LECTURE NOTES
OF
DIGITAL SIGNAL PROCESSING

Collection of the lectures notes of professor Federica Battisti.

Edited By

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Abstract

In this document I have tried to reorder the notes of the Digital Signal Processing course held by Professor Federica Battisti at DEI of the University of Padua during the first semester of the 2020-21 academic year of the master's degree in Physics of Data.
The notes are fully integrated with the material provided by the professor in the Moodle platform. In addition, I will integrate them, as best as possible, with the books recommended by the professor.
There may be formatting errors, wrong marks, missing exponents and even missing parts, since I'm still working on them. If you find errors or if you have any suggestions, let me know (ypu can send an e-mail at rocco.ardino@studenti.unipd.it, labeled with **DSP::TYPO/SUGGESTION**) and I will correct/integrate them, so that this document can be a good study support. However, these notes are not to be intended as a substitute of the lectures held by the professor or of lecture notes made by other people.

Padova, Thursday 12$^{\text{th}}$ November, 2020
Rocco Ardino

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3.1 Continuous-time Fourier transform

Let us start with the definition of this very important tool

Definition 1: Fourier transform of a continuous-time signal

he CTFT of a continuous-time signal $x_a(t)$ is given by:

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}\mathrm{d}t \tag{3.1}$$

often referred to as the Fourier spectrum or simply the spectrum of the continuous-time signal.

Definition 2: Inverse Fourier transform of a continuous-time signal

he inverse CTFT of a Fourier transform $X_a(j\Omega)$ is given by:

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{+j\Omega t}\mathrm{d}\Omega \tag{3.2}$$

often referred to as the Fourier integral.

A CTFT pair will be denoted as:

$$x_a(t) \longleftrightarrow X_a(j\Omega) \tag{3.3}$$

Note that Ω is real and denotes the continuous-time angular frequency variable in radians. In general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$. It can be expressed in the polar form as:

$$X_a(j\Omega) = |X_a(j\Omega)|e^{j\theta_a(\Omega)} \tag{3.4}$$

where $\theta_a(\Omega) = \arg X_a(j\Omega)$. The quantity $|X_a(j\Omega)|$ is called the magnitude spectrum and the quantity $\theta_a(\Omega)$ is called the phase spectrum. Both spectra are real function of Ω and in general the CTFT $X_a(j\Omega)$ exists if $x_a(t)$ satisfies the Dirichlet conditions:

- the signal $x_a(t)$ has a finite number if discontinuities and a finite number of maxima and minima in any finite interval;
- the signal is absolutely integrable, i.e.:

$$\int_{-\infty}^{\infty} |x_a(t)|\mathrm{d}t < \infty \tag{3.5}$$

If the Dirichlet conditions are satisfied, then:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{+j\Omega t}\mathrm{d}\Omega \tag{3.6}$$

converges to $x_a(t)$ except at values of t where $x_a(t)$ has discontinuities. Moreover, it can be showed that if $x_a(t)$ is absolutely integrable, then proving the existence of the CTFT reduces to proving:

$$|X_a(t\Omega)| < \infty \tag{3.7}$$

3.1.1 Energy density spectrum

The total energy E_x of a finite energy continuous-time complex signal $x_a(t)$ is given by:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |x_a(t)|^2\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x_a(t)x_a^*(t)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x_a(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a^*(j\Omega)e^{-j\Omega t}\mathrm{d}\Omega\right]\mathrm{d}t \end{aligned} \tag{3.8}$$

Interchanging the order of the integration we get:

$$\begin{aligned} E_x &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a^*(j\Omega)\left[\int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}\mathrm{d}t\right]\mathrm{d}\Omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a^*(j\Omega)X_a(j\Omega)\mathrm{d}\Omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_a(j\Omega)|^2\mathrm{d}\Omega \end{aligned} \tag{3.9}$$

Hence:

$$\int_{-\infty}^{\infty} |x(t)|^2\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_a(j\Omega)|^2\mathrm{d}\Omega \tag{3.10}$$

The above relation is more commonly known as the Parseval's relation for finite-energy continuous-time signals. The quantity $|X_a(j\Omega)|^2$ is called the energy density spectrum of $x_a(t)$ and it is usually denoted as:

$$S_{xx}(\Omega) = |X_a(j\Omega)|?2 \tag{3.11}$$

The energy over a specified range of frequencies $\Omega_a \leq \Omega \leq \Omega_b$ can be computed using:

$$E_{x,r} = \frac{1}{2\pi}\int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega)\mathrm{d}\Omega \tag{3.12}$$

3.1.2 Band-limited continuous-time signals

A full-band, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty \leq \Omega \leq \infty$. A band-limited continuous-time signal has a spectrum that is limited to a portion of the frequency range $-\infty \leq \Omega \leq \infty$. An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \leq |\Omega| \leq \Omega_b$ can be computed using:

$$X_a(j\Omega) = \begin{cases} 0 & 0 \leq |\Omega| < \Omega_a \\ 0 & \Omega_b < |\Omega| < \infty \end{cases} \tag{3.13}$$

However, an ideal band-limited signal cannot be generated in practice.
Band-limited signals are classified according to the frequency range where most of the signal's is concentrated:

- a lowpass, continuous-time signal has a spectrum occupying the frequency range $0 < |\Omega| \leq \Omega_p < \infty$, where Ω_p is called the bandwidth of the signal;
- a highpass, continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_p \leq |\Omega| < \infty$, where the bandwidth of the signal is from Ω_p to ∞;
- a bandpass, continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_L \leq |\Omega| \leq \Omega_H < \infty$, where $\Omega_H - \Omega_L$ is the bandwidth.

3.1.3 Discrete-time fourier transform

Let us introduce the definition of this concept.

Definition 3: Discrete-time Fourier transform

The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{3.14}$$

where in general $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as:

$$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega}) \tag{3.15}$$

$X_{\text{re}}(e^{j\omega})$ and $X_{\text{im}}(e^{j\omega})$ are respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of ω. $X(e^{j\omega})$ can alternately be expressed as:

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right|e^{j\theta(\omega)} \tag{3.16}$$

where $\theta(\omega) = \arg X(e^{j\omega})$. $\left|X(e^{j\omega})\right|$ and $\arg X(e^{j\omega})$ are called respectively magnitude function and phase function. Both quantities are again real functions of ω. In many applications, the DTFT is called the Fourier spectrum. Likewise, $\left|X(e^{j\omega})\right|$ and $\theta(\omega)$ are called respectively the magnitude and phase spectra.
For a real sequence $x[n]$, $\left|X(e^{j\omega})\right|$ and $X_{\text{re}}(e^{j\omega})$ are even functions of ω, whereas, $\theta(\omega)$ and $X_{\text{im}}(e^{j\omega})$ are odd functions of ω. Note also that $X(e^{j\omega}) = \left|X(e^{j\omega})\right|e^{j\theta(\omega+2\pi k)} = \left|X(e^{j\omega})\right|e^{j\theta(\omega)}$ for any integer k. The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT. Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the range of values $-\pi \leq \theta(\omega) < \pi$, called the principal value.

Example 1: DTFT of the unit sample sequence

The DTFT of the unit sample sequence $\delta[n]$ is given by:

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \delta[0] = 1 \tag{3.17}$$

Example 2: DTFT of a causal sequence

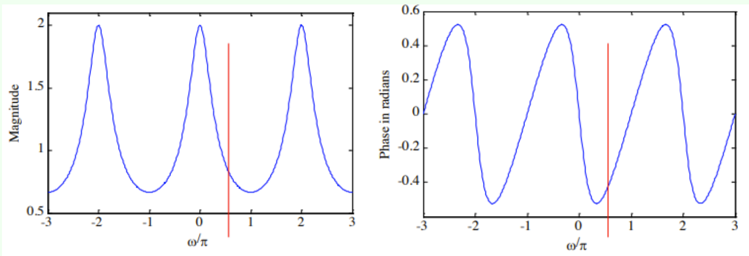
Consider the causal sequence:

$$x[n] = \alpha^n \mu[n], \qquad |\alpha| < 1 \tag{3.18}$$

Its DTFT is given by:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n \\ &= \frac{1}{1-\alpha e^{-j\omega}} \end{aligned} \tag{3.19}$$

as $\left|\alpha e^{-j\omega}\right| = |\alpha| < 1$. If we take for example $\alpha = 0.5$, we get the plot below for the magnitude and phase of the DTFT.



$|X(e^{j\omega}) = |X(e^{-j\omega})||$ $\theta(\omega)$=- $\theta(-\omega)$

The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω. It is also a periodic function of ω with a period 2π:

$$X\left(e^{j(\omega+2\pi k)}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(e^{j\omega}) \tag{3.20}$$

Therefore:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{3.21}$$

represents the Fourier series representation of the periodic function. As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \mathrm{d}\omega \tag{3.22}$$

Proof. Consider:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega \ell} \right) e^{j\omega n} \mathrm{d}\omega \tag{3.23}$$

The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e. $X(e^{j\omega})$ exists. Then:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty}x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}\mathrm{d}\omega &= \sum_{\ell=-\infty}^{\infty}x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\omega(n-\ell)}\mathrm{d}\omega\right)\\ &= \sum_{\ell=-\infty}^{\infty}x[\ell]\frac{\sin\left(\pi(n-\ell)\right)}{\pi(n-\ell)}\\ &= \sum_{\ell=-\infty}^{\infty}x[\ell]\delta[n-\ell]\\ &= x[n] \end{aligned} \tag{3.24}$$

For the convergence condition, an infinite series of the form:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n} \tag{3.25}$$

may or may not converg. Therefore, let us consider:

$$X_k(e^{j\omega}) = \sum_{n=-k}^{k}x[n]e^{-j\omega n} \tag{3.26}$$

Then for uniform convergence of $X_k(e^{j\omega})$:

$$\lim_{k\to\infty} X_k(e^{j\omega}) = X(e^{j\omega}) \tag{3.27}$$

Now, if $x[n]$ is an absolutely summable sequence, i.e., if $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$, then:

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-k}^{k}x[n]e^{-j\omega n}\right| \leq \sum_{n=-k}^{k}|x[n]| < \infty \tag{3.28}$$

for all values of ω. Thus, the absolute summability of $x[n]$ is a sufficient condition for the existence of the DTFT $X(e^{j\omega})$. ■

Example 3: Absolute summability condition

he sequence $x[n] = \alpha^n\mu[n]$ for $|\alpha| < 1$ is absolutely summable as:

$$\sum_{n=-k}^{k}|\alpha^n|\mu[n] = \sum_{n=0}^{\infty}|\alpha^n| = \frac{1}{1-|\alpha|} < \infty \tag{3.29}$$

and its DTFT $X(e^{j\omega})$ therefore converges to $\frac{1}{1-\alpha e^{j\omega}}$ uniformly.

Note that since:

$$\sum_{n=-\infty}^{\infty}|x[n]|^2 \leq \left(\sum_{n=-\infty}^{\infty}|x[n]|\right)^2 \tag{3.30}$$

an absolutely summable sequence has always a finite energy. However, a finite-energy sequence is not necessarily absolutely summable.

Example 4: Absolute summability

The sequence:

$$x[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases} \tag{3.31}$$

has finite energy equal to:

$$E_x = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6} \tag{3.32}$$

but $x[n]$ is not absolutely summable.

In order to represent a finite energy sequence $x[n]$ that is not absolutely summable by a DTFT $X(e^{j\omega})$, it is necessary to consider a mean-square convergence of $X(e^{j\omega})$:

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} \left|X(e^{j\omega}) - X_k(e^{j\omega})\right|^2 \mathrm{d}\omega = 0 \tag{3.33}$$

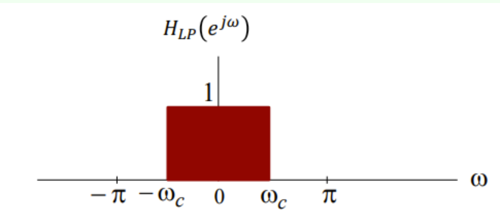
The total energy of the error $X(e^{j\omega}) - X_k(e^{j\omega})$ must approach zero at each value of ω as k goes to ∞. In such a case, the absolute value of the error $\left|X(e^{j\omega}) - X_k(e^{j\omega})\right|$ may not go to zero as k goes to ∞ and the DTFT is no longer bounded.

Example 5: Mean-square convergence

Consider the DTFT:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & 0 \leq |\omega| \leq \omega_c \\ 0 & \omega < |\omega| \leq \pi \end{cases} \tag{3.34}$$

showed in the plot below.



The inverse DTFT of $H_{LP}(e^{j\omega})$ is given by:

$$h_{LP}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} \mathrm{d}\omega = \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \frac{\sin(\omega_c n)}{\pi n} \tag{3.35}$$

for $-\infty < n < \infty$- The energy of $h_{LP}[n]$ is given by $\frac{\omega_c}{\pi}$. $h_{LP}[n]$ is a finite-energy sequence, but it is not absolutely summable. In fact, the result:

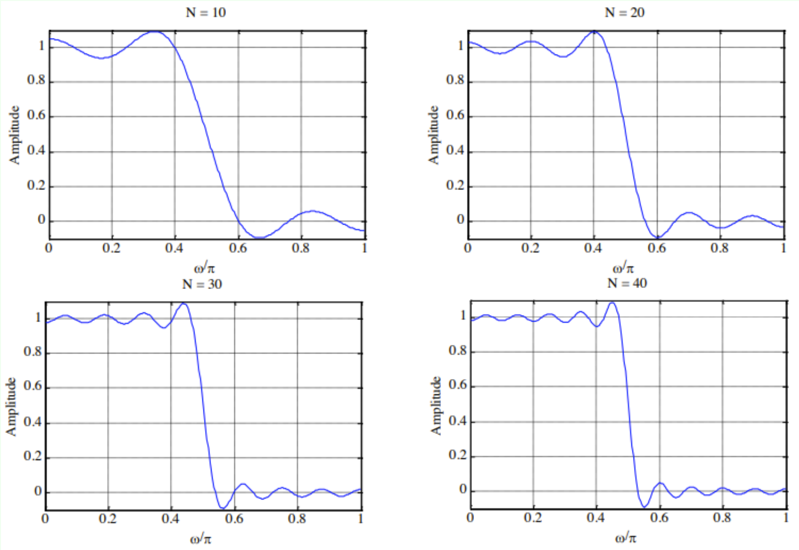
$$\sum_{n=-k}^{n=k} h_{LP}[n] e^{-j\omega n} = \sum_{n=-k}^{n=k} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n} \tag{3.36}$$

does not uniformly converge to $H_{LP}(e^{j\omega})$ for all values of ω, but converges to $H_{LP}(e^{j\omega})$ in the mean-square sense.

The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function:

$$H_{LP,k}(e^{j\omega}) = \sum_{n=-k}^{k} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n} \tag{3.37}$$

for various values of k, as showed below.



As can be seen from these plots, independently of the value of k there are ripples in the plot of $H_{LP,k}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$. The number of ripples increases as k increases with the height of the largest ripple remaining the same for all values of k. As k goes to ∞, the condition:

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega}) - H_{LP,k}(e^{j\omega})\right|^2 \mathrm{d}\omega = 0 \tag{3.38}$$

holds indicating the convergence of $H_{LP,k}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$.
The oscillatory behaviour of $H_{LP,k}(e^{j\omega})$ approximating $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity is known as the Gibbs phenomenon

The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable. Examples of such sequences are the unit step sequence $\mu[n]$, the sinusoidal sequence $\cos(\omega_0 n + \varphi)$ and the exponential sequence $A\alpha^n$. For this type of sequences, a DTFT representation is possible using the Dirac delta function $\delta(\omega)$.
A Dirac delta function $\delta(\omega)$ is a function of ω with infinite height, zero width, and unit area. It is the limiting form of a unit area pulse function p_Δ as Δ goes to zero satisfying:

$$\lim_{\Delta\to 0} \int_{-\infty}^{\infty} p_\Delta(\omega)\mathrm{d}\omega = \int_{-\infty}^{\infty} \delta(\omega)\mathrm{d}\omega \tag{3.39}$$

Example 6: Dirac δ application

onsider the complex exponential sequence:

$$x[n] = e^{j\omega_0 n} \tag{3.40}$$

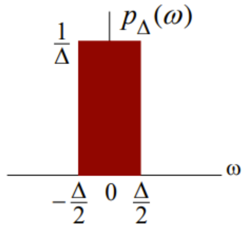


Figure 3.1: Plot and area of $p_\Delta(\omega)$ function.

Its DTFT is given by:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2k\pi) \tag{3.41}$$

where $\delta(\omega)$ is an impulse function of ω and $-\pi \leq \omega_0 \leq \pi$. The function in Eq. 3.41 is periodic in ω with a period 2π and it is called periodic impulse train. In order to verify that $X(e^{j\omega})$ given above is indeed the DTFT of $x[n] = e^{j\omega_0 n}$ we compute the inverse DTFT of $X(e^{j\omega})$:

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)e^{j\omega n}\mathrm{d}\omega \\ &= \int_{-\pi}^{\pi} \delta(\omega - \omega_0)e^{j\omega n}\mathrm{d}\omega \\ &= e^{j\omega_0 n} \end{aligned} \tag{3.42}$$

where we have used the sampling property of the impulse function $\delta(\omega)$.

Last but not least, we list in Table 3.1 a set of commonly used DTFT pairs.

Sequence	**DTFT**
$\delta[n]$	1
1	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
$\mu[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$\alpha^n\mu[n] \quad (\lvert\alpha\rvert < 1)$	$\frac{1}{1 - \alpha e^{-j\omega}}$

Table 3.1: Commonly used DTFT pairs.

3.1.4 DTFT properties

There are a number of important properties of the DTFT that are useful in signal processing applications. These are listed here in Figures 3.2, 3.3, 3.4 without proof, since it is quite straigthforward to derive them. We illustrate the applications of some of the DTFT properties.

Lecture 9.
Tuesday 27th October, 2020.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{\mathrm{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\mathrm{cs}}[n]$	$X_{\mathrm{re}}(e^{j\omega})$
$x_{\mathrm{ca}}[n]$	$jX_{\mathrm{im}}(e^{j\omega})$

Figure 3.2: DTFT symmetry relations for a complex sequence $x[n]$.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{\mathrm{re}}(e^{j\omega}) + jX_{\mathrm{im}}(e^{j\omega})$
$x_{\mathrm{ev}}[n]$	$X_{\mathrm{re}}(e^{j\omega})$
$x_{\mathrm{od}}[n]$	$jX_{\mathrm{im}}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{\mathrm{re}}(e^{j\omega}) = X_{\mathrm{re}}(e^{-j\omega})$
	$X_{\mathrm{im}}(e^{j\omega}) = -X_{\mathrm{im}}(e^{-j\omega})$
	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Figure 3.3: DTFT symmetry realtions for a real sequence $x[n]$.

Example 7: DTFT properties

We determine the DTFT $Y(e^{j\omega})$ of:

$$y[n] = (n+1)\alpha^n \mu[n], \qquad |\alpha| < 1 \tag{3.43}$$

Let $x[n] = \alpha^n \mu[n]$, with $\alpha < 1$. We can therefore write:

$$y[n] = nx[n] + x[n] \tag{3.44}$$

The DTFT of $x[n]$ is given by:

$$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} \tag{3.45}$$

Using the differentiation property of the DTFT, we observe that the DTFT of $nx[n]$ is given by:

$$j\frac{\mathrm{d}X(e^{j\omega})}{\mathrm{d}\omega} = j\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} \tag{3.46}$$

Type of Property	Sequence	Discrete-Time Fourier Transform
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation in frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega}) H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)})\, d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega}) H^*(e^{j\omega})\, d\omega$	

Figure 3.4: DTFT general properties.

Next, using thhe linearity property of the DTFT, we arrive at:

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{(1-\alpha e^{-j\omega})^2} \tag{3.47}$$

Example 8: DTFT properties

We determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$, defined by:

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1] \tag{3.48}$$

The DTFT of $\delta[n]$ is 1. Using the time-shifting property of the DTFT, we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$. Using the linearity property we then obtain the frequency-domain representation of $d_0 v[n] + d_1 v[n-1]$ as:

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega} \tag{3.49}$$

Solving the above equation we get:

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}} \tag{3.50}$$

3.1.5 Energy density spectrum

The total energy of a finite-energy sequence $g[n]$ is given by:

$$E_g = \sum_{n=-\infty}^{\infty} |g[n]|^2 \tag{3.51}$$

From Parseval's relation we observe that:

$$E_g = \sum_{n=-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|G(e^{j\omega})\right|^2 \mathrm{d}\omega \tag{3.52}$$

The quantity:

$$S_{gg}(\omega) = \left|G(e^{j\omega})\right|^2 \tag{3.53}$$

is called the energy density spectrum. The area under this curve in the range $-\pi \leq \omega \leq \pi$ divided by 2π is the energy of the sequence.

3.1.6 Band-limited discrete-time signals

Since the spectrum of a discrete-time signal is a periodic function of ω with a period 2π, a full-band signal has a spectrum occupying the frequency range $-\pi \leq \omega \leq \pi$. A band-limited discrete-time signal has a spectrum that is limited to a portion of the frequency range $-\pi \leq \omega \leq \pi$.
An ideal band-limited signal has a spectrum that is zero outside a frequency range $0 < \omega_a \leq |\omega| \leq \omega_b < \pi$, that is:

$$X(e^{j\omega}) = \begin{cases} 0 & 0 \leq |\omega| < \omega_a \\ 0 & \omega_b < |\omega| < \pi \end{cases} \tag{3.54}$$

However, an ideal band-limited discrete-time signal cannot be generated in practice. A classification of a band-limited discrete-time signal is based on the frequency range where most of the signal energy is concentrated. A lowpass discrete-time real signal has a spectrum occupying the frequency range $0 < |\omega| \leq \omega_p < \pi$ and has a bandwidth of ω_p.
A highpass discrete-time real signal has a spectrum occupying the frequency range $0 < \omega_p \leq |\omega| < \pi$ and has a bandwidth of $\pi - \omega_p$.
A bandpass discrete-time real signal has a spectrum occupying the frequency range $0 < \omega_L \leq |\omega| \leq \omega_H < \pi$ and has a bandwidth of $\omega_H - \omega_L$.

Example 9: Band-limited discrete-time signals

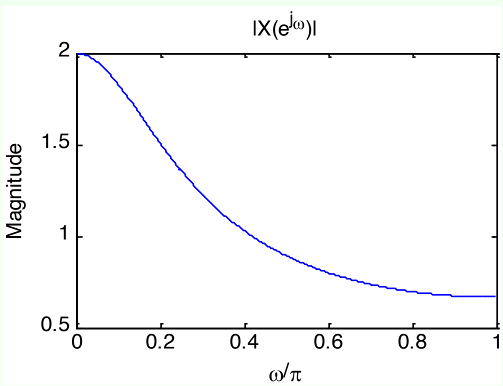
Consider the sequence:

$$x[n] = (0.5)^n \mu[n] \tag{3.55}$$

The DTFT is:

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}} \tag{3.56}$$

and the magnitude spectrum is showed below.



It can be showed that 80% of the energy of this lowpass signal is contained in the frequency range $0 \leq |\omega| \leq 0.5081\pi$. Hence, we can define the 80% bandwidth to be 0.5081π radians.

Returning to the energy density spectrum, we consider some other examples introducing also the concept of band-limited signals.

Example 10: Energy density spectrum

We compute the energy of the sequence:

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n}, \qquad -\infty < n < \infty \tag{3.57}$$

Here:

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega})\right|^2 \mathrm{d}\omega \tag{3.58}$$

where:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases} \tag{3.59}$$

Therefore:

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \mathrm{d}\omega = \frac{\omega_c}{\pi} < \infty \tag{3.60}$$

Hence, $h_{LP}[n]$ is a finite-energy lowpass sequence.

3.2 Linear convolution using DTFT

An important property of the DTFT is given by the convolution theorem. It states that if $y[n] = x[n] * h[n]$, then the DTFT $Y(e^{j\omega})$ of $y[n]$ is given by:

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \tag{3.61}$$

An implication of this result is that the linear convolution $y[n]$ of the sequences $x[n]$ and $h[n]$ can be performed as follows:

- compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively;
- form the DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$;
- compute the IDFT $y[n]$ of $Y(e^{j\omega})$.

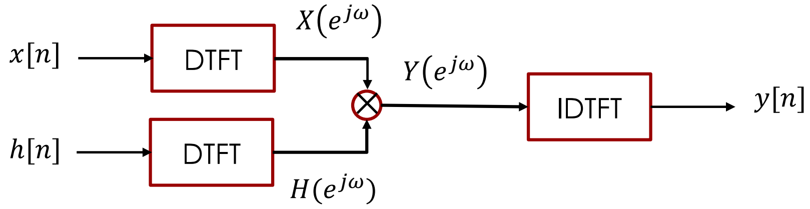


Figure 3.5: Scheme of the computation of linear convolution $y[n]$ of the sequences $x[n]$ and $h[n]$.

Note that in numerical computation, when the computed phase function is outside the range $[-\pi, \pi]$, the phase is computed modulo 2π, to bring the computed value to this range. Thus, the phase functions of some sequences exhibit discontinuities of 2π radians in the plot. For example, there is a discontinuity of 2π at $\omega = 0.72$ in the phase response below:

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-2j\omega} - 0.033e^{-3j\omega} + 0.008e^{-4j\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-2j\omega} + 1.6e^{-3j\omega} + 0.41e^{-4j\omega}} \tag{3.62}$$

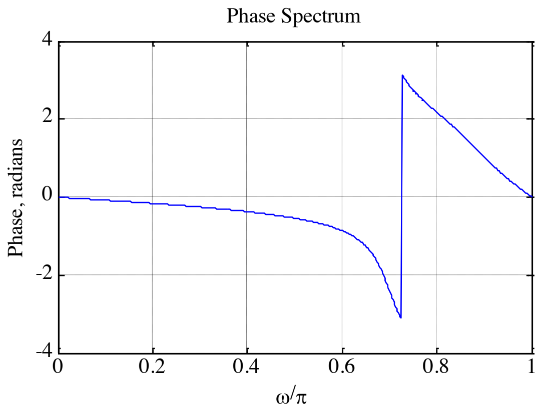


Figure 3.6: Discontinuity in the phase response of Eq. 3.62.

In such cases, often an alternate type of phase function that is continuous function of ω is derived from the original phase function by removing the discontinuities of 2π. Process of discontinuity removal is called unwrapping the phase and the unwrapped phase function will be denoted as $\theta_c(\omega)$.
For example, the unwrapped phase function of the DTFT in Eq. 3.62 is showed in Figure 3.7.

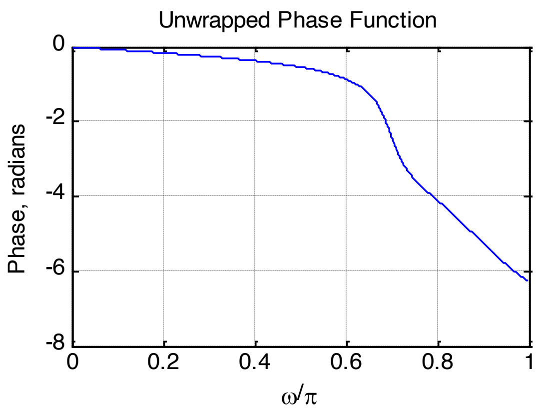


Figure 3.7: Unwrapped phase function of Eq. 3.62.

The conditions under which the phase function will be a continuous function of ω is next derived. Now consider:

$$\ln X(e^{j\omega}) = \ln \left|X(e^{j\omega})\right| + j\theta(\omega) \tag{3.63}$$

where:

$$\theta(\omega) = \arg\left\{X(e^{j\omega})\right\} \tag{3.64}$$

From $\ln X(e^{j\omega})$ we can also compute $\frac{\mathrm{d}\ln X(e^{j\omega})}{\mathrm{d}\omega}$:

$$\frac{\mathrm{d}\ln X(e^{j\omega})}{\mathrm{d}\omega} = \frac{\mathrm{d}\ln \left|X(e^{j\omega})\right|}{\mathrm{d}\omega} + j\frac{\mathrm{d}\theta(\omega)}{\mathrm{d}\omega} \tag{3.65}$$

Thus, $\frac{\mathrm{d}\theta(\omega)}{\mathrm{d}\omega}$ is given by the imaginary part of:

$$\frac{1}{X(e^{j\omega})}\left[\frac{\mathrm{d}X_{\mathrm{re}}(e^{j\omega})}{\mathrm{d}\omega}+j\frac{\mathrm{d}X_{\mathrm{im}}(e^{j\omega})}{\mathrm{d}\omega}\right] \tag{3.66}$$

Hence:

$$\frac{\mathrm{d}\theta(\omega)}{\mathrm{d}\omega}=\frac{1}{|X(e^{j\omega})|^2}\left[X_{\mathrm{re}}(e^{j\omega})\frac{\mathrm{d}X_{\mathrm{im}}(e^{j\omega})}{\mathrm{d}\omega}-X_{\mathrm{im}}(e^{j\omega})\frac{\mathrm{d}X_{\mathrm{re}}(e^{j\omega})}{\mathrm{d}\omega}\right] \tag{3.67}$$

The phase function can thus be defined unequivocally by its derivative:

$$\theta(\omega)=\int_0^{\omega}\left[\frac{\mathrm{d}\theta(\eta)}{\mathrm{d}\eta}\right]\mathrm{d}\eta \tag{3.68}$$

with the constraint $\theta(0)=0$.
The phase function defined by Eq. 3.68 is called the unwrapped phase function of $X(e^{j\omega})$ and it is a continuous function of ω. Therefore, $\ln X(e^{j\omega})$ exists. Moreover, the phase function will be an odd function of ω if:

$$\frac{1}{\pi}\int_0^{2\pi}\left[\frac{\mathrm{d}\theta(\eta)}{\mathrm{d}\eta}\right]\mathrm{d}\eta=0 \tag{3.69}$$

If the above constraint is not satisfied, then the computed phase function will exhibit absolute jumps greater than π.

3.3 The frequency response

Most discrete-time signals encountered in practice can be represented as a linear combination of a very large, maybe infinite, number of sinusoidal discrete-time signals of different angular frequencies. Thus, knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of the superposition property.
An important property of an LTI system is that for certain types of input signals, called eigen functions, the output signal is the input signal multiplied by a complex constant. We consider here one such eigen function as the input.
Consider the LTI discrete-time system with an impulse response $\{h[n]\}$. Its input-output relationship in the time-domain is given by the convolution sum:

$$y[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k] \tag{3.70}$$

If the input is of the form:

$$x[n]=e^{j\omega n},\qquad -\infty<n<\infty \tag{3.71}$$

then it follows that the output is given by:

$$y[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=\left(\sum_{k=-\infty}^{\infty}h[k]e^{-j\omega k}\right)e^{j\omega n} \tag{3.72}$$

Now, let:

$$H(e^{j\omega})=\sum_{k=-\infty}^{\infty}h[k]e^{-j\omega k} \tag{3.73}$$

Then we can write:

$$y[n] = H(e^{j\omega})e^{j\omega n} \tag{3.74}$$

Thus for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega})$. Thusm $e^{j\omega n}$ is an eigen function of the system.
The quantity $H(e^{j\omega})$ is called the frequency response of the LTI discrete-time system. $H(e^{j\omega})$ provides a frequency-domain description of the system and is precisely the DTFT of the impulse response $\{h[n]\}$ of the system. $H(e^{j\omega})$, in general, is a complex function of ω with a period 2π. It can be expressed in terms of its real and imaginary parts:

$$H(e^{j\omega}) = H_{\text{re}}(e^{j\omega}) + jH_{\text{im}}(e^{j\omega}) \tag{3.75}$$

or, in terms of its magnitude and phase:

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right|e^{j\theta(\omega)} \tag{3.76}$$

where:

$$\theta(\omega) = \arg\left\{H(e^{j\omega})\right\} \tag{3.77}$$

The function $\left|H(e^{j\omega})\right|$ is called the magnitude response and the function $\theta(\omega)$ is called the phase response of the LTI discrete-time system. Design specifications for the LTI discrete-time system, in many applications, are given in terms of the magnitude response or the phase response or both.
In some cases, the magnitude function is specified in decibels as:

$$g(\omega) = 20\log_{10}\left|H(e^{j\omega})\right|\text{dB} \tag{3.78}$$

where $G(\omega)$ is called the gain function. The negative of the gain function $A(\omega) = -G(\omega)$ is called the attenuation or loss function.
Note that magnitude and phase functions are real functions of ω, whereas the frequency response is a complex function of ω. If the impulse response $h[n]$ is real then it follows that the magnitude function is an even function of ω:

$$\left|H(e^{j\omega})\right| = \left|H(e^{-j\omega})\right| \tag{3.79}$$

and the phase function is an odd function od ω:

$$\theta(\omega) = -\theta(-\omega) \tag{3.80}$$

Likewise, for a real impulse response $h[n]$, $H_{\text{re}(e^{j\omega})}$ is even and $H_{\text{im}(e^{j\omega})}$ is odd.

Example 11: M-point moving average filter

Consider the M-point moving average filter with an impulse response given by:

$$h[n] = \begin{cases} \frac{1}{M} & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases} \tag{3.81}$$

Its frequency response is then given by:

$$H(e^{j\omega}) = \frac{1}{M}\sum_{n=0}^{M-1} e^{-j\omega n} \tag{3.82}$$

Performing all the calculations:

$$\begin{aligned}
H(e^{j\omega}) &= \frac{1}{M}\left(\sum_{n=0}^{\infty} e^{-j\omega n} - \sum_{n=M}^{\infty} e^{-j\omega n}\right) \\
&= \frac{1}{M}\left(\sum_{n=0}^{\infty} e^{-j\omega n}\right)\left(1 - e^{-jM\omega}\right) \\
&= \frac{1}{M}\frac{1 - e^{-jM\omega}}{1 - e^{-j\omega}} \\
&= \frac{1}{M}\frac{\sin\left(\frac{M\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{(M-1)\omega}{2}}
\end{aligned} \tag{3.83}$$

Thus, the magnitude response of the M-point moving average filter is given by:

$$\left|H(e^{j\omega})\right| = \left|\frac{1}{M}\frac{\sin\left(\frac{M\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right| \tag{3.84}$$

and the phase response is given by

$$\theta(\omega) = -\frac{(M-1)\omega}{2} + \pi \sum_{k=1}^{\lfloor \frac{M}{2} \rfloor} \mu\left[\omega - \frac{2\pi k}{M}\right] \tag{3.85}$$

Note that the frequency response also determines the steady-state response of an LTI discrete-time system to a sinusoidal input.

Example 12: Steady-state response

We determine the steady-state output $y[n]$ of a real coefficient LTI discrete-time system with a frequency response $H(e^{j\omega})$ for an input:

$$x[n] = A\cos(\omega_0 n + \varphi), \qquad -\infty < n < \infty \tag{3.86}$$

We can express the input $x[n]$ as:

$$x[n] = g[n] + g^*[n] \tag{3.87}$$

where:

$$g[n] = \frac{1}{2}Ae^{j\varphi}e^{j\omega_0 n} \tag{3.88}$$

Now the output of the system for an input $e^{j\omega_0 n}$ is simply $H(e^{j\omega_0})e^{j\omega_0 n}$. Because of linearity, the response $v[n]$ to an input $g[n]$ is given by:

$$v[n] = \frac{1}{2}Ae^{j\varphi}H(e^{j\omega_0})e^{j\omega_0 n} \tag{3.89}$$

Likewise, the output $v^*[n]$ to the input $g^*[n]$ is:

$$v^*[n] = \frac{1}{2}Ae^{-j\varphi}H(e^{-j\omega_0})e^{-j\omega_0 n} \tag{3.90}$$

Combining the last two equations we get:

$$\begin{aligned}
y[n] &= v[n] + v^*[n] \\
&= \frac{1}{2}Ae^{j\varphi}H(e^{j\omega_0})e^{j\omega_0 n} + \frac{1}{2}Ae^{-j\varphi}H(e^{-j\omega_0})e^{-j\omega_0 n} \\
&= \frac{1}{2}A\big|H(e^{j\omega_0})\big|\Big\{e^{j\theta(\omega_0)}e^{j\varphi}e^{j\omega_0 n} + e^{-j\theta(\omega_0)}e^{-j\varphi}e^{-j\omega_0 n}\Big\} \\
&= A\big|H(e^{j\omega_0})\big|\cos\left(\omega_0 n + \theta(\omega_0) + \varphi\right)
\end{aligned} \tag{3.91}$$

Thus, the output $y[n]$ has the same sinusoidal waveform as the input with two differences:

- the amplitude is multiplied by $\big|H(e^{j\omega_0})\big|$, the value of the magnitude function at $\omega = \omega_0$;
- the output has a phase lag relative to the input by an amount $\theta(\omega_0)$, the value of the phase function at $\omega = \omega_0$.

The expression for the steady-state response developed earlier assumes that the system is initially relaxed before the application of the input $x[n]$. In practice, excitation $x[n]$ to a discrete-time system is usually a right-sided sequence applied at some sample index $n = n_0$. Now, we develop the expression for the output for such an input. Without any loss of generality, assume $x[n] = 0$ for $n < 0$. From the input-output relation in Eq. 3.70, we observe that for an input:

$$x[n] = e^{j\omega n}\mu[n] \tag{3.92}$$

the output is given by:

$$y[n] = \left(\sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)}\right)\mu[n] = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}\mu[n] \tag{3.93}$$

The output for $n < 0$ is $y[n] = 0$, while for $n \geq 0$ it is given by:

$$\begin{aligned}
y[n] &= \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n} \\
&= H(e^{j\omega})e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}
\end{aligned} \tag{3.94}$$

The first term on the RHS is the same as that obtained when the input is applied at $n = 0$ to an initially relaxed system and it is the steady-state response:

$$y_{\text{sr}}[n] = H(e^{j\omega})e^{j\omega n} \tag{3.95}$$

The second term on the RHS is called the transient response:

$$y_{\text{tr}}[n] = -\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n} \tag{3.96}$$

To determine the effect of the above term on the total output response, we observe:

$$|y_{\text{tr}}[n]| = \left|\sum_{k=n+1}^{\infty} h[k]e^{-j\omega(k-n)}\right| \leq \sum_{k=n+1}^{\infty} |h[k]| \leq \sum_{k=0}^{\infty} |h[k]| \tag{3.97}$$

For a causal, stable LTI IIR discrete-time system, $h[n]$ is absolutely summable. As a result, the transient response $y_{\mathrm{tr}}[n]$ is a bounded sequence. Moreover, as $n \to \infty$:

$$\sum_{k=n+1}^{\infty} |h[k]| \to 0 \tag{3.98}$$

and hence, the transient response decays to zero as n gets very large.
For a causal FIR LTI discrete-time system with an impulse response $h[n]$ of length $N+1$, $h[n] = 0$ for $n > N$. Hence, $y_{\mathrm{tr}}[n] = 0$ for $n > N-1$. Here the output reaches the steady-state value $y_{\mathrm{sr}}[n] = H(e^{j\omega})e^{j\omega n}$ at $n = N$.

3.4 The concept of filtering

One application of an LTI discrete-time system is to pass certain frequency components in an input sequence without any distortion (if possible) and to block other frequency components. Such systems are called digital filters and one of the main subjects of discussion in this course.
The key to the filtering process is the Fourier transform

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\mathrm{d}\omega \tag{3.99}$$

It expresses an arbitrary input as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences. Thus, by appropriately choosing the values of the magnitude function $\left|H(e^{j\omega})\right|$ of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these components can be selectively heavily attenuated or filtered with respect to the others.
To understand the mechanism behind the design of frequency-selective filters, consider a real-coefficient LTI discrete-time system characterized by a magnitude function:

$$\left|H(e^{j\omega})\right| \approx \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases} \tag{3.100}$$

We apply to the system an input:

$$x[n] = A\cos(\omega_1 n) + B\cos(\omega_2 n), \qquad 0 < \omega_1 < \omega_c < \omega_2 < \pi \tag{3.101}$$

Because of linearity, the output of this system is of the form:

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1)) + B\left|H(e^{j\omega_2})\right|\cos(\omega_2 n + \theta(\omega_2)) \tag{3.102}$$

As $\left|H(e^{j\omega_1})\right| \approx 1$ and $\left|H(e^{j\omega_2})\right| \approx 0$, the output reduces to:

$$y[n] \approx A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1)) \tag{3.103}$$

Thus, the system acts like a lowpass filter.
Now we consider an example of design of a very simple digital filter.

Example 13: Design of a simple digital filter

The input consists of a sum of two sinusoidal sequences of angular frequencies 0.1 rad/sample and 0.4 rad/sample. We need to design a highpass filter that will pass the high-frequency component of the input but block the low-frequency component.

For simplicity, assume the filter to be an FIR filter of length 3 with an impulse response:

$$h[0] = h[2] = \alpha \tag{3.104}$$
$$h[1] = \beta \tag{3.105}$$

The convolution sum description of this filter is then given by:

$$\begin{aligned} y[n] &= h[0]x[n]h[1]x[n-1] + h[2]x[n-2] \\ &= \alpha x[n] + \beta x[n-1] + \alpha x[n-2] \end{aligned} \tag{3.106}$$

$y[n]$ and $x[n]$ are, respectively, the output and the input sequences.
The desiign objective is to choose suitable values of α and β so that the output is a sinusoidal sequence with a frequency of 0.4 rad/sample.
Now, the frequency response of the FIR filter is given by:

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} \\ &= \alpha(1 + e^{-j2\omega}) + \beta e^{-j\omega} \\ &= 2\alpha\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)e^{-j\omega} + \beta e^{-j\omega} \\ &= (2\alpha\cos\omega + \beta)e^{-j\omega} \end{aligned} \tag{3.107}$$

The magnitude and phase functions are:

$$\left|H(e^{j\omega})\right| = 2\alpha\cos\omega + \beta \tag{3.108}$$
$$\theta(\omega) = -\omega \tag{3.109}$$

In order to block the low-frequency component, the magnitude function at $\omega = 0.1$ should be equal to zero. Likewise, to pass the high-frequency component, the magnitude function at $\omega = 0.4$ should be equal to one. Thus, the two conditions that must be satisfied are:

$$\left|H(e^{j0.1})\right| = 2\alpha\cos(0.1) + \beta = 0 \tag{3.110}$$
$$\left|H(e^{j0.4})\right| = 2\alpha\cos(0.4) + \beta = 1 \tag{3.111}$$

Solving the above two equations we get:

$$\alpha = -6.76195 \tag{3.112}$$
$$\beta = 13.456335 \tag{3.113}$$
$$\tag{3.114}$$

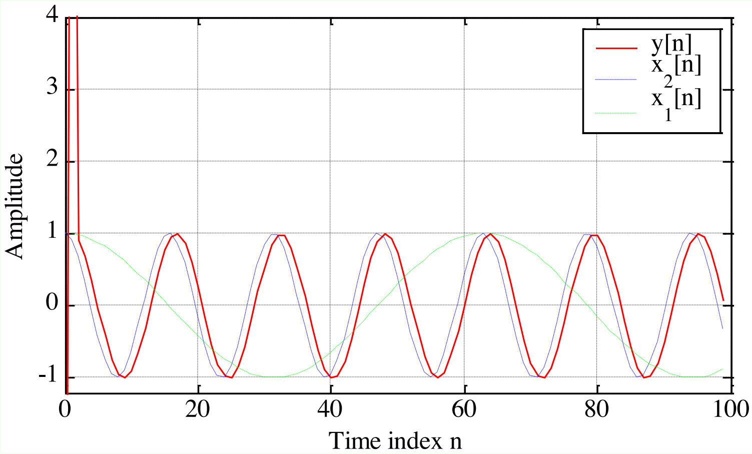
Thus the output-input relation of the FIR filter is given by:

$$y[n] = -6.76195(x[n] + x[n-2]) + 13.456335x[n-1] \tag{3.115}$$

where the input is:

$$x[n] = \{\cos(0.1n) + \cos(0.4n)\}\mu[n] \tag{3.116}$$

A plot of the signals of interests is showed below.



The first seven samples of the output are showed below as well.

n	$\cos(0.1n)$	$\cos(0.4n)$	$x[n]$	$y[n]$
0	1.0	1.0	2.0	-13.52390
1	0.9950041	0.9210609	1.9160652	13.956333
2	0.9800665	0.6967067	1.6767733	0.9210616
3	0.9553364	0.3623577	1.3176942	0.6967064
4	0.9210609	-0.0291995	0.8918614	0.3623572
5	0.8775825	-0.4161468	0.4614357	-0.0292002
6	0.8253356	-0.7373937	0.0879419	-0.4161467

From this table, it can be seen that, neglecting the least significant digit:

$$y[n] = \cos(0.4(n-1)), \qquad n \geq 2 \tag{3.117}$$

Computation of the present value of the output requires the knowledge of the present and two previous input samples. Hence, the first two output samples, $y[0]$ and $y[1]$, are the result of assumed zero input sample values at $n = -1$ and $n = -2$. Therefore, first two output samples constitute the transient part of the output. Since the impulse response is of length 3, the steady-state is reached at $n = N = 2$. Note also that the output is delayed version of the high-frequency component $\cos(0.4n)$ of the input, and the delay is one sample period.

Lecture 12. *Thursday 5th November, 2020.*

3.5 Discrete Fourier Transform

We have discussed the DTFT for a discrete-time function given by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{3.118}$$

and the IDTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega \tag{3.119}$$

The pair and their properties and applications have some limitations. The input signal is usually aperiodic and may be finite in length.

Moreover, we often do not have an infinite amount of data which is required by DTFT. For example in a computer we cannot calculate uncountable infinite (continuum) of

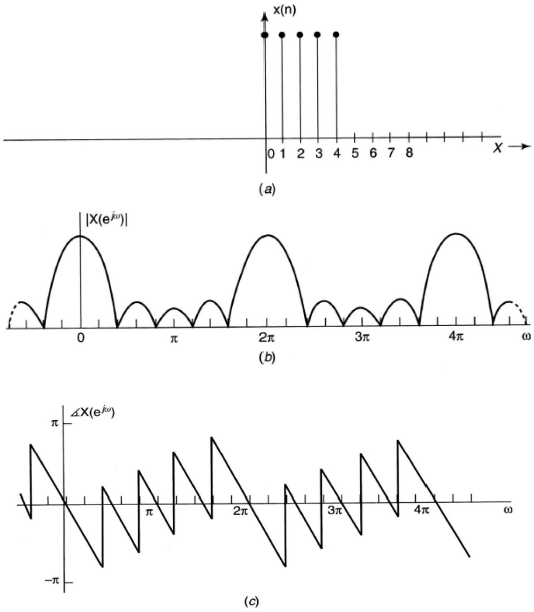


Figure 3.8: In order from top to bottom, a finite-length signal, its magnitude spectrum, its phase spectrum.

frequencies as required by DTFT. Thus, we use DTF to look at finite segment of data. We only observe the data through a window:

$$x_0[n] = x[n]w_R[n] \tag{3.120}$$

$$w_R[n] = \begin{cases} 1 & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases} \tag{3.121}$$

In this case, the $x_0[n]$ is just a sampled data between $n = 0$, $n = N-1$ (so, N points). The solution to our problems is given by the Discrete Fourier Transform (DFT).

Definition 4: Discrete Fourier Transform (DFT)

The simplest relation between a length-N sequence $x[n]$, defined for $0 \leq n \leq N-1$, and its DTFT $X(e^{j\omega})$ is obtained by uniformly sampling on the ω-axis between $0 \leq \omega \leq 2\pi$ at $\omega_k = \frac{2\pi k}{N}$, for $0 \leq k \leq N-1$. From the definition of the DTFT we thus have:

$$X[k] = \left[X(e^{j\omega})\right]_{\omega = \frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi k \frac{n}{N}} \tag{3.122}$$

Note that $X[k]$ is also a length-N sequence in the frequency domain and it is called the Discrete Fourier Transform (DFT) of the sequence $x[n]$. Using the notation $W_N = e^{-j\frac{2\pi}{N}}$, the DFT is usually expressed as:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad 0 \leq k \leq N-1 \tag{3.123}$$

Definition 5: Inverse Discrete Fourier Transform (IDFT)

The Inverse Discrete Fourier Transform (IDFT) is given by:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad 0 \leq n \leq N-1 \tag{3.124}$$

To verify the above expression we multiply both sides of the above equation by $W_N^{\ell n}$ and sum the result from $n=0$ to $n=N-1$, resulting in:

$$\begin{aligned}\sum_{n=0}^{N-1} x[n]W_N^{\ell n} &= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right)W_N^{\ell n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} X[k]W_N^{-(k-\ell)n} \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} X[k]W_N^{-(k-\ell)n}\end{aligned} \tag{3.125}$$

Making use of the identity:

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} = \begin{cases} N & k-\ell = rN,\ n \in \mathbb{Z} \\ 0 & \text{otherwise}\end{cases} \tag{3.126}$$

we observe that the right-hand-side of the last equation is equal to $X[\ell]$. Hence:

$$\sum_{n=0}^{N-1} x[n]W_N^{\ell n} = X[\ell] \tag{3.127}$$

Example 14: Discrete Fourier Transform

Consider the length-N sequence:

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & 1 \leq n \leq N-1\end{cases} \tag{3.128}$$

Its N-point DFT is given by:

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} = x[0]W_N^0 = 1 \tag{3.129}$$

with $0 \leq k \leq N-1$.

Example 15: Discrete Fourier Transform

Consider the length-N sequence:

$$y[n] = \begin{cases} 1 & n = m \\ 0 & 0 \leq n \leq m-1,\ m+1 \leq n \leq N-1\end{cases} \tag{3.130}$$

Its N-point DFT is given by:

$$Y[k] = \sum_{n=0}^{N-1} y[n]W_N^{kn} = y[m]W_N^{km} = W_N^{km} \tag{3.131}$$

with $0 \le k \le N-1$.

Example 16: Discrete Fourier Transform

Consider the length-N sequence defined for $0 \le n \le N-1$:

$$g[n] = \cos\left(\frac{2\pi rn}{N}\right), \qquad 0 \le r \le N-1 \tag{3.132}$$

Using trigonometic identities, we can rewrite:

$$g[n] = \frac{1}{2}\left(e^{j2\pi r\frac{n}{M}} + e^{-j2\pi r\frac{n}{N}}\right) = \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right) \tag{3.133}$$

The N-point DFT of $g[n]$ is thus given by:

$$G[k] = \sum_{n=0}^{N-1} g[n]W_N^{kn} = \frac{1}{2}\left(\sum_{n=0}^{N-1} W_N^{-(r-k)n} + \sum_{n=0}^{N-1} W_N^{(r+k)n}\right) \tag{3.134}$$

with $0 \le k \le N-1$. Making use of the identity:

$$\sum_{n=0}^{N-1} W_N?{-}(k-\ell)n = \begin{cases} N & k-\ell = rN,\ r \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{3.135}$$

we get:

$$\begin{cases} \frac{N}{2} & k = r \\ \frac{N}{2} & k = N-r \\ 0 & \text{otherwise} \end{cases} \tag{3.136}$$

with $0 \le k \le N-1$.

3.5.1 Matrix relations

The DFT samples defined by:

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}, \qquad 0 \le k \le N-1 \tag{3.137}$$

can be expressed in matrix form as:

$$\mathbf{X} = \mathbf{D}_N\mathbf{x} \tag{3.138}$$

where:

$$\mathbf{X} = [X[0]\ X[1]\ \ldots\ X[N-1]]^T \tag{3.139}$$

$$\mathbf{x} = [x[0]\ x[1]\ \ldots\ x[N-1]]^T \tag{3.140}$$

and $\mathbf{D}_N$ is the $N \times N$ DFT matrix given by:

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \tag{3.141}$$

Likewise, the IDFT relation given by:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad 0 \le n \le N-1 \tag{3.142}$$

can be expressed in matrix form as:

$$\mathbf{x} = \mathbf{D}_N^{-1}\mathbf{X} \tag{3.143}$$

where $\mathbf{D}_N^{-1}$ is the $N \times N$ IDFT matrix, given by:

$$\mathbf{D}_N^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix} = \frac{1}{N}\mathbf{D}_N^{*} \tag{3.144}$$

3.5.2 DTFT from DFT by interpolation

The N-point DFT $X[k]$ of a length-N sequence $x[n]$ is simply the frequency samples of its DTFT $X(e^{j\omega})$ evaluated at N uniformly spaced frequency points:

$$\omega = \omega_k = \frac{2\pi k}{N}, \qquad 0 \le k \le N-1 \tag{3.145}$$

Given the N-point DFT $X[k]$ of a length-N sequence $x[n]$, its DTFT $X(e^{j\omega})$ cane be uniquely determined from $X[k]$. Thus:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{N-1} x[n]e^{-k\omega n} \\ &= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right]e^{-j\omega n} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\underbrace{\sum_{n=0}^{N-1} e^{-j\left(\omega-\frac{2\pi k}{N}\right)n}}_{S} \end{aligned} \tag{3.146}$$

To develop a compact expression for the sum S, let $r = e^{-j\left(\omega-\frac{2\pi k}{N}\right)}$. Then:

$$S = \sum_{n=0}^{N-1} r^n \tag{3.147}$$

From the above:

$$\begin{aligned} rS &= \sum_{n=1}^{N} r^n = 1 + \sum_{n=1}^{N-1} r^n r^N - 1 \\ &= \sum_{n=0}^{N-1} r^n + r^N - 1 = S + r^N - 1 \end{aligned} \tag{3.148}$$

or, equivalently:

$$S - rS = (1-r)S = 1 - r^N \tag{3.149}$$

Hence:

$$\begin{aligned}
S &= \frac{1-r^N}{1-r} \\
&= \frac{1-e^{-j(\omega n-2\pi k)}}{1-e^{-j\left(\omega-\frac{2\pi k}{N}\right)}} \\
&= \frac{\sin\left(\dfrac{\omega N 2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N-2\pi k}{2N}\right)} e^{-j\left(\frac{\omega-2\pi k}{N}\right)\left(\frac{N-1}{2}\right)} \qquad (3.150)
\end{aligned}$$

Therefore:

$$X(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \frac{\sin\left(\dfrac{\omega N 2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N-2\pi k}{2N}\right)} e^{-j\left(\frac{\omega-2\pi k}{N}\right)\left(\frac{N-1}{2}\right)} \qquad (3.151)$$

3.5.3 Sampling the DTFT

Consider a sequence $x[n]$ with a DTFT $X(e^{j\omega})$. We sample $X(e^{j\omega})$ at N equally spaced points $\omega_k = \frac{2\pi k}{N}$, $0 \le k \le N-1$, developing the N frequency samples $\left\{X(e^{j\omega_k})\right\}$. These N frequency samples can be considered as an N-point DFT $Y[k]$ whose N-point IDFT is a length-N sequence $y[n]$. Now:

$$X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell} \qquad (3.152)$$

Thus:

$$Y[k] = X(e^{j\omega_k}) = X(e^{j\frac{2\pi k}{N}}) = \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j2\pi k\frac{\ell}{N}} = \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} \qquad (3.153)$$

An IDFT of $Y[k]$ yields:

$$\begin{aligned}
y[n] &= \frac{1}{N}\sum_{k=0}^{N-1} Y[k] W_N^{-kn} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} W_N^{-kn} \\
&= \sum_{\ell=-\infty}^{\infty} x[\ell]\left[\sum_{k=0}^{N-1} W_N^{-k(n-\ell)}\right] \\
&= \sum_{m=-\infty}^{\infty} x[n+mN] \qquad (3.154)
\end{aligned}$$

with $0 \le n \le N-1$, where in the last passage the identity in Eq. 3.135 is employed. Thus, $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0 \le n \le N-1$.

To apply the last result to finite-length sequences, we assume that the samples outside the specified range are zeros. Thus, if $x[n]$ is a length-M sequence with $M \le N$, then $y[n] = x[n]$ for $0 \le n \le N-1$. If $M > N$, there is a time-domain aliasing of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$.

Example 17: Aliasing

Let $x[n] = \{0, 1, 2, 3, 4, 5\}$. By sampling its DTFT $X(e^{j\omega})$ at $\omega_k = \frac{2\pi k}{4}$, with $0 \le k \le 3$, and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by:

$$y[n] = x[n] + x[n+4] + x[n-4], \qquad 0 \le n \le 3 \tag{3.155}$$

We get $y[n] = \{4, 6, 2, 3\}$. $x[n]$ cannote be recovered from $y[n]$.

3.5.4 DFT properties

Like the DTFT, the DFT also satisfies a number of properties that are useful in signal processing applications. Some of these properties are essentially identical to those of the DTFT, while some others are somewhat different. A summary of the DFT properties are given in Figures 3.9, 3.10 and 3.11.

Length-N Sequence	**N-point DFT**
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k \rangle_N]$
$x^*[\langle -n \rangle_N]$	$X^*[k]$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{pcs}}[k] = \frac{1}{2}\{X[\langle k \rangle_N] + X^*[\langle -k \rangle_N]\}$
$j\,\mathrm{Im}\{x[n]\}$	$X_{\mathrm{pca}}[k] = \frac{1}{2}\{X[\langle k \rangle_N] - X^*[\langle -k \rangle_N]\}$
$x_{\mathrm{pcs}}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{\mathrm{pca}}[n]$	$j\,\mathrm{Im}\{X[k]\}$

Note: $x_{\mathrm{pcs}}[n]$ and $x_{\mathrm{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\mathrm{pcs}}[k]$ and $X_{\mathrm{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

Figure 3.9: Symmetry relations of DFT for a complex sequence $x[n]$.

Length-N Sequence	**N-point DFT**
$x[n]$	$X[k] = \mathrm{Re}\{X[k]\} + j\,\mathrm{Im}\{X[k]\}$
$x_{\mathrm{pe}}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{\mathrm{po}}[n]$	$j\,\mathrm{Im}\{X[k]\}$
	$X[k] = X^*[\langle -k \rangle_N]$
	$\mathrm{Re}\,X[k] = \mathrm{Re}\,X[\langle -k \rangle_N]$
Symmetry relations	$\mathrm{Im}\,X[k] = -\mathrm{Im}\,X[\langle -k \rangle_N]$
	$\lvert X[k] \rvert = \lvert X[\langle -k \rangle_N] \rvert$
	$\arg X[k] = -\arg X[\langle -k \rangle_N]$

Note: $x_{\mathrm{pe}}[n]$ and $x_{\mathrm{po}}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

Figure 3.10: Symmetry relations of DFT for a real sequence $x[n]$.

3.5.5 Circular shift of a sequence

This property is analogous to the time-shifting property of the DTFT but with a difference. COnsider length-N squences defined for $0 \le n \le N-1$. Sample values

Type of Property	Length-N Sequence	N-point DFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_o \rangle_N]$	$W_N^{k n_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o n} g[n]$	$G[\langle k - k_o \rangle_N]$
Duality	$G[n]$	$N g[\langle -k \rangle_N]$
N-point circular convolution	$\sum_{m=0}^{N-1} g[m] h[\langle n - m \rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N} \sum_{m=0}^{N-1} G[m] H[\langle k - m \rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$	

Figure 3.11: General properties of DFT.

of such sequences are equal to zero for values of $n < 0$ and $n \geq N$. If $x[n]$ is such a sequence, then for any arbitrary integer n_0, the shifted sequence $x_1[n] = x[n - n_0]$ is no longer defined for the range $0 \leq n \leq N - 1$. We thus need to define another type of shift that will always keep the shifted sequence in the range $0 \leq n \leq N - 1$. The desired shift, called the circular shift, is defined using a modulo operation:

$$x_c[n] = x[\langle n - n_0 \rangle_N] \tag{3.156}$$

For $n_0 > 0$ (right circular shift), the above equation implies:

$$x_c[n] = \begin{cases} x[n - n_0] & n_0 \leq n \leq N - 1 \\ x[N - n_0 + n] & 0 \leq n \leq n_0 \end{cases} \tag{3.157}$$

An illustration of the concept of circular shift is showed in Figure 3.12. As it is possible to observe, a right circular shift by n_0 is equivalent to a left circular shift by $N - n_0$ sample periods. A circular shift by an integer number n_0 greater than N is equivalent to a circular shift by $\langle n_0 \rangle_N$.

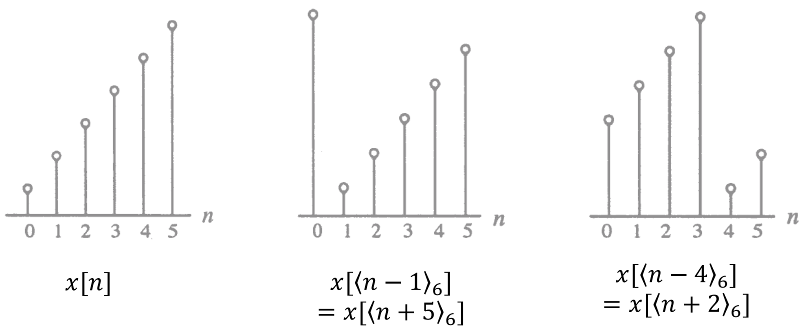


Figure 3.12: Illustration of circular shift.

3.5.6 Circular convolution

This operation is analogous to linear convolution, but with a difference. Consider two length-N sequences, $g[n]$ and $h[n]$, respectively. Their linear convolution results in a length-$(2N - 1)$ sequence $y_L[n]$ given by:

$$y_L[n] = \sum_{m=0}^{N-1} g[m]h[n - m], \qquad 0 \leq n \leq 2N - 2 \tag{3.158}$$

In computing $y_L[n]$ we have assumed that both length-N sequences have been zero-padded to extend their lengths to $2N-1$. The longer form of $y_L[n]$ results from the time-reversal of the sequence $h[n]$ and its linear shift to the right. The first nonzero value of $y_L[n]$ is $y_L[0] = g[0]h[0]$ and the last nonzero value is $y_L[2N-2] = g[N-1]h[N-1]$.
To develop a convolution-like operation resulting in a length-N sequence $y_C[n]$, we need to define a circular time-reversal, and then apply a circular time-shift. Resulting operation, called a circular convolution, is defined by:

$$y_C[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N], \qquad 0 \le n \le N-1 \tag{3.159}$$

Since the operation defined involves two length-N sequences, it is often referred to as an N-point circular convolution, denoted as:

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3.5.7 DFT of real sequences

In most practical applications, sequences of interest are real. In such cases, the symmetry properties of the DFT can be exploited to make the DFT computations more efficient.
Let $g[n]$ and $h[n]$ be two length-N real sequences with $G[k]$ and $H[k]$ denoting their respective N-point DFTs. These two N-point DFTs can be computed efficiently using a single N-point DFT. Now, define a complex length-N sequence:

$$x[n] = g[n] + jh[n] \tag{3.160}$$

Hence, $g[n] = \mathrm{Re}\{x[n]\}$ and $h[n] = \mathrm{Im}\{x[n]\}$. Let $X[k]$ denote the N-point DFT of $x[n]$. Then, we arrive at:

$$G[k] = \frac{1}{2}\{X[k] + X^*[\langle -k\rangle_N]\} \tag{3.161}$$

$$H[k] = \frac{1}{2j}\{X[k] - X^*[\langle -k\rangle_N]\} \tag{3.162}$$

Note that for $0 \le k \le N-1$:

$$X^*[\langle -k\rangle_N] = X^*[\langle N-k\rangle_N] \tag{3.163}$$

Example 18: DFT of real sequences

We compute the 4-point DFTs of the two real sequences $g[n]$ and $h[n]$:

$$g[n] = \{1, 2, 0, 1\} \tag{3.164}$$

$$h[n] = \{2, 2, 1, 1\} \tag{3.165}$$

Then $x[n] = g[n] + jh[n]$ is given by:

$$x[n] = \{1 + j2, 2 + j2, 1 + j\} \tag{3.166}$$

Its DFT $X[k]$ is:

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -1 \end{bmatrix} \begin{bmatrix} 1+j2 \\ 2+j2 \\ j \\ 1+j \end{bmatrix} = \begin{bmatrix} 4+j6 \\ 2 \\ -2 \\ j2 \end{bmatrix} \tag{3.167}$$

From the above:

$$X^*[k] = [4 - j6, 2, -2, -j2] \tag{3.168}$$

Hence:

$$X^*[\langle 4-k\rangle_4] = [4-j6, -j2, -2, 2] \tag{3.169}$$

Therefore:

$$G[k] = [4, 1-j, -2, 1+j] \tag{3.170}$$
$$H[k] = [6, 1-j, 0, 1+j] \tag{3.171}$$

verifying the results derived in Lecture 12

Now, let $v[n]$ be a length-$2N$ real sequence with a $2N$-point DFT $V[k]$. Define two length-N real sequences $g[n]$ and $h[n]$ as follows. Let $G[k]$ and $H[k]$ denote their respective N-point DFTs:

$$\begin{cases} g[n] = v[2n] \\ h[n] = v[2n+1] \end{cases} \qquad 0 \leq n \leq N \tag{3.172}$$

We define a length-N complex sequence $x[n] = g[n] + jh[n]$ with an N-point DFT $X[k]$. Then, as showed earlier:

$$G[k] = \frac{1}{2}\{X[k] + X^*[\langle -k\rangle_N]\} \tag{3.173}$$
$$H[k] = \frac{1}{2j}\{X[k] - X^*[\langle -k\rangle_N]\} \tag{3.174}$$

Now, for $0 \leq k \leq 2N-1$:

$$\begin{aligned} V[k] &= \sum_{n=0}^{2N-1} v[n] W_{2N}^{nk} \\ &= \sum_{n=0}^{N-1} v[2n] W_{2N}^{2nk} + \sum_{n=0}^{N-1} v[2n+1] W_{2N}^{(2n+1)k} \\ &= \sum_{n=0}^{N-1} g[n] W_{N}^{nk} + \sum_{n=0}^{N-1} W_{N}^{nk} W_{2N}^{k} \\ &= \sum_{n=0}^{N-1} g[n] W_{N}^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} h[n] W_{N}^{nk} \\ &= G[\langle k\rangle_N] + W_{2N}^{k} H[\langle k\rangle_N] \end{aligned} \tag{3.175}$$

Example 19: DFT of real sequences

Let us determine the 8-point DFT $V[k]$ of the length-8 real sequence:

$$v[n] = \{1, 2, 2, 2, 0, 1, 1, 1\} \tag{3.176}$$

We form two length-4 real sequences as follows:

$$g[n] = v[2n] = \{1, 2, 0, 1\} \tag{3.177}$$
$$h[n] = v[2n+1] = \{2, 2, 1, 1\} \tag{3.178}$$

Now:

$$V[k] = G[\langle k\rangle_4] + W_8^k H[\langle k\rangle_4] \qquad 0 \leq k \leq 7 \tag{3.179}$$

Substituting the values of of the 4-point DFTs $G[k]$ and $H[k]$ computed earlier, we get:

$$V[0] = G[0] + H[0] = 4 + 6 = 10 \tag{3.180}$$
$$V[1] = G[1] + W_1^0 H[1] = (1-j) + e^{-j\frac{\pi}{4}}(1-j) = 1 - j2.4142 \tag{3.181}$$
$$V[2] = G[2] + W_2^0 H[2] = -2 + e^{-j\frac{3\pi}{4}} \cdot 0 = -2 \tag{3.182}$$
$$V[3] = G[3] + W_3^0 H[3] = (1+j) + e^{-j\frac{3\pi}{4}}(1+j) = 1 - j0.4142 \tag{3.183}$$
$$V[4] = G[0] + W_4^0 H[0] = 4 + e^{-j\pi} \cdot 6 = -2 \tag{3.184}$$
$$V[5] = G[1] + W_5^0 H[1] = (1-j) + e^{-j\frac{5\pi}{4}}(1-j) = 1 + j0.4142 \tag{3.185}$$
$$V[6] = G[2] + W_6^0 H[2] = -2 + e^{-j\frac{3\pi}{2}} \cdot 0 = -2 \tag{3.186}$$
$$V[7] = G[3] + W_7^0 H[3] = (1+j) + e^{-j\frac{7\pi}{4}}(1+j) = 1 + j2.4142 \tag{3.187}$$

3.6 Linear convolution using the DFT

Linear convolution is a key operation in many signal processing applications. Since a DFT can be efficiently implemented using FFT algorithms, it is of interest to develop methods for the implementation of linear convolution using the DFT.
Let $g[n]$ and $h[n]$ be two finite-length sequences of length N and M, respectively. Moreover, we denote with $L = N + M + 1$. Then, we define two length-L sequences:

$$g_e[n] = \begin{cases} g[n] & 0 \le n \le N-1 \\ 0 & N \le n \le L-1 \end{cases} \tag{3.188}$$

$$h_e[n] = \begin{cases} h[n] & 0 \le n \le M-1 \\ 0 & M \le n \le L-1 \end{cases} \tag{3.189}$$

Then:

$$y_L[n] = g[n] * h[n] = y_C[n] = g_e[n] \overset{L}{*} h_e[n] \tag{3.190}$$

The corresponding implementation scheme is illustrated in Figure 3.13.

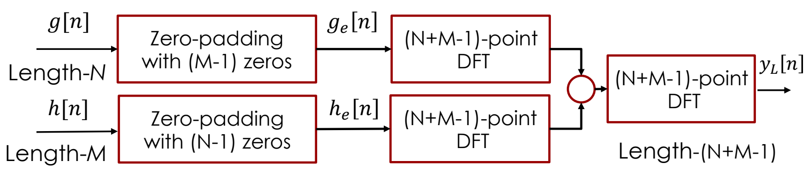

Figure 3.13: Scheme of linear convolution of two finite-length sequences.

We next consider the DFT-based implementation of:

$$y[n] = \sum_{\ell=0}^{M-1} h[\ell]x[n-\ell] = h[n] * x[n] \tag{3.191}$$

where $h[n]$ is a finite-length sequence of length M and $x[n]$ is an infinite length (or a finite length sequence of length much greater that M). We first segment $x[n]$, assumed to be a causal sequence here without any loss of generality, into a set of contiguous finite-length subsequences $x_m[n]$ of length N each:

$$x[n] = \sum_{m=0}^{\infty} x_m[n - mN] \tag{3.192}$$

where:

$$x_m[n] = \begin{cases} x[n+mN] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \tag{3.193}$$

Thus, we can write:

$$y[n] = h[n] * x[n] = \sum_{m=0}^{\infty} y_m[n-mN] \tag{3.194}$$

where:

$$y_m = h[n] * x_m[n] \tag{3.195}$$

Since $h[n]$ is of length M and $x_m[n]$ is of length N, the linear convolution $h[n] * x_m[n]$ is of length $N+M-1$.
As a result, the desired linear convolution $y[n] = h[n] * x[n]$ has been broken up into a sum of infinite numbers of short-length linear convolutions of length $N+M-1$ each:

$$y_m[n] = h[n] * x_m[n] \tag{3.196}$$

Each of these short convolutions can be implemented using the DFT-based method discussed earlier, where now the DFTs (and the IDFT) are computed on the basis of $N+M-1$ points.
There is one more subtlety to take care of before we can implement:

$$y[n] = \sum_{m=0}^{\infty} y_m[n-mN] \tag{3.197}$$

using the DFT-based approach. Now the first convolution in Eq. 3.197, namely $y_0 = h[n] * x_0[n]$, is of length $N+M-1$ abd us defined for $0 \le n \le N+M-2$. The second short convolution, namely $y_1[n] = h[n] * x_1[n]$, is also of length $N+M-1$ but it is deined for $N \le n \le 2N+M-2$. There is an overlap of $M-1$ samples between these two short linear convolutions. Likewisem the third short convolution, namely $y_2[n] = h[n]*x_2[n]$, is also of length $N+M-1$ but is defined for $2N \le b \le 3N+M-2$. Thus, there is an overlap of $M-1$ samples between $h[n] * x_1[n]$ and $h[n] * x_2[n]$.
In general, there will be an overlap of $M-1$ samples between the samples of the short convolutions $h[n] * x_{r-1}[n]$ and $h[n] * x_r[n]$ for $(r-1)N \le n \le rN+M-1$.
This process is illustrated in Figures 3.14 and 3.15 for $M = 5$ and $N = 7$.
Therefore, $y[n]$ obtained by a linear convolution of $x[n]$ and $h[n]$ is given by:

$$y[n] = y_0[n] \qquad 0 \le n \le 6 \tag{3.198}$$
$$y[n] = y_0[n] + y_1[n-7] \qquad 7 \le n \le 10 \tag{3.199}$$
$$y[n] = y_1[n-7] \qquad 11 \le n \le 13 \tag{3.200}$$
$$y[n] = y_1[n-7] + y_2[n-14] \qquad 14 \le n \le 17 \tag{3.201}$$
$$y[n] = y_2[n-14] \qquad 18 \le n \le 20 \tag{3.202}$$
$$\vdots \qquad\qquad \vdots$$

The above procedure is called the overlap-add method since the results of the short linear convolutions overlap and the overlapped portions are added to get the correct final result.
In implementing the overlap-add method using the DFT, we need to compute two $(N+M-1)$-point DFTs and one $(N+M-1)$-point IDFT since the overall linear

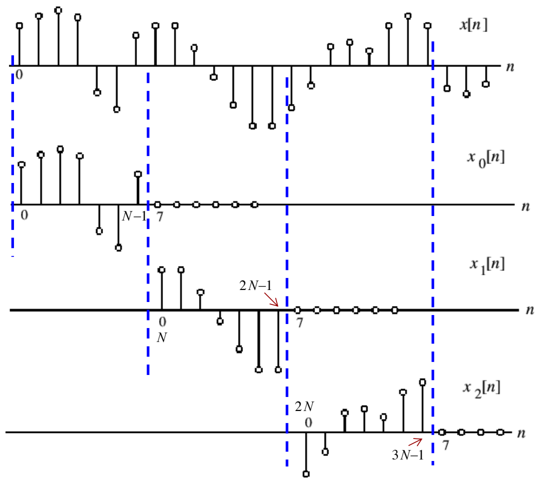

Figure 3.14: Overlap-add method for $M = 5$ and $N = 7$.

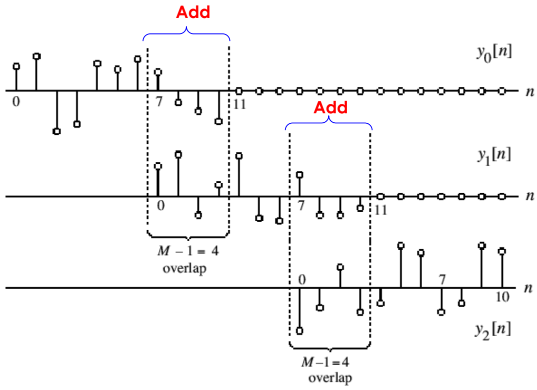

Figure 3.15: Overlap-add method for $M = 5$ and $N = 7$.

convolution was expressed as a sum of short-length linear convolutions of length $N + M - 1$ each. It is possible to implement the overall linear convolution by performing instead circular convolution of length shorter than $N + M - 1$. To this end, it is necessary to segment $x[n]$ into overlapping blocks $x_m[n]$, keep the terms of the circular convolution of $h[n]$ with $x_m[n]$ that corresponds to the terms obtained by a linear convolution of $h[n]$ and $x_m[n]$, and throw away the other parts of the circular convolution.
To understand the correspondence between the linear and circular convolutions, consider a length-4 sequence $x[n]$ and a length-3 sequence $h[n]$. Let $y_L[n]$ denote the result of a linear convolution of $x[n]$ with $h[n]$. The six samples of $y_L[n]$ are given by:

$$y_L[0] = h[0]x[0] \tag{3.203}$$
$$y_L[1] = h[0]x[1] + h[1]x[0] \tag{3.204}$$
$$y_L[2] = h[0]x[2] + h[1]x[1] + h[2]x[0] \tag{3.205}$$
$$y_L[3] = h[0]x[3] + h[1]x[2] + h[2]x[1] \tag{3.206}$$
$$y_L[4] = h[1]x[3] + h[2]x[2] \tag{3.207}$$
$$y_L[5] = h[2]x[3] \tag{3.208}$$

If we append $h[n]$ with a single zero-valued sample and convert it into a length-4 sequence $h_e[n]$, the 4-point circular convolution $y_C[n]$ of $h_e[n]$ and $x[n]$ is given by:

$$y_C[0] = h[0]x[0] + h[1]x[3] + h[2]x[2] \tag{3.209}$$
$$y_C[1] = h[0]x[1] + h[1]x[0] + h[2]x[3] \tag{3.210}$$
$$y_C[2] = h[0]x[2] + h[1]x[1] + h[2]x[0] \tag{3.211}$$
$$y_C[3] = h[0]x[3] + h[1]x[2] + h[2]x[1] \tag{3.212}$$

If we compare the expressions for the samples of $y_L[n]$ with the samples of $y_C[n]$, we observe that the first 2 terms of $y_C[n]$ do not correspond to the first 2 terms of $y_L[n]$, whereas the last 2 terms of $y_C[n]$ are precisely the same as the third and the forth terms of $y_L[n]$.
In general, if we consider the N-point circular convolution of a length-M sequence $h[n]$ with a length-N sequence $x[n]$ with $N > M$, the first $M-1$ samples of the circular convolution are incorrect and are rejected. The remaining $N-M+1$ samples correspond to the correct samples of the linear convolution of $h[n]$ with $x[n]$.
Now we consider an infinitely long or very long sequence $x[n]$. We break it up as a collection of smaller length (length-4) overlapping sequences $x_m[n]$ as $x_m[n] = x[n+2m]$, with $0 \leq n \leq 3$, $0 \leq m \leq \infty$. Next, we form:

$$w_m[n] = h[n] \overset{4}{*} x_m[n] \tag{3.213}$$

or, equivalently

$$w_m[0] = h[0]x_m[0] + h[1]x_m[3] + h[2]x_m[2] \tag{3.214}$$
$$w_m[1] = h[0]x_m[1] + h[1]x_m[0] + h[2]x_m[3] \tag{3.215}$$
$$w_m[2] = h[0]x_m[2] + h[1]x_m[1] + h[2]x_m[0] \tag{3.216}$$
$$w_m[3] = h[0]x_m[3] + h[1]x_m[2] + h[2]x_m[1] \tag{3.217}$$

Computing the above for $m = 0, 1, 2, 3, \ldots$, and substituting the values of $x_m[n]$, we arrive at:

$$\begin{aligned}
w_0[0] &= h[0]x[0] + h[1]x[3] + h[2]x[2] && \leftarrow \text{Reject}\\
w_0[1] &= h[0]x[1] + h[1]x[0] + h[2]x[3] && \leftarrow \text{Reject}\\
w_0[2] &= h[0]x[2] + h[1]x[1] + h[2]x[0] = y[2] && \leftarrow \text{Save}\\
w_0[3] &= h[0]x[3] + h[1]x[2] + h[2]x[1] = y[3] && \leftarrow \text{Save}\\
w_1[0] &= h[0]x[2] + h[1]x[5] + h[2]x[4] && \leftarrow \text{Reject}\\
w_1[1] &= h[0]x[3] + h[1]x[2] + h[2]x[5] && \leftarrow \text{Reject}\\
w_1[2] &= h[0]x[4] + h[1]x[3] + h[2]x[2] = y[4] && \leftarrow \text{Save}\\
w_1[3] &= h[0]x[5] + h[1]x[4] + h[2]x[3] = y[5] && \leftarrow \text{Save}\\
w_2[0] &= h[0]x[4] + h[1]x[5] + h[2]x[6] && \leftarrow \text{Reject}\\
w_2[1] &= h[0]x[5] + h[1]x[4] + h[2]x[7] && \leftarrow \text{Reject}\\
w_2[2] &= h[0]x[6] + h[1]x[5] + h[2]x[4] = y[6] && \leftarrow \text{Save}\\
w_2[3] &= h[0]x[7] + h[1]x[6] + h[2]x[5] = y[7] && \leftarrow \text{Save}
\end{aligned}$$

It should be noted that tho determine $y[0]$ and $y[1]$ we need to form $x_{-1}[n]$, setting $x_{-1}[0] = x_{-1}[1] = 0$, $x_{-1}[2] = x[0]$, $x_{-1}[3] = x[1]$, and compute:

$$w_{-1}[n] = h[n] \overset{4}{*} x_{-1}[n] \qquad 0 \leq n \leq 3 \tag{3.218}$$

then reject $w_{-1}[0]$ and $w_{-1}[1]$, and save $w_{-1}[2] = y[0]$ and $w_{-1}[3] = y[1]$.

In general, let $h[n]$ be a length-N sequence. Let $x_m[n]$ denote the m^{th} section of an infinitely long sequence $x[n]$ of length N and defined by:

$$x_m[n] = x[n + m(N - m + 1)] \qquad 0 \leq n \leq N - 1 \tag{3.219}$$

with $M < N$. Let $w_m[n] = h[n] \overset{N}{*} x_m[n]$. Then, we reject the first $M - 1$ samples of $w_m[n]$ and "about" the remaining $M - M + 1$ samples of $w_m[n]$ to form $y_L[n]$, namely the linear convolution of $h[n]$ and $x[n]$. If $y_m[n]$ denotes the saved portion of $w_m[n]$, i.e.:

$$y_m[n] = \begin{cases} 00 & 0 \leq n \leq M - 2 \\ w_m[n] & M - 1 \leq n \leq N - 2 \end{cases} \tag{3.220}$$

then:

$$y_L[n + m(N - M + 1)] = y_m[n] \qquad M - 1 \leq n \leq N - 1 \tag{3.221}$$

The approach is called overlap-save method since the input is segmented into overlapping sections and parts of the results of the circular convolutions are saved and abutted to determine the linear convolution result. The process is illustrated in Figure 3.16.

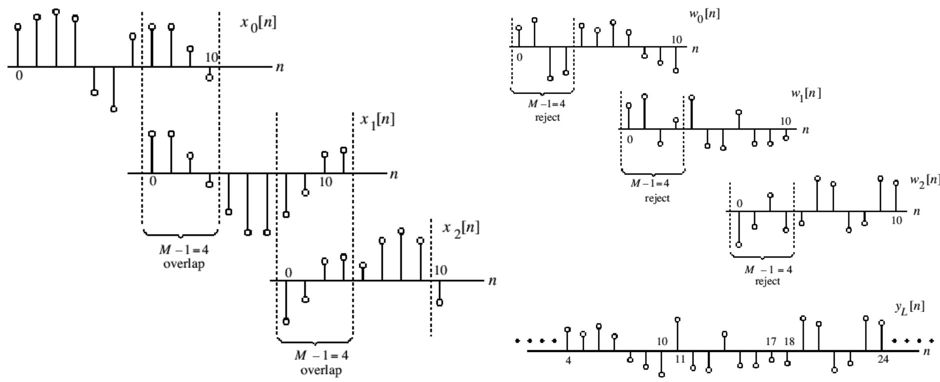

Figure 3.16: Illustration of the overlap-save method.

Chapter 4

The Z transform

Lecture 14.
Thursday 12th
November, 2020.

We have seen that the DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems. However, because of the convergence condition, in many cases, the DTFT of a sequence may not exist. As a result, it is not possible to make use of such frequency-domain characterization in these cases.
A possible solution and alternative is a generalization of the DTFT, which leads to the z-transform. The ladder may exist for many sequences for which the DTFT does not exist. Moreover, use of z-transform techniques permits simple but powerful algebraic manipulations. Consequently, z-transform has become an important tool in the analysis and design of digital filters

4.1 The definition

Definition 6: Z-transform

For a given sequence $g[n]$, its z-transform $G(z)$ is defined as:

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} \tag{4.1}$$

where $z = \mathrm{Re}[z] + j\,\mathrm{Im}[z]$ is a complex variable.

If we let $z = re^{j\omega}$, then the z-transform reduces to:

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n} \tag{4.2}$$

The above can be interpreted as the DTFT of the modified sequence $\{g[n]r^{-n}\}$. For $r = 1$ (i.e., $|z| = 1$), the z-transform reduces to its DTFT, provided the ladder exists.
Like the DTFT, there are conditions on the convergence of the infinite series like:

$$\sum_{n=-\infty}^{\infty} g[n]z^{-n} \tag{4.3}$$

For a given sequence, the set R of values of z for which its z-transform converges is called the region of convergence (ROC).
From our earlier discussion on the uniform convergence of the DTFT, it follows that the series:

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n} \tag{4.4}$$

converges if $\{g[n]r^{-n}\}$ is absolutely summable, i.e. if:

$$\sum_{n=-\infty}^{\infty} \left|g[n]r^{-n}\right| < \infty \tag{4.5}$$

In general, the ROC R of a z-transform of a sequence $g[n]$ is an annular region of the z-plane, namely:

$$R_{g^-} < |z| z R_{g^+} \tag{4.6}$$

where $0 \leq R_{g^-} < R_{g^+} < \infty$.

Example 20: Z-transform calculation

We determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC. Now:

$$X(z) = \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} \tag{4.7}$$

The above power series converges to:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \qquad \left|\alpha z^{-1}\right| < 1 \tag{4.8}$$

ROC is the annular region $|z| > |\alpha|$.

Example 21: Z-transform calculation

The z-transform $\mu(z)$ of the unit step sequence $\mu[n]$ can be obtained from:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \qquad \left|\alpha z^{-1}\right| < 1 \tag{4.9}$$

By setting $\alpha = 1$:

$$\mu(z) = \frac{1}{1 - z^{-1}} \qquad \left|z^{-1}\right| < 1 \tag{4.10}$$

ROC is the annular region $1 < |z| < \infty$. Note that the unit step sequence $\mu[n]$ is not obsolutely summable, and hence its DTFT does not converge uniformly.

Example 22: Z-transform calculation

Consider the anti-causal sequence:

$$y[n] = -\alpha^n \mu[-n-1] \tag{4.11}$$

Its z-transform is given by:

$$\begin{aligned} Y(z) &= -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^m \\ &= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^m = -\frac{-\alpha^{-1} z}{1 - \alpha z^{-1}} \\ &= \frac{1}{1 - \alpha z^{-1}} \end{aligned} \tag{4.12}$$

for $\left|\alpha^{-1} z\right| < 1$. ROC is the annular region $|z| < |\alpha|$.

Note that the z-transforms of the two sequences $\alpha^n\mu[n]$ and $-\alpha^n\mu[-n-1]$ are identical even though the two parent sequences are different. The only way a unique sequence can be associated with a z-transform is by specifying its ROC.
Another important point is that the DTFT $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the z-transform $G(z)$ of $g[n]$ includes the unit circle. However, the existence of the DTFT does not always imply the existence of the z-transform.

Example 23: Z-transform

Th finite energy sequence:

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n} \qquad -\infty < n < \infty \tag{4.13}$$

has a DTFT given by:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases} \tag{4.14}$$

which converges in the mean-square sense. However, $h_{LP}[n]$ does not have a z-transform as it is not absolutely summable for any value of r.

Some commonly used z-transform pairs are listed in Figure 4.1.

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
$\alpha^n\mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$(r^n\cos\omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
$(r^n\sin\omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\|z\| > r$

Figure 4.1: Common z-transform pairs.

4.2 Rational z-transforms

In the case of LTI discrete-time systems we are concerned with in this course, all pertinent z-transforms are rational functions of z^{-1}, that is, they are rations of two polinomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1}z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1}z^{-(N-1)} + d_N z^{-N}} \tag{4.15}$$

The degree of the numerator polynomial $P(z)$ is M and the degree of the denominator polynomial $D(z)$ is N. An alternate representation of a rational z-transform is as a ratio of two polynomials in z:

$$G(z) = z^{(N-M)} \frac{p_0 z^M + \cdots + p_{M-1} z + p_M}{d_0 z^N + \cdots + d_{N-1} z + d_N} \tag{4.16}$$

Again, a rational z-transform can be alternately written in factored form as:

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1 - \lambda_\ell z^{-1})} = z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)} \tag{4.17}$$

We have as roots:

- $z = \xi_\ell$, roots of the numerator polynomial. These values of z are known as the zeros of $G(z)$;
- $z = \lambda_\ell$, roots of the denominator polynomial. These values of z are known as the poles of $G(z)$.

Example 24: Zeros and poles

The z-transform:

$$\mu(z) = \frac{1}{1 - z^{-1}} \qquad |z| > 1 \tag{4.18}$$

has a zero at $z = 0$ and a pole at $z = 1$.

Example 25: ROC of a rational z-transform

The z-transform $H(z)$ of the sequence $h[n] = (-0.6)^n \mu[n]$ is given by:

$$H(z) = \frac{1}{1 + 0.6 z^{-1}} \qquad |z| > 0.6 \tag{4.19}$$

Here the ROC is just outside the circle going through the point $z = -0.6$.

A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20 \log_{10} |G(z)|$ as showed in Figure 4.2 for:

$$G(z) = \frac{1 - 2.4 z^{-1} + 2.88 z^{-2}}{1 - 0.8 z^{-1} + 0.64 z^{-2}} \tag{4.20}$$

Observe that the magnitude plot exhibits very large peaks around the points $z = 0.4 \pm j0.6928$, which are the poles of $G(z)$. It also exhibits very narrow and deep wells around the location of the zeros at $z = 1.2 \pm j1.2$.
ROC of a z-transform is an important concept. Without its knowledge, there is no unique relationship between a sequence and its z-transform. Hence, the z-transform must always be specified with its ROC. Moreover, there is a relationship between the ROC of the z-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability.
Another important distiction is that a sequence can be one of the following types: finite-length, right-sided, left-sided and two-sided. In general, the ROC depends on the type of the sequence of interest.

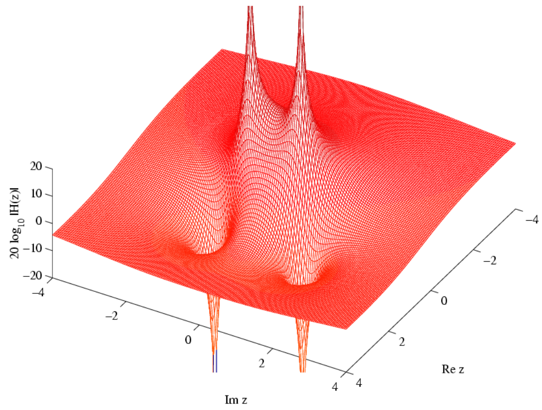

Figure 4.2: Log-magnitude plot for $G(z)$ in Eq. 4.20.

Example 26: Finite-length sequence z-transform

Consider a finite-length sequence $g[n]$ defined for $-M \leq n \leq N$, where M and N are non-negative integers and $|g[n]| < \infty$. Its z-transform is given by:

$$G(z) = \sum_{n=-M}^{N} g[n]z^{-n} = \frac{\sum_{n=0}^{N+M} g[n-M]z^{N+M-n}}{z^N} \tag{4.21}$$

Note that $G(z)$ has M zeros and N poles. As can be seen from the expression for $G(z)$, the z-transform of a finite-length bounded sequence converges everywhere in the z-plane except possibly at $z = 0$ and/or at $z = \infty$.

Example 27: Right-sided sequence z-transform

A right-sided sequence with nonzero sample values for $n \geq 0$ is sometimes called a causal sequence. So, consider a causal sequence $u_1[n]$. Its z-transform is given by:

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n]z^{-n} \tag{4.22}$$

It can be showed that $U_1(z)$ converges exterior to a circle with $|z| = R_1$, including the point $z = \infty$.
On the other hand, a right-sided sequence $u_2[n]$ with nonzero sample values only for $n \geq -M$ with M non-negative has a z-transform $U_2(z)$ with M poles at $z = \infty$. The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, exluding the point $z = \infty$.

Example 28: Left-sided sequence z-transform

A left-sided sequence with nonzero sample values for $n \leq 0$ is sometimes called anticausal sequence. So, consider an anticausal sequence $v_1[n]$. Its z-transform is given by:

$$V_1(z) = \sum_{n=-\infty}^{0} v_1[n]z^{-n} \tag{4.23}$$

It can be showed that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the

point $z = 0$.
On the other hand, a left-sided sequence with nonzero sample values only for $n \leq N$ with N non-negative has a z-transform $V_2(z)$ with N poles at $z = 0$. The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$.

Example 29: Two-sided sequence z-transform

The z-transform of a two-sided sequence $w[n]$ can be expressed as:

$$W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n} = \sum_{n=0}^{\infty} w[n]z^{-n} + \sum_{n=-\infty}^{-1} w[n]z^{-n} \tag{4.24}$$

The first term on the RHS can be interpreted as the z-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$. The second term of the RHS can be interpreted as the z-transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$. If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$. If $R_5 > R_6$, there is no overlap and the z-transform does not exist.
In particular, let us consider as example the two-sided sequence:

$$u[n] = \alpha^n \tag{4.25}$$

where α can be either real or complex. Its z-transform is given by:

$$U(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n} \tag{4.26}$$

The first term on the RHS converges for $|z| > |\alpha|$, whereas the second term converges for $|z| < |\alpha|$. There is no overlap between these two regions, hence the z-transform of $u[n] = \alpha^n$ does not exist.

The ROC of a rational z-transform cannot contain any pole (since it is infinite at a pole) and is bounded by the poles. To show that the z-transform is bounded by the poles, assume that the z-transform $X(z)$ has simple poles at $z = \alpha$ and $z = \beta$. Assume that the corresponding sequence $x[n]$ is a right-sided sequence. Then, $x[n]$ has the form:

$$x[n] = (r_1\alpha^n + r_2\beta^n)\mu[n - N_0] \qquad |\alpha| < |\beta| \tag{4.27}$$

where N_0 is a positive or negative integer. Now, the z-transform of the right.sided sequence $\gamma^n \mu[n - N_0]$ exists if:

$$\sum_{n=N_0}^{\infty} \left|\gamma^n z^{-n}\right| < \infty \tag{4.28}$$

for some z. The condition in Eq. 4.28 holds for $|z| > |\gamma|$, but not for $|z| \leq |\gamma|$. Therefore, the z-transform of Eq. 4.27 has a ROC defined by $|\beta| < |z| \leq \infty$.
Likewise, the z-transform of a left-sided sequence:

$$x[n] = (r_1\alpha^n + r_2\beta^n)\mu[-n - N_0] \qquad |\alpha| < |\beta| \tag{4.29}$$

has a ROC define by $0 \leq |z| < |\alpha|$.

4.3 Inverse z-transform

Firstly, we recall that, for $z = re^{j\omega}$, the z-transform $G(z)$ given by:

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n} \tag{4.30}$$

is the DTFT of the modified sequence $g[n]r^{-n}$. Accordingly, the inverse DTFT is thus given by:

$$g[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{j\omega})e^{j\omega n}\mathrm{d}\omega \tag{4.31}$$

By making a change of variable $z = re^{j\omega}$, the previous equation can be converted into a contour integral given by:

$$g[n] = \frac{1}{2\pi j}\oint_{C'} G(z)z^{n-1}\mathrm{d}z \tag{4.32}$$

where C' is a counterclockwise contour of integration defined by $|z| = r$. But the integral remains unchanged when is replaced with any contour C encircling the point $z = 0$ in the ROC of $G(z)$. The contour integral can be evaluated using the Cauchy's residue theorem resulting in:

$$g[n] = \sum \mathrm{Res}_C\left[G(z)z^{n-1}\right] \tag{4.33}$$

Eq. 4.33 needs to be evaluated at all values of n and is not pursued here.
A rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has an ROC that is exterior to a circle. Here, it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g[n]$ by summing the inverse transform of the individual simpler terms in the expansion. A rational $G(z)$ can be expressed as:

$$G(z) = \frac{P(z)}{D(z)} = \frac{\displaystyle\sum_{i=0}^{M} p_i z^{-i}}{\displaystyle\sum_{i=0}^{N} d_i z^{-i}} \tag{4.34}$$

If $M \geq N$, then $G(z)$ can be re-expressed as:

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)} \tag{4.35}$$

where the degree of $P_1(z)$ is less than N. The rational function $\frac{P_1(z)}{D(z)}$ is called a proper fraction. To develop the proper fraction part $\frac{P_1(z)}{D(z)}$ from $G(z)$, a long division of $P(z)$ by $D(z)$ should be carried out in a reverse order until the remainder polynomial $P_1(z)$ is of lower degree that that of the denominator $D(z)$.

Example 30: Inverse transform by partial-fraction expansion

Consider:

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}} \tag{4.36}$$

By long division in reverse order we arrive at:

$$G(z) = -3.5 + 1.5z^{-1} + \underbrace{\frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}}_{\text{Proper fraction}} \tag{4.37}$$

In most practical cases, the rational z-transform of interest $G(z)$ is a proper fraction with simple poles. Let the poles of $G(z)$ be at $z = \lambda_k$, with $1 \leq k \leq N$. A partial-fraction expansion of $G(z)$ is then of the form:

$$G(z) = \sum_{\ell=1}^{N} \left(\frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \right) \tag{4.38}$$

The constants ρ_ℓ in the partial-fraction expansion are called the residues and are given by:

$$\rho_\ell = \big[(1 - \lambda_\ell z^{-1})G(z)\big]_{z=\lambda_\ell} \tag{4.39}$$

Each term of the sum in partial-fraction expansion has a ROC given by $|z| > |\lambda_\ell|$ and thus has an inverse transform of the form $\rho_\ell(\lambda_\ell)^n \mu[n]$. Therefore, the inverse transform $g[n]$ of $G(z)$ is given by:

$$g[n] = \sum_{\ell=1}^{N} \rho_\ell(\lambda_\ell)^n \mu[n] \tag{4.40}$$

Note that the approach in Eq. 4.40 with a slight modification can also be used to determine the inverse of a rational z-transform of a noncausal sequence.

Example 31: Inverse transfrom of a causal sequence

Let the z-transform $H(z)$ of a causal sequence $h[n]$ be given by:

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1 + 2z^{-1}}{(1 - 0.2z^{-1})(1 + 0.6z^{-1})} \tag{4.41}$$

A partial-fraction expansion of $H(z)$ is then of the form:

$$H(z) = \frac{\rho_1}{1 - 0.2z^{-1}} + \frac{\rho_2}{1 - 0.6z^{-1}} \tag{4.42}$$

Now:

$$\rho_1 = \big[(1 - 0.2z^{-1})H(z)\big]_{z=0.2} = \left[\frac{1 + 2z^{-1}}{1 + 0.6z^{-1}}\right]_{z=0.2} = 2.75 \tag{4.43}$$

$$\rho_2 = \big[(1 + 0.6z^{-1})H(z)\big]_{z=-0.6} = \left[\frac{1 + 2z^{-1}}{1 - 0.2z^{-1}}\right]_{z=-0.6} = -1.75 \tag{4.44}$$

Hence:

$$H(z) = \frac{2.75}{1 - 0.2z^{-1}} - \frac{1.75}{1 + 0.6z^{-1}} \tag{4.45}$$

The inverse transform of the above is therefore given by:

$$h[n] = 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n] \tag{4.46}$$

In case $G(z)$ has multiple poles, the partial-fraction expansion is of slightly different form. Let the pole at $z = v$ be of multiplicity L and the remaining $N - L$ poles be simple and at $z = \lambda_\ell$, for $1 \le \ell \le N - L$. Then, the partial-fraction expansion of $G(z)$ is of the form:

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \sum_{\ell=1}^{N-L} \frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{(1 - v z^{-1})^i} \tag{4.47}$$

where the constants γ_i are computed using:

$$\gamma_i = \frac{1}{(L-i)!(-v)^{L-i}} \frac{\mathrm{d}^{L-i}}{\mathrm{d}(z^{-1})^{L-i}} \left[(1 - v z^{-1}) G(z)\right]_{z=v} \qquad 1 \le i \le L \tag{4.48}$$

The residues ρ_ℓ are calculated as before.

4.4 Z-transform properties

A list of properties of the z-transform is showed in Figure 4.3.

Property	Sequence	z -Transform	ROC
	$g[n]$	$G(z)$	$\mathcal{R}_g$
	$h[n]$	$H(z)$	$\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\|\alpha\|\mathcal{R}_g$
Differentiation of $G(z)$	$n g[n]$	$-z \dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j} \oint_C G(v) H(z/v) v^{-1}\, dv$	Includes $\mathcal{R}_g \mathcal{R}_h$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j} \oint_C G(v) H^*(1/v^*) v^{-1}\, dv$		

Note: If $\mathcal{R}_g$ denotes the region $R_{g^-} < |z| < R_{g^+}$ and $\mathcal{R}_h$ denotes the region $R_{h^-} < |z| < R_{h^+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g^+} < |z| < 1/R_{g^-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g^-}R_{h^-} < |z| < R_{g^+}R_{h^+}$.

Figure 4.3: Properties of the z-transform.

Example 32: Z-transform properties

Consider the two-sided sequence:

$$v[n] = \alpha^n \mu[n] - \beta^n \mu[-n-1] \tag{4.49}$$

Let $x[n] = \alpha^n \mu[n]$ and $y = -\beta^n \mu[-n-1]$ with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms. Now:

$$X(z) = \frac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha| \tag{4.50}$$

$$Y(z) = \frac{1}{1-\beta z^{-1}} \qquad |z| < |\beta| \tag{4.51}$$

Using the linearity property we arrive at:

$$V(z) = X(z) + Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}} \tag{4.52}$$

The ROC of $V(z)$ is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |\beta|$. We have that:

- if $|\alpha| < |\beta|$, then there is an overlap and the ROC is an annular region $|\alpha| < |z| < |\beta|$;
- if $|\alpha| > |\beta|$, then there is no overlap and $V(z)$ does not exist.

Example 33: Z-transform properties

We determine the z-transform and its ROC of the causal sequence:

$$x[n] = r^n(\cos(\omega_0 n))\mu[n] \tag{4.53}$$

We can express $x[n] = v[n] + v^*[n]$, where:

$$v[n] = \frac{1}{2} r^n e^{j\omega_0 n}\mu[n] = \frac{1}{2}\alpha^n \mu[n] \tag{4.54}$$

The z-transform of $v[n]$ is given by:

$$V(z) = \frac{1}{2}\frac{1}{1-\alpha z^{-1}} = \frac{1}{2}\frac{1}{1-re^{j\omega_0 z^{-1}}} \qquad |z| > |\alpha| = r \tag{4.55}$$

Using the conjugation property, we obtain the z-transform of $v^*[n]$ as:

$$V^*(z^*) = \frac{1}{2}\frac{1}{1-\alpha^* z^{-1}} = \frac{1}{2}\frac{1}{1-re^{-j\omega_0 z^{-1}}} \qquad |z| > |\alpha| \tag{4.56}$$

Finally, using the linearity property we get:

$$X(z) = V(z) + V^*(z^*) = \frac{1}{2}\left(\frac{1}{1-re^{j\omega_0 z^{-1}}} + \frac{1}{1-re^{-j\omega_0 z^{-1}}}\right) \tag{4.57}$$

or:

$$X(z) = \frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2 z^{-2}} \qquad |z| > r \tag{4.58}$$

Example 34: Z-transform properties

We determine the z-transform $Y(z)$ and the ROC of the sequence:

$$y[n] = (n+1)\alpha^n \mu[n] \tag{4.59}$$

We can write $y[n] = nx[n] + x[n]$ where:

$$x[n] = \alpha^n \mu[n] \tag{4.60}$$

Now, the z-transform $X(z)$ of $x[n] = \alpha^n \mu[n]$ is given by:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \qquad |z| > |\alpha| \tag{4.61}$$

Using the differentiation property, we arrive at the z-transform of $nx[n]$ as:

$$-z\frac{\mathrm{d}X(z)}{\mathrm{d}z} = \frac{\alpha z^{-1}}{1 - \alpha z^{-1}} \qquad |z| > |\alpha| \tag{4.62}$$

Using the linearity property we finally obtain:

$$Y(z) = \frac{1}{1 - \alpha z^{-1}} + \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} = \frac{1}{(1 - \alpha z^{-1})^2} \qquad |z| > |\alpha| \tag{4.63}$$

Bibliography

[1] Michael E. Peskin *Concepts of Elementary Particle Physics*. Oxford Master Series in Particle Physics, Astrophysics and Cosmology April 2, 2019