In $\mathcal{L}$ there are 4 discrete symmetries/transformations:

- Trivial: $\mathbb{1}$
- Space inversion I_S such as: $I_S x^\mu = g_{\mu\nu} x^\mu = x_\mu = (x_0, -\vec{\mathbf{x}})$
- Time inversion I_t: $I_t x^\mu = -g_{\mu\nu} x^\nu = (-x_0, \vec{\mathbf{x}})$
- Space-time inversion I_{st}: $I_{st} = I_s I_t = I_t I_s \qquad I_{st} x^\mu = -x^\mu$

We can define $\mathcal{I} = \{\mathbb{1}, I_s, I_t, I_{st}\}$, which is a non-invariant subgroup. It's abelian though, and not connected. So, the $\mathcal{L}$ components can be summarized in a table such as:

$\mathcal{L}$ components	$\det(\Lambda)$	$\Lambda^0{}_0$	Discrete transformation
$\mathcal{L}_+^\uparrow$	1	≥ 1	$\mathbb{1}$
$\mathcal{L}_-^\uparrow$	-1	≥ 1	$I_s = g$
$\mathcal{L}_+^\downarrow$	1	≤ -1	$I_t = -g$
$\mathcal{L}_-^\downarrow$	-1	≤ -1	$I_{st} = -\mathbb{1}$

Table 1: Components of $\mathcal{L}$

$\mathcal{L}_+^\uparrow$ group contains two types of continuous transformations:

- **Space rotations**, which depend on 3 d.o.f., so on the three angles of rotation:

$$R = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R_{ij} & \\ 0 & & & \end{array}\right)$$

- **Pure Lorentz transformations (boosts)**. For example:

$$\left.\begin{aligned} t' &= \gamma\left(t - \tfrac{v}{c^2}x\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \iff x'^\mu \equiv L_1{}^\mu{}_\nu x^\nu \qquad L_1 = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. We can also consider $\gamma = \cosh\psi$ in order to have a rotation of a complex angle $\theta = i\psi$.

- **Generic boost**:

$$L^\mu{}_\nu = \left(\begin{array}{c|c} \gamma & \beta_j\gamma \\ \hline -\beta_i\gamma & \delta^i{}_j - \frac{\beta^i\beta_j}{\beta^2}(\gamma - 1) \end{array}\right)$$

  So $\mathcal{L}_+^\uparrow$ depends on 6 d.o.f.. Moreover, $\mathcal{L}_+^\uparrow \ni \Lambda$, we have: $\Lambda = LR$.

We are going to enstablish a connection between $SO(1,3)$ and $SL(2,\mathbb{C})$. So we want to find a homomorphism $SO(1,3) \longrightarrow SL(2,\mathbb{C})$:

$$x^\mu = (x^0, x^1, x^2, x^3) \longrightarrow X = \sigma_\mu x^\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \qquad \begin{aligned} \sigma_\mu &\equiv (\mathbb{1}, \vec{\sigma}) \\ \sigma^\mu &\equiv (\mathbb{1}, -\vec{\sigma}) \end{aligned}$$

We get the following results from trace computing:

$$\operatorname{tr}(\sigma^\mu\sigma_\nu) = 2g_{\mu\nu}$$
$$\operatorname{tr}(\sigma_i\sigma_j) = 2\delta_{ij}$$

Take now $A \in SL(2, \mathbb{C})$:

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \begin{array}{c} \alpha, \beta, \gamma, \delta \in \mathbb{C} \\ \det(\Lambda) = 1 \Longrightarrow \alpha\delta - \beta\gamma = 1 \end{array}$$

If we transform the matrix X trough the matrix A, we get:

$$X' = AXA^\dagger \Longrightarrow \Lambda^\mu{}_\nu = \frac{1}{2}\,\mathrm{tr}\Big(\sigma^\mu A \sigma_\nu A^\dagger\Big) \tag{1}$$

What we have found is that through the homomorphism we have a 2-to-1 correspondence between $SL(2, \mathbb{C})$ and $\mathcal{L}_+^\uparrow$, in fact A and $-A$ are associated to Λ.
Concerning the Lie algebra of the group, there are 6 generators:

- **Rotations**:

$$J_k \equiv i\,\frac{\partial R_k}{\partial \varphi}\bigg|_{\varphi=0} \qquad R = \overbrace{\mathbb{1} - i\delta\varphi\vec{\mathbf{J}}\cdot\vec{\mathbf{n}}}^{\text{Infinitesimal rotation}} = \overbrace{e^{-i\varphi\vec{\mathbf{J}}\cdot\vec{\mathbf{n}}}}^{\text{Finite rotation}} \tag{2}$$

- **Boosts**:

$$K_l \equiv i\,\frac{\partial L_l}{\partial \psi}\bigg|_{\psi=0} \qquad L = \overbrace{\mathbb{1} - i\delta\psi\vec{\mathbf{k}}\cdot\vec{\nu}}^{\text{Infinitesimal boost}} = \overbrace{e^{-i\varphi\vec{\mathbf{k}}\cdot\vec{\nu}}}^{\text{Finite boost}} \tag{3}$$

The commutators are:

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \tag{4a}$$
$$[K_i, K_j] = (-1)\varepsilon_{ijk}J_k \tag{4b}$$
$$[J_i, K_j] = i\varepsilon_{ijk}K_k \tag{4c}$$

We can construct the antisymmetric tensor $M_{\mu\nu}$ with the previous generators:

$$M_{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix} \tag{5}$$

where $J_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}$ and $K_i = M_{0i}$. Concerning the Lie algebra of the group:

$$[M_{\mu\nu}, M_{\lambda\rho}] = i(g_{\lambda\mu}M_{\rho\nu} + g_{\rho\nu}M_{\lambda\mu} - g_{\lambda\nu}M_{\rho\mu} - g_{\rho\mu}M_{\lambda\nu}) \tag{6}$$

We can write in general Λ thorugh the antisymmetric tensor (whose elements are real) $\omega_{\mu\nu}$:

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

An infinitesimal transformation can be written as:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

$$\Lambda^T g \Lambda = g \;\; : \qquad (\delta^\mu{}_\rho + \omega^\mu{}_\rho)g_{\mu\nu}(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) = g_{\rho\sigma} \Longrightarrow g_{\mu\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} = g_{\rho\sigma}$$

Considering that the tensor $\omega_{\mu\nu}$ is antisymmetric, it holds:

$$\omega_{\mu\nu} = \frac{1}{2}(\omega_{\mu\nu} - \omega_{\nu\mu}) = \frac{1}{2}\omega_{\alpha\beta}(g^\alpha{}_\mu g^\beta{}_\nu - g^\alpha{}_\nu g^\beta{}_\mu) \equiv -\frac{1}{2}i\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu\nu}$$

$$(M^{\alpha\beta})_{\mu\nu} = i(g^\alpha{}_\mu g^\beta{}_\nu - g^\alpha{}_\nu g^\beta{}_\mu) \tag{7}$$

So we can write again the tensor Λ:

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \frac{i}{2}\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu\nu} \Longrightarrow \Lambda = \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta} \longrightarrow e^{-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}}$$

Now we focu our attention on irreducible representations, in particular, we consider Casimir operators:

$$C_1 = \frac{1}{2}M^{\mu\nu}M_{\mu\nu} = \vec{\mathbf{J}}^2 - \vec{\mathbf{K}}^2 \tag{8a}$$

$$C_2 = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}M_{\mu\nu}M_{\rho\sigma} = -\vec{\mathbf{J}}\cdot\vec{\mathbf{K}} \tag{8b}$$

Alternatively:

$$\left.\begin{aligned} M_i &\equiv \tfrac{1}{2}(J_i + iK_i) \equiv J^+ \\ N_i &\equiv \tfrac{1}{2}(J_i - iK_i) \equiv J^- \end{aligned}\right\} \Longrightarrow \begin{aligned} [M_i, N_j] &= 0 \\ [M_i, M_j] &= i\varepsilon_{ijk}M_k \\ [N_i, N_j] &= i\varepsilon_{ijk}N_k \\ [M^2, M_i] &= 0 \\ [N^2, N_i] &= 0 \end{aligned}$$

Concerning the algebra:

$$so(1,3) \simeq su(2) \oplus su(2)$$
$$SO(1,3) \simeq SU(2) \otimes SU(2)$$

In terms of representations and noticing that $SO(3) \sim SU(2) \longrightarrow D^{(j)}$:

$$D^{(M,N)} \equiv D^{(M,0)} \oplus D^{(0,N)}$$

So, starting from $D^{(\frac{1}{2},\frac{1}{2})} \equiv D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$:

$$D^{(j_1,j_2)} \otimes D^{(j_1',j_2')} = D^{(j_1+j_1',j_2+j_2')} \oplus D^{(j_1+j_1',j_2+j_2'-1)} \oplus \dots D^{(|j_1-j_1'|,|j_2-j_2'|)}$$

Let's consider $D^{(M,N)}$:

(M,N)	dim	Field (particle)
$(0,0)$	1	Scalar
$(\frac{1}{2},0)$	2	Left handed spinor
$(0,\frac{1}{2})$	2	Right handed spinor
$(\frac{1}{2},\frac{1}{2})$	4	Vector
$(1,0)$	3	Self dual (2-form) field
$(0,1)$	3	Anti-self dual (2-form) field
$(1,1)$	9	Traceless symmetric tensor field

Table 2: Examples

0.1 Poincaré transformations $\mathcal{P} \ni (a, \Lambda)$

The inhomogeneous Lorentz transformations, or Poincaré transformations, connect the space-time coordinates of any two frames of reference whose relative velocity is constant. In general, a Poincaré transformation can be written as a generalization of a Lorentz transformation in the form:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu} \tag{9}$$

where a^μ stand for the components of a vector in $\mathbb{R}^4$. We will denote this kind of transformation by (a, Λ). They form a group.
In general, an infinitesimal translation is given by:

$$(\delta a, \mathbb{1}) = \mathbb{1} - i\delta a_\mu P^\mu \tag{10}$$

with the four operators P^μ the infinitesimal generators of the translations. A finite translation instead is given by the exponentiation:

$$(a, \mathbb{1}) = e^{-ia_\mu P^\mu} \tag{11}$$

In order to get the commutation relations of the infinitesimal generators, it is convenient to use a 5×5 formalism. So we write (a, Λ) as a 5×5 matrix:

$$(a, \Lambda) = \left(\begin{array}{c|c} \Lambda & a \\ \hline 0 & 1 \end{array}\right) \qquad y^\mu = \begin{pmatrix} x^\mu \\ 1 \end{pmatrix}$$

Therefore, we can rewrite the transformation:

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda x + a \\ 1 \end{pmatrix} \tag{12}$$

Expressing in a similar way the infinitesimal transformation in eq. 10, one can obtain explicitly the four generators:

$$\begin{gathered} P_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{gathered} \tag{13}$$

In the same representation, the generators J_i and K_i are replaced by 5×5 matrices, that can be obtained by the combination of two block: the first one is $(J_i)_{4\times4}$ or $(K_i)_{4\times4}$, the second one is 1.
Concerning the Lie algebra, the following commutation relations hold:

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\lambda\rho}, M_{\mu\nu}] &= -i(g_{\lambda\mu}M_{\rho\nu} + g_{\rho\nu}M_{\lambda\mu} - g_{\lambda\nu}M_{\rho\mu} - g_{\rho\mu}M_{\lambda\nu}) \\ [M_{\mu\nu}, P_\rho] &= -i(g_{\mu\nu}P_\rho - g_{\mu\rho}P_\mu) \end{aligned} \tag{14}$$

The subgroup $\mathcal{P}_+^\uparrow$ has $10 = 6 + 4$ generators.
Now, we introduce the Casimir operators C_1 and C_2, and W_μ, namely **Pauli-Lubarsky tensor**:

$$C_1 = P^2 = P^\mu P_\mu \tag{15}$$

$$C_2 = W^2 = W_\mu W^\mu \tag{16}$$

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\sigma\tau}M^{\nu\sigma}P^\tau \tag{17}$$

It follows immediately that:

- $W_\mu P^\mu = 0 = \varepsilon_{\mu\nu\rho\tau}M^{\nu\sigma}P^\tau P^\mu$
- $[P_\mu, W_\mu] = 0$
  $[M_{\mu\nu}, W_\sigma] = -i(g_{\nu\sigma}W_\mu - g_{\mu\sigma}W_\nu)$
  $[W_\mu, W_\nu] = i\varepsilon_{\mu\nu\sigma\tau}W^\sigma P^\tau$