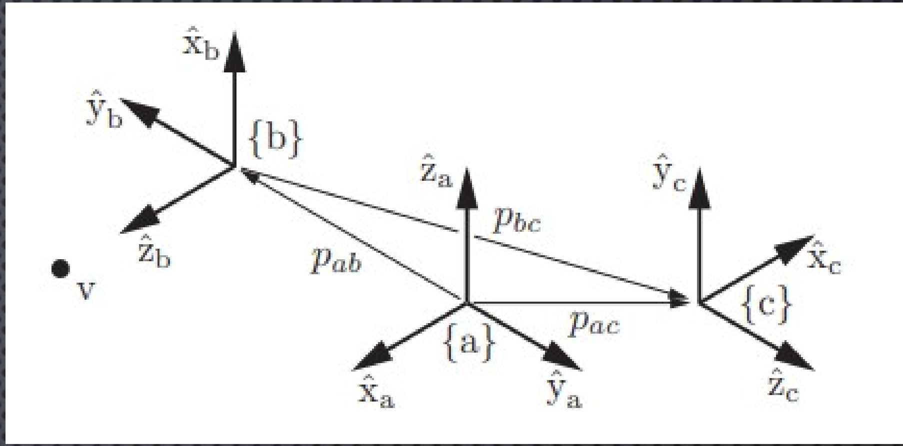
$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}.$$

- THE PRODUCT OF TWO TRANSFORMATION MATRICES IS ALSO A TRANSFORMATION MATRIX
- THE MULTIPLICATION OF TRANSFORMATION MATRICES IS ASSOCIATIVE, SO THAT $(T_1T_2)T_3 = T_1(T_2T_3)$, BUT GENERALLY NOT COMMUTATIVE: $T_1T_2 \neq T_2T_1$

USES OF TRANSFORMATION MATRICES

- TO REPRESENT THE CONFIGURATION (POSITION AND ORIENTATION) OF A RIGID BODY
- TO CHANGE THE REFERENCE FRAME IN WHICH A VECTOR OR FRAME IS REPRESENTED
- TO DISPLACE A VECTOR OR FRAME

EXAMPLE



- REPRESENTING A CONFIGURATION
  - ROTATION MATRICES

$$R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad R_{sc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

  - POSITION

$$p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

$$T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

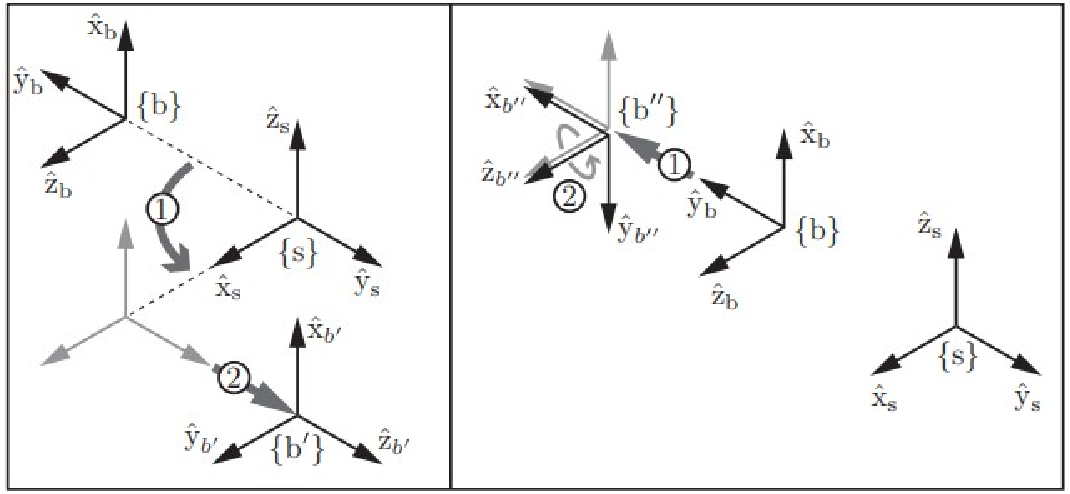
SUBSCRIPT CANCELLATION

- CHANGING THE REFERENCE FRAME OF A VECTOR OR A FRAME
  - BY A SUBSCRIPT CANCELLATION RULE ANALOGOUS TO THAT FOR ROTATIONS, FOR ANY THREE REFERENCE FRAMES {A}, {B}, AND {C}, AND ANY VECTOR V EXPRESSED IN {B} AS V_B, WHERE V_A IS THE VECTOR V EXPRESSED IN {A}.

$$T_{ab}T_{bc} = T_{a\cancel{b}}T_{\cancel{b}c} = T_{ac}$$
$$T_{ab}v_b = T_{a\cancel{b}}v_{\cancel{b}} = v_a,$$

- INVERSE OF A TRANSFORMATION MATRIX

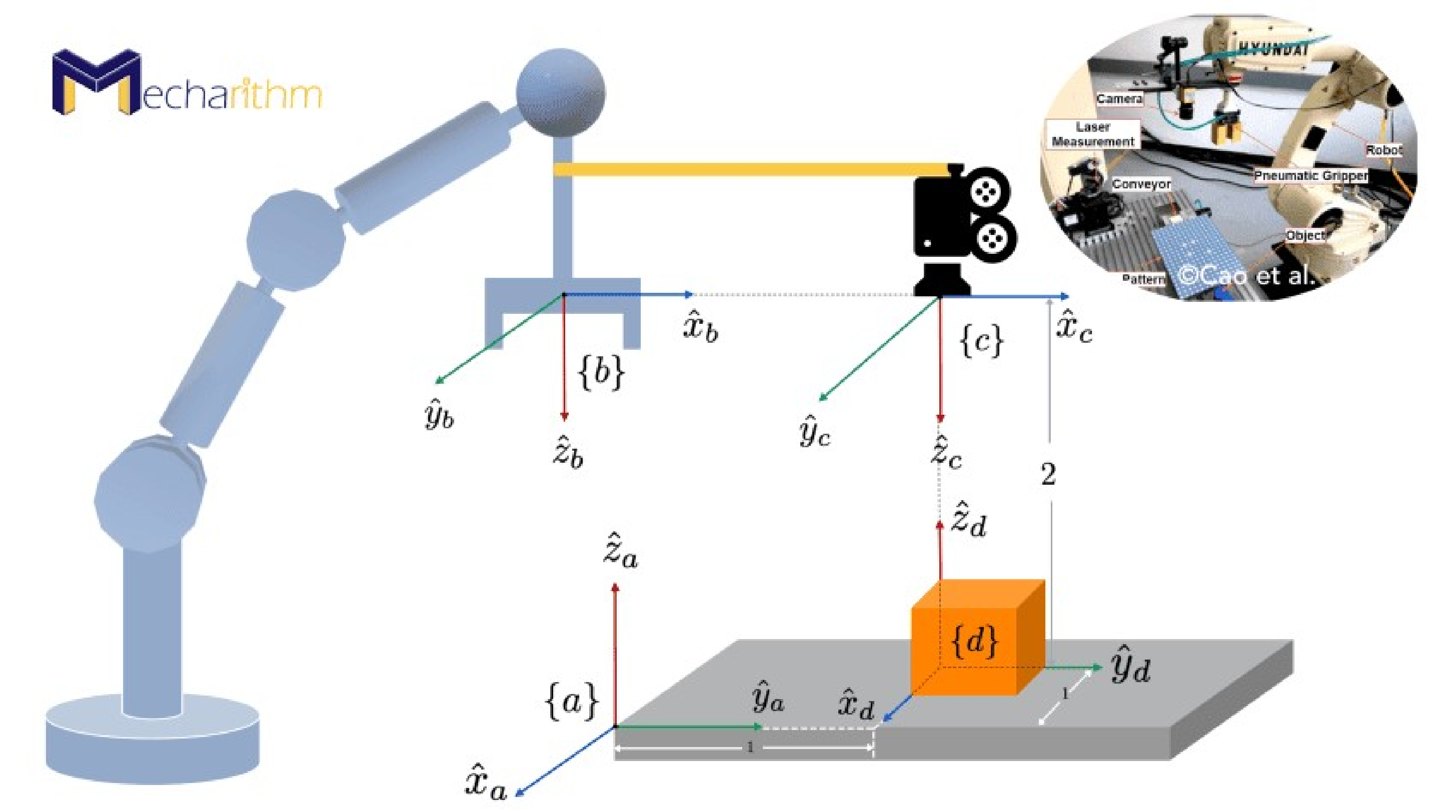
$$T_{de} = T_{ed}^{-1}$$



$$
\begin{aligned}
T_{sb'} &= TT_{sb} = \mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega},\theta)T_{sb} && \text{(fixed frame)} \\
&= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR_{sb} & Rp_{sb} + p \\ 0 & 1 \end{bmatrix} \\
T_{sb''} &= T_{sb}T = T_{sb}\,\mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega},\theta) && \text{(body frame)} \\
&= \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{sb}R & R_{sb}p + p_{sb} \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

FIXED AND BODY TRANSFORMATIONS

TRANSFORMATIONS IN MANIPULATORS



QUATERNIONS

QUATERNIONS

- EXTENDING THE NOTION OF COMPLEX NUMBERS TO HIGHER DIMENSION.
- QUATERNIONS ARE A NON-COMMUTATIVE NUMBER SYSTEM THAT EXTENDS THE COMPLEX NUMBERS.
- VISUALIZING QUATERNIONS

- THE COMPLEX NUMBERS C FORM A PLANE.
- THEIR OPERATIONS ARE VERY RELATED TO TWO-DIMENSIONAL GEOMETRY.
- IN PARTICULAR, MULTIPLICATION BY A UNIT COMPLEX NUMBER:
  - $|Z|^2 = 1$
    WHICH CAN ALL BE WRITTEN:
    $Z = E^{I\Theta}$

    GIVES A ROTATION:
    $R_Z (W) = ZW$

    BY ANGLE Θ.

- HOW DOES THIS WORK?
- $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, \quad i^2 = -1\}$
- ANY COMPLEX NUMBER HAS A LENGTH, GIVEN BY THE PYTHAGOREAN FORMULA:

$$|a + bi| = \sqrt{a^2 + b^2}.$$

- WE CAN ADD AND SUBTRACT IN C. FOR EXAMPLE:

$$a + bi + c + di = (a + c) + (b + d)i.$$

- WE CAN ALSO MULTIPLY, WHICH IS MUCH MESSIER:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

QUATERNION ALGEBRA

- IT FOLLOWS $i^2 = j^2 = k^2 = ijk = -1.$
- THE QUATERNIONS A $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$
- SUPPOSE WE HAVE TWO UNIT QUATERNIONS, P AND Q, WITH SOME VECTOR U.

$$\mathbf{q} = \cos\alpha + \boldsymbol{u}\sin\alpha$$
$$\mathbf{p} = \cos\beta + \boldsymbol{u}\sin\beta$$

$$\mathbf{r} = \mathbf{pq} = \cos(\alpha + \beta) + \boldsymbol{u}\sin(\alpha + \beta)$$

REPRESENTING A QUATERNION

- A QUATERNION CONSISTS OF ONE SCALAR AND A 3-ELEMENT UNIT VECTOR.
- COMMON REPRESENTATIONS
  - $q = w + xi + yj + zk$
  - $q = q_0 + q_1 i + q_2 j + q_3 k$
    - Q_0 IS A SCALAR VALUE REPRESENTING AN ANGLE OF ROTATION
    - Q_1, Q_2, AND Q_3 CORRESPOND TO AN AXIS OF ROTATION ABOUT WHICH THE ANGLE IS PERFORMED
- ALTERNATIVE REPRESENTATIONS
  - $q = (q_0, q_1, q_2, q_3)$
  - $q = (q_0, \mathbf{q}) = q_0 + \mathbf{q}$

QUATERNION TO ROTATION MATRIX

$$R(Q) = \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{bmatrix}$$

- GIVEN A QUATERNION, YOU CAN FIND THE CORRESPONDING THREE DIMENSIONAL ROTATION MATRIX USING THE FOLLOWING FORMULA

WHY QUATERNIONS?

- BETTER COMPUTATIONALLY
  - FOR QUATERNIONS VERSUS A 3X3 ROTATION MATRIX, THE QUATERNION HAS THE ADVANTAGE IN SIZE (4 SCALARS VS. 9) AND SPEED (QUATERNION MULTIPLICATION IS MUCH FASTER THAN 3X3 MATRIX MULTIPLICATION).

- NO GIMBAL LOCK

GIMBAL LOCK

EULER'S THEOREM

- ANY TWO INDEPENDENT ORTHONORMAL COORDINATE FRAMES CAN BE RELATED BY A SEQUENCE OF ROTATIONS (NOT MORE THAN THREE) ABOUT COORDINATE AXES, WHERE NO TWO SUCCESSIVE ROTATIONS MAY BE ABOUT THE SAME AXIS.
- WE CAN REPRESENT AN ORIENTATION WITH 3 NUMBERS
- THIS GIVES US 12 REDUNDANT WAYS TO STORE AN ORIENTATION USING EULER ANGLES.
- WHAT ARE THE 12 WAYS?

- ASSUMING WE LIMIT OURSELVES TO 3 ROTATIONS WITHOUT SUCCESSIVE ROATIONS ABOUT THE SAME AXIS, WE COULD USE ANY OF THE FOLLOWING 12 SEQUENCES:
  - XYZ
  - XZY
  - XYX
  - XZX
  - YXZ
  - YZX
  - YXY
  - YZY
  - ZXY
  - ZYX
  - ZXZ
  - ZYZ

VISUAL DEMONSTRATION

GIMBAL LOCK

SINGULARITY

- A ROBOT SINGULARITY IS A CONFIGURATION IN WHICH THE ROBOT END-EFFECTOR BECOMES BLOCKED IN CERTAIN DIRECTIONS
- SINGULARITIES IN MANIPULATORS

REFERENCES

- MODERN ROBOTICS – KEVIN LYNCH
- INTRODUCTION TO ROBOTICS: MECHANICS AND CONTROL – J J CRAIG

THANK YOU!