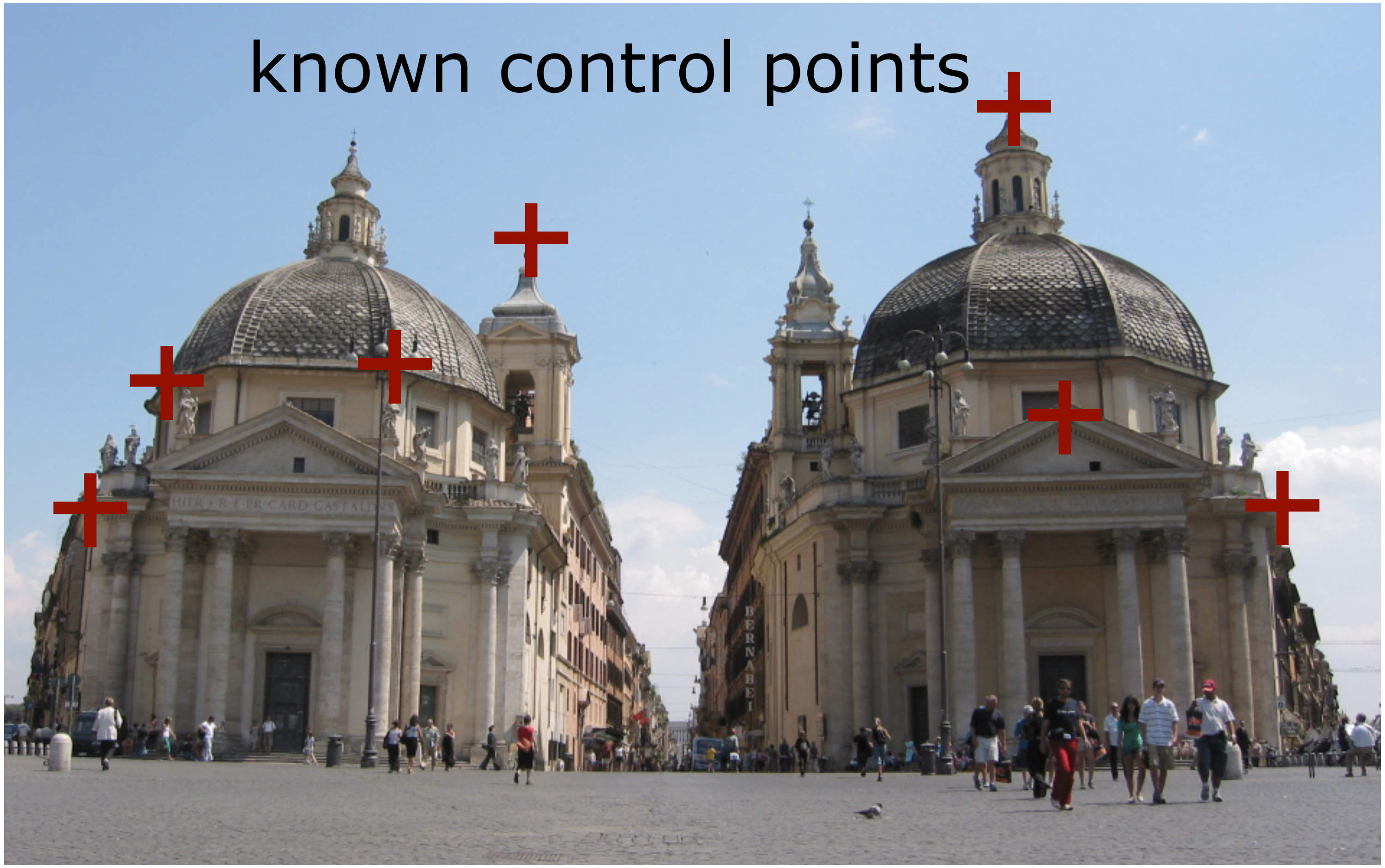
Photogrammetry I

Camera Calibration: Direct Linear Transform and Zhang's Method

Cyrill Stachniss

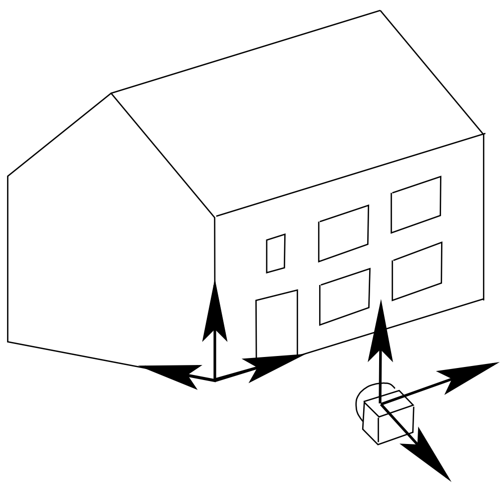
The slides have been created by Cyrill Stachniss.
Partial slides courtesy by Wenzel, Frank, and Bennewitz.

Estimating Camera Parameters Given the Geometry



Possible Applications

- Camera calibration (obtain intrinsics)
- Estimating the pose of a camera given knowledge about the 3D scene
- Estimating the object location given the known pose of a camera



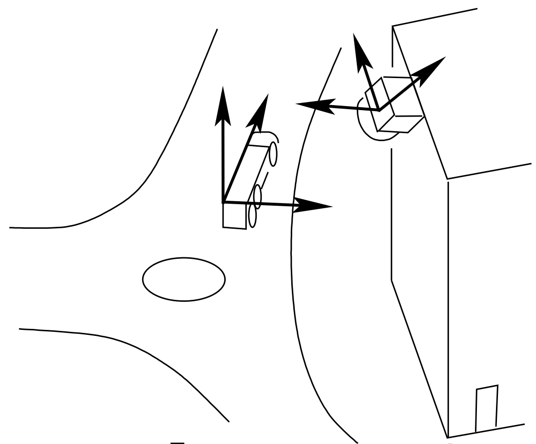


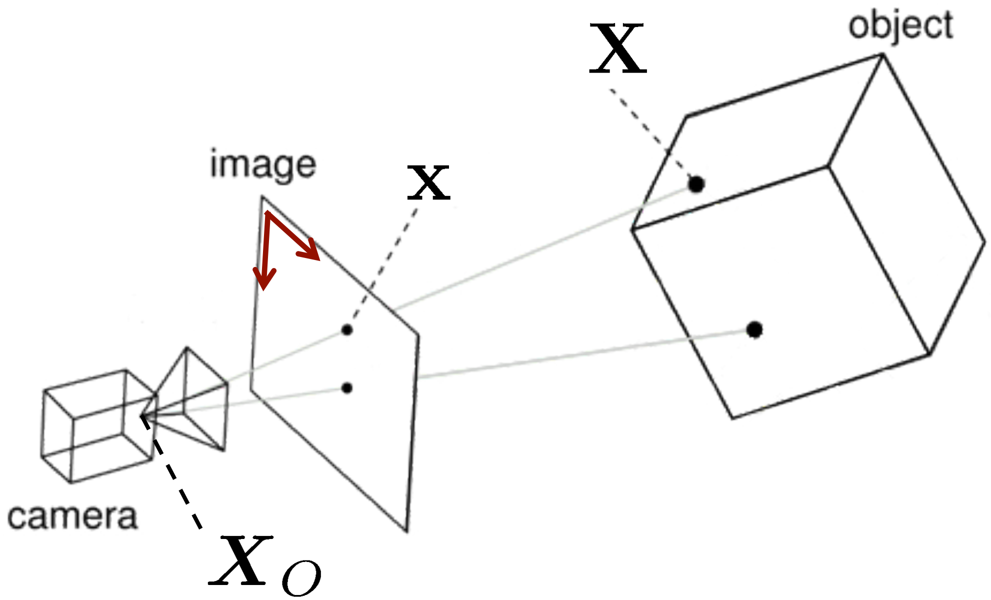
Image courtesy: Förstner

Estimate Ex- and Intrinsics

- **Wanted:** Extrinsic and intrinsic parameters of a camera
- **Given:** Coordinates of object points (control points)
- **Observed:** Coordinates (x, y) of those object points in an image

Mapping

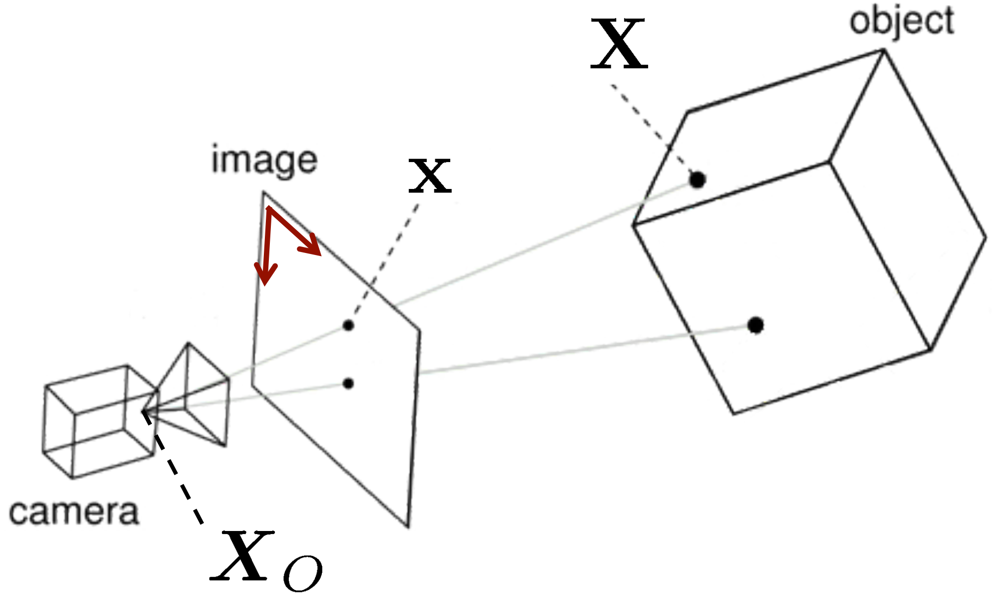
Direct linear transform (DLT) maps any object point $\mathbf{X}$ to the image point $\mathbf{x}$



Mapping

Direct linear transform (DLT) maps any object point $\mathbf{X}$ to the image point $\mathbf{x}$

$$
\begin{aligned}
\mathbf{x} &= \mathsf{K}R[I_3 | -\boldsymbol{X}_O]\mathbf{X} \\
&= \mathsf{P}\,\mathbf{X}
\end{aligned}
$$



Mapping

Direct linear transform (DLT) maps any object point $\mathbf{X}$ to the image point $\mathbf{x}$

$$\begin{aligned}
\underset{3\times1}{\mathbf{x}} &= \underset{3\times3}{\mathsf{K}}\ \underset{3\times3}{R}\ \underbrace{[\underset{3\times3}{I_3} \mid -\underset{3\times1}{\boldsymbol{X}_O}]}_{3\times4}\ \underset{4\times1}{\mathbf{X}} \\
&= \underset{3\times4}{\mathsf{P}}\ \underset{4\times1}{\mathbf{X}}
\end{aligned}$$

Camera Parameters

$$\mathbf{x} = \mathsf{K}R[I_3 | -\boldsymbol{X}_O]\mathbf{X} = \mathsf{P}\ \mathbf{X}$$

- **Interior Orientation (I.O.)**
  - Intrinsic parameters of the camera
  - Given through K
- **Exterior Orientation (E.O.)**
  - Extrinsic parameters of the camera
  - Given through $\boldsymbol{X}_O$ and R
- Projection matrix $\mathsf{P} = \mathsf{K}R[I_3 | -\boldsymbol{X}_O]$ contains both I.O. and E.O.

Direct Linear Transform (DLT)

Compute the **11 intrinsic and extrinsic parameters (I.O. and E.O.)**

$$\mathbf{x} = \mathsf{K} R [I_3 | -\boldsymbol{X}_O] \mathbf{X}$$

observed image point

c, s, m, x_H, y_H

3 rotations

3 translations

control point coordinates (given)

Spatial Resection (Projective 3-Point Algorithm)

Given the intrinsic parameters, compute the **6 extrinisic parameters (E.O.)**

$$\mathbf{x} = \mathsf{K}R[I_3| - \boldsymbol{X}_O]\mathbf{X}$$

observed image points

c, s, m, x_H, y_H (given)

3 rotations

3 translations

control point coordinates (given)

How Many Points Are Needed?

$$\mathbf{x} = \mathsf{P}\ \mathbf{X}$$

Each point gives **???** observation equations

How Many Points Are Needed?

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathsf{P} \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix}$$

Each point gives **???** observation equations

How Many Points Are Needed?

$$\begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} = \mathsf{P} \begin{bmatrix} U/T \\ V/T \\ W/T \\ 1 \end{bmatrix}$$

Each point gives **???** observation equations

How Many Points Are Needed?

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathsf{P} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Each point gives **two** observation equations, one for each image coordinate

$$\begin{aligned} x &= \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \\ y &= \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \end{aligned}$$

Spatial Resection vs. DLT

- **Calibrated camera**
  - 6 unknowns
  - We need at least **3 points**
  - Problem solved by **spatial resection**
- **Uncalibrated camera**
  - 11 unknowns
  - We need at least **6 points**
  - Assuming the model of an **affine camera**
  - Problem solved by **DLT**

DLT: Direct Linear Transform

Computing the Orientation of an Uncalibrated Camera Using $\geq$ 6 Known Points

DLT: Problem Specification

- Task: Estimate the 11 elements of P
- Given:
  - 3D coordinates $\mathbf{X}_i$ of $I \geq 6$ object points
  - Observed image coordinates $\mathbf{x}_i$ of an uncalibrated camera with the mapping

$$\mathbf{x}_i = \mathsf{P}\,\mathbf{X}_i \qquad i = 1, \ldots, I$$

Rearrange the DLT Equation

$$\mathbf{x}_i = \underset{3\times 4}{\mathsf{P}}\mathbf{X}_i = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \mathbf{X}_i$$

Rearrange the DLT Equation

$$\mathbf{x}_i = \underset{3\times4}{\mathsf{P}}\,\mathbf{X}_i = \begin{bmatrix} \boxed{p_{11} \quad p_{12} \quad p_{13} \quad p_{14}} \\ \boxed{p_{21} \quad p_{22} \quad p_{23} \quad p_{24}} \\ \boxed{p_{31} \quad p_{32} \quad p_{33} \quad p_{34}} \end{bmatrix} \mathbf{X}_i$$

$$= \begin{bmatrix} \boxed{\mathbf{A}^\mathsf{T}} \\ \boxed{\mathbf{B}^\mathsf{T}} \\ \boxed{\mathbf{C}^\mathsf{T}} \end{bmatrix} \mathbf{X}_i$$

Rearrange the DLT Equation

$$\mathbf{x}_i = \underset{3\times 4}{\mathsf{P}}\,\mathbf{X}_i = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \mathbf{X}_i$$

$$= \begin{bmatrix} \mathbf{A}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} \\ \mathbf{C}^\mathsf{T} \end{bmatrix} \mathbf{X}_i$$

$$\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = \begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{X}_i \\ \mathbf{B}^\mathsf{T}\mathbf{X}_i \\ \mathbf{C}^\mathsf{T}\mathbf{X}_i \end{bmatrix}$$

Rearrange the DLT Equation

$$\mathbf{x}_i = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = \begin{bmatrix} \mathbf{A}^\top \mathbf{X}_i \\ \mathbf{B}^\top \mathbf{X}_i \\ \mathbf{C}^\top \mathbf{X}_i \end{bmatrix}$$

$\Rightarrow$

$$x_i = \frac{u_i}{w_i} = \frac{\mathbf{A}^\top \mathbf{X}_i}{\mathbf{C}^\top \mathbf{X}_i} \qquad y_i = \frac{v_i}{w_i} = \frac{\mathbf{B}^\top \mathbf{X}_i}{\mathbf{C}^\top \mathbf{X}_i}$$

Rearrange the DLT Equation

$$x_i = \frac{u_i}{w_i} = \frac{\mathbf{A}^\mathsf{T}\mathbf{X}_i}{\mathbf{C}^\mathsf{T}\mathbf{X}_i} \quad \Rightarrow \quad x_i\,\mathbf{C}^\mathsf{T}\mathbf{X}_i - \mathbf{A}^\mathsf{T}\mathbf{X}_i = 0$$

$$y_i = \frac{v_i}{w_i} = \frac{\mathbf{B}^\mathsf{T}\mathbf{X}_i}{\mathbf{C}^\mathsf{T}\mathbf{X}_i} \quad \Rightarrow \quad y_i\,\mathbf{C}^\mathsf{T}\mathbf{X}_i - \mathbf{B}^\mathsf{T}\mathbf{X}_i = 0$$

Leads to an system of equation, which is **linear in the parameters $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$**

$$\begin{aligned} -\mathbf{X}_i^\mathsf{T}\mathbf{A} \qquad\qquad\qquad & +x_i\,\mathbf{X}_i^\mathsf{T}\mathbf{C} = 0 \\ -\mathbf{X}_i^\mathsf{T}\mathbf{B} \quad & +y_i\,\mathbf{X}_i^\mathsf{T}\mathbf{C} = 0 \end{aligned}$$

Estimating the Elements of P

- Collect the elements of P within a vector $\boldsymbol{p}$

$$\boldsymbol{p} = (p_k) = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix} = \mathrm{vec}(\mathsf{P}^{\mathsf{T}})$$

rows of P as column-vectors, one below the other (12x1)

Estimating the Elements of P

- Rewrite
$$
\begin{aligned}
-\mathbf{X}_i^{\mathsf{T}}\mathbf{A} \qquad\qquad\qquad +x_i\,\mathbf{X}_i^{\mathsf{T}}\mathbf{C} &= 0 \\
-\mathbf{X}_i^{\mathsf{T}}\mathbf{B} \quad +y_i\,\mathbf{X}_i^{\mathsf{T}}\mathbf{C} &= 0
\end{aligned}
$$
- as
$$
\begin{aligned}
\boldsymbol{a}_{x_i}^{\mathsf{T}}\boldsymbol{p} &= 0 \\
\boldsymbol{a}_{y_i}^{\mathsf{T}}\boldsymbol{p} &= 0
\end{aligned}
$$
- with

$$
\begin{aligned}
\boldsymbol{p} &= (p_k) = \mathrm{vec}(\mathsf{P}^{\mathsf{T}}) \\
\boldsymbol{a}_{x_i}^{\mathsf{T}} &= (-\mathbf{X}_i^{\mathsf{T}}, \mathbf{0}^{\mathsf{T}}, x_i\mathbf{X}_i^{\mathsf{T}}) \\
&= (-X_i, -Y_i, -Z_i, -1, 0, 0, 0, 0, x_iX_i, x_iY_i, x_iZ_i, x_i) \\
\boldsymbol{a}_{y_i}^{\mathsf{T}} &= (\mathbf{0}^{\mathsf{T}}, -\mathbf{X}_i^{\mathsf{T}}, y_i\mathbf{X}_i^{\mathsf{T}}) \\
&= (0, 0, 0, 0, -X_i, -Y_i, -Z_i, -1, y_iX_i, y_iY_i, y_iZ_i, y_i)
\end{aligned}
$$

Verifying Correctness

$$\boldsymbol{a}_{x_i}^{\mathsf{T}}\boldsymbol{p} = (-X_i,\, -Y_i,\, -Z_i,\, -1, 0,\, 0,\, 0,\, 0, x_iX_i,\, x_iY_i,\, x_iZ_i,\, x_i) \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix}$$

Verifying Correctness

$$
\boldsymbol{a}_{x_i}^{\mathsf{T}} \boldsymbol{p} \quad = \quad (-X_i,\ -Y_i,\ -Z_i,\ -1, 0,\ 0,\ 0,\ 0, x_i X_i,\ x_i Y_i,\ x_i Z_i,\ x_i)
\begin{array}{c}
\begin{bmatrix} \boxed{\begin{matrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \end{matrix}} \\ \boxed{\begin{matrix} p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \end{matrix}} \\ \boxed{\begin{matrix} p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{matrix}} \end{bmatrix}
\\
\begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{bmatrix}
\end{array}
\begin{array}{c}
\boldsymbol{A} \\ \\ \\ \\ \boldsymbol{B} \\ \\ \\ \\ \boldsymbol{C} \\ \\ \\ \\ \\ \\ \\
\end{array}
$$

Verifying Correctness

$$
\begin{aligned}
\boldsymbol{a}_{x_i}^{\top}\boldsymbol{p} &= \left(\boxed{-X_i,\ -Y_i,\ -Z_i,\ -1}, \boxed{0,\ 0,\ 0,\ 0}, \boxed{x_iX_i,\ x_iY_i,\ x_iZ_i,\ x_i}\right) \left[\begin{array}{c} p_{11}\\ p_{12}\\ p_{13}\\ p_{14}\\ p_{21}\\ p_{22}\\ p_{23}\\ p_{24}\\ p_{31}\\ p_{32}\\ p_{33}\\ p_{34} \end{array}\right] \begin{array}{l} \\ \boldsymbol{A}\\ \\ \\ \\ \boldsymbol{B}\\ \\ \\ \\ \boldsymbol{C}\\ \\ \\ \end{array}\\
&= \left(-\mathbf{X}_i^{\top},\ \mathbf{0},\ x_i\mathbf{X}_i^{\top}\right) \begin{bmatrix} \boldsymbol{A}\\ \boldsymbol{B}\\ \boldsymbol{C} \end{bmatrix}
\end{aligned}
$$

Verifying Correctness

$$
\begin{aligned}
\boldsymbol{a}_{x_i}^{\mathsf{T}}\boldsymbol{p} &= (-X_i,\ -Y_i,\ -Z_i,\ -1, 0,\ 0,\ 0,\ 0, x_iX_i,\ x_iY_i,\ x_iZ_i,\ x_i) \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} \\
&= (\quad -\mathbf{X}_i^{\mathsf{T}}, \quad \mathbf{0}, \quad x_i\mathbf{X}_i^{\mathsf{T}} \quad) \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{bmatrix} \\
&= \quad -\mathbf{X}_i^{\mathsf{T}}\mathbf{A} \quad + x_i\,\mathbf{X}_i^{\mathsf{T}}\mathbf{C}
\end{aligned}
$$

Verifying Correctness

$$
\begin{aligned}
\boldsymbol{a}_{y_i}^\top \boldsymbol{p} &= (0,\ 0,\ 0,\ 0,\ -X_i,\ -Y_i,\ -Z_i,\ -1,\ y_iX_i,\ y_iY_i,\ y_iZ_i,\ y_i) \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} \\
&= (\quad \mathbf{0}, \qquad -\mathbf{X}_i^\top, \qquad y_i\mathbf{X}_i^\top \quad) \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{bmatrix} \\
&= \qquad -\mathbf{X}_i^\top \mathbf{B} \qquad + y_i\, \mathbf{X}_i^\top \mathbf{C}
\end{aligned}
$$

Estimating the Elements of P

- For each point, we have

$$\boldsymbol{a}_{x_i}^{\mathsf{T}} \boldsymbol{p} = 0$$

$$\boldsymbol{a}_{y_i}^{\mathsf{T}} \boldsymbol{p} = 0$$

- Collecting everything together

$$\begin{bmatrix} \boldsymbol{a}_{x_1}^{\mathsf{T}} \\ \boldsymbol{a}_{y_1}^{\mathsf{T}} \\ \cdots \\ \boldsymbol{a}_{x_i}^{\mathsf{T}} \\ \boldsymbol{a}_{y_i}^{\mathsf{T}} \\ \cdots \\ \boldsymbol{a}_{x_I}^{\mathsf{T}} \\ \boldsymbol{a}_{y_I}^{\mathsf{T}} \end{bmatrix} \boldsymbol{p} = \underset{2I \times 12}{\mathsf{M}} \underset{12 \times 1}{\boldsymbol{p}} \stackrel{!}{=} 0$$

Estimating the elements of P

- In case of redundant observations, we get contradictions

$$\mathsf{M}\ \boldsymbol{p} = \boldsymbol{w}$$

- Find $\boldsymbol{p}$ such that it minimizes

$$\begin{aligned} \Omega &= \boldsymbol{w}^{\mathsf{T}}\boldsymbol{w} \\ \Rightarrow \quad \widehat{\boldsymbol{p}} &= \arg\min_{\boldsymbol{p}}\ \boldsymbol{w}^{\mathsf{T}}\boldsymbol{w} \\ &= \arg\min_{\boldsymbol{p}}\ \boldsymbol{p}^{\mathsf{T}}\mathsf{M}^{\mathsf{T}}\mathsf{M}\boldsymbol{p} \end{aligned}$$

with $||P||_2 = \sum_{ij} p_{ij}^2 = \|\boldsymbol{p}\| = 1$

Comments

$$\widehat{\boldsymbol{p}} = \arg\min_{\boldsymbol{p}} \ \boldsymbol{w}^{\mathsf{T}} \boldsymbol{w}$$

- A Least-Squares problem
- An algebraic minimization problem
  - Contradictions just algebraic,
    not directly related to geometric entities
  - Due to homogeneity scale of $\boldsymbol{p}$ is arbitrary, so the one of $\boldsymbol{w}$, is too
  - Require $\|\boldsymbol{p}\| = 1$

Solving the Linear System

- Solving a system of linear equations of the form $A\,\boldsymbol{x} = \mathbf{0}$ is equivalent to finding the null space of A
- Thus, we can apply the SVD to solve M $\boldsymbol{p} \stackrel{!}{=} \mathbf{0}$

Solution

$$\mathsf{M}\ \boldsymbol{p} \stackrel{!}{=} \mathbf{0}$$

- Singular value decomposition (SVD)

$$\underset{2I\times 12}{\mathsf{M}} = \underset{2I\times 12}{U}\ \underset{12\times 12}{\mathsf{S}}\ \underset{12\times 12}{V^{\mathsf{T}}} = \sum_{i=1}^{12} s_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathsf{T}}$$

with properties $U^{\mathsf{T}} U = I_{12}$, $V^{\mathsf{T}} V = I_{12}$
and $s_1 \geq s_2 \geq \ldots \geq s_{12}$

Solution

- Linear system $\mathsf{M}\,\boldsymbol{p} = \boldsymbol{w} \stackrel{!}{=} \mathbf{0}$
- Minimize $\boldsymbol{\Omega} = \boldsymbol{w}^\mathsf{T}\boldsymbol{w}$
- using SVD $\mathsf{M} = \mathsf{U}\,\mathsf{S}\,\mathsf{V}^\mathsf{T}$
- Applying SVD leads to:

$$\begin{aligned}
\Omega &= \boldsymbol{p}^\mathsf{T}\mathsf{M}^\mathsf{T}\mathsf{M}\boldsymbol{p} \\
&= \boldsymbol{p}\mathsf{V}\mathsf{S}\mathsf{U}^\mathsf{T}\mathsf{U}\mathsf{S}\mathsf{V}^\mathsf{T}\boldsymbol{p} \\
&= \boldsymbol{p}^\mathsf{T}\mathsf{V}\mathsf{S}^2\mathsf{V}^\mathsf{T}\boldsymbol{p} \\
&= \boldsymbol{p}^\mathsf{T}\left(\sum_{i=1}^{12} s_i^2\boldsymbol{v}_i\boldsymbol{v}_i^\mathsf{T}\right)\boldsymbol{p}
\end{aligned}$$

Solution

$$\Omega = \boldsymbol{p}^{\mathsf{T}} \left(\sum_{i=1}^{12} s_i^2 \boldsymbol{v}_i \boldsymbol{v}_i^{\mathsf{T}} \right) \boldsymbol{p}$$

- Due to orthogonality of *V*

$$\boldsymbol{v}_i \boldsymbol{v}_j^{\mathsf{T}} = \begin{cases} 0 & \text{if } \ i \neq j \\ 1 & \text{if } \ i = j \end{cases}$$

- If we choose $\boldsymbol{p} = \boldsymbol{v}_i$

$$\Omega = \boldsymbol{v}_i^{\mathsf{T}} \left(s_i^2 \boldsymbol{v}_i \boldsymbol{v}_i^{\mathsf{T}} \right) \boldsymbol{v}_i = s_i^2 \, \boldsymbol{v}_i^{\mathsf{T}} \boldsymbol{v}_i \boldsymbol{v}_i^{\mathsf{T}} \boldsymbol{v}_i = s_i^2$$

- Choosing $\boldsymbol{p} = \boldsymbol{v}_{12}$ (the singular vector belonging to the smallest singular value s_{12}) minimizes Ω

Solution

- Estimate of $\boldsymbol{p}$ is given by

$$\widehat{\boldsymbol{p}} = \begin{bmatrix} \widehat{\mathbf{A}} \\ \widehat{\mathbf{B}} \\ \widehat{\mathbf{C}} \end{bmatrix} = \boldsymbol{v}_{12}$$

- and leads to the estimated projection matrix

$$\widehat{\mathsf{P}} = \begin{bmatrix} \widehat{\mathbf{A}}^{\mathsf{T}} \\ \widehat{\mathbf{B}}^{\mathsf{T}} \\ \widehat{\mathbf{C}}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} \widehat{p}_1 & \widehat{p}_2 & \widehat{p}_3 & \widehat{p}_4 \\ \widehat{p}_5 & \widehat{p}_6 & \widehat{p}_7 & \widehat{p}_8 \\ \widehat{p}_9 & \widehat{p}_{10} & \widehat{p}_{11} & \widehat{p}_{12} \end{bmatrix}$$

Solution

- M is of rank 11, if
  - Number of points ≥ 6
  - Assumption: no gross errors

Critical Surfaces
(DE: Gefährliche Orte)

- M is of rank 11, if
  - Number of points ≥6
  - Assumption: no gross errors
- **No solution**, if all points $\mathbf{X}_i$ are located on a **plane**

$$
\mathsf{M} = \begin{bmatrix} \dots \\ \boldsymbol{a}_{x_i}^{\mathsf{T}} \\ \boldsymbol{a}_{y_i}^{\mathsf{T}} \\ \dots \end{bmatrix}
$$

e.g., assume all Z_i=0

$$
= \begin{bmatrix} & & & & & \dots & & & & & & \\ -X_i & -Y_i & -Z_i & -1 & 0 & 0 & 0 & 0 & x_iX_i & x_iY_i & x_iZ_i & x_i \\ 0 & 0 & 0 & 0 & -X_i & -Y_i & -Z_i & -1 & y_iX_i & y_iY_i & y_iZ_i & y_i \\ & & & & & \dots & & & & & & \end{bmatrix}
$$

Critical Surfaces
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- M is of rank 11, if
  - Number of points ≥6
  - Assumption: no gross errors
- **No solution**, if all points $\mathbf{X}_i$ are located on a **plane**

$$
\mathsf{M} = \begin{bmatrix} \dots \\ \boldsymbol{a}_{x_i}^{\mathsf{T}} \\ \boldsymbol{a}_{y_i}^{\mathsf{T}} \\ \dots \end{bmatrix}
= \begin{bmatrix} & & & & & \dots & & & & & & \\ -X_i & -Y_i & \boxed{0} & -1 & 0 & 0 & \boxed{0} & 0 & x_i X_i & x_i Y_i & \boxed{0} & x_i \\ 0 & 0 & \boxed{0} & 0 & -X_i & -Y_i & \boxed{0} & -1 & y_i X_i & y_i Y_i & \boxed{0} & y_i \\ & & & & & \dots & & & & & & \end{bmatrix}
$$

Critical Surfaces
(DE: Gefährliche Orte)

- M is of rank 11, if
  - Number of points ≥6
  - Assumption: no gross errors
- **No solution**, if
  - All points $\mathbf{X}_i$ are located on a **plane**
  - (All points $\mathbf{X}_i$ and projection center $\boldsymbol{X}_o$ are located on a twisted cubic curve)

Decomposition of P

- We have $\widehat{\mathrm{P}}$, how to obtain $\widehat{\mathrm{K}}, \widehat{R}, \widehat{\boldsymbol{X}_O}$?

Decomposition of P

- We have $\widehat{\mathsf{P}}$, how to obtain $\widehat{\mathsf{K}}, \widehat{R}, \widehat{\boldsymbol{X}}_O$?
- Structure of the projection matrix

$$\widehat{\mathsf{P}} = \widehat{\mathsf{K}}\widehat{R}\,[I_3 | -\widehat{\boldsymbol{X}}_O] = [\underset{3\times 3}{\widehat{\mathsf{H}}_\infty} | \underset{3\times 1}{\widehat{\mathbf{h}}}]$$

- with

$$\widehat{\mathsf{H}}_\infty = \widehat{\mathsf{K}}\widehat{R} \qquad \widehat{\mathbf{h}} = -\widehat{\mathsf{K}}\widehat{R}\mathbf{X}_O$$

Decomposition of P

$$\widehat{\mathsf{H}}_\infty = \widehat{\mathsf{K}}\widehat{R} \qquad \widehat{\mathbf{h}} = -\widehat{\mathsf{K}}\widehat{R}\mathbf{X}_O$$

- We get the projection center

$$\widehat{\boldsymbol{X}_O} = -\widehat{\mathsf{H}}_\infty^{-1}\,\widehat{\mathbf{h}}$$

- QR decomposition of $\widehat{\mathsf{H}}_\infty^{-1}$ yields rotation and calibration matrix

$$\widehat{\mathsf{H}}_\infty^{-1} = (\widehat{\mathsf{K}}\,\widehat{R})^{-1} = \widehat{R}^{-1}\,\widehat{\mathsf{K}}^{-1} = \widehat{R}^{\mathsf{T}}\,\widehat{\mathsf{K}}^{-1}$$

- Due to homogeneity normalize

$$\widehat{\widehat{K}} = \frac{1}{\widehat{\mathsf{K}}_{33}}\widehat{\mathsf{K}}$$

Decomposition of P

- Decomposition $\widehat{\mathsf{H}}_\infty^{-1} = \widehat{R}^{\mathsf{T}}\, \widehat{\mathsf{K}}^{-1}$ results in $\widehat{\mathsf{K}}$ with **positive** diagonal elements
- To get **negative camera constant** $\widehat{c}$, choose

$$\widehat{\mathsf{K}} \leftarrow \widehat{\mathsf{K}} R(z,\pi) \qquad \widehat{R} \leftarrow R(z,\pi)\widehat{R}$$

  - Using

$$R(z,\pi) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

decomposition still holds $\widehat{\mathsf{H}}_\infty = \widehat{\mathsf{K}} R(z,\pi)\; R(z,\pi)\widehat{R} = \widehat{\mathsf{K}}\widehat{R}$

DLT in a Nutshell

1. Vectorize P : $\boldsymbol{p} = (p_k) = \operatorname{vec}(\mathsf{P}^{\mathsf{T}})$

DLT in a Nutshell

1. Vectorize P : $\boldsymbol{p} = (p_k) = \text{vec}(\mathsf{P}^\mathsf{T})$
2. Build the M for the linear system

$$\mathsf{M} = \begin{bmatrix} \boldsymbol{a}_{x_1}^\mathsf{T} \\ \boldsymbol{a}_{y_1}^\mathsf{T} \\ \cdots \\ \boldsymbol{a}_{x_I}^\mathsf{T} \\ \boldsymbol{a}_{y_I}^\mathsf{T} \end{bmatrix} \qquad \underbrace{\mathsf{M}}_{2\mathrm{I}\times 12}\ \underbrace{\boldsymbol{p}}_{12\times 1} \stackrel{!}{=} 0$$

with

$$\begin{aligned} \boldsymbol{a}_{x_i}^\mathsf{T} &= (-X_i,\ -Y_i,\ -Z_i,\ -1, 0,\ 0,\ 0,\ 0, x_iX_i,\ x_iY_i,\ x_iZ_i,\ x_i) \\ \boldsymbol{a}_{y_i}^\mathsf{T} &= (0,\ 0,\ 0,\ 0, -X_i,\ -Y_i,\ -Z_i,\ -1, y_iX_i,\ y_iY_i,\ y_iZ_i,\ y_i) \end{aligned}$$

DLT in a Nutshell

3. Solve by SVD $\mathsf{M} = U\, S\, V^{\mathsf{T}}$
   Solution is last column of V

$$\widehat{\boldsymbol{p}} = \begin{bmatrix} \widehat{\mathbf{A}} \\ \widehat{\mathbf{B}} \\ \widehat{\mathbf{C}} \end{bmatrix} = \boldsymbol{v}_{12} \;\Rightarrow\; \widehat{\mathsf{P}} = \begin{bmatrix} \widehat{\mathbf{A}}^{\mathsf{T}} \\ \widehat{\mathbf{B}}^{\mathsf{T}} \\ \widehat{\mathbf{C}}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} \widehat{p}_1 & \widehat{p}_2 & \widehat{p}_3 & \widehat{p}_4 \\ \widehat{p}_5 & \widehat{p}_6 & \widehat{p}_7 & \widehat{p}_8 \\ \widehat{p}_9 & \widehat{p}_{10} & \widehat{p}_{11} & \widehat{p}_{12} \end{bmatrix}$$

DLT in a Nutshell

3. Solve by SVD $\mathsf{M} = U\, S\, V^{\mathsf{T}}$
   Solution is last column of V

$$\widehat{\boldsymbol{p}} = \begin{bmatrix} \widehat{\mathbf{A}} \\ \widehat{\mathbf{B}} \\ \widehat{\mathbf{C}} \end{bmatrix} = \boldsymbol{v}_{12} \;\Rightarrow\; \widehat{\mathsf{P}} = \begin{bmatrix} \widehat{\mathbf{A}}^{\mathsf{T}} \\ \widehat{\mathbf{B}}^{\mathsf{T}} \\ \widehat{\mathbf{C}}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} \widehat{p}_1 & \widehat{p}_2 & \widehat{p}_3 & \widehat{p}_4 \\ \widehat{p}_5 & \widehat{p}_6 & \widehat{p}_7 & \widehat{p}_8 \\ \widehat{p}_9 & \widehat{p}_{10} & \widehat{p}_{11} & \widehat{p}_{12} \end{bmatrix}$$

4. If individual parameters are needed

$$\widehat{\mathsf{P}} = \widehat{\mathsf{K}}\widehat{R}\,[I_3 | -\widehat{\boldsymbol{X}}_O] = [\underbrace{\widehat{\mathsf{H}}_\infty}_{3\times 3} | \underbrace{\widehat{\mathbf{h}}}_{3\times 1}]$$

$$\widehat{\mathsf{H}}_\infty = \widehat{\mathsf{K}}\widehat{R} \qquad \widehat{\mathbf{h}} = -\widehat{\mathsf{K}}\widehat{R}\mathbf{X}_O$$

$$\widehat{\boldsymbol{X}}_O = -\widehat{\mathsf{H}}_\infty^{-1}\,\widehat{\mathbf{h}} \qquad \widehat{\mathsf{H}}_\infty^{-1} = \widehat{R}^{\mathsf{T}}\,\widehat{\mathsf{K}}^{-1} \qquad \widehat{\widehat{K}} = \frac{1}{\widehat{\mathsf{K}}_{33}}\widehat{\mathsf{K}}$$

Discussion DLT

- We realize $\mathsf{P} \leftrightarrow (\mathsf{K}, R, \boldsymbol{X}_O)$ both ways
- We are free to choose sign of c
- Solution is instable if the control points lie approximately on a plane
- Solution is statistically not optimal (no uncertainties of point coordinates)

DLT Summary

- We can estimate the camera parameters given control points
- **Uncalibrated camera**
  - **DLT**
  - Using **≥6 points**
  - Direct solution

R.S. will be the topic of next week’s lecture

Camera Calibration Using a 2D Checkerboard

- Observed 2D pattern (checkerboard)
- **Known size and structure**

Trick for Checkerboard Calibration

- **Set the world coordinate system to the corner of the checkerboard**
- All points on the checkerboard lie in the X/Y plane, i.e., Z=0

Simplification

- The Z coordinate of all points on the checkerboard is equal to zero

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ \boxed{Z} \\ 1 \end{bmatrix}$$

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- We can delete the 3rd column of the extrinsic parameter matrix

Simplification

- The Z coordinate of all points on the checkerboard is equal to zero
- Deleting the 3rd column of the extrinsic parameter matrix leads to

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Setting Up the Equations for Determining the Parameter

$$\mathsf{H} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_{\mathsf{K}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t}]}$$

Setting Up the Equations for Determining the Parameter

$$\mathsf{H} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_{\mathsf{K}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t}]}$$

One point generates the equation:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathsf{H} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \mathsf{K}[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t}] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Setting Up the Equations for Determining the Parameter

- For multiple points, we obtain

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \underset{3\times 3}{\mathsf{H}} \begin{bmatrix} X_i \\ Y_i \\ 1 \end{bmatrix} \qquad i = 1, \ldots, I$$

How to proceed?

Setting Up the Equations for Determining the Parameter

- For multiple points, we obtain

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \underset{3\times 3}{\mathsf{H}} \begin{bmatrix} X_i \\ Y_i \\ 1 \end{bmatrix} \qquad i = 1, \ldots, I$$

- Analogous to **steps 1-3 of the DLT**

DLT-Like Estimation

- We estimate a 3x3 homography instead of a 3x4 projection matrix
- Rest is identical (instead of Z coord.)
- We use $\boldsymbol{a}_{x_i}^{\mathsf{T}} \boldsymbol{h} = 0$

  $\boldsymbol{a}_{y_i}^{\mathsf{T}} \boldsymbol{h} = 0$
- with

$$
\begin{aligned}
\boldsymbol{h} &= (h_k) = \mathrm{vec}(\mathsf{H}^{\mathsf{T}}) \\
\boldsymbol{a}_{x_i}^{\mathsf{T}} &= (-X_i,\ -Y_i,\ -\cancel{Z_i},\ -1, 0,\ 0,\ \cancel{0},\ 0, x_i X_i,\ x_i Y_i,\ x_i \cancel{Z_i},\ x_i) \\
\boldsymbol{a}_{y_i}^{\mathsf{T}} &= (0,\ 0,\ \cancel{0},\ 0, -X_i,\ -Y_i,\ -\cancel{Z_i},\ -1, y_i X_i,\ y_i Y_i,\ y_i \cancel{Z_i},\ y_i)
\end{aligned}
$$

DLT-Like Estimation

- We estimate a 3x3 homography instead of a 3x4 projection matrix
- Rest is identical (instead of Z coord.)
- We use $\boldsymbol{a}_{x_i}^{\mathsf{T}} \boldsymbol{h} = 0$

$$\boldsymbol{a}_{y_i}^{\mathsf{T}} \boldsymbol{h} = 0$$

- with

$$\begin{aligned}
\boldsymbol{h} &= (h_k) = \operatorname{vec}(\mathsf{H}^{\mathsf{T}}) \\
\boldsymbol{a}_{x_i}^{\mathsf{T}} &= (-X_i,\, -Y_i,\, -1, 0,\, 0,\, 0, x_i X_i,\, x_i Y_i,\, x_i) \\
\boldsymbol{a}_{y_i}^{\mathsf{T}} &= (0,\, 0,\, 0, -X_i,\, -Y_i,\, -1, y_i X_i,\, y_i Y_i,\, y_i)
\end{aligned}$$

DLT-Like Estimation

- Solving a system of linear equation leads to an estimate of H
- We need to identify at least 4 points as H has 8 DoF and each point consists of two observations

We estimated H and now we need to compute K from H

Computing K Given H

$$\mathsf{H} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_{\mathsf{K}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t}]}$$

no rotation matrix, thus QR decomposition is not applicable as for DLT

Computing K Given H is Different From the DLT Solution

- Homography H has only 8 DoF
- No direct decomposition as in DLT
- Exploit constraints on the intrinsic parameters

$$\mathsf{H} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_{\mathsf{K}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t}]}$$

$$[\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3] = \mathsf{K}[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t}]$$

Exploiting Constraints for Determining the Parameter

$$\mathsf{H} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_{\mathsf{K}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t}]}$$

$$[\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3] = \mathsf{K}[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t}]$$

$$\Rightarrow \quad \boldsymbol{r}_1 = \mathsf{K}^{-1}\boldsymbol{h}_1 \qquad \boldsymbol{r}_2 = \mathsf{K}^{-1}\boldsymbol{h}_2$$

Exploiting Constraints for Determining the Parameter

$$\mathsf{H} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3] = \underbrace{\begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}}_{\mathsf{K}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}}_{[\boldsymbol{r}_1,\, \boldsymbol{r}_2,\, \boldsymbol{t}]}$$

$$[\boldsymbol{h}_1,\, \boldsymbol{h}_2,\, \boldsymbol{h}_3] = \mathsf{K}[\boldsymbol{r}_1,\, \boldsymbol{r}_2,\, \boldsymbol{t}]$$

$$\boldsymbol{r}_1 = \mathsf{K}^{-1}\boldsymbol{h}_1 \qquad \boldsymbol{r}_2 = \mathsf{K}^{-1}\boldsymbol{h}_2$$

As $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$ form an orthonormal basis

$$\boldsymbol{r}_1^T \boldsymbol{r}_2 = 0 \qquad \|\boldsymbol{r}_1\| = \|\boldsymbol{r}_2\| = 1$$

Exploiting Constraints

$$\boldsymbol{r}_1 = \mathsf{K}^{-1}\boldsymbol{h}_1 \qquad \boldsymbol{r}_2 = \mathsf{K}^{-1}\boldsymbol{h}_2$$

$$\xrightarrow{\boldsymbol{r}_1^T\boldsymbol{r}_2 = 0} \qquad \boldsymbol{h}_1^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_2 = 0$$

Exploiting Constraints

$$\boldsymbol{r}_1 = \mathsf{K}^{-1}\boldsymbol{h}_1 \qquad \boldsymbol{r}_2 = \mathsf{K}^{-1}\boldsymbol{h}_2$$

$\boldsymbol{r}_1^T\boldsymbol{r}_2 = 0$

$$\Longrightarrow \quad \boldsymbol{h}_1^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_2 = 0$$

$\|\boldsymbol{r}_1\| = \|\boldsymbol{r}_2\| = 1$

$$\Longrightarrow \quad \boldsymbol{h}_1^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_1 = \boldsymbol{h}_2^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_2$$

Exploiting Constraints

$$\boldsymbol{r}_1 = \mathsf{K}^{-1}\boldsymbol{h}_1 \qquad \boldsymbol{r}_2 = \mathsf{K}^{-1}\boldsymbol{h}_2$$

$$\xRightarrow{\ \boldsymbol{r}_1^T\boldsymbol{r}_2 = 0\ } \qquad \boldsymbol{h}_1^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_2 = 0$$

$$\xRightarrow{\ \|\boldsymbol{r}_1\| = \|\boldsymbol{r}_2\| = 1\ } \qquad \boldsymbol{h}_1^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_1 = \boldsymbol{h}_2^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_2$$

$$\boldsymbol{h}_1^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_1 - \boldsymbol{h}_2^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_2 = 0$$

Exploiting Constraints

$$\boldsymbol{r}_1 = \mathsf{K}^{-1}\boldsymbol{h}_1 \qquad \boldsymbol{r}_2 = \mathsf{K}^{-1}\boldsymbol{h}_2$$

$$\boxed{\boldsymbol{h}_1^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_2 = 0}$$

$$\boldsymbol{h}_1^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_1 = \boldsymbol{h}_2^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_2$$

$$\boxed{\boldsymbol{h}_1^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_1 - \boldsymbol{h}_2^T\mathsf{K}^{-T}\mathsf{K}^{-1}\boldsymbol{h}_2 = 0}$$

Exploiting Constraints

$$\boldsymbol{h}_1^T \underline{\mathsf{K}^{-T}\mathsf{K}^{-1}} \boldsymbol{h}_2 = 0$$

$$\boldsymbol{h}_1^T \underline{\mathsf{K}^{-T}\mathsf{K}^{-1}} \boldsymbol{h}_1 - \boldsymbol{h}_2^T \underline{\mathsf{K}^{-T}\mathsf{K}^{-1}} \boldsymbol{h}_2 = 0$$

- Define symmetric and positive definite matrix $\mathsf{B} := \mathsf{K}^{-T}\mathsf{K}^{-1}$

Exploiting Constraints

$$\boldsymbol{h}_1^T \underline{\mathsf{B}} \boldsymbol{h}_2 = 0$$

$$\boldsymbol{h}_1^T \underline{\mathsf{B}} \boldsymbol{h}_1 - \boldsymbol{h}_2^T \underline{\mathsf{B}} \boldsymbol{h}_2 = 0$$

- Define symmetric and positive definite matrix $\mathsf{B} := \mathsf{K}^{-T}\mathsf{K}^{-1}$

Exploiting Constraints

$$\boldsymbol{h}_1^T \underline{\mathsf{B}} \boldsymbol{h}_2 = 0$$

$$\boldsymbol{h}_1^T \underline{\mathsf{B}} \boldsymbol{h}_1 - \boldsymbol{h}_2^T \underline{\mathsf{B}} \boldsymbol{h}_2 = 0$$

- Define symmetric and positive definite matrix $\mathsf{B} := \mathsf{K}^{-T}\mathsf{K}^{-1}$
- The calibration matrix can be recovered through Cholesky decomp.

$$\mathsf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}$$

$$\text{chol}(\mathsf{B}) = \mathsf{A}\mathsf{A}^T$$

$$\mathsf{A} = \mathsf{K}^{-T}$$

Exploiting Constraints

- Define a vector $\boldsymbol{b} = (b_{11}, b_{12}, b_{13}, b_{22}, b_{23}, b_{33})$ of unknowns

$$\mathsf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}$$

Exploiting Constraints

- Define a vector $\boldsymbol{b} = (b_{11}, b_{12}, b_{13}, b_{22}, b_{23}, b_{33})$ of unknowns

$$\mathsf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}$$

- Construct a system of linear equations $V\boldsymbol{b} = 0$ using $\boldsymbol{h}_i^T \mathsf{B} \boldsymbol{h}_j = \boldsymbol{v}_{ij}^T \boldsymbol{b}$ ($\boldsymbol{v}_{ij}$ see next slide) exploiting the constraints:

$$\boldsymbol{v}_{12}^T \boldsymbol{b} = 0$$

(first constraint)

$\boldsymbol{r}_1^T \boldsymbol{r}_2 = 0$

$$\boldsymbol{v}_{11}^T \boldsymbol{b} - \boldsymbol{v}_{22}^T \boldsymbol{b} = 0$$

(second constraint)

$\|\boldsymbol{r}_1\| = \|\boldsymbol{r}_2\| = 1$

The Matrix V

- The matrix V is given as

$$V = \begin{pmatrix} \boldsymbol{v}_{12}^T \\ \boldsymbol{v}_{11}^T - \boldsymbol{v}_{22}^T \end{pmatrix} \quad \text{with} \quad \boldsymbol{v}_{ij} = \begin{bmatrix} h_{1i}h_{1j} \\ h_{1i}h_{2j} + h_{2i}h_{1j} \\ h_{3i}h_{1j} + h_{1i}h_{3j} \\ h_{2i}h_{2j} \\ h_{3i}h_{2j} + h_{2i}h_{3j} \\ h_{3i}h_{3j} \end{bmatrix}$$

elements of H

- For one image, we obtain

$$\begin{pmatrix} \boldsymbol{v}_{12}^T \\ \boldsymbol{v}_{11}^T - \boldsymbol{v}_{22}^T \end{pmatrix} \boldsymbol{b} = \boldsymbol{0}$$

The Matrix V

- For multiple images, we stack the matrices to a 2n x 6 matrix

$$\begin{array}{rl} \begin{array}{r} \text{image 1} \rightarrow \\ \\ \\ \text{image n} \rightarrow \\ \\ \end{array} & \begin{pmatrix} \boldsymbol{v}_{12}^{\mathsf{T}} \\ \boldsymbol{v}_{11}^{\mathsf{T}} - \boldsymbol{v}_{22}^{\mathsf{T}} \\ \dots \\ \boldsymbol{v}_{12}^{\mathsf{T}} \\ \boldsymbol{v}_{11}^{\mathsf{T}} - \boldsymbol{v}_{22}^{\mathsf{T}} \end{pmatrix} \boldsymbol{b} = \mathbf{0} \end{array}$$

- We need to solve the linear system $V\boldsymbol{b} = 0$ to obtain $\boldsymbol{b}$ and thus K

Solving the Linear System

- The system $Vb = 0$ has a trivial solution which (invalid matrix B)
- Impose additional constraint $\|b\| = 1$

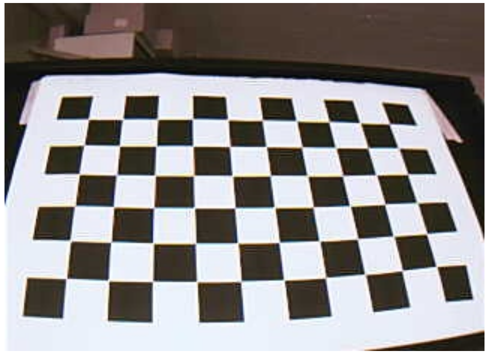
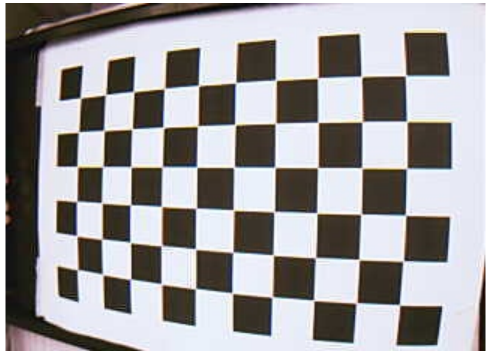
Solving the Linear System

- The system $Vb = 0$ has a trivial solution which (invalid matrix B)
- Impose additional constraint $\|b\| = 1$
- Real measurements are noisy
- Find the solution that minimizes the squares error

$$b^* = \arg\min_{b} \|Vb\| \text{ with } \|b\| = 1$$

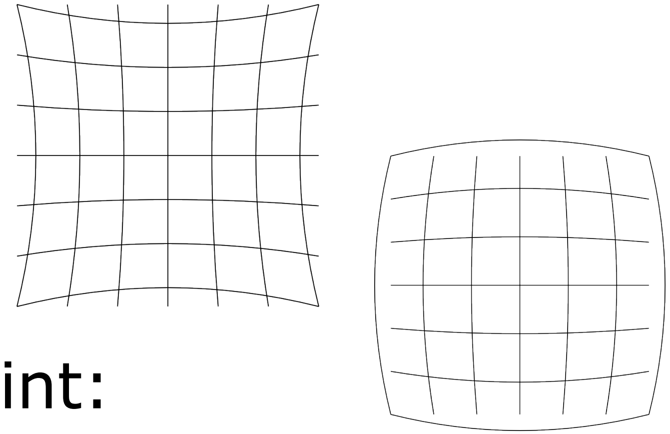
- Eigenvector/Eigenvalue problem similar to the DLT computation

What is Needed?

- We need at least **4 points per plane** to compute the matrix H
- Each **plane** gives us **two equations**
- Since B has 6 DoF, we need at least **3 different views of a plane**
- Solve $V\boldsymbol{b} = 0$ to compute K

Example Lens Distortion Model



Non-linear effects:

- Radial distortion
- Tangential distortion

- Compute the corrected image point:

(1) $$x' = x/z \qquad y' = y/z$$

(2) $$x'' = x'(1+k_1r^2+k_2r^4)+2p_1x'y'+p_2(r^2+2x'^2)$$
$$y'' = y'(1+k_1r^2+k_2r^4)+p_1(r^2+2y'^2)+2p_2x'y'$$

where $r^2 = x'^2 + y'^2$ k_1, k_2 : radial distortion coefficients

p_1, p_2 : tangential distortion coefficients

(3) $$u = f_x \cdot x'' + c_x \qquad v = f_y \cdot y'' + c_y$$

Error Minimization

Lens distortion can be calculated by minimizing a non-linear error function

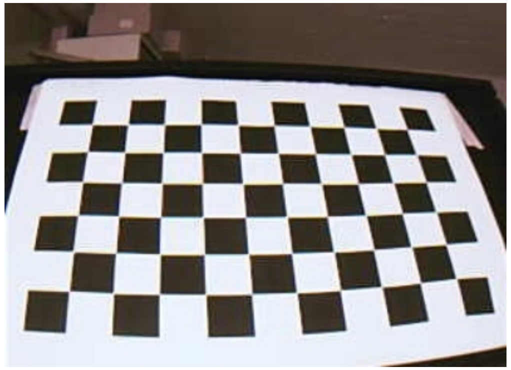
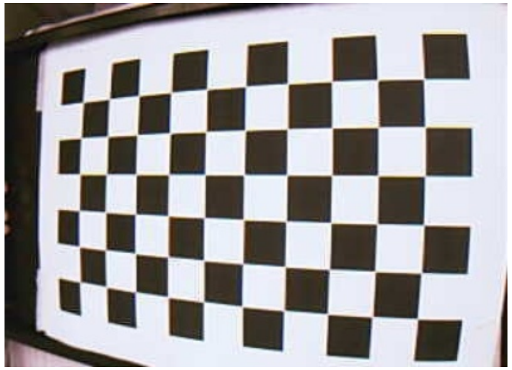
$$\min_{(\mathsf{K},\boldsymbol{q},R_n,\mathrm{t}_n)} \sum_n \sum_i \|\mathbf{x}_{ni} - \hat{\mathbf{x}}(\mathsf{K},\mathbf{q},R_n,\mathbf{t}_n,\mathbf{X}_{ni})\|^2$$

...linearize to obtain a quadratic function, compute derivative, set it to 0, solve linear system, iterate...

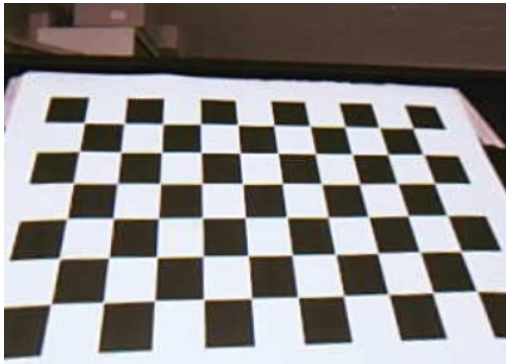
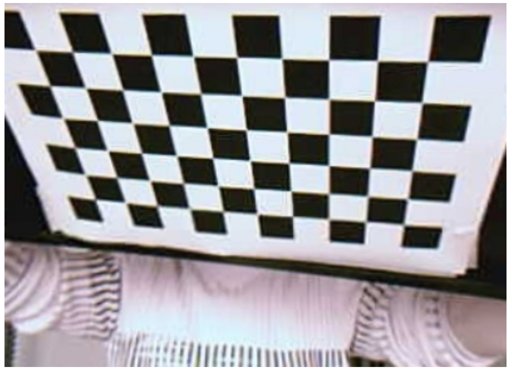
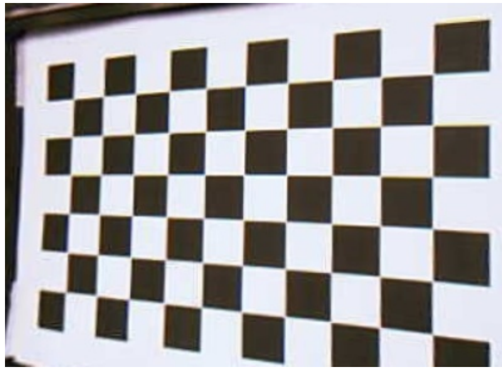
(solved using Levenberg-Marquardt)

Example Results

- Before calibration:

- After calibration:

Summary on Camera Calibration Using a Checkerboard

- Pinhole camera model (first step)
- Non-linear model for lens distortion (second step)
- Approach to 2D camera calibration that
  - accurately determines the model parameters
  - is easy to realize

Outlook for Next Week: Spatial Resection

- Task: Estimate the six extrinsic parameters (E.O.) $\boldsymbol{X}_O, R$
- Given: 3D coordinates $\mathbf{X}_i$ of object points $I \geq 3$
- Observed: Image coordinates $\mathbf{x}_i$ of a calibrated camera

Literature

- Förstner, Scriptum Photogrammetrie I, Chapter 13.3
- Zhang, A Flexible New Technique for Camera Calibration, MSR-TR-98-71

Slide Information

- The slides have been created by Cyrill Stachniss as part of the photogrammetry and robotics courses.
- **I tried to acknowledge all people from whom I used images or videos. In case I made a mistake or missed someone, please let me know.**
- The photogrammetry material heavily relies on the very well written lecture notes by Wolfgang Förstner and the Photogrammetric Computer Vision book by Förstner & Wrobel.
- Parts of the robotics material stems from the great Probabilistic Robotics book by Thrun, Burgard and Fox.
- If you are a university lecturer, feel free to use the course material. If you adapt the course material, please make sure that you keep the acknowledgements to others and please acknowledge me as well. To satisfy my own curiosity, please send me email notice if you use my slides.

Cyrill Stachniss, cyrill.stachniss@igg.uni-bonn.de