Lecture 12.
Thursday 5th November, 2020.

0.1 Discrete Fourier Transform

We have discussed the DTFT for a discrete-time function given by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{1}$$

and the IDTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega \tag{2}$$

The pair and their properties and applications have some limitations. The input signal is usually aperiodic and may be finite in length.

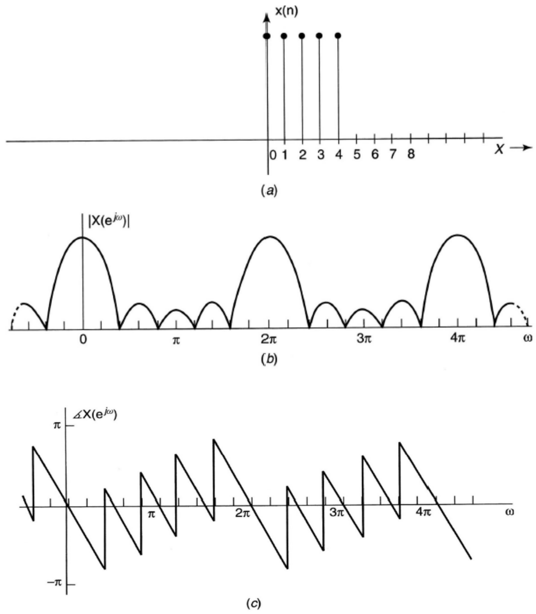

Figure 1: In order from top to bottom, a finite-length signal, its magnitude spectrum, its phase spectrum.

Moreover, we often do not have an infinite amount of data which is required by DTFT. For example in a computer we cannot calculate uncountable infinite (continuum) of frequencies as required by DTFT. Thus, we use DTF to look at finite segment of data. We only observe the data through a window:

$$x_0[n] = x[n]w_R[n] \tag{3}$$

$$w_R[n] = \begin{cases} 1 & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

In this case, the $x_0[n]$ is just a sampled data between $n = 0$, $n = N-1$ (so, N points). The solution to our problems is given by the Discrete Fourier Transform (DFT).

Definition 1: Discrete Fourier Transform (DFT)

The simplest relation between a length-N sequence $x[n]$, defined for $0 \leq n \leq N-1$, and its DTFT $X(e^{j\omega})$ is obtained by uniformly sampling on the ω-axis between $0 \le \omega \le 2\pi$ at $\omega_k = \frac{2\pi k}{N}$, for $0 \le k \le N-1$. From the definition of the DTFT we thus have:

$$X[k] = \left[X(e^{j\omega})\right]_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi k \frac{n}{N}} \tag{5}$$

Note that $X[k]$ is also a length-N sequence in the frequency domain and it is called the Discrete Fourier Transform (DFT) of the sequence $x[n]$. Using the notation $W_N = e^{-j\frac{2\pi}{N}}$, the DFT is usually expressed as:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad 0 \le k \le N-1 \tag{6}$$

Definition 2: Inverse Discrete Fourier Transform (IDFT)

The Inverse Discrete Fourier Transform (IDFT) is given by:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad 0 \le n \le N-1 \tag{7}$$

To verify the above expression we multiply both sides of the above equation by $W_N^{\ell n}$ and sum the result from $n = 0$ to $n = N-1$, resulting in:

$$\begin{aligned}
\sum_{n=0}^{N-1} x[n] W_N^{\ell n} &= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right) W_N^{\ell n} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} X[k] W_N^{-(k-\ell)n} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} X[k] W_N^{-(k-\ell)n}
\end{aligned} \tag{8}$$

Making use of the identity:

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} = \begin{cases} N & k-\ell = rN,\ n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

we observe that the right-hand-side of the last equation is equal to $X[\ell]$. Hence:

$$\sum_{n=0}^{N-1} x[n] W_N^{\ell n} = X[\ell] \tag{10}$$

Example 1: Discrete Fourier Transform

Consider the length-N sequence:

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & 1 \le n \le N-1 \end{cases} \tag{11}$$

Its N-point DFT is given by:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = x[0] W_N^0 = 1 \tag{12}$$

with $0 \leq k \leq N-1$.

Example 2: Discrete Fourier Transform

Consider the length-N sequence:

$$y[n] = \begin{cases} 1 & n = m \\ 0 & 0 \leq n \leq m-1,\ m+1 \leq n \leq N-1 \end{cases} \tag{13}$$

Its N-point DFT is given by:

$$Y[k] = \sum_{n=0}^{N-1} y[n] W_N^{kn} = y[m] W_N^{km} = W_N^{km} \tag{14}$$

with $0 \leq k \leq N-1$.

Example 3: Discrete Fourier Transform

Consider the length-N sequence defined for $0 \leq n \leq N-1$:

$$g[n] = \cos\left(\frac{2\pi r n}{N}\right), \qquad 0 \leq r \leq N-1 \tag{15}$$

Using trigonometic identities, we can rewrite:

$$g[n] = \frac{1}{2}\left(e^{j2\pi r \frac{n}{M}} + e^{-j2\pi r \frac{n}{N}}\right) = \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right) \tag{16}$$

The N-point DFT of $g[n]$ is thus given by:

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn} = \frac{1}{2}\left(\sum_{n=0}^{N-1} W_N^{-(r-k)n} + \sum_{n=0}^{N-1} W_N^{(r+k)n}\right) \tag{17}$$

with $0 \leq k \leq N-1$. Making use of the identity:

$$\sum_{n=0}^{N-1} W_N ?-(k-\ell)n = \begin{cases} N & k-\ell = rN,\ r \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

we get:

$$\begin{cases} \frac{N}{2} & k = r \\ \frac{N}{2} & k = N-r \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

with $0 \leq k \leq N-1$.

0.1.1 Matrix relations

The DFT samples defined by:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad 0 \leq k \leq N-1 \tag{20}$$

can be expressed in matrix form as:

$$\mathbf{X} = \mathbf{D}_N \mathbf{x} \tag{21}$$

where:

$$\mathbf{X} = [X[0]\ X[1]\ \dots\ X[N-1]]^T \tag{22}$$

$$\mathbf{x} = [x[0]\ x[1]\ \dots\ x[N-1]]^T \tag{23}$$

and $\mathbf{D}_N$ is the $N \times N$ DFT matrix given by:

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \tag{24}$$

Likewise, the IDFT relation given by:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad 0 \le n \le N-1 \tag{25}$$

can be expressed in matrix form as:

$$\mathbf{x} = \mathbf{D}_N^{-1}\mathbf{X} \tag{26}$$

where $\mathbf{D}_N^{-1}$ is the $N \times N$ IDFT matrix, given by:

$$\mathbf{D}_N^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix} = \frac{1}{N}\mathbf{D}_N^* \tag{27}$$

0.1.2 DTFT from DFT by interpolation

The N-point DFT $X[k]$ of a length-N sequence $x[n]$ is simply the frequency samples of its DTFT $X(e^{j\omega})$ evaluated at N uniformly spaced frequency points:

$$\omega = \omega_k = \frac{2\pi k}{N}, \qquad 0 \le k \le N-1 \tag{28}$$

Given the N-point DFT $X[k]$ of a length-N sequence $x[n]$, its DTFT $X(e^{j\omega})$ cane be uniquely determined from $X[k]$. Thus:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{N-1} x[n]e^{-k\omega n} \\ &= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right]e^{-j\omega n} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\underbrace{\sum_{n=0}^{N-1} e^{-j\left(\omega-\frac{2\pi k}{N}\right)n}}_{S} \end{aligned} \tag{29}$$

To develop a compact expression for the sum S, let $r = e^{-j\left(\omega-\frac{2\pi k}{N}\right)}$. Then:

$$S = \sum_{n=0}^{N-1} r^n \tag{30}$$

From the above:

$$\begin{aligned} rS &= \sum_{n=1}^{N} r^n = 1 + \sum_{n=1}^{N-1} r^n r^N - 1 \\ &= \sum_{n=0}^{N-1} r^n + r^N - 1 = S + r^N - 1 \end{aligned} \tag{31}$$

or, equivalently:

$$S - rS = (1-r)S = 1 - r^N \tag{32}$$

Hence:

$$\begin{aligned} S &= \frac{1-r^N}{1-r} \\ &= \frac{1-e^{-j(\omega n - 2\pi k)}}{1-e^{-j\left(\omega - \frac{2\pi k}{N}\right)}} \\ &= \frac{\sin\left(\dfrac{\omega N 2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N - 2\pi k}{2N}\right)} e^{-j\left(\frac{\omega - 2\pi k}{N}\right)\left(\frac{N-1}{2}\right)} \end{aligned} \tag{33}$$

Therefore:

$$X(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \frac{\sin\left(\dfrac{\omega N 2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N - 2\pi k}{2N}\right)} e^{-j\left(\frac{\omega - 2\pi k}{N}\right)\left(\frac{N-1}{2}\right)} \tag{34}$$

0.1.3 Sampling the DTFT

Consider a sequence $x[n]$ with a DTFT $X(e^{j\omega})$. We sample $X(e^{j\omega})$ at N equally spaced points $\omega_k = \frac{2\pi k}{N}$, $0 \le k \le N-1$, developing the N frequency samples $\left\{X(e^{j\omega_k})\right\}$. These N frequency samples can be considered as an N-point DFT $Y[k]$ whose N-point IDFT is a length-N sequence $y[n]$. Now:

$$X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell} \tag{35}$$

Thus:

$$Y[k] = X(e^{j\omega_k}) = X(e^{j\frac{2\pi k}{N}}) = \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j2\pi k\frac{\ell}{N}} = \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} \tag{36}$$

An IDFT of $Y[k]$ yields:

$$\begin{aligned} y[n] &= \frac{1}{N}\sum_{k=0}^{N-1} Y[k] W_N^{-kn} \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} W_N^{-kn} \\ &= \sum_{\ell=-\infty}^{\infty} x[\ell] \left[\sum_{k=0}^{N-1} W_N^{-k(n-\ell)}\right] \\ &= \sum_{m=-\infty}^{\infty} x[n+mN] \end{aligned} \tag{37}$$

with $0 \leq n \leq N-1$, where in the last passage the identity in Eq. 18 is employed. Thus, $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0 \leq n \leq N-1$.
To apply the last result to finite-length sequences, we assume that the samples outside the specified range are zeros. Thus, if $x[n]$ is a length-M sequence with $M \leq N$, then $y[n] = x[n]$ for $0 \leq n \leq N-1$. If $M > N$, there is a time-domain aliasing of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$.

Example 4: Aliasing

Let $x[n] = \{0, 1, 2, 3, 4, 5\}$. By sampling its DTFT $X(e^{j\omega})$ at $\omega_k = \frac{2\pi k}{4}$, with $0 \leq k \leq 3$, and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by:

$$y[n] = x[n] + x[n+4] + x[n-4], \qquad 0 \leq n \leq 3 \tag{38}$$

We get $y[n] = \{4, 6, 2, 3\}$. $x[n]$ cannote be recovered from $y[n]$.

0.1.4 DFT properties

Like the DTFT, the DFT also satisfies a number of properties that are useful in signal processing applications. Some of these properties are essentially identical to those of the DTFT, while some others are somewhat different. A summary of the DFT properties are given in Figures 2, 3 and 4.

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k \rangle_N]$
$x^*[\langle -n \rangle_N]$	$X^*[k]$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{pcs}}[k] = \frac{1}{2}\{X[\langle k \rangle_N] + X^*[\langle -k \rangle_N]\}$
$j\,\mathrm{Im}\{x[n]\}$	$X_{\mathrm{pca}}[k] = \frac{1}{2}\{X[\langle k \rangle_N] - X^*[\langle -k \rangle_N]\}$
$x_{\mathrm{pcs}}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{\mathrm{pca}}[n]$	$j\,\mathrm{Im}\{X[k]\}$

Note: $x_{\mathrm{pcs}}[n]$ and $x_{\mathrm{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\mathrm{pcs}}[k]$ and $X_{\mathrm{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

Figure 2: Symmetry relations of DFT for a complex sequence $x[n]$.

0.1.5 Circular shift of a sequence

This property is analogous to the time-shifting property of the DTFT but with a difference. COnsider length-N squences defined for $0 \leq n \leq N-1$. Sample values of such sequences are equal to zero for values of $n < 0$ and $n \geq N$. If $x[n]$ is such a sequence, then for any arbitrary integer n_0, the shifted sequence $x_1[n] = x[n - n_0]$ is no longer defined for the range $0 \leq n \leq N-1$. We thus need to define another type of shift that will always keep the shifted sequence in the range $0 \leq n \leq N-1$. The desired shift, called the circular shift, is defined using a modulo operation:

$$x_c[n] = x[\langle n - n_0 \rangle_N] \tag{39}$$

Length-N Sequence	N-point DFT
$x[n]$	$X[k] = \text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\}$
$x_{\text{pe}}[n]$	$\text{Re}\{X[k]\}$
$x_{\text{po}}[n]$	$j\,\text{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$
	$\text{Re}\,X[k] = \text{Re}\,X[\langle -k\rangle_N]$
	$\text{Im}\,X[k] = -\,\text{Im}\,X[\langle -k\rangle_N]$
	$\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$
	$\arg X[k] = -\arg X[\langle -k\rangle_N]$

Note: $x_{\text{pe}}[n]$ and $x_{\text{po}}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

Figure 3: Symmetry relations of DFT for a real sequence $x[n]$.

Type of Property	Length-N Sequence	N-point DFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_o\rangle_N]$	$W_N^{kn_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o n} g[n]$	$G[\langle k - k_o\rangle_N]$
Duality	$G[n]$	$N g[\langle -k\rangle_N]$
N-point circular convolution	$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle k-m\rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$	

Figure 4: General properties of DFT.

For $n_0 > 0$ (right circular shift), the above equation implies:

$$x_c[n] = \begin{cases} x[n - n_0] & n_0 \le n \le N-1 \\ x[N - n_0 + n] & 0 \le n \le n_0 \end{cases} \tag{40}$$

An illustration of the concept of circular shift is showed in Figure 5. As it is possible to observe, a right circular shift by n_0 is equivalent to a left circular shift by $N - n_0$ sample periods. A circular shift by an integer number n_0 greater than N is equivalent to a circular shift by $\langle n_0\rangle_N$.

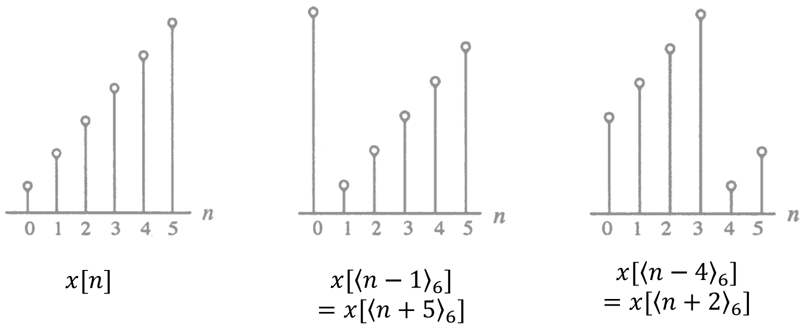

Figure 5: Illustration of circular shift.

0.1.6 Circular convolution

This operation is analogous to linear convolution, but with a difference. Consider two length-N sequences, $g[n]$ and $h[n]$, respectively. Their linear convolution results in a length-$(2N-1)$ sequence $y_L[n]$ given by:

$$y_L[n] = \sum_{m=0}^{N-1} g[m]h[n-m], \qquad 0 \leq n \leq 2N-2 \tag{41}$$

In computing $y_L[n]$ we have assumed that both length-N sequences have been zero-padded to extend their lengths to $2N-1$. The longer form of $y_L[n]$ results from the time-reversal of the sequence $h[n]$ and its linear shift to the right. The first nonzero value of $y_L[n]$ is $y_L[0] = g[0]h[0]$ and the last nonzero value is $y_L[2N-2] = g[N-1]h[N-1]$.
To develop a convolution-like operation resulting in a length-N sequence $y_C[n]$, we need to define a circular time-reversal, and then apply a circular time-shift. Resulting operation, called a circular convolution, is defined by:

$$y_C[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N], \qquad 0 \leq n \leq N-1 \tag{42}$$

Since the operation defined involves two length-N sequences, it is often referred to as an N-point circular convolution, denoted as: