Bundle Adjustment

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Structure from Motion (SfM)

- Recovering the 3D geometry or “structure” of the scene and the camera motion from a set of 2D images when a camera is subject to “motion”
  - The structure refers to the 3D world coordinates of the captured points
  - The motion refers to the 3D world coordinates from which various images were captured from the camera(s)

Bundle Adjustment (BA)

- This is an optimisation algorithm to solve the problem of SfM
- The non-linear least squares constraint problem which we discussed previously is used to model the cost function in the Bundle Adjustment setting

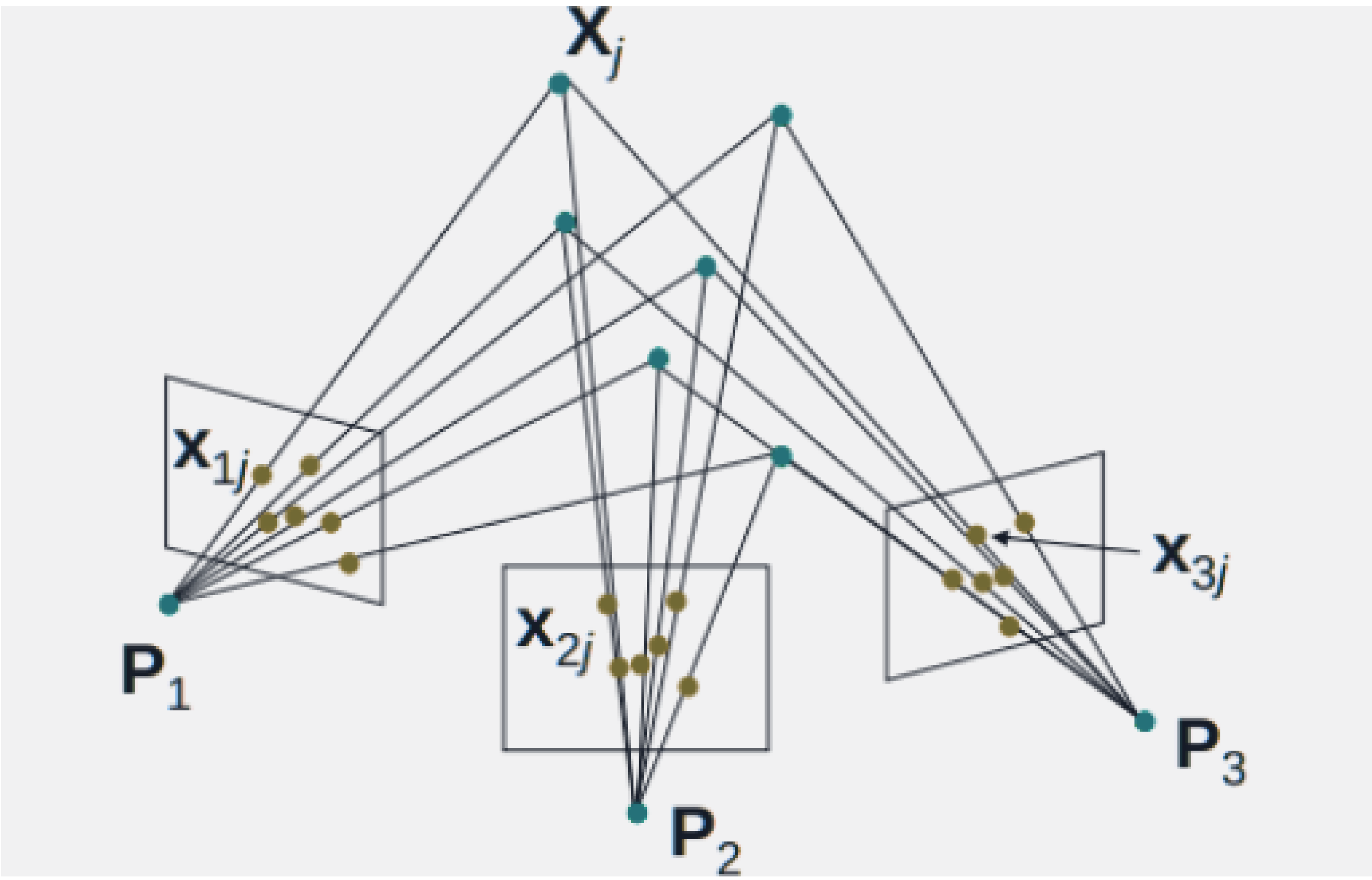
Initialisation for BA

- The following approaches help provide initialisation to BA: (brief overview)
  - Real world information like IMU, Odometry from which we obtain our initial guess for poses. Then by triangulation, obtain initial guess of 3D points as intial guess for BA
  - Fundamental Matrix estimation using the 8 point algorithm on the first pair of images to get the relative poses. Again, by triangulation we can obtain guesses for the 3D points from P3P
  - The relative poses and 3D point estimates can also be obtained by just pairwise fundamental matrix estimation

Reprojection Error

- What we have:
  - m cameras
  - n points
  - Known correspondences across all these cameras and points

- Unknowns:
  - m matrices P_i
  - n coordinates X_i

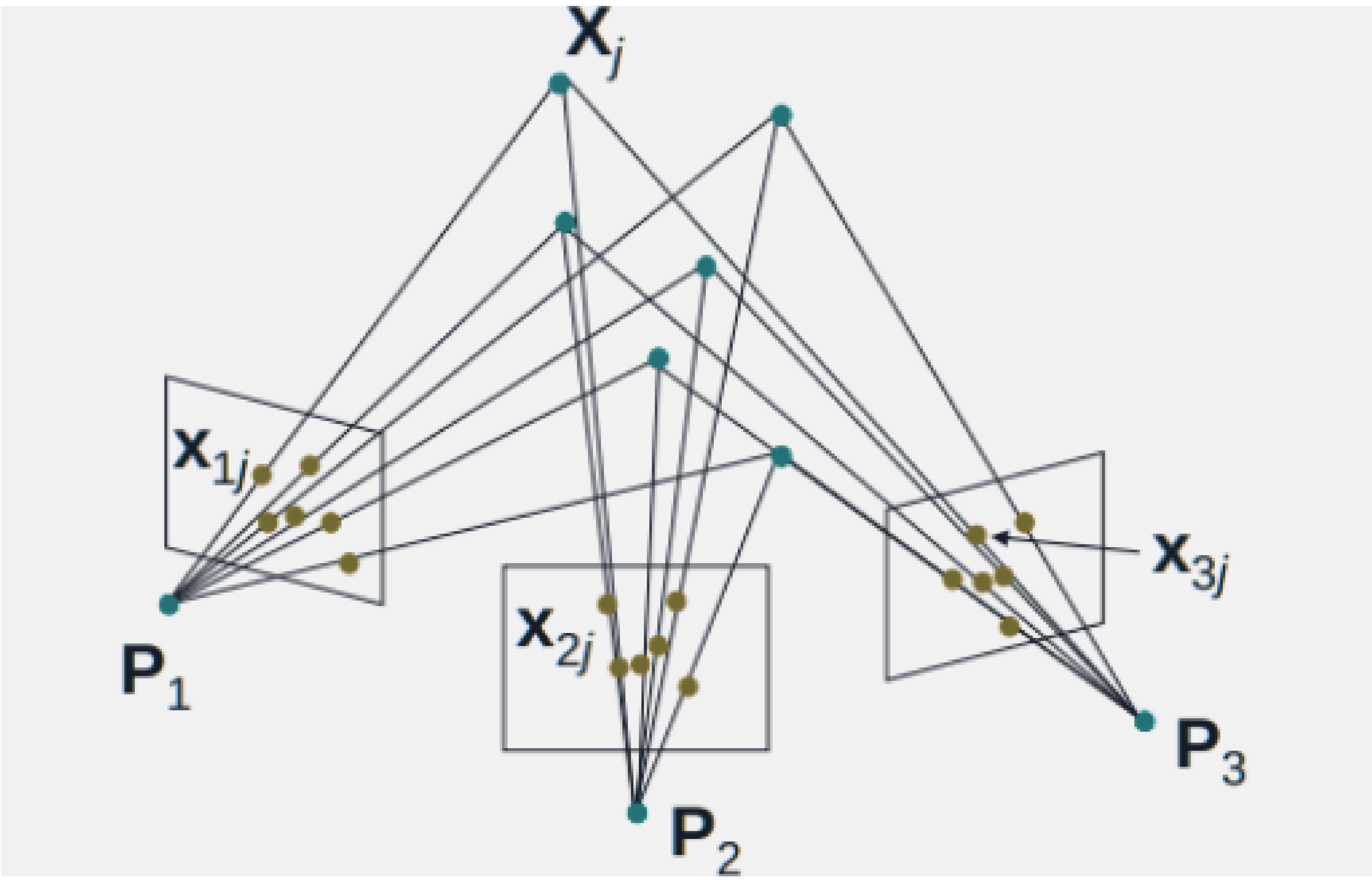


- Known:
  - Data association, i.e., correspondences across images which relate the pixels that correspond to the same real world point taken from various cameras and hence, viewpoints

Reprojection Error

- What we have:
  - m cameras
  - n points
  - Known correspondences across all these cameras and points
- The camera model expressed in homogeneous coordinates is given by the following set of equations

$$\vec{\mathrm{x}}_{ij} = \lambda_{ij} P_i \vec{X}_j, \quad 1 \leq i \leq m, 1 \leq j \leq n$$



- And finally, the reprojection error is given by $\|\lambda_{ij} P_i \vec{X}_j - \vec{\mathrm{x}}_{ij}\|^2$, which denotes how closely the estimated 3D point resembles its actual projection
  - The lambdas are scaling factors to ensure that the third coordinate in the image coordinates in homogeneous system is 1

An Example

- Given that we have 10k images with 1k points each. Each point is seen 10 times on avg
- How many known and unknown parameters do we have?
- Known parameters: 20M
  - These are the image coordinates: 2 * 10k * 1k = 20M known params
  - Why are we multiplying by 2? Pixel coordinates
- Unknown parameters: ~13M
  - No of world points = (10k * 1k) / 10 = 1M. Division by 10 as the same point is repeated in 10 images on average. Hence, we have 1M * 3 = 3M world point params
    - Why are we multiplying by 3? World coordinates
  - For each of the 10k images, we have 10k corresponding orientations of the camera that captured the image. Hence, 10k * 6 = 60k orientation params
  - For each of the image points, there is a single corresponding scaling factor. Hence, the scaling factors contribute to 10k * 1k = 10M scaling params

Can we do better?

- Get rid of the scale parameters
- How? Convert from homogeneous coordinate system to Euclidean coordinate system

Cost Function

$$\arg\min_{\vec{X}j,P_i} \sum_{i=1}^{M}\sum_{j=1}^{N}\left(\left[\frac{P_{i11}X_j + P_{i12}Y_j + P_{i13}Z_j + P_{i14}}{P_{i31}X_j + P_{i32}Y_j + P_{i33}Z_j + P_{i34}} - x_{ij}\right]^2 + \left[\frac{P_{i21}X_j + P_{i22}Y_j + P_{i23}Z_j + P_{i24}}{P_{i31}X_j + P_{i32}Y_j + P_{i33}Z_j + P_{i34}} - y_{ij}\right]^2\right)$$

$$\arg\min_{\vec{X}j,P_i} \sum_{i=1}^{M}\sum_{j=1}^{N}\left\|\frac{P_{(1:2)i}\vec{X}_j}{P_{3i}\vec{X}_j} - \vec{x}_{ij}\right\|^2$$

Cost Function

$$\arg\min_{\vec{X}j,P_i} \sum_{i=1}^{M}\sum_{j=1}^{N} \left(\left[\frac{P_{i11}X_j + P_{i12}Y_j + P_{i13}Z_j + P_{i14}}{P_{i31}X_j + P_{i32}Y_j + P_{i33}Z_j + P_{i34}} - x_{ij}\right]^2 + \left[\frac{P_{i21}X_j + P_{i22}Y_j + P_{i23}Z_j + P_{i24}}{P_{i31}X_j + P_{i32}Y_j + P_{i33}Z_j + P_{i34}} - y_{ij}\right]^2 \right)$$

$$\arg\min_{\vec{X}j,P_i} \sum_{i=1}^{M}\sum_{j=1}^{N} \left\| P_i\vec{X}_j - \vec{x}_{ij} \right\|^2$$

$$\arg\min_{\vec{X}j,P_i} \sum_{i=1}^{M}\sum_{j=1}^{N} \left\| \frac{P_{(1:2)i}\vec{X}_j}{P_{3i}\vec{X}_j} - \vec{x}_{ij} \right\|^2$$

$$\arg\min_{Xj,P_i} \sum_{i=1}^{M}\sum_{j=1}^{N} \left\| \widehat{\vec{x}}_{ij} - \vec{x}_{ij} \right\|^2$$

Reprojection Error

- P_i is the projection matrix of the ith view and X_j is the world coordinate of the jth world point
- x_{ij} is the known observation while $\hat{x}_{ij}$ is the predicted projection of the jth world point in the ith view

What do we finally have?

- We have 2mn equations in total (2 for each match)
- This is a non-linear optimisation problem wherein we minimise the sum of the squared reprojection errors of the reconstructed n 3D points over m images
- How do we solve this familiar looking formulation of a non-linear least squares cost function? **Levenberg-Marquardt Algorithm**

$$\left(\mathbf{J}^{\top}\mathbf{J} + \lambda\mathbf{I}\right)\Delta\mathbf{k} = -\mathbf{J}^{\top}\mathbf{r}(\mathbf{k})$$

- Here r is the residual vector, which is the difference vector between the observed image coordinate vector and the predicted projection of the corresponding world point through the view

$$\hat{\vec{x}}_{ij} = P_i \vec{X}_j$$

Objective Function:

$$F(k) = r^T r$$

$$\arg\min \sum_{i=1}^{M} \sum_{j=1}^{N} \left\| \hat{\vec{x}}_{ij} - \vec{x}_{ij} \right\|^2$$

Understanding RESIDUAL VECTOR

Residual vector

$$r = \begin{bmatrix} \vec{r}_{11} \\ \vec{r}_{12} \\ \vec{r}_{13} \\ \vdots \\ \vec{r}_{1N} \\ \vec{r}_{21} \\ \vdots \\ \vec{r}_{2N} \\ \vdots \\ \vec{r}_{M1} \\ \vdots \\ \vec{r}_{MN} \end{bmatrix} \quad \begin{bmatrix} f_{11} \\ g_{11} \\ f_{12} \\ g_{12} \\ \vdots \\ f_{1N} \\ g_{1N} \\ f_{21} \\ g_{21} \\ \vdots \\ f_{2N} \\ g_{2N} \\ \vdots \\ f_{M1} \\ g_{M1} \\ \vdots \\ f_{MN} \\ g_{MN} \end{bmatrix} \longrightarrow \begin{bmatrix} \hat{x}_{11} - x_{11} \\ \hat{y}_{11} - y_{11} \\ \hat{x}_{12} - x_{12} \\ \hat{y}_{12} - y_{12} \\ \vdots \\ \hat{x}_{1N} - x_{1N} \\ \hat{y}_{1N} - y_{1N} \\ \vdots \end{bmatrix}$$

2N

2N

⋮

M times

2MN terms

(Side note: Note that $\dfrac{\partial f_{11}}{\partial k} = \dfrac{\partial \hat{x}_{11}}{\partial k}$)

$$\hat{\vec{x}}_{ij} = P_i \vec{X}_j$$

Residual vector

Objective Function :

$$F(k) = r^T r$$

$$\arg\min \sum_{i=1}^{M} \sum_{j=1}^{N} \left\| \hat{\vec{x}}_{ij} - \vec{x}_{ij} \right\|^2$$

$$J = \frac{\partial r}{\partial k} \qquad \Big[\underbrace{P_1}_{P_{1_{11}} P_{1_{12}} P_{1_{13}} \dots P_{1_{34}}} \quad P_2 \quad \dots \quad \underbrace{P_M}_{P_{M_{11}} P_{M_{11}} \dots P_{M_{34}}} \quad \underbrace{\vec{X}_1}_{X_1 Y_1 Z_1} \quad \vec{X}_2 \quad \dots \dots \quad \underbrace{\vec{X}_N}_{X_N Y_N Z_N} \Big]$$

$\boxed{\text{INDIVIDUAL JACOBIANS: } J_{ij_m},\ G_{ij_m},\ J_{ij_s},\ G_{ij_s}} \Rightarrow$

'm'otion

$$J_{ij_m} = \left[\frac{\partial f_{ij}}{\partial P_i}\right] = \left[\frac{\partial f_{ij}}{\partial P_{i_{11}}} \ \dots\dots \ \frac{\partial f_{ij}}{\partial P_{i_{34}}} \right]$$

$$G_{ij_m} = \left[\frac{\partial g_{ij}}{\partial P_i}\right] = \left[\frac{\partial g_{ij}}{\partial P_{i_{11}}} \ \dots\dots \ \frac{\partial g_{ij}}{\partial P_{i_{34}}} \right]$$

's'tructure

$$J_{ij_s} = \left[\frac{\partial f_{ij}}{\partial \vec{X}_j}\right] = \left[\frac{\partial f_{ij}}{\partial X_j} \ \ \frac{\partial f_{ij}}{\partial Y_j} \ \ \frac{\partial f_{ij}}{\partial Z_j} \right]$$

$$G_{ij_s} = \left[\frac{\partial g_{ij}}{\partial \vec{X}_j}\right] = \left[\frac{\partial g_{ij}}{\partial X_j} \ \ \frac{\partial g_{ij}}{\partial Y_j} \ \ \frac{\partial g_{ij}}{\partial Z_j} \right]$$

So, from above, notice when we write J_{12_m} simply (Notice 'm') means derivative w.r.t P_1. And J_{12_s} would mean derivative w.r.t $\vec{X}_2$.

$$J_{2MN,\,12M+3N} = \frac{\partial r}{\partial k}$$

MOTION | STRUCTURE

$$= \Big[\underbrace{\frac{\partial r}{\partial p_1}\;\frac{\partial r}{\partial p_2}\;\cdots\cdots\;\frac{\partial r}{\partial p_M}}_{M\ \text{times}}\;\Big|\;\underbrace{\frac{\partial r}{\partial \vec{X}_1}\;\frac{\partial r}{\partial \vec{X}_2}\cdots\cdots\frac{\partial r}{\partial \vec{X}_N}}_{N\ \text{times}}\Big]$$

$$= \begin{array}{c} 2N\left\{\begin{array}{c}\\ \\ \\ \\ \\ \\ \end{array}\right. \\ 2N\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right. \\ \\ \\ \\ \end{array}
\left[\begin{array}{cccc|cccc}
\underset{1\times 12}{J_{11m}} & \underset{1\times 12}{0} & \underset{1\times 12}{0}\cdots & \underset{1\times 12}{0} & \underset{1\times 3}{J_{11s}} & \underset{1\times 3}{0}\cdots & \cdots & \underset{1\times 3}{0} \\
\underset{1\times 12}{G_{11m}} & 0 & \cdots & 0 & G_{11s} & 0 & \cdots & 0 \\
J_{12m} & \vdots & \cdots & 0 & 0 & J_{12s} & \cdots & 0 \\
G_{12m} & \vdots & & & 0 & G_{12s} & \cdots & 0 \\
\vdots & \vdots & \cdots & 0 & \vdots & & & \\
J_{1Nm} & \vdots & \cdots & 0 & 0 & \cdots & \cdots & J_{1Ns} \\
G_{1Nm} & & \cdots & 0 & 0 & \cdots & \cdots & G_{1Ns} \\
\hline
0 & J_{21m} & \cdots & 0 & J_{21s} & 0 & \cdots & 0 \\
0 & G_{21m} & \cdots & 0 & G_{21s} & 0 & \cdots & 0 \\
\vdots & \vdots & & & 0 & J_{22s} & \cdots & \\
 & & & & \vdots & G_{22s} & \cdots & \\
0 & G_{2Nm} & \cdots & 0 & 0 & \vdots & & J_{2Ns} \\
 & & & & & & & G_{2Ns} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & J_{M1m} & J_{M1s} & 0 & \vdots & 0 \\
0 & \vdots & \vdots & G_{M1m} & G_{M1s} & 0 & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & 0 & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & J_{MNs} \\
0 & & \vdots & G_{MNm} & 0 & \vdots & & G_{MNs}
\end{array}\right]$$

Update vector

$$\Delta k = \left[\begin{array}{c}
\left.\begin{array}{c}\delta p_{1_{11}} \\ \vdots \\ \delta p_{1_{34}}\end{array}\right\} 1^{st}\ \text{Camera} \\
\left.\begin{array}{c}\delta p_{2_{11}} \\ \vdots \\ \delta p_{2_{34}}\end{array}\right\} 2^{nd}\ \text{camera} \\
\vdots \\
\left.\begin{array}{c}\delta p_{M_{11}} \\ \vdots \\ \delta p_{M_{34}}\end{array}\right\} M^{th}\ \text{Cam} \\
\hline
\left.\begin{array}{c}\delta X_1 \\ \delta Y_1 \\ \delta Z_1\end{array}\right\} 1^{st}\ \text{3D point} \\
\vdots \\
\left.\begin{array}{c}\delta X_N \\ \delta Y_N \\ \delta Z_N\end{array}\right\} N^{th}\ \text{3D point}
\end{array}\right]$$

$$\overbrace{\Big(\underbrace{J^T J}_{(12M+3N,\,2MN)\ \times\ (2MN,\,12M+3N)} + \lambda I\Big)}^{(12M+3N,\,12M+3N)}\;\underbrace{\Delta k}_{(12M+3N,\,1)} = -\underbrace{J^T}_{(12M+3N,\,2MN)}\;\underbrace{r(k)}_{(2MN,\,1)}$$

Questions?

Acknowledgements

- Borrowed most of the slides from the SfM notion doc by Sai Shubodh

References

- Muti view geometry by Hartley and Zisserman
- RRC Summer School 2023
- Bundle Adjustment – Part I Introduction & Application - Cyril Stachniss
- Bundle Adjustment – Part II Numerics of BA - Cyril Stachniss
- Lecture: Photogrammetry I & II (2021, Uni Bonn, Cyrill Stachniss) - Lectures 53, 54, 55

Further Readings

- Incremental SfM, Failure cases of SfM, Multi View Stereo, and much more
  - Structure from motion, L. Lazebnik, N. Snavely, M. Herbert
  - Structure from Motion, Derek Hoiem

Thank You!