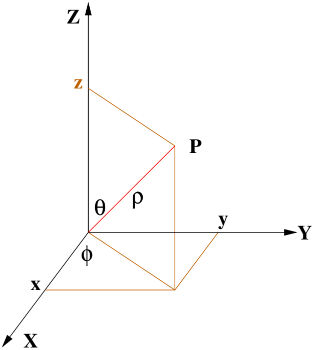
Point Representation

- A point is represented using 2 or 3 numbers $(x, y, [z])$ that are the projections on to the respective coordinate axes.
- Fundamental shape-defining primitive in most Graphics APIs. Everything else is built from it!
- Represented using byte, short, int, float, double, etc.
- The scale and unit are application dependent. Could be angstroms or lightyears!
- Points undergo transformations:
  Translations, Rotations, Scaling, Shearing.

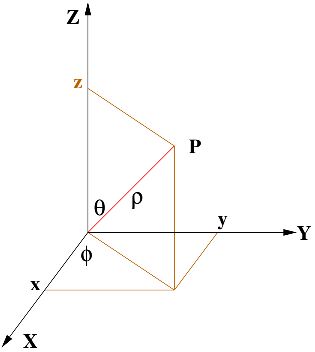
3D Coordinates

- Cartesian: (x, y, z).
- Polar: (ρ, θ, ϕ)
- $z =$
  $y =$
  $x =$
- $\rho =$
  $\phi =$
  $\theta =$
- Elevation: θ, Azimuthal: ϕ



3D Coordinates

- Cartesian: (x, y, z).
- Polar: (ρ, θ, ϕ)
- $z = \rho \cos \theta$
  $y = \rho \sin \theta \sin \phi$
  $x = \rho \sin \theta \cos \phi$
- $\rho^2 = x^2 + y^2 + z^2$
  $\phi = \tan^{-1}(y/x)$
  $\theta = \tan^{-1}(\sqrt{x^2 + y^2}/z)$
- Elevation: θ, Azimuthal: ϕ



Translation

- Translate a point $P = (x, y, [z])$ by $(a, b, [c])$.
- Points coordinates become $P' = (?, ?, ?)$.
- In vector form, $P' =?$.

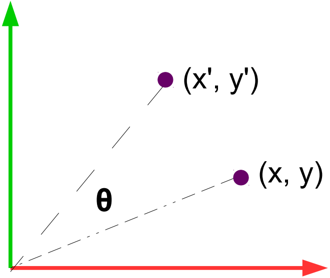
Translation

- Translate a point $P = (x, y, [z])$ by $(a, b, [c])$.
- Points coordinates become $P' = (x + a, y + b, [z + c])$.
- In vector form, $P' = P + T$, where $T = (a, b, [c])$.
- Distances, angles, parallelism are all maintained.

2D Rotation

- Rotate about origin CCW by θ.
- $x' = ?, \quad y' = ?$
- Matrix notation: $P' = R\ P$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



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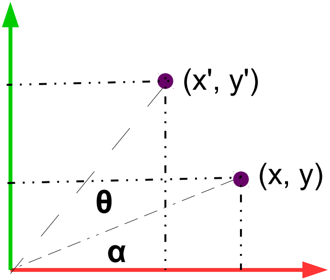
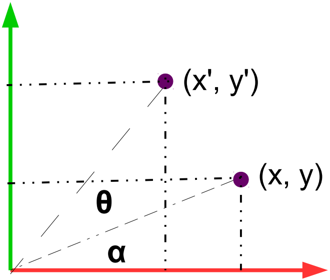


2D Rotation

- Rotate about origin CCW by θ.
- $x' = x\cos\theta - y\sin\theta$,
  $y' = x\sin\theta + y\cos\theta$.



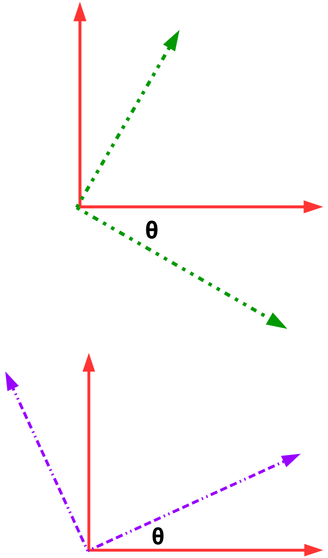
- Matrix notation: $P' = R\ P$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Rotation: Observations

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Orthonormal: $R^{-1} = R^T$
- Rows: vectors that **rotate to** coordinate axes
- Cols: vectors coordinate axes **rotate to**
- Invariants: distances, angles, parallelism.



3D Rotations

- Rotation could be about any axis in 3D!
- About Z-axis: Just like 2D rotation case. Z-coordinates don't change anyway.
- X-Y coordinates change exactly the same way as in 2D.
- CCW +ve, when looking into the **arrowhead**

$$R_z(\theta) = ??$$

3D Rotations

- Rotation could be about any axis in 3D!
- About Z-axis: Z-coordinates don't change anyway

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- CCW +ve; orthonormal; length preserving
- Rows: vectors that rotate onto axes; columns: vectors that axes rotate into....

3D Rotations

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- CCW +ve; orthonormal
- Rows: vectors that rotate onto axes; columns: vectors that axes rotate into....
- Rotation about an arbitrary axis, for example, $[1, 1, 1]^\intercal$??
  Coming soon ....

Non-uniform Scaling

- Scale along X, Y, Z directions by s, u, and t.
- $x' = s\,x, \quad y' = u\,y, \quad z' = t\,z$.
- We are more comfortable with $P' = S\,P$ or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} s & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Invariants: parallelism, ratios of lengths in any direction
  (Angles also for uniform scaling.)

Shearing

- Add a little bit of x to y or other combinations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & x_y & x_z \\ y_x & 1 & y_z \\ z_x & z_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- One of $x_y, x_z, y_x, y_z, z_x, z_y \neq 0$. Rectangles can become parallelograms.
- Invariants: parallelism, ratios of lengths in any direction.

Reflection

- Negative entries in a matrix indicate reflection.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Reflection needn't be about a coordinate axis alone

General Transformation

- Rotation, scaling, shearing, and reflection: **Matrix-vector** product. Vectors get tranformed into other vectors
- Translation alone is a **vector-vector** addition
- Sequence of: Translation, rotation, scaling, translation and rotation: $\mathbf{P'} = \mathbf{R_2}\ [\mathbf{S}\ \mathbf{R_1}\ (\mathbf{P} + \mathbf{t_1}) + \mathbf{t_2}]$
- If translation is also a matrix-vector product, we can combine above transformations into a single matrix: $\mathbf{P'} = \mathbf{R_2}\ \mathbf{T_2}\ \mathbf{S}\ \mathbf{R_1}\ \mathbf{T_1}\ \mathbf{P} = \mathbf{M}\ \mathbf{P}$
- How?

General Transformation

- Rotation, scaling, shearing, and reflection: **Matrix-vector** product. Vectors get tranformed into other vectors
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- How? Answer: **homogeneous coordinates**!

Homogeneous Coordinates

- Add a *non-zero scale factor* w to each coordinate.
  A 2D point is represented by a vector $[x \;\; y \;\; w]^\mathsf{T}$
- $[x \;\; y \;\; w]^\mathsf{T} \equiv (x/w,\; y/w)$.
- Translate $[x \;\; y]^\mathsf{T}$ by $[a \;\; b]^\mathsf{T}$ to get $[x+a \;\; y+b]^\mathsf{T}$

$$\begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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$$\begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Now, translation is also: $\mathbf{P}' = \mathbf{T}\;\mathbf{P}$, a matrix-vector product and a linear operation.

Homogeneous Coordinates

- Add a *non-zero scale factor* w to each coordinate.
  A 2D point is represented by a vector $[x \;\; y \;\; w]^\intercal$
- $[x \;\; y \;\; w]^\intercal \equiv (x/w, \;\; y/w)$.
- Now, translation is also: $\mathbf{P'} = \mathbf{T} \; \mathbf{P}$
- For a point: Rotation followed by translation followed by scaling, followed by another rotation: $\mathbf{P'} = \mathbf{R_2 S T R_1} \; \mathbf{P}$.
- Similarly for 3D. Points represented by: $[x \; y \; z \; w]^T$.
- All matrices are 3×3 in 2D. Last row is $[0 \;\; 0 \;\; 1]$.
- All matrices are 4×4 in 3D. Last row is $[0 \;\; 0 \;\; 0 \;\; 1]$.

Homogeneous Representation

- Convert to a 4-vector with a scale factor as fourth.
  $(x, y, z) \equiv [kx \;\; ky \;\; kz \;\; k]^\mathsf{T}$ for any $k \neq 0$.
- Translation, rotation, scaling, shearing, etc. become linear operations represented by 4×4 matrices.
- What does $[x\ y\ z\ 0]^\mathsf{T}$ mean?
- $[a\ b\ c\ d]^\mathsf{T}$ could be a point or a plane. Lines are specified using two such vectors, either as join of two points or intersection of two planes!

Transformation Matrices: Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- CCW +ve; orthonormal; length preserving; rows give direction vectors that rotate onto each axis; columns ....

Translation, Scaling, Composite

$$T(a,b,c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad S(a,b,c) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- A sequence of transforms can be represented using a composite matrix: $\mathbf{M} = \mathbf{R_x T R_y S T} \cdots$
- Operations are not commutative, but are associative.
- $\mathbf{RT}$ and $\mathbf{TR}$??

Rotation and Translation

- $T_{4\times4} = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$ and $R_{4\times4} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$
- $T\ R = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = \ ?$
- $R\ T = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \ ?$

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- $TR = R\ T$ if: (a) $\mathbf{R} = \mathbf{I}$ or (b) $\mathbf{t} = \mathbf{0}$ or (c) $\mathbf{Rt} = ?$

Rotation and Translation

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- When is $\mathbf{Rt} = \mathbf{t}$? eigenvector of $\mathbf{R}$

Rotation and Translation

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- $TR = R\ T$ if: (a) $\mathbf{R} = \mathbf{I}$ or (b) $\mathbf{t} = \mathbf{0}$ or (c) $\mathbf{Rt} = \mathbf{t}$
- When is $\mathbf{Rt} = \mathbf{t}$? when $\mathbf{t}$ is an eigenvector of $\mathbf{R}$

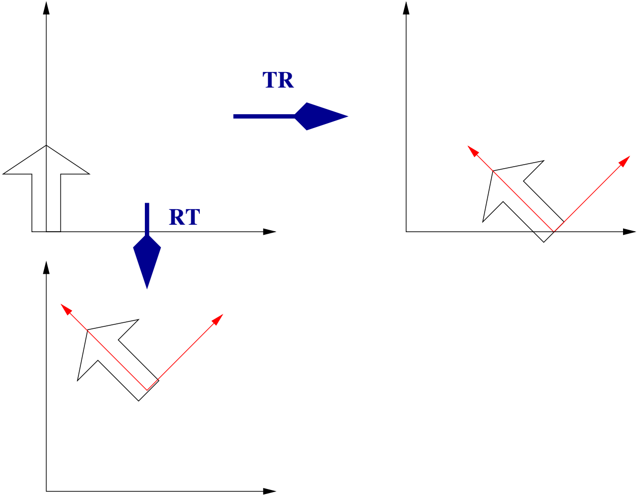
Commutativity

- Translations are commutative: $\mathbf{T_1 T_2 = T_2 T_1}$
- Scaling is commutative: $\mathbf{S_1 S_2 = S_2 S_1}$
- Are rotations commutative? $\mathbf{R_1 R_2 \stackrel{?}{=} R_2 R_1}$
- What would be an example?
  Consider the effect on Z-axis of:

Commutativity

- Translations are commutative: $\mathbf{T_1 T_2 = T_2 T_1}$
- Scaling is commutative: $\mathbf{S_1 S_2 = S_2 S_1}$
- Are rotations commutative? $\mathbf{R_1 R_2 \neq R_2 R_1}$
- Consider the effect on Z-axis of
  $\mathbf{R_x(90) R_y(90)}$ and $\mathbf{R_y(90) R_x(90)}$
- $\mathbf{RT \neq TR}$. (If translation is not parallel to rotation axis)
- Consider: $\mathbf{R}(\pi/\mathbf{4})$ and $T(5,0)$.
  Where does the origin $(0,0)$ go in $\mathbf{TR}$ and $\mathbf{RT}$?

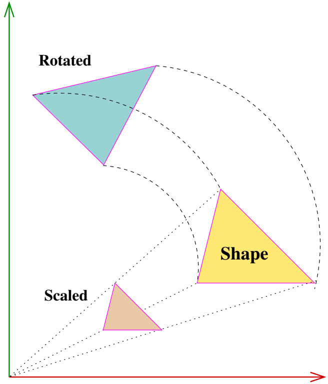
TR and RT



TR keeps it on X axis to $(5, 0)$. **RT** takes it to $(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}})$.

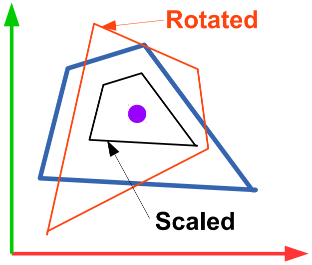
Objects Away from Origin

- Object “translates” when rotated or scaled!!
- Default: Perform these **about the origin**
- How do we transform points “in place”?
- Rotate or scale about the centroid of the object. Or about an arbitrary point
- How?



Transformations About A Point

- Rotating about point P
  – Align P with origin
  – Rotate/scale about origin
  – Translate back
- Rotation:
  $\mathbf{R_C}(\theta) = \mathbf{T}(\mathbf{C})\ \mathbf{R}\ \mathbf{T}(-\mathbf{C})$
- Scaling:
  $\mathbf{S_C}() = \mathbf{T}(\mathbf{C})\ \mathbf{S}()\ \mathbf{T}(-\mathbf{C})$
- A transformation $\mathbf{M}$:
  $\mathbf{M_C} = \mathbf{T}(\mathbf{C})\ \mathbf{M}\ \mathbf{T}(-\mathbf{C})$



R, T Operations on Points

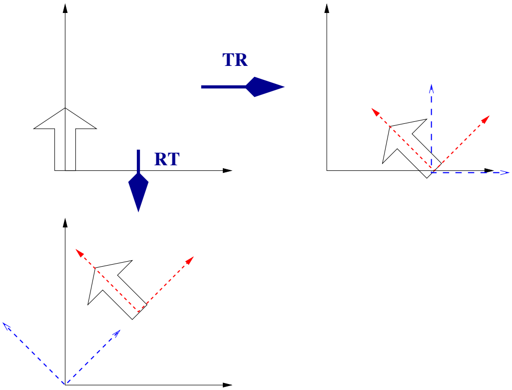
- **T(5,0) R($\pi/4$)**: Impact on a point:
  - R($\pi/4$): **(Point stays at $(\mathbf{0}, \mathbf{0})$)**
  - T$(5, 0)$: **(Point goes to $(\mathbf{5}, \mathbf{0})$)**
- **R($\pi/4$) T(5,0)**: Impact on the point:
  - T$(5, 0)$: **(Point moves to $(\mathbf{5}, \mathbf{0})$)**
  - R($\pi/4$). **(Point rotates about origin)**
- All points on the shape undergo the same motions and get new coordinates

R, T Operaions on Frames

- **T(5,0) R($\pi/4$)**: Impact on coordinate frame:
  - T$(5,0)$: **(Origin shifted to $(\mathbf{5},\mathbf{0})$)**
  - R$(\pi/4)$. **(Axes rotated at new origin)**
- **R($\pi/4$) T(5,0)**: Impact on coordinate frame:
  - R$(\pi/4)$: **(Axes rotate by 45 degrees))**
  - T$(5,0)$. **(Point moves to $(\mathbf{5},\mathbf{0})$ in new axes)**
- Frames move around, giving new coordinates to points on objects!!

Relating Coordinate Frames

- $T(5,0)$ and $R(\pi/4)$
- Start: Black axes
  Next: Blue axes
  Last: Red axes



Points and Frames in General

- Points go through changes in a common coordinate frame when a sequence of transformations is **viewed from right to left**
- Coordinate system goes through the same transformations when the sequence is **viewed from left to right**
- Composite transformations $P' = \mathbf{M_1 M_2 M_3}\ P$ relates the coordinates in successive coordinate frames as we go from left to right, starting with $\mathbf{X'Y'}$ coordinate frame to finally the $\mathbf{XY}$ frame.

Transforming the World Reference

- Consider $P_4 = \mathbf{M_4 M_3 M_2 M_1} \; P_0$
- Point P_0 undergoes 4 operations and get coordinates P_4
- Imagine the point having coordinates P_1, P_2, P_3 after operations $\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}$
- We can also visualize coordinate frames $\mathbf{\Pi}_4, \mathbf{\Pi}_3, \mathbf{\Pi}_2, \mathbf{\Pi}_1, \mathbf{\Pi}_0$ in which a point has coordinates P_4 to P_0 respectively
- Operation $\mathbf{M}_i$ represents a change in coordinates from $\mathbf{\Pi}_i$ to $\mathbf{\Pi}_{i-1}$, resulting in new labels for the point.

Let us look at Ourselves

- Model IIIT Campus as a whole. Campus is our “world”
- Global coordinate frame Π_G for the campus: at the Gate
- Buildings: Himalaya, Vindhya, Bakul, Parul, ..., Palash. Each has a natural coordinate frame. Π_H is Himalaya’s
- Himalaya has several rooms: H105, H204, H205, H304, etc., with own coordinate frames. Π_C is of H205 (our class)
- H205 has 55 desks, with coord frames Π_{Di} for desk i
- Desks are identical in geometry; the coord frame Π_{Di} places each in its location.

Consider a Desk

- Consider a corner point P of desk 37, with coordinates $(200, 30, 100)$ in $\mathbf{\Pi}_{D37}$. That is: $P_{D37} = (200, 30, 100)$
- Since our world is the campus, we have to ultimately describe everything in the coordinate frame $\mathbf{\Pi}_G$

$$P_G = \mathbf{M}_{\mathbf{GH}}\ \mathbf{M}_{\mathbf{HC}}\ \mathbf{M}_{\mathbf{CD37}}\ P_{D37}$$

- $\mathbf{M}_{\mathbf{GH}}$ aligns $\mathbf{\Pi}_G$ to $\mathbf{\Pi}_H$. $\mathbf{M}_{\mathbf{HC}}$ aligns $\mathbf{\Pi}_H$ to $\mathbf{\Pi}_C$. $\mathbf{M}_{\mathbf{CD37}}$ aligns $\mathbf{\Pi}_C$ to $\mathbf{\Pi}_{D37}$
- $P_G = \mathbf{M}_{\mathbf{GH}} \overset{P_H}{|}\ \mathbf{M}_{\mathbf{HC}} \overset{P_C}{|}\ \mathbf{M}_{\mathbf{CD37}} \overset{P_{D37}}{|}\ P$ (for any point P on Desk37)
- We can place a given desk in any **building, room, place**!

Walking on Stage

- Person walking horizontally on stage, with swinging arms
- How does the hand-tip move w.r.t each student? **How?**
- Student knows own position in room's reference frame
- Start at a student's eye. (That provides the viewpoint!)
- Align to room's reference frame using $\mathbf{M_1}$. Different matrix for each student, but everyone same now....
- Walk: pure translation. $\mathbf{M_2}$ aligns to person coord frame
- Arm swing: Simple harmonic motion with angle $\theta(t)$

Simpler viewpoints in newer coord frames.