Robust Implicit Networks via Non-Euclidean Contractions

Anonymous Author(s)
Affiliation
Address
email

Comparison of well-posedness conditions

In this note, we study implicit neural networks and we compare the well-posedness condition proposed in [Anonymous, 2021] with the well-posedness condition for MON proposed in [Winston and Kolter, 2020] on sets of 2×2 matrices.

We fix $\gamma > 0$ and compare the two sets

$$\Gamma_2 = \{A \in \mathbb{R}^{2\times 2} \mid \mu_2(A) \leq \gamma\} \quad \text{and} \quad \Gamma_\infty = \{A \in \mathbb{R}^{2\times 2} \mid \mu_\infty(A) \leq \gamma\}. \tag{1}$$

Since $-I_2$ belongs to both sets (recall $\mu(-I_n) = -1$ for all norms), the two sets have non-empty intersection, i.e., they overlap. Next we show that these two sets are generally distinct and neither is a subset of the other.

We consider three example subsets of $\mathbb{R}^{2\times 2}$ parametrized by two parameters:

(i) for $\mathcal{A}_1 = \left\{ \begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$, we note

$$\mu_2(A) \leq \gamma \implies a + b + \sqrt{2a^2 + 2b^2} \leq 2\gamma, \tag{2}$$
$$\mu_\infty(A) \leq \gamma \implies a + |a+b| \leq \gamma \text{ and } b \leq \gamma. \tag{3}$$

The left image in Figure 1 shows the region (3) as the red area and the region (2) as the blue area for $\gamma = 1$. Clearly, the regions $\mathcal{A}_1 \cap \Gamma_2$ and $\mathcal{A}_1 \cap \Gamma_\infty$ are overlaping but neither is a subset of the other.

(ii) for $\mathcal{A}_2 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$, we note

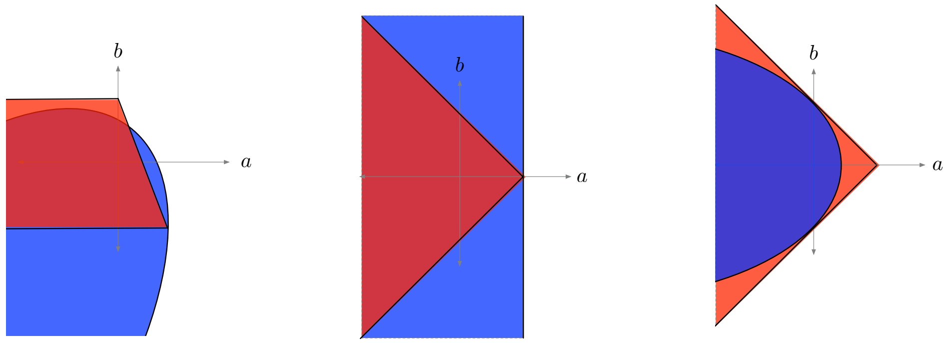
$$\mu_2(A) \leq \gamma \implies a \leq \gamma, \tag{4}$$
$$\mu_\infty(A) \leq \gamma \implies a + |b| \leq \gamma. \tag{5}$$

As illustrated in the middle image in Figure 1, $(\mathcal{A}_2 \cap \Gamma_2) \supset (\mathcal{A}_2 \cap \Gamma_\infty)$.

(iii) for $\mathcal{A}_3 = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$, we note

$$\mu_2(A) \leq \gamma \implies \lambda_{\max}\left(\begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & 0 \end{bmatrix} \right) \leq \gamma \implies a + \sqrt{a^2 + b^2} \leq \gamma, \tag{6}$$
$$\mu_\infty(A) \leq \gamma \implies a + |b| \leq \gamma. \tag{7}$$

As illustrated in the right image of Figure 1, $(\mathcal{A}_3 \cap \Gamma_2) \subset (\mathcal{A}_3 \cap \Gamma_\infty)$.



$$\begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix} \qquad \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

Figure 1: Comparison of well-posed sets Γ_2 (in blue) and Γ_∞ (in red) in equation (1) at $\gamma = 1$, when restrited to the matrix sets $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$.

References

Anonymous. Robust implicit networks via non-Euclidean contractions. In Advances in Neural Information Processing Systems, May 2021. Submitted.

E. Winston and J. Z. Kolter. Monotone operator equilibrium networks. In Advances in Neural Information Processing Systems, 2020. URL https://arxiv.org/abs/2006.08591.