0.0.1 Irreducible representations of $\mathcal{P}_+^\uparrow$

For physical applications, we are interested in those irreducible representations in which the operators P_μ and $M_{\mu\nu}$ are hermitian, since they correspond to dynamical variables, i.e. in the unitary and hence infinite-dimensional irreducible representations of the Poincaré group $\mathcal{P}$, with a particular concern on the restricted group $\mathcal{P}_+^\uparrow$.
We consider now a state $|j,m\rangle$ and we apply the operators P^μ and P^2 to it:

$$P^\mu |j,m\rangle = p^\mu |j,m\rangle \tag{1a}$$
$$P^2 |j,m\rangle = M^2 |j,m\rangle \tag{1b}$$

where p^μ and M^2 are the corresponding eigenvalues. In particular, M^2 is related to the mass of a particle. There are two cases to analyze:

- $\boldsymbol{p^2 = m^2 > 0}$:
  In this case $\frac{p_0}{|p_0|}$, namely the sign of hte energy, is also an invariant of the group $\mathcal{P}_+^\uparrow$. In the four-momentum space, the eigenstates of $p^2 = m^2 > 0$ with $p_0 > 0$, which correspond to physical states, are represented by the points in the upper branch of the hyperboloid in fig. 1. Under a

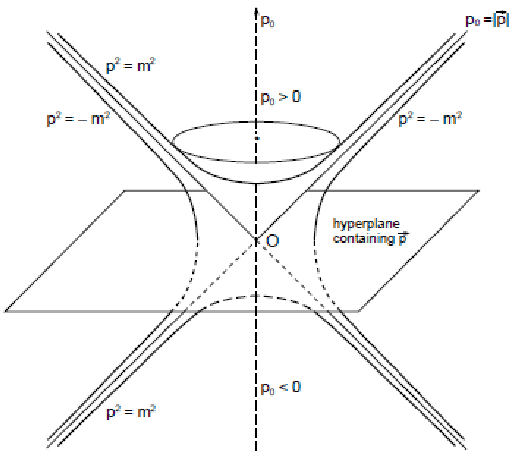

Figure 1: Hyperboloid $p^2 = m^2$ in the four-momentum space.

transformation of $\mathcal{P}_+^\uparrow$ the representative point moves on the same branch of the hyperboloid.
The physical meaning of W_μ is made clear by considering its components:

$$W_0 = \vec{\mathbf{P}} \cdot \vec{\mathbf{J}} \tag{2a}$$
$$\vec{\mathbf{W}} = P_0 \vec{\mathbf{J}} - \vec{\mathbf{P}} \times \vec{\mathbf{K}} \tag{2b}$$

It is convenient to go to a special frame, i.e. the rest frame where $\vec{\mathbf{p}} = 0$. We have:

$$W^\mu = m(0, J^1, J^2, J^3) = m(0, \vec{\mathbf{J}})$$

so W^μ reduces to the components of the total angular momentum $\vec{\mathbf{J}}$, which is the spin in case of a particle. By computing W^2:

$$W^2 = W^\mu W_\mu = -m^2 \vec{\mathbf{J}} \cdot \vec{\mathbf{J}} = -m^2 \vec{\mathbf{J}}^2$$

Taking a state $|p, j_3\rangle$, we have:

$$W_3 |p, j_3\rangle = m j_3 |p, j_3\rangle \tag{3a}$$
$$W^2 |p, j_3\rangle = -m^2 j(j+1) |p, j_3\rangle \tag{3b}$$

If the physical system is a particle, m is the mass of the paticle, which will be identified by m^2, j, j_3.

- **$p^2 = 0 \Longrightarrow$ Massless case**

  There isn't a rest frame of reference, but there is a special one such that $p^\mu = (\overbrace{p_0}^{\omega_0}, 0, 0, \overbrace{p_0}^{\omega_0})$ and $p^2 = 0$. Since we are interested in the physical case of a massless particle, we take also $W^2 = 0$, so $W^\mu = (W^0, W^1,)$