Linear Algebra

Outline

This is to get you revised with the important subset of topics. We picked specific questions from various sources, covering a set of topics that may require you to dive a little deeper into certain topics and proofs, such as ranks, SVD, ...etc.

If you are willing to do a deep review, we suggest you speed-run through all the PSets from https://web.mit.edu/18.06/www/psets.shtml.

PSet

Row Reduction, LU Factorization

4. [15 pts] "In principle, one can avoid temporary failures by reordering the equations at the start"
(a) Give a 3×3 matrix P so that PA swaps the last two rows of the 3×3 matrix A and leaves the first row unchanged.
(b) Out of interest, what do AP, PAP, and P^2A do to A?
(c) Use elimination (and fix the temporary failure) to find the elimination matrices E_{21}, E_{31}, and E_{32} and an upper-triangular matrix U so that

$$E_{32}PE_{31}E_{21}\underbrace{\begin{bmatrix} 3 & 4 & 1 \\ 3 & 4 & 2 \\ 1 & 0 & 1 \end{bmatrix}}_{=A} = U.$$

(d) Show that $E_{32}PE_{31}E_{21} = E_{32}(PE_{31}P)(PE_{21}P)P$. (Note that $PE_{31}P$ and $PE_{21}P$ look like elimination matrices.)
(e) Explain why (d) means that one can avoid temporary failures altogether by reordering the rows of A at the start.
(f) Explain why a computer does not reorder the equations at the start to prevent temporary failure.
(g) Calculate the $PA = LU$ factorization of A.

Rank, NullSpace

Q1

Show the following:

12. rank $\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$ = rank $\mathbf{AB}$. (Note the order!)

Q2

2.5.2.- Find the $N(\mathsf{A})$, $R(\mathsf{A})$, $N(\mathsf{A}^T)$ and $R(\mathsf{A}^T)$ for the matrix A below and express them as the span of the smallest possible set of vectors, where

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{bmatrix}.$$

EigenValues and EigenVectors

9.1.3. **Eigenvalue multiplicities.** We now introduce two numbers that give information regarding the size of eigenspaces. The first number determines the maximum possible size of an eigenspace, while the second number characterizes the actual size of an eigenspace.

Definition 9.1.6. *Let $\lambda_i \in \mathbb{F}$, for $i = 1, \cdots, k$ be all the eigenvalues of a linear operator $\boldsymbol{T} \in L(V)$ on a vector space V over $\mathbb{F}$. Express the characteristic polynomial p associated with $\boldsymbol{T}$ as follows,*

$$p(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k} q(\lambda), \quad \text{with} \quad q(\lambda_i) \neq 0,$$

*and denote by $s_i = \dim E_{\lambda_i}$, the dimension of the eigenspaces corresponding to the eigenvalue λ_i. Then, the numbers r_i are called the **algebraic multiplicity** of the eigenvalue λ_i; and the numbers s_i are called the **geometric multiplicity** of the eigenvalue λ_i.*

Q1

9.1.3.- Let $\mathsf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by

$$\mathsf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & h \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) Find all the eigenvalues and their corresponding algebraic multiplicities of the matrix A.

(b) Find the value(s) of $h \in \mathbb{R}$ such that the matrix A above has a two-dimensional eigenspace, and find a basis for this eigenspace.

(c) Set $h = 1$, and find a basis for all the eigenspaces of matrix A above.

Change of Basis

Source: https://users.math.msu.edu/users/gnagy/teaching/la.pdf

5.5. Change of basis

In this Section we summarize the main calculations needed to find the change of vector and linear transformation components under a change of basis. We provide simple formulas to compute this change efficiently.

5.5.1. **Vector components.** We have seen in Sect. 4.4 that every vector $\boldsymbol{v}$ in a finite dimensional vector space V with an ordered basis $\mathcal{V}$, can be expressed in a unique way as a linear combination of the basis elements, with $\mathsf{v}_v = [\boldsymbol{v}]_v$ denoting the coefficients in that linear combination. These components depend on the basis chosen in V. Given two different ordered bases $\mathcal{V}$, $\tilde{\mathcal{V}} \subset V$, the components v_v and $\mathsf{v}_{\tilde{v}}$ associated with a vector $\boldsymbol{v} \in V$ are in general different. We now use the matrix-vector product to find a simple formula relating v_v to $\mathsf{v}_{\tilde{v}}$.

Let us recall the following notation. Given an n-dimensional vector space V, let $\tilde{\mathcal{V}}$ and $\mathcal{V}$ be two ordered bases of V given by

$$\tilde{\mathcal{V}} = (\tilde{\boldsymbol{v}}_1, \cdots, \tilde{\boldsymbol{v}}_n) \quad \text{and} \quad \mathcal{V} = (\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n).$$

Let $\boldsymbol{I} : V \to V$ be the identity transformation, that is, $\boldsymbol{I}(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$, and introduce the *change of basis matrices*

$$\mathsf{I}_{\tilde{v}v} = \left[\tilde{\mathsf{v}}_{1v}, \cdots, \tilde{\mathsf{v}}_{nv}\right] \quad \text{and} \quad \mathsf{I}_{v\tilde{v}} = \left[\mathsf{v}_{1\tilde{v}}, \cdots, \mathsf{v}_{n\tilde{v}}\right],$$

where we denoted, as usual, $\tilde{\mathsf{v}}_{iv} = [\tilde{\boldsymbol{v}}_i]_v$ and $\mathsf{v}_{i\tilde{v}} = [\boldsymbol{v}_i]_{\tilde{v}}$, for $i = 1, \cdots, n$. Since the sets $\mathcal{V}$ and $\tilde{\mathcal{V}}$ are bases of V, the matrices $\mathsf{I}_{v\tilde{v}}$ and $\mathsf{I}_{\tilde{v}v}$ are invertible, and it is not difficult to show that $(\mathsf{I}_{v\tilde{v}})^{-1} = \mathsf{I}_{\tilde{v}v}$. Finally, introduce the following notation for the change of basis matrices,

$$\mathsf{P} = \mathsf{I}_{\tilde{v}v} \quad \text{and} \quad \mathsf{P}^{-1} = \mathsf{I}_{v\tilde{v}}.$$

Theorem 5.5.1. *Let V be a finite dimensional vector space, let $\tilde{\mathcal{V}}$ and $\mathcal{V}$ be two ordered bases of V, and let $P = \mathsf{I}_{\tilde{v}v}$ be the change of basis matrix. Then, the components $\mathsf{x}_{\tilde{v}}$ and x_v of any vector $\boldsymbol{x} \in V$ in the ordered bases $\tilde{\mathcal{V}}$ and $\mathcal{V}$, respectively, are related by the linear equation*

Q1

5.5.1.- Let $\mathcal{U} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ be an ordered basis of $\mathbb{R}^2$ given by

$$\boldsymbol{u}_1 = 2\boldsymbol{e}_1 - 9\boldsymbol{e}_2, \quad \boldsymbol{u}_2 = \boldsymbol{e}_1 + 8\boldsymbol{e}_2,$$

where $\mathcal{S} = (\mathbf{e}_1, \mathbf{e}_2)$ is the standard ordered basis of $\mathbb{R}^2$.

(a) Find both change of basis matrices I_{us} and I_{su}.

(b) Given the vector $\boldsymbol{x} = 2\boldsymbol{u}_1 + \boldsymbol{u}_2$, find both x_s and x_u.