Trajectory Planning

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Definitions

A **system** is a set of points in the space X.

A **configuration** of a system is the location of every points in the system

Configuration space ($\mathcal{C}$) is a **metric space** comprising all the configurations of the system – space spanned by the configuration variables.

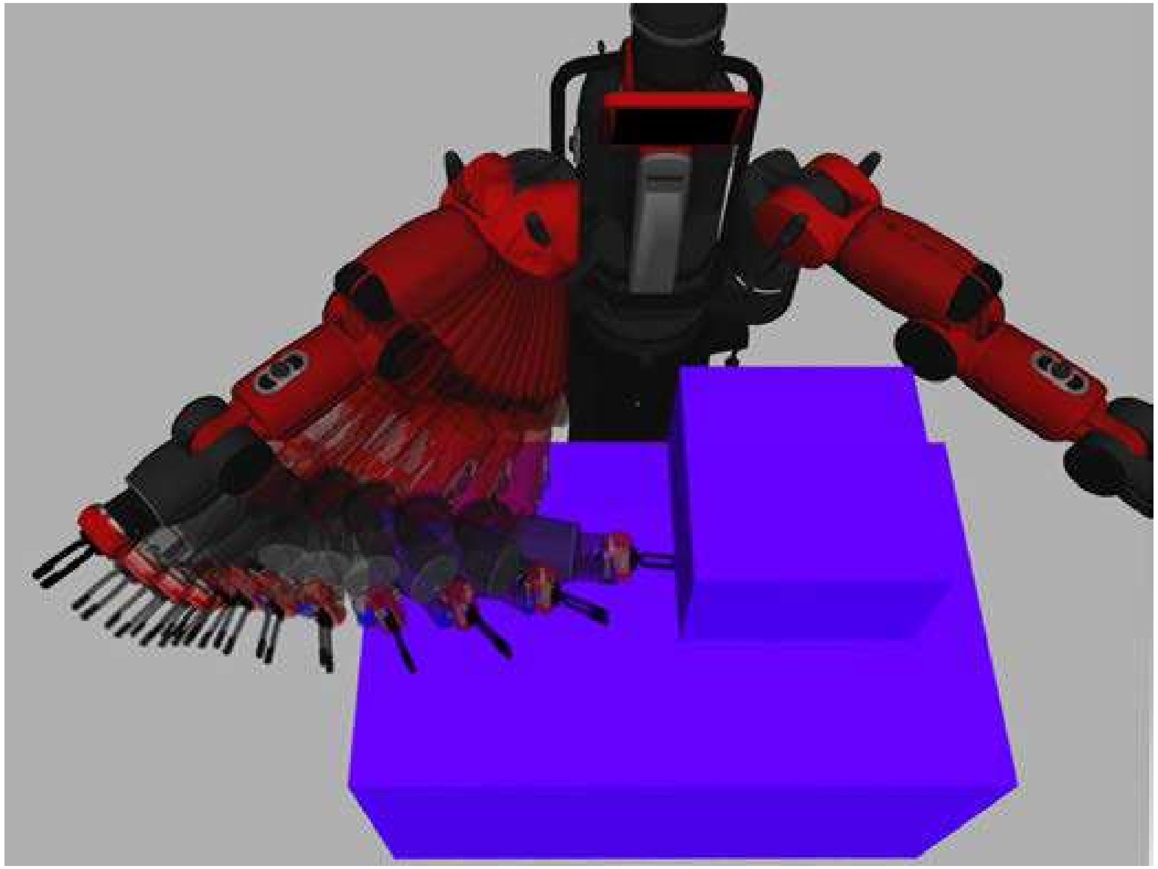
Degrees of freedom of a system is the dimension of $\mathcal{C}$-space – min. no. of real numbers required to specify the configuration.

- The **configuration** of a robot is a complete specification of the positions of every point of the robot.

- A minimum number n of real-valued coordinates needed to represent the Configuration is no. of **Degrees of Freedom (DoF)**

Introduction

- **Motion planning,** finding the robot's motion from a start state to a goal state that avoids obstacles in the environment and satisfies other constraints, such as joint limits or torque limits.



Introduction

- **Path Planning**:
  - subproblem of the general motion planning problem.
  - purely geometric problem of finding a collision-free path from a start configuration to a goal configuration.
  - Not necessarily concerned with the system dynamics, constraints on the motion, or control inputs.
  - The path returned by the path planner can be time-scaled to create a feasible **trajectory**.

- **Path** $\theta(s)$ maps a scalar parameter s, assumed to be 0 at the start of the path and 1 at the end to a robot's configuration space Θ

$$\theta(s): [0,1] \rightarrow \Theta$$

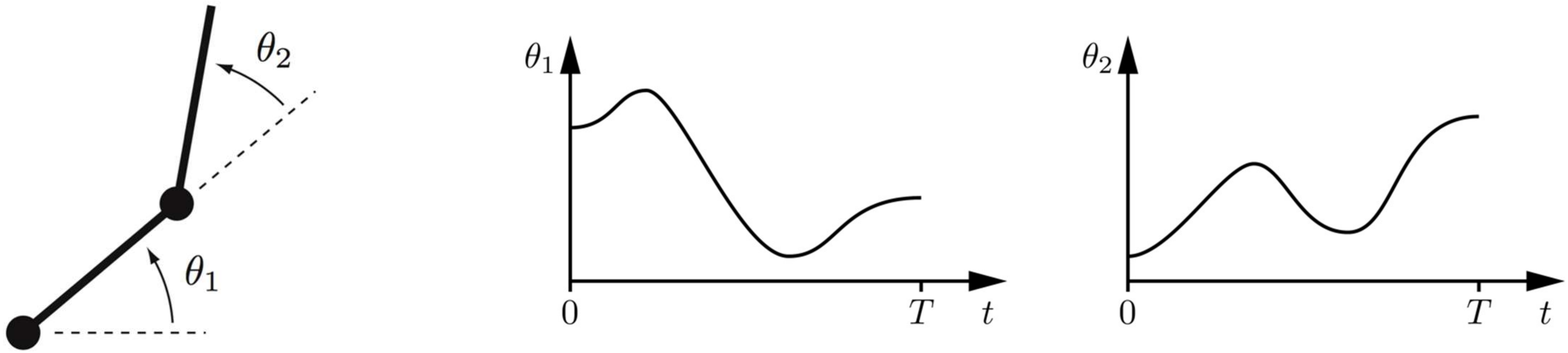
$$\text{With } \theta(0) = \theta_{init} \text{ and } \theta(1) = \theta_{final}$$

Trajectory: A specification of the configuration as a function of time.

$$\theta(t), t \in [0, \mathrm{T}]$$

With $\theta(t_0) = \theta_{init}$ and $\theta(t_f) = \theta_{final}$

$T = t_f - t_0$ is the time taken to execute the trajectory

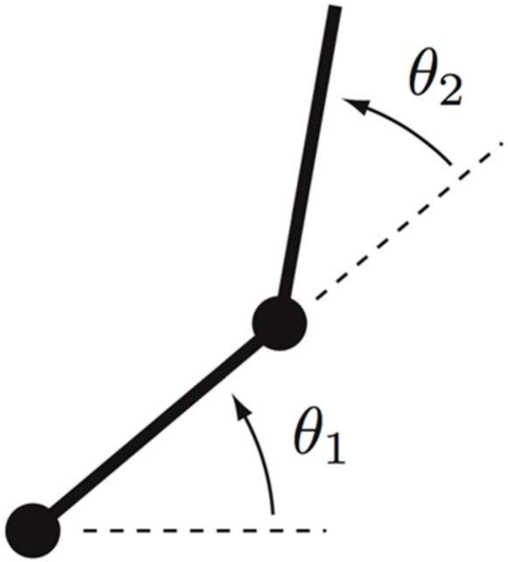


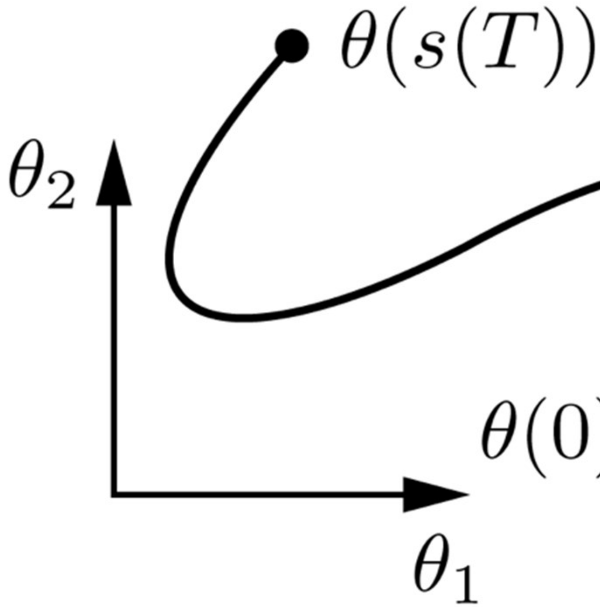
Sometimes we may decouple the speed from the C-Space.

- **Path:** A specification of the configuration as a function of a path parameter.

$$\theta(s), \ s \in [0, 1]$$

- **Time scaling:** A mapping $s(t)$: $[0, \ T] \rightarrow [0, 1]$, from time to the path parameter.
  - Assigns a value s to each time $t \in [0, \ T]$
  - How fast the path is followed?





- Motion as a function of $\theta(s)$ and $s(t)$:

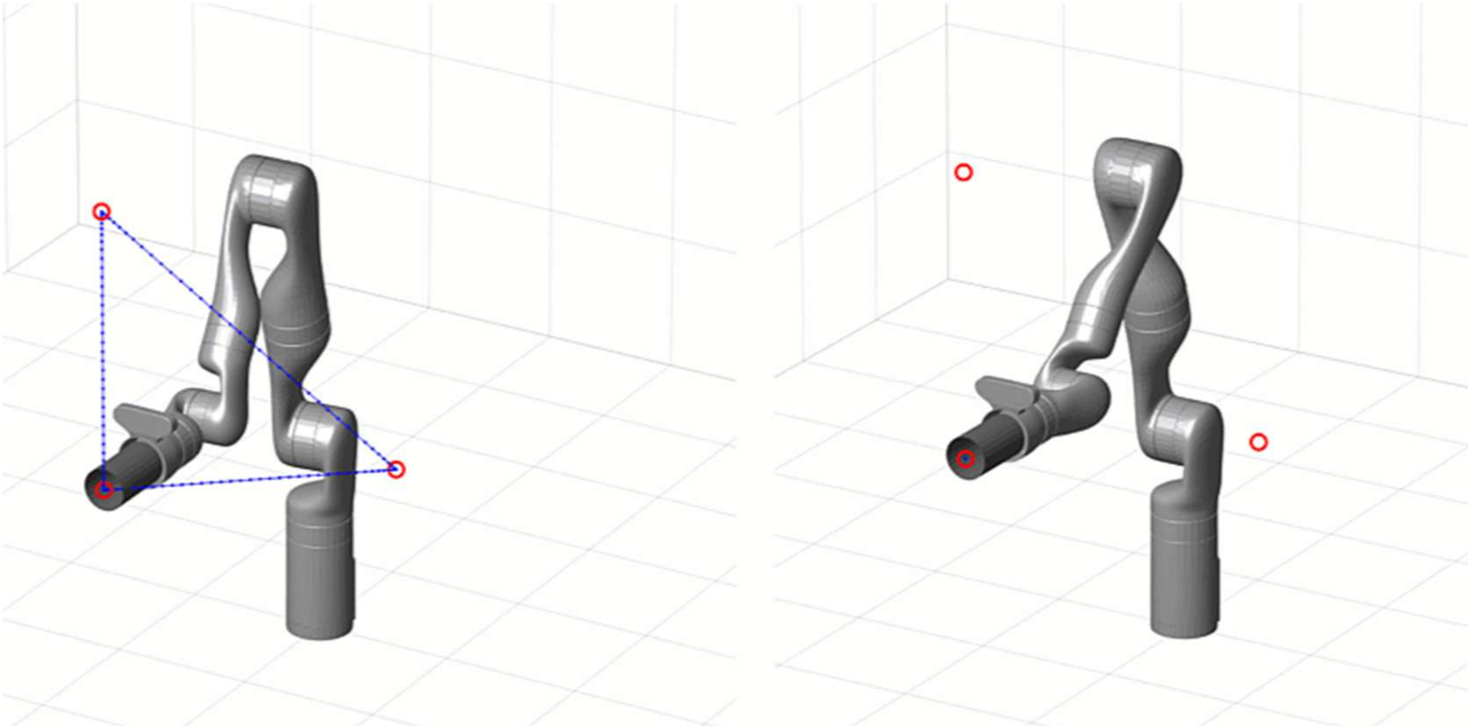
$$\dot{\theta} = \frac{d\theta}{ds}\dot{s},$$
$$\ddot{\theta} = \frac{d\theta}{ds}\ddot{s} + \frac{d^2\theta}{ds^2}\dot{s}^2$$

Both $\theta(s)$ and $s(t)$ must be twice-differentiable to ensure that the robot's acceleration is well-defined.

Task Space vs. Joint Space

Task space means the waypoints and interpolation are on the Cartesian pose

Joint space means the waypoints and interpolation are directly on the joint positions



[Left] Task-space trajectory [Right] Joint-space trajectory

https://blogs.mathworks.com/

Trajectory Generation

- A polynomial is the sum of variables raised to powers and multiplied by coefficients.

$$\sum_{i=0}^{n} a_i x^i$$

- The degree or order of the polynomial is the highest power of the variables.

- Polynomials of different degrees are

$f(x) = x + 1$	Linear
$f(x) = x^2 + x + 1$	Quadratic
$f(x) = x^3 + x^2 + x + 1$	Cubic
$f(x) = x^4 + x^3 + x^2 + x + 1$	Quartic

Parametric Equations

- Given two vectors, interpolate along the line that joins the tip of two vectors.

$$x(s) = (1 - s)x_0 + s x_f$$
$$y(s) = (1 - s)y_0 + s y_f$$

Where $0 \le s \le 1$.

Let $\boldsymbol{p_0} = (x_0, y_0), \boldsymbol{p_f} = (x_f, y_f)$, and $\boldsymbol{p}(s) = (x(s), y(s))$

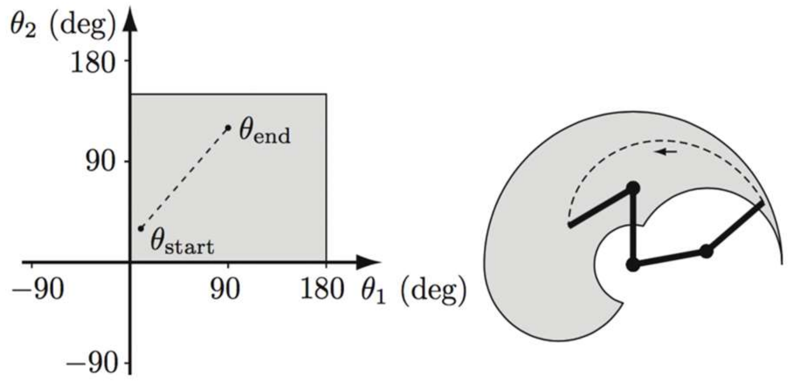
$$\boldsymbol{p}(s) = (1 - s)\boldsymbol{p_0} + s\boldsymbol{p_f}$$

Point-to-Point Trajectories

- straight line in joint space

$$\theta(s) = \theta_{\text{start}} + s(\theta_{\text{end}} - \theta_{\text{start}})$$

Where $s \in [0, 1]$



With derivatives

$$\frac{d\theta}{ds} = \theta_{\text{end}} - \theta_{\text{start}}$$

$$\frac{d^2\theta}{ds^2} = 0$$

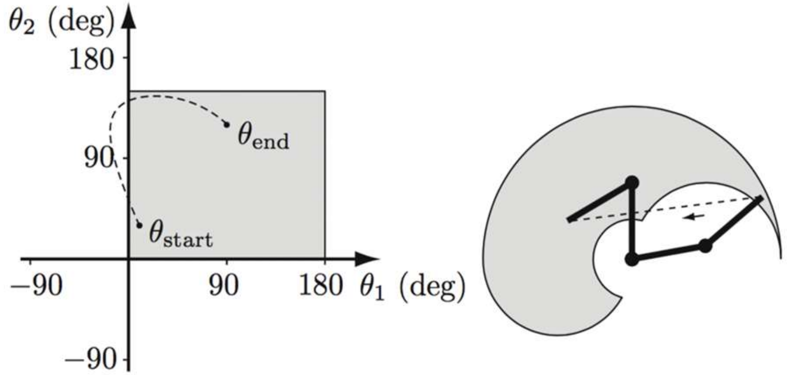
- Straight lines in JS generally do not yield straight-line motion of the end-effector in TS

Point-to-Point Trajectories

- If task-space straight-line motions are desired

$$X(s) = X_{\text{start}} + s(X_{\text{end}} - X_{\text{start}})$$

Where $s \in [0, 1]$

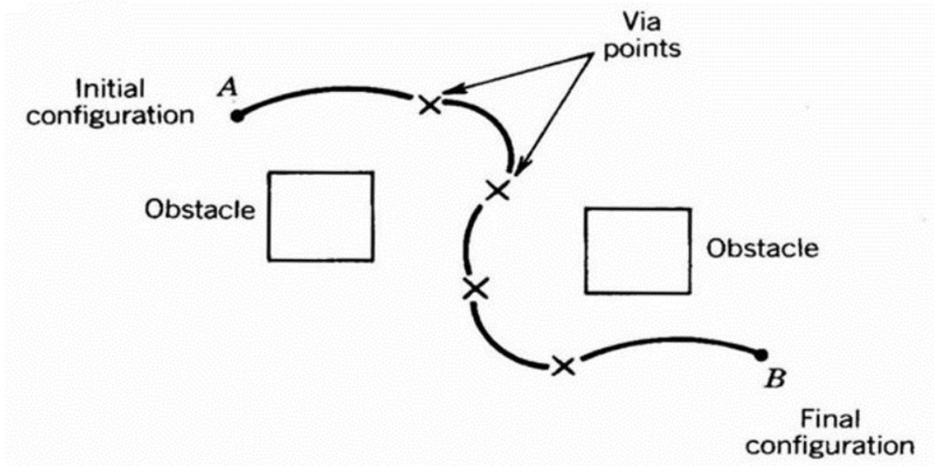


Caveats:

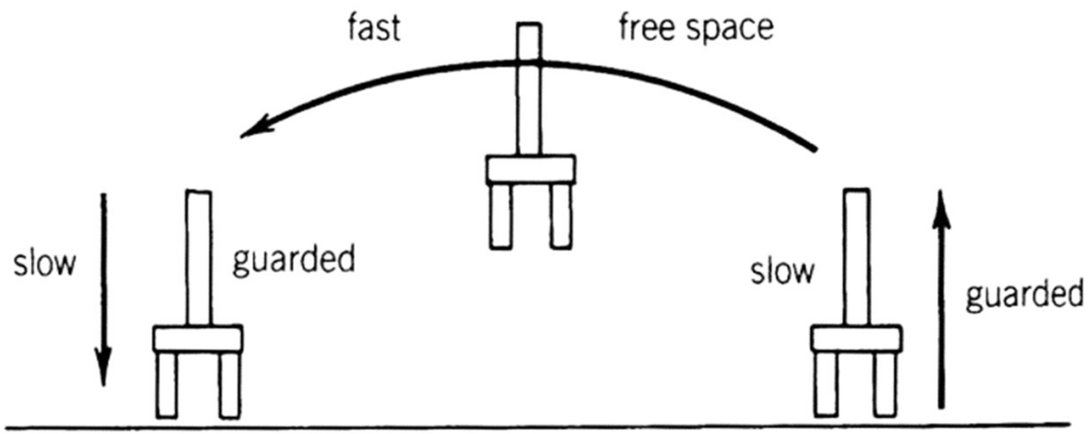
1. If the path passes near a kinematic singularity, the joint velocities may become unreasonably high
2. the robot's reachable TS may not be convex in X coordinates - two reachable endpoints may be reachable but not the intermediate points

Trajectory Planning

- Often, path planning algorithms give the sequence of **via points** along the path



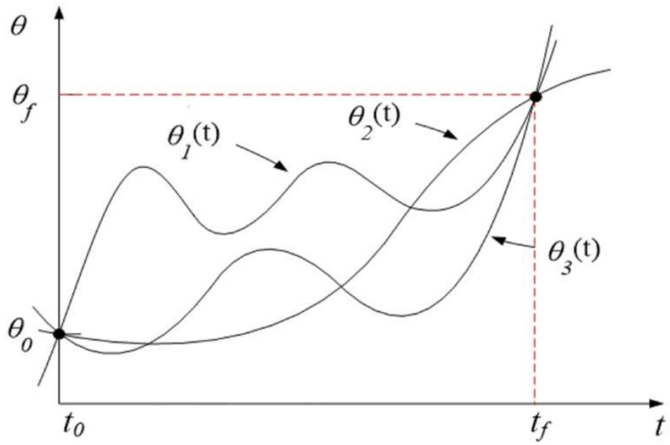
Via points to plan motion around obstacles



Guarded and free motions

Problem:

- Find a trajectory that connects an initial to a final configuration while satisfying other specified constraints at the endpoints (e.g., velocity and/or acceleration constraints)



- One way to generate smooth curves between two configurations is by a polynomial function of t.

- To satisfy n constraints, a polynomial with n independent coefficients is required.

- 1DoF:
  - What kind of time function to choose?
  - Position -> Velocity -> Acceleration -> Jerk
  - Acceleration is proportional to Torque or Force

Cubic polynomial

- A convenient form for the time scaling $s(t)$ is a cubic polynomial of time

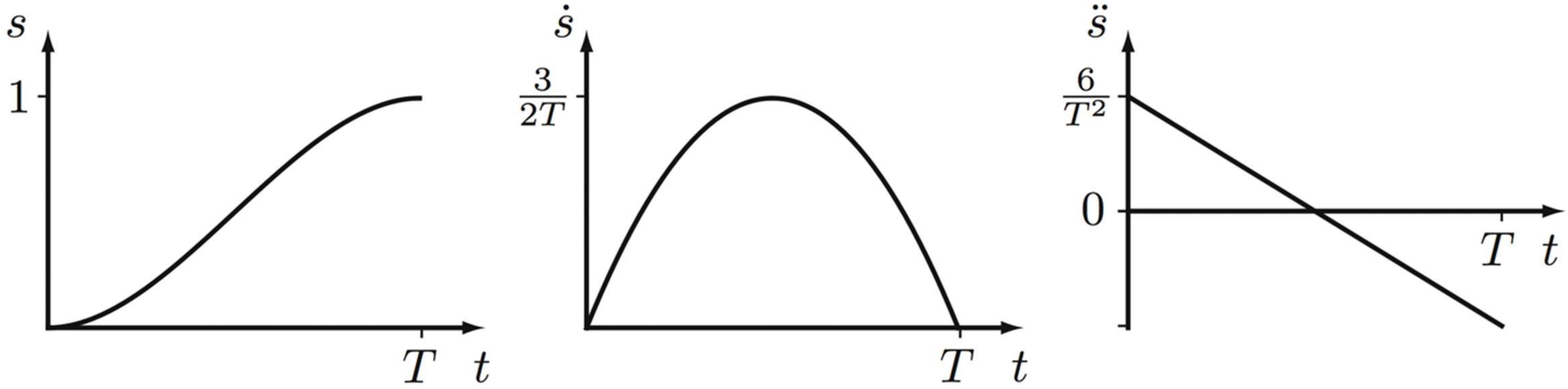
$$s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\dot{s}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

- point-to-point motion in time T imposes the initial and terminal constraints

$$s(0) = 0 \qquad \dot{s}(0) = 0$$
$$s(T) = 1 \qquad \dot{s}(T) = 0$$

$$a_0 = 0,\ a_1 = 0,\ a_2 = \frac{3}{T^2},\ a_3 = -\frac{2}{T^3}$$



Cubic polynomial

- A convenient form for the time scaling $s(t)$ is a cubic polynomial of time

$$s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

4 unknowns – need four constraints

$$\dot{s}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

- point-to-point motion in time T imposes the initial and terminal constraints

$$s(t_0) = a_0 + a_1 t_0 + a_2 {t_0}^2 + a_3 {t_0}^3$$

$$\dot{s}(t_0) = a_1 + 2a_2 t_0 + 3a_3 {t_0}^2$$

$$s(t_f) = a_0 + a_1 t_f + a_2 {t_f}^2 + a_3 {t_f}^3$$

$$\dot{s}(t_f) = a_1 + 2a_2 t_f + 3a_3 {t_f}^2$$

Cubic polynomial

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & t_0 & {t_0}^2 & {t_0}^3 \\ 0 & 1 & 2t_0 & 3{t_0}^2 \\ 1 & t_f & {t_f}^2 & {t_f}^3 \\ 0 & 1 & 2t_f & 3{t_f}^2 \end{bmatrix}^{-1} \begin{bmatrix} s(t_0) \\ \dot{s}(t_0) \\ s(t_f) \\ \dot{s}(t_f) \end{bmatrix}$$

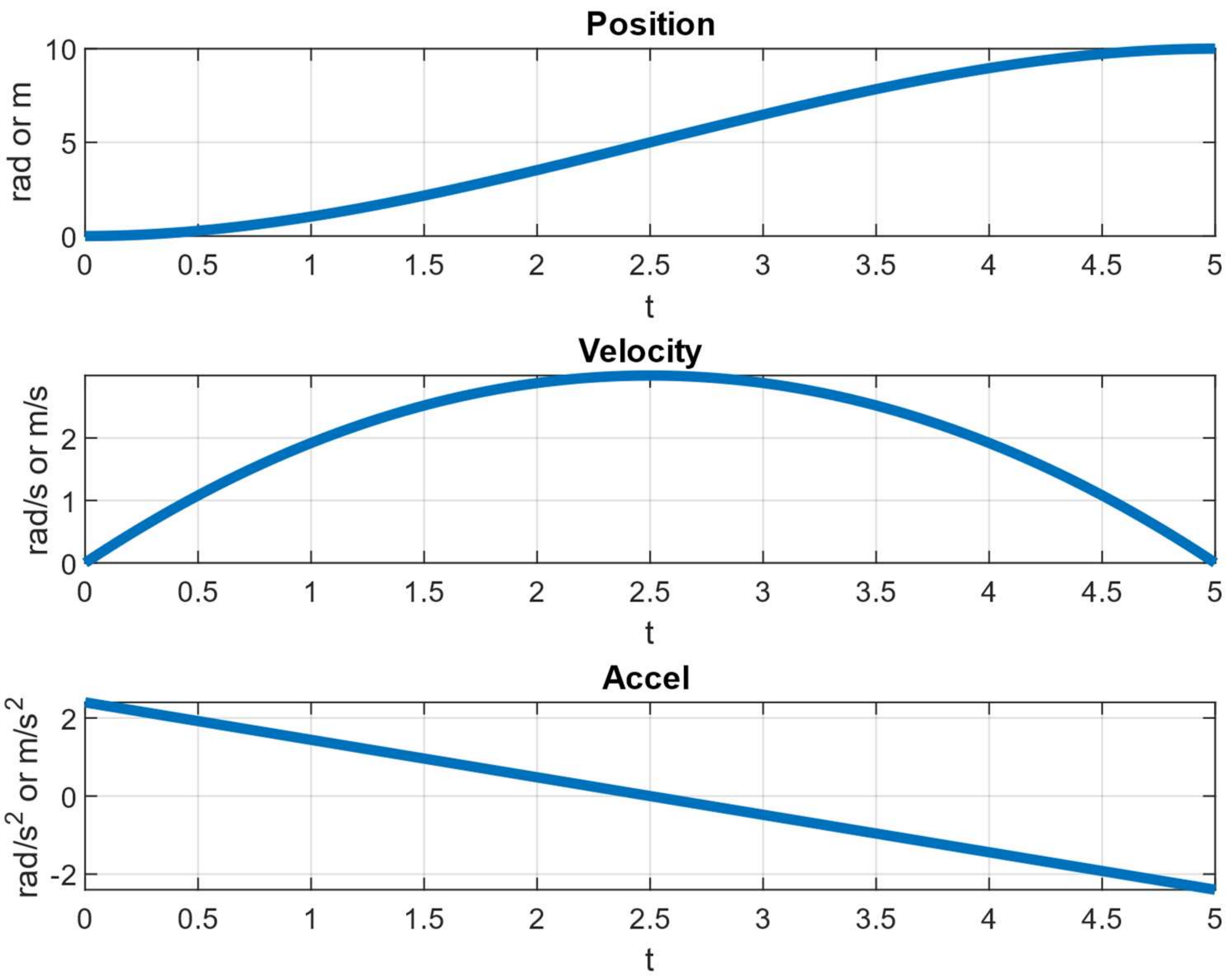
$$\boldsymbol{a} = \boldsymbol{M}\boldsymbol{s}$$

$$\mathrm{Det}(\boldsymbol{M}) = (t_f - t_0)^4$$

unique solution as long as $t_f \neq t_0$

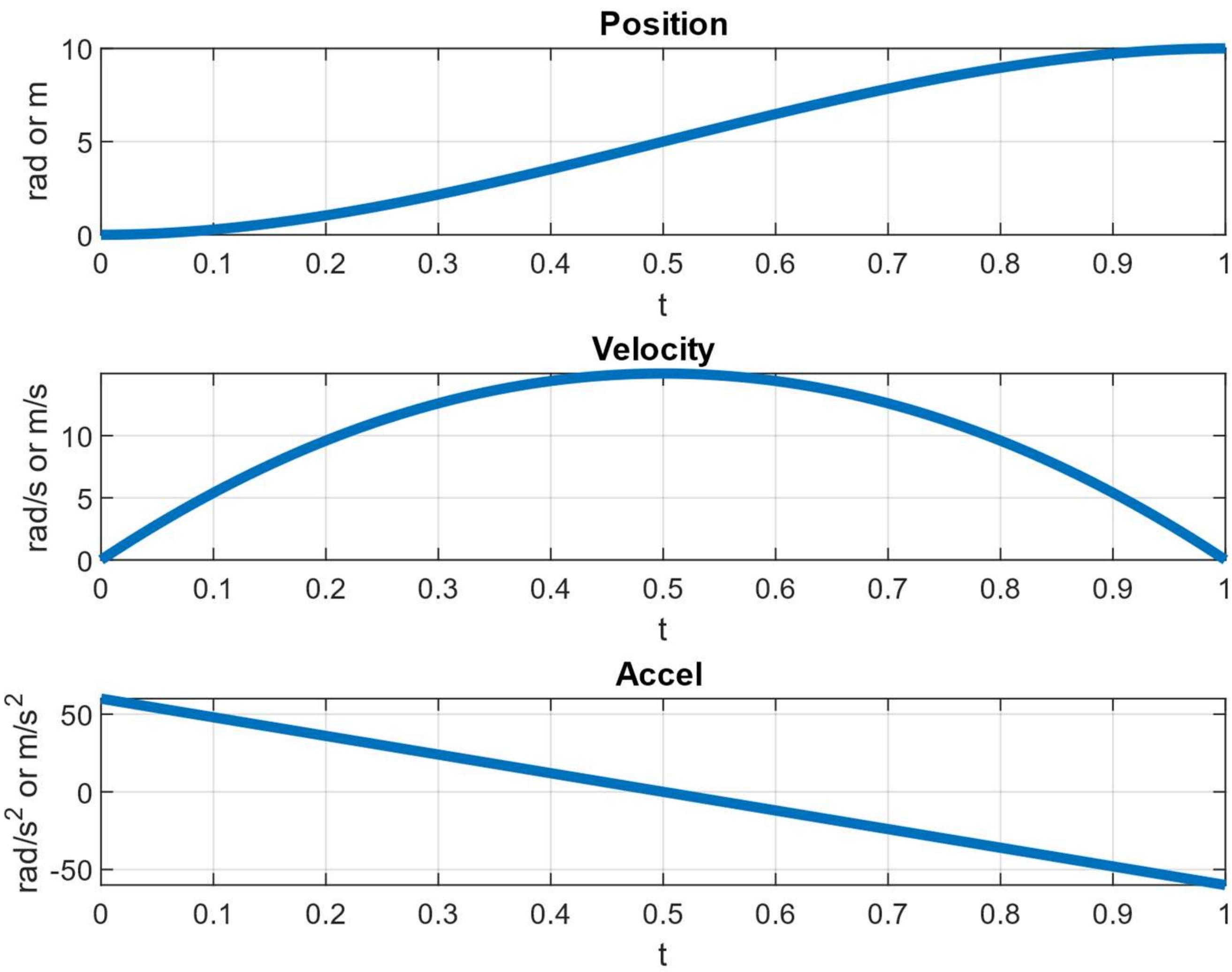
Cubic polynomial

[q0,qdot0,qf,qdotf,t0,tf]= [0,0,10,0,0,5]



Cubic polynomial

$$[q0,qdot0,qf,qdotf,t0,tf] = [0,0,10,0,0,1]$$



Issues with Cubic polynomial

- Cubic trajectory gives continuous positions and velocities at the start and final times but **discontinuities in the acceleration**.

- It leads to an **impulsive jerk**, which may excite vibrational modes in the robot and reduce tracking accuracy

Remedy:

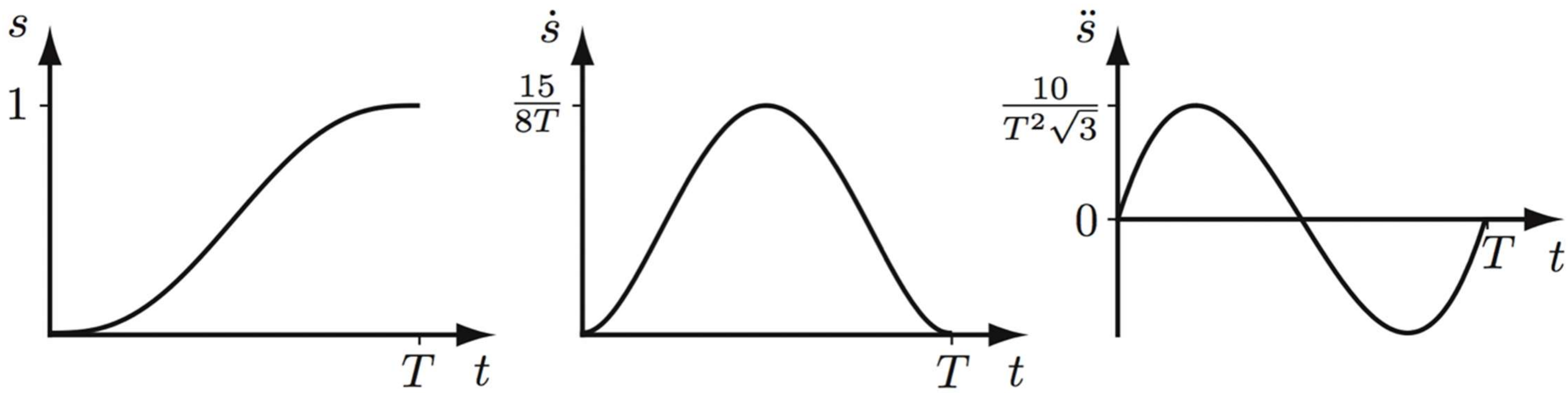
- specify constraints on the acceleration as well
- Need six coefficients – thus **quintic polynomial**

Quintic polynomial

$$s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ a_4 t^4 + a_5 t^5$$

$$s(0) = 0 \qquad \dot{s}(0) = 0 \qquad \ddot{s}(0) = 0$$
$$s(T) = 1 \qquad \dot{s}(T) = 0 \qquad \ddot{s}(T) = 0$$

$$\boldsymbol{a} = \boldsymbol{M}\boldsymbol{s}$$
$$Det(\boldsymbol{M}) = \ -4(t_f - t_0)^9$$



Quintic polynomial

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = [M]^{-1} \begin{bmatrix} s(t_0) \\ \dot{s}(t_0) \\ \ddot{s}(t_0) \\ s(t_f) \\ \dot{s}(t_f) \\ \ddot{s}(t_f) \end{bmatrix}$$

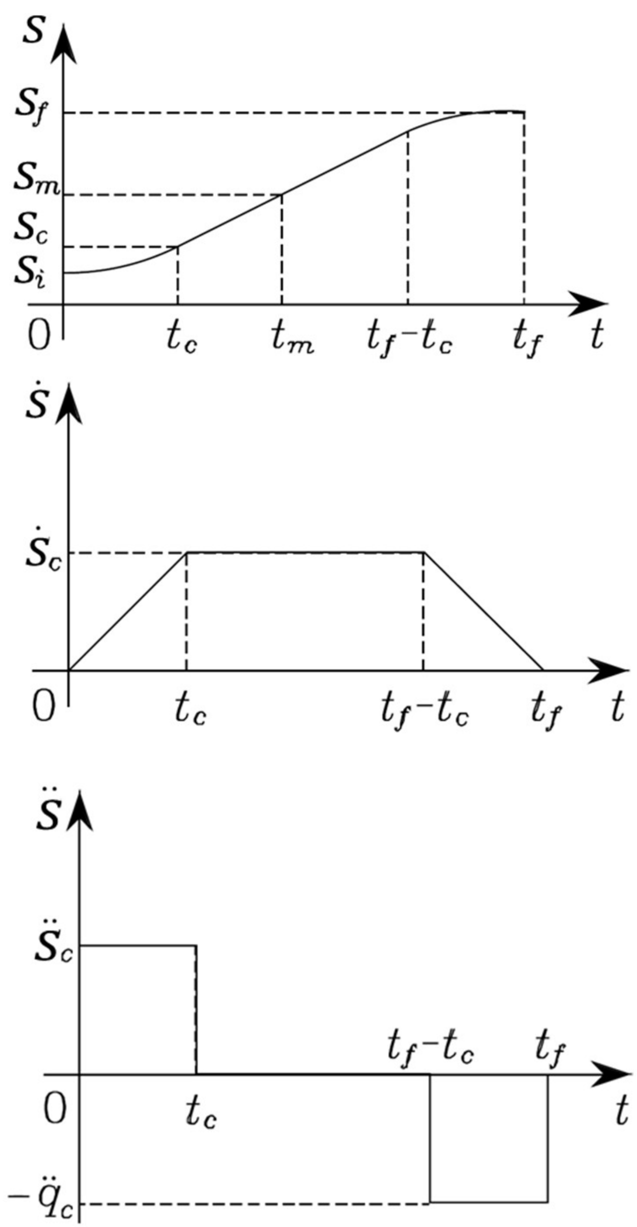
$$M = \begin{pmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 \\ 0 & 1 & 2\,t_0 & 3\,t_0^2 & 4\,t_0^3 & 5\,t_0^4 \\ 0 & 0 & 2 & 6\,t_0 & 12\,t_0^2 & 20\,t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2\,t_f & 3\,t_f^2 & 4\,t_f^3 & 5\,t_f^4 \\ 0 & 0 & 2 & 6\,t_f & 12\,t_f^2 & 20\,t_f^3 \end{pmatrix}$$

Linear Segments with Parabolic Blends

- Linear segment connected with two parabolic segments
  - $t_0 \rightarrow t_c$: Quadratic polynomial and
  - this results in a linear ramp velocity.

  - At t_c (blend time), switch to a linear function, resulting in constant velocity.

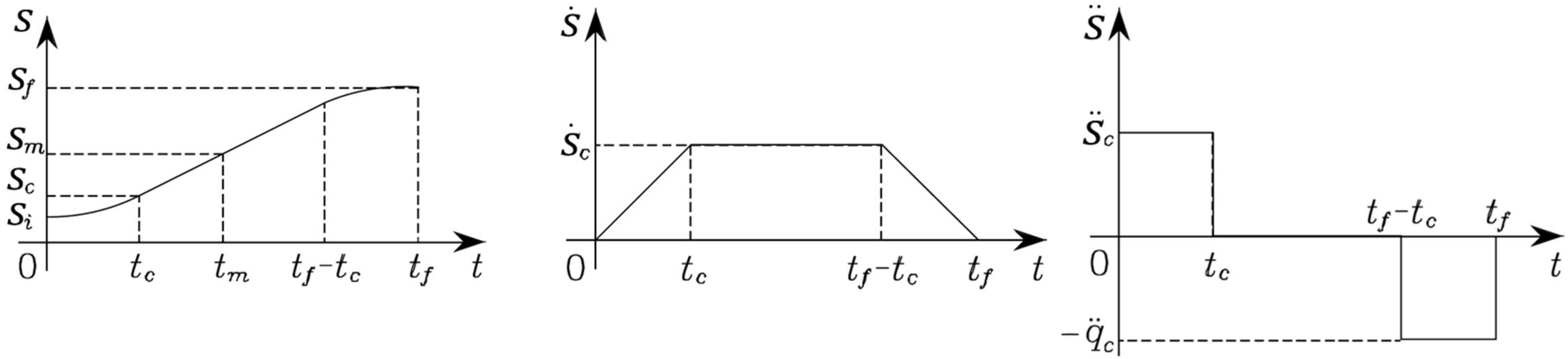
  - $t_f - t_c \rightarrow t_f$: The trajectory switches to quadratic polynomial so that the velocity is linear.
- It has a Trapezoidal velocity profile
- Constant acceleration and deceleration.



Linear Segments with Parabolic Blends

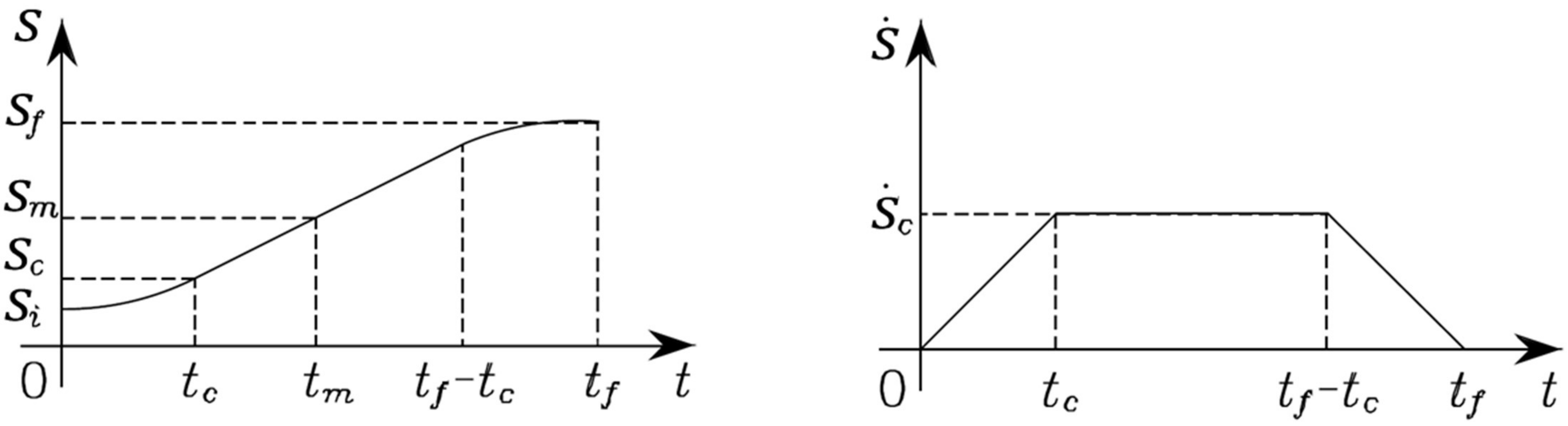
- Assumptions:
  - $\dot{s}(t_0) = \dot{s}(t_f) = 0$
  - Constant accelerations have the same time duration; $|\ddot{s}_c| = \text{constant}$
  - Symmetric trajectories with respect to the average point,

$$s(t_m) = \frac{s(t_0)+s(t_f)}{2} \text{ at } t_m = \frac{t_f}{2}$$



Linear Segments with Parabolic Blends

- 8 Coefficients, 5 parameters $s(t_0), s(t_f), \dot{s}(t_0), \dot{s}(t_f)$ are known, so extra conditions are needed.



  - Continuity conditions $s_1(t_c) = s_2(t_c)$ and $s_2(t_f - t_c) = s_3(t_f - t_c)$
  - Smoothness conditions $\dot{s}_1(t_c) = \dot{s}_2(t_c)$ and $\dot{s}_2(t_f - t_c) = \dot{s}_3(t_f - t_c)$
  - Nine equations and nine unknowns, we get.

Linear Segments with Parabolic Blends

Nine equations

$$s_1(t_0) = s(t_0)$$
$$\dot{s_1}(t_0) = 0$$
$$s_1(t_c) = s_2(t_c)$$
$$\dot{s_2}(t_c) = V_d$$
$$s_2(t_f - t_c) = s_3(t_f - t_c)$$
$$\dot{s}_1(t_c) = \dot{s}_2(t_c)$$
$$\dot{s}_2(t_f - t_c) = \dot{s}_3(t_f - t_c)$$
$$s_3(t_f) = s(t_f)$$
$$\dot{s_3}(t_f) = 0$$

Nine unknowns

$$a_0 \text{ to } a_7 \text{ and } t_c$$

Linear Segments with Parabolic Blends

$t_0 \rightarrow t_c$: Quadratic polynomial

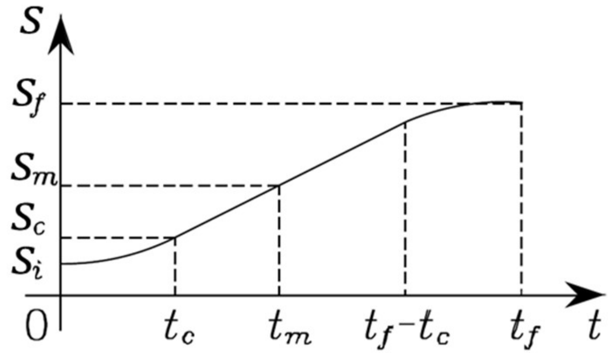
$$s(t_0) = a_0 + a_1 t_0 + a_2 {t_0}^2 \qquad \text{(L.1)}$$

$$\dot{s}(t_0) = a_1 + 2a_2 t_0 \qquad \text{(L.2)}$$

Since $\dot{s}(t_0)$=0, From (L.1) and (L.2)

$$a_0 = s(t_0)$$

$$a_1 = 0$$



At $t = t_c$

$$s(t_c) = s(t_0) + 0\, t_c + a_2 {t_c}^2 \qquad \text{(L.3)}$$

$$\dot{s}(t_c) = 0 + 2a_2 t_c \qquad \text{(L.4)}$$

Linear Segments with Parabolic Blends

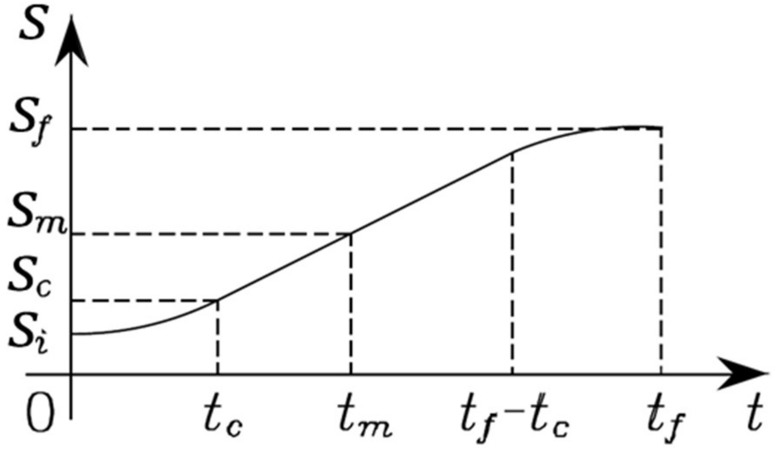
- At end of the parabolic segment at time t_c with constant acceleration $\ddot{s}(t_c)$, (L.4) becomes

$$\ddot{s}(t_c) = 2a_2 \qquad \text{(L.5)}$$

From (L.5)

$$a_2 = \frac{\ddot{s}(t_c)}{2} \qquad \text{(L.6)}$$

W.K.T $\ddot{s}(t_c) = \dot{s}(t_c)/t_c$



- Therefore, the required trajectory between 0 and t_c is given by

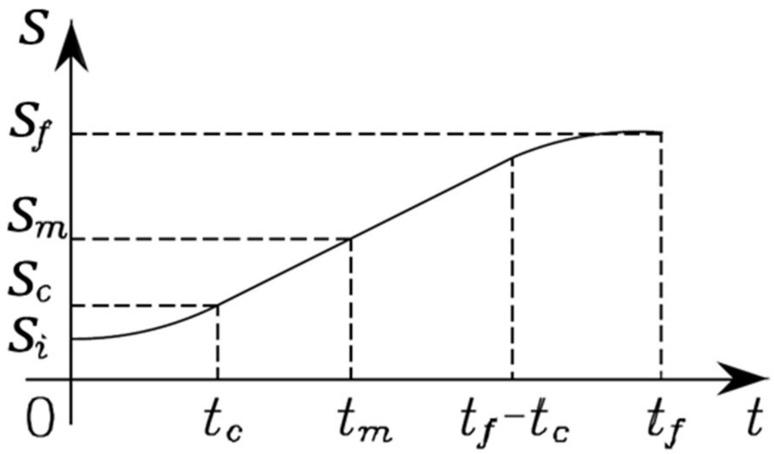
Substitute $a_0 = s(t_0)$, $a_1 = 0$, $a_2 = \frac{\dot{s}(t_c)}{2t_c}$ in

$$s(t) = a_0 + a_1 t + a_2 t^2$$
$$\dot{s}(t) = a_1 + 2a_2 t$$
$$\ddot{s}(t_c) = 2a_2 \quad \Rightarrow$$

$$s(t) = s(t_0) + \frac{\dot{s}(t_c)}{2t_c} t^2 \qquad 0 \le t \le t_c$$

$$\dot{s}(t) = \frac{\dot{s}(t_c)}{t_c} t$$

$$\ddot{s}(t_c) = \frac{\dot{s}(t_c)}{t_c}$$



Linear Segments with Parabolic Blends

Let's find t_c

Substitute a_2 in (L.3)

$$s(t_c) = s(t_0) + \frac{\ddot{s}(t_c)}{2} {t_c}^2 \tag{L.7}$$

- The trajectory must satisfy some constraints: The velocity at the end of the parabolic segment must be equal to the (constant) velocity of the linear segment

$$\ddot{s}(t_c)t_c = \frac{s(t_m) - s(t_c)}{t_m - t_c} \tag{L.8}$$

or

$$V_d = \dot{s}(t_c) = \frac{s(t_m) - s(t_c)}{t_m - t_c}$$

Linear Segments with Parabolic Blends

Sub (L.7) in (L.8) and use $s(t_m) = \frac{s(t_0)+s(t_f)}{2}$ at $t_m = \frac{t_f}{2}$

$$\ddot{s}(t_c){t_c}^2 - \ddot{s}(t_c)t_f t_c + s(t_f) - s(t_0) = 0 \qquad \text{(L.9)}$$

$${t_c}^2 - t_f t_c + \frac{\left(s(t_f)-s(t_0)\right)}{\ddot{s}(t_c)} = 0$$

For a given t_f , $s(t_0)$, and $s(t_f)$ the solution for t_c is

$$t_c = \frac{t_f}{2} - \frac{1}{2}\sqrt{\frac{{t_f}^2\ddot{s}(t_c) - 4(s(t_f)-s(t_0))}{\ddot{s}(t_c)}} \qquad \text{(L.10)}$$

Linear Segments with Parabolic Blends

- Acceleration is then subject to the constraint

$$t_f{}^2\ddot{s}(t_c) \geq 4\left(s(t_f) - s(t_0)\right) \qquad \text{(L.11)}$$

$$|\ddot{s}(t_c)| \geq \frac{|s(t_f)-s(t_0)|}{\left(\frac{t_f}{2}\right)^2} \qquad \text{(L.12)}$$

The acceleration should be guaranteed to complete within half of the time duration

When the acceleration $|\ddot{s}(t_c)| = \frac{4|s(t_f)-s(t_0)|}{t_f{}^2}$ is chosen, results in **triangular profile**. (See (L.10))

Given $s(t_0), s(t_f)$ and t_f, and average transition velocity, the constraint in (L.12) allows the imposition of a value of acceleration consistent with the trajectory.

Linear Segments with Parabolic Blends

W.K.T $\ddot{s}(t_c) = \dot{s}(t_c)/t_c$, Substitute it in (L.9) and simplify it

$$t_c = \frac{s(t_0) - s(t_f) + \dot{s}(t_c)t_f}{\dot{s}(t_c)} \tag{L.13}$$

The resulting acceleration is

$$\ddot{s}(t_c) = \frac{\dot{s}(t_c)^2}{s(t_0) - s(t_f) + \dot{s}(t_c)t_f}$$

W.K.T $\dot{s}(t_c)$ is given

Linear Segments with Parabolic Blends

$t_c \rightarrow t_f - t_c$: Linear function

$$s(t) = a_3 + a_4 t \tag{L.14}$$

$$\dot{s}(t) = a_4 \tag{L.15}$$

Impose the constraint: $s_1(t_c) = s_2(t_c)$

$$a_0 + a_1 t_c + a_2 {t_c}^2 = a_3 + a_4 t_c$$

$$a_3 = a_0 + a_1 t_c + a_2 {t_c}^2 - a_4 t_c$$

Substitute a_3 and a_4 in (L.14)

$$s(t) = a_0 + a_1 t_c + a_2 {t_c}^2 - a_4 t_c + \dot{s}(t) t$$

$$s(t) = s(t_0) + \dot{s}(t_c)(t - \frac{t_c}{2}) \qquad t_c < \mathrm{t} \leq t_f - t_c$$

Linear Segments with Parabolic Blends

- $t_f - t_c \to t_f$: Quadratic Function

$$s(t) = a_5 + a_6 t + a_7 t^2 \qquad \text{(L.16)}$$

$$\dot{s}(t) = a_6 + 2a_7 t \qquad \text{(L.17)}$$

- *Impose the constraints:*

$$\dot{s}_2(t_f - t_c) = \dot{s}_3(t_f - t_c)$$
$$\dot{s}_3(t_f) = 0$$
$$s_3(t_f) = s(t_f)$$

$$\dot{s}_2(t_f - t_c) \Rightarrow \dot{s}_2(t_c) = a_6 + 2a_7(t_f - t_c)$$

$$\dot{s}_3(t_f) \Rightarrow 0 = a_6 + 2a_7 t_f$$

$$s(t_f) = a_5 + a_6 t_f + a_7 {t_f}^2$$

Linear Segments with Parabolic Blends

Subs. $a_5 = s(t_f) - \frac{\dot{s}(t_c){t_f}^2}{2t_c}$, $a_6 = \frac{\dot{s}(t_c)t_f}{t_c}$, $a_7 = -\frac{\dot{s}(t_c)}{2t_c}$ in

$$s(t) = s(t_f) - \frac{\dot{s}(t_c)}{2t_c}(t_f - t)^2 \qquad t_f - t_c < \mathrm{t} \le t_f$$

The sequence of polynomials is

$$s(t) = \begin{cases} s(t_0) + \frac{\dot{s}(t_c)}{2t_c}t^2 & 0 \le \mathrm{t} \le t_c \\ s(t_0) + \dot{s}(t_c)(t - \frac{t_c}{2}) & t_c < \mathrm{t} \le t_f - t_c \\ s(t_f) - \frac{\dot{s}(t_c)}{2t_c}(t_f - t)^2 & t_f - t_c < \mathrm{t} \le t_f \end{cases}$$

Linear Segments with Parabolic Blends

- To avoid triangular and rectangular profiles: $0 < t_c < \frac{t_f}{2}$

Using (L.13), i.e., $t_c = \frac{s(t_0) - s(t_f) + \dot{s}(t_c) t_f}{\dot{s}(t_c)}$, the cruise velocity is subjected to the constraint

$$\frac{s(t_f) - s(t_0)}{t_f} < \dot{s}(t_c) \leq \frac{2(s(t_f) - s(t_0))}{t_f}$$

We have to choose $\dot{s}(t_c)$ based on this constraint

S-curve profile and trapezoidal profile

- The rapid acceleration induces powerful vibrations.
- Increased settling time and reduced accuracy.