THE HISTORY OF MATHEMATICAL EDUCATION

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1. The Association and Mathematical Education. The inclusion of a survey of mathematical education as a part of the semi-centennial celebration of the Mathematical Association of America is appropriate for many reasons. For example, the by-laws of the Association state that one of its objectives is "to assist in promoting the interests of mathematics in America, especially in the collegiate field . . . by conducting investigations for the purpose of improving the teaching of mathematics. . . . "

In pursuit of its goals the Association has published the Carus Monographs of which the fifth, *The History of Mathematics in America before 1900*, by David Eugene Smith and Jekuthiel Ginsburg implicitly contains much of significance for the history of mathematical education in the country, especially at the college level. However, the past concern of the Association for mathematical education at the secondary level should neither be ignored nor allowed to lapse. It was particularly significant that the Association was mentioned on page one of the first volume of *The Mathematics Teacher* to be issued under the aegis of the National Council of Teachers of Mathematics. In 1921 the newly founded Council took over this journal from the Association of Teachers of Mathematics in the Middle States and Maryland. In the first issue of that year C. M. Austin, first president of the National Council, explained, in words which have a familiar ring, the needs and reasons for the founding of this new organization as follows: "During this same period (the previous ten years) high school mathematics courses have been assailed on every hand. So-called educational reformers have tinkered with the courses and they, not knowing the subject and its values, in many cases have thrown out mathematics altogether or made it entirely elective. Individual teachers and local organizations have made a fine defense to be sure, but there could be no concerted action. Finally, the Mathematical Association of America came to the rescue and appointed a committee to study the situation and to make recommendations. Already two valuable reports have been issued and others are in preparation. The pity of it is that this work, wholly in the realm of the secondary schools, should have to be done by an organization of college teachers. True, they have generously called in high school teachers to help, but the fact is that it remained for the college people to initiate the work. They could do it because they possessed a live, vigorous organization [1]. (The final report of the committee of the Association to which Austin referred was the famous "1923 report.")

It is, I believe, altogether appropriate that this Association which is still live and vigorous should continue its interest in mathematical education, not only by recognizing it in the program of this, its semi-centennial, but also in the continuing concern showed by the work of its Committee on the Undergraduate Program which has a panel on teacher training.

I was, therefore, greatly pleased when the Committee asked me to speak on this topic which combines two of my major interests, the history of mathematics and mathematical education. The Committee neither defined these terms nor laid upon me any restrictions as to the time, period, or level of mathematical education which I should consider. I have chosen to consider education to refer to consciously and deliberately planned programs for instruction, and to attempt the task of a broad survey of the major, philosophical-psychological problems of mathematical education from its beginning until today. Such a task, of course, automatically suggests a chronological approach. But I hope that while following a chronological theme I can avoid the impression that I consider the history of either mathematics or mathematical education to be at all well embodied in a sequence of names and dates. I hope, then, that I can point out and trace with you a number of continuing and pervasive themes which I see in the teaching of mathematics through the ages. Many of the problems related to these themes are still with us, and I hope that the function of this talk may be more than a mere interesting and nostalgic review of the past. I hope that a survey of the changing objectives and philosophies of both mathematics and mathematical education can bring into sharper focus the needs and controversies of our time.

2. The Earliest Mathematical Education—Motivation. Although Seidenberg has recently elaborated a conjecture that counting originated in the rituals of prehistoric community life around 10,000 years B.C., which rituals then would no doubt have been taught to successive generations [2], our first knowledge of formal education comes from the Babylonian clay tablets and Egyptian papyri of the early historic period. The Ahmes papyrus of 1650 B. C. was copied from papyri composed as much as two hundred years earlier and gives evidence of having some of the qualities of a copybook or textbook. Not only have excavations in the Near East revealed rooms that appear to have been school classrooms, but also Babylonian cuneiform mathematical texts have, again, schoolbook properties. Scribes, clerks, and priests had to be trained to keep records associated with the calendar, land, taxation, and commerce. Amongst the problems in these ancient texts, however, one often discerns a recreational or theoretical situation such as the recently deciphered Babylonian tablet Plimpton 322 which contains a table of Pythagorean triples. So, from our vantage point in time, we can read both practical motives for formal instruction and evidence of the indulgence of intellectual curiosity in the records of the very earliest days.

However, we turn to Greece, and in particular to Athens of the 6th and 7th centuries B. C., for our first knowledge of formal schooling. In the time of Solon, the state paid tuition for the children of men killed in battle. There was a publically appointed supervisor of education, and parents were required to see that their sons were educated. There were two types of education for two classes of citizens: The liberal arts for the free citizens (here, in fact, is the origin of our word "liberal education"—an education for the free), and education in the so-called practical arts for the lower classes or slaves.

There are a number of interesting anecdotes about early Greek education that are also related to the teaching of Greek mathematics. For instance, there are two versions of the account of the first person to teach mathematics for money. One dating back to 520 B. C., says that a Pythagorean had lost his property and when this misfortune befell him he was allowed by the Brotherhood to make money by teaching geometry. Another story is that Hippocrates, of 460 B. C., who had been a merchant, was captured by pirates. Learning geometry while he was delayed in Athens by a lawsuit in which he attempted to recover his loss, he later taught it to support himself during the remainder of this time. Aristotle gives a slightly different version of this, saying that Hippocrates had been defrauded of his money because, although he was a good geometer, he was stupid and incompetent in the business of ordinary life [3]. This is the earliest citation of evidence in the perennially recurring debate as to the so-called disciplinary value of a mathematical education and its role in teaching people to think well in nonmathematical situations. In more recent times, psychologists have associated this question with the question of "transfer of training."

Pythagoras himself is the hero of the story to the effect that upon his return from Egypt he was eager to transplant to Greece what he had learned and what he had added to Egyptian mathematics. In other words, he was an eager teacher with no students. So he initially paid students the equivalent of six cents a proposition for each theorem learned. He continued this until the student was so interested and enmeshed in the intrigues of mathematics that he was willing to continue even if he had to pay rather than be paid [4]. Here, then, is a fore-taste of another current dispute; namely, does one need to foresee rather immediate profit in order to be well motivated, or can we trust to the intrinsic interest of the subject itself as motivation for the learning of our mathematics students. We have all been questioned by the student who says, "What is the good of all this?" Euclid is supposed to have answered such a question by turning to a slave and saying, "Give the student sixpence since he must need gain by what he learns" [5].

There is another problem that is classical in its antiquity and modern in its occurrence. This is the problem of the student who, impatient of proof and of the slow process of building understandings, wants the formula, the handbook, the quickly achieved result. The story is varyingly told of Alexander the Great speaking to Menaechmus or of Ptolemy speaking to Euclid. It is to the effect that the king wished to learn geometry in a hurry. The geometer's reply was, "For roads over the country there are royal roads and roads for common citizens, but in geometry there is one road for all" [6].

3. The Earliest Mathematical Education—Methods. In any event, all of the three famous Greek philosophers made contributions to the history of mathematical education. Listen to Socrates for a moment or two. This is from Jowlett's translation of Plato's *Dialogues* as quoted by J. W. A. Young who himself deserves mention in a history of mathematical education presented to the Mathematical Association of America. After writing a dissertation in group theory in 1892, he taught at the University of Chicago where he helped to organize the Chicago section of the American Mathematical Society and became a Professor of the Pedagogy of Mathematics. His book, *The Teaching of Mathematics in the Elementary and Secondary School*, first published in 1906, went through many editions. Young chose to illustrate questioning procedures by this quotation in which Socrates is teaching a demonstration lesson. His class is an illiterate slave and his observer is Meno:

Soc. Tell me, boy, do you know that a figure like this is a square?
Boy. I do.
Soc. And do you know that a square figure has these four lines equal?
Boy. Certainly.
Soc. And these lines which I have drawn through the middle of the square are also equal?
Boy. Yes.
Soc. A square may be of any size?
Boy. Certainly.
Soc. And if one side of the square be of two feet and the other side be of two feet, how much will the whole be? Let me explain: If in one direction the space was of two feet and in the other direction of one foot, the whole would be of two feet taken once?
Boy. Yes.
Soc. But since this side is also of two feet, there are twice two feet?
Boy. There are.
Soc. Then the square is of twice two feet?
Boy. Yes.
Soc. And how many are twice two feet? Count and tell me.
Boy. Four, Socrates.
Soc. And might there not be another square twice as large as this, and having, like this, the lines equal?
Boy. Yes.
Soc. And of how many feet will that be?
Boy. Of eight feet.
Soc. And now try and tell me the length of the line which forms the side of that double square: this is two feet—what will that be?
Boy. Clearly, Socrates, that will be double.
Soc. Do you observe, Meno, that I am not teaching the boy anything but only asking him questions; and now he fancies that he knows how long a line is necessary in order to produce a figure of eight square feet; does he not?

I'll go no further with this, but note that the "Socratic method" of teaching in which students were forced to accept a conclusion, often one which differed from their original intuition, by an ingenious sequence of questions, is not the same as modern heuristic procedures in which a student is led, sometimes by himself, to *discover* a new relationship or principle for himself. However, there are significant common elements in the methodologies of Socrates and of heuristic teaching.

Though, like Socrates, Plato was no mathematician himself, the famous painting by Raphael of Plato's Academy, a fresco to be found in the Vatican, illustrates a number of mathematical debts to Plato. I have time only to enumerate them rather hurriedly. The first of these is the concept of the academy. The picture suggests that the academy students were not young and immature, but were more nearly faculty members and graduate students. It was an early "institute for advanced study." In it, we see our first examples of group research projects and of specialization into particular subject fields of mathematics.

A second Platonic concept which has been important in the history of mathematics was elaborated in his *Republic* in which he prescribed the training for the various classes which made up the citizens of his "ideal" republic. Mathematics was a major subject in the training of the philosopher-king class in the period from age 20 to 30. The major function of this mathematics was to serve as training for the mind, for mental discipline, a goal claimed for mathematical instruction over many following centuries, and even today.

Mathematics to Plato consisted of arithmetic, geometry, astronomy, and music. These four subjects ultimately became the famous quadrivium of the Middle Ages to which were joined the three subjects termed the trivium—grammar, rhetoric, and logic, to make the curriculum which dominated education for many centuries.

Having thus seen the origins of a liberal education and of a stress on pure or abstract mathematics as the mental discipline which prepares persons to be philosopher-kings and to solve problems in all areas, it is not too surprising that we should find at this time the first major critic of mathematical education in the person of Isocrates. He criticized the concepts of over-specialization and of mathematics as a mental discipline in the following words:

> "I recommend those who are embarking on the study of geometry and astronomy to devote all their energy and intelligence to them. For I declare that, even if these disciplines cannot make better men of them, they have at least the advantage of keeping young people out of mischief; and I am convinced that no more useful or more suitable activity than these could be found for them. But at a later age, after the examinations which qualify one for the rights of an adult, I maintain that these activities are less appropriate. I would, in fact, say that some of those who have carried the study of these disciplines to the point of teaching them in their turn do not know how to put to proper use the sciences they possess and prove in all circumstances of life to be less judicious than their pupils and even—though I hardly dare suggest it—than their own servants" [7].

Aristotle also believed that the liberal education of its free citizens was essential to the building of a cohesive community and a sound state. However, he felt that small children should gain moral and physical education through play until about the age of 14 when they were were ready for or capable of intellectual pursuits. Here, then, we find an early recognition of the problem of what is called "readiness" in educational jargon which merely asserts that the children who are too young or too ill-prepared cannot understand or profit from the study of abstractions.

4. Mathematical Education in the Middle Ages. The first treatise on educational methods can be attributed to the Roman Quintillian of 35 to 100 A.D. in a book entitled, *Institutes of Oratory*. Briefly, he argued that teachers should take into account the individual differences amongst their students, that students should be given some choice as to the topics studied, in order that their studies might more nearly coincide with the student's particular talent, that student interest was a very important factor in facilitating learning and that interest was aroused by competition and awards, not by punishment (the MAA sponsors contests!). He advocated the study of a foreign language as a device for learning one's own language and stressed mathematics strongly for its teaching of methods of proof. This, you'll note, is a slightly different objective for mathematics instruction than the disciplinary value which I have mentioned previously. Quintillian also stressed memorization as his major teaching technique without seeing anything contradictory between this teaching method and the goal of teaching methods of proof! Quintillian was rediscovered in the Renaissance and some of the blame for the stress on pure memorative learning in the universities of the Middle Ages can be placed upon him.

The monastery schools of the Dark Ages gave way to cathedral schools. As the number of students in these schools increased, it became necessary to appoint a church official to oversee them, and since the school was often in the chancel it was natural to call such a person the *chancellor*. As students organized for mutual protection against a dictatorial chancellor, their groups were called universities. Next faculties were forced to unite to protect themselves against the students, and the earliest forerunner of our modern degrees was the license to teach which also made one a member of the university of professors. The very first formal instruction in pedagogy, however, seems to have been given by the Jesuits of the 16th and 17th century to those members of their order who were going on to teach. In these centuries a double type of education began to develop. One was for middle class students who were often taught by private teachers in the vernacular and with a heavy stress on applications and the vocational uses of mathematics, especially as related to commerce, surveying, and navigation. The other type of education, for upper class students, aimed to create "Christian gentlemen," teaching them little mathematics and with a heavy stress on humanistic studies. This latter was especially true in England and Italy with more emphasis on science and mathematics in France.

However, with the Renaissance we find persons such as Roger Bacon criticizing scholastic education which stressed the reading of Latin and depended upon memorization. Bacon suggested that more observation and study of science and mathematics should replace the narrow study of Aristotle which he felt led to a concealment of real ignorance by a pretense of knowledge.

Even the Reformation plays some role in an indirect way in the development of our subject. A heavy emphasis on so-called "original sin" very naturally led to a heavy emphasis on corporal punishment as something that was good for both mind and body. This idea of a general treatment, which was "good for mind and body" encouraged the notion of certain types of study being good "mental discipline," and thereby supported both Latin and mathematics for this purpose.

Mathematics has fared better than Latin throughout because it has been varyingly endorsed for its practical utility, for its value as mental discipline, and for its incidental teaching of methods of proof. Because of these claims for mental discipline and training in logic, it has also been stressed as an element of a liberal education. However, different aspects of mathematics were also included in the practical education of the middle class and even opposing philosophies were cited to support mathematical instruction. The philosophical materialism of Bacon and Locke which stressed learning through senses supported mathematics for its practical utility. The rational idealism of Descartes and Leibniz, which stressed that we learn through the reason, also supported an emphasis on the teaching of mathematics.

A slow change in mathematical education is revealed by a glance at some texts of the 17th and 18th centuries. The little book *Mathematicae Totius* by Pater Peter Galtrucius Aurelianensis published at Cambridge in 1683 had sections devoted to arithmetic, geometry, astronomy, chronology (the calendar) "gnomonicae" (the sun-dial), geography, optics, and music. Note that this includes the quadrivium and four additional practical topics. All this was embraced in 305 small pages! However, not to give a wrong impression, we must note that Newton, whose *Principia* appeared in 1687, had studied and taught at Cambridge and that his teacher, Isaac Barrow, had still earlier included preliminary notions leading up to the calculus in his lectures on optics, before resigning his position in favor of his promising young pupil. The ancient works of Apollonius were well known there as were the more modern works of Descartes and Fermat. However, these were not the basic fare of the secondary schools or the undergraduate curriculum.

That the elementary mathematical content of the curriculum on the continent was similar to that just displayed for England is revealed by looking at the table of contents of a *Compendium of Elementary Mathematics Composed for the Use of Young Students* by Christian Wolf of the University of Halle published in 1742. His two volumes included the eight topics of Galtrucius and then added trigonometry, mechanics, hydrostatics, "aerometria," hydraulics, perspective, "pyrotechnia," military architecture, civil architecture, and algebra in 900 small pages! Wolf also represents a growing interest in pedagogical problems which developed, especially in the German universities, at this time.

Wolf, for example, supported a lecture method as against the mere reading and exposition of a standard text which was to be memorized. He also supported the teaching of German versus Latin, allowed some electives in the curriculum, and stressed freedom for the faculty in teaching and for research.

5. The Nineteenth Century—Texts and Journals. The nineteenth century, however, was the period of real development of concern for pedagogy, educational aims and methods. The first book which I have been able to find which dealt explicitly with the teaching of mathematics was published in Paris in "An XIV" (1805). It was *Essais sur l'Enseignement en Général et sur celui des Mathématiques en Particulier* by S. F. Lacroix. In 1810 Gergonne began publication of *Annales de Mathématique Pure et Appliquée* and in the prospectus for this journal, in its first pages, he stated that "these *Annales* will be principally consecrated to pure mathematics and to all the searches which have for their object the perfecting and simplifying of teaching." However, he adds that it will also contain notes on the art of conjecturing, on political economy, military art, chronology, mineralogy, civil architecture, fortification, and the mechanic arts, together with problems and theorems to be demonstrated. These latter were to be included *"to stimulate young geometers."*

As one peruses the pages of this famous journal, he would hardly identify it as an educational journal, but would nevertheless be struck by the frequency with which mediocre and good mathematicians were publishing both short notes and longer articles of a pedagogical nature, especially, of course, dealing with expository and logical problems. For example, in the first issue Gergonne himself published an article on "the identity between the products which resulted from the same factors differently multiplied amongst themselves." This amounts to an attempt to prove commutativity and associativity for the integers without, however, using these words or recognizing these properties as axioms for an abstract system. There is also a discussion of an argument by D'Alembert and others that when an equation has two roots, one positive and one negative, the positive root actually solves the problem in a direct sense and the negative root is only a solution if the statement of the problem is modified in such a manner as to "render additive" that which was subtractive and vice versa.

There is also a proof that all similar triangles are right triangles which involves the use of the cosine law and division by a factor which would have been zero under the conditions of the problem. To this Gergonne has added an editorial note discussing and elaborating the problem of division by zero, the situations in which it arises, and stating that text book writers are most culpable not only if they make such an error but also if they fail to make a point of the difficulty explicitly.

The strength of the French influence upon mathematical education in general and upon American mathematical education in particular is further attested to by a number of additional items. Although the first journal with a major emphasis upon the pedagogy of mathematics appears to be the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* founded by J. C. V. Hoffman of the Gymnasium at Freiberg and published in Leipzig beginning in 1870, France was not far behind. *L'Enseignement Mathématique* was founded in 1899 by C. A. Laisant and H. Fehr. This was the first journal devoted *solely* to the teaching of mathematics and in few years became the official journal of the International Commission on the Teaching of Mathematics which was appointed at the Fourth International Congress of Mathematicians at Rome in 1908. Laisant was also the author of one of the early books on the teaching of mathematics, *La Mathématique, Philosophie-Enseignement* (1898), but I found a book on methodology published in 1896 by Felix Dauge, *Cours de Méthodologie Mathématique*, to be most interesting. Its introduction stated that it was actually a second edition, with a different name, of a book first published in 1883 and used in a *"l'école normale"* which had since been discontinued and replaced by the inclusion of a course in methodology in the program of the doctorate in science and mathematics. Time does not permit a detailed analysis of any of these books, but since I was led to Dauge by a citation in the book by J. W. A. Young, I think it may be regarded as more influential in this country than the others. Dauge began by stressing the importance of the intimate connections among the different branches of mathematics, especially between algebra and geometry. He moved on to stress the importance of being concerned for the objectives of mathematical instruction which he puts under two categories, for applications to the needs of society and for personal careers, and secondly, for a *"moyen d'exercice d'esprit."* Dauge went on to say that actually there are only a few general precepts that can be given with reference to the teaching of any subject, that the major process of helping people to develop into good teachers is to furnish them with examples of good teaching in their classes. However, he did propose the following general rules: (1) Don't present only rigorous reasoning to youth, (2) resist sacrificing rigor to brevity in order to include more topics in the course, (3) teach fundamental principles with care but don't insist on too much abstraction nor digress too far into discussions of truth, being, and existence which will only confuse, not clarify, the mathematical concepts for students. Dauge stated that (4) at the beginning it is not necessary to prove all propositions, that it is possible to postulate those requiring more obscure reasoning or difficult proofs, that making students learn such proofs by heart does not actually make the proposition more clear. However, he stressed that (5) memory is important, that students should learn by heart some basic theorems, not only for the sake of the theorems, but to accustom the students to mathematical language and to the need to express themselves with precision and elegance. He warned against (6) extending memory, however, to the memorization of *proofs* and suggests that teachers should ask questions frequently in order to be sure that the student sees how the parts of proofs fit together. He advises that (7) when a student finds himself stopped on a proof, the professor should restrain himself from immediately pointing out the sequence of ideas; let the student find it for himself, an error corrected may be more profitable than several theorems proved. Other words of advice deal with very mundane affairs such as drawing clear diagrams, but also included gems of wisdom that could still be taken to heart, such as (8) do not lecture too long, the students won't pay attention and will be confused by too many terms. (9) Frame some questions to be sure that the students understand. He goes on to urge that although he had warned against consideration of metaphysical questions with the students, the professor should be informed of and concerned for such. Further, the professor should be an inspiration to the students, should not assign work that's above them, should present applications of the theory that is being developed, should sustain their attention by keeping an eye on them during his explanations, and, without departing from a proper and correct demeanor, should give proof of his good will towards the students, encouraging them with words, even for those who worked and did not succeed and punishing only in grave cases where he is certain it is needed! Above all, the instructor should maintain a spirited and zealous attitude himself.

For those who are close to current new materials in the elementary and secondary school, it is interesting to note that Dauge recommended the study of different systems of numeration in order that students might better understand the numeration system to base 10. However, he had been anticipated in this recommendation by DeMorgan in 1831 and was seconded in it by Laisant. A good share of Dauge's book, as was true of those of Lacroix, Laisant, and DeMorgan, was devoted to detailed consideration of certain typical aspects of subject matter; a favorite topic is the treatment of negative numbers, a topic of which DeMorgan wrote with great firmness. DeMorgan insisted that it was absurd to consider that any number could be less than zero. It is a little difficult, I think, for us to realize that considerable controversy over the nature of negative numbers was carried on through the greater share of the nineteenth century not only with respect to pedagogical problems, but also with respect to philosophical problems as to their very existence.

The book in which Augustus DeMorgan argued against negative numbers was *On the Study and Difficulties of Mathematics* published by the Society for the Diffusion of Useful Knowledge in 1831. This antedates all other methodological books except Lacroix. However, apart from this, the English do not seem to have been as concerned for the pedagogy of mathematics as were French and German mathematicians and educators.

DeMorgan stressed the value of mathematics as mental discipline, and an interesting feature of his essays, though not a surprising one, is his stress on the importance of teaching a unit on logic earlier in the course for secondary school students. He also made a strong plea for abolishing the rote memorization of rules and of proofs. He concluded this plea with the prediction that non-rote memoriter teaching may "excite some to become inquirers who would otherwise have been workers of rules and followers of dogma." He, too, advises teachers against entering into purely metaphysical discussions of the foundations of algebra, but insists that they themselves should be alert to these and devotes some time to the absurdities of such expressions as the negative a, square root of negative a, and zero divided by zero. He ends his introduction with the statement, "The number of mathematics students, increased as it has been of late years, would be much augmented if those who hold the highest rank in science would condescend to give more effective assistance in clearing the elements of difficulties which they present."

DeMorgan does present a certain ambivalence not unknown today when in successive paragraphs he stresses that *understanding* of algebra is a most important objective which is not, however, achieved by reading but by doing, and then goes on to state, "It is a remarkable fact that the first elements of mathematics, a science which demonstrates its results with more certainty than any others, contain difficulties that have been the subject of discussions for centuries." These difficulties, he says, are not such as would suggest themselves to the beginner. The young student sees or fancies he sees the truth of every result which can be stated in a few words, or arrived at by a few simple operations. This arises, he says, from the fact that the young student from his earliest infancy learns no fact from his own observation, deduces no truth by the exercise of his own reason, and thus a habit of examination is not formed, and he is fully prepared to believe in the truth of any rule that is set before him. I hardly need point out that in 1831 DeMorgan was stressing the importance of understanding, of student derivation or "discovery" and thought as opposed to memorization.

6. Mathematical Education in the United States. Having noted some of the highlights of mathematical education in Germany, France, and England, the countries having the greatest influence on early American mathematics, it is time we considered developments in the western hemisphere and the United States. The relative recency of mathematical progress in this country may be brought out by citing a few little-known facts. The first book with mathematical content printed in the Americas was printed in Mexico City in 1556. It was Juan Diez Freyle, *Summario compendioso—de plata y ore*, which had for its major concern the treatment of the value of and conversions between different coinages of different purities of gold and silver ore. How consistent this is with the period of Spanish-American conquest and exploration! Written by a missionary, a member of a religious order, but dealing with the value of gold ore!

It is rather interesting, however, that this first book with mathematical content, aimed at this very practical problem of arithmetic, did also contain some algebra, although there was no apparent very close connection with the rest of the book or its purpose.

I will not elaborate on the other early works, but in the Western Hemisphere prior to 1700 there were 7 Mexican and 4 Peruvian books. These books, though they contained mathematics in their titles and content, dealt with military matters, with navigation, surveying, with the calendar and the determination of fast (feast) days and religious celebrations.

The first book with mathematical content printed in North America was John Hill's *The Young Secretary's Assistant*, which appeared in Boston in 1703. As its title implies, it dealt with how to write a business letter, how to keep books, etc. The secretary to the man who was sending out ships to trade, for instance between the Indies and Europe, had to know a little bit of arithmetic, of weights, measures and coinages and how to translate them into different currencies.

The first mathematics book written by a person in this country was Isaac Greenwood's *Arithmetic, Vulgar and Decimal*, printed in 1729.

The first book with any substantial geometric content was Hawney's *Complete Measurer*, printed in 1801. It dealt largely with simple surveying, weights, measures, cordwood, barrels of wine, etc.

The points are that mathematics was very late in arriving in North America, it was imported, and it stressed practical uses.

Now turning to the early schools, and their programs. The predecessor of our secondary schools was the Latin Grammar School. The Boston Latin Grammar School was established in 1635. It was a private school for a middle class clientele, preparing for the professions, ministry and law, medicine and college.

A little later we have the beginning of the academies. Benjamin Franklin founded one in 1751, but they did not really develop until sometime later, flourishing from 1787 to 1870. They were a sort of a common man's version of the Latin Grammar school. That is, they still were private, but they put a little more stress on preparation for commercial enterprises, business, etc.

About the mathematics curriculum of these grammar schools and academies I would quickly note three things: their purposes, methods of presentation, and content. There were two purposes: *practicality*, the uses of arithmetic and geometry in mensuration, calculating money in different currencies, etc., and "*mental discipline*," the idea was that it's good for your mental muscles to wrestle with Latin and arithmetic.

Their methods of presentation involved the presentation of many rules, straight-forward statements, often in italics, to be memorized, a worked example or so for each, and then problems to be done. This may be illustrated by a quotation from Quackenbos' *Elementary Arithmetic*, copyright in 1821, but used by my father in 1892. Page 78 begins the discussion of fractions as follows: "Halves and halves, thirds and thirds, etc. can be added just as we add pears and pears, dollars and dollars." That was paragraph 130 and there was just a bit more discussion in it of adding fractions with common denominators.

Paragraph 131 on the same small page read, "Halves, thirds, etc. cannot be thus directly added any more than we can add pears and dollars." This was followed by an example of how to add halves and thirds. Following the example came a rule. Following the rule, were "exercises for the slate." And that was it! On two small pages he covered adding fractions with and without common denominators, gave examples and a set of problems.

Although this procedure of stating rules, doing examples and giving exercises was typical, there were, of course, persons trying to do other things. Warren Colburn wrote *Intellectual Arithmetic Upon the Inductive Method of Instruction* in 1821. I took down from my shelf his *Introduction to Algebra Upon the Inductive Method of Instruction*, which in the copy I have is dated 1830. Its preface, in part, read:

> "The first object of the author of the following treatise has been to make the transition from arithmetic too algebra as gradual as possible. The book therefore commences with practical questions in simple equations such as the learner might readily solve without the aid of algebra . . . "
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> "The most simple combinations are given first, then those which are more difficult. The learner is expected to derive most of his knowledge by solving the examples himself; . . . "
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> "In fact, explanations rather embarrass than aid the learner because he is apt to trust too much to the man, and neglect to employ his own power; and because the explanation is not made in the way that would naturally suggest itself to him if he were left to examine the subject by himself. . . . This method besides giving the learner confidence, . . . is much more interesting because he seems to be constantly making new discoveries . . . "
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> "In the ninth article the learner is taught to generalize particular cases and to form rules. . . . The learner should solve every question. When the learner is directed to turn back and to do in a new way, something he has done before, let him not fail to do it, for it will be necessary for his future progress. . . . "

The content of the grammar school curriculum and the nature of beginning college work is further delineated by taking a glance at college entrance requirements. Harvard required arithmetic for admission in 1807 and algebra in 1820. Harvard was followed by Yale in 1847, and by Princeton in 1848. In geometry Yale led all the rest. It required geometry in 1865. Princeton, Michigan, and Cornell had geometry as an entrance requirement in 1868, Harvard added geometry in 1870, but when Harvard did do it they required logarithms as well.

An extract from the code of laws governing the University of Michigan in 1837 shows implicitly both its mathematics curriculum and three important influences upon early mathematics education in this country, that of (1) the French, (2) the United States Military Academy, (3) Charles Davies. The texts in mathematics listed for that year were:

Davies' *Arithmetic*
Davies' *Bourdon's Algebra*
Davies' *Legendre's Geometry*
Davies' *Surveying*
Davies' *Descriptive Geometry*
Bridges' *Conic Sections*
Gregory's *Mathematics for Practical Men*

Davies seems nearly to have made a clean sweep of textbook adoptions at the University of Michigan in 1837—(at this time I believe there were seven students!), and indeed his books were extremely popular and went through many editions. In later editions the names of Bourdon and Legendre were dropped and many people forgot that these were translations from the French, originally for use at the United States Military Academy which also brought over several

French mathematicians as teachers. Charles Davies was chairman of the department there for a number of years. As his textbooks became popular he added to them the first book on the teaching of mathematics published in this country, *The Logic and Utility of Mathematics with the Best Methods of Instruction Explained and Illustrated*, published in New York in 1857. It is interesting that, along with DeMorgan, he stressed the importance of explicit instruction in logic. The introduction to this book read, in part:

> "The following is not a series of speculations. It is but an analysis of that system of mathematical instruction which has been steadily pursued at the Military Academy over a quarter of a century, and which has given to that institution its celebrity as a school of mathematical science."
>
> "It is of the essence of that system that a principle be taught before it is applied to practice, . . . the union of the French and English systems of mathematics. . . . "

7. Foreign Influences on Mathematical Education in the United States. However, I should not overemphasize the French influence upon American mathematical education. In methodology Colburn was influenced by a Swiss educator, Pestalozzi, although the French also played a part in the spread of inductive methods. As recently as 1920 the title page of an arithmetic book of which John Dewey was listed as an author bore the following quotation from Laisant, who was a Belgian:

> "The problem is always the same: to interest the pupil, to induce research, to give him the notion continually, the illusion if you please, that he is discovering for himself, that which is being taught him."

Further, although all American mathematicians know something of Felix Klein, many may not fully appreciate his rather direct effect upon American mathematics. I refer to the effect upon graduate study and research of his presence in this country for the Congress of 1893 and the colloquium at Evanston which followed, and also the publication in this country of his pedagogical volumes, *Elementary Mathematics from an Advanced Standpoint* in a translation by one of the Association's founders, Earle Raymond Hedrick. Finally, in this later period we need to consider the effects in this country of the reform movement in Britain. In particular, E. H. Moore, who spoke out strongly for reform in mathematical education in his American Mathematical Society retiring presidential address in 1902, was greatly influenced by John Perry. For the background of Perry's views I quote from a recent pamphlet from the British Ministry of Education, *Teaching Mathematics in the Secondary Schools*, which states, "Mathematics hardly existed in schools until the time of Queen Victoria, whose reign began in 1837."

In 1861 The Royal Commission on Nine Public Schools said, "The chief honors and prizes in our public schools go for the Classics which are to the great number of the boys, intrinsically more attractive than Mathematics . . . but mathematics, at least, has established a title of respect as an instrument of mental discipline." Another quotation from this same group refers to a situation which has been much more significant in England than in this country, but it may become increasingly significant in this country. This is the relationship between the secondary schools and the college entrance examinations, or as they are called in this report, "the external examinations." The quotation is, "The introduction of mathematics as an integral part of the school curriculum coincided with the establishment of external examinations. This circumstance tended to set a pattern of mathematics education before much thought had been given to the matter." The curriculum in England at that time included algebra, geometry, surveying, navigation, mensuration and trigonometry.

Following this, toward the end of the 19th century, student failures in mathematics and Latin increased and colleges complained of the poor preparation of their students.

These quotations and facts point out the following "morals": (1) College criticisms have motivated establishing and changing external examinations which in turn have very significant effects on the school curriculum. (2) The changing nature of the school population has also been a factor in the design of school programs. (3) Practical and intellectual objectives have always had a place in our school programs, but have varied in their relative importance.

John Perry spoke in 1901 to the British Association for the Advancement of Science in England, and out of his speech came a committee that reported a year later. Its report did make significant changes in the examinations and syllabus for geometry in England in the direction of allowing more flexibility in answers rather than reproduction of Euclid. I'm not going to go into that, but among Perry's own emphases were: the use of the laboratory method of teaching, stress on the correlation between mathematics and science, acceptance of some theorems without proof, and discussion of practical uses.

I cannot resist the temptation to read to you a quotation from this discussion in 1901. You will be interested that among persons who contributed to this discussion, either written comments or through their own presence, were Mrs. Boole, Miss Scott from Bryn Mawr, Horace Lamb, David Eugene Smith, Lord Kelvin, and Oliver Heaviside. Here is the quotation from Heaviside:

> "Boys are not philosophers and logicians. . . . Now the prevalent idea of mathematical works is that you must understand the reason first before you practice. This is fuss and fiddlesticks. . . . I know mathematical processes that I have used with success for a very long time of which neither I nor anyone else understands the scholastic logic. . . . There is something wanting, no matter how logical people may pretend. . . . Geometry should be entirely observational and experimental at first. . . . "

I hardly need note that this attitude has its modern advocates.

8. More Recent Developments in the United States. J. W. A. Young called it a remarkable coincidence that a marked impetus toward reform was given in two countries, in two successive years, in addresses before major scientific bodies, Perry in 1901, and E. H. Moore in 1902. Moore called for more "fusion" or "correlation" between mathematics and other subjects, especially science, and for more concern for concrete and developmental work in teaching in the secondary schools.

In addition to these events and the "1923 report" [8] noted earlier there were other landmarks in the progress of mathematical education in the United States in the twentieth century. Some of these were: 1894—the Committee on College Entrance Requirements; 1911–1918, the International Commission on the Teaching of Mathematics. The almost tumultuous progress of the past decade was anticipated by the activities of the Commission on Post-War Plans of the National Council of Teachers of Mathematics which operated from 1944–1947. Recent vigorous activity was then touched off in 1952, prior to Sputnik, by the publication of a report *General Education in School and College* by a committee from three eastern colleges and nearby preparatory schools, and the founding of the University of Illinois Committee on School Mathematics. Although the current N.S.F. institutes were anticipated by summer programs for secondary school teachers at Duke University, 1953 marked the beginning of National Science Foundation support with the Summer Conference on Collegiate Mathematics held at the University of Colorado.

However, I have chosen in this talk to slight these developments of the twentieth century and the last decade, because they are familiar to many, and the literature on them is adequate and reasonably available [9]. I have preferred to use my time to present a longer range point of view. But, since this is the semi-centennial celebration of the Mathematical Association of America, and since some of its charter members present here today were active in the second of two projects it sponsored which were of great significance for mathematical education, I believe a bit more should be said of these two reports.

In 1923 the National Committee on Mathematical Requirements of the Mathematical Association of America, which was appointed by E. R. Hedrick in 1916, brought in its famous report after seven years of work. It classified the objectives of mathematical instruction as practical, disciplinary and cultural, and stressed the role of the function concept in the teaching of mathematics. Most texts for years thereafter cited this report in their prefaces and honored the then current interpretation of the "function concept" by stressing the interrelationships between tables, graphs, formulas, and equations. Its stress on the function concept and general mathematics reflected the influences of Felix Klein and E. H. Moore respectively. The representatives of the Association on this Committee were: J. W. Young, A. R. Crathorne, C. N. Moore, E. H. Moore, D. E. Smith, H. W. Tyler. The other seven members of the committee represented secondary school teachers and organizations.

Two groups reported in 1940. One had been appointed by the Progressive Education Association, and the other was the Joint Committee of the Mathematical Association of America and the National Council of Teachers of Mathematics appointed to study the place of mathematics in secondary education.

The first group expounded a long range philosophy. They stressed the importance of teaching problem solving and the nature of mathematical structure and wrote about data, approximation, function, operation, proof, and symbolism.

This report suffered some in its influence for being associated with the Progressive Education Association, I believe. There were many very good and rather modern ideas in the report.

The Joint Committee had the same number of representatives from the Association as from the National Council. The Association representatives were: K. P. Williams, A. A. Bennett, H. E. Buchanan, F. L. Griffin, C. A. Hutchinson, H. F. MacNiesh and U. G. Mitchell.

The Joint Committee stressed individual differences. It recommended a two or more "track" curriculum. This was again a product of the time, the condition of the schools, and the condition of mathematics in the schools. They also wrote about transfer of training. The notion of transfer of training was for a time substantially discredited by psychologists. It is now beginning to be regarded again as an important educational element but with several theories, some new, to describe and explain it and the conditions under which it takes place.

In conclusion, I have omitted much: (1) the names of many writers whose accounts I have read, (2) expositions of the reports of some famous committees which I listed and others which I did not list, (3) an elaboration of the role of the Mathematical Association of America in mathematical education.

I hope that my recurring reference to recurring themes was not too boring and, more important, not misunderstood. I do *not* mean to imply that there is nothing new in mathematical education (although it is a pity how often truths have to be rediscovered before they can be extended and applied). I *do* mean that looking at how fundamental ideas and perceptions have changed may shed light on the future and suggest what are the real invariants in good mathematical education.

For example Euclid's geometry and modern Euclidean geometry both are based on axioms and definitions, *but* the modern view of the nature and source of these and the connections between them and the physical world is significantly different from Euclid's. I think Newton may have had some concept of the relationship between the physical world and a mathematical model, *but* I don't believe that his view was our view. I believe that students may be brought to understand axiomatics and mathematical models better if they are shown something of their growth and the changing nature of our perceptions of them.

Similarly, today mathematical education is faced with many important problems which are not essentially new, but which, likewise, are not merely the same old ones out of which they developed. Some of these are:

(1) What are the *objectives* of mathematical instruction? We need to recognize that they are not the same for all and that our school population includes *all.*

(2) What is the *role of applications* and *physical models* in both clarifying and motivating instruction?

(3) What is the *level of rigor* which is sound and desirable at different stages of the student's development?

(4) How do we *teach the excitement and beauty* of mathematics, as well as its theorems?

(5) Can we teach students to *discover* or to be more creative?

(6) What are *the relative roles of intuition, induction and deduction* and how do we communicate them?

I hope a historical survey of these continuing themes may direct and stimulate your thinking and suggest that there will always be a continuing problem of cooperating to adapt our curricula and teaching procedures to the changing philosophies and definitions of mathematics itself, as well as to the changing needs of our culture—both intellectual and practical, and to our uncertain but growing knowledge of the psychology of learning, applying, and extending knowledge.

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