***Lecture 20.***
*Thursday $10^{th}$ December, 2020.*

## 0.1 Linear-phase FIR transfer functions

It is impossible to design an IIR transfer function with an exact linear-phase. However, it is always possible to design a FIR transfer function with an exact linear-phase response. We now develop the forms of the linear-phase FIR transfer function $H(z)$ with real impulse response $h[n]$. Let:

$$H(z) = \sum_{n=0}^{N} h[n]z^{-n} \tag{1}$$

If $H(z)$ is to have a linear-phase, its frequency response must be of the form:

$$H(e^{j\omega}) = e^{j(c\omega+\beta)}\breve{H}(\omega) \tag{2}$$

where $c$ and $\beta$ are constants, and $\breve{H}(\omega)$, called the amplitude response, also called the zero-phase response, is a real function of $\omega$. For a real impulse response, the magnitude response $\left|H(e^{j\omega})\right|$ is an even function of $\omega$, i.e.:

$$\left|H(e^{j\omega})\right| = \left|H(e^{-j\omega})\right| \tag{3}$$

Since $\left|H(e^{j\omega})\right| = \left|\breve{H}(\omega)\right|$, the amplitude response is then either an even function or an odd function of $\omega$, i.e.:

$$\breve{H}(-\omega) = \pm\breve{H}(\omega) \tag{4}$$

The frequency response satisfies the relation:

$$H(e^{j\omega}) = H^*(e^{-j\omega}) \tag{5}$$

or, equivalently, the relation:

$$e^{j(c\omega+\beta)}\breve{H}(\omega) = e^{-j(c(-\omega)+\beta)}\breve{H}(\omega) \tag{6}$$

If $\breve{H}(\omega)$ is an even function, then the above relation leads to:

$$e^{j\beta} = e^{-j\beta} \tag{7}$$

implying that either $\beta = 0$ or $\beta = \pi$. From:

$$H(e^{j\omega}) = e^{j(c\omega+\beta)}\breve{H}(\omega) \tag{8}$$

we have:

$$\breve{H}(\omega) = e^{-j(c\omega+\beta)}H(e^{j\omega}) \tag{9}$$

Substituting the value of $\beta$ in Eq. 9, we get:

$$\breve{H}(\omega) = \pm e^{-jc\omega}H(e^{j\omega}) = \pm\sum_{n=0}^{N} h[n]e^{-j\omega(c+n)} \tag{10}$$

Replacing $\omega$ with $-\omega$ in Eq. 10, we get:

$$\breve{H}(-\omega) = \pm \sum_{\ell=0}^{N} h[\ell] e^{j\omega(c+\ell)} \tag{11}$$

Making a change of variable $\ell = N - n$, we rewrite Eq. 11 as:

$$\breve{H}(-\omega) = \pm \sum_{n=0}^{N} h[N-n] e^{j\omega(c+N-n)} \tag{12}$$

Now, as $\breve{H}(\omega) = \breve{H}(-\omega)$, we have:

$$h[n] e^{-j\omega(c+n)} = h[N-n] e^{j\omega(c+N-n)} \tag{13}$$

Eq. 13 leads to the condition:

$$h[n] = h[N-n] \qquad 0 \le n \le N \tag{14}$$

with $c = -\frac{N}{2}$. Thus, the FIR filter with an even amplitude response will have a linear phase if it has a symmetric impulse response. If $\breve{H}(\omega)$ is an odd function of $\omega$, then from:

$$e^{j(c\omega+\beta)} \breve{H}(\omega) = e^{-j(-c\omega+\beta)} \breve{H}(-\omega) \tag{15}$$

we get $e^{j\beta} = -e^{-j\beta}$ as $\breve{H}(-\omega) = -\breve{H}(\omega)$. Eq. 15 is satisfied if $\beta = \pm\frac{\pi}{2}$. Then $H(e^{j\omega}) = e^{j(c\omega+\beta)}$ reduces to:

$$H(e^{j\omega}) = j e^{jc\omega} \breve{H}(\omega) \tag{16}$$

Eq. 16 can be rewritten as:

$$\breve{H}(\omega) = -j e^{-jc\omega} H(e^{j\omega}) = -j \sum_{m=0}^{N} h[n] e^{-j\omega(c+n)} \tag{17}$$

Again, as $\breve{H}(\omega) = \breve{H}(-\omega)$, from Eq. 17 we get:

$$\breve{H}(-\omega) = j \sum_{\ell=0}^{N} h[\ell] e^{j\omega(c+\ell)} \tag{18}$$

Making a change of variable $\ell = N - n$, we rewrite Eq. 18 as:

$$\breve{H}(-\omega) = j \sum_{\ell=0}^{N} h[\ell] e^{j\omega(c+\ell)} \tag{19}$$

Equating the RHS of Eq. 19 with the RHS of Eq. 17, we arrive at the condition for linear phase as:

$$h[n] = h[N-n] \qquad 0 \le n \le N \tag{20}$$

with $c = -\frac{N}{2}$. Therefore, a FIR filter with an odd amplitude response will have linear-phase response if it has an antisymmetric impulse response.
Since the length of the impulse response can be either even or odd, we can define four types of linear-phase FIR transfer functions. In particular, for an antisymmetric FIR filter of odd length, namely $N$ even, $h\left[\frac{N}{2}\right] = 0$. We examine in the following discussion each of the four cases, sketched in Figure 1.

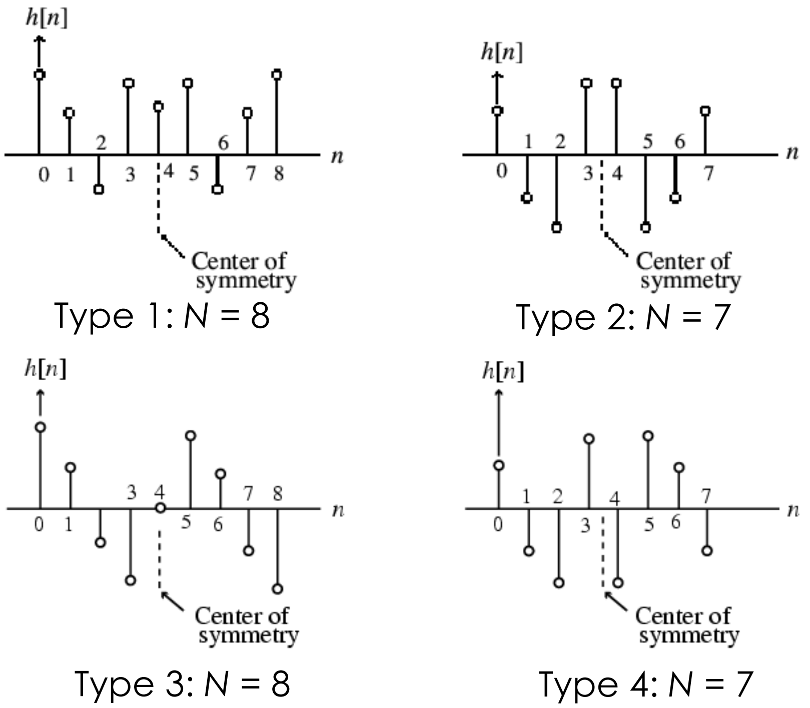



**Figure 1:** The four types of linear-phase FIR transfer functions.

### 0.1.1 Symmetric impulse response with odd length

In this case, the degree $N$ is even. In the following discussion we assume also $N = 8$ for simplicity. Therefore, the transfer function $H(z)$ is given by:

$$H(z) = h[0]+h[1]z^{-1}+h[2]z^{-2}+h[3]z^{-3}+h[4]z^{-4}+h[5]z^{-5}+h[6]z^{6}+h[7]z^{-7}+h[8]z^{-8} \tag{21}$$

Because of symmetry, we have:

$$h[0] = h[8] \tag{22}$$
$$h[1] = h[7] \tag{23}$$
$$h[2] = h[6] \tag{24}$$
$$h[3] = h[5] \tag{25}$$

Thus, we can write:

$$\begin{aligned} H(z) &= h[0](1+z^{-8}) + h[1](z^{-1}+z^{-7}) + h[2](z^{-2}+z^{-6}) + h[3](z^{-3}+z^{-5}) + h[4]z^{-4} \\ &= z^{-4}\big\{h[0](z^{4}+z^{-4}) + h[1](z^{3}+z^{-3}) + h[2](z^{2}+z^{-2}) + h[3](z+z^{-1}) + h[4]\big\} \end{aligned} \tag{26}$$

The corresponding frequency response is then given by:

$$H(e^{j\omega}) = e^{-j4\omega}\{2h[0]\cos(4\omega) + 2h[1]\cos(3\omega) + 2h[2]\cos(2\omega) + 2h[3]\cos(\omega) + h[4]\} \tag{27}$$

The quantity inside the braces is a real function of $\omega$ and can assume positive or negative values in the range $0 \le |\omega| \le \pi$. The phase function is given by:

$$\theta(\omega) = -4\omega + \beta \tag{28}$$

where $\beta$ is either 0 or $\pi$, and hence, it is a linear function of $\omega$. The group delay is given by:

$$\tau(\omega) = -\frac{\mathrm{d}\theta(\omega)}{\mathrm{d}\omega} = 4 \tag{29}$$

indicating a constant group delay of 4 samples.
In the general case for Type 1 FIR filters, the frequency response is of the form:

$$H(e^{j\omega}) = e^{-jN\frac{\omega}{2}}\breve{H}(\omega) \tag{30}$$

where the amplitude response $\breve{H}(\omega)$, also called the zero-phase response, is of the form:

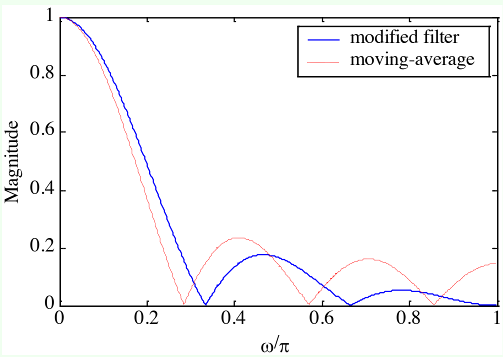
$$\breve{H}(\omega) = h\left[\frac{N}{2}\right] + 2\sum_{n=1}^{\frac{N}{2}} h\left[\frac{N}{2} - n\right]\cos(\omega n) \tag{31}$$

**Example 1: Symmetric impulse response with odd length**

We consider:

$$H_0(z) = \frac{1}{6}\left[\frac{1}{2} + z^{-1} + z^{-3} + z^{-4} + z^{-5} + \frac{1}{2}z^{-6}\right] \tag{32}$$

which is seen to be a slightly modified version of a length-7 moving-average FIR filter.
This transfer function has a symmetriic impulse response and therefore a linear phase response. A plot of the magnitude response of $H_0(z)$ along with that of the 7-point moving-average filter is showed below.




Note the improved magnitude response obtained by simply changing the first and the last impulse response coefficients of a moving-average (MA) filter. It can be showed that we can express:

$$H_0(z) = \frac{1}{2}(1 + z^{-1}) \cdot \frac{1}{6}(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}) \tag{33}$$

which is seen to be a cascade of a 2-point MA filter with a 6-point MA filter. Thus, $H_0(z)$ has a double zero at $z = -1$, i.e. $\omega = \pi$.

### 0.1.2 Symmetric impulse response with even length

In this case, the degree $N$ is odd and we assume $N = 7$ for simplicity for the following discussion. Therefore, the transfer function is of the form:

$$H(z) = h[0]+h[1]z^{-1}+h[2]z^{-2}+h[3]z^{-3}+h[4]z^{-4}+h[5]z^{-5}+h[6]z^{6}+h[7]z^{-7} \tag{34}$$

Making use of the symmetry of the impulse response coefficients, the transfer function can be rewritten as:

$$\begin{aligned} H(z) &= h[0](1+z^{-7}) + h[1](z^{-1}+z^{-6}) + h[2](z^{-2}+z^{-5}) + h[3](z^{-3}+z^{-4}) \\ &= z^{-\frac{7}{2}}\left\{h[0](z^{\frac{7}{2}}+z^{-\frac{7}{2}}) + h[1](z^{\frac{5}{2}}+z^{-\frac{5}{2}}) + h[2](z^{\frac{3}{2}}+z^{-\frac{3}{2}}) + h[3](z^{\frac{1}{2}}+z^{-\frac{1}{2}})\right\} \end{aligned} \tag{35}$$

The corresponding frequency response is given by:

$$H(e^{j\omega}) = e^{-j\frac{7\omega}{2}}\left\{2h[0]\cos\left(\frac{7\omega}{2}\right) + 2h[1]\cos\left(\frac{5\omega}{2}\right) + 2h[2]\cos\left(\frac{3\omega}{2}\right) + 2h[3]\cos\left(\frac{\omega}{2}\right)\right\} \tag{36}$$

As before, the quantity inside the braces is a real function of $\omega$ and can assume positive or negative values in the range $0 \leq |\omega| \leq \pi$. Here, the phase function is given by:

$$\theta(\omega) = -\frac{7}{2}\omega + \beta \tag{37}$$

where $\beta$ is either 0 or $\pi$. A a result, the phase is also a linear function of $\omega$ and the corresponding group delay is:

$$\tau(\omega) = \frac{7}{2} \tag{38}$$

indicating a group delay of $\frac{7}{2}$ samples.
The expression for the frequency response in the general case for Type 2 FIR filters is of the form:

$$H(e^{j\omega}) = e^{-jN\frac{\omega}{2}}\breve{H}(\omega) \tag{39}$$

where the amplitude response is given by:

$$\breve{H}(\omega) = 2\sum_{n=1}^{\frac{N+1}{2}} h\left[\frac{N+1}{2} - n\right]\cos\left(\omega\left(n - \frac{1}{2}\right)\right) \tag{40}$$

### 0.1.3 Antisymmetric impulse response with odd length

In this case, the degree $N$ is even and we assume $N = 8$ for simplicity for the following discussion. Therefore, applying the symmetry condition we get:

$$H(z) = z^{-4}\left\{h[0](z^{4}-z^{-4}) + h[1](z^{3}-z^{-3}) + h[2](z^{2}-z^{-2}) + h[3](z-z^{-1})\right\} \tag{41}$$

The corresponding frequency response is given by:

$$H(e^{j\omega}) = e^{-j4\omega}e^{j\frac{\pi}{2}}\left\{2h[0]\sin(4\omega) + 2h[1]\sin(3\omega) + 2h[2]\sin(2\omega) + 2h[3]\sin(\omega)\right\} \tag{42}$$

It also exhibits a linear phase response given by:

$$\theta(\omega) = -4\omega + \frac{\pi}{2} + \beta \tag{43}$$

where $\beta$ is either 0 or $\pi$. The group delay here is:

$$\tau(\omega) = 4 \tag{44}$$

indicating a constant group delay of 4 samples.
The expression for the frequency response in the general case for Type 3 FIR filters is of the form:

$$H(e^{j\omega}) = e^{-jN\frac{\omega}{2}}\breve{H}(\omega) \tag{45}$$

where the amplitude response is given by:

$$\breve{H}(\omega) = 2\sum_{n=1}^{\frac{N}{2}} h\left[\frac{N}{2} - n\right]\sin(\omega n) \tag{46}$$

### 0.1.4 Antisymmetric impulse response with even length

In this case, the degree $N$ is even and we assume $N = 7$ for simplicity for the following discussion. Therefore, applying the symmetry condition we get:

$$H(z) = z^{\frac{7}{2}}\left\{h[0](z^{\frac{7}{2}} - z^{-\frac{7}{2}}) + h[1](z^{\frac{5}{2}} - z^{-\frac{5}{2}}) + h[2](z^{\frac{3}{2}} - z^{-\frac{3}{2}}) + h[3](z^{\frac{1}{2}} - z^{-\frac{1}{2}})\right\} \tag{47}$$

The corresponding frequency response is given by:

$$H(e^{j\omega}) = e^{-j\frac{7\omega}{2}}e^{j\frac{\pi}{2}}\left\{2h[0]\sin\left(\frac{7\omega}{2}\right) + 2h[1]\sin\left(\frac{5\omega}{2}\right) + 2h[2]\sin\left(\frac{3\omega}{2}\right) + 2h[3]\sin\left(\frac{\omega}{2}\right)\right\} \tag{48}$$

It again exhibits a linear phase response given by:

$$\theta(\omega) = -\frac{7}{2}\omega + \frac{\pi}{2} + \beta \tag{49}$$

where $\beta$ is either 0 or $\pi$. The group delay is constant and is given by:

$$\tau(\omega) = \frac{7}{2} \tag{50}$$

The expression for the frequency response in the general case for Type 4 FIR filters is of the form:

$$H(e^{j\omega}) = e^{-jN\frac{\omega}{2}}\breve{H}(\omega) \tag{51}$$

where the amplitude response is given by:

$$\breve{H}(\omega) = 2\sum_{n=1}^{\frac{N+1}{2}} h\left[\frac{N+1}{2} - n\right]\sin\left(\omega\left(n - \frac{1}{2}\right)\right) \tag{52}$$

### 0.1.5 General form of frequency response

In each of the four types of linear-phase FIR filters, the frequency response is of the form:

$$H(e^{j\omega}) = e^{-jN\frac{\omega}{2}}e^{j\beta}\breve{H}(\omega) \tag{53}$$

The amplitude response $\breve{H}(\omega)$ for each type can become negative over certain frequency ranges, typically in the stopband.

**Example 2: General form of frequency response**

We consider the causal Type 1 FIR transfer function:

$$H_1(z) = -1 + 2z^{-1} - 3z^{-2} + 6z^{-3} - 3z^{-4} + 2z^{-5} - z^{-6} \tag{54}$$

Its amplitude and phase responses are given by:

$$\breve{H}_1(\omega) = 6 - 6\cos(\omega) + 4\cos(2\omega) - 2\cos(3\omega) \tag{55}$$

$$\theta_1(\omega) = -3\omega \tag{56}$$

Next, we consider the causal Type 1 FIR transfer function:

$$H_2(z) = 1 - 2z^{-1} + 3z^{-2} - 6z^{-3} + 3z^{-4} - 2z^{-5} + z^{-6} \tag{57}$$

Its amplitude and phase responses are given by:

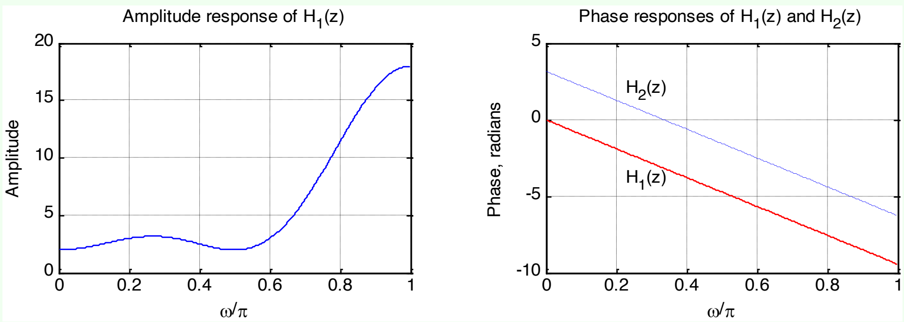
$$\breve{H}_2(\omega) = -\breve{H}_1(\omega) \tag{58}$$

$$\theta_2(\omega) = -3\omega + \pi \tag{59}$$

Note that:

$$\left|H_1(e^{j\omega})\right| = \left|H_2(e^{j\omega})\right| \tag{60}$$

Hence, $H_1(z)$ and $H_2(z)$ have identical magnitude responses but phase responses differing by $\pi$, as showed in the figure below.




**Example 3: General form of frequency response**

We consider the causal Type 1 FIR transfer function:

$$H_3(z) = 1 - 2z^{-1} + 3z^{-2} - 3z^{-4} + 2z^{-5} - z^{-6} \tag{61}$$

Its amplitude and phase responses are given by:

$$\breve{H}_3(\omega) = -6\sin(\omega) + 4\sin(2\omega) + 2\sin(3\omega) \tag{62}$$

$$\theta_3(\omega) = -3\omega + \frac{\pi}{2} \tag{63}$$

Next, we consider the causal Type 1 FIR transfer function:

$$H_4(z) = -1 + 2z^{-1} - 3z^{-2} + 3z^{-4} - 2z^{-5} + z^{-6} \tag{64}$$

Its amplitude and phase responses are given by:

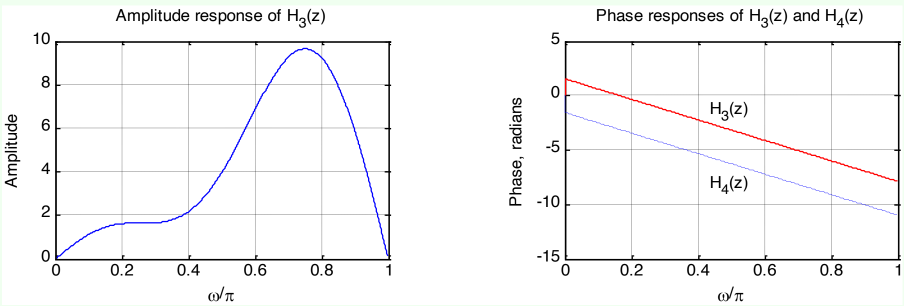
$$\breve{H}_4(\omega) = -\breve{H}_3(\omega) \tag{65}$$

$$\theta_4(\omega) = -3\omega - \frac{\pi}{2} \tag{66}$$

Note that:

$$\left|H_3(e^{j\omega})\right| = \left|H_4(e^{j\omega})\right| \tag{67}$$

Hence, $H_3(z)$ and $H_4(z)$ have identical magnitude responses but phase responses differing by $\pi$, as showed in the figure below.




Now, in general, the magnitude and phase responses of the linear-phase FIR are given by:

$$\left|H(e^{j\omega})\right| = \left|\breve{H}(\omega)\right| \tag{68}$$

$$\theta(\omega) = \begin{cases} -\frac{N\omega}{2} + \beta & \breve{H}(\omega) \geq 0 \\ -\frac{N\omega}{2} + \beta - \pi & \breve{H}(\omega) < 0 \end{cases} \tag{69}$$

The group delay in each case is:

$$\tau(\omega) = \frac{N}{2} \tag{70}$$

Note that, even though the group delay is constant, since in general $\left|H(e^{j\omega})\right|$ is not a constant, the output waveform is not a replica of the input waveform.
A FIR filter with a frequency response that is a real function of $\omega$ is often called a zero-phase filter. Such a filter must have a noncausal impulse response: a zero-phase filter needs to have a purely real-valued frequency response, and, consequently, it must have an impulse response that is even with respect to the time index $n = 0$, i.e., it is non-causal.

### 0.1.6 Zero locations

Let us consider first a FIR filter with a symmetric impulse response:

$$h[n] = h[N - n] \tag{71}$$

Its transfer function can be written as:

$$H(z) = \sum_{n=0}^{N} h[n]z^{-n} = \sum_{n=0}^{N} h[N-n]z^{-n} \tag{72}$$

By making a change of variable $m = N - n$, we can write:

$$H(z) = \sum_{n=0}^{N} h[N-n]z^{-n} = \sum_{m=0}^{N} h[m]z^{-N+m} = z^{-N} \underbrace{\sum_{m=0}^{N} h[m]z^{m}}_{H(z^{-1})} \tag{73}$$

Hence, for a FIR filter with a symmetric impulse response of length $N+1$ we have:

$$H(z) = z^{-N}H(z^{-1}) \tag{74}$$

A real-coefficient polynomial $H(z)$ satisfying the aboce condition is called a mirror-image polynomial (MIP).
Now, let us consider first an FIR filter with an antisymmetric impulse response:

$$h[n] = -h[N-n] \tag{75}$$

Its transfer function can be written as:

$$H(z) = \sum_{n=0}^{N} h[n]z^{-n} = -\sum_{n=0}^{N} h[N-n]z^{-n} \tag{76}$$

By making a change of variable $m = N - n$, we can write:

$$H(z) = -\sum_{n=0}^{N} h[N-n]z^{-n} = -\sum_{m=0}^{N} h[m]z^{-N+m} = -z^{-N}H(z^{-1}) \tag{77}$$

Hence, the transfer function $H(z)$ of an FIR filter with an antisymmetric impulse response satisfies the condition:

$$H(z) = z^{-N}H(z^{-1}) \tag{78}$$

A real-coefficient polynomial $H(z)$ satisying the above condition is called antimirror-image polynomial (AIP).
Now, it follows from the relation $H(z) = \pm z^{-N}H(z^{-1})$ that if $z = \xi_0$ is a zero of $H(z)$, so is $z = \frac{1}{\xi_0}$. Moreover, for an FIR filter with a real impulse response, the zeros of $H(z)$ occur in complex conjugate pairs. Hence, a zero at $z = \xi_0$ is associated with a zero at $z = \xi_0^*$. Thus, a complex zero that is not on the unit circle is associated with a set of 4 zeros given by:

$$z = re^{\pm j\varphi}, \quad \frac{1}{r}e^{\pm j\varphi} \tag{79}$$

A zero on the unit circle appear as a pair:

$$z = e^{\pm j\varphi} \tag{80}$$

as its reciprocal is also its complex conjugate. Since a zero at $z = \pm 1$ is its own reciprocal, it can appear only singly.
Now, a Type 2 FIR filter satisfies:

$$H(z) = z^{-N}H(z^{-1}) \tag{81}$$

with degree $N$ odd. Hence, $H(-1) = (-1)^{-N}H(-1) = -H(-1)$, implying $H(-1) = 0$, i.e., $H(z)$ must have a zero at $z = -1$.
Likewise, a Type 3 or 4 FIR filter satisfies:

$$H(z) = -z^{-N}H(z^{-1}) \tag{82}$$

Thus:

$$H(1) = (-1)^{-N}H(1) = -H(1) \tag{83}$$

implying that $H(z)$ must have a zero at $z = 1$. On the other hand, only the Type 3 FIR filter is restricterd to have a zero at $z = -1$ since here the degree $N$ is even and hence:

$$H(-1) = -(-1)^{-N}H(-1) = -H(-1) \tag{84}$$

Typical zero locations are showed in Figure 2.

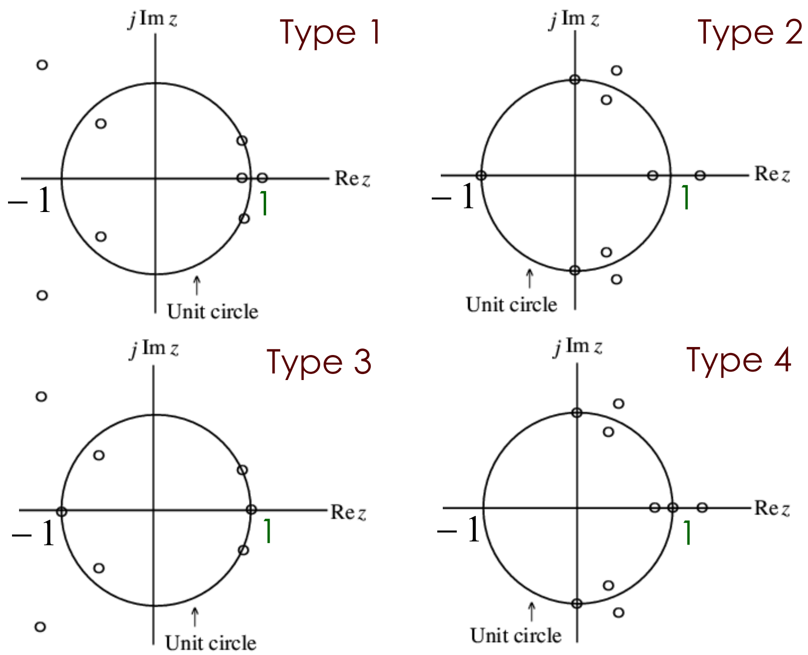



**Figure 2:** Typical zero locations.

So, to summarize:

- Type 1 FIR filter: either an even number or no zeros at $z = 1$ and $z = -1$;
- Type 2 FIR filter: either an even number or no zeros at $z = 1$ and an odd number of zeros at $z = -1$;
- Type 3 FIR filter: an odd number of zeros at $z = 1$ and $z = -1$;
- Type 4 FIR filter: an odd number of zeros at $z = 1$ and either an even number or no zeros at $z = -1$.

The presence of zeros at $z = \pm 1$ leads to the following limitations on the use of these linear-phase transfer functions for designing frequency-selective filters:

- a Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero;

- a Type 3 FIR filter has zeros at both $z = 1$ and $z = -1$, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter;
- a Type 4 FIR filter is not appropriate to design lowpass and bandstop filters due to the presence of a zero at $z = 1$;
- a Type 1 FIR filter has no such restrictions and can be used to design almost any type of filter.