0.1 Continuous-time Fourier transform

Let us start with the definition of this very important tool

> **Definition 1: Fourier transform of a continuous-time signal**
>
> he CTFT of a continuous-time signal $x_a(t)$ is given by:
>
> $$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}\mathrm{d}t \tag{1}$$
>
> often referred to as the Fourier spectrum or simply the spectrum of the continuous-time signal.

> **Definition 2: Inverse Fourier transform of a continuous-time signal**
>
> he inverse CTFT of a Fourier transform $X_a(j\Omega)$ is given by:
>
> $$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega)e^{+j\Omega t}\mathrm{d}\Omega \tag{2}$$
>
> often referred to as the Fourier integral.

A CTFT pair will be denoted as:

$$x_a(t) \longleftrightarrow X_a(j\Omega) \tag{3}$$

Note that Ω is real and denotes the continuous-time angular frequency variable in radians. In general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$. It can be expressed in the polar form as:

$$X_a(j\Omega) = |X_a(j\Omega)|e^{j\theta_a(\Omega)} \tag{4}$$

where $\theta_a(\Omega) = \arg X_a(j\Omega)$. The quantity $|X_a(j\Omega)|$ is called the magnitude spectrum and the quantity $\theta_a(\Omega)$ is called the phase spectrum. Both spectra are real function of Ω and in general the CTFT $X_a(j\Omega)$ exists if $x_a(t)$ satisfies the Dirichlet conditions:

- the signal $x_a(t)$ has a finite number if discontinuities and a finite number of maxima and minima in any finite interval;

- the signal is absolutely integrable, i.e.:

$$\int_{-\infty}^{\infty} |x_a(t)|\mathrm{d}t < \infty \tag{5}$$

If the Dirichlet conditions are satisfied, then:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega)e^{+j\Omega t}\mathrm{d}\Omega \tag{6}$$

converges to $x_a(t)$ except at values of t where $x_a(t)$ has discontinuities. Moreover, it can be showed that if $x_a(t)$ is absolutely integrable, then proving the existence of the CTFT reduces to proving:

$$|X_a(t\Omega)| < \infty \tag{7}$$

0.1.1 Energy density spectrum

The total energy E_x of a finite energy continuous-time complex signal $x_a(t)$ is given by:

$$
\begin{aligned}
E_x &= \int_{-\infty}^{\infty} |x_a(t)|^2 \mathrm{d}t \\
&= \int_{-\infty}^{\infty} x_a(t) x_a^*(t) \mathrm{d}t \\
&= \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} \mathrm{d}\Omega \right] \mathrm{d}t
\end{aligned}
\tag{8}
$$

Interchanging the order of the integration we get:

$$
\begin{aligned}
E_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) \left[\int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} \mathrm{d}t \right] \mathrm{d}\Omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) X_a(j\Omega) \mathrm{d}\Omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \mathrm{d}\Omega
\end{aligned}
\tag{9}
$$

Hence:

$$
\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \mathrm{d}\Omega
\tag{10}
$$

The above relation is more commonly known as the Parseval's relation for finite-energy continuous-time signals. The quantity $|X_a(j\Omega)|^2$ is called the energy density spectrum of $x_a(t)$ and it is usually denoted as:

$$
S_{xx}(\Omega) = |X_a(j\Omega)|?2
\tag{11}
$$

The energy over a specified range of frequencies $\Omega_a \leq \Omega \leq \Omega_b$ can be computed using:

$$
E_{x,r} = \frac{1}{2\pi} \int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega) \mathrm{d}\Omega
\tag{12}
$$

0.1.2 Band-limited continuous-time signals

A full-band, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty \leq \Omega \leq \infty$. A band-limited continuous-time signal has a spectrum that is limited to a portion of the frequency range $-\infty \leq \Omega \leq \infty$. An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \leq |\Omega| \leq \Omega_b$ can be computed using:

$$
X_a(j\Omega) = \begin{cases} 0 & 0 \leq |\Omega| < \Omega_a \\ 0 & \Omega_b < |\Omega| < \infty \end{cases}
\tag{13}
$$

However, an ideal band-limited signal cannot be generated in practice.
Band-limited signals are classified according to the frequency range where most of the signal's is concentrated:

- a lowpass, continuous-time signal has a spectrum occupying the frequency range $0 < |\Omega| \leq \Omega_p < \infty$, where Ω_p is called the bandwidth of the signal;

- a highpass, continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_p \leq |\Omega| < \infty$, where the bandwidth of the signal is from Ω_p to ∞;

- a bandpass, continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_L \leq |\Omega| \leq \Omega_H < \infty$, where $\Omega_H - \Omega_L$ is the bandwidth.

0.1.3 Discrete-time fourier transform

Let us introduce the definition of this concept.

> **Definition 3: Discrete-time Fourier transform**
>
> The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by:
>
> $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{14}$$
>
> where in general $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as:
>
> $$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega}) \tag{15}$$

$X_{\text{re}}(e^{j\omega})$ and $X_{\text{im}}(e^{j\omega})$ are respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of ω. $X(e^{j\omega})$ can alternately be expressed as:

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right|e^{j\theta(\omega)} \tag{16}$$

where $\theta(\omega) = \arg X(e^{j\omega})$. $\left|X(e^{j\omega})\right|$ and $\arg X(e^{j\omega})$ are called respectively magnitude function and phase function. Both quantities are again real functions of ω. In many applications, the DTFT is called the Fourier spectrum. Likewise, $\left|X(e^{j\omega})\right|$ and $\theta(\omega)$ are called respectively the magnitude and phase spectra.
For a real sequence $x[n]$, $\left|X(e^{j\omega})\right|$ and $X_{\text{re}}(e^{j\omega})$ are even functions of ω, whereas, $\theta(\omega)$ and $X_{\text{im}}(e^{j\omega})$ are odd functions of ω. Note also that $X(e^{j\omega}) = \left|X(e^{j\omega})\right|e^{j\theta(\omega+2\pi k)} = \left|X(e^{j\omega})\right|e^{j\theta(\omega)}$ for any integer k. The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT. Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the range of values $-\pi \leq \theta(\omega) < \pi$, called the principal value.

> **Example 1: DTFT of the unit sample sequence**
>
> The DTFT of the unit sample sequence $\delta[n]$ is given by:
>
> $$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \delta[0] = 1 \tag{17}$$
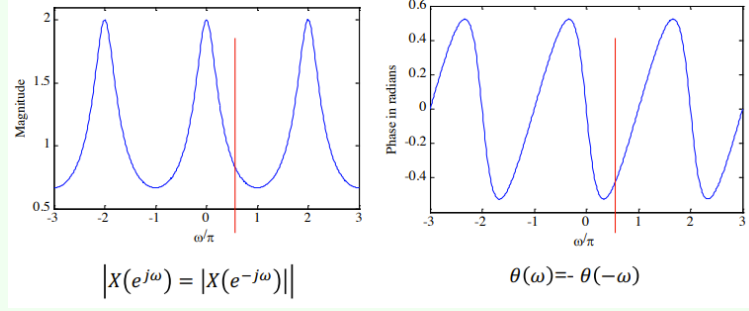
> **Example 2: DTFT of a causal sequence**
>
> Consider the causal sequence:
>
> $$x[n] = \alpha^n \mu[n], \qquad |\alpha| < 1 \tag{18}$$
>
> Its DTFT is given by:
>
> $$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n \mu[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n \\ &= \frac{1}{1-\alpha e^{-j\omega}} \end{aligned} \tag{19}$$

as $\left|\alpha e^{-j\omega}\right| = |\alpha| < 1$. If we take for example $\alpha = 0.5$, we get the plot below for the magnitude and phase of the DTFT.

$$\left|X(e^{j\omega}) = |X(e^{-j\omega})|\right| \qquad\qquad \theta(\omega) = -\,\theta(-\omega)$$

The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω. It is also a periodic function of ω with a period 2π:

$$X\left(e^{j(\omega+2\pi k)}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega}) \tag{20}$$

Therefore:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{21}$$

represents the Fourier series representation of the periodic function. As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega \tag{22}$$

Proof. Consider:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega \ell}\right) e^{j\omega n}\mathrm{d}\omega \tag{23}$$

The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e. $X(e^{j\omega})$ exists. Then:

$$
\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega \ell}\right) e^{j\omega n}\mathrm{d}\omega &= \sum_{\ell=-\infty}^{\infty} x[\ell]\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-\ell)}\mathrm{d}\omega\right) \\
&= \sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin\left(\pi(n-\ell)\right)}{\pi(n-\ell)} \\
&= \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell] \\
&= x[n]
\end{aligned} \tag{24}
$$

For the convergence condition, an infinite series of the form:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{25}$$

may or may not converg. Therefore, let us consider:

$$X_k(e^{j\omega}) = \sum_{n=-k}^{k} x[n]e^{-j\omega n} \tag{26}$$

Then for uniform convergence of $X_k(e^{j\omega})$:

$$\lim_{k \to \infty} X_k(e^{j\omega}) = X(e^{j\omega}) \tag{27}$$

Now, if $x[n]$ is an absolutely summable sequence, i.e., if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, then:

$$\left| X(e^{j\omega}) \right| = \left| \sum_{n=-k}^{k} x[n]e^{-j\omega n} \right| \leq \sum_{n=-k}^{k} |x[n]| < \infty \tag{28}$$

for all values of ω. Thus, the absolute summability of $x[n]$ is a sufficient condition for the existence of the DTFT $X(e^{j\omega})$. ∎

Example 3: Absolute summability condition

he sequence $x[n] = \alpha^n \mu[n]$ for $|\alpha| < 1$ is absolutely summable as:

$$\sum_{n=-k}^{k} |\alpha^n| \mu[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1 - |\alpha|} < \infty \tag{29}$$

and its DTFT $X(e^{j\omega})$ therefore converges to $\frac{1}{1 - \alpha e^{j\omega}}$ uniformly.

Note that since:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left(\sum_{n=-\infty}^{\infty} |x[n]| \right)^2 \tag{30}$$

an absolutely summable sequence has always a finite energy. However, a finite-energy sequence is not necessarily absolutely summable.