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## CSE 483: Mobile Robotics

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### Non-holonomic Trajectory Planning (Bernstein Basis method)

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This document discusses the theory of non holonomic trajectory planning using Bernstein polynomial along with the brief description of the associated topics.

## 1 What is a Nonholonomic motion

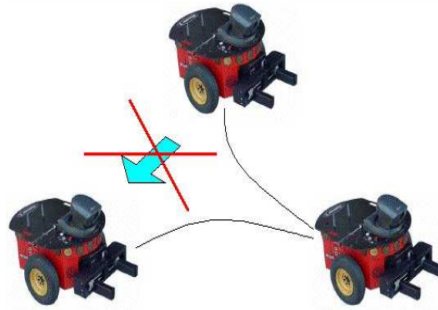


Figure 1:  
Non holonomic characteristic of wheeled robots.

Non-holonomic systems are characterized by constraint equations which involves the time derivatives of the system configuration variables. In a configuration space  $Q \subset R^n$ , the configuration of a mechanical system can be uniquely described by an n-dimensional vector of generalized coordinates.

$$q = (q_1, q_2, q_3, \dots, q_n)^T$$

The generalized velocity at a generic point of a trajectory  $q(t) \subset Q$  is the tangent vector given by

$$\dot{q} = (\dot{q}_1, \dot{q}_2, \dot{q}_3, \dots, \dot{q}_n)^T$$

For a non holonomic systems, these equations are non integrable as they typically arise when the system has less controls than configuration variables (**underactuated systems**). As a result, a nonholonomic mechanical system cannot move in arbitrary directions in its configuration space. For instance, a unicycle has two controls (linear( $v$ ) and angular( $w$ ) velocities), while it moves in a 3-dimensional configuration space( $x, y, \theta$ ). As a consequence, any path in the configuration space does not necessarily correspond to a feasible path for the system. In other words, for a non-holonomic systems, the **line integrals depend not just on the start and end points but also the path taken.**

The state transition matrix representation of a holonomic system is of the form

$$\dot{x} = f(x, u) \tag{1}$$

Since, equation 1 is non-integrable, we can approximate the integration using numerical integration methods, say Euler's method, which gives

$$x_{new} \approx x + f(x, u)\Delta t,$$

This shows that the new state  $x_{new}$  is constrained due to the choice of  $f$ .

Figure 1 shows one of the feasible paths (represented with lines) of a non-holonomic mobile robot to move between 2 states.

## 2 Differential Drive Robots

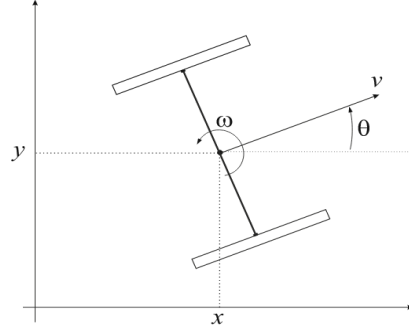


Figure 2:  
Differential drive wheeled robot

Consider a differential drive nonholonomic mobile robot in a two-dimensional, free-space environment, as shown in figure 2. It is assumed that the robot cannot slip in lateral direction,

generalized coordinates :  $q = (x, y, \theta)^T$

Nonholonomic constraints :  $\dot{x} \sin \theta - \dot{y} \cos \theta = 0 \implies \dot{y} = \dot{x} \tan \theta$

$$y = \int \dot{x} \tan \theta dt \quad (2)$$

With 2 control inputs as  $(v, w)$ , the kinematics model of the system is given by:-

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \quad (3)$$

## 3 Motion Planning with Bernstein Polynomials

Since, equation 2 is non integrable, we can approximate the functions  $\dot{x}$  and  $\tan \theta$  with the bernstein polynomials and solve the integral.

NOTE: The Bernstein polynomials are advantageous over other approximation techniques like taylor series as the former and its derivatives polynomials uniformly approximates  $f$  and  $\dot{f}$ , respectively. It holds true for higher order derivatives as well. Moreover, they are the most numerically stable basis.

### 3.1 Bernstein basis polynomial

Let  $f(x)$  be a real-valued function defined and bounded on the interval  $[0,1]$ , then  $B_n(f)$  is the polynomial on  $[0,1]$ .

$$B_n(f(x); t) = \sum_{i=0}^n f\left(\frac{i}{n}\right) C_n^i t^i (1-t)^{n-i} \quad (4)$$

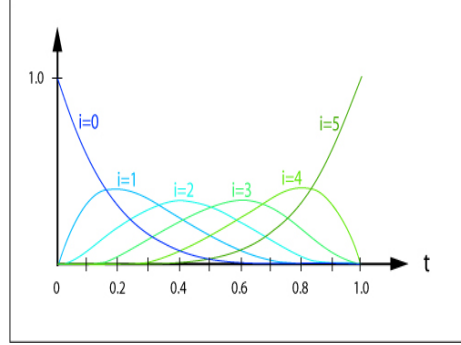


Figure 3:  
Bernstein basis polynomials for  $n=5$ .

If function  $f(x)$  is continuous on  $[0,1]$ , then the Bernstein polynomial  $B_n(f(x))$  tends uniformly to  $f$  as  $n \rightarrow \infty$ . Bernstein basis polynomials with  $n=5$ , are shown in figure 3.

For the given initial state  $(x_{t_0}, y_{t_0}, \theta_{t_0})$  and final state  $(x_{t_f}, y_{t_f}, \theta_{t_f})$ , (more state constraints can be added to the system), with the start time  $t_0$  and end time  $t_f$  of the trajectory of a nonholonomic system,  $y(t)$  and  $x(t)$  can be related as,

$$y(t) = x(t) \tan \theta(t),$$

where, the functions  $x(t)$  and  $\tan \theta(t)$  can be approximated as a linear combination of Bernstein basis polynomials as following-

$$x(t) \approx B_n(x(t)) = B_x(\mu(t)) = \sum_{i=0}^5 W_{x_i} B_i(\mu(t)), \quad (5)$$

Similarly,

$$\tan \theta(t) = k(t) \approx B_n(k(t)) = B_k(\mu(t)) = \sum_{i=0}^5 W_{k_i} B_i(\mu(t)) \quad (6)$$

where,

$$B_i(\mu(t)) = C_i^n (1-\mu)^i \mu^{n-i}$$

$$\mu(t) = \frac{t-t_0}{t_f-t_0}$$

Differentiating equation 5 w.r.t time gives

$$\dot{x}(t) = \dot{B}_x(\mu(t)) = \sum_{i=0}^5 W_{x_i} \dot{B}_i(\mu(t)) \quad (7)$$

using above 2 equations, equation 2 can be rewritten as-

$$y(t) = y_0 + \int_{t_0}^t \left( \sum_{i=0}^5 W_{x_i} \dot{B}_i(\mu(t)) \right) \left( \sum_{i=0}^5 W_{k_i} B_i(\mu(t)) \right) dt \quad (8)$$

The bernstein coefficients of the polynomials and their derivatives for n=5, at time  $t = t_0$  and  $t = t_f$  are shown in the tables below.

Bernstein coefficients	$t = t_0$	$t = t_f$
$B_0(\mu) = {}^5C_0(1-\mu)^5\mu^0$	1	0
$B_1(\mu) = {}^5C_1(1-\mu)^4\mu$	0	0
$B_2(\mu) = {}^5C_2(1-\mu)^3\mu^2$	0	0
$B_3(\mu) = {}^5C_3(1-\mu)^2\mu^3$	0	0
$B_4(\mu) = {}^5C_4(1-\mu)^1\mu^4$	0	0
$B_5(\mu) = {}^5C_5(1-\mu)^0\mu^5$	0	1

Bernstein coefficients derivatives	$t = t_0, \mu = 0$	$t = t_f, \mu = 1$
$\dot{B}_0(\mu) = {}^5C_0 \frac{-5(1-\mu)^4}{(t_f-t_0)}$	$\frac{-5}{(t_f-t_0)}$	0
$\dot{B}_1(\mu) = {}^5C_1 \frac{-4\mu(1-\mu)^3+(1-\mu)^4}{(t_f-t_0)}$	$\frac{5}{(t_f-t_0)}$	0
$\dot{B}_2(\mu) = {}^5C_2 \frac{-3\mu^2(1-\mu)^2+2(1-\mu)^3\mu}{(t_f-t_0)}$	0	0
$\dot{B}_3(\mu) = {}^5C_3 \frac{-2\mu^3(1-\mu)+3(1-\mu)^2\mu^2}{(t_f-t_0)}$	0	0
$\dot{B}_4(\mu) = {}^5C_4 \frac{-\mu^4+4(1-\mu)\mu^3}{(t_f-t_0)}$	0	$\frac{-5}{(t_f-t_0)}$
$\dot{B}_5(\mu) = {}^5C_5 \frac{5\mu^4}{(t_f-t_0)}$	0	$\frac{5}{(t_f-t_0)}$

With the given state constraints(i.e. position, velocity, acceleration, etc.) of the robot at different instants, the unknown, **time independent weight parameters ( $W_{x_0}, W_{x_1}, W_{x_2}, \dots, W_{x_5}$ ) and ( $W_{k_0}, W_{k_1}, W_{k_2}, \dots, W_{k_5}$ )** can be determined.

### 3.2 Finding $W_{x_0}, W_{x_1}, \dots, W_{x_5}$

Given Constraints :  $(x_{t_0}, y_{t_0}), (x_{t_c}, y_{t_c}), (x_{t_f}, y_{t_f}), (\dot{x}_{t_0}, \dot{y}_{t_0}), (\dot{x}_{t_c}, \dot{y}_{t_c}), (\dot{x}_{t_f}, \dot{y}_{t_f})$

Using equations 5, known parameters can be represented as a linear combination of bernstein basis polynomial as follows-

$$x(t_0) = W_{x_0}B_0(\mu(t_0)) + W_{x_1}B_1(\mu(t_0)) + W_{x_2}B_2(\mu(t_0)) + W_{x_3}B_3(\mu(t_0)) + W_{x_4}B_4(\mu(t_0)) + W_{x_5}B_5(\mu(t_0)) \quad (9)$$

$$x(t_f) = W_{x_0}B_0(\mu(t_f)) + W_{x_1}B_1(\mu(t_f)) + W_{x_2}B_2(\mu(t_f)) + W_{x_3}B_3(\mu(t_f)) + W_{x_4}B_4(\mu(t_f)) + W_{x_5}B_5(\mu(t_f)) \quad (10)$$

Putting values of bernstein polynomial coefficients (from tables) in the equations 9 and 10, gives

$$W_{x_0} = x(t_0) = x_{t_0} \quad (11)$$

$$W_{x_5} = x(t_f) = x_{t_f} \quad (12)$$

Using the remaining constraints, all weights  $W_{x_1}, W_{x_2}, W_{x_3}, W_{x_4}$  can be evaluated.

$$\begin{bmatrix} x_{t_c} - W_{x_0}B_0(\mu(t_c)) - W_{x_5}B_5(\mu(t_c)) \\ \dot{x}_{t_0} - W_{x_0}\dot{B}_0(\mu(t_0)) - W_{x_5}\dot{B}_5(\mu(t_0)) \\ \dot{x}_{t_f} - W_{x_0}\dot{B}_0(\mu(t_f)) - W_{x_5}\dot{B}_5(\mu(t_f)) \\ \dot{x}_{t_c} - W_{x_0}\dot{B}_0(\mu(t_c)) - W_{x_5}\dot{B}_5(\mu(t_c)) \end{bmatrix} = \begin{bmatrix} B_1(\mu(t_c)) & B_2(\mu(t_c)) & B_3(\mu(t_c)) & B_4(\mu(t_c)) \\ \dot{B}_1(\mu(t_0)) & \dot{B}_2(\mu(t_0)) & \dot{B}_3(\mu(t_0)) & \dot{B}_4(\mu(t_0)) \\ \dot{B}_1(\mu(t_f)) & \dot{B}_2(\mu(t_f)) & \dot{B}_3(\mu(t_f)) & \dot{B}_4(\mu(t_f)) \\ \dot{B}_1(\mu(t_c)) & \dot{B}_2(\mu(t_c)) & \dot{B}_3(\mu(t_c)) & \dot{B}_4(\mu(t_c)) \end{bmatrix} \begin{bmatrix} W_{x_1} \\ W_{x_2} \\ W_{x_3} \\ W_{x_4} \end{bmatrix} \quad (13)$$

$$A_x = B_x W_x \quad (14)$$

$$W_x = \text{pinv}(B_x)A_x \quad (15)$$

### 3.3 Finding $W_{k_0}, W_{k_1}, \dots, W_{k_5}$

Expanding equation 8,

$$y(t) = y_0 + \int_{t_0}^t (W_{k_0} \cdot f_0(t, t_0, t_f, W_{x_1}, W_{x_2}, \dots, W_{x_5}) + (W_{k_1} \cdot f_1(t, t_0, t_f, W_{x_1}, W_{x_2}, \dots, W_{x_5}) + \dots + (W_{k_5} \cdot f_5(t, t_0, t_f, W_{x_1}, W_{x_2}, \dots, W_{x_5})) \quad (16)$$

with the weight parameters  $W_{x_1} \dots W_{x_5}$  calculated above, equation 16 further reduces to,

$$y(t) = y_0 + W_{k_0}F_0(t) + W_{k_1}F_1(t) + W_{k_2}F_2(t) + W_{k_3}F_3(t) + W_{k_4}F_4(t) + W_{k_5}F_5(t) \quad (17)$$

$$\text{where } F_i(t) = \int_{t_0}^t f_i(t, t_0, t_f, W_{x_1}, W_{x_2}, \dots, W_{x_5}) dt \quad (18)$$

Our objective is to get weights,  $W_{K_0}, W_{K_1}, W_{K_2}, W_{K_3}, W_{K_4}, W_{K_5}$

$$k(t_0) = k_0 = W_{k_0}B_0(\mu(t_0)) + W_{k_1}B_1(\mu(t_0)) + W_{k_2}B_2(\mu(t_0)) + W_{k_3}B_3(\mu(t_0)) + W_{k_4}B_4(\mu(t_0)) + W_{k_5}B_5(\mu(t_0)) \quad (19)$$

which gives,

$$W_{k_0} = k_0 \quad (20)$$

$$k(t_f) = k_f = W_{k_0}B_0(\mu(t_f)) + W_{k_1}B_1(\mu(t_f)) + W_{k_2}B_2(\mu(t_f)) + W_{k_3}B_3(\mu(t_f)) + W_{k_4}B_4(\mu(t_f)) + W_{k_5}B_5(\mu(t_f)) \quad (21)$$

which gives,

$$W_{k_5} = k_f \quad (22)$$

Using any 4 out of remaining constraints on y and k we can form a full rank matrix for  $B_k$  as:

$$\begin{bmatrix} \dot{k}_{t_0} - W_{k_0}\dot{B}_0(\mu(t_0)) - W_{k_5}\dot{B}_5(\mu(t_0)) \\ \dot{k}_{t_f} - W_{k_0}\dot{B}_0(\mu(t_f)) - W_{k_5}\dot{B}_5(\mu(t_f)) \\ y_0 - W_{k_0}F_0 - W_{k_5}F_5 \\ y_f - W_{k_5}F_0 - W_{k_5}F_5 \end{bmatrix} = \begin{bmatrix} \dot{B}_{k_1}(\mu(t_0)) & \dot{B}_2(\mu(t_0)) & \dot{B}_3(\mu(t_0)) & \dot{B}_4(\mu(t_0)) \\ \dot{B}_{k_1}(\mu(t_f)) & \dot{B}_2(\mu(t_f)) & \dot{B}_3(\mu(t_f)) & \dot{B}_4(\mu(t_f)) \\ F_1(t_0) & F_2(t_0) & F_3(t_0) & F_4(t_0) \\ F_1(t_f) & F_2(t_f) & F_3(t_f) & F_4(t_f) \end{bmatrix} \begin{bmatrix} W_{k_1} \\ W_{k_2} \\ W_{k_3} \\ W_{k_4} \end{bmatrix} \quad (23)$$

$$A_k = B_k W_k \quad (24)$$

$$W_k = pinv(B_k) A_k \quad (25)$$

Depending on the number of given constraints i.e., rank of the matrices  $B_x$  ( $p \times q$ ) and  $B_k$  ( $m \times n$ ), the system can be categorized as under constrained, critically constrained and over constrained.

**1. Critically constrained:**

$$\begin{aligned} p=q, \text{rank}(B_x)=p &\implies pinv(B_x) = B_x^{-1} \\ m=n, \text{rank}(B_k)=m &\implies pinv(B_k) = B_k^{-1} \end{aligned}$$

**2. Under constrained:**

$$\begin{aligned} p < q, \text{rank}(B_x) = p &\implies pinv(B_x) = (B_x^T B_x)^{-1} B_x^T \\ m < n, \text{rank}(B_k) = m &\implies pinv(B_k) = (B_k^T B_k)^{-1} B_k^T \end{aligned}$$

**3. Over constrained:**

$$\begin{aligned} p > q, \text{rank}(B_x) = q &\implies pinv(B_x) = B_x^T (B_x B_x^T)^{-1} \\ m > n, \text{rank}(B_k) = n &\implies pinv(B_k) = B_k^T (B_k B_k^T)^{-1} \end{aligned}$$

Once, the weight parameters  $W_k$  and  $W_x$  are determined, the trajectory in 2D plane can be estimated as following,

$$x(t) = W_{x_0} B_0(\mu(t)) + W_{x_1} B_1(\mu(t)) + W_{x_2} B_2(\mu(t)) + W_{x_3} B_3(\mu(t)) + W_{x_4} B_4(\mu(t)) + W_{x_5} B_5(\mu(t)) \quad (26)$$

$$y(t) = y_0 + W_{k_0} F_0(t) + W_{k_1} F_1(t) + W_{k_2} F_2(t) + W_{k_3} F_3(t) + W_{k_4} F_4(t) + W_{k_5} F_5(t) \quad (27)$$

and the orientation of the robot can be defined by-

$$\theta(t) = \arctan((W_{k_0} B_0(\mu(t)) + W_{k_1} B_1(\mu(t)) + W_{k_2} B_2(\mu(t)) + W_{k_3} B_3(\mu(t)) + W_{k_4} B_4(\mu(t)) + W_{k_5} B_5(\mu(t))) \quad (28)$$