

---

# Robust Implicit Networks via Non-Euclidean Contractions

---

Anonymous Author(s)

Affiliation

Address

email

## 1 Comparison of well-posedness conditions

2 In this note, we study implicit neural networks and we compare the well-posedness condition proposed  
3 in [Anonymous, 2021] with the well-posedness condition for MON proposed in [Winston and Kolter,  
4 2020] on sets of  $2 \times 2$  matrices.

5 We fix  $\gamma > 0$  and compare the two sets

$$\Gamma_2 = \{A \in \mathbb{R}^{2 \times 2} \mid \mu_2(A) \leq \gamma\} \quad \text{and} \quad \Gamma_\infty = \{A \in \mathbb{R}^{2 \times 2} \mid \mu_\infty(A) \leq \gamma\}. \quad (1)$$

6 Since  $-I_2$  belongs to both sets (recall  $\mu(-I_n) = -1$  for all norms), the two sets have non-empty  
7 intersection, i.e., they overlap. Next we show that these two sets are generally distinct and neither is a  
8 subset of the other.

9 We consider three example subsets of  $\mathbb{R}^{2 \times 2}$  parametrized by two parameters:

10 (i) for  $\mathcal{A}_1 = \left\{ \begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ , we note

$$\mu_2(A) \leq \gamma \implies a + b + \sqrt{2a^2 + 2b^2} \leq 2\gamma, \quad (2)$$

$$\mu_\infty(A) \leq \gamma \implies a + |a + b| \leq \gamma \text{ and } b \leq \gamma. \quad (3)$$

11 The left image in Figure 1 shows the region (3) as the red area and the region (2) as the blue  
12 area for  $\gamma = 1$ . Clearly, the regions  $\mathcal{A}_1 \cap \Gamma_2$  and  $\mathcal{A}_1 \cap \Gamma_\infty$  are overlapping but neither is a  
13 subset of the other.

14 (ii) for  $\mathcal{A}_2 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ , we note

$$\mu_2(A) \leq \gamma \implies a \leq \gamma, \quad (4)$$

$$\mu_\infty(A) \leq \gamma \implies a + |b| \leq \gamma. \quad (5)$$

15 As illustrated in the middle image in Figure 1,  $(\mathcal{A}_2 \cap \Gamma_2) \supset (\mathcal{A}_2 \cap \Gamma_\infty)$ .

16 (iii) for  $\mathcal{A}_3 = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ , we note

$$\mu_2(A) \leq \gamma \implies \lambda_{\max} \left( \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & 0 \end{bmatrix} \right) \leq \gamma \implies a + \sqrt{a^2 + b^2} \leq \gamma, \quad (6)$$

$$\mu_\infty(A) \leq \gamma \implies a + |b| \leq \gamma. \quad (7)$$

17 As illustrated in the right image of Figure 1,  $(\mathcal{A}_3 \cap \Gamma_2) \subset (\mathcal{A}_3 \cap \Gamma_\infty)$ .

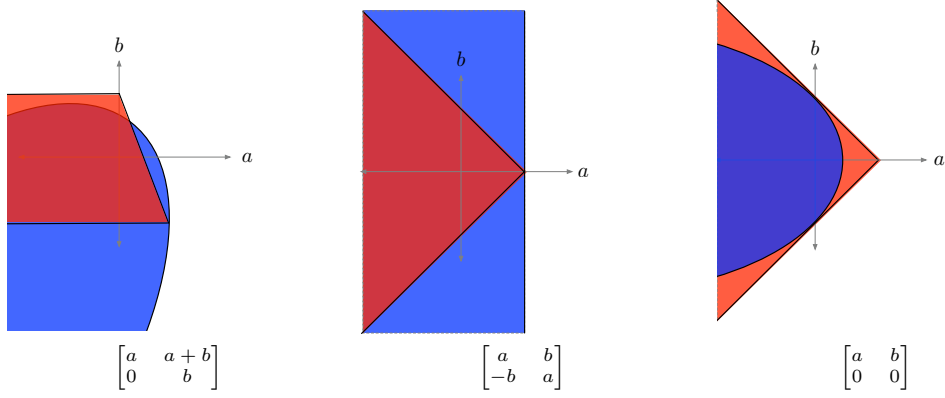


Figure 1: Comparison of well-posed sets  $\Gamma_2$  (in red) and  $\Gamma_\infty$  (in blue) in equation (1) at  $\gamma = 1$ , when restricted to the matrix sets  $\mathcal{A}_1$ ,  $\mathcal{A}_2$  and  $\mathcal{A}_3$ .

## References

- Anonymous. Robust implicit networks via non-Euclidean contractions. In Advances in Neural Information Processing Systems, May 2021. Submitted.
- E. Winston and J. Z. Kolter. Monotone operator equilibrium networks. In Advances in Neural Information Processing Systems, 2020. URL <https://arxiv.org/abs/2006.08591>.