Lecture 8.
Thursday 22nd October, 2020.

0.1 Continuous-time Fourier transform

Let us start with the definition of this very important tool

Definition 1: Fourier transform of a continuous-time signal

he CTFT of a continuous-time signal $x_a(t)$ is given by:

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}\mathrm{d}t \tag{1}$$

often referred to as the Fourier spectrum or simply the spectrum of the continuous-time signal.

Definition 2: Inverse Fourier transform of a continuous-time signal

he inverse CTFT of a Fourier transform $X_a(j\Omega)$ is given by:

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{+j\Omega t}\mathrm{d}\Omega \tag{2}$$

often referred to as the Fourier integral.

A CTFT pair will be denoted as:

$$x_a(t) \longleftrightarrow X_a(j\Omega) \tag{3}$$

Note that Ω is real and denotes the continuous-time angular frequency variable in radians. In general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$. It can be expressed in the polar form as:

$$X_a(j\Omega) = |X_a(j\Omega)|e^{j\theta_a(\Omega)} \tag{4}$$

where $\theta_a(\Omega) = \arg X_a(j\Omega)$. The quantity $|X_a(j\Omega)|$ is called the magnitude spectrum and the quantity $\theta_a(\Omega)$ is called the phase spectrum. Both spectra are real function of Ω and in general the CTFT $X_a(j\Omega)$ exists if $x_a(t)$ satisfies the Dirichlet conditions:

- the signal $x_a(t)$ has a finite number if discontinuities and a finite number of maxima and minima in any finite interval;
- the signal is absolutely integrable, i.e.:

$$\int_{-\infty}^{\infty} |x_a(t)|\mathrm{d}t < \infty \tag{5}$$

If the Dirichlet conditions are satisfied, then:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{+j\Omega t}\mathrm{d}\Omega \tag{6}$$

converges to $x_a(t)$ except at values of t where $x_a(t)$ has discontinuities. Moreover, it can be showed that if $x_a(t)$ is absolutely integrable, then proving the existence of the CTFT reduces to proving:

$$|X_a(t\Omega)| < \infty \tag{7}$$

0.1.1 Energy density spectrum

The total energy E_x of a finite energy continuous-time complex signal $x_a(t)$ is given by:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |x_a(t)|^2 \mathrm{d}t \\ &= \int_{-\infty}^{\infty} x_a(t) x_a^*(t) \mathrm{d}t \\ &= \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} \mathrm{d}\Omega \right] \mathrm{d}t \end{aligned} \tag{8}$$

Interchanging the order of the integration we get:

$$\begin{aligned} E_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) \left[\int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} \mathrm{d}t \right] \mathrm{d}\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) X_a(j\Omega) \mathrm{d}\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \mathrm{d}\Omega \end{aligned} \tag{9}$$

Hence:

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \mathrm{d}\Omega \tag{10}$$

The above relation is more commonly known as the Parseval's relation for finite-energy continuous-time signals. The quantity $|X_a(j\Omega)|^2$ is called the energy density spectrum of $x_a(t)$ and it is usually denoted as:

$$S_{xx}(\Omega) = |X_a(j\Omega)|?2 \tag{11}$$

The energy over a specified range of frequencies $\Omega_a \leq \Omega \leq \Omega_b$ can be computed using:

$$E_{x,r} = \frac{1}{2\pi} \int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega) \mathrm{d}\Omega \tag{12}$$

0.1.2 Band-limited continuous-time signals

A full-band, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty \leq \Omega \leq \infty$. A band-limited continuous-time signal has a spectrum that is limited to a portion of the frequency range $-\infty \leq \Omega \leq \infty$. An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \leq |\Omega| \leq \Omega_b$ can be computed using:

$$X_a(j\Omega) = \begin{cases} 0 & 0 \leq |\Omega| < \Omega_a \\ 0 & \Omega_b < |\Omega| < \infty \end{cases} \tag{13}$$

However, an ideal band-limited signal cannot be generated in practice.
Band-limited signals are classified according to the frequency range where most of the signal's is concentrated:

- a lowpass, continuous-time signal has a spectrum occupying the frequency range $0 < |\Omega| \leq \Omega_p < \infty$, where Ω_p is called the bandwidth of the signal;

- a highpass, continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_p \leq |\Omega| < \infty$, where the bandwidth of the signal is from Ω_p to ∞;

- a bandpass, continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_L \leq |\Omega| \leq \Omega_H < \infty$, where $\Omega_H - \Omega_L$ is the bandwidth.

0.1.3 Discrete-time fourier transform

Let us introduce the definition of this concept.

Definition 3: Discrete-time Fourier transform

The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{14}$$

where in general $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as:

$$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega}) \tag{15}$$

$X_{\text{re}}(e^{j\omega})$ and $X_{\text{im}}(e^{j\omega})$ are respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of ω. $X(e^{j\omega})$ can alternately be expressed as:

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right|e^{j\theta(\omega)} \tag{16}$$

where $\theta(\omega) = \arg X(e^{j\omega})$. $\left|X(e^{j\omega})\right|$ and $\arg X(e^{j\omega})$ are called respectively magnitude function and phase function. Both quantities are again real functions of ω. In many applications, the DTFT is called the Fourier spectrum. Likewise, $\left|X(e^{j\omega})\right|$ and $\theta(\omega)$ are called respectively the magnitude and phase spectra.
For a real sequence $x[n]$, $\left|X(e^{j\omega})\right|$ and $X_{\text{re}}(e^{j\omega})$ are even functions of ω, whereas, $\theta(\omega)$ and $X_{\text{im}}(e^{j\omega})$ are odd functions of ω. Note also that $X(e^{j\omega}) = \left|X(e^{j\omega})\right|e^{j\theta(\omega+2\pi k)} = \left|X(e^{j\omega})\right|e^{j\theta(\omega)}$ for any integer k. The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT. Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the range of values $-\pi \leq \theta(\omega) < \pi$, called the principal value.

Example 1: DTFT of the unit sample sequence

The DTFT of the unit sample sequence $\delta[n]$ is given by:

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \delta[0] = 1 \tag{17}$$

Example 2: DTFT of a causal sequence

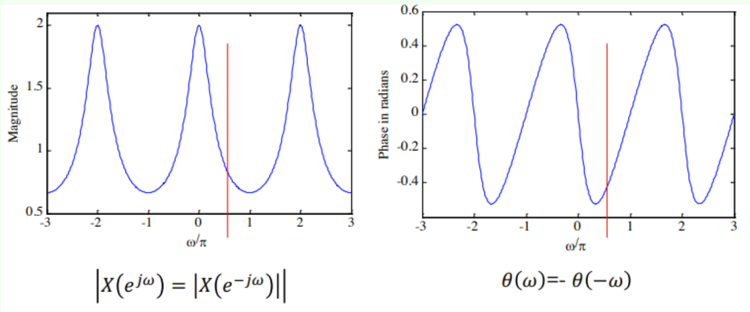
Consider the causal sequence:

$$x[n] = \alpha^n \mu[n], \qquad |\alpha| < 1 \tag{18}$$

Its DTFT is given by:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n \mu[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n \\ &= \frac{1}{1-\alpha e^{-j\omega}} \end{aligned} \tag{19}$$

as $\left|\alpha e^{-j\omega}\right| = |\alpha| < 1$. If we take for example $\alpha = 0.5$, we get the plot below for the magnitude and phase of the DTFT.



The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω. It is also a periodic function of ω with a period 2π:

$$X\left(e^{j(\omega+2\pi k)}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega}) \tag{20}$$

Therefore:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{21}$$

represents the Fourier series representation of the periodic function. As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega \tag{22}$$

Proof. Consider:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}\mathrm{d}\omega \tag{23}$$

The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e. $X(e^{j\omega})$ exists. Then:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}\mathrm{d}\omega &= \sum_{\ell=-\infty}^{\infty} x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\ell)}\mathrm{d}\omega\right) \\
&= \sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin\left(\pi(n-\ell)\right)}{\pi(n-\ell)} \\
&= \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell] \\
&= x[n]
\end{aligned} \tag{24}$$

For the convergence condition, an infinite series of the form:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{25}$$

may or may not converg. Therefore, let us consider:

$$X_k(e^{j\omega}) = \sum_{n=-k}^{k} x[n]e^{-j\omega n} \tag{26}$$

Then for uniform convergence of $X_k(e^{j\omega})$:

$$\lim_{k\to\infty} X_k(e^{j\omega}) = X(e^{j\omega}) \tag{27}$$

Now, if $x[n]$ is an absolutely summable sequence, i.e., if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, then:

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-k}^{k} x[n]e^{-j\omega n}\right| \leq \sum_{n=-k}^{k} |x[n]| < \infty \tag{28}$$

for all values of ω. Thus, the absolute summability of $x[n]$ is a sufficient condition for the existence of the DTFT $X(e^{j\omega})$. ■

Example 3: Absolute summability condition

he sequence $x[n] = \alpha^n \mu[n]$ for $|\alpha| < 1$ is absolutely summable as:

$$\sum_{n=-k}^{k} |\alpha^n|\mu[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty \tag{29}$$

and its DTFT $X(e^{j\omega})$ therefore converges to $\frac{1}{1-\alpha e^{j\omega}}$ uniformly.

Note that since:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left(\sum_{n=-\infty}^{\infty} |x[n]|\right)^2 \tag{30}$$

an absolutely summable sequence has always a finite energy. However, a finite-energy sequence is not necessarily absolutely summable.

Example 4: Absolute summability

The sequence:

$$x[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases} \tag{31}$$

has finite energy equal to:

$$E_x = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6} \tag{32}$$

but $x[n]$ is not absolutely summable.

In order to represent a finite energy sequence $x[n]$ that is not absolutely summable by a DTFT $X(e^{j\omega})$, it is necessary to consider a mean-square convergence of $X(e^{j\omega})$:

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} \left|X(e^{j\omega}) - X_k(e^{j\omega})\right|^2 \mathrm{d}\omega = 0 \tag{33}$$

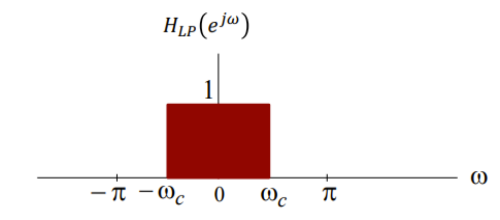
The total energy of the error $X(e^{j\omega}) - X_k(e^{j\omega})$ must approach zero at each value of ω as k goes to ∞. In such a case, the absolute value of the error $\left|X(e^{j\omega}) - X_k(e^{j\omega})\right|$ may not go to zero as k goes to ∞ and the DTFT is no longer bounded.

Example 5: Mean-square convergence

Consider the DTFT:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & 0 \leq |\omega| \leq \omega_c \\ 0 & \omega < |\omega| \leq \pi \end{cases} \tag{34}$$

showed in the plot below.



The inverse DTFT of $H_{LP}(e^{j\omega})$ is given by:

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\mathrm{d}\omega = \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \frac{\sin(\omega_c n)}{\pi n} \tag{35}$$

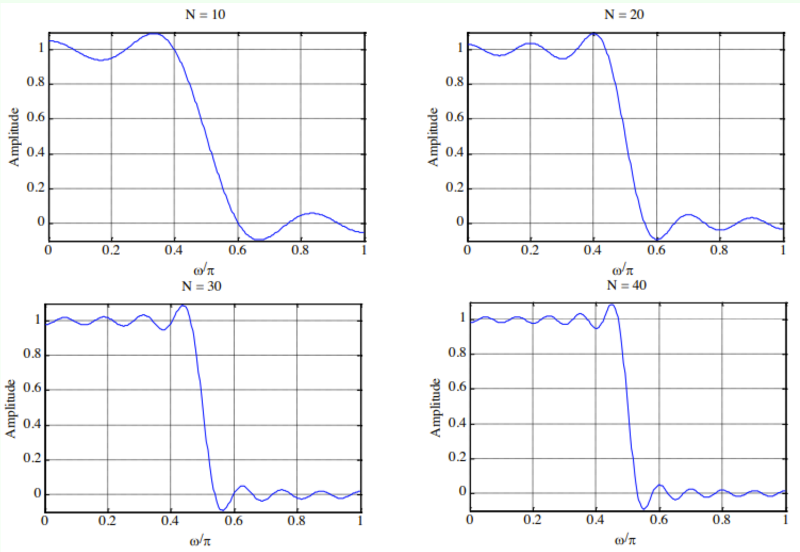
for $-\infty < n < \infty$- The energy of $h_{LP}[n]$ is given by $\frac{\omega_c}{\pi}$. $h_{LP}[n]$ is a finite-energy sequence, but it is not absolutely summable. In fact, the result:

$$\sum_{n=-k}^{n=k} h_{LP}[n]e^{-j\omega n} = \sum_{n=-k}^{n=k} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n} \tag{36}$$

does not uniformly converge to $H_{LP}(e^{j\omega})$ for all values of ω, but converges to $H_{LP}(e^{j\omega})$ in the mean-square sense.
The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function:

$$H_{LP,k}(e^{j\omega}) = \sum_{n=-k}^{k} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n} \tag{37}$$

for various values of k, as showed below.



As can be seen from these plots, independently of the value of k there are ripples in the plot of $H_{LP,k}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$. The number of ripples increases as k increases with the height of the largest ripple remaining the same for all values of k. As k goes to ∞, the condition:

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega}) - H_{LP,k}(e^{j\omega})\right|^2 \mathrm{d}\omega = 0 \tag{38}$$

holds indicating the convergence of $H_{LP,k}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$.
The oscillatory behaviour of $H_{LP,k}(e^{j\omega})$ approximating $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity is known as the Gibbs phenomenon

The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable. Examples of such sequences are the unit step sequence $\mu[n]$, the sinusoidal sequence $\cos(\omega_0 n + \varphi)$ and the exponential sequence $A\alpha^n$. For this type of sequences, a DTFT representation is possible using the Dirac delta function $\delta(\omega)$.
A Dirac delta function $\delta(\omega)$ is a function of ω with infinite height, zero width, and unit area. It is the limiting form of a unit area pulse function p_Δ as Δ goes to zero satisfying:

$$\lim_{\Delta\to 0} \int_{-\infty}^{\infty} p_\Delta(\omega)\mathrm{d}\omega = \int_{-\infty}^{\infty} \delta(\omega)\mathrm{d}\omega \tag{39}$$

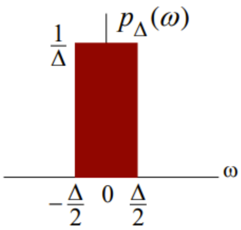


Figure 1: Plot and area of $p_\Delta(\omega)$ function.

Example 6: Dirac δ application

onsider the complex exponential sequence:

$$x[n] = e^{j\omega_0 n} \tag{40}$$

Its DTFT is given by:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2k\pi) \tag{41}$$

where $\delta(\omega)$ is an impulse function of ω and $-\pi \le \omega_0 \le \pi$. The function in Eq. 41 is periodic in ω with a period 2π and it is called periodic impulse train. In order to verify that $X(e^{j\omega})$ given above is indeed the DTFT of $x[n] = e^{j\omega_0 n}$ we compute the inverse DTFT of $X(e^{j\omega})$:

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)e^{j\omega n}\mathrm{d}\omega \\ &= \int_{-\pi}^{\pi} \delta(\omega - \omega_0)e^{j\omega n}\mathrm{d}\omega \\ &= e^{j\omega_0 n} \end{aligned} \tag{42}$$

where we have used the sampling property of the impulse function $\delta(\omega)$.

Last but not least, we list in Table 1 a set of commonly used DTFT pairs.

Sequence	DTFT
$\delta[n]$	1
1	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
$\mu[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$\alpha^n \mu[n] \quad (\lvert\alpha\rvert < 1)$	$\frac{1}{1-\alpha e^{-j\omega}}$

Table 1: Commonly used DTFT pairs.

0.1.4 DTFT properties

There are a number of important properties of the DTFT that are useful in signal processing applications. These are listed here in Figures 2, 3, 4 without proof, since it is quite straigthforward to derive them. We illustrate the applications of some of the DTFT properties.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{\mathrm{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\mathrm{cs}}[n]$	$X_{\mathrm{re}}(e^{j\omega})$
$x_{\mathrm{ca}}[n]$	$jX_{\mathrm{im}}(e^{j\omega})$

Figure 2: DTFT symmetry relations for a complex sequence $x[n]$.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$
$x_{\text{ev}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{od}}[n]$	$jX_{\text{im}}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{\text{re}}(e^{j\omega}) = X_{\text{re}}(e^{-j\omega})$
	$X_{\text{im}}(e^{j\omega}) = -X_{\text{im}}(e^{-j\omega})$
	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Figure 3: DTFT symmetry realtions for a real sequence $x[n]$.

Type of Property	Sequence	Discrete-Time Fourier Transform
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation in frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\,d\omega$	

Figure 4: DTFT general properties.