Geometric Transforms in Robotics

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RRC Robotics Summer School
IIIT-H

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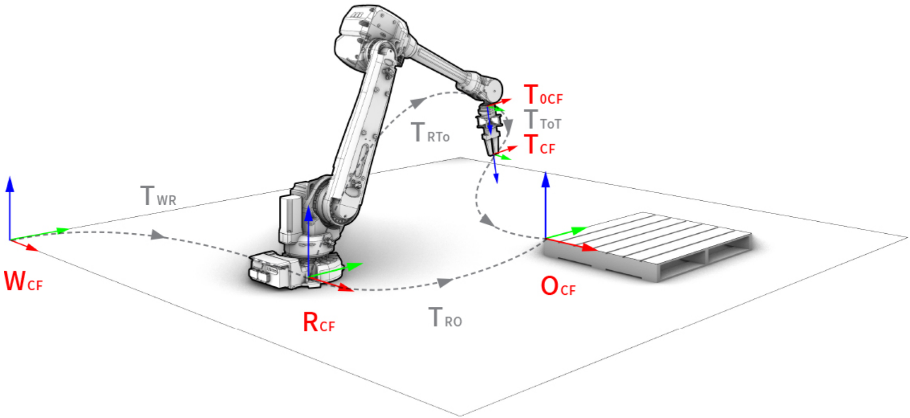
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LIVOX MID-360

Outline

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3. Transformations
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5. Translation
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7. Rotation
8. Rotation
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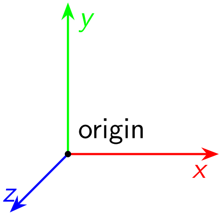
   - Euler-Angle Representation
   - Axis-Angle Representation
   - Quaternions
9. Homogeneous Transformation
10. Inverse Transformation
11. Composite Transformation
12. Denavit-Hartenberg Convention

Coordinate Frames

Point + 3 Axes (mutually perpendicular vectors)

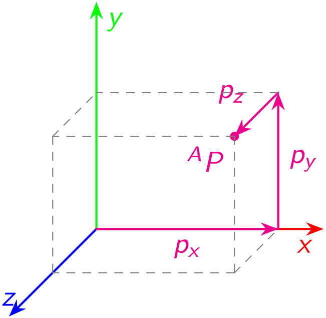


Reference: map, odom, sensors (imu, camera, lidar)
Body (or) Objects: base_link, joint, end-effector

Points

A location in space. By convention a point P in a frame A is represented as:

$^{A}P = [p_x, p_y, p_z]^T$



Points

Points can be represented using different conventions:

- Cartesian: $[x, y, z]$
- Cylindrical: $[r, \theta, z]$
- Spherical: $[\rho, \theta, \phi]$

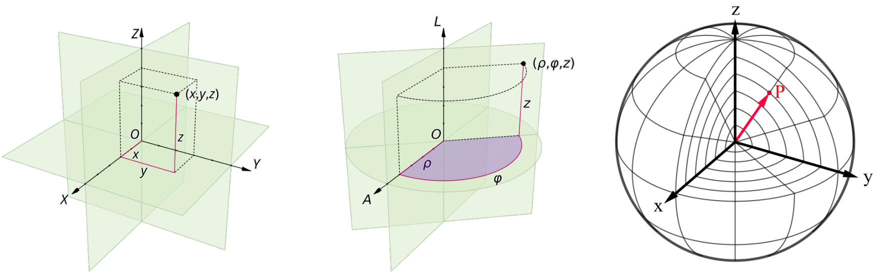


Figure: Overall caption for all three subfigures.

Transformations

Functions that map an n-d vector to another n-d vector.

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

Represented as matrix operations.

- Translations
- Rotations
- Homogenous Transformations

Transformations occur due to a **movement** or due to a **change in perspective**.

Active vs Passive Transformations

Active Transformation
Point P changes to point P'

Passive Transformation
Point P changes frame of reference. ${}^{A}P$ to ${}^{B}P$

Translation

A translation moves a point or frame in space without changing its orientation.
Consider Frame A with origin at $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and Frame B with origin at $\begin{bmatrix} t_x & t_y & t_z \end{bmatrix}$. t_{AB} is the vector from A to B. The transformation from A to B is given by:

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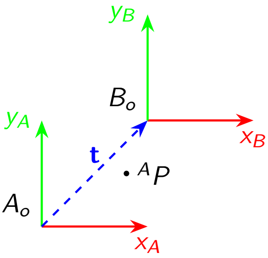
$$^{B}P = \ ^{A}P +^{A} P_{B_O}$$



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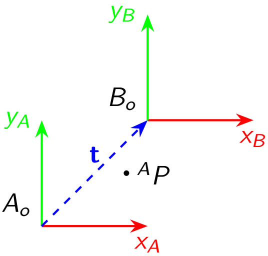
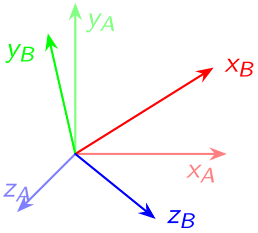


Note that the point stays the same, but the frame in which the point is represented has changed.

Rotation

Consider two frames A and B that have the same origin but are rotated with respect to each other.



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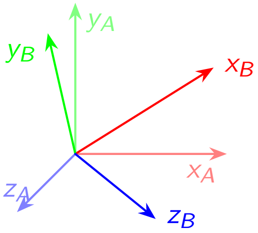
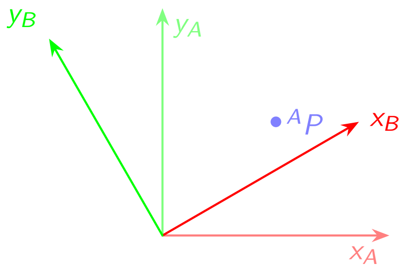


The rotation of B relative to A can be represented by simply stacking the vectors that define the axes of B in terms of A.

$$
{}^{A}R_B = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix}
$$

Rotation



Writing the axes of Frame B in terms of the axes of Frame A,

$$ {}^{A}\hat{X}_B = \begin{bmatrix} cos\theta & sin\theta \end{bmatrix} $$

$$ {}^{A}\hat{Y}_B = \begin{bmatrix} -sin\theta & cos\theta \end{bmatrix} $$

Stacking these together,

$$ {}^{A}R_B = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} $$

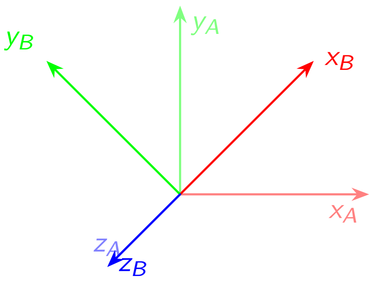
Properties of 2D Rotation Matrix

Properties of the 2D Rotation Matrix:

- $|R| = 1$
- $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$
- $R(-\theta) = R(\theta)^T = R(\theta)^{-1}$
- $\|R(\theta)x\| = \|x\|$

The set of 2x2 rotation matrices form the special orthogonal group **SO(2)** as they are the only set of 2x2 orthogonal matrices with a positive determinant.

Rotation in 3D: Rotating Around the z-axis



$$x_B = \begin{bmatrix} \cos\theta & \sin\theta & 0 \end{bmatrix}$$
$$y_B = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \end{bmatrix}$$
$$z_B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$^{A}R_B = R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

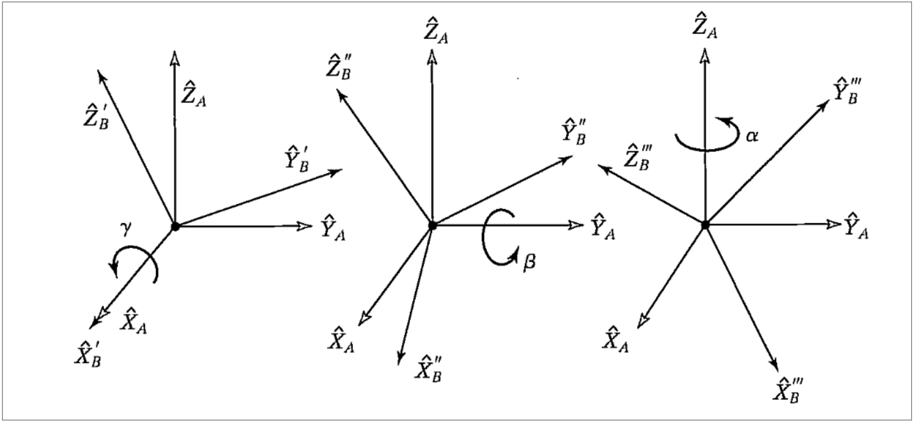
Properties of 3D Rotation Matrices

- $|R| = 1$
- $R(\theta_1)R(\theta_2) \neq R(\theta_2)R(\theta_1)$ (in general)
- $R(-\theta) = R(\theta)^T = R(\theta)^{-1}$
- $\|R(\theta)x\| = \|x\|$

These matrices from the **SO(3)** group.

Fixed-Angle Representation

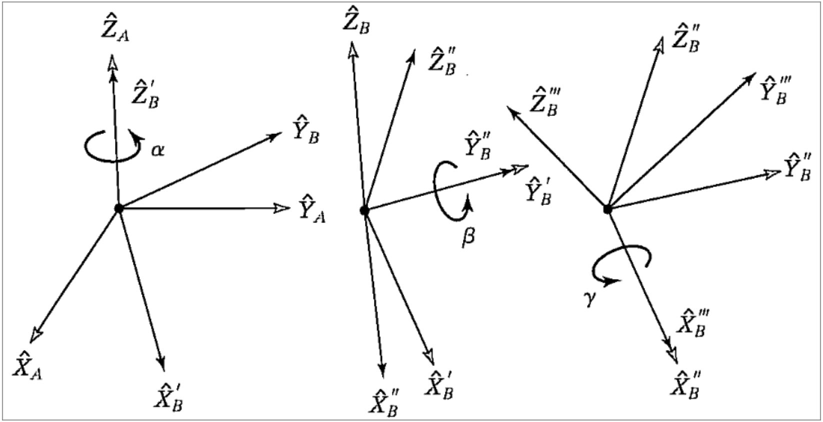
Rotations are described with respect to the axis of a fixed-frame.



$$R = R_Z(\alpha)R_Y(\beta)R_X(\gamma)$$

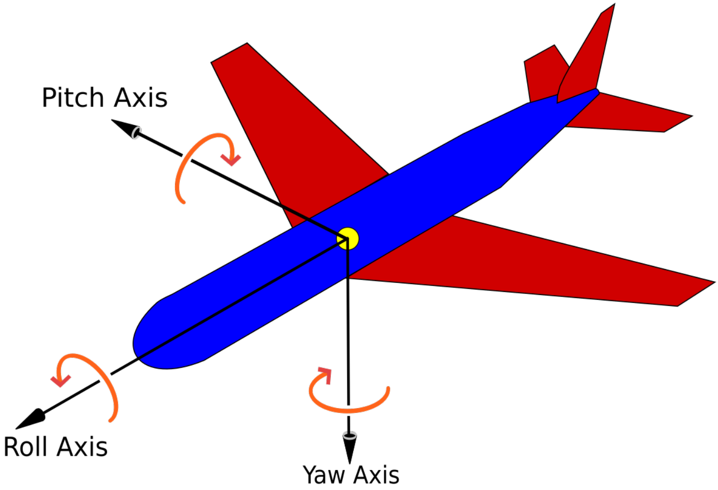
Euler-Angle Representation

Rotations are described with respect to the axis of the moving frame.



$$R = R_Z(\alpha)R_Y(\beta)R_X(\gamma)$$

Roll, Pitch and Yaw



Video

Gimbal Lock

Gimbal lock occurs when two of the three rotation axes in a sequence become aligned, causing a loss of one degree of freedom.

When does it happen?

- Typically arises in **Euler** or **Fixed** angle representations.
- Most common in `XYZ` or `ZYX` sequences when the middle angle is near $\pm 90^\circ$.

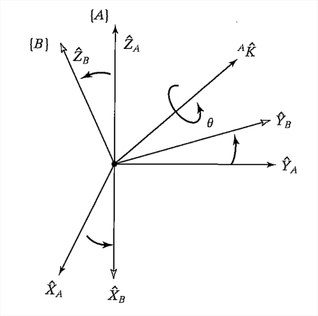
Consequences:

- The system can no longer rotate independently around all three axes.
- Control or interpolation becomes unstable or ambiguous.

Gimbal Lock on the Apollo 13

Axis-Angle Representation

Rotations are described with respect to an axis (K) and an angle (θ).



$$\mathbf{R_K}(\theta) = \begin{bmatrix} C_\theta + k_x^2 V_\theta & k_x k_y V_\theta - k_z S_\theta & k_x k_z V_\theta + k_y S_\theta \\ k_y k_x V_\theta + k_z S_\theta & C_\theta + k_y^2 V_\theta & k_y k_z V_\theta - k_x S_\theta \\ k_z k_x V_\theta - k_y S_\theta & k_z k_y V_\theta + k_x S_\theta & \cos\theta + k_z^2 V_\theta \end{bmatrix}$$

$$S_\theta = \sin\theta, \quad C_\theta = \cos\theta, \quad V_\theta = 1 - \cos\theta$$

Quaternions

$$q = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ k_x \sin\left(\frac{\theta}{2}\right) \\ k_y \sin\left(\frac{\theta}{2}\right) \\ k_z \sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1$$

Learn more about Qauternions from this 3Blue1Brown video.

Homogeneous Transformation

Given any two arbitrary frames A and B, ther exists a transformation between then that is a combination of Rotation and Translation.

$$^{B}P = {}^{B}\mathbf{R}_{A}{}^{A}P +^{B} V_{A_o}$$

To combine them into a single operation, we use **homogeneous coordinates**. A homogeneous coordinate is formed from a 3D point by adding a fourth coordinate which is set to 1. Now, **a point in a frame A**, is a 4D vector represented as:

$$^{A}P = \begin{bmatrix} p_x & p_y & p_z & 1 \end{bmatrix}^T$$

Homogeneous Transformation

Now we can combine the rotation and translation operation into one matrix-multiplication operation.

$$ {}^{B}\mathbf{T}_{A} = \begin{bmatrix} {}^{B}\mathbf{R}_{A} & {}^{B}V_{A_o} \\ \mathbf{0} & 1 \end{bmatrix} $$

And any point in Frame A can be transformed to Frame B by:

$$ {}^{B}P = {}^{B}\mathbf{T}_{A}\,{}^{A}P $$

Inverse Transformations

$$({}^{B}T_{A})^{-1} = {}^{A}T_{B}$$

Given a homogeneous transformation:

$$T = \begin{bmatrix} \mathbf{R} & \mathbf{V} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Using properties of rotation matrices:

- $\mathbf{R}^{-1} = \mathbf{R}^T$ (orthonormal)
- Translation inverts with rotated negative: $-\mathbf{R}^T\mathbf{V}$

Therefore, the inverse is:

$$T^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{V} \\ \mathbf{0} & 1 \end{bmatrix}$$

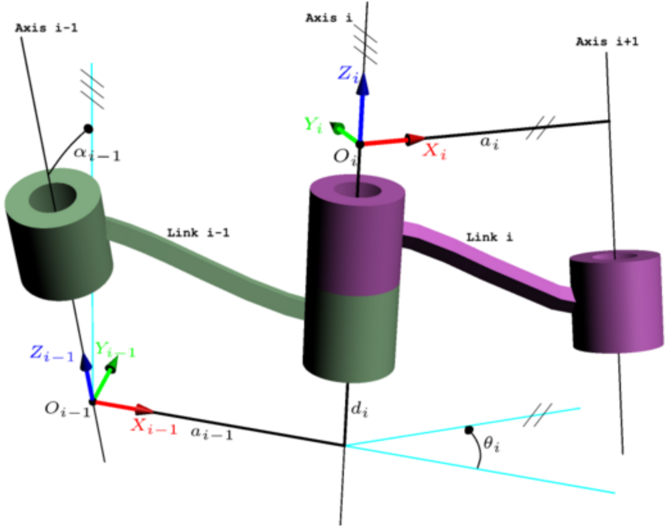
Composite Transformations

Transforms between frames can be chained by pre-multiplication.

$$ {}^{D}\mathbf{T}_{A} = {}^{D}\mathbf{T}_{C}\,{}^{C}\mathbf{T}_{B}\,{}^{B}\mathbf{T}_{A} $$

Denavit-Hartenberg Convention

A convention to represent the transformation between joints in Robotic Arms.



DH Frame Assignment Rules

- **Frame** i is attached to the **distal end** of **Link** i.
- z_i is aligned with the axis of **Joint** i; the direction chosen defines the positive sense of the joint variable θ_i.
- The **common normal** between z_{i-1} and z_i defines the x_i axis.
- The **origin of Frame** i is placed where this normal intersects the z_i axis.
- The y_i axis is defined by:

  $$y_i = z_i \times x_i$$

  Since it must be orthogonal to both axes to complete a right-handed coordinate frame.

Denavit-Hartenberg Parameters (DH)

- **Link length (a_i)**: Distance along the x_i axis from origin of frame i to the z_{i-1} axis.
- **Link twist (α_i)**: Angle between z_{i-1} and z_i axes measured about x_i.
- **Joint distance (d_i)**: Distance along z_{i-1} to intersection with x_i.
- **Joint angle (θ_i)**: Angle between x_{i-1} and x_i measured about z_{i-1}.

DH Transformation Matrix

The transformation from frame $i-1$ to frame i is built from four basic transforms:

$$^{i-1}T_i = T_z(\theta_i)\, T_z(d_i)\, T_x(a_i)\, T_x(\alpha_i)$$

$$^{i-1}T_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where:

$$C\theta_i = \cos\theta_i, \quad S\theta_i = \sin\theta_i, \quad C\alpha_i = \cos\alpha_i, \quad S\alpha_i = \sin\alpha_i$$

Overall transformation from base to end-effector:

$$^{0}T_n = \prod_{i=1}^{n} {}^{i-1}T_i$$

References I

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[2] S. Hutchinson, *Coordinate transformations*, https://motion.cs.illinois.edu/RoboticSystems/CoordinateTransformations.html, Accessed: 2025-05-19, 2020.

[3] S. Lavalle, *Steve lavalle lectures on coordinate frames and transformations*, https://www.youtube.com/watch?v=tgbXCwjlcaE&list=PL_ezWOhnpakMojiJGm-YiCz5zr4GpuLG_&index=9, YouTube video, Accessed: 2025-05-19, 2021.

[4] J. Fantl, *Rediscovering quaternions*, https://jasonfantl.com/posts/Space-of-3D-Rotations/, Accessed: May 20, 2025, 2025.

References II

[5] *Intro2robotics lecture 5a: Forward kinematics: Denavit-hartenberg convention*, https://www.youtube.com/watch?v=S1JZfWhZFsI, YouTube video, Accessed: May 20, 2025, 2023.

Link to collab demo