

Chapter 1

The Z transform

Lecture 14.
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We have seen that the DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems. However, because of the convergence condition, in many cases, the DTFT of a sequence may not exist. As a result, it is not possible to make use of such frequency-domain characterization in these cases. A possible solution and alternative is a generalization of the DTFT, which leads to the z-transform. The latter may exist for many sequences for which the DTFT does not exist. Moreover, use of z-transform techniques permits simple but powerful algebraic manipulations. Consequently, z-transform has become an important tool in the analysis and design of digital filters

1.1 The definition

Definition 1: Z-transform

For a given sequence $g[n]$, its z-transform $G(z)$ is defined as:

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} \quad (1.1)$$

where $z = \text{Re}[z] + j \text{Im}[z]$ is a complex variable.

If we let $z = re^{j\omega}$, then the z-transform reduces to:

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n} \quad (1.2)$$

The above can be interpreted as the DTFT of the modified sequence $\{g[n]r^{-n}\}$. For $r = 1$ (i.e., $|z| = 1$), the z-transform reduces to its DTFT, provided the latter exists. Like the DTFT, there are conditions on the convergence of the infinite series like:

$$\sum_{n=-\infty}^{\infty} g[n]z^{-n} \quad (1.3)$$

For a given sequence, the set R of values of z for which its z-transform converges is called the region of convergence (ROC).

From our earlier discussion on the uniform convergence of the DTFT, it follows that the series:

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n} \quad (1.4)$$

converges if $\{g[n]r^{-n}\}$ is absolutely summable, i.e. if:

$$\sum_{n=-\infty}^{\infty} |g[n]r^{-n}| < \infty \quad (1.5)$$

In general, the ROC R of a z-transform of a sequence $g[n]$ is an annular region of the z -plane, namely:

$$R_{g^-} < |z| < R_{g^+} \quad (1.6)$$

where $0 \leq R_{g^-} < R_{g^+} < \infty$.

Example 1: Z-transform calculation

We determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC. Now:

$$X(z) = \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} \quad (1.7)$$

The above power series converges to:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \quad |\alpha z^{-1}| < 1 \quad (1.8)$$

ROC is the annular region $|z| > |\alpha|$.

Example 2: Z-transform calculation

The z-transform $\mu(z)$ of the unit step sequence $\mu[n]$ can be obtained from:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \quad |\alpha z^{-1}| < 1 \quad (1.9)$$

By setting $\alpha = 1$:

$$\mu(z) = \frac{1}{1 - z^{-1}} \quad |z^{-1}| < 1 \quad (1.10)$$

ROC is the annular region $1 < |z| < \infty$. Note that the unit step sequence $\mu[n]$ is not absolutely summable, and hence its DTFT does not converge uniformly.

Example 3: Z-transform calculation

Consider the anti-causal sequence:

$$y[n] = -\alpha^n \mu[-n - 1] \quad (1.11)$$

Its z-transform is given by:

$$\begin{aligned} Y(z) &= - \sum_{n=-\infty}^{-1} \alpha^n z^{-n} = - \sum_{m=1}^{\infty} \alpha^{-m} z^m \\ &= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^m = -\frac{-\alpha^{-1} z}{1 - \alpha z^{-1}} \\ &= \frac{1}{1 - \alpha z^{-1}} \end{aligned} \quad (1.12)$$

for $|\alpha^{-1} z| < 1$. ROC is the annular region $|z| < |\alpha|$.

Note that the z-transforms of the two sequences $\alpha^n \mu[n]$ and $-\alpha^n \mu[-n-1]$ are identical even though the two parent sequences are different. The only way a unique sequence can be associated with a z-transform is by specifying its ROC.

Another important point is that the DTFT $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the z-transform $G(z)$ of $g[n]$ includes the unit circle. However, the existence of the DTFT does not always imply the existence of the z-transform.

Example 4: Z-transform

The finite energy sequence:

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n} \quad -\infty < n < \infty \quad (1.13)$$

has a DTFT given by:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & 0 \leq |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases} \quad (1.14)$$

which converges in the mean-square sense. However, $h_{LP}[n]$ does not have a z-transform as it is not absolutely summable for any value of r .

Some commonly used z-transform pairs are listed in Figure 1.1.

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
$\alpha^n \mu[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z > \alpha $
$(r^n \cos \omega_o n) \mu[n]$	$\frac{1 - (r \cos \omega_o) z^{-1}}{1 - (2r \cos \omega_o) z^{-1} + r^2 z^{-2}}$	$ z > r$
$(r^n \sin \omega_o n) \mu[n]$	$\frac{(r \sin \omega_o) z^{-1}}{1 - (2r \cos \omega_o) z^{-1} + r^2 z^{-2}}$	$ z > r$

Figure 1.1: Common z-transform pairs.

1.2 Rational z-transforms

In the case of LTI discrete-time systems we are concerned with in this course, all pertinent z-transforms are rational functions of z^{-1} , that is, they are ratios of two polynomials in z^{-1} :

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \dots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \dots + d_{N-1} z^{-(N-1)} + d_N z^{-N}} \quad (1.15)$$

The degree of the numerator polynomial $P(z)$ is M and the degree of the denominator polynomial $D(z)$ is N . An alternate representation of a rational z-transform is as a

ratio of two polynomials in z :

$$G(z) = z^{(N-M)} \frac{p_0 z^M + \cdots + p_{M-1} z + p_M}{d_0 z^N + \cdots + d_{N-1} z + d_N} \quad (1.16)$$

Again, a rational z-transform can be alternately written in factored form as:

$$G(z) = \frac{p_0 \prod_{\ell=1}^M (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^N (1 - \lambda_\ell z^{-1})} = z^{(N-M)} \frac{p_0 \prod_{\ell=1}^M (z - \xi_\ell)}{d_0 \prod_{\ell=1}^N (z - \lambda_\ell)} \quad (1.17)$$

We have as roots:

- $z = \xi_\ell$, roots of the numerator polynomial. These values of z are known as the zeros of $G(z)$;
- $z = \lambda_\ell$, roots of the denominator polynomial. These values of z are known as the poles of $G(z)$.

Example 5: Zeros and poles

The z-transform:

$$\mu(z) = \frac{1}{1 - z^{-1}} \quad |z| > 1 \quad (1.18)$$

has a zero at $z = 0$ and a pole at $z = 1$.

Example 6: ROC of a rational z-transform

The z-transform $H(z)$ of the sequence $h[n] = (-0.6)^n \mu[n]$ is given by:

$$H(z) = \frac{1}{1 + 0.6z^{-1}} \quad |z| > 0.6 \quad (1.19)$$

Here the ROC is just outside the circle going through the point $z = -0.6$.

A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20 \log_{10} |G(z)|$ as showed in Figure 1.2 for:

$$G(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}} \quad (1.20)$$

Observe that the magnitude plot exhibits very large peaks around the points $z = 0.4 \pm j0.6928$, which are the poles of $G(z)$. It also exhibits very narrow and deep wells around the location of the zeros at $z = 1.2 \pm j1.2$.

ROC of a z-transform is an important concept. Without its knowledge, there is no unique relationship between a sequence and its z-transform. Hence, the z-transform must always be specified with its ROC. Moreover, there is a relationship between the ROC of the z-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability.

Another important distinction is that a sequence can be one of the following types: finite-length, right-sided, left-sided and two-sided. In general, the ROC depends on the type of the sequence of interest.

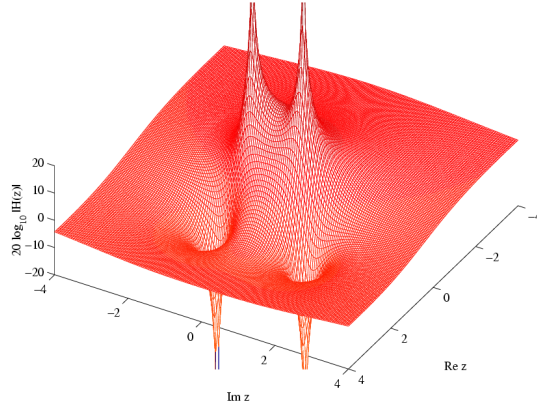


Figure 1.2: Log-magnitude plot for $G(z)$ in Eq. 1.20.

Example 7: Finite-length sequence z-transform

Consider a finite-length sequence $g[n]$ defined for $-M \leq n \leq N$, where M and N are non-negative integers and $|g[n]| < \infty$. Its z-transform is given by:

$$G(z) = \sum_{n=-M}^N g[n]z^{-n} = \frac{\sum_{n=0}^{N+M} g[n-M]z^{N+M-n}}{z^N} \quad (1.21)$$

Note that $G(z)$ has M zeros and N poles. As can be seen from the expression for $G(z)$, the z-transform of a finite-length bounded sequence converges everywhere in the z-plane except possibly at $z = 0$ and/or at $z = \infty$.

Example 8: Right-sided sequence z-transform

A right-sided sequence with nonzero sample values for $n \geq 0$ is sometimes called a causal sequence. So, consider a causal sequence $u_1[n]$. Its z-transform is given by:

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n]z^{-n} \quad (1.22)$$

It can be showed that $U_1(z)$ converges exterior to a circle with $|z| = R_1$, including the point $z = \infty$.

On the other hand, a right-sided sequence $u_2[n]$ with nonzero sample values only for $n \geq -M$ with M non-negative has a z-transform $U_2(z)$ with M poles at $z = \infty$. The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, excluding the point $z = \infty$.

Example 9: Left-sided sequence z-transform

A left-sided sequence with nonzero sample values for $n \leq 0$ is sometimes called anticausal sequence. So, consider an anticausal sequence $v_1[n]$. Its z-transform is given by:

$$V_1(z) = \sum_{n=-\infty}^0 v_1[n]z^{-n} \quad (1.23)$$

It can be showed that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the

point $z = 0$.

On the other hand, a left-sided sequence with nonzero sample values only for $n \leq N$ with N non-negative has a z-transform $V_2(z)$ with N poles at $z = 0$. The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$.

Example 10: Two-sided sequence z-transform

The z-transform of a two-sided sequence $w[n]$ can be expressed as:

$$W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n} = \sum_{n=0}^{\infty} w[n]z^{-n} + \sum_{n=-\infty}^{-1} w[n]z^{-n} \quad (1.24)$$

The first term on the RHS can be interpreted as the z-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$. The second term of the RHS can be interpreted as the z-transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$. If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$. If $R_5 > R_6$, there is no overlap and the z-transform does not exist.

In particular, let us consider as example the two-sided sequence:

$$u[n] = \alpha^n \quad (1.25)$$

where α can be either real or complex. Its z-transform is given by:

$$U(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n} \quad (1.26)$$

The first term on the RHS converges for $|z| > |\alpha|$, whereas the second term converges for $|z| < |\alpha|$. There is no overlap between these two regions, hence the z-transform of $u[n] = \alpha^n$ does not exist.

The ROC of a rational z-transform cannot contain any pole (since it is infinite at a pole) and is bounded by the poles. To show that the z-transform is bounded by the poles, assume that the z-transform $X(z)$ has simple poles at $z = \alpha$ and $z = \beta$. Assume that the corresponding sequence $x[n]$ is a right-sided sequence. Then, $x[n]$ has the form:

$$x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu[n - N_0] \quad |\alpha| < |\beta| \quad (1.27)$$

where N_0 is a positive or negative integer. Now, the z-transform of the right-sided sequence $\gamma^n \mu[n - N_0]$ exists if:

$$\sum_{n=N_0}^{\infty} |\gamma^n z^{-n}| < \infty \quad (1.28)$$

for some z . The condition in Eq. 1.28 holds for $|z| > |\gamma|$, but not for $|z| \leq |\gamma|$. Therefore, the z-transform of Eq. 1.27 has a ROC defined by $|\beta| < |z| \leq \infty$. Likewise, the z-transform of a left-sided sequence:

$$x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu[-n - N_0] \quad |\alpha| < |\beta| \quad (1.29)$$

has a ROC defined by $0 \leq |z| < |\alpha|$.

1.3 Inverse z-transform

Firstly, we recall that, for $z = re^{j\omega}$, the z-transform $G(z)$ given by:

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n} \quad (1.30)$$

is the DTFT of the modified sequence $g[n]r^{-n}$. Accordingly, the inverse DTFT is thus given by:

$$g[n]r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(re^{j\omega})e^{j\omega n}d\omega \quad (1.31)$$

By making a change of variable $z = re^{j\omega}$, the previous equation can be converted into a contour integral given by:

$$g[n] = \frac{1}{2\pi j} \oint_{C'} G(z)z^{n-1}dz \quad (1.32)$$

where C' is a counterclockwise contour of integration defined by $|z| = r$. But the integral remains unchanged when is replaced with any contour C encircling the point $z = 0$ in the ROC of $G(z)$. The contour integral can be evaluated using the Cauchy's residue theorem resulting in:

$$g[n] = \sum \text{Res}_C [G(z)z^{n-1}] \quad (1.33)$$

Eq. 1.33 needs to be evaluated at all values of n and is not pursued here.

A rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has an ROC that is exterior to a circle. Here, it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g[n]$ by summing the inverse transform of the individual simpler terms in the expansion. A rational $G(z)$ can be expressed as:

$$G(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^M p_i z^{-i}}{\sum_{i=0}^N d_i z^{-i}} \quad (1.34)$$

If $M \geq N$, then $G(z)$ can be re-expressed as:

$$G(z) = \sum_{\ell=0}^{M-N} \eta_{\ell} z^{-\ell} + \frac{P_1(z)}{D(z)} \quad (1.35)$$

where the degree of $P_1(z)$ is less than N . The rational function $\frac{P_1(z)}{D(z)}$ is called a proper fraction. To develop the proper fraction part $\frac{P_1(z)}{D(z)}$ from $G(z)$, a long division of $P(z)$ by $D(z)$ should be carried out in a reverse order until the remainder polynomial $P_1(z)$ is of lower degree than that of the denominator $D(z)$.

Example 11: Inverse transform by partial-fraction expansion

Consider:

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}} \quad (1.36)$$

By long division in reverse order we arrive at:

$$G(z) = -3.5 + 1.5z^{-1} + \underbrace{\frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}}_{\text{Proper fraction}} \quad (1.37)$$

In most practical cases, the rational z-transform of interest $G(z)$ is a proper fraction with simple poles. Let the poles of $G(z)$ be at $z = \lambda_k$, with $1 \leq k \leq N$. A partial-fraction expansion of $G(z)$ is then of the form:

$$G(z) = \sum_{\ell=1}^N \left(\frac{\rho_{\ell}}{1 - \lambda_{\ell}z^{-1}} \right) \quad (1.38)$$

The constants ρ_{ℓ} in the partial-fraction expansion are called the residues and are given by:

$$\rho_{\ell} = [(1 - \lambda_{\ell}z^{-1})G(z)]_{z=\lambda_{\ell}} \quad (1.39)$$

Each term of the sum in partial-fraction expansion has a ROC given by $|z| > |\lambda_{\ell}|$ and thus has an inverse transform of the form $\rho_{\ell}(\lambda_{\ell})^n \mu[n]$. Therefore, the inverse transform $g[n]$ of $G(z)$ is given by:

$$g[n] = \sum_{\ell=1}^N \rho_{\ell}(\lambda_{\ell})^n \mu[n] \quad (1.40)$$

Note that the approach in Eq. 1.40 with a slight modification can also be used to determine the inverse of a rational z-transform of a noncausal sequence.

Example 12: Inverse transform of a causal sequence

Let the z-transform $H(z)$ of a causal sequence $h[n]$ be given by:

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})} \quad (1.41)$$

A partial-fraction expansion of $H(z)$ is then of the form:

$$H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1-0.6z^{-1}} \quad (1.42)$$

Now:

$$\rho_1 = [(1-0.2z^{-1})H(z)]_{z=0.2} = \left[\frac{1+2z^{-1}}{1+0.6z^{-1}} \right]_{z=0.2} = 2.75 \quad (1.43)$$

$$\rho_2 = [(1+0.6z^{-1})H(z)]_{z=-0.6} = \left[\frac{1+2z^{-1}}{1-0.2z^{-1}} \right]_{z=-0.6} = -1.75 \quad (1.44)$$

Hence:

$$H(z) = \frac{2.75}{1-0.2z^{-1}} - \frac{1.75}{1+0.6z^{-1}} \quad (1.45)$$

The inverse transform of the above is therefore given by:

$$h[n] = 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n] \quad (1.46)$$

In case $G(z)$ has multiple poles, the partial-fraction expansion is of slightly different form. Let the pole at $z = v$ be of multiplicity L and the remaining $N - L$ poles be simple and at $z = \lambda_\ell$, for $1 \leq \ell \leq N - L$. Then, the partial-fraction expansion of $G(z)$ is of the form:

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \sum_{\ell=1}^{N-L} \frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} + \sum_{i=1}^L \frac{\gamma_i}{(1 - vz^{-1})^i} \quad (1.47)$$

where the constants γ_i are computed using:

$$\gamma_i = \frac{1}{(L-i)!(-v)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} [(1 - vz^{-1})G(z)]_{z=v} \quad 1 \leq i \leq L \quad (1.48)$$

The residues ρ_ℓ are calculated as before.

1.4 Z-transform properties

A list of properties of the z-transform is showed in Figure 1.3.

Property	Sequence	z -Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	\mathcal{R}_g \mathcal{R}_h
Conjugation	$g^*[n]$	$G^*(z^*)$	\mathcal{R}_g
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	\mathcal{R}_g , except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$ \alpha \mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z \frac{dG(z)}{dz}$	\mathcal{R}_g , except possibly the point $z = 0$ or ∞
Convolution	$g[n] \otimes h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j} \oint_C G(v)H(z/v)v^{-1} dv$	Includes $\mathcal{R}_g \mathcal{R}_h$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j} \oint_C G(v)H^*(1/v^*)v^{-1} dv$		

Note: If \mathcal{R}_g denotes the region $R_{g-} < |z| < R_{g+}$ and \mathcal{R}_h denotes the region $R_{h-} < |z| < R_{h+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g+} < |z| < 1/R_{g-}$ and $\mathcal{R}_g \mathcal{R}_h$ denotes the region $R_{g-} R_{h-} < |z| < R_{g+} R_{h+}$.

Figure 1.3: Properties of the z-transform.

Example 13: Z-transform properties

Consider the two-sided sequence:

$$v[n] = \alpha^n \mu[n] - \beta^n \mu[-n - 1] \quad (1.49)$$

Let $x[n] = \alpha^n \mu[n]$ and $y = -\beta^n \mu[-n-1]$ with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms. Now:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \quad |z| > |\alpha| \quad (1.50)$$

$$Y(z) = \frac{1}{1 - \beta z^{-1}} \quad |z| < |\beta| \quad (1.51)$$

Using the linearity property we arrive at:

$$V(z) = X(z) + Y(z) = \frac{1}{1 - \alpha z^{-1}} + \frac{1}{1 - \beta z^{-1}} \quad (1.52)$$

The ROC of $V(z)$ is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |\beta|$. We have that:

- if $|\alpha| < |\beta|$, then there is an overlap and the ROC is an annular region $|\alpha| < |z| < |\beta|$;
- if $|\alpha| > |\beta|$, then there is no overlap and $V(z)$ does not exist.

Example 14: Z-transform properties

We determine the z-transform and its ROC of the causal sequence:

$$x[n] = r^n (\cos(\omega_0 n)) \mu[n] \quad (1.53)$$

We can express $x[n] = v[n] + v^*[n]$, where:

$$v[n] = \frac{1}{2} r^n e^{j\omega_0 n} \mu[n] = \frac{1}{2} \alpha^n \mu[n] \quad (1.54)$$

The z-transform of $v[n]$ is given by:

$$V(z) = \frac{1}{2} \frac{1}{1 - \alpha z^{-1}} = \frac{1}{2} \frac{1}{1 - r e^{j\omega_0} z^{-1}} \quad |z| > |\alpha| = r \quad (1.55)$$

Using the conjugation property, we obtain the z-transform of $v^*[n]$ as:

$$V^*(z^*) = \frac{1}{2} \frac{1}{1 - \alpha^* z^{-1}} = \frac{1}{2} \frac{1}{1 - r e^{-j\omega_0} z^{-1}} \quad |z| > |\alpha| \quad (1.56)$$

Finally, using the linearity property we get:

$$X(z) = V(z) + V^*(z^*) = \frac{1}{2} \left(\frac{1}{1 - r e^{j\omega_0} z^{-1}} + \frac{1}{1 - r e^{-j\omega_0} z^{-1}} \right) \quad (1.57)$$

or:

$$X(z) = \frac{1 - (r \cos \omega_0) z^{-1}}{1 - (2r \cos \omega_0) z^{-1} + r^2 z^{-2}} \quad |z| > r \quad (1.58)$$

Example 15: Z-transform properties

We determine the z-transform $Y(z)$ and the ROC of the sequence:

$$y[n] = (n+1) \alpha^n \mu[n] \quad (1.59)$$

We can write $y[n] = nx[n] + x[n]$ where:

$$x[n] = \alpha^n \mu[n] \quad (1.60)$$

Now, the z-transform $X(z)$ of $x[n] = \alpha^n \mu[n]$ is given by:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \quad |z| > |\alpha| \quad (1.61)$$

Using the differentiation property, we arrive at the z-transform of $nx[n]$ as:

$$-z \frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{1 - \alpha z^{-1}} \quad |z| > |\alpha| \quad (1.62)$$

Using the linearity property we finally obtain:

$$Y(z) = \frac{1}{1 - \alpha z^{-1}} + \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} = \frac{1}{(1 - \alpha z^{-1})^2} \quad |z| > |\alpha| \quad (1.63)$$