Example 1. Examples of Lie groups that we are going to use and their dimension:

- $\mathrm{GL}(N,\mathbb{C})$
  $M \in \mathrm{GL}(N,\mathbb{C}),\ \det\{M\} \neq 0$ (M: $N \times N$ complex matrices)
  $n \equiv \dim \mathrm{GL}(N,\mathbb{C}) = \#\text{Free parameters} = 2N^2$

- $\mathrm{SL}(N,\mathbb{C}) \subseteq \mathrm{GL}(N,\mathbb{C}),\ \det\{M\} = 1$
  $\dim \mathrm{SL}(N,\mathbb{C}) \equiv 2N^2 - 1$

- $\mathrm{GL}(N,\mathbb{R})$, $n = N^2$

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- $\mathrm{SL}(N,\mathbb{R})$, $n = N^2 - 1$

- $\mathrm{U}(N)$, $n = N^2$

- $\mathrm{SU}(N)$, $n = N^2 - 1$

- $\mathrm{O}(N)$, $n = \frac{1}{2}N(N-1)$

- $\mathrm{SO}(N)$, $n = \frac{1}{2}N(N-1)$

- $U(l, N-l)$: **Complex unitary matrices** such that

$$g = \left(\begin{array}{c|c} \mathbb{1}_{l\times l} & \\ \hline & \mathbb{1}_{(N-l)\times(N-l)} \end{array}\right) \qquad \begin{array}{l} UgU^\dagger = g \qquad n = N^2 \\ g_{kk} = \begin{cases} 1 & 1 \leq k \leq l \\ -1 & l < k \leq N \end{cases} \end{array}$$

- $O(l, N-l)$: **Pseudo-orthogonal group**:

$$OgO^T = g \qquad n = \frac{1}{2}N(N-1)$$

  Example: Lorentz group $O(1,3)$.

0.1 Rotation group: $O(3)$ and $SU(2)$

A spatial rotation in 3 dimensions can be described by the transformation:

$$F' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R\vec{r} \qquad r'_i = R_{ij} r_j \tag{1}$$

The R matrix must preserve the distance from the origin, so the lenght of the vectors:

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$$

$$(\vec{r}')^T \vec{r}' = (\vec{r})^T \vec{r} \iff \vec{r}^T R^T R \vec{r} = \vec{r}^T \vec{r} \Longrightarrow R^T R = \mathbb{1}_{3\times 3}$$

$R \in O(3)$ and $O(3)$ is a group. In fact:

1. $(R_1 R_2)^T R_1 R_2 = R_2^T R_1^T R_1 R_2 = \mathbb{1}$

2. $e = \mathbb{1}$

3. Inverse element: R^{-1}, $\det\{R\} \neq 0$

0.1.1 Fundamental representation

It is given by the matrices acting on $\mathbb{R}^3 = \mathbb{V}$. So a rotation around z-axis tansforms $\vec{v}$ as:

$$\vec{v}' = \begin{pmatrix} v'_x \\ v'_y \\ v'_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$\vec{v}' = R_z(\theta)\vec{v}$$

Analogously, the rotation matrices around x and y axes are:

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \qquad R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}$$

It is important to remark that $O(3)$ is a non-abelian Lie group, in fact in general:

$$R_x(\phi)R_z(\theta) \neq R_z(\theta)R_x(\phi)$$

0.1.2 Generators of the group

By the application of the definition of generator, we obtain:

$$J_x \equiv J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J_y \equiv J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad J_z \equiv J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We note that $J_i, i = 1, 2, 3$ is Hermitian, so $J_i^\dagger = J_i$. The commutator of the generators is:

$$[J_i, J_j] = i\varepsilon_{ijk}J_k = if_{ijk}J_k \tag{2}$$

where ε_{ijk} is the Levi-Civita pseudotensor, defined as:

$$\varepsilon_{ijk} = \begin{cases} 1 & ijk = \text{Even permutation of } 123 \\ -1 & ijk = \text{Odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

0.1.3 Finite and infinitesimal rotations

An infinitesimal rotation is described by the transformation through a rotation matrix by an infinitesimal angle $\delta\theta$. By expanding in Taylor series:

$$R_z(\delta\theta) = \mathbb{1} + i\delta\theta J_z \tag{4}$$

If we want to pass to a finite transformation, we have to consider the exponential representation:

$$R_z(\theta) = \mathrm{e}^{i\theta J_z} \tag{5}$$

For a general rotation around an axis $\vec{n} = (n_x, n_y, n_z)$:

$$R_{\vec{n}}(\theta) = \mathrm{e}^{i\vec{\sigma}\cdot\vec{J}} = \mathrm{e}^{i\theta\vec{n}\cdot\vec{J}} \tag{6}$$

0.1.4 Special Unitary 2 × 2 matrices

Now we move to the group $SO(2)$, namely the group of special unitary 2×2 matrices ($UU^\dagger = U^\dagger U = \mathbb{1}$, $(U^\dagger) = (U^{-1})$, with $\det\{U\} = 1$). We can represent its element with:

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad a, b \text{ complex numbers such that } |a|^2 + |b|^2 = 1 \Longrightarrow n = 3 \text{ dof} \tag{7}$$

Thr generators of the group are denoted by $\Sigma^a = \frac{1}{2}\sigma^a$, where σ^a are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They are Hermitian and

$$[\Sigma^a, \Sigma^b] = i\varepsilon_{abc}\Sigma^c$$

f^{abc} is the same of $O(3)$ and there is correspondence between the matrices U and R. Therefore, we can make an homomorphism $O(3)$ versus $SU(2)$:

$$\vec{\mathbf{r}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto h = \vec{\sigma} \cdot \vec{\mathbf{r}} = x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

We know that $r'_i = R_{ij}r_j$ and a matrix in $SU(2)$ $h' = UhU^\dagger$, therefore we can decompose these matrices as:

$$h' = UhU^\dagger = Ur_j\sigma_jU^\dagger = r_jU\sigma_jU^\dagger$$

$$\begin{aligned} h' = UhU^\dagger &= Ur_j\sigma_jU^\dagger = r_jU\sigma_jU^\dagger \\ &= \vec{\mathbf{r}}' \cdot \vec{\sigma} = r'_k\sigma_k = R_{kj}r_j\sigma_k \end{aligned}$$

$$\begin{aligned} \Longrightarrow r_jU\sigma_jU^\dagger &= R_{kj}r_j\sigma_k \\ \sigma_iR_{kj}\sigma_k &= \sigma_iU\sigma_jU^\dagger \\ R_{jk}\sigma_i\sigma_k &= \sigma_iU\sigma_jU^\dagger \end{aligned}$$

Now we use the property $\mathrm{tr}(\sigma_i\sigma_k) = 2\delta_{ik}$ to obtain a correspondence between generators:

$$R_{ij} = \frac{1}{2}\,\mathrm{tr}\Big(\sigma_iU\sigma_jU^\dagger\Big) \tag{8}$$

Both $U, -U \longmapsto \mathbb{R}$, so the correspondence is 2-to-1 and that's why it's not a isomorphism. Only $SU(2)/\mathbb{Z}_2 \simeq O(3)$ is an isomorphism (check on Costa-Fogli). We can use their common algebra to identify their irreducible representations. Let's take the angular momentum representations, their are characterized by:

$$\begin{aligned} J^2\,|j, m\rangle &= j(j+1)\,|j, m\rangle \\ J_z\,|j, m\rangle &= m\,|j, m\rangle \end{aligned}$$

$\forall j$ there are $2j + 1$ states, with $-j \leq m \leq j$
Additional representation is equivalent to the composition of angular momenta:

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |j_1, m_1, j_2, m_2\rangle$$

So:

$$J_1 \otimes J_2 = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} J$$

and in this case we get the irreducible representation (they are already block diagonal):

$$D^{j_1 j_2}(R) \equiv D(R)^{j_1} \otimes D(R)^{j_2} = D(R)^{j_1+j_2} \oplus D(R)^{j_1+j_2-1} \oplus \cdots \oplus D(R)^{|j_1-j_2|}$$

and we can decompose through the Clebsh-Gordan coefficients and we have to remember that j can be either integer or non-integer.

$$|j_1, m_1, j_2, m_2\rangle = \sum_{|j_1-j_2| \leq j \leq j_1+j_2} |J, M\rangle \langle J, M| j_1, m_1, j_2, m_2\rangle$$

0.1.5 3-D representation of $SU(2)$: adjoint representation

Starting from the algebra and the commutators relations:

$$[\Sigma^i, \Sigma^j] = i\varepsilon_{ijk}\Sigma^k$$

$$(\mathbb{T}^i)_{jk} \equiv i\varepsilon_{ijk}$$

$$\mathbb{T}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \mathbb{T}^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad \mathbb{T}^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

They correspond to the J_i of $O(3)$ (stranger link). The fundamental representation is 2×2, this one (i.e. the adjoint) is 3×3.

0.2 (Homogeneous) Lorentz group $\mathcal{L}$

Consider $\Lambda \in O(1,3)$. They are the relativistic transformations between inertial frame of reference. If $\mathbb{V}$ is the Minkowski space, then:

$$x'^\mu = \Lambda^\mu_\nu x^\nu = \begin{cases} \text{controvariant} & x^\mu = (x^0, x^1, x^2, x^3) = (ct, \vec{\mathbf{x}}) \\ \text{covariant} & x_\mu = (x_0, -x_1, -x_2, -x_3) = (ct, -\vec{\mathbf{x}}) \end{cases} \tag{9}$$

We want them to preserve the lenght on Minkowski space, i.e. $x'^2 = x^2$, where $x^2 = x^\mu x_\mu = g_{\mu\nu}x^\mu x^\nu$ and $g^\mu{}_\nu = \delta^\mu\ \nu$. The invariance of the norm implies:

$$g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho Lambda^\nu{}_\sigma x^\rho x^\sigma = g_{\rho\sigma}x^\rho x^\sigma$$

$$[1] \qquad \Lambda^\mu{}_\rho g_{\mu\nu}\Lambda^\nu{}_\sigma = g_{\rho\sigma} \iff \Lambda^T g \Lambda = g \qquad [2]$$

Taking the determinant of the equation [2], we get:

$$\det\big(\Lambda^T g \Lambda\big) = \det(g) \Longrightarrow \det\big(\Lambda^T\big)\det(\Lambda) = 1 \qquad (\det(\Lambda))^2 = 1$$

So there is degeneration: $\det(\Lambda) = \pm 1$. If we take equation [2] with $\rho = \sigma = 0$, we get:

$$(\Lambda^0{}_0)^2 - \sum (\Lambda^i{}_0)^2 = g_{00} = 1 \Longrightarrow (\Lambda^0{}_0)^2 \geq 1 \Longrightarrow \begin{cases} \Lambda^0{}_0 \geq 1 \\ \Lambda^0{}_0 \leq -1 \end{cases}$$

Therefore there are 4 (disjoint) disconnected subsets.
We show now that $\mathcal{L}$ is a group by verifing the three properties:

- $\Lambda_1, \Lambda_2 \in \mathcal{L} \Longrightarrow \Lambda_3 = \Lambda_1\Lambda_2 \in \mathcal{L}$:
  $(\Lambda_1\Lambda_2)^T g (\Lambda_1\Lambda_2) = \Lambda_2^t \Lambda_1^T g \Lambda_1 \Lambda_2 = g$

- $e = \mathbb{1}_{4\times 4} \in \mathcal{L} \qquad \mathbb{1}^T g \mathbb{1} = g$
- Inverse: Λ^{-1}
  $\Lambda^{-1\ T} g \Lambda^{-1} = g \iff \Lambda^{\mu}{}_{\alpha} g_{\mu\nu} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta} \qquad [3]$
  $(\Lambda^{-1})^T \Lambda^T g \Lambda \Lambda^{-1} = (\Lambda^{-1})^T g \Lambda^{-1} \Longrightarrow \Lambda^{-1} \in \mathcal{L}$

In order to find the explicit form of Λ^{-1}, we have to rewrite [3] and remember that $g_{\mu\nu}\Lambda^{\mu}{}_{\alpha} = \Lambda_{\mu\alpha}$:

$$\begin{aligned}
g_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} &= g_{\alpha\beta} \\
\Lambda_{\nu\alpha}\Lambda^{\nu}{}_{\beta} &= g_{\alpha\beta} \\
\Lambda_{\nu}{}^{\alpha}\Lambda^{\nu}{}_{\beta} &= g^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} \Longrightarrow (\Lambda^{-1})^{\alpha}{}_{\nu} = \Lambda^{\nu}{}_{\beta}
\end{aligned}$$

$$\Lambda^{\mu}{}_{\nu} = \left(\begin{array}{c|ccc}
\Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \hline
\Lambda^1{}_0 & & & \\
\Lambda^2{}_0 & & \Lambda^i{}_j & \\
\Lambda^3{}_0 & & &
\end{array}\right) \longrightarrow (\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu} = \left(\begin{array}{c|ccc}
\Lambda^0{}_0 & -\Lambda^1{}_0 & -\Lambda^2{}_0 & -\Lambda^3{}_0 \\ \hline
-\Lambda^0{}_1 & & & \\
-\Lambda^0{}_2 & & (\Lambda^i{}_j)^T & \\
-\Lambda^0{}_3 & & &
\end{array}\right)$$