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Employing genetic 'moments' in the history of mathematics in classroom activities

Vassiliki Farmaki · Theodorus Paschos

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Abstract The integration of history into educational practice can lead to the development of activities through the use of genetic 'moments' in the history of mathematics. In the present paper, we utilize Oresme's genetic ideas – developed during the fourteenth century, including ideas on the velocity–time graphical representation as well as geometric transformations and reconfigurations – to develop mathematical models that can be employed for the solution of problems relating to linear motion. The representation of distance covered as the area of the figure between the graph of velocity and the time axis employed in these activities, leads on naturally to the study of problems on motion by means of functions, as well as allowing for the use of tools (concepts and propositions) from Euclidean geometry of relevance to such problems. By employing simple geometric transformations, equivalent real life problems are obtained which lead, in turn, to a simple classification of all linear motion-related problems. When applied to a wider range of motion problems, this approach prepares the way for the introduction of basic Calculus concepts (such as integral, derivative and their interrelation); in fact, we would argue that it could be beneficial to teach the basic concepts and results of Calculus from an early grade by employing natural extensions of the teaching methods considered in this paper.

Keywords Integration of history · Motion problems · Functional approach
Graphical representations · Transformations · Euclidean geometry

1 Introduction

Problems in context have an important role to play in the development of cognitive functions via the mathematization of real-life situations. Employing Oresme's ideas, and

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taking Duval's theoretical framework into account, we designed a series of activities for solving problems relating to uniform motion. We adapted the idea of integrating the history of mathematics into the educational praxis, utilizing 'genetic' ideas which represent crucial steps in the construction of various mathematical concepts (Furinghetti and Somaglia 1998; Tzanakis and Arcavi 2000).

The design of these activities is inspired by the method employed to study motion during the later Middle Ages (fourteenth century) by Calculators at Merton College, Oxford, but also by N. Oresme at Paris. Basic to our approach is the realization, historically attested to, that this medieval methodology is in fact the genetic "moment" for such fundamental mathematical concepts as function, graphical representation and the integral (Clagett 1959, 1968; Gravemeijer and Doorman 1999; Kaput 1994). 'Reconstructed' in a modern version, Oresme's geometrical model of motion corresponds to the graph of velocity vs. time. In this graph – which we call 'holistic' to distinguish it from the usual functional graph of distance vs. time–velocity, time and distance covered appear simultaneously, with distance represented as the area of the figure between the curve and the time axis. In some cases, at least, this representation allows motion problems to be solved in the graphical–geometrical context through the application of tools (concepts and propositions) from Euclidean geometry. In particular, simple geometric transformations and 'reconfigurations' (Duval 2002) performed on the velocity–time graph lead, on the one hand, to equivalent mathematical formulation and hence to equivalent real problems, and on the other hand to the solution of the problems.

The main goal of this educational design is to help students improve their motivation, from an early stage on, and to move towards the joint development of algebraic, geometric, and analytical thought through the solving of problems based on the velocity–time graph, as this leads students to: (1) obtain the geometric solution to a problem based solely on Euclidean geometry, using simple geometric transformations; (2) obtain an algebraic solution to the problems as well, thus making the transition from a geometric/graphic to an algebraic approach; and (3) realize the intuitive connection between velocity and distance covered in the same graph by relating the distance covered to the areas of the corresponding rectangles, and thus be better placed to grasp the essential point of the Fundamental Theorem of Calculus at a later level.

This paper presents experimental data for 58 fifteen year-old students. The classroom application consisted of a self-contained course of seven sessions aimed at investigating the students' ability to understand these mathematical concepts and results via the suggested methodology at this early age/grade. The classroom activities were introduced employing the theoretical framework laid out in Brousseau's didactical situations (Brousseau 1997). We shall describe the content of the teaching approach by analyzing the aims of the sessions; the central points of the students' discourse in the classroom, and excerpts from two student interviews concerning their solution to a uniform motion problem. We have also included observations, about the students' mental operations, obtained from an analysis of the data collected.

2 The integration of history into educational practice

Many researchers and educators believe that the history of mathematics can play a valuable role in teaching and learning. They argue that history helps students understand that mathematics is not a fixed and finalized knowledge system, but an on-going process with close connection to other branches of science (Furinghetti and Somaglia 1998; Tzanakis and Arcavi 2000). It helps students understand that mathematical concepts are invented, modified and extended through a problem-solving process; that mistakes, doubts, intuitive arguments, controversies and alternative approaches to problems are not only legitimate, but actually an integral part of mathematics in the making (Grugnetti and Rogers 2000, p. 45).

There is the history of documents and the history of ideas. The philosophical reflection on – and interpretation of – historical data identifies and explains educational choices. From the philosophical point of view, mathematics must be seen as a human activity that both takes place in individual cultures and stands outside any particular one. From the cultural point of view, mathematical evolution comes about from the sum of many contributions across different cultures. From the interdisciplinary point of view, students find their understanding of mathematics and other subjects enriched through the history of mathematics (Fauvel and van Maanen 2000, p. xvii).

We designed a series of teaching activities with this in mind, utilizing genetic ideas in the study of motion – a field of study that falls between mathematics and physics – during the later Middle Ages. We used a teaching approach which has been described as a *genetic approach* to teaching and learning (Tzanakis and Arcavi 2000, pp. 208–211). In a genetic approach, the emphasis is less on how to use theories, methods and concepts and more on why they provide an answer to specific mathematical problems and questions without, however, disregarding the 'technical' role of mathematical knowledge. Our research followed the general sense of this approach: (1) we identified the genetic historical 'moments' – which emerged in embryonic form during the fourteenth century – of basic mathematical concepts including functions of continuous variation and their graphical representations, but also the derivative, the definite integral and the way in which these relate geometrically and kinetically; and (2) we reconstructed these crucial ideas into a modern version suitable for classroom use, in agreement with the description:

> ...in a reconstruction in which history enters implicitly, a teaching sequence is suggested in which use may be made of concepts, methods and notations that appeared later than the subject under consideration, always bearing in mind that the overall didactic aim is to understand mathematics in its modern form (Tzanakis and Arcavi 2000, p. 210).

We expressed Oresme's original geometric representation of motion employing 'modern' (in the sense that they appeared later than Oresme's model) Cartesian axes and algebraic methods and notations, which we then used to solve suitable uniform motion problems.

3 Conceptual development of mathematics and students' mathematical thinking and learning

Some researchers in mathematics education argue that the historical development of a mathematical topic tells us something about how an individual might learn – or fail to learn – that topic. This perspective is consistent with the more general view that the way in which mankind developed mathematical knowledge may have some similarities to the way in which individuals acquire mathematical knowledge (Freudenthal 1973, 1991). The link between historical developments in mathematical thinking and students' learning of mathematics has often been described in terms of a psychological version of biological recapitulationism: "ontogeny recapitulates phylogeny" (cf. Radford 2000, p. 145; Kaput 1994, p. 83–84).

Piaget and Garcia (1989) have sought a parallel between the mechanisms of the historical evolution of certain major ideas and the mechanisms by which concepts develop within individuals. These mechanisms, which they consider invariable and omnipresent in time and space, are described as processes that lead from the *intra-object* (meaning the analysis of objects) to the *inter-object* (meaning the analysis of the transformations and relations of objects) and then to the *trans-object* (meaning the constructions of structures). The Piaget and Garcia thesis thus allows for three stages in the historical development of geometry: (1) the *intra-figural* period, during which the object of study are the geometrical properties of figures seen as internal relations between elements of figures; (2) the *inter-figural* period, characterized by efforts to find relationships between the figures. This manifests itself specifically in the search for transformations relating the figures according to various forms of correspondence; and (3) the *trans-figural* period, characterized by the predominance of structures (Piaget and Garcia 1989, p. 109).

Kaput (1994, p. 83) proposes a link to this perspective, suggesting that the historical development of Calculus constitutes a succession of three periods/roots: (1) geometric issues related to computations of areas, volumes, and tangents; (2) a mix of practical and theoretical interest involving the characterization and theoretical exploitation of the continuous variation of physical quantities; and (3) inherently theoretical concerns with the foundations of Calculus.

Vygotsky has a different point of view. He argues that thinking developed as the result of two lines or processes of development: a biological process and a historical (or cultural) one. One of the fundamental differences between his and the Piaget and Garcia's approach concerns the epistemological role of culture. In Vygotsky, culture not only provides the specific forms of scientific concepts, it also modifies the activity of mental functions overall. For him, "ontogenesis in any form or degree does not repeat or produce phylogenesis or is its parallel" (Vygotsky 1997, p. 19). Discussion of these points of view amongst researchers reveals the complexity of the relationship between phylogenesis and ontogenesis, and raises serious epistemological questions and reservations with relation to the nature of the relationship between individual cognitive development and interpretations of the historical evolution of mathematics; as Kaput (1994, p. 83) cautions, "there are many reasons to be careful in using this relationship."

Brousseau's (1983, 1997) 'epistemological obstacles' perspective, Radford's (1997) 'socio-cultural' perspective, and Boero's 'voices and echoes game' (Boero et al. 1997), have contributed to the debate concerning the epistemological assumptions underlying the linking of contemporary students' learning processes (psychological domain) to the conceptual development of mathematics (historical domain).

We generally agree with the Vygotskian view that what happened in the past and what is likely to happen in classrooms are two different phenomena, based on widely differing cultural, sociological, psychological and didactical environments. Hence, it would be an over-simplification to believe that we can determine a hypothetical learning trajectory for individuals guided solely by historical study and an evolutionary analysis of mathematical thinking. We believe we can apply a more modest and restrictive epistemological assumption: that "knowledge is conceived as a culturally mediated cognitive praxis resulting from the activities in which people engage" (Radford 2000, p. 163). From this point of view, the classroom is considered as a micro-space of the general space of culture, and students' understanding of mathematics is seen as a process of cultural intellectual appropriation of meanings and concepts along the lines of student and teacher activities.

In our approach, the historical analysis aims to present both the obstacles encountered in the development of various concepts, and the ideas and methods by which these obstacles have been overcome historically. Suitably reconstructed, this historical background can be valuable in designing teaching activities that will help overcome students' difficulties (which are reasonably expected to bear some relation to those encountered in the past), and hence provide a suitable framework for the learning of these mathematical concepts.

Historically, the study of motion proved crucial both for the emergence of the concept of function and its representation, with which the mathematicians studied continuous variations, and for the invention of the basic concepts of Calculus, including the derivative and the integral (Boyer 1959, Chaps. 4, 5). In this paper, we describe a teaching sequence based on motion problems in which Oresme's geometrical model (detailed in the next section) of the velocity–time graphical representation plays a central role, along with his fundamental idea of describing the distance covered over a given time by the area of the figure in this representation. Oresme's idea, which led to the interrelation in the same figure of velocity, time and distance covered – and connecting, as it does, kinematic considerations with two dimensional geometry – proved to be a stepping stone towards the emergence of the definite integral concept. We posit that if this Oresmian interrelation between velocity, time, and distance covered were used systematically in the development of a suitable educational design stretching over several years of schooling, it could function as an essential learning link with the Fundamental Theorem of Calculus.

Krysinska and Hauchart (2000, p. 244) used motion situations to model the interdependence between a car's stopping distance and its velocity as a means of introducing students to the concept of function, representing the instantaneous velocity graphically using Oresme's 'vertical sticks.' Schneider (1988) designed a didactic project which employed the analysis of the evolution of the central concepts of calculus in terms of epistemological obstacles, and allowed students to test their previous beliefs through discussion and disagreement, and to become aware of their limitations. She argues that the difficulties students face understanding curvilinear area, tangent lines and instantaneous flow bring the same epistemological obstacles to light which plagued mathematicians in the past: the students' failure to mentally separate mathematics from an illusory 'sensible' world of magnitudes. She suggests that Newton's kinematic arguments could inspire the design of several activities leading to the Fundamental Theorem of Calculus (Schneider 2000, p. 247).

4 Genetic historical moments in the study of motion during the fourteenth century

The efforts thirteenth and fourteenth-century mathematicians directed at the 'quantification of variable qualities' played a fundamental role in the evolution of mathematics, and especially in the development of Calculus. It was a period of practical and theoretical interest, involving the characterization and theoretical exploitation of the continuous variation of physical quantities (Kaput 1994). 'This consisted in the idea of studying change quantitatively, and thus admitting into mathematics the concept of variation' (Boyer 1959, p. 71).

William Heytesbury, Richard Swineshead and John Dumbleton, mathematicians/logicians – known as Calculators – at Merton College, Oxford (1330–1340), studied the motion of bodies and introduced the concept of a functional relation between variation in velocity and time. The intensity of velocity, described by them as an arithmetic value ('degree') corresponding to a given instant in a time interval, led to the intuitive emergence of instantaneous velocity. They used letters to symbolize variable quantities, defined several kinds of motion (uniform, uniform acceleration, etc.), proposed theorems concerning motion and proved these mathematically (Clagett 1959). Their prolix proofs were based on Euclidean geometry, but also used geometric series. Symbolic algebra appeared, of course, after the historical period under consideration.

Swineshead defines *uniform motion* in *De motu* as:

> Uniform local motion is one in which in every equal part of the time an equal distance is described (Clagett 1959, p. 243).

W. Heytesbury in *Rules for Solving Sophisms* (*Regule solvendi sophismata*) defines *instant velocity* and *uniform accelerated motion*, as follows:

> ... the velocity at any given instant will be measured (*attendetur*) by the path which would be described by a moving point if, in a period of time, it were moved uniformly at the same degree of velocity with which it is moved in that given instant (Clagett 1959, p. 236).
>
> For any motion whatever is uniformly accelerated (*uniformiter intenditur*) if, in each of any equal parts of the time whatsoever, it acquires an equal increment (*latitudo*) of velocity (Clagett 1959, p. 237).

Although the Calculators clearly described the interrelation between time (*extensio*) and the intensity (*intensio*) of velocity, they did not appeal to the geometrical intuition that was to act as an intermediary between these early attempts to study the problem of variation and the final formulation given by Calculus.

Nicole Oresme takes pride of place among those who recognized the need for a geometric representation of variation. He invented and constructed a bridge between the 'algebraic' consideration of the Merton scholars and a geometrical context for representing and studying qualities and motion.

> The work of Oresme therefore makes most effective use of geometrical diagrams and intuition, and of a coordinate system, to give his demonstrations a convincing simplicity. This graphical representation given by Oresme marked a step toward the development of calculus... It was the study of geometrical problems and the attempt to express these in terms of numbers that suggested the derivative and the integral and made the elaboration of these concepts possible (Boyer 1959, p. 80).

Oresme represented the variations in *De configurationibus qualitatum* (1362), treating them as quantities of qualities such as velocity using geometrical figures. As examples of Oresme's technique, consider the rectangle and right triangle in Fig. 1. Each figure measures the quantity of some quality (velocity). Line AB in either case represents the "extension" (time) of the quality. But in addition to extension, the "intensity" of the quality from point to point in the base line AB has to be represented; this Oresme represented by erecting lines perpendicular to the base line, the length of the lines varying as the intensity varies. Thus at every point along AB there is some intensity of the quality, and the sum of all these lines is the figure representing the quality globally.

Now the rectangle ABDC represents a uniform quality, since the lines AC, EF, BD represent the intensities of the quality at points A, E, and B (E being any point at all on AB) are equal, and thus the intensity of the quality is uniform throughout. In the case of the right triangle ABC, it is equally apparent that the lengths of the perpendicular lines representing intensities uniformly increase in length from zero at point A to BC at B, in accordance with Merton College's definition of uniformly accelerating motion. It is worth noting that by designating the limiting line CD (or AC in the case of the triangle) as the "line of summit" or the "line of intensity," Oresme expresses the essential information of motion in each

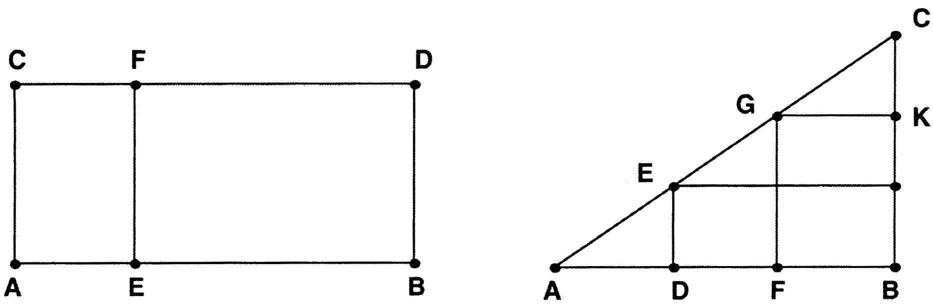


Fig. 1 Examples of variations treated as quantities of qualities using geometrical figures

case, since the perpendicularity of the lines of intensity determines the remaining sides of the figure. Thus, Oresme's representation thus exhibits the 'graph of velocity' in the modern sense. This is a genetic "moment" for the embryonic form of the function concept and its graphical representation.

Oresme also invented and introduced the concept of "total velocity" as an entity having two dimensions – longitude and latitude – one associated with the extension (time), the other with its intensity expressed in degrees. Thus the quantity of total velocity is imagined as a two-dimensional surface whose area represents the distance covered (Clagett 1959, 1968).

Oresme also studied a number of cases of infinite series by means of geometric transformations, comparing and transforming areas and volumes where one or more dimensions or parts are extended or divided indefinitely (Clagett 1968, p. 413–435).

Here is a geometrical example of his method for summing two series, given in *De configurationibus qualitatum*, chapter III viii (Clagett 1968, pp. 413–414):

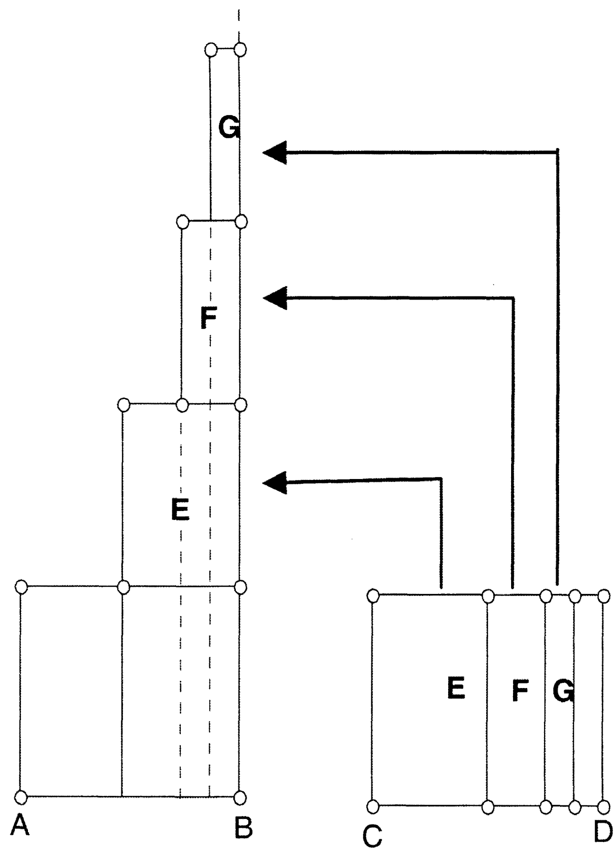
$$1+\frac{1}{2}\times 2+\frac{1}{4}\times 3+\ldots+\frac{n}{2^{n-1}}+\ldots=4, \text{ and } 2+\left[1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}+\ldots\right]=4$$

Oresme achieves this by taking two equal squares of one square foot in area each with base lines AB and CD, respectively (Fig 2); he divides the second square with base CD into proportional parts according to the ratio 2/1. The first proportional part of the second square is then placed over the second half of the first square, the second proportional part over that first proportional part which has been placed over AB, the third over the second, and so on, in the manner indicated in Fig. 2. Then he divides the base AB into proportional parts according to the same ratio 2/1, so that over the successive proportional parts of the base AB we have areas whose altitudes increase gradually towards infinity. Now the whole surface standing on line AB is precisely four times that part of it standing on the first proportional part of the same line AB.

Oresme points out:

> In the same way, if some mobile were moved with a certain velocity in the first proportional part of some period of time, divided in such a way, and in the second part it were moved twice as rapidly, and in the third three times,..., and increasing in this way successively to infinity, the total velocity would be precisely four times the velocity of the first part, so that the mobile in the whole hour would traverse precisely four times what it traversed in the first half of the hour (Clagett 1968, pp. 413–414).

Fig. 2 Oresme's figure (*arrow* added)



It is obvious that these geometrical transformations led Oresme to invent a motion in which the velocity is a step function (in fact, one with an infinite multitude of steps) and which is equivalent to a uniform motion with respect to distance covered.

In our case study, we exploit (1) the definitions of uniform motion and instantaneous velocity and the functional relation between velocity and time given by the Calculators; (2) the representation of uniform motion by a rectangle, and concepts such as the line of intensity and the distance covered as an area given by Oresme, of which the velocity-time graphical representation plays a central role in the modern version, constructing a model for solving problems of uniform motion; and (3) the transformations used by Oresme, which guided us in our design of a series of activities aimed at formulating equivalent motion problems suitable for solution using geometric transformations.

5 Designing the activities

It is certainly a worldwide trend in high school curricula to emphasise numerical and graphical representations when students are introduced to functions in algebra or analysis for the first time. Several research studies have addressed the issue of representations and suggest a theoretical context for them (cf. Goldin and Janvier 1998; Goldin and Kaput 1996; Kaput 1987; von Glasersfeld 1987), while others focus on the translation from one representation of a concept to another (Duval 2002; Slavit 1997). Recently, researchers have focused their attention to the debate concerning semiotics, signs, symbolizing, representations and the concomitant processes of meaning production (Hitt 2002).

Duval's (2002) thesis, which deals with different ways of representing a mathematical concept, provides a suitable theoretical background for the role of representations in the educational praxis. Duval uses the term *register* to describe a system of representations aimed not only at reproduction and communication, but also at objectification. In each register, it is possible to modify a representation *within* that register according to its syntax (*treatment of the representation*), just as it is possible to transform representations from one register to another (*conversion of representations*). Any progress in the learning of mathematics requires coordination between the registers of representation (Duval 2002, p. 319). He argues that we must distinguish two kinds of cognitive operations in mathematical thinking: *processing* and *conversion*. 'Processing' concerns the transformations that are generated within the same register of representation; these include the arithmetical or algebraic computations, as well as the geometric transformations which can be described as intrinsic gestalt transformations of configurations generating geometric figures. 'Conversion' concerns the transformations that lead to a change in register; these include the 'translation' of the representation of an object into a different representation of the same object in another register, such as for example the transformation of equations into Cartesian graphs (Duval 2002, pp. 317–318).

Duval argues that when analyzing any form of visualization it is essential to be aware of the existence of several registers of representation providing specific ways of processing each register. Thus, if geometrical figures depend on a register, we must obtain specific visual operations that are peculiar to this register, allowing us to change any initial geometrical figure into another one while retaining the properties of the initial figure. Duval calls *reconfiguration* "the most typical operation – a specific figural processing which provides figures with a heuristic function – with which you can divide the whole given figure into parts of various shapes (bands, rectangles...) and you can combine these parts in another whole figure or you can make new subfigures appear..." (Duval 2002, p. 326).

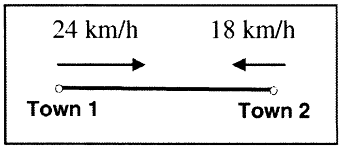
Utilizing Oresme's ideas and taking Duval's theoretical framework into account, we designed a series of activities for solving problems relating to uniform motion; we describe these activities in some detail below in connection with two such problems (in Sections 5.1 and 5.2). In solving these problems, the emphasis is given on the velocity–time graph, and on the relation between 'the area below the graph' and the distance covered in a period of time. We will call this approach *holistic* to distinguish it from the 'traditional' algebraic and 'functional' approaches generally in use, which are based on different representations (Yerushalmy and Gilead 1999). An attractive feature of the 'holistic' functional approach using the velocity–time graph, is that the solution of uniform motion problems is based on Euclidean geometry, and specifically on the equality of the areas of geometric figures and the invariance of these areas under translation (Farmaki et al. 2004). In addition, the geometric transformations that may be applied to the velocity graph lead to equivalent mathematical situations, and therefore to equivalent real-life problems; consequently, the 'reconfiguration' operation prompts the students to realize that the same geometrical model can solve seemingly different – but actually equivalent – problems. In fact, this realization may lead to a classification of equivalent uniform motion problems.

We now turn to the examples of our methodology.

Table for the trip

	time (h)	speed(km/h)	distance(km)
Out	t	24	$24t$
Back	$7-t$	18	$18(7-t)$

Traveling path



$$24t = 18(7-t)$$

Fig. 3 A traditional algebraic solution (cited in Yerushalmy and Gilead 1999)

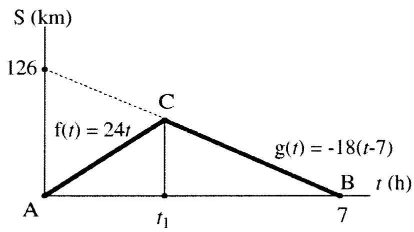
5.1 First example

Consider the following motion problem (cited in Yerushalmy and Gilead 1999): "A cyclist travelled from town 1 to town 2 at an average speed of 24 km/h. Arriving at town 2, she immediately turned back and travelled to town 1 at an average speed of 18 km/h. The whole trip took 7 h. How long was the trip in each direction?"

A. Traditional approach (algebraic)
   Figure 3
B. Functional approach (S, t)
   Figure 4
C. The 'holistic' functional approach (U, t)
   Figure 5

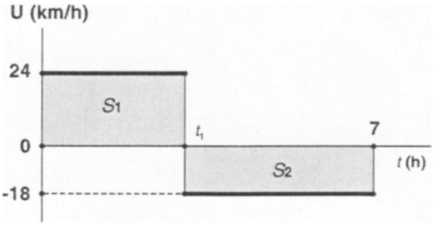
According to the holistic representation of the problem (Figs. 5 and 6a) let t_1 hours be the time taken for the cyclist to travel from town 1 to town 2. By the assumptions of the problem, the areas S_1 and S_2 are equal, since each represents the distance between town 1 and town 2. Sketching the rectangle OABC with area D, as shown in Fig. 6b, we have, according to our hypothesis, that $S_1+D=S_2+D$. We know that the area S_2+D is the area of the rectangle with sides 7 and 18, and is thus equal to 126 ($S_2+D=18\times7=126$). On the other hand, S_1+D is the area of the rectangle with sides 42, (24+18) and t_1, respectively. Thus, from $S_1+D=S_2+D$, we get $126=42\cdot t_1$ and consequently $t_1=3(h)$.

Fig. 4 A functional approach (cited in Yerushalmy and Gilead 1999)



$f(t) = g(t)$

$24t = -18(t-7)$



$$S(t) = \begin{cases} 24t, t \in [0, t_1] \\ 24t_1 - 18(t - t_1), t \in [t_1, 7] \end{cases}$$

t_1 is the time for direct travel.

$$S_1 = S_2.$$

Fig. 5 A functional approach based on the 'holistic' graph (U, t)

5.2 Second example

We present the following motion problem in order to clarify the model for the geometric solution we propose:

> A cyclist traveled from town A to town B at an average speed of 44 km/h. Arriving at town B, she immediately turned back and traveled to town A at an average speed of 18 km/h. If the return time was twice the going out time plus an hour, how long was the trip in each direction?

Figure 7a shows the representation of the problem by the graph (U, t). Let t_1 be the time taken in going from town A to town B. From the problem data, we know that the areas S_1 and S_2 of the two rectangles OABG and GEDF are equal ($S_1 = S_2$). By reflection in the t-axis of the rectangle GEDF and translation by t_1 to the left, we have the graphical representation of an equivalent motion problem (Fig. 7b).

The areas S_1 and S_2 of the corresponding rectangles OABG and OHKL represent the distance between the towns A and B, according to our hypothesis. We partition the rectangle OHKL into the rectangles OHIG, GIMN and NMKL. We choose the partition so that the bases of OHIG, GIMN and NMKL have lengths t_1, t_1, and 1, respectively (Fig. 8).

Translating the rectangle GIMN by t_1 to the left and 18 upwards and translating the rectangle NMKL by $2t_1$ to the left and 36 upwards, we obtain an equivalent mathematical structure (Fig. 8). According to Duval (2002), this is a 'reconfiguration' operation.

Now by subtraction of the common part ORTG from the figures OABG and OZDPTG, we can see that the areas of the rectangles RZDP and RABT are equal (Fig. 9).

We know the sides of rectangle RZDP. Thus $(RZDP) = (RP)(RZ) = 1 \times 18 = 18$. The sides of rectangle RABT are: $(RA) = 8$ and (RT), which represents the unknown time t_1. From the equality $(RZDP) = (RABT)$ we get: $8t_1 = 18 \Rightarrow t_1 = 18/8$ (h).

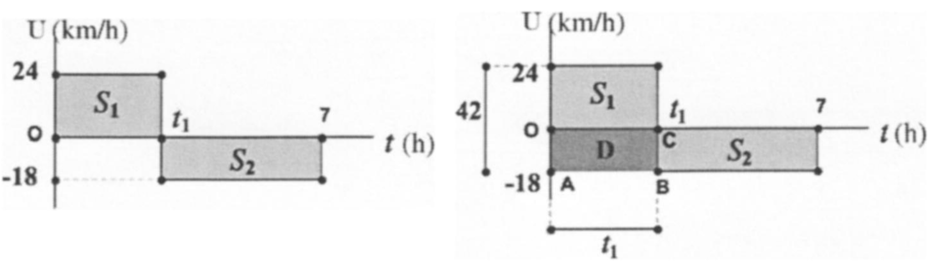


Fig. 6 (a) Holistic representation of the problem. (b) Sketch of the rectangle OABC with area D

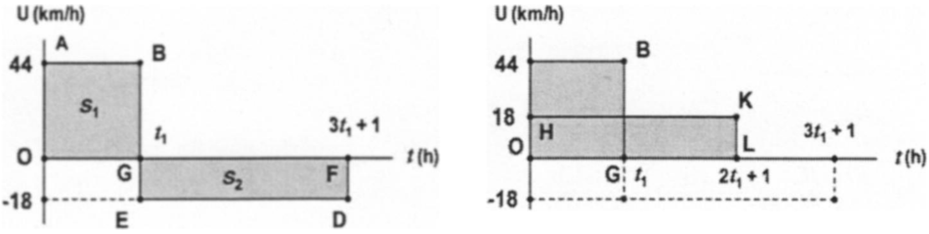


Fig. 7 (a) Representation of the problem by the graph (U, t). (b) Graphical representation of an equivalent motion problem by reflection in the t-axis of the rectangle GEDF and translation by t_1 to the left

6 The teaching experiment – the educational goals and content of the classroom activities

We applied our teaching approach to 58 fifteen year-old students in a high school, divided into three classes. The course consisted of seven 1-h teaching sessions based on the theoretical context of Brousseau's didactic situations about learning and teaching in a didactic milieu (Brousseau 1997). These include situations of *action*, in which the students try various methods to solve given problems; situations of *formulation* and *communication*, where students exchange and compare observations between themselves; situations of *validation*, in which the students try to explain phenomena or to verify a theoretical conjecture; and situations of *institutionalisation*, where the results of the classroom discourse assumes the form of the officially accepted terminology, definitions, theorems, etc. by means of the teacher's authority (Brousseau 1997, pp. 8–18).

During their previous year of study, the students had been taught how to solve simple first degree equations using the classical algebraic method and formulae for calculating the areas of simple figures (square, rectangle). In the experimental teaching the students worked in pairs in the classroom using worksheets. They worked together on completing and formulated their observations in order to communicate their personal experience, discussing various alternative answers as they sought the correct one. Once the students had achieved a consensus, the teacher validated those answers that were acceptable. In some cases, teacher's questioning helped guide the students towards renegotiating conflicting or

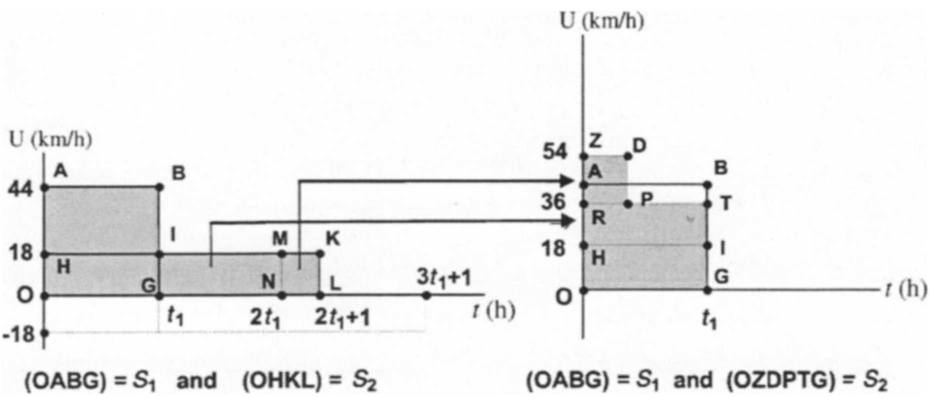


Fig. 8 Translation of the rectangle GIMN by t_1 to the left and 18 *upwards* and translation of the rectangle NMKL **by $2t_1$ to the left** and 36 *upwards* to obtain an equivalent mathematical structure

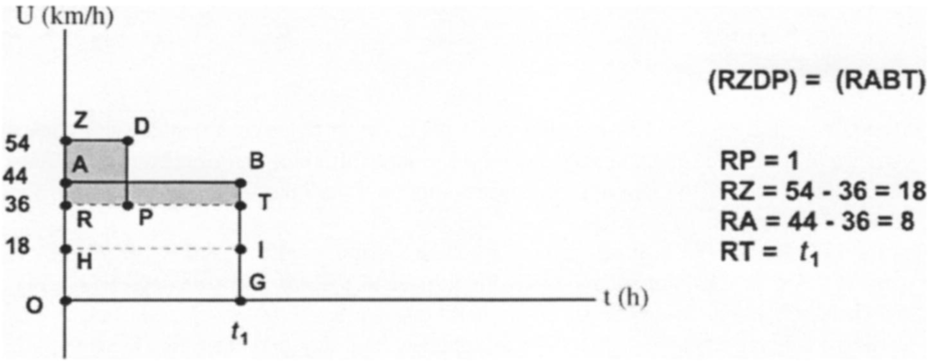


Fig. 9 Subtraction of the common part ORTG from the figures OABG and OZDPTG, showing that the areas of the rectangles RZDP and RABT are equal

erroneous answers. The teacher only provided the answers when the students encountered difficulties they could not overcome.

We will briefly describe the goals and content of the seven sessions. We recorded the discussions between the students during the sessions. In the *first four sessions*, we tried to activate their mathematical reflections via the "holistic" graphical representation of simple uniform motion problems. We consider this approach to be important because: (1) It allows three functional variables to be represented differently on the same graph, since velocity and time are represented by lines, and distances covered are represented by the areas of rectangles. The representation of different variables in different ways in the same context may extend the students' conceptual framework with regard to the mathematization of real-life situations. (2) The representation of the covered distance by an area, and the interrelation of velocity with distance on the same graph, constitutes the students' first contact with the concepts of Calculus.

In the *next three sessions* we introduced two important issues: (1) the *equivalent* motion problems, and the treatment of the *geometric solution* to these problems using geometric transformations, and (2) the shift from the geometrical/graphical to the algebraic context, exploiting the "holistic" graph in order to achieve the *algebraic solution* to the problems. In more detail:

6.1 The velocity–time graphical representation of uniform motion

First teaching session The didactical goals of the activities were to represent the constant velocity of a mobile object during a given time interval using Cartesian axes. The students faced initial difficulties representing time and velocity on the axes. The choice of different units of measurement on each axis led them to realise that different variables can be represented on each Cartesian axis independently. Many students represented the instantaneous velocity by drawing a segment perpendicular to the time axis. They realized they can draw perpendicular segments of equal length at every instant on the time axis in a manner similar to Oresme's. The students thus constructed the 'line of intensity,' representing the velocity during a time interval in the Oresme's sense, by drawing the 'summit line' – formed from all the upper endpoints – parallel to the time axis. The only modification from Oresme's model is that it is applied on Cartesian axes. In an earlier research project, Vergnaud and Errecalde (1980) studied how 10–13 year-old students translated quantities that are given arithmetically onto graphs using line segments. They discovered that most students tended to express numbers as intervals rather than as points on an order line.

Second teaching session The teaching goals were: (1) to represent the piecewise constant velocity of a mobile object during a given time interval taking the direction of the object into account, and (2) to represent the uniform motion of two mobile objects using the same velocity–time graph.

Careful selection of uniform motion problems encouraged the students to discuss the representation of the 'signed' motion with respect to a fixed point of departure using a velocity–time graph. More specifically, when considering "How can the constant velocity of two mobile objects whose directions are opposite and the speed counters show the same value be represented?," they accepted George's statement, to the effect that: "...I think we can draw a segment beneath and parallel to the time axis. This is the positive velocity of the transition over a time interval, while another parallel segment below the time axis is the negative velocity of return. That means that positive is above and the negative is below the time axis." The students' intuition led them to represent the direction of motion on the velocity–time graph, selecting positive or negative values for the velocity using the Cartesian axes.

One difficulty in particular provoked extensive discussion: the confusion between the time axis and the straight line along which the motion itself took place. A few students mistook the change in time as a change in the position of the moving body and consequently represented the motion wrongly on the velocity–time axis. This confusion is similar to that noted by Lacasta Zabalza (1995).

6.2 The distance covered as the area of the rectangles on the velocity–time graph

Third teaching session Initially, the teacher asked the students to calculate the distance covered by a mobile object moving at a constant velocity of 10 km/h over 2 h, then to represent the motion using a velocity–time graph, and finally to compare the area of the rectangle on the graph with the distance covered. As they estimated the same value of the area and the distance, the students were asked to interrelate the distance covered by a mobile object moving at a constant positive velocity with the area of the rectangle between the graph of velocity and the time axis in various similar problems. The students realized that when motion is uniform, the distance covered is proportional to time, and can thus be calculated as a product of the constant velocity and the time interval in which the motion takes place. They also realized that the area of the rectangle on the velocity–time graph may be calculated from the same product, because its sides represent velocity and time, respectively. The classroom discussion helped the students realize that to determine the distance covered by a mobile object moving at a positive constant velocity, it is enough to calculate the area of the corresponding rectangle on the velocity–time graph.

The interrelation between the distance covered and the area of the figure on the velocity–time graph is not self-evident. It implies the passage from the velocity function to the integral, from the 'rate' to the 'total': a shift that proved a difficult conceptual leap to make historically, and was made intuitively by Oresme, who made the connection between kinematic arguments and Euclidean geometry.

The students were led to make this fundamental connection via a guided activity inspired by Oresme's idea.

Fourth teaching session The primary aim was to encourage the students to induce that distance covered is independent of the sign of the velocity. The teacher gave the students various velocity–time graphs relating to uniform motion problems (with positive and negative constant velocities) and asked them to calculate the distances covered. For example, to calculate the distance covered through the direct and return motion of an object traveling from town A to town B with different constant velocities in each direction. The discussion became more lively as questions arose from the case in which velocity is negative and the rectangle which represents the motion lies below the time axis. "Are there negative rectangle's sides?," "What happens in this case with the area?," were some of the questions. John said: "Since the velocity is negative, the product with the time interval is negative. Is the area of the rectangle negative?... No,... something is wrong here." Some students said: "yes, the area is negative," some others said that was impossible. The teacher proposed accepting the opinion expressed by some students that the velocity is negative concerning the direction of motion, but the area must be considered as positive because the areas of geometrical figures are always positive, and also that the distance covered is always positive and independent of direction.

At the end of the session, the teacher proposed a game to the students with a sheet of paper and the Cartesian axes on the blackboard. He moved the paper along the time axis several times in such a way that one side of the rectangle was always a line segment on the axis, and asked the students to observe what changes and what remains constant when shifts of this sort – representing a *translation* or *reflection* with respect to the time axis – occur. The students realised that the rectangle (sides and area) remained constant throughout such transformations, and consequently that: (1) there is no change in the average speed, time interval, or distance covered; and (2) that there is a possible change in the direction of the motion and the time of departure.

6.3 Equivalence via geometric transformations

Fifth teaching session The classroom discussion left the students with a good understanding of the invariance of area under translations. A typical example is described in Fig. 10.

The students were encouraged to try several other geometric transformations. They worked using translations and reflections that lead to equivalent situations with respect to the distance covered. In order to determine the velocity, time or distance covered on the velocity–time graph in the case of simple uniform motion, the students had initially (in the previous sessions) focused on the sides and the area of the rectangle. "No consideration is given... to the transformations of these figures within the space" (Piaget and Garcia 1989, p. 109). The students' efforts to search for transformations to "find relationships between the figures according to various forms of correspondence" (Piaget and Garcia 1989, p. 109) are essential in this sense for passing from the intra-figural to the inter-figural stage of thought.

The teacher encouraged the students to interpret the graphs in Fig. 10, and to describe real problems of motion corresponding to each graph. For instance, with relation to Fig. 10c, Sophie said:

> Two cyclists started their journey at the same time, traveling from town 1 to town 2 at an average speed of 24 km/h and 18 km/h, respectively. The first cyclist arrived in t_1 and the time of the second cyclist was $7-t_1$. How long were the cyclists' trips?

Sophie came to the following realization: "If we solve one of the problems represented above, it is not necessary to solve all the others, because the sides and the area of the rectangles remain the same." However, few of the students understood Sophie's statement.

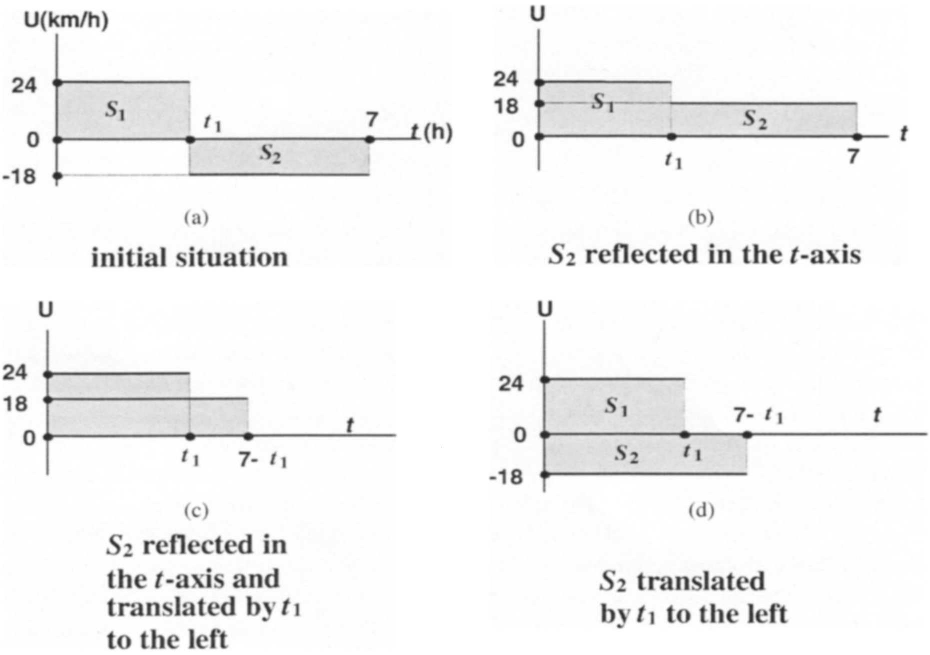


Fig. 10 Example of invariance of area under translations. **a** Initial situation, **b** S_2 reflected in the t-axis, **c** S_2 reflected in the t-axis and translated by t_1 to the *left*, **d** S_2 translated by t_1 to the *left*

Sophie's thinking reveals complex cognitive operations: on the one hand, she handles the geometrical transformations within the graphical representations register – a 'processing' operation according to Duval (2002), while she also makes the 'conversion' (Duval 2002, p. 318) from the graphical context to the interpretation and word formulation of uniform motion problems in the real world. Furthermore, Sophie makes a key cognitive step: using the geometrical transformations, she not only finds relationships between the figures in the graphical context, she also realizes the equivalence of the corresponding uniform motion problems. She classifies a range of real problems that result from the treatment in the mathematical context, making the conversion to the real world.

6.4 Algebraic and geometrical solutions to uniform motion problems

Sixth and seventh teaching sessions In the last two sessions the students worked on the solution of uniform motion problems. Using the velocity–time graph, they were encouraged to seek and find both algebraic and geometrical solutions.

In the case of *algebraic* solutions, the students had to construct an equation, taking the assumptions of the problem into account translated into an equality involving areas of rectangles represented on the graph, and then solve it. Many students found it difficult to choose a suitable unknown magnitude and represent it by a letter–historically, this, too, was involved a difficult conceptual switch from concrete to more abstract situations (Radford 2000, p. 151). Some students made mistakes when constructing the equation, others found it hard to use it to find the correct solution to several problems.

In the case of the *geometrical* solutions, the students had to make appropriate transformations of the figures on the graph to solve the problem. The teacher started with the following problem:

> *A cyclist begins his trip from town A to town B. He covers the first half of the distance between town A and town B in 3 h at a constant velocity of 20 km/h and continues his trip at a constant velocity of 15 km/h. How long does his trip take?*

Many students focused on the equality of the area of the rectangles on the graph, constructing the equation $20\times3=15\times t$. Teacher intervention led the students to take the observation into account that the unknown quantity is one side of a rectangle, while both the area and the other side are known.

The teacher asked the students to look for a geometrical solution to the following problem by trying several transformations of the figures on the graph in order to obtain a rectangle whose area and the length of one side are known: *"Two cyclists moving in opposite directions began their trip at the same instant. Their average speeds are 30 km/h and 20 km/h, respectively. They cover the same distance, but the second cyclist traveled an hour longer than the first. How long did the trip take for each cyclist?"*

The students drew the velocity vs. time graph without difficulty (Fig. 11a). The students found it hard to choose the appropriate geometric transformation for the rectangles on the graph that could lead them to the solution. The teacher guided the students to center their attention on the equivalent problem represented by a graph obtained by reflection of the rectangle under the time axis on the initial graph (Fig. 11b).

The students accepted the idea proposed by two members of the class that the solution is obtained from the abstraction of the common part of the two rectangles. From the equivalence of the shadowed rectangles, we can then find the unknown t by simple division.

The teacher encouraged students in their attempts to find a geometrical solution to several uniform motion problems. At the end of the course, the teacher asked the students to compare the two methods. Most of them thought the algebraic method more 'economical,' though several declared a preference for the geometric approach.

Analyzing the students' cognitive operations in the last two sessions, we observed that the students initially focused on the translation from the graphical to the algebraic register constructing first degree equations to solve the given problems. According to Duval (2002, pp. 317–318), this is a 'conversion' ('processing' operations) to a new context within which the students tried several algebraic computations.

The students used geometrical transformations on the velocity–time graph to search for the solution to the given problems within the graphical/geometrical register. Through their

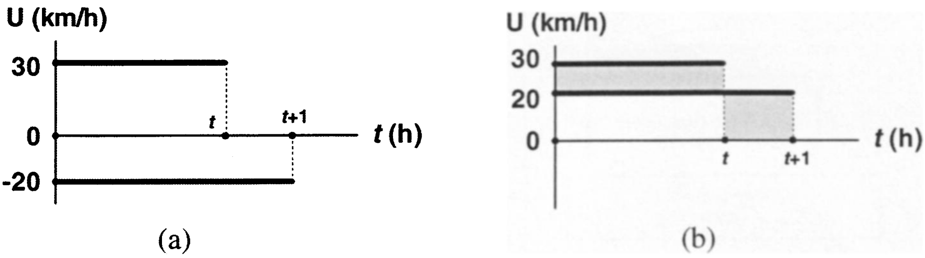


Fig. 11 **a** Velocity vs. time graph. **b** Equivalent problem represented by a graph obtained by reflection of the rectangle under the time axis on the initial graph

'reconfiguration' of the rectangles on the graph, "a specific figural processing which provides figures with a heuristic function, possibly giving an insight to the solution of a problem" (Duval 2002, p. 326), the students tried to determine the unknown as a side of a rectangle whose area and other side are known quantities.

7 Results

7.1 The interviews

Some students were interviewed individually. The interviews were recorded and transcribed in full for detailed analysis. Our aim was to investigate: students' difficulties, the degree to which they understood 'holistic' graphical representations and the equivalences of the motion problems using geometric transformations, and their ability to solve uniform motion problems using both algebraic and geometric methods.

At the end of the course, the teacher gave the students the problem described in Example 5.1 above and asked them: (1) to represent the motion using a graph of velocity versus time, and (2) to calculate the time of the transition and the time of the return using both the algebraic and geometric method.

The solutions given by Emily and Sophie are presented below, along with excerpts from their interviews. Analysing the data and taking Duval's perspective into account, we can schematically represent the students' cognitive operations when solving first degree equations resulting from uniform motion problems in the graphical and algebraic registers, respectively, as follows (Fig. 12):

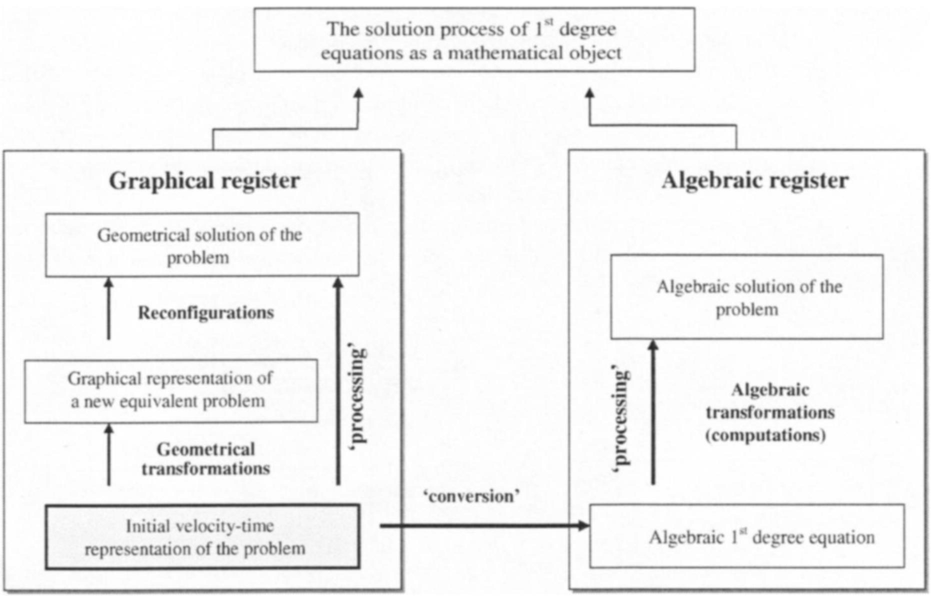


Fig. 12 Students' cognitive operations when solving first degree equations resulting from uniform motion problems in the graphical and algebraic registers

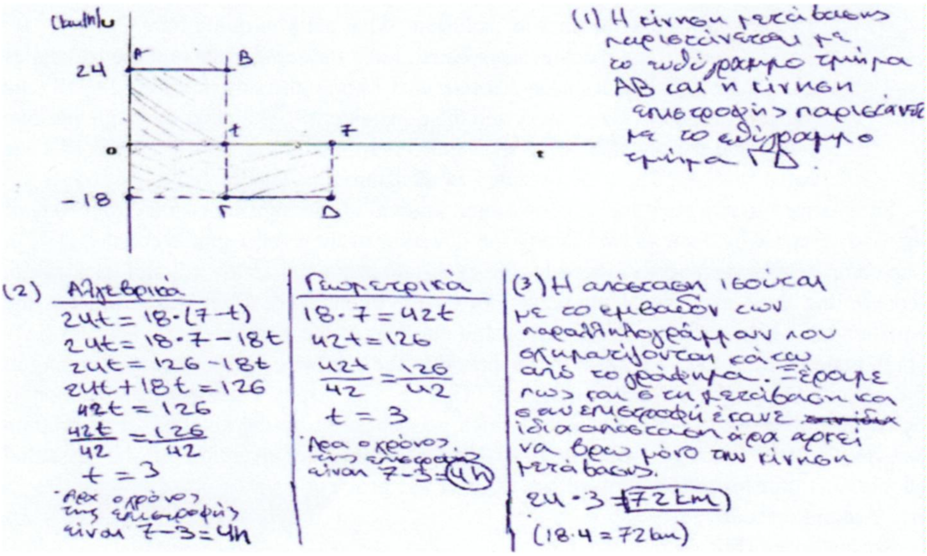


Fig. 13 Emily (1) represents velocity vs. time and says: "The motion of the transition is represented by the linear segment AB and the motion of the return by the segment ΓΔ"; (2) she calculates the time the trip took using two methods, the "algebraic" and the "geometric" one; and (3) comments: "The distance covered is equal to the area of the rectangles under the graph. We know that the distances of the transition and the return are equal, hence it is sufficient to find the distance of the transition"

A. First episode

Emily wrote (Fig. 13)

We asked Emily why she chose to represent the transition time by t on the time axis.

[1] Emily: We know that the total traveling time is 7 h. We don't know the time of transition, which I represent by t in the time interval between 0 and 7 h.

[2] Teacher: Why do you represent the motion here (*indicating the time interval [0, t]*) with this rectangle above the time axis, and here (indicating *the time interval [t, 7]*), by the rectangle under the time axis?

[3] E.: We consider the velocity in this direction to be positive, so we draw the segment AB above the time axis. In the opposite direction, [we draw] *ΓΔ* under the axis.

[4] T.: Here, I see two rectangles. What do these represent?

[5] E.: The first rectangle represents the trip from town 1 to town 2, and the second represents the travel from town 2 to 1. It is the same distance, so the areas [of the rectangles] are the same.

[6] T.: I see an algebraic solution. What did you do?

[7] E.: I found an equality between the distances traveled in both directions.

[8] T.: How?

[9] E.: I said that $24 \times t$ is equal to $18 \times (7-t)$.

[10] T.: What is $24 \times t$?

[11] E.: It is the area of the rectangle here (*shows the first rectangle*), and $18 \cdot (7-t)$ is the area of the rectangle under the time axis.

[12] T.: And...

[13] E.: I solved the equation and found the time.

[14] T.: Let's go to the "geometric" solution. What are you doing here?

[15] E.: The areas of the rectangles are equal. I add the same piece to these rectangles and obtain equality. It makes it easier that I know the side here is... 24+18 (*she indicates it on the worksheet*), and the other side is *t*. The other rectangle has one side equal to 7, which is the total time taken, and the other 18. Hence 18×7 is equal to $42\times t$. Then *t* is obtained by division.

Emily uses *t* to represent the unknown time interval of the transition on the time axis of the (*u*–*t*) graph which she draws, taking the direction of the cyclist into account ([1]–[3]). She compares the distances covered by the cyclist from town 1 to town 2 and back again, representing them as areas of the rectangles on the graph ([5]). Then she formulates the equation taking the equality of the corresponding areas of the rectangles into account ([6]–[11]), making the conversion from the graphical to the algebraic register and proceeding to the solution by algebraic computations ([13], Fig. 12). Emily's geometrical solution is based on the reconfiguration operation which was obtained by the addition of a common part to the two rectangles on the (*u*–*t*) graph ([15]). The invention of this essential geometrical transformation allowed her to solve the problem.

B. Second episode

Sophie wrote (Fig. 14):

We discussed Sophie's answer with her:

[1] Teacher: On the worksheet you wrote: "We add the same area to each rectangle...." Why did you do this?

[2] Sophie: We take a rectangle which has a particular area and we add it to these two rectangles (*shows the rectangles on the initial graph*).

[3] T.: Why?

[4] S.: Because if these two areas are equal, nothing will change if we add this part.

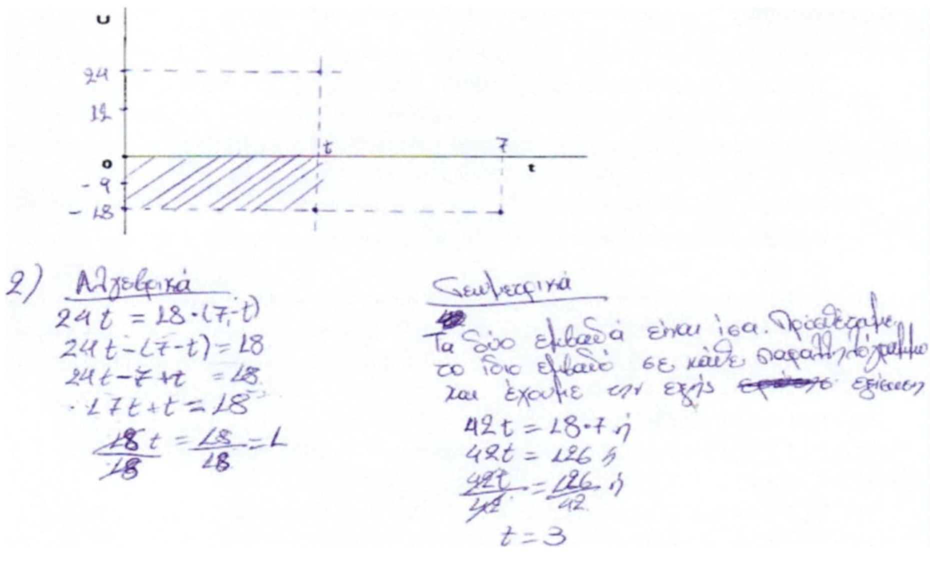


Fig. 14 Sophie represents the motion scenario by the velocity-time graph above. She tried an "algebraic" solution using the initial graphical representation as it appears on the worksheet. She tried a "geometric" solution successfully, on which she wrote: *"The two areas are equal. We add the same area to each rectangle and obtain the equation 24×t=18×7"*

[5] T.: OK, but why do it?

[6] S.: In order to make things easier... I get a simpler equation. I have a rectangle with one side 24 plus 18, that is, 42 and the other side is the time t. We also have the rectangle with one side equal to 18 and the other side equal to 7. The areas of these rectangles are equal. Now it is easy to find the time.

[7] T.: Let's go to the "algebraic" method. When you say "algebraic" what exactly do you mean?

[8] S.: I search for elements on the graph in order to construct the equation. I say 24 km/h multiplied by t hours equals 18 km/h multiplied by (7-t) hours, which is the time interval of the return.

[9] T.: Here, in your "algebraic" method, you are trying to solve the equation $24t=18(7-t)$.

[10] S.: It's wrong.

Subsequently, Sophie said she prefers the "geometrical" method; and when we pointed out that she must invent the addition of a particular figure in order to solve this problem geometrically, she added: "That's not difficult." Then,

[11] T.: Which of the two do you think is the easier method: the algebraic or the geometrical?

[12] S.: The geometrical method. I prefer the geometrical method because I have the figures and I can do things on them.

Sophie describes the reconfiguration operation in the same way as Emily: on the rectangles by which the geometrical solution to the problem is obtained ([1]–[6]). She fully identifies the area of the rectangles on the graph with the distance covered. In order to solve the given problem in the algebraic context, Sophie constructs the equation based on the equality of the rectangles' area on the initial graph ([8]–[9]). Sophie's difficulties did not lie in constructing the equation, but in calculating the solution process algebraically ([8]–[10], Fig. 14). She knows that the algebraic solution is not correct ([10]), probably because she has compared it with the geometric one. Could this be the reason she prefers the 'easier' geometric method? However, Sophie explains her choice by saying she has "to do things on the figures" ([11]–[12]), which is not the case with algebraic computations.

7.2 Some observations on the data collected

To briefly summarize our observations on the data collected:

(1) Most of the students have realized through the classroom discourse that: (a) when motion is uniform, the velocity remains constant, meaning that the motion can be represented by a rectangle in which one side represents the velocity and the other the time interval of the motion; (b) following the definition of a 'signed' motion with respect to a fixed point of departure, the velocity of direct motion is positive and that of return negative; and (c) the areas of the rectangles on the velocity graph (above or below the time axis) are positive and represent the distance covered, irrespective of direction.

Many students succeeded in interpreting and expressing several (u–t) graphs using a possible motion scenario, by passing on to real-life situations.

(2) A large percentage of the students (22 of 58) can use transformations of the rectangles on the graph. They realize that, through these transformations, equivalent motion problems are derived.

(3) A few of them (12 of 22) can use transformations and 'reconfigurations' of the figures to solve the particular problems. They also understand that by using the same

mathematical model, they can solve a class of equivalent motion problems obtained by transformations. They are able to solve the problems using both the 'geometrical' and the 'algebraic' methods.

8 Discussion

We have developed our didactical approach as a contribution to the integration of the history of mathematics into teaching, in the belief that "The past never dies" (Fauvel 1991), and that "History is an engine of mathematical wit" (Fauvel and van Maanen 2000, p. xiii). Euclidean geometry is certainly such a 'past' and such a 'history'; Oresme's ideas touch on both Euclidean geometry and modern mathematics, and should be put to appropriate use in educational practice, not only for providing historical perspective and helping students appreciate how mathematical ideas have come about and evolved in the past, and how many everyday concepts and current problem-solving techniques have been around for quite a long time, but also – most importantly – to inspire a chain of teaching activities with specific goals leading to the essence of mathematical knowledge.

We subscribe to the view that if the learning process is to lead to a real understanding, then students must, among other things, have the ability to represent a mathematical entity in various registers. The ability to identify the same concept in different representation systems, and the flexibility to shift from one representation to another, allow students to develop a deep understanding (Duval 2002; Even 1998).

We realise that designing the activities and aims of our educational approach around Oresme's ideas and Duval's framework might be considered overly ambitious. However, our experiments with a small group of relatively young/early grade students demonstrated that these activities can lead directly and simply to the students understanding concepts and representations which lie at the heart of advanced mathematical thought. The design of the teaching sequences relies on problem situations of uniform motion connecting with velocity and distance, and is therefore based on the students' everyday experience. The element of student/milieu relation provides them with the means to act in a meaningful manner, to reason and ultimately to expand their reality, which in fact grows in tandem with their mathematical development.

We believe that the didactic approach and mathematical model described has applications beyond the uniform motion problems considered in this work, if properly planned. Applied to students of a more advanced level, for instance, it could lead to the solution of a wider range of problems related to uniform motion, where two constant velocities U_1 and U_2 and a known linear relationship between the distances covered and the corresponding times are assumed (e.g., $S_2 = \kappa S_1 + \lambda$ and $t_2 = \mu t_1 + \nu$). Then, this approach could in turn be conceived as the first part of a series of educational activities designed to be more wide-ranging and to involve the representation of various more general motion problems relating to uniform and non-uniform acceleration, by means of the velocity–time graph, and aiming to gradually introduce the students to basic concepts of Calculus, and specifically the definite integral and the Fundamental Theorem.

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