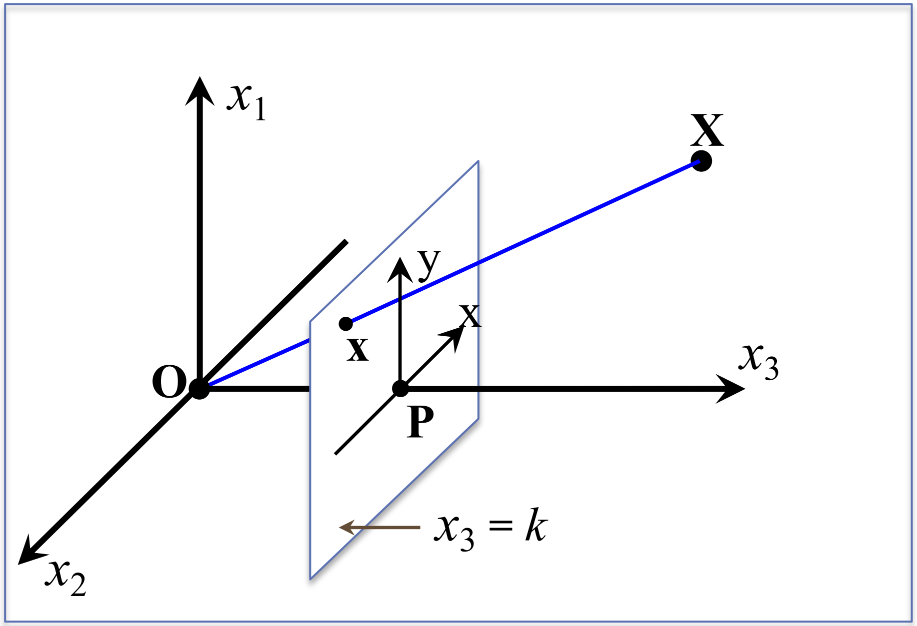
CSE578: Computer Vision

Projective Geometry a Review



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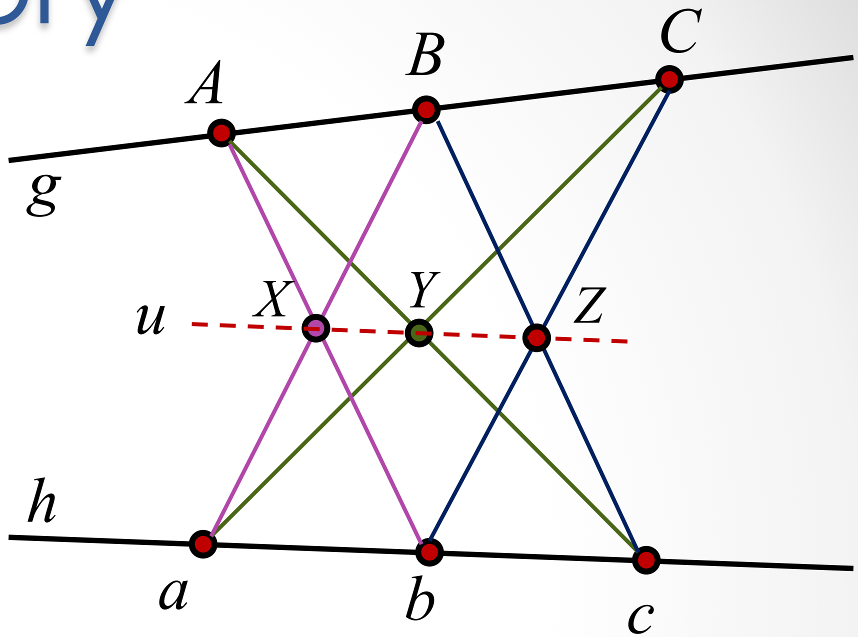
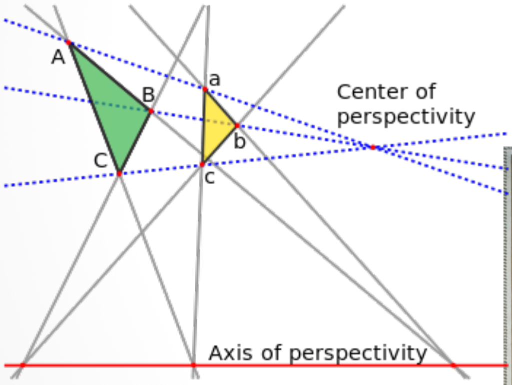
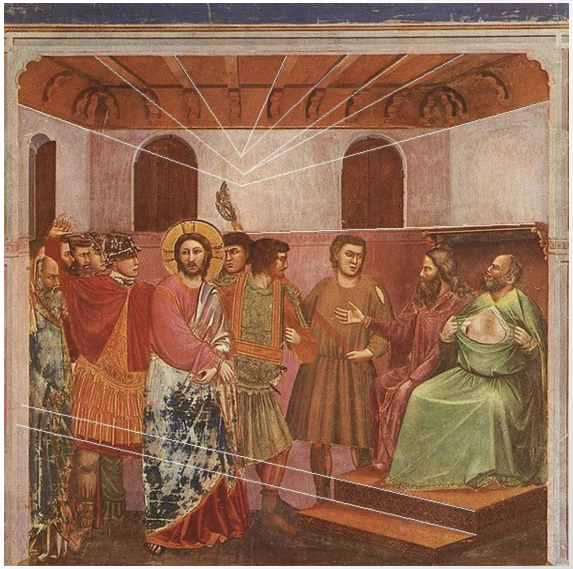
History



- Pappus' Theorem
  - Pappus of Alexandria (400 AD)
- Desargues' Theorem
  - Girard Desargues (1600 AD)



- Renaissance Art
- Jean-Victor Poncelet (1788 –1867)



Points and Lines in $\mathcal{P}^2$

- Points represented by: $\mathbf{x} = [x \;\; y \;\; 1]^{\mathrm{T}}$.
- Consider the line equation: $ax + by + c = 0$.
- $[a\ b\ c][x\ y\ 1]^{\mathrm{T}} = \mathbf{l}.\mathbf{x} = \mathbf{l}^{\mathrm{T}}\,\mathbf{x} = 0$, where $\mathbf{l} = [a\ b\ c]^{\mathrm{T}}$.
- Lines are represented by 3-vectors, just like points.

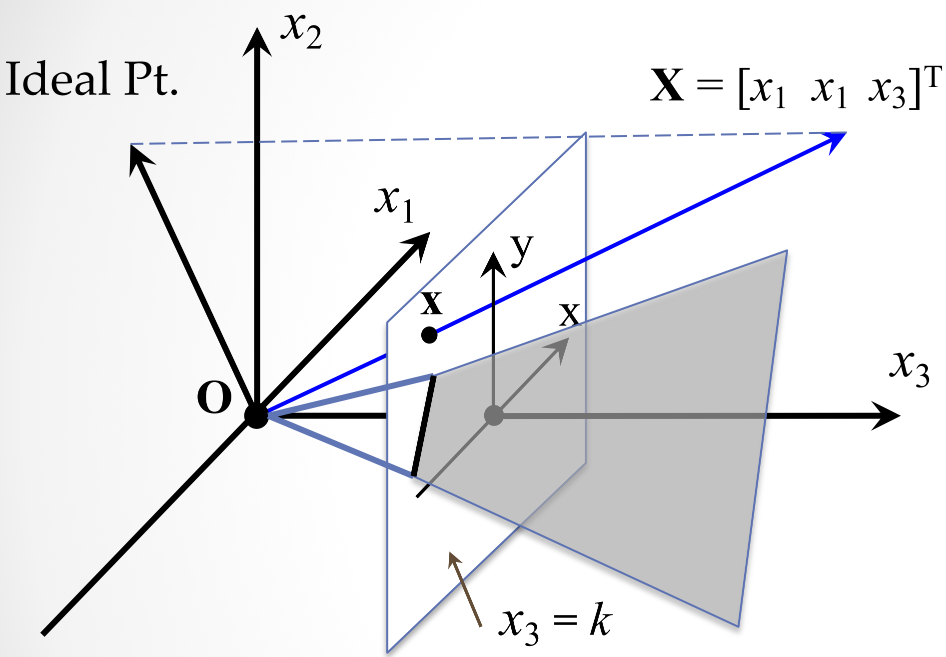
  Overall scale is unimportant.
- What does $\mathbf{l}^{\mathrm{T}}\,\mathbf{x} = 0$ describe?
  - All points $\mathbf{x}$ on a fixed line $\mathbf{l}$?
  - All lines $\mathbf{l}$ passing through a fixed point $\mathbf{x}$?

Points/Line at Infinity

- $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^\mathrm{T}$ represents $[{}^{x_1}/_{x_3} \quad {}^{x_2}/_{x_3}]$
- What happens when $x_3 \rightarrow 0$?
- Becomes **point at infinity**, or **vanishing point** or ideal point in the direction (x_1, x_2).
- Points at infinity can be handled like any other point in projective geometry
- $[x \quad y \quad 0]^\mathrm{T}$ are all points at infinity on the plane.
- What do they form together?
- What is the representation of $\mathbf{l}_\infty$?

$$\mathbf{l}_\infty = [0 \quad 0 \quad 1]^\mathrm{T}$$

Visualizing Projective Geometry of a Plane



- $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^T$ represents rays from the origin in 3-space.
- The plane can be any cross section ⊥r to x_3.
- Ideal points are rays on the $x_3 = 0$ plane.
- Lines are planes passing through the origin.

- Line at infinity, $\mathbf{l}_\infty$, corresponds to $x_3 = 1,\, x_1{,}x_2{=}0$.

Line joining 2 points

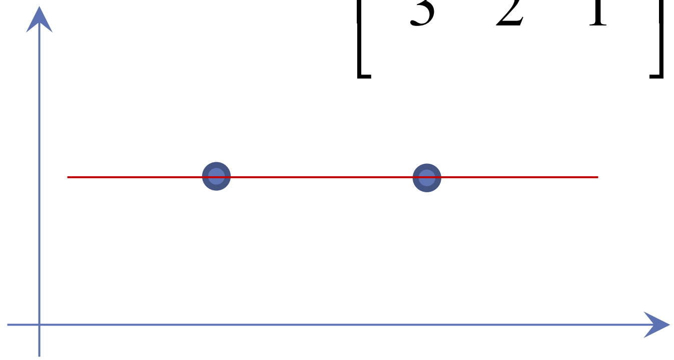
- Let $\mathbf{p}$ and $\mathbf{q}$ be points. We have: $\mathbf{l}^{\mathrm{T}}\mathbf{p} = \mathbf{l}^{\mathrm{T}}\mathbf{q} = 0$.
- Equation of $\mathbf{l}$: $y = y_1 + \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$

  or: $(y_2 - y_1)x - (x_2 - x_1)y + (x_2 y_1 - x_1 y_2) = 0$

  or: $\mathbf{l} = [(y_2 - y_1) \quad -(x_2 - x_1) \quad (x_2 y_1 - x_1 y_2)]^{\mathrm{T}}$
- Considering them as vectors in 3-space, we want to find a vector $\mathbf{l}$ orthogonal to both $\mathbf{p}$ and $\mathbf{q}$.
- The cross-product $\mathbf{x} \times \mathbf{y}$ is a solution. Thus, $\mathbf{l} = \mathbf{p} \times \mathbf{q}$.
- $\mathbf{p}\times\mathbf{q} = [(y_2 - y_1) \quad -(x_2 - x_1) \quad (x_2 y_1 - x_1 y_2)]^{\mathrm{T}}$

Example: Line connecting 2 points

- Line through (5,2) and (3,2): $\begin{bmatrix} i & j & k \\ 5 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$
- i.e., $y = 2$.



- Ideal point of the line $[0 \;\; 1 \;\; -2]^T$ is $[1 \;\; 0 \;\; 0]^T$.
  This is same as $[0 \;\; 1 \;\; k]^T$ for any k.
- Line joining $[3 \;\; 4 \;\; 0]^T$ and $[2 \;\; 3 \;\; 0]^T$ is $[0 \;\; 0 \;\; 1]^T$ or $\mathbf{l}_\infty$.

Point of Intersection of 2 lines

- Lines $\mathbf{l}$, $\mathbf{m}$ intersect at a point $\mathbf{x}$ with $\mathbf{l}^{\mathrm{T}}\mathbf{x} = \mathbf{m}^{\mathrm{T}}\mathbf{x} = 0$.
- $\mathbf{x} = \mathbf{l} \times \mathbf{m}$.
- $\mathbf{l}$: $a_1 x + b_1 y + c_1 = 0$; and $\mathbf{m}$: $a_2 x + b_2 y + c_2 = 0$.
- $x = (b_2c_1 - b_1c_2)/(a_2b_1 - a_1b_2)$.
- $y = (a_1c_2 - a_2c_1)/(a_2b_1 - a_1b_2)$.
- $\mathbf{x} = [(b_2c_1 - b_1c_2)\ \ (a_1c_2 - a_2c_1)\ \ (a_2b_1 - a_1b_2)]^{\mathrm{T}} = \mathbf{l} \times \mathbf{m}$.
- Duality at work: points and lines are interchangeable.

Example: Intersection of Lines

- Intersection of x=1 and y=2: $\begin{bmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
- Same as: $(1,2)$.

- Intersection of x=1 and x=2: $\begin{bmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

- Ideal point of the line $\mathbf{l} = [a \;\; b \;\; c]^{\mathrm{T}}$ is $[\mathrm{b} \;\; -a \;\; 0]^{\mathrm{T}}$
- This is $\mathbf{l} \times \mathbf{l}_\infty$, the intersection of l with line at infinity!

Conics: 2nd order Entities

- General quadratic entity:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

- Rewrite using homogeneous coordinates as:

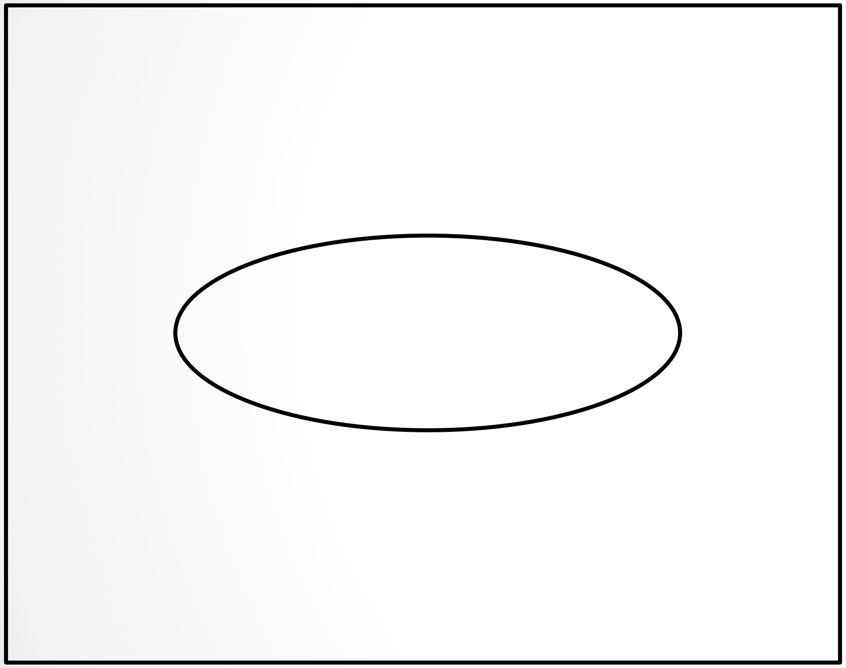
$$ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0.$$

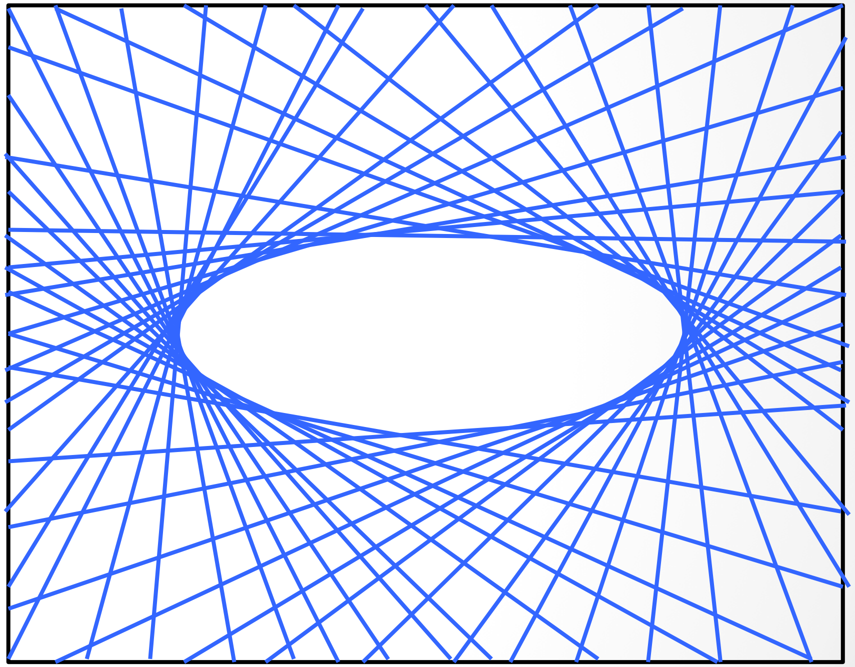
- Rewrite as:

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

- A symmetric C represents a conic: $\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} = 0$. Covers circle, ellipse, parabola, hyperbola, etc.
- Degenerate conics include a line $(a = b = c = 0)$ and two lines when $\mathbf{C} = \mathbf{l}\mathbf{m}^{\mathrm{T}} + \mathbf{m}\mathbf{l}^{\mathrm{T}}$.

Point and Line Conics





Properties of Conics

- $\mathbf{l} = \mathbf{C}\mathbf{x}$ gives the tangent line to the conic at $\mathbf{x}$.
  - A point $\mathbf{x}$ on the conic is on line $\mathbf{l} = \mathbf{C}\mathbf{x}$ as $\mathbf{x}^{\mathrm{T}} (\mathbf{C}\mathbf{x}) = \mathbf{0}$.
  - If $\mathbf{l}$ intersects the conic in another point $\mathbf{y}$;
    - $\mathbf{y}^{\mathrm{T}}\mathbf{C}\mathbf{y} = \mathbf{0}$ as $\mathbf{y}$ is on the conic; and
    - $(\mathbf{C}\mathbf{x})^{\mathrm{T}}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{y} = \mathbf{0}$ as $\mathbf{y}$ is on the line.
    - Thus, $\mathbf{C}\mathbf{y}$ is a line joining $\mathbf{x}$ and $\mathbf{y}$.
    - That is $\mathbf{C}\mathbf{y} = \mathbf{C}\mathbf{x}$ or $\mathbf{x} = \mathbf{y}$.
- Dual Conic: Conic defined by its tangent lines.
  - $\mathbf{l}^{\mathrm{T}}\mathbf{C}^*\mathbf{l} = \mathbf{0}$ or $\mathbf{l}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{l} = \mathbf{0}$ gives the set of lines tangential to $\mathbf{C}$.
- Point of tangency of $\mathbf{l}$ and $\mathbf{C}$ is given by: $\mathbf{C}^{-\mathrm{T}}\mathbf{l}$ or $\mathbf{C}^{-1}\mathbf{l}$.
  - Consider the point $\mathbf{x} = \mathbf{C}^{-1}\mathbf{l}$ on line $\mathbf{l}$.
  - It is also on the conic:

$$\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} = (\mathbf{C}^{-1}\mathbf{l})^{\mathrm{T}}\,\mathbf{C}\,(\mathbf{C}^{-1}\mathbf{l}) = \mathbf{l}^{\mathrm{T}}\mathbf{C}^{-\mathrm{T}}\,(\mathbf{C}\mathbf{C}^{-1})\,\mathbf{l} = \mathbf{l}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{l} = \mathbf{0}$$

Summary

- Point and line representations: $[x\ y\ w]^T$ and $[a\ b\ c]^T$
- Points and lines are duals of each other
  - Intersection of two lines: $\boldsymbol{l} \times \boldsymbol{m}$
  - Line joining to points: $\boldsymbol{x} \times \boldsymbol{y}$
- Points at infinity $[x\ y\ 0]^T$ forms line at infinity $[0\ 0\ 1]^T$
  - Ideal Points or Vanishing points
- Visualization of points and lines (rays from origin)
- Conics are given by $x^T C x = 0$, where C is symmetric
  - Cx is tangent to C at x
- Conics and tangent lines are duals of each other
  - Tangent line-set representation of $x^T C x = 0$: $l^T C^{-1} l = 0$.